Dipartimento di Matematica e fisica

THE GREEN FUNCTION OF A QUASILINEAR p–LAPLACE OPERATOR AND APPLICATION TO THE QUASILINEAR BREZIS–NIRENBERG PROBLEM

Tesi di dottorato in Matematica
XXXIV ciclo

Candidato: Firma
Sabina Angeloni

Relatore di tesi: Firma
Prof. Pierpaolo Esposito

Coordinatore: Firma
Prof. Alessandro Giuliani

Anno Accademico 2021-2022
Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre
Largo San Leonardo Murialdo 1, 00146 Roma

Contents

Chapter 1

Introduction

Given $1 < p < N$ and a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with $x \in \Omega$, we are interested to nonnegative distributional solutions of

$$-\Delta_p G_\lambda - \lambda G_\lambda^{p-1} = 0 \qquad \text{in } \Omega \setminus \{x\}, \tag{1.1}$$

where $\Delta_p(\cdot) = \operatorname{div}\big(|\nabla(\cdot)|^{p-2}\nabla(\cdot)\big)$ is the p-Laplace operator and $\lambda < \lambda_1$. Here, to have (1.1) meaningful, $G_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x\}) \cap W^{1,p-1}(\Omega)$ and λ_1 is the first eigenvalue of $-\Delta_p$ given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{1.2}$$

By elliptic regularity theory (see [30, 33]) we have $G_\lambda \in L^\infty_{\mathrm{loc}}(\Omega\setminus\{x\})$ and then $G_\lambda \in C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x\})$ for some $\alpha \in (0,1)$. If G_λ is singular at x, application of Theorem 1 in [31] guarantees that there exist positive constants C', C'' such that

$$C' \leq \frac{G}{\Gamma} \leq C'' \qquad \text{in } \Omega. \tag{1.3}$$

Here Γ is the fundamental solution of p-laplacian in $\mathbb{R}^N$, that is

$$\Gamma(y) = \frac{C_0}{|y-x|^{\frac{N-p}{p-1}}}, \qquad C_0 = \frac{p-1}{N-p}(N\omega_N)^{-\frac{1}{p-1}} \tag{1.4}$$

ω_N being the measure of the unit ball in $\mathbb{R}^N$. Then, using Theorem 3 in [31], we have that G_λ solves

$$-\Delta_p G_\lambda - \lambda G_\lambda^{p-1} = K\delta_x \qquad \text{in } \Omega,$$

where δ_x is the Dirac measure at x and $K > 0$ is a constant. Without loss of generality, consider the case $K = 1$. Thus we deal with the problem

$$\begin{cases} -\Delta_p G_\lambda - \lambda G_\lambda^{p-1} = \delta_x & \text{in } \Omega \\ G_\lambda \geq 0 & \text{in } \Omega \\ G_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

A *Green function* G_λ is a distributional solution of problem (1.5).

Let us first discuss the case $\lambda = 0$. By a combination of scaling arguments and regularity estimates, Kichenassamy and Veron [23] showed that in the singular situation G differs from Γ by a locally bounded function $H = G - \Gamma$ in Ω (where the index λ is omitted since $\lambda = 0$). The function G, whose existence as a solution of $-\Delta_p G = \delta_x$ can be established in many different ways (see for example [23, 30]), turns out to be unique thanks to a simple argument based on the property $|\nabla H| = o(|\nabla \Gamma|)$ as $y \to x$. As noticed in [23], the same approach via scaling arguments leads to a continuity property of H at x.

The first aim of the present thesis is to establish the Hölder continuity of the so-called *regular part* $H_\lambda = G_\lambda - \Gamma$ at x and to include the case $\lambda < \lambda_1$. Hölder properties will represent a crucial ingredient in the second part of this thesis, when we will assume $\lambda > 0$ to treat the quasilinear Brezis-Nirenberg problem [20] in the most difficult low-dimensional case $N < p^2$.

The main underlying idea to show the regularity result is to consider H_λ as the solution of

$$-\Delta_p(\Gamma + H_\lambda) + \Delta_p \Gamma = \lambda G_\lambda^{p-1} \qquad \text{in } \Omega \tag{1.6}$$

and to apply the Moser iterative scheme in [30] to derive Hölder estimates on H_λ thanks to the coercivity of the operator $-\Delta_p(\Gamma + H) + \Delta_p \Gamma$ in H.
In the case $\lambda \neq 0$ we first establish an existence result for the problem (1.5), where G_λ is found as a solution obtained as limit of approximations (the so-called SOLA, see for example [4, 5]). It will be useful to decompose the Green function as $G_\lambda = \chi\Gamma + \hat{H}_\lambda$, where χ is a cut-off function which is equal to 1 in $B_\xi(x) \subset \Omega$ for some radius $\xi > 0$ and equal to 0 near the boundary.

We have that, in a weak sense, $-\Delta_p(\chi\Gamma) = \delta_x + g$, with g vanishing near x and $g = \Delta_p(\chi\Gamma) = \mathrm{div}(|\nabla(\chi\Gamma)|^{p-2}\nabla(\chi\Gamma))$ away from x. Thus $g \in L^\infty(\Omega)$ and $-\Delta_p G_\lambda + \Delta_p(\chi\Gamma) - \lambda G_\lambda^{p-1} = g$. In particular, one may consider $\hat{H}_\lambda$ as the weak solution of the problem

$$\begin{cases} -\Delta_p(\chi\Gamma + \hat{H}_\lambda) + \Delta_p(\chi\Gamma) = \lambda G_\lambda^{p-1} + g & \text{in } \Omega \\ \hat{H}_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

It is worth noting that $H_\lambda = \hat{H}_\lambda$ in $B_\xi(x)$, being $\chi = 1$. It turns out an integrability condition on $\nabla\hat{H}_\lambda$ which in particular reads as $\nabla\hat{H}_\lambda \in L^p(\Omega)$ for $p \geq 2$ and which always guarantees (even when $1 < p < 2$) $\nabla\hat{H}_\lambda \in L^{\bar{q}}(\Omega)$, $\bar{q} = \frac{N(p-1)}{N-1}$. Since $\nabla\Gamma \in L^q(\Omega)$ for all $q < \bar{q}$, the exponent $\bar{q}$ represents the threshold situation and the assumption $\nabla H_\lambda \in L^{\bar{q}}(\Omega)$ will reveal crucial to use appropriate functions of H_λ as test functions in the weak formulation of (1.6) when running the Moser iterative scheme. The main regularity result in this thesis reads as follows.

Theorem 1.0.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda < \lambda_1$ and $1 < p < N$. Assume*

$$p > \max\left\{2 - \frac{1}{N}, \sqrt{N}, \frac{N}{2}\right\} = \begin{cases} \frac{3}{2} & \text{if } N = 2 \\ \sqrt{3} & \text{if } N = 3 \\ \frac{N}{2} & \text{if } N \geq 4 \end{cases} \tag{1.8}$$

when $\lambda \neq 0$. Problem (1.5) *has a solution G_λ with*

$$\nabla(G_\lambda - \Gamma) \in L^{\bar{q}}(\Omega), \qquad \bar{q} = \frac{N(p-1)}{N-1}, \tag{1.9}$$

which is unique for $p \geq 2$ in the class of solutions satisfying (1.9). *Moreover, the regular part $H_\lambda = G_\lambda - \Gamma$ is Hölder continuous at x_0.*

Let us discuss assumption (1.8) when $\lambda \neq 0$. Since

$$\Gamma \in L^q(\Omega) \qquad \text{for all } 1 \leq q < \bar{q}^*,\ \bar{q}^* = \frac{N(p-1)}{N-p}, \tag{1.10}$$

notice that $p > \frac{N}{2}$ gives $G_\lambda^{p-1} \in L^q(\Omega)$ for some $q > \frac{N}{p}$, a natural condition arising in [30] to prove L^∞-bounds. Condition $G_\lambda^p \in L^1(\Omega)$, ensured by $p > \sqrt{N}$, guarantees that distributional solutions of $-\Delta_p G_\lambda - \lambda G_\lambda^{p-1} = 0$ are in $W^{1,p}(\Omega)$. The technical condition $p > 2 - \frac{1}{N}$ guarantees that $\bar{q} = \frac{N(p-1)}{N-1} > 1$.

As already mentioned, the regularity result established by Theorem 1.0.1 will be crucial to study the quasilinear Brezis-Nirenberg problem in the second part of this thesis.
Given $2 \leq p < N$ and a bounded domain $\Omega \subset \mathbb{R}^N$, we are concerned with the existence of a function u satisfying

$$\begin{cases} -\Delta_p u - \lambda u^{p-1} = u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

where $p^* = \frac{Np}{N-p}$ and λ is a real number in $(0, \lambda_1)$, λ_1 being defined by (1.2). Solutions of (1.11) correspond to critical points of the functional

$$\Phi[u] = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*} - \frac{\lambda}{p}\int_\Omega |u|^p, \quad u \in W_0^{1,p}(\Omega).$$

It is not possible to obtain critical points of Φ using variational methods, because Φ does not in general satisfy the Palais-Smale condition. Indeed p^* is the critical Sobolev exponent and the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is non-compact. Thus other arguments will be needed.
It is known that problem (1.11) admits a solution in the semilinear case for $N \geq 4$ (Brezis and Nirenberg [6]) and also when $p^2 \leq N$ (Guedda and Veron [20]). In the case of lower dimension, the situation could change: according to [6], the semilinear problem with $N = 3$ admits a solution when Ω is a ball for $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. It is possible to extend this result to a general domain Ω with $\lambda \in (\lambda_*, \lambda_1)$, for some $\lambda_* > 0$.
Our goal is to treat the quasilinear case $2 \leq p < N$, following the same ideas as in Brezis-Nirenberg [6] and Guedda-Veron [20]. We define for $\lambda \in \mathbb{R}$

$$S_\lambda = \inf\left\{ \frac{\|\nabla u\|_p^p - \lambda\|u\|_p^p}{\|u\|_{p^*}^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}. \tag{1.12}$$

Then $S = S_0$ is the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. It is known that S is independent of Ω and is never achieved (see [20]). Observing that S_λ decreases from S to 0 as λ ranges in $[0, \lambda_1)$, then the critical parameter λ_* can be defined as

$$\lambda_* = \inf\{\lambda \in (0, \lambda_1) : S_\lambda < S\}. \tag{1.13}$$

Since $S_\lambda = S$ for $\lambda \in [0, \lambda_*)$ and $S_\lambda < S_0$ for $\lambda \in (\lambda_*, \lambda_1)$, then S_λ is not attained for $\lambda \in [0, \lambda_*)$. For what concerns the case $\lambda = \lambda_*$, we will work under the following hypothesis:

$$S_{\lambda_*} \text{ is not achieved.} \tag{1.14}$$

Then the second main result of the thesis reads as follows.

Theorem 1.0.2. *Assume $2 \leq p < N$ with $p > \max\{\sqrt{N}, \frac{N}{2}\}$. Then the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ do hold, where*

(i) there exists $x \in \Omega$ such that $H_\lambda(x, x) > 0$, where $H_\lambda(\cdot, x)$ denotes the regular part of the Green function G_λ with pole at x

(ii) $S_\lambda < S_0$

(iii) S_λ is attained.

Moreover, the implication $(iii) \Rightarrow (i)$ does hold under the assumption (1.14) *and in particular $\lambda_* > 0$.*

Under the assumption (1.14), in the proof of Theorem 1.0.2 we will show that $H_{\lambda_*}(x, x) = 0$ for some $x \in \Omega$, a stronger property which implies the validity of the implication $(iii) \Rightarrow (i)$. Since S_0 is not attained, notice that (1.14) always holds in the case $\lambda_* = 0$ and then $\lambda_* > 0$ follows by the property $H_0(y, y) < 0$ for all $y \in \Omega$. Moreover, since

$$\sup_{y \in \Omega} H_{\lambda_*}(y, y) = \max_{y \in \Omega} H_{\lambda_*}(y, y) = 0 \tag{1.15}$$

and $H_\lambda(y, y)$ is strictly increasing in λ for all $y \in \Omega$ (see Appendix A.3), under the assumption (1.14) the critical parameter λ_* is the first unique value of $\lambda > 0$ attaining (1.15) and can be rewritten as

$$\lambda_* = \sup\{\lambda \in (0, \lambda_1) \colon H_\lambda(y, y) < 0 \text{ for all } y \in \Omega\}.$$

In the rest of this chapter we state the main known results on singular solutions of the p-Laplace equation and then on Brezis-Nirenberg type problem in the case $1 < p^2 \leq N$.

In Chapter 2 we first prove the existence of Green functions. We show also the global L^∞-regularity of H_λ and we discuss some structural properties of problem (1.6) which will be repeatedly used throughout the thesis.
In Chapter 3 we give the proof of Theorem 1.0.1 about the Hölder continuity of H_λ at the pole. This regularity result can be extended to the whole domain Ω when $1 < p \leq 2$, as shown at the end of the chapter.
Chapter 4 deals with existence issues related to the Brezis-Nirenberg type problem and the proof of Theorem 1.0.2.
In the Appendix we present the proof of some technical estimates, involving the operator $-\Delta_p(\Gamma + H) + \Delta_p\Gamma$ in H. Moreover we extend Theorem 1 in [4] to non-homogeneous boundary values and we give the proof of some propositions omitted in Chapter 2.

1.1 Main known results for singular solutions of the p-Laplace equation

This section deals with the known results on the p-Laplace equation (1.5) with $\lambda = 0$. We focus on the article [23] by Kichenassamy and Veron.

A function $u \in W^{1,p}(\Omega)$ is *p-harmonic* in Ω if

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dx = 0 \qquad \forall \varphi \in C_0^1(\Omega).$$

This means that u is a weak solution of the *p-Laplace equation*

$$-\Delta_p u = 0 \qquad \text{in } \Omega, \tag{1.16}$$

which formally corresponds to the Euler-Lagrange equation for

$$\int_\Omega |\nabla u|^p dx.$$

In their paper, Kichenassamy and Veron continued the work of Serrin on the singularity problem associated to the equation (1.16), by improving (1.3). Assuming $0 \in \Omega$ and u p-harmonic in Ω', where $\Omega' = \Omega \setminus \{0\}$, their aim is to describe the behaviour of u near 0 and to find an equation for u in Ω.

1.1.1 The Isotropy Theorem

Let Ω be an open subset of $\mathbb{R}^N$, containing 0, and let $\Omega' = \Omega\backslash\{0\}$. We assume that p is a real number such that $1 < p < N$ and Γ is the fundamental solution for the p-laplacian.[1]

This section deals with the proof of the Isotropy Theorem. The method used by the authors consists of a combination of scaling arguments and regularity estimates, together with a sharp maximum principle.

Theorem 1.1.1. *Let u be a p-harmonic function in Ω' such that $\frac{u(x)}{\Gamma(x)}$ remains bounded in some neighborhood of 0. Then there exists a real number γ such that*

$$u - \gamma\Gamma \in L^\infty_{\mathrm{loc}}(\Omega). \tag{1.17}$$

Moreover, when $\gamma \neq 0$ the following relation holds

$$\lim_{x\to 0} |x|^{\frac{N-p}{p-1}+|\alpha|} D^\alpha (u-\gamma\Gamma)(x) = 0 \tag{1.18}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N \geq 1$, $|\nabla u|^{p-1} \in L^1_{\mathrm{loc}}(\Omega)$ and u satisfies the following equation (even if $\gamma = 0$)

$$-\Delta_p u = \gamma|\gamma|^{p-2}\delta_0 \tag{1.19}$$

in the sense of the distributions in Ω.

Without any loss of generality we may assume[2] $\bar{B}_1 \subset \Omega$. The following elementary estimates will be useful in the sequel.

Lemma 1.1.2. *Under the hypotheses of Theorem 1.1.1, there exist two constants $\alpha = \alpha(N,p) \in (0,1)$ and $C = C(N,p,u) \geq 0$ such that for any x, x' satisfying $0 < |x| \leq |x'| \leq 1$ we have*

$$|\nabla u(x)| \leq C|x|^{-1}\Gamma(x), \tag{1.20}$$

$$|\nabla u(x) - \nabla u(x')| \leq C|x-x'|^\alpha |x|^{-1-\alpha}\Gamma(x). \tag{1.21}$$

[1] We shall frequently write $\Gamma(r)$ for $\Gamma(x)$ whenever $|x| = r$.

[2] For ease of notation, we continue to write B_r to indicate the ball of radius r centered at 0, that is $B_r(0)$.

In the proof of Theorem 1.1.1, we will also use the following strict comparison principle due to Tolksdorf (see [32]).

Lemma 1.1.3. *Let G be a connected open subset of $\mathbb{R}^N$. Assume that u_1 and u_2 are p-harmonic functions in G such that $u_1 \geq u_2$ in G.*
If u_1 and u_2 are not identical in G, then

$$u_1 > u_2 \quad \text{in } G. \tag{1.22}$$

As a consequence we obtain the following sharp maximum principle.

Corollary 1.1.4. *Let G be a connected open subset of $\mathbb{R}^N$ which does not contain 0 and let u be p-harmonic in G. If $\frac{u}{\Gamma}$ or $u - \Gamma$ achieves its maximum in G, then $\frac{u}{\Gamma}$ is constant.*

At this point we are able to prove Theorem 1.1.1.

Proof of Theorem 1.1.1. We define γ^+ and γ^- by

$$\gamma^+ = \limsup_{x\to 0} \frac{u(x)}{\Gamma(x)}, \qquad \gamma^- = \liminf_{x\to 0} \frac{u(x)}{\Gamma(x)}.$$

If $\gamma^+ = \gamma^- = 0$ then $\lim_{x\to 0} \frac{u(x)}{\Gamma(x)} = 0$. So we can assume that $\gamma^+ > 0$. Let β be the supremum of u on ∂B_1. Thus the function $u_\beta = u - \beta$ still satisfies

$$\limsup_{x\to 0} \frac{u_\beta(x)}{\Gamma(x)} = \gamma^+ \quad \text{and} \quad \sup_{x\in\partial B_1} u_\beta(x) = 0$$

and for the sake of simplicity we still call it u. Now we define the following function on $[0, 1]$:

$$\tilde{\gamma}(r) = \sup_{r\leq |x|\leq 1} \frac{u(x)}{\Gamma(x)}.$$

We observe that the function $\tilde{\gamma}$ is nonnegative, being nonincreasing and such that $\tilde{\gamma}(1) = 0$. Moreover, if there exist some $r \in (0, 1]$ and some y with $r < |y| < 1$ such that $\tilde{\gamma}(r) = \frac{u(y)}{\Gamma(y)}$, then $u(x) = \tilde{\gamma}(x)\Gamma(x)$ for any x in $G_r = \{\xi \colon r \leq |\xi| \leq 1\}$. With the aid of Corollary 1.1.4 we obtain that $\tilde{\gamma}$ is constant in G_r. As a consequence, recalling that $\tilde{\gamma}(1) = 0$, we get $\tilde{\gamma} = 0$

on $[r,1]$ and $u=0$ in G_r. Thus $\tilde{\gamma}$ is constructed as a nonincreasing function with the following properties:

$$\left.\begin{array}{l} \tilde{\gamma}(r) = \sup\limits_{|x|=r} \dfrac{u(x)}{\Gamma(x)} \\ \lim\limits_{r\to 0^+} \tilde{\gamma}(r) = \gamma^+ \end{array}\right\}$$

and there exists x_r such that $|x_r| = r$ and

$$\tilde{\gamma}(r) = \frac{u(x_r)}{\Gamma(x_r)}.$$

At this point we define the function u_r on $\Lambda_r = \{\xi \colon 0 < |\xi| < \frac{1}{r}\}$ by

$$u_r(\xi) = \frac{u(r\xi)}{\Gamma(r)}.$$

The function u_r is p-harmonic in Λ_r and we have from Lemma 1.1.2

$$\left.\begin{array}{l} |u_r(\xi)| \le C\left|\dfrac{\Gamma(r\xi)}{\Gamma(r)}\right| \le C\Gamma(\xi), \\ |\nabla u_r(\xi)| \le \dfrac{r}{\Gamma(r)}|(\nabla u)(r\xi)| \le C|\xi|^{\frac{1-N}{p-1}}, \\ |\nabla u_r(\xi) - \nabla u_r(\xi')| \le C|\xi-\xi'|^{\alpha}(\min(|\xi|,|\xi'|))^{\frac{1-N}{p-1}-\alpha}, \end{array}\right\} \tag{1.23}$$

where C does not depend on r. From the Arzelà-Ascoli's Theorem there exist a p-harmonic function v defined in $\mathbb{R}^N \setminus \{0\}$ and a sequence $r_n \to 0$ such that

$$u_{r_n} \to v \quad \text{ in } C^1_{\mathrm{loc}}(R^N \setminus \{0\}).$$

Moreover we have

$$\frac{u_r(\xi)}{\Gamma(\xi)} = \frac{u(r\xi)}{\Gamma(r)\Gamma(\xi)} = \frac{u(r\xi)}{\Gamma(r\xi)}\frac{\Gamma(r\xi)}{\Gamma(r)\Gamma(\xi)} = \frac{u(r\xi)}{\Gamma(r\xi)\Gamma(1)} \le \frac{\gamma^+}{\Gamma(1)}.$$

If we set $\xi_r = \frac{x_r}{r}$, then

$$\frac{u_r(\xi_r)}{\Gamma(\xi_r)} = \frac{\tilde{\gamma}(r)}{\Gamma(1)}.$$

From the compactness of ∂B_1 we can suppose that there exists $\xi_0 \in \partial B_1$ such that $\lim_{r_n\to 0}\xi_{r_n} = \xi_0$. This yields

$$\frac{v(\xi_0)}{\Gamma(\xi_0)} = \frac{\gamma^+}{\Gamma(1)} \quad \text{ and } \quad \frac{v(\xi)}{\Gamma(\xi)} \le \frac{\gamma^+}{\Gamma(1)}.$$

Application of Lemma 1.1.3 gives

$$\frac{v(\xi)}{\Gamma(\xi)} = \frac{\gamma^+}{\Gamma(1)}.$$

Then, recalling the definition of v, we obtain

$$\lim_{r\to 0} u_r(\xi) = v(\xi) = \gamma^+ \frac{\Gamma(\xi)}{\Gamma(1)}, \tag{1.24}$$

uniformly on every compact subset of $\mathbb{R}^N \setminus \{0\}$. In particular, if we set $x = r\xi$ with $\xi \in \partial B_1$, we have

$$\lim_{x\to 0} \frac{u(x)}{\Gamma(x)} = \gamma^+ = \gamma.$$

In order to prove the boundedness of $u - \gamma\Gamma$ we consider, for $\varepsilon > 0$, the following p-harmonic functions in $B_1 \setminus \{0\}$:

$$v_\varepsilon^+(x) = (\gamma + \varepsilon)\Gamma(x) - (\gamma + \varepsilon)\Gamma(1) + \sup_{x\in\partial B_1} u(x), \tag{1.25}$$

$$v_\varepsilon^-(x) = (\gamma - \varepsilon)\Gamma(x) - (\gamma - \varepsilon)\Gamma(1) + \inf_{x\in\partial B_1} u(x). \tag{1.26}$$

By definitions (1.25) and (1.26) we have that $(u - v_\varepsilon^+)^+ = 0 = (v_\varepsilon^- - u)^-$ on ∂B_1. Moreover $v_\varepsilon^-(x) \le u(x) \le u_\varepsilon^+(x)$ in $\bar{B}_1 \setminus \{0\}$. Letting ε to 0 we obtain (1.17). Indeed

$$\inf_{x\in\partial B_1} u(x) - \gamma\Gamma(1) \le u - \gamma\Gamma \le \sup_{x\in\partial B_1} u(x) - \gamma\Gamma(1).$$

At this point we want to show (1.18). Using (1.23) and (1.24) we have

$$\lim_{r\to 0} \nabla u_r(\xi) = \frac{\gamma}{\Gamma(1)} \nabla\Gamma(\xi), \tag{1.27}$$

uniformly on every compact subset of $\mathbb{R}^N \setminus \{0\}$. Settin $x = r\xi$ with $|\xi| = 1$, we get (1.18) forn any α such that $|\alpha| = 1$. From (1.27) we also get that there exists r_0 such that ∇u_r never vanishes on $\bar{G} = \{\xi\colon \frac{1}{2} \le |\xi| \le 2\}$ for $0 < r \le r_0$. Therefore u_r satisfies a nondegenerate elliptic equation in G and is C^∞. Using the same device as in Lemma 1.1.2 we deduce

$$|D^\alpha u_r(\xi)| \le |\xi|^{\frac{p-N}{p-1} - |\alpha|}$$

for any multi-indices α such that $|\alpha| > 2$. This implies (1.18).

In order to prove (1.19), we observe that Green formula gives

$$\int_{|x|>r} \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dx = -\int_{|x|=r} \varphi|\nabla u|^{p-2}u_\nu dS$$

for any $\varphi \in C_0^1(\Omega)$ and $0 < r < 1$. As

$$\nabla u(x) \sim -\gamma(N\omega_N)^{-\frac{1}{p-1}}|x|^{\frac{1-N}{p-1}}\frac{x}{|x|} \quad \text{as } x \to 0,$$

we finally get

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle dx = \gamma|\gamma|^{p-2}\varphi(0).$$

This concludes the proof of Theorem 1.1.1. □

Remark 1.1.5. It is possible to prove something stronger than the boundedness of $u-\gamma\Gamma$: we are able to show that $u(x)-\gamma\Gamma(x)$ admits a limit as x tends to 0. In order to prove this result, we look for the point where the bounded function $u-\gamma\Gamma$ achieves its supremum on $\bar{B}_1$. If it achieves this supremum in B_1 then it is constant from Lemma 1.1.3 and everything is done. So we can suppose that it is not constant. Then we have two different possibilities: either this supremum is achieved at 0 and

$$\sup_{x\in\bar{B}_1}(u(x)-\gamma\Gamma(x)) = \limsup_{x\to 0}(u(x)-\gamma\Gamma(x)), \tag{1.28}$$

or it is achieved for $|x| = 1$. We will discuss the cases separately.
Assume (1.28) and let λ be the value of this supremum. We define

$$\lambda(r) = \sup_{r\le|x|\le 1}(u(x)-\gamma\Gamma(x)) = \sup_{|x|=r}(u(x)-\gamma\Gamma(x)),$$

the function $\lambda(r)$ being non-increasing. Moreover there exists x_r such that $|x_r| = r$ and $\lambda(r) = u(x_r)-\gamma\Gamma(x_r)$. At this point, for $\xi \in \Lambda_r$ and $0 < r \le 1$ in such a way that $0 < r|\xi| \le 1$, we set

$$v_r(\xi) = u(r\xi) - \gamma\Gamma(r). \tag{1.29}$$

The function $v_r(\xi)$ is p-harmonic and bounded on any compact subset of Λ_r as we have $\Gamma(r\xi) = \Gamma(r)\Gamma(\xi)$ and

$$\gamma\Gamma(\xi)+C_1-|u(r\xi)-\gamma\Gamma(r\xi)| \le v_r(\xi) \le \gamma\Gamma(\xi)+C_2+|u(r\xi)-\gamma\Gamma(r\xi)|. \tag{1.30}$$

We observe that the definition of Γ implies the improvement of estimates in Lemma 1.1.2, that is

$$|\nabla u(x)| \le C|x|^{-1},$$

$$|\nabla u(x) - \nabla u(x')| \le C|x - x'|^{\alpha}|x|^{-1-\alpha},$$

for $0 < |x| \le |x'| \le 1$. Returning to (1.29) we have

$$|\nabla v_r(\xi)| \le C|\xi|^{-1},$$

$$|\nabla v_r(\xi) - \nabla v_r(\xi')| \le C|\xi - \xi'|^{\alpha}|\xi|^{-1-\alpha},$$

for $0 < |\xi| \le |\xi'| \le \frac{1}{r}$. As a consequence, the set of functions $\{v_r\}$ is relatively compact in the C^1 topology of any compact subset of Λ_r: there exist $v \in C^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$ and a sequence $r_n \to 0$ such that v_{r_n} converges to v in this topology. Moreover we can assume that $\xi_{r_n} = \frac{x_{r_n}}{r_n}$ converges to some $\xi_0 \in \partial B_1$. As we have $u(r\xi) - \gamma\Gamma(r\xi) \le \lambda$ and $u(r_n\xi_{r_n}) - \gamma\Gamma(r_n\xi_{r_n}) \to \lambda$ as $n \to +\infty$, we deduce

$$v(\xi) \le \gamma\Gamma(\xi) + \lambda \quad \text{and} \quad v(\xi_0) = \gamma\Gamma(\xi_0) + \lambda. \tag{1.31}$$

Application of Corollary 1.1.4 gives that $v = \gamma\Gamma + \lambda$ and v_r converges to $\gamma\Gamma + \lambda$ as r goes to 0 in the C^1 topology of any compact subset of $\mathbb{R}^N \setminus \{0\}$. Returning to (1.29), this means that

$$\lim_{x \to 0}(u(x) - \gamma\Gamma(x)) = \lambda. \tag{1.32}$$

Assume now $\sup_{x\in B_1}(u(x) - \gamma\Gamma(x)) = \sup_{|x|=1}(u(x) - \gamma\Gamma(x))$. We perform the scaling transformation (1.29) and there exists a p-harmonic function v and a sequence $r_n \to 0$ such that v_{r_n} converges to v in the C^1 topology of any compact subset of $\mathbb{R}^N \setminus \{0\}$. Moreover, from (1.30), v satisfies

$$\tilde{C}_1 \le v(\xi) - \gamma\Gamma(\xi) \le \tilde{C}_2.$$

We look at the points where $v - \gamma\Gamma$ achieves its supremum in $\mathbb{R}^N \setminus \{0\}$. If this supremum is achieved at some ξ_0, then $v - \gamma\Gamma$ is equal to some constant λ. Returning to (1.29), we obtain

$$\lim_{r_n \to 0}(u(r_n\xi) - \gamma\Gamma(r_n\xi)) = \lambda$$

in the $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ topology. For $\varepsilon > 0$ fixed, there exists n_0 such that for $n \geq n_0$ we have

$$\gamma\Gamma(r_n\xi) + \lambda - \varepsilon \leq u(r_n\xi) \leq \gamma\Gamma(r_n\xi) + \lambda + \varepsilon,$$

for $\xi \in \partial B_1$. Application of the maximum principle in $\{x \colon r_n < |x| < r_{n_0}\}$ yields

$$\gamma\Gamma(x) + \lambda - \varepsilon \leq u(x) \leq \gamma\Gamma(x) + \lambda + \varepsilon.$$

As ε is arbitrary we obtain again (1.32). So we are left with the case where the supremum of $v - \gamma\Gamma$ is achieved either at 0 or at infinity. In the second case we perform the inversion ϑ of $\mathbb{R}^N \setminus \{0\}$ defined by $\vartheta(x) = \frac{x}{|x|^2}$ which leaves equation invariant and exchanges 0 and infinity. Let $\tilde{v}$ be v or $v \circ \vartheta$ and assume that $\tilde{v}$ achieves its supremum ν at 0. If $\tilde{v} = v$ then we have

$$\lim_{\xi \to 0}(v(\xi) - \gamma\Gamma(\xi)) = \nu,$$

which implies

$$\lim_{\xi \to 0} \lim_{r_n \to 0}(u(r_n\xi) - \gamma\Gamma(r_n\xi)) = \nu.$$

With the maximum principle as above we get (1.32). If $\tilde{v} = v \circ \vartheta$, then

$$\lim_{|\xi| \to +\infty}(v(\xi) - \gamma\Gamma(\xi)) = \nu,$$

which means

$$\lim_{|\xi| \to +\infty} \lim_{r_n \to 0}(u(r_n\xi) - \gamma\Gamma(r_n\xi)) = \nu.$$

For $\varepsilon > 0$ there exists $K > 0$ such that for $|\xi| = K$ we have

$$\nu - \varepsilon \leq \lim_{r_n \to 0}(u(r_n\xi) - \gamma\Gamma(r_nK) \leq \nu + \varepsilon.$$

With the aid of the maximum principle as before we deduce (1.32).

1.1.2 The singular Dirichlet problem

We assume that Ω is a bounded open subset of $\mathbb{R}^N$, containing 0, with a regular boundary $\partial\Omega$. This section deals with the problem of finding a distributional solution of the following problem

$$\begin{cases} -\Delta_p u = K\delta_0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.33}$$

Here $1 < p < N$, $K \in \mathbb{R}$ and $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. We are going to prove the following result.

Theorem 1.1.6. *There exists a unique function $u \in C^{1,\alpha}(\Omega')$ such that $|\nabla u|^{p-1} \in L^1(\Omega)$, $\nabla u \in L^p(\Omega \setminus B_r)$ for $r > 0$ small enough and*

$$\frac{u}{\Gamma} \in L^\infty(\Omega), \tag{1.34}$$

satisfying (1.33)*. Moreover the following estimates hold:*

$$u - |K|^{\frac{1}{p-1}} \mathrm{sgn}(K)\Gamma \in L^\infty(\Omega), \tag{1.35}$$

$$\nabla(u - |K|^{\frac{1}{p-1}} \mathrm{sgn}(K)\Gamma) = o(|x|^{\frac{1-N}{p-1}}), \tag{1.36}$$

$$D^\alpha(u - |K|^{\frac{1}{p-1}} \mathrm{sgn}(K)\Gamma) = o(|x|^{\frac{p-N}{p-1} - |\alpha|}) \tag{1.37}$$

if $K \neq 0$ for any multi-indices α with $|\alpha| \geq 1$.

Proof. (*Uniqueness*) Let u_1 and u_2 be solutions of (1.33). From Theorem 1.1.1 they both satisfy

$$\lim_{x \to 0} \frac{u_i(x)}{\Gamma(x)} = |K|^{\frac{1}{p-1}} \mathrm{sgn}(K).$$

Moreover they also satisfy (1.35) and (1.37), so we have

$$u_1 - u_2 \in L^\infty(\Omega),$$

$$\nabla(u_1 - u_2) = o(|x|^{\frac{1-N}{p-1}}) \quad \text{as } x \to 0.$$

From the equation we have, for all $r > 0$ small enough,

$$\int_{\Omega \setminus B_r} \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla(u_1 - u_2) \rangle dx$$
$$= -\int_{|x|=r} (u_1 - u_2)(|\nabla u_1|^{p-2} \partial_\nu u_1 - |\nabla u_2|^{p-2} \partial_\nu u_2) dS, \tag{1.38}$$

and the right-hand side of (1.38) goes to 0 as $r \to 0$. As for the left-hand side, it is greater than

$$\begin{cases} C|\nabla(u_1 - u_2)|^p & \text{if } p \geq 2, \\ C(1 + |\nabla u_1| + |\nabla u_2|)^{p-2} |\nabla(u_1 - u_2)|^2 & \text{if } 1 < p \leq 2. \end{cases} \tag{1.39}$$

Thus $\nabla(u_1 - u_2) = 0$ a.e. in Ω and so $u_1 = u_2$ a.e. in Ω.

(*Existence*) If $K = 0$, (1.33) is the classical Dirichlet problem. For this reason we assume $K \neq 0$ and, without any loss of generality, $K > 0$. For $\varepsilon > 0$ small enough, let u_ε be the solution of

$$\begin{cases} -\Delta_p u_\varepsilon = 0 & \text{in } \Omega \setminus B_\varepsilon \\ u_\varepsilon = K^{\frac{1}{p-1}}\Gamma(\varepsilon) & \text{on } \partial B_\varepsilon \\ u_\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{1.40}$$

Such a u_ε can be obtained by minimizing the functional $\int_{\Omega\setminus B_\varepsilon} |\nabla v|^p dx$ in $W^{1,p}(\Omega \setminus B_\varepsilon)$, with the boundary conditions of (1.40). We define

$$\Lambda = K^{\frac{1}{p-1}} \sup_{\partial\Omega} |\Gamma| + \sup_{\partial\Omega} |g| > 0.$$

Application of the maximum principle to the functions u_ε, $K^{\frac{1}{p-1}}\Gamma - \Lambda$ and $K^{\frac{1}{p-1}}\Gamma + \Lambda$ yields

$$K^{\frac{1}{p-1}}\Gamma - \Lambda \leq u_\varepsilon \leq K^{\frac{1}{p-1}}\Gamma + \Lambda \quad \text{in } \Omega \setminus B_\varepsilon. \tag{1.41}$$

If G is any compact subset of Ω', then there exists $\eta > 0$ small enough such that $G \subset \Omega \setminus B_\eta$. For $0 < \varepsilon < \frac{1}{2}\eta < \eta$, (1.41), (1.22) and [25] imply the following estimates

$$\|\nabla u_\varepsilon\|_{C^{1,\alpha}(G)} \leq C, \tag{1.42}$$

$$\|\nabla u_\varepsilon\|_{L^p(\Omega\setminus B_\eta)} \leq C, \tag{1.43}$$

where C does not depend on ε. Then there exists a subsequence ε_n going to 0 and a function u which is p-harmonic in Ω' such that

$$u_{\varepsilon_n} \to u \quad \text{in } C^1_{\mathrm{loc}}(\Omega') \qquad \text{and} \qquad \nabla u_{\varepsilon_n} \rightharpoonup \nabla u \quad \text{in } L^p_{\mathrm{loc}}(\bar{\Omega} \setminus \{0\}).$$

Thus the boundary condition on $\partial\Omega$ is preserved and u also satisfies (1.43) in Ω'. Application of Theorem 1.1.1 gives

$$-\Delta_p u = K\delta_0 \qquad \text{in } \Omega,$$

in the sense of distribution. Hence u is the solution of (1.33) and u_ε converges to u. Moreover the following properties hold

$$\begin{cases} (1 + |\nabla u| + |\nabla\Gamma|)^{p-2}|\nabla(u - |K|^{\frac{1}{p-1}}\mathrm{sgn}(K)\Gamma)|^2 \in L^1(\Omega) & \text{if } 1 < p \leq 2, \\ \nabla(u - |K|^{\frac{1}{p-1}}\mathrm{sgn}(K)\Gamma) \in L^p(\Omega) & \text{if } p \geq 2, \end{cases}$$

in virtue of (1.39).

This concludes the proof of Theorem 1.1.6. □

1.2 Main results for Brezis-Nirenberg type problem when $p^2 \leq N$

This section deals with the existence result related to problem

$$\begin{cases} -\Delta_p u - \lambda u^{p-1} = u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.44}$$

with $1 < p^2 \leq N$ and $0 < \lambda < \lambda_1$, proved by Guedda and Veron in [20].

Theorem 1.2.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. Assume $1 < p^2 \leq N$. Then problem (1.44) admits a solution in $W_0^{1,p}(\Omega)$ for any $\lambda \in (0, \lambda_1)$.*

In order to prove Theorem 1.2.1, some preliminary regularity results are needed. Then the proof follows ideas and techniques of Brezis-Nirenberg [6], Aubin [2] and Trudinger [34].

1.2.1 Uniform $C^{1,\alpha}$ estimates

Let Ω be as in Theorem 1.2.1 and $p > 1$. Consider the following problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.45}$$

f being a given function defined on Ω. If the data are not regular enough, recalling the definition of the operator Δ_p, it is traditional to approximate (1.45) by the following non-degenerate problem

$$\begin{cases} -\operatorname{div}((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.46}$$

where ε is a positive constant.

Proposition 1.2.2. *Assume $f_\varepsilon \in C^1(\bar{\Omega})$ and $u_\varepsilon \in C^{3-\delta}(\bar{\Omega})$. Then there exist $\alpha = \alpha(p, N) \in (0, 1)$ and $C = C(p, N, \Omega, \|f_\varepsilon\|_\infty) > 0$ such that*

$$\|u_\varepsilon\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \qquad \textit{for any } \varepsilon \in (0, 1).$$

Proof. See Proposition 1.1 in [20]. □

Next consider the problem

$$\begin{cases} -\Delta_p u + K(x)|u|^{p-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.47}$$

Proposition 1.2.3. *Assume $1 < p < N$, $f \in L^{\frac{N}{p}}(\Omega)$ and $K \in L^{\frac{N}{p}}(\Omega)$. There exists $C_t = C_t(t, p, N, \Omega, K, \|f\|_{\frac{N}{p}})$ such that*

$$\|u\|_{L^t(\Omega)} \leq C_t \tag{1.48}$$

for any $u \in W_0^{1,p}(\Omega)$ solution of (1.47) and $t \in [1, +\infty)$.

Proof. See Proposition 1.2 in [20]. □

Remark 1.2.4. Notice that (1.48) still holds for solutions $u_\varepsilon \in W_0^{1,p}(\Omega)$ of

$$-\mathrm{div}((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon) + K(x)|u_\varepsilon|^{p-2}u_\varepsilon = f \quad \text{in } \Omega,$$

and C_t is independent of $\varepsilon \in (0, 1]$.

Proposition 1.2.5. *Assume $1 < p < N$ and $f \in L^s(\Omega)$ for some $s > \frac{N}{p}$. There exists $C = C(N, p, |\Omega|)$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^s(\Omega)}^{\frac{1}{p-1}}$$

for any $u \in W_0^{1,p}(\Omega)$ solution of (1.45).

Proof. See Proposition 1.3 in [20]. □

As an application we get Corollary 1.2.6, involving the problem

$$\begin{cases} -\Delta_p u = g(\cdot, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.49}$$

which represents the main regularity result of this section.

Corollary 1.2.6. *Assume $1 < p < N$. Let g be a continuous function in $\bar{\Omega} \times \mathbb{R}$ which satisfies*

$$|g(x,r)| \leq C|r|^{p^*-1} + D \quad \textit{for all } (x,r) \in \bar{\Omega} \times \mathbb{R}, \tag{1.50}$$

C and D being real constants. If $u \in W_0^{1,p}(\Omega)$ solves (1.49), then $u \in C^{1,\alpha}(\bar{\Omega})$.

Proof. Setting $K(x) = \operatorname{sign}(u)g(x,u)/(1+|u|^{p-1})$, we have from (1.50) that

$$|K(x)| \leq \frac{C|u|^{p^*-1} + D}{1+|u|^{p-1}} \leq C'|u|^{p^*-p} + D',$$

C', D' being constants. As $p^* - p = \frac{p^2}{N-p}$ and $u \in L^{p^*}(\Omega)$ we deduce that $K \in L^{\frac{N}{p}}(\Omega)$. The equation in (1.49) rewrites as

$$-\Delta_p u = K(x)|u|^{p-2}u + \operatorname{sign}(u)K(x).$$

Application of Proposition 1.2.3 yields $u \in \bigcap_{1 \leq t < +\infty} L^t(\Omega)$, in view of $K|u|^{p-2}u + \operatorname{sign}(u)K \in \bigcap_{1 \leq t < +\infty} L^t(\Omega)$. Then, from Proposition 1.2.5 we deduce that $u \in L^\infty(\Omega)$. Finally Proposition 1.2.2 gives the $C^{1,\alpha}$-regularity of u in $\bar{\Omega}$. $\square$

At this point we are able to prove the following nonexistence result.

Corollary 1.2.7. *Assume $1 < p < N$. Let Ω be starshaped with respect to some point. Then the equation*

$$-\Delta_p u = |u|^{p^*-1} \quad \textit{in } \Omega,$$

admits no nonzero solution in $W_0^{1,p}(\Omega)$.

Proof. As $u \in W_0^{1,p}(\Omega)$, $u \geq 0$ in Ω and $u \in L^\infty(\Omega)$ from Corollary 1.2.6. If u is nonzero then $u > 0$ in Ω and $\partial_\nu u > 0$ on $\partial\Omega$ (see [32] or [36]), which implies $\langle x, \nu \rangle = 0$ on $\partial\Omega$ in view of the Pohozaev identity for the p-laplacian (see [12]). Hence Ω cannot be bounded otherwhile we consider the smallest ball with center 0 containing Ω; such a ball is tangent to $\partial\Omega$ at x_0 and $\langle x_0, \nu_{x_0} \rangle > 0$, a contradiction. $\square$

1.2.2 Proof of the existence theorem

We are going to prove Theorem 1.2.1. We define for $\lambda \in \mathbb{R}$

$$S_\lambda = \inf\left\{\int_\Omega (|\nabla u|^p - \lambda |u|^p)dx \colon u \in W_0^{1,p}(\Omega), \int_\Omega |u|^{p^*}dx = 1\right\}. \tag{1.51}$$

Then

$$S_0 = S = \inf\left\{\int_\Omega |\nabla u|^p dx \colon u \in W_0^{1,p}(\Omega), \int_\Omega |u|^{p^*}dx = 1\right\}$$

is the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Lemma 1.2.8. *S is independent of Ω and is never achieved.*

Proof. The fact that S is independent of Ω is clear as $\frac{\|\nabla u\|_p}{\|u\|_{p^*}}$ is independent of the scaling transformation $k \mapsto u_k(x) = u(kx)$.
Assume by contradiction that S is achieved by some $u \in W_0^{1,p}(\Omega)$. We can suppose $u \geq 0$ in Ω and, if B is a ball containing Ω, we define $\hat{u}$ in B by

$$\hat{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } B \setminus \Omega. \end{cases}$$

As $\hat{u}$ achieves S in B there exists a Lagrange multiplier μ such that

$$-\Delta_p \hat{u} = \mu \hat{u}^{p^*-1} \quad \text{in } B$$

and $\hat{u} \in W_0^{1,p}(B)$. As $\int_B \hat{u}^{p^*} = 1$ and $\int_B |\nabla \hat{u}|^p = \mu$, we deduce that $\mu > 0$ ($\mu = S$) which is impossible from Corollary 1.2.7. □

Remark 1.2.9. When Ω is replaced by $\mathbb{R}^N$ then S is achieved by the functions

$$U_a(x) = \left(Na\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}} (a + |x|^{\frac{p}{p-1}})^{\frac{p-N}{p}} \tag{1.52}$$

for some $a > 0$ (see [19]). Moreover the functions U_a are the only positive solutions in $D^{1,p}(\mathbb{R}^N)$ of[3]

$$-\Delta_p u = u^{p^*-1} \quad \text{in } \mathbb{R}^N,$$

as proved by Damascelli-Merchán-Sciunzi and Vétois (see [9] and [37]).

[3]We recall that $D^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}$.

Lemma 1.2.10. *Let $1 < p^2 \leq N$. Then for any $\lambda > 0$ we have $S_\lambda < S$.*

Proof. Following Aubin's method [2] we define

$$Q_\lambda(u) = \frac{\|\nabla u\|_p^p - \lambda \|u\|_p^p}{\|u\|_{p^*}^p}$$

for $u \in W_0^{1,p}(\Omega)$, $u \neq 0$. We assume $0 \in \Omega$ and we define

$$u_\varepsilon(x) = \frac{\Phi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \tag{1.53}$$

for some $\varepsilon \in (0,1]$ and $\Phi \in C_0^\infty(\Omega)$, $0 \leq \Phi \leq 1$ and $\Phi = 1$ in some neighborhood of 0. The idea is to estimate $Q_\lambda(u_\varepsilon)$.

Step 1. We claim that

$$\|\nabla u\|_p^p = K\varepsilon^{-\frac{N-p}{p}} + O(1), \tag{1.54}$$

where $K = K(N,p) > 0$. From (1.53) we have

$$\nabla u_\varepsilon(x) = \frac{\nabla \Phi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} + \frac{p-N}{p-1} \frac{x\Phi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N}{p}} |x|^{\frac{p-2}{p-1}}}.$$

As $\Phi = 1$ in a neighborhood of 0 we obtain

$$\int_\Omega |\nabla u_\varepsilon|^p = \left(\frac{N-p}{p-1}\right)^p \int_\Omega \frac{|x|^{\frac{p}{p-1}} \Phi^p(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} dx + O(1).$$

Then, writing $\Phi^p = 1 + \Phi^p - 1$, we get (1.54) with

$$K = \left(\frac{N-p}{p-1}\right)^p \int_{\mathbb{R}^N} \frac{|x|^{\frac{p}{p-1}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} dx = L\|\nabla U_1\|_p^p, \tag{1.55}$$

U_1 being defined as in (1.52) and $L = L(N,p)$.

Step 2. We claim that

$$\|u_\varepsilon\|_{p^*}^p = \frac{K}{S}\varepsilon^{-\frac{N-p}{p}} + O(1), \tag{1.56}$$

K being as in (1.55). From (1.53) we have

$$\begin{aligned}
\int_\Omega u_\varepsilon^{p^*} &= \int_\Omega \frac{\Phi^{p^*}(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} dx \\
&= \int_\Omega \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} + \int_\Omega \frac{\Phi^{p^*}(x) - 1}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} dx \\
&= \varepsilon^{-\frac{N}{p}} \int_{\mathbb{R}^N} \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^N} + O(1),
\end{aligned}$$

and finally we get (1.56) as

$$\int_{\mathbb{R}^N} \frac{dx}{(1+|x|^{\frac{p}{p-1}})^N} = L^{\frac{p^*}{p}} \|U_1\|_{p^*}^{p^*} \quad \text{and} \quad \|\nabla U_1\|_p^p = S\|U_1\|_{p^*}^p.$$

Step 3. We claim that

$$\|u_\varepsilon\|_p^p = K_1 \varepsilon^{-\frac{N-p^2}{p}} + O(1) \quad \text{if } 1 < p^2 < N, \tag{1.57}$$

$$\|u_\varepsilon\|_p = K_1 \log \frac{1}{\varepsilon} + O(1) \quad \text{if } p^2 = N, \tag{1.58}$$

where $K_1 = K_1(N, p) > 0$. In the first case we have

$$\begin{aligned} \int_\Omega u_\varepsilon^p &= \int_\Omega \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{N-p}} + \int_\Omega \frac{\Phi^p(x) - 1}{(\varepsilon + |x|^{\frac{p}{p-1}})^{N-p}} dx \\ &= \int_{\mathbb{R}^N} \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{N-p}} + O(1), \end{aligned}$$

and we get the desired result with

$$K_1 = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^{\frac{p}{p-1}})^{N-p}}.$$

In the second case we have

$$\int_\Omega u_\varepsilon^p = O(1) + \int_\Omega \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{p(p-1)}}.$$

Setting $I(\varepsilon) = \int_\Omega \frac{dx}{(\varepsilon+|x|^{\frac{p}{p-1}})^{p(p-1)}}$, it is clear that there exist $0 < R_1 < R_2$ such that

$$\int_{|x|\le R_1} \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{p(p-1)}} \le I(\varepsilon) \le \int_{|x|\le R_2} \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{p(p-1)}}.$$

Moreover, for a fixed $R > 0$ we have

$$\int_{|x|\le R} \frac{dx}{(\varepsilon + |x|^{\frac{p}{p-1}})^{p(p-1)}} = \omega_N \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \frac{s^{p^2-1}}{(1 + s^{\frac{p}{p-1}})^{p(p-1)}} dx,$$

ω_N being the measure of the unit sphere in $\mathbb{R}^N$. Hence $I(\varepsilon) = \frac{\omega_N(p-1)}{p} \log \frac{1}{\varepsilon} + O(1)$, which gives (1.58).

Step 4. From Steps 1-3 we obtain

$$Q_\lambda(u_\varepsilon) = \begin{cases} S - \lambda \frac{K_1}{K} \varepsilon^{p-1} + O(\varepsilon^{\frac{N-p}{p}}) & \text{if } 1 < p^2 < N \\ S - \lambda \frac{K_1}{K} \varepsilon^{p-1} \log \frac{1}{\varepsilon} + O(\varepsilon^{p-1}) & \text{if } p^2 = N. \end{cases}$$

As $S_\lambda \leq Q_\lambda(u_\varepsilon) < S$ we get our result. □

Lemma 1.2.11. *If* $0 < S_\lambda < S$, *then* S_λ *is achieved.*

Proof. Following Aubin [2] and Trudinger [34] we consider the following functional on $W_0^{1,p}(\Omega) \setminus \{0\}$

$$Q_\lambda^t(u) = \frac{\|\nabla u\|_p^p - \lambda \|u\|_p^p}{\|u\|_t^p}$$

for $p \leq t \leq p^*$, and we set

$$S^t = \inf\big\{Q_\lambda^t(u) \colon u \in W_0^{1,p}(\Omega) \setminus \{0\}\big\}$$

which always exists.

Step 1. For $p \leq t < p^*$ there exists $u_t \in W_0^{1,p}(\Omega)$ satisfying

$$\begin{cases} -\Delta_p u_t = \lambda u_t^{p-1} + S^t u_t^{t-1} & \text{in } \Omega \\ u_t > 0, \quad Q_\lambda^t(u_t) = S^t, \quad \|u_t\|_t = 1. \end{cases} \tag{1.59}$$

Indeed the compactness of the embedding $W_0^{1,p}(\Omega)$ into $L^t(\Omega)$ implies that the infimum of Q_λ^t is achieved on the unit sphere in $L^t(\Omega)$ by a nonnegative function $u_t \in W_0^{1,p}(\Omega)$ which solves

$$-\Delta_p u_t - \lambda u_t^{p-1} = C u_t^{t-1}$$

and $C = S^t$. From Vazquez strict maximum principle we get $u_t > 0$ in Ω.

Step 2. The function $t \mapsto S^t$ is continuous on the left from $[p, p^*]$ into $(0, +\infty)$.

In order to prove this claim, first we notice that $S^t > 0$ as $S_\lambda > 0$. For w fixed in $W_0^{1,p}(\Omega) \setminus \{0\}$, we have that $t \mapsto Q_\lambda^t(w)$ is continuous from $[p, p^*]$ into $(0, +\infty)$. Hence $t \mapsto S^t$ is upper semi-continuous, that is $S^t \geq \limsup_{t' \to t} S^{t'}$. Set $\varepsilon > 0$ and $\Phi \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$Q_\lambda^{t'}(\Phi) < S^{t'} + \varepsilon, \quad S^t \leq Q_\lambda^t(\Phi),$$

where $p \leq t' < t \leq p^*$. As

$$\|\Phi\|_{t'} \leq \|\Phi\|_t |\Omega|^{\frac{1}{t'} - \frac{1}{t}},$$

we get

$$Q_\lambda^{t'}(\Phi) \geq Q_\lambda^t(\Phi) |\Omega|^{p(\frac{1}{t} - \frac{1}{t'})}.$$

Thus $S^{t'} + \varepsilon \geq S^t |\Omega|^{p(\frac{1}{t} - \frac{1}{t'})}$, which implies the continuity on the left as

$$\liminf_{t' \nearrow t} S^{t'} + \varepsilon \geq S^t \geq \limsup_{t' \nearrow t} S^{t'}.$$

Step 3. The set of functions $u_t \in W_0^{1,p}(\Omega)$ satisfying (1.59) is bounded in $W_0^{1,p}(\Omega)$ independently of $t \in [p, p^*)$.
Indeed, it is clear that the L^p norm of u_t is bounded independently of $t \in [p, p^*)$ as

$$\|u_t\|_p \leq \|u_t\|_t |\Omega|^{\frac{1}{p} - \frac{1}{t}} \leq C,$$

and

$$\|\nabla u_t\|_p^p \leq \lambda C^p + S^t.$$

Moreover if we fix u_0 in $W_0^{1,p}(\Omega) \setminus \{0\}$, then $S^t \leq Q_\lambda^t(u_0)$. As $t \mapsto Q_\lambda^t(u_0)$ is continuous, there exists M such that

$$S^t \leq \max_{p \leq t \leq p^*} Q_\lambda^t(u_0) = M, \quad t \in [p, p^*],$$

which implies the uniform boundedness of $\{u_t\}$ in $W_0^{1,p}(\Omega)$.

Step 4. The set of functions $\{u_t\}$ is bounded in $C^{1,\alpha}(\bar{\Omega})$.
In order to prove this claim, we set $K_t = S^t - u_t^{t-p}$ in such a way that (1.59) rewrites as

$$-\Delta_p u_t - K_t u_t^{p-1} = 0.$$

Step 3 and Propositions 1.2.2 - 1.2.5 yield that $u_t \in C^{1,\alpha}(\bar{\Omega})$. Moreover, from Step 1 we get

$$\left(\frac{p}{p+\beta}\right)^p (\beta+1) \int_\Omega |\nabla u_t^{1+\frac{\beta}{p}}|^p = \lambda \int_\Omega u_t^{p+\beta} + D^t \int_\Omega u_t^{t+\beta}$$

for any $\beta > 0$. Setting $v = u_t^{1+\frac{\beta}{p}}$, then $\int_\Omega u_t^{t+\beta} \le (\int_\Omega v^t)^{\frac{p}{t}}$ in view of Hölder inequality and $\|u_t\|_t = 1$. As a consequence we get

$$S\Big(\frac{p}{p+\beta}\Big)^p(\beta+1)\Big(\int_\Omega v^{p^*}\Big)^{\frac{p}{p^*}} \le \lambda\int_\Omega v^p + S^t|\Omega|^{p(\frac{1}{t}-\frac{1}{p^*})}\Big(\int_\Omega v^{p^*}\Big)^{\frac{p}{p^*}}.$$

As $\big(\frac{p}{p+\beta}\big)^p(\beta+1) = 1 - \frac{p-1}{2p\beta^2} + O(\beta^2)$, $0 < S_\lambda < S$ and $\lim_{t\nearrow p^*} S^t = S_\lambda$ from Step 2, there exist $\varepsilon > 0$, $\beta > 0$ with $p+\beta \le p^*$ and $t_0 \in (p, p^*)$ such that for any $t \in (t_0, p^*)$

$$\varepsilon\Big(\int_\Omega u_t^{(1+\frac{\beta}{p})p^*}\Big)^{\frac{p}{p^*}} \le \varepsilon\int_\Omega u_t^{p+\beta}.$$

Hence the set of functions $\{u_t\}$ is bounded in $L^{p^*+\frac{\beta N}{N-p}}(\Omega)$. Finally, propositions 1.2.5 and 1.2.2 imply the $C^{1,\alpha}(\bar\Omega)$ boundedness.

Step 5. We are able to end the proof. From Step 4 there exists an increasing sequence $\{t_n\}$ and a function $u \in C^{1,\alpha}(\bar\Omega)$ vanishing on $\partial\Omega$ such that u_{t_n} converges to u in $C^{1,\alpha'}(\bar\Omega)$ for any $\alpha' \in (0,\alpha)$. From Step 2 we have that $\lim_{n\to+\infty} S^{t_n} = S_\lambda$. In order to prove that $u \ne 0$, we get from (1.59)

$$\int_\Omega(|\nabla u_t|^p - \lambda u_t^p) = S^t \ge S\Big(\int_\Omega u_t^{p^*}\Big)^{\frac{p}{p^*}} - \lambda\int_\Omega u_t^p.$$

Letting $t = t_n$ go to p^* we obtain

$$S - \lambda\int_\Omega u^p \le S_\lambda,$$

that is

$$S - S_\lambda \le \lambda\int_\Omega u^p,$$

and then $u \ne 0$. Hence

$$-\Delta_p u - \lambda u^{p-1} = S_\lambda u^{p^*-1}, \tag{1.60}$$

$u \in W_0^{1,p}(\Omega)$ and $u > 0$ in Ω. □

Proof of Theorem 1.2.1. As $\lambda > 0$ we have $S_\lambda < S$ and as $\lambda < \lambda_1$ we have $S_\lambda > 0$. If u achieves the infimum (1.51) we can assume $u \ge 0$ and we have (1.60). Replacing u by ku we obtain a solution of problem (1.44). □

Chapter 2

Existence of Green functions and main properties

This chapter deals with existence and uniqueness results related to (1.5), with $\lambda < \lambda_1$ and $1 < p < N$. We will use the decomposition $G_\lambda = \chi\Gamma + \hat{H}_\lambda$ of the Green function in order to have $\hat{H} = 0$ on $\partial\Omega$, as already explained in the Introduction.
For the sake of simplicity, we will assume the pole being at 0. All the results that we are going to prove can be easily generalized to the case of pole at x.

In Section 2.1 we prove the existence of a solution G_λ of problem (1.5). In particular, we adopt the notion of *solution obtained by limits of approximations* (SOLA), which are solutions obtained via an approximation scheme using solutions of regularized problems. We follow the approach of the SOLA used by Boccardo and Gallouet in [4] and [5]. In the case $\lambda \neq 0$ we have to require the technical condition $p > 2 - \frac{1}{N}$ to guaranteee that $\bar{q} = \frac{N(p-1)}{N-1} > 1$. Then we show that a SOLA G_λ of (1.5) satisfies condition (1.9).
Section 2.2 is devoted to the global L^∞-regularity of the regular part H_λ of any solution to (1.5) satisfying the natural condition (1.9).
Finally, Section 2.3 deals with uniqueness issues. We will show that, if $p \geq 2$, the solution of (1.5) is unique among those satisfying (1.9).

2.1 The existence of a SOLA G_λ

Let $\{f_j\}$ be a sequence in $C_0^\infty(\Omega)$ of nonnegative functions converging to δ_0 in the distribution sense, with $\|f_j\|_1$ uniformly bounded, and such that

$$f_j \to 0 \quad \text{uniformly in } K \Subset \Omega \setminus \{0\} \qquad \text{as } j \to +\infty. \tag{2.1}$$

It is possible to choose such a f_j satisfying (2.1) by setting $f_j(x) = jf(jx)$ where $f \in C_0^\infty(\Omega)$, supp$f \subset B_1$ and $\int_\Omega f(y)dy = 1$.
It is known by [24] that there exists a weak solution G_j of the problem[1]

$$\begin{cases} -\Delta_p G_j - \lambda G_j^{p-1} = f_j & \text{in } \Omega \\ G_j \geq 0 & \text{in } \Omega \\ G_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

that is $G_j \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla G_j|^{p-2}\langle \nabla G_j, \nabla \varphi\rangle - \lambda \int_\Omega G_j^{p-1}\varphi = \int_\Omega f_j \varphi \qquad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.3}$$

We can assume $G_j \geq 0$. Indeed G_j minimizes the functional

$$J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \frac{\lambda}{p}\int_\Omega |v|^p - \int_\Omega f_j v, \qquad v \in W_0^{1,p}(\Omega).$$

and it is easy to check that if u minimizes J, then the same holds for $|u|$, being $f_j \geq 0$.

Lemma 2.1.1. *Let G_j be a solution of (2.2). Assume $p > \max\{\sqrt{N}, 2 - \frac{1}{N}\}$ if $\lambda \neq 0$. Then G_j^{p-1} is uniformly bounded in $L^1(\Omega)$.*

Proof. Assume by contradiction that $\|G_j\|_{p-1} \to +\infty$. We define

$$\hat{G}_j = \frac{G_j}{\|G_j\|_{p-1}}.$$

By the equation in (2.2) we have that G_j solves

$$-\Delta_p \hat{G}_j - \lambda \hat{G}_j^{p-1} = \frac{f_j}{\|G_j\|_{p-1}^{p-1}} \quad \text{in } \Omega. \tag{2.4}$$

[1] In the sequel we will omit the dependence on λ for ease of notation.

We observe that the right-hand side of (2.4) is bounded in $L^1(\Omega)$ and converges to 0. Application of the results in [4] yields that there exists $\hat{G}$ such that, up to a subsequence,

$$\hat{G}_j \to \hat{G} \quad \text{in } W_0^{1,q}(\Omega) \qquad \forall q < \bar{q} = \frac{N(p-1)}{N-1}.$$

Moreover G solves

$$-\Delta_p \hat{G} - \lambda \hat{G}^{p-1} = 0 \qquad \text{in } \Omega. \tag{2.5}$$

Using $\hat{G}$ as test function in the weak formulation of (2.5), we obtain

$$\int_\Omega |\nabla \hat{G}|^p - \lambda \int_\Omega \hat{G}^p = 0.$$

Recalling the definition of λ_1, we have

$$\int_\Omega |\nabla \hat{G}|^p \geq \lambda_1 \int_\Omega \hat{G}^p.$$

Thus

$$(\lambda_1 - \lambda) \int_\Omega \hat{G}^p \leq 0,$$

from which follows $\hat{G} = 0$ a.e., since $\lambda < \lambda_1$. Application of Sobolev inequality gives $\|\hat{G}_j\|_{p-1} \to 0$, contradicting the definition of $\hat{G}_j$ according to which $\|\hat{G}_j\|_{p-1} = 1$.

It is worth noting that $\hat{G}$ is admissible as test function in the weak formulation of (2.5), when $\lambda \neq 0$. Indeed $\hat{G} \in W_0^{1,q}(\Omega)$ for all $q < \bar{q}$. By Sobolev embedding $\hat{G} \in L^s(\Omega)$ for all $s < \bar{q}^* = \frac{N(p-1)}{N-p}$. Since $p^2 > N$, then $p < \bar{q}^*$ and so $\hat{G} \in L^p(\Omega)$. We use $T_l(\hat{G})$ as test function in the weak formulation of (2.5), T_l being defined as in (2.39). By definition (2.39), we have that $\nabla(T_l(\hat{G})) = \nabla \hat{G}$ in $\{|\hat{G}| \leq l\}$ and $\nabla(T_l(\hat{G})) = 0$ outside this set. Moreover $|T_l(\hat{G})| \leq \hat{G}$. Finally we obtain

$$\int_{\{|\hat{G} \leq l\}} |\nabla \hat{G}|^p \leq \lambda \int_\Omega \hat{G}^p. \tag{2.6}$$

Since the right-hand side of (2.6) is uniformly bounded in l, we can pass to the limit $l \to +\infty$ in this inequality and we obtain

$$\int_\Omega |\nabla \hat{G}|^p \leq C.$$

Thus $\hat{G} \in W_0^{1,p}(\Omega)$. Our claim is so established. $\square$

A generalization of Lemma (2.1.1) can be found in Appendix (see Lemma A.2.3). At this point we are going to prove the existence of a SOLA of problem (1.5).

Theorem 2.1.2. *Assume $p > \max\{\sqrt{N}, 2 - \frac{1}{N}\}$ if $\lambda \neq 0$. Then there exists a distributional solution G_λ of problem* (1.5). *Moreover, $G_\lambda \in W_0^{1,q}(\Omega)$ for all $q < \bar{q}$ and $G_\lambda \geq 0$.*

Proof. Setting $h_j = f_j + \lambda G_j^{p-1}$, by Lemma 2.1.1 we deduce that $\|h_j\|_1$ is uniformly bounded and $h_j \in W^{-1,p'}(\Omega) \cap L^1(\Omega)$. Therefore, from the application of the results in [4] we have that, up to a subsequence, there exists G_λ such that

$$G_j \to G_\lambda \qquad \text{in } W_0^{1,q}(\Omega). \tag{2.7}$$

Thus

$$G_j \to G_\lambda \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \nabla G_j \to \nabla G_\lambda \quad \text{a.e. in } \Omega. \tag{2.8}$$

The continuous embedding of $W_0^{1,q}(\Omega)$ in $L^{q^*}(\Omega)$ for $q < N$ implies that

$$G_j \to G_\lambda \qquad \text{in } L^r(\Omega), \qquad r < \bar{q}^* = \frac{N(p-1)}{N-p}.$$

Since $p - 1 < \frac{N(p-1)}{N-p}$, we have that

$$G_j \to G_\lambda \qquad \text{in } L^{p-1}(\Omega).$$

Moreover

$$|\nabla G_j|^{p-2} G_j \to |\nabla G_\lambda|^{p-2} \nabla G_\lambda \qquad \text{in } L^1(\Omega).$$

Therefore we can pass to the limit in (2.3) for $\varphi \in C_0^1(\Omega)$ and we obtain that the limit function G_λ is a distributional solution of problem (1.5). Moreover $G_\lambda \in W_0^{1,q}(\Omega)$ for all $q < \bar{q}$ and $G_\lambda \geq 0$. □

Let G_λ be a solution of (1.5), given by Theorem (2.1.2). Consider the decomposition $G_\lambda = \chi\Gamma + \hat{H}_\lambda$. Our aim now is to define $\hat{H}_\lambda$ as limit of suitable regularized functions and to prove that $\hat{H}_\lambda$ satisfies condition (1.9).
Let Γ_j be the solution of problem

$$\begin{cases} -\Delta_p \Gamma_j = f_j & \text{in } \Omega \\ \Gamma_j = \Gamma & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

f_j being defined as in (2.2). For any fixed j, there exists unique Γ_j solution of the problem (2.9). This solution is obtained using the variational method, by minimizing the functional

$$J(v) = \frac{1}{p}\int_{\Omega} |\nabla v|^p - \int_{\Omega} f_j v, \qquad v \in W^{1,p}(\Omega),\ v = \Gamma \text{ on } \partial\Omega.$$

The strict convexity of the function $t \to |t|^p$ and the fact that $f_j \geq 0$ imply the uniqueness of the minimum of J, from which it follows the uniqueness of the solution of (2.9).
The following proposition allows us to consider $\{\Gamma_j\}$ as a sequence of regularized function of Γ.

Proposition 2.1.3. *Let* $\{\Gamma_j\}$ *be a sequence of solutions of* (2.9). *Then* $\Gamma_j \to \Gamma$ *in* $W^{1,q}(\Omega)$ *for all* $q < \bar{q}$.

Proof. Using the results in [5], we obtain the precompactness of $\{\Gamma_j\}$ in $W^{1,q}(\Omega)$ for all $q < \bar{q}$. Moreover any limit $\tilde{\Gamma}$ of Γ_j is a solution of the limit problem

$$\begin{cases} -\Delta_p \tilde{\Gamma} = \delta_0 & \text{in } \Omega \\ \tilde{\Gamma} = \Gamma & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

and $\tilde{\Gamma} \in W^{1,q}(\Omega)$ for all $q < \bar{q}$.
Application of the uniqueness result due to Kichenassamy and Veron (see [23]) guarantees the uniqueness of the solution of problem (2.10), and so $\tilde{\Gamma} = \Gamma$, yielding that

$$\Gamma_j \to \Gamma \quad \text{in } W^{1,q}(\Omega) \qquad \forall q < \bar{q}.$$

□

At this point, let $\hat{H}_j$ be defined by

$$G_j = \chi\Gamma_j + \hat{H}_j. \tag{2.11}$$

We recall by (2.8) that

$$\nabla G_j \to \nabla G_\lambda \quad \text{a.e. in } \Omega$$

and, up to a subsequence,

$$\Gamma_j \to \Gamma \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \nabla\Gamma_j \to \nabla\Gamma \quad \text{a.e. in } \Omega.$$

in view of Proposition 2.1.3. Thus

$$\nabla(\chi\Gamma_j) \to \nabla(\chi\Gamma) \quad \text{a.e. in } \Omega.$$

So we have that

$$\nabla\hat{H}_j = \nabla G_j - \nabla(\chi\Gamma_j) \to \nabla G_\lambda - \nabla(\chi\Gamma) = \nabla\hat{H}_\lambda \quad \text{a.e. in } \Omega.$$

Then the following result holds.

Proposition 2.1.4. *Let $\hat{H}_\lambda = G_\lambda - \chi\Gamma$, where G_λ is a SOLA of* (1.5)*. Under the hypothesis of Theorem 2.1.2, then $\hat{H}_\lambda \in W_0^{1,\bar{q}}(\Omega)$. In particular, if $p > 2$, we have a stronger result that is $\hat{H}_\lambda \in W_0^{1,p}(\Omega)$.*

Proof. By definition (2.11), it follows that $\hat{H}_j$ solves the problem

$$\begin{cases} -\Delta_p(\chi\Gamma_j + \hat{H}_j) + \Delta_p(\chi\Gamma_j) = \lambda G_j^{p-1} + g_j & \text{in } \Omega \\ \hat{H}_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

where $g_j = f_j + \Delta_p(\chi\Gamma_j)$. In particular, g_j vanishes near 0 and $g = \Delta_p(\chi\Gamma_j)$ away from 0; by the properties of f_j, it follows that we have a good control on Γ_j away from 0 and thus $g \in L^\infty(\Omega)$.

We multiply equation in (2.12) by $\hat{H}_j$ and we integrate on Ω. Hence

$$\int_\Omega \langle |\nabla(\chi\Gamma_j + \hat{H}_j)|^{p-2}\nabla(\chi\Gamma_j + \hat{H}_j) - |\nabla(\chi\Gamma_j)|^{p-2}\nabla(\chi\Gamma_j), \nabla\hat{H}_j\rangle$$
$$= \lambda \int_\Omega G_j^{p-1}\hat{H}_j + \int_\Omega g_j\hat{H}_j. \tag{2.13}$$

The right-hand side of (2.13) is uniformly bounded. This fact is a consequence of the boundedness of g_j if $\lambda = 0$. If $\lambda \neq 0$ we use that G_j is uniformly bounded in $L^r(\Omega)$ for all $r < \bar{q}^*$, and also in $L^p(\Omega)$, being $N < p^2$. We study the left-hand side of (2.13) and we apply the estimates in Appendix (A.1). If $p \leq 2$ we have that

$$C' \int_\Omega (|\nabla(\chi\Gamma_j)| + |\nabla\hat{H}_j|)^{p-2}|\nabla\hat{H}_j|^2$$
$$\leq \int_\Omega \langle |\nabla(\chi\Gamma_j + \hat{H}_j)|^{p-2}\nabla(\chi\Gamma_j + \hat{H}_j) - |\nabla(\chi\Gamma_j)|^{p-2}\nabla(\chi\Gamma_j), \nabla\hat{H}_j\rangle.$$

On the other hand, if $p > 2$ we observe that

$$C'' \int_\Omega |\nabla \hat{H}_j|^p \le \int_\Omega \langle |\nabla(\chi\Gamma_j + \hat{H}_j)|^{p-2}\nabla(\chi\Gamma_j + \hat{H}_j) - |\nabla(\chi\Gamma_j)|^{p-2}\nabla(\chi\Gamma_j), \nabla\hat{H}_j\rangle.$$

Thus the following estimates hold uniformly in j:

$$\int_\Omega (|\nabla(\chi\Gamma_j)| + |\nabla\hat{H}_j|)^{p-2}|\nabla\hat{H}_j|^2 \le C \qquad \text{if } p \le 2, \tag{2.14}$$

$$\int_\Omega |\nabla\hat{H}_j|^p \le C \qquad \text{if } p > 2. \tag{2.15}$$

Application of Fatou's lemma to (2.14) and (2.15) yield

$$\int_\Omega (|\nabla(\chi\Gamma)| + |\nabla\hat{H}_\lambda|)^{p-2}|\nabla\hat{H}_\lambda|^2 \le C \qquad \text{if } p \le 2, \tag{2.16}$$

$$\int_\Omega |\nabla\hat{H}_\lambda|^p \le C \qquad \text{if } p > 2. \tag{2.17}$$

Using (2.17) and the fact that $\bar{q} < p$, we have that $\hat{H}_\lambda \in W_0^{1,\bar{q}}(\Omega)$ if $p > 2$. Now we prove that (2.16) guarantees that $\hat{H}_\lambda \in W_0^{1,\bar{q}}(\Omega)$ also when $p \le 2$. We observe that

$$\int_\Omega |\nabla\hat{H}_\lambda|^{\bar{q}} = \int_\Omega (|\nabla(\chi\Gamma)| + |\nabla\hat{H}_\lambda|)^{\frac{(p-2)\bar{q}}{2}}|\nabla\hat{H}_\lambda|^{\bar{q}}(|\nabla(\chi\Gamma)| + |\nabla\hat{H}_\lambda|)^{\frac{(2-p)\bar{q}}{2}}.$$

Application of Hölder inequality with exponents $\frac{2}{\bar{q}} = \frac{2(N-1)}{N(p-1)}$ and $\left(\frac{2}{\bar{q}}\right)' = \frac{2(N-1)}{3N-2-Np}$ yields

$$\int_\Omega |\nabla\hat{H}_\lambda|^{\bar{q}}$$
$$\le \left(\int_\Omega (|\nabla(\chi\Gamma)|+|\nabla\hat{H}_\lambda|)^{p-2}|\nabla\hat{H}_\lambda|^2\right)^{\frac{N(p-1)}{2(N-1)}} \left(\int_\Omega (|\nabla(\chi\Gamma)|+|\nabla\hat{H}_\lambda|)^{\frac{(2-p)N(p-1)}{3N-2-Np}}\right)^{\frac{3N-2-Np}{2(N-1)}}. \tag{2.18}$$

At this point we observe that

$$\int_\Omega (|\nabla(\chi\Gamma)| + |\nabla\hat{H}_\lambda|)^{\frac{(2-p)N(p-1)}{3N-2-Np}} < +\infty$$

since $\frac{(2-p)N(p-1)}{3N-2-Np} < \bar{q}$. Indeed this inequality is equivalent to $N > \frac{2}{3-p}$ being $p \le 2$, which holds in view of $N > p > \frac{2}{3-p}$. This fact together with (2.16) and (2.18) finally yield

$$\int_\Omega |\nabla\hat{H}_\lambda|^{\bar{q}} \le C.$$

□

As a consequence, recalling the definition of both H_λ and $\hat{H}_\lambda$ as explained in the Introduction, we get the following result.

Corollary 2.1.5. *Under the hypothesis of Theorem 2.1.2, there exists a distributional solution G_λ of problem* (1.5) *satisfying* (1.9).

2.2 The global L^∞-regularity of H_λ

The aim is now to prove the L^∞-regularity of H_λ as solution of (1.6). We include also the case $1 < p < 2$, which is particularly meaningful when $\lambda = 0$ providing a different proof than in [23]. The singular character of equation (1.6) has to be controlled thanks to the assumption $\nabla H_\lambda \in L^{\bar{q}}(\Omega)$.
Before turning to the main result of this section, we discuss the main properties related to problem (1.6), which will be repeatedly used throughout the thesis.

2.2.1 Structural properties of the problem

The first result deals with the asymptotic behaviour of ∇H_λ at the pole.

Proposition 2.2.1. *Let H_λ be a solution of* (1.6)*. Then*

$$|\nabla H_\lambda| = O(|\nabla \Gamma|) \qquad \text{as } x \to 0. \tag{2.19}$$

Proof. Let R be a small positive radius. We set

$$\Omega_R = \left\{ z \colon z = \frac{y}{R},\, y \in \Omega \right\} \tag{2.20}$$

and we define, for $z \in \Omega_R$,

$$G_{\lambda,R}(z) = R^{\frac{N-p}{p-1}} G_\lambda(Rz),$$

$$\Gamma_R(z) = R^{\frac{N-p}{p-1}} \Gamma(Rz) = R^{\frac{N-p}{p-1}} \frac{C_0}{|Rz|^{\frac{N-p}{p-1}}} = \Gamma(z),$$

$$H_{\lambda,R}(z) = R^{\frac{N-p}{p-1}} H_\lambda(Rz).$$

Let $\varphi_R \in C_0^1(\Omega_R)$ be such that $\varphi_R(z) = \varphi(Rz)$ for some $\varphi \in C_0^1(\Omega)$. We observe that

$$\begin{aligned}\int_{\Omega_R} |\nabla G_{\lambda,R}|^{p-2}\langle \nabla G_{\lambda,R}, \nabla\varphi_R\rangle &= \int_{\Omega_R} R^N |\nabla G_\lambda(Rz)|^{p-2}\langle \nabla G_\lambda(Rz), \nabla\varphi(Rz)\rangle dz\\ &= \int_\Omega |\nabla G_\lambda|^{p-2}\langle \nabla G_\lambda, \nabla\varphi\rangle\\ &= \lambda\int_\Omega G_\lambda^{p-1}\varphi + \varphi(0),\end{aligned}$$

using in the last passage the weak formulation of problem (1.5). We have

$$\begin{aligned}\int_\Omega G_\lambda^{p-1}\varphi &= \int_{\Omega_R} R^N G_\lambda^{p-1}(Rz)\varphi(Rz)dz\\ &= R^p\int_{\Omega_R} R^{N-p}G_\lambda^{p-1}(Rz)\varphi(Rz)dz\\ &= R^p\int_{\Omega_R} G_{\lambda,R}^{p-1}\varphi_R.\end{aligned}$$

Moreover $\varphi(0) = \varphi_R(0)$. Thus, for all $\varphi_R \in C_0^1(\Omega_R)$

$$\int_{\Omega_R} |\nabla G_{\lambda,R}|^{p-2}\langle \nabla G_{\lambda,R}, \nabla\varphi_R\rangle = \lambda R^p\int_{\Omega_R} G_{\lambda,R}^{p-1}\varphi_R + \varphi_R(0).$$

Then $G_{\lambda,R}$ solves the problem

$$\begin{cases} -\Delta_p G_{\lambda,R} - \lambda R^p G_{\lambda,R}^{p-1} = \delta_0 & \text{in } \Omega_R\\ G_{\lambda,R} \geq 0 & \text{in } \Omega_R\\ G_{\lambda,R} = 0 & \text{on } \partial\Omega_R.\end{cases} \tag{2.21}$$

With the aid of the decomposition $G_\lambda = \Gamma + H_\lambda$, we write

$$G_{\lambda,R}(z) = R^{\frac{N-p}{p-1}}G_\lambda(Rz) = R^{\frac{N-p}{p-1}}(\Gamma(Rz)+H_\lambda(Rz)) = \Gamma(z)+H_{\lambda,R}(z), \tag{2.22}$$

in view of the definition of $G_{\lambda,R}, \Gamma_R$ and $H_{\lambda,R}$. Moreover, condition (1.3) yields

$$C' \leq \frac{G_\lambda(Rz)}{\Gamma(Rz)} \leq C'',$$

that is

$$C' \leq \frac{R^{\frac{N-p}{p-1}}G_\lambda(Rz)}{\Gamma(z)} \leq C''.$$

Thus, condition (1.3) is invariant by scaling and we obtain

$$C' \le \frac{G_R}{\Gamma} \le C'' \quad \text{in } \Omega_R. \tag{2.23}$$

At this point, recalling (2.21), we have that $G_{\lambda,R}$ solves

$$\begin{cases} -\Delta_p G_{\lambda,R} = \lambda R^p G_{\lambda,R}^{p-1} & \text{in } \Omega_R \setminus \{0\} \\ G_{\lambda,R} \ge 0 & \text{in } \Omega_R \setminus \{0\} \\ G_{\lambda,R} = 0 & \text{on } \partial\Omega_R, \end{cases}$$

and $G_{\lambda,R}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_3 \setminus \{0\})$ for R small in view of (2.23). Application of regularity results in Tolksdorf [33] yields that $G_{\lambda,R}$ is uniformly bounded in $C^{1,\alpha}_{\mathrm{loc}}(B_2 \setminus \{0\})$ for R small. In particular

$$\|\nabla G_{\lambda,R}\|_{\infty,\partial B_1} \le C. \tag{2.24}$$

Using (2.22), we write $\nabla G_{\lambda,R} = \nabla\Gamma + \nabla H_{\lambda,R}$. As a consequence, the boundedness of $\nabla\Gamma$ on ∂B_1 leads to

$$\|\nabla H_{\lambda,R}\|_{\infty,\partial B_1} \le C. \tag{2.25}$$

Observing that

$$\sup_{z\in\partial B_1} |\nabla H_{\lambda,R}(z)| = \sup_{z\in\partial B_1} |R^{\frac{N-1}{p-1}} \nabla H_\lambda(Rz)| = \sup_{y\in\partial B_R} |y|^{\frac{N-1}{p-1}} |\nabla H_\lambda(y)|,$$

we finally arrive at

$$|\nabla H(y)| \le \frac{\tilde{C}}{|y|^{\frac{N-1}{p-1}}} = C|\nabla\Gamma(y)| \qquad \text{for all } |y| = R,$$

in view of (2.25). This concludes the proof. □

To complete this section, we show a consequence to the structural properties of problem (1.6). Before turning to a description of results, we point out the weak formulation of problem (1.6), that is

$$\int_\Omega \langle |\nabla(\Gamma + H_\lambda)|^{p-2}\nabla(\Gamma + H_\lambda) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \nabla\varphi\rangle = \lambda \int_\Omega G_\lambda^{p-1}\varphi \qquad \forall\varphi \in C_0^1(\Omega). \tag{2.26}$$

and $H_\lambda = -\Gamma$ on $\partial\Omega$.

Remark 2.2.2. In (2.26) we consider $\varphi \in C_0^1(\Omega)$ in order to make sense of the left-hand side in the equality. On the other hand, we observe that $\varphi \in W_0^{1,p}(\Omega)$ such that $0 \notin \mathrm{supp}\varphi$ can be used as test function in (2.26). Indeed the fact that 0 does not belong to the support of φ guarantees that integration is made away from the singular point; then, the regularity properties of φ make sense of the formulation. In what follows, we will consider *admissible* the functions with these properties.

Now we prove a lemma which, applied to our problem, will allows us to derive estimates starting from the weak formulation (2.26). For later convenience, let us write the result in a sufficiently general way.

Let A be an open subset of Ω such that $0 \in A$ and let η be a nonnegative smooth function supported in A. For $\varepsilon > 0$, consider a sequence of smooth functions $\{\eta_\varepsilon\}$, converging to η as $\varepsilon \to 0$, with the following properties

$$\eta_\varepsilon = \eta \text{ outside } B_\varepsilon, \quad 0 \leq \eta_\varepsilon \leq \eta, \quad \eta_\varepsilon = 0 \text{ in } B_{\frac{\varepsilon}{2}}, \quad \max|\nabla \eta_\varepsilon| \sim \frac{1}{\varepsilon}. \tag{2.27}$$

Lemma 2.2.3. *Let $\mathcal{H}$ be a solution of*

$$-\Delta_p(\Gamma + \mathcal{H}) + \Delta_p \Gamma = \mathcal{G} \qquad \text{in } A \tag{2.28}$$

such that $\nabla \mathcal{H}$ is L^p-integrable away from 0 and $\mathcal{G} \in L^1(A)$. Let η and η_ε be as previously described. Consider a bounded function $\Psi\colon \mathbb{R} \to \mathbb{R}$ such that $\Psi' > 0$ is bounded. Assume $\eta = 0$ either $\Psi(\mathcal{H}) = 0$ on ∂A. Then we have the following estimates:

- Case $p \leq 2$:

$$\begin{aligned}
&\int_A \eta_\varepsilon^2 \big[|\nabla \Gamma| + |\nabla \mathcal{H}|\big]^{p-2} \Psi'(\mathcal{H}) |\nabla \mathcal{H}|^2 \\
&\leq C\bigg(\int_A \eta_\varepsilon |\nabla \eta_\varepsilon| |\Psi(\mathcal{H})| \big[|\nabla \Gamma| + |\nabla \mathcal{H}|\big]^{p-2} |\nabla \mathcal{H}| + \int_A |\mathcal{G}| \eta_\varepsilon^2 |\Psi(\mathcal{H})|\bigg),
\end{aligned} \tag{2.29}$$

- Case $p > 2$:

$$\begin{aligned}
&\int_A \eta_\varepsilon^p \Psi'(\mathcal{H}) |\nabla \mathcal{H}|^p + \int_A \eta_\varepsilon^p \Psi'(\mathcal{H}) |\nabla \Gamma|^{p-2} |\nabla \mathcal{H}|^2 \\
&\leq C\bigg(\int_A \eta_\varepsilon^{p-1} |\nabla \eta_\varepsilon| |\Psi(\mathcal{H})| \big[|\nabla \Gamma|^{p-2} + |\nabla \mathcal{H}|^{p-2}\big] |\nabla \mathcal{H}| + \int_A |\mathcal{G}| \eta_\varepsilon^p |\Psi(\mathcal{H})|\bigg).
\end{aligned} \tag{2.30}$$

Proof. We observe that the function

$$\varphi = \begin{cases} \eta_\varepsilon^2 \Psi(\mathcal{H}) & \text{if } p \le 2 \\ \eta_\varepsilon^p \Psi(\mathcal{H}) & \text{if } p > 2 \end{cases}$$

is admissible in (2.28), in view of the properties of η_ε and Ψ.

Assume $p \le 2$. Using φ as test function in (2.28) we obtain

$$\int_A \langle |\nabla(\Gamma + \mathcal{H})|^{p-2}\nabla(\Gamma + \mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \nabla(\eta_\varepsilon^2 \Psi(\mathcal{H}))\rangle = \int_A \mathcal{G}\eta_\varepsilon^2 \Psi(\mathcal{H}). \tag{2.31}$$

We have that

$$\nabla(\eta_\varepsilon^2 \Psi(\mathcal{H})) = 2\eta_\varepsilon \nabla\eta_\varepsilon \Psi(\mathcal{H}) + \eta_\varepsilon^2 \Psi'(\mathcal{H})\nabla\mathcal{H}.$$

Thus (2.31) rewrites as

$$\begin{aligned} &\int_A \langle |\nabla(\Gamma + \mathcal{H})|^{p-2}\nabla(\Gamma + \mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, 2\eta_\varepsilon \nabla\eta_\varepsilon \Psi(\mathcal{H})\rangle \\ &\qquad + \int_A \langle |\nabla(\Gamma + \mathcal{H})|^{p-2}\nabla(\Gamma + \mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \eta_\varepsilon^2 \Psi'(\mathcal{H})\nabla\mathcal{H}\rangle \\ &\qquad\qquad = \int_A \mathcal{G}\eta_\varepsilon^2 \Psi(\mathcal{H}). \end{aligned} \tag{2.32}$$

Application of estimates (A.1) and (A.2) in the Appendix A.1 with $x = \nabla\Gamma$ and $y = \nabla\mathcal{H}$ yields

$$\begin{aligned} &\int_A \eta_\varepsilon^2 \big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2} \Psi'(\mathcal{H})|\nabla\mathcal{H}|^2 \\ &\qquad \le C\bigg(\int_A \eta_\varepsilon |\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| + \int_A |\mathcal{G}|\eta_\varepsilon^2|\Psi(\mathcal{H})|\bigg). \end{aligned}$$

This proves inequality (2.29).

Now we assume $p > 2$. Using φ as test function in (2.28) we obtain

$$\int_A \langle |\nabla(\Gamma + \mathcal{H})|^{p-2}\nabla(\Gamma + \mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \nabla(\eta_\varepsilon^p \Psi(\mathcal{H}))\rangle = \int_A \mathcal{G}\eta_\varepsilon^p \Psi(\mathcal{H}). \tag{2.33}$$

We observe that

$$\nabla(\eta_\varepsilon^p \Psi(\mathcal{H})) = p\eta_\varepsilon^{p-1} \nabla\eta_\varepsilon \Psi(\mathcal{H}) + \eta_\varepsilon^p \Psi'(\mathcal{H})\nabla\mathcal{H}.$$

Similarly to the case $p \le 2$, we rewrite (2.33) as

$$\int_A \langle |\nabla(\Gamma+\mathcal{H})|^{p-2}\nabla(\Gamma+\mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, p\eta_\varepsilon^{p-1}\nabla\eta_\varepsilon\Psi(\mathcal{H})\rangle$$
$$+ \int_A \langle |\nabla(\Gamma+\mathcal{H})|^{p-2}\nabla(\Gamma+\mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \eta_\varepsilon^{p}\Psi'(\mathcal{H})\nabla\mathcal{H}\rangle$$
$$= \int_A \mathcal{G}\eta_\varepsilon^p\Psi(\mathcal{H}). \quad (2.34)$$

We study the first term in the left-hand side of (2.34). Application of (A.4) in the Appendix A.1 with $x = \nabla\Gamma$ and $y = \nabla\mathcal{H}$ yields

$$\left|\int_A \langle |\nabla(\Gamma+\mathcal{H})|^{p-2}\nabla(\Gamma+\mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, p\eta_\varepsilon^{p-1}\nabla\eta_\varepsilon\Psi(\mathcal{H})\rangle\right|$$
$$\le C\int_A \eta_\varepsilon^{p-1}|\nabla\eta_\varepsilon|\big[|\nabla\Gamma|^{p-2} + |\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}||\Psi(\mathcal{H})|.$$

For what concerns the second term in the left-hand side of (2.34), we have

$$\langle |\nabla(\Gamma+\mathcal{H})|^{p-2}\nabla(\Gamma+\mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \nabla\mathcal{H}\rangle \ge C'(|\nabla\mathcal{H}|^p + |\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2),$$

in view of (A.3). Then one may write

$$\int_A \langle |\nabla(\Gamma+\mathcal{H})|^{p-2}\nabla(\Gamma+\mathcal{H}) - |\nabla\Gamma|^{p-2}\nabla\Gamma, \eta_\varepsilon^{p}\Psi'(\mathcal{H})\nabla\mathcal{H}\rangle$$
$$\ge \tilde{C}\Big(\int_A \eta_\varepsilon^p\Psi'(\mathcal{H})|\nabla\mathcal{H}|^p + \int_A \eta_\varepsilon^p\Psi'(\mathcal{H})|\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2\Big).$$

With the aid of (2.34) we get

$$\int_A \eta_\varepsilon^p\Psi'(\mathcal{H})|\nabla\mathcal{H}|^p + \int_A \eta_\varepsilon^p\Psi'(\mathcal{H})|\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2$$
$$\le C\Big(\int_A \eta_\varepsilon^{p-1}|\nabla\eta_\varepsilon|\big[|\nabla\Gamma|^{p-2} + |\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}||\Psi(\mathcal{H})| + \int_A |\mathcal{G}|\eta_\varepsilon^p|\Psi(\mathcal{H})|\Big),$$

that is (2.30). $\square$

In addition to the above result, we shall prove that (2.29) and (2.30) hold letting $\varepsilon \to 0$, provided $\nabla\mathcal{H} \in L^{\bar{q}}(A)$.

Corollary 2.2.4. *Assume $\nabla\mathcal{H} \in L^{\bar{q}}(A)$. Under the hypothesis of Lemma 2.2.3, we have the following estimates.*

- Case $p \le 2$:

$$\int_A \eta^2 \big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2} \Psi'(\mathcal{H})|\nabla\mathcal{H}|^2 \le C\Big(\int_A \eta|\nabla\eta||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| + \int_A |\mathcal{G}|\eta^2|\Psi(\mathcal{H})|\Big), \tag{2.35}$$

- Case $p > 2$:

$$\int_A \eta^p \Psi'(\mathcal{H})|\nabla\mathcal{H}|^p + \int_A \eta^p \Psi'(\mathcal{H})|\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2 \le C\Big(\int_A \eta^{p-1}|\nabla\eta||\Psi(\mathcal{H})|\big[|\nabla\Gamma|^{p-2}+|\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}|+\int_A |\mathcal{G}|\eta^p|\Psi(\mathcal{H})|\Big). \tag{2.36}$$

Proof. Assume $p \le 2$. Application of Fatou's lemma gives

$$\int_A \eta^2 \big[|\nabla\Gamma|+|\nabla\mathcal{H}|\big]^{p-2} \Psi'(\mathcal{H})|\nabla\mathcal{H}|^2 \le \liminf_{\varepsilon\to 0} \int_A \eta_\varepsilon^2 \big[|\nabla\Gamma|+|\nabla\mathcal{H}|\big]^{p-2} \Psi'(\mathcal{H})|\nabla\mathcal{H}|^2.$$

Since Ψ is bounded and $|\nabla\Gamma| + |\nabla\mathcal{H}| \ge |\nabla\mathcal{H}|$, a simple use of Hölder inequality yields

$$\int_{B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}} \eta_\varepsilon|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| \le \frac{C}{\varepsilon}\int_{B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}} |\nabla\mathcal{H}|^{p-1}$$
$$\le \frac{C}{\varepsilon}|B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}|^{\frac{1}{N}}\Big(\int_{B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}} |\nabla\mathcal{H}|^{\bar q}\Big)^{\frac{N-1}{N}} = C\Big(\int_{B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}} |\nabla\mathcal{H}|^{\bar q}\Big)^{\frac{N-1}{N}},$$

in view of the properties of η_ε. This implies that

$$\int_{B_\varepsilon \setminus B_{\frac{\varepsilon}{2}}} \eta_\varepsilon|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| \to 0 \quad \text{as } \varepsilon \to 0,$$

in view of $\nabla\mathcal{H} \in L^{\bar q}(A)$. As a consequence

$$\int_A \eta_\varepsilon|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| \to \int_A \eta|\nabla\eta||\Psi(\mathcal{H})|\big[|\nabla\Gamma| + |\nabla\mathcal{H}|\big]^{p-2}|\nabla\mathcal{H}| \quad \text{as } \varepsilon \to 0.$$

Moreover, Lebesgue's theorem yields

$$\int_A |\mathcal{G}|\eta_\varepsilon^2|\Psi(\mathcal{H})| \to \int_A |\mathcal{G}|\eta^2|\Psi(\mathcal{H})| \quad \text{as } \varepsilon \to 0.$$

Therefore, letting $\varepsilon \to 0$ in (2.29) we obtain (2.35).

Assume $p > 2$. With the aid of Fatou's lemma, we have that

$$\int_A \eta^p \Psi'(\mathcal{H})|\nabla\mathcal{H}|^p \le \liminf_{\varepsilon\to 0} \int_A \eta_\varepsilon^p \Psi'(\mathcal{H})|\nabla\mathcal{H}|^p,$$

$$\int_A \eta^p \Psi'(\mathcal{H})|\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2 \le \liminf_{\varepsilon\to 0} \int_A \eta_\varepsilon^p \Psi'(\mathcal{H})|\nabla\Gamma|^{p-2}|\nabla\mathcal{H}|^2.$$

Similarly to the case $p \le 2$ we observe that

$$\begin{aligned}
&\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}} \eta_\varepsilon^{p-1}|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma|^{p-2}+|\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}| \\
&\qquad\le C\Bigg[\frac{1}{\varepsilon^{1+\frac{(N-1)(p-2)}{p-1}}}\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}| + \frac{1}{\varepsilon}\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}|^{p-1}\Bigg] \\
&\qquad\le C\Bigg[\frac{|B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}|^{1-\frac{1}{\bar q}}}{\varepsilon^{\frac{Np-2N+1}{p-1}}}\bigg(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}|^{\bar q}\bigg)^{\frac{1}{\bar q}} + \frac{|B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}|^{\frac{1}{N}}}{\varepsilon}\bigg(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}|^{\bar q}\bigg)^{\frac{N-1}{N}}\Bigg] \\
&\qquad= C\Bigg[\bigg(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}|^{\bar q}\bigg)^{\frac{1}{\bar q}} + \bigg(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\mathcal{H}|^{\bar q}\bigg)^{\frac{N-1}{N}}\Bigg],
\end{aligned}$$

since $\bar q > p-1$ and $p > 2$. This leads to

$$\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}} \eta_\varepsilon^{p-1}|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma|^{p-2}+|\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}| \to 0 \quad \text{as } \varepsilon \to 0,$$

in view of $\nabla\mathcal{H} \in L^{\bar q}(A)$. As a consequence

$$\begin{aligned}
&\int_A \eta_\varepsilon^{p-1}|\nabla\eta_\varepsilon||\Psi(\mathcal{H})|\big[|\nabla\Gamma|^{p-2}+|\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}| \\
&\qquad\qquad \to \int_A \eta^{p-1}|\nabla\eta||\Psi(\mathcal{H})|\big[|\nabla\Gamma|^{p-2}+|\nabla\mathcal{H}|^{p-2}\big]|\nabla\mathcal{H}| \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Moreover, Lebesgue's theorem implies that

$$\int_A |\mathcal{G}|\eta_\varepsilon^p|\Psi(\mathcal{H})| \to \int_A |\mathcal{G}|\eta^p|\Psi(\mathcal{H})| \quad \text{as } \varepsilon \to 0.$$

Thus, letting $\varepsilon \to 0$ in (2.30), we obtain (2.36). □

2.2.2 Proof of the global boundedness

Our goal in this section is to prove the boundedness of H_λ in Ω.

We recall that $H_\lambda = G_\lambda - \Gamma$. Since both G_λ and Γ are bounded away from 0, then it will be enough to prove the L^∞-regularity of H_λ near 0. We will take different approaches depending on whether $\lambda = 0$ or $\lambda > 0$.

Proposition 2.2.5. *Let H be a solution of problem* (1.6) *with $\lambda = 0$. Assume condition* (1.9)*. Then there exists a constant $c_0 > 0$ such that*

$$-c_0 \leq H \leq 0 \quad \text{in } \Omega. \tag{2.37}$$

Proof. The right-hand side of (1.6) is equal to 0, being $\lambda = 0$. Then H solves

$$\begin{cases} -\Delta_p(\Gamma + H) + \Delta_p \Gamma = 0 & \text{in } \Omega \\ H = -\Gamma & \text{on } \partial\Omega. \end{cases} \tag{2.38}$$

Recalling the definition of Γ, we have that $\Gamma < c_0$ on $\partial\Omega$ for some $c_0 > 0$. For $k, l \in \mathbb{R}$, $k < l$, consider the function $T_{k,l}$ defined as follows

$$T_{k,l}[s] = \begin{cases} k & \text{if } s < k \\ s & \text{if } k \leq s \leq l \\ l & \text{if } s > l. \end{cases} \tag{2.39}$$

For $l > 0$ we set $\Psi(s) = T_{0,l}(s)$. Then Ψ has the properties required in Lemma 2.2.3. In particular, we have

$$\Psi(H) = \begin{cases} 0 & \text{if } H < 0 \\ H & \text{if } 0 \leq H \leq l \\ l & \text{if } H > l. \end{cases}$$

Since $H < 0$ on $\partial\Omega$ guarantees that $\Psi(H) = 0$ on $\partial\Omega$, we can apply Corollary 2.2.4 with $A = \Omega$, $\mathcal{H} = H$, $\mathcal{G} \equiv 0$ and $\eta \equiv 1$. Observing that $\Psi'(s) = 1$ in $\{0 \leq s \leq l\}$ and $\Psi'(s) = 0$ outside this set, we get

$$\int_{\{0 \leq H \leq l\}} \big[|\nabla \Gamma| + |\nabla H|\big]^{p-2} |\nabla H|^2 \leq 0 \qquad \text{if } p \leq 2, \tag{2.40}$$

$$\int_{\{0\le H\le l\}} |\nabla H|^p \le 0 \qquad \text{if } p>2. \tag{2.41}$$

Letting $l \to +\infty$ in (2.40) and (2.41), we obtain

$$\int_{\{H\ge 0\}} \left[|\nabla\Gamma| + |\nabla H|\right]^{p-2} |\nabla H|^2 \le 0 \qquad \text{if } p\le 2, \tag{2.42}$$

$$\int_{\{H\ge 0\}} |\nabla H|^p \le 0 \qquad \text{if } p>2. \tag{2.43}$$

We have $H^+ = H$ in $\{H \ge 0\}$, being $H^+ = \max\{H, 0\}$. As a consequence (2.42) and (2.43) imply that $\nabla H^+ = 0$ a.e. in $\{H \ge 0\}$. Moreover since $\nabla H^+ = 0$ also in $\{H < 0\}$, we have that $\nabla H^+ = 0$ a.e. in Ω. Thus $H^+ = 0$ a.e. in Ω. This implies that $H \le 0$ a.e. in Ω.
At this point we want to prove that $H \ge -c_0$. We set for $l > 0$

$$\Psi(s) = -T_{-l,0}(s + c_0).$$

Then $-\Psi$ has the properties required in Lemma 2.2.3. In particular, we have

$$\Psi(H) = \begin{cases} l & \text{if } H + c_0 < -l \\ -(H+c_0) & \text{if } -l \le H + c_0 \le 0 \\ 0 & \text{if } H + c_0 > 0. \end{cases}$$

Since $H = -\Gamma > -c_0$ on $\partial\Omega$, then $\Psi(H) = 0$ on $\partial\Omega$. With the aid of the same techniques as before, application of Corollary 2.2.4 yields $\nabla(H+c_0)^- = 0$ a.e. in Ω, where $(H + c_0)^- = \max\{-(H + c_0), 0\}$. It follows that $(H + c_0)^- = 0$ a.e. in Ω. Thus $H + c_0 \ge 0$ a.e. in Ω, that is $H \ge -c_0$ a.e. in Ω. $\square$

We are going to prove the global boundedness of H_λ when $\lambda > 0$. It will be useful the following result, known as the Caffarelli-Kohn-Nirenberg inequality (see [7]).

Theorem 2.2.6. *Let $N \ge 1$ and a, b and c be such that*

(i) if $N \ge 3$: $-\infty < a < \frac{N-2}{2}$, $a \le b \le a+1$ and $c = \frac{2N}{N-2+2(b-a)}$

(ii) if $N = 2$: $-\infty < a < 0$, $a < b \le a+1$ and $c = \frac{2}{b-a}$

(iii) if $N = 1$: $-\infty < a < -\frac{1}{2}$, $a + \frac{1}{2} < b \le a+1$ and $c = \frac{2}{-1+2(b-a)}$.

Then there exists a constant $C_{a,b} = C(a,b) > 0$ *such that*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^c}{|x|^{bc}} dx\right)^{\frac{1}{c}} \leq C_{a,b}\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}},$$

for any $u \in C_0^\infty(\mathbb{R}^N)$.

Proposition 2.2.7. *Let* H_λ *be a solution of problem* (1.6) *with* $\sqrt{N} < p$ *and* $\lambda \neq 0$. *Assume condition* (1.9). *Moreover we require* $p \geq \frac{2N}{N+1}$ *if* $p \leq 2$ *and* $p > \frac{N}{2}$ *if* $p > 2$. *Then* H_λ *is bounded in* Ω.

Proof. Since H_λ is bounded away from 0, it will be enough to study the regularity in B_R. We define, for $z \in B_2$,

$$\tilde{\Gamma}(z) = R^{\frac{N-p}{p-1}}\Gamma(Rz) \qquad \text{and} \qquad \tilde{H}_\lambda(z) = R^{\frac{N-p}{p-1}} H_\lambda(Rz)$$

and we set $\tilde{G}_\lambda(z) = G(Rz)$. Thus $\tilde{H}_\lambda$ solves

$$-\Delta_p(\tilde{\Gamma} + \tilde{H}_\lambda) + \Delta_p\tilde{\Gamma} = \lambda R^N \tilde{G}_\lambda^{p-1} \qquad \text{in } B_2,$$

which can be rewritten in the form

$$-\Delta_p(\Gamma + \tilde{H}_\lambda) + \Delta_p\Gamma = \lambda R^N \tilde{G}_\lambda^{p-1} \qquad \text{in } B_2,$$

in view of

$$\nabla\tilde{\Gamma}(z) = R^{\frac{N-p}{p-1}+1}\nabla\Gamma(Rz) = R^{\frac{N-1}{p-1}} \frac{C_0}{|Rz|^{\frac{N-1}{p-1}}} = \frac{C_0}{|z|^{\frac{N-1}{p-1}}} = \nabla\Gamma(z).$$

Let h and h' be real numbers such that $1 \leq h' < h \leq 2$. Let η be a nonnegative smooth function such that

$$\eta = 1 \text{ in } B_{h'}, \quad 0 \leq \eta \leq 1 \text{ in } B_h, \quad \eta = 0 \text{ outside } B_h, \tag{2.44}$$

$$\max|\nabla\eta| \sim (h - h')^{-1}, \tag{2.45}$$

and let η_ε be a sequence converging to η as $\varepsilon \to 0$, satisfying properties (2.27). We set $\mathcal{G} = \lambda R^N G_\lambda^{p-1}$ and $u = |\tilde{H}_\lambda| + k$, where

$$k = \begin{cases} \|\mathcal{G}\|_{\frac{p}{p-1},2}^{\frac{1}{p-1}} & \text{if } p \leq 2 \\ \|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0},2} & \text{if } p > 2, \end{cases}$$

for some $\varepsilon_0 > 0$. We observe that k is well-defined in view of the assumptions on p. Indeed, condition $\sqrt{N} < p$ guarantees that $G \in L^p(\Omega)$. If $p > 2$, the right-hand side of (1.6) belongs to $L^{\frac{N}{p-\varepsilon_0}}(\Omega)$ if $\frac{N(p-1)}{p-\varepsilon_0} < \frac{N(p-1)}{N-p}$, that is $p > \frac{N}{2} + \frac{\varepsilon_0}{2}$.

For any $\beta \geq \beta_0 > 0$ and $l > k$ the function

$$\varphi = \begin{cases} \eta_\varepsilon^2 \mathrm{sgn} \tilde{H}_\lambda[(T_{-l,l}u)^\beta - k^\beta] & \text{if } p \leq 2 \\ \eta_\varepsilon^p \mathrm{sgn} \tilde{H}_\lambda[(T_{-l,l}u)^\beta - k^\beta] & \text{if } p > 2 \end{cases}$$

is admissible in (2.26), the function $T_{-l,l}$ being defined as in (2.39). For ease of notation, we will write $T_l u$ instead of $T_{-l,l}u$. We define the following functions

$$v_l = (T_l u)^{\frac{\beta+1}{2}}, \quad v = u^{\frac{\beta+1}{2}}, \quad w = u^{\frac{\beta-1+p}{p}}. \tag{2.46}$$

We would apply Lemma 2.2.3 or Corollary 2.2.4 with $\mathcal{H} = \tilde{H}_\lambda$ and $\Psi(s) = \mathrm{sgn} s[(T_l(|s|+k))^\beta - k^\beta]$. We observe that

$$\Psi'(s) = \beta(|s|+k)^{\beta-1} \text{ in } [-(l-k), l-k], \quad \text{and} \quad \Psi'(s) = 0 \text{ outside this set.}$$

Moreover $|\Psi(s)| \leq (T_l(|s|+k))^\beta$. We study the cases $p \leq 2$ and $p > 2$ separately.

Case $p \leq 2$. Thanks to (2.19), application of (2.29) gives

$$\begin{aligned} &\beta \int_{\{u \leq l\}} \eta_\varepsilon^2 |\nabla \Gamma|^{p-2} |\nabla \tilde{H}_\lambda|^2 u^{\beta-1} \\ &\qquad \leq C\left(\int_{B_2} \eta_\varepsilon |\nabla \eta_\varepsilon| (T_l u)^\beta [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla \tilde{H}_\lambda| + \int_{B_2} |\mathcal{G}| \eta_\varepsilon^2 (T_l u)^\beta \right), \end{aligned}$$

that is

$$\begin{aligned} &\frac{\beta}{(\beta+1)^2} \int_{B_2} \eta_\varepsilon^2 |\nabla \Gamma|^{p-2} |\nabla v_l|^2 \\ &\leq C\left(\int_{B_2} \eta_\varepsilon (T_l u)^\beta |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u| + \int_{B_2} |\mathcal{G}| \eta_\varepsilon^2 (T_l u)^\beta \right), \end{aligned} \tag{2.47}$$

being $|\nabla \tilde{H}_\lambda| = |\nabla u|$ and using the definition of v_l. We observe that

$$\int_{B_2} \eta_\varepsilon (T_l u)^\beta |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u|$$
$$\leq \int_{\{u \leq l\}} \eta_\varepsilon (T_l u)^\beta |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u|$$
$$+ l^\beta \int_{\{u > l\}} \eta_\varepsilon |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u|. \quad (2.48)$$

Then, using that $|\nabla \Gamma| + |\nabla \tilde{H}_\lambda| \geq |\nabla \Gamma|$ and $p \leq 2$, we can estimate the first term of the right-hand side of (2.48) as follows:

$$\int_{\{u \leq l\}} \eta_\varepsilon (T_l u)^\beta |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u|$$
$$\leq C \int_{\{u \leq l\}} \eta_\varepsilon (T_l u)^{\frac{\beta-1}{2}} |\nabla \eta_\varepsilon| \nabla \Gamma|^{p-2} |\nabla u| (T_l u)^{\frac{\beta+1}{2}}$$
$$= \frac{2C}{\beta+1} \int_{B_2} \eta_\varepsilon v_l |\nabla \eta_\varepsilon| |\nabla \Gamma|^{\frac{2(p-2)}{2}} |\nabla v_l|$$
$$\leq \frac{\beta}{2(\beta+1)^2} \int_{B_2} \eta_\varepsilon^2 |\nabla \Gamma|^{p-2} |\nabla v_l|^2 + \frac{C}{\beta} \int_{\{u \leq l\}} |\nabla \Gamma|^{p-2} |\nabla \eta_\varepsilon|^2 v_l^2,$$

applying in the last passage the Young inequality. Thus (2.47) rewrites as

$$\frac{\beta}{(\beta+1)^2} \int_{B_2} |\nabla \Gamma|^{p-2} |\nabla (\eta_\varepsilon v_l)|^2 \leq C \Bigg[\left(\frac{1}{\beta} + \frac{\beta}{(\beta+1)^2} \right) \int_{\{u \leq l\}} |\nabla \Gamma|^{p-2} |\nabla \eta_\varepsilon|^2 v_l^2$$
$$+ l^\beta \int_{\{u > l\}} \eta_\varepsilon |\nabla \eta_\varepsilon| [|\nabla \Gamma| + |\nabla \tilde{H}_\lambda|]^{p-2} |\nabla u|$$
$$+ \frac{\beta l^{\beta+1}}{(\beta+1)^2} \int_{\{u > l\}} |\nabla \Gamma|^{p-2} |\nabla \eta_\varepsilon|^2 + \int_{B_{2r}} |\mathcal{G}| \eta_\varepsilon^2 (T_l u)^\beta \Bigg]. \quad (2.49)$$

At this point let $a = -\frac{(N-1)(2-p)}{2(p-1)}$. Since $p \leq 2$, then $a \leq 0$. Moreover $a \geq -1$ being $p \geq \frac{2N}{N+1}$. We observe that $|\nabla \Gamma|^{p-2} = \frac{C}{|x|^{2a}}$. Since $\eta_\varepsilon v_l$ is supported away from 0, notice that $\eta_\varepsilon v_l \in H_0^1(B_2)$. Thus we can consider $v_j \in C_0^\infty(B_2)$ converging to $\eta_\varepsilon v_l$ in $H_0^1(B_2)$ as $j \to +\infty$. Application of Caffarelli-Kohn-Nirenberg inequality to v_j gives

$$\left(\int_{B_2} |v_j|^{2_a^*} \right)^{\frac{2}{2_a^*}} \leq \tilde{C} \int_{B_2} |\nabla \Gamma|^{p-2} |\nabla v_j|^2, \quad (2.50)$$

where 2_a^* is defined by

$$2_a^* = \frac{2N(p-1)}{N-p}. \tag{2.51}$$

Now we pass to the limit $j \to +\infty$ in (2.50). By Fatou's lemma we get

$$\left(\int_{B_2} |\eta_\varepsilon v_l|^{2_a^*}\right)^{\frac{2}{2_a^*}} \leq \liminf_{j\to+\infty}\left(\int_{B_2} |v_j|^{2_a^*}\right)^{\frac{2}{2_a^*}}.$$

Being $p \leq 2$, as $j \to +\infty$

$$\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla v_j|^2 \to \int_{B_2} |\nabla\Gamma|^{p-2}|\nabla(\eta_\varepsilon v_l)|^2.$$

Thus (2.50) becomes

$$\left(\int_{B_2} |\eta_\varepsilon v_l|^{2_a^*}\right)^{\frac{2}{2_a^*}} \leq \tilde{C}\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla(\eta_\varepsilon v_l)|^2. \tag{2.52}$$

With the aid of (2.52), inequality (2.49) becomes

$$\begin{aligned}\left(\int_{B_2} |\eta_\varepsilon v_l|^{2_a^*}\right)^{\frac{2}{2_a^*}} &\leq C\left(1+\frac{(\beta+1)^2}{\beta^2}\right)\int_{\{u\leq l\}} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v_l^2 \\ &\quad + \frac{Cl^\beta(\beta+1)^2}{\beta}\int_{\{u>l\}} \eta_\varepsilon|\nabla\eta_\varepsilon|(|\nabla\Gamma|+|\nabla\tilde{H}|)^{p-2}|\nabla u| \\ &\quad + Cl^{\beta+1}\int_{\{u>l\}} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 + \frac{C(\beta+1)^2}{\beta}\int_{B_2} |\mathcal{G}|\eta_\varepsilon^2(T_l u)^\beta. \end{aligned}\tag{2.53}$$

Now we let $\varepsilon \to 0$ at fixed l in (2.53). Application of Fatou's lemma gives

$$\int_{B_2} |\eta v_l|^{2_a^*} \leq \liminf_{\varepsilon\to 0}\int_{B_2} |\nabla\Gamma|^{p-2}|\eta v_l|^{2_a^*}.$$

Recalling the properties of $\nabla\eta_\varepsilon$ and the definition of Γ, we get

$$\begin{aligned}&\int_{\{u\leq l\}} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v_l^2 \\ &\quad = \int_{\{u\leq l\}\setminus B_\varepsilon} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v_l^2 + \int_{\{u\leq l\}\cap B_\varepsilon} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v_l^2 \\ &\quad = \int_{\{u\leq l\}\setminus B_\varepsilon} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v_l^2 + O\left(l^{\beta+1}\frac{C\varepsilon^N}{\varepsilon^{2+\frac{(N-1)(p-2)}{p-1}}}\right) \\ &\quad = \int_{\{u\leq l\}\setminus B_\varepsilon} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v_l^2 + O\big(l^{\beta+1}\varepsilon^{\frac{N-p}{p-1}}\big),\end{aligned}$$

in view of $v_l^2 \le l^{\beta+1}$. As a consequence we obtain that

$$\int_{\{u\le l\}} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v_l^2 \to \int_{\{u\le l\}} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v_l^2 \quad \text{as } \varepsilon\to 0.$$

Similarly, we observe that

$$\begin{aligned}\int_{\{u>l\}} & \eta_\varepsilon|\nabla\eta_\varepsilon|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u|\\ &= \int_{\{u>l\}\setminus B_\varepsilon} \eta|\nabla\eta|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u|\\ &\qquad + \int_{\{u>l\}\cap B_\varepsilon} \eta_\varepsilon|\nabla\eta_\varepsilon|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u|,\end{aligned}$$

and, since $\nabla\eta_\varepsilon = 0$ outside $B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}$, we have that

$$\begin{aligned}\int_{\{u>l\}\cap B_\varepsilon} \eta_\varepsilon|\nabla\eta_\varepsilon|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u| &\le \int_{\{u>l\}\cap(B_\varepsilon\setminus B_{\frac{\varepsilon}{2}})} \eta_\varepsilon|\nabla\eta_\varepsilon||\nabla\tilde H_\lambda|^{p-1}\\ \le \frac{C}{\varepsilon}|B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}|^{\frac1N}\left(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}} |\nabla\tilde H_\lambda|^{\bar q}\right)^{\frac{N-1}{N}} &\le \tilde C\left(\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}} |\nabla\tilde H_\lambda|^{\bar q}\right)^{\frac{N-1}{N}},\end{aligned}$$

in view of $p\le 2$ and $|\nabla\Gamma|+|\nabla\tilde H_\lambda|\ge|\nabla\tilde H_\lambda|$. Using condition (1.9) we obtain

$$\int_{\{u>l\}} \eta_\varepsilon|\nabla\eta_\varepsilon|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u| \to \int_{\{u>l\}} \eta|\nabla\eta|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u|$$

as $\varepsilon\to 0$. Similarly

$$\int_{\{u>l\}} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 \to \int_{\{u>l\}} |\nabla\Gamma|^2|\nabla\eta|^2$$

as $\varepsilon\to 0$. Moreover, Lebesgue's theorem implies that

$$\int_{B_2} |\mathcal G|\eta_\varepsilon^2(T_l u)^\beta \to \int_{B_2} |\mathcal G|\eta^2(T_l u)^\beta \quad \text{as } \varepsilon\to 0.$$

We obtain the following estimate

$$\begin{aligned}\|\eta v_l\|_{2_a^*}^2 \le C\left(1+\frac{(\beta+1)^2}{\beta^2}\right)&\int_{\{u\le l\}} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v_l^2\\ + \frac{Cl^\beta(\beta+1)^2}{\beta}&\int_{\{u>l\}} \eta|\nabla\eta|(|\nabla\Gamma|+|\nabla\tilde H_\lambda|)^{p-2}|\nabla u|\\ + Cl^{\beta+1}\int_{\{u>l\}} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 &+ \frac{C(\beta+1)^2}{\beta}\int_{B_2} |\mathcal G|\eta^2(T_l u)^\beta. \quad (2.54)\end{aligned}$$

We study the second term and the third one in the right-hand side of (2.54). Since $\nabla\eta = 0$ in $B_{h'}$ and $u = |\tilde{H}_\lambda| + k \le C + k \le C'$ away from 0, then

$$\int_{\{u>l\}} \eta|\nabla\eta|(|\nabla\Gamma| + |\nabla\tilde{H}_\lambda|)^{p-2}|\nabla u| = 0 \quad \text{and} \quad \int_{\{u>l\}} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 = 0$$

for $l \gg 1$. As for the last term in the right-hand side of (2.54), since $k \le T_l u$ and recalling the definition of k, we have that

$$\int_{B_2} |\mathcal{G}|\eta^2(T_l u)^\beta = \int_{B_2} |\mathcal{G}|\eta^2(T_l u)^{\beta+1-1} \le \int_{B_2} \frac{|\mathcal{G}|}{k}\eta^2 v_l^2 \le \frac{1}{k}\|\mathcal{G}\|_{\frac{p}{p-1}}\|\eta v_l\|_{2p}^2 = \|\eta v_l\|_{2p}^2.$$

Since $T_l u \nearrow u$ as $l \to +\infty$, by the monotone convergence theorem we get

$$\|\eta v\|_{2_a^*}^2 \;\le\; C\left(1 \;+\; \frac{(\beta+1)^2}{\beta^2}\right)\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla\eta|^2 v^2 \;+\; \frac{C(\beta+1)^2}{\beta}\|\eta v\|_{2p}^2,$$

where v is defined as in (2.46). Recalling the definition of Γ and using that $p \le 2$, we obtain

$$\|\eta v\|_{2_a^*}^2 \le C\left(1 + \frac{(\beta+1)^2}{\beta^2}\right)\|\nabla\eta v\|_2^2 + \frac{C(\beta+1)^2}{\beta}\|\eta v\|_{2p}^2. \tag{2.55}$$

that is

$$\left(\int_{B_{h'}} u^{\frac{(\beta+1)2_a^*}{2}}\right)^{\frac{2}{2_a^*}} \le C\left(1 + \frac{(\beta+1)^2}{\beta^2}\right)(h-h')^{-2}\int_{B_h} u^{\beta+1} + \frac{C(\beta+1)^2}{\beta}\left(\int_{B_h} u^{p(\beta+1)}\right)^{\frac{1}{p}}, \tag{2.56}$$

in view of the properties of η and the definition of v.
At this point we define

$$\tilde{\kappa} = \frac{2_a^*}{2} = \frac{N(p-1)}{N-p}. \tag{2.57}$$

Since $\beta \ge \beta_0 > 0$, estimate (2.56) becomes

$$\|u\|_{\tilde{\kappa}(\beta+1),h'}^{\beta+1} \le C(\beta+1)^2[(h-h')^{-2}\|u\|_{\beta+1,h}^{\beta+1} + \|u\|_{p(\beta+1),h}^{\beta+1}], \tag{2.58}$$

from which follows

$$\|u\|_{\tilde{\kappa}(\beta+1),h'}^{\beta+1} \le C(\beta+1)^2(h-h')^{-2}\|u\|_{p(\beta+1),h}^{\beta+1}, \tag{2.59}$$

being $p > 1$. We put $\mu = p(\beta+1)$ and $\kappa = \frac{\tilde{\kappa}}{p}$ in such a way that $(\beta+1)\tilde{\kappa} = \kappa\mu$. By definition (2.57), we have that $\kappa > 1$ when $p > \sqrt{N}$. Indeed $\kappa > 1$ is equivalent to $\tilde{\kappa} > p$, which gives $p^2 > N$. Taking the $(\beta+1)$-th roots in (2.59) we have

$$\|u\|_{\kappa\mu,h'} \leq \left[\tilde{C}\mu(h-h')^{-1}\right]^{\frac{2p}{\mu}} \|u\|_{\mu,h}. \tag{2.60}$$

The required conclusion follows by iteration of inequality (2.60). We set for $\nu = 0, 1, 2, \ldots$

$$\mu_\nu = \kappa^\nu p(1+\beta_0), \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1}.$$

Hence (2.60) becomes

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\frac{1}{\kappa^\nu}} C_2^{\frac{\nu}{\kappa^\nu}} C_3^{\frac{\nu+1}{\kappa^\nu}} \|u\|_{\mu_\nu,h_\nu},$$

where

$$C_1 = \left(\tilde{C}p(1+\beta_0)\right)^{\frac{2}{1+\beta_0}}, \quad C_2 = \kappa^{\frac{2}{1+\beta_0}}, \quad C_3 = 4^{\frac{1}{1+\beta_0}}.$$

Iteration yields

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\sum_{i=0}^{\nu} \frac{1}{\kappa^i}} C_2^{\sum_{i=0}^{\nu} \frac{i}{\kappa^i}} C_3^{\sum_{i=0}^{\nu} \frac{i+1}{\kappa^i}} \|u\|_{\mu_0,h_0} \leq C\|u\|_{p(1+\beta_0),2},$$

because all series are convergent. Since

$$\|u\|_{\infty,1} = \lim_{\nu\to+\infty} \|u\|_{\mu_\nu,1} \leq \lim_{\nu\to+\infty} \|u\|_{\mu_\nu,h_\nu},$$

letting $\nu \to +\infty$ there results

$$\|u\|_{\infty,1} \leq C\|u\|_{p(1+\beta_0),2},$$

that is

$$\|\tilde{H}_\lambda\|_{\infty,1} \leq C(\|\tilde{H}_\lambda\|_{p(1+\beta_0),2} + k), \tag{2.61}$$

in view of $u = |\tilde{H}_\lambda| + k$. Recalling the definition of $\tilde{H}_\lambda$, we have that

$$\|\tilde{H}_\lambda\|_{\infty,1} = R^{\frac{N-p}{p-1}} \|H_\lambda\|_{\infty,R},$$

$$\|\tilde{H}_\lambda\|_{p(1+\beta_0),2} = R^{\frac{N-p}{p-1}} R^{-\frac{N}{p(1+\beta_0)}} \|H_\lambda\|_{p(1+\beta_0),2R}.$$

Moreover,

$$k = \|\mathcal{G}\|_{\frac{p}{p-1},2} = \lambda R^N \|\tilde{G}_\lambda\|_{p,2}^{p-1} = \lambda R^N \left(\int_{B_{2R}} R^{-N}|G_\lambda(y)|^p dy\right)^{\frac{p-1}{p}} = \lambda R^{\frac{N}{p}} \|G_\lambda\|_{p,2R}^{p-1},$$

in view of the definition of $\tilde{G}_\lambda$. Thus inequality (2.61) becomes

$$R^{\frac{N-p}{p-1}}\|H_\lambda\|_{\infty,R} \le C(R^{\frac{N-p}{p-1}}R^{-\frac{N}{p(1+\beta_0)}}\|H_\lambda\|_{p(1+\beta_0),2R} + R^{\frac{N}{p}}\|G_\lambda\|_{p,2R}^{p-1}). \quad (2.62)$$

Dividing both members of (2.62) by $R^{\frac{N-p}{p-1}}$ we finally obtain

$$\|H_\lambda\|_{\infty,R} \le C(R^{-\frac{N}{p(1+\beta_0)}}\|H_\lambda\|_{p(1+\beta_0),2R} + R^{\frac{p^2-N}{p(p-1)}}\|G_\lambda\|_{p,2R}^{p-1}). \quad (2.63)$$

Recalling that $H_\lambda \in L^s(\Omega)$ for all $s < \bar{q}^*$ and since $p < \bar{q}^*$ in view of $N < p^2$, then we are able to choose β_0 small such that $p(1+\beta_0) < \bar{q}^*$ and then (2.63) gives a bound of H_λ in B_R. This concludes the discussion of the case $p \le 2$.

Case $p > 2$. Application of (2.36) with $\Psi(s)$ as before gives

$$\begin{aligned}&\int_{\{u\le l\}} \beta\eta^p(T_lu)^{\beta-1}|\nabla u|^p + \int_{\{u\le l\}} \beta\eta^p|\nabla\Gamma|^{p-2}|\nabla u|^2(T_lu)^{\beta-1}\\ &\le C\bigg(\int_{B_2}\eta^{p-1}|\nabla\eta|[|\nabla\Gamma|^{p-2}+|\nabla\tilde{H}_\lambda|^{p-2}]|\nabla u|(T_lu)^\beta + \int_{B_2}|\mathcal{G}|\eta^p(T_lu)^\beta\bigg).\end{aligned} \quad (2.64)$$

We study the first term in the right-hand side of (2.64).

$$\begin{aligned}&C\int_{B_2}\eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u|(T_lu)^\beta\\ &= C\int_{\{u\le l\}}\eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u|(T_lu)^\beta + Cl^\beta\int_{\{u>l\}}\eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u|\\ &= C\int_{\{u\le l\}}\eta^{\frac{p}{2}}\eta^{\frac{p-2}{2}}|\nabla\eta||\nabla\Gamma|^{\frac{2(p-2)}{2}}(T_lu)^{\frac{\beta-1}{2}}(T_lu)^{\frac{\beta+1}{2}}|\nabla u|\\ &\qquad + Cl^\beta\int_{\{u>l\}}\eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u|\\ &\le \frac{\beta}{2}\int_{\{u\le l\}}\eta^p|\nabla\Gamma|^{p-2}(T_lu)^{\beta-1}|\nabla u|^2 + \frac{C''}{\beta}\int_{\{u\le l\}}\eta^{p-2}|\nabla\Gamma|^{p-2}|\nabla\eta|^2(T_lu)^{\beta+1}\\ &\qquad + Cl^\beta\int_{\{u>l\}}\eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u|,\end{aligned}$$

by virtue of Young's inequality. Similarly

$$\begin{aligned}&C\int_{B_2}\eta^{p-1}|\nabla\eta||\nabla\tilde{H}_\lambda|^{p-2}|\nabla u|(T_lu)^\beta\\ &= C\int_{\{u\le l\}}\eta^{p-1}|\nabla\eta||\nabla u|^{p-1}(T_lu)^\beta + Cl^\beta\int_{\{u>l\}}\eta^{p-1}|\nabla\eta||\nabla u|^{p-1}\\ &\le \frac{\beta}{2}\int_{\{u\le l\}}\eta^p(T_lu)^{\beta-1}|\nabla u|^p + \frac{C''}{\beta^{p-1}}\int_{\{u\le l\}}|\nabla\eta|^p(T_lu)^{\beta-1+p} + Cl^\beta\int_{\{u>l\}}\eta^{p-1}|\nabla\eta||\nabla u|^{p-1}.\end{aligned}$$

As in the case $p \leq 2$, we observe that

$$\int_{\{u>l\}} \eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}|\nabla u| = 0 \text{ for } l \gg 1,$$

$$\int_{\{u>l\}} \eta^{p-1}|\nabla\eta||\nabla u|^{p-1} = 0 \text{ for } l \gg 1.$$

Then, letting $l \to +\infty$ we get

$$\begin{aligned}
&\beta \int_{B_2} \eta^p u^{\beta-1}|\nabla u|^p + \beta \int_{B_2} \eta^p |\nabla\Gamma|^{p-2} u^{\beta-1}|\nabla u|^2 \\
&\quad \leq C\left(\frac{1}{\beta}\int_{B_2} |\nabla\Gamma|^2|\nabla\eta|^2 u^{\beta+1} + \frac{1}{\beta^{p-1}}\int_{B_2} |\nabla\eta|^p u^{\beta-1+p} + \int_{B_2} |\mathcal{G}|\eta^p u^\beta\right).
\end{aligned} \tag{2.65}$$

Since $k \leq u$, recalling the definition of k, we have that

$$\begin{aligned}
\int_{B_2} |\mathcal{G}|\eta^p u^\beta &= \int_{B_2} |\mathcal{G}|\eta^p u^{\beta-1+p} u^{1-p} \\
&\leq \frac{1}{k^{p-1}}\int_{B_2} |\mathcal{G}|\eta^p w^p = \frac{1}{k^{p-1}}\int_{B_2} |\mathcal{G}|(\eta w)^{\varepsilon_0}(\eta w)^{p-\varepsilon_0} \\
&\leq \frac{1}{k^{p-1}}\|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0}}\|\eta w\|_p^{\varepsilon_0}\|\eta w\|_{p^*}^{p-\varepsilon_0} = \|\eta w\|_p^{\varepsilon_0}\|\eta w\|_{p^*}^{p-\varepsilon_0},
\end{aligned}$$

where we have used the Hölder inequality with exponents $\frac{N}{p-\varepsilon_0}, \frac{p}{\varepsilon_0}, \frac{p^*}{p-\varepsilon_0}$, for some $\varepsilon_0 > 0$ small. The Sobolev embedding together with the Young inequality yield

$$\begin{aligned}
C\int_{B_2} |\mathcal{G}|\eta^p u^\beta &\leq \tilde{C}\|\eta w\|_p^{\varepsilon_0}(\|w\nabla\eta\|_p + \|\eta\nabla w\|_p)^{p-\varepsilon_0} \\
&\leq \frac{\beta p^p}{2(\beta-1+p)^p}\|\eta\nabla w\|_p^p + C'\left(\frac{(\beta-1+p)^p}{\beta}\right)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\|\eta w\|_p^p \\
&\qquad + \frac{C'\beta}{(\beta-1+p)^p}\|w\nabla\eta\|_p^p.
\end{aligned} \tag{2.66}$$

Recalling the definition of w, we get

$$\beta\int_{B_2} \eta^p u^{\beta-1}|\nabla u|^p = \frac{\beta p^p}{(\beta-1+p)^p}\int_{B_2} \eta^p|\nabla w|^p.$$

Moreover, since

$$\beta\int_{B_2}\eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2=\frac{4\beta}{(\beta+1)^2}\int_{B_2}\eta^p|\nabla\Gamma|^{p-2}|\nabla v|^2>0,$$

then estimates (2.65) and (2.66) imply

$$\frac{\beta}{(\beta-1+p)^p}\int_{B_2}|\nabla(\eta w)|^p\le C\Big(\frac{1}{\beta}\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2$$
$$+\Big(\frac{\beta}{(\beta-1+p)^p}+\frac{1}{\beta^{p-1}}\Big)\int_{B_2}|\nabla\eta|^pw^p+\Big(\frac{(\beta-1+p)^p}{\beta}\Big)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\int_{B_2}\eta^pw^p\Big),$$

which can be rewritten as

$$\int_{B_2}|\nabla(\eta w)|^p\le C(\beta-1+p)^{\frac{p^2}{\varepsilon_0}}\Big(\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2+\int_{B_2}|\nabla\eta|^pw^p+\int_{B_2}\eta^pw^p\Big),\tag{2.67}$$

being $\beta\ge\beta_0>0$ and $p>2$. At this point we observe that a simple use of the Hölder inequality gives

$$\int_{B_h}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2\le C(h-h')^{-2}\int_{B_h}u^{\beta+1}\le C(h-h')^{-p}\Big(\int_{B_h}w^p\Big)^{\frac{\beta+1}{\beta-1+p}},$$

since $B_h\subset B_2$ and C being a constant which does not depend on β. Then, recalling the properties of η and being $p>2$, (2.67) yields

$$\int_{B_2}|\nabla(\eta w)|^p\le C(\beta-1+p)^{\frac{p^2}{\varepsilon_0}}(h-h')^{-p}\Big(\int_{B_h}w^p\Big)^{\vartheta_h},$$

where $\vartheta_h\in(0,1]$ is defined by

$$\vartheta_h=\begin{cases}1 & \text{if } \int_{B_h}w^p\ge 1\\ \frac{\beta+1}{\beta-1+p} & \text{otherwise.}\end{cases}$$

Application of Sobolev inequality to ηw yields

$$\|\eta w\|^p_{p^*}\le C(\beta-1+p)^{\frac{p^2}{\varepsilon_0}}(h-h')^{-p}\|w\|^{p\vartheta_h}_{p,h}.$$

We set $\kappa=\frac{N}{N-p}>1$. We finally obtain

$$\|\eta w\|^p_{p\kappa}\le C(\beta-1+p)^{\frac{p^2}{\varepsilon_0}}(h-h')^{-p}\|w\|^{p\vartheta_h}_{p,h}.\tag{2.68}$$

Recalling the definition of w and the properties of η, (2.68) becomes

$$\left(\int_{B_{h'}} u^{(\beta-1+p)\kappa}\right)^{\frac{1}{\kappa}} \leq C(\beta-1+p)^{\frac{p^2}{\varepsilon_0}}(h-h')^{-p}\left(\int_{B_h} u^{\beta-1+p}\right)^{\vartheta_h},$$

that is, defining $\mu = \beta - 1 + p$,

$$\|u\|^{\mu}_{\mu\kappa,h'} \leq C\mu^{\frac{p^2}{\varepsilon_0}}(h-h')^{-p}\|u\|^{\vartheta_h\mu}_{\mu,h}.$$

Taking the μ-th roots, we have

$$\|u\|_{\mu\kappa,h'} \leq C^{\frac{1}{\mu}}\mu^{\frac{p^2}{\varepsilon_0\mu}}(h-h')^{-\frac{p}{\mu}}\|u\|^{\vartheta_h}_{\mu,h}. \tag{2.69}$$

The required conclusion follows by iteration of inequality (2.69) for $\beta \geq 1$ in such a way that $\mu \geq p$. To this purpose, we set for $\nu = 0, 1, 2, \ldots$

$$\mu_\nu = \kappa^\nu p, \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1}.$$

Hence (2.69) becomes

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\frac{1}{\kappa^\nu}} C_2^{\frac{\nu}{\kappa^\nu}}\|u\|^{\vartheta_{h_\nu}}_{\mu_\nu,h_\nu},$$

where $C_1 = 2(Cp^{\frac{p^2}{\varepsilon_0}})^{\frac{1}{p}}$ and $C_2 = 2\kappa^{\frac{p}{\varepsilon_0}}$. Since $\vartheta_h \leq 1$ and $C_1, C_2 > 0$, iteration yields

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\sum_{i=0}^{\nu}\frac{1}{\kappa^i}} C_2^{\sum_{i=0}^{\nu}\frac{i}{\kappa^i}}\|u\|^{\prod_{j=0}^{\nu}\vartheta_{h_j}}_{\mu_0,h_0} \leq C\|u\|^{\bar\vartheta}_{p,2},$$

because both series are convergent, where $\bar\vartheta = \lim_{\nu\to+\infty}\prod_{j=0}^{\nu}\vartheta_{h_j} \in [0,1]$. Similarly to the case $p \leq 2$, this leads to

$$\|\tilde H_\lambda\|_{\infty,1} \leq C(\|\tilde H_\lambda\|^{\bar\vartheta}_{p,2} + k^{\bar\vartheta} + k), \tag{2.70}$$

where

$$\|\tilde H_\lambda\|_{\infty,1} = R^{\frac{N-p}{p-1}}\|H_\lambda\|_{\infty,R},$$
$$\|\tilde H_\lambda\|_{p,2} = R^{\frac{N-p}{p-1}}R^{-\frac{N}{p}}\|H_\lambda\|_{p,2R}.$$

Moreover,

$$k = \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2} = (\lambda R^N)^{\frac{1}{p-1}}\|\tilde G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2} = \lambda^{\frac{1}{p-1}}R^{\frac{N-p+\varepsilon_0}{p-1}}\|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}.$$

Thus inequality (2.70) becomes

$$
\begin{aligned}
R^{\frac{N-p}{p-1}}\|H_\lambda\|_{\infty,R} \le C\Bigg[& R^{\frac{N-p}{p-1}} R^{-\frac{N}{p}}\|H_\lambda\|_{p,2R} \\
& + \left(\lambda^{\frac{1}{p-1}} R^{\frac{N-p+\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}\right)^{\bar\vartheta} + \lambda^{\frac{1}{p-1}} R^{\frac{N-p+\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}\Bigg].
\end{aligned} \tag{2.71}
$$

Dividing both members of (2.71) by $R^{\frac{N-p}{p-1}}$, we finally obtain

$$
\begin{aligned}
\|H_\lambda\|_{\infty,R} \le C\Bigg[& R^{-\frac{N}{p}}\|H_\lambda\|^{\bar\vartheta}_{p,2R} \\
& + \lambda^{\frac{1}{p-1}} R^{\frac{\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}\left((\lambda^{\frac{1}{p-1}} R^{\frac{\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R})^{\bar\vartheta-1} + 1\right)\Bigg],
\end{aligned} \tag{2.72}
$$

which gives the bound of H_λ in B_R. It is worth noting that $\bar\vartheta$ depends in general on u through the position of $\int_{B_{h_j}} w^p = \int_{B_{r(1+2^{-j})}} u^{\kappa^j p}$ with respect to 1. This means that estimate (2.72) is not universal, but it depends on the solution H_λ. □

Remark 2.2.8. In the proof of Proposition 2.2.7 we use different approaches depending on whether $p \le 2$ or $p > 2$. In the case $p \le 2$ we apply Lemma 2.2.3 and we obtain an estimate of $\eta_\varepsilon v_l$ in a weighted H^1_0 space. Thus, we are able to apply Theorem 2.2.6 which yields (2.54). Letting $l \to +\infty$, we arrive at the iteration process. In the case $p > 2$, we apply Corollary 2.2.4 that gives an estimate not depending on ε, from which we deduce the iterative scheme.

Remark 2.2.9. We briefly discuss the continuity of k defined by (3.5). When $N = 2, 3$ the hypothesis $\mathcal{G} \in L^{\frac{p}{p-1}}$ is stronger than $\mathcal{G} \in L^{\frac{N}{p-\varepsilon_0}}$. If $N = 4$, there is no discontinuity in $p = 2$. When $N \ge 5$, since we require $p > \sqrt{N}$, then we have to consider $\mathcal{G} \in L^{\frac{N}{p-\varepsilon_0}}$.

2.3 A uniqueness result when $p \ge 2$

This section is devoted to discuss the uniqueness part in Theorem 1.0.1 when $p \ge 2$, among solutions satisfying the natural condition (1.9). When $\lambda = 0$ maximum and comparison principle in weak or strong form are well known, see for example [36], and have been extended in various forms to the case $\lambda < \lambda_1$ in connection with existence and uniqueness results, see [8, 10, 16, 17] just to quote a few.

To extend the previous uniqueness results to the singular situation, the crucial property is the convexity of the functional

$$I(w) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^p & \text{if } w \geq 0 \text{ and } \nabla(w^{\frac{1}{p}}) \in L^p(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.73}$$

proved in [10] for all $p > 1$. A quantitative form is established here giving a positive lower bound for I'' when $p \geq 2$, crucial to be applied on $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0)$ as $\varepsilon \to 0$.

Lemma 2.3.1. *Let $I : L^1(\Omega) \to (-\infty, +\infty]$ be defined by* (2.73)*, with $p \geq 2$. Then we may write*

$$I''(w)[\varphi,\varphi] = \frac{d}{dt}I'(w_t)[\varphi]\Big|_{t=0^+} = \int_\Omega \varrho(w,\varphi)dx \tag{2.74}$$

where

$$\varrho(w,\varphi) = |\nabla w^{\frac{1}{p}}|^{p-2} w^{\frac{2(1-p)}{p}} \Big[C_1(p)\frac{\varphi}{w}\langle\nabla w, \nabla\varphi\rangle \\ + C_2(p)\frac{\varphi^2}{w^2}|\nabla w|^2 + C_3(p,\alpha)|\nabla\varphi|^2\Big]. \tag{2.75}$$

Moreover, $\varrho(\omega,\varphi) \geq 0$.

Proof. Given w and φ, we let $w_t = w + t\varphi$, $t \in \mathbb{R}$, and we assume $w_t \geq 0$ and $\nabla(w_t^{\frac{1}{p}}) \in L^p(\Omega)$ for $t \geq 0$ small. Since

$$|\nabla w^{\frac{1}{p}}|^p = [|\nabla w^{\frac{1}{p}}|^2]^{\frac{p}{2}},$$

then we can compute

$$\begin{aligned} I'(w)[\varphi] = \frac{d}{dt}I(w_t)\Big|_{t=0^+} &= \frac{1}{2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\frac{d}{dt}[|\nabla w_t^{\frac{1}{p}}|^2]\Big|_{t=0^+} dx \\ &= \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle\nabla w^{\frac{1}{p}}, \nabla\Big(\frac{d}{dt}w_t^{\frac{1}{p}}\Big)\rangle\Big|_{t=0^+} dx \\ &= \frac{1}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle\nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-p}{p}}\varphi)\rangle dx. \end{aligned}$$

Similarly, we can compute

$$I''(w)[\varphi,\varphi] = \frac{p-2}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}[\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-p}{p}}\varphi)\rangle]^2 dx + \frac{1}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla(w^{\frac{1-p}{p}}\varphi)|^2 dx + \frac{1-p}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-2p}{p}}\varphi^2)\rangle dx. \quad (2.76)$$

We expand the expressions $\nabla(w^{\frac{1}{p}})$, $\nabla(w^{\frac{1-p}{p}}\varphi)$ and $\nabla(w^{\frac{1-2p}{p}}\varphi^2)$ in (2.76). The first term in (2.76) becomes

$$\begin{aligned} \frac{p-2}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}[\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-p}{p}}\varphi)\rangle]^2 dx &= \frac{(p-2)(1-p)^2}{p^5}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}\varphi^2 w^{\frac{2(2-3p)}{p}}|\nabla w|^4 dx \\ &+ \frac{p-2}{p^3}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4} w^{\frac{4(1-p)}{p}}[\langle \nabla w, \nabla\varphi\rangle]^2 dx \\ &+ \frac{2(p-2)(1-p)}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi w^{\frac{2-3p}{p}}\langle \nabla w, \nabla\varphi\rangle dx. \end{aligned} \quad (2.77)$$

The second term in (2.76) becomes

$$\begin{aligned} &\frac{1}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla(w^{\frac{1-p}{p}}\varphi)|^2 dx \\ &= \frac{(1-p)^2}{p^4}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi^2 w^{\frac{2(1-2p)}{p}}|\nabla w|^2 + \frac{1}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} w^{\frac{2(1-p)}{p}}|\nabla\varphi|^2 dx \\ &\quad + \frac{2(1-p)}{p^3}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi w^{\frac{2-3p}{p}}\langle \nabla w, \nabla\varphi\rangle dx. \end{aligned} \quad (2.78)$$

The third term in (2.76) becomes

$$\begin{aligned} \frac{1-p}{p^2}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-2p}{p}}\varphi^2)\rangle dx &= \frac{(1-p)(1-2p)}{p^4}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi^2 w^{\frac{2(1-2p)}{p}}|\nabla w|^2 dx \\ &+ \frac{2(1-p)}{p^3}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi w^{\frac{2-3p}{p}}\langle \nabla w, \nabla\varphi\rangle dx. \end{aligned} \quad (2.79)$$

We may write the full expression of $I''(w)[\varphi,\varphi]$, using that $\langle \nabla w, \nabla\varphi\rangle = \cos\alpha|\nabla w||\nabla\varphi|$, as follows

$$
\begin{aligned}
I''(w)[\varphi,\varphi] = C_1(p)\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi w^{\frac{2-3p}{p}}\langle \nabla w,\nabla\varphi\rangle dx \\
+ C_2(p)\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\varphi^2 w^{\frac{2(1-2p)}{p}}|\nabla w|^2 \\
+ C_3(p,\alpha)\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} w^{\frac{2(1-p)}{p}}|\nabla\varphi|^2, \quad (2.80)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(p) &= \frac{2(1-p)}{p^3}(p^2-2p+2) < 0,\\
C_2(p) &= \frac{p-1}{p^4}(p^3-3p^2+5p-2) > 0,\\
C_3(p,\alpha) &= \frac{1}{p^2} + \frac{p-2}{p}|\cos\alpha|^2 > 0.
\end{aligned}
$$

We observe that (2.80) can be written in the following form

$$
\begin{aligned}
I''(w)[\varphi,\varphi] = \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2} w^{\frac{2(1-p)}{p}}\big[C_1(p)\frac{\varphi}{w}\langle \nabla w,\nabla\varphi\rangle \\
+ C_2(p)\frac{\varphi^2}{w^2}|\nabla w|^2 + C_3(p,\alpha)|\nabla\varphi|^2\big]. \quad (2.81)
\end{aligned}
$$

Completing the square, we have that

$$
\begin{aligned}
C_1(p)\frac{\varphi}{w}\langle \nabla w,\nabla\varphi\rangle + C_2(p)\frac{\varphi^2}{w^2}|\nabla w|^2 + C_3(p,\alpha)|\nabla\varphi|^2 \\
= \left(\sqrt{C_2(p)}\frac{\varphi}{w}|\nabla w| + \frac{C_1(p)}{2\sqrt{C_2(p)}}\cos\alpha|\nabla\varphi|\right)^2 \\
+ \left(C_3(p,\alpha) - \frac{(C_1(p))^2}{4C_2(p)}\cos^2\alpha\right)|\nabla\varphi|^2. \quad (2.82)
\end{aligned}
$$

At this point we study the inequality

$$
C_3(p,\alpha) - \frac{(C_1(p))^2}{4C_2(p)}\cos^2\alpha \geq 0, \quad (2.83)
$$

that is, recalling the definition of $C_3(p,\alpha)$,

$$
4C_2(p)\left(\frac{1}{p^2} + \frac{p-2}{p}\cos^2\alpha\right) - (C_1(p))^2\cos^2\alpha \geq 0. \quad (2.84)
$$

Thus we have

$$\frac{4C_2(p)}{p} \geq [(C_1(p))^2 p - 4C_2(p)(p-2)] \cos^2 \alpha. \tag{2.85}$$

The left-hand side of (2.85) is

$$\frac{4C_2(p)}{p} = \frac{4(p-1)}{p^5}(p^3 - 3p^2 + 5p - 2). \tag{2.86}$$

The right-hand side of (2.85) is

$$\begin{aligned} &[(C_1(p))^2 p - 4C_2(p)(p-2)] \cos^2 \alpha \\ &= \left[\frac{4(1-p)^2(p^2-2p+2)^2}{p^5} - \frac{4(p-1)(p-2)(p^3-3p^2+5p-2)}{p^4} \right] \cos^2 \alpha. \end{aligned} \tag{2.87}$$

We multiply both members of (2.85) for $\frac{p^5}{4(p-1)}$: the left-hand side (2.86) becomes

$$p^3 - 3p^2 + 5p - 2, \tag{2.88}$$

while the right-hand side (2.87) becomes

$$\begin{aligned} &[(p-1)(p^4 - 4p^3 + 8p^2 - 8p + 4) - (p^2 - 2p)(p^3 - 3p^2 + 5p - 2)] \cos^2 \alpha \\ &= (p^3 - 4p^2 + 8p - 4) \cos^2 \alpha. \end{aligned} \tag{2.89}$$

Therefore, inequality (2.83) can be rewritten as

$$p^3 - 3p^2 + 5p - 2 \geq (p^3 - 4p^2 + 8p - 4) \cos^2 \alpha. \tag{2.90}$$

Since $p > 1$ then $p^3 - 4p^2 + 8p - 4 > 0$. Thus we may assume $\cos^2 \alpha = 1$ in (2.90) and we need

$$p^3 - 3p^2 + 5p - 2 \geq p^3 - 4p^2 + 8p - 4, \tag{2.91}$$

that is

$$p^2 - 3p + 2 \geq 0, \tag{2.92}$$

which holds for all $p \geq 2$. All these facts imply that

$$I''(w)[\varphi, \varphi] = \int_\Omega \varrho(w, \varphi) dx$$

with $\varrho(w, \varphi) \geq 0$ being defined as in (2.75). $\square$

An application of Lemma 2.3.1 yields the validity of a weak comparison principle, which can be found in Appendix (see Proposition A.3.1).
Before turning to the main result of this section, we show that every solution G_λ of problem (1.5) is strictly positive.

Proposition 2.3.2. *Let G_λ be a solution of* (1.5)*. Then $G_\lambda > 0$ in Ω.*

Proof. We observe that there exists $\varepsilon > 0$ such that $G_\lambda = \Gamma + H_\lambda > 0$ in B_ε, in view of $\Gamma \to +\infty$ as $|y| \to 0$. Moreover, G_λ solves

$$\begin{cases} -\Delta_p G_\lambda - \lambda G_\lambda^{p-1} = 0 & \text{in } \Omega_\varepsilon \\ G_\lambda \geq 0 & \text{in } \Omega_\varepsilon \\ G_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$. If $\lambda \geq 0$ then $\lambda G_\lambda^{p-1} \geq 0$. Application of the strong maximum principle for the p-laplacian yields $G_\lambda > 0$ in Ω_ε. Thus $G_\lambda > 0$ in Ω. If $\lambda < 0$, application of the strong maximum principle for quasilinear elliptic equation in [36] leads to the desired conclusion. $\square$

Theorem 2.3.3. *Assume $p \geq 2$ and $p > \max\{\sqrt{N}, \frac{N}{2}\}$ if $\lambda \neq 0$. Then the solution of problem* (1.5) *is unique among those satisfying condition* (1.9)*.*

Proof. Letting G_1 and G_2 be two solutions of (1.5) satisfying (1.9), by elliptic regularity theory [11, 27, 30, 33] we know that $G_i \in C^{1,\alpha}(\bar{\Omega} \setminus \{0\})$, $i = 1, 2$, for some $\alpha > 0$. By [31] we know that G_i, $i = 1, 2$, satisfies (1.3) and by the strong maximum principle [36] $\partial_n G_i < 0$, $i = 1, 2$, on $\partial\Omega$, where n denotes the outward unit normal vector. Set $w_1 = G_1^p$, $w_2 = G_2^p$, $\varphi = w_1 - w_2$ and $w_s = sw_1 + (1-s)w_2$ for $s \in [0, 1]$. We have that for each $s \in [0, 1]$ there hold $w_s + t\varphi \geq 0$ in Ω and $\nabla(w_s + t\varphi)^{\frac{1}{p}} \in L^p(\Omega)$ for t small, in view of the properties of G_1 and G_2.
Letting I_ε be the functional I defined on $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$, we have that

$$\begin{aligned} {I'}_\varepsilon(w_1)[\varphi] - {I'}_\varepsilon(w_2)[\varphi] = \frac{1}{p}\Bigg[& \int_{\Omega_\varepsilon} |\nabla w_1^{\frac{1}{p}}|^{p-2} \langle \nabla w_1^{\frac{1}{p}}, \nabla (w_1^{\frac{1-p}{p}}(w_1 - w_2)) \rangle \\ & - \int_{\Omega_\varepsilon} |\nabla w_2^{\frac{1}{p}}|^{p-2} \langle \nabla w_2^{\frac{1}{p}}, \nabla (w_2^{\frac{1-p}{p}}(w_1 - w_2)) \rangle \Bigg], \end{aligned}$$

that is

$$I'_\varepsilon(w_1)[\varphi] - I'_\varepsilon(w_2)[\varphi] = \frac{1}{p}\int_{\Omega_\varepsilon}\left(\frac{-\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} - \frac{-\Delta_p w_2^{\frac{1}{p}}}{w_2^{\frac{p-1}{p}}}\right)(w_1 - w_2)$$
$$-\frac{1}{p}\int_{\partial B_\varepsilon(0)}[|\nabla w_1^{\frac{1}{p}}|^{p-2}\langle\nabla w_1^{\frac{1}{p}}, \mathrm{n}\rangle w_1^{\frac{1-p}{p}} - |\nabla w_2^{\frac{1}{p}}|^{p-2}\langle\nabla w_2^{\frac{1}{p}}, \mathrm{n}\rangle w_2^{\frac{1-p}{p}}](w_1 - w_2) \tag{2.93}$$

in view of $\varphi = 0$ on $\partial\Omega$. Since $w_1^{\frac{1}{p}} = G_1$ and $w_2^{\frac{1}{p}} = G_2$, using the equation (1.5) satisfied by G_1 and G_2, we obtain that

$$\int_{\Omega_\varepsilon}\left(\frac{-\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} - \frac{-\Delta_p w_2^{\frac{1}{p}}}{w_2^{\frac{p-1}{p}}}\right)(w_1 - w_2) = \int_{\Omega_\varepsilon}(\lambda - \lambda)(w_1 - w_2) = 0$$

and the second term on the right-hand side of (2.93) can be written as

$$-\frac{1}{p}\int_{\partial B_\varepsilon(0)}\left[\frac{|\nabla G_1|^{p-2}\partial_n G_1}{G_1^{p-1}} - \frac{|\nabla G_2|^{p-2}\partial_n G_2}{G_2^{p-1}}\right](G_1^p - G_2^p).$$

Therefore, (2.93) rewrites as

$$I'_\varepsilon(w_1)[\varphi] - I'_\varepsilon(w_2)[\varphi] = -\frac{1}{p}\int_{\partial B_\varepsilon(0)}\left[\frac{|\nabla G_1|^{p-2}\partial_n G_1}{G_1^{p-1}} - \frac{|\nabla G_2|^{p-2}\partial_n G_2}{G_2^{p-1}}\right](G_1^p - G_2^p). \tag{2.94}$$

Notice that

$$I'_\varepsilon(w_1)[\varphi] - I'_\varepsilon(w_2)[\varphi] = \int_0^1 I''_\varepsilon(w_s)[\varphi, \varphi]ds$$

with $I''_\varepsilon(w_s)[\varphi, \varphi] = \int_{\Omega_\varepsilon}\varrho(w_s, \varphi)$ in view of Lemma 2.3.1. Since $\varrho(w_s, \varphi) \geq 0$ when $p \geq 2$ in view of Lemma 2.3.1, by the Fatou's convergence theorem we deduce that

$$\int_0^1 ds\int_\Omega \varrho(w_s, \varphi) \leq \lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(0)}\left(\frac{|\nabla G_2|^{p-2}\partial_n G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2}\partial_n G_1}{G_1^{p-1}}\right)(G_1^p - G_2^p). \tag{2.95}$$

Now we are going to prove that the right-hand side in (2.95) vanishes. For $i = 1, 2$ notice that $H_i = G_i - \Gamma \in L^\infty(\Omega)$ follows by Proposition 2.2.7 in view of the assumption (1.9) for G_i. Once $H_i \in L^\infty(\Omega)$, we have that H_i satisfies $|\nabla H_i| = o(|\nabla\Gamma|)$ near 0 and then

$$G_i^q = \Gamma^q + O(\Gamma^{q-1}), \quad |\nabla G_i|^{p-2}\partial_n G_i = |\nabla\Gamma|^{p-2}\partial_n\Gamma + o(|\nabla\Gamma|^{p-1}) \quad (2.96)$$

as $x \to 0$ with $q \in \{p-1, p\}$. By (2.96) we deduce that $G_1^p - G_2^p = O(\Gamma^{p-1})$ and

$$\frac{|\nabla G_i|^{p-2}\partial_n G_i}{G_i^{p-1}} = \frac{|\nabla\Gamma|^{p-2}\partial_n\Gamma}{\Gamma^{p-1}} + o\left(\frac{|\nabla\Gamma|^{p-1}}{\Gamma^{p-1}}\right),$$

which imply

$$\Big|\int_{\partial B_\varepsilon(0)}\left(\frac{|\nabla G_2|^{p-2}\partial_n G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2}\partial_n G_1}{G_1^{p-1}}\right)(G_1^p - G_2^p)\Big| = o\left(\int_{\partial B_\varepsilon(0)}|\nabla\Gamma|^{p-1}\right) = o(1)$$

as $\varepsilon \to 0^+$, as claimed. As a consequence, $\varrho(w_s, \varphi) = 0$ for $s \in [0,1]$.
If $p > 2$ then

$$C_3(p,\alpha) - \frac{(C_1(p))^2}{4C_2(p)}\cos^2\alpha > 0$$

and so $\nabla w_1 = \nabla w_2$ a.e. in view of the definition of ϱ. If $p = 2$ we have

$$\begin{aligned}
\varrho(w_s,\varphi) &= w_s^{-1}\left(\sqrt{C_2(2)}\frac{\varphi}{w_s}|\nabla w_s| + \frac{C_1(2)}{2\sqrt{C_2(2)}}\cos\alpha|\nabla\varphi|\right)^2 \\
&= \frac{1}{2}w_s^{-1}\varphi\left(\frac{|\nabla w_s|}{w_s} - \cos\alpha\frac{|\nabla\varphi|}{\varphi}\right)^2 \\
&= \frac{1}{2}w_s^{-1}\varphi(|\nabla\log w_s| - \cos\alpha|\nabla\log\varphi|)^2.
\end{aligned}$$

If $\varphi = w_1 - w_2 = 0$, we end the proof. Otherwise we need

$$|\nabla\log w_s| = \cos\alpha|\nabla\log\varphi|,$$

which implies $\nabla w_1 = c\nabla w_2$ a.e. in Ω, being c a constant. Recalling the definition of w_1 and w_2, we obtain $c = 1$. Thus $\nabla(w_1 - w_2) = 0$ a.e., using that $w_1 = w_2 = 0$ on $\partial\Omega$, we finally arrive at $w_1 = w_2$ a.e., that is $G_1 = G_2$ a.e. in Ω.
The uniqueness result is so proved. □

Chapter 3

The regularity result

In this chapter we prove the main regularity result of this thesis, that is the Hölder continuity of H_λ at the pole. We will continue to assume $\lambda < \lambda_1$, $1 < p < N$ and the pole being at 0. All that we are going to prove hold even if the pole is at $x \in \Omega$ with $x \neq 0$.

In Section 3.1 we will use the Moser iterative scheme in [30] to establish local estimates for the solution H_λ of (1.6) at 0.
Section 3.2 is devoted to the proof of an inequality of Harnack type, which is the crucial tool to show Hölder estimates at 0.
In Section 3.3 we will show that when $1 < p \leq 2$ we are able to extend the regularity result to the whole domain Ω.

3.1 Local bound of H_λ

This section is devoted to the proof of local a priori estimates on H_λ. Such bounds are a consequence of the following rather general proposition.

Proposition 3.1.1. *Let $\mathcal{H} \in L^\infty(B_2)$, with $\nabla\mathcal{H} \in L^{\bar{q}}(B_2)$, be a solution of*

$$-\Delta_p(\Gamma + \mathcal{H}) + \Delta_p\Gamma = \mathcal{G} \qquad \text{in } B_2, \tag{3.1}$$

such that $|\nabla\mathcal{H}| = O(|\nabla\Gamma|)$ if $p \leq 2$ and $\|\mathcal{H}\|_{\infty,2} < 1 - \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2}$ when $p > 2$. Assume $\mathcal{G} \in L^{\frac{p}{p-1}}(B_2)$ when $p \leq 2$ and $\mathcal{G} \in L^{\frac{N}{p-\varepsilon_0}}(B_2)$ for some $\varepsilon_0 > 0$ when $p > 2$. When $\mathcal{G} \neq 0$ we require $p \geq \frac{2N}{N+1}$.

Then we have a universal bound for $\mathcal{H}$ in B_1.

- Case $p \leq 2$:

$$\|\mathcal{H}\|_{\infty,1} \leq C\|\mathcal{H}\|_{1+\beta_0,2} \quad \text{if } \mathcal{G} \equiv 0, \tag{3.2}$$

$$\|\mathcal{H}\|_{\infty,1} \leq C(\|\mathcal{H}\|_{p(1+\beta_0),2} + \|\mathcal{G}\|_{\frac{p}{p-1},2}) \quad \text{if } \mathcal{G} \neq 0. \tag{3.3}$$

- Case $p > 2$:

$$\|\mathcal{H}\|_{\infty,1} \leq C(\|\mathcal{H}\|_{1+\beta_0,2} + \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2}), \tag{3.4}$$

for some $\beta_0 > 0$.

In the proof of Proposition 3.1.1 we will use two lemmas which will be crucial tools to show the Harnack inequality. The first one will apply to problem (1.6) in the case $\lambda = 0$ with $p > 2$ and in the case $\lambda \neq 0$. The second one will apply to problem (1.6) when $\lambda = 0$ and $p \leq 2$.

Given a weight function ϱ, for real numbers $h > 0$, $s \neq 0$ and for $x \in \Omega$ we define

$$\Phi_\varrho(s,h) = \left(\int_{B_h(x)} \varrho |u|^s dx\right)^{\frac{1}{s}}.$$

Notice that for $s \geq 1$ we have $\Phi_\varrho(s,h) = \|u\|_{\varrho,s,h}$, that is the norm of u in the weighted space $L^s(\varrho, B_h(x))$. In particular, $\Phi_1(s,h)$ represents the norm of u in $L^s(B_h(x))$.

Lemma 3.1.2. *Let $\mathcal{H}$ and $\mathcal{G}$ be as in Proposition 3.1.1. We set $u = |\mathcal{H}| + k + \varepsilon'$, with $\varepsilon' > 0$ and k being defined as*

$$k = \begin{cases} \|\mathcal{G}\|_{\frac{p}{p-1},2} & \text{if } p \leq 2 \\ \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2} & \text{if } p > 2, \end{cases} \tag{3.5}$$

for some $\varepsilon_0 > 0$. Let h and h' be real numbers such that $1 \leq h' < h \leq 2$. We define

$$\kappa = \begin{cases} \frac{\tilde{\kappa}}{p} & \text{if } p \leq 2 \\ \frac{N}{N-2} & \text{if } p > 2, \end{cases} \tag{3.6}$$

$\tilde{\kappa}$ being as in (2.57). Then the following estimates hold.

- Case $p \le 2$:

$$\Phi_1(\kappa\mu, h') \le [C\mu(h-h')^{-1}]^{\frac{2p}{\mu}}\Phi_1(\mu, h) \quad \textit{if } \mu \in (0,p)\cup(p,+\infty), \tag{3.7}$$

$$\Phi_1(\kappa\mu, h') \ge [C\mu(h-h')^{-1}]^{\frac{2p}{\mu}}\Phi_1(\mu, h) \quad \textit{if } \mu \in (-\infty,0), \tag{3.8}$$

uniformly for μ *away from* 0 *and* p.

- Case $p > 2$:

$$\Phi_1(\kappa\mu, h') \le [C|\mu+p-2|^{\alpha}(h-h')^{-p}]^{\frac{1}{\mu}}\Phi_1(\mu, h) \quad \textit{if } \mu \in (0,1)\cup(1,+\infty), \tag{3.9}$$

$$\Phi_1(\kappa\mu, h') \ge [C|\mu+p-2|^{\alpha}(h-h')^{-p}]^{\frac{1}{\mu}}\Phi_1(\mu, h) \quad \textit{if } \mu \in (-\infty,2-p)\cup(2-p,0), \tag{3.10}$$

uniformly for μ *away from* $2-p$, 0 *and* 1, *where* $\alpha = 2 + \frac{p(p-\varepsilon_0)}{\varepsilon_0}$.

Proof. Let η and η_ε be nonnegative smooth functions satisfying properties (2.44), (2.45) and (2.27) with $r = 1$. Since $\mathcal{H} \in L^\infty(B_2)$ and $\nabla\mathcal{H} \in L^{\bar{q}}(B_2)$, then for any $\beta \in \mathbb{R}$ the function

$$\varphi = \begin{cases} \eta_\varepsilon^2 \operatorname{sgn}\mathcal{H}(u^\beta - (k+\varepsilon')^\beta) & \text{if } p \le 2 \\ \eta_\varepsilon^p \operatorname{sgn}\mathcal{H}(u^\beta - (k+\varepsilon')^\beta) & \text{if } p > 2 \end{cases}$$

is admissible in (3.1). Notice that if $\beta < 0$ then $u \ge \varepsilon' > 0$ and the problem is well-posed. We are going to prove universal estimates for u provided that β is uniformly away from $1-p$ if $p > 2$, 0 and -1.
We define $v = u^{\frac{\beta+1}{2}}$ and $w = u^{\frac{\beta-1+p}{p}}$. We will apply Lemma 2.2.3 or Corollary 2.2.4 with $\Psi(s) = \operatorname{sgn}(s)\operatorname{sgn}(\beta)[(|s|+k+\varepsilon')^\beta - (k+\varepsilon')^\beta]$. We observe that $\Psi'(s) = |\beta|(|s|+k+\varepsilon')^{\beta-1}$ and $|\Psi(s)| \le (|s|+k+\varepsilon')^\beta$.

Assume $p \le 2$. Application of (2.29) gives

$$\frac{|\beta|}{(\beta+1)^2}\int_{B_2}\eta_\varepsilon^2|\nabla\Gamma|^{p-2}|\nabla v|^2 \le C\left(\int_{B_2}\eta_\varepsilon u^\beta|\nabla\eta_\varepsilon|[|\nabla\Gamma|+|\nabla\mathcal{H}|]^{p-2}|\nabla u| + \int_{B_2}|\mathcal{G}|\eta_\varepsilon^2 u^\beta\right). \tag{3.11}$$

Then, using that $|\nabla\Gamma| + |\nabla\mathcal{H}| \geq |\nabla\Gamma|$ and $p \leq 2$, we can estimate the first term of the right-hand side of (3.11) as follows:

$$
\begin{aligned}
C\int_{B_2} &\eta_\varepsilon u^\beta |\nabla\eta_\varepsilon|[|\nabla\Gamma| + |\nabla\mathcal{H}|]^{p-2}|\nabla u| \\
&\leq C\int_{B_2} \eta_\varepsilon u^{\frac{\beta-1}{2}}|\nabla\eta_\varepsilon||\nabla\Gamma|^{p-2}|\nabla u| u^{\frac{\beta+1}{2}} \\
&= \frac{2C}{|\beta+1|}\int_{B_2} \eta_\varepsilon v|\nabla\eta_\varepsilon||\nabla\Gamma|^{\frac{2(p-2)}{2}}|\nabla v| \\
&\leq \frac{|\beta|}{2(\beta+1)^2}\int_{B_2} \eta_\varepsilon^2|\nabla\Gamma|^{p-2}|\nabla v|^2 + \frac{C'}{|\beta|}\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v^2,
\end{aligned}
$$

applying in the last passage the Young inequality. Thus (3.11) rewrites as

$$
\begin{aligned}
\frac{|\beta|}{(\beta+1)^2}&\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla(\eta_\varepsilon v)|^2 \\
&\leq C\left[\left(\frac{1}{|\beta|} + \frac{|\beta|}{(\beta+1)^2}\right)\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v^2 + \int_{B_2}|\mathcal{G}|\eta_\varepsilon^2 u^\beta\right]. \quad (3.12)
\end{aligned}
$$

At this point let $a = -\frac{(N-1)(2-p)}{2(p-1)}$. Since $p \leq 2$, then $a \leq 0$. Moreover we observe that $|\nabla\Gamma|^{p-2} = \frac{C}{|x|^{2a}}$. Since $\mathcal{H} \in L^\infty(B_2)$ and $\eta_\varepsilon v$ is supported away from 0, notice that $\eta_\varepsilon v \in H_0^1(B_2)$. Thus we can consider $v_j \in C_0^\infty(B_2)$ converging to $\eta_\varepsilon v$ in $H_0^1(B_2)$ as $j \to +\infty$. Application of Caffarelli-Kohn-Nirenberg inequality to v_j with $b = 0$ (see Theorem 2.2.6) gives

$$
\left(\int_{B_2}|v_j|^{2_a^*}\right)^{\frac{2}{2_a^*}} \leq C\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla v_j|^2,
$$

where 2_a^* is defined as in (2.51). Using the same calculations as in the proof of Proposition 2.2.7 when $p \leq 2$, we arrive at

$$
\left(\int_{B_{h'}} u^{\kappa\mu}\right)^{\frac{1}{\kappa\mu}} \leq [C\mu(h-h')^{-1}]^{\frac{2p}{\mu}}\left(\int_{B_h} u^\mu\right)^{\frac{1}{\mu}},
$$

in terms of $\mu = p(\beta+1)$. In order to apply estimates (2.58), (2.59) and (2.60), we require that $\beta \in (-1,0)\cup(0,+\infty)$, i.e. $\mu \in (0,p)\cup(p,+\infty)$. Then (3.7) is proved. As for (3.8), we observe that taking the $(\beta+1)$-th roots with $\beta \in (-\infty,-1)$ in (2.59) implies that the sign in (2.60) is reversed. This gives

$$
\left(\int_{B_{h'}} u^{\kappa\mu}\right)^{\frac{1}{\kappa\mu}} \geq [C\mu(h-h')^{-1}]^{\frac{2p}{\mu}}\left(\int_{B_h} u^\mu\right)^{\frac{1}{\mu}}
$$

for $\mu \in (-\infty, 0)$, that is (3.8).

Now we assume $p > 2$. We assume in addition that $\beta \neq 1 - p$. Application of (2.36) gives

$$\int_{B_2} |\beta|\eta^p u^{\beta-1}|\nabla u|^p + \int_{B_2} |\beta|\eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2$$
$$\leq C\Big(\int_{B_2} \eta^{p-1}|\nabla\eta|[|\nabla\Gamma|^{p-2} + |\nabla\mathcal{H}|^{p-2}]|\nabla u|u^\beta + \int_{B_2} |\mathcal{G}|\eta^p u^\beta\Big). \quad (3.13)$$

Moreover, since $\sup_{B_2\setminus B_1} |\nabla\Gamma|^{p-2} < +\infty$,

$$C\int_{B_2} \eta^{p-1}|\nabla\eta||\nabla\Gamma|^{p-2}u^\beta|\nabla u| = C\int_{B_2} \eta^{\frac{p}{2}}\eta^{\frac{p-2}{2}}|\nabla\eta||\nabla\Gamma|^{\frac{2(p-2)}{2}}u^{\frac{\beta-1}{2}}u^{\frac{\beta+1}{2}}|\nabla u|$$
$$\leq \frac{|\beta|}{2}\int_{B_2} \eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2 + \frac{C'}{|\beta|}\int_{B_2} \eta^{p-2}|\nabla\eta|^2u^{\beta+1}$$
$$\leq \frac{|\beta|}{2}\int_{B_2} \eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2 + \frac{C'}{|\beta|}\int_{B_2} |\nabla\eta|^2u^{\beta+1},$$

by virtue of Young's inequality. Similarly

$$C\int_{B_2} \eta^{p-1}|\nabla\eta||\nabla\mathcal{H}|^{p-2}u^\beta|\nabla u| = C\int_{B_2} \eta^{p-1}|\nabla u|^{p-1}u^{\frac{(\beta-1)(p-1)}{p}}u^{\frac{\beta-1+p}{p}}|\nabla\eta|$$
$$\leq \frac{|\beta|}{2}\int_{B_2} \eta^p|\nabla u|^p u^{\beta-1} + \frac{C''}{|\beta|}\int_{B_2} u^{\beta-1+p}|\nabla\eta|^p.$$

As a consequence, (3.13) can be written as

$$|\beta|\int_{B_2} \eta^p u^{\beta-1}|\nabla u|^p + |\beta|\int_{B_2} \eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2$$
$$\leq C\Big(\frac{1}{|\beta|^{p-1}}\int_{B_2} u^{\beta-1+p}|\nabla\eta|^p + \frac{1}{|\beta|}\int_{B_2} |\nabla\eta|^2u^{\beta+1} + \int_{B_2} |\mathcal{G}|\eta^p u^\beta\Big). \quad (3.14)$$

Since $k \leq u$, recalling the definition of k, we have that

$$\int_{B_2} |\mathcal{G}|\eta^p u^\beta = \int_{B_2} |\mathcal{G}|\eta^p u^{\beta-1+p}u^{1-p}$$
$$\leq \frac{1}{k^{p-1}}\int_{B_2} |\mathcal{G}|\eta^p w^p = \frac{1}{k^{p-1}}\int_{B_2} |\mathcal{G}|(\eta w)^{\varepsilon_0}(\eta w)^{p-\varepsilon_0} \quad (3.15)$$
$$\leq \frac{1}{k^{p-1}}\|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0}}\|\eta w\|_p^{\varepsilon_0}\|\eta w\|_{p^*}^{p-\varepsilon_0} = \|\eta w\|_p^{\varepsilon_0}\|\eta w\|_{p^*}^{p-\varepsilon_0},$$

where we have used the Hölder inequality with exponents $\frac{N}{p-\varepsilon_0}, \frac{p}{\varepsilon_0}, \frac{p^*}{p-\varepsilon_0}$, for some $\varepsilon_0 > 0$. The Sobolev embedding together with the Young inequality yield

$$
\begin{aligned}
C\int_{B_2} |\mathcal{G}|\eta^p u^\beta &\le \tilde{C}\|\eta w\|_p^{\varepsilon_0}(\|w\nabla\eta\|_p + \|\eta\nabla w\|_p)^{p-\varepsilon_0} \\
&\le \frac{|\beta|p^p}{2|\beta-1+p|^p}\|\eta\nabla w\|_p^p + C'\left(\frac{|\beta-1+p|^p}{|\beta|}\right)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\|\eta w\|_p^p \\
&\qquad + \frac{C'|\beta|}{|\beta-1+p|^p}\|w\nabla\eta\|_p^p.
\end{aligned} \tag{3.16}
$$

Recalling the definition of w, we get

$$
|\beta|\int_{B_2} \eta^p u^{\beta-1}|\nabla u|^p = \frac{|\beta|p^p}{|\beta-1+p|^p}\int_{B_2}\eta^p|\nabla w|^p.
$$

Thus, estimates (3.14) and (3.16) imply

$$
\begin{aligned}
|\beta|\int_{B_2}\eta^p|\nabla\Gamma|^{p-2}u^{\beta-1}|\nabla u|^2 &\le C\Bigg(\frac{1}{|\beta|}\int_{B_2}|\nabla\eta|^2u^{\beta+1} \\
+ \frac{|\beta-1+p|^p+|\beta|^p}{|\beta|^{p-1}|\beta-1+p|^p}&\int_{B_2}|\nabla\eta|^p w^p + \left(\frac{|\beta-1+p|^p}{|\beta|}\right)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\int_{B_2}\eta^p w^p\Bigg),
\end{aligned} \tag{3.17}
$$

which can be rewritten, recalling the definition of v, as

$$
\begin{aligned}
\frac{|\beta|}{(\beta+1)^2}\int_{B_2}\eta^p|\nabla v|^2 \le C\Bigg(\frac{1}{|\beta|}\int_{B_2}|\nabla\eta|^2v^2 &+ \frac{|\beta-1+p|^p+|\beta|^p}{|\beta|^{p-1}|\beta-1+p|^p}\int_{B_2}|\nabla\eta|^p w^p \\
&+ \left(\frac{|\beta-1+p|^p}{|\beta|}\right)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\int_{B_2}\eta^p w^p\Bigg),
\end{aligned} \tag{3.18}
$$

in view of $|\nabla\Gamma|^{p-2} \ge \delta > 0$. Since $\|u\|_\infty \le 1$ and $p > 2$, we can consider $w^p = u^{\beta+1}u^{p-2} \le v^2$ in (3.18). Since $|\beta| \ge \beta_0 > 0$, we obtain

$$
\int_{B_2}\eta^p|\nabla v|^2 \le C|\beta-1+p|^\alpha\left(\int_{B_2}|\nabla\eta|^2v^2 + \int_{B_2}|\nabla\eta|^pv^2 + \int_{B_2}\eta^pv^2\right), \tag{3.19}
$$

where $\alpha = 2 + \frac{p(p-\varepsilon_0)}{\varepsilon_0}$. Moreover, letting $\gamma = \eta^{\frac{p}{2}}$, we have that $|\nabla\gamma|^2 = |\frac{p}{2}\eta^{\frac{p}{2}-1}\nabla\eta|^2 \le C|\nabla\eta|^2$. Application of Sobolev inequality to γv yields

$$
\|\gamma v\|_{\frac{2N}{N-2}}^2 \le C\left(\int_\Omega|\nabla\gamma|^2v^2 + \int_\Omega\gamma^2|\nabla v|^2\right).
$$

Using all these facts in (3.19), it turns out that

$$\|\gamma v\|_{2\kappa}^2 \le C|\beta - 1 + p|^\alpha \left(\int_{B_2} |\nabla \eta|^2 v^2 + \int_{B_2} |\nabla \eta|^p v^2 + \int_{B_2} \eta^p v^2 \right). \tag{3.20}$$

Recalling the definition of v and the properties of γ and η, (3.20) becomes

$$\left(\int_{B_{h'}} u^{\kappa(\beta+1)} \right)^{\frac{1}{\kappa}} \le C|\beta - 1 + p|^\alpha (h - h')^{-p} \int_{B_h} u^{\beta+1}.$$

Set $\mu = \beta + 1$. Taking the μ-th roots, we have

$$\left(\int_{B_{h'}} u^{\kappa\mu} \right)^{\frac{1}{\kappa\mu}} \le C^{\frac{1}{\mu}} |\mu + p - 2|^{\frac{\alpha}{\mu}} (h - h')^{-\frac{p}{\mu}} \left(\int_{B_h} u^{\mu} \right)^{\frac{1}{\mu}} \tag{3.21}$$

if $\beta \in (-1, 0) \cup (0, +\infty)$, i.e. $\mu \in (0, 1) \cup (1, +\infty)$, and

$$\left(\int_{B_{h'}} u^{\kappa\mu} \right)^{\frac{1}{\kappa\mu}} \ge C^{\frac{1}{\mu}} |\mu + p - 2|^{\frac{\alpha}{\mu}} (h - h')^{-\frac{p}{\mu}} \left(\int_{B_h} u^{\mu} \right)^{\frac{1}{\mu}} \tag{3.22}$$

if $\beta \in (-\infty, 1 - p) \cup (1 - p, -1)$, i.e. $\mu \in (-\infty, 2 - p) \cup (2 - p, 0)$. This concludes the proof of Lemma 3.1.2. □

Lemma 3.1.3. *Let $\mathcal{H}, h, h'$ be as in Lemma 3.1.2 with $u = |\mathcal{H}| + \varepsilon'$. We assume $\mathcal{G} \equiv 0$ and $p \le 2$. We define*

$$\kappa = \frac{N - 2 + p}{N - p} \quad \text{and} \quad \varrho = |\nabla \Gamma|^{p-2}.$$

Then the following estimates hold:

$$\Phi_\varrho(\kappa\mu, h') \le [C\mu(h - h')^{-1}]^{\frac{2}{\mu}} \Phi_\varrho(\mu, h) \quad \text{if } \mu \in (0, 1) \cup (1, +\infty), \tag{3.23}$$

$$\Phi_\varrho(\kappa\mu, h') \ge [C\mu(h - h')^{-1}]^{\frac{2}{\mu}} \Phi_\varrho(\mu, h) \quad \text{if } \mu \in (-\infty, 0), \tag{3.24}$$

uniformly for μ away from 0 and 1.

Proof. The first part of the proof goes as in Lemma 3.1.2, except that now $\mathcal{G} \equiv 0$. Then (3.12) rewrites as

$$\frac{|\beta|}{(\beta+1)^2} \int_{B_2} |\nabla \Gamma|^{p-2} |\nabla(\eta_\varepsilon v)|^2 \le C \left(\frac{1}{|\beta|} + \frac{|\beta|}{(\beta+1)^2} \right) \int_{B_2} |\nabla \Gamma|^{p-2} |\nabla \eta_\varepsilon|^2 v^2. \tag{3.25}$$

Letting $a = -\frac{(N-1)(2-p)}{2(p-1)}$ and observing that $\eta_\varepsilon v \in H_0^1(B_2)$, we can consider $v_j \in C_0^\infty(B_2)$ converging to $\eta_\varepsilon v$ in $H_0^1(B_2)$ as $j \to +\infty$. Application of Caffarelli-Kohn-Nirenberg inequality to v_j with $b = a - \frac{2a}{N-2a}$, $b \in (a, a+1)$ gives

$$\left(\int_{B_2} |\nabla\Gamma|^{p-2}|v_j|^c\right)^{\frac{2}{c}} \leq C\int_{B_2} |x|^{-2a}|\nabla v_j|^2 = \tilde{C}\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla v_j|^2, \quad (3.26)$$

where c is defined as

$$c = \frac{2N}{N-2+2(b-a)} = \frac{2(N-2a)}{N-2-2a} = \frac{2(N-2+p)}{N-p}$$

in view of

$$b = -\frac{(N-1)(2-p)(N-p)}{2(p-1)(N-2+p)} \quad \text{and} \quad bc = -\frac{(N-1)(2-p)}{p-1} = 2a.$$

Now we pass to the limit $j \to +\infty$ in (3.26). By Fatou's lemma we have

$$\left(\int_{B_2} |\nabla\Gamma|^{p-2}|\eta_\varepsilon v|^c\right)^{\frac{2}{c}} \leq \liminf_{j\to+\infty}\left(\int_{B_2} |\nabla\Gamma|^{p-2}|v_j|^c\right)^{\frac{2}{c}}.$$

Being $p \leq 2$, as $j \to +\infty$

$$\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla v_j|^2 \to \int_{B_2} |\nabla\Gamma|^{p-2}|\nabla(\eta_\varepsilon v)|^2.$$

Thus (3.26) becomes

$$\left(\int_{B_2} |\nabla\Gamma|^{p-2}|\eta_\varepsilon v|^c\right)^{\frac{2}{c}} \leq C\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla(\eta_\varepsilon v)|^2. \quad (3.27)$$

With the aid of (3.27), the inequality (3.25) becomes

$$\left(\int_{B_2} |\nabla\Gamma|^{p-2}|\eta_\varepsilon v|^c\right)^{\frac{2}{c}} \leq C\left(1 + \frac{(\beta+1)^2}{\beta^2}\right)\int_{B_2} |\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2 v^2. \quad (3.28)$$

At this point we let $\varepsilon \to 0$ in (3.28). Application of Fatou's lemma gives

$$\int_{B_2} |\nabla\Gamma|^{p-2}|\eta v|^c \leq \liminf_{\varepsilon\to 0}\int_{B_2} |\nabla\Gamma|^{p-2}|\eta_\varepsilon v|^c.$$

Recalling the properties of $\nabla\eta_\varepsilon$ and the definition of Γ, we get

$$\begin{aligned}\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2v^2 &= \int_{B_2\setminus B_\varepsilon}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2+\int_{B_\varepsilon\setminus B_{\frac{\varepsilon}{2}}}|\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2v^2\\ &= \int_{B_2\setminus B_\varepsilon}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2+O\Big(\|v\|_\infty^2\frac{C\varepsilon^N}{\varepsilon^{2+\frac{(N-1)(p-2)}{p-1}}}\Big)\\ &= \int_{B_2\setminus B_\varepsilon}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2+O\big(\varepsilon^{\frac{N-p}{p-1}}\big).\end{aligned}$$

As a consequence we obtain that

$$\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta_\varepsilon|^2v^2\to\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2\quad\text{as }\varepsilon\to 0.$$

Thus we have the following estimate

$$\Big(\int_{B_2}|\nabla\Gamma|^{p-2}|\eta v|^c\Big)^{\frac{2}{c}}\le C\Big(1+\frac{(\beta+1)^2}{\beta^2}\Big)\int_{B_2}|\nabla\Gamma|^{p-2}|\nabla\eta|^2v^2,$$

that is

$$\Big(\int_{B_{h'}}|\nabla\Gamma|^{p-2}u^{\frac{(\beta+1)c}{2}}\Big)^{\frac{2}{c}}\le C\Big(1+\frac{(\beta+1)^2}{\beta^2}\Big)(h-h')^{-2}\int_{B_h}|\nabla\Gamma|^{p-2}u^{\beta+1},\tag{3.29}$$

in view of the properties of η and the definition of v. Observing that $\kappa=\frac{c}{2}$, setting $\mu=\beta+1$ and recalling the definition of ϱ, estimate (3.29) becomes

$$\Big(\int_{B_{h'}}\varrho u^{\kappa\mu}\Big)^{\frac{1}{\kappa}}\le C\mu^2(h-h')^{-2}\int_{B_h}\varrho u^\mu,\tag{3.30}$$

being $|\beta|\ge\beta_0>0$. Taking the μ-th roots in (3.30), we have

$$\Big(\int_{B_{h'}}\varrho u^{\kappa\mu}\Big)^{\frac{1}{\kappa\mu}}\le\big[\tilde C\mu(h-h')^{-1}\big]^{\frac{2}{\mu}}\Big(\int_{B_h}\varrho u^\mu\Big)^{\frac{1}{\mu}}$$

if $\mu\in(0,1)\cup(1,+\infty)$, which yields (3.23), and

$$\Big(\int_{B_{h'}}\varrho u^{\kappa\mu}\Big)^{\frac{1}{\kappa\mu}}\ge\big[\tilde C\mu(h-h')^{-1}\big]^{\frac{2}{\mu}}\Big(\int_{B_h}\varrho u^\mu\Big)^{\frac{1}{\mu}}$$

if $\mu\in(-\infty,0)$, which yields (3.24). $\square$

Proof of Proposition 3.1.1. Assume $p \leq 2$. If $\mathcal{G} \equiv 0$, the required conclusion follows by iteration of inequality (3.23) in Lemma 3.1.3 for $\mu > 1$, which rewrites as

$$\|u\|_{\varrho,\kappa\mu,h'} \leq [C\mu(h-h')^{-1}]^{\frac{2}{\mu}} \|u\|_{\varrho,\mu,h}. \tag{3.31}$$

We set for $\nu = 0, 1, 2, \ldots$

$$\mu_\nu = \kappa^\nu(1+\beta_0), \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1}.$$

Hence (3.31) becomes

$$\|u\|_{\varrho,\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\frac{1}{\kappa^\nu}} C_2^{\frac{\nu}{\kappa^\nu}} C_3^{\frac{\nu+1}{\kappa^\nu}} \|u\|_{\varrho,\mu_\nu,h_\nu},$$

where

$$C_1 = C^{\frac{2}{1+\beta_0}}(1+\beta_0)^{\frac{2}{1+\beta_0}}, \quad C_2 = \kappa^{\frac{2}{1+\beta_0}}, \quad C_3 = 4^{\frac{1}{1+\beta_0}}.$$

Iteration yields

$$\|u\|_{\varrho,\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\sum_{i=0}^{\nu}\frac{1}{\kappa^i}} C_2^{\sum_{i=0}^{\nu}\frac{i}{\kappa^i}} C_3^{\sum_{i=0}^{\nu}\frac{i+1}{\kappa^i}} \|u\|_{\varrho,\mu_0,h_0} \leq C\|u\|_{\varrho,1+\beta_0,2},$$

because all series are convergent. Since

$$\|u\|_{\infty,1} = \lim_{\nu\to+\infty} \|u\|_{\varrho,\mu_\nu,1} \leq \lim_{\nu\to+\infty} \|u\|_{\varrho,\mu_\nu,h_\nu},$$

letting $\nu \to +\infty$ there results

$$\|u\|_{\infty,1} \leq C\|u\|_{\varrho,1+\beta_0,2} \leq C\|u\|_{1+\beta_0,2},$$

in view of $\varrho \leq \varrho_0$. Recalling that $u = |\mathcal{H}| + \varepsilon'$ and letting $\varepsilon' \to 0$, we finally obtain

$$\|\mathcal{H}\|_{\infty,1} \leq C\|\mathcal{H}\|_{1+\beta_0,2},$$

that is (3.2).
On the other hand, if $\mathcal{G} \neq 0$ we apply (3.7) in Lemma 3.1.2. Iteration goes as in the proof of Proposition 2.2.7 when $p \leq 2$, except that now we are working with $\mathcal{H}$ in B_2. Then we finally obtain

$$\|\mathcal{H}\|_{\infty,1} \leq C(\|\mathcal{H}\|_{p(1+\beta_0),2} + k),$$

that is (3.3).

Assume $p > 2$. Application of inequality (3.9) in Lemma 3.1.2 for $\mu > 1$ gives

$$\|u\|_{\mu\kappa,h'} \leq [C|\mu + p - 2|^{\alpha}(h - h')^{-p}]^{\frac{1}{\mu}} \|u\|_{\mu,h}. \tag{3.32}$$

We set for $\nu = 0, 1, 2, \ldots$

$$\mu_\nu = \kappa^\nu(1 + \beta_0), \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1}.$$

Hence (3.32) becomes

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\frac{1}{\kappa^\nu}} C_2^{\frac{\nu}{\kappa^\nu}} \|u\|_{\mu_\nu,h_\nu},$$

where $C_1 = [C2^p(1+\beta_0)^\alpha]^{\frac{1}{1+\beta_0}}$ and $C_2 = (2^p\kappa^\alpha)^{\frac{1}{1+\beta_0}}$. Iteration yields

$$\|u\|_{\mu_{\nu+1},h_{\nu+1}} \leq C_1^{\sum_{i=0}^{\nu} \frac{1}{\kappa^i}} C_2^{\sum_{i=0}^{\nu} \frac{i}{\kappa^i}} \|u\|_{\mu_0,h_0} \leq C\|u\|_{1+\beta_0,2},$$

because both series are convergent. Similarly to the case $p \leq 2$, this leads to

$$\|\mathcal{H}\|_{\infty,1} \leq C(\|\mathcal{H}\|_{1+\beta_0,2} + k),$$

that is (3.4). $\square$

We are going to specialize the argument in Proposition 3.1.1 to a solution H_λ of problem (1.6). Our goal is to obtain a local bound of H_λ, as shown by Serrin in Theorem 1 (see [30]).

Corollary 3.1.4. *Let H_λ be a solution of problem* (1.6) *such that condition* (1.9) *holds. If $\lambda \neq 0$, assume*

$$p > \begin{cases} \sqrt{N} & \text{if } N = 2, 3 \\ \frac{N}{2} & \text{if } N \geq 4. \end{cases}$$

We define

$$p_0 = \begin{cases} p(1+\beta_0) & \text{if } \lambda > 0 \text{ and } p \leq 2 \\ 1 + \beta_0 & \text{otherwise,} \end{cases}$$

β_0 being as in the previous section.
Then we have a bound of H_λ in B_R for $R > 0$ small.

- Case $p \leq 2$:

$$\|H_0\|_{\infty,R} \leq CR^{-\frac{N}{p_0}}\|H_0\|_{p_0,2R} \qquad \textit{if } \lambda = 0, \tag{3.33}$$

$$\|H_\lambda\|_{\infty,R} \leq C(R^{-\frac{N}{p_0}}\|H_\lambda\|_{p_0,2R} + R^{\frac{p^2-N}{p(p-1)}}\|G_\lambda\|^{p-1}_{p,2R}) \qquad \textit{if } \lambda \neq 0. \tag{3.34}$$

- Case $p > 2$:

$$\|H_0\|_{\infty,R} \leq CR^{-\frac{N}{p_0}}\|H_0\|_{p_0,2R} \qquad \textit{if } \lambda = 0. \tag{3.35}$$

$$\|H_\lambda\|_{\infty,R} \leq C(R^{-\frac{N}{p_0}}\|H_\lambda\|_{p_0,2R} + R^{\frac{\varepsilon_0}{p-1}}\|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}) \qquad \textit{if } \lambda \neq 0. \tag{3.36}$$

Proof. Consider H_λ in B_{2R}. We define, for $z \in B_2$,

$$\tilde{\Gamma}(z) = R^{\frac{N-p}{p-1}}\Gamma(Rz) \qquad \text{and} \qquad \tilde{H}_\lambda(z) = R^{\frac{N-p}{p-1}}H_\lambda(Rz),$$

and we set $\tilde{G}_\lambda(z) = G_\lambda(Rz)$. By Proposition 2.2.7 this scaling implies that $\tilde{H}_\lambda \in L^\infty(B_2)$ and $\nabla\tilde{H}_\lambda \in L^{\bar{q}}(B_2)$. In particular $\tilde{H}_\lambda$ solves

$$-\Delta_p(\tilde{\Gamma} + \tilde{H}_\lambda) + \Delta_p\tilde{\Gamma} = \lambda R^N \tilde{G}_\lambda^{p-1} \qquad \text{in } B_2,$$

which can be rewritten in the form

$$-\Delta_p(\Gamma + \tilde{H}_\lambda) + \Delta_p\Gamma = \lambda R^N \tilde{G}_\lambda^{p-1} \qquad \text{in } B_2,$$

by using that

$$\nabla\tilde{\Gamma}(z) = R^{\frac{N-p}{p-1}+1}\nabla\Gamma(Rz) = R^{\frac{N-1}{p-1}}\frac{C_0}{|Rz|^{\frac{N-1}{p-1}}} = \frac{C_0}{|z|^{\frac{N-1}{p-1}}} = \nabla\Gamma(z).$$

Moreover $|\nabla\tilde{H}_\lambda| = O(|\nabla\Gamma|)$. Notice that if $\lambda \neq 0$, setting $\mathcal{G} = \lambda R^N\tilde{G}_\lambda^{p-1}$, we have that $\mathcal{G} \in L^{\frac{p}{p-1}}(B_2)$, in view of $p > \sqrt{N}$. When $p > 2$, condition $p > \frac{N}{2}$ guarantees that $\mathcal{G} \in L^{\frac{N}{p-\varepsilon_0}}(B_2)$ for some ε_0 small. Therefore, we are able to apply Proposition 3.1.1.
Assume $p \leq 2$. If $\lambda = 0$, application of (3.2) to $\tilde{H}_0$ yields

$$\|\tilde{H}_0\|_{\infty,1} \leq C\|\tilde{H}_0\|_{p_0,2}. \tag{3.37}$$

Observing that

$$\|\tilde{H}_0\|_{\infty,1} = R^{\frac{N-p}{p-1}}\|H_0\|_{\infty,R},$$

$$\|\tilde{H}_0\|_{p_0,2} = R^{\frac{N-p}{p-1}} R^{-\frac{N}{p_0}} \|H_0\|_{p_0,2R},$$

inequality (3.37) becomes

$$R^{\frac{N-p}{p-1}} \|H_0\|_{\infty,R} \leq C R^{\frac{N-p}{p-1}} R^{-\frac{N}{p_0}} \|H_0\|_{p_0,2R}. \tag{3.38}$$

Dividing both members of (3.38) by $R^{\frac{N-p}{p-1}}$ we obtain (3.33).
On the other hand, if $\lambda \neq 0$ we apply (3.3) to $\tilde{H}_\lambda$ and we get

$$\|\tilde{H}_\lambda\|_{\infty,1} \leq C(\|\tilde{H}_\lambda\|_{p_0,2} + \|\mathcal{G}\|_{\frac{p}{p-1},2}). \tag{3.39}$$

Here we have

$$\begin{aligned}
\|\mathcal{G}\|_{\frac{p}{p-1},2} &= \left(\int_{B_2} \lambda^{\frac{p}{p-1}} R^{\frac{Np}{p-1}} |\tilde{G}_\lambda|^p \right)^{\frac{p-1}{p}} = \lambda R^N \|\tilde{G}_\lambda\|_{p,2}^{p-1} \\
&= \lambda R^N \left(\int_{B_2} |\tilde{G}_\lambda(z)|^p dz \right)^{\frac{p-1}{p}} = \lambda R^N \left(\int_{B_2} |G_\lambda(Rz)|^p dz \right)^{\frac{p-1}{p}} \\
&= \lambda R^N \left(\int_{B_{2R}} R^{-N} |G_\lambda(y)|^p dy \right)^{\frac{p-1}{p}} = \lambda R^{\frac{N}{p}} \|G_\lambda\|_{p,2R}^{p-1}.
\end{aligned}$$

Thus inequality (3.39) becomes

$$R^{\frac{N-p}{p-1}} \|H_\lambda\|_{\infty,R} \leq C(R^{\frac{N-p}{p-1}} R^{-\frac{N}{p_0}} \|H_\lambda\|_{p_0,2R} + R^{\frac{N}{p}} \|G_\lambda\|_{p,2R}^{p-1}). \tag{3.40}$$

Dividing both members of (3.40) by $R^{\frac{N-p}{p-1}}$ we obtain (3.34).

Now assume $p > 2$. Application of (3.4) to $\tilde{H}_\lambda$ yields

$$\|\tilde{H}_\lambda\|_{\infty,1} \leq C(\|\tilde{H}_\lambda\|_{p_0,2} + \|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0},2}^{\frac{1}{p-1}}), \tag{3.41}$$

provided $\|\tilde{H}_\lambda\|_{\infty,2} < 1 - \|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0},2}^{\frac{1}{p-1}}$. We have that

$$\begin{aligned}
\|\mathcal{G}\|_{\frac{N}{p-\varepsilon_0},2}^{\frac{1}{p-1}} &= \left(\int_{B_2} |\lambda R^N \tilde{G}_\lambda^{p-1}|^{\frac{N}{p-\varepsilon_0}} \right)^{\frac{p-\varepsilon_0}{N(p-1)}} \\
&= (\lambda R^N)^{\frac{1}{p-1}} \left(\int_{B_2} |\tilde{G}_\lambda(z)|^{\frac{N(p-1)}{p-\varepsilon_0}} dz \right)^{\frac{p-\varepsilon_0}{N(p-1)}} \\
&= (\lambda R^N)^{\frac{1}{p-1}} \left(\int_{B_2} |G_\lambda(Rz)|^{\frac{N(p-1)}{p-\varepsilon_0}} dz \right)^{\frac{p-\varepsilon_0}{N(p-1)}} \\
&= (\lambda R^N)^{\frac{1}{p-1}} \left(\int_{B_{2R}} R^{-N} |G_\lambda(y)|^{\frac{N(p-1)}{p-\varepsilon_0}} dy \right)^{\frac{p-\varepsilon_0}{N(p-1)}} \\
&= \lambda^{\frac{1}{p-1}} R^{\frac{N-p+\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}.
\end{aligned}$$

In particular we have that $\|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2} = O(R^{\frac{N-p+\varepsilon_0}{p-1}})$ in view of $G_\lambda \in L^{\frac{N(p-1)}{p-\varepsilon_0}}(B_{2R})$ for $0 < \varepsilon_0 < 2p - N$ and $\|\tilde{H}_\lambda\|_{\infty,2} = O(R^{\frac{N-p}{p-1}})$: these facts yield that $\|\tilde{H}_\lambda\|_{\infty,2} < 1 - \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2}$ for R small as required to have the validity of (3.41). Inequality (3.41) becomes

$$R^{\frac{N-p}{p-1}}\|H_\lambda\|_{\infty,R} \leq C(R^{\frac{N-p}{p-1}}R^{-\frac{N}{p_0}}\|H_\lambda\|_{p_0,2R} + \lambda^{\frac{1}{p-1}}R^{\frac{N-p+\varepsilon_0}{p-1}}\|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}). \tag{3.42}$$

Dividing both members of (3.42) by $R^{\frac{N-p}{p-1}}$ we obtain (3.35) for $\lambda = 0$ and (3.36) for $\lambda \neq 0$ when R small. $\square$

3.2 The Harnack inequality and Hölder continuity of H_λ at the pole

Proposition 3.2.1. *Let H_λ be a solution of problem* (1.6) *satisfying condition* (1.9)*. Assume* (1.8) *when $\lambda \neq 0$. Then*

$$\max_{B_R} H_0 \leq C \min_{B_R} H_0 \qquad \text{if } \lambda = 0, \tag{3.43}$$

$$\max_{B_R} H_\lambda \leq C(\min_{B_R} H_\lambda + k_R) \qquad \text{if } \lambda \neq 0. \tag{3.44}$$

Here $R > 0$ is a small radius, C is a positive constant depending on the structure of the problem and

$$k_R = \begin{cases} R^{\frac{p^2-N}{p(p-1)}}\|G_\lambda\|^{p-1}_{p,2R} & \text{if } p \leq 2 \\ R^{\frac{\varepsilon_0}{p-1}}\|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R} & \text{if } p > 2. \end{cases}$$

Remark 3.2.2. It is worth noting that in the definition of k_R the exponent of R is positive. Indeed, if $p \leq 2$ we have $\frac{p^2-N}{p(p-1)} > 0$ in view of $p > \sqrt{N}$. If $p > 2$ we have trivially $\frac{\varepsilon_0}{p-1} > 0$.

The proof of Proposition 3.2.1 requires an iteration process which uses Lemma 3.1.2, together with another result that we are going to show.

Lemma 3.2.3. *Under the hypothesis of Proposition 3.2.1, let $\tilde{H}_\lambda$, $\tilde{\Gamma}$ and $\tilde{G}_\lambda$ be as in the proof of Corollary 3.1.4. In particular $\tilde{H}_\lambda$ solves*

$$-\Delta_p(\Gamma + \tilde{H}_\lambda) + \Delta_p\Gamma = \mathcal{G} \qquad \text{in } B_2, \tag{3.45}$$

where $\mathcal{G} = \lambda R^N \tilde{G}_\lambda^{p-1}$. We set $u = \tilde{H}_\lambda - \min_{B_1} \tilde{H}_\lambda + k + \varepsilon'$, with $\varepsilon' > 0$ and k being defined as in (3.5)*. If $v = \log u$ and $\bar{v} = |B|^{-1} \int_B v dx$, we have*

$$\frac{1}{|B|} \int_B |v - \bar{v}| dx \leq \tilde{C} \qquad \forall B \subset B_2,\ B \text{ open ball.} \tag{3.46}$$

Proof. Let $B = B_h(x_0) \subset B_1$. Since $|x_0| < 1 - h$ and $h < 1$ imply $|y| \leq |y - x_0| + |x_0| < 1 + \frac{h}{2} < 2$ for all $y \in B_{\frac{3}{2}h}(x_0)$, we have that $B_{\frac{3}{2}h}(x_0) \subset B_2$. Let η be a nonnegative smooth function with compact support in $B_{\frac{3}{2}h}(x_0)$ such that

$$\eta = 1 \text{ in } B,\ 0 \leq \eta \leq 1 \text{ in } B_{\frac{3}{2}h}(x_0),\ \eta = 0 \text{ outside } B_{\frac{3}{2}h}(x_0),\ |\nabla \eta| \leq \frac{3}{h}.$$

We define $\eta_\delta = \eta(\delta + |y|^2)^{\frac{(N-1)(p-2)}{4(p-1)} - 1} |y|^{\frac{5}{2}}$, $\delta > 0$. Since $\tilde{H}_\lambda \in L^\infty(B_2)$ and $\nabla \tilde{H}_\lambda \in L^{\bar{q}}(B_2)$, then $\varphi = -\eta_\varepsilon^2 u^{-1}$ is admissible[1] in (3.84) and we are able to apply Corollary 2.2.4 with $\Psi(s) = -\frac{1}{s}$ and η_δ as cut-off function.
Using φ as test function in the weak formulation of (3.84), we obtain

$$\int_{B_2} \eta_\delta^2 |\nabla \Gamma|^{p-2} |\nabla v|^2 \leq C \left(\int_{B_2} \eta_\delta |\nabla \eta_\delta| |\nabla \Gamma|^{p-2} |\nabla v| + \int_{B_2} |\mathcal{G}| \eta_\delta^2 u^{-1} \right), \tag{3.47}$$

in view of $|\nabla \tilde{H}_\lambda| = O(|\nabla \Gamma|)$ if $p > 2$. Application of Young's inequality gives

$$C \int_{B_2} \eta_\delta |\nabla \eta_\delta| |\nabla \Gamma|^{p-2} |\nabla v| \leq \frac{1}{2} \int_{B_2} \eta_\delta^2 |\nabla \Gamma|^{p-2} |\nabla v|^2 + \tilde{C} \int_{B_2} |\nabla \eta_\delta|^2 |\nabla \Gamma|^{p-2}.$$

As a consequence, (3.47) becomes

$$\int_{B_2} \eta_\delta^2 |\nabla \Gamma|^{p-2} |\nabla v|^2 \leq C \left(\int_{B_2} |\nabla \eta_\delta|^2 |\nabla \Gamma|^{p-2} + \int_{B_2} |\mathcal{G}| \eta_\delta^2 u^{-1} \right). \tag{3.48}$$

We deal with the first term in right-hand side of (3.48). We observe that

$$\int_{B_2} |\nabla \eta_\delta|^2 |\nabla \Gamma|^{p-2} = C \Bigg[\int_{B_2} |y| \left(\frac{|y|^2}{\delta + |y|^2} \right)^{-\frac{(N-1)(p-2)}{2(p-1)}} |\nabla \eta|^2 + \int_{B_2} \left(\frac{|y|^2}{\delta + |y|^2} \right)^{2 - \frac{(N-1)(p-2)}{2(p-1)}} \frac{\eta^2}{|y|} \Bigg],$$

[1]Notice that now we are studying the case $\beta = -1$ of the proof of Lemma 3.1.2.

$$\int_{B_2} |\mathcal{G}|\eta_\delta^2 u^{-1} = \int_{B_2} |y|(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2|\mathcal{G}|u^{-1}.$$

using the definitions of η_δ and $\nabla\Gamma$. Thus (3.48) rewrites as

$$\begin{aligned}\int_{B_2} \eta_\delta^2|\nabla\Gamma|^{p-2}|\nabla v|^2 \leq C\Big[&\int_{B_2}|y|\Big(\frac{|y|^2}{\delta+|y|^2}\Big)^{-\frac{(N-1)(p-2)}{2(p-1)}}|\nabla\eta|^2\\ &+\int_{B_2}\Big(\frac{|y|^2}{\delta+|y|^2}\Big)^{2-\frac{(N-1)(p-2)}{2(p-1)}}\frac{\eta^2}{|y|}+\int_{B_2}|y|(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2|\mathcal{G}|u^{-1}\Big].\end{aligned}\tag{3.49}$$

Since $\left(\frac{|y|^2}{\delta+|y|^2}\right)^\alpha \leq C|y|^{-\max\{-2\alpha,0\}}$, we have that

$$\begin{aligned}|y|\Big(\frac{|y|^2}{\delta+|y|^2}\Big)^{-\frac{(N-1)(p-2)}{2(p-1)}} &\leq C|y|^{-\max\{\frac{(N-1)(p-2)}{p-1}-1,-1\}}\in L^1(\Omega)\\ \Big(\frac{|y|^2}{\delta+|y|^2}\Big)^{2-\frac{(N-1)(p-2)}{2(p-1)}}\frac{1}{|y|} &\leq C|y|^{-\max\{\frac{(N-1)(p-2)}{p-1}-3,1\}}\in L^1(\Omega)\end{aligned}\tag{3.50}$$

in view of $\frac{(N-1)(p-2)}{p-1} < N$. Since $\mathcal{G} = \lambda R^p\Gamma^{p-1}(y)[1+O(R^{\frac{N-p}{p-1}}|y|^{\frac{N-p}{p-1}})]$ in view of $\|H_\lambda\|_\infty < +\infty$, when $\lambda \neq 0$ there holds $k \geq CR^{t_0}$ for some $t_0 \leq p$ and $C > 0$ in view of $p > \max\{\sqrt{N}, \frac{N}{2}\}$, where k is given by (3.5), and then

$$\begin{aligned}\int_{B_2}|y|(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2|\mathcal{G}|u^{-1} &\leq \frac{1}{k}\int_{B_2}|y|(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2|\mathcal{G}|\\ &\leq C\int_{B_2}|y|^{p+1-N}(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}}\eta^2.\end{aligned}\tag{3.51}$$

Since

$$|y|^{p+1-N}(\delta+|y|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \leq |y|^{\min\{p+1-N+\frac{(N-1)(p-2)}{p-1},p+1-N\}}\in L^1(\Omega)\tag{3.52}$$

in view of $p+1+\frac{(N-1)(p-2)}{p-1} > 0$ thanks to $p > \sqrt{N} > \frac{1-N+\sqrt{N^2+6N-3}}{2}$, we can use (3.50), (3.52) and the Lebesgue's convergence theorem in (3.49) and (3.51) to get

$$\int_{B_2}\eta^2|y||\nabla v|^2 \leq C\Big(\int_{B_2}|y||\nabla\eta|^2+\int_{B_2}\frac{\eta^2}{|y|}+\int_{B_2}|y|^{p-\frac{N-1}{p-1}}\eta^2\Big)\tag{3.53}$$

thanks to the Fatou's convergence theorem. Since by (3.53) one deduces

$$\int_B |v-\bar v| \le C'h\int_B |\nabla v| \le C'h\Big(\int_B \frac{1}{|y|}\Big)^{\frac12}\Big(\int_B |y||\nabla v|^2\Big)^{\frac12}$$
$$\le Ch\Big(\int_B \frac{1}{|y|}\Big)^{\frac12}\Big(\int_{B_2} |y||\nabla\eta|^2+\int_{B_2}\frac{\eta^2}{|y|}+\int_{B_2}|y|^{p-\frac{N-1}{p-1}}\eta^2\Big)^{\frac12}$$

and $p-\frac{N-1}{p-1}>-1$ thanks to $p>\sqrt{N}$, for $|x_0|<3h$ one has that

$$\int_B |v-\bar v| \le Ch^{\frac{N+1}{2}}\left(h^{N-1}+h^{p-\frac{N-1}{p-1}+N}\right)^{\frac12}=O(h^N)$$

in view of $B_{\frac32 h}(x_0)\subset B_{5h}(0)$, while for $|x_0|\ge 3h$ there holds

$$\int_B |v-\bar v| \le C\left[h^2\Big(\int_B\frac{1}{|y|}\Big)\Big(\int_{B_2}|y||\nabla\eta|^2\Big)+h^{N+1}(h^{N-1}+h^{\min\{p-\frac{N-1}{p-1}+N,N\}})\right]^{\frac12}$$
$$\le C\left[h^2\Big(\frac{h^N}{|x_0|}\Big)\Big(|x_0|h^{N-2}\Big)+h^{2N}\right]^{\frac12}=O(h^N)$$

in view of $\frac{3h}{2}\le\frac{|x_0|}{2}\le|y|\le\frac32|x_0|$ for all $x\in B_{\frac32 h}(x_0)$. The proof is complete.

□

Proof of Proposition 3.2.1. Let $\tilde H_\lambda$ and u be as in Lemma 3.2.3. The proof is divided into 5 steps.

Step 1. Lemma 3.2.3 and John-Nirenberg Lemma (see Lemma 7 in [30]), imply that there exist $\tilde\lambda$ and $\tilde\mu$ depending on N such that

$$\int_{B_2} e^{p_0 v}dx\int_{B_2} e^{-p_0 v}dx \le 2^{2N}\tilde\mu^2,\quad \text{with } p_0=\frac{\tilde\lambda}{\tilde C},$$

$\tilde C$ being the same as in (3.46). Using the definition of Φ and recalling that $v=\log u$, we obtain

$$\Phi(p_0,2)\le C\Phi(-p_0,2). \tag{3.54}$$

In Step 2 and 3 we will apply Lemma 3.1.2 with $\mathcal{H}=\tilde H_\lambda-\min_{B_1}\tilde H_\lambda$ and we will use the iteration process as described in the proof of Proposition 3.1.1.

We will omit some technical calculations, that have already been discussed. We set

$$\kappa = \begin{cases} \frac{N-2+p}{N-p} & \text{if } p \leq 2 \text{ and } \lambda = 0 \\ \frac{N(p-1)}{p(N-p)} & \text{if } p \leq 2 \text{ and } \lambda > 0 \\ \frac{N}{N-2} & \text{otherwise,} \end{cases}$$

$$\mu = \begin{cases} p(\beta+1) & \text{if } p \leq 2, \lambda > 0 \\ \beta+1 & \text{otherwise.} \end{cases}$$

Step 2. Beginning with $\Phi(p_0, 2)$, we iterate estimate (3.7) if $p \leq 2$ and $\lambda \neq 0$, estimate (3.23) if $p \leq 2$ and $\lambda = 0$, estimate (3.9) if $p > 2$. Therefore, application of the iterative scheme with

$$\mu_\nu = \kappa^\nu p_0, \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1},$$

for $\nu = 0, 1, 2, \ldots$, leads to the inequality

$$\max_{B_1} u \leq C\Phi(p_0, 2). \tag{3.55}$$

Step 3. Beginning with $\Phi(-p_0, 2)$, we iterate estimate (3.8) if $p \leq 2$ and $\lambda \neq 0$, estimate (3.24) if $p \leq 2$ and $\lambda = 0$, estimate (3.10) if $p > 2$. Then, the iteration process with

$$\mu_\nu = -\kappa^\nu p_0, \qquad h_\nu = 1 + 2^{-\nu}, \qquad h'_\nu = h_{\nu+1},$$

for $\nu = 0, 1, 2, \ldots$, leads to the inequality

$$\min_{B_1} u \geq C\Phi(-p_0, 2). \tag{3.56}$$

Remark 3.2.4. In order to have (3.7) applicable at each stage, the successive iterates μ_ν must avoid the point $\mu = p$. To accomplish this in a definite way we shall choose a new initial value $p'_0 \leq p_0$ so that the point $\mu = p$ lies midway between some two consecutive iterates of p'_0. The same procedure can be also applied in (3.9) and (3.23) to avoid the point $\mu = 1$. In particular we have that $\Phi(p'_0, 2) \leq C\Phi(p_0, 2)$. In relation to (3.10), we can use a similar argument to avoid the point $\mu = 2 - p$, with $-p'_0 \geq -p_0$ in such a way that $\Phi(-p_0, 2) \leq C\Phi(-p'_0, 2)$.

Step 4. Putting together (3.54), (3.55) and (3.56), recalling that $u = \tilde{H}_\lambda - \min_{B_1} \tilde{H}_\lambda + k + \varepsilon'$ and letting $\varepsilon' \to 0$, we finally obtain

$$\max_{B_1} \tilde{H}_\lambda \leq C(\min_{B_1} \tilde{H}_\lambda + k), \tag{3.57}$$

that is the Harnack inequality for $\tilde{H}_\lambda$ in B_1.

Step 5. If we assume $\lambda = 0$, then $k = 0$ by definition of $\mathcal{G}$. Consequently, (3.57) rewrites as

$$\max_{B_1} \tilde{H}_0 \leq C \min_{B_1} \tilde{H}_0$$

and, by scaling, we obtain (3.79).

Assume $\lambda \neq 0$. We have that

$$k = \begin{cases} \|\mathcal{G}\|_{\frac{p}{p-1},2} & \text{if } p \leq 2 \\ \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0},2} & \text{if } p > 2, \end{cases}$$

that is

$$k = \begin{cases} \lambda R^{\frac{N}{p}} \|G_\lambda\|^{p-1}_{p,2R} & \text{if } p \leq 2 \\ \lambda^{\frac{1}{p-1}} R^{\frac{N-p+\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R} & \text{if } p > 2. \end{cases}$$

Therefore, if $p \leq 2$ then (3.57) yields

$$R^{\frac{N-p}{p-1}} \max_{B_R} H_\lambda \leq C(R^{\frac{N-p}{p-1}} \min_{B_R} H_\lambda + R^{\frac{N}{p}} \|G_\lambda\|^{p-1}_{p,2R}),$$

that is, dividing by $R^{\frac{N-p}{p-1}}$,

$$\max_{B_R} H_\lambda \leq C(\min_{B_R} H_\lambda + R^{\frac{p^2-N}{p(p-1)}} \|G_\lambda\|^{p-1}_{p,2R}).$$

Similarly, if $p > 2$ we obtain

$$\max_{B_R} H_\lambda \leq C(\min_{B_R} H_\lambda + R^{\frac{\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}).$$

This concludes the proof of Proposition 3.2.1. $\square$

We are now ready to establish the Hölder continuity of H_λ at 0.

We set for $r > 0$

$$M(r) = \max_{B_r} H_\lambda \qquad \text{and} \qquad \mu(r) = \min_{B_r} H_\lambda.$$

Then we define the oscillation of H_λ in B_r by

$$\omega(r) = M(r) - \mu(r).$$

Theorem 3.2.5. *Let H_λ be a solution of problem* (1.6) *satisfying condition* (1.9)*. Assume* (1.8) *if $\lambda \neq 0$. Then there exist $\alpha \in (0,1)$ and $\delta_0 > 0$ so that*

$$\omega(\varrho) \leq C_0 \varrho^\alpha \qquad \textit{for all } \varrho \leq \delta_0. \tag{3.58}$$

Proof. We observe that $M(r)$ and $\mu(r)$ are well defined for $0 < r \leq 1$. It follows that both functions

$$H_1 = M - H_\lambda \qquad \text{and} \qquad H_2 = H_\lambda - \mu$$

are nonnegative in B_r.

Remark 3.2.6. We have always worked with a function $\mathcal{H}$ which solves

$$-\Delta(\Gamma + \mathcal{H}) + \Delta\Gamma = \mathcal{G}. \tag{3.59}$$

We observe that H_2 satisfies the same equation as $\mathcal{H}$. On the other hand, H_1 solves

$$-\Delta(\Gamma - H_1) + \Delta\Gamma = \mathcal{G}. \tag{3.60}$$

Even if (3.59) and (3.60) are different equations, it can be easily verified that the respective weak formulations lead to the same estimates, as described in Lemma 2.2.3 and Corollary 2.2.4. Therefore we are able to apply the Harnack inequality to H_1.

Application of Proposition 3.2.1 to H_1 in the open ball B_r gives

$$M - \mu' = \max_{B_{\frac{r}{2}}} H_1 \leq C(\min_{B_{\frac{r}{2}}} H_1 + k_r) = C(M - M' + k_r), \tag{3.61}$$

where $M' = M'(r) = M(r/2)$, $\mu' = \mu'(r) = \mu(r/2)$ and

$$k_r = \begin{cases} r^{\frac{p^2-N}{p(p-1)}} \|G_\lambda\|_p^{p-1} & \text{if } p \leq 2 \\ r^{\frac{\varepsilon_0}{p-1}} \|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0}} & \text{if } p > 2. \end{cases}$$

Similarly, by applying Proposition 3.2.1 to H_2, we have

$$M' - \mu \leq \max_{B_{\frac{r}{2}}} H_2 \leq C(\min_{B_{\frac{r}{2}}} H_2 + k_r) = C(\mu' - \mu + k_r). \tag{3.62}$$

Adding (3.61) and (3.62) and transposing terms then gives

$$M' - \mu' \leq \frac{C-1}{C+1}(M - \mu) + \frac{2Ck_r}{C+1}. \tag{3.63}$$

We define $\vartheta = \frac{C-1}{C+1} < 1$,

$$\tau = \begin{cases} \frac{2C}{C-1}\|G_\lambda\|_p^{p-1} & \text{if } p \le 2 \\ \frac{2C}{C-1}\|G_\lambda\|_{\frac{N(p-1)}{p-\varepsilon_0}} & \text{if } p > 2 \end{cases} \quad \text{and} \quad \sigma = \begin{cases} \frac{p^2-N}{p(p-1)} & \text{if } p \le 2 \\ \frac{\varepsilon_0}{p-1} & \text{if } p > 2. \end{cases}$$

Recalling that $r \le 1$, then (3.63) becomes

$$\omega\left(\frac{r}{2}\right) \le \vartheta(\omega(r) + \tau r^\sigma), \qquad 0 < r \le 1.$$

Since $\omega(r)$ is an increasing function representing the oscillation of H in B_r, it is clear that for any number $s \ge 2$ one has also

$$\omega\left(\frac{r}{s}\right) \le \vartheta(\omega(r) + \tau r^\sigma), \qquad 0 < r \le 1. \tag{3.64}$$

Iteration of relation (3.64) from $r = 1$ to successively smaller radii yields

$$\omega(s^{-\nu}) \le \vartheta^\nu \left[\omega(1) + \tau \sum_{j=0}^{\nu-1} (\vartheta s^\sigma)^{-j}\right], \qquad \text{for } \nu = 1, 2, \dots. \tag{3.65}$$

Now we choose s according to the relation

$$\vartheta s^\sigma = 2,$$

whence from (3.65) follows

$$\omega(s^{-\nu}) \le \vartheta^\nu(\omega(1) + 2\tau). \tag{3.66}$$

For any fixed ϱ, $0 < \varrho \le s^{-1}$, let ν be chosen such that $s^{-\nu-1} < \varrho \le s^{-\nu}$. Thus, by virtue of (3.66) we have

$$\omega(\varrho) \le \omega(s^{-\nu}) \le \vartheta^\nu(\omega(1) + 2\tau) \le C\vartheta^\nu, \tag{3.67}$$

using also the boundedness of $\omega(1)$ and τ. At this point, if γ is defined by the relation $2^{-\gamma} = \vartheta$, then we have $\vartheta = s^{-\alpha}$ where $\alpha = \sigma\frac{\gamma}{\gamma+1} \in (0,1)$. Therefore (3.67) implies

$$\omega(\varrho) \le C\varrho^\alpha, \qquad \varrho \le 2^{-\frac{\gamma+1}{\sigma}},$$

that is (3.58) with $\delta_0 = 2^{-\frac{\gamma+1}{\sigma}}$. $\square$

Corollary 3.2.7. *Under the hypothesis of Theorem 3.2.5, H_λ is Hölder continuous at the pole i.e.*

$$|H_\lambda(y) - H_\lambda(0)| \leq C|y|^\alpha \quad \textit{near } 0, \tag{3.68}$$

α being as in (3.58).

Proof. Assume by contradiction that (3.68) does not hold. Then there exist sequences $y_n \in \Omega$ such that

$$\frac{|H_\lambda(y_n) - H_\lambda(0)|}{|y_n|^\alpha} \to +\infty \quad \text{as } n \to +\infty. \tag{3.69}$$

Since H_λ is bounded, (3.69) implies that

$$y_n \to 0 \quad \text{as } n \to +\infty. \tag{3.70}$$

We define $\varrho_n = |y_n|$. Then $y_n \in B_{\varrho_n}$ and there exists n_0 such that $\varrho_n \leq \delta_0$ for all $n \geq n_0$. Application of (3.58) yields

$$|H_\lambda(y_n) - H_\lambda(0)| \leq C_0 \varrho_n^\alpha \quad \forall n \geq n_0,$$

in contradiction with (3.69). This proves (3.68). □

3.3 A stronger result when $p \leq 2$

In the previous sections we always worked in balls centered at 0. Consequently, the function Γ had radial symmetry and it was singular in 0.
When $p \leq 2$, our aim is to to generalize the Harnack inequality, making it hold in $B_R(y_0)$ with $y_0 \neq 0$. As a consequence, we will show the global Hölder continuity of H_λ when $p \leq 2$.

We define for $y \in B_2$

$$\tilde{\Gamma}(y) = |y_0|^{\frac{N-1}{p-1}} R^{-1} \Gamma(Ry + y_0). \tag{3.71}$$

Then

$$\nabla\tilde{\Gamma}(y) = |y_0|^{\frac{N-1}{p-1}} \nabla\Gamma(Ry + y_0) = \frac{C|y_0|^{\frac{N-1}{p-1}}}{|Ry + y_0|^{\frac{N-1}{p-1}}} = \frac{C}{\left|\frac{Ry}{|y_0|} + \frac{y_0}{|y_0|}\right|^{\frac{N-1}{p-1}}}.$$

Let $r_0 = \frac{|y_0|}{4}$. We observe that if $R \leq r_0$ then $\frac{2R}{|y_0|} \leq \frac{1}{2}$ and $\nabla\tilde{\Gamma}$ does not have singularity in $\bar{B}_2$.

Proposition 3.3.1. *Assume $p \leq 2$. Let $\mathcal{H} \in L^{\infty}(B_2)$, with $\nabla \mathcal{H} \in L^{\bar{q}}(B_2)$, be a solution of*

$$-\Delta_p(\tilde{\Gamma} + \mathcal{H}) + \Delta_p \tilde{\Gamma} = \mathcal{G} \qquad \text{in } B_2, \tag{3.72}$$

such that $|\nabla \mathcal{H}| = O(|\nabla \tilde{\Gamma}|)$ and $\mathcal{G} \in L^{\frac{p}{p-1}}(B_2)$. When $\mathcal{G} \neq 0$ we require $p \geq \frac{2N}{N+1}$. Then we have a universal bound for $\mathcal{H}$ in $B_1(y_0)$. In particular, estimates (3.2) *and* (3.3) *hold.*

The proof of Proposition 3.3.1 is a consequence of the following Lemma.

Lemma 3.3.2. *Let $\mathcal{H}$ be as in Proposition 3.3.1. Assume $p > \sqrt{N}$, $N = 2, 3$ if $\mathcal{G} \neq 0$ and $p \leq 2$. If $\mathcal{G} \neq 0$, estimates* (3.7) *and* (3.8) *of Lemma 3.1.2 hold with $\kappa = \frac{N}{p(N-2)}$. Otherwise*

$$\Phi_1(\kappa\mu, h') \leq [C\mu(h - h')^{-1}]^{\frac{2}{\mu}} \Phi_1(\mu, h) \qquad \text{if } \mu \in (0,1) \cup (1, +\infty), \tag{3.73}$$

$$\Phi_1(\kappa\mu, h') \geq [C\mu(h - h')^{-1}]^{\frac{2}{\mu}} \Phi_1(\mu, h) \qquad \text{if } \mu \in (-\infty, 0), \tag{3.74}$$

with $\kappa = \frac{N}{N-2}$.

Proof. We will follow the proof of Lemma 3.1.2; then some details will be omitted. We will get better estimates due to the regularity of $\nabla \tilde{\Gamma}$ in B_2.

Assume $\mathcal{G} \neq 0$. Using in (3.12) that $\nabla \tilde{\Gamma} = O(1)$ in B_2, we get

$$\int_{B_2} |\nabla(\eta v)|^2 \leq C\left[\left(1 + \frac{(\beta+1)^2}{\beta^2}\right) \int_{B_2} |\nabla \eta|^2 v^2 + \frac{(\beta+1)^2}{|\beta|} \int_{B_2} |\mathcal{G}| \eta^2 u^\beta\right]. \tag{3.75}$$

Notice that we are able to use η instead of η_ε, because of the regularity of the problem in B_2. Since $k \leq u$ we have that

$$\int_{B_2} |\mathcal{G}| \eta^2 u^\beta = \int_{B_2} |\mathcal{G}| \eta^2 u^{\beta-1+1} \leq \int_{B_2} \frac{|\mathcal{G}|}{k} \eta^2 v^2 \leq \frac{1}{k} \|\mathcal{G}\|_{\frac{p}{p-1}} \|\eta v\|_{2p}^2 = \|\eta v\|_{2p}^2.$$

Thus, application of Sobolev inequality and (3.75) yield

$$\|\eta v\|_{\frac{2N}{N-2}}^2 \leq C\left[\left(1 + \frac{(\beta+1)^2}{\beta^2}\right) \|v \nabla \eta\|_2^2 + \frac{(\beta+1)^2}{|\beta|} \|\eta v\|_{2p}^2\right],$$

that is

$$\left(\int_{B_{h'}} u^{\frac{(\beta+1)N}{N-2}}\right)^{\frac{N-2}{N}} \leq C\Bigg[\left(1+\frac{(\beta+1)^2}{\beta^2}\right)(h-h')^{-2}\int_{B_h} u^{\beta+1} + \frac{(\beta+1)^2}{|\beta|}\left(\int_{B_h} u^{p(\beta+1)}\right)^{\frac{1}{p}}\Bigg], \tag{3.76}$$

in view of the properties of η and the definition of v. Thus inequality (3.76) yield (3.7) and (3.8) as in Lemma 3.1.2, with $\kappa = \frac{N}{p(N-2)} > 1$ in view of $\sqrt{N} < p \leq 2$ and $N = 2, 3$.

Assume $\mathcal{G} \equiv 0$. Similarly to the previous case, we obtain

$$\left(\int_{B_{h'}} u^{\frac{(\beta+1)N}{N-2}}\right)^{\frac{N-2}{N}} \leq \tilde{C}\left(1+\frac{(\beta+1)^2}{\beta^2}\right)(h-h')^{-2}\int_{B_h} u^{\beta+1} \tag{3.77}$$

We set $\mu = \beta+1$ and $\kappa = \frac{N}{N-2}$. Since $|\beta| \geq \beta_0 > 0$, then (3.77) rewrites as

$$\left(\int_{B_{h'}} u^{\kappa\mu}\right)^{\frac{1}{\kappa}} \leq [C\mu(h-h')^{-1}]^2\int_{B_h} u^{\mu}. \tag{3.78}$$

Taking the μ-th roots in (3.78) we obtain (3.73) if $\mu \in (0,1)\cup(1,+\infty)$ and (3.74) if $\mu \in (-\infty, 0)$. □

Corollary 3.3.3. *Let H_λ be a solution of problem* (1.6) *with $p \leq 2$ and let λ, p, p_0 be as in Corollary 3.1.4. Then we have a bound of H_λ in $B_R(y_0)$ for $R \leq r_0$ and $y_0 \neq 0$. In particular, estimates* (3.33) *and* (3.34) *hold except that now balls are centered at y_0.*

Proof. We set for $y \in B_2$

$$\tilde{H}_\lambda(y) = |y_0|^{\frac{N-1}{p-1}}R^{-1}H_\lambda(Ry+y_0) \quad \text{and} \quad \tilde{G}_\lambda(y) = G_\lambda(Ry+y_0).$$

By Proposition 2.2.7 this scaling implies that $\tilde{H}_\lambda \in L^\infty(B_2)$ and $\nabla\tilde{H}_\lambda \in L^{\bar{q}}(B_2)$. Moreover

$$-\Delta_p(\tilde{\Gamma}+\tilde{H}_\lambda) + \Delta_p\tilde{\Gamma} = \lambda|y_0|^{N-1}R\tilde{G}_\lambda^{p-1} \qquad \text{in } B_2.$$

Application of Proposition 3.3.1 to $\tilde{H}_\lambda$ with $\mathcal{G} = \lambda|y_0|^{N-1}R\tilde{G}_\lambda^{p-1}$, with the aid of the same arguments used in the proof of Corollary 3.1.4, yields the desired estimates. □

We are going to generalize Proposition 3.2.1, proving the Harnack inequality in $B_R(y_0)$ with $y_0 \neq 0$ and $R \leq r_0$.

Proposition 3.3.4. *Let $p \leq 2$ and $y_0 \neq 0$. Under the hypothesis of Proposition 3.2.1, we have that*

$$\max_{B_R(y_0)} H_0 \leq C \min_{B_R(y_0)} H_0 \qquad \text{if } \lambda = 0, \tag{3.79}$$

$$\max_{B_R(y_0)} H_\lambda \leq C(\min_{B_R(y_0)} H_\lambda + k_R) \qquad \text{if } \lambda \neq 0. \tag{3.80}$$

Here $R > 0$ is a small radius such that $R \leq r_0$, C is a positive constant depending on the structure of the problem and $k_R = R^{2-\frac{N(p-1)}{p}} \|G_\lambda\|_{p,2R}^{p-1}$.

Remark 3.3.5. It is worth noting that in the definition of k_R the exponent of R is positive. Indeed, the condition $2 - \frac{N(p-1)}{p} > 0$ is equal to require $p < \frac{N}{N-2}$, which is satisfied being $\sqrt{N} < p \leq 2$ and $N = 2, 3$.

The proof of Proposition 3.3.4 requires an iteration process which uses Lemma 3.3.2, together with the generalization of Lemma 3.2.3 to the case $y_0 \neq 0$ that we are going to show.

We consider the following scaling for $y \in B_2$:

$$\tilde{H}_\lambda(y) = |y_0|^{\frac{N-1}{p-1}} R^{-1} H_\lambda(Ry + y_0), \tag{3.81}$$

$$\tilde{\Gamma}(y) = |y_0|^{\frac{N-1}{p-1}} R^{-1} \Gamma(Ry + y_0), \tag{3.82}$$

$$\tilde{G}_\lambda(y) = G_\lambda(Ry + y_0). \tag{3.83}$$

Then $\tilde{H}_\lambda$ solves

$$-\Delta_p(\tilde{\Gamma} + \tilde{H}_\lambda) + \Delta_p \tilde{\Gamma} = \mathcal{G} \qquad \text{in } B_2, \tag{3.84}$$

$\mathcal{G}$ being defined as

$$\mathcal{G}(y) = \lambda |y_0|^{N-1} R \tilde{G}_\lambda^{p-1}(y). \tag{3.85}$$

Lemma 3.3.6. *Under the hypothesis of Proposition 3.3.4, let $\tilde{H}_\lambda$ be as in* (3.81), (3.84). *We set $u = \tilde{H}_\lambda - \min_{B_1} \tilde{H}_\lambda + k + \varepsilon'$, with $\varepsilon' > 0$ and $k = \|\mathcal{G}\|_{\frac{p}{p-1},2}$. If $v = \log u$ and $\bar{v} = |B|^{-1} \int_B v dx$, we have*

$$\frac{1}{|B|} \int_B |v - \bar{v}| dx \leq \tilde{C} \qquad \forall B \subset B_2,\ B \text{ open ball.} \tag{3.86}$$

Proof. The proof is the same as in Lemma 3.2.3.
Since $|\nabla\tilde{\Gamma}| \sim C$, then application of (3.48) with η instead of η_δ yields

$$\int_{B_2} \eta^2 |\nabla v|^2 \le C\Big(\int_{B_2} |\nabla \eta|^2 + \int_{B_2} |\mathcal{G}|\eta^2 u^{-1}\Big). \tag{3.87}$$

Moreover, since $k \le u$, recalling the properties of η, we have that

$$\int_{B_2} |\mathcal{G}|\eta^2 u^{-1} \le \frac{1}{k}\int_{B_2} |\mathcal{G}|\eta^2 \le \frac{1}{k}\|\mathcal{G}\|_{\frac{p}{p-1}}\|\eta\|_{2p}^2 = \|\eta\|_{2p}^2 \le |B_{\frac{3}{2}h}|^{\frac{1}{p}} = Ch^{\frac{N}{p}}.$$

As a consequence, (3.87) becomes

$$\int_{B_2} \eta^2|\nabla v|^2 \le C(h^{N-2} + h^{\frac{N}{p}}) \le Ch^{N-2},$$

in view of $N-2 < \frac{N}{p}$ being $\sqrt{N} < p \le 2$. Then we get

$$\begin{aligned}\int_B |v-\bar{v}| &\le C'h\int_B |\nabla v| \le C'h\Big(\int_{B_2}\eta^2|\nabla v|^2\Big)^{\frac{1}{2}}|B_{\frac{3}{2}h}(x_0)|^{\frac{1}{2}}\\ &\le Ch^{1+\frac{N-2}{2}+\frac{N}{2}} = Ch^N,\end{aligned}$$

that is (3.86) with $B = B_h(x_0)$. □

Proof of Proposition 3.3.4. Let $\tilde{H}_\lambda$ be as in (3.81), (3.84). The proof is divided into 5 steps and goes as in Proposition 3.2.1. In particular, in Step 2 and 3, application of Lemma 3.3.2 with $\mathcal{H} = \tilde{H}_\lambda - \min_{B_1}\tilde{H}_\lambda$ is needed, together with the iteration process as described in the proof of Proposition 3.3.1, with

$$\kappa = \begin{cases} \frac{N}{p(N-2)} & \text{if } \lambda > 0 \\ \frac{N}{N-2} & \text{if } \lambda = 0. \end{cases}$$

Step 4 yields

$$\max_{B_1}\tilde{H}_\lambda \le C(\min_{B_1}\tilde{H}_\lambda + k), \tag{3.88}$$

that is the Harnack inequality for $\tilde{H}_\lambda$ in B_1. Finally, Step 5 deals with the rescaling of (3.88). We have that

$$k = \|\mathcal{G}\|_{\frac{p}{p-1},2} = \lambda|y_0|^{N-1}R^{1-\frac{N(p-1)}{p}}\|G_\lambda\|_{p,2R}^{p-1}.$$

Therefore, (3.88) gives

$$|y_0|^{\frac{N-1}{p-1}} R^{-1} \max_{B_R(y_0)} H_\lambda \leq C(|y_0|^{\frac{N-1}{p-1}} R^{-1} \min_{B_R(y_0)} H_\lambda + |y_0|^{N-1} R^{1-\frac{N(p-1)}{p}} \|G_\lambda\|_{p,2R}^{p-1}),$$

that is, dividing by $|y_0|^{\frac{N-1}{p-1}} R^{-1}$,

$$\max_{B_R(y_0)} H_\lambda \leq C(\min_{B_R(y_0)} H_\lambda + \lambda |y_0|^{\frac{(N-1)(p-2)}{p-1}} R^{2-\frac{N(p-1)}{p}} \|G_\lambda\|_{p,2R}^{p-1}).$$

This concludes the proof of Proposition 3.3.4. □

At this point we are able to show the global Hölder continuity of H_λ. By Corollary 3.2.7 we know that H_λ is Hölder continuous at 0.
We set for $z \in \Omega$ and $r > 0$

$$M(r,z) = \max_{B_r(z)} H_\lambda \qquad \text{and} \qquad \mu(r,z) = \min_{B_r(z)} H_\lambda.$$

Then we define the oscillation of H_λ in $B_r(z)$ by

$$\omega(r,z) = M(r,z) - \mu(r,z).$$

Theorem 3.3.7. *Let H_λ be a solution of problem* (1.6) *with $p \leq 2$ satisfying condition* (1.9) *holds. Assume* (1.8) *if $\lambda \neq 0$. Then there exist $\alpha \in (0,1)$ and $\delta_0 > 0$ such that*

$$\omega(r,z) \leq C_1 r^\alpha \qquad \text{for all } r \leq \delta_0 r_0, \tag{3.89}$$

where $r_0 = \frac{|y_0|}{4}$.

Proof. We observe that $M(r,z)$ and $\mu(r,z)$ are well-defined for $0 < r \leq r_0$. Then the functions

$$H_1 = M - H_\lambda \qquad \text{and} \qquad H_2 = H_\lambda - \mu$$

are nonnegative in $B_r(z)$. For $0 < \varrho \leq 1$ we set $\tilde{\omega}(\varrho, z) = \omega(\varrho r_0, z)$ and

$$\tilde{\tau} = \frac{2C}{C-1} r_0^\sigma \|G\|_{\frac{N(p-1)}{p-\varepsilon_0}}.$$

As in the proof of Proposition 3.2.5, application of Proposition 3.3.4 to H_1 and H_2 yields

$$\tilde{\omega}(s^{-\nu}) \leq \vartheta^\nu (\tilde{\omega}(1) + 2\tilde{\tau})$$

for any $s \geq 2$ and $\nu = 1, 2, \ldots$. It follows that

$$\tilde{\omega}(\delta) \leq (\tilde{\omega}(1) + 2\tilde{\tau})\delta^{\alpha}, \qquad \delta \leq \delta_0. \tag{3.90}$$

Letting $r = \delta r_0$, (3.90) rewrites as

$$\omega(r) \leq \left(\frac{\omega(r_0) + \frac{4C}{C-1}\|G\|_{\frac{N(p-1)}{p-\varepsilon_0}} r_0^{\sigma}}{r_0^{\alpha}} \right) r^{\alpha}, \qquad r \leq \delta_0 r_0. \tag{3.91}$$

Recalling the definition of α, we observe that

$$\sigma - \alpha = \sigma\left(1 - \frac{\gamma}{\gamma+1}\right) = \frac{\sigma}{\gamma+1} > 0,$$

being $\gamma > 0$ in view of $2^{-\gamma} = \vartheta < 1$. Moreover, since $r_0 = \frac{|y_0|}{4}$,

$$B_{r_0}(z) \subset B_{r_0+4r_0}(0) = B_{5r_0}(0).$$

This implies that

$$\omega(r_0, z) \leq \omega(5r_0, 0) \leq C_0(5r_0)^{\alpha},$$

with the aid of (3.58). As a consequence, (3.91) yields

$$\omega(r, z) \leq C_1 r^{\alpha} \qquad r \leq \delta_0 r_0.$$

□

Corollary 3.3.8. *Under the hypothesis of Theorem 3.3.7, H_λ is Hölder continuous i.e.*

$$|H_\lambda(z) - H_\lambda(y)| \leq C|z-y|^{\alpha} \quad \forall z, y \in \Omega, \tag{3.92}$$

α being as in (3.58), (3.89).

Proof. Assume by contradiction that (3.92) does not hold. Then there exist sequences $z_n, y_n \in \Omega$ such that

$$\frac{|H_\lambda(z_n) - H_\lambda(y_n)|}{|z_n - y_n|^{\alpha}} \to +\infty \quad \text{as } n \to +\infty. \tag{3.93}$$

Since H_λ is bounded, (3.93) implies that

$$z_n - y_n \to 0 \quad \text{as } n \to +\infty. \tag{3.94}$$

By compactness, there exists $x_0 \in \bar{\Omega}$ such that $z_n \to x_0$ as $n \to +\infty$. Since (3.94) holds, then $y_n \to x_0$ as $n \to +\infty$. Moreover, from classical regularity results we have that $x_0 = 0$. We define $\varrho_n = \max\{|z_n|, |y_n|\}$. Then $z_n, y_n \in B_{\varrho_n}$ and there exists n_0 such that $\varrho_n \le r_0$ for all $n \ge n_0$.
Application of (3.58) yields

$$|H_\lambda(z_n) - H_\lambda(y_n)| \le C_0 \varrho_n^\alpha \quad \forall n \ge n_0.$$

This implies that

$$|z_n - y_n| \ll \varrho_n \quad \forall n \ge n_0, \tag{3.95}$$

i.e. there exists a constant $M \gg 1$ such that $|z_n - y_n| \le M \varrho_n$ for all $n \ge n_0$. It follows that

$$|z_n - y_n| \ll |z_n| \text{ and } |z_n - y_n| \ll |y_n| \quad \forall n \ge n_0. \tag{3.96}$$

Indeed, assume $\varrho_n = |z_n|$. Then

$$|y_n| \ge |z_n| - |z_n - y_n| \ge (M-1)|z_n - y_n| \quad \forall n \ge n_0,$$

in view of (3.95), and we get (3.96). At this point we observe that $y_n \in B_{2|z_n - y_n|}(z_n)$ for all $n \ge n_0$, with $2|z_n - y_n| \le \delta_0 r_0$ in view of (3.96).
Application of (3.89) with respect to z_n gives

$$|H_\lambda(z_n) - H_\lambda(y_n)| \le C_1 (2|z_n - y_n|)^\alpha,$$

in contradiction with (3.93). This proves (3.92). □

Remark 3.3.9. It is worth noting that the global Hölder continuity does not hold when $p > 2$. Indeed, if $p > 2$ we have to require $|y_0|^{\frac{N-1}{p-1}} \ll R$ to guarantee $\|\tilde{H}_\lambda\|_{\infty, B_2} < 1 - \|\mathcal{G}\|^{\frac{1}{p-1}}_{\frac{N}{p-\varepsilon_0}, B_2}$ and to be able to adapt the calculations of Proposition 3.1.1.

Chapter 4

Application to Brezis-Nirenberg type problem

This chapter is devoted to the discussion of existence issues related to the quasilinear Brezis-Nirenberg problem (1.11), in the low-dimensional case $N < p^2$. Throughout this chapter we assume $p \geq 2$ and $0 < \lambda < \lambda_1$.

In Section 4.1 we will introduce an approximated problem whose solution admits an expansion in terms of $H_\lambda(\cdot, x)$, which is the solution of (1.6) with pole at $x \in \Omega$. A priori estimates on approximating solution are needed and for that we will adapt the result proved in the first part of this thesis.
These approximating solutions will be crucial in Section 4.2, where we will show Theorem (1.0.2). The most delicate point is to prove the implication $(iii) \Rightarrow (i)$ under the hypothesis (1.14). A blow-up analysis will be needed, together with the application of integral identities of Pohozaev type.

4.1 The approximated problem

For $x \in \Omega$, we consider the solution $PU_{\varepsilon,\lambda}$ of the problem

$$\begin{cases} -\Delta_p PU_{\varepsilon,\lambda} = \lambda PU_{\varepsilon,\lambda}^{p-1} + U_\varepsilon^{p^*-1} & \text{in } \Omega \\ PU_{\varepsilon,\lambda} > 0 & \text{in } \Omega \\ PU_{\varepsilon,\lambda} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

the function U_ε being the so-called *standard bubble*, defined as

$$U_\varepsilon(y) = C_1 \left(\frac{\varepsilon}{\varepsilon^p + |y-x|^{\frac{p}{p-1}}} \right)^{\frac{N-p}{p}}, \quad \varepsilon > 0, \tag{4.2}$$

where $C_1 = N^{\frac{N-p}{p^2}} \left(\frac{N-p}{p-1}\right)^{\frac{(p-1)(N-p)}{p^2}}$. We know by Remark 1.2.9 that U_ε solves

$$-\Delta_p U_\varepsilon = U_\varepsilon^{p^*-1} \qquad \text{in } \mathbb{R}^N. \tag{4.3}$$

We decompose the solution $PU_{\varepsilon,\lambda}$ of (4.1) as

$$PU_{\varepsilon,\lambda} = U_\varepsilon + \frac{C_1}{C_0} \varepsilon^{\frac{N-p}{p}} H_{\varepsilon,\lambda}, \tag{4.4}$$

C_0 being the same constant as in (1.4). Setting

$$G_{\varepsilon,\lambda} = \frac{C_0}{C_1} \varepsilon^{-\frac{N-p}{p}} PU_{\varepsilon,\lambda} \quad \text{and} \quad \Gamma_\varepsilon = \frac{C_0}{C_1} \varepsilon^{-\frac{N-p}{p}} U_\varepsilon, \tag{4.5}$$

then (4.4) rewrites as

$$G_{\varepsilon,\lambda} = \Gamma_\varepsilon + H_{\varepsilon,\lambda}.$$

In particular, recalling the definition of U_ε, we obtain that

$$\Gamma_\varepsilon = \frac{C_0}{\left(\varepsilon^p + |y-x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}.$$

At this point we observe that

$$-\Delta_p(G_{\varepsilon,\lambda}) = \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} (-\Delta_p PU_{\varepsilon,\lambda}) = \lambda G_{\varepsilon,\lambda}^{p-1} + \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} U_\varepsilon^{p^*-1},$$

in view of (4.1). Thus $G_{\varepsilon,\lambda}$ solves the problem

$$\begin{cases} -\Delta_p G_{\varepsilon,\lambda} = \lambda G_{\varepsilon,\lambda}^{p-1} + f_\varepsilon & \text{in } \Omega \\ G_\varepsilon > 0 & \text{in } \Omega \\ G_{\varepsilon,\lambda} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where $f_\varepsilon = \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} U_\varepsilon^{p^*-1}$.

Lemma 4.1.1. *$f_\varepsilon \rightharpoonup \delta_x$ weakly as $\varepsilon \to 0$ in the sense of measures in Ω.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$. We define $\Omega_\varepsilon = \{z\colon z = \frac{y-x}{\varepsilon^{p-1}}, y \in \Omega\}$. Then

$$\begin{aligned}
\int_\Omega f_\varepsilon \varphi &= \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} \int_\Omega U_\varepsilon^{p^*-1}(y)\varphi(y)dy \\
&= \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} \int_\Omega \frac{C_1^{p^*-1}\varepsilon^{\frac{N(p-1)}{p}+1}}{(\varepsilon^p + |y-x|^{\frac{p}{p-1}})^{\frac{N(p-1)}{p}+1}}\varphi(y)dy \\
&= \left(\frac{C_0}{C_1}\right)^{p-1} \int_{\Omega_\varepsilon} \frac{C_1^{p^*-1}\varphi(\varepsilon^{p-1}z+x)}{(1+|z|^{\frac{p}{p-1}})^{\frac{N(p-1)}{p}+1}}dz \\
&\to \left[\left(\frac{C_0}{C_1}\right)^{p-1} \int_{\mathbb{R}^N} \left(\frac{C_1}{(1+|z|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}\right)^{p^*-1} dz\right]\varphi(x) = \varphi(x)
\end{aligned}$$

as $\varepsilon \to 0$, since Ω_ε converges to the whole space $\mathbb{R}^N$, and in view of the definition of C_0 and C_1. □

Remark 4.1.2. It is known by [24] that there exists a weak solution $G_{\varepsilon,\lambda}$ of (4.6), that is $G_{\varepsilon,\lambda} \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla G_{\varepsilon,\lambda}|^{p-2}\langle \nabla G_{\varepsilon,\lambda}, \nabla\varphi\rangle = \lambda \int_\Omega G_{\varepsilon,\lambda}^{p-1}\varphi + \int_\Omega f_\varepsilon\varphi \qquad \forall \varphi \in W_0^{1,p}(\Omega).$$

With the aid of the definition of $G_{\varepsilon,\lambda}$ and (4.6), we can consider $H_{\varepsilon,\lambda}$ as the weak solution of the following problem

$$\begin{cases} -\Delta_p(\Gamma_\varepsilon + H_{\varepsilon,\lambda}) + \Delta_p\Gamma_\varepsilon = \lambda G_{\varepsilon,\lambda}^{p-1} & \text{in } \Omega \\ H_{\varepsilon,\lambda} = -\Gamma_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

It will be useful also the following decomposition of $G_{\varepsilon,\lambda}$:

$$G_{\varepsilon,\lambda} = \chi\Gamma_\varepsilon + \hat{H}_{\varepsilon,\lambda}. \tag{4.8}$$

Here χ is a cut-off function which is equal to 1 in $B_\xi(x) \subset \Omega$ for some radius $\xi > 0$ and equal to 0 near the boundary. We have that, in a weak sense,

$$\begin{cases} -\Delta_p(\chi\Gamma_\varepsilon + \hat{H}_{\varepsilon,\lambda}) + \Delta_p(\chi\Gamma_\varepsilon) = \lambda G_{\varepsilon,\lambda}^{p-1} + g_\varepsilon & \text{in } \Omega \\ \hat{H}_{\varepsilon,\lambda} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where $g_\varepsilon = f_\varepsilon + \Delta_p(\chi\Gamma_\varepsilon)$.

Lemma 4.1.3. *The function g_ε in* (4.9) *converges uniformly to the function g in* (1.7) *as $\varepsilon \to 0$.*

Proof. We observe that $g_\varepsilon = g = 0$ in $B_\xi(x)$ where $\chi = 1$. In $\Omega \setminus B_\xi(x)$ we have that $\Gamma_\varepsilon \to \Gamma$ does hold uniformly (even for the derivatives). Then, recalling the definition of Γ_ε, we get

$$
\begin{aligned}
g_\varepsilon = f_\varepsilon + \Delta_p(\chi\Gamma_\varepsilon) &= \left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1} U_\varepsilon^{p^*-1} + \Delta_p(\chi\Gamma_\varepsilon) \\
&= \left(\frac{C_1}{C_0}\right)^{\frac{p^2}{N-p}} \varepsilon^p \Gamma_\varepsilon^{p^*-1} + \Delta_p(\chi\Gamma_\varepsilon) \to \Delta_p(\chi\Gamma) = g
\end{aligned}
$$

uniformly as $\varepsilon \to 0$. □

We are going to prove an expansion for $PU_{\varepsilon,\lambda}$, solution of (4.1), which will be crucial in the proof of Theorem 1.0.2.
Before turning to the main result of this section, we show that a uniform estimate holds for $H_{\varepsilon,\lambda}$. We will use the following lemma, the proof of which is the same as in Lemma 2.1.1, provided that j is replaced by ε.

Lemma 4.1.4. *Let $G_{\varepsilon,\lambda}$ be a solution of* (4.6)*. Assume $p > \sqrt{N}$. Then $\|G_{\varepsilon,\lambda}^{p-1}\|_1$ is uniformly bounded.*

Now we are able to prove the uniform local bound of $H_{\varepsilon,\lambda}$.

Proposition 4.1.5. *Let $H_{\varepsilon,\lambda}$ be a solution of* (4.7)*. Assume $p > \max\{\sqrt{N}, \frac{N}{2}\}$. Then there exist positive constants p_0, ε_0 and C, which do not depend on ε, such that for $R \geq \varepsilon^{p-1}$ small*

$$
\|H_{\varepsilon,\lambda}\|_{\infty,R} \leq C(R^{-\frac{N}{p_0}}\|H_{\varepsilon,\lambda}\|_{p_0,2R} + R^{\frac{\varepsilon_0}{p-1}}). \tag{4.10}
$$

Proof. Before turning to the estimates it is worth noting that we are not able to apply Proposition 3.1.1 and Corollary 3.1.4 since Γ is replaced by Γ_ε. Even if the hypothesis $|\nabla H_{\varepsilon,\lambda}| = O(|\nabla\Gamma_\varepsilon|)$ is not needed being $p > 2$, we observe that in the proof of Proposition 3.1.1 we use that $|\nabla\Gamma| \geq \delta > 0$, which does not hold for Γ_ε. Then we adapt the proof as follows.

Inequality (3.19) rewrites as

$$\int_{B_2} |\nabla \tilde{\Gamma}_\varepsilon|^{p-2}\eta^p|\nabla v|^2 \leq C|\beta-1+p|^\alpha\Big(\int_{B_2}|\nabla\eta|^2v^2+\int_{B_2}|\nabla\eta|^pv^2+\int_{B_2}\eta^pv^2\Big), \tag{4.11}$$

where $\alpha = 2+\frac{p(p-\varepsilon_0)}{\varepsilon_0}$ and $\tilde{\Gamma}_\varepsilon(z) = R^{\frac{N-p}{p-1}}\Gamma_\varepsilon(Rz)$. We observe that

$$|\nabla\tilde{\Gamma}_\varepsilon| = R^{\frac{N-1}{p-1}}\frac{|Rz|^{\frac{1}{p-1}}}{(\varepsilon^p+|Rz|^{\frac{p}{p-1}})^{\frac{N}{p}}} = \frac{|z|^{\frac{1}{p-1}}}{(\varepsilon^pR^{-\frac{p}{p-1}}+|z|^{\frac{p}{p-1}})^{\frac{N}{p}}}.$$

Assuming $R\geq\varepsilon^{p-1}$, we have that $|\nabla\tilde{\Gamma}_\varepsilon|^{p-2}\geq\delta|z|^{\frac{p-2}{p-1}}$ for some $\delta>0$. Thus (4.11) yields

$$\int_{B_2}|z|^{\frac{p-2}{p-1}}\eta^p|\nabla v|^2\leq C|\beta-1+p|^\alpha\Big(\int_{B_2}|\nabla\eta|^2v^2+\int_{B_2}|\nabla\eta|^pv^2+\int_{B_2}\eta^pv^2\Big),$$

that is, letting $\gamma=\eta^{\frac{p}{2}}$,

$$\int_{B_2}|z|^{\frac{p-2}{p-1}}|\nabla(\gamma v)|^2\leq C|\beta-1+p|^\alpha\Big(\int_{B_2}|\nabla\eta|^2v^2+\int_{B_2}|\nabla\eta|^pv^2+\int_{B_2}\eta^pv^2\Big). \tag{4.12}$$

Consider $v_j\in C_0^\infty(B_2)$ converging to γv in $H_0^1(B_2)$ as $j\to+\infty$. Application of Caffarelli-Kohn-Nirenberg inequality to v_j with $a=-\frac{p-2}{2(p-1)}$ and $b=0$ (see Theorem 2.2.6) gives

$$\Big(\int_{B_2}|v_j|^c\Big)^{\frac{2}{c}}\leq\tilde{C}\int_{B_2}|z|^{\frac{p-2}{p-1}}|\nabla v_j|^2, \tag{4.13}$$

where $c=\frac{2N}{N-2-2a}=\frac{2N(p-1)}{N(p-1)-p}$. Letting $j\to+\infty$, with the aid of Fatou's lemma and Lebesgue's theorem in (4.13) we obtain

$$\Big(\int_{B_2}|\gamma v|^c\Big)^{\frac{2}{c}}\leq C|\beta-1+p|^\alpha\Big(\int_{B_2}|\nabla\eta|^2v^2+\int_{B_2}|\nabla\eta|^pv^2+\int_{B_2}\eta^pv^2\Big) \tag{4.14}$$

in view of (4.12). Since $c>2$, we are able to apply the iterative scheme and then we obtain

$$\|H_{\varepsilon,\lambda}\|_{\infty,R}\leq C(R^{-\frac{N}{p_0}}\|H_{\varepsilon,\lambda}\|_{p_0,2R}+R^{\frac{\varepsilon_0}{p-1}}\|G_{\varepsilon,\lambda}\|_{\frac{N(p-1)}{p-\varepsilon_0},2R}). \tag{4.15}$$

At this point, Lemma 4.1.4 together with the results in [5] yield the uniform boundedness of $G_{\varepsilon,\lambda}$ in $W_0^{1,q}(\Omega)$ for all $q<\bar{q}$, and then in $L^r(\Omega)$ for all $r<\bar{q}^*$

in view of the Sobolev embedding. Since $p > \frac{N}{2}$, we are able to choose $\varepsilon_0 > 0$ such that $\frac{N(p-1)}{p-\varepsilon_0} < \frac{N(p-1)}{N-p}$. Thus there exists a constant $\tilde{C}$, which does not depend on ε, such that $\|G_{\varepsilon,\lambda}\|_{\frac{N(p-1)}{p-\varepsilon_0}} \leq \tilde{C}$. Therefore, using (4.15), we obtain the desired conclusion. $\square$

Remark 4.1.6. Proposition 4.1.5 holds even if we consider $H_{\varepsilon,\lambda} + c$ instead of $H_{\varepsilon,\lambda}$, c being a constant, in view of the form of problem (4.7).

With the aid of [4] as in Section 2.1 we have that, for all $q < \bar{q}$, $G_{\varepsilon,\lambda} \to G_\lambda$ in $W_0^{1,q}(\Omega)$, $G_\lambda = G_\lambda(\cdot, x)$ being the Green function with pole at x, and

$$H_{\varepsilon,\lambda} \to H_\lambda \quad \text{in } W^{1,q}(\Omega) \qquad \text{as } \varepsilon \to 0, \tag{4.16}$$

where $H_\lambda = H_\lambda(\cdot, x)$ is the regular part of G_λ. Taking into account these facts, we get the following convergence result for $H_{\varepsilon,\lambda}$.

Proposition 4.1.7. *Let $H_{\varepsilon,\lambda}$ and H_λ be solutions of problems* (4.7) *and* (1.6) *respectively. Assume $p > \max\{\sqrt{N}, \frac{N}{2}\}$. Then*

$$\|H_{\varepsilon,\lambda} - H_\lambda\|_\infty \to 0 \qquad \text{as } \varepsilon \to 0. \tag{4.17}$$

Proof. Assume by contradiction that (4.17) does not hold. Then there exist $\delta > 0$ and a sequence ε_n converging to 0 such that $\|H_{\varepsilon_n,\lambda} - H_\lambda\|_\infty \geq \delta$. Then we can find a sequence $x_n \in \Omega$ which satisfies[1]

$$|H_{\varepsilon_n,\lambda}(x_n) - H_\lambda(x_n)| \geq \delta. \tag{4.18}$$

By compactness, there exists $x_0 \in \bar{\Omega}$ such that $x_n \to x_0$ as $n \to +\infty$. Using the regularity of $H_{\varepsilon,\lambda}$ away from the pole x, we get $x_0 = x$. Thus (4.18) gives

$$|H_{\varepsilon_n,\lambda}(x_n) - H_\lambda(x)| \geq \frac{\delta}{2}, \tag{4.19}$$

in view of the continuity of H_λ in Ω.
Application of Proposition 4.1.5 and Remark 4.1.6 with $c = -H_\lambda(x)$ yields

$$\|H_{\varepsilon_n,\lambda} - H_\lambda(x)\|_{\infty,R} \leq C\big(R^{-\frac{N}{p_0}}\|H_{\varepsilon_n,\lambda} - H_\lambda(x)\|_{p_0,2R} + R^{\frac{\varepsilon_0}{p-1}}\big), \quad R \geq \varepsilon_n^{p-1}.$$

[1]For ease of notation we will write $H_\lambda(\cdot)$ instead of $H_\lambda(\cdot, x)$.

As a consequence, for a small radius $R \geq \varepsilon_n^{p-1}$ we obtain

$$|H_{\varepsilon_n,\lambda}(x_n) - H_\lambda(x)| \leq C\big(R^{-\frac{N}{p_0}} \|H_{\varepsilon_n,\lambda} - H_\lambda(x)\|_{p_0,2R} + R^{\frac{\varepsilon_0}{p-1}}\big) \quad \forall n \geq n_0, \tag{4.20}$$

n_0 being chosen in such a way that $x_n \in B_R(x)$ for all $n \geq n_0$. We observe that the right-hand side of (4.20) is uniformly bounded in n. Indeed H_{ε_n} is uniformly bounded in $W^{1,q}(\Omega)$ for all $q < \bar{q}$, and then in $L^r(\Omega)$ for all $r < \bar{q}^*$ by virtue of the Sobolev embedding applied to $\hat{H}_{\varepsilon_n,\lambda}$. Recalling (4.16), we can pass to the limit $n \to +\infty$ in (4.20) and we get

$$\lim_{n\to+\infty} |H_{\varepsilon_n,\lambda}(x_n) - H_\lambda(x)| \leq C\big(R^{-\frac{N}{p_0}} \|H_\lambda - H_\lambda(x)\|_{p_0,2R} + R^{\frac{\varepsilon_0}{p-1}}\big) \tag{4.21}$$

since we are able to choose $p_0 < \bar{q}^*$. Notice that (4.21) holds for any $R > 0$, since $\lim_{n\to+\infty} \varepsilon_n = 0$.
At this point we let $R \to 0$ in (4.21). We study the first term in the right-hand side of (4.21).

$$\begin{aligned}\|H_\lambda - H_\lambda(x)\|_{p_0,2R} &= \Big(\int_{B_{2R}(x)} |H_\lambda(y) - H_\lambda(x)|^{p_0} dy\Big)^{\frac{1}{p_0}} \\ &\leq C\Big(\int_{B_{2R}(x)} |y-x|^{\alpha p_0} dy\Big)^{\frac{1}{p_0}} \leq \tilde{C} R^{\alpha + \frac{N}{p_0}},\end{aligned}$$

in view of the Hölder continuity of H at the pole x. Then (4.21) becomes

$$\lim_{n\to+\infty} |H_{\varepsilon_n,\lambda}(x_n) - H_\lambda(x)| \leq C(R^\alpha + R^{\frac{\varepsilon_0}{p-1}}) \to 0 \quad \text{as } R \to 0, \tag{4.22}$$

since α and ε_0 are positive constants. This concludes the proof, being (4.22) in contradiction with (4.19). □

Corollary 4.1.8. *Assume $p > \max\{\sqrt{N}, \frac{N}{2}\}$. Then the expansion*

$$PU_{\varepsilon,\lambda} = U_\varepsilon + \frac{C_1}{C_0} \varepsilon^{\frac{N-p}{p}} H_\lambda + o\big(\varepsilon^{\frac{N-p}{p}}\big) \tag{4.23}$$

does hold uniformly as $\varepsilon \to 0$.

Proof. We recall that $PU_{\varepsilon,\lambda}$ decomposes as

$$PU_{\varepsilon,\lambda} = U_\varepsilon + \frac{C_1}{C_0} \varepsilon^{\frac{N-p}{p}} H_{\varepsilon,\lambda}.$$

Thus, application of Proposition 4.1.7 yields (4.23). □

4.2 Proof of Theorem 1.0.2

We are going to discuss the implications in Theorem 1.0.2.

Proof of (i) ⇒ (ii) in Theorem 1.0.2. In order to show $S_\lambda < S$, we define

$$Q_\lambda(u) = \frac{\|\nabla u\|_p^p - \lambda \|u\|_p^p}{\|u\|_{p^*}^p} \tag{4.24}$$

for $u \in W_0^{1,p}(\Omega)$, $u \neq 0$. For ease of notation we assume $x = 0$. The idea is to estimate $Q_\lambda(PU_{\varepsilon,\lambda})$, using the expansion (4.23) and the properties of the standard bubble U_ε defined in (4.2).
Since $PU_{\varepsilon,\lambda}$ is a solution of (4.1), we have that

$$\begin{aligned}\|\nabla PU_{\varepsilon,\lambda}\|_p^p - \lambda \|PU_{\varepsilon,\lambda}\|_p^p &= \int_\Omega (-\Delta_p PU_{\varepsilon,\lambda} - \lambda PU_{\varepsilon,\lambda}^{p-1}) PU_{\varepsilon,\lambda} = \int_\Omega U_\varepsilon^{p^*-1} PU_{\varepsilon,\lambda} \\ &= \int_\Omega U_\varepsilon^{p^*} + \frac{C_1}{C_0}\varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1} H_\lambda + o\Big(\varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1}\Big),\end{aligned} \tag{4.25}$$

in view of (4.23). We set $\Omega_\varepsilon = \{z \colon z = \frac{y}{\varepsilon^{p-1}}, y \in \Omega\}$. We observe that

$$\begin{aligned}\int_\Omega U_\varepsilon^{p^*} &= \int_\Omega \frac{C_1^{p^*} \varepsilon^N}{(\varepsilon^p + |y|^{\frac{p}{p-1}})^N} dy \\ &= \int_{\Omega_\varepsilon} \frac{C_1^{p^*}}{(1 + |z|^{\frac{p}{p-1}})^N} dz = \int_{\mathbb{R}^N} U_1^{p^*} + O(\varepsilon^N)\end{aligned}$$

in view of

$$\int_{\{|z| > \varepsilon^{-(p-1)}\}} \frac{1}{|z|^{\frac{Np}{p-1}}} = O(\varepsilon^N).$$

Moreover

$$
\begin{aligned}
\varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1} H_\lambda &= \varepsilon^{\frac{N-p}{p}+N(p-1)+\frac{Np-N+p}{p}-(Np-N+p)} \int_{\Omega_\varepsilon} U_1^{p^*-1}(z) H_\lambda(\varepsilon^{p-1}z)dz \\
&= \varepsilon^{N-p} \int_{\Omega_\varepsilon} U_1^{p^*-1}(z)[H_\lambda(\varepsilon^{p-1}z) - H_\lambda(0) + H_\lambda(0)] \\
&= \varepsilon^{N-p} H_\lambda(0) \int_{\mathbb{R}^N} U_1^{p^*-1} + O\Big(\varepsilon^{N-p+\alpha(p-1)} \int_{\mathbb{R}^n} U_1^{p^*-1}|z|^\alpha\Big) \\
&\qquad + O\Big(\varepsilon^{N-p} \int_{\{|z|>\varepsilon^{-(p-1)}\}} U_1^{p^*-1}\Big) \\
&= \varepsilon^{N-p} H_\lambda(0) \int_{\mathbb{R}^N} U_1^{p^*-1} + O(\varepsilon^{N-p+\alpha(p-1)}) + O(\varepsilon^N) \\
&= \varepsilon^{N-p} H_\lambda(0) \int_{\mathbb{R}^N} U_1^{p^*-1} + O(\varepsilon^{N-p+\alpha(p-1)}),
\end{aligned}
$$

where we have used the Hölder continuity of H_λ with respect to 0 and the fact that $\int_{\mathbb{R}^n} U_1^{p^*-1}|z|^\alpha < +\infty$. As for the last term in (4.25), we get

$$
o\Big(\varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1}\Big) = o(\varepsilon^{N-p}). \tag{4.26}
$$

Indeed

$$
\int_\Omega U_\varepsilon^{p^*-1} = \int_\Omega \frac{C_1^{p^*-1}\varepsilon^{\frac{Np-N+p}{p}}}{\left(\varepsilon^p + |y|^{\frac{p}{p-1}}\right)^{\frac{Np-N+p}{p}}} dy = O\big(\varepsilon^{\frac{Np-N+p}{p}-p}\big)
$$

and then (4.26) follows, in view of

$$
\int_{\{|z|>\varepsilon^{-(p-1)}\}} \frac{1}{|z|^{\frac{p}{p-1}(p^*-1)}} = O\big(\varepsilon^{\frac{Np-N+p}{p}-p}\big)
$$

being $p > \frac{N}{2}$. As a consequence, letting $L = \int_{\mathbb{R}^N} U_1^{p^*-1}$, (4.25) rewrites as

$$
\|\nabla PU_{\varepsilon,\lambda}\|_p^p - \lambda \|PU_{\varepsilon,\lambda}\|_p^p = \int_{\mathbb{R}^N} U_1^{p^*} + L\frac{C_1}{C_0} H_\lambda(0)\varepsilon^{N-p} + o(\varepsilon^{N-p}). \tag{4.27}
$$

At this point we are going to study $\|PU_{\varepsilon,\lambda}\|_{p^*}^p$. With the aid of Taylor expansion, we get

$$
\begin{aligned}
|PU_{\varepsilon,\lambda}|^{p^*} &= U_\varepsilon^{p^*} + p^* U_\varepsilon^{p^*-1}\Big[\frac{C_1}{C_0}\varepsilon^{\frac{N-p}{p}} H_\lambda + o\big(\varepsilon^{\frac{N-p}{p}}\big)\Big] \\
&\qquad + O\Big(U_\varepsilon^{p^*-2}\big[\varepsilon^{\frac{N-p}{p}} H_\lambda + o\big(\varepsilon^{\frac{N-p}{p}}\big)\big]^2 + \big[\varepsilon^{\frac{N-p}{p}} H_\lambda + o\big(\varepsilon^{\frac{N-p}{p}}\big)\big]^{p^*}\Big),
\end{aligned}
$$

in view of (4.23). Then, using the boundedness of H_λ, we obtain

$$\int_\Omega |PU_{\varepsilon,\lambda}|^{p^*} = \int_\Omega U_\varepsilon^{p^*} + p^* \frac{C_1}{C_0} \varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1} H_\lambda + o\Big(\varepsilon^{\frac{N-p}{p}} \int_\Omega U_\varepsilon^{p^*-1}\Big)$$
$$+ O\Big(\varepsilon^{\frac{2(N-p)}{p}} \int_\Omega U_\varepsilon^{p^*-2}\Big) + O\Big(\int_\Omega \big[\varepsilon^{\frac{N-p}{p}} H_\lambda + o\big(\varepsilon^{\frac{N-p}{p}}\big)\big]^{p^*}\Big).$$

We observe that

$$\varepsilon^{\frac{2(N-p)}{p}} \int_\Omega U_\varepsilon^{p^*-2} = \varepsilon^{\frac{2(N-p)}{p}} \int_\Omega \frac{C_1^{p^*-2} \varepsilon^{\frac{Np-2N+2p}{p}}}{(\varepsilon^p + |y|^{\frac{p}{p-1}})^{\frac{Np-2N+2p}{p}}} dy$$
$$= \varepsilon^{\frac{2(N-p)}{p} + 2(N-p+1-\frac{N}{p})} \int_{\Omega_\varepsilon} U_1^{p^*-2} = O(\varepsilon^{2(N-p)}).$$

Using the same calculations as before, we obtain

$$\int_\Omega |PU_{\varepsilon,\lambda}|^{p^*} = \int_{\mathbb{R}^N} U_1^{p^*} + p^* L \frac{C_1}{C_0} H_\lambda(0) \varepsilon^{N-p} + o(\varepsilon^{N-p}). \tag{4.28}$$

At this point, expansions (4.27) and (4.28) yield

$$Q_\lambda(PU_{\varepsilon,\lambda}) = \frac{L_1 + L \frac{C_1}{C_0} H_\lambda(0) \varepsilon^{N-p} + o(\varepsilon^{N-p})}{\big(L_1 + p^* L \frac{C_1}{C_0} H_\lambda(0) \varepsilon^{N-p} + o(\varepsilon^{N-p})\big)^{\frac{N-p}{N}}}, \tag{4.29}$$

where $L_1 = \int_{\mathbb{R}^N} U_1^{p^*}$. With the aid of Taylor expansion, (4.29) gives

$$Q_\lambda(PU_{\varepsilon,\lambda}) = L_1^{\frac{p}{N}} - (p-1) L_1^{\frac{p-N}{N}} L \frac{C_1}{C_0} \varepsilon^{N-p} H_\lambda(0) + o(\varepsilon^{N-p}). \tag{4.30}$$

Recalling the definition of S, we have that

$$S = \frac{\int_{\mathbb{R}^N} |\nabla U_1|^p}{(\int_{\mathbb{R}^N} U_1^{p^*})^{\frac{p}{p^*}}} = \frac{L_1}{L_1^{\frac{p(N-p)}{Np}}} = L_1^{\frac{p}{N}},$$

in view of $-\Delta_p U_1 = U_1^{p^*-1}$ in $\mathbb{R}^N$. Thus (4.30) rewrites as

$$Q_\lambda(PU_{\varepsilon,\lambda}) = S - (p-1) L_1^{\frac{p-N}{N}} L \frac{C_1}{C_0} \varepsilon^{N-p} H_\lambda(0) + o(\varepsilon^{N-p}). \tag{4.31}$$

Since $H_\lambda(0) > 0$ and recalling (1.12), then for $\varepsilon > 0$ small the expansion (4.31) yields $S_\lambda \le Q_\lambda(PU_{\varepsilon,\lambda}) < S$ and we get (ii). $\square$

The implication (ii) implies (iii) in Theorem 1.0.2 is classical and it is given by Lemma 1.2.11.

At this point we assume (1.14). Let $\lambda_n = \lambda_* + \frac{1}{n}$. Since $\lambda_n > \lambda_*$, we have that $S_{\lambda_n} < S$. Thus S_{λ_n} is achieved and, up to a normalization, we can find some smooth positive function $u_n \in W_0^{1,p}(\Omega)$ verifying

$$\begin{cases} -\Delta_p u_n - \lambda_n u_n^{p-1} = u_n^{p^*-1} & \text{in } \Omega \\ \int_\Omega u_n^{p^*} = S_{\lambda_n}^{\frac{N}{p}}. \end{cases} \tag{4.32}$$

Our goal now is to study u_n as $n \to +\infty$. Since $\lambda_* < \lambda_1$, by (4.32) u_n is uniformly bounded in $W_0^{1,p}(\Omega)$ and then, after passing to a subsequence, u_n converges weakly to some u_* in $W_0^{1,p}(\Omega)$. Observing that

$$Q_{\lambda_n}(u) = Q_{\lambda_*}(u) + \frac{(\lambda_* - \lambda_n)\|u\|_p^p}{\|u\|_{p^*}^p} = Q_{\lambda_*}(u) + O\Big(\frac{1}{n}\Big),$$

we deduce that $S_{\lambda_n} \to S_{\lambda_*}$ as $n \to +\infty$. By passing to the limit in (4.32), we obtain that u_* verifies

$$\begin{cases} -\Delta_p u_* - \lambda_* u_*^{p-1} = u_*^{p^*-1} & \text{in } \Omega \\ \int_\Omega u_*^{p^*} \le S_{\lambda_*}^{\frac{N}{p}}. \end{cases} \tag{4.33}$$

Thus

$$S_{\lambda_*} \le Q_{\lambda_*}(u_*) = \Big(\int_\Omega u_*^{p^*}\Big)^{\frac{p}{N}} \le S_{\lambda_*}$$

if $u_0 \neq 0$, which would imply that S_{λ_*} is achieved by u_*. Since this is not possible, we get that $u_* = 0$ and

$$u_n \rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega), \quad u_n \to 0 \text{ in } L^q(\Omega) \text{ for } 1 \le q < p^* \text{ and a.e. in } \Omega, \tag{4.34}$$

in view of the Sobolev embedding theorem.

We observe that $0 < u_n \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ as follows by elliptic regularity theory [11, 30, 33] to (4.32). Using (4.34) and since $\int_\Omega u_n^{p^*} = S_{\lambda_n}^{\frac{N}{p}}$, we have that $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. For this reason we will start a blow-up analysis to describe the behavior of u_n. To this purpose, let x_n be a point of Ω where u_n achieves its maximum. We set

$$\mu_n = \big[u_n(x_n)\big]^{-\frac{p}{N-p}} \quad \text{and} \quad U_n(z) = \mu_n^{\frac{N-p}{p}} u_n(\mu_n z + x_n), \tag{4.35}$$

for $z \in \Omega_n = \{\frac{y-x_n}{\mu_n} : y \in \Omega\}$. In particular μ_n represents the blow-up speed, while U_n is the blow-up profile. Since

$$\int_\Omega u_n^{p^*} \le \big(u_n(x_n)\big)^{p^*-p} \int_\Omega u_n^p = \mu_n^{-p} \int_\Omega u_n^p,$$

it is clear by (4.34) that $\mu_n \to 0$ as $n \to +\infty$. Moreover, U_n satisfies

$$\begin{cases} -\Delta_p U_n - \lambda_n \mu_n^p U_n = U_n^{p^*-1} & \text{in } \Omega_n \\ U_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{4.36}$$

with $0 < U_n \le U_n(0) = 1$ and $\sup_{n\in\mathbb{N}} \int_{\Omega_n} U_n^{p^*} + \int_{\Omega_n} |\nabla U_n|^p < +\infty$. By standard elliptic estimates [11, 30, 33], we get that U_n is uniformly bounded in $C^{1,\alpha}(K \cap \Omega_n)$, for all compact subset K of $\mathbb{R}^N$. Then, for all $z \in \partial\Omega_n$:

$$1 = U_n(0) - U_n(z) = \langle \nabla U_n(\xi_n), -z\rangle \le C|z|.$$

As a consequence, we obtain that

$$\frac{\operatorname{dist}(x_n, \partial\Omega)}{\mu_n} = \operatorname{dist}(0, \partial\Omega_n) \ge \frac{1}{C}.$$

Hence, up to a subsequence, we can assume that

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(x_n, \partial\Omega)}{\mu_n} = L \in \left[\frac{1}{C}, +\infty\right]$$

and then, up to a subsequence,

$$U_n \to U \quad \text{in } C^1_{\text{loc}}(\bar{\Omega}_\infty), \tag{4.37}$$

where Ω_∞ is an half-space so that $\operatorname{dist}(0, \partial\Omega_\infty) = L$ and U solves

$$\begin{cases} -\Delta_p U = U^{p^*-1} & \text{in } \Omega_\infty \\ U = 0 & \text{on } \partial\Omega_\infty, \end{cases}$$

with $U \in D^{1,p}(\Omega_\infty)$ such that $0 \le U \le U(0) = 1$ in Ω_∞. If $L < +\infty$, then $U \in D_0^{1,p}(\Omega_\infty)$ and by [29] results $U = 0$, contradicting $U(0) = 1$. Hence

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(x_n, \partial\Omega)}{\mu_n} = +\infty, \tag{4.38}$$

and we have that U coincides with $U_\infty = (1+\Lambda|y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}}$, $\Lambda = C_1^{-\frac{p^2}{(N-p)(p-1)}}$, in view of (4.2) with $x = 0$ and $\varepsilon = C_1^{\frac{p}{(N-p)(p-1)}}$ to have $U_\infty(0) = 1$. Since

$$U_n(y) = \mu_n^{\frac{N-p}{p}} u_n(\mu_n y + x_n) \to (1+\Lambda|y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} \quad \text{uniformly for } y \in B_R \tag{4.39}$$

as $n \to +\infty$ for all $R > 0$, then

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \int_{B_{R\mu_n}(x_n)} u_n^{p^*} = \int_{\mathbb{R}^N} U_\infty^{p^*} = S_{\lambda_*}^{\frac{N}{p}}. \tag{4.40}$$

As a consequence, recalling the energy information $\lim_{n\to+\infty} \int_\Omega u_n^{p^*} = S_{\lambda_*}^{\frac{N}{p}}$, we obtain

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \int_{\Omega\setminus B_{R\mu_n}(x_n)} u_n^{p^*} = 0.$$

This property will reveal crucial in the blow-up description of u_n.

Up to a subsequence, assume that $x_n \to x \in \bar{\Omega}$. The following lemma will be useful to establish the main technical point in the proof of the implication $(iii) \Rightarrow (i)$ in Theorem 1.0.2, that is a comparison between u_n and the bubble

$$U_n(y) = \frac{\mu_n^{\frac{N-p}{p(p-1)}}}{(\mu_n^{\frac{p}{p-1}} + \Lambda|y-x_n|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \quad \Lambda = C_1^{-\frac{p^2}{(N-p)(p-1)}}. \tag{4.41}$$

Thanks to such a fundamental estimate, we will be able to obtain (i) by using Pohozaev-type identities.

Lemma 4.2.1. *Under the hypothesis of Theorem 1.0.2, let u_n be as in* (4.32). *Then*

$$u_n \to 0 \qquad \text{in } C_{loc}(\bar{\Omega}\setminus\{x\}). \tag{4.42}$$

Moreover, the following pointwise estimates do hold:

$$\lim_{n\to+\infty} \sup_\Omega |y-x_n|^{\frac{N-p}{p}} u_n < +\infty, \tag{4.43}$$

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \sup_{\Omega\setminus B_{R\mu_n}(x_n)} |y-x_n|^{\frac{N-p}{p}} u_n = 0. \tag{4.44}$$

Proof. In order to prove (4.42), assume that $\bar{B}_2 \subset \mathbb{R}^N \setminus \{x\}$. Let h and h' be real numbers such that $1 \le h < h' \le 2$ and let η be a nonnegative smooth function satisfying (2.44) and (2.45). Using $\eta^p u_n^\beta$ with $\beta \ge 1$ as test function in the weak formulation of (4.32), we get

$$\int_\Omega \frac{\beta p^p}{(\beta-1+p)^p}\eta^p \left|\nabla\left(u_n^{\frac{\beta-1+p}{p}}\right)\right|^p + p\int_\Omega u_n^\beta \eta^{p-1}\langle\nabla\eta, \nabla u_n\rangle |\nabla u_n|^{p-2} \\ = \int_\Omega \lambda_n \eta^p u_n^{\beta-1+p} + \int_\Omega \eta^p u_n^{p^*-1+\beta}. \quad (4.45)$$

Set $w_n = u_n^{\frac{\beta-1+p}{p}}$. Observing that

$$u_n^\beta \nabla u_n |\nabla u_n|^{p-2} \le u_n^\beta |\nabla u_n|^{p-1} = w_n u_n^{\frac{(p-1)(\beta-1)}{p}} |\nabla u_n|^{p-1},$$

we obtain

$$p\int_\Omega u_n^\beta \eta^{p-1}\langle\nabla\eta, \nabla u_n\rangle|\nabla u_n|^{p-2} = O\bigg(\int_\Omega \frac{1}{(\beta-1+p)^{p-1}}\eta^{p-1}|\nabla\eta| w_n |\nabla w_n|^{p-1}\bigg).$$

We study the right-hand side of (4.45). Application of Hölder inequality yields, for $\varepsilon_0 > 0$ sufficiently small,

$$\int_\Omega \lambda_n \eta^p u_n^{\beta-1+p} = \int_\Omega \lambda_n \eta^p w_n^p \le C\|\eta w_n\|_p^p \le C\|\eta w_n\|_{p^*}^{p-\varepsilon_0}\|\eta w_n\|_p^{\varepsilon_0},$$

$$\int_\Omega \eta^p u_n^{p^*-1+\beta} \le C\bigg(\int_{\Omega\setminus\bar{B}_2} u_n^{p^*-1}\bigg)^{\frac{p-\varepsilon_0}{N}} \|\eta w_n\|_{p^*}^{p-\varepsilon_0}\|\eta w_n\|_p^{\varepsilon_0}.$$

With the aid of these estimates and since

$$C\|\eta w_n\|_{p^*}^{p-\varepsilon_0}\|\eta w_n\|_p^{\varepsilon_0} \le C'\|\eta w_n\|_p^{\varepsilon_0}(\|\nabla\eta w_n\|_p + \|\eta\nabla w_n\|_p)^{p-\varepsilon_0} \\ \le \frac{\beta p^p}{2(\beta-1+p)^p}\int_\Omega \eta^p|\nabla w_n|^p + C\bigg(\frac{(\beta-1+p)^p}{\beta}\bigg)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\int_\Omega \eta^p w_n^p \\ + C\int_\Omega |\nabla\eta|^p w_n^p + C\int_\Omega \eta^p w_n^p,$$

(4.45) rewrites as

$$\frac{\beta}{(\beta-1+p)^p}\int_\Omega \eta^p|\nabla w_n|^p \le C\int_\Omega |\nabla\eta|^p w_n^p \\ + C\bigg[1 + \bigg(\frac{(\beta-1+p)^p}{\beta}\bigg)^{\frac{p-\varepsilon_0}{\varepsilon_0}}\bigg]\int_\Omega \eta^p w_n^p,$$

in view of $\int_{\Omega\setminus\bar{B}_2} u_n^{p^*-1} = O(1)$. Since $\beta \geq 1$, it follows that

$$\int_\Omega \eta^p |\nabla w_n|^p \leq C(\beta-1+p)^p \bigg[\int_\Omega |\nabla \eta|^p w_n^p + (\beta-1+p)^{\frac{p(p-\varepsilon_0)}{\varepsilon_0}} \int_\Omega \eta^p w_n^p \bigg].$$

Application of Sobolev inequality yields

$$\|\eta w_n\|_{p^*}^p \leq C\bigg[\Big((\beta-1+p)^p + 1 \Big) \int_\Omega |\nabla \eta|^p w_n^p + (\beta-1+p)^{\frac{p^2(p-\varepsilon_0)}{\varepsilon_0}} \int_\Omega \eta^p w_n^p \bigg]. \tag{4.46}$$

At this point we observe that (4.34) implies that $w_n^p = u_n^{\beta-1+p} \to 0$ in $L^1(\bar{B}_2 \cap \Omega)$ if $1 \leq \beta < p^*-p+1$. Then, using (4.46), $w_n^{p^*} = u_n^{\frac{\beta-1+p}{p}p^*} \to 0$ in $L^1(B_1 \cap \Omega)$, yielding $u_n \to 0$ in $L^s(B_1 \cap \Omega)$ for $s < \frac{N}{N-p}p^*$. Now, for ε_0 sufficiently small

$$\begin{aligned}
\int_\Omega (\lambda_n + u_n^{p^*-p}) \eta^p w_n^p &\leq \|\lambda_n + u_n^{p^*-p}\|_{\frac{N}{p-\varepsilon_0}} \|\eta w_n\|_{p^*}^{p-\varepsilon_0} \|\eta w_n\|_p^{\varepsilon_0} \\
&\leq C(1 + \|u_n\|_{\frac{(p^*-p)N}{p-\varepsilon_0}}^{p^*-p}) \|\eta w_n\|_{p^*}^{p-\varepsilon_0} \|\eta w_n\|_p^{\varepsilon_0} \\
&\leq C \|\eta w_n\|_{p^*}^{p-\varepsilon_0} \|\eta w_n\|_p^{\varepsilon_0},
\end{aligned}$$

in view of $\frac{(p^*-p)N}{p} = p^* < \frac{N}{N-p}p^*$. Application of the iterative scheme in [30] as in the proof of Proposition 2.2.7 yields

$$\|u_n\|_{\infty, B_1 \cap \Omega} \leq C \|u_n\|_{p, B_2 \cap \Omega} \to 0,$$

from which follows (4.42).

In order to prove(4.43), assume by contradiction that there exists $y_n \in \Omega$ such that

$$|y_n - x_n|^{\frac{N-p}{p}} u_n(y_n) = \max_\Omega |y - x_n|^{\frac{N-p}{p}} u_n \to +\infty \quad \text{as } n \to +\infty. \tag{4.47}$$

Setting $\nu_n = \big[u_n(y_n)\big]^{-\frac{p}{N-p}}$, we have that $\nu_n \to 0$ and $\frac{|y_n - x_n|}{\nu_n} \to +\infty$ as $n \to +\infty$. Since $|y_n - x_n|^{\frac{N-p}{p}} u_n(y_n) = \big(\frac{|y_n - x_n|}{\mu_n}\big)^{\frac{N-p}{p}} U_n\big(\frac{y_n - x_n}{\mu_n}\big)$ is bounded if $\frac{|y_n - x_n|}{\mu_n}$ is bounded, then $\frac{|y_n - x_n|}{\mu_n} \to +\infty$ as $n \to +\infty$. Set $V_n(z) = \nu_n^{\frac{N-p}{p}} u_n(\nu_n z + y_n)$ for $z \in \tilde{\Omega}_n = \{\frac{y - y_n}{\nu_n} : y \in \Omega\}$. Then $V_n(0) = 1$ and

$$\begin{aligned}
0 \leq V_n(y) &\leq \nu_n^{\frac{N-p}{p}} |\nu_n y + y_n - x_n|^{-\frac{N-p}{p}} |y_n - x_n|^{\frac{N-p}{p}} u_n(y_n) \\
&= \bigg(\frac{|y_n - x_n|}{|\nu_n y + y_n - x_n|} \bigg)^{\frac{N-p}{p}} \leq 2^{\frac{N-p}{p}}
\end{aligned}$$

holds uniformly for $|y| \leq \frac{1}{2}\frac{|y_n - x_n|}{\nu_n}$. Moreover V_n satisfies

$$\begin{cases} -\Delta_p V_n - \lambda_n \nu_n^p V_n^{p-1} = V_n^{p^*-1} & \text{in } \tilde{\Omega}_n \\ V_n = 0 & \text{on } \partial\tilde{\Omega}_n, \end{cases}$$

with $\int_{\tilde{\Omega}_n} V_n^{p^*} \leq 1$ and $\int_{\tilde{\Omega}_n} |\nabla V_n|^p \leq C$. By standard elliptic estimates [11, 30, 33], we get that V_n is uniformly bounded in $C^{1,\alpha}(K \cap \tilde{\Omega}_n)$ for all compact subset K of $\mathbb{R}^N$. Then $1 = V_n(0) - V_n(z) \leq C|z|$ for all $z \in \partial\tilde{\Omega}_n$. As a consequence

$$\frac{\operatorname{dist}(y_n, \partial\Omega)}{\nu_n} = \operatorname{dist}(0, \partial\tilde{\Omega}_n) \geq \frac{1}{C}.$$

Hence, up to a subsequence, we have that

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(y_n, \partial\Omega)}{\nu_n} = L \in \left[\frac{1}{C}, +\infty\right]$$

and $V_n \to V$ in $C^1_{\text{loc}}(\tilde{\Omega}_\infty)$, where $\tilde{\Omega}_\infty$ is an half-space so that $\operatorname{dist}(0, \partial\tilde{\Omega}_\infty) = L$. Moreover, $V \in D_0^{1,p}(\tilde{\Omega}_\infty)$ solves

$$\begin{cases} -\Delta_p V = V^{p^*-1} & \text{in } \tilde{\Omega}_\infty \\ V = 0 & \text{on } \partial\tilde{\Omega}_\infty, \end{cases}$$

with $0 \leq V \leq 2^{\frac{N-p}{p}}$ in $\tilde{\Omega}_\infty$. If $L < +\infty$, by [29] results $V = 0$, contradicting $V(0) = 1$. Hence $L = +\infty$, $V = U_\infty$ and $\lim_{R\to+\infty} \lim_{n\to+\infty} \int_{B_{R\nu_n}(y_n)} u_n^{p^*} = S_{\lambda_*}^{\frac{N}{p}}$. This implies

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \int_\Omega u_n^{p^*} \geq \lim_{R\to+\infty} \lim_{n\to+\infty} \left(\int_{B_{R\mu_n}(x_n)} u_n^{p^*} + \int_{B_{R\nu_n}(y_n)} u_n^{p^*} \right) = 2S_{\lambda_*}^{\frac{N}{p}},$$

which contradicts $\int_\Omega u_n^{p^*} = S_{\lambda_*}^{\frac{N}{p}}$. Thus (4.43) is proved.

A simple adaptation of the proof of (4.43) gives (4.44) In this case we assume by contradiction that there exist $y_n \in \Omega$ and $\delta > 0$ such that

$$|y_n - x_n|^{\frac{N-p}{p}} u_n(y_n) = \sup_{\Omega \setminus B_{R\mu_n}(x_n)} |y - x_n|^{\frac{N-p}{p}} u_n \geq \delta \quad \text{as } n \to +\infty. \tag{4.48}$$

This implies that $y_n \to x$ and then $u_n(y_n) \to +\infty$ as $n \to +\infty$. Using the same notation as before, we have that $\nu_n \to 0$ as $n \to +\infty$ and

$$\frac{x_n - y_n}{\nu_n} \to p \quad \text{as } n \to +\infty,$$

with $p \neq 0, +\infty$ in view of (4.43) and (4.48). Let η be a radius such that $B_{\frac{\eta\nu_n}{2}}(y_n) \subset \Omega \backslash B_{R\mu_n}(x_n)$ for all $R > 0$ provided n is sufficiently large. Assume $z \in \tilde{\Omega}_n \setminus B_\eta(\frac{x_n - y_n}{\nu_n})$. Then it is easy to check that

$$\eta^{\frac{N-p}{p}} V_n(z) \leq \left| z - \frac{x_n - y_n}{\nu_n} \right|^{\frac{N-p}{p}} V_n(z) = |\nu_n z + y_n - x_n|^{\frac{N-p}{p}} u_n(\nu_n z + y_n) \leq C$$

in view of (4.43). Thus

$$V_n \leq C\eta^{-\frac{N-p}{p}} \qquad \text{in } \tilde{\Omega}_n \setminus B_\eta\left(\frac{x_n - y_n}{\nu_n}\right).$$

As a consequence, passing to the limit as $n \to +\infty$ we get

$$V_n \to V \quad \text{in } \Omega_\infty \setminus \{p\}.$$

Since $V \geq 0$ solves $-\Delta_p V = V^{p^*-1}$ in $\tilde{\Omega}_\infty$, by the strong maximum principle in [36] we obtain that $V > 0$ in $\tilde{\Omega}_\infty$ in view of $V(0) = 1$ thanks to $0 \in \tilde{\Omega}_\infty$. In particular

$$\int_{B_{\frac{\eta}{2}}} V^{p^*} > 0. \tag{4.49}$$

Recalling the definition of V_n, we have that

$$\int_{B_{\frac{\eta}{2}}} V_n^{p^*} = \int_{B_{\frac{\eta\nu_n}{2}}(y_n)} u_n^{p^*} = \int_{B_{\frac{\eta\nu_n}{2}}(y_n) \cap B_{R\mu_n}(x_n)} u_n^{p^*} + \int_{B_{\frac{\eta\nu_n}{2}}(y_n) \setminus B_{R\mu_n}(x_n)} u_n^{p^*} = 0,$$

which contradicts (4.49). This concludes the proof of (4.44). □

At this point we give the proof of the comparison between u_n and the bubble U_n defined by (4.41).

Lemma 4.2.2. *Under the hypothesis of Theorem 1.0.2, let u_n be as in* (4.32). *Then there exists $C > 0$ so that*

$$u_n \leq \frac{C\mu_n^{\frac{N-p}{p(p-1)}}}{(\mu_n^{\frac{p}{p-1}} + \Lambda|y - x_n|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \qquad \text{in } \Omega \tag{4.50}$$

does hold for all $n \in \mathbb{N}$.

Proof. Since the inequality (4.50) clearly holds in $B_{R\mu_n}(x_n)$ for all $R > 0$ in view of (4.39), it remains to prove that it is true in $\Omega \setminus B_{R\mu_n}(x_n)$. Then (4.50) is equivalent to establish the estimate

$$u_n \le \frac{C\mu_n^{\frac{N-p}{p(p-1)}}}{|y-x_n|^{\frac{N-p}{p-1}}} \qquad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.51}$$

for some $C, R > 0$ and for all $n \in \mathbb{N}$.
For $\varrho > 0$ and $0 < \delta < \frac{N-p}{p-1}$ we define

$$M_n = \sup_{\partial B_\varrho(x)\cap\Omega} u_n, \qquad \Phi_n = C\frac{\mu_n^{\frac{N-p}{p(p-1)}-\delta} + M_n}{|y-x_n|^{\frac{N-p}{p-1}-\delta}}.$$

The proof of (4.51) is divided into 3 steps.

Step 1. Our aim is to prove the following estimate:

$$u_n \le \Phi_n \qquad \text{in } \Omega \setminus B_{R\mu_n}(x_n). \tag{4.52}$$

We observe that

$$\nabla\Phi_n = C\Big(-\frac{N-p}{p-1} + \delta\Big)\Big(\mu_n^{\frac{N-p}{p(p-1)}-\delta} + M_n\Big)|y-x_n|^{\frac{p-N}{p-1}+\delta-2}(y-x_n).$$

Then

$$|\nabla\Phi_n| = C\Big(\frac{N-p}{p-1} - \delta\Big)\Big(\mu_n^{\frac{N-p}{p(p-1)}-\delta} + M_n\Big)|y-x_n|^{\frac{1-N}{p-1}+\delta}.$$

Easy computations lead to

$$\begin{aligned}
-\Delta_p\Phi_n &= \Big[C\Big(\frac{N-p}{p-1} - \delta\Big)\Big(\mu_n^{\frac{N-p}{p(p-1)}-\delta} + M_n\Big)\Big]^{p-1}\operatorname{div}(|y-x_n|^{-N+\delta(p-1)}(y-x_n)) \\
&= \Big[C\Big(\frac{N-p}{p-1} - \delta\Big)\Big(\mu_n^{\frac{N-p}{p(p-1)}-\delta} + M_n\Big)\Big]^{p-1}\delta(p-1)|y-x_n|^{-N+\delta(p-1)} \\
&= \Big(\frac{N-p}{p-1} - \delta\Big)^{p-1}\delta(p-1)\frac{\Phi_n^{p-1}}{|y-x_n|^p}.
\end{aligned}$$

Thanks to (4.44), we may choose R such that, for all n large,

$$u_n^{p^*-p} \le \frac{\varepsilon}{|y-x_n|^p} \qquad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.53}$$

for some $0 < \varepsilon < 1$. We observe that

$$-\Delta_p u_n - \left(\lambda_n + \frac{\varepsilon}{|y-x_n|^p}\right) u_n^{p-1} = \left(u_n^{p^*-p} - \frac{\varepsilon}{|y-x_n|^p}\right) v_n^{p-1}. \tag{4.54}$$

Let L_n be the operator

$$L_n u = -\Delta_p u - \left(\lambda_n + \frac{\varepsilon}{|y-x_n|^p}\right) u^{p-1}.$$

Then (4.53) and (4.54) yield

$$L_n u_n \leq 0 \qquad \text{in } \Omega \setminus B_{R\mu_n}(x_n). \tag{4.55}$$

We have that

$$\begin{aligned} L_n \Phi_n &= -\Delta_p \Phi_n - \left(\lambda_n + \frac{\varepsilon}{|y-x_n|^p}\right) \Phi_n^{p-1} \\ &= \left[\left(\frac{N-p}{p-1} - \delta\right)^{p-1} \delta(p-1) - (\lambda_n |y-x_n|^p + \varepsilon)\right] \frac{\Phi_n^{p-1}}{|y-x_n|^p}. \end{aligned}$$

Then, for a suitable choice of ϱ, we obtain

$$L_n \Phi_n \geq 0 \qquad \text{in } B_\varrho(x) \cap \Omega \setminus B_{R\mu_n}(x_n). \tag{4.56}$$

Thus $L_n u_n \leq L_n \Phi_n$ in $B_\varrho(x) \cap \Omega \setminus B_{R\mu_n}(x_n)$, in view of (4.55) and (4.56). Since it easily seen that

$$u_n \leq \Phi_n \qquad \text{on } \partial\big(\Omega \cap B_\varrho(x_n) \setminus B_{R\mu_n}(x_n)\big),$$

in view of (4.39) applied on $\partial B_{R\mu_n}(x_n)$, we obtain

$$u_n \leq \Phi_n \qquad \text{in } \Omega \cap B_\varrho(x) \setminus B_{R\mu_n}(x_n). \tag{4.57}$$

Our aim now is to extend the validity of (4.57) from $\Omega \cap B_\varrho(x) \setminus B_{R\mu_n}(x_n)$ to $\Omega \setminus B_{R\mu_n}(x_n)$. We set $A = \Omega \setminus B_\varrho(x)$. Letting $v_n = \frac{u_n}{M_n}$, we have that

$$\begin{cases} -\Delta_p v_n - \lambda_n v_n^{p-1} = f_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega \\ \sup\limits_{\Omega \cap \partial B_\varrho(x)} v_n = 1, & \end{cases} \tag{4.58}$$

where $f_n = \frac{u_n^{p^*-1}}{M_n^{p-1}}$. We observe that the equation in (4.58) can be rewritten as $-\Delta_p v_n - a_n v_n^{p-1} = 0$, with $a_n = \lambda_n + u_n^{\frac{p^2}{N-p}}$. In particular, using (4.42), we have that a_n converges to λ_* in $L^\infty(A)$ as $n \to \infty$, with $\lambda_* < \lambda_1(\Omega) < \lambda_1(A)$. Since $0 \le v_n \le 1$ on ∂A, we are able to apply Lemma A.2.3 in Appendix, yielding $\sup_{n\in\mathbb{N}} \|v_n\|_{p-1,A} < +\infty$. Moreover, since $v_n \in W_g^{1,p}(A)$ for some $g \in L^\infty(a) \cap W^{1,p}(A)$ such that $g \le 1$ on ∂A, application of Theorem A.2.2 in Appendix provides a universal bound on $v_n - g$ in $W_0^{1,q}(A)$ for all $1 \le q < \bar{q}$, and then on v_n in $L^s(A)$ for all $1 \le s < \bar{q}^*$ in view of Sobolev embedding theorem. In particular $\sup_{n\in\mathbb{N}} \|v_n^{p-1}\|_{q_0,A} < +\infty$ for some $q_0 > \frac{N}{p}$ in view of $\frac{\bar{q}^*}{p-1} > \frac{N}{p}$ thanks to $p > \frac{N}{2}$. Let w_n be the solution of

$$\begin{cases} -\Delta_p w_n = f_n & \text{in } A \\ w_n = 0 & \text{on } \partial A. \end{cases} \tag{4.59}$$

Observing that $\sup_{n\in\mathbb{N}} \|f_n\|_{1,A} < +\infty$ thanks to (4.42), with the aid of [5] and the Sobolev embedding theorem we deduce that $\sup_{n\in\mathbb{N}} \|w_n\|_{q,A} < +\infty$ for all $1 \le q < \bar{q}^*$. At this point, using (4.58) and (4.59), we obtain

$$\begin{cases} -\Delta_p v_n + \Delta_p w_n = \lambda_n v_n^{p-1} & \text{in } A \\ v_n - w_n \le 1 & \text{on } \partial B_\varrho(x) \cap A \\ v_n - w_n = 0 & \text{on } \partial\Omega \cap A. \end{cases} \tag{4.60}$$

Using $\varphi = (v_n - w_n - 1)_+^\beta$, $\beta > 0$, as test function in the weak formulation of (4.60), we get the following estimate[2] in terms of $w = (v_n - w_n - 1)_+^{\frac{\beta-1+p}{p}}$:

$$\frac{\beta}{(\beta-1+p)^p} \int_A |\nabla w|^p \le C|A|^{\frac{(q_0-1)(p-1)}{q_0(\beta-1+p)}} \|v_n^{p-1}\|_{q_0,A} \|w\|_{\frac{pq_0}{q_0-1}}^{\frac{\beta p}{\beta-1+p}},$$

as a consequence of Hölder inequality with exponents $\frac{q_0(\beta-1+p)}{(q_0-1)(p-1)}$, q_0 and $\frac{q_0(\beta-1+p)}{(q_0-1)\beta}$. By Sobolev embedding theorem we deduce that

$$\|w\|_{p^*} \le C(\beta-1+p)\|w\|_{\frac{pq_0}{q_0-1}}^{\frac{p}{\beta-1+p}}.$$

[2]Calculations are similar to those in section 2.2.2, based on Serrin's iterative scheme. Then some details will be omitted.

Recalling the definition of w and taking the $\frac{\beta-1+p}{p}$-th roots, we get

$$\|(v_n - w_n - 1)_+\|_{\frac{(\beta-1+p)p^*}{p}} \leq [C(\beta-1+p)]^{\frac{1}{\beta-1+p}} \|(v_n - w_n - 1)_+\|^{\frac{p}{\beta-1+p}}_{\frac{q_0(\beta-1+p)}{q_0-1}},$$

which is equivalent to

$$\|(v_n - w_n - 1)_+\|_{\kappa\mu,A} \leq \left[Cp\frac{q_0-1}{pq_0}\mu\right]^{\frac{p_1}{\mu}} \|(v_n - w_n - 1)_+\|^{\frac{p_1}{\mu}}_{\mu,A},$$

where $\kappa = \frac{(q_0-1)p^*}{pq_0}$, $\mu = \frac{q_0(\beta-1+p)}{q_0-1}$ and $p_1 = \frac{pq_0}{q_0-1}$. Since $q_0 > \frac{N}{p}$, then $\kappa > 1$ and we are able to find $\beta_0 > 0$ so that $\mu < \bar{q}^*$. Application of iteration process finally yields $\|(v_n - w_n - 1)_+\|_{\infty,A} \leq \tilde{C}$ for some universal $\tilde{C} > 0$, and then $\sup_{n\in\mathbb{N}} \|v_n - w_n\|_{\infty,A} < +\infty$. Therefore $\sup_{n\in\mathbb{N}} \|f_n\|_{q,A} < +\infty$ for some $q > \frac{N}{p}$ in view of $\frac{\bar{q}^*}{p-1} > \frac{N}{p}$. With the aid of [30], we obtain that $\sup_{n\in\mathbb{N}} \|w_n\|_{\infty,A} < +\infty$, from which follows that $\sup_{n\in\mathbb{N}} \|v_n\|_{\infty,A} < +\infty$.
As a result

$$\sup_A u_n \leq C \sup_{\Omega\cap B_\varrho(x)} u_n$$

for some positive constant C, and (4.52) is so established.

Step 2. Let us now prove that

$$M_n = o\big(\mu_n^{\frac{N-p}{p(p-1)}-\delta}\big) \qquad \text{for all } 0 < \delta < \frac{N-p}{p-1}. \tag{4.61}$$

Assume that, on the contrary, there exists $\tilde{C} > 0$ such that

$$M_n \geq \tilde{C}\mu_n^{\frac{N-p}{p(p-1)}-\delta}, \tag{4.62}$$

so that we may rewrite (4.52) as

$$u_n \leq \frac{CM_n}{|y-x_n|^{\frac{N-p}{p-1}-\delta}} \qquad \text{in } \Omega\setminus B_{R\mu_n}(x_n). \tag{4.63}$$

We observe that

$$\int_{B_{R\mu_n}(x_n)} f_n = \int_{B_{R\mu_n}(x_n)} \frac{u_n^{p^*-1}}{M_n^{p-1}} = \frac{\mu_n^{\frac{N-p}{p}}}{M_n^{p-1}} \int_{B_R} U_n^{p^*-1} = O\left(\frac{\mu_n^{\frac{N-p}{p}}}{M_n^{p-1}}\right),$$

in view of (4.35) and (4.37). Application of (4.62) gives

$$\int_{B_{R\mu_n}(x_n)} f_n \to 0 \qquad \text{as } n \to +\infty.$$

On the other hand, inequality (4.63) yields

$$\begin{aligned}\int_{\Omega\setminus B_{R\mu_n}(x_n)} f_n &= O\bigg(\int_{\Omega\setminus B_{R\mu_n}(x_n)} \frac{M_n^{p^*-1-p+1}}{|y-x_n|^{(\frac{N-p}{p-1}-\delta)(p^*-1)}}\bigg)\\ &= O\bigg(\int_{\Omega\setminus B_{R\mu_n}(x_n)} \frac{M_n^{\frac{p^2}{N-p}}}{|y-x_n|^{N+\frac{p}{p-1}-\delta(p^*-1)}}\bigg)\\ &= O\bigg(M_n^{\frac{p^2}{N-p}} \max\big\{\mu_n^{-\frac{p}{p-1}+\delta(p^*-1)}, 1\big\}\bigg).\end{aligned}$$

In particular, if $\delta > \frac{p}{(p-1)(p^*-1)} = \delta_0$, we have that

$$\int_{\Omega\setminus B_{R\mu_n}(x_n)} f_n = O\bigg(M_n^{\frac{p^2}{N-p}}\bigg) \to 0 \qquad \text{as } n \to +\infty.$$

Since $\delta_0 < \frac{N-p}{p-1}$, assuming $\delta \in (\delta_0, \frac{N-p}{p-1})$, we have that $\|f_n\|_1 \to 0$ as $n \to +\infty$. By [5] it turns out that v_n converges to a limit function v in $W_0^{1,q}(\Omega)$ for all $1 \le q < \bar{q}$ and in $L^s(\Omega)$ for all $1 \le s < \bar{q}^*$ as $n \to +\infty$, with

$$-\Delta_p v - \lambda_* v^{p-1} = 0 \qquad \text{in } \Omega. \tag{4.64}$$

Using $\varphi = T_l(v_n)$ as test function in the weak formulation of (4.58), we obtain

$$\int_{\{|v_n|\le l\}} |\nabla v_n|^p \le \lambda_n \int_\Omega v_n^p + l\|f_n\|_1. \tag{4.65}$$

Letting $n \to +\infty$ and then $l \to +\infty$ in (4.65), we deduce

$$\int_\Omega |\nabla v|^p \le \lambda_* \int_\Omega v^p < +\infty.$$

Thus $v = 0$, since $v \in W_0^{1,p}(\Omega)$ would solve (4.64) with $\lambda_* < \lambda_1$. But this is impossible in view of the definition of v_n. Hence

$$M_n \ll \mu_n^{\frac{N-p}{p(p-1)}-\delta}, \qquad \delta \in \bigg(\delta_0, \frac{N-p}{p-1}\bigg). \tag{4.66}$$

This estimate is true even if $\delta = \delta_0$. Indeed in this case we get

$$\int_{\Omega\setminus B_{R\mu_n}(x_n)} f_n = O\bigg(M_n^{\frac{p^2}{N-p}} \log \mu_n\bigg) = O\bigg(\mu_n^{\frac{p^2}{p(p-1)}-\bar{\delta}\frac{p^2}{N-p}} \log \mu_n\bigg) \to 0$$

as $n \to +\infty$, for some $\bar{\delta} \in \left(\delta_0, \frac{N-p}{p-1}\right)$ in view of (4.66). Thus

$$M_n \ll \mu_n^{\frac{N-p}{p(p-1)}-\delta}, \qquad \delta \in \left[\delta_0, \frac{N-p}{p-1}\right). \tag{4.67}$$

At this point we set $\delta_1 = \frac{p^2}{(N-p)(p^*-1)}\delta_0$. Assume $\delta \in (\delta_1, \delta_0)$. Then

$$\int_{\Omega \setminus B_{R\mu_n}(x_n)} f_n = O\left(M_n^{\frac{p^2}{N-p}} \mu_n^{-\frac{p}{p-1}+\delta(p^*-1)} \right) = O\left(\mu_n^{-\delta_0 \frac{p^2}{N-p}+\delta(p^*-1)} \right) \to 0$$

as $n \to +\infty$, in view of (4.67). Arguing as before, we obtain

$$M_n \ll \mu_n^{\frac{N-p}{p(p-1)}-\delta}, \qquad \delta \in \left(\delta_1, \frac{N-p}{p-1}\right). \tag{4.68}$$

We can improve this estimate by observing that, if $\delta = \delta_1$,

$$\begin{aligned}
\int_{\Omega \setminus B_{R\mu_n}(x_n)} f_n &= O\left(M_n^{\frac{p^2}{N-p}} \mu_n^{-\frac{p}{p-1}+\frac{p^2}{(N-p)(p^*-1)}\delta_0(p^*-1)} \right) \\
&= O\left(\mu_n^{\frac{p}{p-1}-\bar{\delta}\frac{p^2}{N-p}-\frac{p}{p-1}+\frac{p^2}{N-p}\delta_0} \right) \\
&= O\left(\mu_n^{\frac{p^2}{N-p}(\delta_0-\bar{\delta})} \right) \to 0 \qquad \text{as } n \to +\infty,
\end{aligned}$$

for some $\bar{\delta} \in (\delta_1, \delta_0)$ in view of (4.68). Thus

$$M_n \ll \mu_n^{\frac{N-p}{p(p-1)}-\delta}, \qquad \delta \in \left[\delta_1, \frac{N-p}{p-1}\right). \tag{4.69}$$

We set $\delta_k = \left(\frac{p^2}{(N-p)(p^*-1)}\right)^k \delta_0$. We observe that $\delta_k \to 0$ as $k \to +\infty$, being $\frac{p^2}{(N-p)(p^*-1)} < 1$. With the same arguments than above, we will clearly have

$$M_n \ll \mu_n^{\frac{N-p}{p(p-1)}-\delta}, \qquad \delta \in \left[\delta_k, \frac{N-p}{p-1}\right)$$

Then (4.61) easily follows.

Step 3. Step 1 and Step 2 lead to a weaker form of (4.51), i.e. there exist $C, R > 0$ so that

$$u_n \le \frac{C\mu_n^{\frac{N-p}{p(p-1)}-\delta}}{|y-x_n|^{\frac{N-p}{p-1}-\delta}} \qquad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.70}$$

does hold for all $n \in \mathbb{N}$, $0 < \delta < \frac{N-p}{p-1}$. In order to establish (4.51), we repeat the previus argument for $v_n = \frac{u_n}{\mu_n^{\frac{N-p}{p(p-1)}}}$. We have that

$$\begin{cases} -\Delta_p v_n - \lambda_n v_n^{p-1} = f_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.71}$$

where $f_n = \frac{u_n^{p^*-1}}{\mu_n^{\frac{N-p}{p}}}$. Notice that f_n satisfies

$$f_n \leq \frac{C\mu_n^{\frac{p}{p-1}-(p^*-1)\delta}}{|y-x_n|^{\frac{(N-p)(p^*-1)}{p-1}-(p^*-1)\delta}} \quad \text{in } \Omega \tag{4.72}$$

where we have extendend (4.70) to $B_{R\mu_n}(x_n)$ in view of the definition of f_n. Thus f_n is uniformly bounded in $L^1(\Omega)$. Indeed

$$\int_\Omega f_n = O(1) + O\Big(\int_{\Omega \setminus B_{R\mu_n}(x_n)} \frac{\mu_n^{\frac{p}{p-1}-(p^*-1)\delta}}{|y-x_n|^{\frac{(N-p)(p^*-1)}{p-1}-(p^*-1)\delta}}\Big) = O(1). \tag{4.73}$$

Letting h_n be the solution of

$$\begin{cases} -\Delta_p h_n = f_n & \text{in } \Omega \\ h_n = 0 & \text{on } \partial\Omega, \end{cases}$$

we deduce that $\sup_{n\in\mathbb{N}} \|v_n - h_n\|_\infty < +\infty$, or equivalently

$$\|u_n - \mu_n^{\frac{N-p}{p(p-1)}} h_n\|_\infty = O(\mu_n^{\frac{N-p}{p(p-1)}}). \tag{4.74}$$

For $\alpha > N$ the radial function

$$W(y) = (\alpha - N)^{-\frac{1}{p-1}} \int_{|y|}^\infty \frac{(t^{\alpha-N}-1)^{\frac{1}{p-1}}}{t^{\frac{\alpha-1}{p-1}}} dt$$

is a positive and strictly decreasing solution of $-\Delta_p W = |y|^{-\alpha}$ in $\mathbb{R}^N \setminus \{0\}$ so that

$$\lim_{|y|\to\infty} |y|^{\frac{N-p}{p-1}} W(y) = \frac{p-1}{N-p}(\alpha - N)^{-\frac{1}{p-1}} > 0. \tag{4.75}$$

Taking $\delta > 0$ small in such a way that $\alpha = \frac{(N-p)(p^*-1)}{p-1} - (p^*-1)\delta > N$, then $w_n(y) = \mu_n^{-\frac{N-p}{p-1}} W(\frac{y-x_n}{\mu_n})$ satisfies

$$-\Delta_p w_n = \frac{\mu_n^{\frac{p}{p-1}-(p^*-1)\delta}}{|y-x_n|^{\frac{(N-p)(p^*-1)}{p-1}-(p^*-1)\delta}} \quad \text{in } \mathbb{R}^N \setminus \{x_n\}.$$

Since

$$h_n(y) = \mu_n^{-\frac{N-p}{p(p-1)}} u_n(y) + O(1) = \mu_n^{-\frac{N-p}{p-1}} U_n\left(\frac{y - x_n}{\mu_n}\right) + O(1) \le C_1 w_n(x)$$

for some $C_1 > 0$ and for all $y \in \partial B_{R\mu_n}(x_n)$ in view of (4.39), (4.74) and $W(R) > 0$, we have that $\Psi_n = Cw_n$ satisfies

$$\begin{cases} -\Delta_p \Psi_n \ge f_n & \text{in } \Omega \setminus B_{R\mu_n}(x_n) \\ \Psi_n \ge h_n & \text{on } \partial\Omega \cup \partial B_{R\mu_n}(x_n) \end{cases}$$

for $C = C_0^{\frac{1}{p-1}} + C_1$ thanks to (4.72), and then by weak comparison principle we deduce that

$$h_n \le \Psi_n \le \frac{C}{|y - x_n|^{\frac{N-p}{p-1}}} \quad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.76}$$

for some $C > 0$ in view of (4.75). Inserting (4.74) into (4.76) we finally deduce the validity of (4.51).
The proof of Lemma 4.2.2 is complete. □

Now we are going to conclude the proof of Theorem 1.0.2. We will use the blow-up analysis of u_n discussed above. In particular, the fundamental estimate in Lemma 4.2.2 will allow us to apply a Pohozaev identity in the whole Ω_n in order to exclude boundary blow-up. Then, still by a Pohozaev identity on a ball, we will obtain an information as $n \to +\infty$ from which the statement (i) of Theorem 1.0.2 will follow.

Proof of implication $(iii) \Rightarrow (i)$ in Theorem 1.0.2. It is enough to show that

$$\text{there exists } x \in \Omega \text{ such that } H_{\lambda_*}(x, x) = 0. \tag{4.77}$$

Indeed, Proposition A.3.1 in the Appendix yields the monotonicity of H_λ with respect to λ. As a consequence we have that (4.77) yields $H_\lambda(x, x) > 0$ being $\lambda > \lambda_*$, that is (i).

First we prove that the accumulation point x cannot be on the boundary of Ω. Assume by contradiction that

$$d_n = d(x_n, \partial\Omega) \to 0 \quad \text{as } n \to +\infty. \tag{4.78}$$

We set $\hat{x} = x - \nu_x$, ν_x being the outward unit normal vector at $x \in \partial\Omega$. We apply now the Pohozaev identity to u_n on Ω, obtained by integration of (4.32) against $\langle y - \hat{x}, \nabla u_n \rangle$ on Ω. This leads to

$$\frac{N-p}{p} \int_\Omega |\nabla u_n|^p + \int_{\partial\Omega} |\nabla u_n|^{p-2} \partial_\nu u_n \langle y - \hat{x}, \nabla u_n \rangle = \frac{\lambda_n N}{p} \int_\Omega u_n^p + \frac{N}{p^*} \int_\Omega u_n^{p^*},$$

that is

$$\frac{N-p}{p} \int_\Omega |\nabla u_n|^p + \int_{\partial\Omega} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle = \frac{\lambda_n N}{p} \int_\Omega u_n^p + \frac{N}{p^*} \int_\Omega u_n^{p^*} \quad (4.79)$$

in view of $\nabla u_n = \partial_\nu u_n \nu$ on $\partial\Omega$, being $u_n = 0$ on $\partial\Omega$. On the other hand, using u_n as test function in the weak formulation of (4.32), we obtain

$$\int_\Omega |\nabla u_n|^p - \lambda_n \int_\Omega u_n^p = \int_\Omega u_n^{p^*}. \quad (4.80)$$

Using (4.80) in (4.79), we get

$$\int_{\partial\Omega} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle \leq \lambda_n \int_\Omega u_n^p. \quad (4.81)$$

We study the right-hand side of (4.81). We have that

$$\lambda_n \int_\Omega u_n^p = O\left(\mu_n^{\frac{N-p}{p-1}} \right),$$

in view of Lemma 4.2.2. As for the left-hand side of (4.81), we rewrite it as

$$\int_{\partial\Omega} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle = \int_{\partial\Omega \setminus B_\varrho(x)} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle + \int_{\partial\Omega \cap B_\varrho(x)} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle.$$

Being $|\nabla u_n| \sim \frac{\mu_n^{\frac{N-p}{p(p-1)}}}{|y-x_n|^{\frac{N-1}{p-1}}}$ and using the definition of $\hat{x}$, then

$$\int_{\partial\Omega \setminus B_\varrho(x)} |\partial_\nu u_n|^p \langle y - \hat{x}, \nu \rangle \sim \mu_n^{\frac{N-p}{p-1}} \int_{\partial\Omega \setminus B_\varrho(x)} \frac{1}{|y - x_n|^{\frac{p(N-1)}{p-1}}} \sim \mu_n^{\frac{N-p}{p-1}}.$$

As a consequence, (4.81) gives

$$\int_{\partial\Omega \cap B_\varrho(x)} |\partial_\nu u_n|^p = O\left(\mu_n^{\frac{N-p}{p-1}} \right). \quad (4.82)$$

At this point we set $\tilde{\Gamma}_n(z) = d_n^{\frac{N-p}{p}} u_n(d_n z + x_n)$ for $z \in \Omega_n = \{\frac{y-x_n}{d_n} : z \in \Omega\}$. By Lemma 4.2.2, it turns out that

$$\tilde{\Gamma}_n \leq \frac{d_n^{\frac{N-p}{p}} \mu_n^{\frac{N-p}{p(p-1)}}}{d_n^{\frac{N-p}{p-1}} |z|^{\frac{N-p}{p-1}}} = \frac{\tilde{\mu}_n^{\frac{N-p}{p(p-1)}}}{|z|^{\frac{N-p}{p-1}}}, \tag{4.83}$$

where $\tilde{\mu}_n = \frac{\mu_n}{d_n}$. In particular $\tilde{\mu}_n \to 0$ as $n \to +\infty$ in view of (4.38). Letting $\Gamma_n = \frac{\tilde{\Gamma}_n}{\tilde{\mu}_n^{\frac{N-p}{p(p-1)}}}$, we rewrite (4.83) as

$$\Gamma_n \leq \frac{1}{|z|^{\frac{N-p}{p-1}}} \qquad \text{in } \Omega_n.$$

Moreover, Γ_n solves the following problem

$$\begin{cases} -\Delta_p \Gamma_n - \lambda_n d_n^p \Gamma_n^{p-1} = \tilde{\mu}_n^{\frac{p}{p-1}} \Gamma_n^{p^*-1} & \text{in } \Omega_n \\ \Gamma_n \geq 0 & \text{in } \Omega_n \\ \Gamma_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{4.84}$$

Since Ω is smooth and by (4.78), we have that, up to a rotation,

$$\lim_{n \to +\infty} \Omega_n = \Omega_\infty = \mathbb{R}^{N-1} \times (-\infty, 1).$$

Letting $n \to +\infty$ in (4.84), we get that Γ_n converges to the solution Γ of the limiting problem

$$\begin{cases} -\Delta_p \Gamma = \delta_0 & \text{in } \Omega_\infty \\ \Gamma \geq 0 & \text{in } \Omega_\infty \\ \Gamma = 0 & \text{on } \partial\Omega_\infty. \end{cases} \tag{4.85}$$

Indeed it is clear that $-\Delta_p \Gamma = 0$ in $\Omega_\infty \setminus \{0\}$. Moreover

$$\tilde{\mu}_n^{\frac{p}{p-1}} \int_{B_\delta} \Gamma_n^{p^*-1} = \mu_n^{-\frac{N-p}{p}} \int_{B_{\delta d_n}(x_n)} u_n^{p^*-1} = \int_{B_{\frac{\delta d_n}{\mu_n}}} U_n^{p^*-1},$$

in view of the definition of Γ_n, $\tilde{\mu}_n$ and U_n. Then

$$\int_{\partial B_\delta} \partial_\nu \Gamma = \lim_{n \to +\infty} \tilde{\mu}_n^{\frac{p}{p-1}} \int_{B_\delta} \Gamma_n^{p^*-1} = \int_{\mathbb{R}^N} U_\infty^{p^*-1}.$$

We observe that

$$\partial_\nu \Gamma_n(y) = \frac{1}{\tilde{\mu}_n^{\frac{N-p}{p(p-1)}}} \partial_\nu \tilde{\Gamma}_n(y) = \frac{d_n^{\frac{N-p}{p}+1}}{\tilde{\mu}_n^{\frac{N-p}{p(p-1)}}} \partial_\nu u_n(d_n y + x_n) = \frac{d_n^{\frac{N}{p}}}{\mu_n^{\frac{N-p}{p(p-1)}}} \partial_\nu u_n(d_n y + x_n).$$

With the aid of (4.82) we obtain

$$\int_{\partial\Omega_n \cap B_{\frac{\varrho}{d_n}}} |\partial_\nu \Gamma_n|^p \le C \frac{d_n^N}{\mu_n^{\frac{N-p}{p-1}}} \int_{\partial\Omega_n \cap B_{\frac{\varrho}{d_n}}} |\partial_\nu u_n|^p \le C d_n^N.$$

As a consequence, using Fatou's lemma and (4.78), we have that

$$\int_{\Omega_\infty} |\partial_\nu \Gamma|^p \le 0$$

and then $\partial_\nu \Gamma = 0$ a.e. on $\partial\Omega_\infty$. Application of the Hopf lemma yields $\Gamma \equiv 0$, which is not possible in view of (4.85). Thus $x \notin \partial\Omega$.

The final step is to prove that (4.77) holds. For ease of notation, we assume $x = 0$. Then we are going to show that $H_{\lambda_*}(0) = 0$. Application of the Pohozaev identity to u_n on B_δ leads to

$$\begin{aligned}
\int_{\partial B_\delta} \left(-|\nabla u_n|^{p-1} \langle y, \nabla u_n \rangle \partial_\nu u_n + \frac{|\nabla u_n|^p}{p} \langle y, \nu \rangle - \frac{\lambda_n}{p} u_n^p \langle y, \nu \rangle \right) \\
+ \int_{B_\delta} \left(-\frac{N-p}{p} |\nabla u_n|^p + \frac{\lambda_n N}{p} u_n^p \right) \\
= \int_{\partial B_\delta} \frac{S_{\lambda_n}}{p^*} u^{p^*} \langle y, \nu \rangle - \int_{B_\delta} \frac{S_{\lambda_n}(N-p)}{p} u^{p^*}.
\end{aligned}$$

On the other hand, using u_n as test function in the weak formulation of (4.32), we get

$$\int_{B_\delta} \left(|\nabla u_n|^p - S_{\lambda_n} u_n^{p^*} \right) = \lambda_n \int_{B_\delta} u_n^p + \int_{\partial B_\delta} u_n |\nabla u_n|^{p-2} \partial_\nu u_n.$$

Thus we obtain

$$\begin{aligned}
\frac{\lambda_n(N-p)}{p} \int_{B_\delta} u_n^p &= \frac{N-p}{p} \left[\int_{B_\delta} \left(|\nabla u_n^p - S_{\lambda_n} u^{p^*} \right) - \int_{\partial B_\delta} u_n |\nabla u_n|^{p-2} \partial_\nu u_n \right] \\
&= \int_{\partial B_\delta} \left(-|\nabla u_n|^{p-2} \langle y, \nabla u_n \rangle \partial_\nu u_n + \frac{|\nabla u_n|^p}{p} \langle y, \nu \rangle \right) \\
&\quad - \frac{\lambda_n}{p} \int_{\partial B_\delta} u_n^p \langle y, \nu \rangle + \frac{\lambda_n N}{p} \int_{B_\delta} u_n^p - \int_{\partial B_\delta} \frac{S_{\lambda_n}}{p^*} u^{p^*} \langle y, \nu \rangle \\
&\quad - \frac{N-p}{p} \int_{\partial B_\delta} u_n |\nabla u_n|^{p-2} \partial_\nu u_n,
\end{aligned}$$

that is

$$\lambda_n \int_{B_\delta} u_n^p = \int_{\partial B_\delta} \left(|\nabla u_n|^{p-2} \langle y, \nabla u_n \rangle \partial_\nu u_n - \frac{|\nabla u_n|^p}{p} \langle y, \nu \rangle + \frac{\lambda_n}{p} u_n^p \langle y, \nu \rangle \right)$$
$$+ \int_{\partial B_\delta} \left(\frac{S_{\lambda_n}}{p^*} u^{p^*} \langle y, \nu \rangle + \frac{N-p}{p} u_n |\nabla u_n|^{p-2} \partial_\nu u_n \right). \quad (4.86)$$

Observing that $\frac{u_n}{\mu_n^{\frac{N-p}{p(p-1)}}}$ converges to the solution G_{λ_*} of (1.5) as $n \to +\infty$, passing to the limit in (4.86) we get

$$\lambda_* \int_{B_\delta} G_{\lambda_*}^p = \delta \int_{\partial B_\delta} \left(|\nabla G_{\lambda_*}|^{p-2} (\partial_\nu G_{\lambda_*})^2 - \frac{1}{p} |\nabla G_{\lambda_*}|^p + \frac{\lambda_*}{p} G_{\lambda_*}^p \right)$$
$$+ \frac{N-p}{p} \int_{\partial B_\delta} G_{\lambda_*} |\nabla G_{\lambda_*}|^{p-2} \partial_\nu G_{\lambda_*}. \quad (4.87)$$

At this point we consider $G_{\varepsilon,\lambda_*}$ as in (4.5) with $\lambda = \lambda_*$ and we apply the Pohozaev identity to $G_{\varepsilon,\lambda_*}$ on B_δ. This leads to

$$\int_{B_\delta} \left(-\Delta_p G_{\varepsilon,\lambda_*} - \lambda_* G_{\varepsilon,\lambda_*}^{p-1} \right) \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle = \int_{B_\delta} \left(\frac{C_0}{C_1} \varepsilon^{-\frac{N-p}{p}} \right)^{p-1} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle. \quad (4.88)$$

We study the left-hand side of (4.88). We have that

$$\int_{B_\delta} \left(-\Delta_p G_{\varepsilon,\lambda_*} - \lambda_* G_{\varepsilon,\lambda_*}^{p-1} \right) \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle = \int_{B_\delta} \left(-\frac{N-p}{p} |\nabla G_{\varepsilon,\lambda_*}|^p + \frac{\lambda_* N}{p} G_{\varepsilon,\lambda_*}^p \right)$$
$$+ \int_{\partial B_\delta} \left(-|\nabla G_{\varepsilon,\lambda_*}|^{p-2} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle \partial_\nu G_{\varepsilon,\lambda_*} + \frac{|\nabla G_{\varepsilon,\lambda_*}|^p}{p} \langle y, \nu \rangle - \frac{\lambda_*}{p} G_{\varepsilon,\lambda_*}^p \langle y, \nu \rangle \right).$$

As for the right-hand side, since $G_{\varepsilon,\lambda_*} = \Gamma_\varepsilon + H_{\varepsilon,\lambda_*}$ and $\Gamma_\varepsilon \nabla U_\varepsilon = U_\varepsilon \nabla \Gamma_\varepsilon$,

$$
\begin{aligned}
\int_{B_\delta} G_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle &= \int_{B_\delta} \Gamma_\varepsilon U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle + \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \\
&= \int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla \Gamma_\varepsilon \rangle + \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \\
&= \int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle - \int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla H_{\varepsilon,\lambda_*} \rangle + \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \\
&= \int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle + \int_{B_\delta} \left[N U_\varepsilon^{p^*-1} + (p^*-1) U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \right] H_{\varepsilon,\lambda_*} \\
&\qquad - \delta \int_{\partial B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1} + \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \\
&= \int_{B_\delta} \left(U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle + N U_\varepsilon^{p^*-1} H_{\varepsilon,\lambda_*} + p^* H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \right) \\
&\qquad - \delta \int_{\partial B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle \\
&= \delta \int_{\partial B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} - (p^*-1) \int_{B_\delta} G_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle - N \int_{B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} \\
&= \delta \int_{\partial B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} - (p^*-1) \int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle - N \int_{B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} \\
&\qquad - N(p^*-1) \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1} - p^*(p^*-1) \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle \\
&\qquad + (p^*-1)\delta \int_{\partial B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1},
\end{aligned}
$$

which rewrites as

$$
\begin{aligned}
&\int_{B_\delta} U_\varepsilon^{p^*-1} \langle y, \nabla G_{\varepsilon,\lambda_*} \rangle \\
&= \frac{\delta}{p^*} \int_{\partial B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} - \frac{N}{p^*} \int_{B_\delta} U_\varepsilon^{p^*-1} G_{\varepsilon,\lambda_*} - \frac{N(p^*-1)}{p^*} \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1} \\
&\qquad - (p^*-1) \int_{B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-2} \langle y, \nabla U_\varepsilon \rangle + \frac{\delta(p^*-1)}{p^*} \int_{\partial B_\delta} H_{\varepsilon,\lambda_*} U_\varepsilon^{p^*-1}.
\end{aligned}
$$

Thus (4.88) rewrites as follows:

$$\delta\int_{\partial B_\delta}\left(-|\nabla G_{\varepsilon,\lambda_*}|^{p-2}(\partial_\nu G_{\varepsilon,\lambda_*})^2+\frac{1}{p}|\nabla G_{\varepsilon,\lambda_*}|^p-\frac{\lambda_*}{p}G^p_{\varepsilon,\lambda_*}\right)$$
$$=\left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1}\left[\frac{\delta}{p^*}\int_{\partial B_\delta}U_\varepsilon^{p^*-1}G_{\varepsilon,\lambda_*}-\frac{N}{p^*}\int_{B_\delta}U_\varepsilon^{p^*-1}G_{\varepsilon,\lambda_*}\right.$$
$$-\frac{N(p^*-1)}{p^*}\int_{B_\delta}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-1}-(p^*-1)\int_{B_\delta}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-2}\langle y,\nabla U_\varepsilon\rangle$$
$$\left.+\frac{\delta(p^*-1)}{p^*}\int_{\partial B_\delta}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-1}\right]-\frac{\lambda_*N}{p}\int_{B_\delta}G^p_{\varepsilon,\lambda_*}+\frac{N-p}{p}\int_{B_\delta}|\nabla G_{\varepsilon,\lambda_*}|^p. \tag{4.89}$$

Using $G_{\varepsilon,\lambda_*}$ as test function in the weak formulation of (4.6), we obtain

$$\int_{B_\delta}\left[|\nabla G_{\varepsilon,\lambda_*}|^p-\left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1}U_\varepsilon^{p^*-1}G_{\varepsilon,\lambda_*}\right]$$
$$=\lambda_*\int_{B_\delta}G^p_{\varepsilon,\lambda_*}+\int_{\partial B_\delta}G_{\varepsilon,\lambda_*}|\nabla G_{\varepsilon,\lambda_*}|^{p-2}\partial_\nu G_{\varepsilon,\lambda_*}. \tag{4.90}$$

Using (4.90) in (4.89), we get

$$\delta\int_{\partial B_\delta}\left(-|\nabla G_{\varepsilon,\lambda_*}|^{p-2}(\partial_\nu G_{\varepsilon,\lambda_*})^2+\frac{1}{p}|\nabla G_{\varepsilon,\lambda_*}|^p-\frac{\lambda_*}{p}G^p_{\varepsilon,\lambda_*}\right)$$
$$=-\lambda_*\int_{B_\delta}G^p_{\varepsilon,\lambda_*}+\frac{N-p}{p}\int_{\partial B_\delta}(G_{\varepsilon,\lambda_*}|\nabla G_{\varepsilon,\lambda_*}|^{p-2}\partial_\nu G_{\varepsilon,\lambda_*}+\delta G_{\varepsilon,\lambda_*}f_\varepsilon)$$
$$-(p^*-1)\left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1}\left[\int_{B_\delta}\left(\frac{N}{p^*}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-1}+H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-2}\langle y,\nabla U_\varepsilon\rangle\right)\right]. \tag{4.91}$$

Recalling the definition of U_ε, we have that

$$\left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1}\left[\int_{B_\delta}\left(\frac{N}{p^*}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-1}+H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-2}\langle y,\nabla U_\varepsilon\rangle\right)\right]$$
$$=\left(\frac{C_0}{C_1}\varepsilon^{-\frac{N-p}{p}}\right)^{p-1}\int_{B_\delta}H_{\varepsilon,\lambda_*}U_\varepsilon^{p^*-1}\left(\frac{N-p}{p}+\frac{\langle y,\nabla U_\varepsilon\rangle}{U_\varepsilon}\right)$$
$$=C_0^{p-1}C_1^{p^*-p}\int_{B_\delta}H_{\varepsilon,\lambda_*}\frac{\varepsilon^{\frac{(N-p)(p^*-p)}{p}}}{(\varepsilon^p+|y|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}}}\frac{N-p}{p(p-1)}\frac{(p-1)\varepsilon^p-|y|^{\frac{p}{p-1}}}{\varepsilon^p+|y|^{\frac{p}{p-1}}}$$
$$=C_0^{p-1}C_1^{p^*-p}\int_{B_{\frac{\delta}{\varepsilon^{p-1}}}}H_{\varepsilon,\lambda_*}(\varepsilon^{p-1}z)\frac{(N-p)[(p-1)-|z|^{\frac{p}{p-1}}]}{p(p-1)(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}}.$$

As a consequence, passing to the limit as $\varepsilon \to 0$ in (4.91), it turns out that

$$
\begin{aligned}
\lambda_* \int_{B_\delta} G_{\lambda_*}^p + (p^*-1) C_0^{p-1} C_1^{p^*-p} H_{\lambda_*}(0) \int_{\mathbb{R}^N} \frac{(N-p)[(p-1)-|z|^{\frac{p}{p-1}}]}{p(p-1)(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} \\
= \delta \int_{\partial B_\delta} \left(|\nabla G_{\lambda_*}|^{p-2} (\partial_\nu G_{\lambda_*})^2 - \frac{1}{p} |\nabla G_{\lambda_*}|^p \right) \\
+ \int_{\partial B_\delta} \left(\frac{\lambda_* \delta}{p} G_{\lambda_*}^p + \frac{N-p}{p} G_{\lambda_*} |\nabla G_{\lambda_*}|^{p-2} \partial_\nu G_{\lambda_*} \right) \quad (4.92)
\end{aligned}
$$

in view of

$$
f_\varepsilon = \left(\frac{C_0}{C_1} \varepsilon^{-\frac{N-p}{p}} \right)^{p-1} U_\varepsilon^{p^*-1} = \frac{C_0^{p-1} C_1^{\frac{p^2}{N-p}} \varepsilon^p}{(\varepsilon^p + |y|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}}} \to 0
$$

on ∂B_δ as $\varepsilon \to 0$. At this point we observe that

$$
\begin{aligned}
\int_{\mathbb{R}^N} \frac{|z|^{\frac{p}{p-1}}}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} &= -\frac{p-1}{p(N-\frac{N-p}{p})} \int_{\mathbb{R}^N} \langle z, \nabla \left(\frac{1}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}}} \right) \rangle \\
&= \frac{N(p-1)}{Np-N+p} \int_{\mathbb{R}^N} \frac{1+|z|^{\frac{p}{p-1}}}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left(1 - \frac{N(p-1)}{Np-N+p}\right) \int_{\mathbb{R}^N} \frac{|z|^{\frac{p}{p-1}}}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} \\
= \frac{N(p-1)}{Np-N+p} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}},
\end{aligned}
$$

which implies

$$
\int_{\mathbb{R}^N} \frac{|z|^{\frac{p}{p-1}}}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} = \frac{N(p-1)}{p} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}}.
$$

It follows that

$$
\int_{\mathbb{R}^N} \frac{(p-1)-|z|^{\frac{p}{p-1}}}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} = \left(p-1-\frac{N(p-1)}{p} \right) \int_{\mathbb{R}^N} \frac{1}{(1+|z|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}+1}} < 0, \quad (4.93)
$$

being $p-1-\frac{N(p-1)}{p} = -\frac{(N-p)(p-1)}{p} < 0$. Comparing (4.87) with (4.92) and using (4.93), we deduce that $H_{\lambda_*}(0) = 0$. □

Appendix A

Technical and generalized results

A.1 Some upper and lower bounds

We present details of some technical estimates which will be applied to the operator of problem (1.6).

Lemma A.1.1. *Let p be a real number such that $p > 1$. Then the following estimates hold for any $x, y \in \mathbb{R}^N$.*

- Case $p \leq 2$:

$$\langle |x+y|^{p-2}(x+y) - |x|^{p-2}x, y\rangle \geq (p-1)(|x|+|y|)^{p-2}|y|^2, \tag{A.1}$$

$$\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| \leq 6(|x|+|y|)^{p-2}|y|. \tag{A.2}$$

- Case $p > 2$:

$$\langle |x+y|^{p-2}(x+y) - |x|^{p-2}x, y\rangle \geq (|x|+|y|)^{p-2}|y|^2, \tag{A.3}$$

$$\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| \leq 3^p(|x|^{p-2} + |y|^{p-2})|y| \tag{A.4}$$

Proof. Assume $p \leq 2$. To prove inequality (A.1), we observe that

$$|x+y|^{p-2}(x+y) - |x|^{p-2}x = \int_0^1 \frac{d}{dt}\big(|x+ty|^{p-2}(x+ty)\big)dt$$
$$= y\int_0^1 |x+ty|^{p-2}dt + (p-2)\int_0^1 |x+ty|^{p-4}\langle x+ty, y\rangle(x+ty)dt. \tag{A.5}$$

Thus, using that $1 < p \leq 2$ together with Cauchy-Schwarz inequality and triangle inequality, we obtain the statement:

$$\begin{aligned}
\langle |x+y|^{p-2}(x+y) - |x|^{p-2}x, y\rangle &= |y|^2 \int_0^1 |x+ty|^{p-2}dt \\
&\quad + (p-2)\int_0^1 |x+ty|^{p-4}\langle x+ty, y\rangle^2 dt \\
&\geq |y|^2 \int_0^1 |x+ty|^{p-2}dt + (p-2)|y|^2 \int_0^1 |x+ty|^{p-2}dt \\
&= (p-1)|y|^2 \int_0^1 |x+ty|^{p-2}dt \\
&\geq (p-1)(|x|+|y|)^{p-2}|y|^2.
\end{aligned}$$

To prove inequality (A.2), we notice that if $|x| > 2|y|$ then

$$\begin{aligned}
\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| &= \left|\int_0^1 \frac{d}{dt}\big(|x+ty|^{p-2}(x+ty)\big)dt\right| \\
&\leq 2^{2-p}|x|^{p-2}\left|\int_0^1 \frac{d}{dt}(x+ty)dt\right| = 2^{2-p}|x|^{p-2}|y| \\
&\leq 4^{2-p}(|x|+|y|)^{p-2}|y|.
\end{aligned}$$

On the other hand, when $|x| \leq 2|y|$, we have that

$$\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| \leq 2(|x|+|y|)^{p-1} \leq 6(|x|+|y|)^{p-2}|y|.$$

These last two inequalities yield estimate (A.2).

Assume $p > 2$. Then (A.5) gives

$$\langle |x+y|^{p-2}(x+y) - |x|^{p-2}x, y\rangle \geq |y|^2 \int_0^1 |x+ty|^{p-2}dt \geq (|x|+|y|)^{p-2}|y|^2,$$

that is (A.3). To prove inequality (A.4) we observe that, similarly to the case $p \leq 2$, if $|x| > 2|y|$ then

$$\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| \leq \frac{1}{2^{p-2}}|x|^{p-2}|y| \leq \frac{1}{2^{p-2}}(|x|^{p-2} + |y|^{p-2})|y|.$$

Otherwise we have that

$$\big||x+y|^{p-2}(x+y) - |x|^{p-2}x\big| \leq 2(|x|+|y|)^{p-1} \leq 2\cdot 3^{p-1}(|x|^{p-2} + |y|^{p-2})|y|.$$

Therefore, we arrive at estimate (A.4). □

A.2 Generalization of some results in the thesis

Here we prove that Theorem 1 in [4], which is stated for homogeneous boundary value, can be easily extended to non-homogeneous ones as done in [1] when p=N. Then we show a result which generalizes Lemma (2.1.1), crucial to show the existence of a SOLA in Section 2.1.

Given $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, we set $W^{1,q}_g(\Omega) = g + W^{1,q}_0(\Omega)$ for all $q \geq 1$. When necessary, we can asssume g to minimize $\int_\Omega |\nabla u|^p$ in $W^{1,p}_g(\Omega)$ and then g to be $p-$harmonic in Ω.

Consider the following problem:

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{A.6}$$

where g is nonnegative and p-harmonic in Ω and $f \in L^1(\Omega)$.

Proposition A.2.1. *Let $u \in W^{1,p}_g(\Omega)$ be a solution of* (A.6)*, with g and f as previously described. There exists a constant C, depending on $\|f\|_1$, such that for any k in $\mathbb{N}$ the following estimate holds:*

$$\int_{\{k<|u-g|<k+1\}} |\nabla(u-g)|^p \leq C. \tag{A.7}$$

Proof. Since g is p-harmonic in Ω, we have that

$$\begin{cases} -\Delta_p u + \Delta_p g = f & \text{in } \Omega \\ u - g = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.8}$$

We set $B_k = \{y \in \Omega \colon k < |u(y) - g(y)| < k+1\}$. Using[1] $T_{k,k+1}(u-g)$ as test function in the weak formulation of problem (A.8), we obtain

$$\int_{B_k} |\nabla(u-g)|^2 (|\nabla u| + |\nabla g|)^{p-2} \leq C \tag{A.9}$$

in view of $\nabla[T_{k,k+1}(u-g)] = 0$ outside B_k, for some constant C depending on $\|f\|_1$. If $p \geq 2$ then (A.9) easily leads to (A.7). If $p < 2$, using the Hölder

[1] The function $T_{k,k+1}(\cdot)$ is defined as in (2.39).

inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we get

$$\int_{B_k} |\nabla(u-g)|^p = \int_{B_k} |\nabla(u-g)|^p (|\nabla u| + |\nabla g|)^{\frac{p(p-2)}{2}} (|\nabla u| + |\nabla g|)^{\frac{p(2-p)}{2}}$$
$$\leq \left(\int_{B_k} |\nabla(u-g)|^2 (|\nabla u| + |\nabla g|)^{p-2} \right)^{\frac{2}{p}} \left(\int_{B_k} (|\nabla u| + |\nabla g|)^p \right)^{\frac{2-p}{p}}.$$

As a consequence, since $\frac{2-p}{p} < 1$, we finally have

$$\left(\int_{B_k} |\nabla(u-g)|^p \right)^{p-1} \leq \int_{B_k} |\nabla(u-g)|^2 (|\nabla u| + |\nabla g|)^{p-2} \leq C,$$

in view of (A.9). □

Proceeding as in the proof of Theorem 1 in [4] and using Proposition A.2.1, we arrive at the desired conclusion, that is the following result.

Theorem A.2.2. *Under the hypothesis of Proposition A.2.1, $u \in W^{1,q}(\Omega)$ for all $q < \bar{q} = \frac{N(p-1)}{N-1}$.*

Lemma A.2.3. *Let $p > \max\{2 - \frac{1}{N}, \sqrt{N}\}$. Let $u_n \in W^{1,p}_{g_n}(\Omega)$ be a sequence of solutions to*

$$-\Delta_p u_n - a_n |u_n|^{p-2} u_n = f_n \quad in\ \Omega. \tag{A.10}$$

Assume that

$$\lim_{n\to+\infty} a_n = a \ in\ L^\infty(\Omega), \quad \sup_{n\in\mathbb{N}} \left[\|f_n\|_{L^1(\Omega)} + \|g_n\|_{L^\infty(\Omega)\cap W^{1,p}(\Omega)} \right] < +\infty \tag{A.11}$$

with $\sup_\Omega a < \lambda_1$. Then $\sup_{n\in\mathbb{N}} \|u_n\|_{p-1} < +\infty$.

Proof. Assume by contradiction that

$$\|u_n\|_{p-1} \to +\infty \qquad \text{as } n \to +\infty. \tag{A.12}$$

Setting $\hat{u}_n = \frac{u_n}{\|u_n\|_{p-1}}$, $\hat{f}_n = \frac{f_n}{\|u_n\|_{p-1}^{p-1}}$ and $\hat{g}_n = \frac{g_n}{\|u_n\|_{p-1}}$, we have that $\hat{u}_n$ solves

$$\begin{cases} -\Delta_p \hat{u}_n - a_n |\hat{u}_n|^{p-2} \hat{u}_n = \hat{f}_n & \text{in } \Omega \\ \hat{u}_n = \hat{g}_n & \text{on } \partial\Omega \end{cases} \tag{A.13}$$

with

$$\|\hat{u}_n\|_{p-1} = 1,\ \|\hat{f}_n\|_{L^1(\Omega)} + \|\hat{g}_n\|_{L^\infty(\Omega)\cap W^{1,p}(\Omega)} \to 0 \text{ as } n \to +\infty \tag{A.14}$$

in view of (A.11)-(A.12). Since $h_n = \hat{f}_n + a_n|\hat{u}_n|^{p-2}\hat{u}_n$ is uniformly bounded in $L^1(\Omega)$ and $\hat{g}_n$ is $p-$harmonic in Ω, we can use the results in [4] extended to non-homogeneous boundary values to show that, up to a subsequence, $\hat{u}_n - \hat{g}_n \to \hat{u}$ in $W_0^{1,q}(\Omega)$ for all $1 \le q < \bar{q}$ as $n \to +\infty$, where $\bar{q} > 1$ in view of $p > 2 - \frac{1}{N}$. Moreover $\hat{u}$ solves

$$\begin{cases} -\Delta_p \hat{u} - a|\hat{u}|^{p-2}\hat{u} = 0 & \text{in } \Omega \\ \hat{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.15}$$

In particular, by Sobolev embedding theorem we have that

$$\hat{u}_n \to \hat{u} \qquad \text{in } L^q(\Omega) \tag{A.16}$$

as $n \to +\infty$ for all $1 \le q < \bar{q}^* = \frac{N(p-1)}{N-p}$ as $n \to +\infty$. Using $T_l(\hat{u}_n - \hat{g}_n) \in W_0^{1,p}(\Omega)$ as a test function in (A.13), we get

$$\int_{\{|\hat{u}_n - \hat{g}_n| \le l\}} \langle |\nabla \hat{u}_n|^{p-2}\nabla \hat{u}_n - |\nabla \hat{g}_n|^{p-2}\nabla \hat{g}_n, \nabla(\hat{u}_n - \hat{g}_n)\rangle$$
$$\le \|a_n\|_\infty \int_\Omega |\hat{u}_n|^{p-1}|\hat{u}_n - \hat{g}_n| + l\|\hat{f}_n\|_1$$

in view of the p-harmonicity of $\hat{g}_n$. Since

$$\|a_n\|_\infty \int_\Omega |\hat{u}_n|^{p-1}|\hat{u}_n - \hat{g}_n| + l\|\hat{f}_n\|_1 \to \|a\|_\infty \int_\Omega |\hat{u}|^p$$

as $n \to +\infty$ in view of (A.11), (A.16) and $\bar{q}^* > p$ thanks to $p > \sqrt{N}$, by (A.1) and the Fatou's convergence Theorem we deduce that

$$\delta \int_{\{|\hat{u}| \le l\}} |\nabla \hat{u}|^p \le \|a\|_\infty \int_\Omega |\hat{u}|^p$$

for some $\delta > 0$. Letting $l \to +\infty$ we get that $\hat{u} \in W_0^{1,p}(\Omega)$ in view of $\hat{u} \in L^p(\Omega)$ and then $\hat{u}$ is an admissible test function in (A.15) leading to

$$\int_\Omega |\nabla \hat{u}|^p - \int_\Omega a|\hat{u}|^p = 0.$$

Since $\sup\limits_\Omega a < \lambda_1$ one finally deduce that $\hat{u} = 0$ and then $\hat{u}_n \to 0$ in $L^{p-1}(\Omega)$ thanks to (A.16), in contadiction with $\|\hat{u}_n\|_{p-1} = 1$. $\square$

A.3 A weak comparison principle

Here we discuss the validity of a weak comparison principle, obtained as application of Lemma 2.3.1, which guarantees that $H_\lambda(y, y)$ is strictly increasing in λ for all $y \in \Omega$.

Proposition A.3.1. *Let $p \geq 2$, $a, f_1, f_2 \in L^\infty(\Omega)$ and $g_1, g_2 \in C^1(\bar{\Omega})$. Let $u_i \in C^1(\bar{\Omega})$, $i = 1, 2$, be solutions to*

$$- \Delta_p u_i - a u_i^{p-1} = f_i \quad \text{in } \Omega, \quad u_i = g_i \text{ on } \partial\Omega, \tag{A.17}$$

so that

$$u_i > 0 \text{ in } \Omega, \quad \partial_n u_i < 0 \text{ on } \partial\Omega \cap \{u_i = 0\}. \tag{A.18}$$

If $f_1 \leq f_2$, $g_1 \leq g_2$ and $f_2 \geq 0$ in Ω, then $u_1 \leq u_2$ in Ω.

Proof. Setting $w_1 = u_1^p$, $w_2 = u_2^p$ and $\varphi = (w_1 - w_2)^+$, consider $w_s = s w_1 + (1-s) w_2$ for $s \in [0,1]$. By the properties of u_1 and u_2, it is easily seen that for each $s \in [0,1]$ there hold $w_s + t\varphi \geq 0$ in Ω and $\nabla(w_s + t\varphi)^{\frac{1}{p}} \in L^p(\Omega)$ for t small. Then we can apply (2.74) at $s = 0, 1$ to get

$$\begin{aligned} I'(w_1)[\varphi] - I'(w_2)[\varphi] &= \int_\Omega |\nabla w_1^{\frac{1}{p}}|^{p-2} \langle \nabla w_1^{\frac{1}{p}}, \nabla (w_1^{\frac{1-p}{p}} \varphi) \rangle - \int_\Omega |\nabla w_2^{\frac{1}{p}}|^{p-2} \langle \nabla w_2^{\frac{1}{p}}, \nabla (w_2^{\frac{1-p}{p}} \varphi) \rangle \\ &= \int_\Omega |\nabla u_1|^{p-2} \langle \nabla u_1, \nabla \frac{\varphi}{u_1^{p-1}} \rangle - \int_\Omega |\nabla u_2|^{p-2} \langle \nabla u_2, \nabla \frac{\varphi}{u_2^{p-1}} \rangle. \end{aligned}$$

Since $\varphi \in W_0^{1,p}(\Omega)$ we deduce that

$$I'(w_1)[\varphi] - I'(w_2)[\varphi] = \int_\Omega \left(\frac{f_1}{u_1^{p-1}} - \frac{f_2}{u_2^{p-1}} \right) (u_1^p - u_2^p)^+ \leq 0$$

in view of (A.17) and $f_1 \leq f_2$ with $f_2 \geq 0$. Since

$$I'(w_1)[\varphi] - I'(w_2)[\varphi] = \int_0^1 I''(w_s)[w_1 - w_2, \varphi] ds = \int_0^1 I''(w_s)[\varphi, \varphi] ds$$

in view of $I''(w_s)[w_1 - w_2, \varphi] = I''(w_s)[\varphi, \varphi]$, by Lemma 2.3.1 $I''(w_s)[\varphi, \varphi] = \int_\Omega \varrho(w_s, \varphi)$ with $\varrho(w_s, \varphi) \geq 0$. Then, we deduce that $\varrho(w_s, \varphi) = 0$ for $s \in [0,1]$. If $p > 2$, then $\nabla\varphi = 0$ in Ω. If $p = 2$, $\langle \nabla w_s, \nabla \varphi \rangle = \varphi \frac{|\nabla w_s|^2}{w_s}$, which implies $\langle \nabla (w_1 - w_2), \nabla \varphi \rangle = s\varphi \frac{|\nabla (w_1 - w_2)|^2}{w_s}$ for all $0 \leq s \leq 1$. In both cases $\nabla\varphi = 0$ in Ω and then $w_1 \leq w_2$ in Ω, or equivalently $u_1 \leq u_2$ in Ω. □

Bibliography

[1] J.A. Aguilar Crespo, I. Peral Alonso, *Blow-up behavior for solutions of* $-\Delta_N u = V(x)e^u$ *in bounded domains in* $\mathbb{R}^N$, Nonlinear Anal. **29** (1997), no. 4, 365–384.

[2] T. Aubin, *Equations différentielles non-linèaires et problème de Yamabé concernant la courbure scalaire*, J. Math. Pures Appl. **5** (1976), 269-293.

[3] L. Boccardo, G. Croce, *Esistenza e regolaritá di soluzioni di alcuni problemi ellittici*, Quaderni dell'Unione Matematica Italiana **51**, Pitagora Ed. (2010).

[4] L. Boccardo, T. Gallouet, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149-169.

[5] L. Boccardo, T. Gallouet, *Nonlinear elliptic equations with right hand side measures*, Commun. Part. Diff. Eq. **17** (1992), 641-655.

[6] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437-477.

[7] L. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights*, Composition Math. **53** (1984), 259-275.

[8] M. Cuesta, P. Takac, *A strong comparison principle for the Dirichlet* $p-$*laplacian*, Reaction diffusion systems, Trieste (1995), 79-87.

[9] L. Damascelli, S. Merchán, L. Montoro, B. Sciunzi, *Radial symmetry and applications for a problem involving the* $-\Delta_p(\cdot)$ *operator and critical nonlinearity in* $\mathbb{R}^N$, Adv. Math. **265** (2014), no. 10, 313-335.

[10] J. I. Dìaz, J. E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, Computes Rendus Acad. Sc. Paris, Série I, **305** (1987), 521-524.

[11] E. Dibenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*. Nonlinear Anal. **7** (1983), no. 8, 827–850.

[12] G. Dinca, F, Isaia, *Generalized Pohozaev identity and a non-existence result for the p-laplacian: weak solutions*, Adv. Diff. Eq. **14** (2009), no. 5-6, 497-540.

[13] O. Druet, *Elliptic equations with critical Sobolev exponents in dimension* 3, Ann. I. H. Poincaré **19, 2** (2002), 125-142.

[14] O. Druet, F. Robert and E. Hebey, *Blow-up theory for elliptic PDEs in Riemannian geometry*. Mathematical Notes **45** (2004). Princeton University Press, Princeton, NJ.

[15] P. Esposito, *On some conjectures proposed by Haïm Brezis*, Nonlinear Anal. **56** (2004), no. 5, 751–759.

[16] J. Fleckinger-Pellè, J. Hernàndez, P. Takac, F. De Thélin, *Uniqueness and positivity for solutions of equations with the p-laplacian*, Reaction diffusion systems, Trieste (1995), 141-155.

[17] J. Fleckinger-Pellè, P. Takac, *Uniqueness of positive solutions for nonlinear cooperative systems with the p-laplacian*, Indiana Univ. Math. J. **43** (1994), 1227-1253.

[18] L. Greco, T. Iwaniec, C. Sbordone, *Inverting the p-harmonic operator*, Manuscripta Math. **92** (1997), 249-258.

[19] M. Guedda, L. Veron, *Local and global properties of solutions of quasilinear elliptic equations*, J. Diff. Eq. **76** (1988), 159-189.

[20] M. Guedda, L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879-902.

[21] T. Iwaniec, *p-Harmonic tensors and quasiregular mappings*, Ann. Math. **136** (1992), 589-624.

[22] T. Iwaniec, C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994), 143-161.

[23] S. Kichenassamy, L. Veron, *Singular solutions of the p-Laplace equation*, Math. Ann. **275** (1986), 599-616.

[24] J. Leray, J. L. Lions, *Quelques rèsultats de Visik sur les problèmes elliptiques non linéaires par les mèthodes de Minty et Browder*, Bull. Soc. Math. France **93** (1965), 97-107.

[25] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (1983), 849-858.

[26] Y.Y. Li, *Prescribing scalar curvature on S^N and related problems, Part I*, J. Diff. Eq. **120** (1995), 319-410.

[27] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[28] J. L. Lions, *Quelques méthodes de résolutions des problèmes aux limites non linéaires*, Paris: Dunod-Gauthier-Villars (1969).

[29] C. Mercuri, M. Willem, *A global compactness result for the p-laplacian involving critical nonlinearities*, Discrete and Continuous Dynamical Systems **28** (2010), no. 2, 469-493.

[30] J. Serrin, *Local behaviour of solutions of quasilinear equations*, Acta Math. **111** (1964), 247-302.

[31] J. Serrin, *Isolated singularities of solutions of quasilinear equations*, Acta Math. **113** (1965), 219-240.

[32] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domain with conical boundary points*, Comm. Partial Differ. Equations **8** (1983), 773-817.

[33] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eq. **51** (1984), 126-150.

[34] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Annali Scu norm. Sup. Pisa **22** (1968), 265-274.

[35] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721-747.

[36] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191-202.

[37] J. Vétois, *A priori estimates and application to the symmetry of solutions for critical p-Laplace equations*, J. Diff. Eq. **260** (2016), no. 1, 149-161.