Generic points and multifractal analysis

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Abstract: Let (X,T) be a dynamical system, where X is a compact metric space and $T\colon X \to X$ a map. Questions of interest for physical systems are related to the long time behavior of orbits $T^n x$, when $n \to \infty$, in particular, the long time behavior of the empirical measure

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x}\,.$$

A point x is *generic for a T-invariant measure* μ if $\lim_n \mathcal{E}_n(x) = \mu$ (weak topology), and G_μ is the set of all generic points of μ. From the Ergodic Theorem one gets

$$\mu(G_\mu) = \begin{cases} 1 & \text{if } \mu \text{ is ergodic} \\ 0 & \text{otherwise.} \end{cases}$$

Kakutani proved for product spaces:

If $X = \mathtt{A}^{\mathbf{N}}$, $\mathtt{A}$ a finite set, and T the left-shift, then $G_\mu \neq \emptyset$ for all T-invariant probability measures μ.

To measure the size of the set G_μ it is natural to consider its topological entropy, which is a dimension-like quantity. Inspired by ideas of Equilibrium Statistical Mechanics and Large Deviations Theory one computes $h_{\text{top}}(G_\mu)$, and as a by-product the topological entropy of level-sets of ergodic averages, for a large class of dynamical systems.