CORSO DI DOTTORATO DI RICERCA IN

MATEMATICA

XXXVII (37°) CICLO

Titolo della tesi

Moduli of Cubic fourfolds and reducible OADP surfaces

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To my Parents,

To all those who suffer in silence from invisible illnesses,

"I wanted only to live in accord with the promptings which came from my true self.
Why was that so very difficult?" Demian
Hermann Hesse

Abstract

In this thesis, we explore the intersection of the Hassett divisor $\mathcal{C}_8$, parametrizing smooth cubic fourfolds X containing a plane P with other divisors $\mathcal{C}_i$. Notably we study the irreducible components of the intersections with $\mathcal{C}_{12}$ and $\mathcal{C}_{20}$. These two divisors generically parametrize cubics containing a smooth cubic scroll, and cubics containing a smooth Veronese surface respectively. First, we find all the irreducible components of the two intersections, and describe the geometry of the generic elements in terms of the intersection of P with the other surface. Then we consider the problem of rationality of cubics in these components, either by finding rational sections of the quadric fibration induced by projection from P, or by finding examples of reducible one-apparent-double-point surfaces inside X. Finally, via some Macaulay computations, we give explicit equations for cubics in each component.

Acknowledgments

This work was carried out under the supervision of Prof. Alessandro Verra and Prof. Michele Bolognesi. I am deeply grateful to them for their generosity, patience, and availability, as well as for all the insightful guidance they have provided throughout this work.

Prof. Verra taught several courses at Roma Tre, including one on Cremona transformations and another on K3 and Enriques surfaces. These courses confirmed my desire to pursue a PhD in birational geometry, as Prof. Verra has a unique way of presenting lectures, always emphasizing the geometric intuition behind each concept in algebraic geometry. I am profoundly grateful for the opportunity to work under his supervision and for the numerous discussions we shared, both on campus and online. His dedication, humanity, and passion for teaching mathematics have been truly inspiring. Even after his retirement, he continued to supervise three students simultaneously—myself, Elena Sammarco, and Federico Pieroni—and he still teaches at Roma Tre to this day.

I first contacted Prof. Michele Bolognesi on October 22, 2020. He immediately proposed a meeting during which he introduced me to algebraic geometry and suggested I start reading the first chapters of "Harris' Algebraic Geometry: A First Course". Since then, Michele has consistently offered guidance, whether for PhD applications or for questions arising in my research and studies. He offered to be my Master's thesis advisor and suggested cubic fourfolds as a subject. Whenever I faced difficulties, he was always available to clarify mathematical concepts and recommend references. Beyond his academic expertise, it is his humanity that impresses me most, his patience, generosity and support, both within and beyond the realm of mathematics. Regarding my thesis, I am deeply grateful for all his ideas, which contributed directly to the two main results of this work, Thm. 5.2.1 and Thm. 5.3.1. He also carefully reviewed and commented on my manuscript multiple times, offering invaluable suggestions for improvement during all the writing process. Since 2020, he has continuously guided and supported me. For instance, he helped me with the administrative enrollment procedures at Roma Tre, and he is also the first one who advised me to prioritize my health during challenging periods in 2022.

I am also grateful to Hanine Awada, a former PhD student of Michele, who collaborated with us in developing these results. She introduced me to Macaulay2 [GS02] and provided numerous suggestions for drafting my thesis, especially at the very beginning when I faced the blank page. Their support has been essential throughout this journey. To both of these remarkable individuals, I wish to express my deepest gratitude.

Also, I would like to thank the anonymous reviewers and the members of the defense committee Prof. Andrea Bruno, Prof. Enrico Fatighenti, and Prof. Flaminio Flamini for their time, valuable comments, and constructive feedback, which contributed to improving this work.

I extend my gratitude to the University of Roma Tre for giving me the opportunity to pursue a PhD. Special thanks go to Prof. Angelo Felice Lopez and Prof. Alessandro Giuliani, the PhD coordinators at Roma Tre, for their guidance and help navigating administrative procedures within the university as a foreign student, and I sincerely appreciate all the support they provided throughout my PhD journey. I also warmly thank all the professors with whom I had the chance to discuss algebraic geometry and attend courses in Roma Tre: Prof. Lucia Caporaso, Prof. Margarida Melo, Prof. Margherita Lelli-Chiesa, Prof. Luca Schaffler, Prof. Amos Turchet, and all other professors working in algebraic geometry at the university. I truly enjoyed my time at Roma Tre, particularly the seminars, lunches at Gustami, and the conferences organized in Rome, most notably the one held in honor of Prof. Verra.

My sincere thanks go to all the PhD students at Roma Tre, especially during the first half of my PhD journey. Elena, Simone, Federico, Armando, Luca, Roberta, and Davide, among many others, made this period especially memorable. Elena and Simone were particularly warm and supportive, offering moral encouragement during difficult periods. Without their help, my studies would have been much more challenging. Elena was also my "thesis sister," since Prof. Verra accepted both of us under his supervision, and we shared the same thesis topic: cubic fourfolds. We had numerous discussions on this common subject, even though our specific research problems differed. I fondly remember a conference in Genova in January 2024 that we attended together. During this week, she summarized aspects of the Hodge theory of cubic fourfolds for me just as I was resuming my studies after a six-month break due to health issues. Together, all the PhD students shared countless discussions not only about algebraic geometry but also about life, everyday challenges, dreams, and hopes. We went out together frequently and had a lot of fun, creating memories I will always treasure. I am sincerely grateful for their warmth, hospitality, and friendship.

For the second half of my PhD journey, I am particularly grateful to Wei Chen, who joined the university later. He hosted me many times in Rome, attended several conferences with me around Europe, and provided continuous friendship and support. Wei was always available to discuss mathematics and gave invaluable advice on my thesis, reading it carefully and providing numerous suggestions.

Also, I would like to acknowledge the other PhD students who accompanied me along this journey: Myrla (now a postdoc in Brazil), Muhammad, Martina, Simon. Beyond Roma Tre, I also thank friends from Sapienza and Tor Vergata, including Wiktor, Davide, Nelson and Giusi.

Beyond Rome, I wish to thank Francesca Rizzo, a PhD student in Paris, for her help with moduli spaces of K3 surfaces, and Sasha Viktorova, a postdoc at Leuven, who reviewed my very first draft on the same topic and provided countless suggestions to improve it.

I would also like to acknowledge my friends who have supported me throughout this journey. Long-standing friends Chemoussa, Amina, Raouf, Ange, Anis, Abdellah, Salma, Massil, Tarek, Rougi, Nedal, Manil, Adlane, Neo and DjamelDine have been part of my life for over ten years. I would also like to thank the new friends I have made along the way: Mohand, Nassima, Léa, Younes, Sabine, Mamiil, and many others have enriched this experience. In particular, Massil Hihat assisted me multiple times with LaTeX and even helped me to fund my participation in the Summer Research Institute 2025 in Colorado. I am also deeply grateful to my girlfriend Caro, who has been an immense support in everyday life and encourages me in everything I do. Thank you all so much.

I also wish to acknowledge my Algerian colleagues and mathematician friends: Mohamed-Lamine Messaci, and Houssem Boukhecham. I am equally thankful to my mathematics teachers in Algeria from kindergarten, middle school, and high school: Madam Allam, Madam Abbas, and Madam Bouchefar. Also, my french teacher Madam Ait Belkacem, who always asks for news.

Nor should I forget the invaluable help of Madam Ducamp, for her steady support and understanding during challenging times.

Finally, I want to express my deepest gratitude to my family, whose support has made this journey possible. My grandparents, aunts, and uncles — in particular my aunt Nesrine, whose moral support has been immense. My uncle Mustapha, my uncle Lakhdar, and my aunt Milica have consistently called to check on us since the passing of our father in 2018. My cousin Farida and her family in Rome, who have been incredibly kind and welcomed me into their home several times.

Most importantly, I honor my father, whose extraordinary values, love of knowledge, and dedication to education have shaped who I am today. His guidance, though no longer physically present, continues to inspire me every day. I hope he is proud of me. My mother, who courageously assumed the role of both parents, has been a pillar of strength, providing both financial support and boundless moral encouragement. Her resilience, wisdom, and love have formed the foundation of all I have achieved.

I am equally grateful to my sister Minna and my brother Badis, whose constant support and companionship have been invaluable. They have been my closest allies and best friends throughout this journey. I feel truly fortunate to have such an extraordinary family, whose love, guidance, and example have made everything possible.

To all of you, I extend my heartfelt gratitude. I will carry with me not only the mathematics I have learned but also the friendships, memories, and kindness that have made this journey truly special.

Contents

We work over the field of complex numbers $\mathbb{C}$.

Chapter 1

Introduction

1.1 Cubic fourfolds and rationality

Cubic hypersurfaces in $\mathbb{P}^5$ are among the most studied, and at the same time the most mysterious objects in algebraic geometry. The reason for this is probably the wealth of geometry that they contain, and the fact that there are lots of very basic and classical problems that are still unresolved, for example the rationality of the generic cubic fourfold.
While there are examples of cubic fourfolds known to be rational, we expect that most cubic fourfolds should be irrational. The study of cubic fourfolds has known many recent developments, especially in the last few decades, supported by the Hodge theoretic approach of Hassett [Has99; Has00; HV16].
In this thesis, we work over the field of complex numbers $\mathbb{C}$. A smooth projective variety X_n of dimension n is said to be rational whenever there exists a birational map $X_n \overset{\sim}{\dashrightarrow} \mathbb{P}^n$ i.e. there exist two open subsets $U \subset X_n$ and $V \subset \mathbb{P}^n$ that are isomorphic $U \simeq V$.
The question of rationality has been studied for a long time by classical algebraic geometers. For example, all complex quadric hypersurfaces $Q \subset \mathbb{P}^{n+1}$ are known to be rational. Indeed, up to a projective transformation, we can assume that $p = [1 : 0 : ... : 0] \in Q$. The projection from $p \in Q$ gives a birational morphism into $\mathbb{P}^n$:

$$\pi_p : Q \setminus \{p\} \longrightarrow \mathbb{P}^n$$
$$[x_1 : .. : x_{n+1}] \mapsto [x_2 : ... : x_{n+1}]$$

The rationality of cubic fourfolds is still an open problem. This is not the case for cubic hypersurfaces of dimension less than 4:

Dim1 Smooth cubic curves (genus 1 curves) are irrational.

Dim2 Smooth cubic surfaces are rational (as long as they contain 2 disjoint lines, this is the case on $\mathbb{C}$) [Dol05].

Dim3 Cubic threefolds are irrational (Clemens and Griffiths 1972 [CG72]).

1.1.1 Cubic fourfolds containing a smooth quartic scroll

The first classical examples of rational cubic fourfolds appeared in the work of Morin [Mor40]. He mistakenly stated that a general cubic fourfold is rational. Morin claimed incorrectly that the family of quartic rational normal scrolls inside a general cubic fourfold has dimension 1. Based on this, he concluded that a general cubic must contain such a scroll, and is therefore rational. The work of Morin was revisited by Fano [Fan42], who studied cubic fourfolds containing a smooth quartic rational normal scroll and showed that they are rational. Fano also proved that generically these cubic fourfolds contain also a smooth quintic Del Pezzo surface. Finally, he corrected the statement of Morin. In fact, he proved that for each cubic fourfold X containing a quartic rational normal scroll, the closure of the family of smooth quartic normal scrolls inside X has dimension 2, instead of 1 (see Theorem 4.5.9). We can briefly explain the rationality argument here. The linear system of quadrics in $\mathbb{P}^5$ passing through a quartic rational normal scroll T defines a rational map $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$. When π is restricted to a cubic fourfold X containing T, it defines a birational map from the cubic fourfold onto a smooth quadric in $\mathbb{P}^5$ of dimension 4 [Tre84, Prop. 3]. Hence, X is rational (see Example 11).

The first rational examples of cubic fourfolds not containing a smooth quartic rational normal scroll were given by Tregub in [Tre93, Prop. 3]. The author showed that cubic fourfolds containing a Veronese surface and a plane intersecting in three points are rational. The locus of such cubics has codimension 2 inside the moduli space of cubic fourfolds, denoted by $\mathcal{C}$. We confirm this result in Theorem 5.3.1.

1.1.2 Cubic fourfolds containing two disjoint planes

Another family of classical rational examples consists of cubic fourfolds containing two disjoint planes [Tre93; Tre84]. We can briefly explain the rationality argument here.

Using the notation $\mathbb{P}^5 = \{[u : v : w : x : y : z]\}$, let $P_1 = \{u = v = w = 0\}$ and $P_2 = \{x = y = z = 0\}$ be two disjoint projective planes. Moreover, we define $F_1 = ux^2 + vy^2 + wz^2$ and $F_2 = u^2x + v^2y + w^2z$ to be two homogenous polynomials of degree 3. We have that $X = \{F_1 = F_2\}$ is a smooth cubic fourfold that contains the two planes P_1 and P_2.

The rationality of X is based on an intersection argument. Given $(p_1, p_2) \in P_1 \times P_2$, we have:

$$\langle p_1, p_2\rangle \cap X = \begin{cases} \{p_1, p_2, p_3\} & if \quad \langle p_1, p_2\rangle \not\subset X \\ \langle p_1, p_2\rangle & if \quad \langle p_1, p_2\rangle \subset X \end{cases}$$

where $\langle p_1, p_2\rangle$ is the projective line joining p_1 and p_2, and $\varphi(p_1, p_2) := p_3 \in X$ is the third point arising from the intersection of X with $\langle p_1, p_2\rangle$.

Setting $S = \{(p_1, p_2) \in P_1 \times P_2 \mid \langle p_1, p_2 \rangle \subset X\}$, we have a birational map:

$$\begin{aligned} P_1 \times P_2 \setminus S &\overset{\sim}{\dashrightarrow} X \\ (p_1, p_2) &\mapsto \varphi(p_1, p_2) \end{aligned}$$

In fact, one can see that S is a complete intersection of hypersurfaces of bidegrees $(1, 2)$ and $(2, 1)$ in $P_1 \times P_2$ by expressing it in the form:

$$S = \{F_1(u, v, w; x, y, z) = -F_2(u, v, w; x, y, z) = 0\} \subset P_{1,[x,y,z]} \times P_{2,[u,v,w]}.$$

Moreover, S is a K3 surface as it has trivial canonical bundle ω_S:

$$\omega_S = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1, 2) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(2, 1) \otimes \omega_{\mathbb{P}^2\times\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3, 3) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3, -3) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2};$$

and the irregularity $q := h^{0,1}(S) = \dim(H^1(S, \mathcal{O}_S))$ is 0. The locus of cubic fourfolds containing two disjoint planes has codimension 2 inside $\mathcal{C}$ (see for example [Aue+14]).

1.2 Lattice theory

In Chapter 2, we study general lattice theory. We define the lattice E_8 that is an even, unimodular and positive definite lattice (Ex. 4), the hyperbolic lattice U (Ex. 3), and the lattice associated to the Dynkin diagram A_2 (Ex. 5). We then describe the lattice $\Lambda_{K3} := H^2(S, \mathbb{Z})$ associated to a K3 surface S, and the lattice $L := H^4(X, \mathbb{Z})$ associated to a cubic fourfold $X \subset \mathbb{P}^5$. Using Milnor's theorem (Thm. 2.1.11) and the Hodge bilinear relations (Prop. 2.3.2), one can show that:

$$\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

and

$$L = E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_3.$$

Both the cubic lattice L and the K3 lattice Λ_{K3} play an important role in the study of K3 surfaces and cubic fourfolds. They both contain a lot of non-trivial geometric information (see Sect. 3.3 and Sect. 4.1.1). Moreover, the similarities between these two lattices are consequences of a strong result due to Beauville and Donagi [BD85, Prop. 2]. This is the first result linking the study of cubic fourfolds to K3 surfaces. It states that the Fano variety of lines $F(X)$, of a cubic fourfold X, is a deformation of $S^{[2]}$, where S is a K3 surface of degree fourteen (Ex. 9), and $S^{[2]}$ is the Hilbert scheme of length two subschemes of S.

1.3 Hodge conjecture and the lattice $A(X)$

In the case of a K3 surface S, the Néron-Severi group $\mathrm{NS}(S)$ is an important sub-lattice of Λ_{K3}. Recall that the Lefschetz theorem on $(1,1)$ classes (Thm. 3.3.1) states that $\mathrm{NS}(S) = H^{1,1}(S) \cap H^2(S,\mathbb{Z})$.
A similar phenomenon occurs for cubic fourfolds, the cohomology group carrying interesting geometric data is the middle cohomology lattice $H^4(X,\mathbb{Z})$. More precisely, we will consider the lattice of integral middle Hodge classes:

$$A(X) := H^{2,2}(X) \cap H^4(X,\mathbb{Z}) = H^2(X,\Omega_X^2) \cap H^4(X,\mathbb{Z}).$$

The Hodge conjecture holds for cubic fourfolds [Voi07, Thm. 18], thus $A(X)$ consists of codimension 2 algebraic cycles on X up to rational equivalence. Moreover, we have that algebraic, rational and homological equivalence coincide inside $A(X)$ (Def. B.0.9).

The lattice $A(X)$ is naturally equipped with the integer valued intersection form, denoted $(\cdot,\cdot)$. It is a positive definite lattice. Let h be the class of a hyperplane section on X, then $h^2 \in A(X)$. The generic cubic fourfold X has $\operatorname{rk} A(X) = \operatorname{rk}\langle h^2\rangle = 1$, and a cubic fourfold is said to be *special* whenever $\operatorname{rk} A(X) > 1$. A labelling of a special cubic fourfold is a rank two saturated sub-lattice of algebraic cycles [Has00] denoted K_d:

$$h^2 \in K_d \subset A(X)$$

where d is the discriminant of the intersection form on the sub-lattice K_d.

1.4 Moduli space and Hassett divisors

The study of the moduli space of cubic fourfolds, denoted by $\mathcal{C}$, particularly through GIT and the period map, has seen some very striking advances in recent years [Voi86; Laz10; Loo09]. The study of rationality has been developed in parallel to this.

In particular, Hassett [Has00] has described countably infinitely many irreducible divisors $\mathcal{C}_d \subset \mathcal{C}$ that parametrize *special cubic 4-folds*. One defines a Hassett divisor $\mathcal{C}_d \subset \mathcal{C}$ as the locus of special cubic fourfolds admitting a labelling K_d [Has00]. We have that $\mathcal{C}_d$ is non-empty if and only if $d > 6$ and $d \equiv 0$ or $2 \pmod 6$ [Has00, Sect. 3].

Only very few $\mathcal{C}_d$'s can be defined explicitly in terms of particular surfaces contained in X. For example, $\mathcal{C}_8$ is the locus of cubic fourfolds containing a plane [Voi86, §3]. These cubics are fibered in quadric surfaces over $\mathbb{P}^2$. $\mathcal{C}_{12}$ is defined as the closure of the locus of cubics containing a cubic scroll, and $\mathcal{C}_{14}$ as the closure of the locus of cubics containing a quartic scroll (or equivalently a quintic Del Pezzo surface) [Fan42],[Tre93]. The divisor $\mathcal{C}_{20}$ is the closure of the locus of cubic fourfolds containing a Veronese surface. The divisor $\mathcal{C}_{18}$ is the closure of the locus of cubic fourfolds containing an elliptic ruled surface.

In fact, for each $d \leq 44$, $d \neq 42$, Nuer [HNu15] explicitly describes $\mathcal{C}_d$ as the locus of cubic fourfolds containing a particular surface. The case $d = 42$ has been described by Lai in [Lai17]. He showed that $\mathcal{C}_{42}$ is the closure of the locus of cubic fourfolds containing a degree 9 rational scroll [Lai17, Thm. 0.3].

The only known examples of rational cubic fourfolds are contained in some very specific Hassett divisors. These are the divisors $\mathcal{C}_d \subset \mathcal{C}$ for the first admissible (Thm. 3.4.7) values of d. The discriminant d is said to be admissible if 4 and 9 do not divide d and for any odd prime number $p \equiv -1 \pmod 3$, p does not divide d. The first studied Hassett divisors with d admissible are $\mathcal{C}_{14}$ [BD85; BRS19; Aue+14; RS19], $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ [RS19; RS21; RS23], and more recently the divisor $\mathcal{C}_{42}$ [RS23]. We have that every cubic fourfold inside these Hassett divisors is rational.

1.5 Associated K3 surfaces

A great amount of the recent work around cubic fourfolds confirms a conjecture stated by Kuznetsov (Conj. 4.3.2) [Kuz10]. He claims that all rational examples of cubic fourfolds must be inside the union of the Hassett divisors $\mathcal{C}_d$ when d is admissible. For such a cubic fourfold $X \in \mathcal{C}_d$, there exists a polarized K3 surface S of degree d linked Hodge theoretically to X (Thm. 3.4.7). In this case, S is called an associated K3 surface to X.

For a general overview of this, one can see the work of Hassett [Has99; Has00; HV16]. For further detailed analyses concerning associated K3 surfaces in the case of the first admissible numbers $d = 14, 26, 38, 42$, one can see [BD85; BRS19; RS19; RS23; RS21].

Let us briefly explain this in the case of a cubic fourfold $X \in \mathcal{C}_{14}$ containing a quintic Del Pezzo surface, or equivalently ([Bea00, Prop. 9.2]) a Pfaffian cubic (Def. 4.5.4). The linear system of quadric hypersurfaces in $\mathbb{P}^5$ containing a quintic Del Pezzo surface $W \subset X$ induces a birational map $\varphi : \mathrm{Bl}_W(X) \tilde{\to} \mathbb{P}^4$ (see Rmk. 4.4.8). The inverse birational map φ^{-1} is induced by the linear system of quartics passing through a degree-nine surface $\tilde{S} \subset \mathbb{P}^4$. This surface $\tilde{S}$ is birational to a degree fourteen K3 surface S (the one associated to X). See also [Has00, Prop. 5.1.2 or Prop. 6.1.1] and [BD85].

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1.6 OADP surfaces and Rationality

In chapter 5, we establish the rationality of some cubic fourfolds inside the intersections of Hassett divisors $\mathcal{C}_{20} \cap \mathcal{C}_8$ and $\mathcal{C}_{12} \cap \mathcal{C}_8$ (see Section 5.2.2 and 5.3.2). To this end, one of the techniques we use is to find one apparent double point (OADP) surfaces inside the cubic fourfold.

Definition 1.6.1. *(OADP variety) Let Y be an equidimensional reduced scheme in $\mathbb{P}^{2n+1}$ of dimension n. The scheme Y is called a (generalized) variety with one apparent double point if, through a general point of $\mathbb{P}^{2n+1}$, there passes a unique secant line to Y that is the unique line cutting Y scheme theoretically in a reduced length two scheme.*

This technique is based on Theorem 4.4.10, which establishes rationality of every cubic fourfold containing a (possibly reducible) OADP surface. It was used for example in [BRS19]. For a complete, and modern overview of OADP varieties, one can see the work of Ciliberto, Mella and Russo [CMR04]. In this paper, the authors classify OADP varieties with dimension at most three.

The notion of an OADP variety is a classical one. For example, smooth OADP surfaces in $\mathbb{P}^5$ appeared in the work of Severi [Sev01]. More recently, Russo [Rus00] completed the proof of Severi (see Theorem 4.4.6). He proved that a smooth OADP surface is either a Del Pezzo surface or a quartic rational normal scroll; the latter being a divisor of type (2,1) or (0,2) in the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (see Definition 4.4.4 and Appendix A).

The name OADP variety is usually reserved for the irreducible reduced scheme satisfying the previous condition. This somewhat bizarre name comes from the fact that the projection of Y from a general point into $\mathbb{P}^{2n}$ acquires a unique singular point, which is *double*. This means that its tangent cone is the union of two copies of $\mathbb{P}^n$ intersecting at the singular point.

1.6.1 Why reducible OADP surfaces ?

The Hassett divisor $\mathcal{C}_{14}$ can be described as the closure of the locus of cubic fourfolds containing a smooth quartic rational normal scroll (or equivalently a smooth quintic Del Pezzo surface [Fan42], [Tre93]). These are the only smooth OADP surfaces in $\mathbb{P}^5$ (Thm. 4.4.6). In particular, the rationality argument concerning cubic fourfolds containing a smooth OADP surface (Ex. 11) cannot be extended to establish rationality for cubic fourfolds not contained in the divisor $\mathcal{C}_{14}$. One way to generalize this technique is to consider reducible OADP surfaces, which appear, for instance, inside cubic fourfolds in $\mathcal{C}_{20}$ (see Sect. 5.3.3) and cubics in $\mathcal{C}_{12}$ (see Sect. 5.2.3).

Example 1. *Let $S \subset \mathbb{P}^5$ be a smooth rational normal scroll of degree 3 and $P \subset \mathbb{P}^5$ a plane. When $P \cap S$ is a line of the ruling of S, we get that $S \cup P$ is an OADP surface by Prop. 5.2.3. In Appendix C, Ex. 17* (7)*, we give an explicit equation of a rational cubic fourfold $X \subset \mathbb{P}^5$ containing the OADP surface $S \cup P$. This cubic fourfold is inside the intersection of the Hassett divisors $\mathcal{C}_8 \cap \mathcal{C}_{12}$ (see Section 5.2).*

1.6.2 The divisor $\mathcal{C}_{14}$

The first rationality result concerning cubic fourfolds inside $\mathcal{C}_{14}$ is the argument used by Fano [Fan42] (Ex. 11).

Another rationality result in $\mathcal{C}_{14}$ is the rationality of Pfaffian cubic fourfolds established by Beauville and Donagi [BD85, Prop. 5].

These two arguments only show the rationality of a generic cubic fourfold in $\mathcal{C}_{14}$. More recently, Bolognesi, Russo and Stagliano [BRS19] studied the Hassett divisor $\mathcal{C}_{14}$ using the generalized surfaces with one apparent double point. In this paper, the authors prove for the first time the rationality of every cubic fourfold inside $\mathcal{C}_{14}$ [BRS19, Thm. 3.9 and Thm. 4.4], completing the previous results of Fano, Beauville and Donagi. For this purpose, they prove that each cubic fourfold $X \in \mathcal{C}_{14}$ contains a possibly reducible OADP surface (see Section 4.5.6). One can also see the work of Russo and Stagliano [RS19], where they give a different proof for the rationality of every cubic in $\mathcal{C}_{14}$ [RS19, Thm. 2], and much more.

1.6.3 The divisor $\mathcal{C}_8$

The divisor $\mathcal{C}_8$ is the locus of cubic fourfolds $X \subset \mathbb{P}^5$ containing a plane P ([Voi86, §3]).

The linear projection $\pi_P : X \dashrightarrow \mathbb{P}^2$ with center P, is resolved by blowing up P:

$$\widetilde{\pi}_P : Bl_P(X) \longrightarrow \mathbb{P}^2$$

We obtain the morphism $\widetilde{\pi}_P$, which is a quadric surface bundle. Under mild conditions on the plane (see [ABB14, Prop. 1.2.5]), the quadric surface bundle degenerates on a smooth sextic curve $D \subset \mathbb{P}^2$.

Proposition 1.6.2. *[Has99, Prop. 2.3] Let $q : \mathcal{Q} \to B$ be a quadric surface bundle over a rational projective variety. Let Q denote the class of the generic fiber of q inside $H^4(\mathcal{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q})$. Suppose there is a class $T \in H^4(\mathcal{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q})$ which has odd intersection with Q, then $\mathcal{Q}$ is rational over $\mathbb{C}$.*

In our situation, this is equivalent to saying that there is an odd degree multisection of the quadric bundle $\widetilde{\pi}_P$, and we will use several times Prop. 1.6.2 in Section 5.2.2 and Section 5.3.2 to check that certain classes of cubic fourfolds are rational.

Cubic fourfolds inside $\mathcal{C}_8$ for which the linear projection $\tilde{\pi}_P$ admits a rational section form infinitely countably many irreducible divisors in $\mathcal{C}_8$. These loci have codimension 2 inside $\mathcal{C}$. They parametrize rational cubic fourfolds [Has99, Thm. 4.2].

1.7 Intersections of Hassett divisors

In this thesis, we consider the intersections of $\mathcal{C}_8$ with two other Hassett divisors, notably $\mathcal{C}_{12}$ and $\mathcal{C}_{20}$. By means of lattice theory, we describe all the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{20}$ and $\mathcal{C}_8 \cap \mathcal{C}_{12}$.

This problem has been addressed for the first time in [Aue+14], where the authors investigate the irreducible components of the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$. In particular, they find that this intersection has five irreducible components. In another paper [BRS19], the authors prove the rationality of every cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{14}$, showing rationality of every $X \in \mathcal{C}_{14}$ for the first time (see Section 4.5.4). One can also see the work of Hanine Awada [Awa24], where she studies the intersection of $\mathcal{C}_8$ with $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ respectively.

In [AT14, Thm. 4.1], Addington and Thomas show that $\mathcal{C}_8$ intersects any non-empty other Hassett divisor $\mathcal{C}_d$. Later, Yang and Yu [YY20, Thm. 3.1] generalize this result and prove that every two non-empty Hassett divisors $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$ intersect.

Hence, every non-empty Hassett divisor $\mathcal{C}_d$ contains a union of three codimension-two subvarieties in $\mathcal{C}$, that are $\mathcal{A}_1 = \mathcal{C}_d \cap \mathcal{C}_{14}$, $\mathcal{A}_2 = \mathcal{C}_d \cap \mathcal{C}_{26}$, and $\mathcal{A}_3 = \mathcal{C}_d \cap \mathcal{C}_{38}$ [YY20, Thm. 3.3]. Since every cubic fourfold in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, and $\mathcal{C}_{38}$ is rational (see for instance [RS19]), each subvariety $\mathcal{A}_i \subset \mathcal{C}_d$ parametrizes rational cubic fourfolds. One can also add the intersection $\mathcal{A}_4 = \mathcal{C}_d \cap \mathcal{C}_{42}$, using the rationality in $\mathcal{C}_{42}$ [RS23, Thm. 5.12].

In the same paper [YY20, Thm. 3.7], the authors also prove that for any two non-empty Hassett divisors $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$, the intersection $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{14}$ is non empty.

In [Awa24, Thm. 1.2], Awada studies cubic fourfolds inside $\mathcal{C}_{18}$ that are fibered in sextic Del Pezzo surfaces over $\mathbb{P}^2$. She exhibits five irreducible components of the intersection $\mathcal{C}_{18} \cap \mathcal{C}_{14}$, eight of $\mathcal{C}_{18} \cap \mathcal{C}_{26}$ and eleven of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$.

For the intersection of more than two Hassett divisors, one can see the work of Awada, Bolognesi and Pedrini [ABP20].

1.7.1 Intersections $\mathcal{C}_8 \cap \mathcal{C}_{12}$ and $\mathcal{C}_8 \cap \mathcal{C}_{20}$

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Here are our results.

Theorem 1.7.1. *(= Thm. 5.2.1 and Lemma 5.2.2) There are three irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{12}$ indexed by the value $P \cdot S = \eta \in \{1, 2, 3\}$, where $P \subset X$ is a plane and S the class of a cubic rational normal scroll (that is $S \cdot S = 7$ and $S \cdot h^2 = 3$). For $\eta = 2$, every element in the corresponding irreducible component is rational.*

Theorem 1.7.2. *(= Thm. 5.3.1) There are seven irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{20}$ indexed by $P \cdot V = \gamma \in \{-2, -1, 0, 1, 2, 3, 4\}$, where $P \subset X$ is a plane and V the class of a Veronese surface (that is $V \cdot V = 12$ and $V \cdot h^2 = 4$). For $\gamma = -1, 1$ and 3 each smooth cubic hypersurface belonging to the corresponding irreducible component is rational.*

As the reader can see, we also draw information about the rationality of cubics inside certain irreducible components of these intersections. This is based principally on two different techniques. The first is now quite standard, and it was introduced by Hassett in [Has99]. In fact, it is based on the rationality result of cubic fourfolds inside $\mathcal{C}_8$ verifying Proposition 1.6.2.

The second technique is somehow less commonly used in the realm of cubic fourfolds, it is rooted on [CR11], and has been partially used in [BRS19]. Basically, by using the fact that X contains a couple of independent surfaces, we manage to construct inside X, one-apparent-double-point surface (or shortly OADP) obtained as the union of other surfaces that intersect in some loci. This establishes directly the rationality of the cubic fourfold X (Thm. 4.4.10).

In particular, in the component of $\mathcal{C}_{20} \cap \mathcal{C}_8$, we find cubic fourfolds containing interesting reducible degenerations of the Veronese surface, given by a cubic scroll plus a plane intersecting the scroll along the directrix line. These examples hence lie in $\mathcal{C}_{20} \cap \mathcal{C}_8 \cap \mathcal{C}_{12}$. On the other hand, inside the component of $\mathcal{C}_8 \cap \mathcal{C}_{12}$ where the cubic scroll has intersection 0 with the plane, we find cubics containing a plane intersecting the scroll along a line of the ruling. These reducible surfaces are degenerations of quartic scrolls, and the cubics containing such objects are hence contained in $\mathcal{C}_8 \cap \mathcal{C}_{12} \cap \mathcal{C}_{14}$.

Chapter 2

Lattice theory and Hodge structures

2.1 General lattice theory

Definition 2.1.1. *(lattice) A lattice L is a free $\mathbb{Z}$-module of finite rank r ($\simeq \mathbb{Z}^r$), together with a $\mathbb{Z}$-valued symmetric bilinear form:*

$$(\cdot,\cdot) : L \times L \longrightarrow \mathbb{Z}$$

called the intersection form.

Definition 2.1.2.

1. *The lattice L is called even if for every $x \in L$, we have $(x, x) \in \mathbb{Z}$ even. It is called odd otherwise, i.e. if there exists an $x \in L$ such that (x, x) is odd.*

2. *The matrix of the bilinear form $(\cdot,\cdot)$ on L with respect to a basis is called the Gram matrix, and is denoted M_L or simply L by abuse of notation.*

3. *The determinant of the Gram matrix M_L, which is independent of the choice of the basis, is called the discriminant of L and is denoted $\operatorname{disc}(L)$.*

4. *Two lattices L_1 and L_2 are isometric if the $\mathbb{Z}$-modules are isomorphic, and the isomorphism respects the lattice structure.*

Definition 2.1.3. *(degenerate, non-degenerate, discriminant group) A lattice L is said to be non-degenerate if $\operatorname{disc}(L)$ is non-zero. This is equivalent to saying that the rank of the matrix M_L, is equal to the rank of the $\mathbb{Z}$-module L. Otherwise, the lattice L is said to be degenerate. When L is non degenerate, the natural morphism:*

$$\begin{aligned} i_L : L &\longrightarrow L^* = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \\ v &\mapsto (v, \cdot) \end{aligned}$$

is injective. Moreover, the group $A_L := L^*/L$ is called the discriminant group. Its order is equal to $|\operatorname{disc}(L)|$.

Definition 2.1.4.

1. *When L is non-degenerate, we can define the signature of L, denoted* $\mathrm{sgn}(L)$*, as the signature (p, q) of the Gram matrix M_L (viewed as a real matrix). Where, p is the number of $+1$ and q is the number of -1 in the diagonalization of M_L.*

2. *The index of the lattice L, denoted $\tau(L)$, is equal to $p - q$.*

3. *A lattice L is said to be positive definite if $q = 0$, and negative definite if $p = 0$. It is indefinite if neither of them vanishes.*

Definition 2.1.5. *(unimodular lattice) A lattice L is said to be unimodular if* $\mathrm{disc}(L) = 1$ *or* -1*. This is equivalent to saying that $i_L : L \simeq L^*$ is an isomorphim, or equivalently the group $A_L = L^*/L$ is trivial.*

2.1.1 Examples

Example 2. *Let $n \in \mathbb{N}$, and I_n be the bilinear form defined on $\mathbb{Z}^n \times \mathbb{Z}^n$ by the identity matrix of size $n \times n$. We get the lattice $(\mathbb{Z}^n, I_n)$ that by abuse of notation we simply denote by I_n.*

Let $p, q \in \mathbb{N}$, we define the lattice $I_{p,q}$ to be the free group $\mathbb{Z}^{p+q}$ endowed with the intersection matrix:

$$I_{p,q} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right]$$

It is odd, and unimodular.

Example 3. *The hyperbolic plane lattice U, is the free group $\mathbb{Z}^2$ endowed with the intersection matrix:*

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

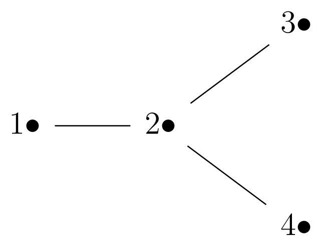
It is an even, and unimodular lattice. It has signature (1,1), and discriminant equal to -1.

Example 4. *The lattice E_8, is defined by the group $\mathbb{Z}^8$ with the following intersection matrix:*

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

It is an even, unimodular, and positive definite lattice. It has discriminant equal to 1.

Example 5. *(Dynkin diagrams ADE) For each Dynkin diagram A_n (for $1 \leq n$), D_n (for $4 \leq n$) and E_n (for $n = 6, 7, 8$), we can define a lattice structure on the free group $\mathbb{Z}^n$. In fact, each Dynkin diagram gives a bilinear form by taking the incidence matrix of the graph. To this matrix, we add $3 \cdot I_n$ to get 2 in the diagonal instead of -1. We illustrate that by the example of D_4:*

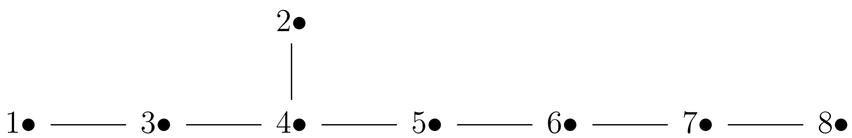


D_4 diagram

We get the incidence matrix $(a_{i,j})_{4\times 4}$ as follows: for each edge between two vertices i and j we put $a_{i,j} = 1$. If there is no edge between the vertices i and j, then $a_{i,j} = 0$. In the diagonal we put 2 instead of -1 (as usually done for the incidence matrix).

$$D_4 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

Taking the Dynkin diagram E_8, we get the same lattice E_8 defined above in Example 4. The graph E_8 is as follows:



E_8 diagram

$$1\bullet \text{ ——— } 2\bullet$$

A_2 diagram

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Remark 2.1.6. *E_8 is the only unimodular lattice of ADE type (coming from a Dynkin diagram).*

2.1.2 Milnor's classification theorem

Let us first introduce some other definitions.

Definition 2.1.7. *(the direct sum) Let L_1 and L_2 be two lattices with corresponding Gram matrices M_1 and M_2, respectively. The $\mathbb{Z}$-module $L_1 \oplus L_2$ has naturally a lattice structure given by the intersection matrix:*

$$L_1 \oplus L_2 = \left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right]$$

such that for two elements $v = (v_1, v_2) \in L_1 \oplus L_2$ and $w = (w_1, w_2) \in L_1 \oplus L_2$:

$$(v, w) = v_1 \cdot M_1 \cdot w_1^\intercal + v_2 \cdot M_2 \cdot w_2^\intercal$$

Definition 2.1.8. *(primitive embedding of lattices) Let $L_1 \hookrightarrow L$ be an injective morphism of lattices, it is called an embedding. When the embedding is of finite index $[L : L_1] := cardinal(L/L_1)$, we have the following formula between the discriminants:*

$$\operatorname{disc}(L_1) = \operatorname{disc}(L)[L : L_1]^2.$$

The embedding $L_1 \hookrightarrow L$ is called primitive when in addition, its cokernel is torsion free.

As an example of a primitive embedding (or primitive sublattice), let us introduce the orthogonal lattice.

Remark 2.1.9. *(the orthogonal lattice $L^\perp$) Let $L_1 \subset L$ be a sublattice. The orthogonal complement to L_1 inside L is the sublattice $L_1^\perp := \{v \in L | (v, L_1) = 0\} \subset L$. We have that $L \cap L^\perp = 0$, hence $L/L_1^\perp = L_1$ is torsion free, and $L_1^\perp$ is a primitive sublattice of L. Moreover, the direct sum $L_1 \oplus L_1^\perp \subset L$ is a sublattice of finite index but in general not primitive. We get the formula:*

$$\operatorname{disc}(L_1) \cdot \operatorname{disc}(L_1^\perp) = \operatorname{disc}(L) \cdot [L : L_1 \oplus L_1^\perp].$$

We end this by introducing the saturation of a sublattice $L_1 \subset L$, denoted $L_1^{sat} := (L_1^{\perp})^{\perp}$. In fact, $L_1 \subset L_1^{sat} \subset L$, and $L_1^{sat} = L_1$ if and only if $L_1 \subset L$ is a primitive sublattice. Furthermore, $L_1^{sat} \subset L$ is the minimal primitive sublattice containing L_1, and $L_1 \subset L_1^{sat}$ is of finite index.

Definition 2.1.10. *(overlattices) An overlattice of a lattice L_1, is simply a lattice L such that there exists an embeding $L_1 \hookrightarrow L$ of finite index. In other words, $L_1 \subset L$ is a sublattice of finite index.*

Theorem 2.1.11. *(Milnor)(classification of unimodular lattices [MH73, Chpt. II]) Let L be an unimodular lattice, that has signature (p, q). We have the following classification:*

1. *if L is odd then L is isomorphic to $I_{p,q}$.*
2. *if L is even and indefinite, then :*
   - $L \simeq U^{\oplus p} \oplus E_8^{\frac{q-p}{8}}$ *if* $p < q$
   - $L \simeq U^{\oplus q} \oplus E_8(-1)^{\frac{p-q}{8}}$ *if* $q < p$
   - $L \simeq U^{\oplus p}$ *if* $q = p$

Milnor's theorem 2.1.11 can be proved using the following theorem. It asserts the existence and uniqueness of an unimodular lattice, with a fixed signature (p, q).

Theorem 2.1.12. *(Existence and uniqueness [MH73, Chpt. II]) Let $(p, q) \in \mathbb{Z}^2$ be two fixed integers. When $p - q = 0$ mod 8, there exists an even, unimodular lattice L with signature (p, q). If p and q are both strictly positive, then L is unique up to isometry.*

2.2 Lattices associated to cubic fourfolds and K3 surfaces

2.2.1 Hodge decomposition

Before we introduce the lattices associated to cubic fourfolds and K3 surfaces, we need the following proposition.

Proposition 2.2.1. *(Hodge decomposition)[Hod51] Let X be a Kähler manifold of dimension n. Then, X admits a Hodge decomposition which is as follows:*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X) = H^q\left(X, \Omega_X^p\right)$, are cohomology groups of the classes of C^∞ complex differential forms, which can be represented by closed forms of type (p, q).

We have the conjugation property:

$$H^{p,q}(X) \simeq \overline{H^{q,p}(X)}.$$

Now, $h^{p,q} := \dim H^q\left(S, \Omega_X^p\right)$ are called the Hodge numbers. The Hodge numbers have symmetry and conjugation properties as follows:

$$h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}.$$

These properties make $H^k(X, \mathbb{Z})$ equipped with an integral Hodge structure (see Def. 2.2.7), as defined by Claire Voisin [Voi02, p. II.7.1.1].

One of our motivating examples is the K3 lattice denoted by Λ_{K3}. This will be viewed again in more details in Section 3.3. First we define K3 surfaces.

Definition 2.2.2. *A K3 surface S is a smooth complex compact simply connected surface that admits a unique nowhere vanishing holomorphic 2-form.*

K3 surfaces are Kähler manifolds [Bar+04, IV.Thm. 3.1 and Thm. 2.10], and hence admit the Hodge decomposition (Prop. 2.2.1). We organize the Hodge numbers of K3 surfaces in the following Hodge diamond. Note that, the most non-trivial cohomology data is carried by the middle cohomology $H^2(S, \mathbb{Z})$.

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

The Hodge diamond of any K3 surface is of the form:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Definition 2.2.3. *(K3 lattice) The K3 lattice is given by the middle cohomology group denoted $\Lambda_{K3} := H^2(S, \mathbb{Z})$. It has rank 22, it is even and unimodular, and has signature $(3, 19)$. Using Milnor's Theorem 2.1.11, we get:*

$$\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

See examples in Section 2.1.1 for the definition of U and E_8.

Definition 2.2.4. *(cubic Fourfold) A cubic fourfold $X \subset \mathbb{P}^5$ is a smooth hypersurface defined by the zero locus of a homogeneous polynomial $F(x, y, z, u, v, t)$ of degree 3.*

Cubic fourfolds are Kähler manifolds, as they are projective [Ara12, Cor. 10.1.7]. Therefore, they admit a Hodge decomposition (see Prop. 2.2.1). A cubic fourfold has the following Hodge diamond. Note that, similarly to K3 surfaces, most non-trivial cohomology data is carried by the middle cohomology group $H^4(X, \mathbb{Z})$.

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & \\
h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\
 & h^{5,0} & & h^{3,2} & & h^{2,3} & & h^{0,5} & \\
 & & h^{6,0} & & h^{3,3} & & h^{0,6} & & \\
 & & & h^{7,0} & & h^{0,7} & & & \\
 & & & & h^{8,8} & & & &
\end{array}$$

We get the Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Let us briefly recall the lattice structure of the middle cohomology $L := H^4(X, \mathbb{Z})$, of a smooth cubic fourfold $X \subset \mathbb{P}^5$. The lattice L has rank $b_4(X) = 23$, it is odd, unimodular, and has signature $(21, 2)$. By Milnor's Theorem 2.1.11, it follows that $L = I_{21,2}$. More precisely, we have the following result.

Definition 2.2.5. *(cubic lattice)*

$$L := H^4(X, \mathbb{Z}) = I_{21,2} \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_3.$$

Where the lattices E_8, U and A_2 are defined in Section 2.1.1. These results will be viewed again in more details in Proposition 4.1.2.

Remark 2.2.6. *Note the similarities between the cubic lattice L and the K3 lattice Λ_{K3}. These are consequences of a strong result, due to Beauville and Donagi [BD85, Prop. 2] explained in the next chapter (see Thm. 3.2.3).*

Next, we state the Torelli theorem for K3 surfaces (Thm. 2.2.11), and for cubic fourfolds (Thm. 2.2.10).

2.2.2 Hodge structures and Torelli theorems

The cubic lattice L and the K3 lattice Λ_{K3} are equipped with integral Hodge structures arising from the Hodge decomposition (Prop. 2.2.1).

Definition 2.2.7. *[Voi02, p. II.7.1.1](Hodge structures) A weight k integral Hodge structure is a lattice V, with a decomposition:*

$$V \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}.$$

Together with a conjugation property:

$$V^{q,p} = \overline{V^{p,q}}.$$

Definition 2.2.8. *(Isomorphism of integral Hodge structures)[Huy16, Chpt. 3. 1.1.iv] Let V be a lattice that has a weight k_1 Hodge structure, and W that has a weight $k_2 = k_1 + 2m$, $m \in \mathbb{Z}$ Hodge structure. These are given by the decompositions $V \otimes \mathbb{C} = \bigoplus_{p+q=k_1} V^{p,q}$, and $W \otimes \mathbb{C} = \bigoplus_{p+q=k_2} W^{p,q}$.*
A morphism of weight m, of integral Hodge structures $V \to W$, is a $\mathbb{Z}$-linear morphism $f : V \to W$, between the $\mathbb{Z}$-modules, such that the $\mathbb{C}$-linear extension $f \otimes \mathbb{C} : V \otimes \mathbb{C} \to W \otimes \mathbb{C}$, satisfies:

$$f(V^{p,q}) \subset W^{p+m,q+m}.$$

The morphism f is said to be an isomorphism of Hodge structures iff f^{-1} exists as a morphism of weight $-m$, of integral Hodge structures.

We also need the following definition before we state Torelli theorems.

Definition 2.2.9. *(isometry) Let* $(V, (\cdot, \cdot))$ *and* $(W, (\cdot, \cdot))$ *be two lattices. A* $\mathbb{Z}$*-linear isomorphism* $f : V \xrightarrow{\sim} W$*, between the* $\mathbb{Z}$*-modules, is said to be an isometry if and only if it preserves the intersection bilinear forms. This means that for each* $a, b \in V$ *we have the following equality:*

$$(a, b) = (f(a), f(b))$$

where, the left side is the evaluation of the intersection pairing associated to the lattice V*, and the right side to the lattice* W*.*

When two cubic fourfolds admit isomorphic integral Hodge structures, they are isomorphic. This is the Torelli theorem (see [Voi86]).

Theorem 2.2.10. *(Torelli Theorem) Let* X_1 *and* X_2 *be two cubic fourfolds. If there exists an isomorphism of integral Hodge structures* $H^4(X_1, \mathbb{Z}) \simeq H^4(X_2, \mathbb{Z})$ *respecting the intersection bilinear forms (isometry) and preserving the hyperplane class* h^2*, then* X_1 *is isomorphic to* X_2*.*

We have a similar result for K3 surfaces, also called Torelli theorem [Huy16, Chpt. 7 Thm. 5.3].

Theorem 2.2.11. *(Torelli Thm for K3 surfaces) Two complex K3 surfaces* S_1 *and* S_2 *are isomorphic if and only if there exists an isomorphism of integral Hodge structures* $H^2(S_1, \mathbb{Z}) \simeq H^2(S_2, \mathbb{Z})$ *such that the intersection bilinear forms are respected (it is an isometry).*

Let S be a K3 surface, and $H^2(S, \mathbb{Z})$ its middle cohomology lattice. Let $\mathrm{O}(H^2(S, \mathbb{Z})) := \{f : H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(S, \mathbb{Z}) \mid f \text{ is an isometry}\}$ be the group of isometries (Def. 2.2.9). We have a natural map:

$$\begin{aligned} \mathrm{Aut}(S) &\longrightarrow \mathrm{O}(H^2(S, \mathbb{Z})) \\ f &\longmapsto f^* \end{aligned}$$

that associates to each automorphism $f : S \xrightarrow{\sim} S$, its corresponding pullback $f^* : H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(S, \mathbb{Z})$. This map is injective.

Proposition 2.2.12. *[Col85](Automorphisms of K3 surfaces) Let f be an automorphism of a complex K3 surface S. If* $f^* = id$ *on* $H^2(S, \mathbb{Z})$*, then* $f = id$*.*

Proof. See [Huy16, Chpt. 15 Prop. 2.1] □

Remark 2.2.13. *Thus, to study and compute* $\mathrm{Aut}(S)$*, one can study the orthogonal group of isometries of* $H^2(S, \mathbb{Z}) \simeq \Lambda_{K3}$*, denoted by* $\mathrm{O}(H^2(S, \mathbb{Z}))$*.*

In addition to this important proposition and the Torelli theorems, both the cubic lattice L and the K3 lattice Λ_{K3} play crucial roles in the study of K3 surfaces and cubic fourfolds. They both contain most non-trivial Hodge theoretic geometric information. This is illustrated in the next chapters (see Sect. 3.3 and Sect. 4.2.2).

Next, we state other important theorems in lattice theory, particularly those that we frequently use to prove our main results in Chapter 5, namely Theorem 5.2.1 and Theorem 5.3.1.

2.3 Useful theorems

Theorem 2.3.1. *[Huy05, Cor. 3.3.16](Hodge index theorem for surfaces) Let S be a compact Kähler surface, then the intersection pairing:*

$$H^2(S,\mathbb{R}) \times H^2(S,\mathbb{R}) \longrightarrow \mathbb{R}, \quad (\alpha,\beta) \longmapsto \int_S \alpha \wedge \beta$$

has signature $(2h^{2,0}(S)+1, h^{1,1}(S)-1)$. Restricted to $H^{1,1}(S)$ it is of signature $(1, h^{1,1}(S)-1)$.

Proposition 2.3.2. *(Hodge-Riemann bilinear relations)[Huy05, Cor. 3.3.18] Let X be a compact complex Kähler manifold of even complex dimension $2n$. The intersection pairing on the middle cohomology of X is of index:*

$$\tau(X) = \sum_{p,q=0}^{2n} (-1)^p h^{p,q}(X).$$

This last proposition is proved using the Hodge-Riemann bilinear relations.

Proposition 2.3.3. *(Sylvester criterion) Let A be a symmetric $n \times n$ real matrix. We have that A is positive definite if and only if every principal minor of A is strictly positive.*

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, and $L := H^4(X,\mathbb{Z})$ the middle cohomology lattice. The following Lemma that we borrow from [YY20, Lemma 2.4], is a strong result that we frequently use in Section 5.2.1 and Section 5.3.1.

Lemma 2.3.4. *[YY20, Lemma 2.4] Let $h^2 \in M \subset L$ be a positive definite saturated sublattice of rank $r \geq 2$. And let $\mathcal{C}_M \subset \mathcal{C}$ be the locus of smooth cubic fourfolds $X \in \mathcal{C}$ such that $h^2 \in M \subset A(X) \subset L$. Then the following three conditions are equivalent:*

1. *there exists no sublattice $h^2 \in K \subset M$ with $K = K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ or $K_6 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$;*
2. *there exists no $r \in M$ such that $(r.r) = 2$ (i.e., M does not represent 2);*
3. *for any $0 \neq x \in M, (x.x) \geq 3$.*

In particular, if M satisfies one of the three equivalent conditions, then $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_K$ for any saturated sublattice $h^2 \in K \subset M$. In this case, $\mathcal{C}_M$ has codimension $r-1$ in $\mathcal{C}$.

Chapter 3

K3 surfaces

3.1 K3 surfaces definitions and examples

First we give another definition of K3 surfaces which is equivalent to definition 2.2.2.

Definition 3.1.1. *A K3 surface is a smooth compact connected surface S, for which the canonical bundle ω_S is trivial, and the irregularity $q := h^{0,1}(S) = \dim(H^1(S, \mathcal{O}_S))$ vanishes.*

Remark 3.1.2. *The fact that $\omega_S = \Omega_S^2$ is trivial, is equivalent to the existence of a unique nowhere vanishing holomorphic 2-form on S. The vanishing of $q = h^{0,1}(S)$ is equivalent to the vanishing of $h^{1,0}(S) := \dim(H^0(S, \Omega_S^1))$. This follows from the following inequality [Bar+04, IV. Lemma 2.6]:*

$$2h^{1,0} \leq h^{1,0} + h^{0,1} \leq 2h^{0,1}$$

which is true for every compact surface.

Example 6. *The first example of a K3 surface, is the quartic surface in $\mathbb{P}^3$. This can be extended, more generally, to the case of complete intersections in projective spaces.*
Let $S \subset \mathbb{P}^{n+2}$ be a complete intersection of degrees $2 \leq d_1 \leq d_2 \leq ... \leq d_n$. Using the canonical bundle formula, we get:

$$\omega_S = (\omega_{\mathbb{P}^{n+2}} \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(d_1) \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(d_2) \otimes ... \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(d_n))|_S = \mathcal{O}_S(-n-3+d_1+d_2+...+d_n).$$

The canonical bundle is trivial iff $n + 3 = d_1 + d_2 + ... + d_n$, so that $2n \leq n + 3$ and $n \leq 3$. Hence, ω_S is trivial iff: $(n, d_1, ..., d_n) = (1, 4)$ or $(2, 2, 3)$ or $(3, 2, 2, 2)$.

In the end, we have that the only complete intersections in projective spaces that are K3 surfaces are:
the quartic surface in $\mathbb{P}^3$; the intersection of a quadric and a cubic in $\mathbb{P}^4$, of degree 6; and the intersection of three quadrics in $\mathbb{P}^5$, of degree 8.

One can prove that the irregularity $q = H^1(S, \mathcal{O}_S)$ also vanishes. This is done using the Lefschetz hyperplane section theorem.

Example 7. *The second example of K3 surfaces are Kummer surfaces.*

We start with an abelian surface $A := \mathbb{C}^2/\Lambda$, *where* $\Lambda = \sum \mathbb{Z}e_i$, *for* $e_i \in \mathbb{R}^4$. *We have a natural involution on* A:

$$\begin{aligned} i : A &\longrightarrow A \\ (x, y) &\mapsto (-x, -y) \end{aligned}$$

This involution has 16 fixed points: we have that $A \simeq (\mathbb{R}/\mathbb{Z})^4$ *as a group. In the group* $\mathbb{R}/\mathbb{Z}$, *the involution* $a \mapsto -a$ *has two fixed points, that are* 0 *and* $\frac{1}{2}$. *Hence, the involution* i, *defined on* $A \simeq (\mathbb{R}/\mathbb{Z})^4$, *has* $16 = 2^4$ *fixed points* $p_1, ..., p_{16}$. *Let* $\pi : \tilde{A} \longrightarrow A$ *be the blow-up of* A *at the 16 fixed points. We have the following diagram:*

$$\begin{tikzcd} \tilde{A} \arrow[r, "\pi"] \arrow[d, "\tilde{p}"'] & A \arrow[d, "p"] \\ \tilde{Z} := \tilde{A}/\tilde{i} \arrow[r, "\tilde{\pi}"] & A/i \end{tikzcd}$$

Where the involution i *on* A, *lifts to an involution* $\tilde{i}$ *on the blow-up* $\tilde{A}$, *and* $\tilde{\pi} : \tilde{Z} \to A/i$ *is a resolution of the 16 singularities. Let* $E_i = \pi^{-1}(p_i)$ *be the exceptional divisors. We have that* $\tilde{i}$ *fixes every* E_i *pointwise.*

In the end, we have that $Kum(A) := \tilde{Z}$ *is a K3 surface called the Kummer surface, it is an important example of K3 surfaces.*

Kum(A) contains 16 disjoint (-2)*-curves that are* $\tilde{p}(E_i)$. *This is in fact a characterization of K3 surfaces that are Kummer surfaces.*

Theorem 3.1.3. *[Nik75, Rmk. 14.3.19] Let* S *be a K3 surface containing 16 disjoint* (-2)*-curves, then* S *is a Kummer surface.*

Example 8. *(the double plane) We start with a double covering of the plane, branched along a smooth sextic curve* $C \subset \mathbb{P}^2$:

$$\pi : S \longrightarrow \mathbb{P}^2$$

For such a covering, we directly have that $\pi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$. *This equality implies that the irregularity* $q(S)$ *vanishes:*

$$H^1(S, \mathcal{O}_S) = H^1(\mathbb{P}^2, \pi_*\mathcal{O}_S) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0.$$

Using the canonical bundle formula for finite coverings, we also get that:

$$\omega_S = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \pi^*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_S.$$

Moreover, S *is smooth because the curve* C *is smooth. Therefore,* S *is a* $K3$ *surface of degree* $(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))^2 = 2$, *where* $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ *is an ample line bundle on* S.

Example 9. *(Grassmannians) First, consider the Plücker embedding:*

$$Gr(r,n) \longrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$$
$$[W] \mapsto \bigwedge^r W$$

By abuse of notation, we simply write $Gr(r,n)$ instead of $Im(Gr(r,n))$ under this embedding. One can show that it is a smooth subvariety of $\mathbf{P} = \mathbb{P}(\bigwedge^r \mathbb{C}^n)$, and has dimension $r(n-r)$. Moreover, we have the canonical bundle formula $\omega_{G(r,n)} = \mathcal{O}_{G(r,n)}(-n) := \mathcal{O}_{\mathbf{P}}(-n)|_{Gr(r,n)}$.

- *Let $Gr(2,6) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^6) = \mathbb{P}^{14}$. This grassmanian has dimension $2(6-2) = 8$ and degree 14. Let $\mathbb{P}^8 \subset \mathbb{P}^{14}$ be a generic linear subspace.*
  The intersection $S = \mathbb{P}^8 \cap Gr(2,6)$ is a smooth K3 surface of degree $(\mathcal{O}_{Gr(2,6)}(1)|_S)^2 = 14$.
  The fact that the canonical class K_S is trivial, comes directly from the canonical bundle formula, and the normal bundle sequence (see [Huy16, Chpt. 1. 4.2]). Indeed, note that $\omega_{Gr(2,6)} = \mathcal{O}_{Gr(2,6)}(-6)$. Hence:

  $$\omega_S = \mathcal{O}_{Gr(2,6)}(-6+1+1+1+1+1+1)|_S = \mathcal{O}_{Gr(2,6)}|_S = \mathcal{O}_S.$$

- *Let $Gr(2,5) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^5) = \mathbb{P}^9$, of dimension 6 and degree 5. Intersecting $Gr(2,5)$ with three generic hyperplanes, and a generic quadric, gives a smooth surface S. Similar computations and arguments as above, show that S is a K3 surface of degree 10.*

Remark 3.1.4. *All K3 surfaces are diffeomorphic. More precisely, they are all diffeomorphic to the quartic K3 surface in $\mathbb{P}^3$. Therefore, they are all simply connected. This can be proved using the theory of periods of K3 surfaces.*

3.2 Link with the Fano variety of lines of a cubic fourfold

The K3 surface $S = \mathbb{P}^8 \cap Gr(2,6)$, of degree 14 defined above (Ex. 9) has an important link with cubic fourfolds. This was pointed out for the first time in a paper by Beauville and Donagi [BD85], which is summarized in Theorem 3.2.3. Let us first recall some definitions.

Definition 3.2.1. *($F(X)$ the Fano variety of lines) Let X be a cubic fourfold. It's Fano variety of lines, is by definition $F(X) := \{l \in Gr(2,6) | \mathbb{P}(l) \subset X\}$. This variety $F(X)$ is a hyperkähler manifold [BD85].*

Hyperkähler manifolds are the generalization of the concept of K3 surfaces to higher dimensions. One of the most known examples of Hyperkähler manifolds is the Hilbert scheme $S^{[m]}$ (see [Bea83, Sect. 6]), where S is a K3 surface. This $S^{[m]}$ parametrizes the 0-dimensional subschemes of S, of length m. It has dimension $2m$.

Definition 3.2.2. *(Hyperkähler Manifolds) A smooth complex Kähler manifold X is said to be Hyperkähler, if it is compact, simply connected, and if it admits a nowhere degenerate holomorphic 2-form ω_X, such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$.*

Theorem 3.2.3. *[BD85, Prop. 2](Fano variety of lines) Let X be a cubic fourfold, $F(X)$ its Fano variety of lines, and let S be the K3 surface of degree 14 (Ex. 9). Then, $F(X)$ is of dimension 4, and it is a deformation of $S^{[2]}$, the Hilbert scheme of length two subschemes of S.*

Remark 3.2.4. *This result will be surprisingly useful when we describe the primitive cohomology of a cubic fourfold $H^4(X, \mathbb{Z})^0$ (see Prop. 4.1.2). In fact, the primitive cohomology is non-unimodular, so that we cannot apply Milnor's theorem (Thm. 2.1.11), which classifies unimodular lattices with a fixed signature (p,q).*

The Hodge diamond of $F(X)$ has the form [BD85]:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 232 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 0 & & & &
\end{array}$$

The non trivial Betti numbers of $F(X)$ are $b_2 = b_6 = 23$ and $b_4 = 276$.

3.3 Lattice theory of K3 surfaces

For any smooth, compact, connected, complex variety X, we have the following short exact sequence, called the exponential sequence:

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{\cdot 2i\pi} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1$$

It induces the exact sequence on cohomology groups:

$$0 \to H^1(X, \mathbb{Z}) \xrightarrow{\alpha} H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

Note that there exists an obvious isomorphism $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^\times)$. By restricting c_1 to its image, we get the short exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{NS}(X) \to 0$$

Where $\mathrm{NS}(X) := \mathrm{Im}\,(c_1) \subset H^2(X, \mathbb{Z})$ is a finitely generated abelian group, called the Neron-Severi group, and the morphism c_1, the first Chern class.

From now, let S be Kähler compact surface. In particular, we have a Hodge decomposition of the middle cohomology, given by $H^2(S, \mathbb{C}) = H^{0,2}(S) \oplus H^{1,1}(S) \oplus H^{2,0}(S)$ (Prop. 2.2.1). Now, consider the map $i^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{C})$, which is induced by the embedding $i : \mathbb{Z}_S \hookrightarrow \mathbb{C}_S$. We aim to address the following question: given the above decomposition of $H^2(S, \mathbb{C})$, how is $\mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbb{Z})$ mapped into this decomposition, via the morphism $i^* \circ c_1$? The answer is that $\mathrm{Pic}(S)$ maps into $H^{1,1}(S) \subset H^2(S, \mathbb{C})$. More precisely, we have the Lefschetz theorem.

Theorem 3.3.1. *[Bar+04, IV. Thm. 2.13](Lefschetz theorem on (1,1)-classes) Let S be a compact complex surface. Then, the image of* $\mathrm{Pic}(S)$ *in* $H^2(S, \mathbb{C})$ *via the map* $i^* \circ c_1$*, is* $H^{1,1}(S) \cap i^*\left(H^2(S, \mathbb{Z})\right)$*. In other words, an element of* $H^2(S, \mathbb{C})$*, is in the image of* $\mathrm{Pic}(S)$*, if and only if it is integral, and can be represented by a real closed* $(1,1)$*-form. In this case,* $\mathrm{NS}(S) = H^{1,1}(S) \cap i^*\left(H^2(S, \mathbb{Z})\right)$.

Remark 3.3.2. *The picard number of S, denoted* $\rho(S) := \mathrm{rank}(\mathrm{NS}(S))$*, is at most equal to* $h^{1,1}(S) = b_2(S) - 2h^{2,0}(S)$*. Hence, for a K3 surfaces S,* $\rho(S) \leq 20$*. Moreover, every number* $\rho(S) \leq 20$ *is realised by some K3 surface S. Note also that when S is projective,* $1 \leq \rho(S)$.

3.3.1 The K3 lattice Λ_{K3} and the Néron-Severi lattice NS(S)

From now, let S be a K3 surface. We have that $H^1(S, \mathcal{O}_S) = 0$, hence the first Chern class morphism $\mathrm{Pic}(S) \xrightarrow{c_1} \mathrm{NS}(S)$, defined above (exact sequence 3.3), is in fact an isomorphism.

Proposition 3.3.3. *[Bar+04, VIII. Prop. 3.6] The map* $\mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbb{Z})$ *is injective, hence maps* $\mathrm{Pic}(S)$ *isomorphically onto the Neron-Severi group* $\mathrm{NS}(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z})$.

By the Hodge decomposition (Prop. 2.2.1), we have that $H^{2,0}(S) \oplus H^{0,2}(S)$, and $H^{1,1}(S)$ are defined over $\mathbb{R}$. This follows from invariance under conjugation:

$$H^{2,0}(S) \simeq \overline{H^{0,2}(S)}$$

and

$$H^{1,1}(S) \simeq \overline{H^{1,1}(S)}$$

The intersection form is positive definite on $H^{2,0}(S) \oplus H^{0,2}(S)$, and has signature $(1, h^{1,1} - 1) = (1, 19)$ on $H^{1,1}(S)$ by the Hodge index theorem (Thm. 2.3.1).

We aim to describe the K3 lattice $\Lambda_{K3} = H^2(S, \mathbb{Z})$, equipped with the cup product. It is a free abelian group of rank $b_2(S) = 22$. The cup product on $H^2(S, \mathbb{R})$ is non-degenerate. It has signature $(2h^{2,0}+1, h^{1,1}-1) = (3, 19)$ by the Hodge index theorem (Thm. 2.3.1). Moreover, the cup-product is unimodular by Poincaré duality, and even.

Now, we can use Milnor's theorem (Thm. 2.1.11), using the fact that the K3 lattice Λ_{K3} is also indefinite:

$$\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Moreover, Λ_{K3} has a discriminant equal to 1. Recall that the lattices U, and E_8 are defined in Section 2.1.1. For more details about Λ_{K3}, one can study [Ser77] or [Huy16, Chpt.1 Prop. 3.5].

Following the discussions above and Prop. 3.3.3, we have two interesting sub-lattices of Λ_{K3}. The first one, is given naturally by restricting the cup-product in $H^2(S, \mathbb{Z})$ to $\mathrm{Pic}(S) = \mathrm{NS}(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z})$, getting the Néron-Séveri non-degenerate lattice. It has signature $(1, \rho(S) - 1)$ when S is projective (such that $1 \leq \rho(S)$).

Definition 3.3.4. *(transcendental lattice) We define the transcendental lattice $T(S)$ to be the orthognal of the Néron-Séveri lattice $\mathrm{NS}(S)$ inside $H^2(S, \mathbb{Z})$.*

$$T(S) = \mathrm{NS}(S)^{\perp}.$$

The lattice $T(S)$ is a primitive sublattice of $H^2(S, \mathbb{Z})$. Moreover, it has signature $(2, 20 - \rho)$. In general $T(S)$ is difficult to compute explicitly.

Example 10. *For $S = V(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subset \mathbb{P}^3$ the quartic K3 surface, we have that:*

$$\mathrm{NS}(S) = U \oplus E_8^{\oplus 2} \oplus I_2(-8).$$

$$T(S) = I_2(8).$$

3.4 Moduli spaces of polarized K3 surfaces

The Moduli space of complex K3 surfaces in general is badly behaved. One has to add some extra data to get good Moduli spaces. To this end, one can consider polarized K3 surfaces.

3.4.1 Polarized K3 surfaces

Let S be a complex surface. Let us recall first that a primitive line bundle on S is a line bundle $\mathcal{L} \in \mathrm{Pic}(S)$, which is not a tensor power of another one.

Definition 3.4.1. *(Polarized K3 surfaces) Let S be a projective K3 surface of degree $2d$. This S is said to be polarized, if it has a fixed ample line bundle $\mathcal{L} \in \mathrm{Pic}(S)$, such that $\mathcal{L}^2 = 2d$. Moreover, $\mathcal{L}$ is called a polarization of S. If $\mathcal{L}$ is primitive, S is said to be primitively polarized.*

Remark 3.4.2.
1. *Many authors (for example [Huy16]) require the primitivity of $\mathcal{L}$ for the definition of a polarized K3 surface $(S, \mathcal{L})$. For simplicity, we will do the same.*
2. *For any (primitively) polarized K3 surface $(S, \mathcal{L})$, we have that $\mathcal{L}^{\otimes 3}$ is very ample [May72, §4 Cor. 6].*
3. *There is a natural equivalent definition of a polarization. One can define a polarized K3 surface S as a couple (S, l), where $l \in H^2(S, \mathbb{Z})$ is the primitive class of a primitive ample line bundle $\mathcal{L} \in \mathrm{Pic}(S) \simeq \mathrm{NS}(S) \simeq H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$. This is given by Lefschetz's theorem on (1,1)-classes (Thm. 3.3.1). We have that $l^2 = 2d$. Fixing an isomorphism $\Lambda_{K3} \simeq H^2(S, \mathbb{Z})$ (Section 3.3), we denote by $\Lambda^0_{K3} := \langle l \rangle^{\perp} \subset \Lambda_{K3}$, the primitive cohomology of S.*

We start with an existence proposition of polarized K3 surfaces. For a complete proof, we refer to [Bea78, VIII. Prop. 15] or [Huy16, Chpt2. Thm. 4.2] for a sketch of the proof.

Proposition 3.4.3. *For each $d \geq 2$, there exists a polarized K3 surface of degree $2d$ in $\mathbb{P}^{d+1}$.*

3.4.2 Moduli space of K3 surfaces

Let $\mathcal{M}_{2d}$ be the moduli functor of polarized K3 surfaces of a fixed degree $2d$ [Huy16, Chpt. 5.1]:

$$\mathcal{M}_{2d} : (Sch/\mathbb{C})^o \longrightarrow Sets$$

The functor associates to each $\mathbb{C}$-Scheme T of finite type, the set $\mathcal{M}_{2d}(T)$:

$$\mathcal{M}_{2d}(T) := \{(f : \mathcal{S} \to T, \mathcal{L})\},$$

consisting of families over T of polarized K3 surfaces of degree $2d$ [Huy16, Chpt. 6.2.4]. More precisely, $\mathcal{L}$ is an invertible sheaf on $\mathcal{S}$. The fibers $\mathcal{S}_t$, $t \in T$ are K3 surfaces polarized by the line bundles $\mathcal{L}|_{\mathcal{S}_t}$, where $\mathcal{L}|_{\mathcal{S}_t}^2 = 2d$.

The Moduli functor $\mathcal{M}_{2d}$ admits a coarse Moduli space M_{2d} that parameterizes polarized K3 surfaces (S, l) of degree $2d$ up to isomorphism. We have the following result proved by Šapiro and Šafarevič in [PŠ71] using the Torelli theorem 2.2.11.

Theorem 3.4.4. *[Huy16, Thm. 5.1.1] The moduli functor $\mathcal{M}_{2d}$ can be coarsely represented by a quasi-projective variety M_d.*

Proposition 3.4.5. *[Huy16, Cor. 6.4.4] The moduli space M_d, of polarized complex K3 surfaces of degree $2d$, is an irreducible quasi-projective variety of dimension 19.*

3.4.3 Polarized K3 surfaces and cubic fourfolds

Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree 3. Before we start studying cubic fourfolds in the next chapter, let us mention another important relation between the middle cohomology lattice $L = H^4(X, \mathbb{Z})$, and the K3 lattice Λ_{K3}. The lattice L and its primitive cohomology lattice L^0 are well-known (see Prop. 4.1.2). Let $h \simeq X \cap \mathbb{P}^4 \subset \mathbb{P}^5$ be the class of a hyperplane section on X, then $L^0 := (h^2)^\perp$ by definition.

For a very general cubic fourfold X, every algebraic surface $T \subset X$ is homologous to a complete intersection, that is a multiple of $h^2 \in L$. To mention the link between polarized K3 surfaces, and cubic fourfolds, we need to define special cubic fourfolds.

Definition 3.4.6. *A cubic fourfold $X \subset \mathbb{P}^5$ is said to be a special cubic fourfold iff it contains an algebraic surface T not homologous to a complete intersection.*

Thus, for a special cubic fourfold X, we have a rank two sublattice:

$\langle h^2, T\rangle \subset L = H^4(X, \mathbb{Z})$, that has even discriminant, denoted by $2d$ (see Prop. 4.2.4).

Hassett proved the following theorem [Has00, Prop. 5.2.2].

Theorem 3.4.7. *Let $X \subset \mathbb{P}^5$ be a special cubic fourfold, then there exists a polarized K3 surface (S, l), with $l \in \Lambda_{K3} = H^2(S, \mathbb{Z})$ of degree $l^2 = 2d$, such that:*

$$L^0 \supset \langle h^2, T\rangle^\perp \xrightarrow{\sim} \langle l\rangle^\perp(-1) =: \Lambda^0_{K3}(-1)$$

if and only if the following conditions on $2d$ are satisfied:
1. 4 and 9 do not divide 2d.
2. for any odd prime number $p \equiv -1$ (mod 3), p do not divide 2d.

In that case $2d$ is said to be admissible.

Remark 3.4.8. *The (-1) means that the intersection form changes its sign, and that the weight of the Hodge structures is shifted by two (Def. 2.2.8).*

This important theorem will be explained more in details in Section 4.3. It will be illustrated by some examples of cubic fourfolds in Section 4.5.

Chapter 4

Cubic fourfolds

4.1 Hodge theory and lattice theory of cubic fourfolds

The classical problem of rationality of the generic cubic fourfold has not been cleared out yet. While this problem is easy to formulate, it is very difficult to solve.

Definition 4.1.1. *(rationality) A general smooth projective variety X of dimension n is said to be rational if there exists a birational map $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$. That means that there exist $U \subset X$ and $V \subset \mathbb{P}^n$ open subsets that are isomorphic.*

Let us now concentrate on the middle cohomology $L := H^4(X, \mathbb{Z})$, of a cubic fourfold $X \subset \mathbb{P}^5$, that contains most nontrivial Hodge theoretic geometric information. This cohomology group L is also called *cohomology lattice* when equipped with the intersection product $(\cdot, \cdot)$. Let $L^0 = H^4_{prim}(X, \mathbb{Z}) := \langle h^2 \rangle^\perp$ be the *primitive cohomology lattice*, where $h \in H^2(X, \mathbb{Z})$ is the *hyperplane class* defined by the embedding $X \subset \mathbb{P}^5$. We recall that L^0 is an even lattice (see [Has00, §2]). By the Lefschetz hyperplane theorem and Poincaré duality, we get:

$$H^2(X, \mathbb{Z}) = \mathbb{Z}h, \quad H^6(X, \mathbb{Z}) = \mathbb{Z}\frac{h^3}{3}.$$

Now we can briefly sketch the computation of the two lattices L and L^0. First of all, we have $(h^2)^2 := (h^2, h^2) = 3$, and therefore L is odd.
By Poincaré duality, L is unimodular. Moreover, we have that the signature of L is equal to $(21, 2)$ using the Hodge-Riemann bilinear relations (see Prop. 2.3.2).

4.1.1 The cubic lattice and Abel-Jacobi map

Considering the discussion above and using Milnor's theorem 2.1.11, we get the following proposition.

Proposition 4.1.2. *[Has00, Prop. 2.1.2] The middle cohomology lattice L is equal to* $I_{21,2}$*. More precisely, we have:*

$$L := H^4(X, \mathbb{Z}) \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_3.$$

The computation of the primitive cohomology L^0 *is harder because it is not unimodular. Thus, Milnor's theorem cannot be applied. Its computation is based on Theorem 3.2.3. We have that:*

$$L^0 := \langle h^2 \rangle^\perp \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2.$$

The lattices U*,* E_8*,* I_3*, and* $I_{21,2}$ *are defined in Section 2.1.1.*

Before we state the proof of this proposition, let us recall that the cubic lattice L has naturally an integral Hodge structure (Def. 2.2.7).

Proof. We will follow the computation of L^0 as it is done by Hassett in [Has00, Prop. 2.1.2]. First, we recall that the Fano variety of lines of a cubic fourfold X is by definition:

$$F := \{l \in Gr(2, 6) \mid \mathbb{P}(l) \subset X\}.$$

The variety F is a smooth hyperkähler manifold of dimension 4 [BD85]. Let $M := H^2(F, \mathbb{Z})$, it is equipped with a symmetric bilinear form (see [BD85] [Bea83]). Let $M^0 := \langle g \rangle^\perp$ be its primitive cohomology, where g is the class of the hyperplane section on F. Let S be the K3 surface of degree fourteen (Ex. 9). Recall that F is a deformation of the Hilbert scheme $S^{[2]}$ of length two subschemes of S (Thm. 3.2.3). This will be crucial for the computation of M and M^0 knowing the K3 lattice Λ_{K3} (Sect. 3.3). The lattice L^0 is deduced from M^0 via the Abel-Jacobi map $\alpha : L = H^4(X, \mathbb{Z}) \longrightarrow M = H^2(F, \mathbb{Z})$ defined as follows.

Let $Z \subset F \times X$ be the universal line $Z := \{(l, x) \in F \times X \mid x \in l\}$, we have the following projections:

$$\begin{array}{ccc} Z & \xrightarrow{p} & F \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

The Abel-Jacobi map is defined by:

$$\alpha = p_* q^* : H^4(X, \mathbb{Z}) \longrightarrow H^2(F, \mathbb{Z}),$$

it is an isomorphism of Hodge structures [BD85, Prop. 4]. Moreover, it verifies:

$$\alpha(h^2) = g.$$

Restricting the Abel-Jacobi map α to the primitive cohomology, we get an isomorphism of Hodge structures $\alpha : L^0 \xrightarrow{\sim} M^0(-1)$, where -1 means that the sign of the bilinear forms is reversed and that the weight is shifted by two (see Def. 2.2.8). This is a result of [BD85].

Proposition 4.1.3. *[BD85, Prop. 6] The Abel-Jacobi map induces an isomorphism between L^0 and M^0. Moreover, for any $x, y \in L^0$ we have $(\alpha(x), \alpha(y)) = -(x, y)$.*

It remains to compute M and M^0. Using Theorem 3.2.3, we get that:

$$M := H^2(F, \mathbb{Z}) \simeq H^2(S^{[2]}, \mathbb{Z}).$$

On the other hand, we have the canonical orthogonal decomposition proved by Beauville [Bea83]:

$$H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta, \text{ where } \delta \text{ is of type (1,1), and } \delta^2 = -2.$$

The final result is deduced knowing the $K3$ lattice Λ_{K3}, described in Section 3.3:

$$M \simeq H^2(S^{[2]}, \mathbb{Z}) = \Lambda_{\mathrm{K3}} \oplus_\perp \mathbb{Z}\delta = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2).$$

Recall that $M^0 := \langle g \rangle^\perp$. Following [Bea83] and [BD85, Prop. 6], we have more precisely $g = 2f - 5\delta$ where $f \in H^2(S, \mathbb{Z})$ is a polarization that verifies $f^2 = 14$. One then finds that :

$$M^0 := \langle g \rangle^\perp = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}.$$

In the end, the proof is completed using the fact that $\alpha : L^0 \xrightarrow{\sim} M^0(-1)$.

□

4.2 The Moduli space $\mathcal{C}$ and Hassett divisors $\mathcal{C}_d$

4.2.1 The moduli space $\mathcal{C}$

Cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by:

$$\mathbb{P}\left(\mathbb{C}[u, v, w, x, y, z]_3\right) \simeq \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3))) \simeq \mathbb{P}^{55}.$$

Smooth cubic hypersurfaces in $\mathbb{P}^5$ correspond to homogeneous cubic polynomials with a non-vanishing discriminant. Hence, cubic fourfolds live inside the complement of a closed hypersurface, they are parametrized by an open Zariski subset $U \subset \mathbb{P}^{55}$.
Cubic fourfolds are isomorphic if and only if they are projectively equivalent by the action of PGL_6 in $\mathbb{P}^5$.
Considering cubic fourfolds up to isomorphism, we get the GIT quotient [MFK94]:

$$\mathcal{C} := U//\mathrm{PGL}_6.$$

This is the coarse Moduli space of the fine Deligne-Mumford moduli stack of cubic fourfolds. We have:

$$\dim(\mathcal{C}) = \dim(U) - \dim\left(\mathrm{PGL}_6\right) = 55 - 35 = 20.$$

Thus, by GIT theory (see [MFK94] Section 4.2 for details), the moduli space $\mathcal{C}$ is a 20-dimensional quasi-projective variety.

4.2.2 Hodge conjecture for cubic 4-folds and the lattice $A(X)$

To study the problem of rationality of cubic fourfolds, we consider the lattice of integral middle Hodge classes of X:

$$A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = H^2(X, \Omega_X^2) \cap H^4(X, \mathbb{Z}),$$

which comes equipped with the integer valued intersection form $(\cdot, \cdot)$. The Hodge-Riemann bilinear relations imply that $A(X)$ is a positive definite lattice. For cubic fourfolds, the (integral) Hodge conjecture holds (see Appendix B.0.11 for more details and [Voi07, Theorem 18]). Moreover, algebraic, rational and homological equivalences coincide for cycles of codimension 2 (Def. B.0.9). The upshot is that the cycle map $\mathrm{CH}^2(X) \to H^4(X, \mathbb{Z})$ is injective, where $\mathrm{CH}^2(X)$ denotes the Chow group of codimension 2 cycles on X up to rational equivalence (see [Col15, §5] or [BP20]).

Definition 4.2.1. *(Chow groups) Let $Z^r(X)$ be the free abelian group generated by codimension-r subvarieties of X, with coefficients in $\mathbb{Z}$. Elements of Z^r are called algebraic cycles of codimension r. We organize these cycles up to rational equivalence $\sim$ (Def. B.0.9) giving the Chow groups $\mathrm{CH}^r(X) := Z^r/\sim$. These groups come with a natural map $\mathrm{CH}^r(X) \longrightarrow H^{2r}(X, \mathbb{Z})$ given by the fundamental class (Def. B.0.5) and called the cycle class map. (see Def. B.0.6 for more details).*

Notably, by the Hodge conjecture, the algebraic cycles are the $(2,2)$-part of $H^4(X, \mathbb{Z})$. The generic cubic fourfold $X \in \mathcal{C}$ has $\mathrm{rk}\, A(X) = \mathrm{rk}\langle h^2 \rangle = 1$, and a cubic fourfold is *special* (Def. 3.4.6) whenever $\mathrm{rk}\, A(X) > 1$. This is equivalent to saying that X contains a surface that is not homologous to the 2-dimensional linear section h^2. Let us call $d(A(X)) \in \mathbb{Z}$ the discriminant of the lattice $A(X)$. This is just the determinant of the Gram matrix.

4.2.3 Special cubic fourfolds

Following the discussion above about the lattice $A(X)$, Hassett [Has00] initiated the study of the Noether-Lefschetz locus:

$$\{X \in \mathcal{C} : \mathrm{rk}(A(X)) > 1\},$$

which is the locus of *special* (Def. 3.4.6) cubic fourfolds. This leads to the following definition of a labelled special cubic fourfold introduced by Hassett.

Definition 4.2.2. *A labelling of a special cubic fourfold is a a rank two saturated (or equivalently primitive) sub-lattice K_d of algebraic cycles [Has00]:*

$$h^2 \in K_d \subset A(X),$$

where d is the discriminant of the intersection form on the sub-lattice K_d. For each special cubic fourfold, we can associate at least one $K_d =< h^2, T >$ as above, where $T \subset X$ is an algebraic surface of degree $d_T = (h^2, T)$. The lattice K_d has a Gram matrix of the form:

$$K_d := \begin{array}{|c|c|c|} \hline & h^2 & T \\ \hline h^2 & 3 & d_T \\ \hline T & d_T & (T,T) \\ \hline \end{array}, \quad d = 3(T,T) - d_T^2 \tag{4.1}$$

(X, K_d) is called a labelled special cubic fourfold.

Remark 4.2.3. *We recall that a sublattice $K_d \subset L$ is said to be saturated iff $K_d^{sat} := (K_d^{\perp})^{\perp} = K_d$ or equivalently iff K_d is primitive (Def. 2.1.8) which means that L/K_d is torsion free.*

Let (X, K_d) be a labelled special cubic fourfold, and $T \subset X$ be a smooth algebraic surface as above. We get:

$$c_1(\mathcal{T}_X) = 3h, \quad c_2(\mathcal{T}_X) = 6h^2.$$

Thus, we can compute the self-intersection as follows [HV16, Sect. 2.2] or [Has00, Sect. 4.1]:

$$\begin{aligned}(T,T) &= c_2\left(\mathcal{N}_{T/X}\right) = c_2(\mathcal{T}_X \mid T) - c_2(\mathcal{T}_T) - c_1(\mathcal{T}_T)\, c_1(\mathcal{T}_X \mid T) + c_1(\mathcal{T}_T)^2 \\ &= 6H^2 + 3HK_T + K_T^2 - \chi(T)\end{aligned}$$

Here $\mathcal{N}_{T/X}$ is the normal bundle of $T \subset X$, and $\mathcal{T}_X$, $\mathcal{T}_T$ are the tangent bundles of X and T, respectively. We also have that $H := h|_T$ and $\chi(T)$ is the topological Euler characteristic.

4.2.4 Hassett divisors $\mathcal{C}_d \subset \mathcal{C}$

The locus in the moduli space $\mathcal{C}$ of special cubic fourfolds with a labelling of discriminant d is denoted $\mathcal{C}_d$. These loci $\mathcal{C}_d \subset \mathcal{C}$ are codimension one subvarieties when they are not empty. They are called Hassett divisors [Has00]. The locus of special cubic fourfolds is the countably infinite union of the irreducible divisors $\mathcal{C}_d$ in $\mathcal{C}$, where d takes certain integer values.

Proposition 4.2.4. *[Has00] $\mathcal{C}_d$ is an non-empty irreducible divisor of $\mathcal{C}$ if and only if $d > 6$ and $d \equiv 0$ or 2 (mod 6); it is empty otherwise.*

Only very few $\mathcal{C}_d$'s can be defined explicitly in terms of particular surfaces contained in X. For example, $\mathcal{C}_8$ is the locus of cubic fourfolds containing a plane [Voi86, §3], $\mathcal{C}_{12}$ is defined as the closure of the locus of cubics containing a cubic scroll, $\mathcal{C}_{14}$ as the closure of the locus of cubics containing a quartic scroll (or equivalently a quintic del Pezzo surface) [Fan42],[Tre93]. $\mathcal{C}_{20}$ is the closure of the locus of cubic fourfolds containing a Veronese surface. The divisor $\mathcal{C}_{18}$ is the closure of the locus of cubic fourfolds containing an elliptic ruled surface. In fact, for each $d \leq 44$, $d \neq 42$, Nuer [HNu15] explicitly describes $\mathcal{C}_d$ as the locus of cubic fourfolds containing a particular surface. The case $d = 42$ has been described by Lai in [Lai17]. He showed that $\mathcal{C}_{42}$ is the closure of the locus of cubic fourfolds containing a degree 9 rational scroll [Lai17, Thm. 0.3].

We study the divisor $\mathcal{C}_8$ and the divisor $\mathcal{C}_{14}$ more in details in Section 4.5.1 and Section 4.5.4, respectively.

4.2.5 Intersection of surfaces on a cubic fourfold

Let $X \subset \mathbb{P}^5$ be a smooth cubic 4-fold. Let $S \subset X$ be a smooth surface and let $P \subset X$ be a plane s.t. the scheme theoretic intersection $S \cap P$ is a smooth curve C of genus g and degree d. Let us now compute the intersection number $S \cdot P \in \mathbb{Z}$ inside X under the previous hypothesis. By [Ful98, Prop. 9.1.1, third formula] we have

$$S \cdot P = c_1(\mathcal{N}_{S/X|C}) \cdot c_1(\mathcal{T}_{P|C})^{-1} \cdot c_1(\mathcal{T}_C) \cap C. \tag{4.2}$$

Let us denote by K_Y the canonical class of an arbitrary smooth projective variety Y. It is straightforward to see that $c_1(\mathcal{T}_C) \cap C = 2 - 2g(C) = -d(d-3)$ and $c_1(\mathcal{T}_{P|C})^{-1} \cap C = K_P \cdot C = -3d$. By using the short exact sequence

$$0 \to \mathcal{T}_{S|C} \to \mathcal{T}_{X|C} \to \mathcal{N}_{S/X|C} \to 0,$$

we get the following equalities:

$$c_1(\mathcal{N}_{S/X|C}) \cap C = c_1(\mathcal{T}_{X|C}) \cap C + c_1(\mathcal{T}_{S|C})^{-1} \cap C = -K_X \cdot C + K_S \cdot C = 3d + K_S \cdot C.$$

By combining the previous calculations with (4.2) we deduce:

$$S \cdot P = K_S \cdot C - d(d-3) = K_S \cdot C + 2 - 2g(C) = -\deg(\mathcal{N}_{C/S}). \tag{4.3}$$

More generally, always from [Ful98, Prop. 9.1.1] (see [BRS19, Prop. 2.8] for this particular form of the statement), we get the following result.

Proposition 4.2.5. *Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface and let $S_1, S_2 \subset X$ be two smooth surfaces such that the scheme theoretic intersection $S_1 \cap S_2$ contains a smooth curve C of degree d and genus g. Then:*

$$mult_C(S_1, S_2) = 3d + K_{S_1} \cdot C + K_{S_2} \cdot C + 2 - 2g, \tag{4.4}$$

where K_{S_i} *denotes the canonical class of* S_i *and* $mult_C(S_1 \cdot S_2)$ *the multiplicity of intersection of* S_1 *and* S_2 *along* C.

4.3 Associated K3 surfaces

Theorem 3.4.7 establishes a Hodge theoretical link between a polarized K3 surface S of degree d, and a special cubic fourfold $X \in \mathcal{C}_d$. Such a K3 surface is said to be associated to X.

Definition 4.3.1. *Let* (X, K_d) *denote a labelled special cubic fourfold. We say that a polarized* K3 *surface* (S, f) *is associated to* (X, K_d) *if there exists an isomorphism of lattices:*

$$H^4(X, \mathbb{Z}) \supset K_d^{\perp} \xrightarrow{\sim} f^{\perp} \subset H^2(S, \mathbb{Z})(-1)$$

respecting Hodge structures. The (-1) means that the intersection form changes its sign, and that the weight of the Hodge structures is shifted by two (see Def. 2.2.8).

In [Kuz10], Kuznetsov asserts a famous conjecture about the rationality of cubic fourfolds. In the Hodge theoretic language, it is equivalent to saying that a special cubic fourfold $X \in \mathcal{C}_d$ is rational if and only if d is admissible (see Thm. 3.4.7).

Conjecture 4.3.2 (Kuznetsov)**.** *A labelled cubic fourfold* $(X, K_d) \in \mathcal{C}_d$ *is rational if and only if* X *admits an associated K3 surface. This is equivalent to the following conditions on the discriminant d:*
1. 4 *and* 9 *do not divide d.*
2. for any odd prime number $p \equiv -1$ *(mod 3), p does not divide d.*

The rationality of cubic fourfolds in $\mathcal{C}_d$ is known for the first admissible numbers 14, 26, 38, and more recently for d=42 confirming the first cases of Kuznetsov's conjecture. On the other hand, for these values of d, there exists explicit realizations of associated K3 surfaces.

The first well studied cubic fourfolds that were shown to be rational by classical algebraic geometers like Morin [Mor40] and Fano [Fan42], are cubic fourfolds containing a smooth quartic rational normal scroll (see Example 11). These cubics are inside $\mathcal{C}_{14}$. They admit an associated K3 surface of degree 14 (Ex. 9). We explain this more in details in Section 4.5.7. More recently in [BRS19], every cubic fourfold in $\mathcal{C}_{14}$ is shown to be rational (see Section 4.5.6).

For the rationality of cubic fourfolds inside $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$, one can see the work of Russo and Stagliano [RS19, Thm. 7 and Thm. 4] or [RS21]. Moreover, the authors exhibit explicit relations between the rationality of these cubics and their associated K3 surfaces in a more recent paper [RS23, Sect. 3.4]. In this last work, they also prove the rationality of every cubic fourfold $X \in \mathcal{C}_{42}$ for the first time [RS23, Thm. 5.12]. Furthermore, they show the existence of a degree 42 K3 surface associated to X [RS23, Sect. 5.5].

4.4 OADP surfaces and rationality of cubic 4-folds

4.4.1 OADP varieties, definitions

In chapter 5, we establish the rationality of some cubic fourfolds inside the intersections of Hassett divisors $\mathcal{C}_{20} \cap \mathcal{C}_8$ and $\mathcal{C}_{12} \cap \mathcal{C}_8$ (see Section 5.2.2 and 5.3.2). To this end, one of the techniques we use is to find OADP surfaces (Def. 4.4.1) inside the cubic fourfold. This technique is based on Theorem 4.4.10, it was used for example in [BRS19]. Varieties with one apparent double point or shortly OADP varieties appeared in the work of many classical algebraic geometers. In 1901, Severi [Sev01] tried to classify smooth OADP surfaces, in particular those inside $\mathbb{P}^5$ for the first time. He proved that the only smooth OADP surfaces contained in $\mathbb{P}^5$ are the quartic rational normal scrolls and the quintic Del Pezzo surfaces. Later, this classification was revisited by many other mathematicians. In 1932, Edge [Edg32] introduced new examples of rational OADP varieties in higher dimensions called after his name.

More recently, Russo [Rus00] completed the proof of Severi to get Theorem 4.4.6. He added to the classification, the quartic rational normal scrolls of type (2,1), namely $S_{2,1} \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (Lemma 4.4.4) that were not considered by classical geometers. Before stating this theorem, let us recall some definitions.

Definition 4.4.1. *(OADP variety) Let Y be an equidimensional reduced scheme in $\mathbb{P}^{2n+1}$ of dimension n. The scheme Y is called a (generalized) variety with one apparent double point if through a general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to Y, that is a unique line cutting Y scheme theoretically in a reduced length two scheme.*

Remark 4.4.2. *Note that this definition does not require the variety Y to be smooth. Indeed, the proof of our results— namely, Theorem 5.2.1 and Theorem 5.3.1, involve reducible OADP surfaces. We discuss this further in Remark 4.4.9.*

Definition 4.4.3. *(Small scheme) A scheme $X \subset \mathbb{P}^N$ is said to be small if, for every linear space $P \subset \mathbb{P}^N$ such that $Y := P \cap X$ has finite length* $\deg(Y)$*, we have* $\dim(\langle Y \rangle) = \deg(Y) - 1$*, that is the* $\deg(Y)$ *points are linearly independent in $\mathbb{P}^N$.*

The name OADP variety is usually reserved for the irreducible reduced scheme satisfying the previous condition. This somehow bizarre name comes from the fact that the projection of Y from a general point into $\mathbb{P}^{2n}$ acquires a unique singular point, which is *double*. This means that its tangent cone is the union of two $\mathbb{P}^n$ intersecting at the singular point.

Let $\mathbf{G} := \mathbb{G}(1, 2n+1)$ be the Grassmannian of lines of $\mathbb{P}^{2n+1}$. We define the *abstract secant variety* denoted $S_Y \subset \mathbf{G} \times \mathbb{P}^{2n+1}$ of a variety $Y \subset \mathbb{P}^{2n+1}$ to be the restriction of the universal family of the Grassmannian $\mathbf{G}$ to the closure of the image of the rational map π which associates to a couple $(p, q) \in Y \times Y$ (with $p \neq q$) the line $\overline{pq}$ spanned by them (Diagrams 4.4.1 below). Let Γ be the graph of π when restricted to $Y \times Y - \Delta_Y$, where Δ_Y is the diagonal, which is the base locus of π. We call *secant variety* $S(Y)$ of Y the variety that parametrizes the secant lines to Y that are inside $\mathbb{P}^{2n+1}$, namely $S(Y) := \pi_2(\Gamma)$. The map π_2 being the natural projection $Y \times Y \times \mathbf{G} \to \mathbf{G}$.

$$\begin{tikzcd}
 & (Y \times Y) \times \mathbf{G} & \\
 & \cup & \\
 & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\begin{matrix} Y \times Y \\ (p,q) \end{matrix} \arrow[rr, dashed, "\pi"] & & \begin{matrix} \mathbf{G} = \mathbb{G}(1, 2n+1) \\ \overline{pq} \end{matrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}^{2n+1} & & & \mathbf{G} \supset S(Y) = \pi_2(\Gamma) \\
S_Y = p_1^{-1}(S(Y)) & \subset & I = \{(x, l) \mid x \in l\} \arrow[ull, "p_2"'] \arrow[ur, "p_1"] & \\
 & & \cap & \\
 & & \mathbb{P}^{2n+1} \times \mathbf{G} &
\end{tikzcd}$$

When Y is an OADP variety, the tautological morphism $p_2 : S_Y \to \mathbb{P}^{2n+1}$ is birational. This means that, by Zariski Main Theorem, the locus of points of $\mathbb{P}^{2n+1}$ through which there passes more than one secant line, has codimension at least two. This is because it is the exceptional locus inside $\mathbb{P}^{2n+1}$. Therefore, its intersection with a cubic hypersurface $X \subset \mathbb{P}^{2n+1}$ has codimension at least one. Consequently, one can view S_Y as a projective line bundle over $Y^{(2)}$; the symmetric product of Y. In this case, the cubic fourfold can be viewed as a section of this fibration, and is hence birational to $Y^{(2)}$, implying its rationality. A similar property holds for reducible OADP varieties.

Now, we state the Theorem of Russo [Rus00, Thm. 3], which classifies smooth OADP surfaces in $\mathbb{P}^5$.

Lemma 4.4.4. *[Rus00, Lemma 4] Let $S_{a,b} \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be a smooth irreducible divisor on the Segre 3-fold given by the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(a,b)$, with $(b > 0)$. $S_{a,b}$ is said to be of type (a,b). If $S_{a,b}$ have one apparent double point, then $(a,b) \in \{(0,2),(1,2),(2,1)\}$.*

Definition 4.4.5. *For $(a,b) \in \{(0,2),(2,1)\}$, the quartic surfaces $S_{a,b} \subset \mathbb{P}^5$ are called rational normal scrolls. The surface $S_{1,2} \subset \mathbb{P}^5$ is called a quintic Del Pezzo surface (see Appendix A for more details about these surfaces).*

Theorem 4.4.6. *[Rus00, Thm. 3] Let S be a smooth non-degenerate surface in $\mathbb{P}^5$ having one apparent double point or OADP surface. Then S is a Del Pezzo surface of degree 5 or one of the two rational normal scrolls of degree 4, namely, $S_{0,2}$ and $S_{2,1}$.*

We end this Section by a Corollary that we borrow from [BRS19].

Corollary 4.4.7. *[BRS19, Cor. 2.7] Let $Y \subset \mathbb{P}^N$ be a non degenerate reduced algebraic set scheme theoretically defined by quadratic forms such that their Koszul syzygies are generated by linear syzygies. If through a general point of $\mathbb{P}^N$ there passes a finite number of secant lines to Y, then $Y \subset \mathbb{P}^N$ is a generalized OADP variety.*

In particular a small algebraic set $Y \subset \mathbb{P}^N$ such that through a general point of $\mathbb{P}^N$ there passes a finite number of secant lines to Y is a generalized OADP variety.

4.4.2 Cubic fourfolds containing a quartic rational normal scroll

Let us motivate the interest in OADP surfaces by the classical example of cubic fourfolds containing a quartic rational normal scroll studied by [Mor40] and revisited by [Fan42].

Example 11. *Let $X \subset \mathbb{P}^5$ be a cubic fourfold containing a quartic rational normal scroll. One can take $S_{2,1} \subset X$ for example. We know that X is rational [HV16, 1.4.Prop.4].*

In fact, the linear system of quadric hypersurfaces in $\mathbb{P}^5$, passing through $S_{2,1}$, and denoted by $\left|I_{S_{2,1}}(2)\right|$ has dimension 6. It defines a rational map $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$.

This rational map induces a birational morphism from $X \subset \mathbb{P}^5$ onto a smooth quadric of dimension 4, denoted $Q_4 \subset \mathbb{P}^5$.

More exactly, φ contracts the bi-secant lines to $S_{2,1}$ that cover the whole space $\mathbb{P}^5$ as $S_{2,1}$ is OADP. This is because the degree of the image by φ of a bi-secant line ℓ to $S_{2,1}$ is 0 as it intersects the base locus in two points. Conversely, the fiber over *the generic point in Q_4 is a projective line intersecting the base locus in two points [BRS19, Prop. 2.5]. Moreover, the generic bi-secant line l to $S_{2,1}$ is tri-secant to the cubic fourfold X. Hence, φ is generically 1 to 1 when restricted to X, as the other two points of the intersection $\ell \cap X$ are inside the base locus $S_{2,1}$.*

(see the work of Fano [Fan42, 4, Thm. 4.3] or Tregub [Tre84, Prop. 3], and [BRS19, Thm. 3.2] for a modern proof).

Remark 4.4.8. *1. A similar argument applies if we replace the quartic rational normal scroll by a quintic Del Pezzo surface S. The linear system of quadrics through S gives a dominant map from $\mathbb{P}^5$ to $\mathbb{P}^4$ [Fan42], [RS21] or [HV16, Sect. 1.5].*

2. Morin [Mor40] mistakenly claimed that a general cubic fourfold is rational. This is because he thought that a general cubic fourfold contains a quartic rational normal scroll, which is incorrect.

Remark 4.4.9. *(Why reducible OADP)*
The surfaces $S_{a,b}$ defined above (Lemma 4.4.4) are the only smooth irreducible OADP surfaces in $\mathbb{P}^5$ (Thm. 4.4.6). These are precisely the surfaces that characterize the Hassett divisor $\mathcal{C}_{14}$ (see Sect. 4.5.4). In particular, the rationality argument using these smooth surfaces $S_{a,b}$ (see Ex. 11) cannot be extended to establish rationality for cubic fourfolds not contained in the divisor $\mathcal{C}_{14}$. Indeed, quartic rational normal scrolls and quintic Del Pezzo surfaces are all contained in cubics inside $\mathcal{C}_{14}$. One way to generalize this technique is to consider reducible OADP surfaces, which can be found, for instance, inside cubic fourfolds in $\mathcal{C}_{20}$ (see Sect. 5.3.3) and cubics in $\mathcal{C}_{12}$ (see Sect. 5.2.3).

4.4.3 Reducible OADP surfaces and rationality

Consider the general setting of smooth cubic hypersurfaces $X \subset \mathbb{P}^{2n+1}$ containing a possibly reducible OADP variety Y, of dimension n (as it is defined in Definition 4.4.1 and [Rus00, Sect. 5]). Let $Y^{(2)}$ be the symmetric product of Y. When Y is irreducible we can establish a birational map:

$$\begin{aligned} \varphi : Y^{(2)} &\dashrightarrow X \\ (v, w) &\mapsto X \cap \ \langle v, w\rangle \end{aligned}$$

where $\langle v, w\rangle$ is the line joining each two distinct points $v, w \in Y$. The rational map φ associates to each $(v, w) \in Y^{(2)}$, the third point of intersection of $\langle v, w\rangle \cap X$, when the line is not contained in X. Since Y is an OADP variety, through a general point $x \in X - Y$ there passes a unique secant line to Y, and thus x has only one preimage via φ. Hence, the map φ is birational. This discussion leads to Theorem 4.4.10, using the fact that $Y^{(2)}$ is rational.

When Y is reducible and has two irreducible components Y_1 and Y_2, the symmetric product $Y^{(2)}$ must be replaced by $Y_1 \times Y_2$, and an analogous argument applies.

Theorem 4.4.10. *[Rus00, Prop. 4 and Prop. 9] Let $X \subset \mathbb{P}^{2n+1}$ be a smooth cubic hypersurface containing an n-dimensional variety Y having one apparent double point. Then:*

1. *If Y is irreducible, then the symmetric product $Y^{(2)}$ is a rational variety. Since X is birational to $Y^{(2)}$, X is rational as well.*

2. *If Y is reducible and has two irreducible components Y_1 and Y_2, then $Y_1 \times Y_2$ is a rational variety. Since X is birational to $Y_1 \times Y_2$, X is rational as well.*

Proof. Suppose that Y is irreducible, then $Y^{(2)}$ is birational to the secant variety of Y denoted $S(Y)$. Moreover, since Y is an OADP variety, $S(Y)$ is rational, as we now prove. Let $H \subset \mathbb{P}^{2n+1}$ be a general hyperplane. As Y is an OADP variety, through a generic point $p \in H$, there exists a unique secant line to Y passing through p, denoted l_p. This defines a rational map:

$$\begin{aligned} H &\dashrightarrow S(Y) \\ p &\mapsto l_p \end{aligned}$$

Since Y is non-degenerate, a general secant line to Y intersects H in a unique point. Therefore, $S(Y)$ is birational to H, and hence rational. A similar argument applies when Y is reducible.

□

Example 12. *Let $S \subset \mathbb{P}^5$ be a smooth rational normal scroll of degree 3 and $P \subset \mathbb{P}^5$ a plane. When $P \cap S$ is a line of the ruling of S, we get that $S \cup P$ is an OADP surface by Prop. 5.2.3. In Appendix C, Ex. 17 (7), we give an explicit equation of a rational cubic fourfold $X \subset \mathbb{P}^5$ containing the OADP surface $S \cup P$. This cubic fourfold is inside the intersection of the Hassett divisors $\mathcal{C}_8 \cap \mathcal{C}_{12}$ (see Section 5.2).*

Example 13. *Let $V \subset \mathbb{P}^5$ be a smooth Veronese surface and P a plane such that the intersection $V \cap P$ is a conic. Then, the union $V \cup P$ is a reducible OADP surface by Prop. 5.3.2. In Appendix C, Ex. 18 (1), we give an explicit equation of a rational cubic fourfold $X \subset \mathbb{P}^5$ containing the OADP surface $V \cup P$. This cubic fourfold is inside $\mathcal{C}_8 \cap \mathcal{C}_{20}$ (see Section 5.3).*

4.5 Hasset divisors $\mathcal{C}_8$, $\mathcal{C}_{12}$, $\mathcal{C}_{14}$ and $\mathcal{C}_{20}$

4.5.1 The divisor $\mathcal{C}_8$ and the quadric surface fibration

The Hassett divisor $\mathcal{C}_8$ (Def. 4.2.4) is defined as the locus of special cubic fourfolds $X \subset \mathbb{P}^5$ containing a plane P (see [Voi86, §3]), it has the following intersection properties:

$$P^2 = 3 \text{ and } P \cdot h^2 = 1.$$

Hence, $X \in \mathcal{C}_8$ has a labelling with this shape:

$$K_8 = \begin{array}{|c|c|c|} \hline & h^2 & \mathrm{P} \\ \hline h^2 & 3 & 1 \\ \hline \mathrm{P} & 1 & 3 \\ \hline \end{array}, \tag{4.5}$$

The linear projection $\pi_P : X \dashrightarrow \mathbb{P}^2$ with center P, is resolved by blowing up P

$$\widetilde{\pi}_P : Bl_P(X) \longrightarrow \mathbb{P}^2$$

into a morphism, that has naturally a quadric bundle structure. Indeed, the general fiber over a point $p \in \mathbb{P}^2$ is the intersection of X with the linear span $\langle P, p\rangle = \mathbb{P}^3 \subset \mathbb{P}^5$. The general intersection $\mathbb{P}^3 \cap X$ is a surface $S = P + Q$ that has degree 3. Hence, the fiber over $p \in \mathbb{P}^2$ inside X must be a quadric surface Q. Under mild conditions on the plane (see [ABB14, Prop. 1.2.5]), the quadric bundle degenerates on a smooth sextic curve $D \subset \mathbb{P}^2$.

4.5.2 Rationality inside $\mathcal{C}_8$

When the quadric surface bundle $\tilde{\pi}_P$ has a rational section, one can show that $Bl_P(X)$ is rational, and hence also X. Indeed, in this case the quadric surface fibration over $\mathbb{P}^2$ is birational to a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$.

Proposition 4.5.1. *[Has99, Prop. 2.3] Let $q : \mathcal{Q} \to B$ be a quadric surface bundle over a rational projective variety. Let Q denote the class of the generic fiber of q inside $H^4(\mathcal{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q})$. If there is a class $T \in H^4(\mathcal{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q})$ which has odd intersection with Q, then $\mathcal{Q}$ is rational over $\mathbb{C}$.*

In our situation, this is equivalent to saying that there is an odd degree multi-section of the quadric bundle $\widetilde{\pi}_P$, and we will use several times this Prop. 4.5.1 in the rest of the thesis to check that certain classes of cubic fourfolds are rational.

Cubic fourfolds inside $\mathcal{C}_8$ for which the linear projection $\tilde{\pi}_P$ admits a rational section, form infinitely countably many irreducible divisors in $\mathcal{C}_8$ that parametrize rational cubic fourfolds.

Theorem 4.5.2. *[Has99, Thm. 4.2] There is a countably infinite collection of divisors $W_i \subset \mathcal{C}_8 \subset \mathcal{C}$, which parametrize rational cubic fourfolds. Each of these is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$.*

Remark 4.5.3. *Note that 8 is not admissible. Therefore, if the conjecture of Kuznetsov (Conj. 4.3.2) is true, then all the rational examples of cubic fourfolds inside $\mathcal{C}_8$ must be inside another Hassett divisor $\mathcal{C}_d$ with d admissible.*

One can see the paper [Add+19], where the authors prove similar results to those stated above. They study cubic fourfolds inside $\mathcal{C}_{18}$ containing an elliptic ruled surface. These cubics are fibered in sextic Del Pezzo surfaces over $\mathbb{P}^2$, they are rational whenever the fibration admits a rational section. The authors show that these rational cubics correspond to an open subset of a countably infinite union of codimension-two loci $\bigcup \tilde{W}_i \subset \mathcal{C}_{18} \subset \mathcal{C}$ [Add+19, Thm. 1].

Let us return to $\mathcal{C}_8$, one could address a natural question: are there examples of rational cubic fourfolds outside these irreducible divisors $W_i \subset \mathcal{C}_8$? The answer is yes.

This was mentioned for the first time in [Aue+14]. The authors study the intersection of the Hassett divisor $\mathcal{C}_8$ with $\mathcal{C}_{14}$, where they exhibit five irreducible components [Aue+14, Thm. A]. These components have codimension 2 inside $\mathcal{C}$. In particular, they prove that there exists an irreducible component $\mathcal{A} \subset \mathcal{C}_8 \cap \mathcal{C}_{14} \subset \mathcal{C}$, for which a general member $X \in \mathcal{A}$ is Pfaffian, with no rational section for the quadric surface bundle $\tilde{\pi}_P$ [Aue+14, Cor. B]. Pfaffian cubic fourfolds are rational [BD85, Prop. 5].

Definition 4.5.4. *(Pfaffian cubics) Let $M = (m_{ij})$ be a skew-symmetric 6×6 matrix, we have the following formula:*

$$\det(M) = \mathrm{Pf}(M)^2$$

where $\mathrm{Pf}(M)$ is a homogeneous form of degree 3 in the entries of M, It is called the Pfaffian of M. The Pfaffian gives rise to the following hypersurface:

$$X = \{\mathrm{Pf}(M) = 0\} \subset \mathbb{P}^5.$$

It is called a Pfaffian cubic fourfold.

One can also study the work of Hanine Awada [Awa24], where she describes the irreducible components of the intersections $\mathcal{C}_8 \cap \mathcal{C}_{26}$ and $\mathcal{C}_8 \cap \mathcal{C}_{38}$ [Awa24, Thm. 1.3]. Using the fact that every cubic fourfold in $\mathcal{C}_{26}$ or in $\mathcal{C}_{38}$ is rational [RS19, Thm. 4 and Thm. 7], she proves the following theorem.

Theorem 4.5.5. *[Awa24, Thm. 1.3]*

1. *Four of the components in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{26}$ contain rational cubic fourfolds whose quadric fibration has no rational section.*

2. *Five of the components in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$ in the moduli space of cubic fourfolds contain rational cubic fourfolds whose quadric fibration has no rational section.*

Remark 4.5.6. *The same question for cubic fourfolds inside $\mathcal{C}_{18}$, that are fibered in sextic Del Pezzo surfaces over $\mathbb{P}^2$, is still open.*

4.5.3 A K3 surface of degree 2

Let $\tilde{X} := Bl_P(X)$ and $f : F_1\left(\tilde{X}/\mathbb{P}^2\right) \to \mathbb{P}^2$ be the variety of lines inside the quadric surface fibration $\tilde{\pi}_P : \tilde{X} \to \mathbb{P}^2$. This means that for each point $p \in \mathbb{P}^2$, the fibre $f^{-1}(p)$ parametrizes the lines inside the quadric surface $\tilde{\pi}_P^{-1}(p)$. If the fibre $\tilde{\pi}_P^{-1}(p)$ is smooth, then $f^{-1}(p)$ is the disjoint union of two projective lines; It is equal to $\mathbb{P}^1$ otherwise. This allows us to consider the Stein factorization:

$$f : F_1\left(\tilde{X}/\mathbb{P}^2\right) \to S \to \mathbb{P}^2,$$

where S is a double covering of the plane $S \to \mathbb{P}^2$ branched over a sextic curve $D \subset \mathbb{P}^2$. This curve D is the discriminant of the quadric surface bundle $\tilde{\pi}_P$.

We show in Example 8 that such a surface S is in fact a K3 surface when D is smooth. In this case, S has degree 2. We recall that the sextic curve D is the space of points where the quadric surface bundle $\tilde{\pi}_P$ degenerates. If we assume that all the fibres over $\tilde{\pi}_P$ have at most one singular point, then D is smooth and the variety of lines of the quadric surface fibration $F_1\left(\tilde{X}/\mathbb{P}^2\right)$ is a $\mathbb{P}^1$-bundle over the double cover of the plane $S \to \mathbb{P}^2$.

4.5.4 The divisor $\mathcal{C}_{14}$ and Pfaffian cubic fourfolds

The first cubic fourfolds that were shown to be rational are the cubic fourfolds containing a quartic rational normal scroll (Ex. 11), studied for the first time by [Mor40] and then [Fan42]. These cubics are inside the Hassett divisor $\mathcal{C}_{14}$.

Recall that $\mathcal{C}_{14}$ is defined as the locus of special cubic fourfolds $X \subset \mathbb{P}^5$ containing a 2-dimensional algebraic cycle T with the following intersection properties:

$$T^2 = 10 \text{ and } T \cdot h^2 = 4$$

The intersection lattice of $X \in \mathcal{C}_{14}$ has the shape:

$$K_{14} = \begin{array}{|c|c|c|}\hline & h^2 & \mathrm{T} \\ \hline h^2 & 3 & 4 \\ \hline \mathrm{T} & 4 & 10 \\ \hline \end{array}, \tag{4.6}$$

The divisor $\mathcal{C}_{14}$ has many different geometric characterizations summarized in the following theorem 4.5.7.

Theorem 4.5.7. *The divisor $\mathcal{C}_{14}$ can be described as the closure of the locus of Pfaffian cubic fourfolds (Def. 4.5.4), [BD85], [Bea00] or equivalently as the closure of the locus of cubic fourfolds containing one of the following surfaces:*

1. *Smooth rational quartic normal scroll [Fan42], [Tre93].*
2. *Smooth quintic Del Pezzo surface [Fan42], [Tre93].*

Remark 4.5.8. *The equivalence between being a Pfaffian cubic fourfold and being a cubic fourfold containing a quintic Del Pezzo surface was proven by Beauville [Bea00, Prop. 9.2]. In [BD85], the authors proved that the closure of the locus of Pfaffian cubics form a divisor inside $\mathcal{C}$, the moduli space of cubic fourfolds. Tregub and Fano [Tre93; Fan42] also proved that the closure of the locus of cubic fourfolds containing a smooth rational quartic scroll, is exactly the closure of the locus of cubics containing a quintic Del Pezzo surface [Tre93, Prop. 1].*

Let us explain how a general $X \in \mathcal{C}_{14}$ containing a quartic rational normal scroll T, contains also a quintic Del Pezzo surface W. A similar argument applies for the converse.

Let $\Sigma = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be a general Segre threefold containing T as a divisor of type $(2,1)$. Divisors of type $(1,2)$ in Σ are quintic Del Pezzo surfaces (see Appendix A). One can show [Tre93, Prop. 1] that for such a general Σ and a general X containing T, we get the following intersection:

$$\Sigma \cap X = T \cup W \quad \text{where W is a quintic del Pezzo surface.}$$

Indeed, the intersection $S = \Sigma \cap X$ is a degree 9 surface. Moreover, S is a divisor of type $(3,3)$, hence W must be of type $(1,2)$.

4.5.5 The family of quartic scrolls contained in $X \in \mathcal{C}_{14}$

Another important classical result that one can find in the work of Fano [Fan42], is about the family of quartic scrolls contained in a cubic fourfold inside $\mathcal{C}_{14}$. This is well explained in a modern setting in the following theorem.

Theorem 4.5.9. *[BRS19, Thm. 3.7] Let $X \in \mathcal{C}_{14}$ be a cubic fourfold containing a smooth quartic rational normal scroll T, and let $\mathcal{T}$ be the closure of the family of smooth quartic rational normal scrolls contained in X (i.e. the Hilbert scheme of smooth quartic rational normal scrolls contained in X). We have that :*

1. $\dim(\mathcal{T}) = 2$.
2. *there exists a unique irreducible 2-dimensional component $\tilde{S}_T \subseteq \mathcal{T}$ containing T, which is birational to the Hilbert scheme of secant lines to T contained in X.*

Remark 4.5.10. *In [Mor40], Morin mistakenly claimed that $\mathcal{T}$ has dimension 1 for a general cubic fourfold. His proof about the rationality of a general cubic fourfold was based on this false statement.*

4.5.6 Rationality inside $\mathcal{C}_{14}$

The first rationality result concerning cubic fourfolds inside $\mathcal{C}_{14}$, is the argument used by Fano [Fan42]. We explain this in Example 11. Another well studied locus of cubic fourfolds in $\mathcal{C}_{14}$ is the locus of Pfaffian cubic fourfolds. Beauville and Donagi [BD85, Prop. 5] established rationality of these cubics.

These two arguments only show the rationality of a generic cubic fourfold in $\mathcal{C}_{14}$. More recently , Bolognesi, Russo and Stagliano [BRS19] studied the Hassett divisor $\mathcal{C}_{14}$ using the generalized surfaces with one apparent double point (OADP surfaces). In this paper, they prove for the first time the rationality of every cubic fourfold inside $\mathcal{C}_{14}$ [BRS19, Thm. 3.9 and Thm. 4.4], completing the previous results of Fano, Beauville and Donagi. Let us briefly explain the arguments they used to establish their main rationality result. First, the authors prove that every cubic fourfold X inside $\mathcal{C}_{14} - \mathcal{C}_8 \cap \mathcal{C}_{14}$ contains a smooth quartic rational normal scroll [BRS19, Thm. 3.9], and hence X is rational as in Example 11.
On the other hand, they investigate the irreducible components of the intersection $\mathcal{C}_{14} \cap \mathcal{C}_8$. They show that cubic fourfolds in each component contain a small (Def. 4.4.3), reducible OADP surface. Hence, every $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ is rational by Theorem 4.4.10.

Remark 4.5.11. *In a previous paper [Aue+14], the authors described the irreducible components of the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$ using lattice theory computations. The results in [BRS19] are more complete as they prove the rationality of every cubic fourfold in $\mathcal{C}_{14} \cap \mathcal{C}_8$, which was not the case in [Aue+14].*

One can also see the work of Russo and Stagliano [RS19], where they give a different proof for the rationality of every cubic in $\mathcal{C}_{14}$ [RS19, Thm. 2]. They show the same result for the divisors $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$.

In [RS21], the same authors describe explicit birational maps $X \overset{\sim}{\dashrightarrow} \mathbb{P}^4$ from general cubic fourfolds in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ respectively.

4.5.7 The associated K3 surface of degree 14

Let $X \subset \mathbb{P}^5$ be a general Pffafian cubic fourfold and F its Fano variety of lines. From the proof of Proposition 4.1.2, we recall that the Abel Jacobi map $\alpha : H^4(X,\mathbb{Z}) \to H^2(F(X),\mathbb{Z})$ is an isomorphism of Hodge structures. Moreover, Theorem 3.2.3 gives rise to a K3 surface S of degree 14 such that [Bea83; BD85]:

$$H^4(X,\mathbb{Z}) \simeq H^2(F(X),\mathbb{Z}) \simeq H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta$$

We also have that α induces an isomorphism $H^4(X,\mathbb{Z})^0 \simeq H^2(F(X),\mathbb{Z})^0(-1)$ of Hodge structures. Therefore, X has a labelling $h^2 \subset K_{14} \subset A(X)$ such that [Has00, Prop. 5.1.2 or Prop. 6.1.1]:

$$H^4(X,\mathbb{Z})^0 \supset K_{14}^\perp \simeq H^2(S,\mathbb{Z})^0(-1).$$

We can give a more explicit geometric argument. Recall that the linear system of quadric hypersurfaces in $\mathbb{P}^5$ containing a quartic rational normal scroll $T \subset X$, induces a birational map $\varphi : X \overset{\sim}{\dashrightarrow} Q_4$ (see Ex. 11), where Q_4 is a smooth quadric of dimension 4. The base locus of the inverse birational map φ^{-1} is an irreducible surface $\tilde{S} \subset \mathbb{P}^5$ that has a finite number of singular points. Under some hypothesis on X (see [BRS19, Thm. 3.2]), $\tilde{S}$ is a smooth surface of degree 10 and sectional genus 7. In this case, this surface is birational to a smooth K3 surface $S = \mathbb{P}^8 \cap \mathbb{G}(2,6) \subset \mathbb{P}^{14}$ of degree 14 and sectional genus 8 (see Ex. 9). More precisely, $\tilde{S}$ is the projection from a tangent plane to the K3 surface $S \subset \mathbb{P}^8$.
Since $\tilde{S}$ is the base locus of the birational map φ^{-1}, over each point in $\tilde{S}$ there is a rational curve inside X. This induces an embedding of the primitive cohomology $H^2(\tilde{S},\mathbb{Z})^0$ into $H^4(X,\mathbb{Z})^0$.

4.5.8 Cubic fourfolds inside $\mathcal{C}_{12}$

Cubic fourfolds inside $\mathcal{C}_{12}$ have a labelling of the following form (4.7). A general cubic fourfold $X \in \mathcal{C}_{12}$ contains a cubic rational normal scroll [Has00][Lemma 4.1.1].

$K_{12}=$

	h^2	S
h^2	3	3
S	3	7

, (4.7)

4.5.9 Cubic fourfolds inside $\mathcal{C}_{20}$

The intersection lattice of a general cubic fourfold X inside $\mathcal{C}_{20}$ has the following shape (4.8). A general cubic fourfold $X \in \mathcal{C}_{20}$ contains a Veronese surface.

$K_{20} =$

	h^2	V
h^2	3	4
V	4	12

, (4.8)

Tregub [Tre93] showed that cubic fourfolds containing a Veronese surface V and a Plane P intersecting in three points, are rational. The locus of such cubics has codimension 2 inside $\mathcal{C}$. The rationality comes from Proposition 4.5.1. In fact, the proper transform of the Veronese surface $\tilde{V} \subset Bl_P(X)$ is a rational section of the quadric surface fibration $\tilde{\pi}_P$. We confirm this result in Theorem 5.3.1.

Chapter 5

Intersection of Hassett divisors

5.1 Introduction to the problem of intersection of Hassett divisors

In this last chapter, we consider the intersections of $\mathcal{C}_8$ with two other Hassett divisors, notably $\mathcal{C}_{12}$ and $\mathcal{C}_{20}$. The general cubic fourfolds in these two divisors contain, respectively, a cubic scroll and a Veronese surface. By means of lattice theory, we describe all the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{20}$ and $\mathcal{C}_8 \cap \mathcal{C}_{12}$.

We also draw information about the rationality of cubics inside certain irreducible components of these intersections. This is based principally on the two different techniques we exposed respectively in Prop. 4.5.1 and Theorem 4.4.10. The first one establishes the rationality of a cubic fourfold $X \in \mathcal{C}_8$ whenever the quadric surface fibration $\tilde{\pi}_P : X \to \mathbb{P}^2$ admits a rational section. The second technique consists in looking for a possibly reducible OADP surface inside the cubic fourfold $X \subset \mathbb{P}^5$. Such a cubic is then immediately rational.

This problem has been addressed for the first time in [Aue+14]. The authors investigate the irreducible components of the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$. In particular, they find that this intersection has five irreducible components. In another paper [BRS19], the authors prove the rationality of every cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{14}$, showing rationality of every $X \in \mathcal{C}_{14}$ for the first time (see Section 4.5.4). One can also see the work of Awada [Awa24], Addington and Thomas [AT14], and Yang and Yu [YY20].

For the intersection of more than two Hassett divisors, one can see the work of Awada, Bolognesi and Pedrini [ABP20].

We provide more information about these papers in the introduction (see Section 1.7).

5.2 Cubic hypersurfaces in $\mathcal{C}_{12}$

One can define the divisor $\mathcal{C}_{12} \subset \mathcal{C}$ in two possible ways. The first one is as the locus of smooth cubic hypersurfaces having a marking of discriminant 12, the second is as the closure of the locus of cubic hypersurfaces containing a smooth cubic rational normal scroll S. More precisely, we say that a cubic fourfold has a marking of discriminant 12 whenever it contains a surface, whose class S verifies $S \cdot S = 7$ and $S \cdot h^2 = 3$. A smooth cubic rational normal scroll has some possible degenerations. First, it can degenerate to a cone over a twisted cubic; otherwise, it could deform to the union of a quadric surface Q and a plane P such that $Q \cap P = \langle Q \rangle \cap P$ is a line L and $\langle Q \rangle$ is the linear envelope of the quadric surface. The second case can further degenerate to the union of three planes if Q degenerates further to the union of two planes. These three degenerations are small (Def. 4.4.3) varieties, and are contained in smooth cubic hypersurfaces. There are other degenerations obtained by projection but the resulting schemes are not contained in any smooth cubic hypersurface in $\mathbb{P}^5$. Hence every element $X \in \mathcal{C}_{12}$ contains either a cubic rational normal scroll or a reducible surface $Q \cup P$ as above, where Q is a quadric surface, irreducible or not. If $Q \cup P \subset X$, then $X \in \mathcal{C}_{12} \cap \mathcal{C}_8$ and $P \cdot (Q + P) = 3$. Furthermore, we observe that if $S = P \cup Q$, letting $P_1 + Q = h^2$, we see that there exists a plane $P_1 \in < h^2, Q, P >$ such that $P_1 \cdot P = 1$ and $P_1 \cdot (Q + P) = -1$.

5.2.1 Irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{12}$

Theorem 5.2.1. *There are three irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{12}$ indexed by the value $P \cdot S = \eta \in \{-1, 0, 1, 2, 3\}$, where P is a plane and S the class of a cubic rational normal scroll. The components corresponding to -1 and 3 (0 and 2, respectively) coincide (see Lemma 5.2.2). For $\eta = 0$ or 2 every element in the corresponding irreducible component is rational.*

Proof. First of all we observe that every cubic X contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{12}$ has a sublattice $\langle h^2, P, S \rangle \subset A(X)$. The intersection lattice of $\langle h^2, P, S \rangle$ is as follows,

	h^2	P	S
h^2	3	1	3
P	1	3	η
S	3	η	7

, (5.1)

for some $\eta = P \cdot S \in \mathbb{Z}$. Let us denote by M_η the rank 3 sublattice.
Note that $A(X)$ is definite positive by the Hodge-Riemann bilinear relations. Hence, by Sylvester criterion (see Prop. 2.3.3), M_η must have positive determinant. The only integer values of η for which its determinant $-3\eta^2 + 6\eta + 29$ is positive are $\pm 2, \pm 1, 0, 3, 4$.

We will now show that there are no such cubics for $\eta = -2, 4$. By [YY20, Lemma 2.4], the existence of a short root (*i.e.* an integer vector in M_η with norm 2) is equivalent to the emptiness of the component (see Lemma 2.3.4). If $\eta = -2$, we observe that $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is a short root, and if $\eta = 4$ then $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ has also norm 2.

An easy explicit calculation shows that no other component admits short roots, hence by [YY20, Lemma 2.4], for all the other values of η the components are non-empty. Let us denote by $\mathcal{C}_{M_\eta}$ the locus of cubic fourfolds such that there is a primitive embedding of $M_\eta \subset A(X)$ preserving h^2.

To determine whether a component $\mathcal{C}_{M_\eta}$ is irreducible or not, we recall that the rank of $A(X)$ is an upper-semicontinuous function on the moduli space, and the M_η here above have rank 3 for $\eta \in \{-1, 0, 1, 2, 3\}$. Moreover, $rk(A(X)) = k$ is a codimension $k-1$ condition, hence we see that the irreducible components of $\mathcal{C}_{M_\eta}$ correspond to rank 3 overlattices B of M_η which admit primitive embeddings into $H^4(X, \mathbb{Z})$.

If there exists an embedding $M_\eta \hookrightarrow B$, then $|d(M_\eta)| = |d(B)| \cdot [B : M_\eta]^2$. Moreover, depending on the value of η, we have:

ϵ	$d(M_\eta)$
-1	20
0	29
1	32
2	29
3	20

. (5.2)

We observe straight away that 29 is square free, hence M_2 and M_0 cannot have overlattices. For the other components, we will show that any possible overlattice has short roots, so that there are no smooth cubic fourfolds with such a lattice.

Consider h^2 and P as a part of a basis of the overlattice B and let U be a vector that completes this to a basis such that $U = xh^2 + yP + zS$, with x, y, z rational coefficients. Consider now the natural embedding of M_η in B. This can be written explicitly as follows:

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & Z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \\ 0 & 0 & \frac{1}{z} \end{pmatrix} \in \mathcal{M}_{3,3}(\mathbb{Z}).$$

If we take $z = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx$, $y' = yn \in \mathbb{Z}$, then we can write $U = \frac{1}{n}(x'h^2 + y'P + S)$. Moreover, by adding appropriately multiples of h^2 and of P, we can assume that $0 \leq x', y' < n$. An explicit computation of the intersections gives the the following:

$$U.h^2 = \frac{1}{n}(3x' + y' + 3) = a,$$
$$U.P = \frac{1}{n}(x' + 3y' + \eta) = b,$$
$$U.U = \frac{1}{n^2}(3x'^2 + 3y'^2 + 6x' + 2\eta y' + 2x'y' + 7) = c.$$

With this notation, the Gram matrix of B is

$$\begin{pmatrix} 3 & 1 & a \\ 1 & 3 & b \\ a & b & c \end{pmatrix}.$$

Since the discriminant of M_{-1} is 20, $[B : M_{-1}]$ can only be 4 and $n = 2$. The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 1,\ y' = 0$, which gives $a = 3,\ b = 0,\ c = 4$;
- $n = 2,\ x' = 0,\ y' = 1$, which gives $a = 2,\ b = 1,\ c = 2$.

In the first of the two cases, one computes explicitly that the Gram matrix of B has $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as a short root, hence it is not a lattice of a cubic fourfold. In the second case, the Gram matrix has a short root $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, hence the component $\mathcal{C}_{M_{-1}}$ is irreducible.

Let us now consider M_1. Since the discriminant of M_1 is 32, then $[B : M_1]^2$ is either 4 or 16, and $n = 2$ or 4. Hence, the only cases where the matrix above has integer entries are the following two cases:

- $n = 2,\ x' = 1,\ y' = 0$, which gives $a = 3,\ b = 1,\ c = 4$;

- $n = 2,\ x' = 0,\ y' = 1$, which gives $a = 2,\ b = 2,\ c = 3$.

In the first case the corresponding Gram matrix has no short roots. On the other hand one can check easily that the two vectors $(-1, 3, 0)$ and $(0, 3, -1)$ make up a basis for the rank 2, primitive lattice $B_{prim} := (h^2)^\perp \subset H^4_{prim}(X, \mathbb{Z})$. The Gram matrix of B_{prim} is $\begin{pmatrix} 24 & 24 \\ 24 & 25 \end{pmatrix}$, hence the lattice is not even. Since $H^4_{prim}(X, \mathbb{Z})$ is even, B cannot be an overlattice of M_1.

In the second case, the Gram matrix has a short root, that is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The value $n = 4$ does not give a integer valued matrix. This means that $\mathcal{C}_{M_1}$ is irreducible as well.

For the last case that is left, since the discriminant of M_3 is 20, the index $[B : M_3]$ can only be 4 and $n = 2$. The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 0,\ y' = 1$, which gives $a = 2,\ b = 3,\ c = 4$;
- $n = 2,\ x' = 1,\ y' = 0$, which gives $a = 3,\ b = 2,\ c = 4$;

In both cases, the Gram matrix of B has a short root $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Hence $\mathcal{C}_{M_3}$ is irreducible.

5.2.2 Rationality in $\mathcal{C}_8 \cap \mathcal{C}_{12}$

Rationality of $X \in \mathcal{C}_{M_0}$: It is not hard to see that the general element of the family where $\#(P \cap S) = 0$ is also rational. In fact we have $\mathrm{rk}(A(X)) = 3$ and let us set $Q + P = h^2$. Since $S \cap P = \emptyset$, we have $S \cdot Q = 3$ and we deduce that the rational quadric fibration over $\mathbb{P}^2$ determined by projection from P has a rational section, hence it is rational by Prop. 4.5.1. Since the general cubic in this component is rational, by [KT19] this holds for all cubics with $\#(P \cap S) = 0$.

Rationality of $X \in \mathcal{C}_{M_2}$: Every cubic fourfold $X \in \mathcal{C}_{M_2}$ is rational. Let $p \in \mathbb{P}^5$ be a general point, and the projections of S and P off p intersect in 3 points, but 2 of them come from $S \cap P$ in $\mathbb{P}^5$. This means that there is only one secant line through p, that is $S \cup P \subset \mathbb{P}^5$ is a reducible OADP and X is then rational. We also observe that $P \not\subset < S >= \mathbb{P}^4$ since $P \cdot S = 2$ implies that P and S cut only at two points.

□

Lemma 5.2.2. *The irreducible components $\mathcal{C}_{M_{-1}}$ and $\mathcal{C}_{M_3}$ coincide, the same holds for $\mathcal{C}_{M_0}$ and $\mathcal{C}_{M_2}$.*

Proof. By applying [YY23, Algorithm 7.6], we find that the generic element of any irreducible component of the intersection $\mathcal{C}_{12} \cap \mathcal{C}_8$ has an intersection lattice of type

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & \tau \\ 0 & \tau & 4 \end{pmatrix},$$

for certain choice of the basis of the lattice and $\tau \in \mathbb{Z}$. Moreover, lattices obtained from τ and $-\tau$ are isometric. The determinant of this matrix is $32 - 3\tau^2$ hence it takes the value 20 only for $\tau = \pm 2$, and hence there is a unique irreducible component with discriminant 20 and $\mathcal{C}_{M_{-1}}$ and $\mathcal{C}_{M_3}$ coincide.

The same argument holds for $\mathcal{C}_{M_0}$ and $\mathcal{C}_{M_2}$, with values of $\tau = \pm 1$. $\square$

5.2.3 Explicit examples of cubic fourfolds in the irreducible components of $\mathcal{C}_{12} \cap \mathcal{C}_8$

($\mathcal{C}_{M_\epsilon} \neq \emptyset$ for $\epsilon \in \{-1, 0, 1, 2, 3\}$)

First of all, we need two technical results.

Proposition 5.2.3. *Let $Y = \tilde{S} \cup \tilde{P} \subset \mathbb{P}^5$ be a reducible quartic surface given by the union of a plane $\tilde{P}$ and a smooth cubic scroll $\tilde{S}$ intersecting along a line $\tilde{L} = \tilde{S} \cap \tilde{P}$. Then Y is OADP if and only if $\tilde{L}^2 = 0$ (that is, $\tilde{L}$ is a line of the ruling of $\tilde{S}$) and secant defective if and only if $\tilde{L}^2 = -1$ (that is, $\tilde{L}$ is the directrix line of $\tilde{S}$) .*

Proof. Consider a cubic scroll $S \subset \mathbb{P}^4$ and a plane P also in $\mathbb{P}^4$ that intersects S along a line L. Let K_S be the canonical sheaf. We observe that $K_S \cdot L = -2 - L^2$. Then, by [Ful98, Prop. 9.1.1, third formula] , we have that the multiplicity of the scheme theoretic intersection $S \cap P$ along L is given by

$$\begin{aligned} mult_L(S \cap P) &= 5 + K_S \cdot L + K_P \cdot L + 2 \\ &= 5 - 2 - L^2 - 3 + 2 \\ &= 2 - L^2. \end{aligned}$$

This formula gives 2 or 3 depending on whether $L^2 = 0$ or $L^2 = -1$. Suppose now that we have our $Y \subset \mathbb{P}^5$, then the computation above implies that Y is OADP if and only if $\tilde{L}^2 = 0$. In fact, let us consider a general $p \in \mathbb{P}^5$ and call S and P the images of $\tilde{S}$ and $\tilde{P}$ in $\mathbb{P}^4$ via the projection off p. If S and P intersect along a set Ω, then the number of secants to Y through p is equal to $deg(S) \cdot deg(P) - mult_\Omega(S \cap P)$, which gives $3 - mult_L(S \cap P)$ in our particular case. This implies that Y is OADP if $mult_L(S \cap P) = 2$, and secant defective if $mult_L(S \cap P) = 3$. $\square$

Proposition 5.2.4. *Let $Y = \tilde{S} \cup \tilde{P}$ be as in Prop. 5.2.3. Then Y is OADP if and only if it is the flat limit of a family of quartic scrolls in $\mathbb{P}^5$; Y is secant defective if and only if it is the flat limit of Veronese surfaces in $\mathbb{P}^5$.*

Proof. By checking the Hilbert polynomial, one observes that the reducible surface $\tilde{S} \cup \tilde{P}$ can be a degeneration of either a Veronese surface, or a quartic scroll. By using a construction contained in [CMR04, Sect. 3], we will show that either cases can occur, and that, in particular, it is the self intersection of the line $\tilde{S} \cap \tilde{P}$ that determines in which case we are.

Let Σ be a Veronese surface in $\mathbb{P}^5$. Let us consider the family of morphisms $\pi_p : \Sigma \to \mathbb{P}^4$ given by the linear projection off a point $p \in \mathbb{P}^5$, generically not lying on Σ but whose limit $\hat{p}$ lies on the surface. The limit map, i.e. the projection off $\hat{p} \in \Sigma$, is not a morphism. Nevertheless, following [CMR04, Sect. 3], we can resolve it as a morphism $\tilde{\pi}_{\hat{p}}$ by replacing Σ with $\widehat{\Sigma} \cup Z$, where $\widehat{\Sigma}$ is the blow-up of Σ in $\hat{p}$ and Z a $\mathbb{P}^2$ intersecting $\widehat{\Sigma}$ along the exceptional divisor. In particular, $\widehat{\Sigma} \cup Z$ is the flat limit of a family of surfaces where each other element is a Veronese surface. Since Σ does not contain lines, we observe that $\widehat{\Sigma} \cup Z$ is sent isomorphically by $\tilde{\pi}_{\hat{p}}$ onto a reducible surface in $\mathbb{P}^4$ whose components are a cubic scroll S plus a plane B (in fact the degree is 4) that intersects the scroll along the exceptional divisor of $\hat{p}$, which is a line. The fact that S is the cubic scroll is due to the fact that it is exactly the image of the blow-up of $\widehat{\Sigma}$. Moreover, the lines inside S are the projections of the conics inside Σ passing through $\hat{p}$, and they all intersect the exceptional line. This means that the exceptional divisor is the (-1)-curve of S, and by Prop. 5.2.3 this is equivalent to Y being secant defective.

On the other hand, if Σ is a quartic scroll, the argument of [CMR04] runs basically the same way. The only difference is that the morphism $\tilde{\pi}_{\hat{p}}$ that resolves the projection off $\hat{p}$ contracts a line, giving the right morphism from the (blown-up) quartic scroll to the cubic scroll. The images of the lines of the ruling of Σ do not intersect the exceptional line, that just replaces the contracted fiber, and hence has zero self-intersection. By Prop. 5.2.3, this is equivalent to Y being OADP.

□

Remark 5.2.5. *This degeneration of the Veronese surface has been considered also in [GR03, Thm. 2.5].*

$S \cap P$ is a finite set.

In the appendix C, via an easy Macaulay calculation [GS02], we display examples of smooth cubic hypersurfaces $X \subset \mathbb{P}^5$ containing a smooth rational normal scroll S of degree 3 and a plane P such that $S \cap P$ is a scheme of length ϵ, $\epsilon \in \{0, \dots, 3\}$ consisting exactly of ϵ reduced points, see Example 17 $(3, 4, 5, 6)$. A general element $X \in \mathcal{C}_{M_\epsilon}$, $\epsilon \in \{0, 1, 2, 3\}$, has $\mathrm{rk}(A(X)) = 3$.

$S \cap P$ is a conic.

In Example 17 (2) , we construct $X \subset \mathbb{P}^5$ a smooth cubic hypersurface containing a smooth rational normal scroll S of degree 3 and a plane P spanned by a conic $C \subset S$. By Eq. (4.3), we have $S \cdot P = -1$, hence $X \in \mathcal{C}_{M_{-1}}$. In this case $S \cup P \subset < S >= \mathbb{P}^4$ is a degenerate reducible surface whose hyperplane sections have arithmetic genus one, since they are given by a twisted cubic plus one of its secant lines.

$S \cap P$ **is a line of the ruling.**

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface containing a smooth rational normal scroll S of degree 3 and a plane P such that $P \cap S = P \cap < S >$ is a line of the ruling of S. An example of such a cubic fourfold is developed in Ex. 17 (7). By (4.3), in this case we have $S \cdot P = 0$, so that $X \in \mathcal{C}_{M_0}$. In particular such cubic hypersurfaces are rational since $S \cup P$ is a small (Def. 4.4.3), equidimensional OADP surface (see Prop. 5.2.3 for an explanation).

More precisely, $S \cup P$ is a flat limit of a family of quartic scrolls in $\mathbb{P}^5$ (see Prop. 5.2.4), hence the corresponding cubics fourfold belong to $\mathcal{C}_8 \cap \mathcal{C}_{12} \cap \mathcal{C}_{14}$. These examples of cubic fourfolds hence lie also in the irreducible component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ where the plane intersects the scroll in 3 points (see [Aue+14; BRS19]), in fact $P \cdot (S \cup P) = 0 + 3$. Indeed, all cubics in this component have Gram matrix of discriminant 29, which is the same as the discriminant of those in $\mathcal{C}_{M_0} \subset \mathcal{C}_8 \cap \mathcal{C}_{12}$.

$S \cap P$ **is the directrix of** S**.**

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface containing a smooth rational normal scroll S of degree 3 and a plane P such that $P \cap S = P \cap < S >$ is the directrix line of S. We produce such examples in the Appendix (Ex. 17 (8)). In this case we have $S \cdot P = 1$ by (4.3), hence $X \in \mathcal{C}_{M_1}$. The scheme $S \cup P \subset \mathbb{P}^5$ is a reducible, secant defective surface (see Prop. 5.2.3) which is a flat projective degeneration of a Veronese surface in $\mathbb{P}^5$ (see Prop. 5.2.4) . In particular the secant lines to $S \cup P$ fill two irreducible varieties of dimension 4: $< S >$ and the join of S and P which is a quadric surface with vertex P. As a consequence, these cubics are contained in $\mathcal{C}_{20} \cap \mathcal{C}_8 \cap \mathcal{C}_{12}$ and notably in the component of $\mathcal{C}_{20} \cap \mathcal{C}_8$ where $P \cdot V = P \cdot (S \cap P) = 1 + 3 = 4$ (see Sect. 5.3.3 for a full description of this component).

5.3 Cubic hypersurfaces in $\mathcal{C}_{20}$

The closure of the locus of smooth cubic hypersurfaces $X \subset \mathbb{P}^5$ containing a Veronese surface $V \subset \mathbb{P}^5$ is indicated with $\mathcal{C}_{20}$. Since $V^2 = 12$, the discriminant of a general $X \in \mathcal{C}_{20}$ with $\mathrm{rk}(A(X)) = 2$ is exactly 20.

5.3.1 Irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{20}$

Theorem 5.3.1. *There are seven distinct irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{20}$ indexed by $P \cdot V = \gamma \in \{-2, -1, 0, 1, 2, 3, 4\}$, where P is a plane and V the class of a surface such that $V^2 = 12$ and $V \cdot h^2 = 4$. For $\gamma = -1, 1$ and 3 each smooth cubic hypersurface belonging to the corresponding irreducible component is rational.*

Proof. Let us first observe that every cubic fourfold X contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{20}$ has intersection lattice as follows,

	h^2	P	V
h^2	3	1	4
P	1	3	γ
V	4	γ	12

, (5.3)

where V is the class of a Veronese surface inside the cubic fourfold X. Let us denote by N_γ the lattice with this intersection matrix. The determinant of this matrix is $-3\gamma^2 + 8\gamma + 48$, and the only integer values for which it is positive are $\{-2, -1, 0, 1, 2, 3, 4, 5\}$. These values could possibly give non-empty components of the intersection $\mathcal{C}_8 \cap \mathcal{C}_{20}$. We will denote by $\mathcal{D}_{N_\gamma}$ the locus of cubic fourfolds X such that $N_\gamma \subset A(X)$ is a saturated sublattice.

For $\gamma = 5$, it is straightforward to see that $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is a short root, hence $\mathcal{D}_{N_5}$ is empty.

The other possible values of γ give rise to lattices with no short roots, hence by [YY20, Lemma 2.4] the associated components $\mathcal{D}_{N_\gamma}$ are non-empty. Hence we are left only with the values $-2, -1, 0, 1, 2, 3, 4$. The corresponding discriminants are:

γ	$d(N_\gamma)$
-2	20
-1	37
0	48
1	53
2	52
3	45
4	32

(5.4)

In order to check irreducibility we need to check that there are no overlattices B of N_γ, that embed primitively in $H^4(X,\mathbb{Z})$. If there exists an embedding $N_\gamma \hookrightarrow B$, then $|d(N_\gamma)| = |d(B)| \cdot [B : N_\gamma]^2$. For $\gamma = -1, 1$, the discriminant are square-free, hence N_1 and N_{-1} cannot have proper finite overlattices, and they correspond to irreducible components.

We need some elementary lattice theory to prove the irreducibility of the components with $\gamma = -2,\ 0,\ 2,\ 3,\ 4$.

Consider h^2 and P as a part of a basis of the overlattice B and let U be a vector that completes this to a basis such that $U = xh^2 + yP + zV$, with x, y, z rational coefficients. Consider now the natural embedding of N_γ in B. This can be written explicitly as follows:

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & Z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \\ 0 & 0 & \frac{1}{z} \end{pmatrix} \in \mathcal{M}_{3,3}(\mathbb{Z}).$$

If we take $z = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx$, $y' = yn \in \mathbb{Z}$, then we can write $U = \frac{1}{n}(x'h^2 + y'P + S)$. Moreover, by adding appropriately multiples of h^2 and of P, we can assume that $0 \leq x', y' < n$. An explicit computation of the intersections gives the following:

$$\begin{aligned} U.h^2 &= \frac{1}{n}(3x' + y' + 3) = a, \\ U.P &= \frac{1}{n}(x' + 3y' + \eta) = b, \\ U.U &= \frac{1}{n^2}(3x'^2 + 3y'^2 + 8x' + 2\eta y' + 2x'y' + 12) = c. \end{aligned}$$

With this notation, the Gram matrix of B is

$$\begin{pmatrix} 3 & 1 & a \\ 1 & 3 & b \\ a & b & c \end{pmatrix}.$$

Since the discriminant of N_{-2} is 20, hence $[B : N_{-2}]$ can only be 4 and $n = 2$. The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 0,\ y' = 0$, which gives $a = 2,\ b = -1,\ c = 3$;
- $n = 2,\ x' = 1,\ y' = 1$, which gives $a = 4,\ b = 1,\ c = 6$.

In the first of the two cases, one computes explicitly that the Gram matrix of B has $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as a short root, hence it is not a lattice of a cubic fourfold.

In the second case, the Gram matrix has a short root $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, hence the component $\mathcal{D}_{N_{-2}}$ is irreducible.

Consider now N_0. Since the discriminant of N_0 is 48, hence $[B : N_0]$ can only be 16 (resp. 4) and $n = 4$ (resp. $n = 2$). The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 0,\ y' = 0$, which gives $a = 2,\ b = 0,\ c = 3$;
- $n = 2,\ x' = 1,\ y' = 1$, which gives $a = 4,\ b = 12,\ c = 7$.

In both cases, one computes explicitly that the Gram matrices of B have $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as a short root, hence they are not lattices of a cubic fourfold. Hence the component $\mathcal{D}_{N_0}$ is irreducible.

Let us consider now N_2. Since the discriminant of N_2 is 52, hence $[B : N_2]$ can only be 4 and $n = 2$. The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 0,\ y' = 0$, which gives $a = 2,\ b = 1,\ c = 3$;
- $n = 2,\ x' = 1,\ y' = 1$, which gives $a = 4,\ b = 3,\ c = 8$.

In the first case, the Gram matrix of B has $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as a short root, hence it is not the lattice of a cubic fourfold.
In the second case, the Gram matrix of B has $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ as a short root. Hence the component $\mathcal{D}_{N_2}$ is irreducible.

The discriminant of N_3 is 45, hence $[B : N_3]$ can only be 9 and $n = 3$. The only case where the Gram matrix of B has integer entries is:

- $n = 3,\ x' = 0,\ y' = 2$, which gives $a = 2,\ b = 3,\ c = 4$;

which has $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ as a short root, hence it is not the lattice of a cubic fourfold and the component $\mathcal{D}_{N_3}$ is irreducible.

Last, the discriminant of N_4 is 35, hence $[B : N_4]$ can be 16 (resp. 4) and $n = 4$ (resp. 2). The only cases where the Gram matrix of B has integer entries are:

- $n = 2,\ x' = 0,\ y' = 0$, which gives $a = 2,\ b = 2,\ c = 3$;
- $n = 2,\ x' = 1,\ y' = 1$, which gives $a = 4,\ b = 4,\ c = 9$;

which has $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as a short root for the first case. Hence it is not a lattice of a cubic fourfold.
For the second case, Let us set $a = (1, -3, 0)$ and $b = (0, -4, 1)$ the basis of B_{prim}. Its Gram matrix is $\begin{pmatrix} 24 & 24 \\ 24 & 25 \end{pmatrix}$ which is not even. This shows that $\mathcal{D}_{N_4}$ is an irreducible component.

5.3.2 Rationality in $\mathcal{C}_8 \cap \mathcal{C}_{20}$

Rationality of $X \in \mathcal{D}_{N_{-1}}$: If $\gamma = -1$, then X is rational since all cycles $C = \alpha h^2 + \beta P + \delta V \in A(X)$, with $\alpha, \beta \in \mathbb{Z}$ and δ an odd integer, intersect the quadric surfaces of class $Q = h^2 - P$ in an odd degree zero cycle. This implies rationality of X by Prop. 4.5.1.

Rationality of $X \in \mathcal{D}_{N_3}$: If $\gamma = 3$, then each X in the corresponding irreducible component is rational since it contains a Veronese surface V and a plane P intersecting V in three points. The union $V \cup P$ is a reducible OADP surface. One can see this fact by remarking that the images of V and P in $\mathbb{P}^4$, via the projection off a generic point $p \in \mathbb{P}^5$, intersect in 4 points, but 3 of them come from $V \cap P$ in $\mathbb{P}^5$. This means that only one secant line to $V \cap P$ passes through p in $\mathbb{P}^5$.

Rationality of $X \in \mathcal{D}_{N_1}$: If $\gamma = 1$, then a cubic hypersurface in the corresponding irreducible component is rational. In fact, letting $Q = h^2 - P$ as before, we have $Q \cdot V = 4 - \gamma$ so that the rational quadric fibration defined by projection from P has a rational section and the total space is hence rational by [Has99, Cor. 2.2].

□

5.3.3 Explicit examples of cubic fourfolds in the irreducible components of $\mathcal{C}_{20} \cap \mathcal{C}_8$.

In this section we will describe the explicit examples of smooth cubic fourfolds $X \subset \mathbb{P}^5$ containing a smooth non-degenerate Veronese surface $V \subset \mathbb{P}^5$ and a plane P, such that $V \cdot P = \gamma$. The first crucial observation (that reflects also our lattice theoretical computation) is that there are no such cubics if $\gamma < -1$ or if $\gamma > 3$, as we will see here below in 5.3.3 and 5.3.3. There are cubics that contain surfaces with this intersection theoretical properties, but the Veronese then is not smooth.

$V \cap P$ **is a conic.**

Assume V is a smooth Veronese surface. If $\gamma < 0$, then the cycle $P \cdot V$ must contain a curve. Since V is defined by quadratic equations, the scheme-theoretic intersection $P \cap V$ must be a conic $C \subset V$ (recall that a Veronese surface contains no lines). From the fact that $K_V \cdot C = -3$, we easily deduce $V \cdot P = -1$ by (4.3). An example of such a cubic is given in Example 18 (1). Moreover, this argument also implies that for cubics in the components indexed by $\gamma = -2$, the class of the Veronese will represent singular/reducible degenerations of the surface, since otherwise $V \cdot P = -1$.

Proposition 5.3.2. *The union $W := V \cup P \subset \mathbb{P}^5$ of a Veronese surface and a projective plane, that intersect each other along a conic C, is a reducible OADP surface.*

Proof. Let us denote as usual by K_V and K_P the two canonical sheaves. First of all we observe that by Equation 4.3 we have $V \cdot P = 2 + K_V \cdot C = -1$. Let us project W off a generic point $p \in \mathbb{P}^5$. By the same argument as in Prop. 5.2.3, we compute the multiplicity in C of the intersection of the two surfaces in $\mathbb{P}^4$. We abuse of notation by sticking to the same notation as in $\mathbb{P}^5$. By [Ful98, Prop. 9.1.1, third formula], then we obtain

$$\begin{aligned} mult_C(V \cap P) &= 10 + K_V \cdot C + K_P \cdot C + 2 \\ &= 10 - 6 - 3 + 2 \\ &= 3. \end{aligned}$$

The number of secants to W through p is equal to $deg(V) \cdot deg(P) - mult_C(V \cap P) = 4 - 3 = 1$. This completes the proof. □

Proposition 5.3.3. *Let $W = V \cup P \subset \mathbb{P}^5$ the union of a Veronese surface and a projective plane, that intersect along a conic. Then W is a flat deformation of a del Pezzo quintic surface.*

Proof. Let us consider the rational map $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^6$ given by quadrics through 3 generic base points. The image is a quintic del Pezzo threefold T and the image of a smooth quadric through the 3 points is a quintic del Pezzo surface. Consider the degenerate quadric Q given by the union of a generic plane and the plane through the 3 fixed points. It is straightforward to see that the image of Q is exactly $V \cup P$ and $V \cap P$ is a conic. This concludes the proof. □

This means that the locus of cubics containing $W = V \cup P$ intersecting along a conic makes up a locus that is contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{20} \cap \mathcal{C}_{14}$.

Moreover, this argument also implies that for cubics in the components indexed by $\gamma = -2$ and 4, the class of the Veronese will represent singular/reducible degenerations (for the case $\gamma = 4$ see Sect. 5.3.3 and Prop. 5.2.3).

$V \cap P$ is a finite set, and $\#(V \cap P) \leq 3$

If $\gamma \geq 0$ and we can suppose that $P \cap V$ is finite and that it consists of γ points counted with multiplicity. Since V is scheme theoretically defined by quadratic equations whose first syzygies are generated by the linear ones, we get $0 \leq \gamma \leq 3$ by the last part of [Eis+05, Thm. 1.1]. Explicit equations for these kind of cubics are developed in Ex. 18 (2,3,4,5).

$V \cdot P = 4$ and V is singular

The generic element of the component where $V \cap P = 4$ can not be a smooth Veronese surface. In fact, 4 points on V, that lie all on the same plane, come from 4 points on $\mathbb{P}^2$ that lie on 3 conics, which is not possible. Hence the generic element of this component should be singular or reducible.

From Sect. 5.2.3, we recall that we constructed examples of cubics in $\mathcal{C}_8 \cap \mathcal{C}_{12}$ where $S \cdot P = 1$ and $S \cap P$ is the directrix line of the cubic scroll. By Prop. 5.2.4, these quartic surfaces are flat limits of families of Veronese surfaces. An easy Hilbert polynomial computation shows that whenever the union of a cubic scroll and a plane is a flat degeneration of a Veronese surface, the intersection $S \cap P$ must be a line.

$$P_V(t) = 2t^2 + 3t + 1, \quad P_S(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1, \quad P_P(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1.$$

We get:

$$P_{S \cap P}(t) = P_S(t) + P_P(t) - P_V(t) = t + 1.$$

Moreover the Gram matrices of cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{12}$ with $S \cdot P = 1$ have discriminant 32, exactly like cubics in $\mathcal{C}_{20} \cap \mathcal{C}_8$ with $V \cdot P = 4$. In fact, in this case, the self intersection of $S \cup P$ is 12, and $P \cdot (S \cup P) = 1 + 3 = 4$, and we recognize a "reducible Veronese", and the cubics containing such a configuration of surfaces live inside $\mathcal{D}_{N_4}$. This means that the locus of cubics containing S and P that intersect along the directrix make up a locus that is contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{20} \cap \mathcal{C}_{12}$, which is probably not cut out by other Hassett divisors, and is contained in $\mathcal{D}_{N_4} \subset \mathcal{C}_{20}$.

Appendix A

Rational normal scrolls and the Del Pezzo surface

Definition A.0.1. *[Har92, Ex. 8.17](Rational normal scrolls)*
Let $a, b \in \mathbb{N}$ with $1 \leq a \leq b$, and let $n = a + b + 1$. Take two disjoint linear subspaces $\mathbb{P}^a$ and $\mathbb{P}^b$ that span $\mathbb{P}^n$. We need two isomorphic rational normal curves $C_a \subset \mathbb{P}^a$ and $C_b \subset \mathbb{P}^b$.

Then fix an isomorphism $\varphi : C_b \xrightarrow{\sim} C_a$. One defines the rational normal scroll surface $S(a, b) \subset \mathbb{P}^n$ to be the union of the lines $\langle p, \varphi(p)\rangle$ joining the rational normal curves C_b and C_a. This surface $S(a, b)$ is non-degenerate and has degree $a + b$.
Moreover, the lines $\langle p, \varphi(p)\rangle$ are called the lines of the ruling of $S(a, b)$.

Remark A.0.2. *The rational normal scroll $S(a, b)$ is isomorphic to the Hirzebruch surface $\mathbf{F}_{b-a} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(b - a))$. This surface $\mathbf{F}_{b-a}$ is a ruled surface over $\mathbb{P}^1$, which means that it is birational to $\mathbb{P}^1 \times \mathbb{P}^1$.*

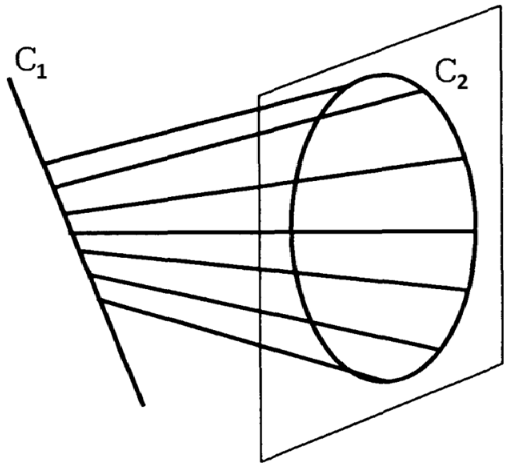
Example 14. *(Quartic scrolls in $\mathbb{P}^5$) A general member of the linear system given by $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2, 1)|$, inside the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (Def. 4.4.5) is isomorphic to $\mathbf{F}_2$. This Hirzebruch surface $\mathbf{F}_2$ is isomorphic to $S(1, 3)$, which is a rational normal quartic scroll.*

On the other hand, a general member of the linear system, given by $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0, 2)|$ is isomorphic to $\mathbb{P}^1\times Q$, where $Q \subset \mathbb{P}^2$ is a conic. This is isomorphic to the Hirzebruch surface $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In the end, a general member of the above linear system is isomorphic to the rational normal quartic scroll $S(2, 2)$.

Example 15. *(The cubic scroll) The cubic scroll $S(2, 1)$ is given by the image of the map:*

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$$
$$[t_0 : t_1 : t_2] \longrightarrow [t_0^2 : t_0t_1 : t_1^2, t_0t_2, t_1t_2, 0]$$

It is the image of the Veronese surface $V \subset \mathbb{P}^5$ under the projection from the point $[0 : 0 : 0 : 0 : 0 : 1]$. The cubic scroll $S(2, 1)$ is also isomorphic to the blow-up of the *plane $\mathbb{P}^2$ at the base point $[0:0:1]$. The ruling lines of $S(2,1)$ are the images of the lines passing through the base point. Moreover, the line C_1 is called the directrix line, it is the image of the exceptional divisor.*



The cubic scroll $S(2,1)$ [Har92].

Example 16. *(Del Pezzo surface) Let us start with the following well-known theorem, classifying surfaces of degree n in $\mathbb{P}^n$.*

Theorem A.0.3. *Let $S \subset \mathbb{P}^n$ be a smooth surface of degree n. Then, S is either isomorphic to a Del Pezzo surface or to a projection into $\mathbb{P}^4$ of the Veronese surface.*

Definition A.0.4. *(Del Pezzo surface) Let $n \in \mathbb{N}$ such that $3 \leq n \leq 8$. A Del Pezzo surface S is the blow-up of the plane $\mathbb{P}^2$ at $9-n$ points in general position.*

The linear system given by $|-K_S|$ corresponds to the linear system of cubic curves in $\mathbb{P}^2$ passing through these points. This gives an embedding of the Del Pezzo surface S into $\mathbb{P}^n$, it has degree n.

Remark A.0.5. *A smooth irreducible divisor on the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, given by $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,2)|$ (see Def. 4.4.5) is isomorphic to the quintic Del Pezzo surface $S \subset \mathbb{P}^5$.*

Appendix B

Hodge conjecture and cubic fourfolds

In the case of a K3 surface S, the Néron Severi group $\mathrm{NS}(S)$ carries most nontrivial geometric data. Recall that the Lefschetz theorem on $(1,1)$ classes (Thm. 3.3.1) states that $\mathrm{NS}(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z})$.
A similar phenomenon occurs for cubic fourfolds $X \subset \mathbb{P}^5$, the cohomology group carrying most nontrivial Hodge theoretic geometric information is the midlle cohomology lattice $H^4(X, \mathbb{Z})$. More precisely, our interest is around the lattice $A(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z})$, which consists of integral middle Hodge classes. It coincides with $\mathrm{CH}^2(X)$, the Chow group of algebraic 2-cycles. This follows from the integral Hodge conjecture that holds for cubic fourfolds [Voi07, Thm. 18].

First, we need to introduce some definitions to understand the Hodge conjecture. It is one of the most important theorems when studying cubic fourfolds from the Hodge theoretic point of view, following Hassett's works [Has00] and [Has99]. The idea of the Hodge conjecture is to unify the analytic complex geometry and the algebraic geometry settings. It links the Chow groups and the Hodge classes given by the cycle class map (Def. B.0.6). Recall the Hodge decomposition (Prop. 2.2.1):

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Let $X \subset \mathbb{P}^N$ be a complex projective variety of complex dimension n. Therefore, analytic and algebraic cycles coincide due to Serre's "GAGA" principle (Serre 1956) [Voi02, Thm. 8.30 and Rmq. 11.18] and [Voi16, Sect. 2.3.2].

Definition B.0.1. *(analytic subset) Let $Z \subset X$ be a closed subset. Z is called an analytic subset if X admits a covering by open sets $U \subset X$ such that $U \cap Z$ is the zero locus of holomorphic functions $f_1, ..., f_i$ on U. For example, submanifolds of X are smooth analytic subsets.*

Remark B.0.2. *An analytic subset $Z \subset X$ is in general non-smooth. To define the dimension of Z, we use the fact that there exists an analytic subset $Z_0 \subset Z$ such that $Z_{smooth} := Z - Z_0$ is a submanifold of X. Moreover, Z is said to be irreducible if Z_{smooth} is connected. In this case, we put $\dim(Z) := \dim(Z_{smooth})$ which has the good property:*

$$\text{for } Z_1 \subset Z_2 \subset X \text{ two analytic subsets we have } \dim(Z_1) \leq \dim(Z_2)$$

Remark B.0.3. *We aim to define a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$ for $Z \subset X$ an analytic subset of complex codimension $r = n - m$, where n is the complex dimension of X.*
To this end, we can assume the analytic subsets $Z \subset X$ to be smooth. Since there exists $Z_0 \subset Z$ such that $Z_{smooth} := Z - Z_0 \subset X - Z_0$ is smooth, the restriction map gives:

$$H^{2r}(X, \mathbb{Z}) \simeq H^{2r}(X - Z_0, \mathbb{Z}).$$

Therefore, when Z is not smooth, we define its cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$ by $[Z] := [Z - Z_0] \in H^{2r}(X - Z_0, \mathbb{Z})$, and then use the above isomorphism to get $[Z] \in H^{2r}(X, \mathbb{Z})$.

From now, we can assume $Z \subset X$ to be a submanifold of complex codimension r. Let $i : Z \hookrightarrow X$ be the inclusion map. We aim to associate to Z a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$. To this end, we use a generator $1_Z \in H^0(Z, \mathbb{Z})$ and map it into the cohomology class we are about to define. More precisely, we need to borrow the following theorem from [Ara12, Thm. 5.5.2].

Theorem B.0.4. *There exists an open neighborhood $Tube_Z$ of Z in X, called a tubular neighbourhood. This possesses a C^∞ map $\pi : Tube_Z \to Z$ that makes $Tube_Z$ a locally trivial bundle over Z, with fibers diffeomorphic to $\mathbb{R}^r$.*

For $Z \subset X$ of complex dimension $m = n - r$, the inclusion map $i : Z \hookrightarrow X$ induces the pullback map $i^* : H^{2m}(X, \mathbb{Z}) \to H^{2m}(Z, \mathbb{Z})$. This factors as follows:

$$H^{2m}(X, \mathbb{Z}) \to H^{2m}(Tube_Z, \mathbb{Z}) \xrightarrow{\sim} H^{2m}(Z, \mathbb{Z})$$

The last map is an isomorphism because the fibers of π are contractible. Moreover, Poincaré duality gives:

$$\begin{tikzcd} H^0(Z, \mathbb{Z}) \arrow[r, "\sim"] \arrow[rr, bend right, "i_Z"'] & H^{2r}(Tube_Z, \mathbb{Z}) \arrow[r] & H^{2r}(X, \mathbb{Z}) \end{tikzcd}$$

Where the first map is called the Thom isomorphism, and the second map is the extension by 0. We can now define $[Z] \in H^{2r}(X, \mathbb{Z})$.

Definition B.0.5. *(fundamental class) The fundamental class of a submanifold $Z \subset X$ of complex codimension r is $[Z] := i_Z(1_Z) \in H^{2r}(X, \mathbb{Z})$.*

In fact, we can describe i_Z more explicitely:

$$\begin{tikzcd}
H^0(Z,\mathbb{Z}) \arrow[r, "\sim"] \arrow[rrr, bend left, "i_Z"] & H^{2m}(Z,\mathbb{Z})^* \arrow[r] & H^{2m}(X,\mathbb{Z})^* \arrow[r, "\sim"] & H^{2r}(X,\mathbb{Z}) \\
1_Z \arrow[r, mapsto] & \int_Z \cdot \arrow[r, mapsto] & \int_Z i^* \cdot \arrow[r, mapsto] & [Z]
\end{tikzcd}$$

with the characterization:

$$\int_Z i^*\alpha = \int_X [Z] \wedge \alpha$$

Definition B.0.6. *(analytic cycles and the cycle map) An analytic cycle of complex codimension $r = n - m$ is a finite formal sum $\sum n_i Z_i$ with $n_i \in \mathbb{Z}$ of analytic subsets $Z_i \subset X$ of codimension r. They form a non-finite rank abelian group denoted by $Z^r(X)$. This group admits the so called cycle map, which is the extension of i_Z by linearity:*

$$i_Z : Z^r \to H^{2r}(X,\mathbb{Z})$$
$$\sum n_i Z_i \mapsto \sum n_i [Z_i]$$

Remark B.0.7. *In the algebraic setting, we use the algebraic cycles instead of the analytic ones. Algebraic cycles form the Chow groups $\mathrm{CH}^r(X)$, that are equal to the groups $Z^r/\sim_{Rat}$ (Def. B.0.9) used above of analytic cycles [Voi02, Rmq. 11.18] and [Voi16, Sect. 2.3.2].*

Considering the Hodge decomposition $H^{2r}(X,\mathbb{C}) = \oplus_{p+q=2r} H^{p,q}(X)$, and the natural inclusion $j : H^{2r}(X,\mathbb{Z}) \hookrightarrow H^{2r}(X,\mathbb{C})$; we want to know how the image of $i_Z : Z^r \to H^{2r}(X,\mathbb{Z})$ is mapped into $H^{2r}(X,\mathbb{C})$ via the natural inclusion j. The answer is given by the following proposition.

Proposition B.0.8. *(analytic/alg cycles and Hodge classes) The image of Z^r inside $H^{2r}(X,\mathbb{C})$ via the map $j \circ i_Z$ is inside $j(H^{2r}(X,\mathbb{Z})) \cap H^{r,r}(X)$. And we simply write by abuse of notation:*

$$i_Z(Z^r) \subset H^{2r}(X,\mathbb{Z}) \cap H^{r,r}(X)$$

The elements inside $H^{2r}(X,\mathbb{Z}) \cap H^{r,r}(X)$ are called integral Hodge classes.

The result is equivalent to saying that the map $j \circ i_Z$ admits a factorization given by the diagram:

$$\begin{tikzcd}
Z^r \arrow[r, "i_Z"] \arrow[drr] & H^{2r}(X,\mathbb{Z}) \arrow[r, "j"] & H^{2r}(X,\mathbb{C}) = \bigoplus_{p+q=2r} H^{p,q} \\
& & H^{r,r}(X) \arrow[u, hook]
\end{tikzcd}$$

Definition B.0.9. *We work inside $X \subset \mathbb{P}^N$, a complex projective variety of complex dimension n.*

1. *Rational equivalence : two algebraic cycles Z_1 and Z_2 of codimension r are said to be rationally equivalent $Z_1 \sim_{Rat} Z_2$ iff there exists an irreducible subvariety $Y \subset X$ of codimension $r-1$ such that $Z_1 = Z_2 + div(\varphi)$ where $\varphi \in K(Y)^*$.*

2. *Algebraic equivalence : an algebraic cycle Z of codimension r is said to be algebraically equivalent to zero $Z \sim_{Alg} 0$ iff there exists a smooth curve C, a cycle $T \subset C \times X$ of codimension r, and two points $a, b \in C$ such that $Z = T(a) - T(b)$.*

3. *Homological equivalence : two algebraic cycles Z_1 and Z_2 of codimension r are said to be homologically equivalent iff they have the same image inside $H^{2r}(X, \mathbb{Z}) \cap H^{r,r}(X)$ via the cycle map (they have the same fundamental class), i.e. $Z_1 \sim_{Hom} Z_2$ iff $i_Z(Z_1) = i_Z(Z_2)$.*

Remark B.0.10. *We have the following relations between these three equivalence relations:*

$$Z_1 \sim_{Rat} Z_2 \implies Z_1 \sim_{Alg} Z_2 \implies Z_1 \sim_{Hom} Z_2$$

In a modern setting, the converse of the above proposition is the integral Hodge conjecture, which appeared for the first time during the 1950 international congress of mathematics in Cambridge, Massachusetts, from 30 August to 6 September.

Conjecture B.0.11. *(Integral Hodge conjecture) Using the same notations as above:*

$$i_Z(Z^r / \sim_{Rat}) = H^{2r}(X, \mathbb{Z}) \cap H^{r,r}(X)$$

In other terms, the algebraic cycles of complex codimension r up to rational equivalence are exactly the (r, r)-part of $H^{2r}(X, \mathbb{Z})$.

Remark B.0.12. *When the integral Hodge conjecture holds, the homological equivalence implies the rational one. Hence, Rational, Algebraic and Homological equivalence coincide.*

$$\begin{tikzcd}
\sim_{Rat} \arrow[dr, Rightarrow] & & \sim_{Hom} \arrow[ll, Rightarrow, "Hodge\ conj"'] \\
& \sim_{Alg} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Remark B.0.13. *When $r = 1$, this is exactly the Lefschetz theorem on (1,1)-classes (Thm. 3.3.1).*
The integral Hodge conjecture is false in general. The Rational Hodge conjecture is still open. It is the same statement but tensoring with $\mathbb{Q}$:

$$i_Z(Z^r) \otimes \mathbb{Q} = H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X).$$

We end this section by stating the fact that the integral Hodge conjecture holds for cubic fourfolds. In particular, concerning the middle integral Hodge classes inside $H^{2,2} \cap H^4(X, \mathbb{Z})$. They coincide with the codimension 2 algebraic cycles up to rational equivalence which are inside the Chow group $\mathrm{CH}^2(X)$.

For a proof of this theorem, one can see [Voi07, Thm. 18] by Claire Voisin or [Zuc77] by Zucker for the original proof about the rational (over $\mathbb{Q}$) Hodge conjecture for cubic fourfolds , done in 1977.

Theorem B.0.14. *The integral Hodge conjecture holds for cubic fourfolds. In particular, we have:*

$$H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = \mathrm{CH}^2(X).$$

Appendix C

Examples of smooth cubic hypersurfaces in $\mathbb{P}^5$ and two Hassett divisors.

We shall construct explicitly, by using the software system Macaulay2 [GS02], smooth cubic hypersurfaces in $\mathbb{P}^5$ which contain a given smooth rational surface and a plane in a certain position with respect to the surface. More precisely, we shall exhibit a smooth cubic hypersurface of $\mathbb{P}^5$ containing a Veronese surface (resp. a cubic rational normal scroll) and a plane intersecting the surface along one of the following schemes: an irreducible conic; a line; a set of $1 \leq i \leq 3$ linearly independent reduced points; the empty scheme. We work over the finite field $\mathbb{F}_{31}$ but our equations hold over fields of characteristic zero. Hence we will set $\mathbb{P}^5 := Proj(\mathbb{F}_{31}[x_0, \ldots, x_5])$ and $\mathbb{P}^2 := Proj(\mathbb{F}_{31}[t_0, \ldots, t_2])$.

C.0.1 Parametrizations of the surfaces

In this section we outline the parametrizations for the three surfaces we consider.

Veronese surface

We get a parametrization of a Veronese surface by the map associated to the linear system of all conics in $\mathbb{P}^2$:

```
i1 : k = ZZ/31;
i2 : P2 = k[t_0..t_2];
i3 : P5 = k[x_0..x_5];
i4 : veroneseMap=map(P2,P5,gens (ideal vars P2)^2);
o4 : RingMap P2 <--- P5
```

The image of our parametrization is defined by the ideal of 2×2 minors of the matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Cubic rational normal scroll

From the definition of rational normal scroll surface $S(a,b) \subset \mathbb{P}^{a+b+1}$, $0 \leq a \leq b$, we deduce an obvious parametrization $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow S(a,b) \subset \mathbb{P}^{a+b+1}$:

```
i5 : scrollMap = (a,b) -> (
      t_0^(b-a+1)*(for j to a list t_0^(a-j)*t_1^j) | t_2*(for j to b list t_0^(b-j)*t_1^j)
      );
```

The image of this parametrization is defined by the ideal of 2×2 minors of the matrix:

$$\begin{pmatrix} x_0 & \cdots & x_{a-1} & x_{a+1} & \cdots & x_{a+b} \\ x_1 & \cdots & x_a & x_{a+2} & \cdots & x_{a+b+1} \end{pmatrix}.$$

In particular, for $(a,b) = (1,2)$, we obtain the map $\mathbb{P}^2 \dashrightarrow S(1,2) \subset \mathbb{P}^4 = V(x_5) \subset \mathbb{P}^5$:

```
i6 : S12Map=map(P2,P5,scrollMap(1,2) | {0});
o6 : RingMap P2 <--- P5
```

Let us construct the cubic scroll surface in different ways. A first one given by a rational map from $\mathbb{P}^2$ to $\mathbb{P}^4$ and then embedded it in $\mathbb{P}^5$ as follows :

```
i7 : needsPackage "SpecialFanoFourfolds";
i8 : cubicScrollMap = map(P2, P5, {t_0^2, t_0*t_1, t_1^2, t_2*t_0, t_1*t_2,0})
i9 : J = kernel cubicScrollMap
i10 : S12 = projectiveVariety J
```

The second way to construct the cubic scroll surface is to see it as the blow up of $\mathbb{P}^2$ in [0:0:1]. For that, we consider the Segre embedding

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^2 &\longrightarrow \mathbb{P}^5 \\ [u:v] \times [t_0 : t_1 : t_2] &\longrightarrow [t_0 u : t_1 u : t_2 u, t_0 v, t_1 v, t_2 v] \end{aligned};$$

the cubic scroll surface is the intersection of the image of $\mathbb{P}^1 \times \mathbb{P}^2$ via the embedding with the hyperplane $V(x_0 - x_4)$ in $\mathbb{P}^5$. Its exceptional divisor is $[u : v] \times [0 : 0 : 1]$.

```
i11 : P12 = k[u,v,t_0..t_2]
i12 : SegreMap = map(P12, P5, {t_0*u, t_1*u, t_2*u, t_0*v, t_1*v,t_2*v})
i13 : I = kernel SegreMap
i14 : H=ideal(x_0-x_4)
i15 : S12=ideal(H,I)
```

The key to find explicit examples of cubic fourfold in each component of the intersection $\mathcal{C}_8 \cap \mathcal{C}_i$ with $i = 12,\ 20$, is to construct "by hand" a plane P that has the needed intersection propriety with the class of surface S_i defining $\mathcal{C}_i$. Then find a smooth cubic hypersurface containing the constructed P and S_i. For that, we need the following method for later use.

```
i14 : randomElementsFromIdeal =  method(TypicalValue => Ideal)
randomElementsFromIdeal(List, Ideal) := Ideal => (L,I)->(
trim ideal((gens I)*random(source gens I, (ring I)^(-L))))
```

A usual trick to find the appropriate plane $P \subset \mathbb{P}^5$ with the needed intersection property with a surface S_i is by using the previous method:

`P = randomElementsFromIdeal({1,1,1},Ideal)`,

and varying the `Ideal` from `point ideal(Si)` to `intersect(point ideal(Si), point ideal(Si))` which gives a priori an idea about the intersection $P.S_i$. Here, `point ideal(Si)` picks a random rational point in S_i. Let us give an example by constructing a plane P that intersects the cubic scroll S_{12}, defined previously by the ideal J, in one point.

```
i16 : a = point J
i17 : p = randomElementsFromIdeal({1,1,1},a)
i18 : P = projectiveVariety p
```

Once the plane is constructed, one needs to define a smooth cubic hypersurface containing the plane and the cubic scroll as follows:

```
i19 : X=random(3,S12+P)
```

Using this method one can find explicit examples of cubic fourfolds in each component of the intersection $\mathcal{C}_8 \cap \mathcal{C}_i$, for $i = 12,\ 20$ as follows.

Example 17. *Here is the collection of examples of cubics containing cubic scroll surface S and a plane P.*

1. $\Pi_1 = (x_0,\, x_1,\, x_2)$; $\Pi_1 \cap S$ *is a line;* $C = -6\,x_0x_1^2 - x_1^3 + 6\,x_0^2x_2 + x_0x_1x_2 + 8\,x_1^2x_2 - 8\,x_0x_2^2 + 15\,x_0x_1x_3 - 2\,x_1^2x_3 + 12\,x_0x_2x_3 - 2\,x_1x_2x_3 - 8\,x_2^2x_3 - 10\,x_1x_3^2 - 6\,x_2x_3^2 - 15\,x_0^2x_4 - 10\,x_0x_1x_4 - 9\,x_1^2x_4 + 11\,x_0x_2x_4 + 8\,x_1x_2x_4 + 10\,x_0x_3x_4 - 15\,x_1x_3x_4 - 13\,x_2x_3x_4 - 10\,x_0x_4^2 + 13\,x_1x_4^2 - 9\,x_0^2x_5 - 12\,x_0x_1x_5 + 8\,x_1^2x_5 + 15\,x_0x_2x_5 - 9\,x_1x_2x_5 - 7\,x_2^2x_5 - 14\,x_0x_3x_5 + 5\,x_1x_3x_5 - 4\,x_2x_3x_5 - 13\,x_0x_4x_5 - 11\,x_1x_4x_5 + 9\,x_2x_4x_5 + 8\,x_0x_5^2 + 2\,x_1x_5^2 - 13\,x_2x_5^2$;

2. $\Pi_2 = (x_3,\, x_4,\, x_5)$; $\Pi_2 \cap S$ *is a smooth conic;* $C = -6\,x_0x_1x_3 + 6\,x_1^2x_3 - 13\,x_0x_2x_3 - 2\,x_1x_2x_3 + 13\,x_2^2x_3 - 9\,x_1x_3^2 + 8\,x_2x_3^2 + 6\,x_0^2x_4 + 7\,x_0x_1x_4 + x_1^2x_4 + x_0x_2x_4 - 13\,x_1x_2x_4 + 9\,x_0x_3x_4 - x_1x_3x_4 - 8\,x_2x_3x_4 - 7\,x_0x_4^2 + 8\,x_1x_4^2 - 4\,x_0^2x_5 - 12\,x_0x_1x_5 - 12\,x_1^2x_5 + x_0x_2x_5 + 4\,x_1x_2x_5 + 15\,x_2^2x_5 - 15\,x_0x_3x_5 + 4\,x_1x_3x_5 + 5\,x_2x_3x_5 - 15\,x_3^2x_5 + 10\,x_0x_4x_5 + 5\,x_1x_4x_5 + 6\,x_2x_4x_5 - 6\,x_3x_4x_5 + 5\,x_4^2x_5 + 3\,x_0x_5^2 + x_1x_5^2 + 14\,x_2x_5^2 - 5\,x_3x_5^2 - 5\,x_4x_5^2 + 6\,x_5^3$;

3. $P = V(x_2 - x_3 + 7\,x_4 + 4\,x_5,\ x_1 + 15\,x_3 - 14\,x_4 - 9\,x_5,\ x_0 - 14\,x_3 + 9\,x_5)$; *$P \cap S$ is empty;* $C = -14\,x_0x_1^2 + 11\,x_1^3 + 14\,x_0^2x_2 - 11\,x_0x_1x_2 - 3\,x_1^2x_2 + 3\,x_0x_2^2 + 6\,x_0x_1x_3 + 13\,x_0x_2x_3 - 3\,x_1x_2x_3 + 7\,x_2^2x_3 + 8\,x_1x_3^2 - 15\,x_2x_3^2 - 6\,x_0^2x_4 - 13\,x_0x_1x_4 + 11\,x_1^2x_4 - 8\,x_0x_2x_4 - 7\,x_1x_2x_4 - 8\,x_0x_3x_4 + 6\,x_1x_3x_4 - 13\,x_2x_3x_4 + 9\,x_0x_4^2 + 13\,x_1x_4^2 - 3\,x_0^2x_5 - 4\,x_0x_1x_5 - 4\,x_1^2x_5 - 10\,x_0x_2x_5 - 14\,x_1x_2x_5 + 12\,x_2^2x_5 - 8\,x_0x_3x_5 + 10\,x_1x_3x_5 + 2\,x_2x_3x_5 + 12\,x_3^2x_5 + 14\,x_0x_4x_5 - 10\,x_1x_4x_5 - 2\,x_2x_4x_5 + 2\,x_3x_4x_5 - 9\,x_4^2x_5 + 9\,x_0x_5^2 - 11\,x_1x_5^2 - 2\,x_2x_5^2 - 13\,x_3x_5^2 - 14\,x_4x_5^2 - 15\,x_5^3$;

4. $P = V(x_2 + 7\,x_3 - 10\,x_4 + 4\,x_5,\ x_1 + 5\,x_3 - 11\,x_4 - 3\,x_5,\ x_0 - 6\,x_3 + 9\,x_4 - 10\,x_5)$; *$P \cap S$ consists of 1 point;* $C = 12\,x_0x_1^2 - 6\,x_1^3 - 12\,x_0^2x_2 + 6\,x_0x_1x_2 + 15\,x_1^2x_2 - 15\,x_0x_2^2 + 2\,x_0x_1x_3 - 7\,x_1^2x_3 + 5\,x_0x_2x_3 - 11\,x_1x_2x_3 - 10\,x_2^2x_3 - 6\,x_1x_3^2 + 4\,x_2x_3^2 - 2\,x_0^2x_4 + 2\,x_0x_1x_4 + 15\,x_1^2x_4 - 4\,x_0x_2x_4 + 10\,x_1x_2x_4 + 6\,x_0x_3x_4 + 11\,x_1x_3x_4 - 14\,x_2x_3x_4 - 15\,x_0x_4^2 + 14\,x_1x_4^2 + 8\,x_0^2x_5 - 11\,x_0x_1x_5 + 3\,x_1^2x_5 + 4\,x_1x_2x_5 - x_2^2x_5 - 5\,x_0x_3x_5 - 2\,x_1x_3x_5 - 9\,x_2x_3x_5 - 8\,x_0x_4x_5 + 10\,x_1x_4x_5 - 11\,x_2x_4x_5 - 12\,x_3x_4x_5 - 14\,x_4^2x_5 + 6\,x_0x_5^2 + 14\,x_1x_5^2 + 7\,x_2x_5^2 + 10\,x_3x_5^2 + 3\,x_4x_5^2 + 8\,x_5^3$;

5. $P = V(x_2 + 15\,x_4 - 13\,x_5,\ x_1 + 15\,x_3 + 3\,x_5,\ x_0 - 15\,x_3 - 10\,x_4 + 14\,x_5)$; *$P \cap S$ consists of 2 points;* $C = 2\,x_0x_1^2 + 7\,x_1^3 - 2\,x_0^2x_2 - 7\,x_0x_1x_2 - 6\,x_1^2x_2 + 6\,x_0x_2^2 + x_0x_1x_3 + 11\,x_1^2x_3 + 11\,x_0x_2x_3 - 8\,x_1x_2x_3 + 11\,x_2^2x_3 - 14\,x_1x_3^2 - 11\,x_2x_3^2 - x_0^2x_4 + 9\,x_0x_1x_4 - 12\,x_1^2x_4 - 11\,x_0x_2x_4 - 11\,x_1x_2x_4 + 14\,x_0x_3x_4 - 13\,x_1x_3x_4 + 2\,x_2x_3x_4 - 7\,x_0x_4^2 - 2\,x_1x_4^2 + 9\,x_0^2x_5 + 7\,x_0x_1x_5 + 8\,x_1^2x_5 - 6\,x_0x_2x_5 + 3\,x_1x_2x_5 + 11\,x_2^2x_5 - 14\,x_0x_3x_5 + 10\,x_1x_3x_5 - 7\,x_2x_3x_5 + 8\,x_3^2x_5 + 5\,x_0x_4x_5 - 6\,x_1x_4x_5 - 13\,x_2x_4x_5 - x_3x_4x_5 + 3\,x_4^2x_5 - x_0x_5^2 + 2\,x_1x_5^2 + 5\,x_2x_5^2 + x_3x_5^2 + 10\,x_4x_5^2$;

6. $P = V(x_5,\ x_1 + 12\,x_2 - 12\,x_3 - 4\,x_4,\ x_0 - 7\,x_2 + 14\,x_3 - 3\,x_4)$; *$P \cap Z$ consists of 3 points;* $C = 6\,x_0x_1^2 + 5\,x_1^3 - 6\,x_0^2x_2 - 5\,x_0x_1x_2 - 13\,x_1^2x_2 + 13\,x_0x_2^2 - 5\,x_0x_1x_3 + 15\,x_1^2x_3 + 13\,x_0x_2x_3 + 7\,x_1x_2x_3 - 12\,x_2^2x_3 + 7\,x_1x_3^2 - 8\,x_2x_3^2 + 5\,x_0^2x_4 + 3\,x_0x_1x_4 + 6\,x_1^2x_4 - 13\,x_0x_2x_4 + 12\,x_1x_2x_4 - 7\,x_0x_3x_4 - 3\,x_1x_3x_4 + 3\,x_2x_3x_4 + 11\,x_0x_4^2 - 3\,x_1x_4^2 - 9\,x_0^2x_5 + 2\,x_0x_1x_5 + 7\,x_1^2x_5 + 14\,x_0x_2x_5 + 13\,x_1x_2x_5 - 5\,x_2^2x_5 - 3\,x_0x_3x_5 + 14\,x_1x_3x_5 - 8\,x_2x_3x_5 + 8\,x_3^2x_5 + 3\,x_0x_4x_5 + 13\,x_1x_4x_5 - 2\,x_2x_4x_5 + 8\,x_3x_4x_5 - 7\,x_4^2x_5 - 9\,x_0x_5^2 - 5\,x_1x_5^2 + 11\,x_2x_5^2 + 5\,x_3x_5^2 + 3\,x_4x_5^2 + 14\,x_5^3$;

7. $P = V(x_3 - x_4 - 2\,x_5,\ x_1 - x_2 - 14\,x_5,\ x_0 - x_2 - 2\,x_5)$; *$P \cap S$ is a line of the ruling of the scroll, i.e. the image of a line passing through* $[0:0:1]$ *via the map of i8;* $C = 6\,x_0x_1^2 + 7\,x_1^3 - 6\,x_0^2x_2 - 7\,x_0x_1x_2 - 15\,x_1^2x_2 + 15\,x_0x_2^2 - 14\,x_0x_1x_3 + 2\,x_1^2x_3 + 3\,x_0x_2x_3 + 15\,x_1x_2x_3 - 7\,x_2^2x_3 - 12\,x_1x_3^2 + 4\,x_2x_3^2 + 14\,x_0^2x_4 - 5\,x_0x_1x_4 + 5\,x_1^2x_4 + 11\,x_0x_2x_4 + 7\,x_1x_2x_4 + 12\,x_0x_3x_4 + 12\,x_1x_3x_4 - 14\,x_2x_3x_4 + 15\,x_0x_4^2 + 14\,x_1x_4^2 - 12\,x_0^2x_5 + 3\,x_0x_1x_5 + 15\,x_1^2x_5 - 13\,x_0x_2x_5 - 11\,x_1x_2x_5 + 10\,x_2^2x_5 - 2\,x_0x_3x_5 - 15\,x_1x_3x_5 - 15\,x_2x_3x_5 - 9\,x_3^2x_5 + 8\,x_0x_4x_5 - 12\,x_1x_4x_5 + 12\,x_2x_4x_5 - 9\,x_3x_4x_5 - 15\,x_4^2x_5 - x_0x_5^2 + 5\,x_1x_5^2 - 15\,x_2x_5^2 - 8\,x_3x_5^2 + 6\,x_4x_5^2 + 15\,x_5^3$;

8. $P = V(x_3 + 11\,x_4,\ x_1 - 7\,x_4,\ x_0 + 8\,x_4)$; *$P \cap S$ is a a directrix line of the cubic scroll surface; the directrix line is the image of the exceptional divisor, which is easily obtained via the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^2$ *as in i12 here above;*

$C = -12\,x_0^3+10\,x_0^2x_1-10\,x_0x_1^2+11\,x_0^2x_2-10\,x_0x_1x_2-2\,x_0^2x_3+3\,x_0x_1x_3-15\,x_1^2x_3+6\,x_0x_2x_3-5\,x_1x_2x_3+8\,x_2^2x_3-15\,x_0x_3^2-14\,x_1x_3^2-11\,x_2x_3^2-x_0^2x_4-9\,x_0x_1x_4+10\,x_1^2x_4+x_0x_2x_4+15\,x_1x_2x_4-5\,x_2^2x_4-11\,x_0x_3x_4-3\,x_1x_3x_4-12\,x_2x_3x_4+15\,x_3^2x_4-15\,x_0x_4^2+14\,x_1x_4^2+10\,x_2x_4^2-4\,x_3x_4^2-3\,x_4^3+10\,x_0^2x_5-6\,x_0x_1x_5-5\,x_1^2x_5-12\,x_0x_2x_5+5\,x_1x_2x_5-9\,x_0x_3x_5+11\,x_1x_3x_5+3\,x_2x_3x_5-15\,x_0x_4x_5-11\,x_1x_4x_5-5\,x_2x_4x_5-11\,x_3x_4x_5-14\,x_0x_5^2+9\,x_1x_5^2+11\,x_4x_5^2$. *See the discussion in Sect. 5.3.3 for more details on this kind of cubics.*

Example 18. *Smooth cubics containing Veronese surface Z and plane P*

1. $\Pi_1 = V(x_0,\, x_1,\, x_2)$; *$\Pi_1 \cap Z$ is a smooth conic of genus 0;* $C = 2\,x_0x_1^2-6\,x_1^3+8\,x_0x_1x_2-3\,x_1^2x_2-4\,x_0x_2^2-12\,x_1x_2^2-15\,x_2^3-2\,x_0^2x_3+6\,x_0x_1x_3+x_1^2x_3-7\,x_0x_2x_3-x_1x_2x_3-8\,x_2^2x_3-x_0x_3^2+13\,x_2x_3^2-8\,x_0^2x_4+10\,x_0x_1x_4-11\,x_1^2x_4+12\,x_0x_2x_4+12\,x_1x_2x_4+3\,x_2^2x_4+12\,x_0x_3x_4-13\,x_1x_3x_4-x_2x_3x_4-13\,x_0x_4^2-6\,x_1x_4^2-10\,x_2x_4^2+4\,x_0^2x_5+10\,x_1^2x_5+15\,x_0x_2x_5+3\,x_1x_2x_5+6\,x_2^2x_5-x_0x_3x_5+7\,x_1x_3x_5-15\,x_2x_3x_5-6\,x_0x_4x_5-6\,x_1x_4x_5-6\,x_2x_4x_5-6\,x_0x_5^2+6\,x_1x_5^2$;

2. $P = V(x_2-6\,x_3-2\,x_4-14\,x_5,\; x_1+8\,x_3-10\,x_4+11\,x_5,\; x_0+11\,x_3-11\,x_4-9\,x_5)$; *$P\cap Z$ is empty;* $C = 7\,x_0x_1^2+12\,x_1^3+2\,x_0x_1x_2-10\,x_1^2x_2-6\,x_0x_2^2+6\,x_1x_2^2+10\,x_2^3-7\,x_0^2x_3-12\,x_0x_1x_3-12\,x_1^2x_3-12\,x_0x_2x_3-11\,x_1x_2x_3+12\,x_0x_3^2-4\,x_2x_3^2-2\,x_0^2x_4-9\,x_0x_1x_4+2\,x_1^2x_4-8\,x_0x_2x_4-5\,x_1x_2x_4+9\,x_0x_3x_4+4\,x_1x_3x_4+9\,x_2x_3x_4+10\,x_0x_4^2-2\,x_1x_4^2+11\,x_2x_4^2-x_3x_4^2-12\,x_4^3+6\,x_0^2x_5+2\,x_0x_1x_5+10\,x_1^2x_5-10\,x_0x_2x_5-10\,x_1x_2x_5+6\,x_2^2x_5-15\,x_0x_3x_5-7\,x_1x_3x_5+11\,x_2x_3x_5+x_3^2x_5+10\,x_0x_4x_5+9\,x_1x_4x_5+9\,x_2x_4x_5+12\,x_3x_4x_5-4\,x_4^2x_5-6\,x_0x_5^2-9\,x_1x_5^2+4\,x_3x_5^2$;

3. $P = V(x_2+13\,x_3+4\,x_4-10\,x_5,\; x_1-5\,x_3+12\,x_4-10\,x_5,\; x_0-11\,x_3-7\,x_4+15\,x_5)$; *$P\cap Z$ consists of 1 point;* $C = -8\,x_0x_1^2+8\,x_1^3+13\,x_0x_1x_2+10\,x_1^2x_2-14\,x_0x_2^2-4\,x_1x_2^2-13\,x_2^3+8\,x_0^2x_3-8\,x_0x_1x_3+13\,x_1^2x_3-8\,x_0x_2x_3+12\,x_1x_2x_3-11\,x_2^2x_3-13\,x_0x_3^2+12\,x_2x_3^2-13\,x_0^2x_4-2\,x_0x_1x_4-14\,x_1^2x_4+12\,x_0x_2x_4+2\,x_1x_2x_4+9\,x_2^2x_4+2\,x_0x_3x_4-12\,x_1x_3x_4-14\,x_2x_3x_4-13\,x_0x_4^2+9\,x_1x_4^2+12\,x_2x_4^2-9\,x_3x_4^2+5\,x_4^3+14\,x_0^2x_5-8\,x_0x_1x_5+2\,x_1^2x_5+13\,x_0x_2x_5-6\,x_1x_2x_5-5\,x_2^2x_5-11\,x_0x_3x_5+5\,x_1x_3x_5+4\,x_2x_3x_5+9\,x_3^2x_5-3\,x_0x_4x_5+15\,x_1x_4x_5-4\,x_2x_4x_5-5\,x_3x_4x_5+14\,x_4^2x_5+5\,x_0x_5^2+4\,x_1x_5^2-14\,x_3x_5^2$;

4. $P = V(x_2+12\,x_3+2\,x_4-7\,x_5,\; x_1-x_3-5\,x_5,\; x_0+7\,x_3-4\,x_4+10\,x_5)$; *$P\cap Z$ consists of 2 points;* $C = 2\,x_0x_1^2+9\,x_1^3-7\,x_0x_1x_2+x_1^2x_2+15\,x_0x_2^2+x_1x_2^2+3\,x_2^3-2\,x_0^2x_3-9\,x_0x_1x_3+4\,x_1^2x_3-x_0x_2x_3-2\,x_1x_2x_3-5\,x_2^2x_3-4\,x_0x_3^2-8\,x_2x_3^2+7\,x_0^2x_4-2\,x_1^2x_4+10\,x_0x_2x_4+13\,x_1x_2x_4+x_2^2x_4+4\,x_0x_3x_4+8\,x_1x_3x_4+5\,x_2x_3x_4+12\,x_0x_4^2-x_1x_4^2-6\,x_2x_4^2+6\,x_3x_4^2-2\,x_4^3-15\,x_0^2x_5-11\,x_0x_1x_5-2\,x_1^2x_5-3\,x_0x_2x_5-8\,x_2^2x_5+13\,x_0x_3x_5-4\,x_1x_3x_5-14\,x_2x_3x_5-6\,x_3^2x_5-x_0x_4x_5-11\,x_1x_4x_5+7\,x_2x_4x_5+2\,x_3x_4x_5+12\,x_4^2x_5+,x_1x_3x_5-7\,x_2x_3x_5-2\,x_0x_4x_5+3\,x_0x_5^2$;

5. $P = V(x_2-13\,x_3-3\,x_4-13\,x_5,\; x_1+9\,x_3+5\,x_4+14\,x_5,\; x_0+x_3-4\,x_4-5\,x_5)$; *$P\cap Z$ consists of 3 points;* $C = -10\,x_0x_1^2+15\,x_1^3-13\,x_0x_1x_2+10\,x_1^2x_2-7\,x_0x_2^2-$

$3\,x_1x_2^2 + 9\,x_2^3 + 10\,x_0^2x_3 - 15\,x_0x_1x_3 + 15\,x_1^2x_3 - 4\,x_1x_2x_3 + 7\,x_2^2x_3 - 15\,x_0x_3^2 - 5\,x_2x_3^2 + 13\,x_0^2x_4 - 10\,x_0x_1x_4 - x_1^2x_4 + 3\,x_0x_2x_4 + 6\,x_1x_2x_4 - 15\,x_2^2x_4 + 5\,x_0x_3x_4 + 5\,x_1x_3x_4 + x_1x_4^2 + 8\,x_2x_4^2 - 3\,x_3x_4^2 + 14\,x_4^3 + 7\,x_0^2x_5 - 9\,x_1^2x_5 - 9\,x_0x_2x_5 + 2\,x_1x_2x_5 - 12\,x_2^2x_5 - 4\,x_0x_3x_5 - x_1x_3x_5 - x_2x_3x_5 + 3\,x_3^2x_5 + 13\,x_0x_4x_5 - 7\,x_1x_4x_5 - 14\,x_2x_4x_5 - 14\,x_3x_4x_5 + 11\,x_4^2x_5 + 12\,x_0x_5^2 + 14\,x_1x_5^2 - 11\,x_3x_5^2$;

6. *The Veronese surface can degenerate into $Z = S(1,2) \cup P$, union of a cubic scroll surface $S(1,2)$ and a plane such that $P \cap S(1,2)$ is the directrix line of the scroll; in this case $\gamma = P\cdot(S(1,2) \cup P) = 4$ (see Sect. 5.3.3); A cubic fourfold in $\mathcal{C}_{14}$ containing this degeneration of Veronese surface is automatically contained in $\mathcal{C}_8 \cap \mathcal{C}_{12}$; $C = -12\,x_0^3 + 10\,x_0^2x_1 - 10\,x_0x_1^2 + 11\,x_0^2x_2 - 10\,x_0x_1x_2 - 2\,x_0^2x_3 + 3\,x_0x_1x_3 - 15\,x_1^2x_3 + 6\,x_0x_2x_3 - 5\,x_1x_2x_3 + 8\,x_2^2x_3 - 15\,x_0x_3^2 - 14\,x_1x_3^2 - 11\,x_2x_3^2 - x_0^2x_4 - 9\,x_0x_1x_4 + 10\,x_1^2x_4 + x_0x_2x_4 + 15\,x_1x_2x_4 - 5\,x_2^2x_4 - 11\,x_0x_3x_4 - 3\,x_1x_3x_4 - 12\,x_2x_3x_4 + 15\,x_3^2x_4 - 15\,x_0x_4^2 + 14\,x_1x_4^2 + 10\,x_2x_4^2 - 4\,x_3x_4^2 - 3\,x_4^3 + 10\,x_0^2x_5 - 6\,x_0x_1x_5 - 5\,x_1^2x_5 - 12\,x_0x_2x_5 + 5\,x_1x_2x_5 - 9\,x_0x_3x_5 + 11\,x_1x_3x_5 + 3\,x_2x_3x_5 - 15\,x_0x_4x_5 - 11\,x_1x_4x_5 - 5\,x_2x_4x_5 - 11\,x_3x_4x_5 - 14\,x_0x_5^2 + 9\,x_1x_5^2 + 11\,x_4x_5^2$.*

C.0.2 Other easy examples

In this last section, we give other easy examples. These examples are generated "by hand". This means that we did not use any method generating random planes $P \subset \mathbb{P}^5$, we chose them manually.

Example 19. *Smooth cubics containing a Veronese surface Z and a plane P.*

1. *$P = V(x_0, x_3, x_5)$; $P \cap Z$ is empty; $C = x_0x_1^2 + 12\,x_0x_1x_2 + x_0x_2^2 - x_0^2x_3 - 8\,x_1^2x_3 + 3\,x_0x_2x_3 - 7\,x_1x_2x_3 + 12\,x_2^2x_3 + 8\,x_0x_3^2 - 15\,x_2x_3^2 - 12\,x_0^2x_4 - 3\,x_0x_1x_4 + 7\,x_0x_2x_4 + 7\,x_0x_3x_4 + 15\,x_1x_3x_4 - 4\,x_0x_4^2 - 4\,x_3x_4^2 - x_0^2x_5 - 7\,x_0x_1x_5 + 10\,x_1^2x_5 + 2\,x_1x_2x_5 - 4\,x_2^2x_5 + 13\,x_0x_3x_5 + 6\,x_2x_3x_5 + 4\,x_3^2x_5 - 2\,x_0x_4x_5 - 6\,x_1x_4x_5 - 12\,x_2x_4x_5 + 10\,x_4^2x_5 + 4\,x_0x_5^2 + 12\,x_1x_5^2 - 10\,x_3x_5^2$*

2. *$P = V(x_0, x_2, x_3)$; $P \cap Z$ consists of 2 points; $C = x_0x_1^2 - 4\,x_0x_1x_2 - 10\,x_1^2x_2 + 14\,x_0x_2^2 - 12\,x_1x_2^2 + 13\,x_2^3 - x_0^2x_3 + x_1^2x_3 - 3\,x_0x_2x_3 - 13\,x_1x_2x_3 - 4\,x_2^2x_3 - x_0x_3^2 + 7\,x_2x_3^2 + 4\,x_0^2x_4 + 13\,x_0x_1x_4 + 15\,x_0x_2x_4 - 9\,x_1x_2x_4 + 14\,x_2^2x_4 + 13\,x_0x_3x_4 - 7\,x_1x_3x_4 - 11\,x_2x_3x_4 + 13\,x_0x_4^2 + 7\,x_2x_4^2 - 14\,x_0^2x_5 - 3\,x_0x_1x_5 - 13\,x_0x_2x_5 - 12\,x_1x_2x_5 - 3\,x_2^2x_5 + 11\,x_1x_3x_5 - 7\,x_2x_3x_5 - 2\,x_0x_4x_5 + 3\,x_0x_5^2$*

3. *$P = V(x_2, x_3, x_4)$; $P \cap Z$ consists of 3 points; $C = x_0x_1x_2 + 14\,x_1^2x_2 - 10\,x_1x_2^2 + 4\,x_2^3 + 9\,x_1^2x_3 - 14\,x_0x_2x_3 + 11\,x_1x_2x_3 + 12\,x_2^2x_3 - 9\,x_0x_3^2 + 15\,x_2x_3^2 - x_0^2x_4 - 13\,x_1^2x_4 + 10\,x_0x_2x_4 - 12\,x_1x_2x_4 - 9\,x_2^2x_4 + 2\,x_0x_3x_4 - 15\,x_1x_3x_4 - 15\,x_2x_3x_4 - 10\,x_0x_4^2 + 11\,x_1x_4^2 - 14\,x_2x_4^2 + 9\,x_3x_4^2 - 11\,x_4^3 - 4\,x_0x_2x_5 - 15\,x_1x_2x_5 + 10\,x_0x_3x_5 + 4\,x_1x_3x_5 - 5\,x_2x_3x_5 - 9\,x_3^2x_5 - 7\,x_0x_4x_5 - 12\,x_1x_4x_5 + 11\,x_3x_4x_5 - 8\,x_4^2x_5 + 8\,x_3x_5^2$*

Example 20. *Smooth cubics containing a Plane P and a cubic scroll S such that the intersection is first a ruling line (Ex. 1), and second the directrix line (Ex. 2).*

1. *First, we take the image (via the map of i8) of the line $L = V(t_0)$ that passes through the point $[0 : 0 : 1]$. It gives the ruling line $l = V(x_0, x_1, x_3, x_5)$. Then, we take the plane $P = V(x_0, x_1, x_3)$ that contains the ruling line l. It remains to verify that P intersects the cubic scroll S in this same line l.*
   We get a smooth cubic 4-fold containing $S \cup P$:
   $C = (x_0x_1^2 - 8\,x_1^3 - x_0^2x_2 + 8\,x_0x_1x_2 - x_1^2x_2 + x_0x_2^2 + x_0x_1x_3 - 10\,x_1^2x_3 - 8\,x_0x_2x_3 - 11\,x_1x_2x_3 - 5\,x_2^2x_3 + 11\,x_1x_3^2 + 11\,x_2x_3^2 - x_0^2x_4 - 13\,x_0x_1x_4 + 11\,x_0x_2x_4 + 5\,x_1x_2x_4 - 11\,x_0x_3x_4 + 9\,x_1x_3x_4 + 4\,x_2x_3x_4 + 11\,x_0x_4^2 - 4\,x_1x_4^2 + 6\,x_0^2x_5 - 14\,x_0x_1x_5 - 4\,x_1^2x_5 + 14\,x_0x_2x_5 - 13\,x_1x_2x_5 - 11\,x_0x_3x_5 - 12\,x_1x_3x_5 - 6\,x_2x_3x_5 + 7\,x_3^2x_5 + x_0x_4x_5 + 15\,x_1x_4x_5 - 9\,x_3x_4x_5 - 11\,x_0x_5^2 + 8\,x_1x_5^2 - 8\,x_3x_5^2)$.

2. *First, we need the equation of the directrix line l which is the image of $[u : v] \times [0 : 0 : 1]$ via the map of i12. We get that $l = V(x_0, x_1, x_3, x_4)$. Then, we take the plane $P = V(x_1, x_3, x_4)$ that contains the directrix line l. It remains to verify that P and S intersect along this same line l.*
   We get a smooth cubic 4-fold containing $P \cup S$:
   $(x_0^2x_1 - 2\,x_0x_1^2 + 10\,x_0x_1x_2 - 8\,x_0^2x_3 + 3\,x_0x_1x_3 - 8\,x_0x_2x_3 - 12\,x_1x_2x_3 + 14\,x_2^2x_3 - 9\,x_0x_3^2 - 14\,x_1x_3^2 + 3\,x_2x_3^2 - 7\,x_0^2x_4 + 3\,x_0x_1x_4 + 2\,x_1^2x_4 - x_0x_2x_4 + 7\,x_1x_2x_4 + 15\,x_2^2x_4 + 11\,x_0x_3x_4 + 8\,x_1x_3x_4 - 6\,x_2x_3x_4 + 9\,x_3^2x_4 - 15\,x_0x_4^2 - 4\,x_1x_4^2 - 6\,x_2x_4^2 + 11\,x_3x_4^2 + 11\,x_4^3 - 5\,x_0x_1x_5 + 14\,x_1^2x_5 - 15\,x_1x_2x_5 - 13\,x_0x_3x_5 + 10\,x_1x_3x_5 + 7\,x_2x_3x_5 - 8\,x_0x_4x_5 - 7\,x_1x_4x_5 - 2\,x_2x_4x_5 + 10\,x_3x_4x_5 + 12\,x_4^2x_5 - 12\,x_1x_5^2 - 7\,x_4x_5^2)$.

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