Università degli Studi di Roma Tre

Dipartimento di Matematica e Fisica

Scuola Dottorale in Scienze Matematiche e Fisiche

XXVIII Ciclo

Ph.D. Thesis

Relations of CR and Dolbeault cohomologies for Matsuki duals orbits in complex flag manifolds

Candidate:
Stefano Marini

Advisor:
Prof. Mauro Nacinovich

Anno Accademico 2015/2016

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Introduction

The aim of this PhD thesis is to investigate relations between the *CR* and the complex geometry of orbits of real forms and of the complexification of their maximal compact subgroups in a complex flag manifold $X \simeq \mathbf{G}/\mathbf{Q}$. The pair of orbits M_+ and M_- which are involved are related by Matsuki duality (see [**33**]). Their homotopies and singular homologies and cohomologies are equal because, by Mostow fibration, their compact intersection M_0 is a deformation retract of both. Here we are interested in their respective CR and Dolbeault cohomologies.

The orbits of real forms, which were first investigated by J.A. Wolf (for a comprehensive introduction to this topic see [**20**,**27**]), are a large class of homogeneous *CR* manifolds. Wolf studied the action of a real form $\mathbf{G}_0$ of a complex semisimple Lie group $\mathbf{G}$ on a $\mathbf{G}$-homogeneous flag manifold X. He showed that $\mathbf{G}_0$ has only finitely many orbits in X, that the union of the open orbits is dense in X, and that there is a unique closed orbit, that is compact (simple proofs of these facts can be found in [**45**]). Moreover, in collaboration with W. Schmid, he proved in [**42**] a vanishing theorem for the cohomology with coefficients in any coherent sheaf $\mathcal{F}$ for the open $\mathbf{G}_0$-orbits and he investigated arc components of general orbits, outlining a framework that includes, as special cases, bounded symmetric domains [**27**]. In [**4**] A. Altomani, C. Medori and M. Nacinovich pursued the investigation of these topics, by starting a systematic study of the subject in the framework of *CR* geometry. Their first topic was the examination of the properties of the closed orbits, that they characterized in terms of *cross-marked Satake diagrams*. Later on, in [**5–11**], they studied the *CR* structure of the general $\mathbf{G}_0$-orbits in X, that they called *parabolic CR manifolds*. They focused on finite type and holomorphic nondegeneracy conditions, canonical $\mathbf{G}_0$-equivariant fibrations and topological properties.

The contents of this work is the following.

Chapter I is a concise introduction to the main notions and tools needed in this thesis, mainly following [**13**, **16**, **17**, **21**, **23**, **24**, **26**, **33**, **40**, **41**, **45**] as general references.

In Chapter II we introduce the Matsuki dual pairs M_+,M_-, their compact intersection M_0, and consider their *CR* structures. Invariant *CR* structures on homogeneous manifolds are described by the datum of a pair $(\mathfrak{g}_0, \mathfrak{q})$ of a real Lie algebra $\mathfrak{g}_0$ and a complex Lie subalgebra $\mathfrak{q}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$, which is called a *CR*-algebra (see [**2**, **34**]). We also recall the definition, given in [**8**], of $\mathfrak{n}$-reductive *CR*- algebras. Using the data of their *CR*-algebras, we compute the *CR*-dimension and codimension of each orbit, pointing out that the immersions $i : M_0 \hookrightarrow M_-$ and $i' : M_+ \hookrightarrow X$ are *CR*-generic (Prop.2.3.1). Moreover M_+ and M_0 have the same *CR*-codimension, so that the scalar Levi form on M_0 is the restiction of that of M_+.

Chapter III focuses on the study of the covariant fibration of G.D. Mostow (cf. [**36**,**37**]). The complex manifold $M_- \simeq \mathbf{K}/\mathbf{V}$ is described as a fiber bundle $\mathbf{K}_0 \times_{\mathbf{I}_0} F_0$

which is covariant with respect to the compact form $\mathbf{K}_0$ of $\mathbf{K}$, over a compact base $M_0 = \mathbf{K}_0/\mathbf{I}_0$. Here, in the context of Matsuki duality, $\mathbf{K}_0$ is a maximal compact subgroup of $\mathbf{G}_0$, $\mathbf{K}$ its complexification and $\mathbf{V} = \mathbf{K} \cap \mathbf{Q}$, $\mathbf{I}_0 = \mathbf{K}_0 \cap \mathbf{Q}$. The right hand side of the isomorphism $M_- \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} F_0$ is the quotient of the Cartesian product $\mathbf{K}_0 \times F_0$ modulo the equivalence relation $(x, v) \sim (x \cdot k, \mathrm{ad}(k^{-1})(v))$, for $x \in \mathbf{K}_0$, $v \in F_0$, and $k \in \mathbf{I}_0$. The aim of Ch.III is to clarify the structure of the typical fiber F_0, which is known to be a *Euclidean* subspace of $\mathbf{K}$ on which $\mathrm{ad}(\mathbf{I}_0)$ acts as a group of linear transforms.

To understand the structure of F_0, it is convenient to gather some information about the noncompact Riemann symmetric space of negative sectional curvature $\mathcal{P}_0(n) = \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$. $\mathbf{K}_0$ has, for some integer $n > 1$, a faithful linear representation in $\mathbf{SU}(n)$, which extends to a faithful linear representation $\mathbf{K} \hookrightarrow \mathbf{SL}_n(\mathbb{C})$. We can therefore identify $\mathbf{K}$ with a closed subgroup of $\mathbf{SL}_n(\mathbb{C})$ and M_- with a submanifold of $\mathbf{SL}_n(\mathbb{C})/\mathbf{V}$. The Mostow fibration of $\mathbf{K}/\mathbf{V}$ can be deduced from that of $\mathbf{SL}_n(\mathbb{C})/\mathbf{V}$.

If the unipotent radical $\mathbf{V}_n$ of the isotropy $\mathbf{V}$ of M_- equals the unipotent radical of a parabolic subgroup, then the typical fiber F_0 can be taken equal to $\exp(\mathfrak{f}_0)$, with $\mathfrak{f}_0 = (\mathfrak{v} + \mathfrak{v}^*)^\perp \cap \mathfrak{p}_0$, where $\mathfrak{p}_0$ is the *hermitian symmetric* summand in the Cartan decomposition of the Lie algebra $\mathfrak{k}$ of $\mathbf{K}$. We say in this case that $\mathbf{V}$ is *HNR*, i.e. has a *horocyclic nilradical*.

In Chapter IV we investigate relations between the *CR* and Dolbeault cohomology groups of the compact intersection and of the complex Matsuki orbit, respectively. The Mostow fibration is used to take the square of the norm of a Euclidean metric on the fibers as an exhaustion function on M_-. For an *HNR* isotropy $\mathbf{V}$, this choice corresponds to minimizing the square of the distance from the base point of an orbit of $\mathbf{V}$ in the symmetric space $\mathbf{K}/\mathbf{K}_0$. This leads to the fact that M_- keeps the pseudoconcavity of M_0. In this way we obtain isomorphisms between some *CR* cohomology groups of M_0 and those of M_-, by using Andreotti-Grauert theory. This construction of the exhaustion function generalizes the proof of the pseudoconvexity of reductive linear complex Lie groups, which can be obtained, using Andreotti-Grauert, from the Cartan decomposition.

The final Chapter V is devoted to discussing some examples.

CHAPTER 1

Preliminaries

1. Real Forms of Complex Lie Groups and Algebras

Let $\mathfrak{g}$ be a complex Lie algebra. A real Lie subalgebra $\mathfrak{g}_0$ is a *real form* of $\mathfrak{g}$ if $\mathbb{C} \otimes \mathfrak{g}_0 = \mathfrak{g}$. The real forms of $\mathfrak{g}$ are in a one-to-one correspondence with the anti-$\mathbb{C}$-linear involutive automorphisms $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ of $\mathfrak{g}$, by

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \sigma(X) = X\} .$$

We call σ the *conjugation* with respect to $\mathfrak{g}_0$. Let $\mathbf{G}$ be a complex Lie group, with Lie algebra $\mathfrak{g}$, and assume that σ lifts to $\mathbf{G}$, (this is always the case when $\mathbf{G}$ is simply connected). Then the subgroup

$$\mathbf{G}_0 = \mathbf{G}^{\sigma} = \{g \in \mathbf{G} \mid \sigma(g) = g\} .$$

In general, we say that a subgroup $\mathbf{G}_0$ is a *real form* of $\mathbf{G}$ iff

- its Lie algebra $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$;
- $\mathbf{G} = \mathbf{G}_0\mathbf{G}^e$, with $\mathbf{G}^e$ the identity component of $\mathbf{G}$.

We recall that a Lie algebra $\mathfrak{g}$ over a field $\mathbb{k}$ is semisimple if and only if its *Killing form*

$$\kappa(X, Y) = \mathrm{trace}(\mathrm{ad}(X), \mathrm{ad}(Y)), \ \forall X, Y \in \mathfrak{g} \tag{1.1}$$

is non-degenerate. A real Lie algebra $\mathfrak{g}_0$ is semisimple iff its complexification $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_0$ is semisimple and compact if its Killing form it is negative definite. If κ is negative semidefinite, then $\mathfrak{g}_0$ is reductive and its Killing form is negative definite on its semisimple ideal and vanishes on its center. Then it is possible to define on $\mathfrak{g}_0$ a negative definite invariant bilinear form and find a compact linear group $\mathbf{G}_0$ with Lie algebra $\mathfrak{g}_0$.

An involutive automorphism $\theta \in \mathrm{Aut}(\mathfrak{g}_0)$ of a real semisimple Lie algebra $\mathfrak{g}_0$ is called a *Cartan involution* if

$$(X, Y) = -\kappa(X, \theta(Y)) = -\kappa(\theta(X), Y), \quad X, Y \in \mathfrak{g}_0 \tag{1.2}$$

is positive definite. As $\theta^2 = \mathrm{id}_{\mathfrak{g}_0}$, we obtain the *Cartan decomposition*

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \tag{1.3}$$

of $\mathfrak{g}_0$ into the sum of its (+1)-eigenspace $\mathfrak{k}_0$ and (–1)-eigenspace $\mathfrak{p}_0$. By (1.2), $\mathfrak{k}_0$ is a compact Lie subalgebra of $\mathfrak{g}_0$. The Cartan involution θ lifts to an involution Θ of any Lie group $\mathbf{G}_0$ with Lie algebra $\mathfrak{g}_0$ and the connected component of the identity $\mathbf{K}_0^e$ of the subgroup $\mathbf{K}_0 = \{a \in \mathbf{G}_0 \mid \Theta(a) = a\}$ is compact and has Lie algebra $\mathfrak{k}_0$. The Cartan decomposition

$$\mathbf{K}_0 \times \mathfrak{p}_0 \ni (x, Y) \xrightarrow{\sim} x \cdot \exp(Y) \in \mathbf{G}_0$$

holds at the group level. Moreover, $\mathbf{K}_0$ is the normalizer in $\mathbf{G}_0$ of $\mathbf{K}_0^e$ and is compact if and only if $\mathbf{G}_0$ has finitely many connected components. In this case $\mathbf{K}_0$ is a maximal compact subgroup of $\mathbf{G}_0$.

Assume that $\mathfrak{g}_0$ is a real form of the complex Lie algebra $\mathfrak{g}$ of a semisimple complex Lie group $\mathbf{G}$. We fix a Cartan involution θ on $\mathfrak{g}_0$ and denote by the same symbol θ its $\mathbb{C}$-linear extension to $\mathfrak{g}$ and by $\mathfrak{k}$, $\mathfrak{p}$ its eigenspaces, wich are the complexifications of $\mathfrak{k}_0$,$\mathfrak{p}_0$, respectively. Using (1.3) we attach to $\mathfrak{g}$ the compact real form

$$\mathfrak{g}_u = \mathfrak{k}_0 \oplus i\mathfrak{p}_0. \tag{1.4}$$

It is the *compact form* associated to θ and is the Lie algebra of a maximal compact subgroup $\mathbf{G}_u$ of $\mathbf{G}$. By costruction the $\mathbb{C}$-linear θ and anti- $\mathbb{C}$-linear σ defined at the beginning are commuting automorphism of $\mathfrak{g}$. Their composition

$$\tau = \sigma \circ \theta = \theta \circ \sigma, \tag{1.5}$$

is the conjugation of $\mathfrak{g}$ with respect to its compact form (1.4).

Notation 1.1.1. For thesis, in such a way that it will be consistent with the work done for matrices in Chapter 3, we use the notation $\tau(X) = -X^*$ for all $X \in \mathfrak{g}$.

2. Root systems

We recall that a *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ is a nilpotent Lie subalgebra equal to its normalizer:

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, H] \subset \mathfrak{h}, \ \forall H \in \mathfrak{h}\}.$$

In particular, if $\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{h}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}'$.

When $\mathfrak{g}$ is reductive, its Cartan subalgebras are commutative and in fact are its maximal Abelian subalgebras.

If $\mathfrak{g}$ is either solvable or with $\mathbb{k}$ algebraically closed, then all its Cartan subalgebras are conjugated by the group $\mathbf{E}$ of its *elementary automorphisms*, which is the group generated by the $\exp(\mathrm{ad}(X))$, for $X \in \mathfrak{g}$ and $\mathrm{ad}(X)$ nilpotent (see e.g. [**17**, Ch.VII,§2,3]).

Assume that $\mathfrak{g}$ is semisimple and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Its dual $\mathfrak{h}^*$ can be identificated with $\mathfrak{h}$ by the non-degenerate form (1.1). The *roots* of $\mathfrak{g}$ with respect $\mathfrak{h}$ a re the element of the set $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$, of nonzero $\alpha \in \mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H) X, \ \forall H \in \mathfrak{h}\} \neq 0.$$

Each $\mathfrak{g}_\alpha$ is one-dimensial and is called the *root subspace* of α. For an $\mathrm{ad}(\mathfrak{h})$-stable subspace $\mathfrak{q}$ of $\mathfrak{g}$ we write $\mathcal{R}(\mathfrak{q}, \mathfrak{h})$ for the roots of $\mathfrak{h}$ in $\mathfrak{q}$, i.e. the subset

$$\mathcal{R}(\mathfrak{q}, \mathfrak{h}) = \{\alpha \in \mathcal{R}(\mathfrak{g}, \mathfrak{h}) \mid \mathfrak{g}_\alpha \subset \mathfrak{q}\}.$$

In a root system $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$ we can always choose a set of *positive roots*, wich is a subset $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ of $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$ such that:

- for each $\alpha \in \mathcal{R}(\mathfrak{g}, \mathfrak{h})$ exactly one of the roots α, $-\alpha$ belongs to $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$;
- if $\alpha,\beta \in \mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ and $\alpha + \beta \in \mathcal{R}(\mathfrak{g}, \mathfrak{h})$, then $\alpha + \beta \in \mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$.

This choice yields a partial ordering "$<$" on $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$, in which $\alpha \prec \beta$ if $\beta - \alpha$ is a root in $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$. Having chosen a set $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ of positive roots, the elements of its complement $\mathcal{R}^-(\mathfrak{g}, \mathfrak{h})$ in $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$ are called negative roots. A root in $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ is *simple* if it is not the sum of two positive roots. The elements $\alpha_1, \ldots, \alpha_k$ of a system Π of simple roots form a basis of $\mathfrak{h}^* \simeq \mathfrak{h}$. Every $\beta \in \mathcal{R}(\mathfrak{g},\mathfrak{h})$ is a linear combination $\beta = \sum_{i=1}^{k} k_i\alpha_i$ of simple roots, with integral coefficients which are either all nonnegative ($\beta \in \mathcal{R}^+(\mathfrak{g},\mathfrak{h})$) or all nonpositive ($\beta \in \mathcal{R}^-(\mathfrak{g},\mathfrak{h})$).

For $\alpha \in \mathcal{R}(\mathfrak{g},\mathfrak{h})$ we define the orthogonal hyperplane to α by

$$L_\alpha = \{\beta \in \mathfrak{h}^* | \kappa(\alpha,\beta) = 0\}, \tag{1.6}$$

and the *reflection*

$$r_\alpha(\beta) = \beta - 2\frac{\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)}\alpha, \tag{1.7}$$

with respect to the hyperplane L_α.

Definition 1.2.1. A *Weyl chamber* C is a connected component of

$$\mathfrak{h}^* \setminus \bigcup_{\alpha\in\mathcal{R}(\mathfrak{g},\mathfrak{h})} L_\alpha .$$

The reflections r_α, for $\alpha \in \mathcal{R}(\mathfrak{g},\mathfrak{h})$, generate a group of isometries of $\mathfrak{h}^*$, which is called the *Weyl group* of $\mathcal{R}(\mathfrak{g},\mathfrak{h})$.

Remark 1.2.2. The r_α, with $\alpha \in \Pi$, are a set of generators of $\mathbf{W}$.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $k = \dim_{\mathbb{C}} \mathfrak{h}$. Let $\Pi = \{\alpha_1, ..., \alpha_k\}$ be a system of simple roots for $\mathcal{R}(\mathfrak{g},\mathfrak{h})$. The *Cartan matrix* $A = (A_{i,j})$ associated to $\mathfrak{g}$ is a $k \times k$ square matrix whose coefficients are integral and can be computed by means of the simple roots by the formula

$$A_{ij} = 2\kappa(\alpha_i,\alpha_j)/\kappa(\alpha_j,\alpha_j), \tag{1.8}$$

where κ is the Killing form, which restricts to a Euclidean scalar product on the real subspace $\mathfrak{h}^*_{\mathbb{R}}$ generated by Π.

Proposition 1.2.3. *A Cartan matrix is a $k \times k$ square matrix $A = (A_{i,j})$ satisfying:*

1. $A_{i.i} = 2$, *for all* $i = 1, \dots, k$;
2. $A_{i,j} \in \{-3,-2,-1,0\}$ *if* $1 \le i \ne j \le k$;
3. $A_{i,j} = 0$ *if and only if* $A_{j,i} = 0$;
4. *there is a diagonal matrix D such that $D\,A\,D^{-1}$ is symmetric and positive.*

Proof. For proofs of these facts see [**16**, Ch.3.4]. □

Note that the Weyl group is determined by the Cartan matrix: if $e_1, \dots, e_k$ is the canonical basis of $\mathbb{R}^n$, the maps $s_1, \dots, s_k$ defined by $s_i(e_j) = e_j - A_{i,j}e_i$ for $j = 1, \dots, k$, corresponding to the reflections with respect to the hyperplanes of the simple roots, generate a finite group isomorphic to the Weyl group $\mathbf{W}$.

The Cartan matrix can also be used to construct the Dynkin diagram.

Definition 1.2.4. The *Dynkin diagram* of $\mathcal{R}$ is a graph on k vertices, corresponding to the simple roots $\{\alpha_1, ..., \alpha_k\}$. The vertices α_i and α_j are connected by

$$A_{ij}A_{ji} = \frac{4\kappa(\alpha_i,\alpha_j)\kappa(\alpha_j,\alpha_i)}{\kappa(\alpha_i,\alpha_i)\kappa(\alpha_j,\alpha_j)}$$

lines. If $\kappa(\alpha_i,\alpha_i) < \kappa(\alpha_j,\alpha_j)$, then an arrow tip is added on the α_j-edge.

One can fix a Cartan subalgebra $\mathfrak{h}$ which is invariant under the conjugation σ defined by a given real form $\mathfrak{g}_0$. Then by duality σ defines a conjugation $\alpha \to \bar{\alpha}$ on $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$. The roots in

$$\mathcal{R}_0 = \{\alpha \in \mathcal{R} \,|\, \bar{\alpha} = -\alpha\},$$

are *imaginary*. They are *compact* when $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ is a maximally vectorial Cartan subalgebra of $\mathfrak{g}_0$. Imaginary roots form a system closed under root addition and form a subsystem of roots. The intersection

$$\Pi_0 = \mathcal{R}_0 \cap \Pi$$

is a basis for $\mathcal{R}_0$. The roots in the complement $\mathcal{R}(\mathfrak{g}, \mathfrak{h}) \setminus \mathcal{R}_0(\mathfrak{g}, \mathfrak{h})$ can be partitioned into the set of *real* roots $\mathcal{R}_{\mathrm{re}}(\mathfrak{g}, \mathfrak{h})$ with $\bar{\alpha} = \sigma(\alpha) = \alpha$ and *complex* roots $\mathcal{R}_{\mathrm{cpx}}(\mathfrak{g}, \mathfrak{h})$ with $\bar{\alpha} \neq \pm\alpha$. In [**15**] Araki showed that $\mathfrak{h}$ and $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ can be chosen in such a way that $\bar{\alpha} \in \mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$ for all noncomplex roots in $\mathcal{R}^+(\mathfrak{g}, \mathfrak{h})$. Then the conjugate $\bar{\alpha}_i$ of a complex simple root is the sum of a complex simple root α_j and of a linear combination of compact roots. Having made this choice, we give the following definition.

Definition 1.2.5. The *Satake diagram* is obtained from the Dynkin diagram in the following way:

- vertices corresponding to the roots of Π_0 are represented by a black circle "●";
- vertices corresponding to the real and complex roots in $\Pi \setminus \Pi_0$ are represented by a white circle "○";
- when $i \neq j$ and $\bar{\alpha}_j \geq \alpha_i$, then the white vertices corresponding to α_i and α_j are joined by an arc "⌢".

Dynkin diagrams characterize the semisimple complex Lie algebras. The Satake diagrams classify their real forms. The compact form is the one with $\mathcal{R}_0(\mathfrak{g}, \mathfrak{h}) = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$, $\mathcal{R}_{\mathrm{re}}(\mathfrak{g}, \mathfrak{h}) = \emptyset$ and the split form the one with $\mathcal{R}_0(\mathfrak{g}, \mathfrak{h}) = \emptyset$, $\mathcal{R}_{\mathrm{re}}(\mathfrak{g}, \mathfrak{h}) = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$.

3. Borel and Parabolic Subgroups

We assume in the following that $\mathbf{G}$ is a complex algebraic, connected and semisimple Lie group, with Lie algebra $\mathfrak{g}$.

Definition 1.3.1. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. A *Borel subgroup* $\mathbf{B}$ of $\mathbf{G}$ is a connected subgroup of $\mathbf{G}$ such that $\mathfrak{b} = \mathrm{Lie}(\mathbf{B})$ is a Borel subalgebra of $\mathfrak{g}$.

Definition 1.3.2. A *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ wich contains a Borel subalgebra. A connected subgroup $\mathbf{Q}$ of $\mathbf{G}$ is *parabolic* if $\mathfrak{q} = \mathrm{Lie}(\mathbf{Q})$ is parabolic in $\mathfrak{g}$.

We recall that if $\mathbf{Q}$ is parabolic in $\mathbf{G}$, then

$$\mathbf{Q} = \mathbf{N}_{\mathbf{G}}(\mathfrak{q}) = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}\}. \tag{1.9}$$

A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$. Conversely, if $\mathfrak{h}$ is a Cartan subalgebra and $\mathcal{R}^+ (\mathfrak{g}, \mathfrak{h})$ a positive system of roots, then

$$\mathfrak{b} = \mathfrak{h} + \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha \tag{1.10}$$

is a Borel subalgebra.

Proposition 1.3.3. *All Borel subgroups are* $\mathbf{G}$*-conjugate.* □

We keep the notation of §1. In particular, θ is the complexification of the involution associated to a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and σ, τ are the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and to the corresponding compact form $\mathfrak{g}_u = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$.

Proposition 1.3.4. *The intersection of two Borel subgroups contains a Cartan subgroup of* $\mathbf{G}$*. The intersection of two Borel subalgebras contains a Cartan subalgebra of* $\mathfrak{g}$*.*

Proof. Every Borel subgroup $\mathbf{B}$ decomposes into the direct product $\mathbf{B} = \mathbf{H}\mathbf{N}$ of a Cartan subgroup $\mathbf{H}$ and of its unipotent radical $\mathbf{N}$. We claim that, if $\mathbf{B}'$ is another Borel subgroup, then the intersection $\mathbf{B} \cap \mathbf{B}'$ contains a Cartan subalgebra. Indeed, by Proposition 1.3.3, we can find $g \in \mathbf{G}$ so that $\mathbf{B} = \mathrm{ad}(g)(\mathbf{B}')$. By the *Bruhat decomposition* (see e.g. [**23**, Ch.IX, Thm.1.4]) we may write $g = b'wb$, with $b, b' \in \mathbf{B}$ and $w \in \mathbf{N}_{\mathbf{G}}(\mathbf{H})$ so $\mathbf{B} = \mathrm{ad}(w\,b)\mathbf{B}'$. Then $b^{-1}\mathbf{H}\,b$ is a Cartan subgroup contained in $\mathbf{B}' \cap \mathbf{B}$. Indeed

$$\begin{aligned}\mathbf{B} = n\,w\,b\,\mathbf{B}'\,b^{-1}w^{-1}n^{-1} &\Rightarrow \mathbf{B} = w\,b\,\mathbf{B}'b^{-1}w^{-1} \Rightarrow \mathbf{H} \subset w\,b\mathbf{B}'b^{-1}w^{-1}\\ &\Rightarrow \mathbf{H} = w^{-1}\mathbf{H}\,w \subset b\,\mathbf{B}'b^{-1} \Rightarrow b^{-1}\mathbf{H}\,b \subset \mathbf{B}' \cap \mathbf{B}.\end{aligned}$$

In particular, if $\mathfrak{b}$, $\mathfrak{b}'$ are Borel subalgebras, and $\mathbf{B}$, $\mathbf{B}'$ the correpsonding analytic subgroups, the Lie algebra $\mathfrak{h}$ of a Cartan subgroup $\mathbf{H} \subset \mathbf{B}\cap\mathbf{B}'$ is contained in $\mathfrak{b} \cap \mathfrak{b}'$. □

Corollary 1.3.5. *Every Borel subalgebra of* $\mathfrak{g}$ *contains a Cartan subalgebra of* $\mathfrak{g}_0$ *and two Cartan subalgebras of* $\mathfrak{g}_0$ *which are contained in the same Borel subalgebra of* $\mathfrak{g}$ *are* $\mathbf{G}_0$*-conjugate.*

Proof. In a solvable Lie algebra, any two Cartan subalgebras are conjugate by an element of its unipotent radical [**45**, Theorem 1.13]. Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$. Since $\sigma(\mathfrak{b})$ is Borel, by Proposition 1.3.4, there is a Cartan subalgebra $\mathfrak{h}$ contained in $\mathfrak{b} \cap \sigma(\mathfrak{b})$. Then $\mathfrak{h}$ and $\sigma(\mathfrak{h})$ are two Cartan subalgebras of the solvable Lie algebra $\mathfrak{b} \cap \sigma(\mathfrak{b})$ and there is an element $X \in \mathfrak{n} \cap \sigma(\mathfrak{n})$ such that $\mathrm{Ad}(\exp(X))(\mathfrak{h}) = \sigma(\mathfrak{h})$. We have

$$\begin{aligned}\mathfrak{h} &= \sigma^2(\mathfrak{h}) = \sigma(\mathrm{Ad}(\exp(X))(\mathfrak{h})) = \mathrm{Ad}(\exp(\sigma(X)))(\sigma(\mathfrak{h}))\\ &= \mathrm{Ad}(\exp(\sigma(X))) \circ \mathrm{Ad}(\exp(X))(\mathfrak{h}) = \mathrm{Ad}(\exp(\sigma(X))\exp(X))(\mathfrak{h}).\end{aligned}$$

Thus $\mathrm{Ad}(\exp(\sigma(X)) \cdot \exp(X)) = e_{\mathbf{G}}$, being a unipotent element of the normalizer of $\mathfrak{h}$. Since the exponential in injective on the nilpotent subalgebra $\mathfrak{n} \cap \sigma(\mathfrak{n})$, from

$$\mathrm{Ad}(\exp(\sigma(X)) \cdot \exp(X)) = \exp(\mathrm{ad}(\sigma(X))) \cdot \exp(\mathrm{ad}(X)) = e_{\mathbf{G}}$$

we obtain that $\sigma(X) = -X$. Set $\mathfrak{h}' = \mathrm{Ad}(\exp(\frac{1}{2}X))(\mathfrak{h})$. Then

$$\begin{aligned}\sigma(\mathfrak{h}') &= \sigma(\mathrm{Ad}(\exp(\tfrac{1}{2}X))(\mathfrak{h})) = (\mathrm{Ad}(\exp(-\tfrac{1}{2}X)))(\sigma(\mathfrak{h}))\\ &= (\mathrm{Ad}(\exp(-\tfrac{1}{2}X)))(\mathrm{Ad}(\exp(X)))(\mathfrak{h}) = \mathfrak{h}'\end{aligned}$$

Then $\mathfrak{h}'_0 = \{H \in \mathfrak{h}' \mid \sigma(H) = H\}$ is a real form of $\mathfrak{h}'$ and hence a Cartan subalgebra of $\mathfrak{g}_0$. Two complete the proof, we note that, if $\mathfrak{b}$ is Borel, then $\mathfrak{b} \cap \mathfrak{g}_0 \subset \mathfrak{g}_0$ is solvable. Hence all its Cartan subalgebras are $\mathbf{G}_0$-conjugate and, by the first part of the proof are also Cartan subalgebras of $\mathfrak{g}_0$. □

Proposition 1.3.6. *Let* $\mathfrak{b}$ *be a Borel subalgebra. There is a* $g \in \mathbf{G}_0$ *so that* $\mathrm{Ad}(g)(\mathfrak{b})$ *contains a Cartan subalgebra stable under both* σ *and* θ*. There also exists a* $k \in \mathbf{K}$ *so that* $\mathrm{Ad}(k)(\mathfrak{b})$ *contains a Cartan subalgebra stable under both* σ *and* θ*.*

Proof. This follows from the fact that each Cartan subalgebra σ-stable is $\mathbf{G}_0$ conjugate to a θ-stable Cartan subalgebra. See [**30**, Prop. 6.59]. □

Fix a Cartan subalgebra $\mathfrak{h}$ contained in a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, and let $\mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system. Denote by $\mathfrak{h}_{\mathbb{R}}$ the real subspace of $\mathfrak{h}$ on wich the roots are real valued. The choice of a Weyl chamber $C \subset \mathfrak{h}_{\mathbb{R}}$ defines a partial order $\prec$ on the dual space $\mathfrak{h}^*_{\mathbb{R}}$. The parabolic algebra $\mathfrak{q}$ is the direct sum of $\mathfrak{h}$ and the root spaces contained in $\mathfrak{q}$, i.e.

$$\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in Q} \mathfrak{g}_\alpha, \tag{1.11}$$

where $Q = \{\alpha \in \mathcal{R} \,|\, \mathfrak{g}_\alpha \subset \mathfrak{q}\}$. The fact that $\mathfrak{q}$ is parabolic means that one can choose a Weyl chamber C in such a way that

$$\alpha \in Q \text{ for all } \alpha \succ 0, \tag{1.12}$$

making the roots subset Q parabolic, i.e. it is closed under root addition and $Q \cup (-Q) = \mathcal{R}$.

A Weyl chamber C for wich (1.12) holds true is called *fit* for Q. We denote by $\mathfrak{C}(\mathcal{R})$ the set of all Weyl chambers for $\mathcal{R}$ and by $\mathfrak{C}(\mathcal{R}, Q)$ the subset of those that are fit for Q. Let $C \in \mathfrak{C}(\mathcal{R})$ and $\Pi(C)$ be the corresponding system of C-positive simple roots. All $\alpha \in \mathcal{R}$ are linear combinations of elements of the basis Π:

$$\alpha = \sum_{\alpha_i \in \Pi} k_{\alpha_i} \alpha_i, \quad k_{\alpha_i} \in \mathbb{Z}. \tag{1.13}$$

We define the *support* of α with respect to Π, denoted by $\mathrm{supp}(\alpha)$, to be the set of $\alpha_i \in \Pi$ for wich $k_{\alpha_i} \neq 0$. Having fixed $C \in \mathfrak{C}(\mathcal{R}, Q)$, we associated to $\mathfrak{q}$ the subset Φ of Π consisting of the simple C-positive roots α for wich $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{q}$. Then Q and $\mathfrak{q}$ are completely determinated by Φ, indeed:

$$Q = \{\alpha \in \mathcal{R} \mid \alpha \succ 0\} \cup \{\alpha \in \mathcal{R} \mid \alpha \prec 0,\ \mathrm{supp}(\alpha) \cap \Phi = \emptyset\} = Q^r \cup Q^n, \tag{1.14}$$

with

$$Q^r = \{\alpha \in Q \mid -\alpha \in Q\} = \{\alpha \in \mathcal{R} \mid \mathrm{supp}(\alpha) \cap \Phi = \emptyset\}, \tag{1.15}$$

$$Q^n = \{\alpha \in Q \mid -\alpha \notin Q\} = \{\alpha \in \mathcal{R} \mid \alpha \succ 0,\ \mathrm{supp}(\alpha) \cap \Phi \neq \emptyset\}, \tag{1.16}$$

$$Q^{-n} = \mathcal{R} \setminus Q = \{\alpha \in \mathcal{R} \mid \alpha \prec 0,\ \mathrm{supp}(\alpha) \cap \Phi \neq \emptyset\}. \tag{1.17}$$

We write Q_Φ for the parabolic set Q associated to $\Phi \subset \Pi$. For the parabolic subalgebra $\mathfrak{q}$ associated to Q_Φ we have the decomposition

$$\mathfrak{q}_\Phi = \mathfrak{h} + \sum_{\alpha \in Q_\Phi} \mathfrak{g}_\alpha = \mathfrak{q}_r \oplus \mathfrak{q}_n, \tag{1.18}$$

where

$$\mathfrak{q}_n = \sum_{\alpha \in Q^n_\Phi} \mathfrak{g}_\alpha \quad \text{is the } \textit{nilradical} \text{ of } \mathfrak{q}, \tag{1.19}$$

$$\mathfrak{q}_r = \mathfrak{h} \oplus \sum_{\alpha \in Q^r_\Phi} \mathfrak{g}_\alpha \quad \text{is a } \textit{reductive complement} \text{ of } \mathfrak{q}_n \text{ in } \mathfrak{q}_\Phi, \tag{1.20}$$

and

$$\mathfrak{q}_{-n} = \sum_{\alpha \in Q^{-n}_\Phi} \mathfrak{g}_\alpha \quad \text{is a complement of } \mathfrak{q}_\Phi \text{ in } \mathfrak{g}. \tag{1.21}$$

We explicitly note that the $\mathfrak{q}_\Phi$ in (1.18) contains the Borel subalgebra $\mathfrak{b}$ defined in (1.10) and

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{q}_{-n}.$$

All Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ are equivalent, modulo inner automorphisms. Having fixed $\mathfrak{h}$, and hence $\mathcal{R}$, all bases of simple roots of $\mathcal{R}$ are equivalent modulo the transpose of inner automorphisms of $\mathfrak{g}$ normalizing $\mathfrak{h}$. Thus, after picking a Weyl chamber C, and having fixed in this way a system Π of C-positive simple roots, the correspondence

$$\Phi \leftrightarrow \mathfrak{q}_\Phi$$

is one-to-one between subsets Φ of Π and complex parabolic Lie subalgebra of $\mathfrak{g}$, modulo inner automorphisms.

We can single out the equivalence class of a parabolic subalgebra of $\mathfrak{g}$ (or subgroup of $\mathbf{G}$) by adding on the Dynkin diagram a cross ”$\times$” under the vertices corresponding to the roots of Φ.

We denote by $\mathbf{Q}^r$ and $\mathbf{Q}^n$, the connected and simply connected Lie subgroups of $\mathbf{G}$ with Lie algebras (1.20) and (1.19), respectively.

Thus the choice of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{q}$, yields the Chevalley decomposition of the parabolic subgroup $\mathbf{Q}$:

Proposition 1.3.7. *Let $\mathbf{Q}$ be the parabolic subgroup of $\mathbf{G}$, corresponding to the complex parabolic Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. With the notation above we have a Chevalley decomposition*

$$\mathbf{Q} = \mathbf{Q}^r \ltimes \mathbf{Q}^n. \tag{1.22}$$

Moreover let $\mathfrak{c} \subset \mathfrak{h}$ be the center of $\mathfrak{q}^r$:

$$\mathfrak{c} = \mathfrak{z}(\mathfrak{q}^r) = \{H \in \mathfrak{h} \mid \mathrm{ad}(H)(\mathfrak{q}^r) = 0\},$$

then the reductive complement $\mathbf{Q}^r$ is characterized by:

$$\mathbf{Q}^r = \mathbf{Z}_\mathbf{G}(\mathfrak{c}) = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(H) = H,\ H \in \mathfrak{c}\}. \tag{1.23}$$

Proof. Any complex parabolic subgroup can also be viewed as a *real* parabolic subgroup. Then (1.22) reduces the to the Langlands decomposition $\mathbf{Q} = \mathbf{MAN}$, with $\mathbf{Q}^r = \mathbf{MA}$. Thus the statement is a consequence of [**30**, Prop. 7.82(a)]. Note that $\mathfrak{q}^r$ is the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$ and its own normalizer. This yields the inclusion $\mathbf{Q}^r \subset \mathbf{N}_\mathbf{G}(\mathfrak{q}^r)$. Since $\mathbf{N}_\mathbf{G}(\mathfrak{q}^r)$ is semi-algebraic, it has finitely many connected components. Thus its intersection with $\mathbf{Q}^n$ is discrete and finite, and thus trivial because $\mathbf{Q}^n$ is connected, simply connected and unipotent. □

4. Complex Flag Manifolds

A $\mathbf{G}$-*homogeneous manifold* M is the datum of a smooth manifold M together with a transitive action of a Lie group $\mathbf{G}$ on M. Having fixed a point $p \in M$, the manifold X can be identified with the coset space $\mathbf{G}/\mathbf{I}_p$, where $\mathbf{I}_p\{g \in \mathbf{G} \mid g \cdot p = p\}$ is a closed, but not necessarily a connected, subgroup of $\mathbf{G}$, that is called the *stabilizer* of p. Conversely, a closed subgroup $\mathbf{I}_p$ of a Lie group $\mathbf{G}$ defines an homogeneous manifold $\mathbf{G}/\mathbf{I}_p$.

Fix a complex semisimple Lie group $\mathbf{G}$ and a closed parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$, with Lie algebra $\mathfrak{q} \subset \mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Consider the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. For all $g \in \mathbf{G}$, $\mathrm{Ad}(g)(\mathfrak{q})$ is still a parabolic subalgebra of $\mathfrak{g}$. By (1.9) the parabolic subgroup $\mathbf{Q}$ is the stabilizer of $\mathfrak{q}$ for this action. Thus the homogeneous manifold,

$$M = \mathbf{G}/\mathbf{Q}, \tag{1.24}$$

has a natural embedding, as a projective algebraic variety, in the Grassmannian of ℓ-dimensional subspaces of $\mathfrak{g}$, for $\ell = \dim_C \mathfrak{q}$, given by

$$M \ni x \cdot \mathbf{Q} \longrightarrow \mathrm{Ad}(x)(\mathfrak{g}) \in \mathfrak{Gr}_\ell(\mathfrak{g}).$$

We call M a *generalized flag manifold*. It is known (see e.g. [**27**, Lemma 1.15]), that M is a compact complex projective variety. Since M is compact and $\mathbf{G}$-homogeneous, we know, by a theorem of D.Montgomery (see [**35**]), that a maximal compact subgroup $\mathbf{G}_u$ of $\mathbf{G}$ acts transitively on M and we can identify M with the $\mathbf{G}_u$-homogeneous space

$$M \simeq \mathbf{G}_u/(\mathbf{G}_u \cap \mathbf{Q}).$$

Moreover we have :

Lemma 1.4.1. *Let $M = \mathbf{G}/\mathbf{Q}$ be a homogeneus compact manifold. Then we can find a maximal compact subgroup $\mathbf{G}_u$ of $\mathbf{G}$. and a reductive Levi factor $\mathbf{Q}^r$ of $\mathbf{Q}$ such that*

$$M \simeq \mathbf{G}_u/\left(\mathbf{G}_u \cap \mathbf{Q}^r\right). \tag{1.25}$$

Proof. We have $\mathbf{G}_u \cap \mathbf{Q}^r \subset \mathbf{G}_u \cap \mathbf{Q}$ since $\mathbf{Q}^r \subset \mathbf{Q}$. To prove the opposite inclusion, we fix a compact form $\mathbf{Q}_u$ whose intersection with $\mathbf{Q}$ contains a maximal torus. If τ is the conjugation with respect to $\mathbf{Q}_u$, than one can take $\mathbf{Q} \cap \tau(\mathbf{Q})$ as a reductive Levi-Chevalley factor $\mathbf{Q}^r$ in $\mathbf{Q}$, so that $\mathbf{G}_u \cap \mathbf{Q} \subset \mathbf{Q} \cap \tau(\mathbf{Q}) = \mathbf{Q}^r$. □

Let τ be the conjugation with respect to a compact form $\mathbf{G}_u$ of $\mathbf{G}$.

Lemma 1.4.2. *Let $M = \mathbf{G}/\mathbf{Q}$ be a $\mathbf{G}$-homogeneous complex compact manifold and $\mathfrak{t}$ a τ-invariant Cartan subalgebra contained in $\mathfrak{q} = \mathrm{Lie}(\mathbf{Q})$. Let $\mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{t})$ be the corresponding root system and $Q = \mathcal{R}(\mathfrak{q}, \mathfrak{h})$ the parabolic set of $\mathfrak{q}$. Then we can find an element $\Upsilon \in \mathfrak{t}_u = \mathfrak{t} \cap \mathfrak{g}_u$ such that*

$$Q^r = \{\alpha \in \mathcal{R} \mid \alpha(\Upsilon) = 0\} \ \text{and}\ Q^n = \{\alpha \mid \alpha(i\Upsilon) > 0\}. \tag{1.26}$$

Proof. Note that the roots in $\mathcal{R}$ take purely imaginary values on the elements of $\mathfrak{t}_u$. Fix a system $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of simple roots of $\mathcal{R}$, in such a way that the corresponding set $\mathcal{R}^+$ of positive roots is contained in Q.

By relabelling the simple roots, we can assume that $Q = Q_\Phi$ for the subset $\Phi = \{\alpha_1, \dots, \alpha_k\} \subset \Pi$, as in (1.18). We define a basis $\{Z_1, ..., Z_k\}$ for $\mathfrak{t}_u^{**} \simeq \mathfrak{t}_u$ by requiring that $\alpha_j(Z_i) = i$ for $1 \leq j \leq k$ and $\alpha_j(Z_i) = 0$ for $k < j \leq n$. Then

$$\Upsilon = \sum_{j=1}^{k} i Z_j$$

verifies the statement. □

The following proposition is a direct consequence of the above lemma:

Proposition 1.4.3. *If $\Upsilon \in \mathfrak{t}_u$ satisfies (1.26), then we can identify M with adjoint $\mathbf{G}_u$-orbit O of Υ in $\mathfrak{g}_u$, by the map*

$$M \ni x \cdot \mathbf{Q} \longrightarrow \Upsilon_x \doteq \mathrm{Ad}(x)(\Upsilon) \in O \subset \mathfrak{g}_u. \tag{1.27}$$

Proof. By Lemma 1.4.1 the map (1.27) is well defined and onto.

By Lemma1.4.2, $\mathfrak{q}_r = \{Z \in \mathfrak{g} \mid [Z, \Upsilon] = 0\}$. The statement follows because $\mathbf{Q}$ is connected and therefore also its maximal compact subgroup $\mathbf{Q} \cap \mathbf{G}_u$ is connected. □

Remark 1.4.4. Fix $x \in M$, with $x \simeq g\mathbf{Q}$. We denote $\mathbf{Q}_x = g\mathbf{Q}g^{-1}$ and $\mathfrak{q}_x = \mathrm{Ad}(g)(\mathfrak{q})$. Let $\mathbf{Q}_x^r = g\mathbf{Q}^r g^{-1}$, then it is the Levi factor of $\mathbf{Q}_x$, with Lie algebra $\mathfrak{q}_x = \mathrm{Ad}(g)\mathfrak{q}$. However, there are some remarks to be made regarding Cartan subalgebras. We have $\Upsilon \in \mathfrak{t} \subset \mathfrak{q}_r \subset \mathfrak{q}$, and in similar way we can find a $\Upsilon_x \in \mathfrak{t}' \subset \mathfrak{q}_x^r \subset \mathfrak{q}_x^r$. However it is not always possible to take $\mathfrak{t}' = \mathrm{Ad}(g)\mathfrak{t}$, since this is not well defined (except when $\mathbf{Q}$ is a Borel subgroup, and $\mathfrak{q}_x^r = \mathfrak{t}'$). Moreover, every $\mathbf{Q}_x$ contains a σ-stable maximal torus and a τ-stable maximal torus, but in general it is not true for $\mathbf{Q}_x$ to contain a torus stable under both σ and τ .

In the following we describe the complex tangent bundle on M with its complex structure. For $x \simeq g \cdot Q \in M$, we have the decomposition

$$\mathfrak{g} = \mathfrak{q}_x + \mathfrak{q}_x^{-n} = \mathfrak{q}_x^r + \mathfrak{q}_x^n + \mathfrak{q}_x^{-n}, \tag{1.28}$$

that it is not intrinsic to x, but depends on the Levi decomposition and hence on the initial choice of $\mathfrak{t}$. We can write $Z \in \mathfrak{g}$ as

$$Z = \pi_{\mathfrak{q}_x^r}(Z) + \pi_{\mathfrak{q}_x^n}(Z) + \pi_{\mathfrak{q}_x^{-n}}(Z),$$

with $\pi_{(\cdot)}$ the standard projection over the respective component in (1.28).

Moreover we have the indentification

$$T_x M \simeq \mathfrak{g}/\mathfrak{q}_x \simeq \mathfrak{q}_x^{-n}.$$

By the fact that M is a complex projective manifold we can think at $\mathfrak{q}_x^{-n}$ and $\sigma(\mathfrak{q}_x^{-n})$, respectively, as the holomophic and anti-holomophic tangent bundle of M in x. For $x_0 \simeq e_{\mathbf{G}} \cdot \mathbf{Q}$ we have $T_{x_0}^{0,1}M \simeq \mathfrak{g}/\mathfrak{q} \simeq \mathfrak{q}^{-n}$ and $T_{x_0}^{1,0}M \simeq \sigma(\mathfrak{q}^{-n}) = \mathfrak{q}^n$, but in general $\sigma(\mathfrak{q}_x^{-n}) \neq \mathfrak{q}_x^n$.

By Lemma 1.4.1 follows :

Lemma 1.4.5. *For every $x \in M$ the map*

$$I_x : \mathfrak{g}/\mathfrak{q}_x \to \mathfrak{g}_u/\mathfrak{g}_u \cap \mathfrak{q}_x^r \tag{1.29}$$

defined as

$$I_x(Z + \mathfrak{q}_x) \doteq \pi_{\mathfrak{q}_x^{-n}}(Z) + \tau(\pi_{\mathfrak{q}_x^{-n}}(Z)) + \mathfrak{g}_u \cap \mathfrak{q}_x^r = \pi_{\mathfrak{q}_x^{-n}}(Z) + \pi_{\mathfrak{q}_x^n}(Z) + \mathfrak{g}_u \cap \mathfrak{q}_x^r$$

is an isomorphism of vector spaces.

Proof. The map (1.29) is well defined. If $Z_1 + \mathfrak{q}_x = Z_2 + \mathfrak{q}_x$ then $\pi_{\mathfrak{q}_x^r}(Z_1) = \pi_{\mathfrak{q}_x^r}(Z_1)$, and hence $I_x(Z_1 + \mathfrak{q}_x) = I_x(Z_2 + \mathfrak{q}_x)$. Moreover we have that $\pi_{\mathfrak{q}_x^r}(Z) + \tau(\pi_{\mathfrak{q}_x^r}(Z)) \in \mathfrak{g}_u$. Finally, since $Z - \left(\pi_{\mathfrak{q}_x^r}(Z_1) + \tau\left(\pi_{\mathfrak{q}_x^r}(Z_1)\right)\right)$, then I_x inverts the map induced by inclusion. □

For $Z \in \mathfrak{q}_x^{-n}$, we denote by $Z^\star \in \mathfrak{X}(M)$ the infinitesimal generator of the one-parameter group $\phi\,(t, x) = exp\,(tZ)\,x$ of diffeomorphism of M.

Proposition 1.4.6. *For $x \in M$ and $Z \in \mathfrak{q}_x^{-\mathfrak{n}}$ we have $Z^\star = (Z + \tau(Z))^\star$*

Using Lemma 1.4.5 we can define a complex structure $\mathrm{J}_x \in \mathrm{End}\,(T_xM)$ on $T_xM \simeq \mathfrak{g}_u/\mathfrak{g}_u \cap \mathfrak{q}_x^r$ as follow:

$$\mathrm{J}_x\left(\pi_{\mathfrak{q}_x^{-n}}(Z) + \pi_{\mathfrak{q}_x^n}(Z) + \mathfrak{g}_u \cap \mathfrak{q}_x^r\right) \doteq i\pi_{\mathfrak{q}_x^{-n}}(Z) - i\pi_{\mathfrak{q}_x^n}(Z) + \mathfrak{g}_u \cap \mathfrak{q}_x^r \tag{1.30}$$

with $Z \in \mathfrak{g}$.

5. Matsuki's Dual Orbits

Let $\mathbf{G}$ be a connected semisimple complex Lie group, $\mathbf{G}_0$ a real form of $\mathbf{G}$, corresponding to a conjugation σ. Fix a Cartan involution θ commuting with σ and denote by $\tau = \sigma \circ \theta = \theta \circ \sigma$ the conjugation corresponding to the compact form $\mathbf{G}_u = \{x \in \mathbf{G} \mid \tau(x) = x\}$ of $\mathbf{G}$.

We begin by considering the *full flag manifold* $N = \mathbf{G}/\mathbf{B}$, with $\mathbf{B} \subset \mathbf{G}$ a Borel subgroup. We can identify N with the set $\mathscr{B}$ of all Borel subalgebras of $\mathfrak{g}$, via the adjoint action of $\mathbf{G}$.

We are interested in the study of the homogeneous submanifolds of N which are the orbits of the left action of a subgroup $\mathbf{G}'$ of $\mathbf{G}$.

Set

$$\mathbf{K}_0 = \mathbf{G}_0^\theta = \mathbf{G}_0 \cap \mathbf{G}_u, \quad \mathbf{K} = \mathbf{G}^\theta = \{z \in \mathbf{G} \mid \theta(z) = z\}.$$

Then $\mathbf{K}_0$ is a maximal compact subgroup of $\mathbf{G}_0$ and $\mathbf{K}$ its complexification.

Definition 1.5.1. For $p \in N = \mathbf{G}/\mathbf{B}$, let us set

$$M_+(p) = G_0 \cdot p \simeq \mathbf{G}_0/\mathbf{S}_0, \qquad \text{with } \mathbf{S}_0 = \{x \in \mathbf{G}_0 \mid x \cdot p = p\}, \tag{1.31}$$

which is a real smooth submanifold of N,

$$M_-(p) = \mathbf{K} \cdot p \simeq \mathbf{K}/\mathbf{V}, \qquad \text{with } \mathbf{V} = \{z \in \mathbf{K} \mid z \cdot p = p\}, \tag{1.32}$$

which is a complex submanifold of N,

$$M_0(p) = \mathbf{K}_0 \cdot p = \mathbf{K}_0/\mathbf{I}_0, \qquad \text{with } \mathbf{I}_0 = \{x \in \mathbf{K}_0 \mid x \cdot p = p\}, \tag{1.33}$$

which is a compact submanifold of N.

Theorem 1.5.2. *There are a finitely many orbits $M_+(p)$ in N.*

Proof. We know (see e.g. [**30**, Prop. 6.64]) that there are only a finite number of conjugacy classes of Cartan subalgebras in $\mathfrak{g}_0$, for the adjoint action of $\mathbf{G}_0$. List them as $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$. Every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is $\mathbf{G}_0$-conjugate to a Borel subalgebra containing one of the $\mathfrak{h}_i$'s of the list. There are only a finite number of Borel subalgebras containing a given Cartan subalgebra. They are parametrized by the Weyl chambers of the corresponding root system, being of the form

$$\mathfrak{b} = \mathfrak{h}_i + \bigoplus_{\alpha \in \mathcal{R}^+(C)} \mathfrak{g}_\alpha$$

for the positive system $\mathcal{R}^+(C)$ corresponding to the Weyl chamber C.

Thus the number of orbits $M_+(p)$ in N is bounded by $k \cdot |\mathbf{W}|$. □

Lemma 1.5.3. *Let* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *be Borel subalgebras containing* θ*-stable Cartan subalgebras of* $\mathfrak{g}_0$*. If* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *are either* $\mathbf{G}_0$*- or* $\mathbf{K}$*-conjugate, then they are* $\mathbf{K}_0$*-conjugate.*

Proof. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$, where $\mathfrak{h}$ and $\mathfrak{h}'$ are the complexifications of θ-stable Cartan subalgebras $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ of $\mathfrak{g}_0$ and $\mathfrak{n}$, $\mathfrak{n}'$ the nilradicals of $\mathfrak{b}$, $\mathfrak{b}'$, respectively.

Assume that $\mathfrak{b} = \mathrm{Ad}(g_0)(\mathfrak{b}')$, for some $g_0 \in \mathbf{G}_0$. The proof in the case where $\mathfrak{b} = \mathrm{Ad}(k)(\mathfrak{b}')$ for some $k \in \mathbf{K}$ is similar, and will be omitted.

Both $\mathrm{Ad}(g_0)(\mathfrak{h}'_0)$ and $\mathfrak{h}_0$ are Cartan subalgebras of $\mathfrak{b} \cap \mathfrak{g}_0$, which is a solvable Lie subalgebra of $\mathfrak{g}_0$. Hence $\mathrm{Ad}(g_0)(\mathfrak{h}'_0) = \mathrm{Ad}(\exp(X_0))(\mathfrak{h}_0)$ for some X_0 in the nilradical of $\mathfrak{b} \cap \mathfrak{g}_0$. For $g_1 = \exp(-X_0) \cdot g_0$, we have $\mathrm{Ad}(g_1)(\mathfrak{h}'_0) = \mathfrak{h}_0$ and $\mathrm{Ad}(g_1)(\mathfrak{g}') = \mathfrak{g}$.

We need to show that $g_1 \in \mathbf{K}_0$. To this aim, we use the Cartan decomposition $\mathbf{G}_0 = \exp(\mathfrak{p}_0) \cdot \mathbf{K}_0$ to write $g_1 = \exp(Y_0) \cdot k_0$, with $Y_0 \in \mathfrak{p}_0$ and $k_0 \in \mathbf{K}_0$. From

$$\mathrm{Ad}(g_1)(\mathfrak{h}'_0) = \mathfrak{h} = \theta(\mathfrak{h}_0) = \theta(\mathrm{Ad}(g_1)(\theta(\mathfrak{h}'_0)) = \mathrm{Ad}(\theta(g_1))(\mathfrak{h}'_0)$$

we obtain that

$$\mathrm{Ad}(\exp(Y_0))(\mathrm{Ad}(k_0)(\mathfrak{h}'_0)) = \mathrm{Ad}(\exp(-Y_0))(\mathrm{Ad}(k_0)(\mathfrak{h}'_0)),$$

i.e. $y_0 = \mathrm{Ad}(\exp(2Y_0))$ normalizes the θ-stable Cartan subalgebra $\mathrm{Ad}(k_0)(\mathfrak{h}'_0)$. This implies that $y_0 \in \mathbf{K}_0$ and thus that $Y_0 = 0$, yielding $g_1 = k_0 \in \mathbf{K}_0$. □

By Lemma 1.5.3, $\mathbf{K}_0$ acts on the set $\mathscr{X}$ of all Borel subalgebras which contain a θ-stable Cartan subalgebra of $\mathfrak{g}_0$.

Corollary 1.5.4. *Each* $\mathbf{G}_0$*-orbit and each* $\mathbf{K}$*-orbit in* $\mathscr{B}$ *meets* $\mathscr{X}$ *in exactly one* $\mathbf{K}_0$*-orbit. There are one-to-one correspondences between* $\mathbf{G}_0$*-orbits in* $N \simeq \mathscr{B}$*,* $\mathbf{K}_0$*-orbits in* $\mathscr{X}$ *and* $\mathbf{K}$*-orbits in* $N \simeq \mathscr{B}$*:*

$$N/\mathbf{G}_0 \simeq \mathscr{B}/\mathbf{G}_0 \leftrightarrow \mathscr{X}/\mathbf{K}_0 \leftrightarrow \mathscr{B}/\mathbf{K} \simeq N/\mathbf{K}.$$

Thus if $\mathfrak{b} \in \mathscr{X}$ *, then* $(\mathbf{G}_0 \cdot \mathfrak{b}) \cap (\mathbf{K} \cdot \mathfrak{b}) = \mathbf{K}_0 \cdot \mathfrak{b}$.

Proof. Every Cartan subalgebra of $\mathfrak{g}_0$ is conjugate to one which is θ-stable (see e.g. [**23**, Ch.IX, Cor.4.2]). This implies that every $\mathbf{G}_0$-orbit in $\mathscr{B}$ intersects $\mathscr{X}$ and Lemma 1.5.3 tells us that the intersection is a $\mathbf{K}_0$-orbit.

A $\mathfrak{b} \in \mathscr{B}$ contains a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, wich is conjugate with a θ-stable Cartan subalgebra $\mathfrak{h}'_0$ of $\mathfrak{g}_0$. Their complexifications $\mathfrak{h} = \mathbb{C} \otimes \mathfrak{h}_0$ and $\mathfrak{h}' = \mathbb{C} \otimes \mathfrak{h}'_0$ are Cartan subalgebras of $\mathfrak{k}$ and hence, since $\mathfrak{k}$ is complex, there is $k \in \mathbf{K}$ such that $\mathrm{Ad}(k)(\mathfrak{h}) = \mathfrak{h}'$. Then $\mathfrak{b}' = \mathrm{Ad}(k)(\mathfrak{b})$ contains the θ-stable Cartan subalgebra $\mathfrak{h}'_0$ and therefore in an element of the intersection of $\mathscr{X}$ with the $\mathbf{K}$-orbit of $\mathfrak{b}$. Again, by Lemma 1.5.3, the intersection of $\mathrm{Ad}(\mathbf{K})(\mathfrak{b})$ with $\mathscr{X}$ is a $\mathbf{K}_0$-orbit. □

Corollary 1.5.5. *There are a finite number of* $\mathbf{K}$*-orbits in* N*.*

A general flag manifold M of $\mathbf{G}$ can be identified with the grassmannian $\mathscr{Q}$ of the parabolic subalgebras $\mathfrak{q}$ which are conjugate to a given parabolic subalgebra $\mathfrak{q}^0$ of $\mathfrak{q}$. Each Borel $\mathfrak{b} \in \mathscr{B}$ is contained in a unique parabolic $\mathfrak{q} \in \mathscr{Q}$, and therefore we obtain a $\mathbf{G}$-equivariant fibration

$$\pi : N \simeq \mathscr{B} \ni \mathfrak{b} \longrightarrow \mathfrak{q} \in \mathscr{Q} \simeq M \quad \text{with} \quad \pi(\mathfrak{b}) = \mathfrak{q} \ \text{ if } \ \mathfrak{b} \subset \mathfrak{q}. \tag{1.34}$$

Writing $M = \mathbf{G}/\mathbf{Q}$ and $N = \mathbf{G}/\mathbf{B}$ with $\mathbf{B} \subset \mathbf{Q}$, the fibration can be described by $\pi(g\mathbf{B}) = g\mathbf{Q}$.

Proposition 1.5.2 and Corollary 1.5.5 imply that $\mathbf{G}_0$ and $\mathbf{K}$ have a finite number of distinct orbits in M.

However, Lemma 1.5.3 may fail, because the reductive factor $\mathfrak{q}_r$ of a parabolic $\mathfrak{q} \notin \mathscr{B}$ may contain Cartan subalgebras of $\mathfrak{g}_0$ which are not $\mathbf{G}_0$-conjugate.

Fix $p \in M \simeq \mathbf{G}/\mathbf{Q}$. If $p = x \cdot \mathbf{Q}$, then the stabilizer of p in $\mathbf{Q}$ is $\mathbf{Q}_p = \mathrm{ad}(x^{-1})(\mathbf{Q})$. Since $\mathbf{K} \cap \mathbf{Q}_p$, $\mathbf{G}_0 \cap \mathbf{Q}_p$ and $\mathbf{K}_0 \cap \mathbf{Q}_p$ are the stabilizer subgroups of p for the actions of $\mathbf{K}$, $\mathbf{G}_0$ and $\mathbf{K}_0$ on M, respectively, then:

$$M_-(p) = \mathbf{K}(p) \simeq \mathbf{K}/\left(\mathbf{K} \cap \mathbf{Q}_p\right), \tag{1.35}$$

$$M_+(p) = \mathbf{G}_0(p) \simeq \mathbf{G}_0/\left(\mathbf{G}_0 \cap \mathbf{Q}_p\right), \tag{1.36}$$

$$M_0(p) = \mathbf{K}_0(p) \simeq \mathbf{K}_0/\left(\mathbf{K}_0 \cap \mathbf{Q}_p\right). \tag{1.37}$$

Set $\mathscr{M}_- = \{M_-(p) \mid p \in M\}$, $\mathscr{M}_+ = \{M_+(p) \mid p \in M\}$, $\mathscr{M}_0 = \{M_0(p) \mid p \in M\}$.

Matsuki's duality (see [**33**]) states that there is a one-to-one correspondence $\mathscr{M}_- \leftrightarrow \mathscr{M}_+$ in which $M_-(p_-) \leftrightarrow M_+(p_+)$ iff $M_-(p_-) \cap M_+(p_+) = M_0(p_0)$ for some $p_0 \in M$.

Theorem 1.5.6 (Matsuki 1988). *There is a bijection between $\mathscr{M}_-$ and $\mathscr{M}_+$ in which $M_-(p') \in \mathscr{M}_-$ and $M_+(p'')$ are related if and only if $M_-(p') \cap M_+(p'') \in \mathscr{M}_0$.*

Proof. The statement follows from the fact that two parabolic subalgebra $\mathfrak{q}, \mathfrak{q}'$, containing a θ-stable Cartan subalgebra defined over $\mathbb{R}$, are $\mathbf{G}_0$- conjugate (or $\mathbf{K}$-conjugate) if and only if that they are $\mathbf{K}_0$-conjugate and the fact that the fibration (1.34) sends orbits to orbits. □

Theorem 1.5.7. *We have:*

(1) *There exist only one closed orbit in $\mathscr{M}_+$ and only one open orbit in $\mathscr{M}_-$.*
(2) *$M_+(p)$ is closed iff $M_-(p)$ is open and $M_+(p) \subset M_-(p)$;*
(3) *$M_-(p)$ is closed iff $M_+(p)$ is open and $M_-(p) \subset M_+(p)$.*

Proof. These results follow from the study of the orbits of the real forms in [**27**], and Matsuki duality. □

6. CR Manifolds

Let M be real manifold of dimensione m, countable at infinity. A *partial complex structure* of type (n, k) on M is the pair (HM, J) consisting of vector subbundle HM of its tangent bundle TM and a smooth fiber-preserving vector bundle isomorphism

$$J : HM \to HM$$

such that:

(1) $J^2 = -Id : HM \to HM$;
(2) $JH_xM = H_xM$, $\forall x \in M$;
(3) $[X, Y] - [JX, JY] \in \Gamma(M, HM)$, $\forall X, Y \in \Gamma(M, HM)$;
(4) $\dim_{\mathbb{R}} H_xM = 2n$, $m = \dim M = 2n + k$.

We say that (HM, J) is *formally integrable* if

$$[X, JY] + [JX, Y] = J([X, Y] - [JX, JY]), \quad \forall X, Y \in \Gamma(M, HM). \tag{1.38}$$

We denote by

$$T^{1,0}M = \{x - iJX | X \in HM\} \quad \text{and} \quad T^{0,1}M = \{x + iJX | X \in HM\},$$

the complex vector subbundle of the complexification $\mathbb{C}HM$ of HM, whose fibers are the eigenspaces corresponding to the eigenvalues i and $-i$ of J, respectively. Then the formal integrability condition of (1.38) can also be expressed by

$$[\Gamma(M, T^{1,0}M), \Gamma(M, T^{1,0}M)] \subset \Gamma(M, T^{1,0}M), \tag{1.39}$$

or, equivalently, by

$$[\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M). \tag{1.40}$$

The following relations hold:

- $T^{1,0}M = \overline{T^{0,1}M}$;
- $T^{1,0}M \cap T^{0,1}M = \{0\}$;
- $T^{1,0}M \oplus T^{0,1}M = \mathbb{C}HM$.

Definition 1.6.1. An abstract *CR manifold* (M, HM, J) of type (n, k) is the triple consisting of a smooth paracompact manifold M of dimension $m = 2n + k$ and a formally integrable partial complex structure (HM, J) of type (n, k) on it. We call n the *CR dimension* and k the *CR codimension* of (M, HM, J).

If $n = 0$, we say that M is *totally real*; if $k = 0$, M is a complex manifold in view of the Newlander-Nirenberg theorem. In the following, we shall often write for simplicity M instead of (M, HM, J).

Definition 1.6.2. A *CR-map* of a *CR* manifold (M_1, H_1, J_1) into a *CR* manifold (M_2, H_2, J_2) is a differentiable map

$$\Phi : M_1 \to M_2$$

such that:

(1) $\Phi_*(H_1) \subset H_2$,
(2) $\Phi_*(J_1X) = J_2\Phi_*(X)$, for every $X \in H_1$.

A *CR-embedding* $\Phi : M \to N$ of an abstract *CR* manifold M into a complex manifold N is a CR-map wich is an embedding.

We say that a CR-embedding $\Phi : M \to N$ is *generic* if the complex dimension of N is $n + k$, where (n, k) is the type of M.

Let M be a *CR manifold* of type (n, k). Its *characteristic bundle* H^0M is the annihilator of HM in T^*M. It is the rank k subbundle of T^*M

$$H^0M = \{\alpha \in T^*M \mid \alpha(X) = 0, \ \forall X \in \Gamma(M, HM)\}.$$

Fix $\alpha \in H^0_xM$ and $X, Y \in H_xM$ and choose $\tilde{\alpha} \in \Gamma(M, H^0M)$ and $\tilde{X}, \tilde{Y} \in \Gamma(M, HM)$ such that $\tilde{\alpha}(x) = \alpha$, $\tilde{X}(x) = X$ and $\tilde{Y}(x) = Y$. Then we have

$$d\tilde{\alpha}(X, Y) = -\alpha([\tilde{X}, \tilde{Y}]), \tag{1.41}$$

with the two sides of this equation only depending on α, X and Y. In this way we associated to $\alpha \in H^0_xM$ a quadratic form

$$L_\alpha(X) = \alpha([J\tilde{X}, \tilde{X}]) = d\tilde{\alpha}(X, JX) \tag{1.42}$$

on H_xM. Since

$$L_\alpha(X) = L_\alpha(JX)$$

this form is Hermitian for the complex structure of H_xM defined by J.

Definition 1.6.3. The Hermitian form L_α defined by (1.42) is called *Levi form* of M at $\alpha \in H^0M$.

We denote by $\lambda(\alpha) = (\lambda^+(\alpha), \lambda^-(\alpha))$ the signature of L_α, considered as a Hermitian form for the complex structure of H_xM defined by J; the integers $\lambda^+(\alpha)$ and $\lambda^-(\alpha)$ are its positive and negative indices of inertia.

Definition 1.6.4. We say that M in *q -pseudoconcave* at $x \in M$ if $\lambda^-(\alpha) \geq q$ for every $\alpha \in H^0_xM \setminus \{0\}$ and that M is *q-pseudoconcave* if it is q-pseudoconcave at all points.

On an m-dimensional complex manifold M se have the Dolbeault complexes of sheaves of germs of smooth forms:

$$(\mathcal{E}^{p,*}, \bar\partial) : \ 0 \to \mathcal{E}^{p,0} \xrightarrow{\ \bar\partial\ } \mathcal{E}^{p,1} \longrightarrow \cdots \longrightarrow \mathcal{E}^{p,m-1} \xrightarrow{\ \bar\partial\ } \mathcal{E}^{p,m} \to 0.$$

Here $\mathcal{E}^{p,j}$ denotes the sheaf of germs of complex valued C^∞ forms of bidegree (p, j) on M. For $U^{\text{open}} \subset M$, we denote by $H^{p,j}(\mathcal{E}(U))$ the cohomology of this complex for sections on U. Likewise, we can define the Dolbeault compexes for sheaves of (p, j)-currents We also have the Dolbeault complexes on the sheaves of germs of currents

$$(\mathcal{D}'^{p,*}, \bar\partial) : \ 0 \to \mathcal{D}'^{p,0} \xrightarrow{\ \bar\partial\ } \mathcal{D}'^{p,1} \longrightarrow \cdots \longrightarrow \mathcal{D}'^{p,m-1} \xrightarrow{\ \bar\partial\ } \mathcal{D}'^{p,m} \to 0.$$

Here, for every $U^{\text{open}} \subset M$ and each pair (p, j), the set $\mathcal{D}'^{p,j}(U)$ of sections of $\mathcal{D}'^{p,j}$ on U is defined as the topological dual of the space $\mathcal{D}^{m-p,m-j}(U)$ of smooth forms, of bidegree $(m - p, m - j)$, having compact support in U, for the standard Schwartz topology. We denote by $H^{p,j}(\mathcal{D}'(U))$ the corresponding cohomology groups. Actually we have the isomorphisms

$$H^j(U, \Omega^p) \simeq H^{p,j}(\mathcal{E}(U)) \simeq H^{p,j}(\mathcal{D}'(U))$$

by the Dolbeault lemma, the elliptic regularity, and the abstract de Rham theorem. Here Ω^p denotes the sheaf of germs of holomorphic p- forms on M, and $H^j(U, \Omega^p)$ are the *Čech* cohomology groups on U , with coefficients Ω^p and closed supports.

Likewise, we can consider the Dolbeault complexes for smooth forms or currents having compact support in U:

$$(\mathcal{D}^{p,*}, \bar\partial) : \ 0 \to \mathcal{D}^{p,0}(U) \xrightarrow{\ \bar\partial\ } \mathcal{D}^{p,1}(U) \longrightarrow \cdots \longrightarrow \mathcal{D}^{p,m-1}(U) \to \xrightarrow{\ \bar\partial\ } \mathcal{D}^{p,m}(U) \to 0,$$

$$(\mathcal{E}'^{p,*}, \bar\partial) : \ 0 \to \mathcal{E}'^{p,0}(U) \xrightarrow{\ \bar\partial\ } \mathcal{E}'^{p,1}(U) \longrightarrow \cdots \longrightarrow \mathcal{E}'^{p,m-1}(U) \to \xrightarrow{\ \bar\partial\ } \mathcal{E}'^{p,m}(U) \to 0$$

where $\mathcal{E}'^{p,j}(U)$ is the space of currents of bidegree (p, j), with compact support contained in U. The corresponding cohomology groups are denoted by $\mathbf{H}^{p,j}(\mathcal{D}(U))$ and $\mathbf{H}^{p,j}(\mathcal{E}'(U))$, respectively. In fact,

$$H^{p,j}(\mathcal{D}(U)) \simeq H^{p,j}(\mathcal{E}'(U)),$$

as they can be interpreted as *Čech* homology groups with coefficients in the same *cosheaf* (see e.g. [**14**]).

Next we consider a C^∞-smooth *CR submanifold* M_0 of type (n,k) , which we assume to be generically and properly embedded in M (so that $n+k=m$). Let $\mathscr{I}_{M_0}$ denote the ideal sheaf in the grassmann algebra $\mathcal{E}$ of germs of complex valued C^∞ forms on N, that is locally generated by function which vanish on M and by their antiholomorphic differentials. This is a graded subsheaf of rings of $\mathcal{E}$ and we denote by $\mathscr{I}^{p,j}_{M_0}$ the intersection $\mathscr{I}_{M_0}\cap\mathcal{E}^{p,j}$. Since

$$\mathscr{I}_{M_0}=\bigoplus_{0\leq p,j\leq m}\mathscr{I}^{p,j}_{M_0}\quad\text{and}\quad\bar\partial(\mathscr{I}^{p,j}_{M_0})\subset\mathscr{I}^{p,j+1}_{M_0},\quad\forall 0\leq p,j\leq m,$$

we obtain subcomplexes of the Dolbeault complexes $(\mathcal{E}^{p,*},\bar\partial)$:

$$(\mathscr{I}^{p,*}_{M_0},\bar\partial):\ 0\to\mathscr{I}^{p,0}_{M_0}\xrightarrow{\ \bar\partial\ }\mathscr{I}^{p,1}_{M_0}\longrightarrow\cdots\longrightarrow\mathscr{I}^{p,m-1}_{M_0}\xrightarrow{\ \bar\partial\ }\mathscr{I}^{p,m}_{M_0}\to 0.$$

and hence quotient complexes $([\mathcal{E}^{p,*}],\bar\partial_{M_0})$, defined by the short exact sequences of fine sheaves complexes:

$$0\longrightarrow(\mathscr{I}^{p,*}_{M_0},\bar\partial)\longrightarrow(\mathcal{E}^{p,*},\bar\partial)\longrightarrow([\mathcal{E}^{p,*},\bar\partial_{M_0})\longrightarrow 0.$$

We denoted by $\bar\partial_{M_0}$ the differential induced by passing to the quotients. Thus the quotient complex is

$$([\mathcal{E}^{p,*}],\bar\partial_{M_0}):\ 0\to[\mathcal{E}^{p,0}]\xrightarrow{\ \bar\partial_{M_0}\ }[\mathcal{E}^{p,1}]\longrightarrow\quad\cdots$$
$$\cdots\quad\longrightarrow[\mathcal{E}^{p,m-1}]\xrightarrow{\ \bar\partial_{M_0}\ }[\mathcal{E}^{p,m}]\to 0.$$

In fact the quotient sheaf $[\mathcal{E}]$ is supported by M_0 and the cohomology groups on $U^{\text{open}}\subset M$ of $([\mathcal{E}^{p,*}],\bar\partial_{M_0})$ only depend on the intersection $U\cap M_0$ and will be denoted by $H^{p,j}([\mathcal{E}](U\cap M_0))$. They are the *smooth* $\bar\partial_{M_0}$ -cohomology groups of $U\cap M_0$, or the *tangential Cauchy-Riemann cohomology groups* of $M_0\cap U$.

Note that $[\mathcal{E}^{p,j}]=0$ when $j>n$. We already observed that $[\mathcal{E}^{p,j}]_p=\{0\}$ for $p\in M\setminus M_0$, and in fact $[\mathcal{E}]$ is isomorphic to a sheaf of free C^∞-modules on M_0, so that the $([\mathcal{E}^{p,*}],\bar\partial_{M_0})$ are complexes of first order partial differential operators on sections of smooth vector bundles on M_0.

In order to define the $\bar\partial_M$-cohomology groups on currents, we first consider the spaces $[\mathcal{D}^{p,j}](M_0\cap U)$ of section in $[\mathcal{E}^{p,j}](M_0\cap U)$ having compact support in $M_0\cap U$. There are short exact sequences

$$0\to\mathscr{I}^{p,j}_{M_0}(U)\cap\mathcal{D}^{p,j}(U)\longrightarrow\mathcal{D}^{p,j}(U)\longrightarrow[\mathcal{D}^{p,j}](U\cap M_0)\to 0. \tag{1.43}$$

Currents are defined by duality:

$$[\mathcal{D}'^{p,j}](U\cap M_0)=\left([\mathcal{D}^{n+k-p,n-j}](U\cap M_0)\right)'.$$

By (1.43), $[\mathcal{D}'^{p,j}](U\cap M_0)$ is isomorphic to annihilator of $\mathscr{I}_{M_0}(U)\cap\mathcal{D}^{n+k-p,n-j}(U)$ in $\mathcal{D}'^{p,k+j}(U)$. Note that in the degree of the current on M corresponding to the current on M_0 there is an index shift equal to the real codimension of M_0. Since the definition of the ideal $\mathscr{I}_{M_0}$ only involves the pull-back of smooth forms to M_0, it follows that the annihilator, and hence $[\mathcal{D}'^{p,j}](M\cap U)$ consists of currents whose coefficients are single layer distributions carried by M_0. In this way we obtain, for each $0 \leq p \leq n+k$, a complex of sheaves

$$([\mathcal{D}'^{p,*}, \bar{\partial}_{M_0}) : \ 0 \to \mathcal{D}'^{p,0} \xrightarrow{\bar{\partial}_{M_0}} \mathcal{D}'^{p,1} \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{D}'^{p,n-1} \xrightarrow{\bar{\partial}_{M_0}} \mathcal{D}'^{p,n} \to 0.$$

We denote by $H^{p,j}([\mathcal{D}'](U \cap M_0)$ its cohomology groups, which are called the and refer to as the *distribution* $\bar{\partial}_M$- cohomology groups of $M_0 \cap U$. The differential operators in this complex are easily described by their identification with

$$\bar{\partial} : (\mathscr{I}_{M_0}(U) \cap \mathcal{D}^{n+k-p,n-j}(U))^0 \to (\mathscr{I}_{M_0}(U) \cap \mathcal{D}^{n+k-p,n-j-1}(U))^0$$

which should be computed in the sense of currents (see e.g. [**41**]).

We can also consider, for $0 \leq p \leq n+k$, the cohomology groups of the corresponding complexes of forms and of currents with compact support:

$$([\mathcal{D}^{p,*}](U \cap M_0), \bar{\partial}_{M_0}) : \ 0 \to [\mathcal{D}^{p,0}](M_0 \cap U) \xrightarrow{\bar{\partial}_{M_0}} [\mathcal{D}^{p,1}](U \cap M_0) \to \cdots$$
$$\cdots \to [\mathcal{D}^{p,n-1}](U \cap M_0) \xrightarrow{\bar{\partial}_{M_0}} [\mathcal{D}^{p,n}](U \cap M_0) \to 0,$$
$$([\mathcal{E}'^{p,*}](U \cap M_0), \bar{\partial}_{M_0}) : \ 0 \to [\mathcal{E}'^{p,0}](M_0 \cap U) \xrightarrow{\bar{\partial}_{M_0}} [\mathcal{E}'^{p,1}](U \cap M_0) \to \cdots$$
$$\cdots \to [\mathcal{E}'^{p,n-1}](U \cap M_0) \xrightarrow{\bar{\partial}_{M_0}} [\mathcal{E}'^{p,n}](U \cap M_0) \to 0,$$

where $[\mathcal{E}'^{p,j}](U \cap M_0)$ is the space consisting of the sections on U of $[\mathcal{D}'^{p,j}]$, having compact support, and we have the identification

$$[\mathcal{E}'^{p,j}](U \cap M_0) \simeq \left(\mathscr{I}^{n+k-p,n-j}_{M_0}(U)\right)^0 .$$

We denote their cohomology groups by

$$H^{p,j}([\mathcal{D}](U \cap M_0)) \text{ and } H^{p,j}([\mathcal{E}'](U \cap M_0)),$$

respectively.

Since the Poincaré lemma may fail for the tangential Cauchy-Riemann complex (see e.g. [**12**]), the cohomology groups computed by the differential complexes can be different from the corresponding Čech groups with coefficients in the sheaves $\Omega^p_{M_0} = \ker\{\bar{\partial}_{M_0} : [\mathcal{E}^{p,0}] \to [\mathcal{E}^{p,1}]\}$ of CR-forms or $\tilde{\Omega}^{p,0} = \ker\{\bar{\partial}_{M_0} : [\mathcal{D}'^{p,0}] \to [\mathcal{D}'^{p,1}]\}$ of CR-currents.

Following we remarks some of the main results for the relation between the Levi form and CR-cohomology group , we gave as general reference [**40**].

We begin with the following proposition:

Proposition 1.6.5. *Let M_0 be a q-pseudoconcave CR-manifold of type (n,k), and M be an $(n+k)$-dimensional tubular neighborhood of M_0. Then there exists a fundamental system U of tubular neighborhoods of M_0, with $U \subset M$, having smooth boundary, and with the Levi form of ∂U having at least q negative and at least* $q-k-1$ *positive eigenvalues.*

Proof. For a proof of this facts see [**40**, Prop.3.2]. □

We recall that a complex manifold M of complex dimension m is called r-pseudoconvex (r-pseudoconcave), in sense of Adreotti (see [**26**, pg.318]), if there is a real valued smooth function ϕ on M, a compact set K and a constant $c_0 \in \mathbb{R}$, such that:

(1) $\phi < c_0$ on $M \setminus K$;

(2) for every $c < c_0$, $\{\phi \le c\}$ is compact in M;
(3) the complex Hessian of ϕ has at least $m - r$ positive ($r + 1$ negative) eigenvalues at each point of $M \setminus K$.

If, in the definition of r-pseudoconvexity, we can choose $K = \emptyset$ the manifold M is called *r-complete*. It is shown in [**13**] that if M is r-pseudoconvex then for any coherent sheaf $\mathcal{F}$ on M we have

$$\dim H^j(M, \mathcal{F}) < \infty \text{ if } j > r.$$

Moreover if M is r-complete then $H^j(M, \mathcal{F}) = 0$ *for* $j > r$. If M is r-pseudoconcave and $\mathcal{F}$ is a locally free coherent sheaf, then

$$\dim H^j(M, \mathcal{F}) < \infty \text{ for } j < r.$$

For an r-pseudoconcave M and any coherent sheaf $\mathcal{F}$ on M, we have

$$\dim H^j_k(M, \mathcal{F}) < \infty \text{ for } j > n + k - r,$$

where H^j_k denotes cohomology with compact supports.

Corollary 1.6.6. *Let M_0 be a compact q-pseudoconcave CR manifold of type (n, k). Then the fundamental system $\{U\}$ of tubular neighborhoods in Proposition* (1.6.5) *can be taken to be $(n - q)$-pseudoconvex and q-pseudoconcave.*

Proof. See [**40**, Corollary 3.3]. □

Theorem 1.6.7. *Let M_0 be a compact q-pseudoconcave CR manifold of type (n, k). Then for all $0 \le p \le n + k$:*

(1) *for $j < q$ and $j > n - q$,* $\dim H^j\left(M_0, \Omega^p_{M_0}\right)$ *and indeed the natural maps*

$$H^j(U, \Omega^p) \to H^j\left(M_0, \Omega^p_{M_0}\right)$$

are isomorphisms, where $\{U\}$ is as in Corollary (1.6.6).

(2) *for $j < q$ there are isomophisms* $H^j\left(M_0, \Omega^p_{M_0}\right) \simeq H^j(M_0, \Omega^p)$.

(3) *for $j > n - q$,* $\dim H^{p,j}(M_0) < \infty$.

Proof. We have

$$H^j\left(M, \Omega^p_{M_0}\right) = \lim_{\substack{\to \\ U \supset M_0}} H^j(U, \Omega^p).$$

By [**13**]

$$H^j(U', \Omega^p) \xrightarrow{\simeq} H^j(U, \Omega^p) \text{ for } j < q \text{ and } j > n - q$$

if $U \supset U$ are tubular neighborhoods as in Corollary (1.6.6), and hence we obtain (1). The isomorphisms in (2) follow by the proof of the abstract deRham theorem, since we have the Poincaré lemma for $\bar{\partial}_{M_0}$ in the correct range [**1**], [**39**]. For (3) see [**41**, p.155]. □

CHAPTER 2

CR Structures of Matsuki's Dual Orbits

References for this chapter are [**2–4**, **9–11**].

1. Homogeneous CR Manifolds

Let $\mathbf{G}_0$ be a real Lie group with Lie algebra $\mathfrak{g}_0$, and denote by $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_0$ its complexification.

Definition 2.1.1. A $\mathbf{G}_0$-*homogeneous* CR *manifold* is a $\mathbf{G}_0$-homogeneous *smooth* manifold endowed with a $\mathbf{G}_0$-invariant CR structure.

Let M be a $\mathbf{G}_0$-homogeneous CR manifold. Fix a point $p_0 \in M$ and denote by $\mathbf{S}_0$ its stabilizer in $\mathbf{G}_0$. The natural projection

$$\pi : \mathbf{G}_0 \longrightarrow \mathbf{G}/\mathbf{S}_0 \simeq M,$$

makes $\mathbf{G}_0$ the total space of a principal $\mathbf{S}_0$-bundle with base M. Denote by $\mathfrak{Z}(\mathbf{G}_0)$ the space of smooth sections of the pullback to $\mathbf{G}_0$ of the bundle $T^{0,1}M$ of anti-holomorphic tangent vectors to M, i.e. the set of complex valued vector fields Z on $\mathbf{G}_0$ such that

$$d\pi^{\mathbb{C}}(Z_g) \in T^{0,1}_{\pi(g)}M, \quad \forall g \in \mathbf{G}_0.$$

Since $T^{0,1}M$ is formally integrable, $\mathfrak{Z}\,(\mathbf{G}_0)$ is formally integrable, i.e.

$$[\mathfrak{Z}(\mathbf{G}_0), \mathfrak{Z}(\mathbf{G}_0)] \subset \mathfrak{Z}(\mathbf{G}_0).$$

Being invariant by left translations, $\mathfrak{Z}(\mathbf{G}_0)$ is generated, as a left $C^\infty\,(\mathbf{G_0}, \mathbb{C})$-module, by its left invariant vector fields. By the formal integrability assumption, the subspace

$$\mathfrak{q} = (d\pi^{\mathbb{C}})^{-1}(T^{0,1}_{x_0}M) \subset \mathfrak{g} = T^{\mathbb{C}}_e\mathbf{G}_0, \tag{2.1}$$

is a $\mathrm{Ad}(\mathbf{S}_0)$-invariant complex Lie subalgebra of $\mathfrak{g}$. By all these observations we have the following Lemma:

Lemma 2.1.2. *Denote by* $\mathfrak{s}_0$ *the Lie algebra of the isotropy subgroup* $\mathbf{S}_0$. *Then (2.1) establishes a one-to-one correspondence between the* $\mathbf{G}_0$*-homogeneus CR structures on* $M = \mathbf{G}_0/\mathbf{S}_0$ *and the* $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{S}_0)$*-invariant complex Lie subalgebras* $\mathfrak{q}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{s}_0$. □

This lemma suggests to introduce the following (see [**34**])

Definition 2.1.3. A *CR algebra* is a pair $(\mathfrak{g}_0, \mathfrak{q})$, consisting of a real Lie algebra $\mathfrak{g}_0$ and a complex Lie subalgebra $\mathfrak{q}$ of its complexification $\mathfrak{g}$, such that the quotient $\mathfrak{g}_0/(\mathfrak{g}_0 \cap \mathfrak{q})$ is a finite dimensional real vector space.

If M is a $\mathbf{G}_0$-homogeneous CR manifold and $\mathfrak{q}$ is defined by (2.1), we say that the CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ is *associated to* M *at* x_0.

The intersection $\mathfrak{q} \cap \bar{\mathfrak{q}}$ is a complex Lie subalgebra of $\mathfrak{g}$, which is the complexification of the Lie algebra $\mathfrak{s}_0 = \mathfrak{q} \cap \mathfrak{g}_0$ of the stabilizer $\mathbf{S}_0$ of x_0 in $\mathbf{G}_0$.

The CR-algebras associated to M at different points are conjugate: if $(\mathfrak{g}_0, \mathfrak{q})$ is associated to M at x_0, and $g \in \mathbf{G}_0$, then $(\mathfrak{g}_0, \mathrm{Ad}(g^{-1})(\mathfrak{q}))$ is associated to M at $g \cdot x_0$.

The CR-dimension and CR-codimension of M can be expressed in terms of the associated CR algebra $(\mathfrak{g}_0, \mathfrak{q})$:

$$CR - \dim\ M = \dim_{\mathbb{C}} \mathfrak{q} - \dim_{\mathbb{C}}(\mathfrak{q} \cap \bar{\mathfrak{q}}); \tag{2.2}$$

$$CR - \text{codimension}\, M = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}}(\mathfrak{q} + \bar{\mathfrak{q}}). \tag{2.3}$$

By Lemma 2.1.2, the datum of its associated CR-algebra at any point characterizes the CR-structure of a $\mathbf{G}_0$-homogeneous CR manifold M.

Let M, N be $\mathbf{G}_0$-homogeneous CR manifolds and $\phi : M \to N$ a $\mathbf{G}_0$-equivariant smooth map. Fix $x_0 \in M$ al let $(\mathfrak{g}_0, \mathfrak{q})$ and $(\mathfrak{g}_0, \mathfrak{e})$ be the CR algebras associated to M at x_0 and to N at $\phi(x_0)$, respectively. Then

$$\mathfrak{q} \cap \bar{\mathfrak{q}} \subset \mathfrak{e} \cap \bar{\mathfrak{e}}. \tag{2.4}$$

PROPOSITION 2.1.4. *A $\mathbf{G}_0$-equivariant map $\phi : M \to N$ is CR if and only if*

$$\mathfrak{q} \subset \mathfrak{e}. \tag{2.5}$$

PROOF. Taking into account (2.4), condition (2.5) is equivalent to the inclusion

$$d\phi^{\mathbb{C}}(T^{0.1}_{x_0} M) \subset T^{0.1}_{\phi(x_0)} N.$$

By the assumption that ϕ and the CR-structures of M and N are $\mathbf{G}_0$-equivariant,this is equivalent to the fact that $d\phi^{\mathbb{C}}(T^{0,1}_x M) \subset T^{0,1}_{\phi(x)} N$ for all $x \in M$ and therefore that ϕ is CR. □

PROPOSITION 2.1.5. *A $\mathbf{G}_0$-equivariant CR-map $\phi : M \to N$ is a CR-submersion if and only if $\mathfrak{e} = \mathfrak{q} + \mathfrak{e} \cap \bar{\mathfrak{e}}$;*

PROOF. A $\mathbf{G}_0$-equivariant CR map ϕ of $\mathbf{G}_0$-homogeneous spaces is always a submersion, because

$$d\phi_e : \mathfrak{g}_0/(\mathfrak{g}_0 \cap \mathfrak{q}) \to \mathfrak{g}_0/(\mathfrak{g}_0 \cap \mathfrak{e})$$

is onto, as $\mathfrak{g}_0 \cap \mathfrak{q} \subset \mathfrak{g}_0 \cap \mathfrak{e}$. It is a CR submersion iff, moreover, $\mathfrak{q} \subset \mathfrak{e}$ and

$$d\phi_e^{\mathbb{C}} : \mathfrak{q}/(\mathfrak{q} \cap \bar{\mathfrak{q}}) \to \mathfrak{e}/(\mathfrak{e} \cap \bar{\mathfrak{e}})$$

is onto. This last condition is equivalent to $\mathfrak{e} = \mathfrak{q} + \mathfrak{e} \cap \bar{\mathfrak{e}}$. □

The fibers of a $\mathbf{G}_0$-equivariant CR submersion are homogeneous CR manifolds. Indeed, if $\mathbf{S}'_0$ is the stabilizer of $\phi(x_0) \in N$, with Lie algebra $\mathfrak{s}'_0 = \mathfrak{e} \cap \mathfrak{g}_0$, then $F_0 = \phi^{-1}(\phi(x_0)) \simeq \mathbf{S}'_0/\mathbf{S}_0$ has $(\mathfrak{s}'_0, \mathfrak{q} \cap \bar{\mathfrak{e}})$ as associated CR algebra at x_0.

COROLLARY 2.1.6. *A CR-subbmersion $\phi : M \to N$ has :*

(1) *totally real fibers if and only if* $\mathfrak{q} \cap \bar{\mathfrak{e}} = \bar{\mathfrak{q}} \cap \mathfrak{e} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ *;*
(2) *complex fibers if and only if* $\mathfrak{q} \cap \bar{\mathfrak{e}} + \bar{\mathfrak{q}} \cap \mathfrak{e} = \mathfrak{e} \cap \bar{\mathfrak{e}}$. □

2. $\mathfrak{v}_n$-reductive CR-algebras

For the results of this section we refer to [**11**].

Let $\mathfrak{k}$ be a reductive complex Lie algebra and

$$\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{s}$$

be its decomposition into the sum of its center $\mathfrak{z} = \{X \in \mathfrak{k} \mid [X, Y] = 0,\ \forall Y \in \mathfrak{k}\}$ and its semisimple ideal $\mathfrak{s} = [\mathfrak{k}, \mathfrak{k}]$.

The Lie algebra $\mathfrak{k}$ admits a faithful linear representation in which all elements of $\mathfrak{z}$ correspond to semisimple matrices. This leads to an intrinsic notion of semisimple and nilpotent elements of $\mathfrak{k}$, as those to which are associated semisimple or nilpotent matrices, respectively. Each element $X \in \mathfrak{k}$ admits a unique Jordan-Chevalley decomposition

(2.6) $$X = X_s + X_n, \quad \text{with } X_s,\ X_n \in \mathfrak{k} \text{ and } X_s \text{ semisimple, } X_n \text{ nilpotent.}$$

A real or complex Lie subalgebra $\mathfrak{v}$ of $\mathfrak{k}$ is called *splittable* if, for each $X \in \mathfrak{v}$, both X_s and X_n belong to $\mathfrak{v}$.

Let $\mathfrak{v}$ be a Lie subalgebra of $\mathfrak{k}$ and rad($\mathfrak{v}$) its radical (i.e. the maximal solvable ideal). We denote by

(2.7) $$\mathfrak{v}_n = \{X \in \text{rad}(\mathfrak{v}) \mid \text{ad}(X) \text{ is nilpotent}\}$$

its nilradical (see [**21**, p.58]). It is the maximal nilpotent ideal of $\mathfrak{v}$. We have (see [**17**, Ch.VII,§5,Prop.7])

Proposition 2.2.1. *Every splittable Lie subalgebra $\mathfrak{v}$ is a direct sum*

(2.8) $$\mathfrak{v} = \mathfrak{v}_r + \mathfrak{v}_n,$$

of its nilradical $\mathfrak{v}_n$ and of a reductive subalgebra $\mathfrak{v}_r$, which is uniquely determined modulo conjugation by elementary automorphisms of $\mathfrak{v}$. □

We assume in the following that $\mathfrak{k}$ is the complexification of a compact Lie algebra $\mathfrak{k}_0$. Since compact Lie algebras are reductive, and the complexification of a reductive real Lie algebra is reductive, $\mathfrak{k}$ is complex reductive. Conjugation in $\mathfrak{k}$ will be taken with respect to the real form $\mathfrak{k}_0$.

Proposition 2.2.2. *For any complex Lie sub algebra $\mathfrak{v}$ of $\mathfrak{k}$, the intersection $\mathfrak{v}\cap\bar{\mathfrak{v}}$ is reductive and splittable. In particular $\mathfrak{v} \cap \bar{\mathfrak{v}} \cap \mathfrak{v}_n = \{0\}$. A splittable $\mathfrak{v}$ admits a Levi-Chevalley decomposition with a reductive Levi factor containing $\mathfrak{v} \cap \bar{\mathfrak{v}}$.*

Proof. We recall that $\mathfrak{v}$ is splittable if and only if rad($\mathfrak{v}$) is splittable. When this is the case, $\mathfrak{v}$ admits a Levi- Chevalley decomposition, and all maximal reductive Lie subalgebras of $\mathfrak{v}$ can be taken as reductive Levi factors. The intersection $\mathfrak{v} \cap \bar{\mathfrak{v}}$ is reductive, because it is the complexification of the compact Lie algebra $\mathfrak{v} \cap \mathfrak{k}_0$. Then the reductive Levi factor in the Levi-Chevalley decomposition of $\mathfrak{v}$ can be taken to contain $\mathfrak{v} \cap \bar{\mathfrak{v}}$. □

We use the notation

$$\mathfrak{i}_0 = \mathfrak{v} \cap \mathfrak{k}_0, \quad \mathfrak{v}_r = \mathfrak{v} \cap \bar{\mathfrak{v}}.$$

The following defintion outlines the class of *CR* algebras that will be considered in this thesis.

Definition 2.2.3. Let $\mathbf{K}_0$ be a compact Lie group with Lie algebra $\mathfrak{k}_0$ and M_0 a $\mathbf{K}_0$-homogeneous CR manifold, with stabilizer $\mathbf{I}_0$ and CR algebra $(\mathfrak{k}_0, \mathfrak{v})$ at a point $x_0 \in M_0$. We say that M_0 and its CR-algebra $(\mathfrak{k}_0, \mathfrak{v})$ are $\mathfrak{v}_n$*-reductive* if

(2.9) $$\mathfrak{v} = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus \mathfrak{v}_n,$$

i.e. if $\mathfrak{v}_r = \mathfrak{v} \cap \bar{\mathfrak{v}}$ is a reductive Levi factor in $\mathfrak{v}$.

If $(\mathfrak{k}_0, \mathfrak{v})$ is $\mathfrak{v}_n$-reductive, then $\mathfrak{v}$ is splittable. Indeed all the elements of $\mathfrak{v}_n$ are nilpotent and those in $\mathfrak{v}\cap\mathfrak{k}_0$ are semisimple. Hence $\mathfrak{v}$ has a system of generators that are either nilpotent or semisimple and therefore is splittable (see [**17**, Ch.VII,§5 ,Thm.1]).

3. Matsuki's dual homogeneous CR manifolds

Having fixed $p_0 \in M \simeq \mathbf{G}/\mathbf{Q}$, we will use the notation M_+, M_- and M_0 for the corresponding submanifolds of M containing p_0, omitting the explicit reference to the point p_0. We assume that the orbits M_+ and M_- are Matsuki-dual, so that M_0 is their compact intersection. We will consider M_+, M_-, M_0 as *homogeneous CR manifolds* of the Lie groups $\mathbf{G}_0, \mathbf{K}$ and $\mathbf{K}_0$, respectively. We begin by considering the principal fibration, given by (1.33),

$$\pi : \ \mathbf{K}_0 \to M_0 \simeq \mathbf{K}_0/\mathbf{I}_0, \tag{2.10}$$

and take the complexification of its differential in e

$$d\pi_e^{\mathbb{C}} : \mathfrak{k} \to T_{p_0}^{\mathbb{C}} M_0. \tag{2.11}$$

With $\mathfrak{v}$ equal to the Lie algebra of $\mathbf{V} \doteq \mathbf{K} \cap \mathbf{Q}_{p_0}$, it was shown in [**8**] that the *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$ is $\mathfrak{v}_n$-reductive in sense of Definition 2.2.3. The projection (2.11) restricts to a $\mathbb{C}$-linear isomorphism of $\mathfrak{v}_n$ onto $T^{0,1}_{p_0} M_0$, and the pullback by (2.11) of the *CR*-structure $T^{0,1}_{p_0} M_0 \cong \mathfrak{v}_{\mathfrak{n}}$ in p_0 is

$$(d\pi_e^{\mathbb{C}})^{-1} \left(T^{0,1}_{p_0} M_0\right) = \mathfrak{v} = \mathfrak{v}_r \oplus \mathfrak{v}_n = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus \mathfrak{v}_n.$$

The pair $(\mathfrak{k}_0, \mathfrak{v})$ is the *CR* algebra associated to M_0 at p_0. Let $\mathfrak{m}_0$ be the linear subspace of $\mathfrak{k}_0$ given by

$$\mathfrak{m}_0 \doteq \{X \in \mathfrak{k}_0 |\, \kappa(X, Y) = 0, \ \forall Y \in \mathfrak{i}_0 \oplus (\mathfrak{v}_n + \bar{\mathfrak{v}}_n) \cap \mathfrak{k}_0\}, \tag{2.12}$$

and let us denote by $\mathfrak{m}$ its complexification. We have

$$CR\text{−dim}(M_0) = \dim_{\mathbb{C}} \mathfrak{v} - \dim_{\mathbb{C}} \mathfrak{v}_r = \dim_{\mathbb{C}} \mathfrak{v}_n, \tag{2.13}$$

$$CR\text{−codim}(M_0) = \dim_{\mathbb{C}} \mathfrak{k} - \dim_{\mathbb{C}}(\mathfrak{v} + \bar{\mathfrak{v}}) = \dim_{\mathbb{C}} \mathfrak{m}, \tag{2.14}$$

for the complexification $\mathfrak{m}$ of the orthogonal of $((\mathfrak{v} + \bar{\mathfrak{v}}) \cap \mathfrak{k}_0)$ in $\mathfrak{k}_0$.

For our purposes it is convenient to describe the totally complex *CR*-structure of M_- by explicitly constructing its *CR* algebra. First we need to consider the complexification

$$\mathfrak{k} \ni X \to (X, \bar{X}) \in \mathfrak{k} \tilde{\oplus} \mathfrak{k} \tag{2.15}$$

of $\mathfrak{k}$, the conjugation $\tilde{\sigma}$ associated to the real form $\mathfrak{k}$ of $\mathfrak{k}\tilde{\oplus}\mathfrak{k}$ being

$$\tilde{\sigma}(X, Y) = (\bar{Y}, \bar{X}). \tag{2.16}$$

Since M_- is a complex manifold, its *real* tangent space has a complex structure J, which corresponds to the differential of the multiplication by i in any holomorphic coordinate chart:

$$M_- \ni p \to J_p \doteq i\cdot \in \mathrm{Hom}_{\mathbb{R}}(T_p M_-, T_p M_-), \quad \text{for } p \in M_-. \tag{2.17}$$

Using (2.15), at the base point $p_0 = (e \cdot V, e \cdot \bar{V})$, we have the complexification of the tangent space

$$T^{\mathbb{C}}_{p_0} M_- \simeq (\mathfrak{k}/\mathfrak{v}) \tilde{\oplus} (\mathfrak{k}/\bar{\mathfrak{v}}), \tag{2.18}$$

and the complex structure defined by (2.17) is described in this identification by the restriction to the real tangent space of

$$J^{\mathbb{C}}_{p_0} : T^{\mathbb{C}}_{p_0} M_- \ni (Z, W) \to (i \cdot Z, -i \cdot W) \in T^{\mathbb{C}}_{p_0} M_-. \tag{2.19}$$

By (2.19) we can define

$$\begin{aligned} T^{1,0}_{p_0}M_- &= \left\{\tfrac{1}{2}((Z,\bar{Z}) - iJ^{\mathbb{C}}(Z,\bar{Z}))\,\middle|\, Z \in \mathfrak{k}/\mathfrak{v}\right\} \\ &= \{(Z,0),\ \text{with } Z \in \mathfrak{k}/\mathfrak{v}\} \\ &= (\bar{\mathfrak{v}}_n \oplus \mathfrak{m})\,\tilde{\oplus}\,0 \end{aligned}$$

and

$$\begin{aligned} T^{0,1}_{p_0}M_- &= \left\{\tfrac{1}{2}((Z,\bar{Z}) + iJ^{\mathbb{C}}(Z,\bar{Z}))\,\middle|\, Z \in \mathfrak{k}/\mathfrak{v}\right\} \\ &= \{(0,\bar{Z}) \text{ with } \bar{Z} \in \mathfrak{k}/\bar{\mathfrak{v}}\} \\ &= 0\,\tilde{\oplus}\,(\mathfrak{v}_n \oplus \mathfrak{m}). \end{aligned}$$

Clearly

$$\tilde{\sigma}(T^{1,0}_{p_0}M_-) = T^{0,1}_{p_0}M_- \quad \text{and} \quad T^{\mathbb{C}}_{p_0}M_- = T^{1,0}_{p_0}M_-\tilde{\oplus}T^{0,1}_{p_0}M_-.$$

We consider M_- as the base of the principal $\mathbf{V}$-bundle

$$\pi : \mathbf{K} \to M_- \simeq \mathbf{K}/\mathbf{V}.$$

Take the complexification of the differential in e of the projection π:

$$d\pi^{\mathbb{C}}_{e_{\mathbb{C}}\oplus e_{\mathbb{C}}} : \mathfrak{k}\,\tilde{\oplus}\,\mathfrak{k} \to T^{\mathbb{C}}_{p_0}M_-. \tag{2.20}$$

Then the pullback of the CR−structure $T^{1,0}_{p_0}M^- \cong T_{p_0}M_- \simeq \bar{\mathfrak{v}}_n \oplus \mathfrak{m}$ is

$$\left(d\pi^{\mathbb{C}}_e\right)^{-1}\left(T^{0,1}_{p_0}M_-\right) = (\mathfrak{v} \oplus \bar{\mathfrak{v}})\,\tilde{\oplus}\,(\mathfrak{v}_{\mathfrak{n}} \oplus \mathfrak{m}) = \mathfrak{v} \oplus \mathfrak{k}. \tag{2.21}$$

Since $\tilde{\sigma}(\mathfrak{v}\,\tilde{\oplus}\,\mathfrak{k}) = \mathfrak{k}\,\tilde{\oplus}\,\bar{\mathfrak{v}}$ then the complexification of the isotropy is

$$(\mathfrak{v}\,\tilde{\oplus}\,\mathfrak{k}) \cap \tilde{\sigma}(\mathfrak{v}\,\tilde{\oplus}\,\mathfrak{k}) = \mathfrak{v}\,\tilde{\oplus}\,\bar{\mathfrak{v}}. \tag{2.22}$$

Then $(\mathfrak{k}, \mathfrak{v} \oplus \mathfrak{k})$ is the CR - algebra of M_- at p_0. Since

$$\mathfrak{v}\,\tilde{\oplus}\,\mathfrak{k} + \tilde{\sigma}(\mathfrak{v}\,\tilde{\oplus}\,\mathfrak{k}) = (\mathfrak{v} \oplus \mathfrak{k}) + (\mathfrak{k} \oplus \bar{\mathfrak{v}}) = \mathfrak{k} \oplus \mathfrak{k},$$

it is totally complex. Indeed

$$CR - \dim M_- = \dim_{\mathbb{C}}(\mathfrak{v} \oplus \mathfrak{k}) - \dim_{\mathbb{C}}(\mathfrak{v} \oplus \bar{\mathfrak{v}}) = \dim_{\mathbb{C}} \bar{\mathfrak{v}}_n + \dim_{\mathbb{C}} \mathfrak{m} \tag{2.23}$$

$$CR - \mathrm{codim} M_- = \dim_{\mathbb{C}} \mathfrak{k} \oplus \bar{\mathfrak{k}} - \dim_{\mathbb{C}}((\mathfrak{v} \oplus \mathfrak{k}) + \tilde{\sigma}(\mathfrak{v} \oplus \mathfrak{k})) = 0. \tag{2.24}$$

For the $\mathbf{G}_0$-orbit M_+ we consider the principal fibration, given by (1.31),

$$\pi : \mathbf{G}_0 \to M_+ \simeq \mathbf{G}_0/\mathbf{S}_0, \tag{2.25}$$

with $\mathbf{S}_0 = \mathbf{G}_0 \cap \mathbf{Q}_{p_0}$, and take the complexification of its differential in e

$$d\pi^{\mathbb{C}}_e : \mathfrak{g} \to T^{\mathbb{C}}_{p_0}M_+. \tag{2.26}$$

We denote the pullback by (2.26) of the CR - structure $T^{0,1}_{p_0}M_+$ in p_0 by

$$\mathfrak{q} = \left(d\pi^{\mathbb{C}}_e\right)^{-1}\left(T^{0,1}_{p_0}M_+\right), \tag{2.27}$$

so that the pair $(\mathfrak{g}_0, \mathfrak{q})$ is the CR - algebra of M_+ at p_0. The set $\mathfrak{q}$ of (2.27) is a complex Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{s}_0$, the Lie algebra of the isotropy $\mathbf{S}_0$. We have

$$CR - \dim M_+ = \dim_{\mathbb{C}} \mathfrak{q} - \dim_{\mathbb{C}}(\mathfrak{q} \cap \bar{\mathfrak{q}}) \tag{2.28}$$

$$CR - \mathrm{codim} M_+ = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}}(\mathfrak{q} + \bar{\mathfrak{q}}). \tag{2.29}$$

Thus, we have

Proposition 2.3.1. *Let M_+ and M_- be Matsuki's dual orbit with $M_0 \simeq M_- \cap M_+$. We have:*

(1) *The immersion* $i_{M_0} : M_0 \to M_-$ *is a generic CR immersion.*

(2) *The immersion* $i_{M_+} : M_+ \to M$ *is a generic CR immersion.*

(3) *The orbits* M_0 *and* M_+ *have same CR−codimension.*

Proof. (1) We have

$$di_{M_0}\left(T_{p_0}M_0^{0,1}\right) = T_{p_0}M_-^{0,1} \cap di_{M_0}\left(T_{p_0}^{\mathbb{C}}M_0\right).$$

The sum of (2.13) with (2.14) equals (2.23). Then i_{M_0} is CR and generic.

(2) Similarly $CR - \dim M_+ + CR - \operatorname{codim} M_+ = \dim_{\mathbb{C}} \bar{\mathfrak{q}}$.

(3) We have $\dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}}(\mathfrak{q} + \bar{\mathfrak{q}}) = \dim_{\mathbb{C}} \mathfrak{k} - \dim_{\mathbb{C}}(\mathfrak{v} + \bar{\mathfrak{v}})$. □

Remark 2.3.2. We note that the immersion $i : M_0 \to M_+$, when $M_0 \neq M_+$, is not CR-generic.

CHAPTER 3

Mostow Fibration

We keep the notation of the previous chapter. By a classical result of G.D. Mostow (see [**36**, **37**]), there is a $\mathbf{K}_0$-equivariant fibration

$$\pi : \mathbf{K}/\mathbf{V} \simeq M_- \to M_0 \simeq \mathbf{K}_0/\mathbf{I}_0.$$

In fact, generalizing the classical Cartan decomposition, Mostow shows (see [**37**, Theorem A, p.473]) that, if $\mathbf{V}$ is a closed subgroup of a Lie group $\mathbf{K}$, and both $\mathbf{V}$ and $\mathbf{K}$ have finitely many connected components, then one can find compact subgroups $\mathbf{I}_0 \subset \mathbf{K}_0 \subset \mathbf{K}$ and *Euclidean subspaces* E and F of $\mathbf{K}$ such that

$$\mathbf{V} = \mathbf{I}_0 \cdot E$$

and

$$\mathbf{K} = \mathbf{K}_0 \cdot F \cdot E$$

as topological direct products, with

$$kFk^{-1} = F, \text{ for all } k \in \mathbf{I}_0.$$

Here $\mathbf{I}_0$ and $\mathbf{K}_0$ are maximal compact subgroups of $\mathbf{V}$ and $\mathbf{K}$, respectively.

By saying that $\mathcal{F} = \{F_p\}$ is a $\mathbf{K}_0$-equivariant fibration of $M_- = \mathbf{K}/\mathbf{V}$ with Euclidean fibers we mean that $\mathbf{K}_0$ acts transitively on $\mathcal{F}$ and that the stabilizer in $\mathbf{K}_0$ of the fiber F_p through $p \in M_-$ acts on F_p as a group of linear isometries for a suitable Euclidean structure on F_p. Each fiber contains *regular points* p which are fixed by its stabilizer. The $\mathbf{K}_0$-orbit of a regular point is a submanifold M_0 of $\mathbf{G}/\mathbf{V}$, which intersects each fiber of $\mathcal{F}$ in a single point, which can be taken as the 0 vector for its Euclidean structure.

Seeking for an exhausting function $\phi : M_- = \mathbf{K}/\mathbf{V} \to \mathbb{R}$, with $\phi \geq 0$ and $\mathbf{K}_0/\mathbf{I}_0 = M_0 = \phi^{-1}(0)$, it is a reasonable approach to fix a Riemannian structure on the fiber of the Mostow fibration and take ϕ equal to the distance from the origin on the fibers.

In this way, using the Andreotti-Grauert theory, we can investigate the relationship of the Dolbeault cohomology of M_- and the inductive limit of the corresponding cohomology groups of the tubular neighborhoods of M_0, which are the real-analytic CR-cohomology groups of M_0.

To this aim, it will be interesting to clarify the structure of the fibers, to understand to what amount the signatures of the scalar Levi forms on M_0 reflects the pseudo-convexity and the pseudo-concavity of the open sets $U_c = \{\phi < c\}$ of the exhaustion.

1. The Mostow fibration of M_-

Let $\mathbf{K}_0$ be a compact Lie group and M_0 a $\mathbf{K}_0$-homogeneous reductive compact homogeneous *CR* manifold. Fix a point $p_0 \in M_0$ to represent M_0 as the quotient $\mathbf{K}_0/\mathbf{I}_0$ of $\mathbf{K}_0$ modulo the stabilizer $\mathbf{I}_0$ of a base point p_0. The *CR* structure of M_0 is assumed to be $\mathbf{K}_0$-equivariant, and therefore is described by the datum of an $\mathbf{I}_0$-invariant complex Lie subalgebra $\mathfrak{v}$ of the complexification $\mathfrak{k}$ of the Lie algebra $\mathfrak{k}_0$ of $\mathbf{K}_0$, subject to the condition that the intersection $\mathfrak{v} \cap \mathfrak{k}_0$ is the Lie algebra $\mathfrak{i}_0$ of the isotropy $\mathbf{I}_0$. The fact that M_0 is reductive means, according to [**8**], that $\mathfrak{v}$ has a Levi-Chevalley decomposition

$$\mathfrak{v} = \mathfrak{v}_r \oplus \mathfrak{v}_n,$$

in which

- $\mathfrak{v}_n$ is the nilradical, consisting of the nilpotent elements of its radical;
- the reductive factor $\mathfrak{v}_r$ can be chosen to be the complexification of $\mathfrak{i}_0$.

Note that the complexification of $\mathfrak{i}_0$ is always reductive, because $\mathfrak{i}_0$ is the Lie algebra of a compact group. Thus our request is that its $\mathbf{I}_0$-invariant complement in $\mathfrak{v}$ consists of its nilradical.

We know from [**8**, Theorem 26] that $\mathfrak{v}$ is the Lie algebra of a closed subgroup $\mathbf{V}$ of the complexification $\mathbf{K}$ of $\mathbf{K}_0$, so that $M_- = \mathbf{K}/\mathbf{V}$ is a $\mathbf{K}$-homogeneous complex manifold, and we have a $\mathbf{K}_0$-equivariant *CR*-embedding $M_0 \hookrightarrow M_-$, which is generic. The subgroup $\mathbf{V}$ is algebraic and admits the Levi-Chevalley decomposition (see [**21**, §6.5])

$$\mathbf{V} = \mathbf{V}_n \rtimes \mathbf{V}_r, \quad \text{with } |\mathbf{V}_n \cap \mathbf{V}_r| < +\infty. \tag{3.1}$$

Since a linear representation of a compact group is completely reducible, $\mathfrak{k}_0$ decomposes into a direct sum of $\mathrm{Ad}(\mathbf{I}_0)$-invariant subspaces:

$$\mathfrak{k}_0 = \mathfrak{i}_0 \oplus ([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0) \oplus \mathfrak{m}_0, \tag{3.2}$$

where the conjugation is taken with respect to the real form $\mathfrak{k}_0$ and $\mathfrak{m}_0$ is the orthogonal of $\mathfrak{i}_0 \oplus ([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0)$ with respect to an $\mathrm{Ad}(\mathbf{K}_0)$-invariant scalar product on $\mathfrak{k}_0$.

The differential of the projection $\pi_0 : \mathbf{K}_0 \to M_0$ restricts to a linear isomorphism of $([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0 \oplus \mathfrak{m}_0) \subset \mathfrak{k}_0 = T_e\mathbf{K}_0$ onto $T_{p_0}M_0$. With $\mathfrak{m} = \mathfrak{m}_0 + i\,\mathfrak{m}_0 \subset \mathfrak{k}$, we obtain by complexification a linear isomorphism of the complement $\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n \oplus \mathfrak{m}$ of $\mathfrak{v}_r$ in $\mathfrak{k}$ onto the complexification $\mathbb{C}T_{p_0}M_0$ of the tangent space of M_0 at p_0, which gives the identifications

$$T_{p_0}M_0 \simeq ([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0) \oplus \mathfrak{m}_0, \quad T^{1,0}_{p_0}M_0 \simeq \bar{\mathfrak{v}}_n, \quad T^{0,1}_{p_0}M_0 \simeq \mathfrak{v}_n,$$
$$H_{p_0}M_0 \simeq [\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0, \quad T_{p_0}M_0/H_{p_0}M_0 \simeq \mathfrak{m}_0.$$

The last one corresponds to representing the quotient TM_0/HM_0 as the orthogonal of HM_0 in TM_0 with respect to a $\mathbf{K}_0$-invariant riemannian metric on M_0.

By the Mostow decomposition of [**37**], we can represent M_- as a Euclidean fiber bundle over M_0. The isotropy $\mathbf{I}_0$ is a maximal compact subgroup of $\mathbf{V}$ and, putting together the Levi-Chevalley decomposition (3.1) and the Cartan decomposition of $\mathbf{V}_r$, we have a diffeomorphism

$$\mathbf{I}_0 \times \mathfrak{i}_0 \times \mathfrak{v}_n \ni (x, Y, Z) \longrightarrow x \cdot \exp(iY) \cdot \exp(Z) \in \mathbf{V}. \tag{3.3}$$

At the Lie algebras level this corresponds to the direct sum decomposition

$$\mathfrak{v} = \mathfrak{i}_0 \oplus (i\,\mathfrak{i}_0) \oplus \mathfrak{v}_n.$$

On the other hand, by complexifying (3.2) we obtain:

$$\mathfrak{k} = \mathfrak{v}_r \oplus (\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n) \oplus (\mathfrak{m}_0 \oplus i\,\mathfrak{m}_0). \tag{3.4}$$

Since[1]

$$\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n = \mathfrak{v}_n \oplus ([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \mathfrak{k}_0),$$

we obtain a direct sum decomposition

$$\mathfrak{k} = \mathfrak{k}_0 \oplus\; i\,\mathfrak{m}_0 \oplus (i\,\mathfrak{i}_0 \oplus \mathfrak{v}_n). \tag{3.5}$$

This makes $\exp(i\mathfrak{m}_0)$ a natural candidate for the Euclidean fiber F of the Mostow fibration.

We recall (see [**36**], [**37**], [**38**]) that, in view of the previous discussion, there is an isomorphism

$$\mathbf{K}_0 \times F \times \mathfrak{i}_0 \times v_n \ni (x, v, Y, Z) \xrightarrow{\sim} x \cdot v \cdot \exp(iY) \cdot \exp(Z) \in \mathbf{K}. \tag{3.6}$$

This shows (see [**37**, p.474]) that

$$M_- = \mathbf{K}/\mathbf{V} \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} F. \tag{3.7}$$

Recall that the right hand side is the quotient of $\mathbf{K}_0 \times F$, modulo the equivalence relation

$$(x, v) \sim (x \cdot k, \mathrm{ad}(k^{-1})(v)), \quad \forall x \in \mathbf{K}_0,\ \forall v \in F,\ \forall k \in \mathbf{I}_0.$$

The diffeomorphism is provided by the last horizontal map in the commutative diagram

$$\begin{array}{ccccc} \mathbf{K}_0 & \longrightarrow & \mathbf{K}_0 \times F & \xrightarrow{(x,v)\to x\cdot v} & \mathbf{K} \\ \downarrow & & \downarrow & & \downarrow \pi \\ M_0 & \longrightarrow & \mathbf{K}_0 \times_{\mathbf{I}_0} F & \xrightarrow{\sim} & M_-, \end{array} \tag{3.8}$$

where $\pi : \mathbf{K} \to M_- = \mathbf{K}/\mathbf{V}$ is the canonical projection, and we have the fibration of M_- over M_0

$$\pi : M_- \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} F \to \mathbf{K}_0/\mathbf{I}_0 \simeq M_0 \tag{3.9}$$

We can consider $\mathbf{K}_0$ as a reductive $\mathbf{K}_0$-homogeneous CR manifold with CR algebra $(\mathfrak{k}_0, \mathfrak{v}_n)$ (see e.g. [**7**]). We have the generic CR-embedding $\mathbf{K}_0 \hookrightarrow \mathbf{K}/\mathbf{V}_n$. Let us denote by $\tilde{M}_-$ the complex homogeneous space $\mathbf{K}/\mathbf{V}_n$. We have a commutative diagram in which, since $\mathfrak{v}_n \subset \mathfrak{v}$, the vertical arrows are CR and holomorphic submersions, respectively:

$$\begin{array}{ccc} \mathbf{K}_0 & \longrightarrow & \tilde{M}_- \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & M_-. \end{array}$$

Note that the first vertical arrow of the diagram is a CR submersion with totally real fibers.

[1]In fact, if $\bar{Z} \in \bar{\mathfrak{v}}_n$, then $\bar{Z} = -Z + (Z + \bar{Z})$, with $Z = \bar{\bar{Z}} \in \mathfrak{v}_n$ and $Z + \bar{Z} \in \mathfrak{k}_0$. The sum is direct because $\mathfrak{v}_n \cap \mathfrak{k}_0 = \{0\}$.

2. Some preliminary remarks

In [**36**, §7] D. Mostow shows that the fiber of a covariant fibration is (essentially) unique. Let us consider the case where $\mathbf{V}$ is a closed algebraic subgroup of $\mathbf{SL}_n(\mathbb{C})$,with maximal compact subgroup $\mathbf{I}_0 = \mathbf{V} \cap \mathbf{SU}(n)$. The typical fiber F_0 is an $\mathbf{I}_0$-invariant Euclidean subspace of $\mathbf{SL}_n(\mathbb{C})$ such that each $\zeta \in \mathbf{SL}_n(\mathbb{C})$ uniquely decomposes into

$$\zeta = u \cdot x \cdot z, \quad \text{with } u \in \mathbf{SU}(n),\ x \in F_0,\ z \in \exp([\mathfrak{p}_0(n) \cap \mathfrak{v}] \oplus \mathfrak{v}_n) \doteq E. \tag{3.10}$$

We want to show that the fiber is essentially determined by its image in $\mathcal{P}_0(n) = \{x \in \mathbb{C}^{n \times n} \mid x^* = x > 0\}$ by the map $\mathbf{SL}_n(\mathbb{C}) \ni x \to x^* x \in \mathcal{P}_0(n)$.

Let us set

$$\Phi_0 = \{x^* x \mid x \in F_0\}. \tag{3.11}$$

We obtain

Proposition 3.2.1. *A necessary condition for F_0 being the fiber of a covariant fibration is that*

$$\{z \in \mathbf{V} \mid z^* \Phi_0 z = \Phi_0\} = \mathbf{I}_0. \tag{3.12}$$

Proof. Assume that $z^* y^* y z = x^* x$, with $z \in \mathbf{V}$ and $x, y \in F_0$. Then $x = uyz$ for some $u \in \mathbf{SU}(n)$. Decompose $z = w \cdot v$, with $w \in \mathbf{I}_0$ and $v \in E$. We obtain

$$x = (u \cdot w) \cdot (w^{-1} \cdot y \cdot w) \cdot v, \quad \text{with } (u \cdot w) \in \mathbf{SU}(n),\ (w^{-1} \cdot y \cdot w) \in F_0,\ v \in E.$$

Due to the uniqueness of the decomposition (3.10), we obtain that $u = w^{-1} \in \mathbf{I}_0$, $x = w^{-1} \cdot y \cdot w \in F_0$ and $v = \mathrm{I}_n$. In particular, $z = w \cdot \mathrm{I}_n = w \in \mathbf{I}_0$. □

Proposition 3.2.2. (3.12)*, together with*

$$\text{the map } F_0 \ni x \longrightarrow x^* x \in \Phi_0 \text{ is a diffeomorphism onto,} \tag{3.13}$$

$$\text{the map } \Phi_0 \times \mathbf{V} \ni (p, z) \longrightarrow z^* p z \in \mathcal{P}_0(n) \text{ is onto,} \tag{3.14}$$

are necessary and sufficient conditions in order that F_0 be the typical fiber of a covariant fibration.

Proof. In fact, for $\zeta \in \mathbf{SL}_n(\mathbb{C})$, by (3.14) there are $z \in \mathbf{V}$ and $p \in \Phi_0$ such that $\zeta^* \cdot \zeta = z^* \cdot p \cdot z$. By (3.13), there is a unique $x \in F_0$ for which $p = x^* x$. From $\zeta^* \cdot \zeta = z^* \cdot x^* \cdot x \cdot z$ we obtain that $\zeta = u \cdot x \cdot z$ with $u \in \mathbf{SU}(n)$. Then we use the decomposition $z = w \cdot v$ with $w \in \mathbf{I}_0$ and $v \in E$ to get the unique decomposition

$$x = (u \cdot w) \cdot (w^{-1} \cdot y \cdot w) \cdot v, \quad \text{with } (u \cdot w) \in \mathbf{SU}(n),\ (w^{-1} \cdot x \cdot w) \in F_0,\ v \in E.$$

This completes the proof. □

3. Geometry of the space of Hermitian symmetric forms

For better understanding the Mostow decomposition of $M_- \simeq \mathbf{K}/\mathbf{V}$, we begin by investigating the structure of $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$, which is a noncompact Hermitian symmetric space of rank $(n-1)$. We consider [**19**] as a general reference.

These comments are relevant because the compact Lie group $\mathbf{K}_0$ has, for some integer $n > 1$, a faithful linear representation in $\mathbf{SU}(n)$, which extends to a linear representation $\mathbf{K} \hookrightarrow \mathbf{SL}_n(\mathbb{C})$. We can therefore identify $\mathbf{K}$ with a closed subgroup of $\mathbf{SL}_n(\mathbb{C})$ and M_- with a submanifold of $\mathbf{SL}_n(\mathbb{C})/\mathbf{V}$.

The linear group $\mathbf{SL}_n(\mathbb{C})$ has the Cartan decomposition

$$\mathbf{SU}(n) \times \mathfrak{p}_0(n) \ni (x, X) \longrightarrow x \exp(X) \in \mathbf{SL}_n(\mathbb{C}), \tag{3.15}$$

where its maximal compact subgroup $\mathbf{SU}(n) = \{x \in \mathbf{SL}_n(\mathbb{C}) \mid x^*x = \mathrm{I}_n\}$ consists of the $n \times n$ unitary matrices with determinant one, and $\mathfrak{p}_0(n)$ is the subspace of traceless Hermitian symmetric matrices in $\mathbb{C}^{n\times n}$.

The quotient $\mathcal{M}_n = \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$ is endowed with a riemannian metric with nonpositive sectional curvature. We can identify $\mathcal{M}_n$ with the set $\mathcal{P}_0(n)$ of positive definite Hermitian symmetric matrices in $\mathbf{SL}_n(\mathbb{C})$ (which is diffeomorphic to $\mathfrak{p}_0(n)$ via the exponential map). In this way $\mathcal{M}_n$ is an open subset of $\mathfrak{p}_0(n)$ and its tangent bundle $T\mathcal{M}_n$ is naturally diffeomorphic to the subbundle of the trivial bundle $\mathcal{P}_0(n) \times \mathfrak{p}(n)$ (here $\mathfrak{p}(n)$ is the space of Hermitian symmetric matrices in $\mathbb{C}^{n\times n}$, with no trace condition)

$$T\mathcal{M}_n \simeq \{(p, X) \in \mathcal{P}_0(n) \times \mathfrak{p}(n) \,|\, \mathrm{trace}(p^{-1}X) = 0\}.$$

In this identification, $\mathbf{SL}_n(\mathbb{C})$ acts on $\mathcal{M}_n$ by

$$\mathbf{SL}_n(\mathbb{C}) \times \mathcal{M}_n \ni (z, h) \longrightarrow zhz^* \in \mathcal{M}_n, \tag{3.16}$$

as a group of isometries, and $\mathbf{SU}(n)$ is the stabilizer of the identity $e = \mathrm{I}_n$, that we choose as the base point.

The metric on $\mathcal{M}_n$ is associated to the restriction to $\mathfrak{p}_0(n)$ of the canonical symmetric bilinear form

$$\langle X|Y\rangle = \mathrm{trace}(XY) \quad \text{for } X, Y \in \mathfrak{sl}_n(\mathbb{C}), \tag{3.17}$$

which is real valued and positive definite on $\mathfrak{p}_0(n)$. This means that at the base point $e = \pi(\mathrm{I}_n)$ the Riemannian metric is defined by

$$g_e(X, Y) = \langle X|Y\rangle, \;\; \text{for } X, Y \in \mathfrak{p}_0(n) \simeq T_e\mathcal{M}_n,$$

and at a point $p = zz^* \in \mathcal{M}_n$, for $z \in \mathbf{SL}_n(\mathbb{C})$, by

$$g_p(X, Y) \doteq g_e(p^{-1}X, p^{-1}Y). \tag{3.18}$$

By the trace's properties, for all $X, Y \in T_p\mathcal{M}_n$, we have

$$\begin{aligned} g_p(X, Y) &= \mathrm{trace}(p^{-1}Xp^{-1}Y) = \mathrm{trace}(X\,p^{-1}Yp^{-1}) \\ &= \mathrm{trace}(z^{-1}X[z^*]^{-1}z^{-1}Y[z^*]^{-1}) \end{aligned}$$

which yields in particular

$$g_p(X, Y) = \langle z^{-1}X(z^*)^{-1}|z^{-1}Y(z^*)^{-1}\rangle, \;\; \text{for } X, Y \in T_p\mathcal{M}_n. \tag{3.19}$$

Thus we have verified

Proposition 3.3.1. $\mathbf{SL}_n(\mathbb{C})$ *acts by* (3.16) *as a group of isometries.*

The exponential map restricts to a diffeomorphism

$$\mathfrak{p}_0(n) \ni X \to \exp(X) \in \mathcal{M}_n$$

and the curves $t \to \exp(tX)$, for $X \in \mathfrak{p}_0(n)$, are the geodesics through the identity in $\mathcal{M}_n$. By translation, the curves in $\mathcal{M}_n$ of the form

$$\mathbb{R} \ni t \to z\exp(tX)z^* \in \mathcal{M}_n, \quad \textit{for } X \in \mathfrak{p}_0(n),\ z \in \mathbf{SL}_n(\mathbb{C}) \tag{3.20}$$

are the complete geodesics issued from $p = zz^*$. We note that, if $z \in \mathbf{SL}_n(\mathbb{C})$ and $z^*z = z_1z_1^*$, with $z_1 \in \mathbf{SL}_n(\mathbb{C})$, then $z_1 = zu$, with $u \in \mathbf{SU}(n)$ and therefore $z_1 \exp(tX)z_1^* = z\exp(t\mathrm{Ad}(u)X)z^*$.

To compute the distance of a point $p \in \mathcal{M}_n$ from the base point e, we observe that $\mathbf{SU}(n)$ is the stabilizer of e and therefore $\mathrm{dist}(p,e) = \mathrm{dist}(z^*p\,z,\ e)$ if $z \in \mathbf{SU}(n)$. Taking $z \in \mathbf{SU}(n)$ such that $p' = zpz^* = \mathrm{diag}(\lambda_1(p),\dots,\lambda_n(p))$, the geodesic segment joining e to p' is

$$[0,1] \ni t \to \exp\left[t\cdot \mathrm{diag}(\log(\lambda_1(h)),\dots,\log(\lambda_n(h)))\right].$$

Since $\mathrm{dist}(p,e) = \mathrm{dist}(p',e)$, this yields the formula for the distance:

$$\mathrm{dist}^2(p,e) = \sum\nolimits_{i=1}^{n} |\log(\lambda_i(p))|^2, \tag{3.21}$$

where $\lambda_i(p) > 0$ are the eigenvalues of p.

Lemma 3.3.2. *If $p_1, p_2 \in \mathcal{P}_0(n)$, then*

$$\mathrm{dist}^2(p_1,p_2) = \sum\nolimits_{i=1}^{n} |\log(\lambda_i(p_1^{-1}p_2)|^2. \tag{3.22}$$

Proof. Let $a \in \mathcal{P}_0(n)$ with $a^2 = p_1$. By Sylvester determinant theorem [**28**] the matrices $p_1^{-1}p_2$ and $a^{-1}p_2a^{-1}$ have the same characteristic polynomial and hence the same eigenvalues. Indeed $p_1^{-1}p_2$ is conjugate to $a(p_1^{-1}p_2)a^{-1} = a^{-1}p_2a^{-1}$. Since $\mathrm{dist}(p_1,p_2) = \mathrm{dist}(e, a^{-1}p_2a^{-1})$, formula (3.22) follows from (3.21). □

3.1. Killing and Jacoby vector fields on $\mathcal{M}_n$. Killing vector fields are the infinitesimal generators of the one parameter groups of isometries. By Prop.3.3.1 the map

$$\mathfrak{sl}_n(\mathbb{C}) \ni Z \longrightarrow \zeta_Z = \{p \to Zp + pZ^*\} \in \mathfrak{X}(\mathcal{M}_n), \tag{3.23}$$

is an isomorphism between $\mathfrak{sl}_n(\mathbb{C})$ and the Lie algebra of Killing vector fields on $\mathcal{M}_n$.

Fix $H \in \mathfrak{p}_0(n)$ and consider the geodesic

$$\gamma_H : \mathbb{R} \ni t \to \exp(tH) \in \mathcal{M}_n,$$

through the base point e. Denote by $\mathcal{J}(H)$ the space of Jacobi vector fields on γ_H, and by $\mathcal{J}_0(H)$ the subspace of those those vanishing for $t = 0$.

For every $Z \in \mathfrak{sl}_n(\mathbb{C})$, the restriction θ_{Z^*}, of ζ_Z, to γ_H is a Jacobi vector field. For $h = \exp(H)$, set $\mathfrak{su}_h(n) = \{Z \in \mathfrak{sl}_n(\mathbb{C}) \mid Z^*h + hZ = 0\}$. It is the Lie algebra of the isotropy group of h for the action (3.16) of $\mathbf{SL}_n(\mathbb{C})$.

Lemma 3.3.3. *The correspondence*

$$\theta : \mathfrak{sl}_n(\mathbb{C}) \ni Z \longrightarrow \theta_Z(t) = \{t \to Z^*\exp(tH) + \exp(tH)Z\} \in \mathcal{J}(H) \tag{3.24}$$

is a linear map with kernel

$$\ker\theta = \{Z \in \mathfrak{su}(n) \mid [H,Z] = 0\} = \mathfrak{su}(n) \cap \mathfrak{su}_h(n). \tag{3.25}$$

We have

$$\dot\theta_Z(t) = \tfrac{1}{2}\left(\exp(tH)[H,Z] - [H,Z^*]\exp(tH)\right) = \tfrac{1}{2}\theta_{[H,Z]}(t), \quad \forall t \in \mathbb{R}. \tag{3.26}$$

In particular, $\dot\theta_Z$ is always orthogonal to $\dot\gamma_H(t)$ along γ_H, whereas θ_Z is orthogonal to $\dot\gamma_H(t)$ along γ_H if and only if

$$\mathrm{trace}\left((Z+Z^*)H\right) = 0. \tag{3.27}$$

Proof. The condition $\theta_Z(0) = 0$ implies that $Z^* + Z = 0$, i.e. that $Z \in \mathfrak{su}(n)$. To compute the covariant derivative $\dot\theta_Z(t)$, we note that for all $X \in \mathfrak{p}_0(n)$ the vector fields

$$\mathbb{R} \ni t \longrightarrow (\exp(tH), \exp(tH/2)X\exp(tH/2)) \in T_{\gamma_H(t)}\mathcal{M}_n$$

are parallel along γ_H. Thus

$$\begin{aligned}\dot\theta_Z(t) &= \frac{d}{ds}\left((\exp(-sH/2)[Z^*\exp([t+s]H) + \exp([t+s]H)Z]\exp(-sH/2))\right)|_{s=0}\\ &= \tfrac{1}{2}\left([Z^*, H]\exp(tH) + \exp(tH)[H, Z]\right).\end{aligned}$$

For $Z \in \mathfrak{su}(n)$, we have $Z^* = -Z$ and therefore $\dot\theta_Z(0) = [H, Z]$, yielding (3.25). Finally, we observe that, since H and $\gamma_H(t)$ commute for all t,

$$\begin{aligned} g_{\gamma_H(t)}(\theta_Z(t), \dot\gamma_H(t)) &= \mathrm{trace}(\gamma_H^{-1}(t)\theta_Z(t)\gamma_H^{-1}(t)H\gamma_H(t)) = \mathrm{trace}\left((Z + Z^*)H\right);\\ g_{\gamma_H(t)}(\dot\theta_Z(t), \dot\gamma_H(t)) &= \mathrm{trace}(\gamma_H^{-1}(t)\dot\theta_Z(t)\gamma_H^{-1}(t)H\gamma_H(t))\\ &= \mathrm{trace}\left([H, Z - Z^*]H\right) = \mathrm{trace}([[H, H], Z - Z^*]) = 0.\end{aligned}$$

This completes the proof. □

To finish up the description of the Jacobi vector fields on γ_H we need to introduce the space

$$\mathfrak{C}(H) = \{X \in \mathfrak{p}_0(n) \mid [H, X] = 0\} = i(\mathfrak{su}(n) \cap \mathfrak{su}_h(n)). \tag{3.28}$$

Lemma 3.3.4. *For $T_0, T_1 \in \mathfrak{C}(H)$,*

$$\{\mathbb{R} \ni t \to J(t) = (T_0 + tT_1)\gamma_H(t)\} \in \mathcal{J}(H), \quad \text{with } J(0) = T_0 \text{ and } \dot J(0) = T_1. \tag{3.29}$$

In particular, $\{T\gamma_H(t) \mid T \in \mathfrak{C}(H)\}$ are the parallel Jacobi vector fields on γ_H. □

Since on the geodesics of negatively curved manifolds there are no pairs of conjugate points, we obtain

Lemma 3.3.5. *Denote by $\mathcal{J}_0(H)$ the space of Jacobi vector fields vanishing at 0. Then the map*

$$\mathcal{J}_0(H) \ni J \longrightarrow J(t) \in T_{\gamma(t)}\mathcal{M}_n \tag{3.30}$$

is a linear isomorphism for every $t \neq 0$. □

Since $\mathcal{M}_n$ has negative sectional curvature, if $J \in \mathcal{J}(H)$, then $t \to \|J(t)\|$ is a nonnegative convex function and therefore there are the three possibilities:

(1) $\|J(t)\|$ has neither a maximum nor a minimum;
(2) $\|J(t)\|$ has a minimum and is unbounded;
(3) $J(t)$ is parallel along γ_H.

For $X, \xi \in \mathfrak{p}_0(n)$, $Z \in \mathfrak{sl}_n(\mathbb{C})$ we shall use the notation:

- J_X is the Jacobi vector field satisfying
$$J_X(0) = 0, \quad \dot J_X(0) = X;$$
- L_ξ is the Jacobi vector field satisfying
$$L_\xi(0) = \xi, \quad \dot L_\xi(0) = 0;$$
- θ_Z is the Jacobi vector field satisfying
$$\theta(0) = Z + Z^*, \quad \dot\theta_Z(0) = \tfrac{1}{2}[H, Z - Z^*].$$

Note that $L_\xi = \frac{1}{2}\theta_\xi$ for $\xi \in \mathfrak{p}_0(n)$ and that θ_Z is the Jacobi associated to the Killing vector field ζ_{Z^*}.

Since H, and hence ad_H, are semisimple, we have a direct sum decomposition

$$\mathfrak{p}_0(n) = \mathrm{Imm}(\mathrm{ad}(iH)) \oplus \mathrm{Ker}(\mathrm{ad}(iH)) = [H, \mathfrak{su}(n)] \oplus \mathfrak{C}(H). \tag{3.31}$$

Therefore we obtain

PROPOSITION 3.3.6. *The space of Jacobi vector fields along γ_H is described by*

$$\mathcal{J}(H) = \{\theta_Z + tT\gamma_H \mid Z \in \mathfrak{sl}_n(\mathbb{C}),\ T \in \mathfrak{C}(H)\}. \tag{3.32}$$

Those vanishing for $t = 0$ are

$$\mathcal{J}_0(H) = \{\theta_Y + tT\gamma_H \mid Y \in i[H, \mathfrak{su}(n)],\ T \in \mathfrak{C}(H)\}. \tag{3.33}$$

For $J(t) = \theta_Z(t) + tT\gamma_H(t)$, with $Z \in \mathfrak{sl}_n(\mathbb{C})$ and $T \in \mathfrak{C}(H)$ we have

$$\begin{cases} J(0) = Z + Z^*, \\ \dot{J}(0) = \frac{1}{2}[H, Z - Z^*] + T, \end{cases} \qquad \begin{cases} \dot{J}(t) = \frac{1}{2}\theta_{[H,Z]+T}(t), \\ \ddot{J}(t) = \frac{1}{4}\theta_{[H,[H,Z]]}(t). \end{cases} \tag{3.34}$$

PROOF. The characterizations of Jacobi vector fields in (3.32) and (3.33) follow from Lemma 3.3.3 and Formula (3.31). To compute their covariant derivatives, we can use the parallel transport $T_{\gamma_H(t)}\mathcal{M}_n \ni X \to \exp(sH/2)X\exp(sH/2) \in T_{\gamma_H(t+s)}\mathcal{M}_n$ along γ_H. Then

$$\begin{aligned} \dot{\theta}_Z(t) &= \left(\frac{d}{ds}\right)_{s=0} [\exp(-sH/2)\{Z^*\exp([t+s]H) + \exp([t+s]H)Z\}\exp(-sH/2)] \\ &= \tfrac{1}{2}[Z^*, H]\exp(tH) + \tfrac{1}{2}\exp(tH)[H, Z] = \tfrac{1}{2}\theta_{[H,Z]}(t). \end{aligned}$$

This yields (3.34). □

LEMMA 3.3.7. *We have*

$$\|J(1)\|^2 = \|J(0)\|^2 + 2(J(0)|\dot{J}(0)) + 2\int_0^1 (1-t)\left(\|\dot{J}(t)\|^2 + (J(t), \ddot{J}(t))\right)dt, \qquad \forall J \in \mathcal{J}(H), \tag{3.35}$$

where norms and scalar products are taken at the corresponding points $\gamma_H(t)$.

PROOF. We use the integral form

$$f(1) = f(0) + f'(0) + \int_0^1 (1-t)f''(t)dt$$

of the reminder in Taylor's expansion of a C^2 function of a real variable, with $f(t) = \|J(t)\|^2_{\gamma(t)}$. □

To better understand Formula (3.35), it is convenient to introduce the symmetric bilinear form

$$(J_1|J_2)_H = \int_0^1 (1-t)\left\{(\ddot{J}_1(t)|J_2(t)) + (J_1|\ddot{J}_2(t)) + 2(\dot{J}_1(t)|\dot{J}_2(t))\right\}dt, \qquad \text{for } J_1, J_2 \in \mathcal{J}(H). \tag{3.36}$$

LEMMA 3.3.8. *Form* (3.36) *is positive semidefinite on $\mathcal{J}(H)$ and*

$$(J|J)_H = 0 \Leftrightarrow J = \theta_\xi \text{ for a } \xi \in \mathfrak{C}(H).$$

PROOF. For $J \in \mathcal{J}(H)$, we have

$$(\ddot{J}, J) = -(R(J, \dot{\gamma}_H)\dot{\gamma}_H|J) \geq 0, \quad \forall t \in \mathbb{R},$$

because $\mathcal{M}_n$ has negative sectional curvature. Thus for each $t \in \mathbb{R}$ the symmetric form

$$\beta(J_1(t), J_2(t)) = g(\ddot{J}_1(t), J_2(t)) + g(J_1(t), \ddot{J}_2(t))$$

is positive semidefinite. Hence, if $(J|J)_H = 0$, then $\dot{J}(t) = 0$ for all t. The statement follows from the fact that the Jacobi fields L_ξ with $\xi \in \mathfrak{C}(H)$ are exactly those which are parallel along γ_H. □

By polarizazion we obtain

LEMMA 3.3.9.

$$(J_1(1)|J_2(1))_h = (J_1(0)|J_2(0))_h + (J_1(0)|\dot{J}_2(0))_h + (J_2(0)|\dot{J}_1(0))_h + (J_1|J_2)_H, \quad \forall J_1, J_2 \in \mathcal{J}(H). \tag{3.37}$$

□

REMARK 3.3.10. To compute explicitly the scalar product of Jacobi vector fields at different points, it is convenient to consider their parallel transport along γ_H to the base point e. The correspondence is

$$T_{\gamma_H(t)}\mathcal{M}_n \ni \theta_Z(t) \longrightarrow Z_t + Z_t^* \in T_e\mathcal{M}_n, \quad \text{with} \quad Z_t = \exp(tH/2)Z\exp(-tH/2).$$

In particular, since $[H, Z]_t = [H, Z_t]$, we obtain

$$\begin{aligned}
\|\dot{\theta}_Z(t)\|^2 &= \tfrac{1}{4}\text{trace}(\{[H, Z_t] + [H, Z_t]^*\}^2) \\
&= \tfrac{1}{4}\text{trace}(\{[H, Z_t] - [H, Z_t^*]\}^2) \\
&= \tfrac{1}{4}\text{trace}([H, Z_t - Z_t^*]^2), \\
(\theta_Z(t)|\ddot{\theta}_Z(t)) &= \tfrac{1}{4}\text{trace}(\{Z_t + Z_t^*\} \cdot \{[H, [H, Z_t]] + [H, [H, Z_t]]^*\}) \\
&= \tfrac{1}{4}\text{trace}(\{Z_t + Z_t^*\} \cdot [H, [H, Z_t + Z_t^*]]) \\
&= -\tfrac{1}{4}\text{trace}([H, Z_t + Z_t^*]^2).
\end{aligned}$$

Since $[H, Z_t^*] = -[H, Z_t]^*$, we obtain

$$\begin{aligned}
\|\dot{\theta}_Z(t)\|^2 + (\theta_Z(t)|\ddot{\theta}_Z(t)) &= \tfrac{1}{4}\text{trace}(([H, Z_t] + [H, Z_t]^*)^2 - ([H, Z_t] - [H, Z_t]^*)^2) \\
&= \text{trace}([H, Z_t] \cdot [H, Z_t]^*).
\end{aligned}$$

With this notation, (3.35) can be rewritten as

$$\begin{aligned}
\|Z_1 + Z_1^*\|^2 &= \|Z_0 + Z_0^*\|^2 + 2(H|[Z_0, Z_0^*]) + \int_0^1 (1-t) \cdot \text{trace}([H, Z_t] \cdot [H, Z_t]^*)dt \\
&= \|Z_0 + Z_0^*\|^2 + ([H, Z_0]|Z_0^*]) + ([H, Z_0]^*|Z_0) \\
&\quad + \int_0^1 (1-t) \cdot \text{trace}([H, Z_t] \cdot [H, Z_t]^*)dt.
\end{aligned}$$

REMARK 3.3.11. The space $\mathcal{J}(H)$ of Jacobi vector fields along γ_H has a natural symplectic structure, defined by the form

$$\begin{aligned}
(*) \qquad \omega(J_1, J_2) &= \text{trace}(J_1(0)\dot{J}_2(0) - \dot{J}_1(0)J_2(0)) \\
&= (J_1(t)|\dot{J}_2(t)) - (J_2(t)|\dot{J}_1(t)), \quad \forall t \in \mathbb{R},
\end{aligned}$$

where the last expression is constant along γ_H (see [**29**, Prop.1.12.3]).

In particular, if $X \in \mathfrak{p}_0(n)$, $Z \in \mathfrak{sl}_n(\mathbb{C})$ and $\mathrm{trace}(XZ) = 0$, then

$$\frac{D}{dt} g_{\gamma_H(t)}(J_X, L_Z) = g_{\gamma_H(t)}(\dot{J}_X, L_Z) + g_{\gamma_H(t)}(J_X, \dot{L}_Z) = 2g_{\gamma_H(t)}(\dot{J}_X, L_Z),$$

because

$$\omega(J_X, L_Z) = g_e(\dot{J}_X(0), L_Z(0)) - g_{\gamma_H(0)}(J_X, \dot{L}_Z(0)) = \tfrac{1}{2}\mathrm{trace}(X(Z + Z^*)) = 0.$$

Lemma 3.3.12. *If* $Z \in \mathfrak{sl}_n(\mathbb{C})$ *and* $X \in \mathfrak{p}_0(n)$, *then*

$$(J_X(1)|\theta_Z(1))_h = (X|Z + Z^*)_e + (J_X|\theta_Z)_H. \tag{3.38}$$

Proof. This follows by Formula (3.37) of Lemma 3.3.9 because $J_X(0) = 0$. □

Lemma 3.3.13. *Let* $X, \xi \in \mathfrak{p}_0(n)$ *be nonzero matrices such that*

$$(X|\xi)_e = \mathrm{trace}(X\,\xi) = 0. \tag{3.39}$$

Then

$$|(J_X(1)|L_\xi(1))_h| < \|J_X(1)\|_h \cdot \|L_\xi(1)\|_h. \tag{3.40}$$

In particular, $J_X(1)$ *and* $L_\xi(1)$ *cannot be proportional.*

Proof. By Formula (3.37), we have

$$\begin{aligned} \|J_X(1)\|_h^2 &= \|J_X\|_H^2, \\ \|L_\xi(1)\|_h^2 &= \|\xi\|_e^2 + \|L_\xi\|_H^2, \\ (J_X(1)|l_\xi(1))_h &= (J_X|L_\xi)_H. \end{aligned}$$

Then

$$|(J_X(1)|L_\xi(1))_h| \le \|J_X\|_H \cdot \|L_\xi\|_H < \|J_X\|_H \cdot \sqrt{\|\xi\|_e^2 + \|L_\xi\|_H^2}$$

when X and ξ are different from zero. This proves the lemma. □

Let J be any Jacobi vector field along γ_H. We rewrite Formula (3.35) in the form

$$\|J(1)\|_h^2 = \|J(0)\|_e^2 + 2(J(0)|\dot{J}(0))_e + (J|J)_H.$$

Let us take $J = \theta_Z + J_X$ with $Z \in \mathfrak{sl}_n(\mathbb{C})$ and $X \in \mathfrak{p}_0(n)$ with $\mathrm{trace}(XZ) = 0$. Then

$$J(0) = Z + Z^*, \quad \dot{J}(0) = X + \tfrac{1}{2}[H, Z - Z^*]$$

yields

$$\begin{aligned} \|J(1)\|_h^2 &= \|Z + Z^*\|_e^2 + 2(Z + Z^*|X + \tfrac{1}{2}[H, Z - Z^*])_e + (J|J)_H \\ &= \|Z + Z^*\|_e^2 + ([H, Z - Z^*]|Z + Z^*)_e + (J|J)_H \\ &= \|Z + Z^*\|_e^2 + 2(H|[Z, Z^*])_e + (J|J)_H. \end{aligned}$$

Thus we obtain

Lemma 3.3.14. *Let* $Z \in \mathfrak{sl}_n(\mathbb{C})$ *and* $X \in \mathfrak{p}_0(n)$, *with* $\mathrm{trace}(XZ) = 0$. *Then*

$$\|\theta_Z(1) - J_X(1)\|_h^2 = \|Z + Z^*\|_e^2 + 2(H|[Z, Z^*])_e + \|\theta_Z - J_X\|_H^2. \tag{3.41}$$ □

An interesting feature of Formula (3.41) is that the first two summands are independent of X. In particular, the left hand side is positive for all Z in a real subspace $\mathfrak{v}'$ of $\mathfrak{sl}_n(\mathbb{C})$ with $\mathfrak{v}' \cap \mathfrak{su}(n) = \{0\}$ when $\|H\|_e$ is sufficiently small.

The Jacobi vector fields can be used to compute the differential of the exponential map. In fact, for $H, X \in \mathfrak{p}_0(n)$, the covariant derivative $\frac{D}{dt}\exp(H+tX)|_{t=0}$ is the value for $t = 1$ of the Jacobian vector field J_X along γ_H, which was taken to satisfy the initial conditions $J_X(0) = 0$ and $\dot{J}_X(0) = X$. By the previous discussion,

$$\begin{aligned} J_X(t) &= [\exp(tH), Y] + tT\exp(tH) \in \mathcal{J}_0(H), \quad \text{where} \\ X &= [Y, H] + T, \text{ with } Y \in \mathfrak{su}(n) \text{ and } T \in \mathfrak{C}(H), \end{aligned} \tag{3.42}$$

and then we get

$$\frac{D}{dt}\exp(H+tX)|_{t=0} = J_X(1) = [\exp(H), Y] + T\exp(H). \tag{3.43}$$

4. Decompositions with Hermitian fibers

Let $\mathbf{V}$ be a closed complex Lie subalgebra of $\mathbf{SL}_n(\mathbb{C})$, which admits a Levi-Chevalley decomposition (3.1). We keep the notation of Section 1, but substitute $i\mathfrak{m}_0$ for $\mathfrak{m}_0$. Then we set

$$\begin{aligned} \mathfrak{m}_0 &= (\mathfrak{v} + \mathfrak{v}^*)^\perp \cap \mathfrak{p}_0(n), \\ \mathfrak{v}' &= \mathfrak{v}_n \oplus (\mathfrak{v} \cap \mathfrak{p}_0(n)). \end{aligned}$$

As we indicated before, $\exp(\mathfrak{m}_0)$ is our favourite candidate for the typical fiber F_0 of a $\mathbf{K}_0$-covariant fibration of $\mathbf{SL}_n(\mathbb{C})/\mathbf{V}$. In fact, we will show that this is the case for some special classes of $\mathbf{V}$. We begin by studying here the map

$$\mathbf{SU}(n) \times \mathfrak{m}_0 \times \mathbf{V} \ni (x, X, z) \to x\exp(X)z \in \mathbf{SL}_n(\mathbb{C})$$

in general.

Proposition 3.4.1. *We can find a real $r > 0$ such that the map*

$$\lambda : \mathfrak{v}' \times \mathfrak{m}_0 \ni (z, X) \longrightarrow \exp(Z^*) \cdot \exp(X) \cdot \exp(Z) \in \mathcal{M}_n \tag{3.44}$$

is a local diffeomorphism for $\|X\|_e < r$.

Proof. By using the diffeomorphisms $p \to z^* \cdot p \cdot z$ of $\mathcal{M}_n$, we can reduce to proving the statement at points $(0, H)$, with $H \in \mathfrak{m}_0$. At those points, to compute the differential we can use the Jacobi vector fields θ_Z and J_X on γ_H. Indeed we have $d\lambda(0, H)(Z, 0) = \theta_Z(1)$ and $d\lambda(0, H)(0, X) = J_X(1)$. Moreover, the maps $\mathfrak{v}' \ni Z \to \theta_Z(1) \in T_h\mathcal{M}_n$ and $\mathfrak{m}_0 \ni X \to J_X(1) \in T_h\mathcal{M}_n$ are injective. By Lemma 3.3.14 we have

$$\|J_X(1) - \theta_Z(1)\|_h^2 \geq \|Z + Z^*\|_e^2 - 2(H|[Z, Z^*])_e, \quad \forall X \in \mathfrak{m}_0.$$

Since $[Z, Z^*] = 0$ when $Z + Z^* = 0$, we have, for some constant $C > 0$,

$$\left|(H|[Z, Z^*])_e\right| \leq C\|H\|_e\|Z + Z^*\|_e^2.$$

The statement follows. □

Corollary 3.4.2. *There is $r > 0$ such that the smooth map*

$$\mathbf{V} \times \mathfrak{m}_0 \ni (z, X) \longrightarrow z^* \cdot \exp(X) \cdot z \in \mathcal{M}_n \tag{3.45}$$

is a submersion for $\|X\| < r$. □

LEMMA 3.4.3. *The map* (3.45) *is onto.*

PROOF. For each $p \in \mathcal{M}_n$, the orbit

$$N_p = \{z^* p\, z \mid z \in \mathbf{V}\} \tag{3.46}$$

of $\mathbf{V}^*$ is a closed smooth submanifold of $\mathcal{M}_n$. There is a point $p_0 \in N_p$ having minimal distance from e, i.e. satisfying

$$\mathrm{dist}(p_0, e) = \min_{q \in N_p} \mathrm{dist}(q, e).$$

We can assume that $p_0 = p$. Let $a = \sqrt{p^{-1}} \in \mathcal{P}_0(n)$, $X = \log(p) \in \mathfrak{p}_0(n)$. Then $[0, 1] \ni t \to \gamma_X(t) = \exp(tX)$ is the geodesic arc joining e to p and $\dot{\gamma}(1) = a^{-1} X\, a^{-1}$ is orthogonal to $T_p N_p = \{Z^* p + pZ \mid Z \in \mathfrak{v}\}$. Since a, a^{-1}, p, X commute, we have

$$\begin{aligned}
(a^{-1} X a^{-1} | Z^* p + pZ)|_h &= \mathrm{trace}(X(aZ^* pa + apZa)) \\
&= \mathrm{trace}(aXZ^* pa + apXZa) \\
&= \mathrm{trace}(aXZ^* a^{-1}) + \mathrm{trace}(a^{-1} XZa) \\
&= \mathrm{trace}(XZ^*) + \mathrm{trace}(XZ) \\
&= \mathrm{trace}(X(Z + Z^*)) = 0, \quad \forall Z \in \mathfrak{v},
\end{aligned}$$

shows that $X \in \mathfrak{m}_0$. In particular, each orbit N_p intersects $\exp(\mathfrak{m}_0)$. Thus, for each $p \in \mathcal{P}_0(n)$, the intersection $N_p \cap \exp(\mathfrak{m}_0)$ contains a point $z^* pz = \exp(X)$, with $z \in \mathbf{V}$ and $X \in \mathfrak{m}_0$. This yields $p = [z^{-1}]^* \exp(X)[z^{-1}]$, proving the Lemma. □

From Lemma 3.4.3 we obtain the surjectivity of the map

$$\mathbf{SU}(n) \times \mathfrak{m}_0 \ni (x, X) \longrightarrow \pi(x \cdot \exp(X)) \in \mathbf{SL}_n(\mathbb{C})/\mathbf{V}, \tag{3.47}$$

where $\pi : \mathbf{SL}_n(\mathbb{C}) \to \mathbf{SL}_n(\mathbb{C})/\mathbf{V}$ is the projection onto the quotient:

PROPOSITION 3.4.4. *The map* (3.47) *is onto.*

PROOF. Let $z \in \mathbf{SL}_n(\mathbb{C})$. Then $z^* z \in \mathcal{M}_n$ and therefore

$$z^* z = v^* \cdot \exp(2X) \cdot v, \quad \text{with } v \in \mathbf{V}, X \in \mathfrak{m}_0.$$

Then

$$x = z \cdot v^{-1} \cdot \exp(-X)$$

satisfies

$$\begin{aligned}
x^* \cdot x &= \exp(-X)[v^{-1}]^* z^* z \cdot v^{-1} \cdot \exp(-X) \\
&= \exp(-X)[v^{-1}]^* v^* \cdot \exp(2X) \cdot v \cdot v^{-1} \cdot \exp(-X) = \mathrm{I}_n.
\end{aligned}$$

Therefore $x \in \mathbf{SU}(n)$ and

$$z = x \cdot \exp(X) \cdot v, \tag{3.48}$$

with $x \in \mathbf{SU}(n)$, $X \in \mathfrak{m}_0$, $v \in \mathbf{V}$. □

We already noticed that, in the situation of §1, where $\mathbf{V}$ is a closed subgoup of the complexification $\mathbf{K}$ of a compact group $\mathbf{K}_0$, we can assume that $\mathbf{K}$ is an algebraic subgroup of $\mathbf{SL}_n(\mathbb{C})$, with $\mathbf{K}_0 = \mathbf{K} \cap \mathbf{SU}(n)$ and $\mathbf{V} \cap \mathbf{SU}(n)$ a maximal compact subgroup of $\mathbf{V}$. We obtain the general statement:

Proposition 3.4.5. *Let* $\mathfrak{f}_0 = \mathfrak{m}_0 \cap \mathfrak{k}$. *Then we have a commutative diagram with surjective arrows*

$$\begin{array}{ccc} \mathbf{K}_0 \times \mathfrak{f}_0 & \longrightarrow & \mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0 \\ & \searrow \quad \swarrow & \\ & \mathbf{K}/\mathbf{V}, & \end{array} \tag{3.49}$$

where the horizontal arrow is the projection onto the quotient, the left arrow is obtained by restricting (3.47)*, and the right arrow is obtained by passing to the quotient.*

Proof. It is sufficient to follow the proof of Proposition 3.4.4 and check that $X \in \mathfrak{f}_0$ and $x \in \mathbf{K}_0$ if we take $z \in \mathbf{K}$.

In fact, for $z \in \mathbf{K}$ we obtain the decomposition

$$z = x \cdot \exp(X) \cdot v, \quad \text{with } x \in \mathbf{K}_0,\ X \in \mathfrak{f}_0,\ v \in \mathbf{V}, \tag{3.50}$$

because $\mathbf{K} \ni z^* z = v^* \exp(2X) v$ implies that $\exp(2X) = [v^*]^{-1} z^* z v^{-1} \in \mathbf{K}$ and hence $X \in \mathfrak{k} \cap \mathfrak{m}_0 = \mathfrak{f}_0$. □

The left arrow of (3.49) *is* actually the Mostow fibration when $\mathbf{V}$ is reductive (see e.g. [**43**]).

Proposition 3.4.6. *If* $\mathbf{V}$ *is reductive, then the natural surjective map*

$$\mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0 \to M_- = \mathbf{K}/\mathbf{V}$$

is a diffeomorphism.

Proof. In this case $\mathbf{V}$ is algebraic and self-adjoint and therefore has the Cartan decomposition

$$\mathbf{V} = (\mathbf{V} \cap \mathbf{SU}(n)) \times \exp(\mathfrak{v} \cap \mathfrak{p}_0(n)).$$

By Lemma 3.3.13, the map

$$(\mathfrak{v} \cap \mathfrak{p}_0(n)) \times \mathfrak{f}_0 \ni (Z, X) \longrightarrow \exp(Z^*) \cdot \exp(X) \cdot \exp(Z) \in \mathbf{V} \cap \mathcal{P}_0(n) = \exp(\mathfrak{v} \cap \mathfrak{p}_0(n))$$

is surjective and is a local diffeomorphism at every point of $(\mathfrak{v} \cap \mathfrak{p}_0(n)) \times \mathfrak{f}_0$. Thus, being a connected covering of a simply connected space, is a diffeomorphism. Hence, for every $\zeta \in \mathbf{K}$, there is a unique pair $(Z, X) \in (\mathfrak{v} \cap \mathfrak{p}_0(n)) \times \mathfrak{f}_0$ such that

$$\zeta^* \cdot \zeta = \exp(Z^*) \cdot \exp(X) \cdot \exp(Z).$$

Therefore $u = \zeta \cdot \exp(-Z) \cdot \exp(-\frac{1}{2}X) \in \mathbf{K}_0$ and we obtain the direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0(n)), \tag{3.51}$$

from which the statement follows. □

Proposition 3.4.6 corresponds to an M_- which is the Stein complexification of a totally real compact M_0. We will be more interested in the case where $\mathbf{V}$ has a nontrivial unipotent radical.

When we know that decomposition (3.47) is unique, we can extract some extra information from the minimal distance characterization of p_0 in the proof of Lemma 3.4.3. For instance, as a corollary of Lemma3.4.3, we obtain the following

Proposition 3.4.7. *For $h \in \mathcal{P}_0(n)$, denote by $D_\ell(h)$ the minor determinant of the first ℓ rows and columns of h. Set $D_0(h) = 1$ and let $0 < \lambda_1(h) \leq \cdots \leq \lambda_n(h)$ be the eigenvalues of h. Then*

$$\text{dist}(h, e) = \sum\nolimits_{\ell=1}^{n} |\log(\lambda_\ell(h))|^2 \geq \sum\nolimits_{\ell=1}^{n} |\log(D_\ell(h)/D_{\ell-1}(h))|^2. \tag{3.52}$$

If h is not diagonal, we have strict inequality.

Proof. We take $\mathbf{V}$ equal to the group of unipotent upper triangular matrices in $\mathbf{GL}_n(\mathbb{C})$. The element $\delta = e^\Delta \in N_h = \{z^* hz \mid z \in \mathbf{V}\}$, with $\Delta \in \mathfrak{p}_0$, at minimal distance from e satisfies $\text{trace}([Z + Z^*]\Delta) = 0$ for all nilpotent upper triangular Z and hence is diagonal. The unique diagonal $\delta = z^* hz$ in N_h is the one obtained by the Gram-Schmidt orthogonalization procedure. The proof is complete. □

The orbit of a point $p \in \mathcal{M}_n$ by the group of unipotent upper triangular matrices of $\mathbf{SL}_n(\mathbb{C})$ is an example of a *horocycle* of maximal dimension in a symmetric space of noncompact type. We will generalize this situation while outlining a class of subroups $\mathbf{V}$ for which $F = \exp(\mathfrak{f}_0)$ can be taken as the fiber of the $\mathbf{K}_0$-covariant fibration.

5. The general case

Before computing in general the typical fiber of the $\mathbf{K}_0$-covariant fibration $\pi : M_- \to M_0$, it is convenient to rehearse some notions introduced in [**8**, §3] to construct what was called there *CR deployment*. We simply assume, in the beginning, that $\mathfrak{k}$ is a reductive Lie algebra over $\mathbb{C}$.

For a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{k}$, we denote by $\mathfrak{r}(\mathfrak{a})$ its radical and by $\mathfrak{n}(\mathfrak{a})$ its ideal consisting of the $\text{ad}_{\mathfrak{k}}$-nilpotent elements of $\mathfrak{r}(\mathfrak{a})$. Starting from any splittable Lie subalgebra $\mathfrak{v}$ of $\mathfrak{k}$ we construct a sequence $\{\mathfrak{v}_{(h)}\}$ by setting recursively

$$\begin{cases} \mathfrak{v}_{(0)} = \mathfrak{v}, \\ \mathfrak{v}_{(h+1)} = \mathbf{N}_{\mathfrak{k}}(\mathfrak{n}(\mathfrak{v}_{(h)})) = \{Z \in \mathfrak{k} \mid [Z, \mathfrak{n}(\mathfrak{v}_{(h)})] \subset \mathfrak{n}(\mathfrak{v}_{(h)})\}, \quad \forall h \geq 0. \end{cases} \tag{3.53}$$

Each Lie subalgebra $\mathfrak{v}_{(h)}$, with $h \geq 1$, is the normalizer in $\mathfrak{k}$ of the ideal of ad-nilpotent elements of the radical of the preceding one. In the paper cited above it is shown that $\mathfrak{v}_{(h)} \subsetneqq \mathfrak{v}_{(h+1)}$ and $\mathfrak{n}(\mathfrak{v}_{(h)}) \subsetneqq \mathfrak{n}(\mathfrak{v}_{(h+1)})$ for all $h \geq 0$ and that the union $\bigcup_{h\geq 0}\mathfrak{v}_{(h)}$ is a parabolic subalgebra $\mathfrak{w}$ of $\mathfrak{k}$, which was called the *parabolic regularization of* $\mathfrak{v}$. This shows that the set

$$\mathfrak{P}(\mathfrak{v}) = \{\mathfrak{q} \mid \mathfrak{q} \text{ is parabolic in } \mathfrak{k} \text{ and } \mathfrak{v} \subset \mathfrak{q},\ \mathfrak{n}(\mathfrak{v}) \subset \mathfrak{n}(\mathfrak{q})\} \tag{3.54}$$

is nonempty.

Let us prove a general lemma on parabolic Lie subalgebras.

Lemma 3.5.1. *If $\mathfrak{q}_1, \mathfrak{q}_2$ are parabolic Lie subalgebras of $\mathfrak{k}$, then*

$$\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2 + \mathfrak{n}(\mathfrak{q}_1)$$

is a parabolic Lie subalgebra of $\mathfrak{k}$.

Proof. By [**17**, Ch.VIII,Prop.10] we know that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. If $\mathcal{R}$ is the set of roots corresponding to $\mathfrak{h}$, each parabolic Lie algebra $\mathfrak{q}_i$

($i = 1, 2$) can be decomposed into a direct sum of $\mathfrak{h}$ and of root spaces (see (1.11)) and there are $A_1, A_2 \in \mathfrak{h}_{\mathbb{R}}$ such that

$$\mathfrak{q}_i = \mathfrak{h} \oplus \sum_{\substack{\alpha\in\mathcal{R},\\ \alpha(A_i)\geq 0}} \mathfrak{k}_\alpha,$$

where $\mathfrak{k}_\alpha = \{Z \in \mathfrak{k} \mid [A, Z] = \alpha(A)Z,\ \forall A \in \mathfrak{h}_{\mathbb{R}}\}$ is the root space of α.

Take $\epsilon > 0$ so small that $\epsilon \cdot |\alpha(A_2)| < \alpha(A_1)$ if $\alpha(A_1) > 0$. Then we have

$$\mathfrak{q} = \mathfrak{h} \oplus \sum_{\substack{\alpha\in\mathcal{R},\\ \alpha(A_1+\epsilon A_2)>0}} \mathfrak{k}_\alpha$$

and this shows that $\mathfrak{q}$ is parabolic. In fact, if $\mathfrak{L}(\mathfrak{q}_i)$ are the $\mathfrak{h}$-invariant reductive summands of $\mathfrak{q}_i$ and $\mathfrak{n}(\mathfrak{q}_i)$ the ideals of nilpotent elements of their radicals, we have

$$\mathfrak{q} = (\mathfrak{L}(\mathfrak{q}_1) \cap \mathfrak{L}(\mathfrak{q}_2)) \oplus (\mathfrak{L}(\mathfrak{q}_1) \cap \mathfrak{n}(\mathfrak{q}_2)) \oplus \mathfrak{n}(\mathfrak{q}_1). \qquad \square$$

Let us assume, from this point, that $\mathfrak{k}$ is the complexification of its compact form $\mathfrak{k}_0$. Using the lemma and parabolic regularization we obtain

Proposition 3.5.2. *We can find* $\mathfrak{q} \in \mathfrak{P}(\mathfrak{v})$ *with* $\mathfrak{L}(\mathfrak{q}) = \mathfrak{q} \cap \bar{\mathfrak{q}}$.

Proof. We can take $\mathfrak{q} = (\mathfrak{w} \cap \overline{\mathfrak{w}}) + \mathfrak{n}(\mathfrak{w})$, for the parabolic regularization $\mathfrak{w}$ of $\mathfrak{v}$. $\square$

This shows that the set

$$\mathfrak{P}_0 = \{\mathfrak{q} \in \mathfrak{P}(\mathfrak{v}) \mid \mathfrak{q} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \oplus \mathfrak{n}(\mathfrak{q})\} \tag{3.55}$$

is nonempty. For $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ we will use $\mathfrak{L}(\mathfrak{q}) = \mathfrak{q} \cap \bar{\mathfrak{q}}$. The parabolic regularizazion produces a *small* $\mathfrak{w}$ and a corresponding smaller $(\mathfrak{w} \cap \overline{\mathfrak{w}}) \oplus \mathfrak{n}(\mathfrak{w})$ in $\mathfrak{P}_0(\mathfrak{v})$. As in [**8**], we are more interested in the *maximal* elements of $\mathfrak{P}(\mathfrak{v})$, because we have (cf. [**8**, Proposition 20])

Proposition 3.5.3. *If* $\mathfrak{q}$ *is any maximal element of* $\mathfrak{P}_0(\mathfrak{v})$*, then*

$$\mathfrak{q} = \mathrm{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})) \quad \textit{and} \quad \mathfrak{n}(\mathfrak{q}) = \sum\nolimits_h \mathrm{ad}^h(\mathfrak{L}(\mathfrak{q}))(\mathfrak{n}(\mathfrak{v})). \tag{3.56}$$

Proof. Let $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ and denote by $\mathfrak{z}$ the center of $\mathfrak{L}(\mathfrak{q})$. Being invariant under conjugation, it is the complexification of the Lie subalgebra $\mathfrak{z}_0$ of a maximal torus $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Set $\mathfrak{z}_{\mathbb{R}} = i\mathfrak{z}_0$. Following the construction of Konstant in [**32**], we consider the set $\mathcal{Z}$ consisting of the nonzero elements ν of the dual $\mathfrak{z}^*_{\mathbb{R}}$ for which

$$\mathfrak{k}_\nu = \{X \in \mathfrak{k} \mid [Z, X] = \nu(Z)X,\ \forall Z \in \mathfrak{z}_{\mathbb{R}}\} \neq \{0\}.$$

This set $\mathcal{Z}$ shares many properties of the root system of a semisimple Lie algebra. With the scalar product defined on $\mathfrak{z}_{\mathbb{R}}$ by the restriction of the trace form of a faithful linear representation of $\mathfrak{k}$ and the corrisponding dual scalar product on $\mathfrak{z}^*_{\mathbb{R}}$, we have

(*i*) $\nu \in \mathcal{Z} \Longrightarrow -\nu \in \mathcal{Z}$, and $\bar{\mathfrak{k}}_\nu = \mathfrak{k}_{-\nu}$,

(*ii*) $\nu_1, \nu_2, \nu_1 + \nu_2 \in \mathcal{Z} \Longrightarrow [\mathfrak{k}_{\nu_1}, \mathfrak{k}_{\nu_2}] = \mathfrak{k}_{\nu_1+\nu_2}$,

(*iii*) $\nu_1, \nu_2 \in \mathcal{Z}$ and $(\nu_1|\nu_2) > 0 \Longrightarrow \nu_1 - \nu_2 \in \mathcal{Z}$,

(*iv*) $\forall \nu \in \mathcal{Z}$, $\mathfrak{k}_\nu$ is an irreducible $\mathfrak{L}(\mathfrak{q})$-module,

(*v*) $\mathfrak{n}(\mathfrak{q}) = \sum_{\nu>0} \mathfrak{k}_\nu$, for some lexicographic order in $\mathcal{Z}$,

(*vi*) $\exists$ a basis $\{\mu_1, \ldots, \mu_\ell\} \subset \mathcal{Z}$ of positive simple roots of $\mathfrak{z}^*_{\mathbb{R}}$.

The Lie algebra $\mathrm{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q}))$ is contained in $\mathfrak{q}$ and is a direct sum

$$\mathrm{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})) = \mathfrak{L}(\mathfrak{q}) \oplus \sum\nolimits_{\nu \in \mathcal{E}} \mathfrak{k}_\nu,$$

for a subset $\mathcal{E}$ of $\mathcal{Z}^+ = \{\nu > 0\}$. Assume that there is a positive simple root μ_i which does not belong to $\mathcal{E}$. Since μ_i is simple, $\mathfrak{q}' = \mathfrak{q} \oplus \mathfrak{k}_{-\mu_i}$ is still a parabolic Lie subalgebra. Let us show that it is an element of $\mathfrak{P}_0(\mathfrak{v})$. We have

$$\mathfrak{q}' = \mathfrak{L}(\mathfrak{q}') \oplus \mathfrak{n}(\mathfrak{q}'), \quad \text{with} \quad \mathfrak{L}(\mathfrak{q}') = \mathfrak{L}(\mathfrak{q}) \oplus \mathfrak{k}_{\mu_i} \oplus \mathfrak{k}_{-\mu_i} \quad \text{and} \quad \mathfrak{n}(\mathfrak{q}') = \sum\nolimits_{\nu \in (\mathcal{Z}^+ \setminus \{\mu_i\})} \mathfrak{k}_\nu.$$

Note that $\mathfrak{L}(\mathfrak{q}') = \mathfrak{q}' \cap \bar{\mathfrak{q}}'$. An element $X \in \mathfrak{n}(\mathfrak{v})$ can be written in a unique way as a sum $X = \sum_{\nu \in \mathcal{E}} X_\nu$ with $X_\nu \in \mathfrak{k}_\nu$. Then $X \in \mathfrak{n}(\mathfrak{q}')$, because $\mathcal{E} \subset \mathcal{Z}^+ \setminus \{\mu_i\}$. This shows that $\mathfrak{n}(\mathfrak{v}) \subset \mathfrak{n}(\mathfrak{q}')$, i.e that $\mathfrak{q}' \in \mathfrak{P}_0(\mathfrak{v})$. Thus, if $\mathfrak{q}$ is maximal in $\mathfrak{P}_0(\mathfrak{v})$, then $\mathrm{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q}))$ contains all $\mathfrak{k}_{\mu_i}$ for $i = 1, \dots, \ell$ and thus is equal to $\mathfrak{q}$, because (*ii*) and the fact that every positive root is a sum o simple positive roots yield that $\mathrm{Lie}(\sum_{i=1}^{\ell} \mathfrak{k}_{\mu_i}) = \mathfrak{n}(\mathfrak{q})$. Finally, it follows from the discussion above that $\mathfrak{n}(\mathfrak{q})$ is the $\mathrm{ad}(\mathfrak{L}(\mathfrak{q}))$-module generated by $\mathfrak{n}(\mathfrak{v})$. □

Analogously, we obtain

Proposition 3.5.4. *If* $\mathfrak{q}$ *ia any maximal element of* $\mathfrak{P}(\mathfrak{v})$*, then*

$$\mathfrak{q} = \mathrm{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathfrak{L}(\mathfrak{q})), \tag{3.57}$$

for any reductive Levi factor $\mathfrak{L}(\mathfrak{q})$ *of* $\mathfrak{q}$*, and* $\mathfrak{n}(\mathfrak{q})$ *is the* $\mathrm{ad}(\mathfrak{L}(\mathfrak{q}))$*-module generated by* $\mathfrak{n}(\mathfrak{v})$. □

5.1. A decomposition Lemma for unipotent Lie groups. We recall that a unipotent Lie group is a connected and simply connected Lie group having a nilpotent Lie algebra. Let $\mathfrak{n}$ be a nilpotent real (or complex) Lie algebra and $\mathbf{N}$ the corresponding unipotent Lie group. We recall that $\exp : \mathfrak{n} \to \mathbf{N}$ is an algebraic diffeomorphism and that each Lie subalgebra $\mathfrak{e}$ of $\mathfrak{n}$ is the Lie algebra of an analytic closed subgroup $\mathbf{E} = \exp(\mathfrak{e})$ of $\mathbf{N}$. The simple version of the Mostow decomposition theorem in this case reads

Proposition 3.5.5. *Let* $\mathbf{N}$ *be a connected and simply connected nilpotent Lie group with Lie algebra* $\mathfrak{n}$ *and* $\mathbf{E}$ *a Lie subgroup of* $\mathbf{N}$ *with Lie algebra* $\mathfrak{e} \subset \mathfrak{n}$*. Then we can find a linear complement* $\mathfrak{l}$ *of* $\mathfrak{e}$ *in* $\mathfrak{n}$ *such that*

$$\mathfrak{l} \times \mathbf{E} \ni (X, x) \longrightarrow \exp(X) \cdot x \in \mathbf{N} \tag{3.58}$$

is a diffeomorphism onto.

Proof. We argue by recurrence on the sum of the dimension n of $\mathfrak{n}$ and of the codimension k of $\mathfrak{e}$ in $\mathfrak{n}$. The statement is indeed trivial when $n = 1$. If $k = 1$, then $\mathfrak{e}$ is an ideal in $\mathfrak{n}$. Indeed, by passing to the quotient, $\mathrm{ad}(\mathfrak{e})$ acts as a group of nilpotent maps, and hence trivially, on the one-dimensional vector space $\mathfrak{n}/\mathfrak{e}$. Any linear complement $\mathfrak{l}$ of $\mathfrak{e}$ in $\mathfrak{n}$, being one-dimensional, is a Lie subalgebra of $\mathfrak{n}$. By [**23**, Ch.VI,Lemma 5.2], the map

$$f : \mathfrak{l} \times \mathfrak{e} \ni (X, Y) \longrightarrow \exp(X) \cdot \exp(Y) \in \mathbf{N}$$

is regular. Since $\mathfrak{e}$ is an ideal, $f(\mathfrak{l} \times \mathfrak{e})$ is a Lie subgroup of $\mathbf{N}$. Since $f(\mathfrak{l} \times \mathfrak{e})$ contains an open neighborhood of $e_{\mathbf{N}}$, it follows that f is onto. Therefore f, being a connected covering of a simply connected manifold, is a diffeomorphism.

Assume now that $k > 1$ and that the statement has already been proved when the codimension of the subalgebra of the nilpotent Lie algebra is less than k or the nilpotent Lie algebra has dimension less than n. Since $\mathfrak{n}$ is nilpotent, its center $\mathfrak{c}$ has positive dimension. If $\mathfrak{e} \cap \mathfrak{c} \ni A \neq 0$, then $\mathbf{A} = \exp(\mathbb{R} \cdot A)$ is a one-dimensional normal subgroup of $\mathbf{N}$ and $\mathbf{E}$. Denote by $\mathfrak{l}'$ the projection into $\mathfrak{n}/(\mathbb{R} \cdot A)$ of a linear complement $\mathfrak{l}$ of $\mathfrak{e}$ in $\mathfrak{n}$. Since $\dim(\mathbf{N}/\mathbf{A}) = n - 1$, by the recursive assumption there is a diffeomorphism

$$f' : \mathfrak{l}' \times (\mathbf{E}/\mathbf{A}) \ni (X', x') \longrightarrow \exp(X') \cdot x' \in \mathbf{N}/\mathbf{A}.$$

Let us show that this implies that (3.58) is also a diffeomorphism. In fact, if $\zeta \in \mathbf{N}$, by the surjectivity of f' there is a pair $(X, y) \in \mathfrak{l} \times \mathbf{E}$ such that $\exp(X) \cdot y = \zeta \cdot a$, for some $a \in \mathbf{A}$. This shows that $\zeta = \exp(X) \cdot (y \cdot a^{-1})$ and therefore (3.58) is onto. If $\zeta = \exp(X_1) \cdot (x_1) = \exp(X_2) \cdot (x_2) \cdot a$, with $X_1, X_2 \in \mathfrak{l}$, $x_1, x_2 \in \mathbf{E}$ and $a \in \mathbf{A}$, then $X_1 = X_2 = X$ because the projection $\mathfrak{l} \to \mathfrak{l}'$ is a linear isomorphism. Moreover, the correspondence $\zeta \to X$ is C^∞-smooth, because f'^{-1} is smooth. Then $\zeta \to x = \exp(-X) \cdot \zeta \in \mathbf{E}$ is also smooth, and $\zeta \to (X, \exp(-X)\zeta)$ yields a smooth inverse of (3.58).

If $\mathfrak{c} \cap \mathfrak{e} = \{0\}$, then we can take a linear complement $\mathfrak{l}$ of $\mathfrak{e}$ in $\mathfrak{n}$ containing $\mathfrak{c}$. Arguing again by recurrence, we obtain a diffeomorphism

$$f' : (\mathfrak{l}/\mathfrak{c}) \times ((\mathbf{E} \cdot \mathbf{C})/\mathbf{C}) \ni (X', x') \longrightarrow \exp(X') \cdot x' \in \mathbf{N}/\mathbf{C}.$$

Then (3.58) is a diffeomorphism. Indeed, $(\mathbf{E} \cdot \mathbf{C})/\mathbf{C} \simeq \mathbf{E}$ and therefore for $\zeta \in \mathbf{N}$ there is a unique $x \in \mathbf{E}$, with $x = \phi(\zeta)$ for a smooth function $\phi : \mathbf{N} \to \mathbf{E}$, such that, for some $Z \in \mathfrak{c}$ and $Y \in \mathfrak{l}$,

$$\zeta \cdot \exp(Z) = \exp(Y) \cdot x \Rightarrow \zeta = \exp(Y + Z) \cdot x.$$

The exponential is a diffeomorphism of $\mathfrak{n}$ onto $\mathbf{N}$. If we denote by $\log : \mathbf{N} \to \mathfrak{n}$ its inverse, we obtain $X = Y + Z = \log(x^{-1}\zeta) \in \mathfrak{l}$ and

$$\mathbf{N} \ni \zeta \to \left(\log([\phi(\zeta)]^{-1}\zeta), \phi(\zeta)\right) \in \mathfrak{l} \times \mathbf{E}$$

is a smooth inverse of (3.58).

This map lifts to the diffeomorphism , because by the recursive assumption, if $\mathfrak{f}$ is a linear complement of $\mathbb{R} \cdot A$ in $\mathfrak{e}$, then

$$\mathfrak{f} \times \mathbb{R} \ni (Y, \lambda) \to \exp(Y) \cdot \exp(\lambda A) = \exp(Y + \lambda A) \in \mathbf{E}$$

is a diffeomorphism. This completes the proof. □

For the applications, we need some precision on the possible choices of $\mathfrak{l}$.

PROPOSITION 3.5.6. *We make the assumptions of Proposition 3.5.5and suppose, moreover, that there is a semisimple Lie group* $\mathbf{A}$ *of automorphisms of* $\mathfrak{n}$ *leaving* $\mathfrak{e}$ *invariant. Then we can choose an* $\mathbf{A}$*-invariant linear complement* $\mathfrak{l}$ *of* $\mathfrak{e}$ *in* $\mathfrak{n}$ *for which* (3.58) *holds true.*

PROOF. We can follow verbatim the arguments in the proof of Proposition 3.5.5, by adding at each step the requirement of $\mathfrak{l}$ being $\mathbf{A}$-invariant. This is possible because of the semisemplicity assumption. In case $\mathfrak{c} \cap \mathfrak{e} = \{0\}$, we can take an $\mathfrak{l}$ containing $\mathfrak{c}$, because $\mathfrak{c}$ is $\mathbf{A}$-invariant. □

5.2. Characterization of the typical fiber. We keep the notation of §1. Fix a parabolic subgroup $\mathbf{Q}$ of $\mathbf{K}$, with Lie algebra $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$. It is well known that

$$\mathbf{K} = \mathbf{K}_0 \cdot \mathbf{Q}. \tag{3.59}$$

Let $\mathbf{L}(\mathbf{Q})$ be the reductive factor of $\mathbf{Q}$ with Lie algebra $\mathfrak{L}(\mathfrak{q}) = \mathfrak{q} \cap \bar{\mathfrak{q}}$, which is the complexification of its maximal compact subgroup $\mathbf{L}_0(\mathbf{Q}) = \mathbf{L}(\mathbf{Q}) \cap \mathbf{K}_0 = \mathbf{Q} \cap \mathbf{K}_0$. The nilradical $\mathfrak{n}(\mathfrak{q})$ of $\mathfrak{q}$ is the Lie algebra of its unipotent radical $\mathbf{Q}_n = \exp(\mathfrak{n}(\mathfrak{q}))$. By using the Cartan decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{k}$ (here $\mathfrak{p}_0 = i\mathfrak{k}_0$ because $\mathfrak{k}$ is the complexification of $\mathfrak{k}_0$), and the Levi-Chevalley decomposition of $\mathbf{Q}$, we obtain the direct product decomposition

$$\mathbf{Q} = \mathbf{L}(\mathbf{Q}) \exp(\mathfrak{n}(\mathfrak{q})) = \mathbf{L}_0(\mathbf{Q}) \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{q}) \cdot \exp(\mathfrak{n}(\mathfrak{q})). \tag{3.60}$$

By applying the decomposition (3.51) with $\mathbf{L}(\mathbf{Q})$ replacing $\mathbf{K}$, we obtain the direct product decomposition

$$\mathbf{L}(\mathbf{Q}) = \mathbf{L}_0(\mathbf{Q}) \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{v}), \tag{3.61}$$

with $\mathfrak{f}_0$ equal to the orthogonal complement of $(\mathfrak{p}_0 \cap \mathfrak{v})$ in $(\mathfrak{p}_0 \cap \mathfrak{q})$.

Lemma 3.5.7. *We have a direct product decomposition*

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \mathbf{Q}_n \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{v}). \tag{3.62}$$

Proof. By (3.60) and (3.61) we obtain the direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{p}_0 \cap \mathfrak{v}) \cdot \mathbf{Q}_n. \tag{3.63}$$

We have

$$\exp(\mathfrak{v} \cap \mathfrak{p}_0) \cdot \mathbf{Q}_n = \mathbf{Q}_n \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0)$$

because $\mathbf{Q}_n$ is a normal subgroup. Indeed, for $X \in \mathfrak{v} \cap \mathfrak{p}_0$ and $z \in \mathbf{Q}_n$, we have

$$z \cdot \exp(X) = \exp(X) \cdot \underbrace{(\exp(-X) \cdot z \cdot \exp(X)}_{\in \mathbf{Q}_n}$$

and

$$\exp(X) \cdot z = \underbrace{(\exp(X) \cdot z \cdot \exp(-X)}_{\in \mathbf{Q}_n} \cdot \exp(X).$$

The fact that the products are direct follows from the direct product decompositions $\mathbf{Q} = \mathbf{L}(\mathbf{Q}) \cdot \mathbf{Q}_n = \mathbf{Q}_n \cdot \mathbf{L}(\mathbf{Q})$. This proves the lemma. □

Lemma 3.5.8. *We can find an* $\mathrm{ad}(\mathbf{I}_0)$*-invariant linear complement* $\mathfrak{l}$ *of* $\mathfrak{v}_n$ *in* $\mathfrak{q}_n$ *to obtain a direct product decomposition*

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{l}) \cdot \exp(\mathfrak{v}_n) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0). \tag{3.64}$$

Proof. The statement is obtained by applying Proposition 3.5.6 to the case where $\mathfrak{n} = \mathfrak{n}(\mathfrak{q})$, $\mathfrak{e} = \mathfrak{v}_n$ and $\mathbf{A} = \mathrm{Ad}(\mathbf{I}_0)$. □

Putting together the results obtained so far, we have:

Theorem 3.5.9. *If* $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$*, then the typical fiber of a Mostow fibration*

$$M_- \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} F_0 \longrightarrow M_0 \simeq \mathbf{K}_0/\mathbf{I}_0 \tag{3.65}$$

can be taken equal to the product

$$F_0 = \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{l}) \tag{3.66}$$

of

$$\mathfrak{f}_0 = \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{p}_0 \cap (\mathfrak{v} + \bar{\mathfrak{v}})^{\perp} \tag{3.67}$$

and of a suitable $\mathrm{Ad}(\mathbf{I}_0)$*-invariant linear complement* $\mathfrak{l}$ *of* $\mathfrak{v}_n = \mathfrak{n}(\mathfrak{v})$ *in* $\mathfrak{q}_n = \mathfrak{n}(\mathfrak{q})$. □

Example 3.5.10. Let $\mathfrak{v}$ be the one dimensional complex Lie subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ generated by a nilpotent matrix of rank $(n-1)$, that we can take equal to

$$Z_0 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Then we can take $\mathfrak{q}$ equal to the Borel subalgebra $\mathfrak{t}_n^+(\mathbb{C})$ of upper triangular complex matrices, $\mathfrak{k}_0$ equal to the algebra $\mathfrak{sd}_n(\mathbb{R})$ of real diagonal traceless matrices, and

$$\mathfrak{l} = \left\{ Z = (z_{i,j}) \in \mathfrak{n}_n^+(\mathbb{C}) \;\middle|\; \sum\nolimits_{1 \leq i < n} z_{i,i+1} = 0 \right\},$$

where $\mathfrak{n}_n^+(\mathbb{C})$ are the upper triangular nilpotent complex $n \times n$ matrices.

Let $\mathcal{P}_0(\mathfrak{k}) = \exp(\mathfrak{p}_0)$ be the Euclidean factor of the Cartan decomposition of $\mathbf{K}$. By employing the linear representation $\mathbf{K} \hookrightarrow \mathbf{SL}_n(\mathbb{C})$, it can be identified to a totally geodesic submanifold of $\mathcal{P}_0(n)$.

Lemma 3.5.11. *Assume that* $\mathbf{N}$ *is a unipotent subgoup of* $\mathbf{K}$*. Then, for every* $p \in \mathcal{P}_0(n)$*, the map*

$$\mathbf{N} \ni z \longrightarrow z^* h z \in N_p = \{z^* p z \mid z \in \mathbf{N}\} \tag{3.68}$$

is a diffeomorphism.

Proof. In fact the stabilizer $\mathbf{Stab}(p)$ of p for the right action

$$\mathbf{K} \times \mathcal{P}_0(\mathfrak{k}) \ni (z, x) \to z^* \cdot x \cdot z \in \mathcal{P}_0(\mathfrak{k})$$

of $\mathbf{K}$ on $\mathcal{P}_0(\mathfrak{k})$ is a compact group and hence $\mathbf{Stab}(p) \cap \mathbf{N} = \{e_{\mathbf{K}}\}$. Thus (3.68) is a diffeomorphism with the image, being the restriction to $\mathbf{N} \simeq \mathbf{N}/\{e_{\mathbf{K}}\}$ of the diffeomorphism $\mathbf{K}/\mathbf{Stab}(p) \to \mathcal{P}_0(\mathfrak{k})$. □

Corollary 3.5.12. *Fix* $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$ *and let* $\mathfrak{k}_0$ *and* $\mathfrak{l}$ *be the corresponding subspaces of* $\mathfrak{k}$ *of Theorem 3.5.9. Then the elements* $X \in \mathfrak{k}_0$ *and* $Z \in \mathfrak{l}$ *of the decomposition*

$$\zeta = u \cdot \exp(X) \cdot \exp(Z) \cdot v, \quad \textit{with} \quad u \in \mathbf{K}_0,\ v \in \exp(\mathfrak{v}_n) \cdot \exp(v \cap \mathfrak{p}_0)$$

are obtained in the following way:

- $[0,1] \ni t \to \exp(2tX)$ *is the geodesic in* $\mathcal{P}_0(\mathfrak{k})$ *joining* $e_{\mathbf{K}}$ *to the unique point* p_0 *of* $S_{\zeta^* \cdot \zeta} = \{z^* \cdot \zeta^* \cdot \zeta \cdot z \mid z \in \mathbf{V} \cdot \mathbf{Q}_n\}$ *at minimal distance from* $e_{\mathbf{K}}$*;*
- Z *is the unique element of* $\mathfrak{l}$ *such that* $\exp(Z^*) \cdot p_0 \cdot \exp(Z)$ *belongs to* $N_{p_0} = \{z^* \cdot p_0 \cdot z \mid z \in \mathbf{V}\}$.

Proof. Indeed the Mostow fibration of $M'_- = \mathbf{K}/(\mathbf{V} \cdot \mathbf{Q}_n)$ can be taken to have a *hermitian* typical fiber $\exp(\mathfrak{k}_0)$ and correspondingly we obtain a unique decomposition

$$\zeta = u \cdot \exp(X) \cdot \xi \cdot \exp(Y) \quad \text{with} \quad \xi \in \mathbf{Q}_n \text{ and } Y \in \mathfrak{v} \cap \mathfrak{p}_0.$$

Next we consider $\xi^* \cdot \exp(2X) \cdot \xi = \xi^* \cdot p_0 \cdot \xi$. By Lemma 3.5.11 and the choice of $\mathfrak{l}$ we know that the element $\xi^* \cdot p_0 \cdot \xi$ of $\{z^* \cdot p_0 \cdot z \mid z \in \mathbf{Q}_n\}$ uniquely decomposes as a product $w^* \cdot \exp(Z^*) \cdot p_0 \cdot \exp(Z) \cdot w$ with $w \in \mathbf{V}_n$ and $Z \in \mathfrak{l}$. This yields the thesis. □

6. Horocyclic Nilradical

In Theorem 3.5.9 we can take $\mathfrak{l} = \{0\}$ when $\mathfrak{v}_n$ is the nilradical of a parabolic $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$.

Theorem 3.6.1. *If $\mathfrak{v}_n$ is the nilradical of a parabolic $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$, then*

$$\mathbf{K}_0 \times \mathfrak{f}_0 \ni (x, X) \longrightarrow [x \cdot \exp(X)] \in M_-, \tag{3.69}$$

where

$$\mathfrak{f}_0 = \mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{p}_0 \cap (\mathfrak{v} + \bar{\mathfrak{v}})^{\perp}, \tag{3.70}$$

induces a diffeomorphism of M_- with

$$\mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0 = \mathbf{K}_0 \times \mathfrak{f}_0 / \sim, \quad (\textit{were} \;\; (x, X) \sim (x \cdot u, \mathrm{Ad}(u^{-1})(X)), \quad \forall u \in \mathbf{I}_0),$$

and

$$M_- \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0 \ni (x, X) \longrightarrow [x] \in M_0 \simeq \mathbf{K}_0/\mathbf{I}_0, \tag{3.71}$$

is a Mostow fibration of M_- over M_0. □

Following the notation in [**44**, p.17], we give the following definition

Definition 3.6.2. Let $\mathbf{V}$ be a closed subgroup of $\mathbf{K}$, which admits a Levi-Chevalley decomposition. If its unipotent radical $\mathbf{V}_n$ is the unipotent radical of a parabolic subgroup, then we say that $\mathbf{V}$ has *horocyclic nilradical*, or simply is *HNR*.

Let us go back to the notation of §1. We observe that by substituting to $\mathbf{V}$ the subgroup $\mathbf{V} \cdot \mathbf{Q}_n$, where $\mathbf{Q}_n$ is the unipotent radical of a parabolic $\mathbf{Q}$ with $\mathfrak{q} \in \mathfrak{P}_0(\mathfrak{v})$, we simply put on the same M_0 a *stronger CR* structure. In fact, the *CR*-algebra at the base point was changed from $(\mathfrak{k}_0, \mathfrak{v})$ to $(\mathfrak{k}_0, \mathfrak{v} + \mathfrak{q}_n)$. The spaces of the realization changed from $M_- = \mathbf{K}/\mathbf{V}$ to an $M'_- = \mathbf{K}/(\mathbf{V} \cdot \mathbf{Q}_n)$, which is the basis of a complex fiber bundle $M_- \to M'_-$, with Stein fibers (cf. [**8**, Thm.4.7]).

CHAPTER 4

Dolbeault and CR Cohomologies

In this chapter we investigate the relationship between the Dolbeault cohomology of M_- and the real-analytic *CR*-cohomology of M_0. The last one is the inductive limit of the corresponding Dolbeault cohomology groups of the tubular neighborhoods of M_0. By using the results of Chapter 3, we are able to construct an exhaustion fuction for M_-, having a complex Hessian whose signature in linked to the pseudoconvexity/pseudoconcavity of M_0. In fact we find conditions for obtaining vanishing thorems fot the *CR* cohomology of M_0 and the Dolbeault cohomololy of the $\mathbf{K}$-orbit M_-, similar to what J.A.Wolf did for the relationship of M_+ to M_0 in the case where M_+ is an open $\mathbf{G}_0$-orbit in [**42**].

We shall actually discuss the slightly more general situation of a homogeneous reductive compact *CR*-manifold M_0.

1. An Exhaustion Function for M_-

In [**22**] H. Grauert showed that a real-analytic manifolds admits a fundamental systems of Stein neighborhoods in any of its complexifications. In particular, the complexification $\mathbf{K}$ of a compact Lie group $\mathbf{K}_0$ is Stein, and the isomorphism provided by the Cartan decomposition

$$\mathbf{K}_0 \times \mathfrak{k}_0 \ni (x, X) \longrightarrow g \cdot \exp(iX) \in \mathbf{K}$$

also provides an exhausting function

$$\mathbf{K} \ni x \cdot \exp(iX) \longrightarrow \|X\|^2 = -\kappa(X, X) \in \mathbb{R},$$

which is zero on $\mathbf{K}_0$, positive on $\mathbf{K} \setminus \mathbf{K}_0$ and strictly pseudo-convex everywhere. Here and in the following we shall denote by κ both the negative definite invariant form of a faithfull unitary representation of $\mathfrak{k}_0$ and its $\mathbb{C}$-bilinear extension to $\mathfrak{k}$. When $\mathfrak{k}_0$ is semisimple, we may take κ equal to its Killing form.

In a similar way, we will obtain from the Mostow decomposition an exhausting function on a complex $\mathbf{K}$-homogeneous space $M_- = \mathbf{K}/\mathbf{V}$. We recall (see §1 of Chapter 3) that the basis M_0 is a reductive $\mathbf{K}_0$-homogeneous compact *CR* manifold, whose *CR* structure is defined by the *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$, for $\mathfrak{v} = \text{Lie}(\mathbf{V})$, with $\mathfrak{v} \cap \overline{\mathfrak{v}}$ being a reductive complement of its nilradical $\mathfrak{v}_n$.

Definition 4.1.1. We say that $M_- \simeq \mathbf{K}/\mathbf{V}$ is *HNR* if its isotropy group $\mathbf{V}$ is *HNR* in the sense of Defintion 3.6.2.

We already noticed that this condition is natural if we consider on M_0 *maximal CR* structures.

If M_- is *HNR*, then by Corollary 3.5.12 we have a direct product decomposition

$$\mathbf{K} = \mathbf{K}_0 \cdot \exp(\mathfrak{k}_0) \cdot \exp(\mathfrak{v}_n) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0) \tag{4.1}$$

with $\mathfrak{p}_0 = i\cdot\mathfrak{k}_0$, $\mathfrak{f}_0 = (\mathfrak{v}+\overline{\mathfrak{v}})^{\perp} \cap \mathfrak{p}_0$ and the $\exp(\mathfrak{f}_0)$-factor is characterized by

$$(4.2)\qquad \begin{cases} \text{if } \zeta = u\cdot\exp(X)\cdot v, \quad \text{with} \quad u \in \mathbf{K}_0,\, X \in \mathfrak{f}_0 \text{ and } v \in \mathbf{V}, \text{ then} \\ \|X\| = \frac{1}{2}\mathrm{dist}(e_{\mathbf{K}}, N_{\zeta^*\zeta}), \quad \text{for} \quad N_{\zeta^*\zeta} = \{v^*\cdot\zeta^*\cdot\zeta\cdot v \mid v \in \mathbf{V}\}. \end{cases}$$

Formula (4.2) is a consequence of the fact that the adjoint of $\mathbf{I}_0 = \mathbf{V}\cap\mathbf{K}_0$ acts as a group of isometries on $\mathfrak{f}_0$.

Thus by passing to the quotient, the map

$$\mathbf{K}_0 \times \mathfrak{f}_0 \ni (x, X) \longrightarrow \|X\|^2 = \kappa(X, X) \in \mathbb{R}.$$

defines a smooth exhausting function (square brackets are used for the equivalence class)

$$(4.3)\qquad \phi : M_- \simeq \mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0 \ni [x, X] \longrightarrow \|X\|^2 \in \mathbb{R}.$$

We have:

Lemma 4.1.2. *If M_- is* HNR*, then the map ϕ defined by* (4.3) *has the properties:*

(1) $\phi \in C^\infty(M_-, \mathbb{R})$ *and* $\phi \geq 0$ *on* M_-*;*
(2) $\phi^{-1}(0) = M_0$ *and* $d\phi \neq 0$ *if* $\phi > 0$ *;*
(3) ϕ *is invariant for the left action of* $\mathbf{K}_0$ *on* M_- *:*

$$\phi(k_0 p) = \phi(p), \quad \forall p \in M_-, \quad \forall k_0 \in \mathbf{K}_0. \qquad \square$$

In the sequel we shall assume that M_- is *HNR* and consider the exhaustion function ϕ defined in (4.3).

Notation 4.1.3. The *level and sublevel sets* of ϕ are the smooth hypersurfaces and domains denoted by

$$(4.4)\qquad U_c = \{p \in M_- \mid \phi(p) = c\} \Subset M_- \quad \text{and} \quad \Omega_c = \{p \in M_- \mid \phi(p) < c\}.$$

2. $\mathbf{K}_0$-Orbits in M_-

We keep the notation of §3.1, §4.1. For an *HNR* manifold M_- we use the Mostow fibration (3.9) and the exhaustion function $\phi \in C^\infty(M_-, \mathbb{R})$ of (4.3).

Since all points of M_- have representatives of the form $x\cdot\exp(X)$ with $x \in \mathbf{K}_0$ and $X \in \mathfrak{f}_0$, then every $\mathbf{K}_0$-orbit intersects the fiber F_0 over the base point p_0 in a point p_X which has a representative $\exp(X)$, for some $X \in \mathfrak{f}_0$. An $x \in \mathbf{K}_0$ stabilizes p_X if and only if $x\cdot\exp(X)$ is still a representative of p_X, and this, by the equivalence relation defining $\mathbf{K}_0 \times_{\mathbf{I}_0} \mathfrak{f}_0$, means that $x \in \mathbf{I}_0$ and $\mathrm{Ad}(x^{-1})(X) = X$. The $\mathbf{K}_0$-orbit M_X through p_X is the homogeneous space $\mathbf{K}_0/\mathbf{I}_X$, where $\mathbf{I}_X$ is the stabilizer in $\mathbf{K}_0$ of p_X, given by

$$\begin{aligned} \mathbf{I}_X &= \{x \in \mathbf{K}_0 \mid x\exp(X) \in \exp(X)\mathbf{V}\} \\ &= \{x \in \mathbf{K}_0 \mid \exp(-X)\,x\,\exp(X) \in \mathbf{V}\} \\ &= \mathbf{K}_0 \cap \mathrm{ad}(\exp(X))(\mathbf{V}). \end{aligned}$$

The isotropy $\mathbf{I}_X$ is a closed Lie subgroup of $\mathbf{K}_0$ with Lie algebra

$$\mathfrak{i}_X = \{Y \in \mathfrak{k}_0 \mid [Y, X] \in \mathfrak{v}\} = \{Y \in \mathfrak{k}_0 \mid [Y, X] \in \mathfrak{i}_0\},$$

because $[Y, X] \in \mathfrak{k}_0$ and $\mathfrak{v}\cap\mathfrak{k}_0 = \mathfrak{i}_0$.

Notation 4.2.1. For $X \in \mathfrak{f}_0$ we denote by

$$(4.5)\qquad M_X \simeq \mathbf{K}_0\cdot p_X \simeq \mathbf{K}_0/\mathbf{I}_X$$

the $\mathbf{K}_0$-orbit through $\exp(X)$, where we recall that $\mathbf{I}_X = \{x \in \mathbf{K}_0 \mid \mathrm{Ad}(x)(X) = X\}$.

Remark 4.2.2. We note that the $\mathbf{K}_0$-orbits M_X in M_- may not be diffeomorphic to M_0. Indeed, as M_0 is a *minimal* $\mathbf{K}_0$-orbit in M_-, an M_X is diffeomorphic if and only if is CR-diffeomorphic to M_0, and this happens if and only if $[\mathrm{ad}(\exp(-X))(\mathbf{K}_0)] \cap \mathbf{V}$ is a maximal compact subgroup of $\mathbf{V}$.

For $X \in \mathfrak{k}_0$, the left translation

$$M_- \ni p \longrightarrow \exp(X) \cdot p \in M_-$$

is a biholomorphism of M_- which transforms M_0 onto a CR-diffeomorphic

$$\tilde{M}_X = \exp(X) \cdot M_0. \tag{4.6}$$

Lemma 4.2.3. *For $X \in \mathfrak{k}_0$, we have*

$$\tilde{M}_X \subset \{\phi \leq \|X\|^2\} = \overline{\Omega}_{\|X\|^2}. \tag{4.7}$$

Proof. Let $\pi : \mathbf{K} \to \mathbf{K}/\mathbf{V} \simeq M_-$ be the canonical projection. Any point of M_0 is $\pi(u)$ for some $u \in \mathbf{K}_0$. Then $p = \exp(X) \cdot \pi(u) = \pi(\exp(X) \cdot u)$. Set $\zeta = \exp(X) \cdot u$. We know that $\phi(p)$ is the square of the half of the distance in $\mathfrak{p}_0$ of the base point $e_{\mathbf{K}}$ from $N_{\zeta^*\zeta} = \{v^* \cdot \zeta^* \cdot \zeta \cdot v \mid v \in \mathbf{V}\}$. Since the point $\zeta^* \cdot \zeta$ belongs to $N_{\zeta^* \cdot \zeta}$ and has distance $2\|X\|$ from $e_{\mathbf{K}}$, (in fact $t \to u^* \cdot \exp(2tX) \cdot u$ is a geodesic joining $e_{\mathbf{K}}$ to $\zeta^* \cdot \zeta$), it follows that $\phi(p) \leq \|X\|^2$. □

We summarize:

Proposition 4.2.4. *Let $c > 0$. Then*

$$U_c = \bigcup_{\substack{X\in\mathfrak{k}_0,\\ \|X\|^2=c}} M_X \ \ \textit{(disjoint union)}, \quad \tilde{M}_X \subset \{\phi \leq \|X\|^2\}, \ \ \forall X \in \mathfrak{k}_0. \tag{4.8}$$ □

In particuar, through each point of the level set U_c, for $c > 0$, we can draw a translate of M_0, which will be CR-diffeomorphic to M_0 and tangent to U_c *from inside*. This means that the boundary U_c of Ω_c is at each point *less convex* than M_0, in a sense that will be made more precise in the following.

3. Levi Forms

Since $\mathfrak{k}$ is the complexification of $\mathfrak{k}_0$ we continue using the notations $\mathfrak{p}_0 = i\mathfrak{k}_0$ and $\tau(Z) = -Z^*$ for the conjugation (1.5).

First we recall how the *vector-valued Levi form* on M_0 can be computed by exploiting its $\mathbf{K}_0$-homogeneity. The quotient $T_{p_0}M_0/H_{p_0}M_0$ at the base point p_0 may be identified with the orthogonal complement $\mathfrak{m}_0$ of $([\mathfrak{v} \oplus \mathfrak{v}^*] \cap \mathfrak{p}_0(n))$ in $\mathfrak{k}_0$. Denote by ϖ the orthogonal projection of $\mathfrak{k}_0$ onto $\mathfrak{m}_0$. Then, via the identification $\mathfrak{v}_n \simeq T^{0,1}_{p_0}M_0$, the vector-valued Levi form at p_0 is defined by

$$L : \mathfrak{v}_n \ni Z \longrightarrow -\varpi(i[Z, Z^*]) \in \mathfrak{m}_0. \tag{4.9}$$

Since $[Z, Z^*] \in \mathfrak{p}_0 = i\mathfrak{k}_0$, the "$i$" factor is needed to get an argument in $\mathfrak{k}_0$.

For the *real* bundle HM_0, we have the identification $H_{p_0}M_0 \simeq (\mathfrak{v}_n \oplus \mathfrak{v}_n^*) \cap \mathfrak{k}_0$.

For $X \in (\mathfrak{v}_n \oplus \mathfrak{v}_n^*) \cap \mathfrak{k}_0$ there is a unique $Y = J_0X \in (\mathfrak{v}_n \oplus \mathfrak{v}_n^*) \cap \mathfrak{k}_0$ such that $X + iJ_0X \in \mathfrak{v}_n$. This defines the partial complex structure

$$J_0 : HM_0 \to HM_0$$

of M_0. Denote by the same symbol J_0 its extension to a C-linear automorphism of $\mathfrak{v}_n \oplus \mathfrak{v}_n$. Then $J_0^2 = -\mathrm{I}$ and $\mathfrak{v}_n$, $\mathfrak{v}_n^*$ are the eigenspaces of the eigenvalues $-i$ and i, respectively. We can rewrite the vector valued Levi form at p_0 as a real quadratic form on $H_{p_0}M_0$. Setting $Z = X + iJ_0X$ in (4.9) we obtain

$$L(X + iJX) = -\varpi(i\,[X + iJ_0X, X - iJ_0X]) = -2\varpi([X, J_0X]), \quad \forall X \in (\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n) \cap \mathfrak{k}_0.$$

The fact that the invariant symmetric bilinear form κ vanishes on the nilpotent Lie subalgebra $\mathfrak{v}_n$ yields, for all $X \in (\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n) \cap \mathfrak{k}_0$,

$$0 = \kappa(X + iJ_0X, X + iJ_0X) = \kappa(X, X) - \kappa(J_0X, J_0X) + 2i\kappa(X, J_0X)$$
$$\Downarrow$$
$$\kappa(X, X) = \kappa(J_0X, J_0X), \quad \kappa(X, J_0X) = 0,$$

showing, as expected, that J_0 is an isometry, product of $\pi/2$-rotations on mutually orthogonal planes of $H_{p_0}M_0$.

To describe the scalar Levi forms of M_0, we use its natural Riemannian metric associated to $(-\kappa)$, which induces the Riesz isomorphism $T_{p_0}M_0 \simeq T^*_{p_0}M_0$ and allows us to represent the fiber of the characteristic bundle at the base point as a subspace of $T_{p_0}M_0$, given by

$$\mathrm{H}^0 M_0 \simeq (\mathfrak{v} \oplus \mathfrak{v}^*)^{\perp} \cap \mathfrak{k}_0 = \mathfrak{m}_0.$$

We associate to $X \in \mathfrak{m}_0$ the co-vector $\alpha \in \mathrm{H}^0 M_0$ defined by

$$\alpha_X(\,\cdot\,) = -\kappa(X, \,\cdot\,).$$

Then the *scalar Levi form* on M_0 is given at the base point by

$$\mathrm{L}_{\alpha_X}(Z) = -\alpha_X(i[Z, Z^*]) = \kappa(X, i\,[Z, Z^*]) \tag{4.10}$$

Assume that M_- is *HNR* and use the notation of §1,2. In this case the typical fiber $\mathfrak{f}_0$ is $i\mathfrak{m}_0$ and the scalar Levi forms can be parametrized by the elements $X \in \mathfrak{f}_0$, by

$$\mathrm{L}_{iX}(Z, -Z^*) = -\kappa(X, [Z, Z^*]), \quad Z \in \mathfrak{v}_n.$$

Remark 4.3.1. When M_0 is the intersection of two Matsuki-dual orbits M_+ and M_-, both M_0 and M_+ are homogeneous *CR* manifolds, of the groups $\mathbf{K}_0$ and $\mathbf{G}_0$, respectively. Then the Levi form L_α, defined in (4.10), is the restriction of the corresponding scalar Levi form on M_+. Indeed, by (3) of Proposition 2.3.1, at points p of M_0 we have $H^0_pM_0 = H^0_pM_+$ and $T^{0,1}_pM_0 \subset T^{0,1}_pM_+$.

To study the Dolbeault cohomology of M_-, we will use Andreotti-Grauert theory. To this aim we need to investigate the Levi convexity/concavity of the level sets (4.8) of the exhausting function ϕ.

Let $\pi : \mathbf{K} \to \mathbf{K}/\mathbf{V} \simeq M_-$ be the canonical projection. It will be convenient to exploit the fact that π is a holomorphic submersion to lift some computation on M_- to $\mathbf{K}$. We use, for the description of the complex structure of $\mathbf{K}$, results from [**31**, Ch.III].

We can associate to $\mathfrak{k}$ two algebras of complex valued holomorphic and anti-holomorphic vector fields. To each $Z \in \mathfrak{k}$ we let correspond the holomorphic and anti-holomorphic vector fields $Z^\sharp$ and $\bar{Z}^\sharp$ on $\mathbf{K}$, defined by

$$\begin{cases} Z^\sharp f(z) = \frac{\partial}{\partial \tau}\Big|_{\tau=0} f(z \cdot \exp(\tau \cdot Z)), \\ \bar{Z}^\sharp f(z) = \frac{\partial}{\partial \bar{\tau}}\Big|_{\tau=0} f(z \cdot \exp(\tau \cdot Z)), \end{cases} \qquad \forall f \in C^\infty(\mathbf{K}, \mathbb{C}),\ \forall z \in \mathbf{K}, \tag{4.11}$$

where τ is a complex variable and, $\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial(\mathrm{Re}\tau)} - \frac{\partial}{\partial(\mathrm{Im}\tau)}\right)$, $\frac{\partial}{\partial\bar{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial(\mathrm{Re}\tau)} + \frac{\partial}{\partial(\mathrm{Im}\tau)}\right)$. Then $\mathfrak{k}^{1,0} = \{Z^\sharp \mid Z \in \mathfrak{k}\}$ and $\mathfrak{k}^{0,1} = \{\bar{Z}^\sharp \mid Z \in \mathfrak{k}\}$ are the algebras of left-invariant holomorphic and anti-holomorphic vector fields, respectively. Note that $(Z^\sharp + \bar{Z}^\sharp)$ is the left-invariant *real* vector field associated to $Z \in \mathfrak{k}$.

This construction allows a simpler representation of the complex structure of $\mathbf{K}$. In fact

$$T^{1,0}_z\mathbf{K} = \{Z^\sharp_z \mid Z \in \mathfrak{k}\} \quad \text{and} \quad T^{0,1}_z\mathbf{K} = \{\bar{Z}^\sharp \mid Z \in \mathfrak{k}\}. \tag{4.12}$$

Since, for $Z \in \mathfrak{k}$ and $f \in C^\infty(M_-, \mathbb{C})$ we have

$$\pi_*(Z^\sharp_z)f = \left(\frac{\partial}{\partial\tau}\right)_{\tau=0} f(\pi(z \cdot \exp(\tau \cdot Z))),$$

the push-forward $\pi_*(Z^\sharp_z)$ vanishes at $\pi(z)$ if and only if $\mathrm{Ad}(z)(Z) \in \mathfrak{v}$. Analogously, $\pi_*(\bar{Z}^\sharp_z) = 0$ vanishes at $\pi(z)$ if and only if $Z \in \mathfrak{v}$. Therefore we obtain

$$\begin{cases} T^{1,0}_{zp_0}M_- = \{\pi_*(Z^\sharp_z) \mid Z \in \mathrm{Ad}(z)(\bar{\mathfrak{v}}_n \oplus \mathfrak{m})\}, \\ T^{0,1}_{zp_0}M_- = \{\pi_*(\bar{Z}^\sharp_z) \mid Z \in \mathrm{Ad}(z)(\bar{\mathfrak{v}}_n \oplus \mathfrak{m})\}. \end{cases} \tag{4.13}$$

We recall that the complex Hessian at $z_0 \in \mathbf{K}$ of a smooth real valued function $\psi \in C^\infty(\mathbf{K}, \mathbb{R})$, computed on a complex vector field θ of type $(1,0)$ which is tangential to the level set $\{\psi(z) = \psi(z_0)\}$, is

$$\begin{aligned} i\partial\bar{\partial}\psi_{z_0}(\theta, \bar{\theta}) &= id\bar{\partial}\psi(\theta, \bar{\theta})(z_0) \\ &= i\{\theta\bar{\partial}\psi(\bar{\theta}) - \bar{\theta}\bar{\partial}\psi(\theta) - \bar{\partial}\psi([\theta, \bar{\theta}])\}(z_0) \\ &= -i\bar{\partial}\psi([\theta, \bar{\theta}])(z_0) = i\partial\psi([\theta, \bar{\theta}])(z_0) \\ &= -\tfrac{1}{2}\, d^c\psi([(\mathrm{Re}\theta), (\mathrm{Im}\theta)])(z_0) \\ &= \tfrac{1}{2} d\psi(J[(\mathrm{Re}\theta), (\mathrm{Im}\theta)])(z_0), \end{aligned}$$

where the third and fourth terms are equal since $\bar{\partial}\psi$ vanishes identically on $(1,0)$-vector fields, $\partial\psi(\bar{\theta}) = d\psi(\bar{\theta})$ because $\bar{\partial}\psi$ and $d\psi$ take the same values on $(0,1)$-vector fields and then $d\psi(\bar{\theta}) = 0$ on $\{\psi(z) = \psi(z_0)\}$ because θ is tangential to the level surface of ψ; the fourth and fifth terms are equal because $d = \partial + \bar{\partial}$ and $d\psi(z_0)$ vanishes on the vector field $[\theta, \bar{\theta}]$ which is tangential to $\{\psi(z) = \psi(z_0)\}$; the last equalities follow from $\partial = \frac{1}{2}(d + i\, d^c)$, so that $\partial\psi(z_0) = \frac{i}{2} d^c\psi(z_0)$ on vector fields that are tangent to $\{\psi(z) = \psi(z_0)\}$. Finally, from $2\partial = d + i\, d^c$ one obtains that for all *real* vector fields ξ we have $(d\psi + id^c\psi)(\xi + iJ\xi) = 0$ and hence $d\psi(J\xi) = -d^c\psi(\xi)$.

Let $\phi \in C^\infty(M_-, \mathbb{R})$ and $\psi = \pi^*\phi$. If $\theta_z \in \mathbb{C}T_z\mathbf{K}$, then

$$\pi_*(\theta_z)\phi = \theta_z\pi^*\phi = \theta_z\psi.$$

Assume now that ϕ is the exhausting function (4.3). We want to compute its complex Hessian at points p of M_- where $\phi(p) > 0$. Since ϕ is $\mathbf{K}_0$-invariant, it suffices to compute its complex Hessian at points $[\exp(X)]$ with $X \in \mathfrak{k}_0$. This reduces the question to computing the complex Hessian of $\psi = \pi^*\phi$ at points $\exp(X)$ for $X \in \mathfrak{k}_0$, $X \neq 0$. The function ψ is defined by

$$\psi(z) = \|X\|^2 \quad \text{if } z = u \cdot \exp(X) \cdot v, \text{ with } u \in \mathbf{K}_0,\ X \in \mathfrak{k}_0,\ v \in \mathbf{V}. \tag{4.14}$$

Let $Z \in \mathfrak{k}$. To compute the *real* derivative with respect to the left invariant vector field $\vec{Z} = \frac{1}{2}(Z^\sharp + \bar{Z}^\sharp)$ of ψ at $\exp(X)$, we consider the function

$$\mathbb{R} \ni t \longrightarrow \exp(X) \cdot \exp(tZ) = \exp(tZ) \cdot \exp(\mathrm{Ad}(\exp(-tZ))(X)) \in \mathbf{K}.$$

We have

LEMMA 4.3.2. *For $Z \in \mathfrak{k}$ and $X \in \mathfrak{f}_0$, we have*

$$\vec{Z}_{\exp(X)}\psi = (X \,|\, Z + Z^*). \tag{4.15}$$

PROOF. Let $Z \in \mathfrak{k}$ and consider, for $t \in \mathbb{R}$, the element $\zeta_t = \exp(tZ) \cdot \exp(X)$ of $\mathbf{K}$. By Theorem 3.6.1 and Proposition 3.4.4, there are uniquely determined $v_t \in \exp(\mathfrak{v} \cap \mathfrak{p}_0) \cdot \exp(\mathfrak{v}_n)$ and $X_t \in \mathfrak{f}_0$ such that

$$\zeta_t^* \zeta_t = \exp(X) \cdot \exp(tZ^*) \cdot \exp(tZ) \cdot \exp(X) = v_t^* \cdot \exp(2X_t) \cdot v_t,$$

Let us look for the solution by setting $X_t = X + tH + 0(t^2)$, $v_t = \exp(tW) + 0(t^2)$, with $H \in \mathfrak{f}_0$ and $W \in (\mathfrak{v} \cap \mathfrak{p}_0) \oplus \mathfrak{v}_n$.

Then

$$\vec{Z}_{\exp(X)}\psi = 2(X|H).$$

Let us compute this scalar product. By differentiating with respect to t and setting $t = 0$ we obtain

$$\exp(X) \cdot (Z + Z^*) \cdot \exp(X) = W^* \cdot \exp(2X) + \exp(2X) \cdot W + J_{2H}(1),$$

where J_{2H} is the Jacobi vector field along the geodesic $t \to \exp(2tX)$ in $\mathcal{P}_0(\mathbf{K})$ which satisfies the initial conditions

$$\begin{cases} J_{2H}(0) = 0, \\ \dot{J}_{2H}(0) = 2H. \end{cases}$$

Then

$$J_{2H}(t) = \exp(2tX) \cdot S - S \cdot \exp(2tX) + 2T \cdot \exp(2tX),$$

with $S \in \mathfrak{k}_0$ and $T \in \mathfrak{C}(X) \cap \mathfrak{p}_0$, where $\mathfrak{C}(X) = \{Y \in \mathfrak{k} \mid [X, Y] = 0\}$, satisfying

$$2H = [X, S] + 2T$$

and therefore

$$(X|H) = \kappa(X, H) = \tfrac{1}{2}\kappa(X, [X, S]) + \kappa(X, T) = (X|T)$$

because $\kappa(X, [X, S]) = \kappa([X, X], S) = 0$.

We obtain the equation

$$\exp(X) \cdot (Z + Z^*) \cdot \exp(X) = (W^* - S + T) \cdot \exp(2X) + \exp(2X) \cdot (W + S + T),$$

that can be rewritten as

$$Z + Z^* = [\exp(X)(W + S)\exp(-X)] + [\exp(X)(W + S)\exp(-X)]^* + 2T.$$

Thus we obtain

$$\begin{aligned} 2(X|T) &= 2\kappa(X, T) \\ &= \kappa(X, Z + Z^*) - \kappa(X, \mathrm{Ad}(\exp(X))(W) + \mathrm{Ad}(\exp(-X))(W^*)) \\ &\qquad - \kappa(X, \mathrm{Ad}(\exp(X))(S) - \mathrm{Ad}(\exp(-X))(S)) \\ &= (X|Z + Z^*) - (X|W + W^*) - (X|S - S) \end{aligned}$$

because $\mathrm{Ad}(\exp(\pm X))(X) = X$ and the form κ is $\mathrm{Ad}(\mathbf{K})$-invariant. Hence

$$2(X|T) = (X|Z + Z^*),$$

since $X \in \mathfrak{k}_0$ is orthogonal to $(\mathfrak{v} + \mathfrak{v}^*) \cap \mathfrak{p}_0$. Thus we obtain (4.15). □

4. Dolbeault and CR Cohomologies

We remark that by Andreotti's theory we know that for every coherent sheaf $\mathcal{F}$ on an r-pseudoncave complex manifold M_- we have

$$\mathbf{H}^j(M, \mathcal{F}) < \infty, \forall j < r.$$

Then we have the following proposition:

Proposition 4.4.1. *For a complex flag manifold $M \simeq \mathbf{G}/\mathbf{Q}$ let M_-be the Matsuki's dual $\mathbf{K}$-orbit of a $\mathbf{G}_0$-orbit M_+, with $M_0 \simeq M_- \cap M_+$ being a compact CR-manifold . If M_- is HNR and M_0 is r-psudoconvave, then for every coherent sheaf $\mathcal{F}$ we have*

$$\dim(\mathbf{H}^j(M_0, \mathcal{F}) \simeq \mathbf{H}^j(M_-, \mathcal{F})) < \infty, \quad \forall j < r. \tag{4.16}$$

In particular,

$$\dim(\mathbf{H}^{p,j}(M_0) \simeq \mathbf{H}^{p,j}(M_-)), \quad \forall j < r. \tag{4.17}$$

Here we used the notation $\mathbf{H}^{p,j}$ for the $\bar{\partial}$ and $\bar{\partial}_{M_0}$-cohomologies on forms of type $(p, *)$. Because of the Poincaré lemma, they coincide with the Čech cohomology with coefficients in the sheaf of germs of holomorphic p-forms.

Proof. Since M_- is *HNR*, then the exhaustion function ϕ in (4.3) is well defined. Then to verify (4.16) we can apply Andreotti-Grauert's theory, showing that each subdomain $\Omega_c = \{\phi < c\}$ is r-pseudoconcave. To this aim, we prove that the complex hessian of ϕ admits at least r negative eigenvalues on the analytic tangent to $U_c = \partial\Omega_c$. We may consider the vector fields tangent to the submanifolds M_X and $\tilde{M}_X$, defined respectively in (4.5) and (4.6). By exploiting the $\mathbf{K}_0$-invariance of ϕ, we can take, without any loss of generality, $p_0 = [\exp(X)] \in U_c$, with $\|X\|^2 = c \in \mathbb{R}$. By Lemma (4.2.3) we have $\tilde{M}_X \subset \overline{\Omega}_c = \{\phi \leq \|X\|^2\}$ and $\tilde{M}_X$ is tangent to U_c at p_0. Since $\tilde{M}_X$ is *CR*-diffeomorphic to M_0, it is r-pseudoconcave. Being $\tilde{M}_X \subset \overline{\Omega}_c$, the restriction of the complex Hessian of ϕ to the analytic tangent to $\tilde{M}_X$ at $[\exp(X)]$ has at least as many negative eigenvalues as the Levi form of $\tilde{M}_X$ in the codirection $Jd\phi([\exp(X)])$ and hence, by the assumption, has at least r negative eigenvalues. This completes the proof. □

CHAPTER 5

Some Examples

1. Examples of orbits of the complex type: $\mathbf{SL}_n(\mathbb{C})$

1.1. Orbits of $\mathbf{SL}_3(\mathbb{C})$ in $\mathcal{F}_1(\mathbb{C}^3)\times\mathcal{F}_2(\mathbb{C}^3)$. The *real* action of $\mathbf{G}_0 = \mathbf{SL}_3(\mathbb{C})$ is defined by $a\cdot(\ell, L) = (a(\ell), \bar{a}(L))$. Since on the maximal compact subgroup $\mathbf{K}_0 = \mathbf{SU}(3)$ we have $\bar{a} = [a^\intercal]^{-1}$, the action $a\cdot(\ell, L) = (a(\ell), [a^\intercal]^{-1}(L))$ of $\mathbf{K}_0$ complexifies to a holomorphic action of $\mathbf{K} = \mathbf{SL}_3(\mathbb{C})$. We identify $\mathbb{C}^3$ with its dual space via the bilinear form $\langle v|w\rangle = w^\intercal v$ and use the notation $W^0 = \{v \in \mathbb{C}^3 \mid \langle v|w\rangle = 0,\ \forall w \in W\}$. Then both $\mathbf{G}_0$ and $\mathbf{K}$ have two distinct orbits in $\mathcal{F}_1(\mathbb{C}^3)\times\mathcal{F}_2(\mathbb{C}^3)$, which are paired by the Matsuki duality:

$$\text{(open)}\quad M+(0) = \{(\ell, L) \mid \ell\cap\overline{L} = \{0\},\quad \leftrightarrow\quad M_-(0) = M_0(0) = \{(\ell, L) \mid \ell = L^0\},$$

$$\text{(minimal)}\, M_+(1) = M_0(1) = \{(\ell, L) \mid \ell\subset\overline{L}\}\quad \leftrightarrow\quad M_-(1) = \{(\ell, L) \mid \ell\cap L^0 = \{0\}\}.$$

We note that $M_+(0)$ is the single open orbit of $\mathbf{G}_0 = \mathbf{SL}_3(\mathbb{C})$ and $M_-(1)$ the single compact orbit of $\mathbf{K} = \mathbf{SL}_3(\mathbb{C})$, which can be identified with the diagonal of $\mathbb{CP}^2\times\mathbb{CP}^2$.

The orbit $M_+(1) = M_0(1)$, that we shall denote by M_0 within this subsection, is the minimal orbit of $\mathbf{G}_0 = \mathbf{SL}_3(\mathbb{C})$. We shall set M_- for its Matsuki dual $M_-(1)$.

Let $\ell_0 = \langle e_1\rangle$ and $L_0 = \langle e_1, e_2\rangle$, so that $p_0 = (\ell_0, L_0) \in M_0$ and $M_0 \simeq \mathbf{SU}(3)/\mathbf{I}_0$ for the stabilizer

$$\mathbf{I}_0 = \{\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \mid |\lambda_i| = 1,\ \lambda_1\cdot\lambda_2\cdot\lambda_3 = 1\} \simeq \mathbf{S}^1\times\mathbf{S}^1$$

of p_0 in $\mathbf{K}_0 = \mathbf{SU}(3)$. The stabilizer V of p_0 in $\mathbf{K} = \mathbf{SL}_3(\mathbb{C})$ is

$$\mathbf{V} = \left\{\left.\begin{pmatrix} z_{1,1} & z_{1,2} & 0\\ 0 & z_{2,2} & 0\\ 0 & z_{3,2} & z_{3,3}\end{pmatrix}\right| z_{i,j}\in\mathbb{C},\quad z_{1,1}z_{2,2}z_{3,3} = 1\right\}.$$

In particular

$$\mathfrak{v}_n = \left\{\left.\begin{pmatrix} 0 & z_1 & 0\\ 0 & 0 & 0\\ 0 & z_2 & 0\end{pmatrix}\right| z_1, z_2\in\mathbb{C}\right\},$$

and M_0 has *CR*-dimension two. We have

$$\mathfrak{i}_0\oplus[\mathfrak{v}_n\oplus\bar{\mathfrak{v}}_n]\cap\mathfrak{k}_0 = \left\{\left.\begin{pmatrix} i\tau_1 & \zeta_{1,2} & 0\\ -\bar{\zeta}_{1,2} & i\tau_2 & \bar{\zeta}_{3,2}\\ 0 & \zeta_{3,2} & i\tau_3\end{pmatrix}\right| \tau_i\in\mathbb{R},\ \tau_1+\tau_2+\tau_3 = 0,\ \ \zeta_{i,j}\in\mathbb{C}\right\}.$$

Thus an $\mathrm{Ad}(\mathbf{I}_0)$-invariant complement of $\mathfrak{i}_0\oplus[\mathfrak{v}_n\oplus\bar{\mathfrak{v}}_n]\cap\mathfrak{k}_0$ can be taken to be

$$\mathfrak{m}_0 = \left\{\left.\begin{pmatrix} 0 & 0 & \theta\\ 0 & 0 & 0\\ -\bar{\theta} & 0 & 0\end{pmatrix}\right| \theta\in\mathbb{C}\right\} \Longrightarrow i\mathfrak{m}_0 = \left\{X(\theta) = \left.\begin{pmatrix} 0 & 0 & \theta\\ 0 & 0 & 0\\ \bar{\theta} & 0 & 0\end{pmatrix}\right| \theta\in\mathbb{C}\right\}$$

We have

$$\exp\begin{pmatrix}0&0&\theta\\0&0&0\\\bar\theta&0&0\end{pmatrix}=\begin{pmatrix}\cosh|\theta|&0&\theta\dfrac{\sinh|\theta|}{|\theta|}\\0&1&0\\\bar\theta\dfrac{\sinh|\theta|}{|\theta|}&0&\cosh|\theta|\end{pmatrix},$$

so that

$$\exp(i\mathfrak{m}_0)=\left\{\begin{pmatrix}\cosh t&0&e^{is}\sinh t\\0&1&0\\e^{-is}\sinh t&0&\cosh t\end{pmatrix}\,\middle|\, t,s\in\mathbb{R}\right\}.$$

Let us consider any matrix $a=(a_{i,j})\in\mathbf{SL}_3(\mathbb{C})=\mathbf{K}$ and the corresponding point $a\cdot p_0=(\ell_a,L_a)$ in M_-. Then ℓ_a is the line in $\mathbb{C}^3$ generated by the first column of a, while $L_a=[a^\intercal]^{-1}(\langle e_1,e_2\rangle)$. Thus we can characterize L_a in the following way. A vector $(z_1,z_2,z_3)^\intercal$ belongs to L_a iff, for some $w_1,w_2\in\mathbb{C}$, we have:

$$[a^\intercal]^{-1}\begin{pmatrix}w_1\\w_2\\0\end{pmatrix}=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\Leftrightarrow a^\intercal\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}w_1\\w_2\\0\end{pmatrix}\Leftrightarrow L_a=\left\{\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\,\middle|\,a_{1,3}z_1+a_{2,3}z_2+a_{3,3}z_3=0\right\}.$$

We note that the necessary and sufficient condition for (ℓ_a,L_a) to belong to M_0 is that $a_{1,1}\bar a_{1,3}+a_{2,1}\bar a_{2,3}+a_{3,1}\bar a_{3,3}=0$. This is the entry in the third line and first column of a^*a.

When $a=a_u\exp(X(\theta))$, with $a_u\in\mathbf{K}_0$, we have $a^*a=[\exp[X(\theta)]^2$ and

$$\left|[\exp[X(\theta)]^2_{3,1}\right|=\left|2\bar\theta\sinh(|\theta|)\cosh(\theta)/|\theta|\right|=\sinh(2|\theta|).$$

The exhausting function can therefore be written as

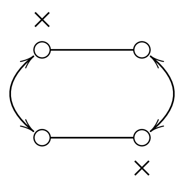
$$\phi([a])=\tfrac{1}{2}\log(\tau+\sqrt{1+\tau^2}),\quad\text{where}\quad\tau=|a_{1,1}\bar a_{1,3}+a_{2,1}\bar a_{2,3}+a_{3,1}\bar a_{3,3}|.$$

The matrix $X(\theta)$ has eigenvalues $e^{-|\theta|},1,e^{|\theta|}$. Let us consider for the matrices $a^*X(\theta)a$, with $a\in\mathbf{SU}(3)$, the scalar product of the first and the third columns. Since $\operatorname{Re}(v^*w)=\frac{1}{2}(\|v+w\|^2-\|v-w\|^2)$, $\operatorname{Im}(v^*w)=\frac{1}{2}(\|v+iw\|^2-\|v-iw\|^2)$, we obtain that an upper bound for $\sinh(\phi([X(\theta)a]))$, when $a\in\mathbf{SU}(3)$, is $\frac{1}{2}(e^{2|\theta|}-e^{-2|\theta|})=\sinh(2|\theta|)=\sinh(\phi(X(\theta)))$. This shows that the translate of M_0 are tangent to the level hypersurfaces $\phi([a])=c>0$ and stay within $\phi([a])\leq c$.

1.2. A special example. We consider the action of $\mathbf{SL}_3(\mathbb{C})$, thought as a real group, on the flag manifold $\mathbb{CP}^2\times\mathbb{CP}^2$ of $\mathbf{SL}_3(\mathbb{C})\times\mathbf{SL}_3(\mathbb{C})$, given, in homogeneous coordinates, by

$$a\cdot(z,w)=(a(z),[a^*]^{-1}(w)),\quad [z],[w]\in\mathbb{CP}^2.$$

In fact in this way we consider on the first factor the standard representation, and on the second the conjugate of the dual representation, so that the minimal orbit corresponds to the pairs (ℓ_1,ℓ_2) consisting of a complex line ℓ_1 and a complex 2-plane ℓ_2 with $\bar\ell_1\subset\ell_2$. Its cross-marked Satake diagram is



The equation for the minimal orbit is

$$M_0 := \{([z],[w] \mid (z|w) = 0\},$$

where $(\cdot|\cdot)$ is the standard Hermitian scalar product in $\mathbb{C}^3$. There is an open orbit in $\mathbb{CP}^2 \times \mathbb{CP}^2$, which is the complement of M_0:

$$M'_+ := \{([z],[w] \mid (z|w) \neq 0\}.$$

A maximal compact subgroup of $\mathbf{SL}_3(\mathbb{C})$ is the special unitary group $\mathbf{SU}(3)$ relative to the chosen Hermitian scalar product. Since $[a^*]^{-1} = a$ for $a \in \mathbf{SU}(3)$, its action on $\mathbb{CP}^2 \times \mathbb{CP}^2$ is

$$a \cdot ([z],[w]) = ([a(z)],[a(w)]), \quad \forall a \in \mathbf{SU}(3), \quad \forall z, w \in \mathbb{C}^3 \setminus \{0\}.$$

The complexification $\mathbf{SL}_3(\mathbb{C})$ of $\mathbf{SU}(3)$ acts then on $\mathbb{CP}^2 \times \mathbb{CP}^2$ by

$$a \cdot ([z],[w]) = ([a(z)],[a(w)]), \quad \forall a \in \mathbf{SL}_3(\mathbb{C}), \quad \forall z, w \in \mathbb{C}^3 \setminus \{0\}.$$

Thus the two orbits Matzuki-dual to M_0 and M'_+ are

$$M_- = \{([z],[w]) \mid z \wedge w \neq 0\} \quad \text{and} \quad M'_- = \{([z],[z]) \mid z \in \mathbb{C}^3 \setminus \{0\}\}.$$

The complement of M_- is the diagonal of $\mathbb{CP}^2 \times \mathbb{CP}^2$, which has codimension two in $\mathbb{CP}^2 \times \mathbb{CP}^2$: hence M_- turns out to be 1-pseudo-concave, as expected. Note that in this case the complement of M^- in $\mathbb{CP}^2 \times \mathbb{CP}^2$ is not a set theoretical complete intersection. In fact $S = \{(z,w) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid z \wedge w = 0\}$ has codimension two, but the minimal number of generators of the ideal $\mathcal{I}_S \subset \mathbb{C}_0[z,w]$ of homogeneous polynomial vanishing on S needs three generators.

1.3. The general case. Let us consider $\mathbf{SL}_n(\mathbb{C})$, the special complex linear group, as a real group $\mathbf{G}_0$. The flag manifolds of its complexification $\mathbf{G} = \mathbf{SL}_n(\mathbb{C}) \times \mathbf{SL}_n(\mathbb{C})$ are the products $M = \mathcal{F}_{k_1,\ldots,k_p}(\mathbb{C}^n) \times \mathcal{F}_{h_1,\ldots,h_q}(\mathbb{C}^n)$ where both $1 \leq k_1 < \cdots < k_p < n$ and $1 \leq h_1 < \cdots < h_q < n$ are increasing sequences of integers, and each factor of the Cartesian product is the flag of linear subspaces of $\mathbb{C}^n$ of the dimensions prescribed by the subscripts. The *real* action of $\mathbf{SL}_n(\mathbb{C})$ on M is described by

$$a \cdot (L_{k_1}, \ldots, L_{k_p}; \ell_{h_1}, \ldots, \ell_{h_q}) = (a(L_{k_1}), \ldots, a(L_{k_p}); \bar{a}(\ell_{h_1}), \ldots, \bar{a}(\ell_{h_q})).$$

The special unitary group $\mathbf{SU}(n) = \{a \in \mathbf{GL}_n(\mathbb{C}) \mid a^*a = \mathrm{I}_n, \ \det(a) = 1\}$ is a maximal compact subgroup $\mathbf{K}_0$ of $\mathbf{SL}_n(\mathbb{C})$. Since $\bar{a} = {}^T a^{-1}$ for $a \in \mathbf{SU}(n)$, the restriction to $\mathbf{SU}(n)$ of the action of $\mathbf{SL}_n(\mathbb{C})$ on $\mathbf{X}$ can be described by

$$a \cdot (L_{k_1}, \ldots, L_{k_p}; \ell_{h_1}, \ldots, \ell_{h_q}) = (a(L_{k_1}), \ldots, a(L_{k_p}); {}^T a^{-1}(\ell_{h_1}), \ldots, {}^T a^{-1}(\ell_{h_q})).$$

The same formula expresses the holomorphic action of the *complex Lie group* $\mathbf{SL}_n(\mathbb{C})$, considered now as the complexification $\mathbf{K}$ of $\mathbf{SU}(n)$. Thus the *complex* orbits of $\mathbf{K}$ correspond to the canonical action of $\mathbf{SL}_n(\mathbb{C})$ on the first factor, and the *dual* action of $\mathbf{SL}_n(\mathbb{C})$ on the second factor.

Let us denote by $\langle z|w\rangle = \sum z_i w_i$ the usual duality pairing in $\mathbb{C}^n$, which is $\mathbb{C}$-bilinear, and by $E^0 = \{z \in \mathbb{C}^n \mid \langle z|w\rangle = 0, \ \forall w \in E\}$ the annihilator of a subset E of $\mathbb{C}^n$ with respect to the duality form.

Then

(1) The orbits of the real form $\mathbf{G}_0 = \mathbf{SL}_n(\mathbb{C})$ are parametrized by the invariants

$$d_{i,j} = \dim_{\mathbb{C}}(L_i \cap \bar{\ell}_j), \quad i = 1, \ldots, p, \quad j = 1, \ldots, q.$$

(2) The orbits of the complex form $\mathbf{K} = \mathbf{SL}_n(\mathbb{C})$ are parametrized by the invariants

$$\delta_{i,j} = \dim_{\mathbb{C}}(L_i \cap \ell_j^0), \quad i = 1, \dots, p, \quad j = 1, \dots, q.$$

1.4. The case $M = \mathcal{F}_p \times \mathcal{F}_q$, $p \leq q$. M is a projective manifold of complex dimension $[(p+q)n - p^2 - q^2]$.

The real orbits $M_+(p,q;k)$ of $\mathbf{G}_0 = \mathbf{SL}_n(\mathbb{C})$ are parametrized by the integer $k = \dim_{\mathbb{C}}(L \cap \bar{\ell})$, with $\nu = \max\{0, n-p-q\} \leq k \leq p$. We denote by $M_-(p,q;k)$ its Matsuki dual and by $M_0(p,q;k)$ the intersection $M_+(p,q;k) \cap M_-(p,q;k)$. Then

$$\begin{cases} M_+(p,q;k) = \{(L,\ell) \in \mathcal{F}_p \times \mathcal{F}_q \mid \dim(L \cap \bar{\ell}) = k\}, \\ M_-(p,q;k) = \{(L,\ell) \in \mathcal{F}_p \times \mathcal{F}_q \mid \dim(L \cap \ell^0) = p-k\}, \\ M_0(p,q;k) \simeq \{(\lambda,\mu,\ell) \in \mathcal{F}_k \times \mathcal{F}_{p-k} \times \mathcal{F}_q \mid \lambda \subset \bar{\ell}, \ \mu \subset \ell^0\}. \end{cases}$$

Indeed, since $\ell^0 = \bar{\ell}^{\perp}$, we have $\mathbb{C}^n = \ell^0 \oplus \bar{\ell}$, and the compact orbit $M_0(p,q;k)$ consists of those pairs (L,ℓ) for which L admits the orthogonal decomposition

$$L = (L \cap \bar{\ell}) \oplus (L \cap \ell^0) = \lambda \oplus \mu.$$

We have canonical fibrations

$$\begin{array}{ccc} M_0(p,q;k) & \longrightarrow & M_0(k,q;k) = M_+(k,q;k) \ \text{(minimal orbit)} \\ \downarrow & & \\ \mathcal{F}_{q-p,q}(\mathbb{C}^n), & & \end{array}$$

where $\mathcal{F}_{q-p,q}(\mathbb{C}^n)$ is the minimal orbit of the complex $\mathbf{K} = \mathbf{SL}_n(\mathbb{C})$ in the complex flag manifold $\mathcal{F}_{q-p}(\mathbb{C}^n) \times \mathcal{F}_q(\mathbb{C}^n)$.

1.5. The minimal orbit. Consider the minimal orbit $M_0 = M_0(p,q;p) = M_+(p,q;p)$ of $\mathbf{G}_0 = \mathbf{SL}_n(\mathbb{C})$ in $M = \mathcal{F}_p(\mathbb{C}^n) \times \mathcal{F}_q(\mathbb{C}^n)$. The elements of $T_{\mathrm{o}}M_0$ (the tangent space to M_0 at the base point are represented by matrices

$$\begin{pmatrix} X & 0 \\ 0 & \bar{X} \end{pmatrix}, \quad \text{where} \quad X = \begin{pmatrix} 0 & 0 & 0 \\ Z_1 & 0 & 0 \\ W & Z_2^* & 0 \end{pmatrix} \quad \text{with} \quad \begin{cases} Z_1 \in \mathbb{C}^{(q-p)\times p}, & Z_2 \in \mathbb{C}^{(n-q)\times(q-p)}, \\ W \in \mathbb{C}^{(n-q)\times p}. \end{cases}$$

The elements with $W = 0$ represent $T^{0,1}_{\mathrm{o}} M_0$, those with $Z_1 = 0$, $Z_2 = 0$ the quotient $T_{\mathrm{o}}M_0/H_{\mathrm{o}}M_0$. We have in fact

$$CR - \dim(M_0) = (q-p)(n+p-q), \quad CR - \operatorname{codim}(M_0) = 2p(n-q).$$

The Levi form can be computed by commuting the matrices in $\mathbf{T}^{0,1}_{\mathrm{o}} M_0$ and their conjugates. From

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ Z_1 & 0 & 0 \\ 0 & Z_2^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \Theta_1 & 0 & 0 \\ 0 & \Theta_2^* & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Z_2^*\Theta_1 - \Theta_2^* Z_1 & 0 & 0 \end{pmatrix}$$

we obtain a Levi form with associated vector-valued Hermitian quadratic form

$$\mathfrak{L}(Z_1, Z_2) = \begin{pmatrix} Z_1^* Z_2 + Z_2^* Z_1 \\ i(Z_1^* Z_2 - Z_2^* Z_1) \end{pmatrix}.$$

Hence all scalar Levi forms that are different from zero have even rank $2r(q-p)$ with $1 \leq r \leq \min\{p, (n-q)\}$ and Witt index $r(q-p)$. Thus M_0 is strongly $(q-p)$-pseudo-concave and weakly $((q-p) \cdot \max\{p, n-q\})$-pseudo-concave.

The dual orbit $M_- = M_-(p,q;p)$ consists of the pairs $(L,\ell) \in M = \mathcal{F}_p(\mathbb{C}^n) \times \mathcal{F}_q(\mathbb{C}^n)$ with $\dim_{\mathbb{C}}(L \cap \ell^0) = 0$. Thus $M_- = M \setminus V$. To describe the algebraic variety V, let us use the embedding $\mathcal{F}_h(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\Lambda^h(\mathbb{C}^n))$ and the isomorphism $\mathcal{F}_q(\mathbb{C}^n) \to \mathcal{F}_{n-q}(\mathbb{C}^n)$ defined by $\ell \to \ell^0$. Then $\mathbf{V}$ is the set

$$V \simeq \{([v_1 \wedge \cdots \wedge v_p], [w_1 \wedge \cdots \wedge w_{n-q}] \mid v_1 \wedge \cdots v_p \wedge w_1 \wedge \cdots \wedge w_{n-q} = 0\}.$$

Let us consider $n \times (n+p-q)$ matrix $A = (v_1, \ldots, v_p, w_1, \ldots, w_{n-q})$. The generators of the ideal that defines V are the $\binom{n}{n+p-q}$ minor determinants of order $(n+p-q)$ of A. The codimension of V is given by the number $(q-p+1)$ of minor determinants of order $(n+p-q)$ that are obtained by bordering a minor determinant of order $(n+p-q-1)$ of A.

According to [**12**,**18**,**24**,**25**,**39**] the cohomology groups of the tangential Cauchy-Riemann complex on M_0 are finite dimensional in degree j for $j < q-p$ and $j > (q-p)(n+p-q-1)$ and infinite dimensional for $j = [(q-p)\cdot\max\{p, n-q\}]$.

We want to describe the Mostow fibration for the minimal orbit of $\mathbf{SL}_n(\mathbb{C})$ in $\mathcal{F}_p \times \mathcal{F}_q$. In this case $M_+ = M_0 = \{(\ell, L) \in \mathcal{F}_p \times \mathcal{F}_q \mid \bar{\ell} \subset L\}$ and $M_- = \{(\ell, L) \mid \ell \cap L^0 = \{0\}\}$.

We consider M_0 as a homogeneous space of $\mathbf{SU}(n)$. We choose the base point $p_0 = (\langle e_1, \ldots, e_p\rangle, \langle e_1, \ldots, e_q\rangle)$. Then

$$\mathbf{I}_0 = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \middle| \begin{matrix} A \in \mathbf{U}(p),\ B \in \mathbf{U}(q-p),\ C \in \mathbf{U}(n-q), \\ \det(A)\det(B)\det(C) = 1 \end{matrix} \right\}.$$

The action of $\mathbf{SL}_n(\mathbb{C})$ on $\mathcal{F}_p \times \mathcal{F}_q$ is $a\cdot(\ell, L) = (a(\ell), [a^\intercal]^{-1}(L))$. Then the stabilizer $\mathbf{V}$ of p_0 is

$$\mathbf{V} = \left\{ \begin{pmatrix} Z_{1,1} & Z_{1,2} & 0 \\ 0 & Z_{2,2} & 0 \\ 0 & Z_{3,3} & Z_{3,3} \end{pmatrix} \middle| \begin{matrix} Z_{1,1} \in \mathbf{GL}_p(\mathbb{C}),\ Z_{2,2} \in \mathbf{GL}_{q-p}(\mathbb{C}),\ Z_{3,3} \in \mathbf{GL}_{n-q}(\mathbb{C}), \\ Z_{1,2} \in \mathbb{C}^{p\times(q-p)},\ Z_{3,2} \in \mathbb{C}^{(n-q)\times(q-p)}, \\ \det(Z_{1,1})\det(Z_{2,2})\det(Z_{3,3}) = 1 \end{matrix} \right\}.$$

We obtain

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & Z_{1,3} \\ 0 & 0 & 0 \\ Z_{3,1} & 0 & 0 \end{pmatrix} \middle| Z_{1,3} \in \mathbb{C}^{p\times(n-q)},\ Z_{3,1} \in \mathbb{C}^{(n-q)\times p} \right\},$$

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & 0 & Z_{1,3} \\ 0 & 0 & 0 \\ -Z_{1,3}^* & 0 & 0 \end{pmatrix} \middle| Z_{1,3} \in \mathbb{C}^{p\times(n-q)} \right\}.$$

We have

$$\exp\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ A^* & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{A\,A^*}) & 0 & A[\sinh(\sqrt{A^*A})/(\sqrt{A^*A})] \\ 0 & \mathrm{I}_{q-p} & 0 \\ A^*[\sinh(\sqrt{A\,A^*})/(\sqrt{A\,A^*})] & 0 & \cosh(\sqrt{A^*A}) \end{pmatrix}.$$

Note that $\mathbf{I}_0$ contains a maximal torus and therefore $\mathfrak{v}$ is *regular* according to [**7**].

1.6. Some special cases. Let $p = q = 1$, with $M = \mathcal{F}_k(\mathbb{C}^n) \times \mathcal{F}_h(\mathbb{C}^n)$. When $h = k$, the minimal orbit $M_0 = \{L = \bar{\ell}\}$ is totally real, and embeds into the Stein manifold $M_- = \{L \cap \ell^0 = \{0\}\}$. To check this, we consider the standard embedding

$\mathcal{F}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\Lambda^k\mathbb{C}^n)$ and the map $\mathbb{P}(\Lambda^k(\mathbb{C}^n)) \to \mathbb{P}(\Lambda^{n-k}(\mathbb{C}^n))$ induced by the polarity. This map can be described in the following way. First we note that the duality pairing on $\mathbb{C}^n$ extends to a duality pairing on $\Lambda^k(\mathbb{C}^n)$:

$$\langle v_1 \wedge \cdots \wedge v_k | w_1 \wedge \cdots \wedge w_k \rangle = \frac{1}{k!} \sum\nolimits_{\sigma \in \mathbf{S}_k} \varepsilon(\sigma) \langle v_1 | w_{\sigma_1} \rangle \cdots \langle v_k | w_{\sigma_k} \rangle.$$

On the other hand, the isomorphism $\Lambda^n(\mathbb{C}^n) \simeq \mathbb{C}$ identifies $\Lambda^{n-k}(\mathbb{C}^n)$ to the dual space of $\Lambda^k(\mathbb{C}^n)$. Thus we obtain a linear isomorphism $\Lambda^k(\mathbb{C}^n) \to \Lambda^{n-k}(\mathbb{C}^n)$, that defines a map on the corresponding projective spaces. Then $M_- \simeq \{(v_1 \wedge \cdots \wedge v_k) \wedge (w_{k+1} \wedge \cdots \wedge w_n) \neq 0\}$ is the complement in M of a hypersurface.

2. Examples of orbits of real form : $\mathbf{SU}(p, q)(\mathbb{C})$

2.1. Example: minimal orbits of $\mathbf{SU}(p, q)$. Let us consider the minimal orbits of $\mathbf{SU}(p.q)$ in the Grassmannian $\mathcal{G}r_\ell(\mathbb{C}^{p+q})$ of ℓ-planes in $\mathbb{C}^{p+q}$. We assume that $1 \leq p \leq q$ and distinguish the three cases in which $\ell \leq p$, or $q \leq \ell < (p+q)$, or $p < \ell < q$ (if $p+1 < q$). We will denote by $\hbar$ a fixed Hermitian symmetric form of signature (p, q) on $\mathbb{C}^{p+q}$.

1. $\boxed{1 \leq \ell \leq p}$ The minimal orbit M_0 consists of the ℓ-planes of $\mathbb{C}^{p+q}$ which are totally isotropic with respect to $\hbar$.

The maximal compact subgroups $\mathbf{K}_0 \simeq \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ of $\mathbf{SU}(p, q)$ correspond to all possible choices of two $\hbar$-orthogonal subspaces $W_p \simeq \mathbb{C}^p$ and $W_q \simeq \mathbb{C}^q$ of $\mathbb{C}^{p+q}$, with $\hbar|_{W_p} > 0$ and $\hbar|_{W_q} < 0$, and $\mathbf{K}_0$ consists of the elements of $\mathbf{SU}(p, q)$ which leave one of, and hence both, the subspaces W_p and W_q invariant. Let π_p and π_q denote the projections associated to the direct sum decomposition $\mathbb{C}^{p+q} = W_p \oplus W_q$. A totally isotropic ℓ-plane α has trivial intersections with both W_p and W_q and therefore $\pi_p(\alpha)$, $\pi_q(\alpha)$ are ℓ-dimensional and α is the graph of a uniquely determined

$$\phi_\alpha : \pi_p(\alpha) \to \pi_q(\alpha), \quad \text{with } \hbar(\phi_\alpha(v), \phi_\alpha(v)) = -\hbar(v, v) \text{ for all } v \in W_p.$$

[We call such a ϕ_α an *isometry*.] Moreover, there is a principal fibration

$$M_0 \ni \alpha \longrightarrow (\pi_p(\alpha), \pi_q(\alpha)) \in \mathcal{G}r_\ell(W_p) \times \mathcal{G}r_\ell(W_q), \quad \text{with structure group } \mathbf{U}(\ell),$$

which can also be viewed as a a presentation of M_0 as a *CR*-lifting of the complex structure of the product of the Grassmannians.

Since $\dim_{\mathbb{C}} \mathcal{G}r_\ell(\mathbb{C}^m) = \ell(m - \ell)$, the *CR* dimension of M_0 is

$$n = \ell(p - \ell) + \ell(q - \ell) = \ell(p + q - 2\ell) = \dim_{\mathbb{C}} \mathcal{G}r_\ell(W_p) \times \mathcal{G}r_\ell(W_q)$$

and its *CR* codimension is

$$k = \dim_{\mathbb{R}} \mathbf{U}(\ell) = \ell^2.$$

The Matsuki dual of M_0 is the open orbit M_- in $\mathcal{G}r_\ell(\mathbb{C}^{p+q})$ of the complexification $\mathbf{K} \simeq \mathbf{S}(\mathbf{GL}_p(\mathbb{C}) \times \mathbf{GL}_q(\mathbb{C}))$ of $\mathbf{K}_0$. It is described by

$$\begin{aligned} M_- &= \{\alpha \in \mathcal{G}r_\ell(\mathbb{C}^{p+q}) \mid \alpha \cap W_p = \{0\}, \quad \alpha \cap W_q = \{0\}\} \\ &= \{\alpha \in \mathcal{G}r_\ell(\mathbb{C}^{p+q}) \mid \pi_p(\alpha) \in \mathcal{G}r_\ell(W_p), \ \pi_q(\alpha) \in \mathcal{G}r_\ell(W_q)\}. \end{aligned}$$

The $\mathbf{U}(\ell)$-principal fibration of M_0 yields a $\mathbf{GL}_\ell(\mathbb{C})$-principal fibration of M_- with the same basis:

$$M_- \ni \alpha \longrightarrow (\pi_p(\alpha), \pi_q(\alpha)) \in \mathcal{G}r_\ell(W_p) \times \mathcal{G}r_\ell(W_q), \quad \text{and structure group } \mathbf{GL}_\ell(\mathbb{C}).$$

The elements α of M_- are the graph of $\mathbb{C}$-linear isomorphisms $\phi_\alpha : \pi_p(\alpha) \to \pi_q(\alpha)$. Using the Hermitian structure of $\pi_p(\alpha)$ and $\pi_q(\alpha)$, we can compiute the adjoint $\phi^*_\alpha : \pi_q(\alpha) \to \pi_p(\alpha)$. The composition $\phi^*_\alpha\phi_\alpha$ is Hermitian symmetric and positive definite on $\pi_p(\alpha)$ and has a positive definite Hermitian symmetric square root $\sqrt{\phi^*_\alpha\phi_\alpha}$. Then $\psi_\alpha = \phi_\alpha \circ (\sqrt{\phi^*_\alpha\phi_\alpha})^{-1} \in \mathbf{U}(\pi_p(\alpha)) \simeq \mathbf{U}(\ell)$ and

$$\varpi(\alpha) = \{z + \psi_\alpha(z) \mid z \in \pi_p(\alpha)\} \in M_0$$

defines the Mostow fibration $\varpi : M_- \to M_0$.

The exhausting function Φ of M_- can be defined by taking the Hermitian symmetric logarithm $\log(\phi^*_\alpha\phi_\alpha)$ and setting

$$\Phi(\alpha) = \tfrac{1}{4}\mathrm{trace}([\log(\phi^*_\alpha\phi_\alpha)]^2).$$

Let us better explain how this fits into our general scheme. Fix an orthonormal basis $e_1, \dots, e_{p+q}$ of $\mathbb{C}^{p+q}$ for $\hbar$ in which the first p vectors $e_1, \dots, e_p$ are a basis of W_p. Then the ℓ-plane $\alpha_0 = \langle e_1 + e_{q+1}, \dots, e_\ell + e_{q+\ell}\rangle$ is a point of M_0. The elements of $\mathbf{K}$ are represented by block matices

$$z = \begin{pmatrix} z_p & 0 \\ 0 & z_q \end{pmatrix} \quad \text{with } z_p \in \mathbf{GL}_p(\mathbb{C}),\ z_q \in \mathbf{GL}_q(\mathbb{C}) \text{ and } \det(z_p)\cdot\det(z_q) = 1.$$

The stabilizer of α is

$$\mathbf{V} = \left\{ z = \begin{pmatrix} a_{1,1} & a_{1,2} & & \\ 0 & a_{2,2} & & \\ & & a_{1,1}^{-1} & b_{1,2} \\ & & 0 & b_{2,2} \end{pmatrix} \in \mathbf{K} \;\middle|\; \begin{array}{ll} a_{1,1} \in \mathbf{GL}_\ell(\mathbb{C}), & a_{2,2} \in \mathbf{GL}_{p-\ell}(\mathbb{C}), \\ b_{2,2} \in \mathbf{GL}_{q-\ell}(\mathbb{C}), & a_{1,2} \in \mathbb{C}^{\ell\times(p-\ell)}, \\ b_{1,2} \in \mathbb{C}^{\ell\times(q-\ell)}. & \end{array} \right\}.$$

Hence the nilradical

$$\mathfrak{v}_n = \left\{ \begin{pmatrix} 0 & Z_1 & & \\ 0 & 0 & & \\ & & 0 & Z_2 \\ & & 0 & 0 \end{pmatrix} \;\middle|\; Z_1 \in \mathbb{C}^{\ell\times(p-\ell)},\ Z_2 \in \mathbb{C}^{\ell\times(q-\ell)} \right\}$$

of its Lie algebra $\mathfrak{v}$ is horocyclic:

2. $\boxed{q \le \ell \le p+q}$ The case of ℓ-planes with $p \le q \le \ell < p+q$ can be treated analogously, as the mapping $\alpha \to \alpha^\perp$ is an anti-CR-isomorphism from the minimal orbit of $\mathbf{SU}(p,q)$ in $\mathcal{G}r_\ell(\mathbb{C}^{p+q})$ onto the minimal orbit of $\mathbf{SU}(p,q)$ in $\mathcal{G}r_{p+q-\ell}(\mathbb{C}^{p+q})$. Here the perpendicular is taken with respect to the fixed Hermitian symmetric form $\hbar$, of signature (p,q).

3. $\boxed{p < \ell < q}$ Finally let us consider the case where $p < \ell < q$. Then the minumal orbit M_0 consists of the ℓ-planes α on which $\hbar$ restricts to a degenerate form of rank $(\ell - p)$. For $\alpha \in M_0$, the intersection $\alpha \cap \alpha^\perp$ is a p-dimensional totally isotropic subspace of $\mathbb{C}^{p+q}$ and therefore, keeping the notation of **1**, establishes an *isometry* of W_p onto a p-dimensional subspace α_p of W_q.

Denote by $\mathcal{G}r_{\ell-p,\ell}(W_q)$ the flag manifold of all pairs $(L_{\ell-p}, L_\ell)$ consisting of an $(\ell-p)$-plane $L_{\ell-p}$ and an ℓ-plane L_ℓ with $L_{\ell-p} \subset L_\ell \subset W_q$. We obtain a fibration

$$M_0 \ni \alpha \longrightarrow (\alpha \cap W_q, \pi_q(\alpha)) \in \mathcal{G}r_{\ell-p,\ell}(W_q)$$

onto a compact complex manifold of dimension

$$(q + p - \ell)(\ell - p) + p(q - \ell) = \ell(q + p - \ell) - p^2.$$

In fact, the *CR* structure on M_0 is the lift of the complex structure of the base $\mathcal{G}r_{\ell-p,\ell}(\mathbb{C}^{p+q})$ for this projection map. The fiber is $\mathbf{U}(p)$, of real dimension p^2 (equal to the *CR*-codimension of M_0). In fact different choices of an *isometry* of W_p onto $\pi_q(\alpha^\perp \cap \alpha) = [\alpha \cap W_q]^\perp \cap \pi_q(\alpha)$ define different points on the fiber of $(\alpha \cap W_q, \pi_q(\alpha)$ in M_0.

Let us choose an orthonormal basis $e_1, \dots, e_{p+q}$ of $\mathbb{C}^{p+q}$, with $e_1, \dots, e_p$ a basis of W_p. Then

$$\alpha_0 = \langle e_1 + e_{p+1}, \dots, e_p + e_{2p}, e_{2p+1}, \dots, e_{p+\ell} \rangle \in M_0.$$

Its stabilizer in $\mathbf{K}$ is

$$\mathbf{V} = \left\{ \left. \begin{pmatrix} a_{2,2}^{-1} & 0 & 0 & 0 \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix} \right| \begin{matrix} a_{2,2} \in \mathbf{GL}_p(\mathbb{C}), & a_{3,3} \in \mathbf{GL}_{\ell-p}(\mathbb{C}), \\ a_{4,4} \in \mathbf{GL}_{q-\ell}(\mathbb{C}) & a_{2,3} \in \mathbb{C}^{p\times(\ell-p)}, \\ a_{2,4} \in \mathbb{C}^{p\times(q-\ell)}, & a_{3,4} \in \mathbb{C}^{(\ell-p)\times(q-\ell)} \end{matrix} \right\}.$$

Also in this case

$$\mathfrak{v}_n = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{2,3} & Z_{2,4} \\ 0 & 0 & 0 & Z_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \begin{matrix} Z_{2,3} \in \mathbb{C}^{p\times(\ell-p)}, \\ Z_{2,4} \in \mathbb{C}^{p\times(q-\ell)}, \\ Z_{3,4} \in \mathbb{C}^{(\ell-p)\times(q-\ell)} \end{matrix} \right\}$$

is horocyclic and we have

$$\mathfrak{m} = \left\{ \left. \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| Z \in \mathfrak{gl}_p(\mathbb{C}) \right\},$$

where we kept the block notation introduced in the description of $\mathbf{V}$ and $\mathfrak{v}_n$.

To give an explicit expression of the exhausting function, we note that

$$M_- = \{\alpha \in \mathcal{G}r_\ell(\mathbb{C}^{p+q}) \mid \alpha \cap W_p = \{0\}, \ \ \dim_{\mathbb{C}} \alpha \cap W_q = \ell - p\}.$$

In particular, to an element $\alpha \in M_-$ we can associate a linear map

$$\phi_\alpha : W_p \to \pi_q(\alpha)/(\alpha \cap W_q) \simeq \pi_q(\alpha^\perp \cap \alpha)$$

and we can consider $\phi_\alpha^* \phi_\alpha$ as a Hermitian symmetric map on W_p. The condition for α being an element of M_0 is that $\phi_\alpha^*\phi_\alpha = \mathrm{I}_{W_p}$. Hence

$$\Phi(\alpha) = \tfrac{1}{4}\operatorname{trace}([\log(\phi_\alpha^*\phi_\alpha)]^2)$$

is an exhausting function for M_-.

2.2. General orbits of $\mathbf{SU}(p,q)$ in the Grassmannian. Let us consider the orbits of the real form $\mathbf{SU}(p,q)$ of $\mathbf{SL}_{p+q}(\mathbb{C})$ in $\mathcal{G}r_\ell(\mathbb{C}^{p+q})$. As in the previous section, we assume that $p \le q$ and fix $\hbar$ of signature (p,q) on $\mathbb{C}^{p+q}$.

The orbits are classified by the signature of the restriction of $\hbar$ to the ℓ-plane $\alpha \in \mathcal{G}r_\ell(\mathbb{C}^{p+q})$: we have orbits $M_+(a,b,c)$ where a is the dimension of $\ker(\hbar|_\alpha)$, b the positive and c the negative index of inertia of $\hbar|_\alpha$. The integers a,b,c are subject to the conditions

$$0 \le a, b \le p, \ 0 \le c \le q, \ a + b + c = \ell.$$

The maximal compact subgroup $\mathbf{K}_0 \simeq \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ of $\mathbf{SU}(p,q)$ is defined by choosing a p-dimensional subspace W_p and an h-orthogonal q-dimensional subspace W_q with $\hbar > 0$ on W_p and $\hbar < 0$ on W_q. Then $\mathbf{K}_0 \simeq \mathbf{S}(\mathbf{U}(p) \bowtie \mathbf{U}(q))$, with the first factor operating on W_p and the second on W_q. The complexification is $\mathbf{S}(\mathbf{GL}_p(\mathbb{C}) \rtimes \mathbf{GL}_q(\mathbb{C}))$. Accordingly, the orbits of $\mathbf{K}$ in $\mathcal{G}r_\ell(\mathbb{C}^{p+q})$ are characterized by a pair of nonnegative integers (d, e) with $d = \dim_{\mathbb{C}}(\alpha \cap W_p)$, $e = \dim_{\mathbb{C}}(\alpha \cap W_q)$, restricted by the conditions

$$0 \leq d \leq p,\ 0 \leq e \leq q,\ d + e \leq \ell.$$

Then we have

$$M_+(a,b,c) = \{\ell \in \mathcal{G}r_m(\mathbb{C}^{p+q}) \mid \hbar|_\ell \text{ has signature } (b,c)\},$$
$$M_-(a,b,c) = \{\ell \in \mathcal{G}r_m(\mathbb{C}^{p+q}) \mid \dim_{\mathbb{C}}(\ell \cap W_p) = b,\ \ \dim_{\mathbb{C}}(\ell \cap W_q) = c\},$$
$$M_0(a,b,c) = M_+(a,b,c) \cap M_-(a,b,c).$$

We note that if L is a totally isotropic subspace of $\mathbb{C}^{p+q}$, then $W_p + L^\perp = \mathbb{C}^{p+q}$ and $W_q + L^\perp = \mathbb{C}^{p+q}$. Indeed, $(W_p + L^\perp)^\perp = W_p^\perp \cap L = W_q \cap L = \{0\}$, and $(W_q + L^\perp)^\perp = W_q^\perp \cap L = W_p \cap L = \{0\}$. Then by the Grassmann intersection formula

$$\begin{cases} \dim_{\mathbb{C}}(L^\perp \cap W_p) = \dim_{\mathbb{C}}(W_p) + \dim_{\mathbb{C}}(L^\perp) - (p+q) = \dim_{\mathbb{C}}(W_p) - \dim_{\mathbb{C}}(L), \\ \dim_{\mathbb{C}}(L^\perp \cap W_q) = \dim_{\mathbb{C}}(W_q) + \dim_{\mathbb{C}}(L^\perp) - (p+q) = \dim_{\mathbb{C}}(W_q) - \dim_{\mathbb{C}}(L). \end{cases}$$

Thus, for every $a = 0, 1, \ldots, p$ we can describe $M_0(a,b,c)$ as a complex fiber bundle over the minimal orbit $M_0(a,0,0)$ (which is a point in the case $a = 0$). Indeed,

$$M_0(a,b,c) = \{L \oplus U \oplus Q \mid L \in M_0(a,0,0),\ \ U \in \mathcal{G}r_b(L^\perp \cap W_p),\ \ Q \in \mathcal{G}r_c(L^\perp \cap W_q)\}.$$

Let us describe the Mostow fibration. If $\ell \in M_(a,b,c)$, then $U = \ell \cap W_p$ has dimension b, $Q = \ell \cap W_q$ has dimension c and the subspace $U \oplus Q$ is non degenerate. Therefore $\ell \cap (U \oplus Q)^\perp$ is an a-dimensional subspace of ℓ, and the restriction of the projections π_p and π_q to L are linear and injective. Hence L can be viewed as the graph of a linear isomorphism $\phi_L : \pi_p(L) \to \pi_q(L)$. The subspaces $\pi_p(L)$ and $\pi_q(L)$ have Euclidean structures defined by $\hbar|_{\pi_p(L)}$ and $-\hbar|_{\pi_q(L)}$, respectively. Thus we can define the adjoint $\phi_L^* : \pi_q(L) \to \pi_p(L)$ and form the composite $\phi_L^* \circ \phi_L : \pi_p(L) \to \pi_p(L)$. This is Hermitian symmetric and positive definite and therefore there is a Hermitian symmetric positive definite $\sqrt{\phi_L^* \circ \phi_L}$. Then $\phi_L \circ [\sqrt{\phi_L^* \circ \phi_L}]^{-1}$ is an isometry of $\pi_p(L)$ onto $\pi_q(L)$ for the restrictions of $\hbar$ and $-\hbar$, respectively. Hence $L' = \{z + \phi_L \circ [\sqrt{\phi_L^* \circ \phi_L}]^{-1}(z) \mid z \in \pi_p(L)\}$ is totally isotropic and the map $M_-(a,b,c) \ni (L + U + Q) \to (L' + U + Q) \in M_0(a,b,c)$ is the Mostow fibration.

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