DIPARTIMENTO DI MATEMATICA E FISICA

Curves on Enriques surfaces and on rational elliptic surfaces

Tesi di dottorato in Geometria Algebrica

XXXVI Ciclo

Candidato:
Simone Pesatori Firma:__________________

Relatore di tesi:
Andreas Leopold Knutsen Firma:__________________

Coordinatore:
Alessandro Giuliani Firma:__________________

Anno Accademico 2023/2024 ,
Matematica e Fisica , Università degli Studi Roma Tre
Largo San Leonardo Murialdo 1, 00146, Roma

Curves on Enriques surfaces and on rational elliptic surfaces

By: Simone Pesatori

Supervisor: Andreas Leopold Knutsen

Acknowledgments

First of all, I want to thank my advisor Andreas Leopold Knutsen. His kindness and enthusiasm have been crucial allies for me during my time in Rome. I am very grateful to him for guiding me in this work, and I would like to thank him for all the invaluable knowledge and the experience he shared with me. I am also grateful for the opportunity of spending a fruitful and beautiful month at the University of Bergen in Fall 2023, during which this thesis took shape.

I also thank Alessandro Verra, Andrea Bruno, Concettina Galati, Rick Miranda, Antonella Grassi, Luca Schaffler, Alice Garbagnati and Lucia Caporaso, for the productive and essential conversations we had during these years.

Simone Pesatori
2023/2024

Introduction

Throughout the Thesis we will work over the complex field $\mathbb{C}$. In the first two chapters of this work we analyze the moduli space of complex rational elliptic surfaces with a section built by Miranda in [44]. Our purpose is to stratify this space in terms of the configurations of singular fibers that the surfaces have. Given a configuration σ of singular fibers, what are the main properties of the stratum parametrizing surfaces with σ as configuration of singular fibers? Is it irreducible? When do two of these strata intersect? When is one contained in another?

The main tool in the construction of this space is the Weierstrass model of the elliptic surfaces with a section. Every surface $X \longrightarrow C$ is associated to a triplet $(A, B, \mathbb{L})$, where $\mathbb{L}$ is a line bundle over the base curve C of nonnegative degree, A is a global section of $4\mathbb{L}$ and B is a global section of $6\mathbb{L}$. Through these data, it is possible to localize the points of C over which the singular fibers lie and detect the types of singular fibers the surface has. Basically, the information is encoded in the zeroes of the sections A, B and $D = 4A^3 + 27B^2 \in H^0(C, 12\mathbb{L})$ and the type of singular fibers over a point $q \in C$ only depends on the triplet

$$(v_q(A), v_q(B), v_q(D)),$$

where $v_q(A)$ is the order of vanishing of the section A at q and the same for $v_q(B)$ and $v_q(D)$.

Since the moduli space we analyze is constructed by using the Weierstrass model, we need to understand how these models behave in families, in order to stratify the moduli space. In the case of the rational elliptic surfaces, the base curve C is isomorphic to $\mathbb{P}^1$, and A, B and D are homogeneous polynomials in two variables. The goal of Chapter 1 is to study the space of the pairs of polynomials of given degrees in terms of the configurations of their multiple zeroes and of their common zeroes. We introduce the notion of *generalized coincident root locus*, that is the main tool to stratify the moduli space of rational elliptic surfaces.

In [49] and [46], Miranda and Persson prove that there is a list of 279 possible configurations of singular fibers for a rational elliptic surface with a section. The tools we use in Chapter 2 do not allow us to take account of the fibers of type I_k with $k > 1$ and of type I_k^*, with $k > 0$, then our classification turns out to be weaker than the one of Miranda and Persson. However, our method controls some fibers, that we call special smooth fibers, that are smooth elliptic fibers with invariant J equal to 1 or 0. Moreover, we take account of the multiplicity of the J-map at the points of the discriminant locus, so that our classification refines the Miranda-Persson one in this sense.

Although our analysis does not control the multiplicative fibers, we describe a strategy to get the complete stratification of the moduli space W_1 involving also the fibers I_k and I_k^*. Unfortunately, this solution is computationally too hard for the softwares we have.

The main objects of Chapter 3 are the complex Enriques surfaces of base change type, a particular kind of Enriques surfaces introduced by Hulek and Schütt in [36]. Starting from a complex rational elliptic surface, they produce nine-dimensional families of Enriques surfaces having a genus 1 pencil with a (possibly singular) rational bisection which splits in two smooth curves in the K3 cover. Much of the chapter is devoted to showing that these bisections are generically nodal (meaning that they have just simple nodes as singularities). The main result of the chapter is showing that the rational bisections on the Enriques surfaces of base change type deform to rational curves over the very general Enriques surface, positively answering a question posed by Galati and Knutsen in [29] about the existence of rational curves on the very general Enriques surface.

A second interesting result we give concerns the Severi varieties of curves on Enriques surfaces. In [13], the authors prove that if a Severi variety of curves on the very general Enriques surface has dimension greater then expected (said nonregular), then the pullback of its general members split in the K3 cover in two linearly equivalent curves, with a prescribed number of nodes. We show that the assumption of very generality is necessary: for instance, we find plenty of Severi varieties that violate the result of Ciliberto, Dedieu, Galati and Knutsen in the Enriques surfaces of base change type.

Another relevant result concerns the classification of the genus 1 pencils over the K3 surfaces covering the Enriques surfaces of base change type. The genus 1 fibrations on K3 surfaces with a non-symplectic involution over a rational elliptic surface or with an Enriques involution have been intensively studied in the last decades (see for example [31] for the rational case and [18] for the Enriques one), but never in the case in which the K3 carries both the involutions. Garbagnati and Salgado, in a series of works (see, for example, [31] and [32]), perform a classification of the genus 1 fibrations on the very general K3 surface covering a rational elliptic surface. Such K3 surfaces form a 10-dimensional family $\mathcal{F}$ in the moduli space of K3 surfaces. In the previously mentioned work by Hulek and

Schütt, the constructed Enriques surfaces form 9-dimensional families and they are covered by K3 surfaces which also cover a rational elliptic surface. Thus, they are subfamilies of $\mathcal{F}$. In other words, the K3 covers are a limit case of the ones considered by Garbagnati and Salgado: they admit both the rational and the Enriques involution, while the very general just carries the rational one. We contribute to this classification by extending the results by Garbagnati and Salgado to the limit case and by expanding it with the addition of new classes of genus 1 pencils. These new classes are given by the pullback of some of the nonregular Severi varieties of curves of the Enriques constructed by Hulek and Schütt that we found. They are particularly interesting because of their behaviour with respect to the rational involution: it sends one of these genus 1 pencils to another one. In the classification by Garbagnati and Salgado the fibrations with this property are the hardest to find, while in the limit case subfamilies we could systematically produce examples of these objects.

CONTENTS

Chapter 1

Polynomial spaces

In this chapter we explore the notion of coincident root locus, considered for the first time by Cayley in [7] and investigated by Chipalkatti in [10] and by Fehér, Némethi and Rimányi in [26]. It is a space that parametrizes polynomials with a given configuration of the multiplicities of their roots. The author in [10] shows that one of these spaces is contained in another if and only if the first is obtained from the second by *collapsing* some roots in a sense that we explain.

In the second part of the chapter we generalize this description to spaces parametrizing pairs of polynomials. We define the generalized coincident root loci, that parametrize pairs of polynomials with given configuration of multiple and (possibly) common roots. The main result of the chapter is Proposition 1.2.14, that states that a generalized coincident root locus is contained in another if and only if the first is obtained from the second by collapsing roots (common or not) of the two polynomials. This will be the main tool to stratify the moduli space of the rational elliptic surfaces: every rational elliptic surface is associated to two particular polynomials and to every stratum of the moduli space of rational elliptic surface we associate a generalized coincident root locus.

1.1 Coincident root loci

Consider a homogeneous degree d polynomial in two variables over the complex numbers, that is, a binary form $F(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$. It splits as a product of linear forms, and it is classical that F has a repeated factor if and only if its discriminant vanishes. Similarly, we can ask for algebraic conditions on the coefficients a_i, so that F has say, a triple factor or two double factors. More generally, we may fix a partition λ of d, and ask for algebraic conditions so that the factors of F have multiplicities as dictated by the parts in λ. The object of this section is to review the method for answering such questions given by Chipalkatti in [10]: sometimes we add or extend some proofs not given by the author in order to make the reader familiar with the ideas and the notations. For instance, Proposition 1.1.2 and Proposition 1.1.3 are just claimed and not proved by the author.

The polynomial $F(x, y)$ as above will be identified with the point $[a_0, \dots, a_d]$ of $\mathbb{P}^d$. Let then $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of d, and consider the subset $\mathbb{P}_\lambda$ of $\mathbb{P}^d$ composed by polynomials F that split as $\prod_{i=1}^{n} L_j^{\lambda_j}$ for some linear forms L_j. We see that $\mathbb{P}_\lambda$ is a projective subvariety of $\mathbb{P}^d$. This is the coincident root locus, the main object of the section.

1.1.1 The definition of the coincident root locus

In the sequel $\lambda = (1^{e_1}, 2^{e_2}, \ldots, d^{e_d})$ denotes a partition of d, having e_r parts of size r for $1 \leq r \leq d$. The number of parts of λ is $\sum_{r=1}^{d} e_r = n$.

Let V denote a two dimensional $\mathbb{C}$-vector space: with the choice of $\{x, y\}$ as basis of V, we identify a point of $\mathbb{P}(\mathrm{Symm}^{e_r} V) \cong \mathbb{P}^{e_r}$ with a degree e_r polynomial $G_r(x, y)$ determined up to scalars.

Consider the maps

$$\begin{aligned} v_r : \mathbb{P}^{e_r} &\longrightarrow \mathbb{P}(\mathrm{Symm}^{re_r} V) \\ G_r &\longmapsto G_r^r \end{aligned}$$

and the multiplication map

$$\begin{aligned} \mu : \prod_{i=1}^{d} \mathbb{P}^{re_r} &\longrightarrow \mathbb{P}^d \\ (H_1, \ldots, H_d) &\longmapsto \prod_{r=1}^{d} H_r. \end{aligned}$$

Finally, consider the composition $f_\lambda = \mu \circ \prod_{r=1}^{d} v_r$,

$$\begin{aligned} Y_\lambda := \prod_{r=1}^{d} \mathbb{P}^{e_r} \longrightarrow \prod_{r=1}^{d} \mathbb{P}^{re_r} &\longrightarrow \mathbb{P}^d \\ (G_1, \ldots, G_d) &\longmapsto \prod_{r=1}^{d} G_r^r \end{aligned}$$

Definition 1.1.1. *The coincident root locus X_λ is defined to be the image of f_λ.*

Proposition 1.1.2 (Chipalkatti)**.** *The coincident root locus X_λ is rational, its dimension is n (the number of parts of λ) and its degree is $\frac{n!}{\prod_r (e_r!)} \prod_r r^{e_r}$.*

Proof. The map f_λ is a generically one to one morphism between projective varieties. In fact, the general polynomial F in X_λ has n distinct roots: if

$$f_\lambda^{-1}(F) = \{(F_1, \dots, F_d), (G_1, \dots, G_d)\},$$

we have that $F_i^i \cdot F_j^j = G_i^i \cdot G_j^j$ for some $F_i \neq G_i$, $F_j \neq G_j$ and $i \neq j$ (every exponent appears just once by construction of f_λ). This implies that $F_i^i = G_j^j$ and $F_j^j = G_i^i$, then $\deg(F_i) \cdot i = \deg(G_j) \cdot j$ and $\deg(F_j) \cdot j = \deg(G_i) \cdot i$. We can have this equalities only if we impose that the polynomials have common multiple roots, but we chose F to be general, thus its preimage $f_\lambda^{-1}(F)$ is composed only by the element $(G_1, \dots, G_d) \in \prod_{i=1}^{d} \mathbb{P}^{e_r}$.
We actually proved that f_λ has finite fibers, too. In fact, if $F \in X_\lambda$ and $f_\lambda^{-1}(F)$ has more than one element, we saw that we need to impose its factors $G_1, \dots, G_d$ having some coincident roots, and we have a finite number of choice to do this.
The map f_λ is a morphism and the domain is a projective variety, then the image X_λ is closed. Hence, X_λ is a closed in a projective space, whence it is projective.
We can conclude that f_λ is birational and thus that its image X_λ has dimension $\sum_{r=1}^{d} e_d = n$.
To prove the degree, we identify $\mathbb{P}^d$ with the set of effective divisors of degree d on $\mathbb{P}^1$.
Let Σ denote a general $(d-n)$-dimensional linear subspace of $\mathbb{P}^d$, it corresponds to a linear series g_d^{d-n} on $\mathbb{P}^1$. The points of $\Sigma \cap X_\lambda$ correspond to those divisors in the series which may be written as $\sum_r r(P_{r,1} + P_{r,2} + \cdots + P_{r,e_r})$, for some points $P_{r,j} \in \mathbb{P}^1$. According to De Jonquières' formula (see [2, p. 359]), the number of such divisors is the coefficient of $t^{e_1} t^{e_2} \cdots t^{e_d}$ in $(1 + t_1 + 2t_2 + \cdots + dt_d)^n$, hence the assertion. □

We want to understand the intersection between these coincident root loci: how do more of them intersect? When is one contained in another? The next proposition gives an exhaustive answer to our questions.

Proposition 1.1.3 (Chipalkatti). *Let λ and μ be two partitions of d. Then, $\mathbb{P}_\mu \subseteq \mathbb{P}_\lambda$ if and only if λ is a refinement of μ.*

Before proving the proposition, we give an intuitive explanation of the phenomenon: say that μ is obtained by λ adding up the first two parts of $\lambda = (\lambda_1, \dots, \lambda_n)$, so that

$\mu = (\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_n)$. Roughly speaking, X_μ is the subset of X_λ corresponding to the polynomials such that the roots of multiplicities λ_1 and λ_2 collapse to a root of multiplicity $\lambda_1 + \lambda_2$. More precisely, the general polynomial F in X_λ splits in the product of linear forms $F = L_1^{\lambda_1} \cdot L_2^{\lambda_2} \cdots L_n^{\lambda_n}$, while the general polynomial G in X_μ is of the form $G = M^{\lambda_1+\lambda_2} \cdot M_3^{\lambda_3} \cdots M_n^{\lambda_n}$ for some linear forms $M, M_2, \ldots, M_n$.

Proof of Proposition 1.1.3. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$: we prove the first direction of the statement just for the case in which μ is obtained by λ adding up the first two parts of λ. The general case will follow from the fact that every refinement of a partition can be obtained in a finite number of addition of two parts. So, we have $\mu = (\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_n)$.
It is not restrictive to suppose that $\lambda_3 = \cdots = \lambda_n = 1$ and that $\lambda_1 \neq \lambda_2$. This just simplifies the notation: now, we can write $\lambda = (1^{n-2}, \lambda_1^1, \lambda_2^1)$, where n as always is the number of parts of λ and $\mu = (1^{n-2}, (\lambda_1 + \lambda_2)^1)$. We want to show that a polynomial $G(x, y) \in X_\mu$ belongs to X_λ. We have the two defining maps

$$\begin{aligned} f_\lambda : \mathbb{P}^{n-2} \times \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^{n-2} \times \mathbb{P}^{\lambda_1} \times \mathbb{P}^{\lambda_2} \longrightarrow \mathbb{P}^d \\ (f, h_1, h_2) &\longmapsto (f, h_1^{\lambda_1}, h_2^{\lambda_2}) \qquad \longmapsto f \cdot h_1^{\lambda_1} \cdot h_2^{\lambda_2} \end{aligned}$$

and

$$\begin{aligned} f_\mu : \mathbb{P}^{n-2} \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^{n-2} \times \mathbb{P}^{\lambda_1+\lambda_2} \longrightarrow \mathbb{P}^d \\ (g, h) &\longmapsto (g, h^{\lambda_1+\lambda_2}) \qquad \longmapsto g \cdot h^{\lambda_1+\lambda_2} \end{aligned}$$

A polynomial $G \in X_\mu$ of the form $g \cdot h^{\lambda_1+\lambda_2}$ belongs to the image of f_λ: in fact, it is sufficient to take F of the form $f \cdot h_1^{\lambda_1} \cdot h_2^{\lambda_2}$ and choose $f = g \in \mathbb{P}^{n-2}$ and $h_1 = h_2 = h \in \mathbb{P}^1$.

To prove the other direction, let $\mu = (\mu_1 \dots, \mu_m)$ and let f be a general polynomial in $\mathbb{P}_\mu$, so that

$$f = \prod_{i=1}^{m} f_i^{\mu_i}$$

for some linear forms f_i. The generality of f implies that $f_i \neq f_j$ for every $i \neq j$. Now, since $f \in \mathbb{P}_\lambda$, it can be written as

$$f = \prod_{i=1}^{n} g_i^{\lambda_i}.$$

Up to multiplication by a scalar, f can be written as a product of linear forms only in one way. Hence, if $g_i \neq g_j$ for every $i \neq j$, we have that, up to reordering the indices, $f_i = g_i$ and $\lambda_i = \mu_i$ for every i, from which $\lambda = \mu$. Let us suppose that $g_{j_1} = \cdots = g_{j_p}$ for some indices $j_1, \dots, j_p$ and that there are no other indices j with $g_j = g_{j_1}$. This implies that $f_i = g_{j_1} = \cdots = g_{j_p}$ and $\lambda_i = \mu_{j_1} + \cdots + \mu_{j_p}$ for an index i. By iterating the process for every set of coincident linear forms in the written $g = \prod_{j=1}^{n} g_j^{\lambda_j}$, we conclude that λ is a refinement of μ.

□

We want to represent the stratification of $\mathbb{P}^d$ in terms of the coincident root loci with a diagram.

Definition 1.1.4. *The diagram of specialization of $\mathbb{P}^d$ is an oriented graph with all the coincident root loci as vertices: for every two coincident root loci X_1 and X_2, there is an arrow from X_1 to X_2 if and only if $X_2 \subset X_1$ and the codimension of X_2 in X_1 is 1.*

Example 1.1.5 (Stratification of $\mathbb{P}^4$)**.** *The partitions of 4 are:* (1^4), $(1^2, 2)$, (2^2), $(1, 3)$ *and* (4).
As showed in Prop 1.1 the dimension of X_λ is the number of parts of λ. The specialization diagram of $\mathbb{P}^4$ is

$$\begin{tikzcd}
& X_{(1^4)} \arrow[d] & \\
& X_{(1^2,2)} \arrow[dl] \arrow[dr] & \\
X_{(2^2)} \arrow[dr] & & X_{(1,3)} \arrow[dl] \\
& X_{(4)} &
\end{tikzcd}$$

Example 1.1.6 (Stratification of $\mathbb{P}^6$). *The partition of 6 are:* (111111), (21111), (2211), (3111), (222), (321), (411), (42), (33), (51) *and* (6). *The specialization diagram of* $\mathbb{P}^6$ *is*

$$\begin{tikzcd}
& X_{(1^6)} \arrow[d] & \\
& X_{(1^4,2)} \arrow[dl] \arrow[dr] & \\
X_{(1^3,3)} \arrow[d] \arrow[dr] & & X_{(2^2,1^2)} \arrow[dll] \arrow[dl] \arrow[d] \\
X_{(1,2,3)} \arrow[d] \arrow[dr] \arrow[drr] & X_{(1^2,4)} \arrow[d] \arrow[dr] & X_{(2^3)} \arrow[d] \\
X_{(3^2)} \arrow[dr] & X_{(1,5)} \arrow[d] & X_{(2,4)} \arrow[dl] \\
& X_{(6)} &
\end{tikzcd}$$

1.2 Generalized coincident root loci

The aim of this section is to generalize the notion of coincident root locus to pairs of polynomials.

Consider two binary forms $F(x,y) = \sum_{i=0}^{n} a_i x^i y^{n-i}$ and $G(x,y) = \sum_{j=0}^{m} b_j x^j y^{m-j}$. They split in product of linear forms and it is classical that F and G have a common factor if and only if their resultant vanishes. Similarly, we can ask for algebraic conditions on the coefficients a_i and b_j, so that F and G have say, two common factors or a common factor that is double for F and triple for G. More generally, we may fix a partition of n, a partition of m and a set of relations R (that is a set of pairs consisting of a part of λ and a part of μ), and ask for algebraic conditions so that the factors of F and G have multiple roots and common factors with multiplicities as dictated by the partitions and the relations.

The polynomials $F(x,y)$ and $G(x,y)$ as above will be identified with the points $[a_0,\ldots,a_n]$ of $\mathbb{P}^n$ and $[b_0,\ldots,b_m]$ of $\mathbb{P}^m$. Let then $\lambda = (\lambda_1,\ldots,\lambda_r)$ be a partition of n, $\mu = (\mu_1,\ldots,\mu_s)$ be a partition of m and $R = \{((\lambda_{i_1},\mu_{j_1}),\ldots,(\lambda_{i_c},\mu_{j_c})\}$ a set of relations. Consider the set $X_{\lambda,\mu,R}$, that we define as the subset of $\mathbb{P}^n \times \mathbb{P}^m$ composed by pairs of polynomials (F,G) that split as products of linear forms $F = L_1^{\lambda_1}\cdots L_r^{\lambda_r}$ and $G = M_1^{\mu_1}\cdots M_s^{\mu_s}$ such that $L_{i_k} = M_{j_k}$ for $k = 1,\ldots,c$, which is a projective subvariety of $\mathbb{P}^n \times \mathbb{P}^m$, as we will show. This is the generalized coincident root locus, the main object of this section.

1.2.1 The definition of the generalized coincident root locus

In the sequel, $\lambda = (\lambda_1,\ldots,\lambda_r)$ is a partition of n, $\mu = (\mu_1,\ldots,\mu_s)$ is a partition of m and $R = \{(\lambda_{i_1},\mu_{j_1}),\ldots,(\lambda_{i_c},\mu_{j_c})\}$ is the set of relations. Then, r and s are the numbers of parts of λ and μ respectively.

Definition 1.2.1. *With the notations as above, we call λ^C the set of parts of λ not involved in the relations, that are the parts of λ not appearing in R. We define μ^C in the same way. Moreover, we call $t := \sum_{k=1}^{c} \lambda_{i_k}$ and $u := \sum_{j=1}^{c} \mu_{j_k}$ the sums of the parts of λ and μ involved in the relations in R.*

We want to define the generalized coincident root locus in a similar way to the definition of the simple coincident root locus we introduced in the previous section. We will define it as the image of a generically one to one morphism from a product of suited projective spaces to $\mathbb{P}^n \times \mathbb{P}^m$. Before giving the final definition, we begin with a simpler setting, making two assumptions. We assume that

(i) λ^C and μ^C are composed only by 1's, so that we have $\lambda = (\lambda_{i_1}, \dots, \lambda_{i_c}, 1, \dots, 1)$ and $\mu = (\mu_{j_1}, \dots, \mu_{j_c}, 1, \dots, 1)$, where the number of 1's in λ^C is $r - c$ and the number of 1's in μ^C is $s - c$;

(ii) there are not repeated relations in R: for instance, R does not contain two relations $(\lambda_{i_{k_1}}, \mu_{j_{k_1}})$ and $(\lambda_{i_{k_2}}, \mu_{j_{k_2}})$ such that $\lambda_{i_{k_1}} = \lambda_{i_{k_2}}$ and $\mu_{j_{k_1}} = \mu_{j_{k_2}}$.

The only purpose of these assumptions is to help the reader in getting familiar with the notations and the definitions. Afterwards, we will remove them.

Now, we consider the map

$$
\begin{aligned}
f_{\lambda,\mu,R} : (\mathbb{P}^1)^c \times \mathbb{P}^{n-t} \times \mathbb{P}^{m-u} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\
(f_1, \dots, f_c, g, p) &\longmapsto (f_1^{\lambda_{i_1}} \cdots f_c^{\lambda_{i_c}} \cdot g, f_1^{\mu_{j_1}} \cdots f_c^{\mu_{j_c}} \cdot p).
\end{aligned}
$$

Definition 1.2.2. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is defined to be the image of $f_{\lambda,\mu,R}$.*

Proposition 1.2.3. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is rational and its dimension is $r + s - c$, where r is the number of parts in λ, s is the number of parts in μ and c is the number of relations in R.*

Proof. The map $f_{\lambda,\mu,R}$ is a generically one to one morphism between projective varieties. Indeed, the general pair $(F, G) = (f_1^{\lambda_{i_1}} \cdots f_c^{\lambda_{i_c}} \cdot g, f_1^{\mu_{i_1}} \cdots f_c^{\mu_{j_c}} \cdot p) \in X_{\lambda,\mu,R}$ is composed by polynomials with no common zeroes other than the ones dictated by R and with no multiple zeroes other than the ones dictated by their partitions. This implies that the general fiber $f_{\lambda,\mu,R}^{-1}(F, G)$ is composed only by the element $(f_1, \dots, f_c, g, p) \in (\mathbb{P}^1)^c \times \mathbb{P}^{n-t} \times \mathbb{P}^{m-u}$. In fact, otherwise, we would have another element $(f_1', \dots, f_c', g', p')$ sent by $f_{\lambda,\mu,R}$ to (F, G). But, under this assumption, the only possible case that does not violate the generality of (F, G) is the one in which, for some i, j, we have $f_i^{\lambda_i} \cdot f_j^{\lambda_j} = f_i'^{\lambda_i} \cdot f_j'^{\lambda_j}$ and $f_i^{\mu_i} \cdot f_j^{\mu_j} = f_i'^{\mu_i} \cdot f_j'^{\mu_j}$. This means that $f_i = f_j'$, $f_j = f_i'$, $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$, but we assumed that there are no repeated relations in R. So, the general fiber is composed only by one element, hence the map is generically one to one. Moreover, $f_{\lambda,\mu,R}$ has finite fibers and this can be seen in the same way we proved the finiteness of the fibers of f_λ in the Proposition 1.1.2.

The map $f_{\lambda,\mu,R}$ is a morphism and the domain is a projective variety, whence the image $X_{\lambda,\mu,R}$ is closed. Hence, $X_{\lambda,\mu,R}$ is a closed in a projective space, than it is a projective variety. Furthermore, since the morphism $f_{\lambda,\mu,R}$ is generically one to one, $X_{\lambda,\mu,R}$ is a rational variety. Its dimension is equal to the one of the domain $(\mathbb{P}^1)^c \times \mathbb{P}^{n-t} \times \mathbb{P}^{m-u}$, that is $c+n-t+m-u = r+s-c$, since $n-t = r-c$ and $m-u = s-c$. □

The next example shows how to construct the generalized coincident root locus for two polynomials F and G of degree n and m, having two common zeroes, one simple for F and G and the other simple for F and double for G. Recall that we assumed F and G having only simple roots other than the common ones.

Example 1.2.4. *Let $\lambda = (1^n)$, $\mu = (1^{m-2}, 2)$ and $R = \{(1,2),(1,1)\}$. We have that $c = 2$, $t = 1+1 = 2$ and $u = 2+1 = 3$. Our defining morphism is*

$$\begin{aligned} f_{\lambda,\mu,R} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2} \times \mathbb{P}^{m-3} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f_1, f_2, g, p) &\longmapsto (f_1 \cdot f_2 \cdot g, f_1 \cdot f_2^2 \cdot p). \end{aligned}$$

The dimension of $X_{\lambda,\mu,R}$ is $n+m-3$.

We want now to remove our two assumptions to give the final definition. We start by removing the condition (i), that ensures λ^C and μ^C to be composed only by 1's. What we do is to exploit the Definition 1.1.1 of simple coincident root locus of the previous section to involve the partitions λ^C and μ^C in our definition.

Recall that r is the number of parts of λ and s is the number of parts of μ, then λ^C has $v := r-c$ parts and μ^C has $w := s-c$ parts. We write the partitions $\lambda^C = (1^{a_1}, 2^{a_2}, \ldots, (n-t)^{a_{n-t}})$ and $\mu^C = (1^{b_1}, 2^{b_2}, \ldots, (m-u)^{b_{m-u}})$. Notice that $\sum_{i=1}^{n-t} a_i = v$ and $\sum_{j=1}^{m-u} b_j = w$

Consider the morphisms defining the coincident root loci $X_{\lambda^C} \in \mathbb{P}^{n-t}$ and $X_{\mu^C} \in \mathbb{P}^{m-u}$:

$$f_{\lambda^C} : \prod_{i=1}^{n-t} \mathbb{P}^{a_i} \longrightarrow \mathbb{P}^{n-t}$$

$$(g_1,\dots,g_{n-t}) \longmapsto \prod_{i=1}^{n-t} g_i^i$$

and

$$f_{\mu^C} : \prod_{j=1}^{m-u} \mathbb{P}^{b_j} \longrightarrow \mathbb{P}^{m-u}$$
$$(p_1,\dots,p_{m-u}) \longmapsto \prod_{j=1}^{m-u} p_j^j.$$

The morphism $f_{\lambda,\mu,R}$ now has to take account of the configuration of the roots of F and G other than the common ones, so that we define it in the following way

$$f_{\lambda,\mu,R} : (\mathbb{P}^1)^c \times \prod_{i=1}^{n-t} \mathbb{P}^{a_i} \times \prod_{j=1}^{m-u} \mathbb{P}^{b_j} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(f_1,\dots,f_c,g_1,\dots,g_{n-t},p_1,\dots,p_{m-u}) \longmapsto (f_1^{\lambda_{i_1}}\cdots f_c^{\lambda_{i_c}} \cdot \prod_{i=1}^{n-t} g_i^i, f_1^{\mu_{j_1}}\cdots f_c^{\mu_{j_c}} \cdot \prod_{j=1}^{m-u} p_j^j).$$

This map can be seen as a generalization of the previous $f_{\lambda,\mu,R}$, so we denote it in the same way because from now on we do not impose the condition (i) and then there will not be ambiguity.

Definition 1.2.5. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is defined to be the image of $f_{\lambda,\mu,R}$.*

Proposition 1.2.6. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is rational and its dimension in $\mathbb{P}^n \times \mathbb{P}^m$ is $c+v+w$, where v is the number of parts in λ^C, w is the number of parts in μ^C and c is the number of relations in R.*

Proof. We just need to prove that $f_{\lambda,\mu,R}$ is generically one to one, since the other steps of the proof are the same of the ones in the proof of the Proposition 1.2.3. Notice that this new $f_{\lambda,\mu,R}$ behaves as the previous one except for the introduction of the factor $\prod_{i=1}^{n-t} \mathbb{P}^{a_i} \times \prod_{j=1}^{m-u} \mathbb{P}^{b_j}$. It is sufficient to notice that $f_{\lambda,\mu,R}$ acts on $\prod_{i=1}^{n-t} \mathbb{P}^{a_i}$ as f_{λ^C} and thus it behaves as a generically one to one morphism, as showed in the proof of the Proposition 1.1.2. The same can be said for the factor $\prod_{j=1}^{m-u} \mathbb{P}^{b_j}$. Hence, by Proposition 1.2.3, $f_{\lambda,\mu,R}$ is birational. The dimension of $X_{\lambda,\mu,R}$ is the one of the domain, that is $r+s-c = c+v+w$. □

In order to produce a birational morphism in full generality, we need to remove the assumption (ii), according to which R has no repeated relations. First of all, we want to explain why we introduced this condition, with an example showing what kind of issues one could face admitting the repetitions.

Example 1.2.7. *Let $\lambda = (1^n)$, $\mu = (1^m)$ and $R = \{(1,1),(1,1)\}$. In this case $t = 1+1 = 2$ and $s = 1+1 = 2$. The morphism*

$$\begin{aligned} f_{\lambda,\mu,R} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2} \times \mathbb{P}^{m-2} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f_1, f_2, g, p) &\longmapsto (f_1 \cdot f_2 \cdot g, f_1 \cdot f_2 \cdot p) \end{aligned}$$

is not generically one to one: in fact, $f_{\lambda,\mu,R}(f_1, f_2, g, p) = f_{\lambda,\mu,R}(f_2, f_1, g, p)$.

Now we remove the assumption (ii) and, in order to avoid the issue of the example, we introduce a new notation for R that compacts the writing of the repeated entries and indicate the number of times that relation appears in R.

Notation 1.2.1. If $(\lambda_i, \mu_j) \in R$ is a repeated relation appearing k times, we just write $(\lambda_i, \mu_j)^k$.

With this notation, $R = \{(\lambda_{i_1}, \mu_{j_1})^{e_1}, \ldots, (\lambda_{i_h}, \mu_{j_h})^{e_h}\}$, with h equal to the number of different relations.

We have $\sum_{k=1}^{h} e_k = c$. Note that, with these new notations, the sums of the parts involved in the relations, called t and u, are now $t = \sum_{k=1}^{h} e_k \lambda_{i_k}$ and $u = \sum_{k=1}^{h} e_k \mu_{i_k}$. Now consider the morphism

$$\begin{gathered} f_{\lambda,\mu,R} : \prod_{k=1}^{h} \mathbb{P}^{e_k} \times \prod_{i=1}^{n-t} \mathbb{P}^{a_i} \times \prod_{j=1}^{m-u} \mathbb{P}^{b_j} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f_1, \ldots, f_h, g_1, \ldots, g_{n-t}, p_1, \ldots, p_{m-u}) \longmapsto \\ (f_1^{\lambda_{i_1}} \cdots f_h^{\lambda_{i_h}} \cdot \prod_{i=1}^{n-t} g_i^i, f_1^{\mu_{j_1}} \cdots f_h^{\mu_{j_h}} \cdot \prod_{j=1}^{m-u} p_j^j). \end{gathered}$$

Definition 1.2.8. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is defined to be the image of $f_{\lambda,\mu,R}$.*

The next example shows how this new definition solves the issue pointed out in the Example 1.2.7.

Example 1.2.9. *Let $\lambda = (1^n)$, $\mu = (1^m)$ and $R = \{(1,1)^2\}$. The morphism*

$$\begin{aligned} f_{\lambda,\mu,R} : \mathbb{P}^2 \times \mathbb{P}^{n-2} \times \mathbb{P}^{m-2} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f, g, p) &\longmapsto (f \cdot g, f \cdot p) \end{aligned}$$

with this new definition is generically one to one.

Now we can state the proposition in the general form.

Proposition 1.2.10. *The generalized coincident root locus $X_{\lambda,\mu,R}$ is rational and its dimension in $\mathbb{P}^n \times \mathbb{P}^m$ is $c + v + w$, where v is the number of parts in λ^C, w is the number of parts in μ^C and $c = \sum_{k=1}^{h} e_k$ is the total number of relations in R.*

Proof. We just need to prove that $f_{\lambda,\mu,R}$ is generically one to one, since the other steps of the proof are the same of the ones in the proof of the Proposition 1.2.6. Again, notice that this new $f_{\lambda,\mu,R}$ behaves as the previous one except for the introduction of the factor $\prod_{k=1}^{h} \mathbb{P}^{e_k}$. Note that $f_{\lambda,\mu,R}$ acts on $\prod_{k=1}^{h} \mathbb{P}^{e_k}$ associating to $(f_1, \dots, f_h)$ the pair of polynomials $(f_1^{\lambda_{i_1}} \cdots f_h^{\lambda_{i_h}}, f_1^{\mu_{j_1}} \cdots f_h^{\mu_{j_h}})$, where $f_1^{\lambda_{i_1}} \cdots f_h^{\lambda_{i_h}} \in \mathbb{P}^t$ and $f_1^{\mu_{j_1}} \cdots f_h^{\mu_{j_h}} \in \mathbb{P}^u$. Remember that $t = \sum_{k=1}^{h} e_k \lambda_{i_k}$ and $u = \sum_{k=1}^{h} e_k \mu_{i_k}$. Suppose that this map is not generically one to one, then a general pair $(F, G) = (f_1^{\lambda_{i_1}} \cdots f_h^{\lambda_{i_h}}, f_1^{\mu_{j_1}} \cdots f_h^{\mu_{j_h}}) \in \mathbb{P}^t \times \mathbb{P}^u$ is such that its fiber is composed by at least two elements $(f_1, \dots, f_h)$ and $(f_1', \dots, f_h')$ in $\prod_{k=1}^{h} \mathbb{P}^{e_k}$. The general pair $(F, G) \in \mathbb{P}^t \times \mathbb{P}^u$ is such that every f_i has distinct roots and such that, for every i, j, f_i and f_j have not common roots: thus it follows, for some i, j, that $f_i^{\lambda_i} \cdot f_j^{\lambda_j} = {f'}_i^{\lambda_i} \cdot {f'}_j^{\lambda_j}$

and $f_i^{\mu_i} \cdot f_j^{\mu_j} = {f'}_i^{\mu_i} \cdot {f'}_j^{\mu_j}$. This means that $f_i = f'_j$, $f_j = f'_i$, $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$, but this implies we have repeated relations other than the ones in A and encoded in the factor $\prod_{k=1}^{h} \mathbb{P}^{e_k}$, violating our construction of $f_{\lambda,\mu,R}$.

$\square$

The final goal is to find a stratification of $\mathbb{P}^n \times \mathbb{P}^m$ in terms of the generalized coincident root loci, so the next step is to understand when a generalized coincident root locus is contained in another. We generalize the notion of coarsening for partitions to coarsening for triplets (λ, μ, R). Let us start with an example.

Example 1.2.11. *Let $\lambda = (1^n)$, $\mu = (1^m)$ and $R = \{(1,1)\}$. We have the morphism*

$$\begin{aligned} f_{\lambda,\mu,R} : \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f, g, p) &\longmapsto (f \cdot g, f \cdot p). \end{aligned}$$

The generalized coincident root locus $X_{\lambda,\mu,R}$ parametrizes pairs of polynomials with a common root. Adding more relations or increasing the multiplicity of the roots in the relations, we expect the corresponding generalized coincident root loci to be subvarieties of $X_{\lambda,\mu,R}$, and it is what actually happens.
For example, if we take $\lambda' = (1^n)$ and $\mu' = (1^{m-2}, 2)$ and $R' = \{(1,2)\}$, we have the morphism

$$\begin{aligned} f_{\lambda',\mu',R'} : \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{m-2} &\longrightarrow \mathbb{P}^n \times \mathbb{P}^m \\ (f_1, g_1, p_1) &\longmapsto (f_1 \cdot g_1, f_1^2 \cdot p_1). \end{aligned}$$

The coincident root locus $X_{\lambda',\mu',R'}$ is a subvariety of $X_{\lambda,\mu,R}$. In fact, a point $(F_1, G_1) = (f_1 \cdot g_1, f_1^2 \cdot h_1) \in X_{\lambda',\mu',R'}$ belongs to $X_{\lambda,\mu,R}$: it is the image of (f, g, p) under $f_{\lambda,\mu,R}$, where $f = f_1$, $g = g_1$ and $p = f_1 \cdot p_1$.
Similarly, taking $\lambda'' = (1^n)$. $\mu'' = (1^m)$ and $R'' = \{(1,1)^2\}$, we have the morphism

$$f_{\lambda'',\mu'',R''} : \mathbb{P}^2 \times \mathbb{P}^{n-2} \times \mathbb{P}^{m-2} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(f_2, g_2, p_2) \longmapsto (f_2 \cdot g_2, f_2 \cdot p_2).$$

The coincident root locus $X_{\lambda'',\mu'',R''}$ is a subvariety of $X_{\lambda,\mu,R}$, since a point $(F_2, G_2) = (f_2 \cdot g_2, f_2 \cdot h_2) \in X_{\lambda'',\mu'',R''}$, with $f_2 = l_1 \cdot l_2$ for some linear forms l_1 and l_2, belongs to $X_{\lambda,\mu,R}$: it is the image of (f, g, p) under $f_{\lambda,\mu,R}$, where $f = l_1$, $g = l_2 \cdot g_2$ and $p = l_2 \cdot p_2$.

In the next definition we generalize the cases described in the Example 1.2.11.

Definition 1.2.12. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n, $\mu = (\mu_1, \ldots, \mu_s)$ a partition of m and $R = \{(\lambda_{i_1}, \mu_{j_1}), \ldots, (\lambda_{i_c}, \mu_{j_c})\}$ a set of relations.*

(a) We say that another set $R' = \{(\lambda_{i_1}, \mu_{j_1}), \ldots, (\lambda_{j_d}, \mu_{j_d})\}$, with $d > c$, is obtained from R by adding relations if $R' \setminus R = \{(\lambda_{i_{c+1}}, \mu_{j_{c+1}}), \ldots, (\lambda_{i_d}, \mu_{j_d})\}$, where

$$\lambda_{i_{c+k}} \in \lambda^C \text{ and } \mu_{j_{c+k}} \in \mu^C,\ k = 1, \ldots, d - c.$$

(b) Collapsing two relations $(\lambda_{i_p}, \mu_{j_p})$ and $(\lambda_{i_q}, \mu_{j_q})$ in R means to consider the new triplet (λ', μ', R'), such that

$$\lambda' = (\lambda_1, \ldots, \widehat{\lambda_{i_p}}, \ldots, \widehat{\lambda_{i_q}}, \ldots, \lambda_{i_p} + \lambda_{i_q}, \ldots, \lambda_r),$$

$$\mu' = (\mu_1, \ldots, \widehat{\mu_{j_p}}, \ldots, \widehat{\mu_{j_q}}, \ldots, \mu_{j_p} + \mu_{j_q}, \ldots, \mu_s)$$

and

$$R' = \{(\lambda_{i_1}, \mu_{j_1}), \ldots, \widehat{(\lambda_{i_p}, \mu_{j_p})}, \ldots, \widehat{(\lambda_{i_q}, \mu_{j_q})}, \ldots, (\lambda_{i_p} + \lambda_{i_q}, \mu_{j_p} + \mu_{j_q}), \ldots, (\lambda_{i_c}, \mu_{j_c})\}.$$

We shall say that a set of relations R' is obtained from R by collapsing relations if it is constructed by iterating the operation of collapsing two relations of R a finite number of times.

(c) *Increasing the multiplicity of a relation* $(\lambda_{i_p}, \mu_{i_p})$ *in* R *from the side of* λ *means to consider the new triplet* (λ', μ', R'), *such that*

$$\lambda' = (\lambda_1, \ldots, \widehat{\lambda_{i_p}}, \ldots, \widehat{\lambda_{i_q}}, \ldots, \lambda_{i_p} + \lambda_{i_q}, \ldots, \lambda_r),$$

$$\mu' = \mu$$

and

$$R' = \{(\lambda_{i_1}, \mu_{j_1}), \ldots, \widehat{(\lambda_{i_p}, \mu_{j_p})}, \ldots, (\lambda_{i_p} + \lambda_{i_q}, \mu_{j_p}), \ldots, (\lambda_{i_c}, \mu_{j_c})\}.$$

Increasing the multiplicity of a relation in A *from the side of* μ *is defined in an analogous way.*

We shall say that a set of relations R' *is obtained from* R *by increasing the multiplicity of relations if it is constructed by iterating the previous two operations a finite number of times.*

Definition 1.2.13. *We say that a triplet* (λ', μ', R') *is a coarsening of* (λ, μ, R) *if* λ' *and* μ' *are coarsenings of* λ *and* μ *respectively, and* R' *is obtained from* R *by adding, collapsing or increasing the multiplicities of its relations.*

Proposition 1.2.14. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r)$ *and* $\mu = (\mu_1, \ldots, \mu_s)$ *be partitions of* n *and* m *respectively, and let* $R = \{(\lambda_{i_1}, \mu_{j_1})^{a_1}, \ldots, (\lambda_{i_h}, \mu_{j_h})^{a_h}\}$ *be a set of relations. A generalized coincident root locus* $X_{\lambda',\mu',R'}$ *is contained in* $X_{\lambda,\mu,R}$ *if and only if the triplet* (λ', μ', R') *is a coarsening of* (λ, μ, R).

Proof. Assume $R = \{(\lambda_1, \mu_1)^{a_1}, \ldots, (\lambda_h, \mu_h)^{a_h}\}$. Since every coarsening is obtained from R by adding, collapsing or increasing the multiplicity of its relations a finite number of times, without loss of generality, to show the first direction, it is sufficient to prove the assertion in the following three cases:

(a) $\lambda' = \lambda$, $\mu' = \mu$ and $R' = R \bigcup \{(\lambda_{c+1}, \mu_{c+1})\}$, where $\lambda_{c+1} \in \lambda^c$ and $\mu_{c+2} \in \mu^c$.

(b) $\lambda' = (\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_r)$, $\mu' = (\mu_1 + \mu_2, \mu_3, \ldots, \mu_s)$ and $R' = \{(\lambda_1 + \lambda_2, \mu_1 + \mu_2)^{h_{1,2}}, (\lambda_3, \mu_3)^{a_3}, \ldots, (\lambda_h, \mu_h)^{a_h}\}$

(c) $\lambda' = (\lambda_1 + \lambda_{c+1}, \lambda_2, \ldots, \lambda_r)$, $\mu' = \mu$
and $R' = \{(\lambda_1 + \lambda_{c+1}, \mu_1)^{a_{1,c+1}}, (\lambda_2, \mu_2)^{a_2}, \ldots, (\lambda_h, \mu_h)^{a_h}\}$, where $\lambda_{c+1} \in \lambda^C$.

In the case (a), assume for simplicity that the new relation $(\lambda_{c+1}, \mu_{c+1})$ appears in R' only once and that λ_{c+1} and μ_{c+1} appear only once in λ^C and μ^C. Then, $e_{c+1} = a_{c+1} = b_{c+1} = 1$. We have the morphisms

$$f_{\lambda,\mu,R} : \prod_{k=1}^{h} \mathbb{P}^{e_k} \times \prod_{i=1}^{n-t} \mathbb{P}^{a_i} \times \prod_{j=1}^{m-u} \mathbb{P}^{b_j} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(f_1, \ldots, f_h, g_1, \ldots, g_{n-t}, p_1, \ldots, p_{m-u}) \longmapsto (f_1^{\lambda_1} \cdots f_h^{\lambda_h} \cdot \prod_{i=1}^{n-t} g_i^i, f_1^{\mu_1} \cdots f_h^{\mu_h} \cdot \prod_{j=1}^{m-u} p_j^j)$$

and

$$f_{\lambda,\mu,R'} : \prod_{k=1}^{h} \mathbb{P}^{e_k} \times \mathbb{P}^1 \times \prod_{i=1}^{n-t-\lambda_{c+1}} \mathbb{P}^{a_i} \times \prod_{j=1}^{m-u-\mu_{c+1}} \mathbb{P}^{b_j} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$
$$(f_1, \ldots, f_h, f, g_1, \ldots, g_{n-t-\lambda_{c+1}}, p_1, \ldots, p_{m-u-\mu_{c+1}}) \longmapsto$$
$$(f_1^{\lambda_1} \cdots f_h^{\lambda_h} \cdot f^{\lambda_{c+1}} \cdot \prod_{i=1}^{n-t-\lambda_{c+1}} g_i^i, f_1^{\mu_1} \cdots f_h^{\mu_h} \cdot f^{\mu_{c+1}} \cdot \prod_{j=1}^{m-u-\mu_{c+1}} p_j^j).$$

A pair $(F, G) = (f_1^{\lambda_1} \cdots f_h^{\lambda_h} \cdot f^{\lambda_{c+1}} \cdot \prod_{i=1}^{n-t-\lambda_{c+1}} g_i^i, f_1^{\mu_1} \cdots f_h^{\mu_h} \cdot f^{\mu_{c+1}} \cdot \prod_{j=1}^{m-u-\mu_{c+1}} p_j^j) \in X_{\lambda,\mu,R'}$ belongs to $X_{\lambda,\mu,R}$: in fact, we can choose $g_{\lambda_{c+1}} = p_{\mu_{c+1}} = f$, where $g_{\lambda_{c+1}} \in \mathbb{P}^1$ is the linear polynomial in $\prod_{i=1}^{n-t} \mathbb{P}^{a_i}$ corresponding to the factor associated to the part $\lambda_{c+1} \in \lambda^C$ and $p_{\mu_{c+1}} \in \mathbb{P}^1$ is the linear polynomial in $\prod_{j=1}^{m-u} \mathbb{P}^{b_j}$ corresponding to the factor associated to the part $\mu_{c+1} \in \mu^C$. $g_{\lambda_{c+1}}$ and $h_{\lambda_{c+1}}$ are linear because we assumed $a_{c+1} = b_{c+1} = 1$.

For the case (b), assume that $(\lambda_1 + \lambda_2, \mu_1 + \mu_2)$ appears once in R' and λ_1, λ_2, μ_1 and μ_2 appear once in λ and μ respectively, so that the exponent in R' associated to the part

$(\lambda_1+\lambda_2,\mu_1+\mu_2)$ that we call for this time $h_{1,2}$ is 1 and $a_1=a_2=b_1=b_2=1$. We have the morphism

$$\begin{gathered} f_{\lambda',\mu',R'}:\mathbb{P}^1\times\prod_{k=3}^{h}\mathbb{P}^{e_k}\times\prod_{i=1}^{n-t}\mathbb{P}^{a_i}\times\prod_{j=1}^{m-u}\mathbb{P}^{b_j}\longrightarrow\mathbb{P}^n\times\mathbb{P}^m\\ (f,f_3\dots,f_h,g_1,\dots,g_{n-t},p_1,\dots,p_{m-u})\longmapsto\\ (f^{\lambda_1+\lambda_2}\cdot f_3^{\lambda_3}\cdots f_h^{\lambda_h}\cdot\prod_{i=1}^{n-t}g_i^i,f^{\mu_1+\mu_2}\cdot f_3^{\mu_3}\cdots f_h^{\mu_h}\cdot\prod_{j=1}^{m-u}p_j^j). \end{gathered}$$

A pair $(F,G)=(f^{\lambda_1+\lambda_2}\cdot f_3^{\lambda_3}\cdots f_h^{\lambda_h}\cdot\prod_{i=1}^{n-t-\lambda_{c+1}}g_i^i,f^{\mu_1+\mu_2}\cdot f_3^{\mu_3}\cdots f_h^{\mu_h}\cdot\prod_{j=1}^{m-u}h_j^j)$ of $X_{\lambda',\mu',R'}$ is contained in $X_{\lambda,\mu,R}$: in fact, we can choose the linear polynomials f_1 and f_2 in such a way that $f_1=f_2=f$ (all the three polynomials are linear because we assumed $h_{1,2}=a_1=a_2=b_1=b_2=1$).

For the case (c), assume that $(\lambda_1+\lambda_{c+1},\mu_1)$ appears once in R' and that λ_1, μ' and λ_{c+1} appear once in λ, μ and λ^C respectively. Then, the exponent in R' associated to the part $(\lambda_1+\lambda_{c+1},\mu_1)$ that we call for this time $h_{1,c+1}$ is 1 and $a_1=b_1=a_{c+1}=1$. We have the morphism

$$\begin{gathered} f_{\lambda',\mu',R'}:\times\prod_{k=1}^{h}\mathbb{P}^{e_k}\times\prod_{i=1}^{n-t-\lambda_{c+1}}\mathbb{P}^{a_i}\times\prod_{j=1}^{m-u}\mathbb{P}^{b_j}\longrightarrow\mathbb{P}^n\times\mathbb{P}^m\\ (f_1,f_2\dots,f_h,g_1,\dots,g_{n-t-\lambda_{c+1}},p_1,\dots,p_{m-u})\longmapsto\\ (f^{\lambda_1+\lambda_{c+1}}\cdot f_2^{\lambda_2}\cdots f_h^{\lambda_h}\cdot\prod_{i=1}^{n-t-\lambda_{c+1}}g_i^i,f^{\mu_1}\cdot f_2^{\mu_2}\cdots f_h^{\mu_h}\cdot\prod_{j=1}^{m-u}p_j^j).\text{ A pair}\\ (F,G)=(f^{\lambda_1+\lambda_{c+1}}\cdot f_2^{\lambda_2}\cdots f_h^{\lambda_h}\cdot\prod_{i=1}^{n-t-\lambda_{c+1}}g_i^i,f^{\mu_1}\cdot f_2^{\mu_2}\cdots f_h^{\mu_h}\cdot\prod_{j=1}^{m-u}p_j^j)\in X_{\lambda',\mu',R'} \end{gathered}$$

is contained in $X_{\lambda,\mu,R}$: in fact, we can chose the linear polynomials f_1 and $g_{\lambda_{c+1}}$, where $g_{\lambda_{c+1}}\in\mathbb{P}^1$ is the linear polynomial in $\prod_{i=1}^{n-t}\mathbb{P}^{a_i}$ corresponding to the factor associated to the part $\lambda_{c+1}\in\lambda^C$, in such a way that $f_1=g_{\lambda_{c+1}}$ (the two polynomials are linear because we assumed $h_{1,c+1}=a_1=1$).

To prove the other direction, let $\lambda'=(\lambda'_1,\dots,\lambda'_{r'})$, $\mu'=(\mu'_1\dots,\mu'_{s'})$ and $R'=$

$\{(\lambda'_{i_1}, \mu'_{j_1})^{a_1}, \ldots, (\lambda'_{i_{h'}}, \mu_{j_{h'}})^{a_{h'}}\}$ and let (F, G) be a general pair of polynomials in $X_{\lambda',\mu',R'}$. We have that $F = f'^{\lambda'_1}_1 \cdots f'^{\lambda'_{h'}}_{h'} \cdot g'^{\lambda'_{h'+1}}_{h'+1} \cdots g'^{\lambda'_{r'}}_{r'}$ and $G = f'^{\mu'_1}_1 \cdots f'^{\mu'_{h'}}_{h'} \cdot p'^{\mu'_{h'+1}}_{h'+1} \cdots p'^{\mu'_{s'}}_{s'}$ in such a way all the linear forms f'_i, g'_j and p'_k are pairwise distinct. The pair (F, G) also belongs to $X_{\lambda,\mu,R}$, so that $F = f_1^{\lambda_1} \cdots f_h^{\lambda_h} \cdot g_{h+1}^{\lambda_{h+1}} \cdots g_r^{\lambda_r}$ and $G = f_1^{\mu_1} \cdots f_h^{\mu_h} \cdot p_{h+1}^{\mu_{h+1}} \cdots p_s^{\mu_s}$. Since there is only one way to write F and G in product of linear forms, if all the polynomials f_i, g_j and p_k are different, we have that $f'_i = f_i$, $g'_i = g_i$ and $p'_i = p_i$ and that $\lambda_i = \lambda'_i$ and $\mu_i = \mu'_i$ for every i, so that $(\lambda, \mu, R) = (\lambda', \mu', R')$. As in the proof of Proposition 1.1.3, we can group all the coincident linear forms involved in the writing of the pair $(F.G)$ as a member of $X_{\lambda,\mu,R}$ and conclude that the triplet (λ', μ', R') is a coarsening of (λ, μ, R). For instance, if $f_i = f_j$ for some i and j, R is obtained from R' by collapsing a relation; if $f_i = g_j$ or $f_i = p_k$, R is obtained from R' by increasing the multiplicity of a relation; if $g_i = p_j$, R is obtained from R' by adding a relation and if $g_i = g_j$ (or $p_i = p_j$) we have that λ' (or μ') is a coarsening of λ (or μ).

□

Corollary 1.2.15. *Given generalized coincident root loci $X_{\lambda',\mu',R'} \subset X_{\lambda,\mu,R}$, the codimension of $X_{\lambda',\mu',R'}$ in $X_{\lambda,\mu,R}$ is equal to the number of simple operations from (λ, μ, R) to obtain (λ', μ', R'), where by simple operation we mean adding a single relation, increasing the multiplicity of a single relation, collapsing a single relation or adding up two parts of λ^C or μ^C.*

Proof. The dimension of $X_{\lambda,\mu,R}$ is $c + v + w$. If (λ', μ', R') is obtained from (λ, μ, R) by adding a single relation, the number of relations of R' will be $c' = c + 1$ and the number parts of λ^C and μ^C will be $v' = v - 1$ and $w' = w - 1$. Hence, the dimension of $X_{\lambda',\mu',R'}$ is $c + v + w - 1$: every time we collapse a relation, the codimension increases by 1.
If (λ', μ', R') is obtained from (λ, μ, R) by collapsing a single relation, we will have $c' = c - 1$, $v' = v$ and $w' = w$. If (λ', μ', R') is obtained from (λ, μ, R) by increasing the

multiplicity of a relation, say from the side of λ, we will have $c' = c$, $v' = v - 1$ and $w' = w$. Finally, if, say, λ' is obtained from λ by adding up two parts of λ^C, we will have $c' = c$, $v' = v - 1$ and $w' = w$. Then, every time we perform a simple operation, the codimension increases by 1. $\square$

We want to represent the stratification of $\mathbb{P}^n \times \mathbb{P}^m$ in terms of the generalized coincident root loci with a diagram.

Definition 1.2.16. *The diagram of specialization of* $\mathbb{P}^n \times \mathbb{P}^m$ *is an oriented graph with all the generalized coincident root loci as vertices: for every two generalized coincident root loci* X_1 *and* X_2*, there is an arrow from* X_1 *to* X_2 *if and only if* $X_2 \subset X_1$ *and the codimension of* X_2 *in* X_1 *is 1.*

Chapter 2

Elliptic surfaces

In this chapter we analyze the moduli space of rational elliptic surfaces in terms of the special fibers the surfaces have. Once recalled the basic notions about elliptic surfaces, their singular fibers and the Weierstrass model they admit, in Section 2.3 we review the construction of the moduli space of rational elliptic surfaces performed by Miranda in [45].

Miranda and Persson in [49] and [46] developed a wonderful theory on the configurations of the singular fibers that a rational elliptic surface can admit: in Section 2.4 we add some other objects in this investigation, taking into account also some special smooth fibers and another invariant of the singular fibers, i.e. the multiplicity of the J-map, not considered by Miranda and Persson. The set of the configurations of singular fibers we consider is listed in Theorem 2.4.13. The weakness of the method we develop in this chapter, to study the aforementioned moduli space, relies in the fact that it does not allow us to consider one specific type of singular fibers, that is, cycles of smooth rational curves. For this reason, we give the notion of suitable rational elliptic surface.

In section 2.5 We exploit the machinery of the generalized coincident root loci presented in Chapter 1 to stratify the moduli space of (suitable) rational elliptic surfaces in terms of the special fibers they can have. We treat the case of isotrivial rational elliptic surfaces, defined as elliptic surfaces such that all the smooth fibers are isomorphic, separately. The main results of this chapter are Theorem 2.5.13 and Corollary 2.5.25, in which we produce the above mentioned stratifications. By taking into account the multiplicity of the J-map allows us to refine the stratification of the strictly semistable locus of the moduli space of rational elliptic surfaces performed by Miranda in [45]: we point out such a refinement in Proposition 2.5.29.

We introduce the main object of the chapter, the elliptic surfaces. For the proofs of the results we refer to [44] and [45].

Definition 2.0.1. *An elliptic curve (E, O) is a smooth projective curve E of genus one, together with a chosen point O called its origin.*

The next result states that every elliptic curve can be embedded in the projective plane $\mathbb{P}^2$ as a cubic curve.

Theorem 2.0.2. *[Weierstrass immersion] Let (E, O) be an elliptic curve. Then there exist rational maps $x, y \in \mathbb{C}(E)$ such that the map*

$$\phi : E \setminus \{O\} \to \mathbb{P}^2,\ \phi = (x, y, 1)$$

gives an isomorphism $\tilde{\phi}$ of E onto a plane cubic defined by the Weierstrass equation

$$Y^2 = X^3 + aX + b,$$

in the affine chart $(X : Y : 1)$ with $a, b \in \mathbb{C}$ and satisfying $\tilde{\phi} \cong \phi$ away from O and $\tilde{\phi}(O) = (0 : 1 : 0)$.

Proof. See [52, Chapter III, Proposition 3.1]. □

Definition 2.0.3. *Suppose X is an algebraic surface (smooth or singular). A genus one fibration is a morphism $f : X \to C$, where C is a smooth curve, such that the general fiber is a smooth curve of genus one and all the fibers are connected. If there is a section $s : C \to X$, we say that $f : X \to C$ is an elliptic fibration (with a given section), and X is an elliptic surface over C. A smooth elliptic surface $f : X \to C$ is relatively minimal if there are no (-1)-curves lying on the fibers of f.*

Given a smooth elliptic surface $X \to C$ over C, we can blow-down all the (-1)-curves along the fibers and get a relative minimal model of X, which is also smooth. We will see that a relatively minimal model of a smooth elliptic surface is unique in a sense described below.

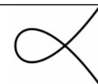
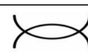
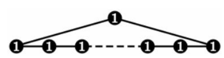
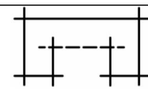
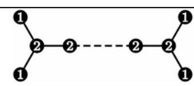
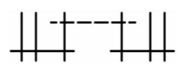
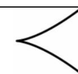
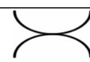
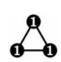
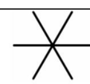
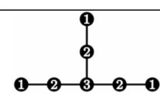
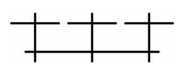
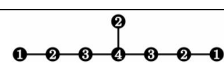
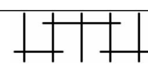
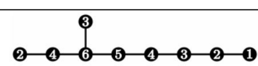
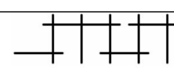
Definition 2.0.4. *Let $f_1 : X_1 \to C$ and $f_2 : X_2 \to C$ be two elliptic surfaces over C. A morphism of elliptic surfaces (over C) is a morphism of surfaces $\psi : X_1 \to X_2$ such that $f_1 = f_2 \circ \psi$. If ψ is also an isomorphism of surfaces, we say that ψ is an isomorphism of elliptic surfaces (over C) and that X_1 and X_2 are isomorphic as elliptic surfaces (over C). Furthermore, X_1 and X_2 are birational as elliptic surfaces (over C) if there is a birational* *map $f : X_1 \to X_2$ such that $f_1 = f_2 \circ f$ and we say that f preserves the elliptic structure of X_1 and X_2.*

Proposition 2.0.5. *Suppose X_1 and X_2 are relatively minimal elliptic surfaces over C, which are birational as elliptic surfaces over C. Let $f : X_1 \to X_2$ be a birational map that preserves the elliptic structure of X_1 and X_2. Then, f is an isomorphism of elliptic surfaces over C.*

Corollary 2.0.6. *Suppose $X \to C$ is an elliptic surface over C, then there is a unique relatively minimal elliptic surface $X_0 \to X$ that is birational to X as elliptic surfaces over C.*

Kodaira classified all possible fibers of a relatively minimal elliptic surface as shown in Table 1 (see [39]). The first column of the table lists the notations of the fiber types. In the second column, a Dynkin diagram represents the intersection matrix of the irreducible components of a fiber. Each solid dot of a Dynkin diagram represents an irreducible component. The number in each solid dot denotes the multiplicity of the corresponding component. There are three types of fibers that are irreducible: a smooth elliptic curve (I_0), a nodal rational curve (I_1) and a cuspidal rational curve (II). For a reducible singular fiber, all its irreducible components are smooth rational curves with self-intersection equal to (-2). Let $f : X \to C$ be a relatively minimal elliptic surface over C with a section $s : C \to X$. The image of the section $S = s(C)$ is a divisor on X. Then for each fiber, S intersects exactly one of its components with multiplicity 1.

TABLE I Kodaira's Classification

	Dynkin Diagram	Fiber	Components
I_0			smooth elliptic curve
I_1			nodal rational curve
I_2			two smooth rational curves
$I_N, N \geq 3$			N smooth rational curves
$I_N^*, N \geq 0$			N+5 smooth rational curves
II			a cuspidal rational curve
III			two smooth rational curves
IV			three smooth rational curves
$IV*$			7 smooth rational curves
III*			8 smooth rational curves
II*			9 smooth rational curves

2.1 The Weierstrass model

We introduce the Weierstrass fibrations. Unless differently specified, the proofs are given by Miranda in [44] and [45].

Definition 2.1.1. *Let X be a surface and C be a smooth curve. A Weierstrass fibration is a flat morphism $f : X \to C$ such that every fiber is either*

- *a smooth genus one curve,*
- *a rational curve with a node, or*
- *a rational curve with a cusp.*

and the general fiber is smooth. Moreover, there is a given section S that does not pass through nodes or cusps of any fiber.

Remark 2.1.2. *Suppose that $f : X \to C$ is a relatively minimal elliptic surface with a section S. We can contract the union of all components of each singular fiber that do not intersect S (see [44]). Such contraction gives a singular surface X' that admits a Weierstrass fibration*

$$X \xrightarrow{contraction} X' \xrightarrow{W.fibration} C$$

The singularities of X' are rational double points of the type denoted by the Dynkin diagrams for the corresponding singular fibers of X, after removing the vertex corresponding to the noncontracted component ([45]). On the other hand, $X \to X'$ is the minimal resolution of the singularities of X'.

Let $f' : X' \to C$ be a Weierstrass fibration obtained from a relatively minimal elliptic surface $f : X \to C$ with a section S. We still denote S the corresponding section of X'. The normal bundle of $S \subset X'$ is denoted by $\mathcal{N}_{S/X'}$. Since $f'_{|S}$ is an isomorphism onto C, $f'_*\mathcal{N}_{S/X'}$ is a line bundle on C. We denote its dual bundle by

$$(f'_*\mathcal{N}_{S/X'})^{-1} = \mathbb{L}.$$

We call $\mathbb{L}$ the fundamental line bundle of the Weierstrass fibration $f' : X' \to C$. (See [44]).
A Weierstrass fibration $X' \to C$, with its fundamental line bundle $\mathbb{L}$, can be realized as a divisor inside a $\mathbb{P}^2$-bundle over C.

Proposition 2.1.3. *X' is isomorphic to the divisor of $Y = \mathbb{P}(\mathcal{O}_C(-2\mathbb{L}) \oplus \mathcal{O}_C(-3\mathbb{L}) \oplus \mathcal{O}_C)$ defined by*

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A \in H^0(C, 4\mathbb{L})$, $B \in H^0(C, 6\mathbb{L})$ and $[x, y, z]$ are the coordinate of the $\mathbb{P}^2$-bundle Y.

Definition 2.1.4. *We say that $(\mathbb{L}, A, B)$ is the Weierstrass data of X'. We call the section $D = 4A^3 + 27B^2 \in H^0(C, 12\mathbb{L})$ the discriminant of the fibration and the divisor (D) on C the discriminant divisor of the elliptic surface $X \to C$, where $X \to X'$ is the minimal resolution of X' (and therefore is a relatively minimal elliptic surface over C). We also call $\mathbb{L}$ the fundamental line bundle of the elliptic surface $X \to C$ and the triplet $(\mathbb{L}, A, B)$ the Weierstrass data of X.*

We have the canonical bundle formula for relatively minimal elliptic surfaces.

Theorem 2.1.5. *Let $f : X \to C$ be a relatively minimal elliptic surface and $\mathbb{L}$ be its fundamental line bundle. Then the canonical bundle of X is*

$$K_X = f^*(K_C + \mathbb{L}),$$

where K_C is the canonical bundle of C. Furthermore, $\deg(\mathbb{L}) = \chi(X)$, where $\chi(X)$ is the Euler characteristic of X.

Corollary 2.1.6. *Let $e(X)$ denote the topological Euler characteristic of X. Then*

$$e(X) = 12 \deg(\mathbb{L})$$

Proof. By the Noether's formula we have

$$12\chi(X) = K_X^2 + e(X).$$

Moreover, 2.1.5 implies that K_X is a multiple of a fiber, then $K_X^2 = 0$. □

We have a classification of minimal elliptic surfaces based on the genus of the base curve and on the degree of the fundamental line bundle.

Lemma 2.1.7. *Let $f : X \to C$ be a minimal elliptic surface with a section with fundamental line bundle $\mathbb{L}$. Let $g = g(C)$ be the genus of C.*

(a) if $g = 0$, then X is
   - *a product of an elliptic curve and $\mathbb{P}^1$ if $\deg(\mathbb{L}) = 0$*
   - *a rational surface if $\deg(\mathbb{L}) = 1$*
   - *a K3 surface is $\deg(\mathbb{L}) = 2$*
   - *a properly elliptic surface if $\deg(\mathbb{L}) \geq 3$*

(b) if $g = 1$, then X is
   - *an abelian surface (in particular a product of two elliptic curves) if $\mathbb{L} = \mathcal{O}_C$*
   - *a hyperelliptic surface if $\mathbb{L}$ is torsion of order 2,3,4 or 6.*
   - *a properly elliptic surface if $\deg(\mathbb{L}) \geq 1$*

(c) if $g = 2$, then X is a properly elliptic surface.

Lemma 2.1.8. *Let $f : X \to C$ be a relatively minimal elliptic surface with a section and $\mathbb{L}$ be its fundamental line bundle. If X is not a product surface, then its Hodge diamond is*

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g+deg(\mathbb{L})-1 & & 10deg(\mathbb{L})+2g & & g+deg(\mathbb{L})-1 \\ & g & & g & \\ & & 1 & & \end{array}$$

If X is a product surface, then its Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g+deg(\mathbb{L}) & & 10deg(\mathbb{L})+2g-2 & & g+deg(\mathbb{L}) \\ & g & & g & \\ & & 1 & & \end{array}$$

Proof. [44] (Lemma IV.1.1) □

2.1.1 The singular fibers

Recall that we can associate to a relatively minimal elliptic surface $f : X \to C$ its Weierstrass model $f' : X' \to C$, together with its defining equation

$$y^2z = x^3 + Axz^2 + Bz^3,$$

where $A \in H^0(C, 4\mathbb{L})$ and $B \in H^0(C, 6\mathbb{L})$. In the previous section we introduced the discriminant $D = 4A^3 + 27B^2$ and the corresponding divisor (D) on C. A, B and D are global sections of some power of the line bundle $\mathbb{L}$, then we can see these sections locally as functions on the base curve C depending on a local parameter t of C. We can write

$$A = A(t)$$
$$B = B(t)$$
$$D = D(t).$$

The next proposition shows that the singular fibers of $X \to C$ lie over the points of the discriminant divisor.

Proposition 2.1.9. *Let $f : X \to C$ be a relatively minimal elliptic surface over C with Weierstrass data $(\mathbb{L}, A, B)$. Then, the discriminant $D = D(t) = 4A(t)^3 + 27B(t)^2$ vanishes over a point $Q = (t_0) \in C$ if and only if the fiber X_Q is singular.*

Proof. See [44, see Theorem 2.1]. □

Remark 2.1.10. *Since D is a degree $12\deg(\mathbb{L})$ line bundle over the curve C, X has $12\deg(\mathbb{L})$ singular fibers counted with multiplicity. In the next part of the section, we explain what counted with multiplicity means.*

We already mentioned that Kodaira in [39] classified all the types of the fibers that can occur in a relatively minimal elliptic surface. Néron in [48] has taken Kodaira's classification one step further and has shown that the type of singular fiber that occur over a point Q depends only on the order of vanishing of A, B and D at Q; moreover, he calculates explicitly, given the triplet $(v_Q(A), v_Q(B), v_Q(D))$, the type of the fiber X_Q. We reproduce these results in the next table. Columns 3, 4, and 5 contain the orders of vanishing of A, B and D at Q. For example, the data $(0, 0, 1)$ corresponds to the simplest singular fiber, the nodal one I_1; $(0, 0, k)$ to a cycle of k rational curves I_k, $(1, 1, 2)$ to a cusp II and so on.

Another fundamental notion that characterizes the elliptic surfaces is the J-map. First of all, we give the Definition of J-invariant for an elliptic curve in Weierstrass form.

Definition 2.1.11. *Let $(E, 0)$ be a smooth elliptic curve defined by the Weierstrass equation*

$$Y^2 = X^3 + aX + b$$

as in Theorem 2.0.2. Then, the J-invariant of E is the complex number

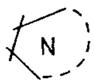
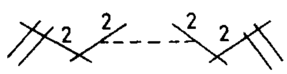
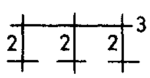
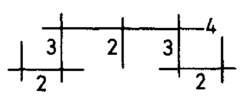
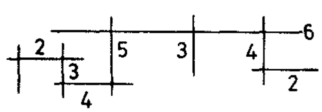
$$J(E) = 4a^3/(4a^3 + 27b^2).$$

The J-map is a map from the base curve C of the fibration to $\mathbb{P}^1$, that associates to a point Q of the curve, the J-invariant of the elliptic curve X_Q, that is the fiber over Q.

Definition 2.1.12. *The J-map of the elliptic surface $X \to C$ is defined as*

$$J : C \longrightarrow \mathbb{P}^1$$
$$Q = t_0 \mapsto J(Q) = J(t_0) = 4A(t_0)^3/(4A(t_0)^3 + 27B(t_0)^2)$$

The J-map extends to the singular fibers and in the table of singular fibers its value and multiplicity are reported for every case, where by multiplicity of the J-map at a point we mean the ramification order of J at that point. For our purposes it is important to take account of this multiplicity: if it is different from the *general case* (in a sense that we will precise in Definition 2.4.1), we will write it next to the Kodaira notation of the singular fibers. The value of J at Q and its multiplicity $m(J)$, as the type of singular fiber that occurs, only depends on the order of vanishing of A, B and D at Q and all the information are encoded in the table of the fibers.

Name	Graph	$v_q(A)$	$v_q(B)$	$v_q(D)$	J	$m(J)$
I_0		0	0	0	$\neq 0, 1, \infty$	–
I_0		0	K	0	1	$2K$
I_0		L	0	0	0	$3L$
I_1		0	0	1	∞	1
I_N		0	0	N	∞	N
I_0^*		2 $L \geqq 3$ 2	3 3 $K \geqq 4$	6 6 6	$\neq 0, 1, \infty$ 0 1	– $3L-6$ $2K-6$
I_N^*		2	3	$N+6$	∞	N
II		$L \geqq 1$	1	2	0	$3L-2$
III		1	$K \geqq 2$	3	1	$2K-3$
IV		$L \geqq 2$	2	4	0	$3L-4$
IV^*		$L \geqq 3$	4	8	0	$3L-8$
III^*		3	$K \geqq 5$	9	1	$2K-9$
II^*		4	5	10	0	2

2.2 Elliptic surfaces over the projective line

From now on, the base curve C will be the projective line $\mathbb{P}^1$. We follow [45]. Let $f : X \to \mathbb{P}^1$ be a relatively minimal elliptic surface, with a section S and fundamental line bundle $\mathbb{L}$.

Proposition 2.2.1. *$\mathbb{L} \cong \mathcal{O}_{\mathbb{P}^1}(N)$ for some $N \geq 0$ and $N = 0$ if and only if X is a product. Moreover, $S \cdot S = -N$.*

Proof. See [45, Cor 2.4] □

Proposition 2.2.1 allows us to be more specific in describing Weierstrass fibrations over $\mathbb{P}^1$ by their Weierstrass form. Let us denote by V_N the vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N))$.

Proposition 2.2.2. *Let $f : X \to \mathbb{P}^1$ be a relatively minimal elliptic surface with section S contracting to the Weierstrass fibration $f' : X' \to \mathbb{P}^1$ and let $N = deg(\mathbb{L})$. Then X' is isomorphic to the divisor of the $\mathbb{P}^2$-bundle $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2N) \oplus \mathcal{O}_{\mathbb{P}^1}(-3N) \oplus \mathcal{O}_{\mathbb{P}^1})$ defined by*

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A \in V_{4N}$ and $B \in V_{6N}$.

Moreover,

(i) $D = 4A^3 + 27B^2 \in V_{12N}$ is not identically zero and D vanishes at $Q \in \mathbb{P}^1$ if and only if the fiber X_Q is singular,

(ii) for every $Q \in \mathbb{P}^1$, either $v_Q(A) \leq 3$ or $v_Q(B) \leq 5$,

(iii) every pair of forms $(A, B) \in V_{4N} \oplus V_{6N}$ satisfying (i) and (ii) defines a Weierstrass fibrations $X' \to \mathbb{P}^1$ with only rational double points as singularities, which resolves to a relatively minimal elliptic surface $X \to \mathbb{P}^1$ with $\mathbb{L} \cong \mathcal{O}_{\mathbb{P}^1}(N)$.

Proof. [45, Cor 2.5]. □

Remark 2.2.3. *In Remark 2.1.10 we noticed that a relatively minimal elliptic surface with a section has* $12\deg(\mathbb{L})$ *singular fibers. If the base curve is* $\mathbb{P}^1$*, the singular fibers are* $12N$ *counted with multiplicity and we will see in the next section that the general elliptic surface over* $\mathbb{P}^1$ *with* $\deg(\mathbb{L}) = N$ *has exactly* $12N$ *singular fibers of type* I_1.

Since $N = 1$, a rational relatively minimal elliptic surface $X \to \mathbb{P}^1$ has Weierstrass data

$$(\mathcal{O}_{\mathbb{P}^1}(1), A, B),$$

where

$$\begin{gathered} A = A(s,t) \in V_4 \text{ is a degree 4 homogeneous polynomial in two variables,} \\ B = B(s,t) \in V_6 \text{ is a degree 6 homogeneous polynomial in two variables} \\ D(s,t) = 4A(s,t)^3 + 27B(s,t)^2 \in V_{12} \text{ is a degree 12 homogeneous polynomial in two} \\ \text{variables,} \end{gathered}$$

where $[s,t]$ are the homogeneous coordinates of the base curve $\mathbb{P}^1$.

2.3 The moduli space of rational elliptic surfaces

For the construction of the moduli space of rational elliptic surfaces with a section, we refer to [45].
Let $T_1 \subset V_4 \oplus V_6$ be the open set of the pairs of forms (A, B) satisfying (i) and (ii) of Proposition 2.2.2. Given the Weierstrass fibration X' as above, the pair (A, B) is unique up to isomorphism, in the following sense. The multiplicative group $\mathbb{C}^*$ acts on T_1 by

$$(\lambda, (A, B)) = (\lambda^2 A, \lambda^3 B).$$

The group $\mathrm{SL}(V_1)$ acts on T_1 in the following way: for $M \in \mathrm{SL}(V_1)$, the action is given by

$$M \cdot (x, y) = M \begin{pmatrix} x \\ y \end{pmatrix}.$$

This action extends to $V_k = \mathrm{Symm}^k(V_1)$ in the obvious manner.

Remark 2.3.1. *The action of* $\mathrm{SL}(V_1)$ *corresponds to change of coordinates in* $\mathbb{P}^1$ *and the action of* $\mathbb{C}^*$ *to the admissible changes of coordinates in the* $\mathbb{P}^2$*-bundle. As discussed in [45, Section 2], these actions commute and then define an action of* $\mathbb{C}^* \times \mathrm{SL}(V_1)$ *on* T_1.

Lemma 2.3.2. *Two pairs of forms in* T_1 *give rise to isomorphic Weierstrass fibrations if and only if they are in the same orbit of* $\mathbb{C}^* \times \mathrm{SL}(V_1)$.

Corollary 2.3.3. *The set of isomorphism classes of relatively minimal rational elliptic surfaces with a section is in one to one correspondence with the set of orbits of* $T_1/(\mathbb{C}^* \times \mathrm{SL}(V_1))$.

In order to put a geometric structure on this set of orbits, Miranda performed a GIT quotient on T_1 obtaining a coarse moduli space. In the next section we present it. Since the actions of $\mathbb{C}^*$ and $\mathrm{SL}(V_1)$ commute, we can consider the quotient by the action of $\mathbb{C}^*$ first.

The construction of the parameter space

We are just considering the weighted action of $\mathbb{C}^*$ on $V_4 \oplus V_6 \setminus \{0\}$ given by

$$(\lambda, (A, B)) = (\lambda^2 A, \lambda^3 B),$$

where $A =^t \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix} \in V_4$ and $B =^t \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \in V_6$ represent the coefficients of the homogeneous polynomials A and B respectively.
This is the action on $V_4 \oplus V_6 \setminus \{0\}$ that defines the weighted projective space

$$\mathbb{P}(2^5, 3^7),$$

where the bihomogeneous coordinates are $[a_0, \ldots, a_4, b_0, \ldots, b_6]$.

Definition 2.3.4. *We denote by* M_1 *this weighted projective space and by* E_1 *the image of* T_1 *under the quotient.*

E_1 is a parameter space for the rational elliptic surfaces with a section.

The construction of the moduli space

The construction of the quotient of M_1 by the action of $\mathrm{SL}(V_1)$ is performed by Miranda in [45] by using the techniques of geometric invariant theory (we refer to [28] for the definitions of stability and semi-stability). By geometric invariant theory we have a diagram

$$\begin{tikzcd} E_{1,s} \arrow[r] \arrow[d, "\pi_1"] & E_{1,ss} \arrow[r] \arrow[d, "\overline{\pi}_1"] & E_1 \\ W_1 \arrow[r] & \overline{W}_1 & \end{tikzcd}$$

where W_1 is open in $\overline{W}_1$. The maps π_1 and $\overline{\pi}_1$ have the following properties:

(i) for $x, y \in E_{1,s}$, $\pi_1(x) = \pi_1(y)$ if and only if x and y are in the same orbit of $\mathrm{SL}(V_1)$,

(ii) for $x, y \in E_{1,ss}$, $\pi_1(x) = \pi_1(y)$ if and only if the closure of the orbits of x and y intersect in $E_{1,ss}$.

The variety W_1 is a geometric quotient of $E_{1,s}$ and it is an orbit space, while $\overline{W}_1$ is not an orbit space and it is the compactification of W_1.

Theorem 2.3.5. *The variety W_1 is a coarse moduli space for stable rational relatively minimal elliptic surfaces with a section.*

The stability and the semistability of a point $r \in M_1$ represented by a pair $(A, B) \in V_4 \oplus V_6$ only depends on the order of vanishing of A and B over the points of $\mathbb{P}^1$, as showed in the next theorem.

Theorem 2.3.6. *(i) The point $r \in E_1$ represented by the pair (A, B) is not semistable if and only if there exists a point $Q \in \mathbb{P}^1$ such that*

$$v_Q(A) > 2 \text{ and } v_Q(B) > 3.$$

(ii) The point $r \in E_1$ represented by the pair (A, B) is not stable if and only if there exists a point $Q \in \mathbb{P}^1$ such that

$$v_Q(A) \geq 2 \text{ and } v_Q(B) \geq 3.$$

Miranda gives a geometric characterization of stability and semi-stability.

Theorem 2.3.7. *Let r be a point of E_1 represented by the pair of forms (A, B). Let X be the rational elliptic fibration defined by (A, B). Then, r is stable if and only if X has a smooth general fiber and X has only reduced fibers.*

Notice that the reduced fibers are the I_k-fibers and the fibers of type II, III and IV.

Theorem 2.3.8. *Let r be a point of M_1 represented by the pair (A, B), and assume that the fibration X defined by (A, B) has a smooth general fiber. Then, r is strictly semi-stable if and only if X has a fiber of type I_N^* for some $N \geq 0$.*

2.4 Special fibers

From now on, the definitions and the results are original. In this section we put emphasis on some of the fibers of $X \to \mathbb{P}^1$. First of all, we go into detail of the theory of the singular fibers. Later, we investigate the smooth fibers over points of $\mathbb{P}^1$ over which the J-map assumes the value 0 or 1, that we will call special smooth fibers. Our main goal is to stratify the compactification $\overline{W}_1$ of the moduli space W_1 in terms of the special fibers the surfaces have, where by special we mean singular or special smooth fibers.

2.4.1 The singular fibers

First of all, we introduce a new notation, already mentioned in Section 2.1.1, for the singular fibers, that take account of the multiplicity of the J-map. Looking at the table of singular fibers, one can notice that more triplets $(v(A), v(B), v(D))$ give rise to the same Kodaira type of singular fiber. For example, all the triplets of type $(L, 1, 2)$, with $1 \leq L \leq 4$ give rise to a cuspidal singular fiber II. The difference between them is the multiplicity of the J-map over the point of the base curve corresponding to the fiber, that is equal to $3L - 2$. For our future purposes, it is important to distinguish these cases.

Definition 2.4.1. *We say that $m(J)$ is general for a certain Kodaira type of singular fiber, if the corresponding data $(v(A), v(B), v(D))$ take the minimum values to be associated to the given Kodaira type.*

In the example above, $(1,1,2)$ is the "minimum triplet", and the general multiplicity of J for the cuspidal fiber is 1.

Notation 2.4.1. Given a singular fiber of a certain Kodaira type with $m(J)=m$, we shall indicate the multiplicity m in subscript of the notation for the given Kodaira type of singular fiber. If $m(J)$ is general, we shall use the usual Kodaira notation.

Remark 2.4.2. *We added the following notations:*

(a) II_4, II_7, II_{10}

(b) III_3, III_5, III_7, III_9

(c) IV_5, IV_8

(d) $I^*_{0_3}$, $I^*_{0_{6,0}}$, $I^*_{0_2}$, $I^*_{0_4}$, $I^*_{0_{6,1}}$

(e) IV^*_4

(f) III^*_3

As already mentioned, the triplets of vanishings that correspond to the cuspidal fibers II are $(L,1,2)$ for $1 \leq L \leq 4$ and $m(J)=3L-2$ and this shows (a). Looking at the table of singular fibers, one can notice that the triplets corresponding to the fiber III are of type $(1,K,3)$, with $2 \leq K \leq 6$, and in this case $m(J)=2K-3$, explaining the notations given in (b). All the other cases are similar, except for the I_0^* one. In that case, the triplets $(4,3,6)$ and $(2,6,6)$ give rise to a fiber I_0^* with $m(J)=6$: in order to distinguish them, we indicate the value of J, that is 0 for the first case and 1 for the second. We shall write $I^*_{0_{6,0}}$ for the first and $I^*_{0_{6,1}}$ for the second.

As observed in Remark 2.2.3, the number of singular fibers of a relatively minimal elliptic surface over $\mathbb{P}^1$ is $12N$, thus a rational elliptic surface X has 12 singular fibers, counted with multiplicity, with respect to the order of vanishing of D over the points of the discriminant divisor. More precisely, if X has m singular fibers over m points $Q_1,\ldots,Q_m$, we must have

$$\sum_{i=1}^{m} v_{Q_i}(D) = 12.$$

The contribution of every Kodaira type of singular fiber is encoded in the table of singular fibers.

Definition 2.4.3. *The singular fibers of type I_k are called multiplicative and all the other are called additive.*

Definition 2.4.4. *Let X be an elliptic surface. We call the configuration of singular fibers of X the set of its singular fibers and we denote this set by $\sigma(X)$.*
Furthermore, we call the configuration of additive fibers of X the set of its singular additive fibers and we denote it by $\mathrm{Add}(X)$.

Notice that the additive fibers are the ones that arise from common zeroes of the polynomials A and B associated to the Weierstrass model of X.

Let us compute the total number of the fibers of type I_k and of type I_k^* on an elliptic surface over $\mathbb{P}^1$.

Proposition 2.4.5. *Let X be an elliptic surface over $\mathbb{P}^1$ with a section. Let i_k denote the number of I_k-fibers and i_k^* the number of I_k^*-fibers in X. We have the formula*

$$\deg(J) = \sum_k k(i_k + i_k^*) = 12N - \sum_Q v_Q(D) - 6i_k^*,$$

where the second sum runs over the points Q such that the singular fiber $X_Q \in \mathrm{Add}(X)$, with X_Q not of type I_k^*.

Proof. The degree of the J-map is equal to the number of its poles, counted with multiplicity, then the first equality is proved. If A and B have no common roots, the degree of $J = 4A^3/(4A^3 + 27B^2)$ is $12N$. Every time A and B have a common root Q, the degree of J decreases by $\min\{v_Q(4A^3), v_Q(4A^3 + 27B^2)\}$. By looking at the table of singular fibers, one can deduce that this latter quantity is equal to $v_Q(4A^3 + 27B^2) = v_Q(D)$ except for the case in which the fiber over Q if of type I_k^*, with $k > 0$: in this case, we have $v_Q(4A^3) = 6$ and $v_Q(D) > 6$. Hence, for every fiber of type I_k^*, the quantity to subtract from $12N$ is 6 and this completes the proof. □

Before deepening the theory of the configuration of singular fibers in a rational elliptic surface, we prove that the general elliptic surface over $\mathbb{P}^1$ with $\deg(\mathcal{L}) = 12N$ has $12N$ singular fibers of type I_1, as promised in Remark 2.2.3.

Proposition 2.4.6. *Let X be a general relatively minimal elliptic surface over $\mathbb{P}^1$ with fundamental line bundle $\mathbb{L}$ such that $\deg(\mathbb{L}) = 12N$. Then, X has $12N$ singular fibers of type I_1.*

Proof. Let $\mathcal{B}_N$ be the open subvariety of the weighted projective space

$$\mathbb{P}_N := \mathbb{P}\,(\underbrace{2,\ldots,2}_{4N+1},\underbrace{3,\ldots,3}_{6N+1}),$$

where the homogeneous degree $12N$ polynomial

$$4(a_0x^{4N} + a_1x^{4N-1}y + \cdots + a_{4N}y^{4N})^3 + 27(b_0x^{6N} + b_1x^{6N-1}y + \cdots + b_{6N}y^{6N})^2$$

has distinct roots. Since $[1,0,\ldots,0,1] \in \mathcal{B}_N$, it is nonempty. Since for a polynomial having distinct roots is an open condition, this is sufficient to conclude that $\mathcal{B}_N$ has dimension $10N+1$ and that the general elliptic surface with a section over $\mathbb{P}^1$ with $\deg(\mathbb{L}) = 12N$ has $12N$ singular fibers of type I_1. $\square$

Miranda and Persson in their works [49] and [44] listed all the possible 279 configurations of singular fibers that can occur in a rational elliptic surface. In their list, they do not take account of the multiplicity of the J-map. In Theorem 2.4.13 we refine their because we take account of the multiplicity of the J-map; on the other hand, we are not able to consider the I_k-fibers different from I_1 and the I_k^*-fibers different from I_0^*.

Fixed the additive fibers, the natural expectation is that the general rational elliptic surface with the given set of additive fibers has $\deg(J)$ distinct fibers of type I_1. We will prove this in Lemma 2.5.1.

Example 2.4.7. *Let X be a rational elliptic surface associated to the pair of forms (A, B), such that A and B have a common simple root. We have $\mathrm{Add}(X) = \{II\}$ and since the degree of J is 10, we expect $\sigma(X) = \{II, 10I_1\}$.*

Before stating the main theorem, we focus on the so-called isotrivial elliptic surfaces.

Isotrivial rational elliptic surfaces

Definition 2.4.8. *An elliptic surface X is said to be isotrivial if all its smooth fibers are isomorphic.*

Remark 2.4.9. *Since two elliptic curves are isomorphic if and only if they have the same J-invariant (see [52, proposition 1.4]), we have that X is isotrivial if and only if the J-map is constant.*

The next proposition lists all the possible configurations of singular fibers that can occur in an isotrivial rational elliptic surface with $J = 1$.

Proposition 2.4.10. *The possible configurations of singular fibers in an isotrivial rational elliptic surface X with a section with $J = 1$ are: $4III$, $I_0^* + 2III$, $2I_0^*$ and $III^* + III$.*

Proof. Since $J = 4A^3/(4A^3 + 27B^2)$, we have that X is isotrivial with $J = 1$ if and only if $B = 0$. Then, we have to take account only of the zeroes of the degree 4 polynomial A. The proposition is proved by noticing that X has a fiber of type III over a simple root of A, of type I_0^* over a double root and III^* over a triple root. □

Notice that, by Proposition 2.2.2 (ii), the degree 4 polynomials with a root of multiplicity 4 do not correspond to any rational elliptic surface.

Here we describe all the possible configurations of singular fibers that can occur in an isotrivial rational elliptic surface with $J = 0$.

Proposition 2.4.11. *The possible configurations of singular fibers in an isotrivial rational elliptic surface X with a section with $J = 0$ are are $6II$, $IV + 4II$, $I_0^* + 3II$, $2IV + 2II$, $IV^* + 2II$, $I_0^* + IV + 2II$, $3IV$, $2I_0^*$, $IV^* + IV$ and $II^* + II$.*

Proof. Since $J = 4A^3/(4A^3 + 27B^2)$, we have that X is isotrivial with $J = 0$ if and only if $A = 0$. Then, we have to take account only on the zeroes of the degree 6 polynomial B.

The proposition is proved by noticing that X has a fiber of type II over a simple root of B, of type IV over a double root, of type I_0^* over a triple root, of type IV^* over a quadruple root and finally of type II^* over a quintuple root. □

Again, notice that the degree 6 polynomials with a sextuple root do not correspond to any rational elliptic surface.

Proposition 2.4.12. *The only possible configuration of singular fibers in an isotrivial rational elliptic surface X with $J \neq 0, 1$ is*

$$2I_0^*$$

Proof. In order for $J = 4A^3/(4A^3 + 27B^2)$ to be constant and different from 0 and 1, we need that

$$4A^3 + 27B^2 = 4cA^3$$

for some $c \in \mathbb{C} \setminus \{0, 1\}$, so that

$$27B^2 = 4(c+1)A^3.$$

This implies that A is of the form $A = \lambda f^2 g^2$ and B of the form $\mu f^3 g^3$ for some linear forms f and g. Looking at the table of singular fibers, a pair (A, B) of this type gives rise to a surface with two singular fibers of type I_o^*. □

Now, we list all the possible configuration of additive fibers, different from I_k^* with k greater than 0, that a rational elliptic surface can have, taking account of the multiplicity of the J map.

Theorem 2.4.13. *All the possible configurations of additive fibers in a rational elliptic surface (having no I_k^*-fibers other than I_0^*) other than the isotrivial ones with $J = 0$ and $J = 1$ for a relatively minimal rational elliptic surface with a section are II, II_4, III, $II + II$, $II + II + II$, $III + II$, $II_4 + II$, III_5, II_7, IV, $II + II + II + II$, $III + II + II$,*

$III_3 + II$, $III + II_4$, $III + III$, $II_4 + II + II$, $IV + II$, $II_7 + II$, $II_4 + II_4$, I_0^*, IV_5, III_5, II_{10}, $III + II + II + II$, $IV + II + II$, $III + III + II$, $III_3 + II + II$, $I_0^* + II$, $III + II_4 + II$, $IV + III$, $IV_5 + II$, $IV + II_4$, $I_{0_2}^*$, $III_5 + II$, $III_3 + II_4$, $III_3 + III$, $III_3 + II + II$, $I_{0_3}^*$, $III + II_7$, IV_8, III_7, $I_0^* + II + II$, $IV + III + II$, $III_3 + II + II + II$, $III + III + II + II$, $I_{0_3}^* + II$, $IV + IV$, $I_{0_2}^* + II$, $I_0^* + III$, $III_3 + III + II$, $III + III + II_4$, $III + III + III$, $I_0^* + II_4$, IV^*, $III_3 + II_4 + II$, $IV_5 + III$, $I_{0_{6,0}}^*$, $I_{0_4}^*$, $III_7 + II$, $III_5 + II + II$, $III_5 + II_4$, $III_5 + III$, $III_3 + II_7$, $IV + III_3$, $III_3 + III_3$, III_9, $IV^* + II$, $I_0^* + IV$, $I_{0_2}^* + II + II$, $I_0^* + III + II$, $I_{0_3}^* + III$, $IV + III_3 + II$, $IV + III + III$, IV_4^*, III^*, $I_{0_4}^* + II$, $I_{0_2}^* + III$, $I_0^* + III_3$, $III_3 + III + II_4$, $I_{0_2}^* + II_4$, $IV_5 + III_3$, $IV + III_5$, $III_5 + II_4 + II$, $I_{0_{6,1}}^*$, $III_9 + II_4$, $III_5 + II_7$, II^*, $IV^* + III$, $III^* + II$, $I_{0_2}^* + IV$, $I_0^* + I_0^*$, $I_{0_3}^* + III_3$, III_3^*, $I_{0_4}^* + II_4$, $IV_5 + III_5$ *and* $III_9 + II_7$.

Proof. We give a direct proof for some configurations.

The configuration $III + III + III$ corresponds to a pair of forms (A, B) such that A and B have three common roots such that for everyone of them $(v(A), v(B), v(D)) = (1, 2, 3)$, and no other common roots. The configuration $IV^* + III + I_1$ corresponds to a pair of forms (A, B) such that A and B have two common roots, one such that $(v(A), v(B), v(D)) = (3, 5, 8)$ the other such that $(v(A), v(B), v(D)) = (1, 2, 3)$ and no other common roots. Consider the configuration $III_5 + II_7 + 7I_1$: it corresponds to a pair of forms (A, B) such that A and B have two common roots, one such that $(v(A), v(B), v(D)) = (1, 4, 3)$ the other such that $(v(A), v(B), v(D)) = (3, 1, 2)$ and no other common roots. The proof is analogous for all the other 93 configurations.

To prove that there are no other configurations of additive fibers, it is sufficient to notice that every configuration of additive fibers corresponds to a given configuration of common roots for a pair of polynomials (A, B). The possible configurations of common roots between a degree 4 polynomial A and a degree 6 polynomial B are 97, while we have a list of

96 configurations of additive fibers. But we have to remove the configuration consisting of two polynomials with a common root that is quadruple for A and sextuple for B (that does not correspond to any elliptic surface), so we have the assertion.

□

2.4.2 Special smooth fibers

In this section we explain how to localize and control some special smooth fibers, that will help us to stratify the spaces parametrizing rational elliptic surfaces with a section.

Definition 2.4.14. *A special smooth fiber of an elliptic surface X is a smooth fiber over which the J-map assumes the value 0 or 1. We call them J_0-fiber and J_1-fiber respectively.*

Lemma 2.4.15. *Let $X \to \mathbb{P}^1$ be a relatively minimal elliptic surface with a section associated to the pair (A, B).*

(a) The J-map assumes the value 1 in correspondence of smooth fibers over points where the form B vanishes but A does not.

(b) The J-map assumes the value 0 in correspondence of smooth fibers over points where the form A vanishes but B does not.

Proof. The proof is a trivial application of the Néron classification of the fibers in terms of the orders of vanishing of A, B and D. The only smooth fibers with $J = 0$ occur in the points such that $(v(A), v(B), v(D)) = (0, K, 0)$ and the only smooth fibers with $J = 1$ occur when the triplet is $(v(A), v(B), v(D)) = (L, 0, 0)$. □

Notation 2.4.2. Looking at the table of singular fibers, one can notice that if $Q \in \mathbb{P}^1$ is a root of A (not in common with B) of multiplicity a, the J_0-fiber X_Q appears with multiplicity $3a$; if $Q \in \mathbb{P}^1$ is a root of B (not in common with A) of multiplicity b, the J_1-fiber X_Q appears with multiplicity $2b$. If the multiplicities are different from 2 and 3, we subscript them. For example, if A has a double root Q, then it appears with multiplicity 6 and we say that the fiber X_Q is a $J_{0,6}$-fiber for X.

We denote by $r(A)$ the number of J_0-fibers and by $r(B)$ the number of J_1-fibers (counted with multiplicity). As showed in the next Proposition, they are invariant for surfaces with the same configuration of singular fibers.

Proposition 2.4.16. *The number of J_0-fibers (counted with multiplicity) is*

$$r(A) = 4 - \sum_{Q \in (D)_a} v_Q(A) - 2z;$$

while the number of J_1-fibers (counted with multiplicity) is

$$r(B) = 6 - \sum_{Q \in (D)_a} v_Q(B) - 3z,$$

where the sums run over the subset $(D)_a$ of the points of the discriminant divisor (D) giving rise to an additive fiber different from I_k^* with $k > 0$ and z indicates the number of I_k^*-fibers with $k > 0$.

Proof. The quantity of J_0-fibers is equal to the number of zeroes of A (counted with multiplicity) that are not not in common with B, thus we have to subtract to 4 the number of zeroes of A not involved in additive fibers. The proof for the number of J_1-fibers is the same. □

Remark 2.4.17. *Given a rational elliptic surface X with $\mathrm{Add}(X)$ as configuration of additive fibers, we could have more configurations of special smooth fibers on X, according to the orders of vanishing of A and B at their roots. The number of possible configurations of special smooth fibers is the product of the number of partitions of $r(A)$ and the number of partitions of $r(B)$. Indeed, for every pair of partitions of $r(A)$ and $r(B)$, we have a different configuration of special smooth fibers.*

Example 2.4.18. *Let us fix $\sigma = \{IV^*\}$ as configuration of additive fibers. The fiber IV^* corresponds to $(v(A), v(B)) = (3,4)$, then $r(A) = 4-3 = 1$ and $r(B) = 6-4 = 2$. We have that the only partition of $r(A) = 1$ is (1) and the two partitions of $r(B) = 2$ are (1^2) and (2). The pair of partitions $((1),(1^2))$ corresponds to surfaces with a singular fibers IV^*, four singular fibers I_1, a J_0-fiber and two J_1-fibers. On the other side, the pair of partitions $((1),(2))$ corresponds to surfaces with a singular fibers IV^*, four singular fibers I_1, a J_0-fiber and a $J_{1,4}$-fiber.*

Definition 2.4.19. *We shall call the configuration of special fibers of X the union of $\mathrm{Add}(X)$ and the set of its special smooth fibers and we denote this set by $\delta(X)$.*

2.5 The stratification

In the last part of the chapter we discuss the problem of the stratification of the boundary of the spaces E_1 and $\overline{W}_1$, constructed by Miranda, in terms of the configurations of special fibers the surfaces have. First of all, we see that working with the configurations of special fibers is not restrictive.

Lemma 2.5.1. *Let δ be a possible configuration of special fibers for a rational elliptic surface. Then, a general rational elliptic surface X with δ as configuration of special fibers, has $\deg(J) I_1$ singular fibers other than the ones in δ.*

Proof. To prove the Lemma, we need to show that the general pair of polynomials (A, B) with configuration of the roots as dictated by δ is such that the discriminant $D = 4A^3 + 27B^2$ has $\deg(J)$ distinct roots other than the multiple ones imposed by δ. It is sufficient to show that for every δ there exists at least one pair (A, B) satisfying this open condition. The set of possible configurations of additive fibers is large and for every one of them there are a lot of compatible sets of special smooth fibers: for this reason, we do not report a pair (A, B) satisfying the condition described above for every δ. We make a couple of examples. Let us consider the configuration

$$\delta = \{II + 3J_0 + J_1 + 2J_{1,4}\}.$$

In this case A and B have a common root, A has three distinct other roots and B has other two double roots and another root. We choose

$$A = xy(x-y)(x+y)$$

and

$$B = x(x-2y)^2(x+2y)^2(x-3y),$$

so that

$$D = 4A^3 + 27B^2 = x^2(4(xy^3(x-y)^3(x+y)^3) + 27((x-2y)^4(x+2y)^4(x-3y)^2)).$$

The first factor has a double root in $[0,1] \in \mathbb{P}^1$, while the second factor has ten distinct roots different from $[0,1]$, proving that the set of pairs (A,B) giving rise to an elliptic surface with configuration of special fibers δ and other ten fibers of type I_1 is nonempty. Let now consider the configuration

$$\delta = \{I_0^* + 2J_0 + 3J_1\}.$$

In this case A and B have a common root that is double for A and triple for B, A has other two distinct roots and B has other three distinct roots. We choose

$$A = x^2y(x-y)$$

and

$$B = x^3(x+y)(x+2y)(x-2y),$$

so that

$$D = 4A^3 + 27B^2 = x^6(4y^3(x-y)^3 + 27(x+y)^2(x+2y)^2(x-2y)^2).$$

The first factor has a sextuple root in $[0,1] \in \mathbb{P}^1$, while the second factor has six distinct roots different from $[0,1]$, proving the the set of pairs (A,B) giving rise to an elliptic surface with configuration of special fibers δ and other six fibers of type I_1 is nonempty.
The proof will be complete once showed that the space parametrizing rational elliptic surface with δ as configuration of special fibers is irreducible: we will prove it for every δ in 2.5.12. □

Notation 2.5.1. Since in this work we do not consider the multiplicative fibers different from I_1 and the I_k^*-fibers different from I_0^*, for the general rational elliptic surface X with δ as configuration of special fibers, Lemma 2.5.1 allows us to call configuration of special fibers and to indicate with $\delta(X)$ also the set consisting on the configuration of special fibers of X and $\deg(J)$ fibers of type I_1.

The machinery we will use to perform the stratification is the one of the generalized coincident root loci presented in Chapter 1. For this reason, as already mentioned, we are not able to control the fibers that came from multiple roots of the discriminant divisor D that not directly come from common zeroes of A and B, namely the ones of type I_k or I_k^* other than I_1 and I_0^*. In Remark 2.5.32, we explore this problem and show a method that solves it theoretically, but computationally too hard to be taken into account.

All the discussion above leads us to give the following Definition.

Definition 2.5.2. *A relatively minimal rational elliptic surface with a section is said to be suitable if it is not isotrivial with $J = 0$ or $J = 1$ and if it does not have multiplicative singular fibers different from I_1 and I_k^*-fibers different from I_0^*.*

Definition 2.5.3. *We call $\tilde{E}_1$ the subset of E_1 parametrizing suitable rational elliptic surfaces.*

Now, we get into the problem of the stratification.

Definition 2.5.4. *Given a possible configuration δ of special fibers, we denote as E_δ the* *subset of $\tilde{E}_1$ corresponding to surfaces with δ as configuration of special fibers and $\overline{E}_\delta$ its Zariski-closure in $\tilde{E}_1$. Furthermore, we call W_δ and $\overline{W}_\delta$ the analogue subsets of $\overline{W}_1$*

The questions we want to answer are: given a configuration δ of special fibers, how is E_δ in the parameter space $\tilde{E}_1$? Is it irreducible? When do two of them intersect? When is one contained in the closure of another? The same questions are posed for $W_\delta \in \overline{W}_1$. We summarize the results with a diagram, that we call the stratification diagram of $\tilde{E}_1$ (and of $\overline{W}_1$).

As exhaustively explained, all the information about the special fibers are encoded in the order of vanishing of A, B and D at their zeroes: stratifying the spaces E_1 and $\overline{W}_1$ in terms of the configurations of special fibers means understanding how the configuration of (possibly common) zeroes of A and B change, while the pair (A, B) moves inside $\tilde{E}_1$ (and $\overline{W}_1$) and how it affects the configuration of zeroes of $D = 4A^3 + 27B^2$.

Example 2.5.5 (The general configuration)**.** *In the general case, we have that both A and B have distinct roots, A and B have not common roots and D has twelve distinct roots. This corresponds to the configuration δ composed by twelve fibers of type I_1, four J_0-fibers and six J_1-fibers.*

$$\delta = 12I_1 + 4J_0 + 6J_1$$

Example 2.5.6 (The first degenerations)**.** *We want to understand what happens in codimension 1. Roughly speaking, we can have three types of degenerations:*

(i) A and B have a common root, that is forced to be double for D. In this case we have the configuration δ_1 composed by a cuspidal fiber II, ten fibers of type I_1, three J_0-fibers and five J_1-fibers. I_1.

$$\delta_1 = II + 10I_1 + 3J_0 + 5J_1$$

(ii) A and B have not common roots but A has a double root. In this case the configuration δ_2 of special fibers is composed by twelve I_1, a $J_{0,6}$-fiber, two J_0-fibers and six J_1-fibers.

$$\delta_2 = 12I_1 + J_{0,6} + 2J_0 + 6J_1$$

(ii) A and B have not common roots but B has a double root. In this case the configuration δ_3 is composed by twelve I_1, four J_0-fibers, a $J_{1,4}$-fiber and four J_1-fibers.

$$\delta_3 = 12I_1 + 4J_0 + J_{1,4} + 4J_1$$

We will see that E_{δ_1}, E_{δ_2} and E_{δ_3} are the three irreducible components of maximal dimension of the boundary of the parameter space $\tilde{E}_1$ (and the same for the moduli space $\overline{W}_1$).

2.5.1 The stratification of the parameter space

The goal of this section is to show the stratification of the space $\tilde{E}_1 \subset E_1$ parametrizing suitable rational elliptic surfaces in terms of the special fibers the surfaces have. We treat the isotrivial cases with $J = 1$ and $J = 0$ separately. We exploit the machinery of the generalized coincident root loci introduced in Chapter 1: we associate a configuration of special fibers $\delta_{\lambda,\mu,R}$ to every triplet (λ, μ, R) composed by a partition λ of 4, a partition μ of 6 and a set of relations R. Hence, we associate a stratum $E_{\delta_{\lambda,\mu,R}}$ of the parameter space $\tilde{E}_1$ to the generalized coincident root locus $X_{\lambda,\mu,R}$ in $\mathbb{P}^4 \times \mathbb{P}^6$. We show that the stratum $E_{\delta_{\lambda,\mu,R}}$ of $\tilde{E}_1$ is rational of dimension one more than $X_{\lambda,\mu,R} \subset \mathbb{P}^4 \times \mathbb{P}^6$; furthermore, we prove that all the intersections between the strata in $\tilde{E}_1$ are reflected by the intersections between their corresponding generalized coincident root loci. The results are collected in Theorem 2.5.12 and Theorem 2.5.13. Finally, we resume all the results in a diagram, called the *stratification diagram* of $\tilde{E}_1$: we notice that this diagram is very similar to the specialization diagram of $\mathbb{P}^4 \times \mathbb{P}^6$ defined in Chapter 1. The precise statement will be presented in Theorem 2.5.15.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of 4, $\mu = (\mu_1, \dots, \mu_s)$ be a partition of 6 and $R = \{(\lambda_{i_1}, \mu_{j_1})^{a_1}, \dots, (\lambda_{i_h}, \mu_{j_h})^{a_h}\}$ be a set of relations. Recall that the generalized coincident root locus $X_{\lambda,\mu,R} \subset \mathbb{P}^4 \times \mathbb{P}^6$ is an irreducible and rational variety representing pairs of polynomials with configurations of zeroes prescribed by λ and μ and common zeroes as dictated by R.

Proposition 2.5.7. *There is a one to one correspondence between the set of the possible configurations of special fibers for a suitable rational elliptic surface and the set of triplets (λ, μ, R) as above, other than $(\lambda = (4), \mu = (6), R = \{(4,6)\})$.*

Proof. The triplet (λ, μ, R) uniquely determines a configuration of special fibers. Indeed, we associate a relation (λ_1, μ_1) to the type of singular fiber such that $v(A) = \lambda_1$ and $v(B) = \mu_1$. Moreover, we associate a part $\lambda_2 \in \lambda^C$ to a J_0-fiber with $m(J) = 2\lambda_2$ and a part $\mu_2 \in \mu^C$ to a J_1-fiber with $m(J) = 3\mu_2$. The case $(\lambda = (4), \mu = (6), R = \{(4,6)\})$ corresponds precisely to that pairs of polynomials that do not satisfy the condition (ii) of the Proposition 2.2.2, then it is not associated to any elliptic surface. The other direction is obvious.

□

Definition 2.5.8. *Given a triplet (λ, μ, R), we denote by $\delta_{\lambda,\mu,R}$ its associated configuration of special fibers and by $E_{\delta_{\lambda,\mu,R}} \subset \tilde{E}_1$ the subset corresponding to surfaces having $\delta_{\lambda,\mu,R}$ as configuration of special fibers.*

Example 2.5.9. *Let $\lambda = (1^4)$, $\mu = (1^2, 2^2)$ and $R = \{(1,1),(1,2)\}$, so that $\lambda^C = (1^2)$ and $\mu^C = (1,2)$. The relation $(1,1)$ indicates that A and B have a common simple zero, corresponding to a singular fibers of X of type II. The relation $(1,2)$ corresponds to a singular fiber of type III. About the special smooth fibers: $\lambda^C = (1^2)$, then X has two J_0-fibers, while $\mu^C = (1,2)$, then X has a J_1-fiber and a $J_{1,4}$-fiber. Since $\deg(J) = 7$, if X is general in $E_{\delta_{\lambda,\mu,R}}$, it has other seven I_1 fibers. The triplet*

$$(\lambda, \mu, R) = ((1^4), (1^2, 2^2), \{(1,1), (1,2)\})$$

is associated to the configuration of special fibers

$$\delta_{\lambda,\mu,R} = II + III + 2J_0 + J_1 + J_{1,4} + 7I_1.$$

Remark 2.5.10 (The correspondence between generalized coincident loci and strata of the parameter space)**.** *Recall that the space E_1 is an open dense subset of the weighted projective space*

$$\mathcal{B} = \mathbb{P}(2^5, 3^7).$$

The strata of $\tilde{E}_1$ *lie in* M_1*, while the coincident root loci are contained in* $\mathbb{P}^4 \times \mathbb{P}^6$*: both spaces represent pairs of polynomials, but in the first case the two polynomials are considered up to multiplication by a scalar, while in the second we just quotient by a single weighted relation. This difference is reflected by the fact that the* M_1 *has dimension 11, while* $\mathbb{P}^4 \times \mathbb{P}^6$ *has dimension 10.*

Consider the morphism

$$\begin{aligned} \Phi : \tilde{E}_1 &\longrightarrow \mathbb{P}^4 \times \mathbb{P}^6 \\ [(A,B)] &\longmapsto ([A],[B]), \end{aligned}$$

where $[(A,B)]$ *denotes the class of the pair* (A,B) *under the weighted action of* $\mathbb{C}^*$*, that is*

$$(\lambda, (A,B)) = (\lambda^2 A, \lambda^3 B);$$

and $[A] \in \mathbb{P}^4$ *and* $[B] \in \mathbb{P}^6$ *represent the class of the polynomials up to the multiplication by a nonzero constant.*

Notice that Φ *is well defined: the locus where the first five coordinates vanish and the locus where the last seven coordinates vanish correspond to isotrivial rational elliptic surfaces with* $J = 0$ *and* $J = 1$ *respectively, and they are disjoint from* $\tilde{E}_1$*. We shall call these spaces* E_A *and* E_B*.*

The fiber

$$\Phi^{-1}([A],[B])$$

is the open subset of the pencil

$$p(A,B) := \{[(\alpha A, \beta B]\} \subset M_1,$$

with $[\alpha,\beta]\in\mathbb{P}^1$*, given by the intersection with* $\tilde{E}_1$*, that we call* $\tilde{p}(A,B)$*.*

The fact that the fibers of Φ *are 1-dimensional is meaningful in the sense of the rational elliptic surfaces. Given a pair of polynomials* $(A,B)\in\mathbb{P}^4\times\mathbb{P}^6$*, the morphism* Φ *separates isomorphism classes of rational elliptic surfaces. What makes the members of* $\tilde{p}(A,B)$ *different is the J-map: given two points* $(\alpha_1 A,\beta_1 B)$ *and* $(\alpha_2 A,\beta_2 B)$ *of* $\tilde{p}(A,B)$*, the corresponding J-maps* J_1 *and* J_2 *are respectively*

$$J_1 = 4\alpha_1^3A^3/(4\alpha_1^3A^3+27\beta_1^2B^2)$$

and

$$J_2 = 4\alpha_2^3A^3/(4\alpha_2^3A^3+27\beta_2^2B^2).$$

They are different unless there exists $\gamma\in\mathbb{C}^*$ *such that* $\alpha_1=\gamma^2\cdot\alpha_2$ *and* $\beta_1=\gamma^3\cdot\beta_2$*, but this is in fact the equivalence relation defining the weighted projective space* M_1*.*

The next Lemma states how to associate a stratum of $\tilde{E}_1$ to a generalized coincident root locus of $\mathbb{P}^4\times\mathbb{P}^6$.

Lemma 2.5.11. *Let* λ *and* μ *be partitions of 4 and 6, let* R *be a set of relations and let* $X_{\lambda,\mu,R}$ *be the generalized coincident root locus associated to the triplet* (λ,μ,R)*. Then*

$$\Phi^{-1}(X_{\lambda,\mu,R}) = E_{\delta_{\lambda,\mu,R}}.$$

Basically the result depends on the fact that the morphism Φ does not affect the configuration of the zeroes of the polynomials. Notice that, since Φ is a morphism and $X_{\lambda,\mu,R}$ is an algebraic variety, this Lemma implies also that $E_{\delta_{\lambda,\mu,R}}$ is an algebraic variety.

Proof. Let $([A],[B])$ be a pair of forms belonging to $X_{\lambda,\mu,R}$. The fiber $\Phi^{-1}([A],[B])$ is either empty (in the case $(\lambda=(4),\mu=(6),R=\{(4,6)\})$), or $\tilde{p}(A,B)$. This latter is in contained in $E_{\delta_{\lambda,\mu,R}}$: let $[\alpha,\beta]\in\mathbb{P}^1$ such that $[(\alpha A,\beta B)]\in\tilde{p}(A,B)$. The polynomials αA and βB have the same (common or not) roots as A and B, thus, by definition of $E_{\delta_{\lambda,\mu,R}}$, we have $[(\alpha A, \beta B)] \in E_{\delta_{\lambda,\mu,R}}$.

On the other side, in the same way one sees that the image $([A],[B])$ of a point $[(A,B)] \in E_{\delta_{\lambda,\mu,R}}$ is composed by forms with the prescribed properties to belong to $X_{\lambda,\mu,R}$. □

For the next Theorem we use the notations of Chapter 1: c is the number of common roots between A and B (the number of relations), r is the number of distinct roots of A (the number of parts of λ) and s is the number of distinct roots of B (the number of parts of μ).

Theorem 2.5.12. *For every triplet (λ, μ, R), the stratum $E_{\delta_{\lambda,\mu,R}} \subset \tilde{E}_1$ is irreducible, rational and of dimension $r+s-c+1$.*

Proof. By Prop 1.2.10 $X_{\lambda,\mu,R}$ is irreducible and rational of dimension $r+s-c$. Furthermore, the fibers of $\Phi : \tilde{E}_1 :\to \mathbb{P} \times \mathbb{P}$ are are birational to lines. We can conclude that $E_{\sigma_{\lambda,\mu,R}}$ is birational to $X_{\lambda,\mu,R} \times \mathbb{P}^1$, and then it is an irreducible rational variety of dimension $r+s-c+1$. □

The next Theorem is the main tool for the stratification of $\tilde{E}_1$

Theorem 2.5.13. *A stratum $E_{\delta_{\lambda,\mu,R}}$ is contained in the closure of another stratum $\overline{E}_{\delta_{\lambda',\mu',R'}}$ if and only if the triplet (λ, μ, R) is a coarsening of the triplet (λ', μ', R') in the sense of Definition 1.2.12.*

Proof. Let $X_{\lambda,\mu,R}$ and $X_{\lambda',\mu',R'}$ be the generalized coincident root loci associated to the triplets (λ, μ, A) and (λ', μ', A'). Consider once again the morphism

$$\Phi : \tilde{E}_1 \longrightarrow \mathbb{P}^4 \times \mathbb{P}^6.$$

By Lemma 2.5.11,

$$\Phi^{-1}(X_{\lambda,\mu,R}) = E_{\delta_{\lambda,\mu,R}}$$

and

$$\Phi^{-1}(X_{\lambda',\mu',R'}) = E_{\sigma_{\lambda',\mu',R'}}.$$

Moreover, we proved in Proposition 1.2.14 that $X_{\lambda,\mu,R} \subset X_{\lambda',\mu',R'}$ if and only if (λ, μ, R) is a coarsening of (λ', μ', R'), from which the assertion follows. □

We know all the strata, all their dimension and all the intersection between them. In other words, we have the complete stratification of $\tilde{E}_1$.

Definition 2.5.14. *We call stratification diagram of $\overline{E}_1$ the oriented graph having all the strata $E_\delta \subset \tilde{E}_1$ as vertices and with an arrow from a stratum E_{δ_1} to another stratum E_{δ_2} when E_{δ_2} is a subvariety of $\overline{E}_{\delta_1}$ of codimension 1.*

We actually already proved the next theorem.

Theorem 2.5.15. *The stratification diagram of $\tilde{E}_1$ is the specialization diagram of $\mathbb{P}^4 \times \mathbb{P}^6$, from which we remove the component $X_{(4),(6),\{(4,6)\}}$.*

Proof. By Proposition 2.5.7, there is a one to one correspondence between the set of the strata of $\tilde{E}_1$ and the coincident root loci in $\mathbb{P}^4 \times \mathbb{P}^6$ other than $X_{(4),(6),\{(4,6)\}}$. Moreover, Theorem 2.5.13 tells to us that the inclusions between strata are preserved by this correspondence, while Theorem 2.5.12 ensures that the codimension between the strata are also preserved. □

In the next example we show a family of rational elliptic surface with 12 I_1 fibers, whose boundary (in codimension 1) has an irreducible component parametrizing surfaces with a II^*-fiber.

Example 2.5.16. *Let $\lambda = (4)$, $\mu = (1,5)$ and $R = \emptyset$.*
The associated configuration of special fibers is

$$\sigma = 12I_1 + J_{0,12} + J_{1,10} + J_1.$$

The stratum $E_{12I_1+J_{0,12}+J_{1,10}+J_1}$ has dimension 4.
Let now (λ', μ', R') be the coarsening of (λ, μ, R) given by $\lambda_1 = \lambda$, $\mu_1 = \mu$ and $R' = \{(4,5)\}$. The associated configuration of special fibers is

$$\sigma_1 = II^* + 2I_1 + J_1$$

and the stratum

$$E_{II^*+2I_1+J_1}$$

has dimension 3 and then it has codimension 1 in

$$E_{12I_1+J_{0,12}+J_{1,10}+J_1}.$$

We complete the description of E_1 by analyzing the components with constant J-map equal to 1 and 0. Recall once again that E_1 is contained in the weighted projective space $\mathbb{P}(2^5, 3^7)$, where the first five coordinates $a_0, \ldots, a_4$ correspond to the coefficients of the degree 4 polynomial $A = a_o x^4 + \cdots + a_4 y^4$ and the last seven correspond to the coefficients of the degree 6 polynomial $B = b_0 x^6 + \cdots + b_6 y^6$. Now, notice that the morphism Φ does not extend to the whole parameter space E_1, since

$$\Phi : E_1 \longrightarrow \mathbb{P}^4 \times \mathbb{P}^6$$

cannot be defined in the locus where the first 5 coordinates vanish and in the locus where the last 7 coordinates vanish.

Definition 2.5.17. *We denote these two loci E_B and E_A respectively.*

As noticed in Remark 2.5.10, E_A and E_B parametrize isotrivial elliptic surfaces.

Lemma 2.5.18. *E_A parametrize isotrivial rational elliptic surfaces with $J = 1$ and is isomorphic to $\mathbb{P}^4$, while E_B parametrizes isotrivial rational elliptic surfaces with $J = 0$ and is is isomorphic to $\mathbb{P}^6$.*

Proof. E_A is the locus in which the last seven coordinates vanish, then it is isomorphic to $\mathbb{P}(2^5) \cong \mathbb{P}^4$; analogously, E_B is isomorphic to $\mathbb{P}(3^7) \cong \mathbb{P}^6$. □

Corollary 2.5.19. *All the strata inside of E_A and E_B representing isotrivial rational elliptic surfaces with $J \equiv 1$ or $J \equiv 0$ are irreducible and rational.*

Proof. By Proposition 2.4.10 and Proposition 2.4.11, the components representing rational elliptic surfaces with $J \equiv 1$ or $J \equiv 0$ are exactly the coincident root loci contained in $\mathbb{P}^4$ and $\mathbb{P}^6$ other than the curves representing polynomials with a quadruple root in $\mathbb{P}^4$ and polynomials with a sextuple root in $\mathbb{P}^6$. By Proposition 1.1.2, they are all irreducible and rational. $\square$

We give the *stratification diagrams* for E_A and E_B. These diagrams are oriented graphs having all the strata in E_A and E_B as vertices and with an arrow from a stratum to another if and only if the second is a subvariety of the first of codimension 1. The strata are listed in Proposition 2.4.10 and Proposition 2.4.11, and the stratification diagrams are exactly the specialization diagrams of $\mathbb{P}^4$ and $\mathbb{P}^6$ respectively, from which we remove the locus corresponding to polynomials with just one root of maximal multiplicity.

The stratification diagram of E_A is

$$\begin{tikzcd}
& E_{4III} \arrow[d] & \\
& E_{I_0^*+2III} \arrow[dl] \arrow[dr] & \\
E_{2I_0^*} & & E_{III^*+III}
\end{tikzcd}$$

and the stratification diagram of E_B is

$$\begin{tikzcd}
& E_{6II} \arrow[d] & \\
& E_{IV+4II} \arrow[dl] \arrow[dr] & \\
E_{I_0^*+3II} \arrow[d] \arrow[dr] & & E_{2IV+2II} \arrow[dll] \arrow[dl] \arrow[d] \\
E_{I_0^*+IV+II} \arrow[d] \arrow[dr] \arrow[drr] & E_{IV^*+2II} \arrow[d] \arrow[dr] & E_{3IV} \arrow[d] \\
E_{2I_0^*} & E_{II^*+II} & E_{IV^*+IV}
\end{tikzcd}$$

2.5.2 The stratification of the moduli space

Recall that W_1 is the coarse moduli space for stable rational relatively minimal elliptic surfaces with a section and that $\overline{W}_1$ is its compactification in the sense of geometric invariant theory. Let us focus on the stable locus W_1 first.

Definition 2.5.20. *We call $\tilde{W}_1$ the open subset of W_1 parametrizing isomorphism classes of stable suitable not isotrivial rational elliptic surfaces.*

By Proposition 2.3.6, a rational elliptic surface is stable if and only if it has only reduced fibers. Then, a stable surface X can have fibers of Kodaira type I_k, II, III or IV. If X is associated to the pair of forms (A, B), then there are no common roots of multiplicity at least 2 for A and at least 3 for B. This leads us to give the following definitions.

Definition 2.5.21. *A triplet (λ, μ, R) is stable if and only if there are not relations $(\lambda_i, \mu_j) \in R$ with $\lambda_i \geq 2$ and $\mu_j \geq 3$.*

In order to exploit the stratification of $\tilde{E}_1$, we want to point out that the orbit of a stable element in a stratum E_δ is all contained in E_δ.

Lemma 2.5.22. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of 4, $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of 6 and $R = \{(\lambda_{i_1}, \mu_{j_1})^{a_1}, \ldots, (\lambda_{i_h}, \mu_{j_h})^{a_h}\}$ be a set of relations such that (λ, μ, R) is a stable triplet. Then, the stratum $E_{\delta_{\lambda,\mu,R}}$ is invariant under the action of $\mathrm{SL}(V_1)$.*

The idea of the proof is that the elements of $\mathrm{SL}(V_1)$ represent change of coordinates on the base curve $\mathbb{P}^1$, whence its action does not change the configuration of the (possibly common) roots for a pair of polynomials.

Proof. Let $[A, B] \in E_{\delta_{\lambda,\mu,R}}$ and let $M \in \mathrm{SL}(V_1)$. A and B split in product of linear forms, so that, according with the triplet (λ, μ, R), we can write

$$A = \sum_{i=1}^{r} (a_i x + b_i y)^{\lambda_i} \text{ and } B = \sum_{j=1}^{s} (c_j x + d_j y)^{\mu_i}.$$

Since the change of coordinates induced by M acts on the coefficients a_i, b_i, c_j and d_j but not on the corresponding powers λ_i and μ_j, we conclude that $(M \cdot [A, B]) \in E_{\delta_{\lambda,\mu,R}}$. $\square$

Definition 2.5.23. *If (λ, μ, R) is a stable triplet, we call $W_{\lambda,\mu,R}$ the image of the stratum $E_{\delta_{\lambda,\mu,R}}$ under the quotient of E_1 by* $\mathrm{SL}(V_1)$.

Proposition 2.5.24. *If (λ, μ, R) is a stable triplet, then $W_{\lambda,\mu,R}$ is irreducible of dimension $r + s - c - 2$.*

Proof. The dimension of $E_{\delta_{\lambda,\mu,R}}$ is $r + s - c + 1$ and since it is composed by stable points, the dimension of its image $W_{\lambda,\mu,A}$ under the quotient by $\mathrm{SL}(V_1)$ is

$$\dim(W_{\lambda,\mu,R}) = \dim E_{\delta_{\lambda,\mu,R}} - \dim(\mathrm{SL}(V_1)) = r + s - c + 1 - 3 = r + s - c - 2.$$

□

Corollary 2.5.25. *Let (λ, μ, R) and (λ', μ', R') be two stable triplets. Then, the stratum $W_{\delta_{\lambda,\mu,R}}$ is contained in the closure of $W_{\delta_{\lambda',\mu',R'}}$ if and only if the triplet (λ, μ, R) is a coarsening of the triplet (λ', μ', R').*

Proof. Since $E_{\delta_{\lambda,\mu,R}} \subset \overline{E}_{\delta_{\lambda',\mu',R'}}$ and they consist of stable points, we have that the inclusion is preserved by the quotient by $\mathrm{SL}(V_1)$, whence $W_{\delta_{\lambda,\mu,R}} \subset \overline{W}_{\delta_{\lambda',\mu',R'}}$. □

This corollary gives us the complete stratification of $\tilde{W}_1$. It is the same of the stratifcation diagram of $\tilde{E}_1$ up to some modifications: we just have to remove the non stable locus, then all the configurations having fibers of Kodaira type I_0^*, IV^*, III^* and II^* (corresponding to non-stable triplets).

The strictly-semistable locus

Let us adopt the following notation: let W_{sss} denote the closed subvariety $\overline{W}_1 \setminus W_1$; similarly let E_{sss} denote the corresponding locus in E_1.

Proposition 2.5.26. *Let r be a point of E_1 represented by the pair (A, B). Then r is in E_{sss} if and only if the associated elliptic surface X has a fiber of type I_N^* for some $N \geq 0$.*

Proof. [45, Proposition 8.2] □

Theorem 2.5.27. *Let (A_1, B_1) and (A_2, B_2) be two points of E_{sss} defining fibrations X_1 and X_2. Let J_1 and J_2 be the values of the J-map of the singular fibers on X_1 and X_2 of type I_N^*, whose existence is ensured by Proposition 2.5.26.*
Then (A_1, B_1) and (A_2, B_2) have the same image in W_{sss} if and only if $J_1 = J_2$.

Proof. [45, Theorem 8.3] □

This theorem implies that surfaces with a fiber of type I_0^* are classified in $\overline{W}_1$ by the finite J-value of that fiber; however, surfaces with a fiber of type I_N^* for $N \geq 1$ are all mapped to one point w_∞ of $\overline{W}_1$. Since the map $E_s \to W_1$ is surjective, Miranda in [45, Section 8] stratified $\overline{W}_1$ as follows:

$$\overline{W}_1 = W_1 \cap Y \cap w_\infty,$$

where points of W_1 classify completely the stable rational Weierstrass fibrations up to isomorphism, $Y \cong \mathbb{A}^1$ classifies the fibrations with a singular fiber of type I_0^* by the J-value of that singular fiber, and w_∞ represents all the fibration with a singular fiber of type I_N^*, with $N \geq 1$.

With our refined list of singular fibers that take account of the J-map and its multiplicity, we can add some details about the contraction of the strictly semistable locus.

Definition 2.5.28. *We call Y_0 and Y_1 the points of Y representing fibrations with a fiber of Kodaira type I_0^* such that the value of the J-map is 0 or 1 respectively.*

Looking at the table of the singular fibers, we can easily prove the following Proposition.

Proposition 2.5.29. *Let (A, B) a point in E_{sss} defining a rational elliptic surface X. If X has a singular fiber of type $I^*_{0_3}$ or $I^*_{0_{6,0}}$, then it is contracted to Y_0. If X has a singular fiber of type $I^*_{0_2}$, $I^*_{0_4}$ or $I^*_{0_{6,0}}$, then it is contracted to Y_1*

These properties of the strictly semistable locus lead us to give the following definitions.

Definition 2.5.30. *A triplet (λ, μ, R) is strictly semistable if there is a relation $(\lambda_i, \mu_j) \in R$ with $\lambda_i = 2$ and $\mu_j \geq 3$ or $\lambda_i \geq 2$ and $\mu_j = 3$.*
A triplet (λ, μ, R) is said unstable if there is a relation $(\lambda_i, \mu_j) \in R$ with $\lambda_i > 2$ and $\mu_j > 3$.

Proposition 2.5.31. *If (λ, μ, R) is a strictly semistable triplet, then the stratum $E_{\lambda,\mu,R}$ is contracted by $\overline{\pi}_1$ to the line Y. Moreover, if there is a relation $(2, L)$ with $L > 3$, $E_{\lambda,\mu,R}$ is contracted to Y_1 and if there is a relation $(K, 3)$ with $K > 2$, $E_{\lambda,\mu,R}$ is contracted to Y_0.*

Proof. $E_{\lambda,\mu,R}$ contains a fiber of Kodaira type I_0^*, then by the Theorem 2.5.27 it is contracted to Y. Proposition 2.5.29 implies the second statement. □

Remark 2.5.32 (The fibers I_k and I_k^*)**.** *We want to describe a strategy to stratify the moduli space of rational elliptic surfaces with a section in terms of their singular fibers, without restrictions on the type of fibers: this time we are involving the fibers I_k and I_k^*. The method we present exploits the stratification of the projective spaces, seen as spaces parametrizing polynomials, in terms of the coincident root loci, that we presented in Chapter 1. In particular, we use the stratification of $\mathbb{P}^{12}$, that will be the space of the discriminants of the fibrations. We actually consider the hypersurface of $\mathbb{P}^{12}$ given by polynomials that are sum of the square of a degree 6 polynomial and the cube of a degree 4 polynomial.*

The stratification of $\mathbb{P}^{12}$ is not sufficient for our purpose for trivial reasons. For instance, consider the partition $(2, 1^{10})$ of 12: the coincident root locus $X_{(2,1^{10})} \subset \mathbb{P}^{12}$ parametrizes polynomials with a double root. Choose a polynomial $D \in X_{(2,1^{10})}$ and assume that D can be written as a sum of the square of a degree 4 polynomial A and the cube of a degree 6 polynomial B: a priori, it is not possible to understand if the double root corresponds to a common root of A and B or not. In other words, given the discriminant D, we are not able to understand if the fiber over the double root of D is of type I_2 or II. In order to separate the cases and refine the stratification, consider the morphism ψ:

$$\psi : E_1 \longrightarrow \mathbb{P}^{12}$$
$$(A, B) \longmapsto D = 4A^3 + 27B^2.$$

The "pullback of the stratification" of the intersection between the coincident root loci of $\mathbb{P}^{12}$ and the hypersurface of the polynomials that are sum of a cube and a square is actually the stratification we are looking for. Coming back to the previous example, we have that $\psi^{-1}(X_{1^{10},2})$ is the union of the two strata E_{I_2} and E_{II} parametrizing fibrations with a I_2-fiber and II-fiber, respectively. In general, fixed a partition λ of 12 such that the intersection between the image of ψ and X_λ is nonempty, through ψ we separate the multiplicative fibers from the additive ones.

From the side of $\mathbb{P}^{12}$ we know very well the strata: for every λ, we know the dimension of X_λ, its degree, its singular locus and we know all the intersections between the strata. This method then completely solves the problem in theory. However, obtaining an explicit solution is computationally too hard. $X_{(1^{10},2)}$ is defined in $\mathbb{P}^{12}$ by a degree 22 polynomial in 13 variables, and the other coincident root loci are huger (for a treatment of the ideals defining the coincident root loci see [10]). Moreover, the hypersurface representing the discriminants of rational elliptic surfaces has degree 3762 (see [Theorem 6.2][55]). Unfortunately, there are no softwares able to give us the equations defining these loci and even less so their preimages under ψ.

Chapter 3

Enriques surfaces of base change type

In Section 3.1 we review and introduce some properties of the elliptic curves and of the rational elliptic surfaces: in particular, after recalling that every rational elliptic surface arises as the blow-up of the plane at nine points that are base points of a pencil of cubic curves, we give the notion of origin cutting linear system, which will be crucial for the rest of the treatment, and lastly we recall what a base change for an elliptic surface is. As an application, we compute the linear classes of the torsion multisections of a general rational elliptic surface.

In Section 3.2 we present the main features of Enriques surfaces of base change type. We firstly describe their construction due to Hulek and Schütt, then we will focus on the rational bisections they have and lastly we show their connection with the origin cutting bisections of some rational elliptic surfaces involved in their construction. Part of the proofs have been performed by Hulek and Schütt: in order to make the reader familiar with the geometric ideas and the notations, we report and sometimes extend them.

Section 3.3 collects most of the original results of the chapter. First of all, we introduce the notion of Severi variety of curves on surfaces and we give the state of the art in the literature, particularly focusing on the case of Enriques surfaces. Then, we relate the existence of the Enriques surfaces of base change type to the nonemptiness of some Severi varieties of some rational surfaces (for instance the plane, the rational elliptic surfaces and the Hirzebruch surfaces). Once introduced the logarithmic Severi varieties, we prove that the above mentioned bisections are actually nodal. After that, we investigate the geometry of the Enriques surfaces of base change type: we find some other particular Severi varieties on them and we focus on the genus 1 pencils of the K3 cover of these surfaces.

In Section 3.5 we show that the rational bisections on the Enriques surfaces of base change type deform to rational curves on the very general Enriques surface.

3.1 Elliptic curves and elliptic surfaces

We briefly recall the notions and the main properties of the elliptic curves and some properties of the elliptic surfaces we did not point out in Chapter 2. As general references the reader might consult [52], [44] and [50]

3.1.1 Elliptic curves

We recall the Definition of a smooth elliptic curve.

Definition 3.1.1. *An elliptic curve (E, O) is a smooth projective curve E of genus one, together with a chosen point O called its origin.*

It is a very well-known fact that every elliptic curve has a group law, that we denote by $\boxplus$, with origin O. We saw in Chapter 2 that every smooth elliptic curve can be embedded in the projective plane as a smooth cubic curve: the next classical result describes how the group law works in this context.

Let (E, O) be a smooth plane cubic: we call P_O the third intersection point between E and the tangent line to E at O.

Proposition 3.1.2 (Group law for cubic curves)**.** *Let D_1 and D_2 be divisors in* $\mathrm{Pic}(\mathbb{P}^2)$ *such that $D_1 \sim D_2$, that*

$$D_{1_{|E}} = a_1 Q_1 + \cdots + a_k Q_k$$

and that

$$D_{2_{|E}} = b_1 R_1 + \cdots + b_m R_m$$

with the a_i's and the b_j's integers. Then

$$a_1 Q_1 \boxplus \cdots \boxplus a_k Q_k = b_1 R_1 \boxplus \cdots \boxplus b_m R_m.$$

In particular, if D is a degree d curve in $\mathbb{P}^2$ intersecting E in $3d$ (not necessarily distinct) points $Q_1, \ldots, Q_{3d}$, then

$$Q_1 \boxplus \cdots \boxplus Q_{3d} = dP_O.$$

If the origin O is an inflection point for the cubic E, then the sum of the intersection points between E and any curve is O.

Definition 3.1.3. *An n-torsion point of* (E, O) *is a point* $Q \in E$ *such that* $Q^{\boxplus n} = O$.

Lemma 3.1.4. *Every smooth elliptic curve has* $n^2 - 1$ *nontrivial n-torsion points.*

Every elliptic curve admits a natural involution (-1), that acts by interchanging opposite points with respect to the origin O. The fixed locus of (-1) is composed by O and the three nontrivial 2-torsion points.

The main object of the chapter, namely the Enriques surfaces of base change type, are constructed starting from a rational elliptic surface: in the previous chapters we focused on their singular fibers and their behaviour in families, while in the next section we are going to investigate their geometry.

3.1.2 Rational elliptic surfaces

As we stated in Lemma 2.1.7, a relatively minimal elliptic surface $S \to \mathbb{P}^1$ with a section E and fundamental line bundle $\mathcal{O}_{\mathbb{P}^1}(N)$ is rational if and only if $N = 1$.

Example 3.1.5. *Let* C_1 *and* C_2 *be two smooth cubic curves in* $\mathbb{P}^2$ *and consider the pencil of cubic curves generated by* C_1 *and* C_2. *It has nine base points counted with multiplicity, corresponding to the nine intersection points between* C_1 *and* C_2. *Let* $e : S = \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ *be the blow-up of* $\mathbb{P}^2$ *at the base points of the pencil of cubics. Then the pullback of the pencil is base point free and induces a morphism* $f : S \to \mathbb{P}^1$. *A general fiber of* f *is the strict transform of a general member of the pencil of cubics, which is a smooth elliptic curve. Then* $f : S \to \mathbb{P}^1$ *is a rational elliptic surface: the section E can be chosen to be the exceptional divisor of the last blow-up of* $S \to \mathbb{P}^2$. *Since E is a* (-1)*-curve in S, the fundamental line bundle of* $S \to \mathbb{P}^1$ *is* $\mathcal{O}_{\mathbb{P}^1}(1)$

The next Lemma states that every rational elliptic surface arises in this way.

Lemma 3.1.6. *Let* $f : S \to \mathbb{P}^1$ *be a relatively minimal rational elliptic surface with a section. Then X is the 9-fold blow-up of the plane* $\mathbb{P}^2$ *at the base points* $P_1, \ldots, P_9$ *of a pencil of generically smooth cubic curves which induces the fibration* f.

Proof. See [44, Lemma IV.1.2]. □

In Chapter 2 we deeply investigated the configuration of singular fibers that a rational elliptic surface can have. From now on, unless differently specified, we will consider rational elliptic surfaces with twelve nodal curves as singular fibers.

Definition 3.1.7. *We say that a rational elliptic surface is general if it has twelve nodal curves as singular fibers.*

We call $E_1, \dots, E_9$ the exceptional divisors over the points $P_1, \dots, P_9$. With this notation, the Picard group of S is

$$\operatorname{Pic}(S) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_9.$$

Let $X \to C$ be an elliptic surface with a chosen section s_0. Then the set of sections is an abelian group with the group addition defined fiber by fiber.

Definition 3.1.8. *The group of the sections of $X \to C$ is called Mordell-Weil group of the elliptic surface, denoted by $\mathrm{MW}(X \to C)$ or simply $\mathrm{MW}(X)$ if the surface has only one elliptic fibration or if it is clear to what fibration we are referring. The chosen section s_0, which is the zero element of $\mathrm{MW}(X)$, is called the zero-section.*

If $S \cong \mathrm{Bl}_{P_1,\dots,P_9}\mathbb{P}^2$ is a general rational elliptic surface, we choose the last exceptional divisor E_9 to be the zero-section of the fibration. This in particular means that for every t in the base $\mathbb{P}^1$, the origin of the fiber F_t is its intersection with E_9. Let us call O_t this point and A_t the third intersection point between F_t and the tangent line to F_t at O_t.

With the previous notations, the Mordell-Weil group of S is

$$\mathrm{MW}(S) \cong \mathbb{Z}^8,$$

and it is generated by the exceptional divisors $E_1, \dots, E_8$. The neutral element is the zero-section E_9.

Remark 3.1.9. *Every rational elliptic surface $S \cong \mathrm{Bl}_{P_1,\dots,P_9}\mathbb{P}^2$ carries a natural involution $(-1) \in \mathrm{Aut}(S)$, that acts fiber by fiber by interchanging opposite points with respect* *to the group law with origin O_t. This involution is known in the literature as the Bertini involution. As pointed out in Section 3.1, for every $t \in \mathbb{P}^1$ such that F_t is smooth, the points fixed by (-1) are O_t and the three nontrivial 2-torsion points of F_t. We will show that the fixed locus of (-1) is the union of the zero-section E_9 and a trisection parametrizing the nontrivial 2-torsion points of any smooth fiber.*

We give the definition of torsion multisection for an elliptic surface $X \to C$ with a given section s_0 that we choose to be the zero-section of the fibration.

Definition 3.1.10 (torsion multisection)**.** *Let $X[m]$ be the closure of the locus in X of points $P_t \in X_t$, with X_t smooth elliptic fiber, such that $P_t^{\boxplus m} = 0_t$.*
We define $X[m]_0 := X[m] - s_0$ to be the m-torsion multisection of X.

It is clear that $X[m]_0$ is an (m^2-1)-section for X.

Remark 3.1.11. *We want to point out that $X[m]_0$ does not intersect the zero-section s_0. The proof is essentially performed by Miranda in [44, Proposition VII.3.2]: the author proves that if a torsion section meets the zero-section in an elliptic fibration, then the two section coincides; the identical argument shows that if a torsion multisection meets the zero-section, then the multisection has the zero-section as irreducible component.*

From now on, the Definitions and the results are original. We introduce the notion of origin cutting linear systems for a general rational elliptic surface S.

Definition 3.1.12. *Let $\mathcal{L} \in \mathrm{Pic}(S)$ be a divisor such that*

$$\mathcal{L}_{|F_t} = \sum_{i=1}^{r} a_{i,t} Q_{i,t}$$

with $a_{i,t} \in \mathbb{Z}$ for every i and for every $t \in \mathbb{P}^1$.
We say that $\mathcal{L}$ is origin cutting if

$$Q_{1,t}^{\boxplus a_{1,t}} \boxplus \cdots \boxplus Q_{r,t}^{\boxplus a_{r,t}} = O_t \text{ for every } t \in \mathbb{P}^1.$$

If $\mathcal{L}$ is effective and such that $\mathcal{L} \cdot F_t = k$, we sometimes refer to it as an origin cutting k-section.

Remark 3.1.13. *By Proposition 3.1.2, for a divisor to be origin cutting only depends on its linear class. For this reason, we can extend the notion of origin cutting divisor to the linear systems. In other words, the origin cutting linear systems consist of divisors cutting each curve of the elliptic pencil in points whose sum in the group law is the origin.*

Lemma 3.1.14. *The torsion multisections are origin cutting.*

Proof. The sum of the $m^2 - 1$ nontrivial m-torsion points of an elliptic curve is the origin of the group law. □

The next Lemma states one of the main properties of these systems.

Lemma 3.1.15. *If an origin cutting k-section B of a general rational elliptic surface $S \cong \mathrm{Bl}_{P_1,\dots,P_9} \mathbb{P}^2$ has a k-ple point Q, then either $Q \in S[k]_0$ or $Q \in E_9$.*

Proof. If $Q \in B$ is k-ple, then $Q^{\boxplus k} = 0$ by definition of origin cutting divisor. □

Remark 3.1.16. *It is obvious that an origin cutting k-section B cannot have a $(k+1)$-ple point. In particular, this implies that an origin cutting bisection can just have double points as singularities.*

The next Proposition describes more precisely the origin cutting linear systems.

Proposition 3.1.17. *Let $S \cong \mathrm{Bl}_{\{P_1,\dots,P_9\}} \mathbb{P}^2$ be a general rational elliptic surface and let $\mathcal{L} \in \mathrm{Pic}(S)$ be an effective divisor. Then, $\mathcal{L}$ is an origin cutting k-section (without E_9 as irreducible component) if and only if it is of the form*

$$\mathcal{L} \sim 3(c+k)L - (c+k)E_1 - \cdots - (c+k)E_8 - cE_9,$$

for some $c \in \mathbb{Z}_+$.

Proof. We have

$$\mathcal{L} \sim aL - b_1E_1 - \cdots - b_8E_8 - cE_9$$

for some integers a, b_i, with $i = 1, \ldots, 8$, and c. Since $\mathcal{L}$ is origin cutting, we have that for every $t \in \mathbb{P}^1$,

$$\mathcal{L}_{|F_t} = Q_{1,t} + \cdots + Q_{k,t}$$

in such a way that

$$Q_{1,t} \boxplus \cdots \boxplus Q_{k,t} = O_t.$$

If we call $\overline{\mathcal{L}}$ and $\overline{F}$ the pushforwards of $\mathcal{L}$ and F under $S \cong \mathrm{Bl}_{\{P_1,\ldots,P_9\}} \mathbb{P}^2 \to \mathbb{P}^2$, we have that

$$\overline{\mathcal{L}}_{|\overline{F}_t} = b_1P_1 + \cdots + b_8P_8 + cP_9 + Q_{1,t} + \cdots + Q_{k,t}.$$

But $\overline{\mathcal{L}}$ has degree a, then

$$b_1P_1 \boxplus \cdots \boxplus b_8P_8 \boxplus cP_9 \boxplus Q_{1,t} \boxplus \cdots \boxplus Q_{k,t} = b_1P_1 \boxplus \cdots \boxplus b_8P_8 = aA_t \text{ for every } t.$$

Furthermore, for every t, by Proposition 3.1.2

$$P_1 \boxplus \cdots \boxplus P_8 = 3A_t:$$

indeed, $P_1, \ldots, P_9$ are the base points of the pencil of cubics and by the choice of E_9 as zero-section, we have that the origin of $\overline{F}_t$ is P_9. Now, the equality $b_1P_1 \boxplus \cdots \boxplus b_8P_8 = aA_t$ implies that $3b_1P_1 \boxplus \cdots \boxplus 3b_8P_8 = 3aA_t$. Moreover, by Proposition 3.1.2, we have $aP_1 \boxplus \cdots \boxplus aP_8 = 3aA_t$, from which

$$(3b_1 - a)P_1 \boxplus \cdots \boxplus (3b_8 - a)P_8 = P_9 = O_t \text{ for every } t,$$

or, equivalently,

$$(3b_1 - a)E_1 \boxplus \cdots \boxplus (3b_8 - a)E_8 = E_9.$$

Since $\mathrm{MW}(S)$ is a free abelian group generated by $E_1, \dots, E_8$ with neutral element E_9, the latter equality implies that $3b_i = a$ for every $i = 1, \dots, 8$. We deduce

$$b_1 = \cdots = b_8 =: b \text{ and } a = 3b.$$

Lastly, since $\mathcal{L}$ is a k-section, we have that

$$3a - 8b - c = b - c = k,$$

so that

$$b = c + k$$

and this completes the proof.

□

As an application, we compute the linear class of the torsion multisection $S[n]_0$ of a general rational elliptic surface $S \cong \mathrm{Bl}_{\{P_1,\dots,P_9\}} \mathbb{P}^2$ for every $n \in \mathbb{Z}_+$.

Proposition 3.1.18. *For every $n \in \mathbb{Z}_+$, the n-torsion multisection is an (n^2-1)-section of the form*

$$S[n]_0 \sim 3(n^2-1)L - (n^2-1)E_1 - \dots - (n^2-1)E_8.$$

Proof. Lemma 3.1.14 ensures that the torsion multisections are origin cutting. Then, for every $n \in \mathbb{Z}_+$, we have that $S[n]_0$ is of the form

$$S[n]_0 \sim 3(n^2+c-1)L - (n^2+c-1)E_1 - \dots - (n^2+c-1)E_8 - cE_9$$

for some $c \in \mathbb{Z}_+$. Moreover, in Remark 3.1.11 we pointed out that the torsion multisections do not intersect the zero-section, so that $c = 0$. □

Remark 3.1.19. *In particular, Proposition 3.1.18 implies that the 2-torsion trisection $S[2]_0$ is*

$$S[2]_0 \sim 9L - 3E_1 - \cdots - 3E_8,$$

as showed by using other methods by Vakil in [55]. Notice that the union of $S[2]_0$ and E_9 is the fixed locus of the Bertini involution (-1).

We now discuss an example relevant to the rest of the treatment.

Example 3.1.20 (origin cutting bisections)**.** *For $k = 2$, the form of an origin cutting bisection $\mathcal{B}_m$ such that $\mathcal{B}_m \cdot E_9 = 2m$ for some $m \in \mathbb{Z}_+$ is*

$$\mathcal{B}_m \sim 6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9.$$

We will be interested in rational members of these bisections: by Proposition 3.1.15, the double points of the bisections belong either to $S[2]_0$ or to E_9.

Lemma 3.1.21. *The origin cutting bisections are invariant with respect to the involution (-1).*

Proof. $\mathcal{B}_m$ cuts every fiber F_t in two points $Q_{1,t}$ and $Q_{2,t}$, in such a way $Q_{1,t} \boxplus Q_{2,t} = O_t$, or, equivalently, $Q_{1,t} = \boxminus Q_{2,t}$. This implies that for every origin cutting bisection $B \in |\mathcal{B}_m|$, we have $(-1)^*(B) = B$. $\square$

We describe the quotient of a general rational elliptic surface by the involution (-1). We call q the quotient map $q : S \to S/(-1)$. This result is classical: see for example [55, Proof of Proposition 3.2], [18, Section 4.4, p.408] or [27, Section 2].

Proposition 3.1.22. *The quotient $S/(-1)$ is isomorphic to the second Hirzebruch surface $\mathbb{F}_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$.*

In the last part of the section we recall what a base change for an elliptic surface is.

3.1.3 Base change of an elliptic surface

Let $f : S \to C$ denote an elliptic surface. In order to apply a base change, we only need another projective curve B mapping surjectively to C. Formally the base change of S from C to B then is defined as a fiber product:

$$\begin{array}{ccc} S \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow \\ S & \longrightarrow & C \end{array}$$

In practice, we simply pull-back the Weierstrass form (or in general the equation) of S over C via the morphism $B \to C$. Clearly this replaces smooth fibers by smooth fibers.
The effect of a base change on the singular fibers depends on the ramification of the morphism $B \to C$. Singular fibers at unramified points are replaced by a fixed number of copies in the base change (the number being the degree of the morphism). If there is ramification, we have to be more careful. Of course the vanishing orders of the polynomials of the Weierstrass form and of the discriminant are multiplied by the ramification index. This suffices to solve the base change problem for multiplicative fibers: a base change of ramification index d replaces a fibre of type I_n by a fibre of type I_{dn}. For additive fibers, however, the pull-back of the Weierstrass form might become non-minimal. For a deeper investigation of the singular fibers under base change, see [50].

3.2 Enriques surfaces of base change type

First of all, we recall the basics about K3 surfaces and Enriques surfaces. As general references the reader might consult [3], [18] or [36].

Definition 3.2.1. *A smooth projective surface X is called K3 surface if X is (algebraically) simply connected with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$.*
An Enriques surface Y is a quotient of a K3 surface X by a fixed point free involution τ. Such an involution is also called Enriques involution.

Every K3 surface X is such that $b_1(X) = q(X) = 0$ and $h^{2,0}(X) = p_g(X) = 1$. It is easy to deduce that $\chi(\mathcal{O}_X) = 2$ and that the topological Euler-Poincaré characteristic $e(X) = 24$.

In a K3 surface X algebraic equivalence is the same as numerical equivalence and the Nerón-Severi group $\mathrm{NS}(X)$ is equipped with the structure of an even integral lattice of signature $(1, \rho(X) - 1)$, where $\rho(X)$ is the Picard number of X. Also the cohomology group $H^2(X, \mathbb{Z})$ is equipped with the structure of a lattice which is even, integral, non-degenerate and unimodular of signature $(3, 19)$. By lattice theory

$$H^2(X, \mathbb{Z}) \cong 3U \oplus 2E_8(-1) = \Lambda$$

where U denotes the hyperbolic plane and E_8 is the unique positive-definite unimodular even lattice of rank 8. The lattice Λ is called the *K3 lattice*. The Néron-Severi group $\mathrm{NS}(X)$ embeds primitively into $H^2(X, \mathbb{Z})$ as a lattice.

On the other hand, let L be a lattice of rank $r \leq 20$ and signature $(1, r-1)$ admitting a primitive embedding into the K3 lattice Λ. Moduli theory ensures that K3 surfaces having Néron-Severi group isometric to L and containing an ample class form a moduli space of dimension $20 - r$. This moduli space is globally irreducible if and only if the primitive embedding $L \hookrightarrow \Lambda$ is unique up to isometries.

Every Enriques surface Y is also such that $b_1(Y) = q(Y) = 0$. It is not simply-connected; its fundamental group is not trivial:

$$\pi_1(Y) = \mathbb{Z}/2\mathbb{Z}.$$

The universal covering

$$g : X \to Y$$

of Y is a K3 surface. Hence we have $e(Y) = \frac{1}{2}e(X) = 12$ and $h^{2,0}(X) = p_g(X) = 0$, as well as $\rho(Y) = b_2(Y) = 10$. Unlike in the K3 case, algebraic and numerical equivalence of divisors do not coincide on an Enriques surface Y: there is two-torsion in $\mathrm{NS}(Y)$ represented by the canonical divisor K_Y. The quotient $\mathrm{NS}(Y)_f$ of $\mathrm{NS}(Y)$ by its torsion subgroup is an even unimodular lattice, which is isomorphic to the so-called *Enriques lattice*:

$$\mathrm{Num}(Y) = \mathrm{NS}(Y)_f \cong U \oplus E_8(-1).$$

Via pull-back under the universal covering, this lattice embeds primitively into $\mathrm{NS}(X)$. Here the intersection form is multiplied by two

$$U(2) \oplus E_8(-2) \hookrightarrow \mathrm{NS}(X).$$

As explained before, such K3 surfaces form a ten-dimensional moduli space that is in fact irreducible (the embedding of $U(2) \oplus E_8(-2)$ into Λ is unique up to isometries because the embedded Enriques lattice is 2-elementary).

Definition 3.2.2. *We say that an Enriques surface Y is Picard very general if its universal covering X is such that*

$$\mathrm{NS}(X) \cong U(2) \oplus E_8(-2)$$

Remark 3.2.3. *An Enriques surface Y is Picard very general if and only if the Picard rank of its universal covering X is equal to* 10*.*

It is a special feature of K3 surfaces and Enriques surfaces that a single surface may admit more than one genus 1 pencil. Exhibiting a genus 1 fibration on a K3 surface is equivalent to finding a connected divisor $D \neq 0$ of self-intersection $D^2 = 0$. Then D or $-D$ is effective by Riemann-Roch. After subtracting the base locus, the linear system of the resulting effective divisor gives a genus 1 pencil.

The Enriques lattice contains the hyperbolic plane; in particular, there is a divisor $D \in \mathrm{NS}(Y)$ with $D^2 = 0$. It follows that either $\pm D$ or $\pm 2D$ induces a genus 1 pencil on Y. Here the factor two comes into play since every genus 1 pencil on an Enriques surface has exactly two fibers of multiplicity two, called *half-fibers*. The canonical divisor can be represented as the difference of the supports of the half-fibers of a genus 1 pencil: if F is a genus 1 pencil of Y and

$$2E_1 = F \text{ and } 2E_2 = F, \text{ then } K_Y \sim E_1 - E_2.$$

We give the definition of ϕ-invariant for a nef line bundle of an Enriques surface, introduced by Cossec in [16].

Definition 3.2.4. *Let $H \in \mathrm{Pic}(Y)$ be a nef divisor of Y. Then, the ϕ-invariant of H is defined to be*

$$\phi(H) := \min\{E \cdot H | E^2 = 0, E > 0\} \in \mathbb{Z}.$$

Here E is a half-fiber of a genus 1 pencil of Y, whence 2ϕ is the minimum of the intersection between H and a genus 1 pencil.

In this work we mainly deal with line bundles with ϕ-invariant equal to 1: the next Theorem due to Cossec and Dolgachev (see [Theorem 2.4.14][18]) recalls the main property of such divisors.

Proposition 3.2.5. *Let $L \in \mathrm{NS}(Y)$ be a big and nef linear system without fixed components. Then, the following are equivalent:*

(i) $\phi(L) = 1$*,*

(ii) $|L|$ *has two base points.*

If a line bundle L is such that $\phi(L) = 1$, we have that $L \cdot E = 1$ for a half-fiber of a genus 1 pencil $|F|$ on Y. Since $F \sim 2E$, we obtain that $L \cdot F = 2$ and therefore any member of the linear system $|L|$ is a bisection for the elliptic pencil $|F|$. Moreover, all the smooth members of $|L|$ are hyperelliptic curves, with the hyperelliptic series cut by the elliptic curves of $|F|$.

The next result due to Galati and Knutsen (see [29, Theorem 1.1]) states a necessary condition for the existence of a rational curve in the very general Enriques surface. Here very general just means that there exists a set that is the complement of a countable union of proper Zariski-closed subsets in the moduli space of Enriques surfaces satisfying the given conditions.

Theorem 3.2.6. *[Galati, Knutsen] Let Y be a very general Enriques surface. If $C \subset Y$ is an irreducible rational curve, then C is 2-divisible in* $\mathrm{Num}(Y)$.

As a consequence of this Theorem, if C is a rational curve on the very general Enriques surface, then $\phi(C) \neq 1$.

3.2.1 K3 surfaces of base change type

Let $S = \mathrm{Bl}_{\{P_1,\dots,P_9\}} \mathbb{P}^2$ denote a rational elliptic surface. We let now

$$g : \mathbb{P}^1 \to \mathbb{P}^1$$

be a morphism of degree two. Denote the ramification points by t_0 and t_∞.

Proposition 3.2.7. *The pull-back X of S via g is a K3 surface*

Proof. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d, "g"] \\ S \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

The pull-back $\tilde{E}$ of any section E of S is a section for the induced elliptic fibration on X and $\tilde{E}^2 = -2$. Hence X is an elliptic surface over $\mathbb{P}^1$ having a section with self-intersection -2 (or, equivalently, the degree of the fundamental line bundle of the Weierstrass model of X is 2) and then by Lemma 2.1.7 it is a K3 surface. □

With abuse of notation, we denote by g also the double cover $X \to S$ and we denote by $\tilde{E}_i$ the pull-backs of the exceptional divisors E_i of S. With this notation, $\tilde{E}_9$ is the zero-section for the induced elliptic fibration on X.

Moreover, we denote by S_t the fiber on S over a point $t \in \mathbb{P}^1$ and by X_t and X_{-t} the two components of its preimage on X. Since $S_t \cong X_t \cong X_{-t}$, if $Q_t \in S_t$, we denote the two points in its preimage $g^{-1}(Q_t)$ by $\tilde{Q}_t$ and $\tilde{Q}_{-t}$. Sometimes, we refer to the pair X_t and X_{-t} as *twin fibers*, to the pair $\tilde{Q}_t$ and $\tilde{Q}_{-t}$ as *twin points in twin fibers* and to the pair $\tilde{Q}_t$ and $\boxminus\tilde{Q}_{-t}$ as *opposite points in twin fibers* (with respect to $\tilde{E}_9$).

Let ι denote the deck transformation for g, i.e. $\iota \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g = g \circ \iota$. Then ι induces an automorphism of X that we shall also denote by ι. The quotient X/ι is exactly the rational elliptic surface S we started with.

Remark 3.2.8. *We obtain a ten-dimensional family of elliptic K3 surfaces: eight dimensions from the rational elliptic surfaces and two dimensions from the base change, given by the choice of the two ramification points of the corresponding double cover $\mathbb{P}^1 \to \mathbb{P}^1$.*

Definition 3.2.9. *We say that such a base change $g : X \to S$ is very general if S is general, S_{t_0} and S_{t_∞} are smooth elliptic curves and ι^* acts as the identity on $\mathrm{NS}(X)$.*

In this case we also say that X is base change very general as K3 surface of base change type.

Proposition 3.2.10. *A base change very general K3 surface X does not carry any Enriques involution.*

Proof. We have that $\mathrm{NS}(X) \cong U \oplus E_8(-2)$ (see [36, Section 3.2]). Twice the Enriques lattice $U(2) \oplus E_8(-2)$ does not embed primitively into $U \oplus E_8(-2)$: indeed, U cannot be realized as primitive sublattice of $U(2)$. Hence, X cannot admit an Enriques involution. □

3.2.2 Enriques surfaces of base change type

We saw that a very general K3 surface of base change type does not admit any Enriques involution. Hulek and Schütt in [36] impose a geometric condition on the base change $g : X \to S$ that allows to construct (a countable number of) families of K3 surfaces with Enriques involution.

In order to exhibit K3 surfaces with Enriques involution within our family of K3 surfaces of base change type, we need the following Lemma, that summarizes the discussion in [36, Section 3.3].

Lemma 3.2.11 (Hulek-Schütt)**.** *Let S be a general rational elliptic surface, let X be a K3 surface of base change type obtained as the double cover of S and let $g : X \to S$ the quotient map. Moreover, let R be a section for the elliptic fibration on X given by the pullback of the elliptic fibration on S. Then*

- *either R is invariant with respect to ι^*,*
- *or R is anti-invariant with respect to ι^* (meaning that $\iota^*(R) = (-1)^*(R)$,where (-1) indicates the involution on X acting fiber by fiber by interchanging opposite points with respect to the zero-section $\tilde{E}_9$).*

Moreover, in the former case, R is the pull-back of a section $E \in \mathrm{MW}(S)$ and it cuts twin points in twin fibers, while in the latter R cuts opposite points in twin fibers.

We denote by $\boxplus R \in \mathrm{Aut}(X)$ the automorphism of X given by the translation by R fiber by fiber. It acts as an automorphism on every smooth elliptic fiber and it is classical that it can be extended to the singular fibers (see, for example, [50, Section 7.6], or [17, Section 1.1]).

Proposition 3.2.12. *Let R be an anti-invariant section with respect to i^*. Then, the quotient $g : X \to S$ identifies R with its opposite section $\boxminus R$ with respect to $\tilde{E}_9$, as well as $R^{\boxplus k} := R \boxplus \cdots \boxplus R$ with $R^{\boxminus k} := \boxminus R \boxminus \cdots \boxminus R$.*

Proof. For every $t \in \mathbb{P}^1$, if R cuts a fiber X_t in a point $\tilde{Q}_t$, the opposite section $\boxminus R$ cuts X_t in the opposite point $\boxminus \tilde{Q}_t$. Moreover, R intersects the twin fiber X_{-t} in $\boxminus \tilde{Q}_{-t}$, while $\boxminus R$ cuts X_{-t} in $\tilde{Q}_{-t}$. To complete the proof, it is sufficient to notice that g identifies $\tilde{Q}_t$ and $\tilde{Q}_{-t}$ as well as $\boxminus \tilde{Q}_t$ and $\boxminus \tilde{Q}_{-t}$. In the same way one can prove that $R^{\boxplus k}$ is identified with $R^{\boxminus k}$. $\square$

Remark 3.2.13. *If there exists $R \in \mathrm{MW}(X \to \mathbb{P}^1)$ that is anti-invariant with respect to ι^*, then $X \to S$ is not very general in the sense of Definition 3.2.9. Indeed, by Proposition 3.2.12 we have $\iota^*(R) = \boxminus R \nsim R$, whence ι^* does not act as the identity on $\mathrm{NS}(X)$.*

Proposition 3.2.14 (Hulek-Schütt)**.** *Let $R \in \mathrm{MW}(X)$ be an anti-invariant section with respect to ι^* and consider the automorphism of X given by*

$$\tau := \iota \circ (\boxminus R).$$

Then, $\tau \in \mathrm{Aut}(X)$ is an involution and it is an Enriques involution if and only if R does not intersect $\tilde{E}_9$ along X_{t_0} and X_{t_∞}.

Proof. The automorphism τ acts on a point $x_t \in X_t$ in the following way:

$$\tau(x_t) = (\iota \circ \boxminus R)(x_t) = \iota(x_t \boxminus R_t) = (x_{-t} \boxplus R_{-t}) \in X_{-t}.$$

On the other hand,

$$\tau(x_{-t} \boxplus R_{-t}) = (\iota \circ \boxminus R)(x_{-t} \boxplus R_{-t}) = \iota(x_{-t}) = x_t,$$

from which we deduce that τ is an involution on X. We have to check whether τ has any fixed points on X. Since ι exchanges fibers while translation by P fixes them, we clearly have

$$\mathrm{Fix}(\tau) \subset \mathrm{Fix}(\iota) = \{X_{t_0}, X_{t_\infty}\}.$$

On the fixed fibers, say at t_0, the involution τ acts as

$$\tau(x) = x \boxplus R_{t_0}, \text{ where } R_{t_0} = R \cap X_{t_0}.$$

Hence deciding whether τ acts freely amounts to checking which points on the fixed fibers R specialises to. If the fixed fibers are smooth, there are fixed points (even fixed fibers) if and only if $R \cap \tilde{E}_9 \cap (X_{t_0} \cup X_{t_\infty}) \neq \emptyset$.
The case of non-smooth fibers requires some additional attention. For the complete proof, see [36, Section 3.4]. □

Definition 3.2.15. *We denote by $Y = X/\tau$ the Enriques surface obtained with the construction described in Theorem 3.2.14. We say that Y is an Enriques surfaces of base change type and we denote by f the quotient $X \to Y$.*

Remark 3.2.16. *The given elliptic fibration on X induces a genus 1 pencil on Y. Here the smooth fiber Y_t of Y at t is isomorphic to the fibers X_t and X_{-t} at $g^{-1}(t)$ as genus 1 curves or to the fiber of the rational elliptic surface S_t.*

Lemma 3.2.17. *[Hulek, Schütt] The sections $\tilde{E}_9$ and R of the specified elliptic fibration on X are identified under the quotient $f : X \to Y$ and give a rational bisection for the induced genus 1 fibration on Y.*

Proof. Let X_t and X_{-t} be two twin fibers. The involution τ acts on a point $x_t \in X_t$ in the following way:

$$\tau(x_t) = \iota \circ \boxminus R(x_t) = \iota(x_t \boxminus R_t) = x_{-t} \boxplus R_{-t}.$$

A point $0_t \in \tilde{E}_9$ is sent to $0_{-t} \boxplus R_{-t} = R_{-t}$ and a point $R_t \in R$ is sent to $R_{-t} \boxminus R_{-t} = 0_{-t}$, where in this latter case the sign is changed because R is ι^* anti-invariant, whence $\iota(R_t) = \boxminus R_{-t}$. Since R and $\tilde{E}_9$ are rational, then their image is. Finally, we have $f(R) \cdot Y_t = \frac{1}{2}(R + \tilde{E}_9)(X_t + X_{-t}) = \frac{1}{2}(4) = 2$, where the second equality holds since $\tilde{E}_9$ and R are sections for the elliptic pencil $|X_t|$ on X, so that the image of R and $\tilde{E}_9$ is a bisection for the pencil $|Y_t|$ on Y. □

The construction strongly depends on the choice of the ι^* anti-invariant section R. One could ask if such a section actually exists. The proof of their existence is performed by Hulek and Schütt in a strongly lattice-theoretical way (see [36, Section 3]) and we omit it. In particular, this Theorem collects their results in this context.

Theorem 3.2.18 (Hulek, Schütt)**.** *For every nonnegative integer $m \in \mathbb{Z}_+$, there exists a 9-dimensional family Σ_m of K3 surfaces of base change type such that, for every $X_m \in \Sigma_m$,*

$$\mathrm{NS}(X_m) \cong U \oplus E_8(-2) \oplus < -4(m+1) >.$$

Moreover, X_m covers an Enriques surface of base change type such that $R_m \cdot \tilde{E}_9 = 2m$, where $\tilde{E}_9$ is the zero-section of the elliptic fibration induced by the base change construction and R_m is an anti-invariant section with respect to the involution giving rise to the base change.

Recall that X_m is constructed as a double cover $g : X_m \to S$, where S is a rational elliptic surface, and therefore X_m inherits the structure of elliptic fibration given by the pullback of the elliptic fibration on S via g. Let us call F the class of a fiber in the Néron-Severi group $\mathrm{NS}(X_m)$. Recall that if $S \cong \mathrm{Bl}_{P_1,\dots,P_9}\mathbb{P}^2$, we denote by E_i, with $i = 1, \dots, 9$, the exceptional divisors over the points P_i and by $\tilde{E}_i$ their pullbacks under g. Finally, if we choose E_9 to be the zero-section for the elliptic fibration on S, the section $\tilde{E}_9$ will be the zero-section for the induced elliptic fibration on X_m. Here, the hyperbolic lattice U is generated by the class of a fiber F and the zero-section $\tilde{E}_9$ (more precisely, by F and $F + \tilde{E}_9$), the sublattice $E_8(-2)$ is generated, for example, by the classes $\tilde{E}_1 - \tilde{E}_2$, $\tilde{E}_2 - \tilde{E}_3$, $\tilde{E}_3 - \tilde{E}_4$, $\tilde{E}_4 - \tilde{E}_5$, $\tilde{E}_5 - \tilde{E}_6$, $\tilde{E}_6 - \tilde{E}_7$, $\tilde{E}_7 - \tilde{E}_8$ and $\tilde{E}_8 - \tilde{E}_1$, while the generator of $< -4(m+1) >$ is the class $R_m - \tilde{E}_9 - (2 + 2m)F$ (see [36, Section 3.7]).

Remark 3.2.19. *The group of sections* $\mathrm{MW}(X_m)$ *is generated by the pullbacks* $\tilde{E}_i$ *under the quotient* $g : X \to S$ *of the generators* E_i *of* $\mathrm{MW}(S)$, *for* $i = 1 \ldots, 8$, *and* R_m. *One could choose* $R_m^{\boxplus k}$ *(with k odd for reasons that we explain in Remark 3.2.26) to define the Enriques involution. To avoid confusion in the notations, we will choose the section* R_m *to be non-divisible in* $\mathrm{MW}(X_m)$.

Corollary 3.2.20. *The Enriques surfaces of base change type are not Picard very general.*

Proof. By Theorem 3.2.18, the Néron-Severi group of $X_m \in \Sigma_m$ is

$$\mathrm{NS}(X_m) \cong U \oplus E_8(-2) \oplus < -4(m+1) >,$$

and in particular $\rho(X_m) = 11$. □

Remark 3.2.21. *We obtained that* X_m *is not base change very general and that* Y_m *is not Picard very general. Theorem 3.2.18 motivates the respective definitions we gave: a countable set of codimension 1 families in the moduli of K3 surfaces of base change type X such that* i^* *does not act as the identity on* $\mathrm{NS}(X)$ *is the family of* Σ_m*'s; a countable set of codimension 1 families in the moduli of K3 surfaces X with an Enriques involution such that* $\rho(X) \neq 10$ *is again the family of the* Σ_m*'s.*

We will focus on Σ_m and on the geometry of its members in the next sections.

Definition 3.2.22. *We denote by* $B_{Y,m} := f(R_m) = f(\tilde{E}_9)$ *the induced rational bisection on* Y_m *and we say that* $B_{Y,m}$ *is an m-special curve for* Y_m. *Sometimes, we shall say that* Y_m *is an m-special Enriques surface and that the induced genus 1 pencil on* Y_m *is an m-special genus 1 pencil.*

Remark 3.2.23. $B_{Y,m}$ *has arithmetic genus m. Indeed,*

$$B_{Y,m}^2 = \tfrac{1}{2}(R_m + \tilde{E}_9)^2 = \tfrac{1}{2}(R_m^2 + \tilde{E}_9^2 + 2R_m \cdot \tilde{E}_9) = \tfrac{1}{2}(-2 - 2 + 4m) = 2m - 2,$$

where the third equality holds since R_m and $\tilde{E}_9$ are (-2)-curves with $R_m \cdot \tilde{E}_9 = 2m$, so that

$$p_a(B_{Y,m}) = \tfrac{1}{2}B_{Y,m}^2 - 1 = m.$$

The construction due to Hulek and Schütt is universal in the following sense:

Proposition 3.2.24 (Hulek, Schütt)**.** *Let Y be an Enriques surface with a genus 1 pencil having a rational bisection which splits into sections for the induced fibration on the universal K3 cover X. Then X and Y arise through the base change construction described above.*

Proof. [36, Proposition 3.1]. □

The next Proposition links the rational bisection $B_{Y,m}$ to the rational elliptic surface S we started with to construct Y_m.

Proposition 3.2.25. *Let $S \cong \mathrm{Bl}_{\{P_1,\dots,P_9\}}\mathbb{P}^2$ be a general rational elliptic surface, and let X_m and Y_m be a K3 surface and an Enriques surface obtained by the base change construction. Let then $g : X_m \to S$ and $f : X_m \to Y_m$ denote the corresponding quotients. Finally, let $B_{Y,m}$ be the m-special curve of Y_m and let R_m and $\tilde{E}_9$ be the two components of its preimage under f. Then, $B_{S,m} := g(R_m)$ is a rational bisection for the elliptic pencil on S. Moreover, $B_{S,m} \sim \mathcal{B}_m := 6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9$ is an origin cutting bisection.*

Proof. The section R_m is anti-invariant with respect to ι^*, then it cuts opposite points $\tilde{Q}_t$ and $\boxminus\tilde{Q}_{-t}$ in twin fibers X_t and X_{-t} with respect to $\tilde{E}_9$. Hence, $B_{S,m}$ cuts the fiber $S_t = g(X_t) = g(X_{-t})$ in the opposite points Q_t and $\boxminus Q_t$ with respect to E_9 and then it is an origin cutting bisection. Moreover, since R_m is rational, then $B_{S,m}$ is. □

In the following Remark, we give a geometrical interpretation of the phenomenon.

Remark 3.2.26. *By Proposition 3.2.12, the bisection $B_{S,m} \subset S$ splits in R_m and $\boxminus R_m$ in X_m. This means that $B_{S,m}$ is tangent to the branch locus of g, that is the union of S_{t_0} and S_{t_∞}. Geometrically, since $B_{S,m}$ is a bisection for the elliptic pencil, it carries a $2:1$ map over $\mathbb{P}^1$. By the Riemann-Hurwitz formula, it has two ramification points, that correspond exactly to the two fixed fibers S_{t_0} and S_{t_∞}, to whom $B_{S,m}$ is tangent.*

Since $B_{S,m}$ is origin cutting, it has to be tangent to S_{t_0} and S_{t_∞} along the 2-torsion trisection $S[2]_0$ or along E_9. It is easy to see that it is tangent to the fixed locus along $S[2]_0$: in fact, by Proposition 3.2.14, to produce an Enriques involution τ, R_m cannot intersect E_9 along the fixed locus. For this reason in Remark 3.2.19 we claimed that one could choose also $R_m^{\boxplus k}$ as section anti-invariant with respect to ι^* to define an Enriques involution, but with k odd: if k is even, then $R_m^{\boxplus k}$ intersects $\tilde{E}_9$ along the fixed locus.

The previous Remark leads us to state the following Proposition, that is the converse of Proposition 3.2.25.

Proposition 3.2.27. *Let $S \cong \mathrm{Bl}_{\{P_1,\dots,P_9\}}\mathbb{P}^2$ be a general rational elliptic surface and let $B_{S,m} \sim \mathcal{B}_m := 6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9$ be a rational origin cutting bisection, tangent to two fibers S_{t_0} and S_{t_∞} along the 2-torsion trisection $S[2]_0$. Consider the double covering $g : X_m \to S$ ramified over $S_{t_0} + S_{t_\infty}$. Then, X_m belongs to Σ_m and $B_{S,m}$ splits in two (-2)-curves R_m and $\boxminus R_m$, that are opposite with respect to $\tilde{E}_9 = g^*(E_9)$ and anti-invariant with respect to the involution ι on X_m giving rise to the double covering g. Moreover, we have $R_m \cdot \tilde{E}_9 = 2m$ and if we denote by $f : X_m \to Y_m$ the Enriques quotient given by the involution $\tau = \iota \circ \boxminus R_m$, we have that $\tilde{E}_9$ is identified with R_m by f.*

Proof. The origin cutting bisection $B_{S,m}$ is tangent to the branch locus $S_{t_0} + S_{t_\infty}$ of g, hence it splits in two curves R_m and R'_m on X_m. Denoting by S_t a fiber for the elliptic fibration on S and by X_t and X_{-t} the two irreducible components of its preimage $g^{-1}(S_t)$, we have

$$B_{S,m} \cdot S_t = 2, \text{ from which } (R_m + R'_m) \cdot (X_t + X_{-t}) = 4.$$

Since X_t and X_{-t} are members of an elliptic pencil on the K3 surface X_m, every factor of the intersection product $(R_m + R'_m) \cdot (X_t + X_{-t})$ is greater or equal to 1: otherwise, R_m or R'_m would be fibers of the elliptic pencil $|X_t|$ on X_m. Hence, $R_m \cdot X_t = R'_m \cdot X_t = 1$ and whence R_m and R'_m are sections for $|X_t|$, so they are smooth (-2)-curves. Moreover, since $B_{S,m}$ is origin cutting, R_m and R'_m cut opposite points with respect to $\tilde{E}_9$ in every fiber. In other words, $R'_m = \boxminus R_m$. This also proves that R_m (as well as R'_m) is anti-invariant with respect to ι: indeed, $\iota^*(R_m) = \boxminus R_m = (-1)^*(R_m)$, where (-1) indicates the involution on X_m that acts fiberwise by interchanging opposite points with respect to $\tilde{E}_9$. Since $B_{S,m} \subset S$ is tangent to the branch locus $S_{t_0} + S_{t_\infty}$ along the 2-torsion trisection $S[2]_0$, we have that R_m intersects the ramification locus $X_{t_0} + X_{t_\infty}$ away from $\tilde{E}_9$ and then, by Theorem 3.2.14, the involution τ is an Enriques involution. Finally, since $B_{S,m} \cdot E_9 = 2m$, we have $(R_m + \boxminus R_m) \cdot \tilde{E}_9 = 4m$, from which $R_m \cdot \tilde{E}_9 = 2m$. □

Corollary 3.2.28. *The set of irreducible rational curves in $|\mathcal{B}_m|$ intersecting E_9 in simple nodes and tacnodes is nonempty.*

Proof. First of all, since $g^{-1}(B_{S,m}) = R_m + \boxminus R_m$, we have that morphism g restricted to R_m is birational. Now, R_m and $\boxminus R_m$ cut $\tilde{E}_9$ in the same points: in fact, R_m and $\boxminus R_m$ cut a fiber X_t in opposite points, and if R_m meets X_t along $\tilde{E}_9$, the intersection point is the origin of the group law of the fiber and whence it coincides with its opposite point. This corresponds to a double point of $B_{S,m}$ along E_9. Since $R_m \cdot \tilde{E}_9 = 2m$, we have that $B_{S,m} \cdot E_9 = 2m$ and that this latter intersection is composed by (possibly not ordinary) double points. The singularities of $B_{S,m}$ along E_9 can just be (possibly not ordinary) nodes but not cusps. Otherwise, it could not split. This means that the possible singularities are simple nodes and tacnodes. □

Notice that if $B_{S,m}$ admitted a tacnode along E_9 in the fiber S_t on S, then the curves R_m (as well as $\boxminus R_m$) and $\tilde{E}_9$ would be tangent in X_m at the two preimages of the tacnode in the fibers X_t and X_{-t}. Since the m-special curve $B_{Y,m}$ is the image of R_m and $\tilde{E}_9$ under the Enriques quotient f and since f is étale and identifies the fibers X_t and X_{-t}, also $B_{Y,m}$ would admit a tacnode in the fiber $Y_t = f(X_t) = f(X_{-t})$ on Y_m at the image of the two points at which R_m and $\tilde{E}_9$ are tangent. One of the aims of the next section is to prove that generically this does not happen, meaning that generically the m-special curves are nodal.

Remark 3.2.29 (Description of the families)**.** *The ϕ invariant of every m-special curve $B_{Y,m}$ is equal to 1: indeed, as proved in Lemma 3.2.17, $B_{Y,m}$ is a bisection for an elliptic pencil $|F_m|$ on Y_m. Now, consider a half-fiber E_m of the pencil $|F_m|$: since $B_{Y,m} \cdot F_m = 2$ and $2E_m \sim F_m$, we have $B_{Y,m} \cdot E_m = 1$.*

There is a subcase of this setting that has been investigated extensively before, namely nodal Enriques surfaces. In general, an Enriques surface is called nodal if it contains a nodal curve, i.e. a curve of self-intersection -2, thus necessarily rational and smooth. On the K3 cover, such a curve splits into two disjoint smooth rational curves, again with self-intersection -2. It was proved by Cossec in [16] that the property of being nodal always translates to genus 1 fibrations. Namely for an Enriques surface Y, it is equivalent to contain a smooth (-2)-curve and to admit a special genus 1 fibration, i.e. a genus 1 fibration with a smooth (-2)-curve curve as bisection. For the case $P \cap E_9 = \emptyset$, whence $m = 0$, the construction due to Hulek and Schütt thus leads exactly to nodal Enriques surfaces.

Σ_1 parametrizes K3 surfaces covering Enriques surfaces with a 1-special curve B_Y, that is a rational curve of arithmetic genus 1. In other words, $B_Y^2 = 0$ and then either $|B_Y|$ or $|2B_Y|$ is a genus 1 pencil on Y_1, with B_Y as one of its singular fibers. As usual $f : X_1 \to Y_1$ indicates the Enriques quotient. Since B_Y splits in X_1 in two (-2)-curves meeting at two points, it is 2:1 covered by a member of $|f^*(B_Y)|$ and thus it is a half-fiber of the genus 1 pencil $|2B_Y|$. The Y_1's are precisely the Enriques surfaces having a genus 1 pencil with a

nodal half-fiber: it is not surprising that they live in a subfamily of codimension 1 in the moduli space of the Enriques surfaces.

By Theorem 3.2.6, every rational curve in a very general Enriques surface is 2-divisible. For $m \geq 2$, we have that Σ_m parametrizes K3 surfaces covering Enriques surfaces Y_m having at least a linear system L with $p_a(L) = m$ and $\phi(L) = 1$, with a rational member. Thus, the peculiarity of Y_m is the existence of a not 2-divisible linear system of arithmetic genus m having a rational member.

We conclude the section with the following Lemma, stating that the general $X_m \in \Sigma_m$ covers an Enriques surface Y_m without (-2)-curves.

Lemma 3.2.30. *Let X_m be a general member of Σ_m, with $m \geq 1$. Then, Y_m is nonnodal, meaning that there are no smooth rational curves on Y_m.*

Proof. Let $X_m \in \Sigma_m$ be general. Σ_0 is the nine-dimensional irreducible family in the moduli space of K3 surfaces parametrizing K3 surfaces covering a nodal Enriques surface, while Σ_m is the irreducible nine-dimensional family parametrizing K3 surfaces covering an Enriques surface with an m-special curve. Suppose that Y_m admits a smooth rational curve: since Σ_m and Σ_0 are irreducible of the same dimension, they coincide in an open subset. In other words, the general member X_m of Σ_m belongs to Σ_0 and the general member X_0 of Σ_0 belongs to Σ_m. But since

$$\mathrm{NS}(X_m) \cong U \oplus E_8(-2) \oplus < -4(m+1) >$$

and

$$\mathrm{NS}(X_0) \cong U \oplus E_8(-2) \oplus < -4 >,$$

this is a contradiction unless $m = 0$. $\square$

3.3 Severi varieties

First of all, we recall the basics about Severi varieties of curves on surfaces. Roughly speaking, the Severi varieties parametrize curves with a prescribed number of nodes in a fixed linear system. Let S be a smooth complex projective surface and L a line bundle on S such that the complete linear system $|L|$ contains smooth, irreducible curves (such a line bundle, or linear system, is often called a Bertini system). Let

$$p := p_a(L) = \tfrac{1}{2}L \cdot (L + K_S) + 1$$

be the arithmetic genus of any curve in $|L|$.

Definition 3.3.1. *For any integer $0 \leq \delta \leq p$, consider the locally closed, functorially defined subscheme of $|L|$*

$$V_{|L|,\delta}(S) \text{ or simply } V_{|L|,\delta}$$

parametrizing irreducible curves in $|L|$ having only δ nodes as singularities: this is called the Severi variety of δ-nodal curves in $|L|$. We will let $g := p - \delta$ be the geometric genus of the curves in $V_{|L|,\delta}$.

We will also consider, for any given integer g such that $0 \leq g \leq p_a(L)$, the locally closed subscheme of $|L|$

$$V_g^{|L|}(S) \text{ or simply } V_g^{|L|}$$

whose geometric points parametrize reduced and irreducible curves C having geometric genus g, i.e. such that their normalizations have genus g. We shall call such a family the *equigeneric Severi variety* of genus g curves in $|L|$. When $\delta = p_a(L) - g$, we have $V_{|L|,\delta} \subset V_g^L$. It is well-known that, if $V_{|L|,\delta}$ is nonempty, then all of its irreducible components V have dimension $\dim(V) \geq \dim|L| - \delta$. If $V_{|L|,\delta}$ is smooth of dimension $\dim|L| - \delta$ at $[C]$ it is said to be *regular at* $[C]$. An irreducible component V of $V_{|L|,\delta}$ will be said to be *regular* if the condition of regularity is satisfied at any of its points, equivalently, if it is smooth of dimension $\dim|L| - \delta$.

Severi varieties were introduced by Severi in [51], where he proved that all Severi varieties of irreducible δ-nodal curves of degree d in $\mathbb{P}^2$ are nonempty and smooth of the expected dimension. Severi also claimed irreducibility of such varieties, but his proof contains a gap. The irreducibility was proved by Harris in [34].

Severi varieties on other surfaces have received much attention in recent years, especially in connection with enumerative formulas computing their degrees (see for example [4], [6] and [8]. Nonemptiness, smoothness, dimension and irreducibility for Severi varieties have been widely investigated on various rational surfaces (see, e.g., [30], [53] and [54], as well as K3, Enriques and abelian surfaces (see, e.g.,[11], [37], [38], [42], [5], [47], [13] and [14]).

In this Section, we contribute to the study of the Severi varieties of curves on Enriques surfaces and rational elliptic surfaces. We give the state of the art about the Severi varieties of curves on these surfaces. About the Enriques case, we refer to [13] and [14]. The next result states the main property of the Severi varieties on Enriques surfaces.

Let Y be an Enriques surfaces, X be its K3 cover, $f : X \to Y$ denote the quotient map and τ denote the Enriques involution. Let now V be an irreducible component of a Severi varieties of δ-nodal curves on Y. Ciliberto, Dedieu, Galati and Knutsen in [13, Proposition 1] prove that if V is regular, then the curves in V are covered by irreducible curves of X, while if V is nonregular, then each curve C of V splits in X in two curves C' and C''. Moreover, they show that if Y is very general in moduli, then C' and C'' are linearly equivalent. The precise statement is described in their following Proposition.

Proposition 3.3.2 (Ciliberto, Dedieu, Galati, Knutsen)**.** *Let L be a Bertini linear system, with $L^2 > 0$, on a smooth Enriques surface Y. Then the Severi variety $V_{|L|,p}(S)$ is smooth and every irreducible component $V \subseteq V_{|L|,\delta}(S)$ has either dimension $g-1$ or g; in the former case the component is regular. Furthermore, with the notation introduced above,*

(a) for any curve C in a $(g-1)$-dimensional irreducible component V, $f^{-1}(C)$ is irreducible;

(b) for any g-dimensional component V, there is a line bundle L' on X with $(L')^2 = 2(p-d)-2$ and $L' \cdot \tau^* L' = 2d$ for some integer d satisfying

$$\tfrac{p-1}{2} \leq d \leq \delta,$$

such that $f^*L = L' \otimes \tau^* L'$, and the curves parametrized by $V \subseteq V_{|L|,\delta}(S)$ are the birational images under f of the curves in $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by τ transversely (in $2d$ points). In other words, for any $[C] \in V$, we have $f^{-1}C = Y + \tau(Y)$, with $[Y] \in V_{|L'|,\delta-d}(X)$ and $[\tau(Y)] \in V_{|\tau^* L'|,\delta-d}(X)$ intersecting transversely.

Furthermore, if $L' \cong \tau^* L'$, which is the case if Y is very general in moduli, then $d = \frac{p-1}{2}$ and $L \sim 2M$, for some $M \in \mathrm{Pic}(Y)$ such that $M^2 = d$.

Recall that any rational elliptic surface is isomorphic to the blow-up of $\mathbb{P}^2$ in nine points that are base points of a pencil of cubics. There are some general results about the Severi varieties on curves in blown-up planes (for example [30]), but in the setting in which the blown-up points are in general position. This does not cover our case, in which $P_1, \dots, P_9$ are base points of a pencil of cubic curves.

From now on, most of the results are original (except for the digression about the logarithmic Severi varieties).

3.3.1 The m-special curves

As noticed at the end of the previous section, proving that $B_{Y,m}$ is nodal, consequently that R_m (as well as $\boxminus R_m$) and E_9 intersect transversely, is equivalent to proving that the singularities of $B_{S,m}$ lying along E_9 are simple nodes.

Definition 3.3.3. *Let $S \cong \mathrm{Bl}_{\{P_1 \dots, P_9\}} \mathbb{P}^2$ be a general rational elliptic surface and let $S[2]_0$ be the 2-torsion trisection. We call $V^{S[2]_0}_{\mathcal{B}_m}(S) \subset V_{|\mathcal{B}_m|,4m+2}(S)$ the Severi variety of irreducible rational curves in $|\mathcal{B}_m|$ with two simple intersection points with $S[2]_0$.*

The Definition of $V^{S[2]_0}_{\mathcal{B}_m}(S)$ is motivated by the following Remark.

Remark 3.3.4. *In the Example 3.1.20 we showed that an origin cutting bisection for a rational elliptic surface $S \to \mathbb{P}^1$ is of the form*

$$\mathcal{B}_m \sim 6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9.$$

The arithmetic genus of $\mathcal{B}_m$ is

$$p_a(\mathcal{B}_m) = \frac{1}{2}[(6m+5)(6m+4) - 8(2m+2)(2m+1) - 2m(2m-1)]) = 4m+2.$$

and by Proposition 3.1.15 the double points of a curve in $|\mathcal{B}_m|$ belong to $S[2]_0$ or to E_9. Moreover,

$$\mathcal{B}_m \cdot E_9 = 2m \text{ and } \mathcal{B}_m \cdot S[2]_0 = 6(m+1).$$

Let now $B_{S,m}$ be a rational member of $\mathcal{B}_m$ and let us assume that the singularities of $B_{S,m}$ are simple nodes. We have $p_a(B_{S,m}) = 4m+2$, whence it has $4m+2$ nodes. Since

$$B_{S,m} \cdot E_9 + B_{S,m} \cdot S[2]_0 = 8m+6,$$

two free intersection points Q_1 and Q_2 between $B_{S,m}$ and $S[2]_0 + E_9$ remain. Since $B_{S,m}$ can just have double points as singularities (see Remark 3.1.16) and $B_{S,m} \cdot E_9$ and $B_{S,m} \cdot S[2]_0$ are even, we have that Q_1 and Q_2 belong both to E_9 or to $S[2]_0$. In the proof of Corollary 3.2.28 we saw that the rational bisections of our interest are the ones intersecting E_9 in double points, so we consider the case in which the two points belong to $S[2]_0$.

In order to prove that $V^{S[2]_0}_{\mathcal{B}_m}(S)$ is nonempty, we transfer the problem to the quotient $S/(-1) \cong \mathbb{F}_2$. Once again, the double points of $B_{S,m}$ all belong to $E_9 + S[2]_0$, that is the ramification locus of the quotient map (as seen in Remark 3.1.19). Hence, the image of $B_{S,m}$, which we will see in Lemma 3.3.7 below is a smooth rational curve, is tangent to the branch locus at the images of the singular points.

For this reason, we introduce the so-called *logarithmic Severi varieties*, parametrizing nodal curves with given tangency conditions to a fixed curve. The definition and the main results are given by Dedieu in [21] and they are based on the works of Caporaso and Harris (see for example [8]).

Let us denote by $\underline{N}$ the set of all the sequences $\alpha = [\alpha_1, \alpha_2, \dots]$ of nonnegative integers with all but finitely many α_i non-zero. In practice we shall omit the infinitely many zeroes at the end. For a sequence $\alpha \in \underline{N}$, we let

$$|\alpha| = \alpha_1 + \alpha_2 + \dots$$
$$\mathcal{I}\alpha = \alpha_1 + 2\alpha_2 + \dots + n\alpha_n + \dots$$

Definition 3.3.5. *Let S be a smooth projective surface, $T \subset S$ a smooth, irreducible curve and L be a line bundle or a divisor class on S with arithmetic genus p. Let γ be an integer satisfying $0 \leq \gamma \leq p$, let $\alpha \in \underline{N}$ such that*

$$\mathcal{I}\alpha = L \cdot T.$$

We denote by $V_{\gamma,\alpha}(S, T, L)$ the locus of curves in L such that

- *C is irreducible of geometric genus γ and algebraically equivalent to L,*
- *denoting by $\mu : \tilde{C} \to S$ the normalization of C composed with the inclusion $C \subset S$, there exist $|\alpha|$ points $Q_{i,j} \in C$, $1 \leq j \leq \alpha_i$ such that*

$$\mu^* T = \sum_{1 \leq j \leq \alpha_i} i Q_{i,j}.$$

Theorem 3.3.6 (Dedieu)**.** *Let V be an irreducible component of $V_{\gamma,\alpha}(S, T, L)$, $[C]$ a general member of V and $\mu : \tilde{C} \to S$ its normalization as in the Definition 3.3.5. Let now $Q_{i,j}$, $1 \leq j \leq \alpha_i$ points in $\tilde{C}$ such that*

$$\mu^* T = \sum_{1 \leq j \leq \alpha_i} i Q_{i,j}$$

and set

$$D = \sum_{1 \leq j \leq \alpha_i} (i-1) Q_{i,j}.$$

(i) If $-K_S \cdot C_i - \deg \mu_* D_{|C_i|} \geq 1$ for every irreducible component C_i of C, then

$$\dim(V) = -(K_S + T) \cdot L + \gamma - 1 + |\alpha|$$

(ii) If $-K_S \cdot C_i - \deg \mu_* D_{|C_i|} \geq 2$ for every irreducible component C_i of C, then

(a) the normalization map μ is an immersion, except possibly at the points $Q_{i,j}$;

(b) the points $Q_{i,j}$ of $\tilde{C}$ are pairwise distinct.

In our new setting, the surface we are focusing on is $\mathbb{F}_2$. We call f the class of the members of the ruling and e the special section such that $e^2 = -2$. We describe the images of the origin cutting bisections under the quotient $q : S \to \mathbb{F}_2$.

Lemma 3.3.7. *$\mathcal{B}_m$ is sent to $b_m \sim 2(m+1)f + e$ and the branch locus consists of two curves e and s_2, where e is the special section and we have $s_2 \sim 6f + 3e$.*

Proof. Recall that the ramification locus consists of the union of E_9 and $S[2]_0 \sim 9L - 3E_1 - \cdots - 3E_8$. To prove the Lemma, it is sufficient to describe the map $q^* : \mathrm{Pic}(\mathbb{F}_2) \to \mathrm{Pic}(S)$. The quotient map q sends the cubics of the pencil to the lines of the ruling, then $q^*(f) = F \sim 3L - E_1 - \cdots - E_9$. Moreover, E_9 is sent to e and it belongs to the branch locus, then $q^*(e) = 2E_9$. Now, $\mathcal{B}_m \subset S$ is an origin cutting bisection and then it cuts a fiber on S in opposite points with respect to E_9: since q identifies exactly the opposite points with respect to E_9, we have that $q(\mathcal{B}_m)$ is a section for the ruling f and $q^*q_*(\mathcal{B}_m) = \mathcal{B}_m$. Finally, $\mathcal{B}_m \sim 6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9$ is sent to $b_m \sim 2(m+1)f + e$, and the component of the ramification locus $S[2]_0 \sim 9L - 3E_1 - \cdots - 3E_8 \sim 3F + 3E_9$ is sent to $6f + 3e$. □

The linear system b_m consists of sections, thus each irreducible member is a smooth rational curve and we shall omit the genus in the notation of the Severi varieties of curves in it. We are interested in the images of curves of $V^{S[2]_0}_{\mathcal{B}_m}(S)$: the singular points of $B_{S,m}$ along $E_9 + S[2]_0$ become tangency points between $q(B_{S,m}) \sim b_m$ and $e + s_2$. For this reason, we set

$$\alpha = [2, 4m+2].$$

As we explained in Remark 3.3.4, $V^{S[2]_0}_{\mathcal{B}_m}(S)$ consists of curves having two simple intersection points with $S[2]_0$, so that we are interested in the members of $V_\alpha(\mathbb{F}_2, e + s_2, b_m)$ totally tangent to e with even order at every intersection point and possibly intersecting s_2 transversely in two points. We call $V^{s_2}_{b_m}(\mathbb{F}_2)$ the logarithmic Severi variety contained in $V_\alpha(\mathbb{F}_2, e + s_2, b_m)$ parametrizing such curves. In the following Theorem we prove that the general member of $V^{s_2}_{b_m}(\mathbb{F}_2)$ is simply tangent to e and to s_2 at every intersection point except for two points in s_2 at which the intersection is transverse.

Theorem 3.3.8. *$V_{\mathcal{B}_m}^{S[2]_0}(S)$ is nonempty of dimension 1.*

Proof. First of all, we prove that the logarithmic Severi variety $V_{b_m}^{s_2}(\mathbb{F}_2)$ of curves on $\mathbb{F}_2$ is nonempty and that its general member is tangent to $e+s_2$ in pairwise distinct points. Finally, we notice that there is a 1:1 correspondence between $V_{\mathcal{B}_m}^{S[2]_0}(S)$ and $V_{b_m}^{s_2}(\mathbb{F}_2)$.

The nonemptiness follows from Corollary 3.2.28: indeed, let C be an irreducible curve in the set of rational curves in $\mathcal{B}_m$ intersecting E_9 in simple nodes and tacnodes. Since C is a bisection, it cannot have triple points, whence it has just double points as singularities. Moreover, since C is origin cutting, all of its double points other than the ones lying along E_9 lie along $S[2]_0$. The image $q(C) \sim 2(m+1)f+e$ is totally tangent to the special section e and to the trisection s_2 except for (possibly) two points, whence $q(C) \in V_{b_m}^{s_2}(\mathbb{F}_2)$.

Every irreducible curve of b_m is a section and then it is smooth, so it is isomorphic to its normalization. Let $[c]$ be a general member of $V_{b_m}^{s_2}(\mathbb{F}_2)$. We have $\alpha_1 = 2$ and $\alpha_2 = 4m+2$, so we let $Q_{1,1}, Q_{1,2}, Q_{2,1}, \dots, Q_{2,4m+2}$ be points in c such that, denoting by $\mu : c \to \mathbb{F}_2$ the inclusion $c \subset \mathbb{F}_2$, we have $\mu^*(s_2+e) = Q_{1,1}+Q_{1,2}+2Q_{2,1}+\cdots+2Q_{2,4m+2}$ as in Definition 3.3.5. We set $D = Q_{2,1}+\cdots+Q_{2,4m+2}$: by construction, $d := \deg D_{|c} = 4m+2$. The anti-canonical class of $\mathbb{F}_2$ is

$$-K_{\mathbb{F}_2} \sim |4f+2e|.$$

We have

$$-K_{\mathbb{F}_2} \cdot c - d = (4f+2e)(2(m+1)f+e) - d = 4m+4-(4m+2) = 2.$$

Thus, by Theorem 3.3.6(ii-b), we conclude that the points at which c is tangent to e and s_2 are pairwise distinct and that c intersects s_2 transversely in two distinct points (distinct also from the ones at which it is tangent).

By construction, the pullback of a member of $V^{s_2}_{b_m}(\mathbb{F}_2)$ belongs to $V^{S[2]_0}_{\mathcal{B}_m}(S)$. This in particular implies that $V^{S[2]_0}_{\mathcal{B}_m}(S)$ is nonempty. Furthermore, any member of $V^{S[2]_0}_{\mathcal{B}_m}(S)$ belongs to the set of irreducible rational members of $\mathcal{B}_m$ intersecting E_9 in simple nodes and tacnodes and then it is sent to a member of $V^{s_2}_{b_m}(\mathbb{F}_2)$.

Finally, by Theorem 3.3.6(i), we have

$$\dim(V^{S[2]_0}_{\mathcal{B}_m}(S)) = \dim(V^{s_2}_{b_m}(\mathbb{F}_2)) = -(K_S + T)\cdot L + \gamma - 1 + |\alpha| =$$
$$-(|-4f - 2e + 6f + 4e|)\cdot |(2m+2)f + e| - 1 + 4m + 4 =$$
$$-|2f + 2e| \cdot |(2m+2)f + e| + 4m + 3 = -4m - 2 + 4m + 3 = 1.$$

□

Remark 3.3.9. *It is not surprising that $V^{S[2]_0}_{\mathcal{B}_m}(S)$ is actually a curve: for every rational elliptic surface S and for every m, we expected that we could construct a one-dimensional family of Enriques surfaces of base change type. These curves parametrize the pairs $(Y_m, \mathcal{E}_{Y_m})$ of a m-special Enriques surface Y_m and its genus 1 pencil $\mathcal{E}_{Y_m}$ induced by the one of S.*

In Theorem 3.3.8, we showed that the Severi variety $V^{S[2]_0}_{\mathcal{B}_m}(S)$ is nonempty for a general rational elliptic surface S. In other words, for every such S and for every nonnegative integer m, there exists a one-dimensional family of origin cutting rational bisections intersecting the zero-section E_9 in m simple nodes. By Proposition 3.2.27, these origin cutting bisections can be used to construct Enriques surfaces of base change type. Let $[B_{S,m}]$ be a general element in $V^{S[2]_0}_{\mathcal{B}_m}(S)$ and denote as usual by $g : X_m \to S$ the base change induced by $B_{S,m}$ with the procedure described in Proposition 3.2.27, by $f : X_m \to Y_m$ the corresponding Enriques quotient and by R_m the irreducible component of $g^{-1}(B_{S,m})$ identified with $\tilde{E}_9 = g^{-1}(E_9)$ by f. Since $B_{S,m}$ intersects E_9 in simple nodes, we have that the intersection between R_m and $\tilde{E}_9$ is transverse and therefore that the induced m-special curve on Y_m is nodal. We proved the following Theorem, which finally states that the general m-special Enriques surface constructed starting from a fixed general rational elliptic surface S is such that $B_{Y,m}$ is nodal.

Theorem 3.3.10. *Let $S = \mathrm{Bl}_{\{P_1,\dots,P_9\}}\mathbb{P}^2$ be a general rational elliptic surface. Denote by $q : S \to \mathbb{F}_2$ the quotient map over the second Hirzebruch surface $\mathbb{F}_2$ and let C be a general member of $V^{s_2}_{b_m}(\mathbb{F}_2)$. Let us denote $B_{S,m} \in V^{S[2]_0}_{\mathcal{B}_m}(S)$ the preimage of C under q and S_{t_0} and S_{t_∞} the fibers to whom $B_{S,m}$ is tangent. Let now X_m and Y_m be the K3 surface and the Enriques surface obtained by the base change construction with S_{t_0} and S_{t_∞} as fixed fibers. Let then $g : X_m \to S$ and $f : X_m \to Y_m$ denote the corresponding quotients. Finally, denote by $R_m \in \mathrm{MW}(X_m)$ the component of $g^{-1}(B_{S,m})$ identified with $\tilde{E}_9$ by f and $B_{Y,m}$ the corresponding m-special curve on Y.*
Then, $B_{Y,m}$ is nodal.

In the last part of the section we show that the m-special curves are not unique: for every section E in the Mordell-Weil group of S, there exists a nodal rational curve $B_{Y_E} \in Y_m$ of arithmetic genus m, which we will refer to also as m-special curve.

Lemma 3.3.11. *Let $E \in \mathrm{MW}(S)$ be a section for the rational elliptic surface S. Then, $\tilde{E} = g^{-1}(E) \in \mathrm{MW}(X_m)$ is identified with $\tilde{E} \boxplus R_m$ by f and their image $B_{Y_E} := f(\tilde{E}) = f(\tilde{E} \boxplus R_m)$ is a nodal rational curve of arithmetic genus m.*

Proof. Let us denote by $\boxplus\tilde{E} \in \mathrm{Aut}(X)$ the automorphism of X given by the translation by $\tilde{E}$. The curve $\tilde{E}_9$ is sent to $\tilde{E}$ and R_m to $\tilde{E} \boxplus R_m$ by the translation $\boxplus\tilde{E}$. To complete the proof of the first part of the statement, it is sufficient to show that $\boxplus\tilde{E}$ commutes with τ. Let $Q_t \in X_t$: $\boxplus\tilde{E}(\tau(Q_t)) = \boxplus\tilde{E}(Q_{-t} \boxplus R_{m_{-t}}) = Q_{-t} \boxplus R_{m_{-t}} \boxplus \tilde{E}_{-t} = \iota(Q_t + \tilde{E}_t - R_{m_t}) = \tau(Q_t + \tilde{E}_t) = \tau(\boxplus\tilde{E}(Q_t))$. Since $\tilde{E} + \tilde{E} \boxplus R_m$ is obtained by translating $E_9 + R_m$ for $\tilde{E}$ and the transversality of the corresponding intersections is preserved under the automorphism $\boxplus\tilde{E}$, we conclude that B_{Y_E} is nodal. $\square$

We compute some intersection numbers between these sections on the K3 surface X_m.

Lemma 3.3.12. *For every $E \in \mathrm{MW}(S)$, $\tilde{E} \boxplus R_m \cdot \tilde{E} \boxminus R_m = 8m + 6$.*

Proof. Since the translation by $\tilde{E}$ is an automorphism of the K3 surface X_m, it is sufficient to compute the intersection product $R_m \cdot \boxminus R_m$.

We have that

$$B_{S,m}^2 = \tfrac{1}{2}(R_m + \boxminus R_m)^2 = -2 + R_m \cdot \boxminus R_m)$$

and

$$B_{S,m}^2 = (6(m+1)L - 2(m+1)E_1 - \cdots - 2(m+1)E_8 - 2mE_9)^2 = 8m+4,$$

from which $R_m \cdot \boxminus R_m = 8m+6$. □

Remark 3.3.13. *The intersection between R_m and $\boxminus R_m$ consists of $8m+6$ distinct points: $8m+4$ are the preimages of the $4m+2$ nodes of $B_{S,m}$ and the two remaining intersection points correspond to the two points at which $B_{S,m}$ is tangent to the branch locus $S_{t_0} \cup S_{t_\infty}$ of the double cover $g : X \to S$.*

We are able to find other rational curves on Y_m. Recall that the Mordell-Weil group of X is generated by $\tilde{E}_i$, with $i \in \{1, \dots, 8\}$, and R_m, with neutral element $\tilde{E}_9$. Then, every section in $\mathrm{MW}(X_m)$ is of the form $\tilde{E} \boxplus R_m^{\boxplus k}$, with $E \in \mathrm{MW}(S)$.

Lemma 3.3.14. *Every section $\tilde{E} \boxplus R_m^{\boxplus k} \in \mathrm{MW}(X_m)$ is identified by $f : X_m \to Y_m$ with $\tilde{E} \boxplus R_m^{\boxplus(1-k)} \in \mathrm{MW}(X_m)$. In particular, the section $R_m^{\boxplus k}$ is identified with $R_m^{\boxplus(1-k)}$.*

Proof. The proof is quite similar to the one of Lemma 3.3.11. It remains to show that $R_m^{\boxplus k}$ is identified with $R_m^{\boxplus(1-k)}$. Let X_t and X_{-t} be two twin fibers. The involution τ acts on a point $x_t \in X_t$ in the following way:

$$\tau(x_t) = \iota \circ \boxminus R_m(x_t) = \iota(x_t \boxminus R_{m_t}) = x_{-t} \boxplus R_{m_{-t}}.$$

A point $R_{m_t}^{\boxplus k} \in R_m^{\boxplus k}$ is sent to $\iota(R_{m_t}^{\boxplus(k-1)}) = R_{m_{-t}}^{\boxplus(1-k)} \in R_m^{\boxplus(1-k)}$, while a point $R_{m_t}^{\boxplus(1-k)} \in R_m^{\boxplus(1-k)}$ is sent to $\iota(R_{m_t}^{\boxminus k}) = R_{m_{-t}}^{\boxplus k}$. □

Definition 3.3.15. *We call $B_{E,k,m}$ the image $f(\tilde{E} \boxplus R_m^{\boxplus k}) = f(\tilde{E} \boxplus R_m^{\boxplus(1-k)}) \subset Y_m$*

Proposition 3.3.16. *For every $k \in \mathbb{Z}_+$ and for every $E \in \mathrm{MW}(S)$, the curve $B_{E,k,m} \subset Y_m$ is rational of arithmetic genus*

$$p_a(B_{E,k,m}) = (4k^2 - 4k + 1)m + 4k^2 - 4k.$$

Proof. First of all, since $R_m^{\boxplus k}$ is rational, then its image is. We have that

$$\begin{aligned} p_a(B_{E,k,m}) &= \tfrac{1}{2}B_{E,k,m}^2 + 1 = \tfrac{1}{2}\left(\tfrac{1}{2}(R_m^{\boxplus k} + R_m^{\boxplus(1-k)})^2\right) = \\ &= \tfrac{1}{4}(-2 - 2 + 2R_m^{\boxplus k} \cdot R_m^{\boxplus(1-k)}) = \tfrac{1}{2}R_m^{\boxplus k} \cdot R_m^{\boxplus(1-k)}. \end{aligned}$$

Moreover,

$$R_m^{\boxplus k} \cdot R_m^{\boxplus(1-k)} = R_m \cdot \tilde{E}_9 + R_m \cdot X_m[2k-1]_0,$$

where $S[2k-1]_0$ is the $(2k-1)$-torsion multisection for the elliptic fibration on S and $X_m[2k-1]_0 \subset X_m$ is its preimage under g. Indeed, the intersection product

$$R_m^{\boxplus k} \cdot R_m^{\boxplus(1-k)}$$

is the same as

$$R_m^{\boxplus(2k-1)} \cdot \tilde{E}_9$$

and since $\tilde{E}_9$ is the zero-section, this latter intersection is composed by the $(2k-1)$-torsion points (trivial or not) in every fiber. By Proposition 3.1.18, we have

$$S[2k-1]_0 \sim 12k(k-1)L - 4k(k-1)E_1 - \cdots - 4k(k-1)E_8.$$

Finally,

$$R_m \cdot \tilde{E}_9 + R_m \cdot X_m[2k-1]_0 = (8k^2 - 8k + 2)m + 8k^2 - 8k,$$

from which

$$p_a(B_{E,k,m}) = (4k^2 - 4k + 1)m + 4k^2 - 4k.$$

□

3.3.2 Nonregular Severi varieties on Enriques surfaces of base change type

In the previous Section, we defined the regular components of Severi varieties of curves on an algebraic surface to be the smooth components of the expected dimension. Regarding the Enriques surfaces, in Proposition 3.3.2 the authors prove that the regular components have dimension $g-1$ and that the nonregular ones have dimension g, where g is the geometric genus of the general curve of the Severi variety. Moreover, the authors show that if a component of a Severi variety of curves on a very general Enriques surface is nonregular, then its members split in two linearly equivalent curves in the K3 cover.

We show that the assumption of very generality is necessary: in fact, in the Enriques surfaces of base change type (that form a countable set of 9-dimensional subfamilies in the moduli space of Enriques surfaces), there are plenty of nonregular Severi varieties violating the result of Ciliberto, Dedieu, Galati and Knutsen.

Definition 3.3.17. *We call special nonregular component a nonregular component of a Severi variety of curves on an Enriques surface such that its members split in two nonlinearly equivalent curves in the K3 cover. We call special nonregular curve (nonregular) every member of a special nonregular (nonregular) component.*

Proposition 3.3.2 does not say anything about the equigeneric Severi varieties of curves on Y. Let C be an irreducible curve on an Enriques surface Y. We denote by $\nu_C : \overline{C} \to C$ the normalization of C and define $\eta_C := \mathcal{O}_C(K_S) = \mathcal{O}_C(-K_S)$, a nontrivial 2-torsion element in $\mathrm{Pic}^0(C)$, and $\eta_{\overline{C}} := \nu^*\eta_C$. By standard results on covering of complex manifolds (see [13, Section 2] or [3, Section 1.17]), two cases may happen:

- $\eta_{\overline{C}} \ncong \mathcal{O}_{\overline{C}}$ and $f^{-1}(C)$ is irreducible;
- $\eta_{\overline{C}} \cong \mathcal{O}_{\overline{C}}$ and $f^{-1}(C)$ consists of two irreducible components conjugated by the Enriques involution.

Following [22, proofs of Thm 4.2 and Cor 2.7], the dimension of the equigeneric Severi variety of genus g curves in $|L| = |C|$ at the point $|C|$ satisfies the inequality

$$\dim_{[C]} V_g^{|L|} \geq h^0(\omega_{\tilde{C}} \otimes \eta_{\tilde{C}}) = \begin{cases} g-1 \text{ if } \eta_{\tilde{C}} \neq \omega_{\tilde{C}} \\ g \text{ if } \eta_{\tilde{C}} = \omega_{\tilde{C}} \end{cases}$$

Proposition 3.3.2 states exactly that the latter is in fact an equality when C is nodal. Since by the results due to Dedieu and Sernesi (see [22]) the equigeneric Severi varieties of genus g curves in a K3 surfaces have dimension g, we actually have that equality holds also for non-nodal curves on Y that split in the K3 cover. In other words, an irreducible component V of an equigeneric Severi variety of curves of genus g in an Enriques surface Y such that the curves in V split in the K3 cover has dimension g. Notice that we cannot say anything about the dimension of an irreducible component of an equigeneric Severi varieties of curves whose general member is 2:1 covered by irreducible curves of the K3 cover. This discussion leads us to give the following definition.

Definition 3.3.18. *We shall call nonregular a component of an equigeneric Severi variety of curves on an Enriques surface such that every curve splits in the K3 cover. Furthermore, we shall call special nonregular a nonregular component of an equigeneric Severi variety of curves on an Enriques surface such that its members split in two nonlinearly equivalent curves in the K3 cover.*

As in the nodal case, a nonregular irreducible component V of an equigeneric Severi variety of curves of genus g has dimension g. The definition of special nonregular component for a equigeneric Severi variety of curves is justified by the following Lemma, stating that the fact that a nonregular curve on a very general Enriques surface Y splits in the K3 cover in two linearly equivalent curves holds in general and not only for nodal curves.

Part of the ideas we will use to show the existence of some special nonregular components of some equigeneric Severi varieties of curves are similar to the ones developed by Ciliberto and Dedieu in [12] and presented by Dedieu in [21, Section 5.2].

Lemma 3.3.19. *Let Y be a Picard very general Enriques surface and X be is its universal cover. Moreover, let τ denote the Enriques quotient. Then, τ acts as the identity on* $\mathrm{NS}(X) \cong U(2) \oplus E_8(-2)$.

Proof. This is a very well-known fact. See, for example, [25, Section 11]. □

The first example of special nonregular curves are the m-special curves.

Proposition 3.3.20. *Let $B_{Y,m}$ be an m-special curve on an m-special Enriques surface Y_m. Then, $B_{Y,m}$ is a special nonregular curve.*

Proof. First of all, every nodal rational curve is nonregular: the geometric genus of the rational curves is 0, then they necessarily belong to nonregular components. As we proved in Theorem 3.3.10, $B_{Y,m}$ is nodal, then it is nonregular. Now, $B_{Y,m}$ splits in two (-2)-curves (that are never linearly equivalent) in the K3 cover, whence it is special nonregular. □

Definition 3.3.21. *Let D be an irreducible effective divisor on S. We denote by*

$$D_Y := f_*(g^*(D))$$

the pushforward as cycle via f in Y_m of the pull-back under g of D in X_m.
Furthermore, we denote by

$$|D|_Y \subset f_*(g^*|D|)$$

the family of the divisors D_Y's with $D \in |D|$ irreducible, different from an elliptic fiber and such that $g^{-1}(D)$ is irreducible.

Remark 3.3.22. *We will see in Proposition 3.3.25 that if the preimage $g^{-1}(D)$ on X_m of an irreducible effective divisor D on S different from an elliptic fiber is irreducible, then $g^{-1}(D)$ is sent birationally to its image $f(g^{-1}(D))$ on Y_m. For this reason, we do not have to worry about the multiplicity of the morphism f in the definition of D_Y as cycle.*

Notice that $g^*|D|$ is in general noncomplete:

$$h^0(g^*\mathcal{O}_S(D)) = h^0(g_*g^*\mathcal{O}_S(D)) = h^0(\mathcal{O}_S(D)) + h^0(\mathcal{O}_S(D+K_S)),$$

whence we have

$$h^0(g^*\mathcal{O}_S(D)) = h^0(D)$$

if and only if $\mathcal{O}_S(D+K_S) = \mathcal{O}_S(D-F))$ does not have global sections, where $F \sim (-K_S)$ is the class of a fiber.

The aim of the section is to prove that $|D|_Y$ is a special nonregular equigeneric family of curves for every effective divisor D on S.

Lemma 3.3.23. *Let D be an irreducible effective divisor on S different from an elliptic fiber. Then, the general curve of $g^*|D|$ is irreducible.*

Proof. Let C be a curve in S. It splits in the K3 cover if and only if it intersects the branch locus $S_{t_0} + S_{t_\infty}$ with even order at every intersection point and this is not the case for the general curve of a complete linear system $|D|$. □

Lemma 3.3.24. *$R_m \in \mathrm{MW}(X_m)$ has infinite order.*

Proof. It is a very well known fact that any two torsion section of a smooth relatively minimal elliptic surface are disjoint (see, for example, [44, Proposition VII.3.2]). Let us assume R_m of l-torsion for some $l \in \mathbb{Z}_+$: since the opposite of an l-torsion point is an l-torsion point, also $\boxminus R_m$ is an l-torsion section. But by Lemma 3.3.12 $R_m \cdot \boxminus R_m = 8m+6$, whence we have a contradiction and R_m has infinite order in $\mathrm{MW}(X_m)$. □

Proposition 3.3.25. *Let D be an effective divisor on S different from an elliptic fiber. Then, for every irreducible curve $C \in |D|$ such that $g^{-1}(C) \subset X_m$ is irreducible, we have that $g^{-1}(C)$ is identified by f with $g^{-1}(C) \boxplus R_m \subset X_m$. In particular, $|D|_Y \subset f_*(g^*|D|)$ is contained in an equigeneric nonregular Severi variety of curves on Y_m.*

Proof. $g^{-1}(C)$ cuts twin points in opposite fibers: indeed, if

$$C_{|S_t} = a_1 Q_{1,t} + \cdots + a_k Q_{k,t},$$

then

$$g^{-1}(C)_{|X_t} = a_1 Q_{1,t} + \cdots + a_k Q_{k,t}$$

and

$$g^{-1}(C)_{|X_{-t}} = a_1 Q_{1,-t} + \cdots + a_k Q_{k,-t}.$$

For every $j \in \{1, \ldots, k\}$, we have

$$\tau(Q_{j,t}) = (\iota \circ \boxminus R_m)(Q_{j,t}) = \iota(Q_{j,t} \boxminus R_{m_t}) = Q_{j,-t} \boxplus R_{m_{-t}} \in g^{-1}(C) \boxplus R_m.$$

Now, $(g^{-1}(C) \boxplus R_m)_{|X_t} = a_1(Q_{1,t} \boxplus P_t) + \cdots + a_k(Q_{k,t} \boxplus R_{m_t})$: for every $j \in \{1, \ldots, k\}$, we have

$$\tau(Q_{j,t} \boxplus R_{m_t}) = (\iota \circ \boxminus R_m)(Q_{j,t} \boxplus P_t) = \iota(Q_{j,t}) = Q_{j,-t} \in g^{-1}(C).$$

In order to prove that $|D|_Y$ is nonregular, we need to show that $g^{-1}(C)$ and $g^{-1}(C) \boxplus R_m$ do not coincide. Suppose $g^{-1}(C) = g^{-1}(C) \boxplus R_m$: for every t and for every $j \in \{1, \ldots, k\}$, $Q_{j,t} \boxplus R_m \in g^{-1}(C)$ and, by iterating, $Q_{j,t} \boxplus R_m^{\boxplus r} \in g^{-1}(C)$ for every $r \in \mathbb{Z}_+$. This is a contradiction because the number of intersection points between C and X_t is finite and by Lemma 3.3.24 R_m has infinite order in $\mathrm{MW}(X_m)$.

□

Theorem 3.3.26. *For every irreducible effective divisor D on S different from an elliptic fiber, $|D|_Y$ is contained in an equigeneric special nonregular Severi variety of curves on Y_m.*

Proof. In the proof of Proposition 3.3.25 we saw that every $\tilde{C} \in |D|_Y$ splits in the K3 X_m in $g^{-1}(C) + g^{-1}(C) \boxplus R_t$ for a member $C \in |D|$ which is irreducible with $g^{-1}(C)$ irreducible. For every $t \in \mathbb{P}^1$, we set $g^{-1}(C)_{|X_t} = a_1 Q_{1,t} + \cdots + a_k Q_{k,t}$. Suppose that $g^{-1}(C) \sim g^{-1}(C) \boxplus R_t$: it follows that for every $t \in \mathbb{P}^1$, the degree $a_1 + \cdots + a_k$ divisors $a_1 Q_{1,t} + \cdots + a_k Q_{k,t}$ and $a_1 Q_{1,t} \boxplus R_{m_t} + \cdots + a_k Q_{k,t} \boxplus R_{m_t}$ are linearly equivalent in the elliptic curve X_t. In other words, we have the equality

$$Q_{1,t}^{\boxplus a_1} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} = Q_{1,t}^{\boxplus a_1} \boxplus R_{m_t} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxplus R_{m_t} = Q_{1,t}^{\boxplus a_1} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxplus R_{m_t}^{\boxplus k}$$

for every t, which implies that R_m is a k-torsion section. Since by Lemma 3.3.24 R_m has infinite order in $\mathrm{MW}(X_m)$, this is a contradiction. □

Now, we want to understand when $|D|_Y$ is a component of a an equigeneric special nonregular Severi variety of curves and when $|D|_Y$ it is strictly contained in such a component.

Proposition 3.3.27. *Let D be a k-section for the elliptic fibration on S of arithmetic genus p. Then, $|D|_Y$ is a special nonregular component of an equigeneric Severi variety of curves on Y_m if and only if $p = 0$.*

Proof. Since S is a rational surface and D is effective, we have $\chi(\mathcal{O}_S) = 1$ and $h^1(D) = h^2(D) = 0$. Hence, by Riemann-Roch for surfaces we have

$$\dim |D| = \tfrac{1}{2}(D^2 - D \cdot K_S) = \tfrac{1}{2}(D^2 + D \cdot S_t) = p + k - 1.$$

Then, $\dim |D|_Y = \dim |D| = p + k - 1$. Furthermore, since D is a k-section, it intersects the branch locus $S_{t_0} + S_{t_\infty}$ of $g : X_m \to S$ in $2k$ points, so that by the Riemann-Hurwitz formula, the arithmetic genus of $g^{-1}(D)$ is

$$p_a(g^{-1}(D)) = 2p + k - 1.$$

Since $g^{-1}(D)$ is sent birationally to its image $f(g^{-1}(D))$ in $|D|_Y$, the geometric genus of the members of $|D|_Y$ is again $2p + k - 1$. Finally, by the discussion above, we have that a nonregular component of an equigeneric Severi variety of curves in an Enriques surface has the dimension equal to the geometric genus of its members. Hence, we conclude that $|D|_Y$ is a special nonregular component if and only if

$$p + k - 1 = 2p + k - 1,$$

then if and only if the genus of D is zero; otherwise, $|D|_Y$ is just contained in such a component. □

A remarkable class of examples of linear system of genus zero on a rational elliptic surface S is given by the conic bundles, which as we will see in the next section induce genus 1 fibrations on the K3 surface X.

With the same arguments used to prove Proposition 3.3.25 and Theorem 3.3.26, one can easily show that for every irreducible curve $C \in |D|$ different from an elliptic fiber and such that $g^{-1}(C)$ irreducible, the curve $g^{-1}(C) \boxplus R_m^{\boxplus k}$ is identified by f with $g^{-1}(C) \boxplus R_m^{\boxplus(1-k)}$ and conclude that the subfamily of $f_*(g^*|D| \boxplus R_m^{\boxplus k})$ consisting of the divisors on Y_m obtained starting from an irreducible curve $C \in |D|$ different from an elliptic fiber and such that $g^{-1}(C)$ is irreducible is contained in an equigeneric special nonregular Severi variety of curves on Y.

We produced a lot of examples of equigeneric special nonregular Severi varieties of curves for every Enriques surface of base change type. It could be interesting to compute the arithmetic genus of the curves in $|D|_Y$ that we denote by $p(D_Y)$. The last part of this subsection is devoted to introduce some tools through which we develop a method to compute $p(D_Y)$.

Recall that $B_{S,m} = g(R_m) = g(\boxminus R_m) \in S$.

Definition 3.3.28. *Let D be an irreducible effective divisor on S. We denote by*

$$D \boxplus B_{S,m} := g_*(g^*(D) \boxplus R_m)$$

the pushforward as cycle via g in S of the translation by R_m of the pullback under g of D in X_m.

Moreover, we denote by

$$|D| \boxplus B_{S,m} := g_*(g^*|D| \boxplus R_m)$$

the family of the divisors $D \boxplus B_{S,m}$ with $D \in |D|$ without an elliptic fiber as irreducible component.

Remark 3.3.29. *Proposition 3.3.30 will ensure that, for every effective divisor D on S without an elliptic fiber as irreducible component, $g^{-1}(D) \boxplus R_m$ on X_m is sent birationally* *to its image $g(g^{-1}(D) \boxplus R_m)$ on Y_m. For this reason, we do not have to worry about the multiplicity of the morphism g in the Definition of $D \boxplus B_{S,m}$ as cycle.*

Lemma 3.2.12 and Lemma 3.3.11 can be generalized to every effective divisor D on S.

Proposition 3.3.30. *Let D be an effective divisor on S. Then, for every $C \in |D|$ without an elliptic fiber as irreducible component, we have that $g^{-1}(C) \boxplus R_m$ is identified by g with $g^{-1}(C) \boxminus R_m$ in S.*

Proof. As in the proof of Proposition 3.3.25, the preimage $g^{-1}(C)$ cuts twin points in opposite fibers. For every $t \in \mathbb{P}^1$ we let

$$g^{-1}(C)_{|X_t} = a_1 Q_{1,t} + \cdots + a_k Q_{k,t}$$

and

$$g^{-1}(C)_{|X_{-t}} = a_1 Q_{1,-t} + \cdots + a_k Q_{k,-t}.$$

Now, $(g^{-1}(C) \boxplus R_m)_{|X_t} = a_1(Q_{1,t} \boxplus R_{m_t}) + \cdots + a_k(Q_{k,t} \boxplus R_{m_t})$: for every $j \in \{1, \ldots, k\}$ we have $\iota(Q_{j,t} \boxplus R_{m_t}) = (Q_{j,-t} \boxminus R_{m_{-t}}) \in g^{-1}(C) \boxminus R_m$. In the same way one can prove that ι sends every point of $g^{-1}(C) \boxminus R_m$ to points belonging to $g^{-1}(C) \boxplus R_m$. As in the proof of Proposition 3.3.25, if $g^{-1}(C) \boxminus R_m$ and $g^{-1}(C) \boxplus R_m$ were the same curve, the section R_m would be a torsion section. $\square$

Notice that for a section $E \in \mathrm{MW}(S)$, we have the two notations $B_{S,m} \boxplus E$, denoting the translation of $B_{S,m}$ by E, and $E \boxplus B_{S,m}$, which is the pushforward via $g : X \to S$ of the translation by R_m of the pullback under g of E, that do not indicate the same object.

Lemma 3.3.31. *Let $E \in \mathrm{MW}(S)$ be a section for the rational elliptic surface S. Then, $E \boxplus B_{S,m} = 2(B_{S,m} \boxplus E)$.*

Proof. Since $g^{-1}(D \boxplus E) = g^{-1}(D) \boxplus \tilde{E}$ for every effective divisor D on S, we have that $g^{-1}(B_{S,m} \boxplus E) = R_m \boxplus \tilde{E} + \boxminus R_m \boxplus \tilde{E}$. Moreover, by Proposition 3.3.30 we have

$$g(\tilde{E} \boxplus R_m) = g(\tilde{E} \boxminus R_m) = B_{S,m} \boxplus E,$$

from which

$$E \boxplus B_{S,m} = g_*(\tilde{E} \boxplus R_m + \tilde{E} \boxminus R_m) = g_*(\tilde{E} \boxplus R_m) + g_*(\tilde{E} \boxminus R_m) = 2B_{S,m} \boxplus E.$$

□

Remark 3.3.32. *We proved that every curve C of $|D| \boxplus B_{S,m}$ splits in X_m: it means that C intersects S_{t_0} and S_{t_∞} with even order at every intersection point. In other words, for every effective divisor D on S, there exists a family of splitting curves $|D| \boxplus B_{S,m}$. The existence of the family $|D| \boxplus B_{S,m}$ is due to the presence of the rational bisection $B_{S,m}$ and then to the choice of the nongeneral pair (S_{t_0}, S_{t_∞}).*

We want to compute the arithmetic genus $p(D_Y)$ of the curves in $|D|_Y$ and also the arithmetic genus of the curves in $|D| \boxplus B_{S,m}$ that we denote by $p(D \boxplus B_{S,m})$ for every effective D on S.

Proposition 3.3.33. *Let D be a k-section on S. Then,*

$$p(D_Y) = p(g^*(D)) + \tfrac{1}{2}(g^*(D) \cdot (g^*(D) \boxplus R_m)) \text{ and}$$
$$p(D \boxplus B_{S,m}) = p(g^*(D)) + \tfrac{1}{2}((g^*(D) \boxplus R_m)(\cdot g^*(D) \boxminus R_m)) - k.$$

Proof. We have that

$$p(D_Y) = \tfrac{1}{2}D_Y^2 + 1.$$

Let $C \in |D|$: we have

$$2D_Y^2 = (g^{-1}(C) + g^{-1}(C) \boxplus R_m)^2 = g^{-1}(C)^2 + (g^{-1}(C) \boxplus R_m)^2 + 2g^{-1}(C) \cdot g^{-1}(C) \boxplus P = 2p(g^{-1}(C)) - 2 + 2p(g^{-1}(C) \boxplus R_m) - 2 + 2g^{-1}(C) \cdot g^{-1}(C) \boxplus R_m = 4p(g^{-1}(C)) - 4 + 2g^{-1}(C) \cdot g^{-1}(C) \boxplus R_m$$

and thus

$$p(D_Y) = p(g^*(D)) + \tfrac{1}{2}(g^*(D) \cdot g^*(D) \boxplus R_m).$$

Moreover, we have that

$$p(D \boxplus B_{S,m}) = \tfrac{1}{2} D \boxplus B_{S,m} \cdot (D \boxplus B_{S,m} + K_S) + 1.$$

Let $C \in |D|$: we have

$$2(D \boxplus B_{S,m})^2 = (g^{-1}(C) \boxplus R_m + g^{-1}(C) \boxminus R_m)^2 =$$
$$2p(g^{-1}(C) \boxplus R_m) - 2 + 2p(g^{-1}(C) \boxminus R_m) - 2 + 2(g^{-1}(C) \boxplus R_m) \cdot (g^{-1}(C) \boxminus R_m) =$$
$$4p(g^{-1}(C)) - 4 + 2(g^{-1}(C) \boxplus R_m) \cdot (g^{-1}(C) \boxminus R_m).$$

Furthermore, since D is a k-section,

$$(D \boxplus B_{S,m}) \cdot K_S = -2k.$$

Finally,

$$p(D \boxplus B_{S,m}) = p(g^*(D)) + \tfrac{1}{2}(g^*(D) \boxplus R_m) \cdot (g^*(D) \boxminus R_m) - k.$$

□

The next part of the Section is devoted to showing a method to perform the computation of the arithmetic genus of $|D|_Y$ for some explicit divisors D on S.

Proposition 3.3.34. $D \cdot (D \boxplus B_{S,m}) = g^*(D) \cdot (g^*(D) \boxplus R_m)$

Proof. First of all, we prove that for every $C \in |D|$,

$$g^{-1}(C) \cdot (g^{-1}(C) \boxplus R_m) = g^{-1}(C) \cdot (g^{-1}(C) \boxminus R_m).$$

Let $Q_t \in X_t$ be a point belonging to the intersection between $g^{-1}(C)$ and $g^{-1}(C) \boxplus R_m$. Thus, there exists another point $W_t \in X_t \cap g^{-1}(C)$ such that $W_t \boxplus R_{m_t} = Q_t$, or, equivalently, $W_t = Q_t \boxminus R_{m_t}$. Analogously one can prove that for every point in the set $g^{-1}(C) \cap g^{-1}(C) \boxminus R_m$ there corresponds a point in the set $g^{-1}(C) \cap g^{-1}(C) \boxplus$

R_m. Now, the morphism f has degree 2, then $g^{-1}(C)\cdot(g^{-1}(C \boxplus B_{S,m})) = g^{-1}(C)\cdot(g^{-1}(C)\boxplus R_m + g^{-1}(C)\boxminus R_m) = 2C\cdot(C\boxplus B_{S,m})$ and then we have the assertion. □

Now, computing the intersection $(D\cdot D\boxplus B_{S,m})$ for a given divisor D is not trivial and we will need a formula due to Cantat and Dolgachev. In [17], the authors provide a formula to compute the linear class of the translation by a section of a divisor on a rational elliptic surface.

Proposition 3.3.35 (Cantat, Dolgachev)**.** *Let S be a rational elliptic surface, $E \in \mathrm{MW}(S)$ be a section and $D \sim aL - b_1E_1 - \cdots - b_9E_9$ an effective divisor in on S. Then, denoting as $t_E(D)$ the translation of D by E, we have*

$$t_E(D) \sim D - (D\cdot K_S)(E-E_9) + [D\cdot(E-E_9) - \tfrac{1}{2}(D\cdot K_S)(E-E_9)^2]K_S.$$

Proof. See [17, Proof of Proposition 2.10] □

Lemma 3.3.36. *Let D be a k-section of S such that there exist k sections $E_1,\dots,E_k \in$ $\mathrm{MW}(S)$ with $D \sim E_1+\cdots+E_k$. Then,*

$$D\boxplus B_{S,m} \sim B_{S,m}\boxplus E_1 + \cdots + B_{S,m}\boxplus E_k$$

.

Proof. The proof is an application of Lemma 3.3.31 □

Remark 3.3.37. *We are able to compute the intersection $D\cdot D\boxplus B_{S,m}$ for every explicit effective line bundle $D \sim aL - b_1E_1 - \cdots - b_9E_9$ decomposable in sections. In fact, if $D \sim E_1+\cdots+E_k$, since*

$$D\boxplus B_{S,m} = B_{S,m}\boxplus E_1 + \cdots + B_{S,m}\boxplus E_k,$$

thanks to the formula of Proposition 3.3.35, we can compute the linear class $B_{S,m}\boxplus E_j$ for every $j = 1,\dots,k$ and we can easily compute the product

$$D \cdot (D \boxplus B_{S,m}) = (E_1 + \cdots + E_k) \cdot (B_{S,m} \boxplus E_1 + \cdots + B_{S,m} \boxplus E_k).$$

Example 3.3.38. *Let $D \sim L$ be the net of the strict transforms of the lines of $\mathbb{P}^2$ in the rational elliptic surface S. We want to compute the arithmetic genus of $L_Y = f_*(g^*\mathcal{O}_S(L))$ in Y_m.*

L is a 3-section of the given elliptic pencil and we can decompose it, for example, in the sum of the three sections $L \sim E_1 + E_2 + (L - E_1 - E_2)$.

By applying the formula of Cantat-Dolgachev, one can see that

$$B_{S,m} \boxplus E_1 \sim 6(m+1)L - 2mE_1 - 2(m+1)E_2 - \cdots - 2(m+1)E_9,$$

$$B_{S,m} \boxplus E_2 \sim 6(m+1)L - 2(m+1)E_1 - 2mE_2 - 2(m+1)E_3 - \cdots - 2(m+1)E_9$$

and

$$B_{S,m} \boxplus (L - E_1 - E_2) \sim$$
$$(6m+8)L - (2m+4)E_1 - (2m+4)E_2 - 2(m+1)E_3 - \cdots - 2(m+1)E_9.$$

Now we can compute the product

$$L \cdot L \boxplus B_{S,m} = (E_1 + E_2 + (L - E_1 - E_2)) \cdot ((6(m+1)L - 2mE_1 - 2(m+1)E_2 - \cdots - 2(m+1)E_9) + 6(m+1)L - 2(m+1)E_1 - 2mE_2 - 2(m+1)E_3 - \cdots - 2(m+1)E_9 + (6m+8)L - (2m+4)E_1 - (2m+4)E_2 - 2(m+1)E_3 - \cdots - 2(m+1)E_9) = 18m + 20.$$

The arithmetic genus $p(L_Y)$ is equal to

$$p(L_Y) = \tfrac{1}{2} D \cdot D \boxplus B_{S,m} + p(g^*(D)) = 9m + 10 + p(g^*(D)).$$

Since L is a 3-section, it intersects the branch locus $S_{t_0} + S_{t_\infty}$ in 6 points, so that by the Riemann-Hurwitz formula, $p(g^*(D)) = 2$, and, finally,

$$p(L_Y) = 9m + 12.$$

Since the genus of L is zero, by Proposition 3.3.27 we have that $|L|_Y$ is a special non-regular component of an equigeneric (with genus 2) Severi variety of curves in Y_m.

3.4 Genus 1 pencils on K3 surfaces with a non-symplectic involution

In what follows, we consider K3 surfaces with a non-symplectic involution σ.

Definition 3.4.1. *Let X be a K3 surface, then* $H^{2,0}(X, \mathbb{Z}) = \mathbb{Z}\omega_X$ *where* ω_X *is a nowhere vanishing symplectic form. An involution* σ *on* X *is called symplectic if it does preserve the symplectic structure on X, i.e.,* $\sigma(\omega_X) = \omega_X$.
If σ *does not preserve the symplectic structure on X, i.e.* $\sigma(\omega_X) = -\omega_X$*, it is called non-symplectic*

The following classical result states that an involution on a K3 surface can be of three types.

Proposition 3.4.2. *Let X be a K3 surface and* $\sigma \in \mathrm{Aut}(X)$ *an involution. Then one of the following holds*

(i) X/σ *is birational to a K3 surface, if* σ *is symplectic*

(ii) X/σ *is an Enriques surface if* σ *is non-symplectic and* $\mathrm{Fix}(\sigma) = \emptyset$

(iii) X/σ *is a rational surface if* σ *is non-symplectic and* $\mathrm{Fix}(\sigma) \neq \emptyset$

For a reference, see [33, section 3], where the authors also provide a deep analysis of the geometry of the quotient $X \to X/\sigma$ for every kind of fixed locus of σ. In the same work, the authors collect their contribution on the classification of the genus 1 fibrations on K3 surfaces with a non-symplectic involution (given for example in [31] and [32]). Most of their results hold under the hypothesis that σ^* acts trivially on $\mathrm{NS}(X)$.

In particular, in the previous sections, we focused on the case (ii) and on the case (iii), in which $\mathrm{Fix}(\sigma)$ is empty or consists of two elliptic curves lying in the same pencil. We give our contribution by analyzing the genus 1 fibrations on a K3 surface $X_m \in \Sigma_m$: here X_m carries both an involution with two linearly equivalent elliptic curves as fixed locus (giving rise to a rational elliptic surface as quotient) and an Enriques involution. In this setting, neither of the two involution acts trivially on $\mathrm{NS}(X_m)$.

In general, we distinguish the genus 1 pencils on the K3 surfaces according to the action of the involution on their fibers, and study the corresponding linear systems on the quotient surface.

Let $\epsilon : X \to \mathbb{P}^1$ be a genus 1 pencil on X. We denote by $[F]$ the class of the fiber of this fibration and by F_t the fiber over the point $t \in \mathbb{P}^1$. The action of σ on the genus 1 pencil ϵ can be of three different types:

(1) σ preserves the fibers of the fibration ϵ, i.e. $i(F_t) = F_t$ for every $t \in \mathbb{P}^1$. In this case the action of σ on the basis of the fibration is trivial.
(2) σ does not preserve the fibers of the fibration, but σ^* preserves the fibration and in particular the class of a fiber, i.e. $i(F_t) = F_{t'}$ for certain values $t \neq t'$, but $i^*([F]) = F$. In this case σ restricts to an involution of the basis of the fibration. We say that σ preserves the fibration.
(3) σ does not preserve the fibration and in particular $i^*([F]) = [F']$ where $[F']$ is the class of another genus 1 pencil on X, $\epsilon' : X \to \mathbb{P}^1$ which is not ϵ. In particular $[F] \nsim [F']$.

Definition 3.4.3. *We say that a genus 1 pencil on X is of type 1), 2), 3) with respect to σ if σ preserves the fibers, preserves the fibration but not the fibers, and does not preserve the fibration respectively.*

We are interested in the cases σ is an Enriques involution or X/σ is a rational elliptic surface. Our purpose is to identify the linear systems on X/σ which induce genus 1 fibrations on X and to compare the two cases in the situation in which X carries both the involutions, the Enriques and the rational one.

For the definitions and the results about the situation in which X carries the rational involution but not the Enriques one, we refer to [31] and [32].

Let $S \cong \mathrm{Bl}_{\{P_1,\dots,P_9\}} \mathbb{P}^2$ be a rational elliptic surface and denote by

$$\epsilon_S : S \to \mathbb{P}^1$$

the elliptic fibration. Let now X be a K3 surface obtained from S with the base change construction, with S_{t_0} and S_{t_∞} as fixed fibers and denote by

$$\epsilon_X : X \to \mathbb{P}^1$$

the elliptic fibration induced by ϵ_S. We call again ι the involution on X such that $X/\iota \cong S$.

Since a part of the genus 1 fibrations on X which are not induced by ϵ_S are induced by conic bundles on S, we recall the definition of conic bundle on a rational surface. The definition we give is the same as the ones in [31, Definition 3.1] and in [43, Definition 2.7].

Definition 3.4.4. *A conic bundle on the surface S is a surjective morphism $g : S \to \mathbb{P}^1$ such that the generic fiber is a smooth rational curve.*

We give an interpretation of the definition above in terms of classes of divisors on $\mathrm{Pic}(S)$. In the following we denote by S_t the fiber of the elliptic fibration ϵ_S over $t \in \mathbb{P}^1$.
Let D be a smooth fiber of a conic bundle on S. Since the class of D gives the class of a fiber, it is nef and has trivial self-intersection. Moreover, since it is rational, the adjunction Formula implies that $D \cdot K_S = -2$ and since $S_t \sim -K_S$, one obtains $D \cdot S_t = 2$. On the other hand, given a base point free class D with the above intersection properties, the induced map $|D| : S \to \mathbb{P}^1$ is a conic bundle. From the above, on a surface endowed with an elliptic fibration, we have the following equivalent definition of conic bundle.

Definition 3.4.5. *A conic bundle on S is a base point free class D in* $\mathrm{Pic}(S)$ *such that*

(i) $D \cdot S_t = 2$,

(ii) $D^2 = 0$.

Proposition 3.4.6 (Garbagnati, Salgado)**.** *Each conic bundle on S induces a genus 1 fibration on X.*

Proof. Let D be a conic bundle on S. $(g^*(D))^2 = 2D^2 = 0$, then $g^*(D) = 0$ and since it is clearly an effective class, it is a genus 1 pencil on X. □

We saw above two constructions which produce genus 1 pencils on X: they can be induced by the elliptic fibration ϵ_S or by conic bundles on S; but there might be also several other genus 1 pencils on X which are neither induced by conic bundles nor by ϵ_S. Garbagnati and Salgado determine what are the conditions on the fibers of S which ensure that all the genus 1 fibrations on X are induced either by ϵ_S or by conic bundles on S and they describe the other admissible linear systems on S which can induce genus 1 fibrations on X.

For the proofs of the following results, see [31, Section 4].

Theorem 3.4.7 (Garbagnati, Salgado)**.** *Let $\epsilon : X \to \mathbb{P}^1$ be a genus 1 fibration on X. If ϵ is of type 1) with respect to ι, then ϵ is induced by a conic bundle on S.*
If ϵ is of type 2) with respect to ι then ϵ coincides with ϵ_X.
If ϵ is of type 3) with respect to ι then ϵ is induced by a 1-dimensional linear system on $S \cong \mathrm{Bl}_{\{P_1 \dots, P_9\}} \mathbb{P}^2$, whose pushforward in $\mathbb{P}^2$ is noncomplete.

The next result due to Garbagnati and Salgado shows that genus 1 fibrations of type 3) cannot appear if ι is the identity on the Neron–Severi group of X.

Corollary 3.4.8 (Garbagnati, Salgado)**.** *If the fixed fibers S_{t_0} and S_{t_∞} of ϵ_S are smooth genus 1 curves and ι^* is the identity on $\mathrm{NS}(X)$, then a genus 1 pencil on X which is not ϵ_X is induced by a conic bundle on S.*
The above conditions can be realized only if ϵ_S is an elliptic fibration without reducible fibers.

Remark 3.4.9. *Corollary 3.4.8 implies that if X is a base change very general K3 surface (Definition 3.2.9), then every genus 1 pencil on X is induced by a conic bundle except for ϵ_X.*

The next Proposition describes the genus 1 pencils on a K3 surface covering a Picard very general Enriques surface with respect to the Enriques involution.

Proposition 3.4.10. *Let Y be a Picard very general Enriques surface and let X be its K3 cover; let τ denote the Enriques involution and let $f : X \to Y$ denote the Enriques quotient. Let $\epsilon : X \to \mathbb{P}^1$ be a genus 1 pencil and let $[F]$ denote its algebraic class in $\mathrm{NS}(X)$. Then, ϵ is of type 2) with respect to τ.*

Proof. Since Y is very general, by Lemma 3.3.19 τ^* acts as the identity on $\mathrm{NS}(X)$, then $\tau^*([F]) = [F]$. Let suppose that τ acts as the identity on the base $\mathbb{P}^1$. Then, since it is base point free, by the Riemann-Hurwitz formula we have that every smooth fiber F_t is sent by f to a smooth genus 1 curve Y_t on Y. Thus, the genus 1 pencil $|Y_t|$ is a Severi variety of genus 1 curves of dimension 1, then it is nonregular. But since Y is very general and the members of $|Y_t|$ are covered by irreducible curves, this is a contradiction and then τ acts on the base $\mathbb{P}^1$ as a nontrivial involution, proving that ϵ is of type 2) with respect to τ. $\square$

Remark 3.4.11. *We saw that the K3 surfaces $X_m \in \Sigma_m$ carry both the involutions. As we pointed out in the previous section, X_m is not base change very general and Y_m is not Picard very general: since $\iota^*(R_m) = \boxminus R_m \neq R_m$ in $\mathrm{NS}(X_m)$, ι^* does not act as the identity on $\mathrm{NS}(X_m)$; the Picard number of X_m is 11.*

In other words, the K3 surfaces covering the Enriques surfaces of base change type are out of the range of the general results due to Garbagnati and Salgado and of the classical results about the Picard very general Enriques surfaces.

In the next part of the section, we prove that the assumptions of very generality (in both the senses) are necessary for the stated results. We find a lot of genus 1 pencils in every $X_m \in \Sigma_m$ behaving very differently with respect to the ones described on the base change very general K3 surfaces and on the K3 surfaces covering a Picard very general Enriques surface.

Lemma 3.4.12. *Let S be a general rational elliptic surface, $g : X \to S$ be a base change such that the fixed fibers S_{t_0} and S_{t_∞} are smooth and let D be a conic bundle on S. Then, $g^*|D|$ is a genus 1 pencil on X having at most eight irreducible singular fibers and least eight reducible singular fibers consisting of two rational curves meeting at two points.*

Proof. Every irreducible singular fiber of the elliptic pencil $g^*|D|$ on X arises from a smooth member $C \in |D|$ tangent to the fixed locus in one point. The conic bundle $|D|$ cut a hyperelliptic series on S_{t_0} and on S_{t_∞}. Since S_{t_0} and S_{t_∞} are smooth elliptic curves, there are four ramification points for both hyperelliptic series. Hence, there are four members of $|D|$ tangent to S_{t_0} and four members of $|D|$ tangent to S_{t_∞}. If the set of the members of $|D|$ tangent to S_{t_0} and the one of the four tangent members to S_{t_∞} are disjoint, the pencil $g^*|D|$ will have eight irreducible singular fibers of type I_1. If some of the tangent members coincide, the induced singular fibers will be less and more complicated (see [31] for a precise statement and for a deeper investigation of the singular fibers of the conic bundles).

A reducible fiber of the conic bundle, that consists of a chain of rational curves with negative self-intersection, induces a reducible singular fiber on $g^*|D|$. If S is general, there are no (-2)-curves lying on S and therefore the only singular fibers that can appear in $|D|$ are pairs of (-1)-curves meeting at one point, giving rise to singular fibers consisting of two smooth rational curves meeting at two points (see [31, Theorem 5.3] for a reference about the singular fibers induced by the ones of a conic bundle). Since every genus 1 pencil on a K3 surface is such that the sum of Euler characteristics of its singular fibers is equal to 24 and the contribution given by the irreducible singular fibers to $g^*|D|$ is at most 8 (the Euler characteristic of a rational nodal curve is equal to 1), we need at least other eight singular fibers of type I_2 (whose Euler characteristic is equal to 2), whence $|D|$ admits at least eight reducible singular fibers that consist of two (-1)-curves meeting at one point. $\square$

The next results show that the assumption ι^* acting as the identity on $\mathrm{NS}(X)$ is necessary for Corollary 3.4.8. For a K3 surface $X_m \in \Sigma_m$, we can systematically find examples of genus 1 pencils on X_m of type 3) with respect to ι, even if S is general and S_{t_0} and S_{t_∞} are smooth.

Proposition 3.4.13. *Let $X_m \in \Sigma_m$ and let D be a conic bundle on S.*
Then, $g^*|D| \boxplus R_m$ is a genus 1 pencil of type 3) on X_m with respect to ι.

Proof. By Proposition 3.3.30 $g^*|D| \boxplus R_m$ is identified with $g^*|D| \boxminus R_m$ by g_*. Now, let $C \in |D|$: for every $t \in \mathbb{P}^1$, we set $g^{-1}(C)_{|X_t} = a_1 Q_{1,t} + \cdots + a_k Q_{k,t}$. We have that

$g^*|D| \boxplus R_m \nsim g^*|D| \boxminus R_m$: suppose that $g^{-1}(C) \boxplus R_m \sim g^{-1}(C) \boxminus R_m$. It follows that for every $t \in \mathbb{P}^1$, the degree $a_1 + \cdots + a_k$ divisors $a_1 Q_{1,t} \boxplus R_{m_t} + \cdots + a_k Q_{k,t} \boxplus R_{m_t}$ and $a_1 Q_{1,t} \boxminus R_{m_t} + \cdots + a_k Q_{k,t} \boxminus R_{m_t}$ are linearly equivalent in the elliptic curve X_t. In other words, we have the equality

$$Q_{1,t}^{\boxplus a_1} \boxplus R_{m_t} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxplus R_{m_t} = Q_{1,t}^{\boxplus a_1} \boxminus R_{m_t} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxminus R_{m_t},$$

from which

$$Q_{1,t}^{\boxplus a_1} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxplus R_{m_t}^{\boxplus k} = Q_{1,t}^{\boxplus a_1} \boxplus \cdots \boxplus Q_{k,t}^{\boxplus a_k} \boxplus R_{m_t}^{\boxplus k}$$

and thus

$$R_{m_t}^{\boxplus k} = R_{m_t}^{\boxminus k},$$

or equivalently

$$R_{m_t}^{\boxplus 2k} = O_t,$$

for every t, which is a contradiction since by Lemma 3.3.24 R_m has infinite order in $\mathrm{MW}(X_m)$.

□

We are able to find other genus 1 pencils of type 3) with respect to ι by starting from a genus 1 pencil of the Enriques surface Y_m covered by $X_m \in \Sigma_m$.

Proposition 3.4.14. *Let Y_m be a general m-special Enriques surface of base change type, and let p_Y be a genus 1 pencil on Y_m different from the one induced by ϵ_S. Thus, $\epsilon := f^*(p_Y)$ is a genus 1 pencil on the universal cover X_m of Y_m that is of type 3) with respect to ι.*

Proof. Let us assume $m > 1$. Since by Proposition 3.2.30 there are no (-2)-curves on Y_m, we have that p_Y has just irreducible singular fibers (since all the components of a

reducible fiber are smooth rational curves) and the same for ϵ. By Lemma 3.4.12, every genus 1 pencil on X_m induced by a conic bundle on S has reducible singular fibers, thus ϵ cannot be of type 1). If $m = 1$, it could happen that one of the singular fibers of p_Y is a nodal half-fiber, producing a singular fiber of ϵ consisting of two (-2)-curves. But Lemma 3.4.12 states that every genus 1 pencil on X_m induced by a conic bundle on S has at least eight reducible singular fibers.
If $m = 0$, since Y is general in Σ_0, p_Y has at most one reducible fiber (see [24, Theorem 6.5.5]), while by Lemma 3.4.12 any genus 1 pencil on X induced by a conic bundle has at least eight reducible singular fibers.
Moreover, by Theorem 3.4.7 the only genus 1 pencil of type 2) is ϵ_X, the elliptic fibration induced by ϵ_S. Since by hypothesis p_Y is not the pushforward of ϵ_X, we can conclude that ϵ is of type 3) with respect to ι.

□

We are able to show the first examples of genus 1 pencils on a K3 surface that covers an Enriques surface not of type 2) with respect to the Enriques involution.

Proposition 3.4.15. *Let $X_m \in \Sigma_m$ and D be a conic bundle on S. Then, $g^*|D|$ is a genus 1 pencil of type 3) with respect to τ and $|D|_Y = f_*(g^*|D|)$ is an equigeneric special nonregular Severi variety of curves on Y_m.*

Proof. By Proposition 3.3.25 $g^*|D|$ is identified with $g^*|D| \boxplus R_m$ by f_*, and by 3.3.26 we have $g^*(D) \nsim g^*(D) \boxplus R_m$. Finally, since the genus of D is zero, the proof is an application of Proposition 3.3.27. □

Remark 3.4.16. *This result allows us to find other rational curves on Y_m. In fact, since the singular fibers of $g^*|D|$ are rational, also their images on Y_m are rational.*

By applying the Cantat-Dolgachev Formula, we can compute the arithmetic genus of D_Y for some conic bundles D on S.

Proposition 3.4.17. *Let D be a conic bundle on S of the form $L - E_i$, with $i \in \{1, \dots, 9\}$. Then, D_Y has arithmetic genus $4m + 5$.*

Proof. Without loss of generality, we choose the conic bundle $D \sim L - E_1$.

D is a bisection of the elliptic pencil and we can decompose it in $D \sim L - E_1 \sim (L - E_1 - E_2) + E_2$.

By applying the formula due to Cantat-Dolgachev, one can see that

$$B_{S,m} \boxplus E_2 \sim 6(m+1)L - 2(m+1)E_1 - 2mE_2 - 2(m+1)E_3 - \dots - 2(m+1)E_9$$

and

$$B_{S,m} \boxplus (L - E_1 - E_2) \sim$$
$$(6m+8)L - (2m+4)E_1 - (2m+4)E_2 - 2(m+1)E_3 - \dots - 2(m+1)E_9.$$

Now it is easy to compute the product

$$D \cdot (D \boxplus B_{S,m}) = (E_2 + (L - E_1 - E_2)) \cdot ((6(m+1)L - 2(m+1)E_1 - 2mE_2 - 2(m+1)E_3 - \dots - 2(m+1)E_9) + (6m+8)L - (2m+4)E_1 - (2m+4)E_2 - 2(m+1)E_3 - \dots - 2(m+1)E_9)) =$$

$$= 8m + 8.$$

Finally, the arithmetic genus $p(D_Y)$ is equal to

$$\frac{1}{2} D \cdot (D \boxplus B_{S,m}) + p(g^*(D)) = 4m + 4 + 1 = 4m + 5.$$

$\square$

The same result holds for every conic bundle of the form $2L - E_{i_1} - \dots - E_{i_4}$, with $i_j \in \{1, \dots, 9\}$ and for every conic bundle of the form $3L - 2E_{i_1} - E_{i_2} - \dots - E_{i_6}$, with $i_j \in \{1, \dots, 9\}$. The proof is analogous to the one given for the conic bundles $L - E_i$ and we omit it: it is just an application of the Cantat-Dolgachev formula. We are not able to prove that for every conic bundle D on S the arithmetic genus of D_Y is $4m+5$, but the computations we did lead us to formulate the following Conjecture.

Conjecture 3.4.1. Let D be a conic bundle on S. Then, D_Y has arithmetic genus $4m+5$.

Notice that to prove the Conjecture, it would be sufficient to show that every conic bundle can be obtained as pushforward of a conic bundle of the form $L-E_i$, $2L-E_{i_1}-\cdots-E_{i_4}$ or $3L-2E_{i_1}-E_{i_2}-\cdots-E_{i_6}$ under an automorphism of S.

3.5 Rational curves on the very general Enriques surface

In this section we prove that the very general Enriques surface admits rational curves of arbitrarily large arithmetic genus with $\phi=2$. We saw that an m-special curve $B_{Y,m}$ is such that $\phi(B_{Y,m})=1$. A consequence of Theorem 3.2.6 is that a rational curve in the very general Enriques surfaces cannot have $\phi=1$. What happens is that two m-special curves on Y_m deform together to give rise to an irreducible rational curve with $\phi=2$. We will actually prove the phenomenon by looking at the deformations of the K3's that cover our special and very general Enriques surfaces. In particular, we exploit the "regeneration" results given by Chen, Gounelas and Liedtke in [20] about curves on K3 surfaces. We show that the union of two suitable rational curves on X_m deforms to an irreducible rational curve on the very general K3 surface with an Enriques involution. Actually, our proof holds for every K3 surface that is a double covering of a smooth quadric.

3.5.1 Double coverings of quadrics

First of all, we review and comment the model of the K3 surfaces covering an Enriques surface as a double cover of $\mathbb{P}^1\times\mathbb{P}^1$ presented by Barth, Peters, Hulek and Van de Ven in [3].

Let $Q\cong\mathbb{P}^1\times\mathbb{P}^1$ be a smooth quadric in $\mathbb{P}^3$ and let $((x_o:x_1),(y_0:y_1))$ be bihomogeneous coordinates on it. We define the involution σ on Q by

$$\sigma((x_o:x_1),(y_0:y_1))=((x_o:-x_1),(y_0:-y_1)).$$

Theorem 3.5.1. *[(Horikawa's representation of nonnodal Enriques surfaces)] Let Y be a nonnodal Enriques surface, $f:X\to Y$ be the universal covering, τ be the Enriques involution and $\epsilon_1,\epsilon_2\subset Y$ be two half pencils with $\epsilon_1\cdot\epsilon_2=1$. If τ and σ are defined* *as above, then there is a σ-invariant bihomogeneous polynomial of bidegree $(4,4)$ in $(x_o, x_1), (y_0, y_1)$ with zero-set B on the smooth quadric Q, such that the universal covering X of Y is the minimal resolution of the double covering of Q ramified over B. The curve B is reduced with at worst simple singularities and does not contain any fixed point of τ. The involution τ on X is induced by the involution σ on Q. The two rulings of Q define the two genus 1 pencils $f^*(\epsilon_1)$ and $f^*(\epsilon_2)$ on X.*

Proof. See [3, Theorem 18.1]. □

We denote by $\pi : X \to Q$ the described double covering. Theorem 3.5.1 has a converse: as shown by the authors in [3, V, Section 23], given a σ-invariant curve as in the Theorem, the K3 surface X and an Enriques surface $Y = X/\tau$ can be constructed from it.

We want now to apply this model to the case of Enriques surfaces of base change type. In particular, we will see that also the general nodal Enriques surface admits such a model with B smooth, completing the description due to Barth, Peters, Hulek and Van de Ven.

Sometimes, we shall write $B_{\epsilon_1,\epsilon_2}$ to indicate the branch locus of the double cover $\pi : X \to Q$ associated to the two genus 1 pencils ϵ_1 and ϵ_2. Recall that an Enriques surface is said to be nonnodal if it does not admit any smooth rational curve.

Lemma 3.5.2. *Let Y_m be an m-special Enriques surface and let ϵ_Y be an m-special genus 1 pencil of Y_m. Then, there exists a half-fiber $\epsilon \subset Y$ such that $\epsilon_Y \cdot \epsilon = 1$.*

Proof. The result holds for every genus 1 pencil on an Enriques surface, see for example [15, Section 3]. □

Lemma 3.5.3. *If Y is nonnodal, then $B_{\epsilon_1,\epsilon_2}$ is smooth for every choice of ϵ_1 and ϵ_2. In particular, the assertion holds for a general m-special Enriques surface Y_m.*

Proof. Suppose B has a singular point q, that by Theorem 3.5.1 turns out to be a simple singularity. Let $L_1 \in |\mathcal{O}(1,0)|$ and $L_2 \in |\mathcal{O}(0,1)|$ be the two members of the rulings intersecting B at q. Let us denote by $\pi' : X' \to Q$ the double covering of Q ramified over B. Since the branch locus B is singular at q, also X' has a singularity at q. The K3 surface X, that is the universal covering of Y, is the minimal resolution of X': let us denote by $E_q \subset X$ the exceptional locus over q and by $\tilde{L}_1 \subset X$ and $\tilde{L}_2 \subset X$ the strict transforms of $(\pi')^{-1}(L_1) \subset X'$ and $(\pi')^{-1}(L_2) \subset X'$ respectively. Since q is a simple singularity, E_q consists of a chain of (-2)-curves. As usual f denote the Enriques quotient: the genus 1 pencil $f^*(\epsilon_1)$ on X, that is the pullback of the ruling $|\mathcal{O}(1,0)|$, has a reducible member consisting of the union of E_q and $\tilde{L}_1$, while the genus 1 pencil $f^*(\epsilon_2)$ on X has $E_q + \tilde{L}_2$ as singular fiber. This proves that both the pencils $f^*(\epsilon_1)$ and $f^*(\epsilon_2)$ have a member consisting of a configuration of at least two (-2)-curves. Since they share the exceptional locus E_q, these singular members cannot be invariant with respect to τ. So τ sends, say, $E_q + \tilde{L}_1$ to another member of $f^*(\epsilon_1)$ and hence the genus 1 pencil ϵ_1 on Y has a member consisting of at least two smooth rational curves. Since Y is nonnodal, this is a contradiction. $\square$

Recall that Σ_0 is the nine-dimensional family parametrizing K3 surfaces that cover a nodal Enriques surface. Let $S \cong \mathrm{Bl}_{\{P_1\dots,P_9\}}\mathbb{P}^2$ be a general rational elliptic surface, Y_0 and X_0 be a nodal Enriques surface and a K3 surface in Σ_0 obtained with the base change construction ramified over two smooth fibers of S, and let ϵ_{Y_0} and ϵ_{X_0} be the genus 1 pencils on Y_0 and X_0 induced by the one of S. Recall that $\tilde{E}_9 \in \mathrm{MW}(X_0)$ denote the zero-section of ϵ_{X_0} and that $R_0 \in \mathrm{MW}(X_0)$ denote the section of ϵ_{X_0} identified with $\tilde{E}_9$ by the Enriques quotient.

Lemma 3.5.4. *The linear system $|\tilde{E}_9 + \tilde{E}_1 \boxplus R_0|$ is a genus 1 pencil on X_0 of type 2) with respect to τ. Consequently, $f_*|\tilde{E}_9 + \tilde{E}_1 \boxplus R_0|$ is an elliptic pencil on Y_0.*

Proof. First of all, by applying the Cantat-Dolgachev formula, one can see that

$$\tilde{E}_9 \cdot \tilde{E}_1 \boxplus R_0 = 2,$$

proving that $|\tilde{E}_9 + \tilde{E}_1 \boxplus R_0|$ is a genus 1 pencil. Now, since

$$\tau(\tilde{E}_9) = R_0 \text{ and } \tau(\tilde{E}_1 \boxplus R_0) = \tilde{E}_1,$$

we have that

$$\tau(\tilde{E}_9 + \tilde{E}_1 \boxplus R_0) = \tilde{E}_1 + R_0.$$

Moreover, we have

$$(\tilde{E}_9 + \tilde{E}_1 \boxplus R_0) \cdot (R_0 + \tilde{E}_1) = 0,$$

whence $\tilde{E}_9 + \tilde{E}_1 \boxplus R_0$ and $R_0 + \tilde{E}_1$ are linearly equivalent. This proves that

$$|\tilde{E}_9 + \tilde{E}_1 \boxplus R_0|$$

is of type 2) with respect to τ. □

We conclude the section by showing that also a general nodal Enriques surface admits a model as quotient of a K3 surface that is a double covering of a smooth quadric Q ramified over a smooth curve. Thanks to Lemma 3.5.4, we can choose ϵ_{Y_0} and $\epsilon_{R_0} := f_*|\tilde{E}_9 + \tilde{E}_1 \boxplus R_0|$ to produce the model of X_0 as a double covering of Q. Indeed, since $\epsilon_{X_0} \cdot (\tilde{E}_9 + \tilde{E}_1 \boxplus R_0) = 2$ (both $\tilde{E}_9$ an $\tilde{E}_1 \boxplus R_0$ are sections for ϵ_{X_0}), we have that a half-fiber of ϵ_{Y_0} meets a half-fiber of ϵ_{R_0} exactly in one point.

Proposition 3.5.5. *Let Y_0 be a general nodal Enriques surface and let X_0 be its universal cover. Let $\pi : X_0 \to Q$ be the double covering associated to the half-fibers ϵ_{Y_0} and ϵ_{R_0}. Then, the branch locus $B_{\epsilon_{Y_0}, \epsilon_{R_0}} \subset Q$ is smooth.*

Proof. As in the proof of Lemma 3.5.3, the existence of a singular point of B would imply the presence of a (-2)-curve as a component of a fiber of ϵ_{Y_0}. But since ϵ_{Y_0} is a special genus 1 pencil (it admits a smooth rational bisection) and Y_0 is general nodal, this is a contradiction (see [24, Theorem 6.5.5 (vii)]). □

3.5.2 Rational curves

Let Y be a very general Enriques surface, X be its universal cover and Q be a smooth quadric surface. Let $f : X \to Y$ denote the Enriques quotient, $\pi : X \to Q$ denote the quotient over Q associated to two half-fibers on Y and B denote the branch locus of π. We investigate the problem of the existence of rational curves on Y. Let us suppose Y has an irreducible rational curve C.

Lemma 3.5.6. *If $C \subset Y$ is an irreducible rational curve, then $f^{-1}(C)$ consists of two linearly equivalent rational curves C_1' and C_2' on X.*

Proof. As in Section 3.3, we denote by $\nu_C : \overline{C} \to C$ the normalization of C and define $\eta_C := \mathcal{O}_C(K_S) = \mathcal{O}_C(-K_S)$, a nontrivial 2-torsion element in $\mathrm{Pic}^0(C)$, and $\eta_{\overline{C}} := \nu^*\eta_C$. By standard results on covering of complex manifolds (see [13, Section 2] or [3, Section 1.17]), two cases may happen:

- $\eta_{\overline{C}} \ncong \mathcal{O}_{\overline{C}}$ and $f^{-1}(C)$ is irreducible;
- $\eta_{\overline{C}} \cong \mathcal{O}_{\overline{C}}$ and $f^{-1}(C)$ consists of two irreducible components conjugated by the Enriques involution τ.

Since C is rational, its normalization $\overline{C}$ is isomorphic to $\mathbb{P}^1$, whence any degree 0 line bundle on it is linearly equivalent to $\mathcal{O}_{\overline{C}}$: it implies that $f^{-1}(C)$ consists of two rational curves C_1' and C_2'. Moreover, since Y is very general, we have that τ^* acts as the identity on $\mathrm{NS}(X)$, so that $C_1' \sim C_2'$.

□

The images $C_1'' := \pi(C_1')$ and $C_2'' := \pi(C_2')$ on Q have to be rational curves. In order to better understand the curves $C \subset Y$ and $C' := C_1' + C_2' \subset X$, we begin our analysis by starting from $C_i'' \subset Q$. The curves C_i'' belong to the linear system $|\mathcal{O}_Q(m,n)|$, with $i = 1, 2$, for some nonnegative integers m and n.

Lemma 3.5.7. *Suppose that $m = 1$ or $n = 1$. Then, $\pi^{-1}(C_i'') = C_i'$, for $i = 1, 2$, with C_i' irreducible and rational on X. Equivalently, there exists an intersection point between C_i''*

and the branch locus B of $\pi : X \to Q$ at which the intersection has odd order.

Proof. The first statement is equivalent to the second since a curve in Q splits in the K3 cover X if and only if it intersects the branch locus B with even order at every intersection point. Suppose that $\pi^{-1}(C_i'')$ consists of two curves C_i' and $\tilde{C}_i'$. Now, if $m = 1$ or $n = 1$, we have that the curve C_i'' is a section for one of the rulings $\mathcal{O}_Q(0,1)$ or $\mathcal{O}_Q(1,0)$. Hence, we have, say, $C_i'' \cdot \mathcal{O}_Q(1,0) = 1$, from which $(C_i' + \tilde{C}_i') \cdot \pi^*\mathcal{O}_Q(0,1) = 2$. Since $C_i' \cdot \pi^*\mathcal{O}_Q(0,1) = 0$ would imply that C_i' is a member of the elliptic pencil $|\pi^*\mathcal{O}_Q(0,1)|$ and the same for $\tilde{C}_i'$, we have $C_i' \cdot \mathcal{O}_Q(0,1) = \tilde{C}_i' \cdot \mathcal{O}_Q(0,1) = 1$, so that the rational curves C_i' and $\tilde{C}_i'$ are sections for the elliptic pencil $|\pi^*\mathcal{O}_Q(0,1)|$ and therefore smooth (-2)-curves. Since X covers a very general Enriques surface Y, this is a contradiction. $\square$

Proposition 3.5.8. *Suppose that $m = 1$ or $n = 1$ and denote by p an intersection point between C_i'' and B at which the intersection has odd order k. Then, there exists another intersection point p' between C_i'' and B at which the intersection has odd order k'. Furthermore, $C_i'' \in |\mathcal{O}_Q(m,n)|$ is a rational curve tangent to B at $r := 2m + 2n - \frac{k-1}{2} - \frac{k'-1}{2} - 1$ (possibly coincident) points $q_1, \ldots, q_r$ with even order.*

Proof. In order to show the existence of p', it is sufficient to notice that the intersection number $C_i'' \cdot B = 4m + 4n$ is even. We have

$$C_i'^2 = 2C_i''^2 = 2(\mathcal{O}_Q(m,n)^2) = 2(2mn) = 2(2mn + 1 - 1),$$

from which the arithmetic genus of C_i' is

$$p_a(C_i') = 2mn + 1.$$

Since m or n is equal to 1, the curve C_i'' is smooth as well as the branch locus B, and therefore the singularities of C_i' come from points at which C_i'' is tangent to B. Suppose C_i'' is tangent to B at a point s with order k_s and consider local coordinates (x, y) for Q in such a way B has local equation $y = 0$ at s: the curve C_i'' has local equation $y - x^{k_s}$ at s. If the double covering is given by $(x,t) \mapsto (x,t^2)$ in the local coordinates (x,y,t), we have that C_i' has local equation $t^2 - x^{k_s}$ in a neighborhood of $\pi^{-1}(s)$. This implies that C_i' has an A_{k_s-1} singularity at $\pi^{-1}(s)$. Now, if k_s is even, an A_{k_s-1} singularity imposes $\frac{k_s}{2}$ conditions on the genus of C_i', while if k_s is odd an A_{k_s-1} singularity imposes $\frac{k_s-1}{2}$ conditions. Since C_i' is rational, its geometric genus $g(C_i')$ is

$$g(C_i') = 2mn + 1 - \tfrac{k-1}{2} - \tfrac{k'-1}{2} - \sum_{j=1}^{r'} \tfrac{k_j}{2} - \sum_{j=r'+1}^{r} \tfrac{k_j-1}{2} = 0,$$

where k_j indicates the intersection order between C_i'' and B at the point q_j, the first sum runs over the points at which the intersection has even order and the second sum runs over the points at which the intersection has odd order. We have

$$\sum_{j=1}^{r'} \tfrac{k_j}{2} + \sum_{j=r'+1}^{r} \tfrac{k_j-1}{2} = 2mn + 1 - \tfrac{k-1}{2} - \tfrac{k'-1}{2} =$$

$$= 2m + 2n - 1 - \tfrac{k}{2} - \tfrac{k'}{2} + 1 = 2m + 2n - \tfrac{k}{2} - \tfrac{k'}{2},$$

where the second equality holds since for $m = 1$ or $n = 1$, we have $2mn + 1 = 2m + 2n - 1$. But $C_i'' \cdot B = \mathcal{O}(m,n) \cdot \mathcal{O}(4,4) = 4m + 4n$, so that we have

$$\sum_{j=1}^{r} k_j = 4m + 4n - k - k',$$

from which

$$\sum_{j=1}^{r} \tfrac{k_j}{2} = 2m + 2n - \tfrac{k}{2} - \tfrac{k'}{2}.$$

Hence, $r' = r$ and the sum $\sum_{j=r'+1}^{r} \frac{k_j-1}{2}$ is empty, which means that the intersection order between C_i'' and B at q_j is even for every q_j. $\square$

Remark 3.5.9. *In the situation of our interest, at least one between m and n will be equal to 1: in this case, all the irreducible curves of $|\mathcal{O}_Q(m,n)|$ are smooth, so they are isomorphic* *to their normalization. In other words, the condition of Proposition 3.5.8 is equivalent to requiring that* C_i' *belongs to the logarithmic Severi variety* $V_{0,\alpha}(\mathbb{P}^1 \times \mathbb{P}^1, B, \mathcal{O}_Q(m,n))$, *where* $\alpha = [2, 2mn+1]$.

Corollary 3.5.10. *If* $C \subset Y$ *is a rational curve, then its arithmetic genus* $p(C)$ *is such that*

$$p(C) \equiv_4 1.$$

Proof. By Theorem 3.2.6, C is 2-divisible in $\mathrm{Num}(Y)$, i.e., there exists an effective divisor D on Y such that $C \sim 2D$. We have

$$p_a(C) = \tfrac{1}{2}C^2 + 1 = \tfrac{1}{2}(4D^2) + 1 = 2D^2 + 1.$$

Since for an Enriques surface D^2 is even for every effective divisor D, we have the assertion. □

By reversing the process, to prove the existence of rational curves on Y, it is sufficient to find a curve in the linear system $|\mathcal{O}_Q(m,n)|$ satisfying the conditions of Proposition 3.5.8.

The aim of the last part of the thesis is to show that the logarithmic Severi variety $V_{0,\alpha}(\mathbb{P}^1 \times \mathbb{P}^1, B, \mathcal{O}(m,n))$, with $\alpha = [2, 2m+2n-1]$, is nonempty for values of m and n such that mn is arbitrarily large with $m = 1$ or $n = 1$, or, equivalently, that the very general Enriques surface Y admits rational curves of arbitrarily large arithmetic genus. From now on, we shall write just V_α to indicate $V_{0,\alpha}(\mathbb{P}^1 \times \mathbb{P}^1, B, \mathcal{O}(m,n))$ (or $V_{\alpha,B}$ when we want to focus on the curve B).

Lemma 3.5.11. *If* $V_{0,\alpha}$ *is nonempty, then its dimension is 0.*

Proof. Suppose that there exists $[C''] \in V_{0,\alpha}$. Since $m = 1$ or $n = 1$, we have that C'' is a smooth rational curve and therefore it is isomorphic to its normalization. We have $\alpha_1 = 2$ and $\alpha_2 = 2m + 2n - 1$, so we let $Q_{1,1}, Q_{1,2}, Q_{2,1}, \ldots, Q_{2,2m+2n-1}$ be points in C'' such that, denoting by $\mu : C'' \to Q$ the inclusion $C'' \subset Q$, we have $\mu^*(B) = Q_{1,1} + Q_{1,2} + 2Q_{2,1} + \cdots + 2Q_{2,2m+2n-1}$ as in Definition 3.3.5. We set $D = Q_{2,1} + \cdots + Q_{2,2m+2n-1}$: by construction, $d := \deg D_{|C''} = 2m + 2n - 1$. We have that

$$-(K_{\mathbb{P}^1\times\mathbb{P}^1})\cdot C'' - \deg \mu_* D_{|C''} = \mathcal{O}(2,2)\cdot\mathcal{O}(m,n) - (2n+2m-1) =$$
$$2m+2n-2n-2m+1 = 1.$$

Then, by Theorem 3.3.6

$$\dim(V_{0,\alpha}) = -(K_{\mathbb{P}^1\times\mathbb{P}^1} + \mathcal{O}(4,4))\cdot\mathcal{O}(m,n) - 1 + |\alpha| =$$
$$-2m-2n-1+2m+2n+1 = 0.$$

□

The next result describes the images of the sections $\tilde{E}_9$ and R_m of a K3 surface $X_m \in \Sigma_m$ in Q under π.

Proposition 3.5.12. *Let $X_m \in \Sigma_m$ be a K3 surface covering an m-special Enriques surface Y_m, let ϵ_X and ϵ_Y be the genus 1 pencils on X_m and Y_n respectively induced by the base change construction, let ϵ be a genus 1 pencil on Y_m with $\epsilon_Y \cdot \epsilon = 4$ (or, equivalently, such that two respective half-fibers intersect in one point) and let $\pi : X_m \to Q$ be the model associated to the pencils ϵ_Y and ϵ. Then, $\pi(\tilde{E}_9)$ and $\pi(R_m)$ belong to the linear system $|\mathcal{O}_Q(1, k(m))|$ for some nonnegative integer $k(m)$. Moreover, they are totally tangent to $B_{\epsilon_Y,\epsilon}$.*

We point out that for what follows, we shall need the existence of two sections for ϵ_X intersecting each other in $2m$ points and such that they are interchanged by τ but not by σ. Without loss of generality, we can suppose that these two sections are $\tilde{E}_9$ and R_m. Suppose that $\tilde{E}_9 + R_m$ is invariant under the involution σ: since σ acts as the identity on the base $\mathbb{P}^1$ of the elliptic pencil ϵ_X, the restriction of σ to any smooth fiber X_t is an involution identifying the point R_{m_t} with the point O_t. Since $R_m \in \mathrm{MW}(X_m)$ has infinite order, in particular it is not a 2-torsion section: hence, there is an open subset $Z_1 \subset \mathbb{P}^1$ for which for every $t \in Z_1$, the point R_{m_t} where the section R_m meets X_t is not a 2-torsion point. The only involution of an elliptic curve that sends the origin to a chosen point Q, that is not 2-torsion, is the composition $\boxplus Q \circ (-1)$ (or, equivalently, $(-1) \circ \boxminus Q$) between (-1) and the translation by Q. Now, consider the sections $\tilde{E}_1$ and $\tilde{E}_1 \boxplus R_m$ in $\mathrm{MW}(X_m)$, where $E_1 \in \mathrm{MW}(S)$. Since $\tilde{E}_1$ has infinite order in $\mathrm{MW}(X_m)$, we can also consider the open subset $Z_2 \subset \mathbb{P}^1$ for which for every $t \in Z_2$, the intersection E_{1_t} between E_1 and the fiber X_t is not a 2-torsion point. By Lemma 3.3.11, $\tilde{E}_1$ and $\tilde{E}_1 \boxplus R_m$ are interchanged by τ and $\tilde{E}_1 \cdot (\tilde{E}_1 \boxplus R_m) = 2m$. Moreover, let $t \in Z_1 \cap Z_2$: the involution σ restricted to X_t acts as $\boxplus R_{m_t} \circ (-1)_t$, whence it sends the point $\tilde{E}_{1_t}$ to $\boxminus \tilde{E}_{1,t} \boxplus R_{m_t}$. This implies that σ does not identify the two sections $\tilde{E}_1$ and $\tilde{E}_1 \boxplus R_{m_t}$ unless $E_{1_t} = \boxminus E_{1_t}$ for every t, but we constructed the open subset $Z_1 \cap Z_2$ of the base $\mathbb{P}^1$ in such a way for every $t \in Z_1 \cap Z_2$ the point $\tilde{E}_{1_t}$ is not a 2-torsion point.

Proof of Proposition 3.5.12. Every (-2)-curve s of X is identified by π with another (-2)-curve: if, otherwise, the morphism π restricted to s was 2:1 to the image $\pi(s)$, we would have $\pi(s)^2 = -1$, but this is a contradiction since by Lemma 3.5.3 the branch locus $B_{\epsilon_Y,\epsilon}$ of π is smooth. Hence, the image of s is a rational curve intersecting $B_{\epsilon_Y,\epsilon}$ with even order at every intersection point. Since $\tilde{E}_9$ is a section for ϵ_X, the quotient map π identifies it with another section s_1 and their image $\pi(\tilde{E}_9) = \pi(s_1)$ is a section for $\pi_*(\epsilon_X) = \mathcal{O}(0,1)$, so that $\pi(\tilde{E}_9) \in |\mathcal{O}(1, k(m))|$ for some $k(m) \in \mathbb{Z}_+$. The same happens to the section R_m: it is identified by π with another section s_2. Furthermore, since τ interchanges $\tilde{E}_9$ and R_m, whence σ interchanges $\pi(\tilde{E}_9)$ and $\pi(R_m)$. Finally, since σ preserves the linear equivalence, also $\pi(R_m) = \pi(s_2) \in |\mathcal{O}(1, k(m))|$. $\square$

Lemma 3.5.13. *$s_1 \cdot s_2 \geq 2m$ and thus $k(m) \geq m$.*

Proof. We have

$$\sigma(\pi(\tilde{E}_9)) = \sigma(\pi(R_m))$$

and

$$\pi^{-1}(\pi(E_9)) = \tilde{E}_9 + s_1 \text{ and } \pi^{-1}(\pi(R_m)) = R_m + s_2,$$

whence τ interchanges s_1 and s_2. By Lemma 3.3.14 a section

$$\tilde{E} \boxplus R_m^{\boxplus k} \in \mathrm{MW}(X_m)$$

is interchanged by τ with the section

$$\tilde{E} \boxplus R_m^{\boxplus(1-k)} \in \mathrm{MW}(X_m).$$

Furthermore,

$$\tilde{E} \boxplus R_m^{\boxplus k} \cdot \tilde{E} \boxplus R_m^{\boxplus(1-k)} = R_m^{\boxplus k} \cdot R_m^{\boxplus(1-k)} =$$
$$= \tilde{E}_9 \cdot R_m^{\boxplus(1-2k)} = \tilde{E}_9 \cdot R_m + R_m \cdot X_m[2k-1]_0 \geq \tilde{E}_9 \cdot R_m = 2m.$$

Finally,

$$2\mathcal{O}(1, k(m))^2 = 4k(m)$$

and

$$2\mathcal{O}(1, k(m))^2 = (\tilde{E}_9 + s_1) \cdot (R_m + s_2) = 2m + \tilde{E}_9 \cdot s_2 + s_1 \cdot R_m + s_1 \cdot s_2 \geq$$
$$4m + \tilde{E}_9 \cdot s_2 + s_1 \cdot R_m \geq 4m,$$

from which

$$k(m) \geq m.$$

□

We proved the following Corollary.

Corollary 3.5.14. *For every m, there exists a family of curves $\mathfrak{B}_m \subset |\mathcal{O}(4,4)|$ such that for every $B_m \in \mathfrak{B}_m$, the logarithmic Severi variety $V_{0,\alpha}(\mathbb{P}^1 \times \mathbb{P}^1, B_m, \mathcal{O}(1, k(m)))$ is nonempty for some $k(m) \geq m$, with $\alpha = [2, 2k(m)+1]$.*

Remark 3.5.15. *Actually, if $B_m \in \mathfrak{B}_m$, we have that $V_{\alpha,B_m} \neq \emptyset$ even for $\alpha = [0, 2k(m)+2]$, but for our purpose the nonemptiness of V_{α,B_m} for $\alpha = [2, 2k(m)+1]$ is sufficient.*

The next result states that on the very general Enriques surface there are rational curves of arbitrarily large arithmetic genus and $\phi = 2$.

Theorem 3.5.16. *Let Y be a very general Enriques surface. Then, for every $n \in \mathbb{Z}$, there is $\overline{n} \in \mathbb{Z}$, with $\overline{n} \geq n$, such that there exists a rational curve $C_{\overline{n}} \subset Y$ of arithmetic genus $\overline{n}$ and with $\phi(C_{\overline{n}}) = 2$.*

To prove Theorem 3.5.16, we need the following definitions and results about stable maps on families of K3 surfaces. We follow [40] for the original Definitions of stable maps and their moduli space and [20] for the regeneration results about curves on families of K3 surfaces.

Definition 3.5.17. *Let V be a complex algebraic variety.*
A stable map is a structure $(C, x_1, \dots, x_k, f)$ consisting of a connected compact reduced curve C with $k \geq 0$ pairwise distinct marked nonsingular points x_i and at most ordinary double singular points, and a map $f : C \to V$ having no nontrivial first order infinitesimal automorphisms, identical on V and $x_1, \dots, x_k$.

The condition of stability means that every irreducible component of C of genus 0 (resp. 1) which maps to a point must have at least 3 (resp. 1) special (i.e. marked or singular) points on its normalization. There is a notion of isomorphism of stable maps and of moduli space of stable maps due to Kontsevich: $\overline{M}_p(V, \beta)$ denotes the moduli stack of stable maps to V of curves of arithmetic genus $p \geq 0$ with $k \geq 0$ marked points such that $f_*[C] = \beta$.

The next result, due to Chen, Gounelas and Liedtke, is our main tool to prove Theorem 3.5.16.

Theorem 3.5.18. *Let $\mathcal{F} \to \mathcal{V}$ be a smooth projective family of K3 surfaces over an irreducible base scheme $\mathcal{V}$ and let $\mathcal{H}$ be a line bundle on $\mathcal{F}$. Then, every irreducible component $M \subset \overline{M}_p(\mathcal{F}/\mathcal{V}, \mathcal{H})$ satisfies*

$$\dim M \geq p + \dim \mathcal{V}.$$

Proof. See [20, Theorem 2.11]. □

We are able to prove Theorem 3.5.16.

Proof of Theorem 3.5.16. We consider a one-dimensional family $\mathcal{Q}$ over an irreducible curve $\mathcal{C}$ such that for every $t \in \mathcal{C}$,

$$\mathcal{Q}_t \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover, we consider the line bundle over $\mathcal{Q}$

$$\mathcal{R} := (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(4,4))$$

defined in the following way: for every $t \in \mathcal{C}$, the restriction of $\mathcal{R}$ to $\mathcal{Q}_t$ is

$$\mathcal{R}_t \cong (\mathcal{Q}_t, \mathcal{O}(4,4)).$$

For every $m \in \mathbb{Z}_+$, we consider another line bundle $\mathcal{H}^m_{\mathcal{Q}}$ over $\mathcal{Q}$ defined in the following way: for every $t \in \mathcal{C}$, the restriction of $\mathcal{H}^m_{\mathcal{Q}}$ to $\mathcal{Q}_t$ is

$$\mathcal{H}^m_{\mathcal{Q},t} \cong (\mathcal{Q}_t, \mathcal{O}(1, k(m))).$$

Moreover, we fix a point $t_0 \in \mathcal{C}$ and consider a curve $B_m \in \mathcal{R}_{t_0} = (\mathcal{Q}_{t_0}, \mathcal{O}(4,4))$ such that $V_{0,\alpha}(\mathcal{Q}_{t_0}, B_m, \mathcal{O}(1, k(m))) \neq \emptyset$ (then a curve B_m belonging to the family $\mathfrak{B}_m \subset |\mathcal{O}(4,4)|$). Finally, we let $\mathcal{W}$ with

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[dr] & \mathcal{R} \arrow[d] \\ & \mathcal{C} \end{tikzcd}$$

be a section of the relative line bundle $\mathcal{R}$, such that $\mathcal{W}_{t_0} = B_m$.

Let now $\mathcal{X} \to \mathcal{C}$ be the family over $\mathcal{C}$ of K3 surfaces such that for every $t \in \mathcal{C}$, $\mathcal{X}_t$ is the K3 surface obtained as the minimal resolution of the double cover of $\mathcal{Q}_t$ ramified over $\mathcal{W}_t$ and let π

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[dr] & \mathcal{Q} \arrow[d] \\ & \mathcal{C} \end{tikzcd}$$

be the relative double covering. For every $m \in \mathbb{Z}_+$, we define the line bundle on $\mathcal{X}$

$$\mathcal{H}^m_{\mathcal{X}} := \pi^* \mathcal{H}^m_{\mathcal{Q}}$$

to be the pullback of $\mathcal{H}^m_{\mathcal{Q}}$ via π. In other words, the restriction of $\mathcal{H}^m_{\mathcal{X}}$ to every $\mathcal{X}_t$ is the pullback of $(\mathcal{Q}_t, \mathcal{O}(1, k(m)))$.

Since $V_{0,\alpha}(\mathcal{Q}_{t_0}, B_m, \mathcal{O}(1, k(m))) \neq \emptyset$, the line bundle $\mathcal{H}^m_{\mathcal{X},t_0}$ admits a special member consisting of two smooth rational curves, that we call $\tilde{E}_9$ and s_1 to be consistent with the notations of Proposition 3.5.12.

We want to show that this curve deforms in the family $\mathcal{X} \to \mathcal{C}$ as a rational curve by using Theorem 3.5.18.

Let L be the union of two copies of $\mathbb{P}^1$ meeting at one point and consider the map $f : L \to \mathcal{X}_{t_0}$ such that $f(L) = \tilde{E}_9 + s_1$. Since there are no contracted components, (f, L) is a stable map. Now, consider the moduli stack

$$\overline{M}_o(\mathcal{X}/\mathcal{C}, \mathcal{X}_{\mathcal{H}}).$$

We proved that it is nonempty, whence by Theorem 3.5.18 every irreducible component of it has dimension $\dim \mathcal{C}$, that is 1. By Lemma 3.5.11, the fibers over the points of $\mathcal{C}$ have dimension 0, thus, for every t, the K3 surface $\mathcal{X}_t$ is such that the line bundle $\mathcal{X}_{\mathcal{H},t}$ has a rational member, that is sent to a rational member of $\mathcal{Q}_t$.

We actually proved that for every $B \in |\mathcal{O}(4,4)|$, $V_{0,\alpha}(\mathcal{Q}, B, \mathcal{O}(1, k(m))) \neq \emptyset$. Since $\mathcal{W}$ is an arbitrary section of the relative line bundle $\mathcal{R}$ over $\mathcal{Q}$, in particular we have that, for every $m \in \mathbb{Z}$, the very general Enriques surface admits a rational curve $B_{k(m)}$ of arithmetic genus $4k(m) + 1$. The ϕ-invariant of all these curves is 2 and it is computed by the genus 1 pencil on Y induced by the ruling $\mathcal{O}(0,1)$ of Q. Indeed, with he notation of Lemma 3.5.6 and Proposition 3.5.8, we have

$$B_{k(m)} \cdot f_* \pi^* \mathcal{O}(0,1) = \tfrac{1}{2}(C'_1 + C''_2) \cdot \pi^* \mathcal{O}(0,1) = (C''_1 + C''_2) \cdot \mathcal{O}(0,1) =$$

$$= \mathcal{O}(2, 2k(m)) \cdot \mathcal{O}(0, 1) = 2.$$

$\square$

References

[1] K. Ascher, D. Bejleri. Compact moduli of elliptic K3 surfaces. Geometry & Topology 27 (2023) 1891–1946.

[2] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris. Geometry of Algebraic Curves, Volume I. Grundlehren der mathematischen Wissenschaften, No. 267. Springer–Verlag, New York, 1985.

[3] W. Barth, K. Hulek, C. Peters, A, Van de Ven. Compact complex surfaces. Erg. der Math. und ihrer Grenzgebiete, 3. Folge, Band 4. Springer (1994), Berlin.

[4] A. Beauville, Counting rational curves on K3 surfaces, Duke Math. J. 97 (1999), 99–108

[5] A. Bruno, M. Lelli-Chiesa, Irreducibility of Severi varieties on K3 surfaces, arXiv:2112.09398

[6] J. Bryan, G. Oberdieck, R. Pandharipande, Q. Yin, Curve counting on abelian surfaces and threefolds, Alg. Geom. 5 (2018), 398–463

[7] A. Cayley. A memoir on the conditions for the existence of given systems of equalities among the roots of an equation. Philosophical Transactions of the Royal Society of London, vol. cxlvii:pages 727–731, 1857.

[8] L. Caporaso, J. Harris, Counting plane curves of any genus, Invent. Math. 131 (1998), 345–392.

[9] L. Chiantini, C. Ciliberto, On the Severi varieties of surfaces in $\mathbb{P}^3$, J. Algebraic Geom. 8 (1999), 67–83.

[10] J. Chipalkatti. On equations defining Coincident Root loci. Journal of algebra 267 (2003), no. 1, 246–271.

[11] X. Chen, Rational curves on K3 surfaces, J. Algebraic Geom. 8 (1999), 245–278.

[12] C. Ciliberto, T. Dedieu. Double covers and extensions, Kyoto J. Math. 64 (2024), no. 1, 75–94.

[13] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, A note on Severi varieties of nodal curves on Enriques surfaces, in ”Birational Geometry and Moduli Spaces”, Springer INdAM Series 39 (2018), 29–36.

[14] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, Nonemptiness of Severi varieties on Enriques surfaces. Forum Math. Sigma 11 (2023), Paper No. e52, 32 pp

[15] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, Irreducible unirational and uniruled components of moduli spaces of polarized Enriques surfaces, Math. Z. 303 (2023), no. 3, Paper No. 73, 34 pp.

[16] F. Cossec. On the Picard group of Enriques Surfaces, Math. Ann. 271 (1985), 577–600.

[17] S. Cantat, I. Dolgachev. Rational surfaces with a large group of automorphism. Journal of the american mathematical society, 2012, 863–905.

[18] F. Cossec, I. Dolgachev. Enriques Surfaces 1. University of Michigan.

[19] P. Comparin, A. Garbagnati. Van Geemen–Sarti involutions and elliptic fibrations on K3 surfaces double cover of P2, Journal of Mathematical Society of Japan 66 (2014) 479-522.

[20] X. Chen, F. Gounelas, C. Liedtke. Curves on K3 surfaces, Duke Mathematical Journal (2022), 3283-3362

[21] T. Dedieu. Geometry of logarithmic Severi varieties at a general point. https://hal.archives-ouvertes.fr/hal-02913705.

[22] T. Dedieu, E. Sernesi, Equigeneric and equisingular families of curves on surfaces, Publ. Mat. 61 (2017),175–212

[23] I. Dolgachev, S.Kondo. The rationality of the moduli spaces of Coble surfaces and of nodal Enriques surfaces. Izv. Ross. Akad. Nauk Ser. Mat.77(2013), no.3, 77–92

[24] I. Dolgachev, S.Kondo. Enriques Surfaces 2. University of Michigan.

[25] I. Dolgachev, S.Kondo. Moduli spaces of K3 surfaces and complex ball quotients. Arithmetic and geometry around hypergeometric functions, Progress in Mathematics, 260, Birkhäuser, Basel, 2007, pp. 43-100

[26] L. Fehér, A. Némethi, R. Rimányi. Coincident root loci of binary forms. Michigan Math. J. 54 (2006), no. 2, 375–392.

[27] R. Donagi, B. Ovrut, T. Pantev, D. Waldram. Spectral involutions on rational elliptic surfaces. Adv. Theor. Math. Phys. 5 (2001), no. 3, 499–561. L. Feher, A. Nemethi, R. Rimanyi. Coincident root loci of binary forms. Michigan Math. J. 54 (2006), no. 2, 375–392.

[28] D. Mumford. Geometric invariant theory. Berlin, Gottingen, Heidelberg: Springer 1963

[29] C. Galati, A.L. Knutsen. Rational curves and Seshadri constants on Enriques surfaces. Proc. Amer. Math. Soc. 152 (2024), 3165-3175

[30] G.-M. Greuel, C. Lossen, E. Shustin, Geometry of families of nodal curves on the blown-up projective plane, Trans. Amer. Math. Soc. 350 (1998), 251–274.

[31] A. Garbagnati, C. Salgado. Linear systems on rational elliptic surfaces and elliptic fibrations on K3 surfaces. J. Pure Appl. Algebra 223 (2019), no. 1, 277–300.

[32] A. Garbagnati, C. Salgado. Elliptic fibrations on K3 surfaces with a non-symplectic involution fixing rational curves and a curve of positive genus. Rev. Mat. Iberoam.36(2020), no.4, 1167–1206.

[33] A. Garbagnati, C. Salgado. Elliptic fibrations and involutions on K3 surfaces, Assoc. Women Math. Ser., 32 Springer, Cham, 2024, 293–322.

[34] J. Harris, On the Severi problem, Invent. Math. 84 (1986), 445–461

[35] G. Heckmann, E. Looijenga. The moduli space of rational elliptic surfaces. Mathematical Society of Japan, Tokyo, 2002, 185–248.

[36] K. Hulek, M. Schütt. Enriques surfaces and Jacobian elliptic K3 surfaces. Math. Z. 268 (2011), no. 3-4, 1025-1056 Math. Soc. Japan, 58 (2006), 665–680.

[37] A. L. Knutsen, M. Lelli-Chiesa, Genus two curves on abelian surfaces, Ann. Sci. Ec. Norm. Sup´er. ´ (4) 55 (2022), no. 4, 905–918.

[38] A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, Severi Varieties and Brill-Noether theory of curves on abelian surfaces, J. Reine Angew. Math. 749 (2019), 161–200.

[39] K. Kodaira. On compact analytic surfaces: II. Annals of Mathematics, pages 563–626, 1963.

[40] M. Kontsevich. Enumeration of rational curves via torus actions. Progr. Math., 129, Birkhäuser Boston, Inc., Boston, MA, 1995, 335–368.

[41] S. Kurmann (2011). Some remarks on equations defining coincident root loci. Journal of Algebra Volume 267, Issue 1, 1 September 2003, Pages 246-271

[42] H. Lange, E. Sernesi, Severi varieties and branch curves of abelian surfaces of type (1, 3), International J. of Math. 13, (2002), 227–244.

[43] D. Loughran, C. Salgado. Rank jumps on elliptic surfaces and the Hilbert property. Annales de l'Institut Fourier, Volume 72 (2022) no. 2, pp. 617-638

[44] R. Miranda. The basic theory of elliptic surfaces. Dottorato Ric. Mat.[Doctorate in Mathematical Research] ETS Editrice, Pisa, 1989, vi+108 pp.

[45] R. Miranda, The moduli of Weierstrass fibrations over P1. Math. Ann. 255 (1981), no. 3, 379–394.

[46] R. Miranda. Persson's list of singular fibers for a rational elliptic surface. Math. Z. 205 (1990), no. 2, 191–211.

[47] S. Mori, S. Mukai, The uniruledness of the moduli space of curves of genus 11, Springer-Verlag, Berlin, 1983, 334–353.

[48] A. Neron. Modeles minimaux des varieties abeliennes sur les corps locaux et globaux. Inst. Hautes Études Sci. Publ. Math.(1964), no.21, 128 pp.

[49] U. Persson. Configurations of Kodaira fibers on rational elliptic surfaces. Math. Z. 205 (1990), no. 1, 1–47.

[50] M. Schütt, T. Shioda. Elliptic Surfaces. Adv. Stud. Pure Math., 2010

[51] F. Severi, Vorlesungen ¨uber Algebraische Geometrie, Johnson Pub. (reprinted, 1968; 1st. ed., Leipzig 1921).

[52] J. H. Silverman. The arithmetic of elliptic curves, Grad. Texts in Math., 106 Springer-Verlag, New York, 1986, xii+400 pp.

[53] A. Tannenbaum, Families of algebraic curves with nodes, Compositio Math. 41 (1980), 107–126.

[54] D. Testa, The irreducibility of the moduli spaces of rational curves on del Pezzo surfaces, J. Algebraic Geom. 18 (2009), 37–61.

[55] R. Vakil. Twelve points on the projective line, branched covers, and rational elliptic fibrations. Math Ann 320, 33–54 (2001)