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by

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Phase transitions and coarse graining for a system of particles in the continuum

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Introduction

A central problem in equilibrium statistical mechanics is the derivation of phase diagrams of fluids in which gas, liquid and solid regions are present and separated by coexistence curves in the pressure-temperature plane (see Fig. 1). The challenge in the study of *systems of interacting particles in the continuum* is therefore to find a good compromise between a mathematical model which we are able to analyze rigorously and a model which has a non trivial structure eventually responsible for the complex thermodynamic behaviour these systems happen to have. Hence one should understand which are the relevant features of the microscopic model that would give rise to the physical phenomena we want to look at, such as phase transitions. This work is devoted to prove rigorously the existence of a liquid-vapor branch in the diagram, when considering a system of particles in $\mathbb{R}^d$ interacting with a reasonable potential with both long and short range contributions.

There are several ways to characterize phase transitions of first order with order parameter the density. One is to say that there is a "forbidden interval" of densities, say (ρ', ρ''), so that if we put a mass $\rho|\Lambda|$ of fluid in the region Λ (where $|\Lambda|$ denotes the volume of Λ) with $\rho \in (\rho', \rho'')$, then the fluid is separated into a part with density ρ' and another one with density ρ''. Another one is to say that the free energy density is linear in (ρ', ρ'') and therefore it is no more strictly convex in the forbidden interval.

Intermolecular forces are often described by Lennard-Jones interactions

$$V(r) = a\, r^{-12} - b\, r^{-6}, \qquad a, b > 0 \tag{0.0.1}$$

where r is the intermolecular distance (the molecules being represented by points of $\mathbb{R}^d$).

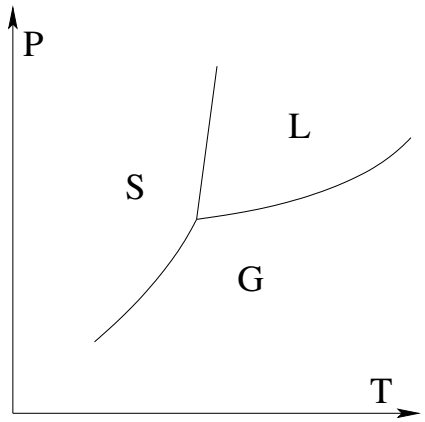

Figure 1: Phase diagram of fluids.

There is general agreement on the validity of the conjecture that simple fluids whose intermolecular forces are described by potentials like (0.0.1) behave according to the phase diagram discussed above. Despite serious efforts no rigorous proof of this conjecture has yet been found. The difficulty of handling Lennard-Jones (or more realistic) interactions has promoted the introduction of several simplified models mostly designed to investigate the vapor-liquid coexistence line, as the model introduced by Lebowitz, Mazel and Presutti, [38] and [39], hereafter shorthanded by LMP. Other cases in which liquid-vapor phase transitions are proved are the Widom-Rowlinson model of two component fluids, [62], studied by Ruelle [54] and the one dimensional continuum systems with long range interaction studied by Johansson [33].

What makes the LMP theory strong is the good compromise between realistic models of fluids and mathematically treatable systems. In fact the LMP hamiltonian, with a two body repulsive interaction and a four body attractive part, is supposed to mimic the interaction (0.0.1). The model is defined through Kac potentials, i.e. functions which scale in the following way: $J_\gamma(r,r') = \gamma^d J(\gamma r, \gamma r')$, where $J(r,r')$ is a symmetric, translation invariant (i.e. $J(r,r') = J(0,r'-r)$) smooth probability kernel supposed for simplicity to vanish for $|r-r'| \geq 1$. Thus the range of the interaction has order γ^{-1} (for both repulsive and attractive potentials) and the "Kac scaling parameter" γ is assumed to be small. This choice of the potentials makes the LMP model a perturbation of the mean field, in the sense that when taking the thermodynamic limit followed by the limit $\gamma \to 0$ the free energy is equivalent to the free energy in the van der Waals description.

Because of the long range interaction of the Kac potential, the LMP hamiltonian a priori excludes the appearance of spatial patterns and the model only describes the vapor-liquid transition, the solid phase being absent from the picture. Motivated by this consideration we studied a variant of the LMP model obtained by adding to the hamiltonian a hard core potential of radius R independent of γ. This kind of interaction acts on a scale much smaller than γ^{-1}. In fact, physically one expects that for densities much larger than the typical gas phase density particles feel strongly the repulsive part of the interaction. Hence, the reasons which lead us to implement the LMP result for this new model are of different nature. One is due to the fact that we want to deal with a more realistic interaction and one is related to the hope that one can get closer to answer to a fundamental open problem of statistical mechanics: how to get to the solid phase. However, as we said at the beginning of the introduction, this work is mainly devoted to prove liquid-vapor phase transition, the solid phase being beyond our reach and at present still considered very hard. To be more specific we prove perturbativly that by adding a hard core interaction to the LMP model, with the hard core radius R sufficiently small, the LMP phase transition is essentially unaffected.

Note that we would have several problems in dealing with a stable hamiltonian where the hard core interaction is the only repulsive potential, i.e. without the four body Kac repulsive interaction. Indeed there is a poor control of the hard spheres gas, an obstacle at present unsurmountable. It may very well be that it exhibits a order-disorder phase transition by itself, as suggested by numerical computations in $d = 3$ dimensions. While in the regime of small densities it is known, due to cluster expansion results, that the entropy of the hard spheres is to leading order the same as in the ideal gas and their difference can be expressed as a convergent power series of the density, nothing is known for large values of the density. Unfortunately the values of the density appearing at the transition are exceeding the radius of convergence of the cluster expansion and thus go beyond the region in which we have a good control of the entropy for hard spheres.

Going back to the models in the continuum, it is indeed surprising that the theory of equilibrium statistical mechanics does not yet include a rigorous derivation of the phase diagram. Historically the first proof of liquid-vapor type phase transitions was given for lattice systems, with the pioneering "argument of Peierls", [52], in which is shown coexistence of

different phases for the Ising model. In general discrete systems are much better understood, also thanks to the development of the Pirogov-Sinai technique, [50], (hereafter shorthanded by P-S) which applies in the low temperature regime. The key is to look at configurations of the system for positive but small values of the temperature as of perturbations of the two ground states, i.e. the configuration with all $+$ and the configuration with all $-$, and therefore study the so-called *abstract contour model* instead of the original one. In this new description the goal is to prove Peierls bounds and hence phase transitions.

The extension of such theory from the lattice to the continuum systems seems very difficult, since the ground states are a lot more complex. Instead of being constant (as for the Ising model) they will have a periodic structure as solids have. Therefore, when we raise the temperature the ground states can evolve in too many ways as they have so many degrees of freedom and it would be difficult to study the excitations of the system. In this scenario, by proving closeness to the ground states one could prove existence of a solid phase and of the formation of periodic patterns.

If instead one wants to take advantage of the techniques developed for the Ising model, then we should look at the liquid-vapor coexistence line, where the liquid phase is characterized by a density essentially constant and the same happens for the gaseous phase. In the LMP model what is perturbed are indeed the homogeneous states with densities $\rho_{\beta,\pm}$ which appear in the mean field limit $\gamma \to 0$. In this way by choosing γ small enough one is able to adapt the P-S theory for the low temperature regime to the continuum model and prove coexistence of liquid and gas phases as small perturbations at finite γ from the mean field behavior. The idea of studying phase transitions by perturbative techniques around mean field, had already been successfully exploited in lattice systems, where the Peierls argument applies directly because of the spin flip symmetry of the models ([5], [4], [10]).

In our model we want to study perturbations of the LMP model by adding a hard core interaction, i.e. allowing the point particles to have a "dimension" (or analogously forbidding the point particles to be too close), the volume ϵ of the particles being the new small parameter. Hence we have two small parameters, ϵ and γ, where the first is considered to be fixed, while the latter is chosen small accordingly to the former. Again we have to think to the mean field as the unperturbed model, for which we can carry out an heuristic analysis (see Chapter 3) which shows that the critical temperature $\beta_{c,\epsilon}$ for the model where the par-

ticles are replaced by hard spheres is smaller than that for LMP. This is due to the fact that introducing forbidden zones for the particles the entropy becomes smaller and this helps the liquid-vapor transition to occur.

In the rest of this introduction we present a brief outline of our strategy, which at this stage will result necessarily a little vague. Our proof will follow P-S theory in the version proposed by Zahradník, [63], which involves the notion of cutoff weights. The analysis requires first of all the notion of *coarse graining* and contours which are introduced in Section 4.1. To do a coarse graining for our model means to divide the space into cells whose size is much larger than the radius of the hard core interaction but still smaller than the range of the Kac potential γ^{-1}. With this choice significant density fluctuations inside a single cell are quite infrequent and we can reduce our analysis to the study of the model in the restricted ensembles, i.e. where the admissible configurations are those whose empirical density in each cube is close to the macroscopic density of one of the two phases. This equivalence between models is true once we take into account the weight of the contours of the corresponding abstract contour model. Hence we are dealing with a coarse grained system in which the configurations we look at are those chosen in the restricted ensembles, roughly speaking those which should be seen under the effects of a double well potential once we restrict to its minima. In this scenario we are able to compute the effective hamiltonian for the coarse grained system with a multi-canonical constraint (given by the fixed density in each cell). This computation involves an integration over the positions of the particles in each cell leading to a new Gibbs measure which depends only on the cells variables.

The computations which lead to the effective hamiltonian are in general very hard, nevertheless due to the choice of the interaction they can be carried out. The crucial point here is to show convergence for cluster expansion in the canonical ensemble (note that the fixed density in each cell means a canonical constraint). This was not known before, as it was only proved in the grand canonical ensemble, and is the content of the paper [53] written in collaboration with D. Tsagkarogiannis. The paper in fact is for more general models, since the only requirement is that the interaction is stable and tempered. We prove in the low density - high temperature regime the validity of the cluster expansion for the canonical partition function and that its convergence is uniform in the volume. Furthermore in the same

regime the free energy can be expressed as a series in powers of the density which tends to Mayer's virial expansion in the thermodynamic limit. For simplicity we report this result in the Appendix A directly for the hard spheres model, where convergence holds for $\rho\epsilon$ small enough, leaving out the computation of the infinite volume limit. Analogously we carry out a cluster expansion for the LMP model plus hard core in the coarse grained system in Chapter 7. The crucial observation is that the Kac interaction, due to its long range nature, results more or less constant inside each box and therefore the part of the interaction which we have to integrate over the boxes is the remaining part which is small.

Once we are left with a coarse grained description we still need to prove the most delicate point which is the exponential decay of correlations in the restricted ensembles. More precisely we want to see that the dependence of the particle density in each box on the boundary condition decays exponentially with the distance from the boundary. The difficulty of this part is related to the nature of the system, which has no analogue of the spin flip symmetry shown in the Ising model. Hence after putting ourselves in the abstract contour model, we will have to deal with expressions which involve not only the support of the contour, i.e. a "surface term", but also "bulk terms" involving its whole interior. To solve this point, in Chapter 8 a Dobrushin uniqueness theorem is proved through an analysis of the Vaserstein distance between the two Gibbs measures with the same hamiltonian but with different boundary conditions.

Contents

1. The particle model

In this Chapter we define the microscopic particle model that we want to study.

1.1 Phase configurations and phase space

- We consider a system of identical point particles in $\mathbb{R}^d$, $d \geq 2$, and call *particle configuration* a countable, locally finite collection of points in $\mathbb{R}^d$. The *phase space* $\mathcal{Q}^\Lambda$ is the collection of all particle configurations in a bounded region Λ. We simply write $\mathcal{Q}$ when $\Lambda \equiv \mathbb{R}^d$. The particle configurations are denoted by q and sometimes by q_Λ when we want to specify that they are in $\mathcal{Q}^\Lambda$. We write $q = (q_1, ...q_n)$ to indicate a configuration of $|q| = n$ particles positioned at points $q_1, ..., q_n$ (the order is unimportant) of $\mathbb{R}^d$.

- Given any symmetric function $f \in L^\infty(\mathcal{Q}^\Lambda)$, Λ a bounded measurable set of $\mathbb{R}^d$, we define the *free measure* $\nu^\Lambda(dq)$ so that:

$$\int_{\mathcal{Q}^\Lambda} \nu^\Lambda(dq) f(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} dq_1 \cdots dq_n \, f(q_1, ..., q_n) \tag{1.1.1}$$

- We will sometimes use the following compact notation, writing

$$\frac{1}{n!} \sum_{i_1 \neq .. \neq i_n} F(q_{i_1}, ..., q_{i_n}) \equiv \int dq^{\otimes n} F \tag{1.1.2}$$

where we can interpret $dq^{\otimes n}$ as a point measure on $\mathbb{R}^{dn}$:

$$dq^{\otimes n} \equiv q^{\otimes n}(dr_1 ... dr_n) = \frac{1}{n!} \sum_{i_1 \neq .. \neq i_n} \delta_{q_{i_1}}(r_1) \, dr_1 \cdots \delta_{q_{i_n}}(r_n) \, dr_n \tag{1.1.3}$$

$\delta_r(r')$ being the delta function at r. Note that this measure represents the direct product of n point measures $dq^{\otimes 1}$, except for the $n!$ taking into account the fact that particles are indistinguishable and for deleting all the terms where the same particle appears

more than once. We can also define the analogous measure in the case of a profile density $\rho \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$:

$$d\rho^{\otimes n}(r_1, ...r_n) = \frac{1}{n!}\rho(r_1)\cdots\rho(r_n)dr_1\cdots dr_n \tag{1.1.4}$$

- We conclude this section by observing that there is a natural notion of addition of two particle configurations, $q + q'$, being the configuration which collects all the particles of q and q'. Notice that the notion is well defined because particles are indistinguishable.

1.2 Kac potentials and energy

As we explained already at the mean field level, we want to study a model which is a perturbation of the LMP model obtained by adding a hard core interaction on a scale much smaller with respect to the long range of the LMP interaction.

The local version of the mean field LMP energy is an example of many body Kac potentials, while the hard core potential is defined already on a microscopic scale.

The idea behind Kac potentials is that one would like to describe phase transitions as a perturbation of the mean field. The notion of "scaling" here plays an important role. We have three basic lengths:

- the hard core radius R which is considered small but of order 1,
- the range of the interaction of the Kac potentials, denoted by γ^{-1}, which is very large since it mimics locally a mean field,
- the size of the regions, which is even larger, eventually tending to infinity.

Let us consider the energy (3.1.2) and, instead of setting the density $\rho(r)$ identically constant and equal to the total density $n/|\Lambda|$ (where n is the number of particles and $|\Lambda|$ the volume of the region Λ), let us relax this assumption to a local condition. To do this we first define a local particle density at $r \in \mathbb{R}^d$ given a configuration q and we use the notation $\rho(r; q)$. It is local in the sense that it depends only on the restriction of q to a neighborhood of r. Then in analogy with (3.1.3) we define the hamiltonian for the LMP model as:

$$H(q) = \int e(\rho(r; q))\, dr, \tag{1.2.1}$$

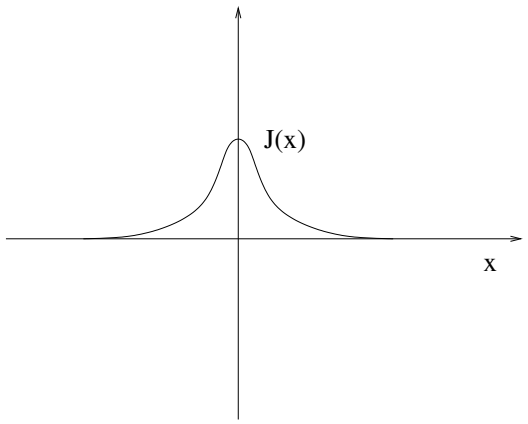

Figure 1.1: The function $J(0, x)$.

where $e(\rho(r; q))$ is the local energy density and its integral represent the total energy of the configuration q.

1.2.1 Local particle density

For $\rho(r; q)$ we use the following definition:

$$J_\gamma * q(r) := \sum_{q_i \in q} J_\gamma(r, q_i). \tag{1.2.2}$$

To fix ideas, first think of $J_\gamma(r, r')$ as the indicator of the ball $B_{\gamma^{-1}R_d}(r)$ of radius $\gamma^{-1}R_d$ divided by its volume:

$$J_\gamma(r, r') = \frac{\mathbb{1}_{|r-r'| \le \gamma^{-1}R_d}}{|B_{\gamma^{-1}R_d}(r)|} \tag{1.2.3}$$

where R_d is the radius of the ball in $\mathbb{R}^d$ having a unit volume and $|B_{\gamma^{-1}R_d}(r)| = \gamma^{-d}$. In such a case:

$$J_\gamma * q(r) = \frac{|q \cap B_{\gamma^{-1}R_d}(r)|}{|B_{\gamma^{-1}R_d}(r)|} \tag{1.2.4}$$

is the "empirical" particle density in a ball $B_{\gamma^{-1}R_d}(r)$. The Kac parameter γ is thought very small, so that many particles of the given configuration contribute to the average $J_\gamma * q(r)$.

It is convenient to generalize the definition of local particle density by introducing more general convolution kernel as it may be convenient to have $J_\gamma(r, r')$ smooth rather than discontinuous.

The main property of Kac potentials is that they depend on the scale parameter γ in the following way:

$$J_\gamma(r, r') = \gamma^d J(\gamma r, \gamma r') \tag{1.2.5}$$

where J is a symmetric, translation invariant, smooth probability kernel which we suppose to vanish for $|r - r'| \geq 1$ (see Fig. 1.1). This implies of course:

$$\int dr J_\gamma(r, r') = \int dr J(r, r') = 1. \tag{1.2.6}$$

With this definition, (1.2.2) becomes the empirical density of a configuration q around a point r weighted with the probability kernel J_γ, which involves only the particles which are in a ball of radius $\gamma^{-1}R_d$ with centre r since it has compact support. Note that the scaling property (1.2.5) means that when γ is small the model has a large but finite interaction range $2\gamma^{-1}R_d$, but a small interaction intensity between any couple of particles.

The notation $J_\gamma * q(r)$ reminds one of convolutions and indeed the r.h.s. of (1.2.2) may be interpreted as the convolution of the kernel J_γ with the sum of Dirac deltas $\sum \delta(r - q_i)$ at the positions q_i of the particles in the configuration q.

1.2.2 LMP hamiltonian

Let us now define the LMP part of the hamiltonian as the integral of the energy density as in (1.2.1):

$$H_{\gamma,\lambda}(q) = \int e(J_\gamma * q(r))\, dr \tag{1.2.7}$$

where the choice for $e(\rho)$ in the LMP model is given by (3.1.3) (the quartic term to ensure stability).

Note that the LMP interaction can be written in terms of one, two and four body potentials in the following way:

$$H_{\gamma,\lambda}(q) = -\lambda|q| - \frac{1}{2!}\sum_{i\neq j} J_\gamma^{(2)}(q_i, q_j) + \frac{1}{4!}\sum_{i_1\neq\ldots\neq i_4} J_\gamma^{(4)}(q_{i_1}, ..., q_{i_4}), \tag{1.2.8}$$

where

$$\begin{aligned} J_\gamma^{(2)}(q_i, q_j) &= \int J_\gamma(r, q_i) J_\gamma(r, q_j)\, dr \\ J_\gamma^{(4)}(q_{i_1}, ..., q_{i_4}) &= \int J_\gamma(r, q_{i_1}) \cdots J_\gamma(r, q_{i_4})\, dr, \end{aligned} \tag{1.2.9}$$

i.e. $J_\gamma^{(2)}$ is the convolution with J_γ with itself and $J_\gamma^{(4)}$ is the convolution of 4 kernels. Note that in the case of non smooth kernel as in (1.2.3), we have explicitly:

$$J_\gamma^{(n)}(q_1, ..., q_n) = \gamma^{nd} \int dr \prod_{j=1}^{n} \mathbb{1}_{|r-q_j| \leq \gamma^{-1} R_d} \tag{1.2.10}$$

for $n = 2, 4$. From (1.2.6) we have that potentials $J_\gamma^{(n)}$ $(n = 2, 4)$ have the following general properties:

$$\int dq_n \, J_\gamma^{(n)}(q_1, ..., q_n) = J_\gamma^{(n-1)}(q_1, ..., q_{n-1}), \qquad \int dq_2 \, J_\gamma^{(2)}(q_1, q_2) = 1 \tag{1.2.11}$$

and therefore

$$\int dq_2 \cdots dq_n \, J_\gamma^{(n)}(q_1, ..., q_n) = 1. \tag{1.2.12}$$

Furthermore, they depend on the scaling parameter γ in the following way:

$$J_\gamma^{(n)}(q_1, ..., q_n) = \gamma^{d(n-1)} J_\gamma^{(n)}(\gamma q_1, ..., \gamma q_n). \tag{1.2.13}$$

Recalling the notation (1.1.3) we can also write the LMP hamiltonian in the form:

$$H_{\gamma,\lambda}(q) = -\lambda |q| - \int dq^{\otimes 2} J_\gamma^{(2)} + \int dq^{\otimes 4} J_\gamma^{(4)}. \tag{1.2.14}$$

It is convenient to give a definition of the LMP hamiltonian when instead of a configuration ρ is given a continuous density $\rho \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$:

$$H_{\gamma,\lambda}(\rho) = -\lambda \int dr \rho(r) - \int d\rho^{\otimes 2} J_\gamma^{(2)} + \int d\rho^{\otimes 4} J_\gamma^{(4)} \tag{1.2.15}$$

where the measure is defined in (1.1.4).

1.2.3 Hard core interaction

In the model with hard cores the phase space is restricted like in (A.1.2). This is equivalent to add an interaction which is $= \infty$ when the particles get too much close with each other and is 0 when the particles are far, i.e. a potential $V^{\mathrm{hc}} : \mathbb{R}^d \to \mathbb{R}$ such that:

$$V^{\mathrm{hc}}(q_i, q_j) = \begin{cases} +\infty & \text{if } |q_i - q_j| \leq R \\ 0 & \text{if } |q_i - q_j| > R \end{cases} \tag{1.2.16}$$

where $|q_i - q_j|$ denotes the euclidean distance between the two particles in q_i and q_j. R is the radius of the hard spheres and their volume is $\epsilon = V_d(R)$, i.e. the volume of the d-dimensional sphere of radius R. The hard core potential depends on q_i, q_j only through their distance.

Note that the potential (A.1.2) is stable and tempered. Stability comes from its positivity and temperedness follows from

$$C(\epsilon) := \int dq \, |e^{-\beta V^{\mathrm{hc}}(q)} - 1| = \int dq \, \mathbb{1}_{\{|q| \leq R\}} = \epsilon < \infty. \tag{1.2.17}$$

1.2.4 LMP-hard core hamiltonian

The hamiltonian of the model we want to consider is therefore the following:

$$H_{\gamma,R,\lambda}(q) = \int e(J_\gamma * q(r)) \, dr + H^{\mathrm{hc}}(q) \tag{1.2.18}$$

where

$$H^{\mathrm{hc}}(q) := \sum_{i<j} V^{\mathrm{hc}}(q_i, q_j). \tag{1.2.19}$$

Given two configurations q and $\bar{q}$, we will use the following notation,

$$H_{\gamma,R,\lambda}(q|\bar{q}) = H_{\gamma,R,\lambda}(q + \bar{q}) - H_{\gamma,R,\lambda}(\bar{q}) \tag{1.2.20}$$

to represent the energy of the particle configuration q in the field generated by $\bar{q}$ and

$$U_{\gamma,R,\lambda}(q, \bar{q}) = H_{\gamma,R,\lambda}(q + \bar{q}) - H_{\gamma,R,\lambda}(q) - H_{\gamma,R,\lambda}(\bar{q}) \tag{1.2.21}$$

to indicate the interaction energy between q and $\bar{q}$.

1.3 Gibss measures

The Gibbs grand canonical measure in the bounded measurable region Λ in $\mathbb{R}^d$ and boundary conditions $\bar{q} \in \mathcal{Q}^{\Lambda^c}$ is the probability measure on $\mathcal{Q}^\Lambda$ given by the Gibbs formula:

$$\mu^\Lambda_{\gamma,\beta,R,\lambda}(dq|\bar{q}) = Z^{-1}_{\gamma,\beta,R,\lambda}(\Lambda|\bar{q}) e^{-\beta H_{\gamma,R,\lambda}(q|\bar{q})} \nu^\Lambda(dq) \tag{1.3.1}$$

where β is the inverse temperature, $Z_{\gamma,\beta,R,\lambda}(\Lambda|\bar{q})$ is the grand canonical partition function (defined as the normalization factor for $\mu^\Lambda_{\gamma,\beta,R,\lambda}(dq|\bar{q})$ to be a probability) and $\nu^\Lambda(dq)$ the free measure defined in (1.1.1).

The infinite volume Gibbs measures are probabilities $\mu_{\gamma,\beta,R,\lambda}$ on $\mathcal{Q}$, such that for any measurable region Λ

$$\mu_{\gamma,\beta,R,\lambda}(dq^{\Lambda}|\bar{q}^{\Lambda^c}) = \mu^{\Lambda}_{\gamma,\beta,R,\lambda}(dq^{\Lambda}|\bar{q}^{\Lambda^c}); \qquad \text{for } \mu_{\gamma,\beta,R,\lambda} \text{ almost all } \bar{q}^{\Lambda^c} \tag{1.3.2}$$

where the expression on the l.h.s. is the conditional probability given the configuration $\bar{q}^{\Lambda^c}$ outside Λ.

2. Main results

In this chapter we state our main theorem, Theorem 2.1.1, on the existence of two distinct Gibbs states for suitable choice of chemical potential and temperature. It extends to the model with the hard core interaction what is already proven for the LMP model (see [39]).

2.1 The main theorem

We shall consider the LMP model with additional hard core interaction (for definition see Section 1.2.4), hereafter shorthanded by LMP-hc model. If the hard core radius R is sufficiently small the LMP phase transition is essentially unaffected.

Theorem 2.1.1. *Consider the LMP-hc model in dimensions $d \geq 2$. For such a model there are R_0, $\beta_{c,R}, \beta_{0,R}$ (as in Proposition 3.2.1) and for any $0 < R \leq R_0$ and $\beta \in (\beta_{c,R}, \beta_{0,R})$ there is $\gamma_{\beta,R} > 0$ so that for any $\gamma \leq \gamma_{\beta,R}$ there is $\lambda_{\beta,\gamma,R}$ such that:*

There are two distinct DLR measures $\mu^{\pm}_{\beta,\gamma,R}$ with chemical potential $\lambda_{\beta,\gamma,R}$ and inverse temperature β and two different densities: $0 < \rho_{\beta,\gamma,R,-} < \rho_{\beta,\gamma,R,+}$.

In this work we will only prove the existence of two distinct states, which are interpreted as the two pure phases of the system: $\mu^{+}_{\beta,\gamma,R}$ describes the liquid phase with density $\rho_{\beta,\gamma,R,+}$ while $\mu^{-}_{\beta,\gamma,R}$ describes the vapor phase, with the smaller density $\rho_{\beta,\gamma,R,-}$.

Remark. $\rho_{\beta,\gamma,R,\pm}$ and $\lambda_{\beta,\gamma,R}$ have limit as $\gamma \to 0$, the limit being $\rho_{\beta,R,-} < \rho_{\beta,R,+}$ and $\lambda(\beta, R)$ which are respectively densities and chemical potential for which there is a phase transition in the mean field model (see the next chapter).

Remark. The DLR measures $\mu^{\pm}_{\beta,\gamma,R}$ have several properties one can prove:

- the measures are obtained as weak limit of "diluted Gibbs measures" (see Section 4.3)
- the measures are translational invariant,
- any other translational invariant measure is a convex combination of the two,
- the measures have a trivial σ-algebra at infinity.

We refer to [52], Chapter 12, for the derivation of such properties which are carried out for the LMP model, believing they are not affected too much by the presence of the hard core interaction. We however skip the proof since it is not in the original purpose of this work.

2.2 Phase transitions in the canonical ensemble

From Theorem 2.1.1 it is clear that we prove phase transitions using a gran canonical description of the equilibrium states. This implies that we either see a configuration with density $\rho_{\beta,\gamma,R,+}$ or with density $\rho_{\beta,\gamma,R,-}$ and there is no coexistence of the two phases in the same configuration q but only on statistical average, except when the chemical potential is chosen to be equal to $\lambda_{\beta,\gamma,R}$.

The aim of this section is to reformulate the condition that there is a phase transition in the system using a canonical description, intead of a gran canonical one. When we see phase transitions in the canonical ensemble, they look different. In fact, the canonical constraint forces the two phases to coexist in each configuration (which is typical, i.e. discarding a certain number of bad configurations) for any value of $\rho \in (\rho_{\beta,\gamma,R,-}, \rho_{\beta,\gamma,R,+})$.

We already gave the definition of gran canonical Gibbs measure in (1.3.1) of Section 1.3. Analogously we can introduce the canonical Gibbs measure as the probability measure on $\mathcal{Q}^\Lambda$, where each configuration $q \in \mathcal{Q}^\Lambda$ is such that $|q| = N$, defined as

$$\mu^{\text{can}}_{\beta,N,\Lambda}(dq) = \frac{1}{Z^{\text{can}}_{\beta,N,\Lambda}} \frac{1}{N!} e^{-\beta H(q)} \mathbb{1}_{\{q \in \mathcal{Q}^\Lambda\}} dq \tag{2.2.1}$$

where dq is the Lebesgue measure and where we drop for simplicity the dependence of γ and R from the hamiltonian and therefore from the measure. Indeed we can forget about our model and give a formulation of the problem for a general model.

We do not discuss the case of phase transitions in the canonical ensemble, but we hope to address this issue in a subsequent work. Nevertheless the picture we should have in mind

can be made more clear with a thermodynamic description of phase transitions in terms of "forbidden intervals". We have a phase transition if there is a interval of densities, say (ρ', ρ''), so that if we put a mass $\rho|\Lambda|$ of fluid in the region Λ (where $|\Lambda|$ denotes the volume of Λ) with $\rho \in (\rho', \rho'')$, then the fluid is separated into a part with density ρ' and another one with density ρ''. It does not exist an equilibrium state with homogeneous density ρ, but inside Λ one can go from one phase which occupies a region Λ' to another phase in $\Lambda \setminus \Lambda'$ and see a phase transition.

Hence we can reformulate the condition that there is a phase transition in the sense discussed above. The formulation involves "coarse graining", a notion which will be extensively used in the sequel.

Definition 2.2.1 (Partition $\mathcal{D}^{(\ell)}$ of $\mathbb{R}^d$)**.** For any $\ell \in \{2^n, n \in \mathbb{Z}\}$ and any $i \in \ell\mathbb{Z}^d$, we set

$$C_i^{(\ell)} = \{r \in \mathbb{R}^d : i_k \leq r_k < i_k + \ell, k = 1, ..., d\} \tag{2.2.2}$$

(r_k and i_k being the kth coordinate of r and i) and call

$$\mathcal{D}^{(\ell)} = \{C_i^{(\ell)}, i \in \ell\mathbb{Z}^d\} \tag{2.2.3}$$

the corresponding partition of $\mathbb{R}^d$.

We also denote $C_r^{(\ell)}$ the cube of $\mathcal{D}^{(\ell)}$ which contains r. Since ℓ has the form 2^n, each cube of $\mathcal{D}^{(\ell)}$ is the union of cubes of $\mathcal{D}^{(\ell')}$ for $\ell > \ell'$; $\mathcal{D}^{(\ell)}$ is then coarser than $\mathcal{D}^{(\ell')}$ and $\mathcal{D}^{(\ell')}$ finer than $\mathcal{D}^{(\ell)}$.

A function $f(r)$ is $\mathcal{D}^{(\ell)}$-measurable if it is constant on each cube $C_i^{(\ell)}, i \in \ell\mathbb{Z}^d$; a region Λ is $\mathcal{D}^{(\ell)}$-measurable if its characteristic function is $\mathcal{D}^{(\ell)}$-measurable, or, equivalently, it is union of cubes of $\mathcal{D}^{(\ell)}$.

We want to define also empirical density profiles of particle configurations for every fixed partition $\mathcal{D}^{(\ell)}$:

Definition 2.2.2 (Empirical density)**.** Given a particle configuration q we define its empirical density profile on the scale ℓ the following average:

$$\rho_\ell(r; q) = \frac{|q \cap C_r^{(\ell)}|}{\ell^d} \tag{2.2.4}$$

where is the number of particles of the configuration q which are in the cube $C_r^{(\ell)}$.

From Definition 2.2.2 we can see that the empirical density $\rho_\ell(r;q)$ is a non negative function constant on the atoms of $\mathcal{D}^{(\ell)}$. Hence it is $\mathcal{D}^{(\ell)}$-measurable. We shall use the empirical density $\rho_\ell(r;q)$ to quantify the definition of phase transition given from the viewpoint of the"forbidden interval" (ρ', ρ'').

We will have to restrict to "large probability events" since the canonical Gibbs measure gives positive probability to all open sets and therefore for any Λ there is a positive probability to see configurations which are really atypical. Moreover even in the typical configurations we should expect to see somewhere deviations from the expected behavior. We thus choose an accuracy parameter $\epsilon > 0$ and define the sets

$$\mathcal{G} = \{r \in \Lambda : |\rho_\ell(r;q) - \rho'| > \epsilon \quad \text{or} \quad |\rho_\ell(r;q) - \rho''| > \epsilon\} \tag{2.2.5}$$

then the set of typical configurations in the phase transition case will be denoted by

$$G^{\text{pt}}_{\rho,\Lambda,\epsilon,\ell} := \{q \in \mathcal{Q}^\Lambda : |\mathcal{G}| \leq \epsilon|\Lambda|\}. \tag{2.2.6}$$

Definition 2.2.3. (ρ', ρ'') is a forbidden density interval if for any $\rho \in (\rho', \rho'')$ and any increasing sequence of cubes Λ invading $\mathbb{R}^d$:

$$\lim_{\epsilon\to 0}\lim_{\ell\to\infty}\lim_{\Lambda\nearrow\mathbb{R}^d} \mu^{\text{can}}_{\beta,\lfloor\rho|\Lambda|\rfloor,\Lambda}\left[G^{\text{pt}}_{\rho,\Lambda,\epsilon,\ell}\right] = 1. \tag{2.2.7}$$

Thus in a first scenario ρ is far from the forbidden interval and with large probability $\rho_\ell(r;q)$ is close to ρ in a large number of boxes and we do not have a phase transition. In the other scenario if ρ is inside the forbidden interval then with large probability in a large number of boxes $\rho_\ell(r;q)$ will be either equal to ρ' or to ρ'' being different from ρ. In this second case we will have a phase transition.

Besides Theorem 2.1.1 we want to conclude this chapter with the following conjecture:

Conjecture 2.2.4. *Consider the LMP-hc model in dimensions $d \geq 2$. For such a model there are R_0, $\beta_{c,R}, \beta_{0,R}$ (as in Proposition 3.2.1) and for any $0 < R \leq R_0$ and $\beta \in (\beta_{c,R}, \beta_{0,R})$ there is $\gamma_{\beta,R} > 0$ so that for any $\gamma \leq \gamma_{\beta,R}$ there is an interval $(\rho_{\beta,\gamma,R,-}, \rho_{\beta,\gamma,R,+})$ such that* (2.2.7) *holds true (where $\mu^{can}_{\beta,\lfloor\rho|\Lambda|\rfloor,\Lambda}$ is referred to the LMP-hc model).*

2.3 The idea of Pirogov-Sinai

The idea of the proof is based on an argumend proposed by Pirogov-Sinai, which we will discuss better in the next chapters and which relies (as for the Ising model) on the possibility to rewrite the partition function of the model as the partition function of an abstract contour model.

When dealing with phase transitions for systems in the continuum one has to think to some analogies with models on the lattice, for instance the Ising model, which are understood a lot better. In our case we want to establish a correspondence with phase transitions in the Ising model at small temperatures for which one can prove Peierls bounds (see for instance Chapter 9 of [52]). There are two reasons for this strategy: one is due to the fact that we want to follow a perturbative argument, the other to the fact that reducing our continuum model to a model on a lattice excitations have a discrete structure. In the Ising model the small parameter is β^{-1} and the limit Gibbs measures concentrate respectively on perturbations of the ground states, which are the all plus and all minuses spin configurations at zero temperature. Analogously in the LMP-hc model the small parameter of the theory is γ and the measures concentrate on perturbations of the homogeneous states with densities $\rho_{\beta,\gamma,R,\pm}$ which appear in the mean field limit $\gamma \to 0$. Also the proofs have analogies, they are both based on probability estimates on contours, defined as regions which separate plus and minus islands, namely where the configurations look typical of the two phases. The analogue in Ising of the chemical potential is the magnetic field h: due to the spin flip symmetry in Ising the critical magnetic field is $h = 0$. In LMP instead there is no symmetry between the two phases and as a consequence the critical value of the chemical potential where there is a phase transition has to be chosen and is part of the problem. Hence, the proof of Theorem 2.1.1 is perturbative, the mean field states being the ground states and the small parameter being the inverse of the interaction length.

Here we want just to discuss the physical picture to have in mind while talking of phase transitions. There is a special value $\lambda_{\beta,\gamma,R}$ of the chemical potential where the two phases coexist. This means that the two minima of the free energy are equal and hence by selecting the boundary conditions one gets one of the two possible phases. If instead we vary λ away from $\lambda_{\beta,\gamma,R}$ in agreement with the selected boundary conditions then we run into the one phase regions: one of the two phases present at $\lambda_{\beta,\gamma,R}$ persists becoming stable and the other

one becomes metastable and disappears. We should think of a stable phase as a homogeneous sea with rare and small islands of the other phase, separated from the sea by *contours*. On the other side, when varying λ in opposition to the boundary conditions then we see the formation of a large island of the truly stable phase. This island is large as the cost of the surface of the contour is overcome by the volume gain of having the correct phase in the bulk. So in this case large islands are quite probable in a typical configuration and this make impossible a real computation which involves expansions. The theory of Pirogov-Sinai forbids this scenario, hence the existence of such big contours, by definition of *metastable* states.

The metastable phase defined with the constraint that contours cannot be too large has a free energy larger than in the truly stable phase. Thus stable and metastable phases are recognized by their free energies and the right chemical potential is selected by requiring equality of the two and phase coexistence then follows. As a consequence of this approach we will need a computation of the free energy of states where contours are rare, since either we are in a stable phase where they are improbable, or we are in a metastable phase where they are rare by definition.

To implement this idea we shall follow the approach of Milos Zahradnik, which introduces the notion of *cutoff weights* for the contours, which we will develop in Section 4.6.

3. The mean field model

Mean field models are those in which the interaction range is of the same order as the size of the region where the system is confined. This means in our case that two particles in the system interact in the same way independently of their location. Under such assumption one can see that the mean field model exhibits a phase transition once it reaches a certain critical temperature and that the phase transition is described through the density which is the order parameter.

3.1 The model

Let us start by describing the model in its mean field version.

First of all observe that the hard-core interaction restricts the phase space in the following way:

$$\mathcal{X}_{\Lambda}^{R} = \bigcup_{n} \mathcal{X}_{n,\Lambda}^{R}, \qquad \mathcal{X}_{n,\Lambda}^{R} = \{(r_1, ...r_n) \in \Lambda^n : \min_{i \neq j} |r_i - r_j| > R\} \tag{3.1.1}$$

where Λ is a bounded region. The number of particles in finite volumes is then bounded by the "close packing" density $\rho_{\text{cp}} = \rho_{\text{cp}}(R)$. The constraint (A.1.2) is due to a pair interaction which is equal to $+\infty$ whenever the two particles are at distance $\leq R$ (R being the "hard-core lenght") and equal to zero otherwise. We will use ϵ to indicate the volume of the hard spheres, $\epsilon = c_d R^d$, c_d a coefficient depending on the dimension d.

The mean field hamiltonian given a configuration $q \equiv (r_1, ..., r_n)$ in $\mathcal{X}_{n,\Lambda}^{R}$ is:

$$H_{\Lambda,R,\lambda}^{\text{mf}}(q) = |\Lambda| e_{\lambda}(\rho) \tag{3.1.2}$$

where $\rho = |q|/|\Lambda|$ and $e_{\lambda}(\cdot)$ has the meaning of an energy density and has the same form as in the LMP model. It depends only on the total particle density ρ, with the constraint $\rho \in [0, \rho_{\text{cp}}]$. The label R in the hamiltonian is to remember of the hard core condition.

λ is interpreted as a chemical potential, so that $e_\lambda(\rho) = -\lambda\rho + e_0(\rho)$. To mimic a fluid of hard spheres, $e_0(\rho)$ should be a decreasing function of the density ρ from 0 till when the energy reaches its minimum, which corresponds to an optimal disposition of the molecules. An increase of ρ causes an increase of the energy, which will diverge when $\rho \to \rho_{\rm cp}$. The choice which fulfills such requirements is (as already chosen for the LMP model):

$$e_\lambda(\rho) = -\lambda\rho - \frac{\rho^2}{2} + \frac{\rho^4}{4!}. \tag{3.1.3}$$

The first term $(-\lambda\rho)$, is the energy of the chemical potential λ, the term $-\rho^2/2$ comes from the attractive pair interaction and the term $\rho^4/4!$ is the four body repulsive interaction, which has been added in the LMP model to stabilize the potential.

The mean field canonical partition function is defined in the following way:

$$Z^{\rm mf}_{n,\Lambda} = \frac{1}{n!}\int_{\mathcal{X}^R_{n,\Lambda}} e^{-\beta|\Lambda|e_\lambda(n/|\Lambda|)} = \exp\Big\{-\beta\Big(-\frac{n^2}{2|\Lambda|} + \frac{n^4}{4!|\Lambda|^3} - \lambda n + \frac{|\Lambda|}{\beta} f^{\rm hc}_\Lambda\big(\frac{n}{|\Lambda|}\big)\Big)\Big\}, \tag{3.1.4}$$

where

$$f^{\rm hc}_\Lambda\big(\frac{n}{|\Lambda|}\big) = -\frac{1}{|\Lambda|}\log Z^{\rm hc}_{n,\Lambda}, \qquad Z^{\rm hc}_{n,\Lambda} := \frac{1}{n!}\int_{\mathcal{X}^R_{n,\Lambda}} dq_1\cdots dq_n \tag{3.1.5}$$

while the grand canonical one is $\Xi^{\rm mf}_\Lambda = \sum_{n=0}^\infty Z^{\rm mf}_{n,\Lambda}$. We omit the proof that in the thermodynamic limit $|\Lambda|, n \to \infty$ the canonical mean field free energy is:

$$\phi_{\beta,\lambda,R}(\rho) = \lim_{|\Lambda|,n\to\infty:\frac{n}{|\Lambda|}\to\infty} \frac{-1}{\beta|\Lambda|}\log Z^{\rm mf}_{n,\Lambda} = e_\lambda(\rho) + \frac{1}{\beta} f^{\rm hc}(\rho) \tag{3.1.6}$$

where

$$f^{\rm hc}(\rho) := \lim_{|\Lambda|,n\to\infty:\frac{n}{|\Lambda|}\to\infty} f^{\rm hc}_\Lambda\big(\frac{n}{|\Lambda|}\big) = \rho(\log\rho - 1) - \sum_{n\geq 1}\frac{\beta_n}{n+1}\rho^{n+1}. \tag{3.1.7}$$

The existence of the thermodynamic limit in (3.1.7) follows from general arguments, while due to a cluster expansion argument (see for instance Theorem A.1.1 of Appendix A or for more general interactions [53]) one can prove that the limit function $f^{\rm hc}$ is analytic and convex. Furthermore, the β_n are the Mayer's coefficients:

$$\beta_n = \frac{1}{n!}\sum_{\substack{g\in\mathcal{B}_{n+1}\\ V(g)\ni\{1\}}}\int_{(\mathbb{R}^d)^n}\prod_{\{i,j\}\in E(g)}(e^{-\beta V^{\rm hc}(q_i-q_j)}-1)dq_2\dots dq_{n+1}, \quad q_1\equiv 0. \tag{3.1.8}$$

Note that the hard-core free energy tends to ∞ for $\rho \to \rho_{\rm hc}$.

The grand canonical pressure is the Legendre transform of $\phi_{\beta,0,R}(\rho)$:

$$p_{\beta,\lambda,R} = \lim_{|\Lambda|\to\infty} -\frac{1}{\beta|\Lambda|}\log\Xi^{\rm mf}_\Lambda = \sup_{\rho\in[0,\rho_{\rm cp}]}\{\lambda\rho - \phi_{\beta,0,R}(\rho)\} = \sup_{\rho\in[0,\rho_{\rm cp}]}\{\lambda\rho - e_0(\rho) - \frac{1}{\beta}f^{\rm hc}(\rho)\} \tag{3.1.9}$$

3.2 Thermodynamics of the mean field

In this subsection we will state the main properties of $\phi_{\beta,\lambda,R}(\rho)$ as a function of β, λ and ρ. We will see that the mean field model has a phase transition, in this picture appearing as a loss of convexity of $\phi_{\beta,\lambda,R}(\rho)$, occurring at a value of β which is a correction of order ϵ of the critical temperature of the LMP model.

The critical points of $\phi_{\beta,\lambda,R}(\rho)$ as a function of ρ namely the solutions of the mean field equation:

$$\frac{d}{d\rho}\Big\{e_\lambda(\rho) + \frac{1}{\beta}f^{\mathrm{hc}}(\rho)\Big\} = 0 \tag{3.2.1}$$

have the form:

$$\rho = \exp\Big\{ - \beta e'_\lambda(\rho) - f'_R(\rho)\Big\} := K_{\beta,\lambda,R}(\rho), \tag{3.2.2}$$

where we define $f_R(\rho)$ to be the second term of (3.1.7), i.e. the free energy minus the entropy of the free system.

We start by studying the convexity properties of $\phi_{\beta,\lambda,R}(\rho)$ as a function of ρ for fixed β and λ; since $\phi_{\beta,\lambda,R}(\rho) = -\lambda\rho + \phi_{\beta,0,R}(\rho)$ they are independent of λ and in the next proposition we set $\lambda = 0$.

Note that a multiplicity of minimizers of $\phi_{\beta,0,R}(\cdot)$ corresponds to a mean field phase transition.

Proposition 3.2.1 (Convexity properties of $\phi_{\beta,0,R}(\cdot)$)**.** *There is a critical inverse temperature $\beta_{c,R}$, such that $\phi_{\beta,0,R}(\rho)$ is convex for $\beta \le \beta_{c,R}$, while for $\beta > \beta_{c,R}$ it has two inflection points $0 < s_-(\beta) < s_+(\beta)$, being concave for $\rho \in (s_-(\beta), s_+(\beta))$ and convex for $\rho \notin (s_-(\beta), s_+(\beta))$.*

Remark. The critical temperature has the following expansion:

$$\beta_{c,R} = \beta_c - \epsilon\beta_c^{2/3} + O(\epsilon^2), \tag{3.2.3}$$

where $\beta_c = 3/2^{3/2}$ is the critical temperature for the LMP model (without hard core interaction) and ϵ is the volume of the hard spheres with radius R.

The fact that the critical temperature is smaller than the LMP one, has the following heuristic meaning. By adding a hard core interaction the point particles of the LMP model have a smaller entropy. This is due to the fact that we have less ways of arranging the

particles in the space since they already occupy a volume ϵ and hence the phase space is smaller. When the entropy is smaller one should have "more easily" the occurrence of a phase transition, as when one decreases the temperature in the system.

Let us prove the expansion (3.2.3). We want to calculate the critical temperature, i.e. the value of β for which the second derivative:

$$\phi''_{\beta,\lambda,R}(s) = \frac{1}{\beta s} - 1 + \frac{s^2}{2} + \frac{1}{\beta} f''_R(s) \tag{3.2.4}$$

is equal to zero.

To fix ideas, let us first study the sign of the function without the contribution $\frac{1}{\beta} f''_R(s)$. So one has to find β_c such that:

$$\phi''_{\beta,\lambda}(s) = \frac{1}{\beta s} - 1 + \frac{s^2}{2} = 0. \tag{3.2.5}$$

This is equivalent to study the sign of:

$$\psi(s,\beta) := 2 - 2\beta s + \beta s^3. \tag{3.2.6}$$

For every fixed value of β we can look at its derivative in s: $\partial_s \psi(s,\beta) = -2\beta + 3\beta s^2$. We have that ψ has two critical points: $s = \pm(2/3)^{1/2}$, but since we are interested to the behaviour for non negative values of s we only consider the positive solution. The function $\psi(s,\beta)$ is first equal to 2 for $s = 0$, then for $0 < s < (2/3)^{1/2}$ is decreasing until it reaches its minimum in $s_0 = (2/3)^{1/2}$ and then it increases. With this behaviour in mind we have to find a value of β for which its minimum crosses the s axis, i.e. such that

$$\psi(s_0,\beta) = 2 - 2\beta\Big(\frac{2}{3}\Big)^{\frac{1}{2}} + \beta\Big(\frac{2}{3}\Big)^{\frac{3}{2}} = 0 \tag{3.2.7}$$

holds. The solution of (3.2.7) is: $\beta_c = (3/2)^{3/2}$ and it is unique.

Let us now go back to the true function $\phi''_{\beta,\lambda,R}(s)$ and think of the hard core part as a perturbation in ϵ:

$$\phi''_{\beta,\lambda,R}(s) = \frac{1}{\beta s} - 1 + \frac{s^2}{2} + \frac{1}{\beta} f''_R(s). \tag{3.2.8}$$

We want again to study the sign of the following function

$$\psi_R(s,\beta) := 2 - 2\beta s + \beta s^3 + 2s f''_R(s) \tag{3.2.9}$$

where:

$$f_R''(s) = -\sum_{n\geq 1} n\beta_n s^{n-1} = -\epsilon - 2\epsilon^2 s + o(\epsilon^3), \qquad f_R'''(s) = -\sum_{n\geq 2} n(n-1)\beta_n s^{n-2} = -2\epsilon^2 + o(\epsilon^3). \tag{3.2.10}$$

Even though one may confuse the inverse temperature β with the n-th Mayer's coefficient β_n, we keep them in agreement with the literature. Let us fix β and study the sign of $\psi_R(s,\beta)$ in s, through its derivative:

$$\begin{aligned} \partial_s \psi_R(s,\beta) &= -2\beta + 3\beta s^2 + 2[-\epsilon - 2\epsilon^2 s] + 2s(-2\epsilon^2) \\ &= -2\beta + 3\beta s^2 - 2\epsilon - 8s\epsilon^2 = 0 \end{aligned} \tag{3.2.11}$$

where we kept at most the second order in ϵ. We can expand equation (3.2.11), supposing that s has the following expression in powers of ϵ: $s = s_0 + s_1\epsilon + ...$, and get:

$$-2\beta + 3\beta(s_0 + s_1\epsilon)^2 - 2\epsilon - 8\epsilon^2(s_0 + s_1\epsilon) = 0. \tag{3.2.12}$$

Hence at the order zero in ϵ we obtain:

$$-2\beta + 3\beta s_0^2 = 0 \quad \Rightarrow s_0 = \Big(\frac{2}{3}\Big)^{1/2} \tag{3.2.13}$$

which gives the same minimum found in the case of the LMP model with no hard core interaction (see discussion before), while at the first order in ϵ:

$$6\beta s_0 s_1 - 2 = 0 \quad \Rightarrow s_1 = \frac{1}{3\beta}\Big(\frac{3}{2}\Big)^{1/2}. \tag{3.2.14}$$

Hence the minimum of (3.2.9) is:

$$s_{0,R} = s_0 + s_1\epsilon + ... = \Big(\frac{2}{3}\Big)^{1/2} + \frac{\epsilon}{3\beta}\Big(\frac{3}{2}\Big)^{1/2} + o(\epsilon^2). \tag{3.2.15}$$

Analogously to what we have done for the case without hard core, we find β for which the minimum of $\psi_R(s,\beta)$ is touching the s axis. We do this perturbatively, considering that β has the following expansion in powers of ϵ: $\beta = b_0 + b_1\epsilon + ...$

$$\psi_R(s_{0,R},\beta) = 2 - 2(b_0 + b_1\epsilon)\Big[\Big(\frac{2}{3}\Big)^{1/2} + \frac{\epsilon}{3(b_0+b_1\epsilon)}\Big(\frac{3}{2}\Big)^{1/2}\Big] + \tag{3.2.16}$$

$$+ (b_0 + b_1\epsilon)\Big[\Big(\frac{2}{3}\Big)^{1/2} + \frac{\epsilon}{3(b_0+b_1\epsilon)}\Big(\frac{3}{2}\Big)^{1/2}\Big]^3 - 2\epsilon\Big[\Big(\frac{2}{3}\Big)^{1/2} + \frac{\epsilon}{3(b_0+b_1\epsilon)}\Big(\frac{3}{2}\Big)^{1/2}\Big] = 0. \tag{3.2.17}$$

If we only keep ϵ as higher order we get:

$$\psi_R(s_{0,R},\beta) = 2 - 2\Big(\frac{2}{3}\Big)^{1/2}(b_0+b_1\epsilon) - \Big(\frac{2}{3}\Big)^{1/2}\epsilon + \Big(\frac{2}{3}\Big)^{3/2}(b_0+b_1\epsilon) + \Big(\frac{2}{3}\Big)^{1/2}\epsilon - 2\epsilon\Big(\frac{2}{3}\Big)^{1/2} = 0 \tag{3.2.18}$$

which gives at the zero order in ϵ:

$$2 - 2\Big(\frac{2}{3}\Big)^{1/2}b_0 + \Big(\frac{2}{3}\Big)^{3/2}b_0 = 0 \qquad \Rightarrow b_0 = \Big(\frac{3}{2}\Big)^{3/2} \tag{3.2.19}$$

i.e. the critical temperature β_c obtained in the LMP model. At the first order in ϵ we get:

$$b_1\Big[\Big(\frac{2}{3}\Big)^{3/2} - 2\Big(\frac{2}{3}\Big)^{1/2}\Big] - 2\Big(\frac{2}{3}\Big)^{1/2} = 0 \qquad \Rightarrow b_1 = -\frac{3}{2} \tag{3.2.20}$$

which gives that the critical value of β for the new model is:

$$\beta_{c,R} = \beta_c - \epsilon\beta_c^{2/3} + O(\epsilon^2) \tag{3.2.21}$$

where β_c is the critical temperature for the LMP model and the correction is of order ϵ, i.e. the volume of the hard shperes. Note that unicity of b_0 and b_1 implies unicity of $\beta_{c,R}$ for each fixed value of ϵ.

Proposition 3.2.2 (Phase transitions)**.** *For any $\beta > \beta_{c,R}$, there is $\lambda(\beta,R)$ so that $\phi_{\beta,\lambda(\beta,R),R}(\cdot)$ has two global minimizers, $\rho_{\beta,R,-} < \rho_{\beta,R,+}$ (and a local maximum at $\rho_{\beta,R,0}$). Moreover, there is c so that for any $\zeta > 0$ small enough:*

$$\phi_{\beta,\lambda(\beta,R),R}(\rho) \geq \phi_{\beta,\lambda(\beta,R),R}(\rho_{\beta,R,\pm}) + c\zeta^2, \qquad \textit{if } \min\{|\rho-\rho_{\beta,R,+}|, |\rho-\rho_{\beta,R,-}|\} \geq \zeta. \tag{3.2.22}$$

Proof.

Let us consider $\phi_{\beta,\lambda,R}(\rho) = -\lambda\rho + \phi_{\beta,0,R}(\rho)$ for $\beta > \beta_{c,R}$. If $\rho_{\beta,R,+}$ and $\rho_{\beta,R,-}$ are the two inflection points of $\phi_{\beta,0,R}(\rho)$, i.e. satisfying (3.2.1), then we can choose λ such that:

$$\phi_{\beta,\lambda,R}(\rho_{\beta,R,+}) - \phi_{\beta,\lambda,R}(\rho_{\beta,R,-}) = -\lambda(\rho_{\beta,R,+} - \rho_{\beta,R,-}) + (\phi_{\beta,0,R}(\rho_{\beta,R,+}) - \phi_{\beta,0,R}(\rho_{\beta,R,-})) = 0 \tag{3.2.23}$$

i.e. such that the two local minimizers become global. This means that by adding a linear function with some slope λ one can have the two minima at the same height. Moreover, by expanding $\phi_{\beta,\lambda,R}(\rho)$ around its minima, we get (3.2.22).

□

Proposition 3.2.3 (Critical points). *For any $\beta > \beta_{c,R}$ there is an interval $(\lambda_-(\beta,R), \lambda_+(\beta,R))$ containing $\lambda(\beta,R)$ and for any λ in the interval $\phi_{\beta,\lambda,R}(\cdot)$ it has two local minima $\rho_{\beta,\lambda,R,\pm}$ which are differentiable functions of λ and $\frac{d}{d\lambda}(\phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,+}) - \phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,-})) = \rho_{\beta,\lambda,R,-} - \rho_{\beta,\lambda,R,+} < 0$. For all $\beta > \beta_{c,R}$,*

$$\frac{d}{d\rho} K_{\beta,\lambda(\beta,R),R}(\rho)\Big|_{\rho=\rho_{\beta,R,\pm}} \equiv K'_{\beta,\lambda(\beta,R),R}(\rho_{\beta,R,\pm}) < 1, \tag{3.2.24}$$

the condition (3.2.24) being equivalent to $\phi''_{\beta,\lambda(\beta),R}(\rho_{\beta,R,\pm}) > 0$. Moreover, there exists $\beta_{0,R} > \beta_{c,R}$ such that

$$K'_{\beta,\lambda(\beta,R),R}(\rho_{\beta,R,\pm}) > -1, \qquad \text{for all } \beta \in (\beta_{c,R}, \beta_{0,R}). \tag{3.2.25}$$

Proof.

Let $s = \rho_{\beta,\lambda,R,\pm}$ be the equilibrium densities, i.e. the solutions of

$$s = \exp\Big\{ -\beta e'_\lambda(s) - \frac{1}{\beta} f'_R(s) \Big\} := K_{\beta,\lambda,R}(s). \tag{3.2.26}$$

We can also write the function $K_{\beta,\lambda,R}$ as a composition of two functions:

$$\begin{aligned} &K_{\beta,\lambda,R}(s) = D^{-1} \circ \psi(s), \qquad \text{where:} \\ &D(s) = \frac{1}{\beta}\log s + \frac{1}{\beta} f'_R(s), \qquad \psi(s) = -e'_\lambda(s). \end{aligned} \tag{3.2.27}$$

To prove (3.2.27) it is enough to write the identity: $D(s) = \psi(s)$ and to use relation (3.2.26). Hence:

$$\Big|\frac{d}{ds} K_{\beta,\lambda(\beta,R),R}(s)\Big| = \Big|\Big(D'(\psi(s))\Big)^{-1} \frac{d\psi}{ds}\Big| = \Big|\beta\Big(-1 + \frac{s^2}{2}\Big)\Big[\frac{1}{s} + f''_R(s)\Big]^{-1}\Big| \tag{3.2.28}$$

and we want to prove the r.h.s. of (3.2.28) is < 1 for all $\beta \in (\beta_{c,R}, \beta_0)$ and $\lambda = \lambda(\beta,R)$.

From Proposition 3.2.1 we have:

$$\phi''_{\beta,\lambda,R}(s) = \frac{1}{\beta s} - 1 + \frac{s^2}{2} + \frac{1}{\beta} f''_R(s) > 0 \qquad \text{for } s = \rho_{\beta,\lambda,R,\pm} \tag{3.2.29}$$

and $\beta > \beta_{c,R}$. Hence, there is $\beta_0 > \beta_{c,R}$ such that for all $\beta \in (\beta_{c,R}, \beta_0)$ we have:

$$\beta s\Big(1 - \frac{s^2}{2}\Big) < 1 + s f''_R(s), \quad \Rightarrow \Big(1 - \frac{s^2}{2}\Big) < \Big|\frac{1 + s f''_R(s)}{\beta s}\Big|. \tag{3.2.30}$$

On the other side, we also have:

$$\phi''_{\beta,\lambda,R}(s) < \frac{2(1 + s f''_R(s))}{\beta s} \tag{3.2.31}$$

hence:

$$-1+\frac{s^2}{2}=\phi''_{\beta,\lambda,R}(s)-\frac{1+sf''_R(s)}{\beta s}<\frac{1+sf''_R(s)}{\beta s} \tag{3.2.32}$$

and therefore:

$$\Big(1-\frac{s^2}{2}\Big)>-\Big|\frac{1+sf''_R(s)}{\beta s}\Big|. \tag{3.2.33}$$

With estimates (3.2.30) and (3.2.33) we prove:

$$\Big|\frac{d}{ds}K_{\beta,\lambda(\beta,R),R}(s)\Big|_{s=\rho_{\beta,R,\pm}}\Big|<1. \tag{3.2.34}$$

□

4. Contours and Peierls bounds

4.1 Phase indicators and contours

In this Chapter we will mainly refer to Definition 2.2.1 for the notion of partition $\mathcal{D}^{(\ell)}$ on scale ℓ. First of all let us give some topological notions for the sets $\mathcal{D}^{(\ell)}$-measurable. Some of the theorems listed in the following sections are reported without proof, as they are the same as for the LMP model and we refer to [52], Section 10.5.

4.1.1 Connected sets

Two sets are connected if their closures have non empty intersection, thus two cubes with a common vertex are connected and connection here is what it is usually called $*$-connection when working in $\mathbb{Z}^d$. A maximal connected component B of a set A is a connected subset of A which is "maximal" in the sense that if $C \subset A$ is connected and $C \supset B$ then $C = B$. Any set is the disjoint union of its maximal connected components. A connected set A is simply connected if its complement is connected.

4.1.2 Outer and inner boundaries of a set

The outer boundary $\delta^{\ell}_{\text{out}}[\Lambda]$ of a $\mathcal{D}^{(\ell)}$-measurable region Λ is the union of all the cubes $C \in \mathcal{D}^{(\ell)}$ not in Λ but connected to Λ. The inner boundary $\delta^{\ell}_{\text{in}}[\Lambda]$ of Λ is the outer boundary of Λ^c. With reference to Fig. 4.6 in Section 4.2, if Λ is the region in the interior of the thick line, $\delta^{\ell}_{\text{out}}[\Lambda]$ is the region between the thick and the dashed external lines; $\delta^{\ell}_{\text{in}}[\Lambda]$ between the thick and the internal dashed lines.

Theorem 4.1.1. *Let Λ be a bounded, connected $\mathcal{D}^{(\ell)}$-measurable region. Then*

- *The maximal connected components of Λ^c are all bounded except one which is unbounded.*

- *Any bounded, maximal connected component of Λ^c is simply connected.*

- *If Λ is bounded and simply connected, then $\delta^\ell_{\text{out}}[\Lambda]$ and $\delta^\ell_{\text{in}}[\Lambda]$ are both connected.*

4.1.3 The scaling parameters

Let us introduce the following scaling parameters:

$$\ell_{1,\gamma} = \gamma^{-1/2}, \quad \ell_{2,\gamma} = \gamma^{-(1-\alpha)}, \quad \ell_{3,\gamma} = \gamma^{-(1+\alpha)}, \quad \zeta = \gamma^a \tag{4.1.1}$$

with $1 \gg \alpha \gg a > 0$ and, for simplicity, we suppose that $\ell_{2,\gamma}, \ell_{3,\gamma}, \gamma^{-1} \in \{2^n, n \in \mathbb{N}_+\}$. We can forget about the first scaling parameter for what follows and use only $\ell_{2,\gamma}, \ell_{3,\gamma}$ (it will be useful in Chapter 6). For simplicity from now on we can drop γ from the notation of the two scale parameters. For γ small, which is the regime we are interested in, $1 \ll \ell_2 \ll \gamma^{-1} \ll \ell_3$, thus in the cubes of $\mathcal{D}^{(\ell_2)}$ there are typically many particles to make statistics reliable; yet the cubes are so small that the interaction felt by particles in the same cube is approximately the same. Local closeness (of the particles density in cubes of $\mathcal{D}^{(\ell_2)}$) to $\rho_{\beta,R,+}$ (or to $\rho_{\beta,R,-}$) with accuracy ζ must extend to regions of size ℓ_3 to call the configuration in the plus (or minus) equilibrium.

4.1.4 The three phase indicators

To quantify the above considerations we introduce three phase indicators: $\eta^{(\zeta,\ell_2)}(q;r)$, $\theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ and $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$. The values of the three functions will describe with increasing degree of accuracy the local phase of the system. As we will see $\theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ and $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ depend on q only via $\eta^{(\zeta,\ell_2)}(q;\cdot)$, so that once the function $\eta^{(\zeta,\ell_2)}(q;\cdot)$ is specified the other two phase indicators are completely determined, we may thus use only $\eta^{(\zeta,\ell_2)}(m;\cdot)$, but the other indicators hopefully make the definitions more clear. Since in the whole sequel ζ and ℓ_2, ℓ_3 are fixed as in (4.1.1) we drop them from the notation writing simply η, θ and Θ.

Definition 4.1.2. Given $\beta \in (\beta_c, \beta_0)$, γ, ζ and α, we define a function $\eta(q;r)$, $r \in \mathbb{R}^d$, $q \in \mathcal{Q}$, with values $\pm 1, 0$ by setting $\eta(q;r) = \pm 1$ if

$$\left|\rho^{(\ell_2)}(q;r) - \rho_{\beta,R,\pm}\right| \leq \zeta, \qquad \text{where} \qquad \rho^{(\ell_2)}(q;r) = \frac{|q \cap C_r^{(\ell_2)}|}{\ell_2^d} \tag{4.1.2}$$

and $\eta(q;r) = 0$ otherwise.

If $\eta^{(\zeta,\ell_2)}(q;r) = \pm 1$ we call the cube $C_r^{(\ell_2)}$ in $\pm$ equilibrium, otherwise $\eta^{(\zeta,\ell_2)}(q;r) = 0$ and in $C_r^{(\ell_2)}$ the system is off equilibrium. Thus $\eta^{(\zeta,\ell_2)}(q;r)$ indicates the phase, or its absence, on the small scale ℓ_2.

Definition 4.1.3. $\theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ and $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ are defined in terms of $\eta^{(\zeta,\ell_2)}(q;\cdot)$ as:

- $\theta^{(\zeta,\ell_2,\ell_3)}(q;r) = \pm 1$ if $\eta^{(\zeta,\ell_2)}(q;r') = \pm 1$ constantly for all r' in $C_r^{(\ell_3)}$.

  $\theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 0$ otherwise.

- $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = \pm 1$ if $\eta^{(\zeta,\ell_2)}(q;r') = \pm 1$ constantly for all r' in $C_r^{(\ell_3)} \cup \delta_{\text{out}}^{\ell_3}[C_r^{(\ell_3)}]$.
  $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 0$ otherwise.

$\theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 1$ means that the phase indicated by $\eta^{(\zeta,\ell_2)}(q;\cdot)$ on the small scale ℓ_2 is constantly the plus phase in the whole [large] cube $C_r^{(\ell_3)}$. Points r close to the boundary of $C_r^{(\ell_3)}$ however may not be in local equilibrium but if $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 1$ then $\theta^{(\zeta,\ell_2,\ell_3)}(q;\cdot) = 1$ in all cubes of $\mathcal{D}^{(\ell_3)}$ connected to $C_r^{(\ell_3)}$ and since $\ell_3 > 1$ all points of $C_r^{(\ell_3)}$ are in local equilibrium. Thus, while $\eta^{(\zeta,\ell_2)}(q;r)$ and $\theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ are "local" in the sense that their values depend only on the restriction of q to $C_r^{(\ell_2)}$ and respectively $C_r^{(\ell_3)}$, this is no longer true for $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ which is "non local" as it depends on the restriction of q to $C_r^{(\ell_3)} \cup \delta_{\text{out}}^{\ell_3}[C_r^{(\ell_3)}]$.

Definition 4.1.4. A point r is correct (for the configuration q) if $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) \neq 0$ and called plus/minus (correct) if $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = \pm 1$. The plus/minus phases of q are the sets of its plus/minus correct points, they are mutually disconnected and the regions in between are "the spatial supports of the contours".

Definition 4.1.5. A contour of q is a pair $\Gamma = \big(\mathrm{sp}(\Gamma), \eta_\Gamma\big)$, where $\mathrm{sp}(\Gamma)$, "the spatial support of Γ", is a maximal connected component of $\{r \in \mathbb{R}^d : \Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 0\}$ and η_Γ is the restriction to $\mathrm{sp}(\Gamma)$ of $\eta^{(\zeta,\ell_2)}(q;\cdot)$.

$\Gamma = \big(\mathrm{sp}(\Gamma), \eta_\Gamma\big)$ is "an abstract contour" if it is a contour of some q.

4.1.5 Geometry of contours

In the sequel we restrict to bounded contours, meaning that their spatial support is bounded. Definitions and properties stated in this paragraph are exemplified and visualized in Section 4.2.

The exterior, $\mathrm{ext}(\Gamma)$, of Γ is the unbounded, maximal connected component of $\mathrm{sp}(\Gamma)^c$. The interior is the set $\mathrm{int}(\Gamma) = \mathrm{sp}(\Gamma)^c \setminus \mathrm{ext}(\Gamma)$; we denote by $\mathrm{int}_i(\Gamma)$ the maximal connected components of $\mathrm{int}(\Gamma)$. Let

$$c(\Gamma) = \mathrm{sp}(\Gamma) \cup \mathrm{int}(\Gamma), \tag{4.1.3}$$

then, by Theorem 4.1.1, $\mathrm{int}_i(\Gamma)$ and $c(\Gamma)$ are both simply connected.

The outer boundaries of Γ are the sets

$$A(\Gamma) := \delta^{\ell_3}_{\mathrm{out}}[\mathrm{sp}(\Gamma)] \cap \mathrm{int}(\Gamma),\ A_{\mathrm{ext}}(\Gamma) := \delta^{\ell_3}_{\mathrm{out}}[c(\Gamma)] \tag{4.1.4}$$

We will also call $A_i(\Gamma) = A(\Gamma) \cap \mathrm{int}_i(\Gamma)$. The inner boundaries of Γ are the sets

$$B_i(\Gamma) = \delta^{\ell_3}_{\mathrm{out}}[A_i(\Gamma)] \cap \mathrm{sp}(\Gamma),\ \ B_{\mathrm{ext}}(\Gamma) = \delta^{\ell_3}_{\mathrm{in}}[c(\Gamma)] \tag{4.1.5}$$

An easy but important consequence of the above definitions is the following theorem (the reader may check its validity in the case of Figure 4.1).

Theorem 4.1.6. *If q has a bounded contour Γ, $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ is a non zero constant on any $A_i(\Gamma)$ and $A_{\mathrm{ext}}(\Gamma)$.*

Proof. $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) \neq 0$ for any $r \in A$, A either $A_{\mathrm{ext}}(\Gamma)$ or $A_i(\Gamma)$, because A is outside $\mathrm{sp}(\Gamma)$ and being connected to $\mathrm{sp}(\Gamma)$ cannot intersect the spatial support of any other contour. We will next prove that $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ is constant and non zero on A. Since A is connected (by Theorem 4.1.1) it will suffice to prove that if two cubes C and C' of $\mathcal{D}^{(\ell_3)}$, are connected

to each other and both in A, then $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r)$ has a constant value on the union $C \cup C'$. Indeed, suppose for instance that $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r) = 1$, $r \in C$; then $\theta^{(\zeta,\ell_2,\ell_3)}(q;r') = 1$ for all $r' \in \delta^{\ell_3}_{\rm out}[C]$ and hence for all $r' \in C'$, thus $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r') = 1$ on C' (as we already know that $\Theta^{(\zeta,\ell_2,\ell_3)}(q;r') \neq 0$ on C'). □

Definition 4.1.7. Γ is a plus, minus, contour if $\Theta(q;r) = \pm 1$ on $A_{\rm ext}(\Gamma)$.

We add a superscript $\pm$ to $A_i(\Gamma)$ writing $A^{\pm}_i(\Gamma)$, $i = 1,.,n_{\pm}$, with $\pm$ chosen so that $\eta_\Gamma = 1$ on $B^+_i(\Gamma)$ and $\eta_\Gamma = -1$ on $B^-_i(\Gamma)$. We also write

$$A^+(\Gamma) = \bigcup_{i=1}^{n_+} A^+_i(\Gamma), \quad A^-(\Gamma) = \bigcup_{i=1}^{n_-} A^-_i(\Gamma) \tag{4.1.6}$$

$\mathrm{int}^{\pm}_i(\Gamma)$ if $\mathrm{int}_i(\Gamma)$ contains $A^{\pm}_i(\Gamma)$,

$$\mathrm{int}^{\pm}(\Gamma) = \bigcup_{i=1}^{n_{\pm}} \mathrm{int}^{\pm}_i(\Gamma), \qquad \mathrm{int}(\Gamma) = \mathrm{int}^+(\Gamma) \cup \mathrm{int}^-(\Gamma) \tag{4.1.7}$$

4.2 Pictures and examples

To visualize the notions introduced in the last section we fix a configuration, compute the phase indicators and draw pictures of the corresponding plus/minus correct regions and contours. Let $S \subset \mathbb{R}^2$ be the coordinate square with bottom left corner the origin and side $21\ell_3$ In Figure 4.1 S is partitioned into 21×21 squares of side ℓ_3, we will denote by C_x, $x \in \mathbb{Z}^2$ the square of side ℓ_3 with bottom left corner $x\ell_3$, x the "lattice coordinate" of $x\ell_3$. We suppose that $\theta^{(\ell_2,\ell_3,\zeta)} \equiv 1$ outside S while the values in S are as in Fig. 4.1.
$\Theta^{(\ell_2,\ell_3,\zeta)}$ is then completely determined. Its values on S are reported in Fig. 4.2, where the plus (marked by $+$), minus (marked by $-$) correct regions and the support of the contours (marked by 0) are separated by the discontinuity lines of $\Theta^{(\ell_2,\ell_3,\zeta)}$. Notice that $\Theta^{(\ell_2,\ell_3,\zeta)} \equiv 1$ on $\delta^{\ell_3}_{\rm in}[S]$, $\delta^{\ell_3}_{\rm in}[S]$.

Thus the configuration we are considering is one of those which arise when computing the plus diluted partition function in S, see next section.

The configuration of Fig. 4.1 has three contours, Γ_1, Γ_2 and Γ_3: $\mathrm{sp}(\Gamma_1)$ is the region between the discontinuity lines of Fig 4.2 which contains $C_{(17,3)}$, $\mathrm{sp}(\Gamma_2)$ the one which contains $C_{(16,17)}$ and $\mathrm{sp}(\Gamma_3)$ the one containing $C_{(5,11)}$ (i.e. the squares marked 0 in Fig 4.2). Γ_1 and

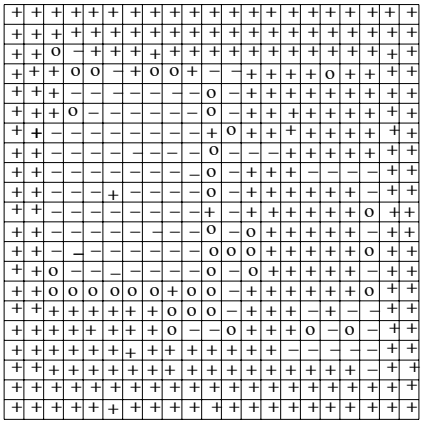

Figure 4.1: Values of $\theta^{(\ell_2,\ell_3,\zeta)}$ on S.

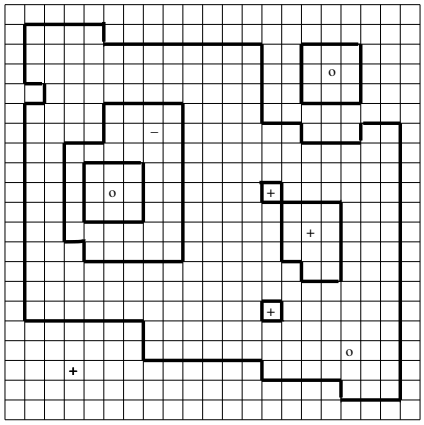

Figure 4.2: Discontinuity lines of $\Theta^{(\ell_2,\ell_3,\zeta)}$ and regions where it is $+$, $-$ and 0.

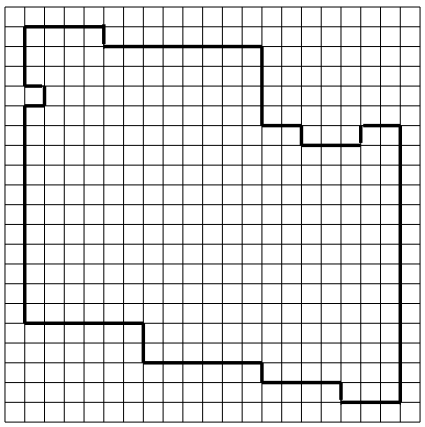

Figure 4.3: The set $c(\Gamma_1)$.

Γ_2 are both external, plus contours, Γ_3 is a minus contour. $c(\Gamma_1)$ is drawn in Fig. 4.3 Γ_1 has three internal parts, $\text{int}^-(\Gamma_1)$, $\text{int}_1^+(\Gamma_1)$ and $\text{int}_2^+(\Gamma_1)$, see Fig. 4.4, $\text{int}^-(\Gamma_1)$ has a contour inside, Γ_3, see Fig. 4.5. The sets $B_{\text{ext}}(\Gamma_1)$ and $A_{\text{ext}}(\Gamma_1)$ are drawn in Fig. 4.6, the sets $B_i(\Gamma_1)$ and $A_i(\Gamma_1)$ are drawn in Fig. 4.7, observe that $B_2(\Gamma_1) \cap B_3(\Gamma_1) \neq \emptyset$.

Values of $\eta^{(\zeta,\ell_2)}$ are reported by blowing up portions of S, see Fig. 4.8 and Fig. 4.9.

4.3 Diluted Gibbs measures

We will prove phase transitions by introducing two classes of boundary conditions and showing that they give rise in the thermodynamic limit to two distinct measures. We consider "diluted Gibbs measures", their definition involves special boundary conditions as well as constraints on the structure of the configurations close to the boundaries. The constraints are such that contours cannot reach the boundaries, for this reason the partition functions are called diluted.

Definition 4.3.1. A configuration $\bar{q}$ is a *plus boundary condition* relative to a bounded $\mathcal{D}^{(\ell_3)}$-measurable region Λ if there is a configuration $q^+ \in \mathcal{Q}_+$, i.e. such that $\eta(q^+; r) = 1$ for all $r \in \mathbb{R}^d$ and $\bar{q}$ and q^+ are the same in the region $\{r \in \Lambda^c : \text{dist}\,(r, \Lambda) \leq 2\gamma^{-1}\}$.

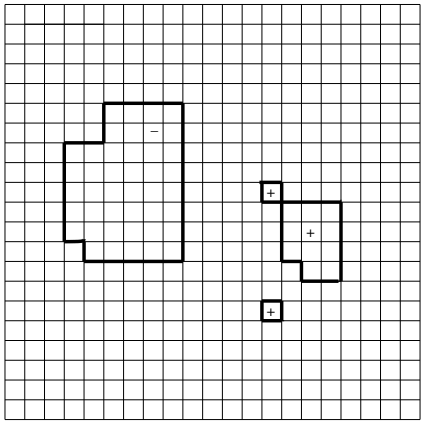

Figure 4.4: $\mathrm{int}^-(\Gamma_1)$ is the region with $-$, $\mathrm{int}_1^+(\Gamma_1)$ is the $+$ region on top left and $\mathrm{int}_2^+(\Gamma_1)$ the one consisting of a single square.

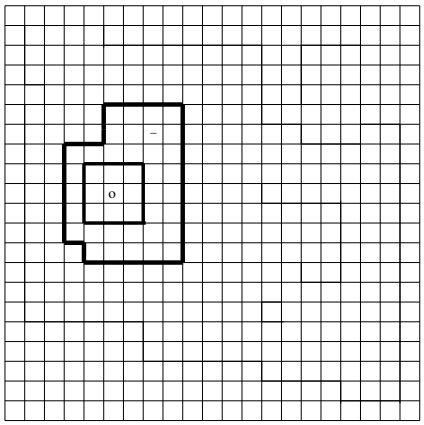

Figure 4.5: $\mathrm{int}^-(\Gamma_1)$ with inside $\mathrm{sp}(\Gamma_3)$.

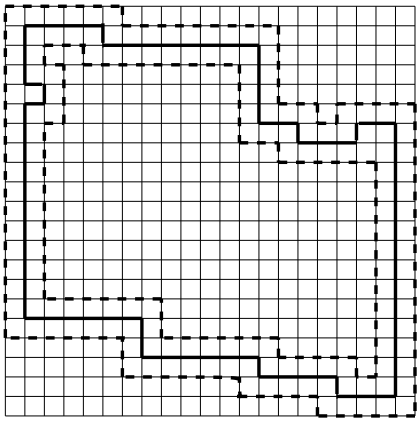

Figure 4.6: $A_{\text{ext}}(\Gamma_1) = \delta^{\ell_3}_{\text{out}}[c(\Gamma)]$ is the region between the external dashed and the thick line; $B_{\text{ext}}(\Gamma_1) = \delta^{\ell_3}_{\text{in}}[c(\Gamma)]$ is the region between the internal dashed and the thick line.

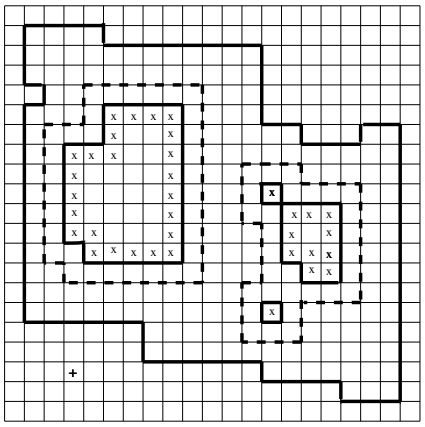

Figure 4.7: $A_i(\Gamma_1) = \delta^{\ell_3}_{\text{in}}[\text{int}_i(\Gamma_1)]$ are the regions made of the x marked squares contained in $\text{int}_i(\Gamma_1)$; $B_i(\Gamma_1) = \delta^{\ell_3}_{\text{out}}[\text{int}_i(\Gamma_1)]$ are the region between the dashed lines and the thick lines inside.

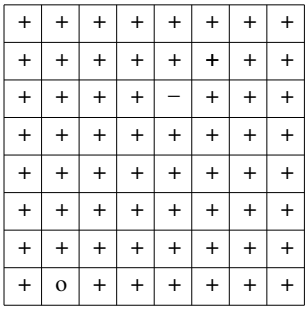

Figure 4.8: Blow up of the squares $C_{(2,6)}$ where $\theta^{(\zeta,\ell_2,\ell_3)} = 0$ divided into squares of side ℓ_2 where the values of $\eta^{(\zeta,\ell_2)}$ are reported.

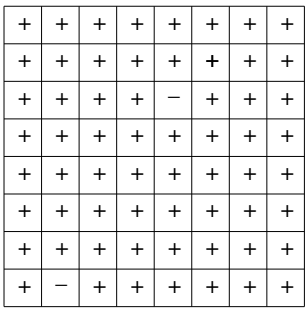

Figure 4.9: Blow up of the squares $C_{(2,7)}$ where $\theta^{(\zeta,\ell_2,\ell_3)} = 0$ divided into squares of side ℓ_2 where the values of $\eta^{(\zeta,\ell_2)}$ are reported.

Definition 4.3.2. The plus diluted Gibbs measure in a bounded $\mathcal{D}^{(\ell_3)}$-measurable region Λ with plus boundary conditions $\bar{q}$ is

$$\mu^{\Lambda,+}_{\gamma,\beta,R,\lambda}(dq_\Lambda|\bar{q}) := \frac{1}{Z^+_{\gamma,\beta,R,\lambda}(\Lambda|\bar{q})} e^{-\beta H_{\gamma,R,\lambda}(q_\Lambda|\bar{q}_{\Lambda^c})} \mathbf{1}_{\Theta((q_\Lambda+q^+_{\Lambda^c});r)=1\ r\in\delta^{\ell_3}_{\rm in}[\Lambda]} \nu^\Lambda(dq_\Lambda) \tag{4.3.1}$$

where $q^+ \in \mathcal{Q}_+$, ν_Λ is defined in Section 1.1 and

$$Z^+_{\gamma,\beta,R,\lambda}(\Lambda|\bar{q}) = \int_{\mathcal{Q}^\Lambda} e^{-\beta H_{\gamma,R,\lambda}(q_\Lambda|\bar{q}_{\Lambda^c})} \mathbf{1}_{\Theta((q_\Lambda+q^+_{\Lambda^c});r)=1\ r\in\delta^{\ell_3}_{\rm in}[\Lambda]} \nu^\Lambda(dq_\Lambda) \tag{4.3.2}$$

is the plus diluted partition function.

Definition 4.3.3. Minus boundary conditions, minus diluted Gibbs measures and minus diluted partition functions are defined analogously.

4.4 Weight of a contour

We refer again to Section 4.1 for notation and properties of contours, in particular for the definition of the sets $\mathrm{sp}(\Gamma)$, $c(\Gamma)$, $\mathrm{int}(\Gamma)$, $A_{\rm ext}(\Gamma)$, $A^\pm(\Gamma)$ and $\mathrm{int}^\pm(\Gamma)$. All these sets in agreement with the notation are independent of which configuration q gives rise to the contour Γ, namely which $q \in \mathcal{Q}(\Gamma)$, where

$$\mathcal{Q}(\Gamma) = \Big\{ q : \Gamma \text{ is a contour for } q \Big\} \tag{4.4.1}$$

Recall finally that Γ is a plus or minus a contour if $\Theta(q;r) = \pm 1$ on $A_{\rm ext}(\Gamma) = \delta^{\ell_+,\gamma}_{\rm out}[c(\Gamma)]$. Given a plus contour Γ and a plus boundary condition q^+ for $c(\Gamma)$, we define the weight $W^+_{\gamma,R,\lambda}(\Gamma;\bar{q})$ of Γ as equal to

$$\frac{\mu^{c(\Gamma)}_{\gamma,\beta,R,\lambda,q^+}\Big(\eta(q_{c(\Gamma)};r) = \eta_\Gamma(r), r\in\mathrm{sp}(\Gamma);\ \Theta(q_{c(\Gamma)};r) = \pm 1, r\in A^\pm(\Gamma)\Big)}{\mu^{c(\Gamma)}_{\gamma,\beta,R,\lambda,q^+}\Big(\eta(q_{c(\Gamma)};r) = 1, r\in\mathrm{sp}(\Gamma);\ \Theta(q_{c(\Gamma)};r) = 1, r\in A^\pm(\Gamma)\Big)} \tag{4.4.2}$$

where the measure $\mu^{c(\Gamma)}_{\gamma,\beta,R,\lambda,q^+}$ has been defined in Section 1.3. Thus the numerator is the probability of the contour Γ conditioned to the outside of $\mathrm{sp}(\Gamma)$ while the denominator is the probability with the same conditioning that the contour Γ is absent and replaced by the plus configurations. The weight $W^-_{\gamma,\lambda}(\Gamma;q^-)$ of a minus contour Γ is defined analogously. Since the range of the interaction is $\le 2\gamma^{-1}$, we obviously have:

Lemma 4.4.1. *The weights* $W^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm})$ *depend only on the restriction of* $q^{\pm}$ *to* $\{r \in c(\Gamma)^c : \mathrm{dist}(r, c(\Gamma)) \leq 2\gamma^{-1}\}$.

4.5 Peierls bounds

The main technical point in the proof of Theorem 2.1.1 is to prove that contours are improbable. In particular we will prove Peierls estimates which prove that the probability of a contour decays exponentially with its volume.

Let us introduce the notation:

$$N_\Gamma = \frac{|\mathrm{sp}(\Gamma)|}{\ell_3^d} \tag{4.5.1}$$

to indicate the number of cubes of the partition $\mathcal{D}^{(\ell_3)}$ contained in $\mathrm{sp}(\Gamma)$. Then we want to prove the following:

Theorem 4.5.1. *It exists* R_0 *such that for any* $R \leq R_0$ *and any* $\beta \in (\beta_{c,R}, \beta_{0,R})$ *there are* c $\gamma_{\beta,R} > 0$ *and* $\lambda_{\beta,\gamma,R}$, $\gamma \leq \gamma_{\beta,R}$, *so that for any* $\pm$ *contour* Γ *and any* $\pm$ *boundary condition* $q^{\pm}$ *relative to* $c(\Gamma)$,

$$W^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm}) \leq \exp\Big\{ - \beta c\, (\zeta^2 \ell_2^d)\, N_\Gamma \Big\} \tag{4.5.2}$$

As a corollary of Theorem 4.5.1 we have:

Theorem 4.5.2. *It exists* R_0 *such that for any* $R \leq R_0$ *and any* $\beta \in (\beta_c, \beta_0)$ *let* c, $\gamma_{\beta,R}$, γ *and* $\lambda_{\beta,\gamma,R}$ *as in Theorem 4.5.1, then for any bounded, simply connected,* $\mathcal{D}^{(\ell_3)}$ *measurable region* Λ, *any plus/ minus boundary condition* $q^{\pm}$ *and any* $r \in \Lambda$,

$$\mu^{\Lambda,\pm}_{\gamma,\beta,R,\lambda_{\beta,\gamma,R},q^{\pm}}(\{\Theta(q;r) = \pm 1\}) \geq \; 1 \; - \; \exp\Big\{ - \beta \frac{c}{2}\, (\zeta^2 \ell_2^d) \Big\}. \tag{4.5.3}$$

Theorem 4.5.2 implies that for any $R \leq R_0$ and γ small enough (chosen according to R) the difference between the diluted Gibbs measures $\mu^{\Lambda,+}_{\gamma,\beta,R,\lambda_{\beta,\gamma,R},q^{\pm}}(dq)$ and $\mu^{\Lambda,-}_{\gamma,\beta,R,\lambda_{\beta,\gamma,R},q^{\pm}}(dq)$ survives in the thermodynamic limit $\Lambda \nearrow \mathbb{R}^d$ and a phase transition then occurs. The implications of Theorem 4.5.2 on the structure of the DLR measures will not be discussed in this work and we give as reference [52], Chapter 12, where it is proved for the LMP model

that indeed there are a plus and a minus distinct DLR measures, that they are translational invariant and that any translational invariant measure is a convex combination of the two. We believe however the structure of these properties is not affected by the addition of the hard core interaction (see remark at the end of Section 2.1).

We end this section by writing the ratio (4.4.2) of probabilities in the definition of the weight of a contour as a ratio of two partition functions. Indeed referring for the sake of definiteness to a plus contour Γ we have

$$W^+_{\gamma,R,\lambda}(\Gamma;q^+) = \frac{\mathcal{N}^+_{\gamma,R,\lambda}(\Gamma,q^+)}{\mathcal{D}^+_{\gamma,R,\lambda}(\Gamma,q^+)} \tag{4.5.4}$$

where:

$$\begin{aligned}\mathcal{N}^+_{\gamma,R,\lambda}(\Gamma,q^+) &= \int_{q_{\mathrm{sp}(\Gamma)}:\eta(q_{\mathrm{sp}(\Gamma)};r)=\eta_\Gamma(r),r\in\mathrm{sp}(\Gamma)} e^{-\beta H_{\gamma,R,\lambda,\mathrm{sp}(\Gamma)}(q_{\mathrm{sp}(\Gamma)}|q^+_{A_{\mathrm{ext}}})} \\ &\times\ Z^-_{\gamma,\beta,R,\lambda}(\mathrm{int}^-(\Gamma)|q_{\mathrm{sp}(\Gamma)})\, Z^+_{\gamma,\beta,R,\lambda}(\mathrm{int}^+(\Gamma)|q_{\mathrm{sp}(\Gamma)})\end{aligned} \tag{4.5.5}$$

$$\begin{aligned}\mathcal{D}^+_{\gamma,R,\lambda}(\Gamma,q^+) &= \int_{q_{\mathrm{sp}(\Gamma)}:\eta(q_{\mathrm{sp}(\Gamma)};r)=1,r\in\mathrm{sp}(\Gamma)} e^{-\beta H_{\gamma,R,\lambda,\mathrm{sp}(\Gamma)}(q_{\mathrm{sp}(\Gamma)}|q^+_{A_{\mathrm{ext}}})} \\ &\times\ Z^+_{\gamma,\beta,R,\lambda}(\mathrm{int}^-(\Gamma)|q_{\mathrm{sp}(\Gamma)})\, Z^+_{\gamma,\beta,R,\lambda}(\mathrm{int}^+(\Gamma)|q_{\mathrm{sp}(\Gamma)}).\end{aligned} \tag{4.5.6}$$

4.6 Cutoff weights

The aim of this section is to introduce the notion of cutoff weights as required in the approach proposed by Milos Zahradnik to the P-S theory (see Section 2.3).

The idea is that in the picture of P-S large contours are less likely to be observed. If to implement this idea one fix a constraint which literally forbids contours larger than some given value then also the stable phase would be modified, as any contour no matter how large has anyways a positive probability to occur. It is thus convenient to allow for all contours, but to give them a weight which is the true one only if the true weight is small, i.e. as in the Peierls bounds; otherwise it is given a fictitiously small value. In the stable phase the cutoff (if properly chosen) is not reached and the state is not modified by this procedure.

We will discuss the plus case, the minus one is completely analogous and not considered explicitly. The starting point is to write the plus diluted partition functions as partition

functions on the *restricted ensembles*

$$\mathcal{Q}_+^\Lambda = \{q : \eta(q;r) = 1, r \in \Lambda\}. \tag{4.6.1}$$

4.6.1 The set of all contours as a graph

- We denote by $\{\Gamma\}^+$ the collection of all plus contours with bounded spatial support. Recalling that two sets are connected if their closures have non empty intersection, we draw edges joining any two elements Γ and Γ' of $\{\Gamma\}^+$ if $\mathrm{sp}(\Gamma)$ is connected to $\mathrm{sp}(\Gamma')$, thus equipping $\{\Gamma\}^+$ with a graph structure. (Notice that two connected contours cannot appear in a configuration q).

- $\mathcal{B}^+$ is the space of all finite subsets of $\{\Gamma\}^+$ made of elements which are mutually disconnected. Elements of $\mathcal{B}^+$ are denoted by $\underline{\Gamma}$.

- $\mathcal{B}^{+,\mathrm{ext}}$ is the subset of $\mathcal{B}^+$ made of mutually "external" contours: namely if $\underline{\Gamma} \in \mathcal{B}^{+,\mathrm{ext}}$ for any $\Gamma \neq \Gamma'$ in $\underline{\Gamma}$, $c(\Gamma) \cap c(\Gamma') = \emptyset$.

- $\{\Gamma\}_\Lambda^+$, $\mathcal{B}_\Lambda^+$ and $\mathcal{B}_\Lambda^{+,\mathrm{ext}}$, (by default Λ denotes a bounded $\mathcal{D}^{(\ell_+,\gamma)}$-measurable region) denote the previous quantities with the additional restriction that all contours have spatial support in $\Lambda \setminus \delta_{\mathrm{in}}^{\ell_+,\gamma}[\Lambda]$, i.e. [the spatial support of the] contours is not connected to Λ^c.

- Let $W^+_{\gamma,R,\lambda}(\emptyset, q^+) = 1$ and let us write

$$W^+_{\gamma,R,\lambda}(\underline{\Gamma}, q^+) = \prod_{\Gamma \in \underline{\Gamma}} W^+_{\gamma,R,\lambda}(\Gamma, q^+), \quad \underline{\Gamma} \in \mathcal{B}^+,\ q^+ \in \mathcal{X}^+ \tag{4.6.2}$$

- If $\underline{\Gamma} \in \mathcal{B}^{+,\mathrm{ext}}$ we write

$$\mathrm{int}(\underline{\Gamma}) = \bigcup_{\Gamma \in \underline{\Gamma}} \mathrm{int}(\Gamma), \quad \mathrm{ext}(\underline{\Gamma}) = \bigcap_{\Gamma \in \underline{\Gamma}} \mathrm{ext}(\Gamma). \tag{4.6.3}$$

4.6.2 Partition functions in restricted ensembles

A central point in P-S theory is a change of measure. The idea is to introduce a new Gibbs measure simpler than the original one, but which gives the same pro

The diluted partition function in a region Λ can be written as a partition function in $\mathcal{Q}_+^\Lambda = \{q \in \mathcal{X}_\Lambda : \eta(q,r) = 1, r \in \Lambda\}$:

Theorem 4.6.1. *For any bounded $\mathcal{D}^{(\ell_3)}$-measurable region Λ and any plus b.c. q^+,*

$$Z^+_{\gamma,\beta,R,\lambda}(\Lambda|q^+) = \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} \int_{q_\Lambda\in\mathcal{Q}^\Lambda_+} W^+_{\gamma,R,\lambda}(\underline{\Gamma}, q_\Lambda)\, e^{-\beta H_{\gamma,R,\lambda}(q_\Lambda|q^+_{\Lambda^c})} \tag{4.6.4}$$

where $q^+_{\Lambda^c}$ is made of all the particles of q^+ which are in Λ^c.

4.6.3 Fictitious weights

Throughout the sequel $\hat{W}^\pm_{\gamma,R,\lambda}(\Gamma; q^\pm)$, $\Gamma \in \{\Gamma\}^+$, $q^\pm$ $\pm$ boundary conditions for $c(\Gamma)$, denote strictly positive numbers which, like the true weights, see Lemma 4.4.1, depend only on the restriction of $q^\pm$ to $\{r \in c(\Gamma)^c : \mathrm{dist}(r, c(\Gamma)) \le 2\gamma^{-1}\}$. Recalling (4.6.4) we then define for any bounded, simply connected $\mathcal{D}^{(\ell_3)}$-measurable region Λ and any $\pm$ b.c. $q^\pm$ for $c(\Gamma)$,

$$\hat{Z}^\pm_{\gamma,\beta,R,\lambda}(\Lambda|q^\pm) = \sum_{\underline{\Gamma}\in\mathcal{B}^\pm_\Lambda} \int_{\mathcal{Q}^\Lambda_+} \hat{W}^\pm_{\gamma,R,\lambda}(\underline{\Gamma}, q_\Lambda)\, e^{-\beta H_{\gamma,R,\lambda}(q_\Lambda|q^\pm_{\Lambda^c})} \tag{4.6.5}$$

(the integral over the free measure $\nu^\Lambda(dq_\Lambda)$) where, if $\underline{\Gamma} = (\Gamma_1, .., \Gamma_n)$,

$$\hat{W}^\pm_{\gamma,R,\lambda}(\underline{\Gamma}, q) = \prod_{i=1}^n \hat{W}^\pm_{\gamma,R,\lambda}(\Gamma_i, q), \quad \underline{\Gamma} = (\Gamma_1, .., \Gamma_n). \tag{4.6.6}$$

By (4.6.4) if the weights of the contours are the true ones, then $\hat{Z}^\pm_{\gamma,\beta,R,\lambda}(\Lambda|q^\pm)$ are the plus and minus diluted partition functions. We should regard (4.6.5) as the definition of $\hat{Z}^\pm_{\gamma,\beta,R,\lambda}(\Lambda|q^\pm)$ as a function of the variables $\{\hat{W}^\pm_{\gamma,R,\lambda}(\Gamma; q)$, Γ a plus, respectively, minus contour, whose spatial support is in $\Lambda\setminus\delta^{\ell_+,\gamma}_{\mathrm{in}}[\Lambda]\}$. We next define $\hat{N}^+_{\gamma,R,\lambda}(\Gamma, q^+)$ and $\hat{D}^+_{\gamma,R,\lambda}(\Gamma, q^+)$ as functions of the weights $\hat{W}^\pm_{\gamma,R,\lambda}(\cdot;\cdot)$ as:

$$\begin{aligned} \hat{N}^+_{\gamma,R,\lambda}(\Gamma, q^+) &= \int_{q_{\mathrm{sp}(\Gamma)}:\eta(q_{\mathrm{sp}(\Gamma)};r)=\eta_\Gamma(r), r\in\mathrm{sp}(\Gamma)} e^{-\beta H_{\gamma,R,\lambda,\mathrm{sp}(\Gamma)}(q_{\mathrm{sp}(\Gamma)}|q^+_{A_{\mathrm{ext}}})} \\ &\times \hat{Z}^-_{\gamma,\beta,R,\lambda}(\mathrm{int}^-(\Gamma)|q_{\mathrm{sp}(\Gamma)})\, \hat{Z}^+_{\gamma,\beta,R,\lambda}(\mathrm{int}^+(\Gamma)|q_{\mathrm{sp}(\Gamma)}) \end{aligned} \tag{4.6.7}$$

$$\begin{aligned}\hat{D}^{+}_{\gamma,R,\lambda}(\Gamma,q^{+}) &= \int_{q_{\mathrm{sp}(\Gamma)}:\eta(q_{\mathrm{sp}(\Gamma)};r)=1,r\in\mathrm{sp}(\Gamma)} e^{-\beta H_{\gamma,R,\lambda,\mathrm{sp}(\Gamma)}(q_{\mathrm{sp}(\Gamma)}|q^{+}_{A_{\mathrm{ext}}})} \\ &\times \hat{Z}^{+}_{\gamma,\beta,R,\lambda}(\mathrm{int}^{-}(\Gamma)|q_{\mathrm{sp}(\Gamma)})\,\hat{Z}^{+}_{\gamma,\beta,R,\lambda}(\mathrm{int}^{+}(\Gamma)|q_{\mathrm{sp}(\Gamma)}). \qquad (4.6.8)\end{aligned}$$

$\hat{N}^{-}_{\gamma,R,\lambda}(\Gamma,q^{-})$, $\hat{D}^{-}_{\gamma,R,\lambda}(\Gamma,q^{-})$ are defined analogously. As said they are regarded as functions of the weights $\hat{W}^{\pm}_{\gamma,R,\lambda}(\cdot;\cdot)$ and like the true weights, also $\hat{N}^{\pm}_{\gamma,R,\lambda}(\Gamma,q^{\pm})$ and $\hat{D}^{\pm}_{\gamma,R,\lambda}(\Gamma,q^{\pm})$ depend only on the restriction of $q^{\pm}$ to $\{r\in c(\Gamma)^{c}:\mathrm{dist}(r,c(\Gamma))\leq 2\gamma^{-1}\}$.

4.6.4 The cutoff weights

We are now ready for the final step, the choice of the weights. We have:

Theorem 4.6.2. *There is a unique choice of $\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm})$ such that for any $\pm$ contour Γ and any $q^{\pm}$,*

$$\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm})=\min\Big\{\frac{\hat{N}^{\pm}_{\gamma,R,\lambda}(\Gamma,q)}{\hat{D}^{\pm}_{\gamma,R,\lambda}(\Gamma,q)},e^{-\beta\frac{c}{100}(\zeta^{2}\ell^{d}_{-,\gamma})N_{\Gamma}}\Big\} \qquad (4.6.9)$$

with $\hat{N}^{\pm}_{\gamma,R,\lambda}(\Gamma,q)$ and $\hat{D}^{\pm}_{\gamma,R,\lambda}(\Gamma,q)$ as in (4.6.7)-(4.6.6) *and* (4.6.5) *and hence dependent on the weights $\hat{W}^{\pm}_{\gamma,R,\lambda}(\cdot;\cdot)$. Moreover the weights $\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm})$ depend only on the restriction of $q^{\pm}$ to $\{r\in c(\Gamma)^{c}:\mathrm{dist}(r,c(\Gamma))\leq 2\gamma^{-1}\}$.*

4.6.5 Recovering the true weights

As anticipated in the introduction to the section, an important point of the scheme is that in within the context of "the fictitious cutoff weights model" we can decide whether it is the true one.

Theorem 4.6.3. *Suppose that for any plus or minus contour Γ*

$$\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm})<e^{-\beta\frac{c}{100}(\zeta^{2}\ell^{d}_{-,\gamma})N_{\Gamma}} \qquad (4.6.10)$$

then

$$\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm})=W^{\pm}_{\gamma,R,\lambda}(\Gamma;q^{\pm}),\quad \hat{Z}^{\pm}_{\gamma,\beta,R,\lambda}(\Lambda|q^{\pm})=Z^{\pm}_{\gamma,\beta,R,\lambda}(\Lambda|q^{\pm}) \qquad (4.6.11)$$

where $W^{\pm}_{\gamma,R,\lambda}(\Gamma;q_{A^{\pm}})$ are the true weights, defined in (4.5.4) *and $Z^{\pm}_{\gamma,\beta,R,\lambda}(\Lambda|q^{\pm})$ the plus minus diluted partition functions.*

Corollary 4.6.4. *To prove Theorem 4.5.1 it suffices to show that for $R < R_0$ and γ small enough, for all Γ*

$$\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm}) = \frac{\hat{N}^{\pm}_{\gamma,R,\lambda}(\Gamma, q^{\pm})}{\hat{D}^{\pm}_{\gamma,R,\lambda}(\Gamma, q^{\pm})} \leq e^{-\beta c\,(\zeta^2 \ell^d_{-,\gamma})\,N_\Gamma} \tag{4.6.12}$$

Proof. By Theorem 4.6.3 if (4.6.12) holds then

$$W^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm}) = \hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm}) \leq e^{-\beta c\,(\zeta^2 \ell^d_{-,\gamma})\,N_\Gamma} \tag{4.6.13}$$

□

5. Outline of the proof

In this chapter we shall present a sketch of the proof of Theorem 4.5.1. We fix the inverse temperature β in the open interval $(\beta_{c,R}, \beta_{0,R})$ and since it is fixed we shall often drop it from the notation. By Corollary 4.6.4 it is enough to prove that for γ small (4.6.12) holds for all contours Γ. Hence we can proceed using the cutoff weights $\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm})$ instead of the original ones $W^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm})$.

The main difficulty in proving (4.5.2) is that both $\hat{N}^{\pm}_{\gamma,R,\lambda}(\Gamma, q^{\pm})$ and $\hat{D}^{\pm}_{\gamma,R,\lambda}(\Gamma, q^{\pm})$ are defined in terms of expressions which involve not only the support of Γ but also its whole interior. They are therefore "bulk quantities" while the desired bound involves only the volume of the support of Γ, which for some contours, at least, is a "surface quantity". The main issue here is to find cancellations of the bulk terms between $\hat{N}^{\pm}_{\gamma,R,\lambda}$ and $\hat{D}^{\pm}_{\gamma,R,\lambda}$. This is easy when special symmetries allow to relate the $+$ and $-$ ensembles, as in the ferromagnetic Ising model. Such simplifications are not present here and this is the main issue which makes continuum models difficult to be studied.

The analysis leading to (4.6.12) has been rather general and it has not used in any essential way the specific features of the LMP-hc model, which instead enter massively in its proof.

The proof immediately follows from the three basic estimates below, which can be regarded as the three main steps.

5.1 Step 1: equality of pressures

Looking at the "bulk terms" involving the interior components of Γ one can see that up to exponential surface correction terms, one can approximate the partition function by:

$$\hat{Z}^{\pm}_{\gamma,\beta,R,\lambda}(\mathrm{int}^{\pm}(\Gamma)|q_{\mathrm{sp}(\Gamma)}) \approx e^{\beta P^{\pm}_{\gamma,R,\lambda}|\mathrm{int}^{\pm}(\Gamma)|} \tag{5.1.1}$$

where $P^{\pm}_{\gamma,R,\lambda}$ is the thermodynamic pressure given, for any van Hove sequence of $\mathcal{D}^{(\ell_3)}$-measurable regions Λ_n and any $\pm$ Λ_n-boundary conditions $q^{\pm}_n$, by the following limit:

$$\lim_{n\to\infty} \frac{1}{\beta|\Lambda_n|} \log \hat{Z}^{\pm}_{\gamma,R,\lambda}(\Lambda_n|q^{\pm}_n) = P^{\pm}_{\gamma,R,\lambda}. \tag{5.1.2}$$

Although (5.1.1) is a rough approximation, equality of pressures is necessary for the bulk terms in $\hat{W}^{\pm}_{\gamma,R,\lambda}(\Gamma; q^{\pm})$ to cancel.

Hence we have to prove that the limit in (5.1.2) exists, that $P^{\pm}_{\gamma,R,\lambda}$ depend continuously on $\lambda \in [\lambda(\beta,R)-1, \lambda(\beta,R)+1]$ and there are c' and $\lambda_{\beta,\gamma,R} \in [\lambda(\beta,R)-1, \lambda(\beta,R)+1]$ so that

$$P^{+}_{\gamma,R,\lambda_{\beta,\gamma,R}} = P^{-}_{\gamma,R,\lambda_{\beta,\gamma,R}} \qquad \text{for all } \gamma \text{ small enough.} \tag{5.1.3}$$

Chapter 6 is devoted to prove (5.1.3).

Existence of the thermodynamic limit is standard and we do not report it in this work. It is indeed more complicate than in the Ising model since there is an additional term in the partition function which takes into account the contribution of the weights of the contours. We refer to [52], Sect. 11.7 for both the existence of the limit and the continuity of the pressures as functions of λ.

The equality of the pressures follows from a coarse graining argument a la Lebowitz-Penrose by which we prove closeness of the pressures to their mean field values $p^{\pm;\mathrm{mf}}_{R,\lambda} := -\phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,\pm})$. Then, since the difference of the plus and minus mean field pressures changes sign when λ varies in an interval around $\lambda(\beta,R)$, the same happens to the difference $P^{+}_{\gamma,R,\lambda_{\beta,\gamma,R}} - P^{-}_{\gamma,R,\lambda_{\beta,\gamma,R}}$. Continuity of $P^{+}_{\gamma,R,\lambda_{\beta,\gamma,R}} - P^{-}_{\gamma,R,\lambda_{\beta,\gamma,R}}$ in λ then implies the existence of a zero. The uniqueness is a much deeper question which is, however, here unessential as we just want to prove existence of phase transitions.

A crucial point in pursuing this program is to do a coarse graining on scale $\gamma^{-1/2}$ with both the Kac and the hard core interactions and prove that the logarithm of the partition function with contours is close to the "mesoscopic" free energy on the given scale. The "mesoscopic" free energy will be the sum of the coarse grained LMP hamiltonian and the hard core free energy. The latter is computed in the canonical ensemble in the Appendix A and is a consequence of [53]. When applying the Lebowitz and Penrose theory, we have no problems for the LMP part of the hamiltonian because for its special structure it is easy to write it as a function of the occupation numbers in each atom of the partition. On the other

side, the hard core interaction has a short range and hence it can make interact two particles from contiguous atoms. We solve this problem with finding an upper and a lower bound.

A final remark is that in Step 1 the choice of cutoff weights is essential. In fact to have Peierls bound (4.5.2) on the contours one needs to fix λ such that the pressures are equal. However, to show equality of pressures one needs to show that the bound (4.5.2) holds and by introducing the cutoff weights one escapes from this loop.

5.2 Step 2: energy estimates

We want to prove that there is $c' > 0$ so that given γ small enough, for $R < R_0$,

$$\frac{\hat{N}^+_{\gamma,R,\lambda_{\beta,R,\gamma}}(\Gamma, q^+)}{\hat{D}^+_{\gamma,R,\lambda_{\beta,R,\gamma}}(\Gamma, q^+)} \leq e^{-\beta(c\zeta^2 - c'\gamma^{1/2-2\alpha d})\ell_2^d N_\Gamma} \frac{e^{\beta I^-_{\gamma,\lambda(\beta,R)}(\mathrm{int}^-(\Gamma))}\hat{Z}^-_{\gamma,R,\lambda_{\beta,\gamma,R}}(\mathrm{int}^-(\Gamma)|\chi^-)}{e^{\beta I^+_{\gamma,\lambda(\beta,R)}(\mathrm{int}^-(\Gamma))}\hat{Z}^+_{\gamma,R,\lambda_{\beta,\gamma,R}}(\mathrm{int}^-(\Gamma)|\chi^+)} \tag{5.2.1}$$

for all plus contour Γ, with $I^\pm_{\gamma,\lambda(\beta,R)}(\Lambda)$ defined in (5.3.3) and where we use the shorthand notation:

$$\chi^\pm_\Delta(r) = \rho_{\beta,\pm}\mathbf{1}_{r\in\Delta}, \quad \chi^\pm = \chi^\pm_{\mathbb{R}^d} \tag{5.2.2}$$

An analogous bound holds for the minus contours.

We do not prove explicitly estimate (5.2.1), but we refer to [52] where it is carried out for the LMP model, assuming the proofs will not be affected too much by the addition of the hard core interaction. We report here only an outline of the proof.

First we should mention that the validity of (5.2.1) is not restricted to the special value $\lambda = \lambda_{\beta,\gamma,R}$, but it holds as well for all $\lambda \in [\lambda(\beta,R) - c\gamma^{1/2}, \lambda(\beta,R) + c\gamma^{1/2}]$ (c a positive constant and γ small enough). However, since we do not need such a strong statement we write (5.2.1) for this value of the chemical potential.

The proof is divided into two steps. In the first step one needs to prove that it is possible to factorize with "negligible error" the estimate in int(Γ) from the one in sp(Γ). In this proof one manage to bound the l.h.s. of (5.2.1) by a product of three factors: one is the fraction appearing on the r.h.s., another factor is a constrained partition function in sp(Γ) and finally the last one is the "negligible error" $e^{\beta c\gamma^{1/2-2\alpha_2 d}\ell_2^d N_\Gamma}$, $c > 0$ a constant. The second step

involves a bound on the above constrained partition function in $\mathrm{sp}(\Gamma)$ which yields the gain factor $e^{-\beta(c\zeta^2 - c'\gamma^{1/2-2\alpha_2 d})\ell_2^d N_\Gamma}$, $c > 0$ another constant. Combining the two one gets .

The main tool used in this part of the proof is a coarse graining argument and an analysis a la Lebowitz and Penrose. The error in doing a coarse graining is bounded by $e^{\beta c\gamma^{1/2}|\mathrm{sp}(\Gamma)|} = e^{\beta c\gamma^{1/2-2\alpha d}\ell_2^d N_\Gamma}$, which is the "negligible factor" mentioned above. Just as in the Ising model the key point is that we can afford to make errors of this size, because it is a small fraction of the gain term in the Peierls bounds. Thus in both steps we have a reduction, after coarse graining, to variational problems with the LMP free energy functional. They involve two different regions, one is at the boundary between interior and spatial support of Γ, the other is in the bulk of the spatial support. In the former we exploit the definition of contours which implies that the boundary of $\mathrm{int}^{\pm}(\Gamma)$ is in the middle of a "large region" (of size ℓ_3) where $\eta(\cdot;\cdot)$ is identically equal to ± 1 respectively. By the strong stability properties of the LMP free energy functional, the minimizers are then proved to converge exponentially to $\rho_{\beta,R,\pm}$ with the distance from the boundaries. Here we use the assumption that $\beta \in (\beta_{c,R}, \beta_{0,R})$, i.e. where the mean field operator $K_{\beta,\lambda(\beta,R),R}$ is a contraction, see Proposition 3.2.3. We then conclude that with a negligible error we have "thick corridors" where the minimizers are equal to $\rho_{\beta,R,\pm}$ thus separating the regions outside and inside the corridors.

After this step we have plus/minus partition functions in $\mathrm{int}^{\pm}(\Gamma)$ with boundary conditions $\rho_{\beta,R,\pm}$ and still a variational problem in the region $\mathrm{sp}(\Gamma)$ with the constraint that profiles should be compatible with the presence of the contour Γ. The analysis of such a minimization problem leads to the gain factor in the Peierls bounds.

5.3 Step 3: surface correction to the pressure

The last step consists in proving that there is $c' > 0$ so that for any γ small enough the following holds: for all plus contours Γ

$$\Big| \log \Big\{ \frac{e^{\beta I^{\pm}_{\gamma,\lambda(\beta,R)}(\mathrm{int}^-(\Gamma))}\, \hat{Z}^{\pm}_{\gamma,R,\lambda_{\beta,\gamma,R}}(\mathrm{int}^-(\Gamma)|\chi^{\pm})}{e^{\beta|\mathrm{int}^-(\Gamma)|P^{\pm}_{\gamma,R,\lambda_{\beta,\gamma,R}}}} \Big\} \Big| \leq c'\gamma^{1/2}\ell_3^d N_\Gamma \tag{5.3.1}$$

where we use the same shorthand notation as in (5.2.2). and where $I^{\pm}_{\gamma,\lambda(\beta,R)}(\Lambda)$ is a surface term:

$$I^{\pm}_{\gamma,\lambda(\beta,R)}(\Lambda) = \int_{\Lambda^c} \{e_{\lambda(\beta,R)}(\rho_{\beta,R,\pm}) - e_{\lambda(\beta,R)}(J_\gamma * \rho_{\beta,R,\pm}\mathbf{1}_{\Lambda^c})\} \tag{5.3.2}$$

$$- \int_{\Lambda} e_{\lambda(\beta,R)}(J_\gamma * \rho_{\beta,R,\pm}\mathbf{1}_{\Lambda^c}). \tag{5.3.3}$$

Analogous bound (with $\mathrm{int}^+(\Gamma)$ instead of $\mathrm{int}^-(\Gamma)$) holds for minus contours.

Hence we want to prove that the leading term in the partition function is $e^{\beta|\mathrm{int}^-(\Gamma)|P^{\pm}_{\gamma,R,\lambda_{\beta,\gamma,R}}}$ which is therefore the same for the two partition functions, and thus the whole point it to estimate the next term, i.e. the surface corrections to the pressure, proving that they are small as $e^{c''\gamma^{1/2}\ell_3^d N_\Gamma}$ at least when the boundary conditions "are perfect", i.e. given by $\chi^{\pm}$.

This is definitely the most delicate point in the whole scheme and it is carried out in Chapters 7 and 8. Furthermore (5.3.1) plus Step 1 (see Section 5.1) would imply:

Theorem 5.3.1 (Surface corrections to the pressure)**.** *It exists R_0 such that for any $R \le R_0$ and any $\beta \in (\beta_{c,R}, \beta_{0,R})$ there are $c > 0$, $\gamma_{\beta,R} > 0$ and $\lambda_{\beta,\gamma,R}$, $\gamma \le \gamma_\beta$, $|\lambda(\beta,R) - \lambda_{\beta,R,\gamma}| \le c\gamma^{1/2}$, so that for any bounded $\mathcal{D}^{(\ell_3)}$-measurable region Λ,*

$$\frac{e^{\beta I^-_{\gamma,\lambda(\beta,R)}(\Lambda)}\hat{Z}^-_{\gamma,R,\lambda_{\beta,\gamma,R}}(\Lambda|\chi^-_{\Lambda^c})}{e^{\beta I^+_{\gamma,\lambda(\beta,R)}(\Lambda)}\hat{Z}^+_{\gamma,R,\lambda_{\beta,\gamma,R}}(\Lambda|\chi^+_{\Lambda^c})} \;\le\; e^{c\gamma^{1/2}|\delta^{\ell_3}_{\mathrm{out}}[\Lambda]|} \tag{5.3.4}$$

In the classical P-S models where we perturb the ground states, the partition functions $\hat{Z}^{\pm}(\mathrm{int}^-(\Gamma))$ have only the contribution of the contour weights (the restricted ensembles are in fact singletons consisting each one of a ground state). The Peierls bounds which again hold by definition (because the weights are cutoff weights!) prove the validity of a cluster expansion, from which (5.3.1) then follows. A natural requirement for the extension of the theory to the continuum where $\mathcal{Q}_\pm$ has a non-trivial structure is to prove the validity of a cluster expansion. We shall prove it in Chapter 7 where the constraints $\mathcal{Q}_\pm$ induce an analysis in the canonical ensemble. The idea is that (5.3.1) is implied by the following theorem (stated for a generic $\mathcal{D}^{(\ell_3)}$ measurable region):

Theorem 5.3.2. *Let Λ be a bounded $\mathcal{D}^{(\ell_3)}$ measurable region. Let x_i be the centers of the*

cubes $C^{(\ell_2)} \in \mathcal{D}^{(\ell_2)}$ *then define*

$$f_{x_1,..,x_n} = \int_{\{r_i \in C^{(\ell_2)}_{x_i}, 1 \le i \le n\}} q^{\otimes n}(dr_1..dr_n) J^{(n)}_\gamma(r_1,..,r_n). \tag{5.3.5}$$

There are positive constants δ, c' *and* c *so that for all* $f_{x_1,..,x_n}$

$$\left| E_{\mu^1}\big(f_{x_1,..,x_n}\big) - E_{\mu^2}\big(f_{x_1,..,x_n}\big) \right| \le c' e^{-c[\gamma^{-\delta}\ell_3^{-1} dist(C^{(\ell_2)}_{x_1}, \Lambda^c)]} \tag{5.3.6}$$

where E_{μ^i}, $i = 1, 2$, *are the expectations with respect to* μ_i *, i.e. finite volume Gibbs measures in* Λ *with b.c.* $\bar{q}^i$.

We compute the expectations in (5.3.6) in two steps: we first fix the number of particles in the cubes $C^{(\ell_2)}$ and integrate over their positions; then, in the second step, we sum over the particle numbers. By its very nature, the Kac assumption makes the first step simple: to first order in fact the energy is independent of the positions of the particles inside each cube. Neglecting the higher orders terms, the energy drops out of the integrals (with fixed particle numbers) which can then be computed explicitly. The result is the phase space volume of the set of configurations with the given particle numbers: this is an entropy factor which, together with the energy, reconstructs the mesoscopic energy functional.

By using cluster expansion techniques, we will show here that it is possible to compute exactly the correction due to the dependence of the energy on the actual positions of the particles in each cube. The correction, which is a function of the particle numbers, is a hamiltonian with many body interactions of all orders. Even thought it has a complicated expression, yet it is still a regular hamiltonian which becomes small as γ gets small.

For what concern the hard core part of the interaction we can use again a cluster expansion technique, using the result [53] obtained for a system with a single canonical constraint and therefore extending it to present case of multi-canonical constraints.

The expansion will involve therefore both the two kind of interactions, giving rise to a complex set of diagrams with multiple links. We nevertheless manage to deal with it proving convergence of the series in the form of Kotecký and Preiss, [34].

Once we are left with an "effective hamiltonian" we still have to sum over the particle numbers. Since we work in a contour model, the particle densities are close to the mean field values $\rho_{\beta,R,\pm}$ so that the marginal of the Gibbs measure over the set $\{\rho^{(\ell_2)}_x\}_x$ is Gibbsian

and it is a small perturbation of a hamiltonian given by the mean field free energy functional restricted to a neighborhood of the mean field equilibrium density (i.e. we do not feel the double well structure). In such a setup we manage to prove the validity of the Dobrushin uniqueness condition (Chapter 8), where we take into account the contribution of the hard cores as a cluster expansion sum, which is small because of the Kotheckỳ-Preiss Theorem.

6. Equality of the pressures

In this chapter we prove that the pressures in the + and − contour models can be made equal by suitably tuning the chemical potential as a function of γ. Hence we select the value $\lambda = \lambda_{\beta,R,\gamma}$ of the chemical potential for which the phase transition will be proved to occur. We first restrict λ to the interval $[\lambda(\beta,R)-1, \lambda(\beta,R)+1]$ and then we shall prove that actually $\lambda_{\beta,R,\gamma}$ lies in the smaller interval $[\lambda(\beta,R)-c''\gamma^{1/2}, \lambda(\beta,R)+c''\gamma^{1/2}]$ for all γ small enough and c'' a positive constant.

6.1 Definitions and results

We remind some definitions and notation.

In this chapter the temperature β is fixed in the open interval $(\beta_{c,R}, \beta_0)$ and $\zeta > 0$ is the accuracy parameter which enters in the definition of the contours.

All the sets are bounded $\mathcal{D}^{(\ell_3)}$- misurable regions of $\mathbb{R}^d$. Plus/minus boundary conditions are denoted by $q^\pm$, where $q^\pm$ may stand either for a particle configuration or a density function. The plus/minus restricted ensembles are denoted by:

$$\mathcal{Q}^\Lambda_\pm = \{q_\Lambda \in \mathcal{Q}^\Lambda : \eta(q_\Lambda; r) = \pm 1, r \in \Lambda\}; \quad \mathcal{Q}_\pm = \mathcal{Q}^{\mathbb{R}^d}_\pm \tag{6.1.1}$$

and the plus/minus partition functions with plus/minus b.c. $q^\pm$ are (like (4.6.5)):

$$\hat{Z}^\pm_{\gamma,\beta,R,\lambda}(\Lambda|q^\pm) = \int_{\mathcal{Q}^\Lambda_\pm} \nu^\Lambda(dq) e^{-\beta H_\gamma(q|q^\pm)} e^{-\beta H^{\rm hc}(q|q^\pm)} X_{\gamma,\lambda,q_\Lambda}(\Lambda) \tag{6.1.2}$$

with $X_{\gamma,\lambda,q_\Lambda}(\Lambda) = \sum_{\underline{\Gamma}\in\mathcal{B}^\pm_\Lambda} \hat{W}(\underline{\Gamma}|q_\Lambda)$ and where $H^{\rm hc}(q|\bar{q}) = H^{\rm hc}(q) + U^{\rm hc}(q,\bar{q})$.

Note that the two partition functions $Z^\pm_{\gamma,R,\lambda}(\Lambda|q^\pm)$ do not differ only by the boundary conditions. They are in fact defined on different phase spaces, i.e. the restricted ensembles $\mathcal{Q}^\Lambda_\pm$, and the contour weights entering in their definition are also different.

The main result in this chapter is the following.

Theorem 6.1.1. *For any $\lambda \in [\lambda(\beta,R)-1, \lambda(\beta,R)+1]$ the following holds.*

- *For any van Hove sequence Λ_n of $\mathcal{D}^{(\ell_3)}$-misurable regions and any sequence $q_n^\pm$ of $\pm$ b.c. the following limits exist:*

$$\lim_{n\to\infty} \frac{1}{\beta|\Lambda_n|} \log \hat{Z}^{\pm}_{\gamma,R,\lambda}(\Lambda_n|q_n^\pm) = P^{\pm}_{\gamma,R,\lambda} \tag{6.1.3}$$

  and they are independent of the sequence $\{\Lambda_n, q_n^\pm\}$.

- *$P^{\pm}_{\gamma,R,\lambda}$ depend continuously on λ and there is a constant $c>0$ such that*

$$P^{+}_{\gamma,R,\lambda} - P^{-}_{\gamma,R,\lambda} > 0 \qquad \text{if } \lambda = \lambda(\beta,R) + c\gamma^{1/2} \tag{6.1.4}$$

$$P^{+}_{\gamma,R,\lambda} - P^{-}_{\gamma,R,\lambda} < 0 \qquad \text{if } \lambda = \lambda(\beta,R) - c\gamma^{1/2} \tag{6.1.5}$$

As an immediate corollary of Theorem 6.1.1 we have:

Theorem 6.1.2. *For all γ small enough there is $\lambda_{\beta,R,\gamma}$ such that*

$$P^{+}_{\gamma,R,\lambda_{\beta,R,\gamma}} = P^{-}_{\gamma,R,\lambda_{\beta,R,\gamma}}, \qquad |\lambda_{\beta,R,\gamma} - \lambda(\beta,R)| \le c\gamma^{1/2}, \tag{6.1.6}$$

with c as in (6.1.4).

6.2 Coarse graining

The equality of pressures follows from a coarse graining argument by which we shall prove closeness of the pressures to their mean field values:

$$\lim_{\gamma\to 0} P^{\pm}_{\gamma,R,\lambda} = p^{\pm;\mathrm{mf}}_{R,\lambda} := -\phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,\pm}). \tag{6.2.1}$$

Since the difference of the plus and minus mean field pressures changes sign when λ varies in an interval around $\lambda(\beta,R)$, the same happens to the difference $P^{+}_{\gamma,R,\lambda} - P^{-}_{\gamma,R,\lambda}$. Then the existence of a zero is proved, once we assure the continuity of $P^{+}_{\gamma,R,\lambda} - P^{-}_{\gamma,R,\lambda}$ in λ.

Existence of the thermodynamic limit for the two pressures $P^{\pm}_{\gamma,R,\lambda}$ is standard and we skip it. It can be proved as for the Ising model, since spin variables are here bounded because of the hard-core interaction. One has only to take care of the contours which give contribution

in the partition function, but their contribution is exponentially small as in (7.4.15). For all the details see [52], Sect. 11.7.

As already said for the equality of pressures we shall use a Lebowitz-Penrose approach via coarse graining. Hence the main result in this section is an estimate on "constrained partition functions" in terms of the mesoscopic free energy functional. Note that the coarse graining in this chapter is intended to be done on a scale $\ell_{1,\gamma}$ (see Section 4.1) where for simplicity we drop γ from the notation.

Definition 6.2.1. Let $\mathcal{M}_\Lambda$, Λ being a $\mathcal{D}^{(\ell_1)}$-misurable region, be the space of non-negative, $\mathcal{D}^{(\ell_1)}$-misurable functions on Λ. Furthermore we define:

$$\rho^{(\ell_1)}(q;r) = \frac{|q \cap C_r^{(\ell_1)}|}{\ell_1^d}, \qquad \rho^{(\ell_1)}(\rho;r) = \frac{1}{\ell_1^d}\int_{C_r^{(\ell_1)}} \rho(r')dr' \tag{6.2.2}$$

and for any "constraint" $\mathcal{B} \subset \mathcal{M}_\Lambda$ we define:

$$Z_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) = \int_{q_\Lambda:\rho^{(\ell_1)}(q_\Lambda,\cdot)\in\mathcal{B}} e^{-\beta H_{\gamma,R}(q_\Lambda|\bar{q}_{\Lambda^c})} \tag{6.2.3}$$

where $\bar{q}$ is a boundary condition.

Note that for the presence of the hard-core interaction in the hamiltonian we have that the density is automatically bounded by the close-packing density ($\rho_{\rm cp}$) of the hard-spheres. Hence:

$$\rho^{(\ell_1)}(q_\Lambda, r) \le \rho_{\rm cp}, \qquad r \in \Lambda \tag{6.2.4}$$

We also suppose that there is c so that for any "constraint" $\mathcal{B} \subset \mathcal{M}_\Lambda$ and for any $\rho_\Lambda \in \mathcal{B}$ there is q_Λ so that:

$$\rho^{(\ell_1)}(q_\Lambda,\cdot) \in \mathcal{B}, \qquad \text{and} \qquad |\rho^{(\ell_1)}(q_\Lambda,\cdot) - \rho_\Lambda| \le c\gamma^{d/2}. \tag{6.2.5}$$

We shall coarse grain with mesh ℓ_1 and call X_Λ the set containing the centres of the cubes $C_x^{(\ell_1)}$ contained in Λ, which from simplicity from now on will be just denoted by C_x. Let us write the n-body coarse grained potentials $\tilde{J}_\gamma^{(n)}$ as follows:

$$\tilde{J}_\gamma^{(n)}(r_1,..,r_n) = \frac{1}{\left|C^{(\ell_1)}\right|^n}\int_{C_{r_1}^{(\ell_1)}} dr'_1 ... \int_{C_{r_n}^{(\ell_1)}} dr'_n J_\gamma^{(n)}(r',...,r'_n). \tag{6.2.6}$$

Analogously we will add a tilde to denote quantities which are computed with $\tilde{J}_\gamma^{(n)}$ instead of $J_\gamma^{(n)}$. In particular $\tilde{H}_{\gamma,R}$ and $\tilde{Z}$ are the coarse grained energy and partition function. For any $\rho_\Lambda \in \mathcal{M}_\Lambda$ we define also the coarse grained mesoscopic free energy functional:

$$\tilde{F}_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\bar{q}) = \tilde{H}_\gamma(\rho_\Lambda|\bar{q}_{\Lambda^c}) + |\Lambda| f^{\rm hc}_{\Lambda,R}(\rho_\Lambda|\bar{q}) = \tilde{H}_\gamma(\rho_\Lambda|\bar{q}_{\Lambda^c}) - \frac{1}{\beta}\int_\Lambda S(\rho_\Lambda) + |\Lambda| f_{\Lambda,R}(\rho_\Lambda|\bar{q}) \tag{6.2.7}$$

where $S(\rho) = -\rho(\log\rho - 1)$ is the entropy and:

$$f_{\Lambda,R}(\rho_\Lambda|\bar{q}) := -\frac{1}{\beta|\Lambda|}\sum_\pi z_R^T(\pi;\rho_\Lambda,\bar{q}) \tag{6.2.8}$$

is the contribution to the free energy coming from cluster expanding the canonical partition function for the hard spheres over the boxes of the partition. See equation (A.4.16) and in general Appendix A where we do a cluster expansion in a volume Λ with periodic boundary conditions. The extension to the case with coarse graining over the boxes with side ℓ_1 is an easy generalization and nothing changes but the fact that the multi-indices are over the particle labels which now refer to a precise box (as we do in the more general case of Kac interaction plus hard core in Section 7.3). In the case with a more general class of interactions, (i.e. stable and tempered), see [53] where we also prove convergence to Mayer's virial expansion at the thermodynamic limit. The notation used in (6.2.7) and (6.2.8) has been introduced according to the mean field model, in order to have the analogous quantities also in the mesoscopic description.

Hence we want to prove the following theorem:

Theorem 6.2.2. *There is $c > 0$ so that for any $\bar{q}$, any R and any $\mathcal{B} \subset \mathcal{M}_\Lambda$:*

$$\Big|\log Z_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) + \beta \inf_{\rho_\Lambda\in\mathcal{B}} \tilde{F}_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\bar{q})\Big| \le c\gamma^{1/2}|\Lambda| \tag{6.2.9}$$

To prove Theorem 6.2.2 we use the following lemma which will be used to replace the partition function in (6.2.9) by its tilde analogue.

Lemma 6.2.3. *There is $c > 0$ so that for all $\bar{q}$, all R and any $\mathcal{B} \subset \mathcal{M}_\Lambda$:*

$$e^{-c\gamma^{1/2}|\Lambda|} \le \frac{Z_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B})}{\tilde{Z}_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B})} \le e^{c\gamma^{1/2}|\Lambda|}. \tag{6.2.10}$$

We skip the proof of Lemma 6.2.3 which is the same in the LMP model without extra hard-core interaction. In the proof we use some energy bounds for the LMP hamiltonian, hence the R-interaction gives no contribution except giving a natural cutoff for the values of the density (we have $\rho_{\rm cp}$ instead of $\rho_{\max}$). For the reference see [52], Sect. 11.1.

Proof of Theorem 6.2.2.

By Lemma 6.2.3 we can replace $Z_{\gamma,R,\lambda,\Lambda,\bar q}(\mathcal{B})$ with its tilde analogue with an error $c\gamma^{1/2}|\Lambda|$; it is therefore sufficient to prove (6.2.9) with the tilde partition function. Analogously as in Sections 7.2 and 8.1 the energy $\tilde H_\gamma(q_\Lambda|\bar q_{\Lambda^c})$ depends only on the number of particles $n_x = |q_\Lambda \cap C_x|$ in the cubes C_x, $x \in X_\Lambda$. We shall write $\tilde H_\gamma(\underline{n}|\bar q_{\Lambda^c})$ for the γ-energy of any configuration whose particle numbers are $\underline{n} = (n_x)_{x\in X_\Lambda} \in \mathbb{N}^{X_\Lambda}$. Denoting by $\rho_x = \gamma^{d/2} n_x$ the density in each cube and by $\rho = (\rho_x)_{x\in X_\Lambda}$ the function in $\mathcal{M}_\Lambda$ which is constant in each cube of the partition, we look for a lower and for an upper bound of $\log \tilde Z_{\gamma,R,\lambda,\Lambda,\bar q}(\mathcal{B})$. While the γ-interaction in $\tilde H_{\gamma,R}$ depends only on the occupation numbers, the hard-core interaction is a short range potential, hence it makes two particles coming from two contiguous boxes interact. To avoid interactions among contiguous boxes we will use upper and lower bound which create independence on the boxes.

For the upper bound we neglect interactions among contiguous boxes. Hence for the positivity of the hard-core interaction we have the following:

$$\tilde Z_{\gamma,R,\lambda,\Lambda,\bar q}(\mathcal{B}) \le \sum_{\underline{n}:\rho\in\mathcal{B}} e^{-\beta \tilde H_\gamma(\underline{n}|\bar q_{\Lambda^c})} \left[\prod_x e^{-\beta\gamma^{-d/2} f_{C_x,R}(\rho|\bar q)} \frac{|C_x|^{n_x}}{n_x!}\right] \tag{6.2.11}$$

where $f_{C_x,R}$ is defined in (6.2.8).

Now we write:

$$\frac{|C_x|^{n_x}}{n_x!} = \exp\{|C_x| S(\rho_x) + c(n_x)\} \tag{6.2.12}$$

where:

$$c(n_x) := \log\Big(\frac{|C_x|^{n_x}}{n_x!}\Big) - |C_x| S(\rho_x), \qquad c(n_x) \le c \log \gamma^{-d/2} \tag{6.2.13}$$

with c a positive constant and having used that $n_x \le \rho_{\rm cp}\ell_2^d$.

Since there are $|\Lambda|\gamma^{d/2}$ values of x and n_x may take at most $\rho_{\rm cp}\ell_2^d$ values, we have $(\rho_{\rm cp}\ell_2^d)^{|\Lambda|\gamma^{d/2}}$ terms in the sum. Hence:

$$\tilde Z_{\gamma,R,\lambda,\Lambda,\bar q}(\mathcal{B}) \le e^{-\beta \inf_{\underline{n}:\rho\in\mathcal{B}} \tilde F_{\gamma,R,\lambda,\Lambda}(\rho|\bar q)} \Big[c' \rho_{\rm cp}\, \ell_2^d\, e^{c\log\gamma^{-d/2}}\Big]^{|\Lambda|\gamma^{d/2}} \tag{6.2.14}$$

and therefore:

$$\log \tilde{Z}_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) \leq -\beta \inf_{\rho_\Lambda \in \mathcal{B}} \tilde{F}_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\bar{q}) + c|\Lambda|\gamma^{d/2} \log \gamma^{-1}. \tag{6.2.15}$$

For the lower bound we neglect those configurations with particles in the frame of width $R/2$ inside each cube C_x. Hence the partition function is larger than the one computed as product over disjointed smaller cubes since in the last one we have reduced the volume of the phase space. Let us call C'_x the cubes of side $\ell_1 - R$, thus we have:

$$\tilde{Z}_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) \geq \sum_{\underline{n}:\rho \in \mathcal{B}} e^{-\beta \tilde{H}_\gamma(\underline{n}|\bar{q}_{\Lambda^c})} \left[\prod_x e^{-\beta(\ell_1 - R)^d f_{C'_x,R}(\rho|\bar{q})} \frac{|C'_x|^{n_x}}{n_x!} \right] \tag{6.2.16}$$

We write again:

$$\frac{|C'_x|^{n_x}}{n_x!} = \exp\{|C'_x| S(\rho_x) + c'(n_x)\} \tag{6.2.17}$$

where:

$$c'(n_x) := \log\left(\frac{|C'_x|^{n_x}}{n_x!} \right) - |C'_x| S(\rho_x), \qquad c'(n_x) \geq -c \log \gamma^{-d/2} \tag{6.2.18}$$

To reconstruct the mesoscopic free energy $\tilde{F}_{\gamma,R,\lambda,\Lambda}$ we replace $|C'_x|$ with $|C_x|$ paying for each cube an error which goes like the surface, i.e. $O(\gamma^{-(d-1)/2} R)$:

$$\tilde{Z}_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) \geq \sum_{\underline{n}:\rho \in \mathcal{B}} e^{-\beta \tilde{H}_\gamma(\underline{n}|\bar{q}_{\Lambda^c})} \left[\prod_x e^{-\beta|C_x| f_{C_x,R}(\rho|\bar{q}) + |C_x| S(\rho_x) - c \log \gamma^{-d/2} - c' \gamma^{-(d-1)/2} R} \right] \tag{6.2.19}$$

We can find a lower bound by choosing only one term in the sum over $\underline{n}$, i.e. the term in which $\tilde{F}_{\gamma,R,\lambda,\Lambda}$ takes its minimum. Hence, having again $|\Lambda|\gamma^{d/2}$ values of x,

$$\log \tilde{Z}_{\gamma,R,\lambda,\Lambda,\bar{q}}(\mathcal{B}) \geq -\beta \inf_{\underline{n}:\rho \in \mathcal{B}} \tilde{F}_{\gamma,R,\lambda,\Lambda}(\rho|\bar{q}) - c \log \gamma^{-1} |\Lambda| \gamma^{d/2} - c' |\Lambda| \gamma^{d/2} \gamma^{-(d-1)/2} R. \tag{6.2.20}$$

Let $\tilde{\rho}$ be the minimizer of $\tilde{F}_{\gamma,R,\lambda,\Lambda}(\cdot|\bar{q})$ on the constraint $\mathcal{B}$. Then by (6.2.5) there is $\underline{n}$ so that ρ obtained from $\underline{n}$ is in $\mathcal{B}$ and furthermore there is c so that $|\rho - \tilde{\rho}| \leq c\gamma^{d/2}$ on Λ. The lower bound (6.2.2) then follows using that $f^{\mathrm{hc}}_{\Lambda,R}$ (see (6.2.7)) is convex and analytic and hence: $|f^{\mathrm{hc}}_{\Lambda,R}(\gamma^{d/2}(n+t)) - f^{\mathrm{hc}}_{\Lambda,R}(\gamma^{d/2} n)| \leq k\gamma^{d/2}$ for all $|t| \leq c$ and $n \leq \rho_{\mathrm{cp}} \ell_2^d$. Note that the LMP energy is differentiable hence it gives a contribution of order $k\gamma^{d/2}$. □

6.3 Equality of the plus and minus pressures

In this section we will complete the proof of (6.1.4).

First of all we have to analyze the variational problem with constraint we have in (6.2.2). Observe that given any $c > 0$ for all γ small enough,

$$|\rho_{\beta,\lambda,R,\pm} - \rho_{\beta,R,\pm}| < \zeta \qquad \text{for } |\lambda - \lambda(\beta, R)| \leq c\gamma^{1/2} \tag{6.3.1}$$

Hence, we prove the following proposition.

Proposition 6.3.1. *Let λ be such that $|\rho_{\beta,\lambda,R,\pm} - \rho_{\beta,R,\pm}| < \zeta$; then for all R,*

$$\inf_{\rho_\Lambda : \eta(\rho_\Lambda;\cdot)\equiv\pm 1} F_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\rho_{\beta,\lambda,R,\pm}\mathbb{1}_{\Lambda^c}) = F_{\gamma,R,\lambda,\Lambda}(\rho_{\beta,\lambda,R,\pm}\mathbb{1}_\Lambda|\rho_{\beta,\lambda,R,\pm}\mathbb{1}_{\Lambda^c}) \tag{6.3.2}$$

$$= \phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,\pm})|\Lambda| + I^{\pm}_{\gamma,R,\lambda}(\Lambda) \tag{6.3.3}$$

where:

$$I^{\pm}_{\gamma,R,\lambda}(\Lambda) = \int_{\Lambda^c} \{e_\lambda(\rho_{\beta,\lambda,R,\pm}) - e_\lambda(\rho_{\beta,\lambda,R,\pm} J_\gamma * \mathbb{1}_{\Lambda^c})\} - \int_\Lambda e_\lambda(\rho_{\beta,\lambda,R,\pm} J_\gamma * \mathbb{1}_{\Lambda^c}). \tag{6.3.4}$$

The same result holds for $\tilde{F}_{\gamma,R,\lambda,\Lambda}$ with J_γ replaced by $\tilde{J}_\gamma$.

Proof .

Denote by ρ the function equal to ρ_Λ on Λ and to ρ_{Λ^c} on Λ^c. Then regarding $\rho_{\Lambda^c} = 0$ on Λ and recalling that $S(0) = 0$, so that $S(\rho_{\Lambda^c})\mathbb{1}_{\Lambda^c}$,

$$F_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\rho_{\Lambda^c}) = \int_{\mathbb{R}^d} \{e_\lambda(J_\gamma * \rho) - e_\lambda(J_\gamma * \rho_{\Lambda^c})\} + \{f^{\mathrm{hc}}_{\Lambda,R}(\rho) - f^{\mathrm{hc}}_{\Lambda,R}(\rho_{\Lambda^c})\} \tag{6.3.5}$$

$$= \int_{\mathbb{R}^d} \{\phi_{\beta,\lambda,R}(J_\gamma * \rho) - \phi_{\beta,\lambda,R}(J_\gamma * \rho_{\Lambda^c})\} - \{f^{\mathrm{hc}}_{\Lambda,R}(J_\gamma * \rho) - f^{\mathrm{hc}}_{\Lambda,R}(\rho)\} \tag{6.3.6}$$

$$+ \{f^{\mathrm{hc}}_{\Lambda,R}(J_\gamma * \rho_{\Lambda^c}) - f^{\mathrm{hc}}_{\Lambda,R}(\rho_{\Lambda^c})\} \tag{6.3.7}$$

where $\phi_{\beta,\lambda,R}$ is defined in (3.1.6) In our case $\rho_{\Lambda^c} = \rho_{\beta,R,\pm}\mathbb{1}_{\Lambda^c}$ so that we can write the integral of the sum as the sum of the integrals, and in the integral with $\{f^{\mathrm{hc}}_{\Lambda,R}(J_\gamma * \rho) - f^{\mathrm{hc}}_{\Lambda,R}(\rho)\}$ we can replace $f^{\mathrm{hc}}_{\Lambda,R}(\rho)$ by $J_\gamma * f^{\mathrm{hc}}_{\Lambda,R}(\rho)$. Then $F_{\gamma,R,\lambda,\Lambda}(\rho_\Lambda|\rho_{\Lambda^c})$ becomes:

$$\int_{\mathbb{R}^d} \{\phi_{\beta,\lambda,R}(J_\gamma * \rho) - \phi_{\beta,\lambda,R}(J_\gamma * \rho_{\Lambda^c})\} + \{J_\gamma * f^{\mathrm{hc}}_{\Lambda,R}(\rho) - f^{\mathrm{hc}}_{\Lambda,R}(J_\gamma * \rho)\} \tag{6.3.8}$$

$$+\{f^{\mathrm{hc}}_{\Lambda,R}(J_\gamma * \rho_{\Lambda^c}) - f^{\mathrm{hc}}_{\Lambda,R}(\rho_{\Lambda^c})\}. \tag{6.3.9}$$

Since $\rho_{\Lambda^c} = \rho_{\beta,R,\pm}\mathbb{1}_{\Lambda^c}$ and $\eta(\rho_\Lambda;\cdot) \equiv \pm 1$, for all γ small enough the first curly bracket is minimized by setting $\rho_\Lambda = \rho_{\beta,R,\pm}$; the second curly bracket by convexity is non-negative (in

the region of convergence of cluster expansion) and vanishes when $\rho_\Lambda = \rho_{\beta,R,\pm}\mathbb{1}_\Lambda$; the third one is independent of ρ_Λ and the first equality of (6.3.2) is proved.

To prove the second equality in (6.3.2) we have to look at the original expression for $F_{\gamma,R,\lambda,\Lambda}(\rho_{\beta,R,\pm}\mathbb{1}_\Lambda|\rho_{\beta,R,\pm}\mathbb{1}_{\Lambda^c})$ and set $\rho_\Lambda = \rho_{\beta,R,\pm}$. Recalling the definition of $I^\pm_{\gamma,R,\lambda}(\Lambda)$ in (6.3.1), we then get:

$$F_{\gamma,R,\lambda,\Lambda}(\rho_{\beta,\lambda,R,\pm}\mathbb{1}_\Lambda|\rho_{\beta,\lambda,R,\pm}\mathbb{1}_{\Lambda^c}) = \int_\Lambda \Big\{e_\lambda(\rho_{\beta,\lambda,R,\pm}) + f^{\rm hc}_{\Lambda,R}(\rho_{\beta,\lambda,R,\pm})\Big\} + I^\pm_{\gamma,R,\lambda}(\Lambda) \quad (6.3.10)$$

$$= |\Lambda|\phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,\pm}) + I^\pm_{\gamma,R,\lambda}(\Lambda) \quad (6.3.11)$$

The same proof works for $\tilde J_\gamma$ and $\tilde F_{\gamma,R,\lambda}$. □

Lemma 6.3.2. *There is a constant c' so that for any R and any λ such that $|\rho_{\beta,\lambda,\pm}-\rho_{\beta,\pm}| \le \zeta$,*

$$|P^\pm_{\gamma,R,\lambda} - p^{\pm;mf}_{R,\lambda}| \le c'\gamma^{1/2} \quad (6.3.12)$$

Proof.

Let Δ_n be an increasing sequence of $\mathcal{D}^{\ell_3}$-misurable cubes which invades $\mathbb{R}^d$. Then using the canonical partition function defined in (6.1.2) (by the existence of thermodynamic limit):

$$P^\pm_{\gamma,R,\lambda} = \lim_{n\to\infty} \frac{1}{\beta|\Delta_n|}\log \hat Z^\pm_{\gamma,R,\lambda}(\Delta_n|\rho_{\beta,\lambda,R,\pm}). \quad (6.3.13)$$

For the upper bound we have:

$$\log \hat Z^\pm_{\gamma,R,\lambda}(\Delta_n|\rho_{\beta,\lambda,R,\pm}) \le \log Z_{\gamma,R,\lambda,\Lambda,\bar q}(\{\eta(\cdot;\cdot)\equiv\pm1\}) + \frac{|\Delta_n|}{\ell_3^d}\log 2 \quad (6.3.14)$$

$$\le -\beta \inf_{\rho_{\Delta_n}:\eta(\rho_{\Delta_n};\cdot)\equiv\pm1} F_{\gamma,R,\lambda,\Delta_n}(\rho_{\Delta_n}|\rho_{\beta,\lambda,R,\pm}\mathbb{1}_{\Delta_n^c}) + c\gamma^{1/2}|\Delta_n| + \frac{|\Delta_n|}{\ell_3^d}\log 2 \quad (6.3.15)$$

$$\le -\beta\phi_{\beta,\lambda,R}(\rho_{\beta,\lambda,R,\pm})|\Delta_n| - \beta I^\pm_{\gamma,R,\lambda}(\Delta_n) + c\gamma^{1/2}|\Delta_n| + |\Delta_n|\gamma^{(1+\alpha)d}\log 2 \quad (6.3.16)$$

$$\le \{\beta\, p^{\pm;\rm mf}_{R,\lambda} + c\gamma^{1/2}\}|\Delta_n| - \beta I^\pm_{\gamma,R,\lambda}(\Delta_n) \quad (6.3.17)$$

where $Z_{\gamma,R,\lambda,\Lambda,\bar q}(\mathcal{B})$ is defined in (6.2.3). The upper bound follows from Lemma 6.2.3, Theorem 6.2.2 and Proposition 6.3.1, because $\lim_{n\to\infty}\frac{I^\pm_{\gamma,R,\lambda}(\Delta_n)}{|\Delta_n|} = 0$.

For the lower bound we use that $X_{\gamma,\lambda,q_\Lambda}(\Lambda) \geq 1$ and get:

$$\log \hat{Z}^{\pm}_{\gamma,R,\lambda}(\Delta_n|\rho_{\beta,\lambda,R,\pm}) \geq \log Z_{\gamma,R,\lambda,\Lambda,\bar{q}}(\{\eta(\cdot;\cdot) \equiv \pm 1\}) \tag{6.3.18}$$

and then the proof proceeds as in the upper bound. □

Lemma 6.3.3. *There are δ and a both positive such that*

$$\frac{d}{d\lambda}(p^{+;mf}_{R,\lambda} - p^{-;mf}_{R,\lambda}) > a, \qquad |\lambda - \lambda(\beta,R)| \leq \delta. \tag{6.3.19}$$

Proof.

$$\frac{d}{d\lambda}(p^{+;\mathrm{mf}}_{R,\lambda} - p^{-;\mathrm{mf}}_{R,\lambda})\Big|_{\lambda=\lambda(\beta,R)} = \rho_{\beta,R,+} - \rho_{\beta,R,-} > 0. \tag{6.3.20}$$

□

We are now going to show that:

Claim Let c be such that $ca > 2c'$, a as in (6.3.19) and c' as in (6.3.12), then for all γ small enough (in particular such that (6.3.1) holds):

$$P^{+}_{\gamma,R,\lambda(\beta,R)+c\gamma^{1/2}} - P^{-}_{\gamma,R,\lambda(\beta,R)+c\gamma^{1/2}} > 0 \tag{6.3.21}$$

$$P^{+}_{\gamma,R,\lambda(\beta,R)-c\gamma^{1/2}} - P^{-}_{\gamma,R,\lambda(\beta,R)-c\gamma^{1/2}} > 0. \tag{6.3.22}$$

Proof.

By Lemmas 6.3.2 and 6.3.3, writing $\lambda'' := \lambda(\beta,R) + c\gamma^{1/2}$, for all γ small enough

$$P^{+}_{\gamma,R,\lambda''} - P^{-}_{\gamma,R,\lambda''} \geq p^{+;\mathrm{mf}}_{R,\lambda''} - p^{-;\mathrm{mf}}_{R,\lambda''} - 2c'\gamma^{1/2} \geq a(\lambda'' - \lambda(\beta,R)) - 2c'\gamma^{1/2} > 0, \tag{6.3.23}$$

where we have done a Taylor expansion for both $p^{+;\mathrm{mf}}_{R,\lambda''}$ and $p^{-;\mathrm{mf}}_{R,\lambda''}$ around $\lambda(\beta,R)$ and used that $p^{+;\mathrm{mf}}_{R,\lambda(\beta,R)} = p^{-;\mathrm{mf}}_{R,\lambda(\beta,R)}$. Analogously, if $\lambda' = \lambda(\beta,R) - c\gamma^{1/2}$,

$$P^{+}_{\gamma,R,\lambda'} - P^{-}_{\gamma,R,\lambda'} \leq p^{+;\mathrm{mf}}_{R,\lambda'} - p^{-;\mathrm{mf}}_{R,\lambda''} + 2c'\gamma^{1/2} \leq -a(\lambda(\beta,R) - \lambda') + 2c'\gamma^{1/2} < 0. \tag{6.3.24}$$

□

7. Cluster expansion

This Chapter and the following will be entirely devoted to prove Theorem 5.3.2. Here we use cluster expansion techniques to compute the expectations (5.3.5) as functions of the number of particles in the cubes $C^{(\ell_2)}$ that we think as of fixed. The aim of this Chapter is to reduce our model to an effective model which depends only on the cell numbers (see Section 7.5).

7.1 Definitions and results

The setup is the following:

- We use the definitions in Section 4.1 and drop the γ in the notation $\ell_{2,\gamma}$ and $\ell_{3,\gamma}$ for simplicity.
- Λ will denote a bounded connected $\mathcal{D}^{(\ell_3)}$ measurable region, S the set of cubes $C^{(\ell_3)}$ in Λ contiguous to Λ^c. $\Delta = \Lambda \setminus S$. Σ is the strip in Λ made of all the cubes $C^{(\ell_3)}$ contiguous to Λ^c and $\Lambda^0 = \Lambda/\Sigma$.
- $f_{x_1,..,x_n}$ is a function defined as

$$f_{x_1,..,x_n} = \int_{\{r_i \in C_{x_i}^{(\ell_2)}, 1 \le i \le n\}} q^{\otimes n}(dr_1..dr_n) J_\gamma^{(n)}(r_1,..,r_n) \tag{7.1.1}$$

  with the condition that $C_{x_i}^{(\ell_2)}$ is in Δ for some $1 \le i \le n$ and where the measure is defined in (1.1.3).
- μ^1 and μ^2 are two probability measures. μ^1 is a finite volume Gibbs measure in Λ with boundary condition $\bar{q}^1$ outside Λ which is moreover conditioned to have all its contours ω spatially supported in Λ^0. μ^2 is the same finite volume Gibbs measure with boundary condition $\bar{q}^2$.

Our aim is to prove that there are positive constants δ, c' and c so that for all $f_{x_1,..,x_n}$

$$\left|E_{\mu^1}\big(f_{x_1,..,x_n}\big) - E_{\mu^2}\big(f_{x_1,..,x_n}\big)\right| \leq c' e^{-c[\gamma^{-\delta}\gamma^{1+\alpha}\mathrm{d}ist(C^{(\ell_2)}_{x_1},\Lambda^c)]} \tag{7.1.2}$$

7.2 Block spin hamiltonians

Let Λ and Λ^0 be as in the previous section and we fix $\bar{q} \in \mathcal{Q}_+^{\Lambda^c}$. In this chapter we will refer interchangeably to the measures μ^1 or μ^2. We split the hamiltonian into the LMP part and the hard-core part, i.e. $H_{\gamma,R,\lambda}(q|\bar{q}) = H_\gamma(q|\bar{q}) + H^{\mathrm{hc}}(q|\bar{q})$ where H_γ shorthands the hamiltonian $H_{\gamma,\lambda_{\gamma,\beta}}$. Furthermore for the hard-core part we give the following definition:

$$H^{\mathrm{hc}}(q|\bar{q}) = H^{\mathrm{hc}}(q) + U^{\mathrm{hc}}(q,\bar{q}) \tag{7.2.1}$$

where $H^{\mathrm{hc}}(q)$ is the energy of the particles inside Λ while (see (1.2.21)) $U^{\mathrm{hc}}(q,\bar{q})$ indicates the interaction energy between a particle in Λ and one in Λ^c (remembering that the configuration outside Λ is fixed $\bar{q} \in \mathcal{Q}_+^{\Lambda^c}$).

We have

$$\hat{Z}^+_{\gamma,R,\lambda}(\Lambda|\bar{q}) := \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} \int_{\mathcal{Q}^\Lambda_+} \nu^\Lambda(dq) e^{-\beta H_\gamma(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q|\bar{q})} \hat{W}(\underline{\Gamma}|q) \tag{7.2.2}$$

which dropping all the suffixes will be denoted by $\hat{Z}(\Lambda|\bar{q})$. $\hat{W}$ stands for the Zahradnik weights with the corresponding parameters.

Our goal is to prove that $\hat{Z}(\Lambda|\bar{q})$ can be written as the partition function of a hamiltonian which depends on variables $\rho_x^{(\ell_2)}$, $x \in X_\Lambda^{(\ell_2)}$, $X_\Lambda^{(\ell_2)}$ the set of centers of cubes $C^{(\ell_2)}$ in Λ. To simplify notation we drop the superscript ℓ_2 writing ρ_x, x tacitly supposed in $X_\Lambda^{(\ell_2)}$. The new energy of a density configuration $\rho = \{\rho_x\}_{x\in X_\Lambda^{(\ell_2)}}$ is defined as

$$h(\rho|\bar{q}) = -\log \sum_{\underline{\Gamma}\in\mathcal{B}^+_\Lambda} \int_{\mathcal{Q}^\Lambda_+} \nu^\Lambda(dq) \mathbf{1}_{\rho^{(\ell_2)}(q)=\rho}\, e^{-\beta H_\gamma(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q|\bar{q})} \hat{W}(\underline{\Gamma}|q) \tag{7.2.3}$$

so that

$$\hat{Z}(\Lambda|\bar{q}) = \sum_\rho e^{-h(\rho|\bar{q})}$$

Setting $n_x = \ell_2^d \rho_x$, we multiply and divide, inside the argument of the log in (7.2.3), by

$$\prod_{x\in X_\Lambda} \frac{\ell_2^{dn_x}}{n_x!}$$

We denote by $\{q_{x,i},\ i=1,..,n_x,\ x\in X_\Lambda\}$, the particles in $C_x^{(\ell_2)}$. Thus particles are now labelled by the pair (x,i), x specifies the cube $C_x^{(\ell_2)}$ to which the particle "belongs", i distinguishes among the particles in $C_x^{(\ell_2)}$. The corresponding free measure, whose expectation is denoted by E_ρ^0, is the product of the probabilities which give uniform distribution to the positions $q_{x,i}$ in their boxes $C_x^{(\ell_2)}$ divided by $n_x!$ since the particles in each box $C_x^{(\ell_2)}$ are indistinguishable. Note that when we change from labeling of all particles in Λ to labeling separately the particles in each box we have to multiply by $\frac{N!}{\prod_{x\in X_\Lambda^{(\ell_2)}} n_x!}$ for all such possibilities. For reasons to be clear in the sequel we define a new prior measure where we weight the previous free measure by only the part of the short range potential which interacts with the boundary configuration $\bar q$ (i.e. $U^{\rm hc}(q,\bar q)$). To be more precise, for the particles in a given box $C_x^{(\ell_2)}$, $x\in X_\Lambda$, the new measure will be:

$$\frac{dq_{x,1}\cdots dq_{x,n_x}e^{-\beta U^{\rm hc}(q^{(C_x)},\bar q)}}{\int dq_{x,1}\cdots dq_{x,n_x}e^{-\beta U^{\rm hc}(q^{(C_x)},\bar q)}}Z_{x,\bar q}(\rho_x) \tag{7.2.4}$$

where $q^{(C_x)}$ denotes the configuration of the particles in $C_x^{(\ell_2)}$, each integral in the denominator is over $C_x^{(\ell_2)}$ with the constraint $\mathcal{Q}_+^\Lambda$ and where:

$$Z_{x,\bar q}(\rho_x)=\int_{\mathcal{Q}_+^\Lambda}\frac{dq_{x,1}}{\ell_2^d}\cdots\frac{dq_{x,n_x}}{\ell_2^d}\ e^{-\beta U^{\rm hc}(q^{(C_x)},\bar q)} \tag{7.2.5}$$

is the extra factor coming from the change of measure and contributing for each cube, with:

$$U^{\rm hc}(q^{(C_x)},\bar q):=\sum_{i=1}^{n_x}\sum_{j=1}^{|\bar q|}V^{\rm hc}(q_{x,i}-\bar q_j). \tag{7.2.6}$$

The corresponding expectation will be denoted by $E^0_{\rho,\bar q}$. We then have

$$h(\rho|\bar q)=-\sum_x\log\frac{\ell_2^{dn_x}}{n_x!}-\sum_{x:\,C_x^{(\ell_2)}\in\partial\Lambda^{\rm int}}\log Z_{x,\bar q}(\rho_x)-\log E^0_{\rho,\bar q}\Big(e^{-\beta H_\gamma(q|\bar q)}e^{-\beta H^{\rm hc}(q)}\sum_{{\rm sp}(\Gamma)\,\subseteq\,\Lambda^0}W(\Gamma|q)\Big) \tag{7.2.7}$$

where $\partial\Lambda^{\rm int}$ is the set of the $\mathcal{D}^{(\ell_2)}$ boxes adjacent to Λ^c (i.e., the interior boxes of Λ). Note that the total normalization is a product of the normalizations in each cube and that, because of the hard-core interaction, $Z_{x,\bar q}(\rho_x)$ for a given box C_x gives the following contribution:

$$Z_{x,\bar q}(\rho_x)=\Big(\int_{C_x}\frac{dq}{\ell_2^d}\mathbb{1}_{q\in C_x^{\bar q}}\Big)^{n_x}=\frac{|C_x^{\bar q}|^{n_x}}{\ell_2^{dn_x}} \tag{7.2.8}$$

where: $C_x^{\bar{q}} = \{r \in C_x : \mathrm{dist}(r, \bar{q}_i) > R, \forall i\}$. This means that, because of the presence of the hard-core, the new measure "reduces" the admissible volume for the particles in each box.

Let $H^{(\ell_2)}(q|\bar{q})$ be the coarse-grained hamiltonian on scale ℓ_2. It is obtained by replacing $J_\gamma^{(n)}$ by $\tilde{J}_\gamma^{(n)}$, where:

$$\tilde{J}_\gamma^{(n)}(r_1, ..., r_n) = \frac{1}{|C^{(\ell_2)}|^n} \int_{C_{r_1}^{(\ell_2)}} dq_1 \cdots \int_{C_{r_n}^{(\ell_2)}} dq_n \, J_\gamma^{(n)}(q_1, ..., q_n) \tag{7.2.9}$$

are the coarse-grained potentials.

It depends only on the particle numbers n_x (or the densities ρ_x) and we can thus write

$$h^0(\rho|\bar{\rho}) = H^{(\ell_2)}(q|\bar{q}), \qquad \rho_x = \rho_x^{(\ell_2)}(q), \ \bar{\rho}_x = \rho_x^{(\ell_2)}(\bar{q}) \tag{7.2.10}$$

Setting

$$\Delta H(q|\bar{q}) = H_\gamma(q|\bar{q}) - H^{(\ell_2)}(q|\bar{q}) \tag{7.2.11}$$

we have

$$h(\rho|\bar{q}) = -\sum_x \log \frac{\ell_2^{dn_x}}{n_x!} - \sum_{x:\, C_x^{(\ell_2)} \in \partial\Lambda^{\mathrm{int}}} \log Z_{x,\bar{q}} + \beta h^0(\rho|\bar{\rho}) + \delta h(\rho|\bar{q}) \tag{7.2.12}$$

where

$$\delta h(\rho|\bar{q}) = -\log E_{\rho,\bar{q}}^0\Big(e^{-\beta\Delta H(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q)} \sum_{\mathrm{sp}(\Gamma) \subseteq \Lambda^0} W(\Gamma|q)\Big) \tag{7.2.13}$$

It is convenient to split $\delta h(\rho|\bar{q})$ in three parts:

$$\delta h(\rho|\bar{q}) = \sum_{i=1}^{3} h^i(\rho|\bar{q}), \qquad h^{1,2}(\rho|\bar{q}) = \sum_{i=1}^{2} h^i(\rho|\bar{q}) \tag{7.2.14}$$

where

$$h^{1,2}(\rho|\bar{q}) = -\log E_{\rho,\bar{q}}^0\big(e^{-\beta\Delta H(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q)}\big) \tag{7.2.15}$$

$$h^3(\rho|\bar{q}) = -\log E_{\rho,\bar{q}}\big(\sum_{\mathrm{sp}(\Gamma) \subseteq \Lambda^0} W(\Gamma|q)\big) \tag{7.2.16}$$

$$E_{\rho,\bar{q}}(f) = \frac{E_{\rho,\bar{q}}^0\big(e^{-\beta\Delta H(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q)} f\big)}{E_\rho^0\big(e^{-\beta\Delta H(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q)}\big)} \tag{7.2.17}$$

7.3 Analysis of $h^{1,2}$

In the sequel $\rho = \{\rho_x\}_{x\in X_\Lambda^{(\ell_2)}}$ is a fixed density configuration and all q will be compatible with ρ, i.e. $|q \cap C_x^{(\ell_2)}| = \rho_x \ell_2^d = n_x$. To simplify notation, we denote here by i the particle labels (instead of the pair (x,i) used before) so that the label i encodes the knowledge of the cube $C_x^{(\ell_2)}$ where q_i is; for such a reason we may also say that a label i is in some $\mathcal{D}^{(\ell_2)}$ measurable set.

The "error part" of the hamiltonian is

$$\begin{aligned}\Delta H(q|\bar q) = &-\sum_{i_1,i_2}[J_\gamma^{(2)}(q_{i_1},q_{i_2}) - \tilde J_\gamma^{(2)}(q_{i_1},q_{i_2})]\\ &+\sum_{i_1,i_2,i_3,i_4}[J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) - \tilde J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4})]\end{aligned} \tag{7.3.1}$$

We have used a shorthand notation: the sum is over distinct labels and runs over the particles of q and $\bar q$ with the condition that at least one of the particles of q should be present. For simplicity we write q_i also when the label i refers to the particle $\bar q_i$. The difference between $\tilde J_\gamma$ and J_γ is small, namely there are constants c_n, $n = 2, 4$, so that for all γ small enough

$$\big|J_\gamma^{(n)}(q_1,..,q_n) - \tilde J_\gamma^{(n)}(q_1,..,q_n)\big| \le c_n \gamma^{\alpha+(n-1)d}\prod_{i=2}^n \mathbf{1}_{|q_1-q_i|\le 2\gamma^{-1}}. \tag{7.3.2}$$

With this in mind we set

$$w_\gamma^{(2)}(q_{i_1},q_{i_2}) = e^{\beta[J_\gamma^{(2)}(q_{i_1},q_{i_2}) - \tilde J_\gamma^{(2)}(q_{i_1},q_{i_2})]} - 1 \tag{7.3.3}$$

$$w_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) = e^{-\beta[J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4}) - \tilde J_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4})]} - 1 \tag{7.3.4}$$

where both are bounded by a constant times the right hand side of (7.3.2). Then we have

$$e^{-\beta\Delta H(q|\bar q)} = \prod_{i_1,i_2}\Big(1 + w_\gamma^{(2)}(q_{i_1},q_{i_2})\Big)\prod_{i_1,i_2,i_3,i_4}\Big(1 + w_\gamma^{(4)}(q_{i_1},q_{i_2},q_{i_3},q_{i_4})\Big) \tag{7.3.5}$$

with same convention as for the sums in (7.3.1). Similarly, for $H^{\rm hc}(q)$ we have

$$e^{-\beta H^{\rm hc}(q)} = \prod_{i_1,i_2}(1 + v_R^{(2)}(q_{i_1},q_{i_2})), \qquad \text{where} \qquad v_R^{(2)}(q_{i_1},q_{i_2}) := e^{-\beta V^{\rm hc}(q_{i_1},q_{i_2})} - 1 \tag{7.3.6}$$

where we only consider the particles inside Λ, i.e. in (7.3.6) for each pair of indices i_1, i_2 we must have $q_{i_1}, q_{i_2} \in \Lambda$. We next expand the products in (7.3.5), (7.3.6) and classify the terms using the following notation:

- Let $L^{(2)} = (i_1, i_2)$ and $L^{(4)} = (i_1, i_2, i_3, i_4)$ denote a pair (resp. a quadruple) of mutually distinct particle labels and, given a labeled particle configuration q, we denote by $q(L^{(2)}) = (q_{i_1}, q_{i_2})$ and $q(L^{(4)}) = (q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4})$ the pair (resp. the quadruple) of the positions of the corresponding particles.

- We will refer to the two types of 2-links by calling them respectively γ-links and R-links.

- Two links are connected if they have a common label.

- (Definition of Θ.) A diagram θ is a collection of 2- and 4-links, i.e., a *ordered* triple $\theta \equiv \Big(\mathcal{L}_R^{(2)}(\theta), \mathcal{L}_\gamma^{(2)}(\theta), \mathcal{L}^{(4)}(\theta)\Big)$, where we denote by $\mathcal{L}_R^{(2)}(\theta)$, $\mathcal{L}_\gamma^{(2)}(\theta)$ and $\mathcal{L}^{(4)}(\theta)$ the set of 2-links (of type R and γ) and of 4-links in θ. Note that one can have a repetition of links, i.e. a same link $L^{(2)}$ can belong to both sets $\mathcal{L}_\gamma^{(2)}(\theta)$ and $\mathcal{L}_R^{(2)}(\theta)$. We use $\mathcal{L}^{(2)}(\theta)$ for the set of 2-links (which eventually contains twice a link when it is both a γ-link and a R-link) and Θ for the set of all such diagrams.

- Two diagrams θ and θ' are compatible ($\theta \sim \theta'$) if the set of their common labels is empty. A compatible collection of diagrams consists of mutually compatible diagrams. When they are not compatible we will use the notation $\nsim$.

- We define $w_{\gamma,R}(\theta, q)$ as the product of $w_\gamma^{(2)}(q(L^{(2)}))$, $v_R^{(2)}(q(L^{(2)}))$ and $w_\gamma^{(4)}(q(L^{(4)}))$ over all the links that contribute to θ:

$$w_{\gamma,R}(\theta, q) := v_R(\mathcal{L}_R^{(2)}(\theta), q) \prod_{L^{(2)} \in \mathcal{L}_\gamma^{(2)}(\theta)} w_\gamma^{(2)}(q(L^{(2)})) \prod_{L^{(4)} \in \mathcal{L}^{(4)}(\theta)} w_\gamma^{(4)}(q(L^{(4)})) \tag{7.3.7}$$

  where for any $L \subset \mathcal{L}_R^{(2)}$ we have also defined:

$$v_R(L, q) := \prod_{L^{(2)} \in L} v_R^{(2)}(q(L^{(2)})) \tag{7.3.8}$$

From (7.2.15), (7.3.5) and (7.3.6) we get:

$$h^{1,2}(\rho|\bar{q}) = -\log E^0_{\rho,\bar{q}}\Big(\sum_{\{\theta_1,\dots,\theta_k\}\sim} \prod_{j=1}^{k} w_{\gamma,R}(\theta_j, q)\Big), \tag{7.3.9}$$

where the sum is over all the compatible collections $\{\theta_1, ..., \theta_k\}$ such that every θ_j, $j = 1, ..., k$, has at least one particle inside Λ.

The goal is to bring (7.3.9) into the form of the partition function of an abstract polymer model for which one can prove the validity of the convergence condition. As far as the γ-links is concerned, due to the smallness of their weight (see (7.3.3) and (7.3.4)) when we multiply by the large number of such structures we will eventually be able to prove convergence. But, for the case of R-links, such labeled graph structure is dangerous since their number grows as e^{n^2} for the case of n vertices. The way out is to consider new types of diagrams where γ links may occur deliberately, but the R-links occur at most as a tree structure. Next, we present this construction in two steps after which we will eventually arrive to a new type of diagram for which we can prove convergence of the corresponding abstract polymer model.

We start with a definition: given a diagram θ and a label i such that $\theta \nsim i$, we provide an abstract rooted graph structure according to the following algorithm. The root of the graph is i. Links of θ which have i as an endpoint are links of the first generation. New links may start at the endpoints of the links of the first generation and they will be links of the second generation and so on. We define the graph distance of the label v (and we will use the notation $d(v)$) as the minimum number of links that are necessary to construct a path connecting v to the root. The m-th generation of the graph is the set of labels I such that $d(v) = m$ for all $v \in I$.

Step 1. We first get rid of all the R-links which appear over γ-links and we extract a subdiagram $\hat{\theta}$. Let $\hat{\Theta} \subset \Theta$ be the set of all the diagrams which do not have double 2-links, i.e. $\hat{\Theta} := \{\hat{\theta} : \mathcal{L}_\gamma^{(2)}(\hat{\theta}) \cap \mathcal{L}_R^{(2)}(\hat{\theta}) = \emptyset\}$.

Definition 7.3.1. We define a map $\phi : \Theta \to \hat{\Theta}$, which to each diagram $\theta \in \Theta$ associates a diagram $\hat{\theta} = \phi(\theta) \in \hat{\Theta}$, such that: $\mathcal{L}_\gamma(\phi(\theta)) = \mathcal{L}_\gamma(\theta)$ and $\mathcal{L}_R^{(2)}(\phi(\theta)) = \mathcal{L}_R^{(2)}(\theta) \setminus (\mathcal{L}_\gamma^{(2)}(\theta) \cap \mathcal{L}_R^{(2)}(\theta))$. Moreover, the set of all the possible diagrams θ can be obtained as the disjoint unions of $\phi^{-1}(\hat{\theta})$ and $\hat{\theta}$, for all $\hat{\theta} \in \hat{\Theta}$.

Given a diagram $\theta \in \Theta$, from Definition 7.3.1 we have:

$$w_{\gamma,R}(\theta, q) = w_{\gamma,R}(\phi(\theta), q) v_R(\mathcal{L}_R^{(2)}(\theta) \setminus \mathcal{L}_R^{(2)}(\phi(\theta)), q). \tag{7.3.10}$$

Step 2. The next step is to obtain a diagram which is at most a tree in R.

Definition 7.3.2. (Partial ordering relation $\prec$ on a diagram θ). For $L_1^{(2)}, L_2^{(2)} \in \mathcal{L}_R^{(2)}(\theta)$ we have that $L_1^{(2)} \prec L_2^{(2)}$ according to lexicographic ordering (i.e., we start by comparing the first index and if the same we compare the next etc.). We say that a diagram is ordered if the set of its R-links is ordered according to this definition.

Definition 7.3.3. (Redundant link). Given an ordered diagram θ, we say that a link $L^{(2)} \in \mathcal{L}_R^{(2)}(\theta)$ is redundant in the following two cases:

- If $L^{(2)} = \{i, j\}$ with $d(i) = d(j)$ (which we will call "horizontal");
- If $L_1^{(2)} = \{i_1, j\}$ with $d(i_1) = d(j) - 1$ and it exists $L_2^{(2)} = \{i_2, j\} \in \mathcal{L}_R^{(2)}(\theta)$, with $d(i_2) = d(j) - 1$, such that: $L_2^{(2)} \prec L_1^{(2)}$ (i.e., $i_2 < i_1$). We call such a link " vertical" and redundant with respect to some other link that has the same endpoint in the generation more distant from the root.

We denote the set of the redundant links of a diagram θ by: $\mathcal{R}_R^{(2)}(\theta)$.

We call $\bar{\Theta} \subset \hat{\Theta}$ the set of diagrams with no double 2-links and with no redundant links. In formulas: $\bar{\Theta} := \{\bar{\theta} : \bar{\theta} \in \hat{\Theta}, \mathcal{R}_R^{(2)}(\bar{\theta}) = \emptyset\}$.

Definition 7.3.4. We define the map $\psi : \hat{\Theta} \to \bar{\Theta}$, which to each diagram $\hat{\theta} \in \hat{\Theta}$ associates a diagram $\psi(\hat{\theta})$ such that: $\mathcal{L}_\gamma^{(2)}(\psi(\hat{\theta})) = \mathcal{L}_\gamma^{(2)}(\hat{\theta})$ and $\mathcal{L}_R^{(2)}(\psi(\hat{\theta})) = \mathcal{L}_R^{(2)}(\hat{\theta}) \setminus \mathcal{R}_R^{(2)}(\hat{\theta})$. Moreover, the set of all possible diagrams $\hat{\theta} \in \hat{\Theta}$ is the disjoint union of all $\psi^{-1}(\bar{\theta})$ and $\bar{\theta}$, for all $\bar{\theta}$.

From (7.3.10), for any $\theta \in \Theta$ we have (recall also the definition in (7.3.8))

$$w_{\gamma,R}(\theta, q) = w_{\gamma,R}(\psi(\phi(\theta)), q)\, v_R^{(2)}(\mathcal{L}_R^{(2)}(\phi(\theta)) \setminus \mathcal{R}_R^{(2)}(\phi(\theta)), q)\, v_R^{(2)}(\mathcal{L}_R^{(2)}(\theta) \setminus \mathcal{L}_R^{(2)}(\phi(\theta)), q) \tag{7.3.11}$$

For $\bar{\theta} \in \bar{\Theta}$ we define:

$$\bar{w}_{\gamma,R}(\bar{\theta}, q) := w_{\gamma,R}(\bar{\theta}, q) \sum_{\hat{\theta}:\psi(\hat{\theta})=\bar{\theta}} v_R^{(2)}(\mathcal{L}_R^{(2)}(\hat{\theta}) \setminus \mathcal{R}_R^{(2)}(\hat{\theta}), q) \sum_{\theta:\phi(\theta)=\hat{\theta}} v_R^{(2)}(\mathcal{L}_R^{(2)}(\theta) \setminus \mathcal{L}_R^{(2)}(\hat{\theta}), q) \tag{7.3.12}$$

which using (7.3.11) yields

$$\begin{aligned}\sum_{\theta\in\Theta} w_{\gamma,R}(\theta,q) &= \sum_{\bar\theta\in\bar\Theta}\sum_{\hat\theta:\psi(\hat\theta)=\bar\theta}\sum_{\theta:\phi(\theta)=\hat\theta} w_{\gamma,R}(\theta,q)\\ &= \sum_{\bar\theta\in\bar\Theta} w_{\gamma,R}(\bar\theta,q)\sum_{\hat\theta:\psi(\hat\theta)=\bar\theta} v_R(\mathcal{L}_R^{(2)}(\hat\theta)\setminus\mathcal{R}_R^{(2)}(\hat\theta),q)\sum_{\theta:\phi(\theta)=\hat\theta} v_R(\mathcal{L}_R^{(2)}(\theta)\setminus\mathcal{L}_R^{(2)}(\hat\theta),q)\\ &= \sum_{\bar\theta\in\bar\Theta}\bar w_{\gamma,R}(\bar\theta,q). \end{aligned} \tag{7.3.13}$$

We define the statistical weight of the diagram $\bar\theta$:

$$z_{\gamma,R}(\bar\theta,\bar q) := E^0_{\rho,\bar q}\big(\bar w_{\gamma,R}(\bar\theta,\cdot)\big) \tag{7.3.14}$$

and we obtain:

$$h^{1,2}(\rho|\bar q) = -\log\sum_{\{\bar\theta_1,\dots,\bar\theta_k\}_\sim}\prod_{j=1}^k z_{\gamma,R}(\bar\theta_j,\bar q) \tag{7.3.15}$$

where the sum is over compatible collections of diagrams as before. Hence, we are in the context of the abstract polymer model on $\bar\Theta$ with activities $z_{\gamma,R}(\bar\theta)$ and we next prove the convergence condition. Given $\bar\theta\in\bar\Theta$, let $|\bar\theta|$ be the number of distinct particle labels in $\bar\theta$ and $|\mathcal{L}_\gamma(\bar\theta)|$ be the number of only the γ-links.

Lemma 7.3.5. *For $\kappa>0$ let*

$$a(\bar\theta) := |\bar\theta|, \quad \textit{and} \quad b(\bar\theta) := \kappa\log\gamma^{-1}|\mathcal{L}_\gamma(\bar\theta)| + |\bar\theta| \tag{7.3.16}$$

Then $\forall k<\alpha$ and for all γ and R small enough,

$$\sum_{\bar\theta\not\sim\bar\theta'}|z_{\gamma,R}(\bar\theta,\bar q)|e^{a(\bar\theta)+b(\bar\theta)} \le a(\bar\theta'). \tag{7.3.17}$$

for any fixed $\bar\theta'$.

Proof. It suffices to prove

$$\sum_{\bar\theta:\ \bar\theta\sim i}|z_{\gamma,R}(\bar\theta,\bar q)|e^{a(\bar\theta)+b(\bar\theta)} \le 1, \tag{7.3.18}$$

for every fixed i. We next bound (7.3.14): given $\bar\theta\in\bar\Theta$ and some $\hat\theta\in\psi^{-1}(\bar\theta)$, in order to bound the last sum in (7.3.12) we recall that there is a one-to-one correspondence between

an element $\theta \in \phi^{-1}(\hat{\theta})$ and the union of $\hat{\theta}$ with the possible extra R-links (which have to be chosen among the links $\mathcal{L}_\gamma^{(2)}(\hat{\theta})$). Hence, recalling (7.3.8),

$$\begin{aligned}\sum_{\theta:\phi(\theta)=\hat{\theta}} v_R(\mathcal{L}_R^{(2)}(\theta) \setminus \mathcal{L}_R^{(2)}(\hat{\theta}), q) &= \sum_{A \subset \mathcal{L}_\gamma^{(2)}(\hat{\theta})} v_R(A, q) = \sum_{A \subset \mathcal{L}_\gamma^{(2)}(\hat{\theta})} \prod_{L^{(2)} \in A} v_R^{(2)}(q(L^{(2)})) \\ &= \prod_{L^{(2)} \in \mathcal{L}_\gamma^{(2)}(\hat{\theta})} [v_R^{(2)}(q(L^{(2)})) + 1] \leq 1 \end{aligned} \tag{7.3.19}$$

where in the last inequality we use the positivity of the potential, i.e., $v_R^{(2)}(q(L^{(2)})) + 1 \leq 1$ for every $L^{(2)}$.

To bound the second sum in (7.3.12) we define the set of all redundant links that one can add to $\bar{\theta}$:

$$\mathbf{R}(\bar{\theta}) = \cup_{\hat{\theta} \in \psi^{-1}(\bar{\theta})} \mathcal{R}_R^{(2)}(\hat{\theta}) \tag{7.3.20}$$

Then,

$$\begin{aligned}\sum_{\hat{\theta}:\psi(\hat{\theta})=\bar{\theta}} v_R(\mathcal{L}_R^{(2)}(\hat{\theta}) \setminus \mathcal{R}_R^{(2)}(\hat{\theta}), q) &= \sum_{A \subset \mathbf{R}(\bar{\theta})} v_R(A, q) = \sum_{A \subset \mathbf{R}(\bar{\theta})} \prod_{L^{(2)} \in A} v_R^{(2)}(q(L^{(2)})) \\ &= \prod_{L^{(2)} \in \mathbf{R}(\bar{\theta})} [v_R^{(2)}(q(L^{(2)})) + 1] \leq 1, \end{aligned} \tag{7.3.21}$$

again by the positivity of the potential. Thus, using (7.3.19) and (7.3.21), from (7.3.12) we have reduced to diagrams which are at most trees in the R-links. To bound the activity, γ-links are bounded uniformly in q using (7.3.2) while R-links can now be integrated independently yielding:

$$z_{\gamma,R}(\bar{\theta}, \bar{q}) \leq E^0_{\rho,\bar{q}}\big(w_{\gamma,R}(\bar{\theta}, \cdot)\big) \leq \prod_{L \in \mathcal{L}_R^{(2)}(\bar{\theta})} \bar{v}_R^{(2)}(L, \bar{q}) \prod_{L \in \mathcal{L}_\gamma^{(2)}(\bar{\theta})} \bar{w}_\gamma^{(2)}(L, \bar{q}) \prod_{L \in \mathcal{L}_\gamma^{(4)}(\bar{\theta})} \bar{w}_\gamma^{(4)}(L, \bar{q}) \tag{7.3.22}$$

where we have used the definitions:

$$\bar{v}_R^{(2)}(L, \bar{q}) := E^0_{\rho,\bar{q}}\big(v_R(q(L), \bar{q})\big), \qquad \bar{w}_\gamma^{(n)}(L, \bar{q}) := \max_q w_\gamma^{(n)}(q(L), \bar{q}),\ n = 2, 4 \tag{7.3.23}$$

Note the difference between $v_R^{(2)}(\cdot)$, $v_R(\cdot, \cdot)$ and $\bar{v}_R^{(2)}(\cdot, \cdot)$; similarly for $w_{\gamma,R}^{(n)}(\cdot, \cdot)$ and $\bar{w}_{\gamma,R}^{(n)}(\cdot, \cdot)$, $n = 2, 4$. From (7.3.2) we have:

$$|\bar{w}_\gamma^{(2)}(L, \bar{q})| \leq c\gamma^{\alpha+d}, \qquad |\bar{w}_\gamma^{(4)}(L, \bar{q})| \leq c\gamma^{\alpha+3d}. \tag{7.3.24}$$

For $\bar{v}_R^{(2)}(L,\bar{q})$ supposing that $L \equiv \{i,j\}$, if both labels correspond to particles in the same box C we have:

$$|\bar{v}_R^{(2)}(L,\bar{q})| \leq \frac{\left| \int_{q_i \in C} dq_i e^{-\beta U^{\mathrm{hc}}(q_i,\bar{q})} \int_{q_j \in C \cap B_R(q_i)} dq_j \; e^{-\beta U^{\mathrm{hc}}(q_j,\bar{q})} (e^{-\beta V^{\mathrm{hc}}(q_i,q_j)} - 1) \right|}{\left| \int_{\mathcal{Q}_+^\Lambda} dq_j \; e^{-\beta U^{\mathrm{hc}}(q_j,\bar{q})} \mathbb{1}_{q_j \in C \setminus \partial_R C^{\mathrm{int}}} \right|^2} \tag{7.3.25}$$

where $U^{\mathrm{hc}}(q,\bar{q})$ is defined in (7.2.6) and $\partial_R C^{\mathrm{int}}$ is the inner boundary of width R of the box C. Calculating the denominator we obtain: $(\ell_2 - 2R)^{2d}$, because when q_j is in $C \setminus \partial_R C^{\mathrm{int}}$ the particle can not interact with the boundary and therefore $U^{\mathrm{hc}}(q_j,\bar{q}) = 0$. The numerator can be bounded by $cR^d \ell_2^d$. Hence,

$$|\bar{v}_R^{(2)}(L,\bar{q})| \leq c \frac{\epsilon}{\ell_2^d} \tag{7.3.26}$$

where ϵ is the volume of a hard-sphere of radius R and c is a constant almost equal to 1. On the other hand, if $q_i \in C_x$ and $q_j \in C_y$ we obtain:

$$\begin{aligned} \left| \int_{q_i \in C_x} \frac{dq_i}{\ell_2^d} \mathbb{1}_{q_i \in \partial_R C_x} \int_{q_j \in C_y \cap B_R(q_i)} \frac{dq_j}{\ell_2^d} (e^{-\beta V^{\mathrm{hc}}(q_i,q_j)} - 1) \right| \leq \\ \frac{\ell_2^d - (\ell_2 - 2R)^d}{\ell_2^d} \frac{R^d}{\ell_2^d} \leq \frac{2R}{\ell_2} \frac{R^d}{\ell_2^d} \leq \frac{2R\epsilon}{\ell_2^{d+1}} \end{aligned} \tag{7.3.27}$$

which is of even lower order.

To prove (7.3.18) we proceed by induction. We first prove (7.3.18) for $\bar{\theta}$ with $\ell(\bar{\theta}) = 1$. Suppose that $|\mathcal{L}_R^{(2)}(\bar{\theta})| = k$, $|\mathcal{L}_\gamma^{(2)}(\bar{\theta})| = n$ and $|\mathcal{L}_\gamma^{(4)}(\bar{\theta})| = m$. The number of vertices in $\bar{\theta}$ is at most $1 + k + n + 3m$. Since the configurations are in $\mathcal{Q}_+$ there are at most $c(\rho_{\beta,+} + \zeta)\gamma^{-d}$ 2-links and at most $c(\rho_{\beta,+} + \zeta)\gamma^{-3d}$ 4-links passing through it. Analogously for the R-links, one can consider that once we fix a label, there are at most $c(\rho_{\beta,+} + \zeta)\ell_2^d$ particles which can interact with it. We have:

$$\begin{aligned} \sum_{\bar{\theta} \nsim i : \ell(\bar{\theta}) = 1} |z_{\gamma,R}(\bar{\theta},\bar{q})| e^{a(\bar{\theta}) + b(\bar{\theta})} \leq \sum_{n+k+m>0} [c(\rho_{\beta,+} + \zeta)\gamma^{-d}]^n \cdot [c(\rho_{\beta,+} + \zeta)\ell_2^d]^k \cdot [c(\rho_{\beta,+} + \zeta)\gamma^{-3d}]^m \\ \gamma^{(\alpha+d)n + (\alpha+3d)m} \cdot \left(\frac{\epsilon}{\ell_2^d}\right)^k \cdot e^{\kappa \log \gamma^{-1}(n+m) + 2(n+k+3m+1)} \leq \\ \leq e^2 \sum_{n+m} \gamma^{-\kappa(n+m)} e^{2(n+3m)} [c(\rho_{\beta,+} + \zeta)\gamma^\alpha]^{n+m} \\ \cdot \sum_k e^{2k} [c(\rho_{\beta,+} + \zeta)\epsilon]^k \end{aligned} \tag{7.3.28}$$

Note that the two sums can start either from 0 or from 1, but not both from 0, since we are supposing that there is at least a point in the first generation. The 2 in the exponential comes from both the contribution of $a(\bar{\theta})$ and $b(\bar{\theta})$ to $|\bar{\theta}|$. Thus, choosing $\kappa < \alpha$, the following inequality holds true for γ and R small enough

$$\sum_{\bar{\theta} \nsim i : \ell(\bar{\theta})=1} |z_{\gamma,R}(\bar{\theta}, \bar{q})| e^{a(\bar{\theta})+b(\bar{\theta})} \leq e^2[1+x][1+y][1+z] - e^2 \leq 1 \tag{7.3.29}$$

where:

$$\begin{aligned} x :=& e^{(\kappa-\alpha)\log\gamma^{-1}+2}[c(\rho_{\beta,+}+\zeta)] \leq 1 \\ y :=& e^{(\kappa-\alpha)\log\gamma^{-1}+6}[c(\rho_{\beta,+}+\zeta)] \leq 1 \\ z :=& e^2[c(\rho_{\beta,+}+\zeta)]\epsilon \leq 1. \end{aligned} \tag{7.3.30}$$

We proceed by induction. We suppose that:

$$X_N := \sum_{\bar{\theta} \nsim i : \ell(\bar{\theta}) \leq N} |z_{\gamma,R}(\bar{\theta}, \bar{q})| e^{a(\bar{\theta})+b(\bar{\theta})} \leq 1 \tag{7.3.31}$$

and we prove it for $\bar{\theta}$ with $\ell(\bar{\theta}) \leq N+1$. Summing over the links of the first generation we have:

$$\begin{aligned} \sum_{\bar{\theta} \nsim i : \ell(\bar{\theta}) \leq N+1} |z_{\gamma,R}(\bar{\theta}, \bar{q})| e^{a(\bar{\theta})+b(\bar{\theta})} \leq & \sum_{n+k+m>0} \sum_{\substack{A \subset \mathcal{L}_R^{(2)} \\ |A|=k}} \sum_{\substack{B \subset \mathcal{L}_\gamma^{(2)} \\ |B|=n}} \sum_{\substack{C \subset \mathcal{L}_\gamma^{(4)} \\ |C|=m}} \prod_{L \in A} |\bar{v}_R^{(2)}(L, \bar{q})| \cdot \prod_{L \in B} |\bar{w}_\gamma^{(2)}(L, \bar{q})| \cdot \\ & \prod_{L \in C} |\bar{w}_\gamma^{(4)}(L, \bar{q})| \cdot [1+X_N]^{1+k+n+3m} e^{2(1+k+n+3m)+\kappa \log\gamma^{-1}(n+m)}, \end{aligned} \tag{7.3.32}$$

where 1 in the square bracket corresponds to the case in which nothing is "growing" from a given endpoint of the first generation link while X_N is the inductive estimate for the case of nonempty subdiagrams growing from the given endpoint. Using again (7.3.31) and (7.3.28), we have:

$$\begin{aligned} \sum_{\bar{\theta} \nsim i : \ell(\bar{\theta})=1} |z_{\gamma,R}(\bar{\theta}, \bar{q})| e^{a(\bar{\theta})+b(\bar{\theta})} \leq & 2e^2 \sum_{n+m} e^{2(n+m)} (2\gamma^{-\kappa})^{(n+3m)} [c(\rho_{\beta,+}+\zeta)\gamma^\alpha]^{n+m} \cdot \\ & \cdot \sum_k (2e^2)^k [c(\rho_{\beta,+}+\zeta)\epsilon]^k. \end{aligned} \tag{7.3.33}$$

Hence, again choosing $\kappa < \alpha$, the following inequality holds true for γ and R small enough

$$\sum_{\bar\theta \nsim i : \ell(\bar\theta)=1} |z_{\gamma,R}(\bar\theta, \bar q)| e^{a(\bar\theta)+b(\bar\theta)} \le 2e^2[1+x'][1+y'][1+z'] - 2e^2 \le 1 \tag{7.3.34}$$

where:

$$x' := 2x \le 1, \quad y' := 8y \le 1, \quad z' := 2z \le 1, \tag{7.3.35}$$

with x, y, z as in (7.3.30). Lemma 7.3.5 is proved. □

By Lemma 7.3.5, it is a standard result (see e.g. [34]) that (7.3.15) becomes:

$$h^{1,2}(\rho|\bar q) = -\sum_{\pi} z^T_{\gamma,R}(\pi; \rho; \bar q) \tag{7.3.36}$$

where π is a collection of non-compatible diagrams (repetitions are allowed) and $z^T_{\gamma,R}(\pi;\rho)$ is the Möbius inversion:

$$z^T_{\gamma,R}(\pi;\rho) := \sum_{\pi' :\, \pi' \subset \pi} (-1)^{|\pi|-|\pi'|} \log \sum_{\substack{\{\bar\theta_1,\dots,\bar\theta_n\}_\sim, \\ \bar\theta_j \in \pi', \forall j}} \prod_{j=1}^{n} z_{\gamma,R}(\bar\theta_j), \quad |\pi| := |\{\bar\theta :\, \bar\theta \in \pi\}| \tag{7.3.37}$$

Note that $z^T_{\gamma,R}(\pi;\rho) = 0$ if π is not connected and from now on we will call π a cluster. Moreover we have that for any fixed diagram $\bar\theta'$

$$\sum_{\pi \nsim \bar\theta'} |z^T_{\gamma,R}(\pi;\rho)| e^{b(\pi)} \le a(\bar\theta'), \text{ where } \quad b(\pi) := \sum_{\bar\theta \in \pi} b(\bar\theta)\pi(\theta) \tag{7.3.38}$$

or

$$\sum_{\pi \ni \bar\theta'} |z^T_{\gamma,R}(\pi;\rho)| e^{b(\pi)} \le |z_{\gamma,R}(\theta')| e^{a(\bar\theta')} \tag{7.3.39}$$

recalling also that $b(\bar\theta) = \kappa \log \gamma^{-1} |\mathcal{L}_\gamma(\bar\theta)| + |\bar\theta|$ for some $\kappa < \alpha$.

Geometrically a cluster is a diagram-like object with possibly some links entering more than once. We underline that despite in diagrams the particles are labelled, the sum on the r.h.s. of (7.3.36) becomes however independent of the labels, depending only on the number of particles in each cube $C^{(\ell_2)}$, i.e., on ρ.

7.4 Analysis of h^3

We fix, for a while, $\underline\Gamma = \{\Gamma_i\}_{i=1}^n$, where $\Gamma_i \equiv (\mathrm{sp}(\Gamma_i), \eta_{\Gamma_i})$ and set $f = W(\underline\Gamma|q) = \prod_{i=1}^n W(\Gamma_i|q)$ in the numerator in (7.2.17). Recall that $W(\Gamma_i|q)$ depends only on $q^{D_i} := q \cap D_i$

where $D_i := \{r \in c(\Gamma_i)^c : \mathrm{dist}(r, c(\Gamma_i)) \le 2\gamma^{-1}\} \in \mathcal{D}^{(\ell_2)}$. We also let $D := \cup_{i=1}^n D_i$. Conditioning on the values of q in D, we are in the same setup as in the previous subsection with Λ replaced by $\Lambda \setminus D$, because $W(\Gamma|q)$ does not depend on $q \cap D^c$. We then have

$$E^0_{\rho,\bar q}\big(e^{-\beta\Delta H(q|\bar q)} e^{-\beta H^{\mathrm{hc}}(q)} W(\Gamma|q)\big) = E^0_{\rho^D,\bar q}\Big(W(\Gamma|q) e^{-h^{1,2}(\rho^{\Lambda\setminus D}|\bar q \cup q^D)}\Big) \tag{7.4.1}$$

with obvious meaning of the symbols.

We write $h^{1,2}$ as a sum of clusters using (7.3.36). As we saw in the previous section, due to the dependence of the prior measure on $\bar q$ (now on both $\bar q$ and q^D), the clusters involving a particle in a neighboring ℓ_2-cell to D will also depend on q^D. We denote the union of the set D with the frame consisting of the neighboring ℓ_2-cells by $D^* \in \mathcal{D}^{(\ell_2)}$.

To distinguish between clusters we introduce $\bar D_i := D_i \cup \{r : \mathrm{dist}(r, D_i) \le \ell_3/4\} \in \mathcal{D}^{(\ell_2)}$ and we call $\mathcal{B}_i$ the set of all clusters π whose points are all in $\bar D_i$. As the distance between contours is $\ge \ell_3$, the sets $\mathcal{B}_i$ are mutually disjoint; we call $\mathcal{B}$ their union. Note that they depend on $\underline{\Gamma}$ through the domain where they are constructed.

By $\mathcal{R}_i$ we denote the set of π which have points both in D_i^* (so that they depend on q^D) and in the complement of $\bar D_i$ (such π are therefore not in $\mathcal{B}_i$). There may be points of $\pi \in \mathcal{R}_i$ which are in D_j^*, $j \ne i$, hence also $\pi \in \mathcal{R}_j$, so that the sets $\mathcal{R}_i$ are not disjoint. We call $\mathcal{R}$ their union.

For any given $\underline{\Gamma}$ we do analogous splitting on the polymers appearing when developing the denominator of (7.2.17) thus defining the sets $\mathcal{B}_i'$, $\mathcal{B}'$, $\mathcal{R}_i'$, $\mathcal{R}'$.

The clusters that appear in the numerator and denominator of (7.2.17) are different, however those not in $\mathcal{B}\cup\mathcal{R}$ (i.e., those that do not involve q^D) are common to the corresponding ones in the denominator (7.2.17) (i.e., those not in $\mathcal{B}'\cup\mathcal{R}'$) and have same statistical weights, hence they cancel in the formula below. We therefore get ($F(q^D)$ is defined later in (7.4.3))

$$\begin{aligned} h^3(\rho|\bar q) = -\log \sum_{\underline{\Gamma}:\, \mathrm{sp}(\underline{\Gamma}) \subset \Lambda^0} E^0_{\rho^D,\bar q}\Big(& F(q^D) W(\underline{\Gamma}|q^D) \\ & \times \exp\Big\{ -\sum_{\pi\in\mathcal{R}} z^T_{\gamma,R}(\pi|\bar q \cup q^D) + \sum_{\pi\in\mathcal{R}'} z^T_{\gamma,R}(\pi|\bar q)\Big\}\Big). \end{aligned} \tag{7.4.2}$$

Recall that the clusters in $\mathcal{R}'$ can have links also in the interior of D^*. The expectation $E^0_{\rho^D,\bar q}$ is over the positions q^D of the particles in D, their numbers being specified by ρ^D and we

also know which particles are in which box. We have also defined:

$$F(q^D) = \prod_{i=1}^{n} F(q^{D_i}), \qquad F(q^{D_i}) := \exp\Big\{ - \sum_{\pi \in \mathcal{B}_i} z^T_{\gamma,R}(\pi | q^{D_i}) + \sum_{\pi \in \mathcal{B}'_i} z^T_{\gamma,R}(\pi) \Big\} \tag{7.4.3}$$

We have used that both sets $\mathcal{B}_i$ and $\mathcal{B}'_i$ are mutually disjoint and the sets $\bar{D}_i$ are in Λ^0 hence away from Λ^c. Note that $F(q^{D_i})$ is a probability density: $E^0_{\rho^{D_i}}(F(q^{D_i})) = 1$, with the normalization being the second sum (over $\pi \in \mathcal{B}'_i$ in (7.4.3)). In fact, calling $D^0_i = \bar{D}_i \setminus D_i$,

$$\exp\Big\{ - \sum_{\pi \in \mathcal{B}_i} z^T_{\gamma,R}(\pi | q^{D_i}) \Big\} = E^0_{\rho^{D^0_i}} \Big(e^{-\beta \Delta H(q^{D^0_i} | q^{D_i})} e^{-\beta H^{\rm hc}(q^{D^0_i})} \Big)$$

where the energy $\Delta H(q^{D^0_i} | q^{D_i})$ is defined as in (7.2.12) with nothing outside $\bar{D}_i$. We then get

$$E^0_{\rho^{D_i}} \Big(\exp\Big\{ - \sum_{\pi \in \mathcal{B}_i} z^T_{\gamma,R}(\pi | q^{D_i}) \Big\} \Big) = E^0_{\rho^{\bar{D}_i}} \Big(e^{-\beta \Delta H(q^{\bar{D}_i})} e^{-\beta H^{\rm hc}(q^{\bar{D}_i})} \Big)$$

which is exactly the second sum in (7.4.3), since:

$$\exp\Big\{ - \sum_{\pi \in \mathcal{B}'_i} z^T_{\gamma,R}(\pi) \Big\} = E^0_{\rho^{\bar{D}_i}} \Big(e^{-\beta \Delta H(q^{\bar{D}_i})} e^{-\beta H^{\rm hc}(q^{\bar{D}_i})} \Big)$$

hence $E^0_{\rho^{D_i}}(F(q^{D_i})) = 1$.

Both $F(q^D)$ and $W(\Gamma | q^D)$ are product of factors depending on the variables q^{D_i}, thus the only interaction in (7.4.2) comes from the exponent in the last term (which represents the interaction between the contours Γ_i). To decouple the variables q^{D_i} we define:

$$f^{\pm}(\pi; \bar{q}, q^D) := e^{\mp z^T_{\gamma,R}(\pi | \bar{q}, q^D)} - 1, \quad \text{for } \pi \in \mathcal{R} \text{ or } \mathcal{R}' \text{ resp.} \tag{7.4.4}$$

and in (7.4.2) we expand the products obtaining:

$$\begin{aligned} \exp\Big\{ - \sum_{\pi \in \mathcal{R}} z^T_{\gamma,R}(\pi | \bar{q}, q^D) + \sum_{\pi \in \mathcal{R}'} z^T_{\gamma,R}(\pi | \bar{q}) \Big\} &= \prod_{\pi \in \mathcal{R}} (1 + f^+(\pi; \bar{q}, q^D)) \prod_{\pi \in \mathcal{R}'} (1 + f^-(\pi; \bar{q})) \\ &= \sum_{\substack{\pi_1, \dots, \pi_k, \\ \pi_i \in \mathcal{R} \cup \mathcal{R}'}} \prod_{i=1}^{k} f^{\pm}(\pi_i; \bar{q}, q^D) \end{aligned} \tag{7.4.5}$$

If we plug it into (7.4.2) we obtain products of $F(q^D) W(\Gamma | q^D)$ and $f^{\pm}(\pi)$ for $\pi \in \mathcal{R}$ or $\mathcal{R}'$. To integrate, it might occur that there is an element $f^{\pm}(\pi)$ with labels which belong to the sets D^*_i and D^*_j $(i \neq j)$, i.e., the "frames" around two contours Γ_i and Γ_j in Γ. Hence, to

formulate the problem into the general context of the abstract polymer model we define as *connected* polymer P a set of contours with "connections" consisting of elements of $\mathcal{R} \cup \mathcal{R}'$ which necessarily "connect" all contours in the given set and "decorations" consisting of clusters in $\mathcal{R} \cup \mathcal{R}'$ not necessarily connecting contours. We denote by $\mathcal{P}$ the space of all such elements:

$$\begin{aligned} \mathcal{P} \quad := \quad & \big\{ P \equiv (\underline{\Gamma}(P), R(P)), \forall \Gamma_i, \Gamma_j \in \underline{\Gamma}(P), \exists \pi \in R(P) \subset \mathcal{R} \cup \mathcal{R}' \\ & \text{connecting } \; D_i^*, D_j^* \in D^*(\underline{\Gamma}) \big\} \end{aligned} \tag{7.4.6}$$

We use $D(P), D^*(P)$ to denote the set of frames corresponding to the contours in P and $R(P)$ to denote the set of clusters. We also introduce $A(\pi)$ to denote the union of the $C^{(\ell_2)}$ cells which correspond to the labels of π. Similarly, let $A(P) := \cup_{\Gamma \in \underline{\Gamma}(P)} D^*(\Gamma) \cup_{\pi \in R(P)} A(\pi)$. We say that two polymers P and P' are compatible, $P \sim P'$, if

$$\begin{aligned} & \overline{\mathrm{sp}(\Gamma)} \cap \overline{\mathrm{sp}(\Gamma')} = \emptyset, \forall \Gamma \in \underline{\Gamma} \,\mathrm{and}\, \Gamma' \in \underline{\Gamma}', \qquad \text{and} \\ & \big\{ D^*(P) \cap A(P') \big\} \cup \big\{ D^*(P') \cap A(P) \big\} = \emptyset \end{aligned}$$

A compatible collection of polymers consists of mutually compatible polymers. The statistical weight of P is

$$\zeta_{\gamma,R}(P) := E^0_{\rho^{D(P)},\bar{q}} \Big(\prod_{D \in D(P)} [F(q^D) W(\Gamma(D)|q^D)] \prod_{\pi \in R(P)} f^{\pm}(\pi; \bar{q}, q^D) \Big) \tag{7.4.7}$$

where $f^{\pm}_{\pi}$ is one of the two appearing in (7.4.4), which one being specified by the clusters in $R(P)$. Expressed in terms of polymers, the expectation in (7.4.2) becomes

$$h^3(\rho|\bar{q}) = -\log \sum_{\substack{\{P_1,\ldots,P_k\}_\sim \\ P_i \in \mathcal{P}}} \prod_{i=1}^{k} \zeta_{\gamma,R}(P_i) \tag{7.4.8}$$

where the sum is over compatible collections of diagrams.

We have an analogue of Lemma 7.3.5. We define $n(\pi)$ and $n(\mathcal{L}_\gamma(\pi))$ to be respectively the number of distinct particle labels and of γ-links in π; $n(P) := n(\cup_{\pi \in R(P)} \pi)$ and $n(\mathcal{L}_\gamma(P)) := n(\cup_{\pi \in R(P)} \mathcal{L}_\gamma(\pi))$ of those in $R(P)$. Moreover, let N_Γ be the total number of cubes $C^{(\ell_3)}$ in $\mathrm{sp}(\Gamma)$, $m(P) = \sum_{\Gamma \in \underline{\Gamma}(P)} N_\Gamma$.

Lemma 7.4.1. *For any $b > 0$ let*

$$|P| := n(P) + m(P), \qquad g_b(P) := b[\log \gamma^{-1} n(\mathcal{L}_\gamma(P)) + \gamma^{2a-(1-\alpha)d} m(P)] \tag{7.4.9}$$

Then there is $b^\star > 0$ so that for all γ and R small enough

$$\sum_{P \not\sim P'} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^\star}(P)} \leq |P'| \tag{7.4.10}$$

for any fixed $P' \in \mathcal{P}$.

Proof. We claim that it suffices to prove that there is $\delta > 0$ (to be determined) so that for all $x \in X_\Lambda^{(\ell_3)}$

$$\sum_{P:\, \mathrm{sp}(\underline{\Gamma}(P)) \supset C_x^{(\ell_3)}} |\zeta_{\gamma,R}(P)| e^{|P|+g_\delta(P)} \leq 1 \tag{7.4.11}$$

It is straightforward that (7.4.11) further implies

$$\sum_{P:\, \mathrm{sp}(\underline{\Gamma}(P)) \supset C_x^{(\ell_3)}} |\zeta_{\gamma,R}(P)| e^{|P|+g_{\delta_1}(P)} \leq \exp\big\{ -\delta_1 \gamma^{2a-(1-\alpha)d} \big\} \tag{7.4.12}$$

where $\delta_1 = \frac{\delta}{2}$ and δ as in (7.4.11). With this result at hand we first prove the claim. Recalling the definition of compatibility, our goal is to bound the following (for some b^* to be determined):

$$\begin{aligned} \sum_{P \not\sim P'} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^\star}(P)} \quad \leq \quad & \sum_{\substack{x \in X_\Lambda^{(\ell_3)}: \\ C_x^{(\ell_3)} \subset \mathrm{sp}(\underline{\Gamma}(P'))}} \sum_{P:\, \mathrm{sp}(\underline{\Gamma}(P)) \cap C_x^{(\ell_3)} \neq \emptyset} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^\star}(P)} \\ & + \sum_{\substack{x \in X_\Lambda^{(\ell_2)}: \\ C_x^{(\ell_2)} \subset A(P') \setminus D^*(P')}} \sum_{P:\, D^*(P) \supset C_x^{(\ell_2)}} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^\star}(P)} \\ & + \sum_{\substack{x \in X_\Lambda^{(\ell_2)}: \\ C_x^{(\ell_2)} \subset D^*(P')}} \sum_{P:\, A(P) \setminus D^*(P) \supset C_x^{(\ell_2)}} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^\star}(P)} \end{aligned} \tag{7.4.13}$$

In the first term, the first sum can be bounded by $m(P')$, while the second by (7.4.12), for $b^* = \delta_1$. In the second term, the first sum is bounded by the number of labels in $R(P')$ (i.e., $n(P')$), while the second sum can be bounded by the second sum of the first term, i.e., again

by (7.4.12). For the third term, the first sum is bounded by $m(P')$ like before. The second sum we further bound it by:

$$\begin{aligned}
&\sup_x \sum_{y\in X_\Lambda:\, y\neq x} \sum_{\substack{P:\, A(P)\supset C_x^{(\ell_2)} \\ \exists \pi\in R(P):\, A(\pi)\supset C_y^{(\ell_2)}}} |\zeta_{\gamma,R}(P)| e^{|P|+g_{b^*}(P)} \\
&\leq \sup_x \sum_{y\in X_\Lambda:\, y\neq x} \sum_{\pi:\, \pi\supset C_x^{(\ell_2)}\cup C_y^{(\ell_2)}} |z^T_{\gamma,R}(\pi)| e^{(1+b^*\log\gamma^{-1})n(\pi)} \sum_{\tilde P:\, D^*(\tilde P')\supset C_y^{(\ell_2)}} |\zeta_{\gamma,R}(\tilde P')| e^{|\tilde P'|+g_{b^*}(\tilde P')} \\
&\leq C \sup_x \sum_{y\in X_\Lambda:\, y\neq x} \exp\big\{-\delta_1\gamma^{2a-(1-\alpha)d}\big\} \qquad (7.4.14)
\end{aligned}$$

for some constant $C>0$. In the second line we wrote the polymer P as the union of a cluster π with $\pi\supset C_x^{(\ell_2)}\cup C_y^{(\ell_2)}$ and of another polymer $\tilde P$ with $D^*(\tilde P)\supset C_y^{(\ell_2)}$. Then using that $|\zeta_{\gamma,R}(P)|\leq |\zeta_{\gamma,R}(\tilde P)|\cdot |z^T_{\gamma,R}(\pi)|$ going to the third line we applied (7.3.38) for $b^*\leq\kappa^*$ and (7.4.12) for $b^*\leq\delta_1$. Then, overall, from (7.4.13) using (7.4.14) and the fact that $\kappa^*<\alpha$, we choose $b^*<\min\{\frac{\alpha}{2},\frac{\delta}{2}\}$ and conclude the proof of the Lemma, pending the proof of (7.4.11).

Proof of (7.4.11)*:* we endow the polymer P with a graph structure with vertices the set $\{\Gamma:\Gamma\in\underline{\Gamma}(P)\}$ and edges $\{\Gamma_i,\Gamma_j\}$ whenever there is at least one cluster $\pi\in R(P)$ such that $A(\pi)\cap D^*(\Gamma_i)\neq\emptyset$ and $A(\pi)\cap D^*(\Gamma_j)\neq\emptyset$. (Recall that a polymer may have more clusters attached to a contour, which however do not end to some other contour.) Choosing some Γ_0 to be the root of the graph, we define the level $\ell(P)$ of the polymer P to be the maximum distance (see the definition given before) of some vertex $\Gamma\in\underline{\Gamma}(P)$ from the root.

We prove (7.4.11) by induction: we first prove it by summing over all polymers P with $\ell(P)=1$. We start with the case $\ell(P)=0$, i.e., when P consists of only a contour that we call it Γ_0. Using the uniform bound:

$$\sup_{q^{D_0}} |W(\Gamma_0|q^{D_0})| \leq \exp\{-\beta c\zeta^2\ell_2^d N_{\Gamma_0}\} \qquad (7.4.15)$$

where $N_{\Gamma_0} = |\mathrm{sp}(\Gamma_0)|/\ell_3^d$ we get (notice that for each Γ_0 we have $3^{|\mathrm{sp}(\Gamma_0)|\ell_2^{-d}}$ choices of η_{Γ_0})

$$\begin{aligned}
&\sum_{P:\ell(P)=0,\,\mathrm{sp}(\underline{\Gamma}(P))\supset C_x^{(\ell_3)}} |\zeta_{\gamma,R}(P)|e^{|P|+g_\delta(P)} \\
&\quad\le \sum_{\mathrm{sp}(\Gamma_0)\supset C_x^{(\ell_3)}} e^{-\beta c\gamma^{2a-(1-\alpha)d}N_{\Gamma_0}} 3^{|\mathrm{sp}(\Gamma_0)|\ell_2^{-d}} e^{(1+\delta\gamma^{2a-(1-\alpha)d})N_{\Gamma_0}} \\
&\quad\le \sum_{\mathrm{sp}(\Gamma_0)\supset C_x^{(\ell_3)}} e^{N_{\Gamma_0}\left(-\beta c\gamma^{2a-(1-\alpha)d}+\log 3\gamma^{-(1+\alpha)d+(1-\alpha)d}+1+\delta\gamma^{2a-(1-\alpha)d}\right)} \\
&\quad\le \sum_{\mathrm{sp}(\Gamma_0)\supset C_x^{(\ell_3)}} \exp\Big\{-N_{\Gamma_0}\gamma^{2a-(1-\alpha)d}[\frac{1}{2}-\delta]\Big\} \\
&\quad\le \exp\Big\{-\gamma^{2a-(1-\alpha)d}[\frac{1}{4}-\delta]\Big\} \qquad (7.4.16)
\end{aligned}$$

for $\delta < 1/4$ and all γ small enough. The last inequality follows from a known estimate (see Errico, Enzo).

We next consider the contribution of links of the first level and prove (choosing $\delta < \kappa^*$)

$$\begin{aligned}
\sum_{\substack{P:\,\mathrm{sp}(\underline{\Gamma}(P))\supset C_x^{(\ell_3)}\\ \ell(P)=1}} |\zeta_{\gamma,R}(P)|e^{|P|+g_\delta(P)} \;\le\; &\sum_{\mathrm{sp}(\Gamma_0)\supset C_x^{(\ell_3)}} \exp\Big\{-N_{\Gamma_0}\gamma^{2a-(1-\alpha)d}[\frac{1}{2}-\delta]\Big\}\times \\
&\Big(1+2\sum_{n\ge\gamma\ell_3/4}\gamma^{-\delta n}\gamma^{\kappa^* n}e^{c_\gamma n}\Big)^{c|D^*(\Gamma_0)|} \qquad (7.4.17)
\end{aligned}$$

where c_γ is defined in (7.4.18) below. A polymer P with $\ell(P) = 1$ consists first of the root, say Γ_0, which contributes the first factor in (7.4.17) and it is obtained as in (7.4.16). Once Γ_0 is fixed, we have a given number of particles in $D^*(\Gamma_0)$, which cannot exceed $c|D^*(\Gamma_0)|$, $c = \rho_{\beta,+} + \zeta$, as we are in a contour model. From each particle it may or may not grow a cluster π, the latter case contributes by 1. Otherwise there are two possible types of clusters π ($\in \mathcal{R}$ or $\mathcal{R}'$), hence the factor 2. Note that $n(\mathcal{L}_\gamma(\pi)) \ge \gamma\frac{\ell_3}{4}$ and that $\mathrm{card}(\mathcal{R}') > \mathrm{card}(\mathcal{R})$ so it suffices to consider the sum over $\pi \in \mathcal{R}'$ (with the factor 2). From each point of the cluster π it may (or may not) start another contour. This can happen only once for each π and in such a way that the new contour does not intersect the contour in the root, hence only for clusters whose labels are far enough (more than a distance of order ℓ_3). Then, for the statistical weight (7.4.7) we recall that $F(q^D)$ is a probability density and bounding all

the other factors uniformly in q^D we obtain:

$$|\zeta_{\gamma,R}(P)| \le \prod_{\Gamma\in\underline{\Gamma}(P)} |W(\Gamma,q^D)| \prod_{\substack{\pi\in R(P):\\ A(\pi)\cap D^*(\Gamma_0)\neq\emptyset}} |f_\pi^\pm|$$

where we bounded by 1 all other factors of the type $|f_\pi^\pm|$ that correspond to clusters which start from the contours of the first generation. Thus, following the above argumentation we have

$$\sum_{\substack{P:\,\mathrm{sp}(\underline{\Gamma}(P))\supset C_x^{(\ell_3)}\\ \ell(P)=1}} |\zeta_{\gamma,R}(P)|e^{|P|+g_\delta(P)} \le \sum_{\mathrm{sp}(\Gamma_0)\supset C_x^{(\ell_3)}} \exp\Big\{-N_{\Gamma_0}\gamma^{2a-(1-\alpha)d}[\frac{1}{2}-\delta]\Big\}\times$$
$$\prod_{i\in D^*(\Gamma_0)}\Big(1+2\sum_{\pi\in\mathcal{R}',\,\pi\ni i}|f_\pi|e^{n(\pi)+n(\mathcal{L}_\gamma(\pi))\delta\log\gamma^{-1}}(1+\sum_{\substack{\Gamma\in\underline{\Gamma}(P):\\ D^*(\Gamma)\cap A(\pi)\neq\emptyset}}|W(\Gamma,q^D)|e^{N_\Gamma(1+\delta\gamma^{2a-(1-\alpha)d})})\Big)$$

which is a quite generous upper bound. To estimate the last parenthesis we use again (7.4.16) and obtain:

$$1+\sum_{\substack{\Gamma\in\underline{\Gamma}(P):\\ D^*(\Gamma)\cap A(\pi)\neq\emptyset}}|W(\Gamma,q^D)|e^{N_\Gamma(1+\delta\gamma^{2a-(1-\alpha)d})} \le 1+\exp\{-\gamma^{2a-(1-\alpha)d}[\frac{1}{4}-\delta]\} =: e^{c_\gamma} \quad (7.4.18)$$

where $c_\gamma\to 0$ as $\gamma\to 0$. For the sum over the clusters π we sum over the values $n(\mathcal{L}_\gamma(\pi))$ recalling that, by definition, $n(\mathcal{L}_\gamma(\pi))\ge\gamma\ell_3/4$. Using (7.3.38) we obtain:

$$\sum_{\substack{\pi:\,\pi\ni i\\ n(\mathcal{L}_\gamma(\pi))=n}}|f_\pi^\pm|e^{n(\pi)+\delta\log\gamma^{-1}n(\mathcal{L}_\gamma(\pi))} = \gamma^{-\delta n}\gamma^{\kappa^* n}\sum_{\substack{\pi:\,\pi\ni i\\ n(\mathcal{L}_\gamma(\pi))=n}}|f_\pi^\pm|e^{n(\pi)}\gamma^{-\kappa^* n} \le C\gamma^{-\delta n}\gamma^{\kappa^* n} \quad (7.4.19)$$

Note that $|f_\pi^\pm|$, defined in (7.4.4), can be bounded by $z_{\gamma,R}^T(\pi;\rho)$ since the latter is small. In this way we prove (7.4.17), details are omitted. By taking $\delta<\kappa^*$, the r.h.s. of (7.4.17) is bounded by 1 for all γ small enough, and with this we conclude the case $\ell(P)=1$.

We then suppose that the inequality holds for $\ell(P)\le N$ for some N and we are going to prove that it then remains true for $N+1$. We sum over the first level and get exactly the same bound as in (7.4.17) except for replacing the factor $e^{c_\gamma n}$ by $[1+e^{c_\gamma}]^n$, the extra 1 coming from the induction assumption. By taking γ small enough this term is bounded by 1 in a similar manner. This proves the induction assumption and (7.4.11). □

By Lemma 7.4.1, using KP we obtain:

$$h^3(\rho|\bar{q}) = -\sum_C \zeta^T_{\gamma,R}(C;\rho) \tag{7.4.20}$$

having denoted by C the new clusters associated to the polymers P and $\zeta^T_{\gamma,R}(C;\rho)$ are the coefficients of the new cluster expansion.

7.5 Reduction to a density model

In this section we will prove the following lemma:

Lemma 7.5.1. *Let $f_{x_1,..,x_n}$ be as in* (7.1.1)*, then:*

$$E_{\mu^i}\big(f_{x_1,..,x_n}\big) = E_{\mu^i}(g_N) + R_i, \qquad i = 1,2 \tag{7.5.1}$$

where g_N is a function of $\{\rho_x\}$ with $x \in X_\Lambda$ contained in one of the cubes $C^{(N\ell_3)}_{x_i}$, $i = 1,..,n$; N is the integer part of N' such that: $dist(C^{(\ell_2)}_{x_i}, \Lambda^c)/2 = dist(C^{(N'\ell_3)}_{x_i}, \Lambda^c)$; R_i are remainder terms. Moreover, there are $\delta > 0$ and constants c_1, c_2, c so that

$$\|g_N\|_\infty \le c_1, \qquad \|R_i\|_\infty \le c_2 e^{-c\gamma^{-\delta}N} \tag{7.5.2}$$

Proof.

Let us start from the identity

$$E_{\mu^i}\big(f_{x_1,..,x_n}\big) = \frac{d}{du}\log Z^{(u)}(\Lambda|\bar{q})\,\Big|_{u=0} \tag{7.5.3}$$

where

$$Z^{(u)}(\Lambda|\bar{q}) = \sum_{\underline{\Gamma}:\,\mathrm{sp}(\underline{\Gamma})\,\subset\,\Lambda^0} \int_{\mathcal{Q}^\Lambda_+} \nu^\Lambda(dq) e^{-\beta H_\gamma(q|\bar{q})} e^{-\beta H^{\mathrm{hc}}(q|\bar{q}) + u f_{x_1,..,x_n}} W(\underline{\Gamma}|q) \tag{7.5.4}$$

By the measurability properties of g_N, the measures μ^i in $E_{\mu^i}(g_N)$ can be replaced by their marginals on $\{\rho_x\}$, so that the proof of (7.1.2) will be reduced, using (7.5.1), to an analysis of the two Gibbs measures with hamiltonian $h(\rho|\bar{q})$ (see (7.2.3)) and different boundary conditions.

We proceed as in Section 7.2, we fix a density profile $\{\rho_x\}$ and study the multi-canonical partition function (7.2.3) with the extra energy term $uf_{x_1,..,x_n}$.

We rewrite the extra factor $e^{uf_{x_1,..,x_n}}$ as:

$$e^{uf_{x_1,..,x_n}} = e^{u\tilde{f}_{x_1,..,x_n}} e^{u(f_{x_1,..,x_n}-\tilde{f}_{x_1,..,x_n})} \tag{7.5.5}$$

where:

$$\tilde{f}_{x_1,..,x_n} := \frac{1}{n!} \sum_{\substack{q_{i_j}\in C_{x_j}^{(\ell_2)} \\ j=1,..,n}} \tilde{J}_\gamma^{(n)}(q_{i_1},...,q_{i_n}) \tag{7.5.6}$$

so that the term $e^{u\tilde{f}_{x_1,..,x_n}}$ does not see the integration and gives an extra contribution to h^0 of the same kind as in (7.2.10). The new energy is given by (7.2.12) with δh as in (7.2.13) and with $u(f_{x_1,..,x_n} - \tilde{f}_{x_1,..,x_n})$ added to ΔH and H_R. We divide again the hamiltonian δh into three parts as in (7.2.14) and we first do the analysis of $h^{1,2}$ and then of h^3.

Following Section 7.3 we expand the extra term, as well as the ones coming from ΔH and H_R, and we rewrite it as:

$$\begin{aligned} e^{u(f_{x_1,..,x_n}-\tilde{f}_{x_1,..,x_n})} &= \prod_{\substack{q_{i_j}\in C_{x_j}^{(\ell_2)} \\ j=1,..,n}} [e^{u\frac{1}{n!}(J_\gamma^{(n)}-\tilde{J}_\gamma^{(n)})(q_{i_1},...,q_{i_n})} - 1 + 1] \\ &:= \prod_{\substack{q_{i_j}\in C_{x_j}^{(\ell_2)} \\ j=1,..,n}} [\tilde{w}_\gamma^{(u)}(q_{i_1},...,q_{i_n}) + 1] \end{aligned} \tag{7.5.7}$$

where we will have contributions for $n = 2$ or $n = 4$. In giving the definition of the new diagrams for the expansion we have to take into account that now we have three kind of possible links (γ-, R- and u- links). Hence, we will give a slightly different definition of a diagram θ: it will be a collection of 2- and 4-links, i.e., a *ordered* quintuple $\theta \equiv \left(\mathcal{L}_R^{(2)}(\theta), \mathcal{L}_\gamma^{(2)}(\theta), \mathcal{L}_\gamma^{(4)}(\theta), \mathcal{L}_{\gamma,u}^{(2)}(\theta), \mathcal{L}_{\gamma,u}^{(4)}(\theta)\right)$, where $\mathcal{L}_{\gamma,u}^{(2)}(\theta)$ and $\mathcal{L}_{\gamma,u}^{(4)}(\theta)$ denote the set of 2-links and of 4-links in θ of the new type, i.e. carrying a weight $\tilde{w}_\gamma^{(u)}$. Furthermore, we will have the same definition as (7.3.7), but with two more terms chosen in the last two set of links. Hence:

$$\begin{aligned} w_{\gamma,R,u}(\theta,q) := v_R(\mathcal{L}_R^{(2)}(\theta),q) \prod_{L^{(2)}\in\mathcal{L}_\gamma^{(2)}(\theta)} w_\gamma^{(2)}(q(L^{(2)})) \prod_{L^{(4)}\in\mathcal{L}_\gamma^{(4)}(\theta)} w_\gamma^{(4)}(q(L^{(4)})) \cdot \\ \cdot \prod_{L^{(2)}\in\mathcal{L}_{\gamma,u}^{(2)}(q(L^{(2)}))} \tilde{w}_\gamma^{(u)}(\theta) \prod_{L^{(4)}\in\mathcal{L}_{\gamma,u}^{(4)}(q(L^{(4)}))} \tilde{w}_\gamma^{(u)}(\theta) \end{aligned} \tag{7.5.8}$$

We proceed as before following the *Step 1*, i.e. getting rid of the R-links which have the same endpoints as other γ-links or/and u-links.

Hence, analogously to (7.3.15), we obtain again:

$$h^{1,2}(\rho|\bar{q}) = -\log \sum_{\{\bar{\theta}_1,...,\bar{\theta}_k\}_\sim} \prod_{j=1}^{k} z_{\gamma,R,u}(\bar{\theta}_j, \bar{q}; x_1, ..., x_n) \tag{7.5.9}$$

where $x_1, ..., x_n$ is the n-uple which appears in the definition of $f_{x_1,..,x_n}$ and where $z_{\gamma,R,u}$ is the obvious generalization of $z_{\gamma,R}$ defined in (7.3.14). Given any $\gamma > 0$, for u small enough the new links satisfy the same bound as the old ones and the analogue of Lemma 7.3.5 holds as well. We then obtain the expression:

$$h^{1,2}(\rho|\bar{q}) = -\sum_{\pi} z^T_{\gamma,R,u}(\pi; \rho; \bar{q}; x_1, ..., x_n) \tag{7.5.10}$$

like in (7.3.36), but where the polymers on the r.h.s are not only those considered in Section 7.2 but also those containing the new u-links. The analysis of [the analogue of] h^3 is unchanged, provided we consider all polymers π, including those with the new u-links. Hence we have:

$$h^3(\rho|\bar{q}) = -\sum_{C} \zeta^T_{\gamma,R,u}(C; \rho; x_1, ..., x_n) \tag{7.5.11}$$

instead of (7.4.20). Thus (7.3.36) and (7.4.20) still hold, with the sums on their r.h.s. including the new polymers. In conclusion

$$\begin{aligned} Z^{(u)}(\Lambda|\bar{q}) = \sum_{\{\rho_x\}} \exp\Big\{ \sum_x \log \frac{\ell_2^{dn_x}}{n_x!} - \sum_{x:\, C_x^{(\ell_2)} \in \partial\Lambda^{\rm int}} \log Z_{x,\bar{q}} - \beta \tilde{h}^0(\rho|\bar{\rho}) \\ + \sum_{\pi \not\in \Lambda^c} z^T_{\gamma,R,u}(\pi; \rho; \bar{q}; x_1, ..., x_n) + \sum_{C \not\in (\Lambda^0)^c\}} \zeta^T_{\gamma,R,u}(C; \rho; x_1, ..., x_n) \Big\} \end{aligned} \tag{7.5.12}$$

where in $\tilde{h}^0$ there are both the contributions coming from the old term h^0 and the new one (see (7.5.6)). By (7.5.3) we have:

$$E_{\mu^i}\big(f_{x_1,..,x_n}\big) = E\bigg(\sum_{\pi \not\in \Lambda^c} \frac{d}{du} z^T_{\gamma,R,u}(\pi; \rho; \bar{q}; x_1, ..., x_n)\Big|_{u=0} + \sum_{C \not\in (\Lambda^0)^c} \frac{d}{du} \zeta^T_{\gamma,R,u}(C; \rho; x_1, ..., x_n)\Big|_{u=0} \bigg) \tag{7.5.13}$$

where the expectation on the r.h.s. is over the Gibbs measure with hamiltonian

$$h(\rho|\bar{q}) = -\sum_x \log \frac{\ell_2^{dn_x}}{n_x!} + \beta h^0(\rho|\bar{\rho}) - \sum_{\pi \not\in \Lambda^c} z^T_{\gamma,R}(\pi; \rho; \bar{q}) - \sum_{C \not\in (\Lambda^0)^c} \zeta^T_{\gamma,R}(C; \rho) \tag{7.5.14}$$

where $z^T_{\gamma,R}$ and $\zeta^T_{\gamma,R}$ are the old weights (since the square bracket term in (7.5.7) gives 1 for $u = 0$). Notice that this expectation is the same as E_{μ^i}, since passing to the Gibbs measure with hamiltonian $h(\rho|\bar{q})$ one has only a dependence over the cell variables $\{\rho_x\}$.

Since the derivative w.r.t. u of the weights, on the r.h.s. of (7.5.13) is computed at $u = 0$, only polymers with a single u-link contribute and the derivative is obtained by replacing, in the computation of the weight, the original square bracket term $[e^{u\frac{1}{n!}(J_\gamma^{(n)} - \tilde{J}_\gamma^{(n)})(q_{i_1},...,q_{i_n})} - 1]$ by $\frac{1}{n!}(J_\gamma^{(n)} - \tilde{J}_\gamma^{(n)})(q_{i_1}, ..., q_{i_n})$. In fact using an analogous of formula (7.3.37) with the new activities we get:

$$\sum_{\pi \not\in \Lambda^c} \frac{d}{du} z^T_{\gamma,R,u}(\pi; \rho; \bar{q}; x_1, ..., x_n)\Big|_{u=0} = \sum_{\pi \not\in \Lambda^c} \tilde{z}^T_{\gamma,R}(\pi; \rho; \bar{q}; x_1, ..., x_n) \tag{7.5.15}$$

where (using the definition $\underline{x} = x_1, ..., x_n$):

$$\tilde{z}^T_{\gamma,R}(\pi; \rho; \bar{q}; \underline{x}) := \sum_{\pi' :\, \pi' \subset \pi} (-1)^{|\pi|-|\pi'|} \log \sum_{\substack{\{\bar{\theta}_1,...,\bar{\theta}_n\}_\sim, \\ \bar{\theta}_j \in \pi', \forall j}} \prod_{j=1}^{n} \tilde{z}_{\gamma,R}(\bar{\theta}_j; \rho; \bar{q}; \underline{x}), \quad |\pi| := |\{\bar{\theta} :\, \bar{\theta} \in \pi\}| \tag{7.5.16}$$

and where:

$$\tilde{z}_{\gamma,R}(\bar{\theta}_j; \rho; \bar{q}; \underline{x}) = E^0_{\rho,\bar{q}}\Big[\bar{w}_{\gamma,R}(\bar{\theta}, \cdot)\frac{1}{n!}(J_\gamma^{(n)} - \tilde{J}_\gamma^{(n)})(\cdot)\Big]. \tag{7.5.17}$$

with the link given by $(J_\gamma^{(n)} - \tilde{J}_\gamma^{(n)})$ belonging to the connected diagram $\bar{\theta}$ and carrying the indices $x_1, ..., x_n$.

Note that since:

$$|\tilde{z}_{\gamma,R}| \leq |z_{\gamma,R}| \tag{7.5.18}$$

where in the r.h.s. there are only the old weights, then (using (7.3.39)):

$$\sum_{\pi \ni \bar{\theta}'} |\tilde{z}^T_{\gamma,R}(\pi; \rho; \bar{q}; \underline{x})| e^{b(\pi)} \leq \sum_{\pi \ni \bar{\theta}'} |z^T_{\gamma,R}(\pi; \rho; \bar{q}; \underline{x})| e^{b(\pi)} \leq |z_{\gamma,R}(\theta'; \rho; \bar{q}; \underline{x})| e^{a(\bar{\theta}')} \tag{7.5.19}$$

hence we can use the corollary of Lemma 7.3.5 also for the new sum (7.5.15).

We divide the sum in (7.5.15) into two parts: the polymers which are entirely contained in the box $C_{x_i}^{(N\ell_3)}$ for $i = 1, ..., n$ (which will contribute to the definition of g_N) and the remaining polymers. Hence we define:

$$g^1_N := \sum_{\substack{\pi \subset C_{x_i}^{(N\ell_3)} \\ \pi \ni \underline{x}}} \tilde{z}^T_{\gamma,R}(\pi; \rho; \bar{q}; \underline{x}) \tag{7.5.20}$$

and using (7.5.19), (7.3.39) and Lemma 7.3.5 we get:

$$||g^1_N||_\infty \leq \sum_{\theta \ni x_i} \sum_{\substack{\pi \ni \theta \\ \pi \subset C_{x_i}^{(N\ell_3)}}} |\tilde{z}^T_{\gamma,R}(\pi; \rho; \bar{q}; \underline{x})| e^{b(\pi)} e^{-b(\pi)} \leq \sum_{\theta \ni x_i} |z_{\gamma,R}(\theta; \rho; \bar{q}; \underline{x})| e^{a(\bar{\theta})} \leq c'_1. \tag{7.5.21}$$

For the remaining terms,

$$R^1 := \sum_{\pi \cap (C_{x_i}^{(N\ell_3)})^c \neq \emptyset} \tilde{z}^T_{\gamma,R}(\pi;\rho;\bar{q};\underline{x}) \tag{7.5.22}$$

instead, we exploit the exponential decay given by (7.3.39):

$$||R^1||_\infty \leq \sum_{\theta \ni x_i} \sum_{\substack{\pi \ni \theta \\ \pi \cap (C_{x_i}^{(N\ell_3)})^c \neq \emptyset}} |\tilde{z}^T_{\gamma,R}(\pi;\rho;\bar{q};\underline{x})| e^{b(\pi)} e^{-b(\pi)} \leq c_2' e^{-c\gamma^{-\alpha}N} \tag{7.5.23}$$

where we remind that: $b(\pi) = \sum_{\theta \in \pi} b(\theta)\pi(\theta)$ and where $|\theta| > N\ell_3\gamma$ (if there is at least one γ-link) in order for the polymers to exit from the box $C_{x_i}^{(\ell_2)}$.

Observe that if in $z_{\gamma,R}(\theta;\rho;\bar{q};\underline{x})$ there are only R-links one has to do the estimate differently, but the result is even better, since for every R-link which crosses a box ℓ_2 we get a term of order $1/\ell_2$, see (7.3.26), in addition to the exponential term $e^{-cN\ell_3 R^{-1}}$. Hence we can keep estimate (7.5.23) as the relevant term.

For the second sum in (7.5.13) we do exactly the same procedure defining:

$$g_N^2 := \sum_{\substack{C \subset C_{x_i}^{(N\ell_3)} \\ C \ni \underline{x}}} \tilde{\zeta}^T_{\gamma,R}(C;\rho;\bar{q};\underline{x}) \tag{7.5.24}$$

for $\tilde{\zeta}^T$ the analogous of $\tilde{z}^T$ (see (7.5.16)) in the case with contours. Then, using Lemma 7.4.1 we get:

$$||g_N^2|| \leq c_1'' \tag{7.5.25}$$

while for the remaining terms:

$$R^2 := \sum_{C \cap (C_{x_i}^{(N\ell_3)})^c \neq \emptyset} \tilde{\zeta}^T_{\gamma,R}(C;\rho;\bar{q};\underline{x}) \tag{7.5.26}$$

we have:

$$||R^2||_\infty \leq \sum_{P \ni x_i} \sum_{\substack{C \ni P \\ C \cap (C_{x_i}^{(N\ell_3)})^c \neq \emptyset}} |\tilde{\zeta}^T_{\gamma,R}(C;\rho;\bar{q};\underline{x})| e^{g_b(C)} e^{-g_b(C)} \leq c_2'' e^{-\gamma^{2a-(1-\alpha)d}N} \tag{7.5.27}$$

where $g_b(C) := \sum_{P \in C} g_b(P)C(P)$ and $g_b(P)$ is defined in (7.4.9). In the last inequality we have used the fact that if a contour exceeds the box $C_{x_i}^{(N\ell_3)}$ then it must have a number of $C^{(\ell_3)}$ cubes which is larger than N.

Hence estimates (7.5.2) and Lemma 7.5.1 are proved with:

$$E_{\mu^i}(g_N) = E_{\mu^i}(g_N^1 + g_N^2), \qquad R_i = E_{\mu^i}(R^1 + R^2). \tag{7.5.28}$$

8. Dobrushin condition

8.1 Setting and Vaserstein distance

Let's consider the space:

$$\mathcal{X}^{\Lambda} = \left\{ \underline{n} = (n_x)_{x \in X_\Lambda} \in \mathbb{N}^{X_\Lambda} : |\ell_2^{-d} n_x - \rho_{\beta,+}| \leq \zeta, \text{ for all } x \in X_\Lambda \right\} \tag{8.1.1}$$

and the following effective hamiltonian:

$$h(\rho|\bar{q}) = \sum_{x \in X_\Lambda} \{-\ell_2^d S(\rho_x) + c(n_x)\} + \beta h^0(\rho|\bar{\rho}) + h^{1,2}(\rho|\bar{q}) + h^3(\rho|\bar{q}) \tag{8.1.2}$$

where ρ_x and n_x are used interchangeably, since:

$$\rho_x = \ell_2^{-d} n_x, \qquad \ell_2 = \gamma^{-1+\alpha}, \qquad \rho = (\rho_x)_{x \in X_\Lambda} \tag{8.1.3}$$

$\bar{q} \in \mathcal{Q}_+^{\Lambda^c}$ is a boundary condition and:

$$c(n_x) = -\log \frac{\ell_2^{dn_x}}{n_x!} + \ell_2^d S(\rho_x) = \log \sqrt{2\pi n_x} + \frac{1}{12 n_x} + 0(n_x^{-3}) \tag{8.1.4}$$

is the remainder of the Stirling formula, where $S(\rho) := -\rho(\log \rho - 1)$ is the entropy. The effective hamiltonian is given by:

$$h(\rho|\bar{q}) = h(\{\rho, \bar{q}\}) - h(\bar{q}) \tag{8.1.5}$$

where the first term on the r.h.s. is the energy of the configuration $\{\rho, \bar{q}\}$ (with ρ in Λ and $\bar{q}$ outside Λ), while the second term is only the energy given by $\bar{q}$ outside Λ. We remind that:

- $h^0(\rho|\bar{\rho})$ is the hamiltonian $H^{(\ell_2)}(q|\bar{q})$, obtained in Section 7.2 (see (7.2.10)) by replacing $J_\gamma^{(n)}$ by $\tilde{J}_\gamma^{(n)}$ in the LMP hamiltonian.

- $h^{1,2}(\rho|\bar{q})$ is the cluster expansion of the quantity in (7.2.15), i.e. the series in (7.3.36) that we recall for simplicity:

$$h^{1,2}(\rho|\bar{q}) = -\sum_{\pi} z^T_{\gamma,R}(\pi;\rho,\bar{q}). \tag{8.1.6}$$

  We separate the contribution of the hard cores from the terms in the sum which contain both R and γ obtaining by an abuse of notation:

$$h^{1,2}(\rho|\bar{q}) = h^R(\rho|\bar{q}) + h^{1,2}_{\gamma,R}(\rho|\bar{q}) \tag{8.1.7}$$

  where:

$$h^R(\rho|\bar{q}) := -\sum_{\pi} z^T_R(\pi;\rho,\bar{q}). \tag{8.1.8}$$

  Note that $h^{1,2}_{\gamma,R}(\rho|\bar{q})$ is at least of order γ.

- $h^3(\rho|\bar{q})$ is the cluster expansion of the quantity in (7.2.16), i.e. the series in (7.4.20):

$$h^3(\rho|\bar{q}) = -\sum_{C} \zeta^T_{\gamma,R}(C;\rho,\bar{q}) \tag{8.1.9}$$

  and analogously to $h^{1,2}$:

$$h^3(\rho|\bar{q}) = h^{R,C}(\rho|\bar{q}) + h^3_{\gamma,R}(\rho|\bar{q}) \tag{8.1.10}$$

  where the notation C reminds of the contribution coming from the contour model:

$$h^{R,C}(\rho|\bar{q}) := -\sum_{C} \zeta^T_R(C;\rho,\bar{q}). \tag{8.1.11}$$

Notice that $h^{1,2}_{\gamma,R}(\rho|\bar{q})$ and $h^3_{\gamma,R}(\rho|\bar{q})$ are at least of order γ.

We want to bound the Vaserstein distance between two Gibbs measures with the same hamiltonian (8.1.2) but with different b.c. $\bar{q}^i$, $i = 1, 2$. It is convenient to define the Vaserstein distance in terms of the following cost function:

$$d(\underline{n}^1, \underline{n}^2) = \sum_{x\in X_\Lambda} d(n^1_x, n^2_x), \qquad d(n^1_x, n^2_x) = |n^1_x - \underline{n}^2_x| \tag{8.1.12}$$

Following Dobrushin, we need to estimate the Vaserstein distance between conditional probabilities at a single site. We thus fix arbitrarily $x \in \Lambda$, $\underline{n}^i$, $i = 1, 2$, in $\mathcal{X}^{\Lambda\setminus x}$, call $\rho^i := \ell_2^{-d}\underline{n}^i$; $\bar{q}^i$ are the b.c. outside Λ. The energy in x plus the interaction with the outside is, as usual,

$$h(\rho_x|\rho^i, \bar{q}^i) = h\big(\{\rho_x, \rho^i\}|\bar{q}^i\big) - h(\rho^i|\bar{q}^i) \tag{8.1.13}$$

where the first term on the r.h.s. is the energy of the configuration $\{\rho_x, \rho^i\}$ (with $\bar{q}^i$ outside Λ). The second term is the energy in $\Lambda \setminus x$ of ρ^i with nothing in x and $\bar{q}^i$ outside Λ. We also set, recalling the decomposition of the energy h into a sum of h^k, $k = 0, 1, 2, 3$ (see (8.1.2)),

$$h^k(\rho_x|\rho^i, \bar{q}^i) = h^k\big(\{\rho_x, \rho^i\}|\bar{q}^i\big) - h^k(\rho^i|\bar{q}^i), \qquad k = 0, 1, 2, 3. \tag{8.1.14}$$

The conditional Gibbs measures are then the following probabilities on $\mathcal{X}^x$ (for $i = 1, 2$):

$$p(n_x|\rho^i, \bar{q}^i) = \frac{1}{Z_x(\rho^i, \bar{q}^i)} \exp\big\{ - h(\rho_x|\rho^i, \bar{q}^i)\big\}, \tag{8.1.15}$$

and their Vaserstein distance is

$$R\Big(p(\cdot|\rho^1, \bar{q}^1), p(\cdot|\rho^2, \bar{q}^2)\Big) := \inf_Q \sum_{n_x^1, n_x^2} Q(n_x^1, n_x^2) d(n_x^1, n_x^2) \tag{8.1.16}$$

where the inf is over all the joint representations Q of $p(\rho_x|\rho^i, \bar{q}^i)$, $i = 1, 2$.

The key bound for the Dobrushin scheme to work is the following:

$$R\Big(p(\cdot|\rho^1, \bar{q}^1), p(\cdot|\rho^2, \bar{q}^2)\Big) \le \sum_{z \in X_\Lambda, z \neq x} r_{\gamma,R}(x, z) d(n_z^1, n_z^2) + \sum_{z \in X_{\Lambda^c}} r_{\gamma,R}(x, z) D_z(\bar{q}^1, \bar{q}^2) \tag{8.1.17}$$

complemented by suitable conditions on the parameters $r_{\gamma,R}(x, z)$, like those proved in the next theorem. $D_z(\bar{q}^1, \bar{q}^2)$ in (8.1.17) is the distance of two configurations, i.e. the number of discrepancies in $C_z^{(\ell_2)}$ between $\bar{q}^1$ and $\bar{q}^2$. In formulas, if we suppose: $\bar{q}^1 = (\bar{q}_1^1, ..., \bar{q}_n^1)$ and $\bar{q}^2 = (\bar{q}_1^2, ..., \bar{q}_{n+p}^2)$,

$$D_z(\bar{q}^1, \bar{q}^2) := p + \min_{\{j_\ell\}} \sum_{\ell=1}^{n} \mathbb{1}_{\bar{q}_\ell^1 \neq \bar{q}_{j_\ell}^2}, \tag{8.1.18}$$

the min being over all the subsets $\{j_\ell\}$ of $\{1, ..., n + p\}$ which have cardinality n.

Theorem 8.1.1. *The bound* (8.1.17) *holds and there are positive constants $u < 1$ and c_i so that for all $x \in \Lambda$,*

$$\sum_z r_{\gamma,R}(x, z) \le u \tag{8.1.19}$$

$$r_{\gamma,R}(x, z) \le c_1 e^{-c_2 \gamma |z-x|}, \quad |z - x| \ge \ell_3 \tag{8.1.20}$$

Proof.

By using triangular inequalities we can reduce to two cases: when we change the single n_z by 1 and when we change the b.c. $\bar{q}^i$ in $C_z^{(\ell_2)}$, $z \in X_{\Lambda^c}$. Let $\underline{n}^1$, $\underline{n}^2$ and $\underline{n}$ be such that:

$$\underline{n}^2 = \underline{n}; \quad n_y^1 = n_y \text{ for all } y \neq z, \quad n_z^1 = n_z + 1 \tag{8.1.21}$$

We also suppose $\bar{q}^1 = \bar{q}^2 =: \bar{q}$.

Let $t \in [0,1]$ and, recalling (8.1.13) and (8.1.14), we define the following interpolated hamiltonians:

$$\begin{aligned} h_t(\rho_x|\underline{n},\bar{q}) &= t\, h(\rho_x|\underline{n}^2,\bar{q}) + (1-t)h(\rho_x|\underline{n}^1,\bar{q}) \\ h_t^k(\rho_x|\underline{n},\bar{q}) &= t\, h^k(\rho_x|\underline{n}^2,\bar{q}) + (1-t)h^k(\rho_x|\underline{n}^1,\bar{q}), \quad k=0,..,3. \end{aligned} \tag{8.1.22}$$

We call $p_t(n_x)$ the density of the corresponding Gibbs measure (without making explicit the dependence on $\underline{n}$ and $\bar{q}$, often omitted in the sequel):

$$p_t(n_x) = \frac{1}{Z_t} \exp\big\{ - h_t(\rho_x)\big\}. \tag{8.1.23}$$

Again p_t is normalized and we call Z_t its normalization, with $Z_1 = Z_x(\rho^1)$ and $Z_0 = Z_x(\rho^2)$.

In general one can prove an upper bound for the Vaserstein distance given in the following lemma:

Lemma 8.1.2. *For every antisymmetric function $\phi_t : \mathcal{X}^x \times \mathcal{X}^x \to \mathbb{R}$, $t \in [0,1]$ such that:*

$$\sum_{n_x^1} \phi_t(n_x^1, n_x^2) = \frac{dp_t}{dt}(n_x^2), \tag{8.1.24}$$

we have:

$$R\Big(p(\cdot|\rho^1), p(\cdot|\rho^2)\Big) \le \frac{1}{2}\int_0^1 dt \sum_{(n_x^1,n_x^2)\in(\mathcal{X}_x)^2} |\phi_t(n_x^1,n_x^2)| d(n_x^1,n_x^2). \tag{8.1.25}$$

Because of bound (8.1.25), our goal is now t construct a function ϕ_t with the above properties.

If we define the following quantities:

$$\begin{aligned} A_t(n_x) &= -\frac{dp_t}{dt}(n_x)\, \mathbb{1}_{\{\frac{dp_t}{dt}(n_x)<0\}} \\ B_t(n_x) &= \frac{dp_t}{dt}(n_x)\, \mathbb{1}_{\{\frac{dp_t}{dt}(n_x)>0\}}, \end{aligned} \tag{8.1.26}$$

we have:

$$\frac{dp_t}{dt}(n_x) = B_t(n_x) - A_t(n_x). \tag{8.1.27}$$

Also, since ϕ_t is antisymmetric,

$$\sum_{n_x} \frac{dp_t}{dt}(n_x) = 0 \tag{8.1.28}$$

and this implies:

$$\sum_{n_x} B_t(n_x) = \sum_{n_x} A_t(n_x) = I \tag{8.1.29}$$

and:

$$\phi_t(n_x^1, n_x^2) = \frac{A_t(n_x^1)B_t(n_x^2) - B_t(n_x^1)A_t(n_x^2)}{I}. \tag{8.1.30}$$

So if we manage to have a sharp cntrol of where $dp_t/dt(n_x)$ is positive or negative we can easily construct $\phi_t(n_x)$ as in (8.6).

In the next section we will investigate the expression of $dp_t/dt(n_x)$ with its sign and, as we will see, it will be easier to write it as a sum of terms each one having 0 sum and apply the above procedure to each of them separately.

8.2 Contribution of h^0

We remind that h^0 is the hamiltonian obtained by replacing $J_\gamma^{(n)}$ by $\tilde{J}_\gamma^{(n)}$, with $\tilde{J}_\gamma^{(n)}$ as in (7.2.9) satisfying bound (7.3.2).

Hence with a more compact notation we can write:

$$h^0(\rho) = -\lambda \sum_x n_x - \frac{1}{2!} \sum_{x_1,x_2} \tilde{J}_\gamma^{(2)}(x_1, x_2) g(n_{x_1}, n_{x_2}) + \frac{1}{4!} \sum_{x_1,...,x_4} \tilde{J}_\gamma^{(4)}(x_1, ..., x_4) g(n_{x_1}, ..., n_{x_4}) \tag{8.2.1}$$

where g is the Poisson polynomial, defined as followed. Let us consider $A = A(X_\Lambda)$ the set of all $x_i \in X_\Lambda$ where every x_i is repeated a certain number of times. Let $I(x_i) \in \{0, 1, 2, ...\}$ be the moltiplicity of each x_i in A and $\text{supp}(A) = \{x_i \in A : I(x_i) > 0\}$ be the support of the set A. Then for each $B \subset A$:

$$g(B) = \prod_{x_i \in \text{supp}(B)} G(n_{x_i}; I(x_i)) \tag{8.2.2}$$

where: $G(a, k) = a(a-1)\cdots(a-k+1)$. By (8.1.14) we have (for $i = 1, 2$):

$$h^0(\rho_x|\rho^i) = -\lambda n_x - \sum_y \tilde{J}_\gamma^{(2)}(x, y) g(n_x, n_y) + \frac{1}{3!} \sum_{x_1,x_2,x_3} \tilde{J}_\gamma^{(4)}(x, x_1, x_2, x_3) g(n_x, n_{x_1}, n_{x_2}, n_{x_3}) \tag{8.2.3}$$

where all the variables on which we sum are varying in $\Lambda \setminus x$ and $n_y = \rho_y^i \ell_2^d$ for all $y \neq x$. Note that from the normalization of $\tilde{J}_\gamma^{(2)}$ (i.e. $\int dr_1 \tilde{J}_\gamma^{(2)}(r_1, r_2) = 1$), one has $\sum_x \tilde{J}_\gamma^{(2)}(x, y) \simeq \ell_2^{-d}$

while $\tilde{J}^{(2)}_\gamma(x,y)$ itself is of order γ^d. One can do the same kind of estimates for the 4-bodies interaction in order to establish the order of the terms in (8.2.3).

Setting $n^1_z - n^2_z = 1$, the interpolated hamiltonian is:

$$\begin{aligned} h^0_t(\rho_x) &= h^0(\rho_x|\rho^2) + t[h^0(\rho_x|\rho^1) - h^0(\rho_x|\rho^2)] = \\ &= -\lambda n_x - \sum_y \tilde{J}^{(2)}_\gamma(x,y) g(n_x, n_y) + \frac{1}{3!} \sum_{x_1,x_2,x_3} \tilde{J}^{(4)}_\gamma(x,x_1,x_2,x_3) g(n_x, n_{x_1}, n_{x_2}, n_{x_3}) \\ &+ t\Big[- \tilde{J}^{(2)}(x,z) n_x + \frac{1}{2!} \sum_{x_1,x_2} \tilde{J}^{(4)}_\gamma(x,z,x_1,x_2) g(n_x, n_{x_1}, n_{x_2}) \Big]. \end{aligned} \tag{8.2.4}$$

Setting:

$$\begin{aligned} \lambda^* = \lambda + \sum_{y \neq x} \tilde{J}^{(2)}_\gamma(x,y) n_y + \frac{1}{3!} \sum_{x_1,x_2,x_3 \neq x} \tilde{J}^{(4)}(x,x_1,x_2,x_3) g(n_{x_1}, n_{x_2}, n_{x_3}) + \\ + t\Big[- \tilde{J}^{(2)}(x,z) + \frac{1}{2!} \sum_{x_1,x_2 \neq x} \tilde{J}^{(4)}_\gamma(x,z,x_1,x_2) g(n_{x_1}, n_{x_2}) \Big]. \end{aligned} \tag{8.2.5}$$

we get:

$$h^0_t(\rho_x) = -\lambda^* n_x + \sum_{i=1}^3 a^0_i \gamma^{d(i-1)} n^i_x \tag{8.2.6}$$

where $a^0_i = a^0_i(\gamma, \underline{n}, \bar{q})$ and there is $c > 0$ so that

$$|a^0_1| \leq c\gamma^d, \qquad |a^0_i| \leq c, \ i > 1 \tag{8.2.7}$$

Similarly:

$$\begin{aligned} \frac{dh^0_t(\rho_x)}{dt} &= - \tilde{J}^{(2)}(x,z) n_x + \frac{1}{2!} \sum_{x_1,x_2} \tilde{J}^{(4)}_\gamma(x,z,x_1,x_2) g(n_x, n_{x_1}, n_{x_2}) \\ &= \tilde{J}^{(2)}_\gamma(x,z) \big(b^0_1 n_x + b^0_2 \gamma^d n^2_x + b^0_3 \gamma^{2d} n^3_x \big), \quad |b^0_i| \leq c \end{aligned} \tag{8.2.8}$$

where we write explicitly the coefficient of n_x:

$$b^0_1 \tilde{J}^{(2)}_\gamma(x,z) = -\tilde{J}^{(2)}_\gamma(x,z) + \frac{1}{2} \sum_{x_1,x_2 \neq x} \tilde{J}^{(4)}_\gamma(x,z,x_1,x_2) g(n_{x_1}, n_{x_2}) := -\tilde{J}^{(2)}_\gamma(x,z) + \frac{1}{2} \tilde{J}^{(2)}_\gamma(x,z|n) \tag{8.2.9}$$

Since:

$$\tilde{J}^{(2)}_\gamma(x,z)(\rho_{\beta,+} - \zeta)^2 \leq \tilde{J}^{(2)}_\gamma(x,z|n) \leq \tilde{J}^{(2)}_\gamma(x,z)(\rho_{\beta,+} + \zeta)^2, \tag{8.2.10}$$

we can write:

$$\tilde{J}^{(2)}_\gamma(x,z)\Big[\frac{1}{2}(\rho^2_{\beta,+}-2\zeta\rho_{\beta,+})-1\Big] \le \frac{1}{2}\tilde{J}^{(2)}_\gamma(x,y|n)-\tilde{J}^{(2)}_\gamma(x,z) \le \tilde{J}^{(2)}_\gamma(x,z)\Big[\frac{1}{2}(\rho^2_{\beta,+}+2\zeta\rho_{\beta,+})-1\Big] \tag{8.2.11}$$

and therefore, neglecting the orders ζ^2:

$$\Big|\frac{1}{2}\tilde{J}^{(2)}_\gamma(x,y|n) - \tilde{J}^{(2)}_\gamma(x,z) - \tilde{J}^{(2)}_\gamma(x,z)\Big(\frac{1}{2}\rho^2_{\beta,+}-1\Big)\Big| \le \zeta\rho_{\beta,+}\tilde{J}^{(2)}_\gamma(x,z) \tag{8.2.12}$$

Hence:

$$\Big|b^0_1 - \Big(-1+\frac{1}{2}\rho^2_{\beta,+}\Big)\Big| \le \zeta\rho_{\beta,+}. \tag{8.2.13}$$

8.3 Contribution of h^R

We want now to consider the contribution of h^R to dp_t/dt. Given a cluster π, let us define the *area* of the polymer $A(\pi) := \cup_{\tilde{\theta}\in\pi} V(\tilde{\theta})$. From (8.1.8) we have ($i = 1, 2$):

$$h^R(\rho_x|\rho^i) = -\sum_{\pi:\, A(\pi)\cap\mathcal{L}_x\neq\emptyset} z^T_R(\pi;\rho_x,\rho^i) \tag{8.3.1}$$

where with $\mathcal{L}_x$, we denote the set of labels of the particles in the box C_x. We remind that $|\mathcal{L}_x| = n_x$. Hence in (8.3.1) we are summing over all the clusters which have at least one vertex in the box labelled with x.

Let us sum first on a generic label j belonging to the box x and then sum over the clusters that contain that label (dividing, however, by the number of points of the clusters which are in the box, i.e. $|A(\pi)\cap\mathcal{L}_x|$):

$$h^R(\rho_x|\rho^i) = \sum_{j\in\mathcal{L}_x}\sum_{\pi:\, A(\pi)\ni j}\frac{1}{|A(\pi)\cap\mathcal{L}_x|}z^T_R(\pi;\rho_x,\rho^i) = n_x\sum_{\pi:\, A(\pi)\ni 1}\frac{1}{|A(\pi)\cap\mathcal{L}_x|}z^T_R(\pi;\rho_x,\rho^i) \tag{8.3.2}$$

where, with the constraint $A(\pi)\ni j$, the condition $A(\pi)\cap\mathcal{L}_x\neq\emptyset$ is automatically satisfied, so we can drop it. Notice that in the second equality we have fixed the label j to be the label 1 (which belongs to C_x) multiplying by a factor n_x.

For reasons that will be clear later, we want to emphasize the dependence on the number of particles n_{x_i} of each box C_{x_i} that the polymers cross. Hence we use the following notation:

$$h^R(\rho_x|\rho^i) = n_x\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x,\, y_i\in X_\Lambda}}\sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N}, m_i>0}}\tilde{V}(y_1,...,y_k;m_1,...,m_k) \tag{8.3.3}$$

where:

$$\tilde{V}(y_1,...,y_k;m_1,...,m_k)=\binom{n_{y_1}-1}{m_1-1}\binom{n_{y_2}}{m_2}\cdots\binom{n_{y_k}}{m_k}\sum_{\substack{\pi:\pi\sim\{y_1,...,y_k;m_1,...,m_k\}\\ A(\pi)\ni 1}}\frac{z_R^T(\pi;\rho_x,\rho^i)}{|A(\pi)\cap\mathcal{L}_x|} \tag{8.3.4}$$

We are first summing over the sequence of boxes $\{y_1,...,y_k\}$ and over the sequence of frequencies $\{m_1,...,m_k\}$ with which the corresponding box is crosses by the polymers and then to sum over all the polymers compatible with these constraints we use the symbol $\sim$. With specifying the boxes and the frequencies, one has to multiply by the binomial coefficients counting the ways to collect m_i labels from each box containing n_{y_i} labels. Notice that we already performed the choice of one label from the box $y_1=x$, hence we have one choice less for that box. Now we write:

$$\tilde{V}(y_1,...,y_k;m_1,...,m_k)=V(y_1,...,y_k;m_1,...,m_k)\tilde{n}_{y_1}^{m_1-1}\cdots\tilde{n}_{y_k}^{m_k} \tag{8.3.5}$$

with:

$$\tilde{n}_{y_i}^{m_i}:=G(n_{y_i};m_i), \tag{8.3.6}$$

$$\tilde{n}_{y_i}^{m_i-1}:=G(n_{y_i}-1;m_i) \tag{8.3.7}$$

where the definition of the quantity $G(a;k)$ is given in Section 8.2 and:

$$V(y_1,...,y_k;m_1,...,m_k)=\frac{1}{(m_1-1)!\prod_{i>1}m_i!}\sum_{\substack{\pi:\pi\sim\{y_1,...,y_k;m_1,...,m_k\}\\ A(\pi)\ni\{1\}}}\frac{z_R^T(\pi;\rho_x,\rho^i)}{|A(\pi)\cap\mathcal{L}_x|}. \tag{8.3.8}$$

Then using (8.3.3) and (8.3.5), and considering that the difference between $h^R(\rho_x|\rho^1)$ and $h^R(\rho_x|\rho^2)$ is nonzero only if we are summing on polymers long enough to go from x to z, the R contribution to dp_t/dt is:

$$\begin{aligned}\frac{dh_t^R}{dt}(\rho_x)=\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x,y_k=z}}\sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N},m_i>0}}V(y_1,...,y_k;m_1,...,m_k)\cdot\\ \cdot\tilde{n}_{y_1}^{m_1}\cdots\tilde{n}_{y_{k-1}}^{m_{k-1}}\left[\tilde{n}_{y_k}^{m_k}-\left(\tilde{n}_{y_k}+1\right)^{m_k}\right]\end{aligned} \tag{8.3.9}$$

Hence, using the approximation:

$$\tilde{n}_{y_k}^{m_k}-\left(\tilde{n}_{y_k}+\frac{1}{\ell_2^d}\right)^{m_k}\simeq -m_k\tilde{n}_{y_k}^{m_k-1} \tag{8.3.10}$$

we have:

$$\frac{dh_t^R}{dt}(\rho_x) = - \sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x, y_k=z}} \sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N}, m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\cdot m_k \tilde{n}_{y_1}^{m_1}\cdots\tilde{n}_{y_{k-1}}^{m_{k-1}}\tilde{n}_{y_k}^{m_k-1} \quad (8.3.11)$$

8.4 Contributions of $h^{1,2}_{\gamma,R}$ and $h^3_{\gamma,R}$

From (8.1.7) and (8.1.8) we have:

$$h^{1,2}_{\gamma,R}(\rho|\bar{q}) = h^{1,2}(\rho|\bar{q}) - h^R(\rho|\bar{q}) = -\sum_\pi z^T_{\gamma,R}(\pi;\rho,\bar{q}) + \sum_\pi z^T_R(\pi;\rho,\bar{q}) \quad (8.4.1)$$

Note that $h^R(\rho|\bar{q})$ is exactly the cluster expansion sum for a model of particles in the canonical ensemble which interact only via a hard-core potential of order R (see [53]), but in this case the multi-indices are defined over the set of all trees (instead of vertices sets). On the other side, $h^{1,2}_{\gamma,R}(\rho|\bar{q})$ is given by the remaining terms, i.e. the sum of diagrams where the γ-links appear either by themselves or attached to some R-link, with the prescription that R-links appear at most as trees in the structures and never form diagrams by themselves.

We now write the hamiltonian as a sum $\sum^*$ which encodes the above constraints and we separate the hamiltonian in two parts, by fixing a large enough integer N and setting:

$$h^1_{\gamma,R}(\rho|\bar{q}) = -\sum^*_{\pi\not\in\Lambda^c, L(\pi)\le N} z^T_{\gamma,R}(\pi;\rho,\bar{q}), \qquad h^2_{\gamma,R}(\rho|\bar{q}) = -\sum^*_{\pi\not\in\Lambda^c, L(\pi)>N} z^T_{\gamma,R}(\pi;\rho,\bar{q}) \quad (8.4.2)$$

where $L(\pi)$ denotes the number of γ-links that appear in the polymer π.

We call $h^1_t(\rho_x)$ (which depends on $\underline{n}$ and $\bar{q}$) the contribution of h^1 to h_t, as in (8.1.22). $h^1_t(\rho_x)$ is therefore the sum over all the polymers with at most N γ-links and such that at least one of the endpoints of π belongs to $C_x^{(\ell_2)}$. It then follows that

$$h^1_t(\rho_x) = \sum_{i=1}^{3N} a^1_i \gamma^{d(i-1)+i\alpha/3} n^i_x, \quad |a^1_i| \le c \quad (8.4.3)$$

To prove (8.4.3) we represent the polymers by a tree structure with root a label of a particle in $C_x^{(\ell_2)}$. When in the tree there is a 2-link, we attach to the descendant the weight $\gamma^{d+\alpha}$ times the characteristic function that it should be at distance $\le \gamma^{-1}$ from the ascendant. If we have a 4-link we attach to each descendant the weight $\gamma^{d+\alpha/3}$ times the characteristic function that it should be at distance $\le \gamma^{-1}$ from the ascendant. Then the weight of the whole polymer is bounded by a constant times the product of these weights.

We call here equivalent two polymers which differ by a permutation of labels such that any label in $C_x^{(\ell_2)}$ is still in $C_x^{(\ell_2)}$ after the permutation. There are finitely many classes of equivalence and summing over all the polymers inside each class we get (8.4.3). Analogously

$$\frac{dh_t^1(\rho_x)}{dt} = \gamma^d \mathbf{1}_{|x-z|\le N\gamma^{-1}} \sum_{i=1}^{3N} b_i^1 \gamma^{d(i-1)+\alpha/3i} n_x^i, \quad |b_i^1| \le c \tag{8.4.4}$$

which is proved by taking as a root the special particle in $C_z^{(\ell_2)}$ and imposing that at least one of the endpoint of the polymer is in $C_x^{(\ell_2)}$. As this is not possible if $|x-z| > N\gamma^{-1}$ (because the polymer has at most N links) we get the characteristic function in (8.4.4), which is then proved by an argument analogous to that used in the proof of (8.4.3).

Calling $h_t^2(\rho_x)$ the contribution of h^2 to h_t, as in (8.1.22), there are two positive constants c and b such that:

$$|h_t^2(\rho_x)| \le c\ell_2^d \gamma^{bN}, \tag{8.4.5}$$

$h_t^2(\rho_x)$ is in fact given by the sum over all the polymers π with at least N links and an endpoint in $C_x^{(\ell_2)}$. (8.4.5) then follows from Lemma 7.3.5 and (7.3.38).

We also have

$$|\frac{dh_t^2(\rho_x)}{dt}| \le c\ell_2^d \gamma^{bN/2} \gamma^{b\gamma|x-z|/2}, \tag{8.4.6}$$

The t derivative is in fact the interaction of the "extra particle" in $C_z^{(\ell_2)}$ with those in $C_x^{(\ell_2)}$. We thus have to sum over all polymers that have as an endpoint the extra particle and that have another endpoint in $C_x^{(\ell_2)}$. They must also have at least N links and since each link connects particles at distance not larger than γ^{-1} there must be at least $\gamma|x-z|$. We thus get an upper bound by summing over all polymers that start from the special particle and have at least the maximum between N and $\gamma|x-z|$, hence (8.4.6).

The other contribution we want to consider in this section is given by h^3. From (8.1.9) and (8.1.10) we have:

$$h_{\gamma,R}^3(\rho|\bar{q}) = h^3(\rho|\bar{q}) - h^{R,C}(\rho|\bar{q}) = -\sum_C \zeta_{\gamma,R}^T(C;\rho,\bar{q}) + \sum_C \zeta_R^T(C;\rho,\bar{q}) \tag{8.4.7}$$

Repeating the same argument as we did for $h_{\gamma,R}^{1,2}$ to the interpolated hamiltonian h_t^3 and its derivative dh_t^3/dt, we get the same expressions as in (8.4.5) and (8.4.6) respectively.

8.5 Contribution of $h^{R,C}$

We want now to consider the contribution of $h^{R,C}$ to dp_t/dt, where $h^{R,C}$ is defined in (8.1.11) and analogously to h^R (see (8.3.1)) we have:

$$h^{R,C}(\rho_x|\rho^i) = - \sum_{\substack{C:\, A(C)\cap\mathcal{L}_x\neq\emptyset \\ A(C)\cap\mathcal{L}_x^c\neq\emptyset}} \zeta_R^T(C;\rho_x,\rho^i). \tag{8.5.1}$$

where $A(C)$ is the set of particles labels in the cluster C. Unlike h^R, this contribution has a much smaller weight in the effective hamiltonian. Indeed, if we look at Lemma 7.4.1. we see that since every polymer which appear in the cluster sum (8.5.1) contains at least one contour, the interpolated hamiltonian can be bounded by:

$$\left|h_t^{R,C}(\rho_x)\right| \leq c\ell_2^d e^{-b\gamma^{2a-(1-\alpha)d}} \tag{8.5.2}$$

On the other side, we write:

$$\frac{dh_t^{R,C}(\rho_x)}{dt} = - \sum_{\substack{C:\, A(C)\cap\mathcal{L}_x\neq\emptyset \\ A(C)\cap\mathcal{L}_z\neq\emptyset}} \zeta_R^T(C;\rho_x,\rho^i) \tag{8.5.3}$$

since, again, the derivative is the interaction of the "extra particle" in $C_z^{(\ell_2)}$ with those in $C_x^{(\ell_2)}$. Thus, repeating the argument for dh_t^3/dt and using Lemma 7.4.1 with $\mathcal{L}_\gamma(P)=\emptyset$ we get:

$$|\frac{dh_t^{R,C}(\rho_x)}{dt}| \leq c\ell_2^d e^{-b\gamma^{2a-(1-\alpha)d}\cdot|x-z|\ell_3^{-1}}. \tag{8.5.4}$$

where we used the fact that to reach the extra particle in $C_z^{(\ell_2)}$ one needs a contour which contains $|x-z|\ell_3^{-1}$ cubes $C^{(\ell_3)}$.

8.6 Construction of ϕ_t

Denoting by $\langle\cdot\rangle_t$ the expectation w.r.t. p_t, we have:

$$\begin{aligned}\frac{dp_t(n_x)}{dt} = p_t(n_x)\Bigg\{ &\frac{dh_t^0(\rho_x)}{dt} - \Big\langle\frac{dh_t^0(\rho_x)}{dt}\Big\rangle_t + \frac{dh_t^R}{dt}(\rho_x) - \Big\langle\frac{dh_t^R}{dt}(\rho_x)\Big\rangle_t \\ &+\frac{dh_t^{R,C}}{dt}(\rho_x) - \Big\langle\frac{dh_t^{R,C}}{dt}(\rho_x)\Big\rangle_t + \sum_{i=1}^3\Big[\frac{dh_t^i}{dt}(\rho_x) - \Big\langle\frac{dh_t^i}{dt}(\rho_x)\Big\rangle_t\Big]\Bigg\}\end{aligned} \tag{8.6.1}$$

and substituting the contributions given in the above sections we get:

$$\frac{dp_t(n_x)}{dt} = p_t(n_x)\Bigg\{ \tilde{J}^{(2)}_\gamma(x,z)\beta\Big(b^0_1[n_x - \langle n_x\rangle_t] + b^0_2\gamma^d[n^2_x - \langle n^2_x\rangle_t] + b^0_3\gamma^{2d}[n^3_x - \langle n^3_x\rangle_t]\Big)$$
$$+\gamma^d \mathbf{1}_{|z-x|\le N\gamma^{-1}} \sum_{i=1}^{3N} b^1_i \gamma^{d(i-1)}\gamma^{\alpha/3i}[n^i_x - \langle n^i_x\rangle_t]$$
$$+\sum_i \mathcal{V}(x,z,i)[\tilde{n}^i_x - \langle \tilde{n}^i_x\rangle_t] + \Big[\frac{dh^{R,C}_t}{dt}(\rho_x) - \Big\langle \frac{dh^{R,C}_t}{dt}(\rho_x)\Big\rangle_t\Big]$$
$$+\Big[\frac{dh^2_t(\rho_x)}{dt} - \Big\langle \frac{dh^2_t(\rho_x)}{dt}\Big\rangle_t\Big] + \Big[\frac{dh^3_t(\rho_x)}{dt} - \Big\langle \frac{dh^3_t(\rho_x)}{dt}\Big\rangle_t\Big]\Bigg\} \tag{8.6.2}$$

where we shortened the notation:

$$\frac{dh^R_t}{dt}(\rho_x) = -\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x, y_k=z}} \sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N}, m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\cdot m_k \tilde{n}^{m_1}_{y_1}\cdots\tilde{n}^{m_{k-1}}_{y_{k-1}}\tilde{n}^{m_k-1}_{y_k}$$
$$:= \sum_m \mathcal{V}(x,z,m)\tilde{n}^m_x. \tag{8.6.3}$$

The r.h.s. of (8.6.2) has $3N+7$ terms that we call consecutively $p_t(n_x)\delta_\ell(n_x)$, $\ell = 1,..,3N+7$:

$$\frac{dp_t(n_x)}{dt} = \sum_{\ell=1}^{3N+7} p_t(n_x)\delta_\ell(n_x) \tag{8.6.4}$$

Let us drop the index x from now on, always remembering that we are working in the box $C^{(\ell_2)}_x$. Since $p_t(n)\delta_\ell(n)$ is centered, i.e. its sum over n is 0, we can construct the function $\phi_t(n,n')$ in analogy with (8.6) as follows. We call $A_\ell(n)$ and $B_\ell(n)$ minus the negative and resp. the positive parts of $p_t(n)\delta_\ell(n)$ and set,

$$\phi_t(n,n') = \sum_{\ell=1}^{3N+7}\phi^{(\ell)}_t(n,n'), \qquad \phi^{(\ell)}_t(n,n') = \frac{[A_\ell(n)B_\ell(n') - A_\ell(n')B_\ell(n)]}{\sum_n A_\ell(n)} \tag{8.6.5}$$

where $\phi_t(n,n')$ obviously satisfies (8.1.24). Substituting in the r.h.s. of inequality (8.1.25) the contributions for $\ell \le 3N+7$ we have:

$$R\Big(p(\cdot|\rho^1), p(\cdot|\rho^2)\Big) \le \frac{1}{2}\sum_{\ell=1}^{3N+7}\int_0^1 \sum_{n,n'} |\phi^{(\ell)}_t(n,n')||n-n'| \tag{8.6.6}$$

- We now want to investigate the terms with $\ell \le 3N+3$. For each term $p_t(n)\delta_\ell(n)$ is equal, up to a multiplicative constant, to

$$p_t(n)[n^k - \langle n^k\rangle_t], \tag{8.6.7}$$

hence, if we let $\bar{n}$ be such that $\bar{n}^k = \langle n^k \rangle_t$, then one can define:

$$A(n) = p_t(n)[\langle n^k \rangle_t - n^k] \mathbb{1}_{n \le \bar{n}} \tag{8.6.8}$$

$$B(n) = p_t(n)[n^k - \langle n^k \rangle_t] \mathbb{1}_{n > \bar{n}} \tag{8.6.9}$$

Since $p_t(n)[n^k - \langle n^k \rangle_t]$ is centered

$$\sum_n A(n) = \sum_n B(n) =: I \tag{8.6.10}$$

where, again, the sum over n above and in the sequel is always, tacitly, restricted to $n \in \mathcal{X}^x$.

Thus the generic contribution of these terms to the r.h.s. of (8.6.6) gives:

$$\begin{aligned} &\frac{1}{2} \sum_{n,n'} I^{-1} |A(n)B(n') - A(n')B(n)||n - n'| \\ &\quad = \sum_{n < \bar{n},\, n' > \bar{n}} I^{-1} A(n) B(n') \big([n' - \bar{n}] + [\bar{n} - n| \big) \\ &\quad = \sum_{n' > \bar{n}} B(n')[n' - \bar{n}] + \sum_{n < \bar{n}} A(n)[\bar{n} - n] \\ &\quad = \sum_n p_t(n)[n^k - \langle n^k \rangle_t][n - \bar{n}] = \sum_n p_t(n)[n^k - \langle n^k \rangle_t][n - \langle n \rangle_t] \end{aligned} \tag{8.6.11}$$

We will prove in Section 8.8 that for all $1 < k \le 3N$

$$\sum_n p_t(n)[n^k - \langle n^k \rangle_t][n - \langle n \rangle_t] \le (\ell_2^d)^k \tag{8.6.12}$$

and for the second moment there are a^* and c such that:

$$\sum_n p_t(n)[n - \langle n \rangle_t]^2 \le (\ell_2^d) \Big\{ \frac{1}{\rho^*} + \ell_2^{-d} h^{R''}(\rho^*) \Big\}^{-1} (1 + c\gamma^{a^\star}) \tag{8.6.13}$$

where ρ^* is the minimum of the functional $F_t(\rho)$ defined in (8.8.4). Hence, by using these bounds for the first $3 + 3N$ terms in (8.6.6) we get:

$$\begin{aligned} &\frac{1}{2} \sum_{\ell=1}^{3N+3} \int_0^1 \sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n - n'| \\ &\qquad \le t \tilde{J}_\gamma^{(2)}(x,z) \Big\{ |b_1^0| \ell_2^d \beta \Big[\frac{1}{\rho^*} + \ell_2^{-d} h^{R''}(\rho^*) \Big]^{-1} (1 + c\gamma^{a^\star}) + c\big[\gamma^d (\ell_2^d)^2 + \gamma^{2d} (\ell_2^d)^3 \big] \Big\} \\ &\qquad + \gamma^{\alpha/3} \gamma^d \mathbf{1}_{|x-z| \le N\gamma^{-1}} c \ell_2^d \sum_{i=1}^{3N} \gamma^{d(i-1)} \ell_2^{d(i-1)} \end{aligned} \tag{8.6.14}$$

There are $u^\star$ and $c = c(N)$ so that for all ζ and γ small enough

$$\frac{1}{2}\sum_{\ell=1}^{3N+3}\int_0^1 \sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| \le \tilde{J}_\gamma^{(2)}(x,z)\ell_2^d u^\star + c\gamma^{\alpha/3}\gamma^d \ell_2^d \mathbf{1}_{|x-z|\le N\gamma^{-1}} \quad (8.6.15)$$

where, because of the contraction property (see (3.2.24) and (3.2.28)), we have:

$$u^* \le \Big|\Big(-1+\frac{1}{2}\rho^2_{\beta,R,+}\Big)\Big|\Big[\frac{1}{\rho_{\beta,R,+}} + f_R{}''(\rho_{\beta,R,+})\Big]^{-1}(1+c\gamma^{a^\star}) < 1, \quad (8.6.16)$$

being $f_R(\rho)$ the mean field hard core free energy (see (3.1.7)) and using (8.8.11) and (8.8.4).

- We now want to estimate the contribution coming from the term $3N+4$ in (8.6.2), i.e.

$$\sum_i \mathcal{V}(x,z,i)[\tilde{n}_x^i - \langle \tilde{n}_x^i\rangle_t] \quad (8.6.17)$$

where we used the notation in (8.6.3). If we write:

$$\tilde{n}_x^i - \langle \tilde{n}_x^i\rangle_t = c(n_x^i - \langle n_x^i\rangle_t) \quad (8.6.18)$$

where $c<1$, we have again a term which is, up to a multiplicative constant of order one, equal to (8.6.7). Hence, it is centered and analogously to (8.6.11) we can write its contribution to (8.6.6) in the following way:

$$\frac{1}{2}\sum_{n,n'}|\phi_\tau^{(\ell)}(n,n')||n-n'| = c\Big|\sum_m \mathcal{V}(x,z,m)\Big|\sum_n p_t(n)[n^m - \langle n^m\rangle_t][n-\langle n\rangle_t] \quad (8.6.19)$$

where in the l.h.s. $\ell = 3N+4$ and where we dropped the index x from n_x. Instead of using directly bound (8.6.12) we write:

$$\sum_n p_t(n)[n^m - \langle n^m\rangle_t][n-\langle n\rangle_t] = \langle n\rangle_t^m \sum_{i=1}^m \sum_n p_t(n)[n-\langle n\rangle_t]^{i+1}\langle n\rangle_t^{-i} \le$$
$$\le \langle n\rangle_t^m \sum_{i=1}^m c\ell_2^{-di}\ell_2^{\frac{d}{2}(i+1)} \le \langle n\rangle_t^m(1+O(\ell_2^{-d})), \quad (8.6.20)$$

where in the last inequality we used (8.8.30) (see Section 8.8).

Since $n, \langle n\rangle_t \in \ell_2^d[\rho_{\beta,+}-\zeta, \rho_{\beta,+}+\zeta]$, we have the following bound:

$$\frac{\langle n\rangle_t}{n} \le \frac{\rho_{\beta,+}+\zeta}{\rho_{\beta,+}-\zeta} \quad (8.6.21)$$

which implies:

$$\langle n\rangle_t \leq n(1+\zeta) \tag{8.6.22}$$

and that if c is a constant then:

$$\langle n\rangle_t^m(1+O(\ell_2^{-d})) \leq c\,\tilde{n}^m(1+\zeta)^m. \tag{8.6.23}$$

Now we want to estimate:

$$c\Big|\sum_m \mathcal{V}(x,z,m)\Big|\tilde{n}^m(1+\zeta)^m := \Big|\frac{d\tilde{h}_t^R}{dt}(\rho_x)\Big| \tag{8.6.24}$$

which up to a multiplicative constant is expression (8.6.3). We have:

$$\Big|\frac{d\tilde{h}_t^R}{dt}(\rho_x)\Big| \leq \Big|\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x,y_k=z}}\sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N},m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\,\tilde{n}_{y_1}^{m_1-1}\tilde{n}_{y_2}^{m_2}\cdots\tilde{n}_{y_k}^{m_k}\Big|\cdot$$
$$\cdot e^{m_1\log(1+\zeta)+\log m_k}e^{d(x,z)}e^{-d(x,z)} \tag{8.6.25}$$

where we use the notation $d(x,z)$ to indicate the euclidean distance between C_x and C_z and where:

$$e^{m_1\log(1+\zeta)+\log m_k}e^{d(x,z)} \leq e^{m_1+m_k}e^{d(x,z)} \tag{8.6.26}$$

if γ is small enough. On the other side one can say that:

$$e^{m_1+m_k}e^{d(x,z)} \leq e^{(1+R)(\sum_{i=1}^k m_i)} \tag{8.6.27}$$

for some constant b. In fact one has $d(x,z) \leq R(\sum_{i=1}^k m_i)$ for a cluster going from C_x to C_z crossing k boxes with multiplicities m_i, $i=1,...,k$. Therefore, expression in (8.6.25) has been bounded by:

$$\Big|\frac{d\tilde{h}_t^R}{dt}(\rho_x)\Big| \leq \quad \Big|\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x,y_k=z}}\sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N},m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\cdot \tag{8.6.28}$$
$$\cdot\,\tilde{n}_{y_1}^{m_1-1}\tilde{n}_{y_2}^{m_2}\cdots\tilde{n}_{y_k}^{m_k}\cdot e^{(1+R)(\sum_{i=1}^k m_i)}e^{-d(x,z)}\Big| \tag{8.6.29}$$

and we can state the following lemma:

Lemma 8.6.1. *It exists a constant C such that:*

$$\Big|\frac{d\tilde{h}_t^R}{dt}(\rho_x)\Big| \leq C\frac{e^{-d(x,z)}}{\ell_2}. \tag{8.6.30}$$

Lemma 8.6.1 will be proven in Section 8.7.

Hence, we have a bound for (8.6.19) which is given by (8.6.30) and we get for $\ell = 3N+4$ the following contribution:

$$\frac{1}{2}\sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| \leq C\frac{e^{-d(x,z)}}{\ell_2}. \tag{8.6.31}$$

- For the term with $\ell = 3N+5$ in (8.6.2) we simply write:

$$\begin{aligned}\sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| &\leq \sum_{n,n'} I^{-1}[A_\ell(n)B_\ell(n') + A_\ell(n')B_\ell(n)](n+n') \\ &\leq \sum_n A_\ell(n)n + \sum_n B_\ell(n)n \leq c(\ell_2^d)^2[c\ell_2^d e^{-b\gamma^{2a-(1-\alpha)d}\cdot|x-z|\ell_3^{-1}}]\end{aligned} \tag{8.6.32}$$

the bracket being the l.h.s. of (8.5.4). We have used that $n \leq c\ell_2^d$ and, consequently, $\mathrm{Card}(\mathcal{X}^x) \leq c\ell_2^d$.

- Analogously to estimate (8.6.32), for last the two terms in (8.6.2), i.e. those with $\ell = 3N+i$, $i = 6, 7$, we have:

$$\sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| \leq c(\ell_2^d)^2[c\ell_2^d\gamma^{bN/2}\gamma^{b\gamma|x-z|/2}] \tag{8.6.33}$$

where in the bracket we used (8.4.6) and again: $n \leq c\ell_2^d$, $\mathrm{Card}(\mathcal{X}^x) \leq c\ell_2^d$. For γ small enough we have:

$$\gamma^{b\gamma|x-z|/2} = \exp\{-\log\gamma^{-1}\frac{b}{2}\gamma|x-z|\} \leq e^{-\gamma|x-z|} \tag{8.6.34}$$

and for N large enough

$$c(\ell_2^d)^2[c\ell_2^d\gamma^{bN/2}] \leq \ell_3^{-d}\gamma \tag{8.6.35}$$

so that:

$$\sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| \leq \gamma\ell_3^{-d}e^{-\gamma|x-z|}. \tag{8.6.36}$$

Putting all the contributions together (see estimates (8.6.15), (8.6.31), (8.6.32), (8.6.36)) we have:

$$\frac{1}{2}\int_0^1 \sum_{\ell=1}^{3N+7}\sum_{n,n'} |\phi_\tau^{(\ell)}(n,n')||n-n'| \leq r_\gamma(x,z) \tag{8.6.37}$$

where

$$r_\gamma(x,z) = \tilde{J}^{(2)}_\gamma(x,z)\ell_2^d u^\star + c\gamma^{\alpha/3}\gamma^d\ell_2^d \mathbf{1}_{|x-z|\le N\gamma^{-1}} + c\ell_2^{-1}e^{-|x-z|} \\ + c\ell_2^{3d}e^{-b\gamma^{2a-(1-\alpha)d_\cdot|x-z|\ell_3^{-1}}} + \gamma\ell_3^{-d}e^{-\gamma|x-z|} \tag{8.6.38}$$

We then have

$$\sum_z r_\gamma(x,z) \le u^\star + cN^d\gamma^{\alpha/3} + c\gamma^{(d+1)(1-\alpha)} + c\gamma^{d[(1-\alpha)d-2a+2\alpha]} + \gamma^{d+1} < 1 \tag{8.6.39}$$

so that (8.1.19) is satisfied. We have used:

$$\gamma\ell_3^{-d}\sum_{z\neq x} e^{-\gamma|x-z|} \le \gamma\ell_3^{-d}\sum_{z\in\mathbb{Z}^d} e^{-\gamma\ell_2|z|} \le \gamma\ell_3^{-d}\int e^{-\gamma\ell_2|r|}dr \le \gamma\ell_3^{-d}(\gamma\ell_2)^{-d} = \gamma^{d+1} \tag{8.6.40}$$

for the last sum and an analogue estimate for the third and the forth term. (8.1.20) follows obviously from (8.6.38), since for $|x-z| > \ell_3$ the first two terms give 0 contribution.

8.7 Proof of Lemma 8.6.1

From the definition of $V(y_1,...,y_k;m_1,...,m_k)$, see (8.3.8), and (8.6.28) we get:

$$\Big|\sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x,y_k=z}}\ \sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N},m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\,\tilde{n}_{y_1}^{m_1-1}\tilde{n}_{y_2}^{m_2}\cdots\tilde{n}_{y_k}^{m_k}\,e^{(1+R)(\sum_{i=1}^k m_i)}\Big| = \\ = \Big|\sum_{\substack{\pi:A(\pi)\ni 1\\ A(\pi)\cap\mathcal{L}_z\neq\emptyset}} \frac{1}{|A(\pi)\cap\mathcal{L}_x|} z_R^T(\pi;\rho_x,\rho^1)e^{b(\pi)}\Big| \tag{8.7.1}$$

where $b(\pi) := \sum_{\theta\in\pi} b(\theta)\pi(\theta)$ with π a generic cluster connecting C_x and C_z having the multiplicities $m_1,...,m_k$, with m_1 multiplicity of the box C_x and m_k multiplicity of C_z. We also remind the definition $b(\theta) = \kappa\log\gamma^{-1}|\mathcal{L}_\gamma(\theta)| + |\theta|$, which for a tree without γ links becomes: $b(\theta) = |\theta|$.

Hence,

$$\Big|\sum_{\substack{\pi:A(\pi)\ni 1\\ A(\pi)\cap\mathcal{L}_z\neq\emptyset}} \frac{1}{|A(\pi)\cap\mathcal{L}_x|} z_R^T(\pi;\rho_x,\rho^1)e^{b(\pi)}\Big| \le \sum_{\substack{\pi:A(\pi)\ni 1\\ A(\pi)\cap\mathcal{L}_z\neq\emptyset}} \frac{1}{|A(\pi)\cap\mathcal{L}_x|}\Big|z_R^T(\pi;\rho_x,\rho^1)\Big|e^{b(\pi)} \tag{8.7.2}$$

An upper bound for the r.h.s. of (8.7.2) can be found by first summing over all the polymers θ such that contain the label 1 and that exit the box C_x with only a label, e.g. the label

2 in the adjacent box C_y, and then over the clusters that contain such polymers. Hence we obtain:

$$n_y \sum_{\substack{\theta:|V(\theta)\cap\mathcal{L}_x^c|=1\\ \theta\ni 1, V(\theta)\cap\mathcal{L}_y=\{2\}}} \sum_{\pi:\pi\ni\theta} \frac{1}{|A(\pi)\cap\mathcal{L}_x|} \Big| z_R^T(\pi;\rho_x,\rho^1)\Big| e^{b(\pi)} \tag{8.7.3}$$

where n_y is given by the fact that we chose the label 2 in C_y among all the labels $\mathcal{L}_y$, with $|\mathcal{L}_y| = n_y$. By the corollary of the cluster expansion (see (7.3.38)), one has:

$$\sum_{\pi:\pi\ni\theta} \Big| z_R^T(\pi;\rho_x,\rho^1)\Big| e^{b(\pi)} \le \Big| z_R(\theta;\rho_x,\rho^1)\Big| e^{a(\theta)} \tag{8.7.4}$$

with $a(\theta) = |\theta|$, and therefore:

$$(8.7.3) \le n_y \sum_{\substack{\theta:|V(\theta)\cap\mathcal{L}_x^c|=1\\ \theta\ni 1, V(\theta)\cap\mathcal{L}_y=\{2\}}} \Big| z_R(\theta;\rho_x,\rho^1)\Big| e^{a(\theta)}. \tag{8.7.5}$$

Let us start bounding $\Big| z_R(\theta;\rho_x,\rho^1)\Big|$ once we fixed θ such that satisfies the constraints in the sum in (8.7.6) and with $|\theta| = n$. Using the definitions in Section 7.3 we have:

$$\begin{aligned}\Big| z_R(\theta;\rho_x,\rho^1)\Big| &= \Big| \int \prod_{i\in V(\theta)} \frac{dq_i}{\ell_2^d} \prod_{\ell\in E(\theta)} v_R^{(2)}(q(\ell))\Big| \le \frac{1}{\ell_2^{d(n-2)}} \Big[\int_{C_x} dq\, v_R^{(2)}(q)\Big]^{n-2} \cdot \\ &\cdot \left| \int_{q_1\in C_x} \frac{dq_1}{\ell_2^d} \mathbb{1}_{q_1\in\partial_R C_x} \int_{q_2\in C_y\cap B_R(q_1)} \frac{dq_2}{\ell_2^d} \Big(e^{-\beta V_R(q_1,q_2)} - 1\Big)\right| \end{aligned} \tag{8.7.6}$$

where we are again using the convention that the label i encodes the knowledge of the box in which the particle q_i is. In bounding the activity $z_R(\theta;\rho_x,\rho^1)$ once we fixed a given polymer, we group all the remaining $n-2$ coordinates in the box C_x once we exclude the label 1 and we change variables by considering only the relative distance with respect to the label 1 or 2, considered as fixed. Hence we obtain $n-2$ equal contributions. We are left with only one link with endpoints 1 and 2 which we integrate in the second term of (8.7.6). Like in (7.3.27) we have:

$$\Big| z_R(\theta;\rho_x,\rho^1)\Big| \le \frac{\epsilon^{n-2}}{\ell_2^{d(n-2)}} \frac{2R\epsilon}{\ell_2^{d+1}} \tag{8.7.7}$$

Hence:

$$n_y \sum_{\substack{\theta:|V(\theta)\cap\mathcal{L}_x^c|=1\\ \theta\ni\{1\}, V(\theta)\cap\mathcal{L}_y=\{2\}}} \Big| z_R(\theta;\rho_x,\rho^1)\Big| e^{a(\theta)} \le 2\ell_2^d \rho_y \sum_{n>1} \binom{n_x-1}{n-2} n^{n-2} \frac{\epsilon^{n-1}}{\ell_2^{d(n-1)}} \frac{R}{\ell_2} e^n \tag{8.7.8}$$

where instead of summing over the polymers θ we sum over the number of vertices of such polymers, but paying a factor which is given by all the choices of $n-2$ labels among the $n_x - 1$ labels of the box C_x (we already chose the label 1 in the box C_x and the label 2 in the box C_y) and by n^{n-2}, i.e. the number of trees which can be constructed over n vertices. We have:

$$(8.7.8) \leq \frac{2R}{\ell_2}\rho_y \sum_{n>1} \frac{n^{n-2}}{(n-2)!}\Big(\frac{n_x}{\ell_2^d}\Big)^{n-2}\epsilon^{n-1}e^n$$
$$\leq \frac{2R\rho_y}{\ell_2 \epsilon \rho_x^2}\sum_{n>1}\frac{e^{2n}}{\sqrt{2\pi n}}\rho_x^n\epsilon^n \tag{8.7.9}$$

where we used the approximation $\binom{n}{k} \simeq \frac{n^k}{k!}$ and Stirling's bound: $n! \geq n^n e^{-n}\sqrt{2\pi n}$. By choosing R small enough, $\rho_x \epsilon < e^{-2}$ and therefore:

$$\Big| \sum_{\substack{\{y_1,...,y_k\}:\\ y_1=x, y_k=z}} \sum_{\substack{\{m_1,...,m_k\}\\ m_i\in\mathbb{N}, m_i>0}} V(y_1,...,y_k;m_1,...,m_k)\,\tilde{\rho}_{y_1}^{m_1-1}\tilde{\rho}_{y_2}^{m_2}\cdots\tilde{\rho}_{y_k}^{m_k} e^{(1+R)(\sum_{i=1}^k m_i)}\Big| \leq \frac{2R\rho_y}{\ell_2\epsilon\rho_x^2} \tag{8.7.10}$$

Hence we can bound the r.h.s. of (8.6.28) with the following quantity:

$$\frac{2R\rho_y}{\ell_2\epsilon\rho_x^2}e^{-d(x,z)} \leq C\frac{e^{-d(x,z)}}{\ell_2} \tag{8.7.11}$$

proving the lemma. □

8.8 Estimates of the k-th moment

To prove the estimates (8.6.12) and (8.6.13) we first write:

$$n^k - \langle n^k\rangle_t = \sum_{i=1}^k\ [\langle n\rangle_t]^{k-i}[n-\langle n\rangle_t]^i \tag{8.8.1}$$

hence we are reduced to estimates of

$$\sum_{n\in\mathcal{X}^x} p_t(n)[n-\langle n\rangle_t]^k. \tag{8.8.2}$$

Letting $\rho = \rho_x$ below, we write the interpolated hamiltonian (8.1.22) with all its terms in the following way:

$$h_t(\rho) = \ell_2^d\Big\{F_t(\rho) + \ell_2^{-d}[h_t^2(\rho)+h_t^3(\rho)]\Big\} + c(n_x) \tag{8.8.3}$$

with:

$$F_t(\rho) = -\beta\lambda^\star\rho + \rho(\log\rho - 1) + \beta\sum_{i=1}^{3} a_i^0\rho^i\gamma^{\alpha d(i-1)} + \sum_{i=1}^{3N} a_i^1\rho^i\gamma^{\alpha d(i-1)+\alpha/3i} + F_t^R(\rho) \tag{8.8.4}$$

where λ^* is defined in (8.2.5) and:

$$F_t^R(\rho) := \ell_2^{-d}\Big(h_t^R(\rho) + h_t^{R,C}(\rho)\Big). \tag{8.8.5}$$

Let us consider $h_t^R(\rho)$ in (8.8.5) and do the following approximation:

$$\ell_2^{-d}h_t^R(\rho) = \ell_2^{-d}\Big(h^R(\rho|\rho^2) + t\frac{dh_t^R}{dt}(\rho)\Big) = \ell_2^{-d}h^R(\rho) + O(\ell_2^{-1}) \tag{8.8.6}$$

where $h^R(\rho|\rho^2)$ is defined in (8.3.1), dh_t^R/dt in (8.3.9) and:

$$h^R(\rho) := -\sum_{\pi: A(\pi)\subset\mathcal{L}_x} z_R^T(\pi;\rho) \tag{8.8.7}$$

is the R-cluster expansion in a single box C_x with 0-boundary conditions. The one-body approximation in the measure is quite reasonable. Let us define:

$$|R(\rho,\rho^2)| := \ell_2^{-d}|h^R(\rho|\rho^2) - h^R(\rho)| = \ell_2^{-d}\Big|\sum_{\substack{\pi: A(\pi)\cap\mathcal{L}_x\neq\emptyset \\ A(\pi)\cap\mathcal{L}_x^c\neq\emptyset}} z_R^T(\pi;\rho)\Big|. \tag{8.8.8}$$

Let us sum first on a generic label j belonging to the box x and then sum over the clusters that contain that label (dividing, however, by the number of points of the clusters which are in the box, i.e. $|A(\pi)\cap\mathcal{L}_x|$):

$$\begin{aligned} |R(\rho,\rho^2)| &= \ell_2^{-d}\Big|\sum_{j\in\mathcal{L}_x}\sum_{\substack{\pi:\, A(\pi)\ni j \\ A(\pi)\cap\mathcal{L}_x^c\neq\emptyset}} \frac{1}{|A(\pi)\cap\mathcal{L}_x|} z_R^T(\pi;\rho,\rho^i)\Big| = \\ &= \ell_2^{-d}\Big|n\sum_{\substack{\pi:\, A(\pi)\ni 1 \\ A(\pi)\cap\mathcal{L}_x^c\neq\emptyset}} \frac{1}{|A(\pi)\cap\mathcal{L}_x|} z_R^T(\pi;\rho,\rho^i)\Big| \le C\frac{R}{\ell_{2\epsilon}} \end{aligned} \tag{8.8.9}$$

where, with the constraint $A(\pi)\ni j$, the condition $A(\pi)\cap\mathcal{L}_x\neq\emptyset$ is automatically satisfied, so we can drop it. Notice that in the second equality we have fixed the label j to be the label 1 (which belongs to C_x) multiplying by a factor n. See (8.7.2) - (8.7.10) for the proof.

Because of estimate (8.5.2) the hamiltonian $h_t^{R,C}$ in the functional (8.8.5) can be neglected, hence:

$$F_t^R(\rho) = \ell_2^{-d} h_t^R(\rho) + O(e^{-b\gamma^{2a-(1-\alpha)d}}) = \ell_2^{-d} h^R(\rho) + O(\ell_2^{-1}). \tag{8.8.10}$$

Note that one can make the following infinite volume approximation:

$$F_t^R(\rho) = \ell_2^{-d} h^R(\rho) + O(\ell_2^{-1}) = f_R(\rho) + O(\ell_2^{-1}) \tag{8.8.11}$$

by paying with an error which is of the same order as the surface and where (see (3.1.7)):

$$f_R(\rho) := -\sum_{n\geq 1} \frac{\beta_n}{n+1} \rho^{n+1} \tag{8.8.12}$$

is the hard core free energy in the mean field model without the entropy part ($\rho(\log\rho - 1)$). Denoting by F_t' and F_t'' derivatives w.r.t. ρ, we have, for all ζ and γ small enough,

$$F_t'(\rho_{\beta,+} + \zeta) > 0, \quad F_t'(\rho_{\beta,+} - \zeta) < 0 \tag{8.8.13}$$

$$F_t''(\rho) > 0, \quad \text{for all } |\rho - \rho_{\beta,+}| \leq \zeta \tag{8.8.14}$$

Then there is $\rho^\star$ so that

$$F_t'(\rho^\star) = 0, \quad |\rho^\star - \rho_{\beta,+}| < \zeta \tag{8.8.15}$$

$\rho^\star$ being the unique minimizer of $F_t(\rho)$ in $|\rho - \rho_{\beta,+}| \leq \zeta$. When $|\rho - \rho^\star| \leq \ell_2^{-\frac{d}{2}+\delta d}$ ($\delta > 0$ is a small parameter which will be specified later) we have, by a Taylor expansion,

$$\ell_2^{-d} h_t(\rho) = F_t(\rho^\star) + \frac{1}{2} F_t''(\rho^\star)(\rho - \rho^\star)^2 + \tilde{R}_t(\rho)(\rho - \rho^\star)^3 + \ell_2^{-d} c(n_x) \tag{8.8.16}$$

where $\tilde{R}_t(\rho)$ contains both the rest of the Taylor expansion and the contributions coming from h_t^2 and h_t^3:

$$\tilde{R}_t(\rho) = \sum_{k=3}^{\infty} F_t^{(k)}(\rho^*)(\rho - \rho^*)^{k-3} + \ell_2^{-d}[h_t^2(\rho) + h_t^3(\rho)](\rho - \rho^*)^{-3}. \tag{8.8.17}$$

There is $C > 0$ such that:

$$\sup_{|\rho-\rho^*|\leq \ell_2^{-d/2+\delta d}} \Big| \sum_{k=3}^{\infty} F_t^{(k)}(\rho^*)(\rho - \rho^*)^{k-3} \Big| \leq C \tag{8.8.18}$$

and therefore:

$$\sup_{|\rho-\rho^*|\leq \ell_2^{-d/2+\delta d}} |\tilde{R}_t(\rho)(\rho - \rho^*)^3| \leq C\,(\ell_2^{-\frac{3}{2}d+3\delta d} + \gamma^b) =: R. \tag{8.8.19}$$

we are supposing γ so small that the set $|\rho-\rho^\star| \le \ell_2^{-d/2+\delta d}$ is contained in the set $|\rho-\rho_{\beta,+}| \le \zeta$. By (8.8.14)

$$\ell_2^{-d} \inf_{|\rho-\rho^\star|\ge \ell_2^{-d/2+\delta d}} h_t(\rho) \ge F_t(\rho^\star) + \frac{1}{2}F_t''(\rho^\star)\ell_2^{-d+2\delta d} - c\ell_2^{-\frac{3}{2}d+3\delta d} - c\ell_2^{-d}\log \ell_2$$
$$\ge F_t(\rho^\star) + \frac{1}{4}F_t''(\rho^\star)\ell_2^{-d+\delta d}. \tag{8.8.20}$$

Calling $n^\star = \ell_2^d \rho^\star$, we will first prove that there is $c > 0$ so that

$$|n^\star - \langle n\rangle_t| \le c. \tag{8.8.21}$$

The small parameter in the following analysis is

$$\xi := \ell_2^{-d/2} \tag{8.8.22}$$

We write:

$$\langle n\rangle_t - n^* = \frac{\sum_n e^{-h_t(\rho)}(n-n^*)}{\sum_n e^{-h_t(\rho)}} = \frac{\Delta}{m_0}\xi^{-1} \tag{8.8.23}$$

where:

$$\Delta = e^{\xi^{-2}F_t(\rho^\star)}[2\pi n^\star]^{1/2}\xi \sum_n e^{-h_t(\rho)}\xi(n-n^\star), \tag{8.8.24}$$

the sum over n is always meant to be extended over $n \in \mathcal{X}^x$, and where:

$$m_0 = e^{\xi^{-2}F_t(\rho^\star)}[2\pi n^\star]^{1/2}\xi \sum_n e^{-h_t(\rho)}. \tag{8.8.25}$$

We write

$$\Delta = \Delta' + \Delta'', \qquad m_0 = m_0' + m_0''$$

the primed variables being obtained by restricting the sum over n in (8.8.24) and (8.8.25) to $\{n : |n-n^\star| \le \xi^{-1-\delta}\}$, while the double primed ones are restricted to $\{n : |n-n^\star| > \xi^{-1-\delta}\}$. By (8.8.16)

$$\Delta' = \xi \sum_{|n-n^\star|\le\xi^{-1-\delta}} [\xi(n-n^\star)]\sqrt{\frac{n^*}{n}}\exp\Big\{-\frac{F_t''(\rho^\star)}{2}[\xi(n-n^\star)]^2$$
$$-\tilde{R}_t(\rho)\xi[\xi(n-n^\star)]^3 - \xi^2\vartheta(\rho)\Big\}$$

$\vartheta(\rho)$ comes from the remainder of the Stirling formula: there is c so that for all ρ

$$|\vartheta(\rho)| \le c$$

We are now going to prove that

$$|\Delta'| \leq c\xi. \tag{8.8.26}$$

Let us shorthand the following quantity:

$$\psi = \tilde{R}_t(\rho)\xi[\xi(n-n^\star)]^3 + \xi^2\vartheta(\rho)$$

and write:

$$e^{-\psi}\sqrt{\frac{n^*}{n}} = \sqrt{\frac{n^*}{n}} + (e^{-\psi}-1)\sqrt{\frac{n^*}{n}}$$

so that we can bound the terms as follows:

$$\Big|\sqrt{\frac{n^*}{n}}\Big| \leq c\xi|\xi(n-n^\star)|$$

$$\Big|(e^{-\psi}-1)\sqrt{\frac{n^*}{n}}\Big| \leq c[\xi^2 + \xi|\xi(n-n^\star)|^3].$$

Hence we have:

$$|\Delta'| \leq c\xi \sum_{|n-n^\star|\leq\xi^{-1-\delta}} e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2}\Big\{\xi^2|\xi(n-n^\star)| + \xi|\xi(n-n^\star)|^4 + \xi|\xi(n-n^\star)|^2\Big\} \leq c\xi$$

which proves (8.8.26). Since:

$$m_0' = \xi \sum_{|n-n^\star|\leq\xi^{-1-\delta}} \sqrt{\frac{n^*}{n}} e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2} e^{-\psi}$$

by analogous procedure

$$\Big|m_0' - \sqrt{\frac{2\pi}{F_t''(\rho)}}\Big| \leq c\xi. \tag{8.8.27}$$

In fact the leading term in m_0' is given by:

$$\xi \sum_{|n-n^\star|\leq\xi^{-1-\delta}} e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2} \leq \xi\sum_n e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2} \leq \xi\int dn\, e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2} + c\xi \tag{8.8.28}$$

By (8.8.20)

$$|\Delta''| \leq \xi \sum_{|n-n^\star|\geq\xi^{-1-\delta}} |\xi(n-n^\star)| \exp\Big\{-\frac{F_t''(\rho^\star)}{4}\xi^{-4\delta}\Big\}$$

so that

$$|\Delta''| \leq \xi(c\xi^{-2})\xi^{-1} \exp\Big\{-\frac{F_t''(\rho^\star)}{4}\xi^{-4\delta}\Big\}$$

as the sum over n contains at most $c\xi^{-2}$ terms and $|\xi(n-n^\star)| \le \xi^{-1}$. Analogously

$$m_0'' \le \xi(c\xi^{-2}) \exp\Big\{ - \frac{F_t''(\rho^\star)}{4} \xi^{-4\delta} \Big\}$$

and we conclude

$$\frac{|\Delta|}{m_0} \le \frac{c\xi + c\xi^{-2} \exp\Big\{ - \frac{F_t''(\rho^\star)}{4} \xi^{-4\delta} \Big\}}{\sqrt{\frac{2\pi}{F_t''(\rho^\star)}} - c\xi - c\xi^{-1} \exp\Big\{ - \frac{F_t''(\rho^\star)}{4} \xi^{-4\delta} \Big\}} \le c\xi$$

and (8.8.21) follows from (8.8.23). Analogously to (8.8.25) we write

$$m_k = e^{\xi^{-2} F_t(\rho^\star)} [2\pi n^\star]^{1/2} \xi \sum_n e^{-h_t(\rho)} [\xi(n - \langle n \rangle_t)]^k \tag{8.8.29}$$

and

$$\sum_n p_t(n) [n - \langle n \rangle_t]^k = \frac{m_k}{m_0} \xi^{-k} \tag{8.8.30}$$

As before (details are omitted) for any there is c so that

$$\frac{|m_k|}{m_0} \le c \tag{8.8.31}$$

then, by (8.8.1), for $k > 1$,

$$\begin{aligned} \sum_n p_t(n) [n^k - \langle n^k \rangle_t][n - \langle n \rangle_t] &\le \sum_{i=1}^k [\langle n \rangle_t]^{k-i} c\xi^{-i-1} \\ &\le \sum_{i=1}^k [c\ell_2^d]^{k-i} [c\ell_2^{d/2}]^{i+1} \le c\ell_2^{dk} \end{aligned}$$

(8.6.12) is proved.

We need a more accurate estimate when $k = 1$ in. We write

$$[n - \langle n \rangle_t]^2 = [n - n^\star]^2 + [n^\star - \langle n \rangle_t]^2 + 2[n - n^\star][n^\star - \langle n \rangle_t] \tag{8.8.32}$$

and use it in (8.8.29), having three contributes for m_2. We write explicitly the main contribution $\tilde{m}_2$, which is given by the first one in (A.1.5) where the sum in m_2 is over $|n - n^*| \le \xi^{-1-\delta}$. Recalling (8.8.21) one can easily see that all the other terms give corrections of order ξ.

$$\tilde{m}_2 = e^{\xi^{-2} F_t(\rho^\star)} [2\pi n^\star]^{1/2} \xi \sum_{|n - n^\star| \le \xi^{-1-\delta}} e^{-h_t(\rho)} [\xi(n - n^*)]^2 \tag{8.8.33}$$

and again:

$$|\tilde{m}_2| \le \xi \sum_{|n-n^\star|\le \xi^{-1-\delta}} \sqrt{\frac{n^*}{n}} e^{-F_t''(\rho^\star)[\xi(n-n^\star)]^2/2} [\xi(n-n^*)]^2 + c\xi \tag{8.8.34}$$

where the $c\xi$ comes from the part arising from $|\psi|$ in the expansion of $e^{-\psi}$. Dividing by m_0, and considering that the leading term in m_0 is given by $\sqrt{\frac{2\pi}{F_t''(\rho)}}$ as proved in (8.8.27), we have:

$$\Big|\frac{m_2}{m_0} - \frac{1}{F_t''(\rho^\star)}\Big| \le c\xi \tag{8.8.35}$$

which implies of course:

$$\sum_n p_t(n)[n - \langle n\rangle_t]^2 \le \xi^{-2}\Big(\frac{1}{F_t''(\rho^\star)} + c\xi\Big). \tag{8.8.36}$$

8.9 Changing boundary conditions

We complete here the proof of Theorem 8.1.1 by showing that for any $\rho = \{\rho_y\}$, $y \in \mathcal{X}^{\Lambda/x}$, and any $\bar{q}^1$ and $\bar{q}^2$

$$R\Big(p(\cdot|\rho,\bar{q}^1), p(\cdot|\rho,\bar{q}^2)\Big) \le \sum_{z\in X_{\Lambda^c}} r_{\gamma,R}(x,z) D_z(\bar{q}^1,\bar{q}^2) \tag{8.9.1}$$

For the sake of definitiness let us suppose that

$$\big|\bar{q}^2 \cap C_z^{(\ell_2)}\big| = m, \;\; \big|\bar{q}^1 \cap C_z^{(\ell_2)}\big| = m+p, \;\; \big|\bar{q}^1 \cap \bar{q}^2 \cap C_z^{(\ell_2)}\big| = n$$

with m, p, n non negative. Then, by (8.1.18)

$$D_z(\bar{q}^1,\bar{q}^2) = p + m - n$$

Let $q_1..q_p$ be particles of $\bar{q}^1$ which are not in $\bar{q}^2$ and

$$\bar{q}_0^1 = \bar{q}^1, \;\; \bar{q}_j^1 = \bar{q}^1 - (q_1 + .. + q_j)$$

meaning that we take out of $\bar{q}^1$ the particles $q_1..q_j$. Then

$$R\Big(p(\cdot|\rho,\bar{q}^1), p(\cdot|\rho,\bar{q}^2)\Big) \le \sum_{j=0}^{p-1} R\Big(p(\cdot|\rho,\bar{q}_j^1), p(\cdot|\rho,\bar{q}_{j+1}^1)\Big) + R\Big(p(\cdot|\rho,\bar{q}_p^1), p(\cdot|\rho,\bar{q}^2)\Big) \tag{8.9.2}$$

Let $q'_1 ... q'_{m-n}$ be the particles in $\bar{q}^1_p$ which are not in $\bar{q}^2$ and $q''_1 ... q''_{m-n}$ the corresponding ones in $\bar{q}^2$ and not in $\bar{q}^1_p$. We set

$$\bar{q}^2_0 = \bar{q}^2, \;\; \bar{q}^2_j = \bar{q}^2 + (q'_1 + .. + q'_j) - (q''_1 + .. + q''_j)$$

so that

$$R\Big(p(\cdot|\rho, \bar{q}^1), p(\cdot|\rho, \bar{q}^2)\Big) \leq \sum_{j=0}^{p-1} R\Big(p(\cdot|\rho, \bar{q}^1_j), p(\cdot|\rho, \bar{q}^1_{j+1})\Big) + \sum_{j=0}^{m-n-1} R\Big(p(\cdot|\rho, \bar{q}^2_j), p(\cdot|\rho, \bar{q}^2_{j+1}))\Big) \tag{8.9.3}$$

The total number of terms in (8.9.3) is exactly equal to $D_z(\bar{q}^1, \bar{q}^2)$, it will thus be enough to prove that each single term is bounded by $r_\gamma(x, z)$. The terms in the first sum are treated exactly as when $z \in \Lambda$. For the terms in the second sum we observe that h^0 is the same in both configurations. The change of the energies h^k, $k \neq 0$, is due to the difference between the interaction of the special particle with the others in the two distinct positions. Such contributions to the Vaserstein distance are again those computed above and we get the previous bound with a factor 2 (we do not need compensations between the two contributions and we just sum them up).

(8.8.35) then follows, once we recall that $D_z(\bar{q}^1, \bar{q}^2) = k + m$, where k is the difference between the number of particles in $C_z^{(\ell_2)}$ and m the optimal number of shifts of particles, and $k + m$ is the same as the number of triangular inequalities used to reduce to the case where we either add a particle or shift one.

8.10 The Dobrushin method for constructing couplings

In the following we refer again to (8.1.1) for the definition of the state space and in general to Section 7.1 and Section 8.1 for the setting.

Claim 8.10.1. *We are going to prove that there exists a joint representation $\mathcal{P}(\underline{n}^1, \underline{n}^2|\bar{q}^1, \bar{q}^2)$ of the marginals of μ^1 and μ^2 on $\mathcal{X}^\Lambda$ such that, for any $x \in X_\Lambda$ and denoting by $\mathcal{E}$ the expectation w.r.t. to $\mathcal{P}$,*

$$\mathcal{E}\big[(d(n^1_x, n^2_x)\big] \leq \sum_{z \in X_\Lambda, z \neq x} r_{\gamma,R}(x, z) \mathcal{E}\big[d(n^1_z, n^2_z)\big] + \sum_{z \notin X_\Lambda} r_{\gamma,R}(x, z) \gamma^{-d} c \tag{8.10.1}$$

For simplicity from now on we will drop the dependence on $\bar{q}^1$ and $\bar{q}^2$ from $\mathcal{P}(\underline{n}^1, \underline{n}^2|\bar{q}^1, \bar{q}^2)$. The existence of such joint representation relies on the algorithm proposed by Dobrushin (for the reference see [52]) based on a local optimization strategy. We can also state it in a more general setting, but for the purpose of this section and to make the approach easier for the reader we will apply such strategy directly to our problem.

The key point of the entire theory is that "local bounds can be made global", as proved in the next theorem.

Let us first suppose that we have already found a "good" coupling for the conditional probabilities: $p(\cdot|\rho^1, \bar{q}^1)$ and $p(\cdot|\rho^2, \bar{q}^2)$ with $\rho^i = \{\rho_y\}$, $y \in \mathcal{X}^{\Lambda/x}$, $i = 1, 2$. We call it $Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(\cdot,\cdot)$ as a probability on $\mathcal{X}^x \times \mathcal{X}^x$. We remind again that the Vaserstein distance is defined through a cost function, i.e. a pseudo-distance on $\mathcal{X}^x$ defined in (8.1.12), and we define: $d_I(\underline{n}^1, \underline{n}^2) := \sum_{x\in I} d(n_x^1, n_x^2)$, $I \subset X_\Lambda$.

Theorem 8.10.2. *Suppose that there are non-negative functions $K_x(\underline{n}^1, \underline{n}^2) = K_x(\rho^1, \bar{q}^1, \rho^2, \bar{q}^2)$ such that:*

$$\sum_{n_x^1, n_x^2} d(n_x^1, n_x^2) Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) \leq K_x(\rho^1, \bar{q}^1, \rho^2, \bar{q}^2) \tag{8.10.2}$$

Then there is a coupling $\mathcal{P}$ of μ^1 and μ^2 such that,

$$\sum_{\underline{n}_1, \underline{n}_2} d_x(\underline{n}^1, \underline{n}^2)\mathcal{P}(\underline{n}^1, \underline{n}^2) \equiv \sum_{\underline{n}_1, \underline{n}_2} d(n_x^1, n_x^2)\mathcal{P}(\underline{n}^1, \underline{n}^2) \leq \sum_{\underline{n}_1, \underline{n}_2} K_x(\rho^1, \bar{q}^1, \rho^2, \bar{q}^2)\mathcal{P}(\underline{n}^1, \underline{n}^2). \tag{8.10.3}$$

Proof.

Let P_0 be a coupling of μ^1 and μ^2. Let us divide $\underline{n}^i$ into n_x^i and ρ^i, $i = 1, 2$, and call

$$P_0(\rho^1, \rho^2) := \sum_{n_x^1, n_x^2} P_0(n_x^1, \rho^1, n_x^2, \rho^2) \tag{8.10.4}$$

namely $P_0(\rho^1, \rho^2)$ is the marginal distribution of the components ρ^1, ρ^2. Define next

$$P(\underline{n}_1, \underline{n}_2) := Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) P_0(\rho^1, \rho^2). \tag{8.10.5}$$

We claim that $P(\underline{n}_1, \underline{n}_2)$ is also a coupling of μ^1 and μ^2: since $Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2)$ is a

coupling of the conditional probabilities,

$$\sum_{\underline{n}^2} P(\underline{n}^1, \underline{n}^2) = \sum_{\rho^2} \sum_{n_x^2} Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) P_0(\rho^1, \rho^2) \tag{8.10.6}$$

$$= \sum_{\rho^2} p(n_x^1|\rho^1, \bar{q}^1) P_0(\rho^1, \rho^2) \tag{8.10.7}$$

$$= \sum_{\rho^2} p(n_x^1|\rho^1, \bar{q}^1) \sum_{n_x^3, n_x^2} P_0(n_x^3, \rho^1, n_x^2, \rho^2) \tag{8.10.8}$$

We first sum over $\underline{n}^2 = (\rho^2, n_x^2)$, and, since P_0 is a coupling, we get:

$$\sum_{\underline{n}^2} P(\underline{n}_1, \underline{n}_2) = \sum_{n_x^3} \mu^1(\rho^1, n_x^3|\bar{q}^1) p(n_x^1|\rho^1, \bar{q}^1) = \mu^1(\underline{n}^1). \tag{8.10.9}$$

The same argument applies to the second component completing the proof that $P(\underline{n}^1, \underline{n}^2)$ is a coupling of μ^1 and μ^2. Since a convex combination of couplings is still a coupling,

$$P_1(\underline{n}^1, \underline{n}^2) := \frac{1}{M} \sum_x Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) P_0(\rho^1, \rho^2) \tag{8.10.10}$$

is also a coupling, where M is the cardinality of the set $\mathcal{X}^\Lambda$. By iteration

$$P_{k+1}(\underline{n}^1, \underline{n}^2) := \frac{1}{M} \sum_x Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) P_k(\rho^1, \rho^2) \tag{8.10.11}$$

are all couplings of μ^1 and μ^2. Then:

$$\mathcal{P}_k(\underline{n}^1, \underline{n}^2) := \frac{1}{k+1} \sum_{h=0}^{k} P_h(\underline{n}^1, \underline{n}^2) \tag{8.10.12}$$

is a coupling and, by letting $k \to \infty$ along a convergent subsequence k_j (the space $\mathcal{X}^\Lambda$ is finite), also

$$\mathcal{P}(\underline{n}^1, \underline{n}^2) := \lim_{k_j \to \infty} \mathcal{P}_{k_j}(\underline{n}^1, \underline{n}^2) \tag{8.10.13}$$

is a coupling. Moreover, since $\lim_{k_j \to \infty} \mathcal{P}_{k_j}(\underline{n}^1, \underline{n}^2) = \mathcal{P}(\underline{n}^1, \underline{n}^2)$,

$$\mathcal{P}(\underline{n}^1, \underline{n}^2) = \lim_{k_j \to \infty} \frac{1}{M} \sum_x Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) \mathcal{P}_{k_j}(\rho^1, \rho^2) \tag{8.10.14}$$

$$= \frac{1}{M} \sum_x Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1, n_x^2) \mathcal{P}(\rho^1, \rho^2), \tag{8.10.15}$$

so that if $y \in \mathcal{X}^\Lambda$

$$\sum_{\underline{n}^1,\underline{n}^2} d_y(\underline{n}^1,\underline{n}^2)\mathcal{P}(\underline{n}^1,\underline{n}^2) \tag{8.10.16}$$

$$= \sum_{\underline{n}^1,\underline{n}^2} d_y(\underline{n}^1,\underline{n}^2)\frac{1}{M}\sum_{x\neq y} Q_{x,\rho^1,\rho^2,\bar{q}^1,\bar{q}^2}(n_x^1,n_x^2)\mathcal{P}(\rho^1,\rho^2) \tag{8.10.17}$$

$$+ \sum_{\underline{n}^1,\underline{n}^2} d_y(\underline{n}^1,\underline{n}^2)\frac{1}{M} Q_{y,\rho^1_{(y)},\rho^2_{(y)},\bar{q}^1,\bar{q}^2}(n_y^1,n_y^2)\mathcal{P}(\rho^1_{(y)},\rho^2_{(y)}) \tag{8.10.18}$$

where $\rho^i_{(y)} := \{\rho_z\}$, $z \in \mathcal{X}^{\Lambda/y}$, $i = 1,2$, and following this notation $\rho^i \equiv \rho^i_{(x)}$. We rewrite the first term on the r.h.s. of (8.10.16) as follows. We fix $x \neq y$ and sum over n_x^1, n_x^2, getting

$$\frac{1}{M}\sum_{x\neq y}\sum_{\rho^1,\rho^2} d_y(\rho^1,\rho^2)\mathcal{P}(\rho^1,\rho^2) = \frac{1}{M}\sum_{x\neq y}\sum_{\underline{n}^1,\underline{n}^2} d_y(\underline{n}^1,\underline{n}^2)\mathcal{P}(\underline{n}^1,\underline{n}^2) \tag{8.10.19}$$

Analogously, in the second term on the r.h.s. of (8.10.16) we write

$$\sum_{\rho^1_{(y)},\rho^2_{(y)}} \mathcal{P}(\rho^1_{(y)},\rho^2_{(y)}) \sum_{n_y^1,n_y^2} Q_{y,\rho^1_{(y)},\rho^2_{(y)},\bar{q}^1,\bar{q}^2}(n_y^1,n_y^2) d(n_y^1,n_y^2) \tag{8.10.20}$$

$$\leq \sum_{\rho^1_{(y)},\rho^2_{(y)}} \mathcal{P}(\rho^1_{(y)},\rho^2_{(y)}) K_y(\rho^1_{(y)},\bar{q}^1,\rho^2_{(y)},\bar{q}^2) \tag{8.10.21}$$

$$= \sum_{\underline{n}^1,\underline{n}^2} \mathcal{P}(\underline{n}^1,\underline{n}^2) K_y(\rho^1_{(y)},\bar{q}^1,\rho^2_{(y)},\bar{q}^2). \tag{8.10.22}$$

Collecting the above estimates we get from (8.10.16)

$$\sum_{\underline{n}_1,\underline{n}_2} d_y(\underline{n}^1,\underline{n}^2)\mathcal{P}(\underline{n}^1,\underline{n}^2) \equiv \sum_{\underline{n}_1,\underline{n}_2} d(n_y^1,n_y^2)\mathcal{P}(\underline{n}^1,\underline{n}^2) \tag{8.10.23}$$

$$\leq \frac{M-1}{M}\sum_{\underline{n}_1,\underline{n}_2} d(n_y^1,n_y^2)\mathcal{P}(\underline{n}^1,\underline{n}^2) + \frac{1}{M}\sum_{\underline{n}_1,\underline{n}_2} \mathcal{P}(\underline{n}^1,\underline{n}^2) K_y(\rho^1_{(y)},\bar{q}^1,\rho^2_{(y)},\bar{q}^2) \tag{8.10.24}$$

hence (8.10.3), once changed y into x.

□

An immediate consequence of Theorem 8.10.2 is the following corollary:

Corollary 8.10.3. *Suppose there are constants C_x and $r(x,z)$ so that* (8.10.2) *holds with*

$$K_x(\rho^1,\bar{q}^1,\rho^2,\bar{q}^2) \leq C_x + \sum_{z\neq x} r(x,z) d(n_z^1,n_z^2). \tag{8.10.25}$$

Let $\mathcal{P}$ as in Theorem 8.10.2 and $v(x) := \sum_{\underline{n}^1,\underline{n}^2} d(n_x^1, n_x^2)\mathcal{P}(\underline{n}^1, \underline{n}^2)$.

Then

$$v(x) \leq \sum_{z \neq x} r(x,z)v(z) + C_x, \tag{8.10.26}$$

hence $v(\cdot) \leq u(\cdot)$, where:

$$u(x) = \sum_{z \neq x} r(x,z)u(z) + C_x. \tag{8.10.27}$$

Proof of (8.12.1).

Let us show how Corollary 8.10.3 implies (8.12.1). By a general theorem (see for instance Theorem 3.2.1.1 of [52]), the inf in (8.1.16) is a min. Hence, there is a coupling of $p(\cdot|\rho^1, \bar{q}^1)$ and $p(\cdot|\rho^2, \bar{q}^2)$ which attains the Vaserstein distance on the l.h.s. of (8.1.17) and (8.10.25) holds with

$$C_x = \sum_{z \notin X_\Lambda} r_{\gamma,R}(x,z) D_z(\bar{q}^1, \bar{q}^2) \tag{8.10.28}$$

and $r(x,z) = r_{\gamma,R}(x,z)$. Hence, using the bound $D_z(\bar{q}^1, \bar{q}^2) \leq c\gamma^{-d}$ and (8.10.26), one obtains (8.12.1) by setting: $v(x) := \sum_{\underline{n}^1,\underline{n}^2} d(n_x^1, n_x^2)\mathcal{P}(\underline{n}^1, \underline{n}^2) = \mathcal{E}\big[(d(n_x^1, n_x^2)\big]$ where the last expression is the expectation w.r. to $\mathcal{P}$.

8.11 Equations for the first moments and decay properties

We study here the linear system of the $|X_\Lambda|$ equations:

$$u(x) = \sum_{z \neq x} r(x,z)u(z) + C_x, \quad x \in X_\Lambda \tag{8.11.1}$$

where C_x must be of the form:

$$C_x = C'_x + \sum_{z \notin X_\Lambda} r(x,z)\psi(z), \quad \sup|\psi(z)| < \infty. \tag{8.11.2}$$

$v(x), x \in X_\Lambda$, is a sub-solution of (8.11.1) if:

$$v(x) \leq \sum_{z \neq x} r(x,z)v(z) + C_x. \tag{8.11.3}$$

Theorem 8.11.1. *Let $r(x,z) \geq 0$ satisfy*

$$\sup_x \sum_{z \neq x} r(x,z) := r < 1. \tag{8.11.4}$$

Then (8.11.1) *has the unique solution*

$$u(x) = \sum_{y \in X_\Lambda} g_\Lambda(x,y) C_y \tag{8.11.5}$$

where

$$g_\Lambda(x,y) := \sum_{n \geq 0} r_\Lambda^n(x,y), \qquad r_\Lambda(x,y) = r(x,y) \mathbb{1}_{x,y \in X_\Lambda} \tag{8.11.6}$$

with $r_\Lambda^0(x,y) := \mathbb{1}_{x=y}$ and $r_\Lambda^n(x,y)$ the n-convolution of $r_\Lambda(x,y)$. The series in (8.11.6) *is convergent and*

$$g_\Lambda(x,y) \leq \frac{1}{1-a_\Lambda}, \qquad a_\Lambda := \sup_{x \in X_\Lambda} \sum_{y \in X_\Lambda, y \neq x} r_\Lambda(x,y) \leq r < 1. \tag{8.11.7}$$

Finally, if v is a sub-solution of (8.11.1)*, then $v \leq u$.*

For the proof of Theorem 8.11.1 see [52], Chapter 3.

Let us now investigate the decay properties of the Green function g_Λ. We define:

$$g(x,y) := \sum_{n \geq 0} r^n(x,y) \tag{8.11.8}$$

and notice that $g_\Lambda \leq g$ point-wise, so that the decay properties of g immediately are reflected in the decay properties of g_Λ and then of u.

To deduce good decay properties of $u(x)$ we use a strong assumption on the decay rate of $r(x'y)$ as $|x-y| \to \infty$. Suppose there is a metric $\delta(x,y)$ such that $\delta(x,y) \geq a > 0$ for all $x \neq y$, and

$$\sup_x \sum_{y \neq x} r(x,y) e^{\delta(x,y)} \leq r' < 1 \tag{8.11.9}$$

then (8.11.9) implies (8.11.4) with $r < e^{-a}$.

Theorem 8.11.2. *Suppose that* (8.11.9) *holds, then for any x, any set $A \subset X_\Lambda$ and any non-negative function f on A,*

$$\sum_{y \in A} g(x,y) f(y) \leq \frac{1}{1-r'} \sup_{y \in A} \{ e^{-\delta(x,y} f(y) \}. \tag{8.11.10}$$

Then we have the following corollary:

Corollary 8.11.3. *Let u satisfy* (8.11.1)*; then, if* (8.11.9) *holds,*

$$u(x) \leq \frac{1}{1-r'} \sup_{y \in X_\Lambda} \{e^{-\delta(x,y)} C'_y\} + \frac{b}{1-r'} \sup_{y \in \Lambda^c} e^{-\delta(x,y)}. \tag{8.11.11}$$

8.12 Conclusions

Coming back to:

$$\mathcal{E}\big[(d(n_x^1, n_x^2)\big] \leq \sum_{z \in X_\Lambda, z \neq x} r_{\gamma,R}(x,z) \mathcal{E}\big[d(n_z^1, n_z^2)\big] + \sum_{z \notin X_\Lambda} r_{\gamma,R}(x,z) \gamma^{-d} c \tag{8.12.1}$$

we can apply Corollary 8.11.3 obtaining that there are three positive constants δ, c, c' so that:

$$\mathcal{E}\big[(d(n_x^1, n_x^2)\big] \leq c \exp\Big\{ - c' \gamma^{-\delta} \ell_3^{-1} \mathrm{dist}(x, \Lambda^c)\Big\} \tag{8.12.2}$$

with c_1, c_2 suitable positive constants.

Proof of (8.12.2).
We have from Theorem 8.1.1:

$$r_{\gamma,R}(x,z) \leq c_1 e^{-c_2 \gamma |z-x|}, \quad |z-x| \geq \ell_3. \tag{8.12.3}$$

First we want to prove that this condition implies (8.11.9). Chosing $\delta(x,y) = \kappa_\gamma |x-y|$,

$$\begin{aligned} &\sup_x \sum_{y \neq x} r(x,y) e^{\delta(x,y)} = \sup_x \Big\{ \sum_{|x-y| \leq \ell_3} r(x,y) e^{\kappa_\gamma |x-y|} + \sum_{|x-y| > \ell_3} r(x,y) e^{\kappa_\gamma |x-y|} \Big\} \leq \\ &\leq e^{c_\gamma \gamma \ell_3} + \sum_{|x-y| > \ell_3} c_1 e^{-c_2 \gamma |x-y| + c_\gamma \gamma |x-y|} < 1 \end{aligned} \tag{8.12.4}$$

where we used: $\kappa_\gamma \equiv c_\gamma \gamma$. In order to bound it with 1 the second term in (8.12.4) implies $c_\gamma < c_2$ while the first one implies $c_\gamma = \gamma^{\alpha'_2}$ with $\alpha'_2 \geq \alpha$. Hence we can apply Corollary 8.11.3 for $v(x) := \mathcal{E}\big[(d(n_x^1, n_x^2)\big]$, $v(x)$ being a sub-solution of (8.11.1), where C_x, being as in (8.10.28), is of the form (8.11.2) with $C'_x \equiv 0$ and $\psi(z) := D_z(\bar{q}_1, \bar{q}_2) \leq c\gamma^{-d}$. Hence we obtain (8.12.2) by putting $\delta := \alpha - \alpha'_2 > 0$. □

Proof of (7.1.2).

In this proof we are going to use Lemma 7.5.1 of Section 7.5 and (8.12.2). We remind that

$$E_{\mu^i}\big(f_{x_1,..,x_n}\big) = E_{\mu^i}(g_N) + R_i, \qquad i = 1,2 \tag{8.12.5}$$

where g_N and R_i satisfy bounds (7.5.2).

We also remind that g_N is defined summing the two contributions (7.5.20) and (7.5.24), and setting $\underline{y} = \{y_1, ..., y_m\}$, $\underline{x} = \{x_1, ..., x_n\}$, we can rewrite it as follows:

$$g_N := \sum_{\substack{\pi \subset C_{x_i}^{(N\ell_3)} \\ \pi \ni \underline{x}}} \tilde{z}^T_{\gamma,R}(\pi;\rho;\bar{q};\underline{x}) + \sum_{\substack{C \subset C_{x_i}^{(N\ell_3)} \\ C \ni \underline{x}}} \tilde{\zeta}^T_{\gamma,R}(C;\rho;\bar{q};\underline{x}) = \sum_{\underline{y} \supset \underline{x}} g_{\underline{y}} \tag{8.12.6}$$

where:

$$g_{\underline{y}} := \sum_{\substack{\pi \subset C_{x_i}^{(N\ell_3)} \\ \pi \ni \underline{y}}} \tilde{z}^T_{\gamma,R}(\pi;\rho;\bar{q};\underline{y}) + \sum_{\substack{C \subset C_{x_i}^{(N\ell_3)} \\ C \ni \underline{y}}} \tilde{\zeta}^T_{\gamma,R}(C;\rho;\bar{q};\underline{y}) \tag{8.12.7}$$

Hence:

$$|E_{\mu^1}(g_N) - E_{\mu^2}(g_N)| = \Big| \sum_{\underline{n}^1,\underline{n}^2} \mathcal{P}(\underline{n}^1,\underline{n}^2|\bar{q}^1,\bar{q}^2)\big[g_N(\underline{n}^1) - g_N(\underline{n}^2)\big]\Big| = \tag{8.12.8}$$

$$= \Big| \sum_{\underline{n}^1,\underline{n}^2} \mathcal{P}(\underline{n}^1,\underline{n}^2|\bar{q}^1,\bar{q}^2) \sum_{\underline{y} \supset \underline{x}} \big[g_{\underline{y}}(\underline{n}^1) - g_{\underline{y}}(\underline{n}^2)\big]\Big| \leq \tag{8.12.9}$$

$$\leq \sum_{\underline{y}} \sum_{\underline{n}^1,\underline{n}^2} \mathcal{P}(\underline{n}^1,\underline{n}^2|\bar{q}^1,\bar{q}^2)\Big|g_{\underline{y}}(\underline{n}^1) - g_{\underline{y}}(\underline{n}^2)\Big| \leq \tag{8.12.10}$$

$$\leq \sum_{\underline{y}} \|g_{\underline{y}}\|_\infty \sum_{\underline{n}^1,\underline{n}^2} \mathcal{P}(\underline{n}^1,\underline{n}^2|\bar{q}^1,\bar{q}^2) \sum_{i=1}^m \mathbb{1}_{n_i^1 \neq n_i^2} \leq \tag{8.12.11}$$

$$\leq \sum_{\underline{y}} \|g_{\underline{y}}\|_\infty \mathcal{E}\big[\sum_{i=1}^m d(n_i^1, n_i^2)\big] \leq \tag{8.12.12}$$

$$\leq \max_i \sum_{\underline{y}} \|g_{\underline{y}}\|_\infty m\, \mathcal{E}\big[d(n_i^1, n_i^2)\big] \tag{8.12.13}$$

Using (8.12.2) and (7.5.2) we get (7.1.2). □

Conclusions and open problems

As we see from Chapters 7 and 8, we manage to carry out some computations which imply the decay of correlations as required by the third step of Chapter 5 (see Section 5.3), by controlling the difference of the two expectations in equation (5.3.6). Hence the hardest part is to calculate these expectations. We first use cluster expansion to reduce our system to a spin block model through the derivation of an effective hamiltonian and then Dobrushin uniqueness theory to prove exponential decay.

Anyways, the exponential decay of correlations does not imply estimate (5.3.1) when the two local functions $f_{x_1,..,x_n}$ defined in (5.3.5) are too close to the boundary of Λ. In that case the difference of expectations becomes of order 1 and we have to change strategy. We shall address this issue in a subsequent work, adapting to our model the techniques developed by Presutti in [52].

Another issue which we hope to address later concerns the σ-algebra at infinity and the derivation of some properties for the two Gibbs measures representing the two phases. While we think these properties can again be derived as in [52] (Chapter 12), to prove phase transition in the canonical ensemble is still an open problem. As we said in Chapter 2, one would like to prove closeness to one of the two phases in the coarse grained picture.

In the more abstract context of a system of particles interacting in the continuum via a stable and tempered pair potential we are able to compute the free energy by cluster expanding the canonical partition function, as we do in [53]. A question arising is wether one is able to compute the correlations and to compare the behaviour in the canonical with that in the gran canonical ensemble, expecting that the decay is not exponential anymore

even though small as 1 over the volume. See [56] for the correlation functions in the gran canonical ensemble.

We want to conclude this section by introducing a problem which seems at first very different from the topic we dealt with in this work and which is related to the quantum version of a system of particles in the continuum. We refer to the set of results obtained by Ginibre in the context of quantum gases, see [24], [25] and [26], where there have been computed the correlation functions performing a cluster expansion as for the virial series. As these results are carried out using the formalism of the gran canonical ensemble we should be able to obtain similar results working in the canonical, adapting to the quantum model the methods in [53]. Hence, a first step in this direction would be to perform a cluster expansion in the canonical ensemble for a quantum gas, while a second more ambitious project would be to prove phase transitions for the same quantum model, to have similar results as we have in the present work. In this regards there is an unpublished paper by Kuna, Merola and Presutti which proves gas-liquid phase transitions for a quantum gas with the Boltzman statistics. The main issue is whether we are able to prove the same transition in the case of a system of bosons when the density is far from the regime in which there is condenstation, using the Ginibre results and [53].

A. Cluster expansion in the canonical ensemble

A.1 Model and result

This chapter is devoted to the theory of cluster expansion in the canonical ensemble when the system is composed of hard spheres in d dimensions. In the following we refer to the paper [53] where the interaction is more general, i.e. stable and tempered, but we report the result directly in the case of hard core interactions for simplicity of notation.

We consider a configuration $\mathbf{q} \equiv \{q_1, \dots, q_N\}$ of N particles (where q_i are the position of the i^{th} particle), confined in a box $\Lambda(\ell) := (-\frac{\ell}{2}, \frac{\ell}{2}]^d \subset \mathbb{R}^d$ (for some $\ell > 0$), which we will also denote by Λ when we do not need to explicit the dependence on ℓ.

We assume periodic boundary conditions since it is a general result (see e.g. [56] and [21]) that the thermodynamic limit is independent of the choice of the boundary conditions. This particular choice in the present section is not essential, periodic boundary conditions are used in order to obtain translation invariance in some cases (see e.g. Lemma A.4.1). Furthermore, our result remains valid with other boundary conditions by slightly changing the proof (see Appendix B for estimates of the finite volume corrections).

We obtain the periodic boundary conditions by covering $\mathbb{R}^d$ with boxes Λ and adding all interactions. Let

$$V^{per}(q_i, q_j) := \sum_{n \in \mathbb{Z}^d} V^{\mathrm{hc}}(q_i - q_j + n\ell) \tag{A.1.1}$$

with

$$V^{\mathrm{hc}}(q_i - q_j) = \begin{cases} +\infty & \text{if } |q_i - q_j| \leq R \\ 0 & \text{if } |q_i - q_j| > R \end{cases} \tag{A.1.2}$$

where $|q_i - q_j|$ denotes the euclidean distance between the two particles in q_i and q_j. R is the radius of the hard spheres and their volume is $\epsilon = V_d(R)$, i.e. the volume of the d-dimensional sphere of radius R. The hard core potential depends on q_i, q_j only through their distance.

The *canonical partition function* of the system with periodic boundary conditions is given by

$$Z_{\beta,\Lambda,N} := \frac{1}{N!}\int_{\Lambda^N} dq_1 \ldots dq_N \, e^{-\beta H_\Lambda(\mathbf{q})}, \tag{A.1.3}$$

where H_Λ is the energy of the system given by

$$H_\Lambda(\mathbf{q}) := \sum_{1\le i<j\le N} V^{per}(q_i, q_j). \tag{A.1.4}$$

Remark. Note that the following is proved also for more general interactions, i.e. when the pair potential is stable and tempered. This means that there exists $B \ge 0$ such that:

$$\sum_{1\le i<j\le N} V(q_i - q_j) \ge -BN, \tag{A.1.5}$$

for all N and all $q_1, ..., q_N$ and the integral

$$C(\beta) := \int_{\mathbb{R}^d} |e^{-\beta V(q)} - 1| dq \tag{A.1.6}$$

is convergent for some $\beta > 0$ (and hence for all $\beta > 0$). Hence the hard core interaction is a particular case where $B = 0$ and $C(\beta) = \epsilon$ independent of the temperature.

Given $\rho > 0$ we define the *thermodynamic free energy* by

$$f_\beta(\rho) := \lim_{|\Lambda|,N\to\infty,\, N=\lfloor\rho|\Lambda|\rfloor} f_{\beta,\Lambda}(N), \text{ where } f_{\beta,\Lambda}(N) := -\frac{1}{\beta|\Lambda|}\log Z_{\beta,\Lambda,N}. \tag{A.1.7}$$

The main result, given in Theorem A.1.1, is that, for values of the hard core volume ϵ small enough, the thermodynamic free energy is an analytic convex function of the density. In addition, the coefficients of the resulting series are given by the well-known irreducible coefficients of Mayer that we will denote by β_n

$$\beta_n := \frac{1}{n!}\sum_{\substack{g\in\mathcal{B}_{n+1}\\ V(g)\ni\{1\}}} \int_{(\mathbb{R}^d)^n} \prod_{\{i,j\}\in E(g)} (e^{-\beta V^{\mathrm{hc}}(q_i-q_j)} - 1) dq_2 \ldots dq_{n+1}, \quad q_1 \equiv 0 \tag{A.1.8}$$

where $\mathcal{B}_{n+1}$ is the set of 2-connected graphs g on $(n+1)$ vertices and $E(g)$ is the set of edges of the graph g. We define a 2-connected graph to be a connected graph which by removing any single vertex and all related edges remains connected. The precise definitions are given in the next section. In the literature such a graph is also known as irreducible. Note the unfortunate coincidence of notation between the inverse temperature β and the irreducible coefficients β_n, which however we keep in agreement with the literature.

Theorem A.1.1. *There exists a constant $c_0 > 0$ independent of N and Λ such that if $\rho\,\epsilon < c_0$ then*

$$\frac{1}{|\Lambda|}\log Z_{\beta,\Lambda,N} = \frac{1}{|\Lambda|}\log\frac{|\Lambda|^N}{N!} + \frac{N}{|\Lambda|}\sum_{n\geq 1} F_{\beta,N,\Lambda}(n), \tag{A.1.9}$$

with $N = \lfloor \rho|\Lambda|\rfloor$ and in the thermodynamic limit

$$\lim_{N,|\Lambda|\to\infty,\, N=\lfloor\rho|\Lambda|\rfloor} F_{\beta,N,\Lambda}(n) = \frac{1}{n+1}\beta_n\rho^{n+1}, \tag{A.1.10}$$

for all $n \geq 1$ and β_n given in (A.1.8)*. Furthermore, there exist constants $C, c > 0$ such that, for every N and Λ, the coefficients $F_{\beta,N,\Lambda}(n)$, $n \geq 1$, (which are given by the explicit formulas in* (A.4.14) *and* (A.4.15)*) satisfy*

$$|F_{\beta,N,\Lambda}(n)| \leq Ce^{-cn}. \tag{A.1.11}$$

Note that Theorem A.1.1 proves the validity of the cluster expansion for the canonical partition function in the regime where the volume ϵ of the hard spheres is small enough. The convergence is uniform in the volume and in the thermodynamic limit it reproduces Mayer's virial expansion providing an alternative and more direct derivation which avoids the deep combinatorial issues present in the original proof. Hence taking the thermodynamic limit term by term, we have the following:

Corollary A.1.2.

$$\beta f_\beta(\rho) = \rho(\log\rho - 1) - \sum_{n\geq 1}\frac{1}{n+1}\beta_n\rho^{n+1}. \tag{A.1.12}$$

In this Appendix we will only prove convergence for the cluster expansion, while we will leave out the proof of the thermodynamic limit, which is however contained in the paper [53].

A.2 Reduction to an abstract polymer model

We view the canonical partition function $Z_{\beta,\Lambda,N}$ as a perturbation around the ideal case, hence normalizing the measure by multiplying and dividing by $|\Lambda|^N$ in (A.1.3) we write

$$Z_{\beta,\Lambda,N} = Z^{ideal}_{\Lambda,N}\, Z^{int}_{\beta,\Lambda,N}, \tag{A.2.1}$$

where

$$Z^{ideal}_{\Lambda,N} := \frac{|\Lambda|^N}{N!} \quad \text{and} \quad Z^{int}_{\beta,\Lambda,N} := \int_{\Lambda^N}\frac{dq_1}{|\Lambda|}\cdots\frac{dq_N}{|\Lambda|}\, e^{-\beta H_\Lambda(\mathfrak{q})}. \tag{A.2.2}$$

For $Z^{int}_{\beta,\Lambda,N}$ we use the idea of Mayer in [44] which consists of developing $e^{-\beta H_\Lambda(\mathbf{q})}$ in the following way

$$e^{-\beta H_\Lambda(\mathbf{q})} = \prod_{1\le i<j\le N}(1+f_{i,j}) = \sum_{E\subset\mathcal{E}(N)}\prod_{\{i,j\}\in E} f_{i,j}, \tag{A.2.3}$$

where $\mathcal{E}(N) := \{\{i,j\} : i,j \in [N],\, i\neq j\}$, $[N] := \{1,...,N\}$ and

$$f_{i,j} := e^{-\beta V^{\rm hc}(q_i-q_j)} - 1 \tag{A.2.4}$$

Note that in the last sum in equation (A.2.3) we have also the term with $E=\emptyset$ which gives 1.

A *graph* is a pair $g \equiv (V(g), E(g))$, where $V(g)$ is the set of *vertices* and $E(g)$ is the set of *edges*, with $E(g) \subset \{U \subset V(g) : |U| = 2\}$. A graph $g = (V(g), E(g))$ is said to be *connected* if for any pair $A, B \subset V(g)$ such that $A \cup B = V(g)$ and $A \cap B = \emptyset$, there is a link $e \in E(g)$ such that $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$. Singletons are considered to be connected. We use $\mathcal{C}_V$ to denote the set of connected graphs on the set of vertices $V \subset [N]$.

Two sets $V, V' \subset [N]$ are called *compatible* (denoted by $V \sim V'$) if $V \cap V' = \emptyset$; otherwise we call them *incompatible* ($\nsim$). This definition induces in a natural way the notion of compatibility between graphs with set of vertices $V(g), V(g') \subset [N]$, i.e., $g \sim g'$ if $V(g) \cap V(g') = \emptyset$.

With these definitions, to any set E in equation (A.2.3) we can associate a graph, i.e., a pair $g \equiv (V(g), E(g))$, where $V(g) := \{i : \exists e \in E \text{ with } i \in e\} \subset [N]$ and $E(g) = E$. Note that the resulting graph does not contain isolated vertices. It can be viewed as the pairwise compatible (non-ordered) collection of its connected components, i.e., $g \equiv \{g_1, \ldots, g_k\}_\sim$ for some k, where each g_l, $l = 1, \ldots, k$, belongs to the set of all connected graphs on at most N vertices and it contains at least two vertices. Hence,

$$e^{-\beta H_\Lambda(\mathbf{q})} = \sum_{\substack{\{g_1,\ldots,g_k\}_\sim \\ g_l \text{ connected}}} \prod_{l=1}^{k} \prod_{\{i,j\}\in E(g_l)} f_{i,j}, \tag{A.2.5}$$

where again the empty collection $\{g_1, ..., g_k\}_\sim = \emptyset$ contributes the term 1 in the sum. Therefore, observing that integrals over compatible components factorize, we get

$$Z^{int}_{\beta,\Lambda,N} := \sum_{\substack{\{g_1,\ldots,g_k\}_\sim \\ g_l \text{ connected}}} \prod_{l=1}^{k} \tilde{z}_\Lambda(g_l) = \sum_{\substack{\{V_1,\ldots,V_k\}_\sim \\ |V_l|\ge 2,\,\forall l}} \prod_{l=1}^{k} z_\Lambda(V_l), \tag{A.2.6}$$

where

$$z_\Lambda(V) := \sum_{g \in \mathcal{C}_V} \tilde{z}_\Lambda(g), \qquad \tilde{z}_\Lambda(g) := \int_{\Lambda^{|g|}} \prod_{i \in V(g)} \frac{dq_i}{|\Lambda|} \prod_{\{i,j\} \in E(g)} f_{i,j}. \tag{A.2.7}$$

We also denote by $|g|$ the *cardinality* of $V(g)$, i.e., $|g| := |V(g)|$. Both expressions in (A.2.6) are in the form of the abstract polymer model which we specify next.

A.3 Abstract polymer model

An *abstract polymer model* $(\Gamma, \mathbb{G}_\Gamma, \omega)$ consists of (i) a set of polymers $\Gamma := \{\gamma_1, ..., \gamma_{|\Gamma|}\}$, (ii) a binary symmetric relation $\sim$ of compatibility between the polymers (i.e., on $\Gamma \times \Gamma$) which is recorded into the compatibility graph $\mathbb{G}_\Gamma$ (the graph with vertex set Γ and with an edge between two polymers γ_i, γ_j if and only if they are an incompatible pair) and (iii) a weight function $\omega : \Gamma \to \mathbb{C}$. Then, we have the following formal relation which will become rigorous by Theorem A.3.1 below (see [34], [6] and [47]):

$$Z_{\Gamma,\omega} := \sum_{\{\gamma_1,\ldots,\gamma_n\}_\sim} \prod_{i=1}^{n} \omega(\gamma_i) = \exp\left\{ \sum_{I \in \mathcal{I}} c_I \omega^I \right\}, \tag{A.3.1}$$

where

$$c_I = \frac{1}{I!} \sum_{G \subset \mathcal{G}_I} (-1)^{|E(G)|}, \tag{A.3.2}$$

or equivalently ([6])

$$c_I = \frac{1}{I!} \frac{\partial^{\sum_\gamma I(\gamma)} \log Z_{\Gamma,\omega}}{\partial^{I(\gamma_1)}\omega(\gamma_1) \cdots \partial^{I(\gamma_n)}\omega(\gamma_n)}\Big|_{\omega(\gamma)=0}. \tag{A.3.3}$$

The sum in (A.3.1) is over the set $\mathcal{I}$ of all multi-indices $I : \Gamma \to \{0, 1, \ldots\}$, $\omega^I = \prod_\gamma \omega(\gamma)^{I(\gamma)}$, and, denoting $\mathrm{supp} I := \{\gamma \in \Gamma : I(\gamma) > 0\}$, $\mathcal{G}_I$ is the graph with $\sum_{\gamma \in \mathrm{supp} I} I(\gamma)$ vertices induced from $\mathcal{G}_{\mathrm{supp} I} \subset \mathbb{G}_\Gamma$ by replacing each vertex γ by the complete graph on $I(\gamma)$ vertices.

Furthermore, the sum in (A.3.2) is over all connected subgraphs G of $\mathcal{G}_I$ spanning the whole set of vertices of $\mathcal{G}_I$ and $I! = \prod_{\gamma \in \mathrm{supp} I} I(\gamma)!$. Note that if I is such that $\mathcal{G}_{\mathrm{supp} I}$ is not connected (i.e., I is not a *cluster*) then $c_I = 0$.

Remark. Note that one can also use the following notation (following [34]):

$$Z_{\Gamma,\omega} := \sum_{\{\gamma_1,\ldots,\gamma_n\}_\sim} \prod_{i=1}^{n} \omega(\gamma_i) = \exp\left\{ \sum_{I \in \mathcal{I}} \omega^T(I) \right\} \tag{A.3.4}$$

where π is a collection of non-compatible diagrams (repetitions are allowed) and $\omega^T(I)$ is given by the Möbius inversion formula:

$$\omega^T(I) := \sum_{I' \colon I' \subset I} (-1)^{|I|-|I'|} \log \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}_\sim, \\ \gamma_j \in I', \forall j}} \prod_{j=1}^{n} \omega(\gamma_j), \quad |I| := |\{\gamma : \gamma \in I\}| \tag{A.3.5}$$

Note that $\omega^T(I) = 0$ if I is not a cluster.

We state the general theorem as in [6], [47] but in a slightly different form. Let

$$L = L(\delta) = \sup_{x \in (0,\delta)} \left\{ \frac{-\log(1-x)}{x} \right\} = \frac{-\log(1-\delta)}{\delta}, \tag{A.3.6}$$

for $\delta \in (0,1)$. Notice that for δ small we have $L = 1 + O(\delta)$. The optimal bound for the convergence radius is beyond our scope, however we hope to come back to this issue in a subsequent work.

Theorem A.3.1. *Assume that there are two non-negative functions $a, c : \Gamma \to \mathbb{R}$ such that for any $\gamma \in \Gamma$, $|\omega(\gamma)| e^{a(\gamma)+c(\gamma)} \leq \delta$ holds, for some $\delta \in (0,1)$. Moreover, assume that for any polymer γ'*

$$\sum_{\gamma \nsim \gamma'} |\omega(\gamma)| e^{a(\gamma)+c(\gamma)} \leq \frac{1}{L} a(\gamma'), \tag{A.3.7}$$

where L is given in (A.3.6)*. Then, for any polymer $\gamma' \in \Gamma$ the following bound holds*

$$\sum_{I \colon I(\gamma') \geq 1} |c_I \omega^I| e^{\sum_{\gamma \in supp I} I(\gamma) c(\gamma)} \leq L |\omega(\gamma')| e^{a(\gamma')+c(\gamma')}, \tag{A.3.8}$$

where c_I are given in (A.3.3)*.*

Proof. Apply Theorem 1 in [6] with activities $\omega(\gamma) e^{c(\gamma)}$. □

In view of (A.2.6) we can represent the partition function $Z^{int}_{\beta,\Lambda,N}$ both as a polymer model on connected graphs with weights $\tilde{z}_\Lambda$ and as a polymer model on $\mathcal{V}(N) := \{V : V \subset \{1,\dots,N\}, |V| \geq 2\}$ with weights z_Λ and compatibility graph $\mathbb{G}_\mathcal{V}$.

A.4 Convergence of the cluster expansion

In this section we check the convergence condition of Theorem A.3.1. We work in the case where polymers are subsets of vertices, which in the abstract polymer formulation is given by the space $(\mathcal{V}(N), \mathbb{G}_\mathcal{V}, \zeta_\Lambda)$. Then, as a corollary of Theorem A.3.1 we prove (A.1.11).

Lemma A.4.1. *There exists a constant $c_0 > 0$ such that for $\rho\epsilon < c_0$ there exist two positive constants a, c and $\delta \in (0,1)$ such that*

$$\sup_{\Lambda\subset\mathbb{R}^d} \sup_{V\in\mathcal{V}(N)} |z_\Lambda(V)| e^{a|V|+c|V|} \leq \delta \tag{A.4.1}$$

holds, where $N = \lfloor \rho|\Lambda| \rfloor$. Moreover, for any set $V' \in \mathcal{V}(N)$:

$$\sup_{\Lambda\subset\mathbb{R}^d} \sum_{V:\ V\not\sim V'} |z_\Lambda(V)| e^{a|V|+c|V|} \leq \frac{1}{L} a|V'|. \tag{A.4.2}$$

where L is given in (A.3.6).

Proof. Let $\alpha := a + c$. To bound $|z_\Lambda(V)|$ we use a version of the tree-graph inequality (proved in this form in [51], Proposition 6.1(a)) which states that for a hard core potential, we have the following bound:

$$\Big| \sum_{g\in\mathcal{C}_n} \prod_{\{i,j\}\in E(g)} f_{i,j} \Big| \leq \sum_{T\in\mathcal{T}_n} \prod_{\{i,j\}\in E(T)} |f_{i,j}|, \tag{A.4.3}$$

where $\mathcal{T}_n$ and $\mathcal{C}_n$ are respectively the set of trees and connected graphs with n vertices. Note that in the case of a stable and tempered interaction we would have a coefficient $e^{2\beta B n}$ on the r.h.s. of (A.4.3) multiplying the sum over trees, where B is the stability constant. Then, considering a fixed V with $|V| = n$,

$$|z_\Lambda(V)| e^{\alpha|V|} \ \leq \ e^{\alpha n} \sum_{T\in\mathcal{T}_n} \int_{\Lambda^n} \frac{dq_1}{|\Lambda|} \cdots \frac{dq_n}{|\Lambda|} \prod_{\{i,j\}\in E(T)} |f_{i,j}|. \tag{A.4.4}$$

Given a rooted tree T let us call $(i_1, j_1), (i_2, j_2), ..., (i_{n-1}, j_{n-1})$ its edges. We have:

$$\begin{aligned}
\int_{\Lambda^n} \frac{dq_1}{|\Lambda|} \cdots \frac{dq_n}{|\Lambda|} \prod_{\{i,j\}\in E(T)} |f_{i,j}| &= \frac{1}{|\Lambda|^n} \int_{\Lambda^n} dq_1 \cdots dq_n \prod_{k=1}^{n-1} |f_{i_k,j_k}| \\
&\leq \frac{1}{|\Lambda|^n} \int_\Lambda dq_{i_1} \int_\Lambda dy_2 \cdots \int_\Lambda dy_n \prod_{k=2}^{n} |e^{-\beta V^{\mathrm{hc}}(y_k)} - 1| \\
&\leq \frac{|\Lambda|}{|\Lambda|^n} \left[\int_\Lambda dx |e^{-\beta V^{\mathrm{hc}}(x)} - 1| \right]^{n-1} =: \frac{|\Lambda|}{|\Lambda|^n} \epsilon^{n-1},
\end{aligned}$$

where we considered q_{i_1} as the root and we used the change of variables:

$$y_k = q_{i_k} - q_{j_k}, \qquad \forall k = 2, ..., n. \tag{A.4.5}$$

We choose $\rho\,\epsilon$ such that:

$$\delta' := \rho e^{\alpha}\epsilon < 1, \quad \alpha = a + c \tag{A.4.6}$$

Then, since the number of all trees in $\mathcal{T}_n$ is n^{n-2}, from (A.4.4) we obtain (recalling that $N = \lfloor \rho|\Lambda| \rfloor$):

$$|z_\Lambda(V)|e^{\alpha|V|} \leq \frac{n^{n-2}}{|\Lambda|^{n-1}} e^{\alpha n}\epsilon^{n-1} \leq \frac{1}{2}\rho\,\epsilon e^{2\alpha}, \tag{A.4.7}$$

by using the bound $2 \leq n \leq N$ and the fact that $\rho e^{\alpha}\epsilon < 1$. Then defining $\delta := \frac{1}{2}\rho\,\epsilon\, e^{2\alpha}$, (A.4.1) is satisfied.

Moreover, for any fixed i we have:

$$\begin{aligned}
\sum_{V:\, V\ni i} |z_\Lambda(V)|e^{\alpha|V|} &\leq \sum_{n\geq 2} \binom{N-1}{n-1} \frac{n^{n-2}}{|\Lambda|^{n-1}} e^{\alpha n}\epsilon^{n-1} \\
&\leq e^{\alpha} \sum_{n\geq 2} \frac{n^{n-2}}{(n-1)!} \left(\frac{N}{|\Lambda|}\right)^{n-1} (e^{\alpha}\epsilon)^{n-1} \\
&\leq \frac{1}{2\sqrt{\pi}} \frac{e^2\delta}{1-e\delta'},
\end{aligned} \tag{A.4.8}$$

where in the last inequality we have used Stirling's bound: $n! \geq n^n e^{-n}\sqrt{2\pi n}$.

Choosing $a = 1$ and δ' such that (for any given $c > 0$)

$$1 + \frac{e^2}{2\sqrt{\pi}} \log(1 - \frac{1}{2}e^{1+c}\delta') \geq e\delta' \tag{A.4.9}$$

we obtain that $\frac{1}{2\sqrt{\pi}} \frac{e^2\delta}{1-e\delta'} \leq \frac{1}{L}$, where L is given in (A.3.6). A sufficient condition for (A.4.9) is that $e^{1+c}\delta' \leq 0.45796$ in which case $c_0 = 0.45796\, e^{-2(1+c)}$ for any given $c > 0$. Then, since $\{V \not\sim V'\} \subset \bigcup_{i\in V'}\{V \ni i\}$ we get (A.4.2) and conclude the proof of the lemma. □

The way we chose to present the cluster expansion as well as its convergence can by no means give the best radius of convergence. Our goal was merely to obtain (giving up the seek for the best radius) the consequence of the cluster expansion theorem, given in (A.3.8), which we use in order to establish (A.1.9).

After proving the convergence condition in Lemma A.4.1, an immediate consequence of Theorem A.3.1 is that for all $V' \in \mathcal{V}(N)$ and by choosing $c(V) := c|V|$ and $a(V) := |V|$ the following bound is true:

$$\sum_{I:\, I(V')\geq 1} |c_I z_\Lambda^I| e^{c\|I\|} \leq L|z_\Lambda(V')|e^{\alpha|V'|}, \quad \|I\| := \sum_{V\in \mathrm{supp} I} I(V)|V|, \tag{A.4.10}$$

where we also remind that $\alpha = 1 + c$.

Proof of (A.1.9) *and* (A.1.11). Let $[N] \equiv \{1, \dots, N\}$ and $A(I) := \cup_{V \in \mathrm{supp} I} V \subset [N]$ be the area of the union of V's in the support of I. Noticing that $c_I \neq 0$ only if $|A(I)| \geq 2$, we have:

$$\begin{aligned}\frac{1}{|\Lambda|}\sum_I c_I z_\Lambda^I &= \frac{1}{|\Lambda|}\sum_{n\geq 1}\sum_{\substack{A\subset[N]\\|A|=n+1}}\sum_{I:\,A(I)=A} c_I z_\Lambda^I \\ &= \frac{N}{|\Lambda|}\sum_{n\geq 1}\frac{1}{n+1}\sum_{\substack{A\ni 1\\|A|=n+1}}\sum_{I:A(I)=A} c_I z_\Lambda^I = \frac{N}{|\Lambda|}\sum_{n\geq 1}\frac{1}{n+1}\sum_{\substack{I:A(I)\ni 1\\|A(I)|=n+1}} c_I z_\Lambda^I. \quad \text{(A.4.11)}\end{aligned}$$

Passing to the second line, we replaced the sum over sets $A \subset [N]$ by N times the sum over classes of equivalence of sets A under permutations that can be pinned down by choosing a point from A and fixing it to equal 1 (over-counting, however, by $|A| = n+1$). This leads to the following definition:

$$F_{\beta,N,\Lambda}(n) := \frac{1}{n+1}\sum_{\substack{I:\,A(I)\ni 1\\|A(I)|=n+1}} c_I z_\Lambda^I \quad \text{(A.4.12)}$$

and hence we obtain the representation (A.1.9). The function $F_{\beta,N,\Lambda}(n)$ is uniformly bounded for all N, Λ as well as absolutely summable over n, namely from (A.4.10) with $V' \equiv \{1\}$ we get:

$$|F_{\beta,N,\Lambda}(n)| \leq \frac{e^{-cn}}{n+1}\sum_{\substack{I:\,A(I)\ni 1\\|A(I)|=n+1}} |c_I z_\Lambda^I| e^{cn} \leq e^{-cn} L e^{\alpha}, \quad \text{(A.4.13)}$$

which concludes the proof of (A.1.11). □

Having proved (A.1.11), by dominated convergence we can look at the thermodynamic limit of each individual term $F_{\beta,N,\Lambda}(n)$. The sum in the definition of these terms does not depend on the labels of the extra n particles (we have already chosen label 1). Thus,

$$F_{\beta,N,\Lambda}(n) = \frac{1}{n+1}\binom{N-1}{n}\sum_{I:\,A(I)=[n+1]} c_I z_\Lambda^I = \frac{1}{n+1} P_{N,|\Lambda|}(n) B_{\beta,\Lambda}(n), \quad \text{(A.4.14)}$$

where

$$P_{N,|\Lambda|}(n) := \frac{(N-1)\dots(N-n)}{|\Lambda|^n} \quad \text{and} \quad B_{\beta,\Lambda}(n) := \frac{|\Lambda|^n}{n!}\sum_{I:\,A(I)=[n+1]} c_I z_\Lambda^I. \quad \text{(A.4.15)}$$

While obviously $P_{N,|\Lambda|}(n) \to \rho^n$, for $B_{\beta,\Lambda}(n)$ is more complicate to show that it will give β_n at the thermodynamic limit. We leave out the investigation of the cancellations which are eventually responsible for the cluster expansion to give at the limit Mayer's virial expansion (see [53]).

Remark. We can also write, using notation (A.3.4) and (A.3.5):

$$\frac{1}{|\Lambda|}\log Z_{\beta,\Lambda,N} = \frac{1}{|\Lambda|}\log\frac{|\Lambda|^N}{N!} + \frac{1}{|\Lambda|}\sum_{\pi} z_\Lambda^T(\pi) \tag{A.4.16}$$

where π is varying in the family of multi-indices over the vertices set $\mathcal{V}(N)$ and

$$z_\Lambda^T(\pi) := \sum_{\pi' :\, \pi' \subset \pi} (-1)^{|\pi|-|\pi'|} \log \sum_{\substack{\{V_1,\dots,V_n\}_\sim,\\ V_j \in I',\, \forall j}} \prod_{j=1}^{n} z_\Lambda(V_j), \quad |\pi| := |\{V : V \in \pi\}|. \tag{A.4.17}$$

B. Finite volume corrections to the free energy

Let us put ourselves in the setting of Appendix A and recall the definition of the thermodynamic free energy:

$$f_\beta(\rho) := \lim_{|\Lambda|,N\to\infty,\, N=\lfloor\rho|\Lambda|\rfloor} f_{\beta,\Lambda}(N), \text{ where } f_{\beta,\Lambda}(N) := -\frac{1}{\beta|\Lambda|}\log Z_{\beta,\Lambda,N}, \tag{B.0.1}$$

$Z_{\beta,\Lambda,N}$ being the canonical partition function defined in (A.1.3). From Corollary A.1.2 we have:

$$\beta f_\beta(\rho) = \rho(\log\rho - 1) - \sum_{n\geq 1}\frac{1}{n+1}\beta_n\rho^{n+1} \tag{B.0.2}$$

where for simplicity we recall the definition of Mayer's coefficients:

$$\beta_n := \frac{1}{n!}\sum_{\substack{g\in\mathcal{B}_{n+1}\\ V(g)\ni\{1\}}}\int_{(\mathbb{R}^d)^n}\prod_{\{i,j\}\in E(g)}(e^{-\beta V^{\mathrm{hc}}(q_i-q_j)}-1)dq_2\dots dq_{n+1}, \quad q_1\equiv 0. \tag{B.0.3}$$

This section is devoted to find a bound for the difference between the free energy (B.0.2) and its finite volume version. The estimate of the difference is used in Section 8.8, where we want to compute the k-th moment of the number of occupation in the site x of our effective model. In this case is essential to approximate the measure by substituting in the exponential the free energy for the hard cores in a single box with zero boundary conditions with the mean field free energy (see (8.8.11)).

In what follows we will consider two different models. One is the model defined in Section A.1 for which we have chosen periodic boundary conditions and the other one is the same model with zero boundary conditions.

Let us denote by $Z^p_{\beta,\Lambda,N}$ and $Z^0_{\beta,\Lambda,N}$ the canonical partition function in the two cases, i.e. *periodic*-b.c. and *zero*-b.c. respectively. Note that for the notation in Appendix A we have $Z^p_{\beta,\Lambda,N} = Z_{\beta,\Lambda,N}$.

We want to prove the following result:

Theorem B.0.2. *It exists a constant $c > 0$ independent of N and Λ such that:*

$$\Big|\frac{1}{|\Lambda|}\log Z^0_{\beta,\Lambda,N} - \beta f_\beta(\rho)\Big| \leq c\frac{\ell^{d-1}}{\ell^d} \tag{B.0.4}$$

Proof.

Here we give a sketch of the proof. It can be divided into two steps:

- The first step consists in showing closeness between the finite volume free energy with zero and periodic boundary conditions, i.e.:

  Lemma B.0.3. *Let $\Lambda := (-\frac{\ell}{2}, \frac{\ell}{2}]^d$ and let C be the frame of width $\delta = 2R$ (where R is the hard core length) around Λ. Then:*

  $$Z^p_{\beta,\Lambda,N} \leq Z^0_{\beta,\Lambda,N} \leq Z^p_{\beta,\Lambda\cup C,N} \tag{B.0.5}$$

- Once we have done the periodic boundary condition approximation, we are in the same case as Appendix A and using convergence of cluster expansion we want to show:

  Lemma B.0.4. *Let $\Lambda := (-\frac{\ell}{2}, \frac{\ell}{2}]^d$. There exist two constants $C', C'' > 0$ such that:*

  $$\Big|\frac{1}{|\Lambda|}\log Z^p_{\beta,\Lambda,N} - \beta f_\beta(\rho)\Big| \leq C'\frac{(\log|\Lambda|)^4}{|\Lambda|}. \tag{B.0.6}$$

  and

  $$\Big|\frac{1}{|\Lambda\cup C|}\log Z^p_{\beta,\Lambda\cup C,N} - \beta f_\beta(\rho)\Big| \leq C''\frac{(\log|\Lambda\cup C|)^4}{|\Lambda\cup C|}. \tag{B.0.7}$$

□

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