

Università degli Studi “Roma Tre”
Dottorato di Ricerca in Matematica, XV Ciclo

Semistar Operations and Multiplicative Ideal Theory

Giampaolo Picozza

THESIS

October 29, 2004

Advisor :

Prof. Marco Fontana

Thesis Committee:

Prof. Valentina Barucci	Università degli Studi di Roma “La Sapienza”
Prof. Paul-Jean Cahen	Université Paul Cezanne (Aix-Marseille III)
Prof. Stefania Gabelli	Università degli Studi “Roma Tre”
Prof. Evan Houston (Reviewer)	University of North Carolina at Charlotte
Prof. K. Alan Loper (Reviewer)	Ohio State University

submitted: June 2004

Contents

Introduction	3
1 Background Results	8
1.1 Semistar operations on integral domains	8
1.2 Examples of semistar operations	11
1.2.1 The d -, e - semistar operations and the trivial extension of a star operation	11
1.2.2 The v and the t semistar operations	12
1.2.3 Semistar operations defined by overrings	13
1.2.4 Spectral semistar operations	15
1.2.5 The semistar operation $v(I)$	15
1.3 Localizing systems and stable semistar operations	17
1.4 The semistar operation $\tilde{\star}$	20
1.5 The semistar operations \star_ι and \star^t	22
1.6 Kronecker function rings and Nagata rings	23
2 Star operations on overrings and semistar operations	26
2.1 Composition of semistar operations	26
2.2 Star operations on overrings and semistar operations	30
2.3 Some applications	38
2.3.1 Semistar operations on valuation domains	39
2.3.2 Semistar operations on Prüfer domains	43
2.3.3 Integral domains with all semistar operations spectral	46
2.3.4 Totally divisorial domains and the semistar operation $v(I)$	48
2.3.5 Integral domains with at most $\dim(D) + 2$ semistar operations	51
3 Semistar invertibility	55
3.1 Semistar invertibility	55
3.2 Quasi- \star -invertibility	60
3.3 Semistar invertibility and the Nagata ring	67

4 Semistar Dedekind domains	71
4.1 Prüfer semistar multiplication domains	71
4.2 Semistar Noetherian domains	77
4.3 The Nagata ring of a semistar Noetherian domain	83
4.4 Semistar almost Dedekind domains	88
4.5 Semistar Dedekind domains	97
Bibliography	111
List of notation	117
Index	119

Introduction

The object of this work is the study of semistar operations on integral domains. Semistar operations were introduced in 1994 by Okabe and Matsuda in order to generalize the classical concept of star operation, as described in [38, Section 32], and hence the related classical theory of ideal systems based on the works of W. Krull, E. Noether, H. Prüfer and P. Lorenzen from 1930's.

The star operations are defined by axioms selected by Krull among the properties satisfied by some classical operations, such as the v -operation, the t -operation and the completion (or w -operation, using Gilmer's notation) with respect to a family of valuation overrings (the definitions are in Chapter 1).

Let us denote by $\mathbf{F}(D)$ the set of nonzero fractional ideals of an integral domain D . We recall that a *star operation* on D is a map $\star : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$, $I \mapsto I^\star$ such that, for all $x \in K$, $x \neq 0$, and for all $I, J \in \mathbf{F}(D)$, the following properties hold:

- (star.1) $D^\star = D$ and $(xI)^\star = xI^\star$.
- (star.2) $I \subseteq J$ implies $I^\star \subseteq J^\star$.
- (star.3) $I \subseteq I^\star$ and $I^{\star\star} := (I^\star)^\star = I^\star$.

Classical examples of star operation are the v -operation and the t -operation.

Star operations have shown to be an essential tool in multiplicative ideal theory, allowing a new approach for characterizing several classes of integral domains. For example, an integrally closed domain D is a Prüfer domain if and only if $I^t = I$ for each nonzero ideal I of D , [38, Proposition 34.12], a domain D is a Krull domain if and only if $(II^{-1})^t = D$, for each nonzero ideal I of D . Other relevant classes of domains, e.g. Mori domains and PvMDs, have been also defined and investigated using star operations.

The following example gives an enlightening motivation for introducing the notion of semistar operation. Consider the map $b : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$, $I \mapsto \bigcap \{IV \mid V \text{ valuation overring of } D\}$, that associates to each nonzero

fractional ideal its completion (or integral closure cf. O. Zariski and P. Samuel [83, Appendix 4] and R. Gilmer [38, Page 302]). This map gives rise to a star operation (called the *b-operation*) if and only if D is integrally closed (cf. [38, Page 398]), since D^b coincides with the integral closure of D , as a consequence of a celebrated theorem by Krull (and so condition (star.1) of the definition of star operation is satisfied if and only if D is integrally closed).

Since the other conditions required by the definition of star operation are easily verified, it is natural to look for a class of operations that includes the “integral closure of ideals” even if D is not supposed to be integrally closed.

So we do not require anymore that D^\star coincides with D and, as a consequence, we need to define \star on the larger set $\overline{F}(D)$ of all D -submodules of the quotient field of D , since the integral closure of a domain is not necessarily a fractional ideal.

These considerations lead to the notion of *semistar operation* (cf. Definition 1.1). By construction, the integral closure of ideals (or, more precisely, of modules) is a semistar operation without any assumption on the integral closure of the domain D . Moreover, the set of star operations can be canonically embedded in the set of semistar operations (cf. Section 1.2.1). Hence, we have a more flexible notion and a larger class of operations, that gives a more appropriate context for approaching several questions of multiplicative ideal theory.

For example, in a series of papers, M. Fontana and K.A. Loper ([27], [28] and [29]) have generalized the classical construction of *Kronecker function rings*. These rings were studied in a general setting by W. Krull in a series of papers published, starting from 1936, with the common title “Beiträge zur arithmetik kommutativer Integritätsbereiche” (cf. also the books by H. Weyl, “Algebraic theory of numbers”, Princeton 1940 and by H. Edwards “Divisor theory”, Birkhäuser, 1990 and, for an axiomatic approach, a recent paper by F. Halter Koch [44]). In the context of star operations, the Kronecker function ring can be constructed only for integrally closed domains and for e.a.b. star operations (cf. [38, Section 32]). The notion of semistar operation allows one to define a Kronecker function ring $\text{Kr}(D, \star)$ for an arbitrary domain D and an arbitrary semistar operation \star . Moreover, $\text{Kr}(D, \star)$ has all the properties of the “classical” Kronecker function ring, since it coincides with the Kronecker function ring of an integrally closed domain determined by D and \star , with respect to an e.a.b. star operation \star_a canonically associated with \star . We will recall this construction in Section 1.6, with other important results concerning the generalization of the classical Nagata ring (see M. Fontana and K.A. Loper [29]).

In this work, we focus our attention on the problem of the characterization of several classes of integral domains, by using the semistar tool. Since the set of semistar operations is larger than the set of star operations, it is clear that the use of semistar operations leads to a finer classification.

In the first chapter we recall the main definitions and results used in this work. In particular, we give several examples of semistar operations and we introduce a class of new semistar operations that generalize the star operation defined by W. Heinzer, J. Huckaba and I. Papick in [48] for introducing the notion of m -canonical ideal (see Section 1.2.5).

In the second chapter our first goal is to measure “the size” of the set of all semistar operations of an integral domain D and to compare this set with the set of all star operations on the overrings of D (this give, in some sense, an idea of how much the use of semistar operations instead of star operations leads, for instance, to a finer characterizations of integral domains).

We will show that the semistar operations on an integral domain D are at least as many as the star operations on all overrings of D . More precisely, we show that there is a bijection between the set of semistar operations and the set of the (semi)star operations on all overrings of D (a (semi)star operation is a semistar operations that restricted to the set of nonzero fractional ideals of D is a “classical” star operation).

Moreover, this bijection suggests a new approach for studying semistar operations: for instance, in some cases, the study of properties of semistar operations on a domain D can be transferred to the study of (semi)star operations on the overrings. This approach may lead to an effective simplification, since (semi)star operations share many properties with star operations and so it is possible to apply to the case of (semi)star operations several results already proven for star operations.

We give some examples of the use of these techniques in Section 2.3. For example, we will apply this method for studying semistar operations on valuation domains (Section 2.3.1), for obtaining characterizations of domains with special properties on the set of semistar operations (domains with every semistar operation of finite type, or stable, or spectral; all these notions are defined in Chapter 1), for evaluating the size of the set of the semistar operations (in particular, in Section 2.3.5 we show that a domain D with $\dim(D) + 2$ semistar operations is a valuation domain or a divisorial pseudo-valuation domain, with some additional properties). Another result that we obtain applying this method is a characterization of totally divisorial domains (see works by S. Bazzoni, L. Salce and B. Olberding) in terms of semistar operations (in particular, the semistar operations introduced in Section 1.2.5).

Important classes of domains are defined using the notion of invertibility of ideals: Dedekind domains are the domains in which each nonzero ideal is invertible, Prüfer domains are the domains in which each nonzero finitely generated ideal is invertible. This notion have been generalized to the case of star operations: in particular the invertibility with respect to the t -operation (the t -invertibility, a nonzero ideal I of an integral domain D is t -invertible if and only if $(II^{-1})^t = D$) has been deeply investigated, since, as we have mentioned above, it allows, for example, to characterize Krull domains. Also

PvMDs are defined using t -invertibility: a domain D is a PvMD if each nonzero finitely generated ideal of D is t -invertible. Clearly, Krull domains can be seen as a generalization of Dedekind domain and PvMDs can be interpreted as a generalization of Prüfer domains. In fact, they share several “ideal theoretic” properties with the domain that they generalize, when we restrict to consider only the set of ideals I such that $I^t = I$.

Thus, it is natural to investigate a further generalization of these classes of domains, considering not only the t -operation, but an arbitrary semistar operation. To do this, it is necessary to develop a theory of invertibility with respect to a semistar operation.

Chapter 3 is devoted to the study of two different notions of invertibility, that we call respectively *semistar invertibility* and *quasi-semistar-invertibility*. The first notion has been introduced by M. Fontana, P. Jara and E. Santos in [26], where they define and study the Prüfer semistar multiplication domains (P \star MD for short, a generalization of Prüfer domains): they say that a nonzero ideal I is semistar invertible with respect to a semistar operation \star on D if $(II^{-1})^\star = D^\star$. This is the direct translation to the semistar case of the classical notion of star invertibility. In Section 3.1 we discuss some properties of semistar invertibility generalizing classical results about invertibility and t -invertibility and show that there are some obstructions in developing this theory having as a model the classical theory. Moreover, while this notion has proven to be useful to introduce P \star MDs, it cannot be used to generalize Dedekind domains in a satisfactory way (this is one of the goals of Chapter 4). So, in Section 3.2, we introduce the notion of quasi-semistar-invertibility: a nonzero ideal I of an integral domain D is quasi- \star -invertible (quasi-semistar-invertible with respect to a semistar operation \star on D) if $(IH)^\star = D^\star$ for some $H \in \overline{F}(D)$. We prove that, in general, these two notions do not coincide and that the second notion of invertibility seems, in some sense, more natural in the semistar context. In Section 3.3, we study the behaviour of semistar and quasi-semistar invertible ideals in the Nagata ring.

Several results exposed in Chapter 3 have been obtained in collaboration with Marco Fontana [31].

As we have mentioned above, one of the goals of Chapter 4 is to obtain a generalization in the semistar context of the theory of Dedekind domains. A Dedekind domain is a Noetherian Prüfer domain. So, in order to introduce a generalization of Dedekind domain, we need to generalize the notions of Noetherian and Prüfer domains to the semistar context.

We have already mentioned that a generalization of Prüfer domain to the semistar setting has been investigated in [26] (cf. also [25]). We recall the main properties of the Prüfer semistar multiplication domains in Section 4.1, and we obtain some new results, in particular concerning the “descent” to subrings of the P \star MD property, using the techniques developed in Chapter 2. We characterize the P \star MDs as a particular type of subrings of PvMDs having

the same field of fractions.

In Section 4.2, we introduce and give some basic result about *semistar Noetherian domains*, that is, integral domains with the ascending chain condition on a distinguished set of ideals (the *quasi- \star -ideals*) determined by a semistar operation \star . This notion generalizes at the same time the notions of Noetherian, Mori and strong Mori domains. Several results in this section, and nearly everything from Section 4.4 are obtained as a joint work with Said El Baghdadi and Marco Fontana [21].

In Section 4.3, we prove a semistar version of the Hilbert Basis Theorem, that generalize both the classical result concerning Noetherian domains and a more recent “Basis-type” theorem proven for strong Mori domains, [80, Theorem 1.13]. As an easy consequence we obtain that an integral domain D is Noetherian with respect to a semistar operation \star if and only if the Nagata ring of D with respect to \star is Noetherian.

In Section 4.5 we introduce the notion of *semistar Dedekind domain* (for short \star -DD), as a semistar Noetherian domain which is a $P\star MD$. We obtain several characterizations of these domains, that generalize in a very satisfactory way the classical ones. For example, we relate this notion with the concept of quasi-semistar-invertibility (Proposition 4.59), we show in Theorem 4.73 that a “semistar translation” of the classical Noether’s Axioms characterizes semistar Dedekind domains and, in Theorem 4.85, we give a semistar analogue (using the “semistar product”) of the characterization of Dedekind domains as the integral domains in which each ideal is product of prime ideals. We have also investigated the “descent” to subrings of the semistar Dedekind property, and shown that semistar Dedekind domains are particular subrings of Krull domains having some “flatness like” properties. In particular, we have shown that each Mori domain not strongly Mori (following the terminology introduced by V. Barucci and S. Gabelli [11, page 105]) is a semistar Dedekind domain for a semistar operation induced by a particular set of t -maximal ideals (see Example 4.82 for the details).

Chapter 1

Background Results

In this chapter we give an overview of some general notions and results concerning semistar operations on integral domains.

1.1 Semistar operations on integral domains

Let D be an integral domain with quotient field K . Let $\overline{\mathbf{F}}(D)$ denote the set of all nonzero D -submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D , i.e. $E \in \mathbf{F}(D)$ if $E \in \overline{\mathbf{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D -submodules of K . Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D)$.

Definition 1.1. A *semistar operation* on D is a map $\star : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

- (\star_1) $(xE)^\star = xE^\star$.
- (\star_2) $E \subseteq F$ implies $E^\star \subseteq F^\star$.
- (\star_3) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

We denote by $\overline{\mathbf{F}}^\star(D)$ the set $\{E^\star \mid E \in \overline{\mathbf{F}}(D)\}$.

The following lemma establishes some basic properties of semistar operations (see [65, Proposition 5] and, for a similar result for star operations, [38, Proposition 32.2]):

Lemma 1.2. *Let D be an integral domain, \star a semistar operation on D . Then, for all $E, F \in \overline{\mathbf{F}}(D)$ and for every subset $\{E_\alpha\} \subseteq \overline{\mathbf{F}}(D)$:*

$$(1) (EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star .$$

$$(2) (E + F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star .$$

$$(3) \bigcap_\alpha E_\alpha^\star = (\bigcap_\alpha E_\alpha)^\star, \text{ if } \bigcap_\alpha E_\alpha \neq 0. \quad \square$$

As a direct consequence of Lemma 1.2(1) we have the following results (see [65, Proposition 6] and [23]):

Proposition 1.3. *Let D be an integral domain, \star a semistar operation on D . Then:*

(1) *Let R be an overring of D . Then R^\star is an overring of D . In particular, D^\star is an overring of D .*

(2) *If $E \in \overline{\mathbf{F}}(D)$ then $E^\star \in \overline{\mathbf{F}}(D^\star)$.*

(3) *If $E \in \mathbf{F}(D)$ then $E^\star \in \mathbf{F}(D^\star)$.*

Proof. (1) $R^\star R^\star \subseteq (R^\star R^\star)^\star = (RR)^\star = R^\star$.

(2) $D^\star E^\star \subseteq (D^\star E^\star)^\star = (DE)^\star = E^\star$.

(3) Since $E \in \mathbf{F}(D)$, there exists a nonzero element $d \in D$ such that $dE \subseteq D$. Then, $dE^\star = (dE)^\star \subseteq D^\star$. Since $d \in D \subseteq D^\star$, we have $E^\star \in \mathbf{F}(D^\star)$. \square

If \star is a semistar operation on D , such that $D^\star = D$, we say that \star is a (semi)star operation.

Remark 1.4. (see [23, Remark 1.1]) If \star is a (semi)star operation on D , then the restriction $\star|_{\mathbf{F}(D)}$ is a star operation on D . The only thing to check is that if $E \in \mathbf{F}(D)$ then $E^\star \in \mathbf{F}(D)$ and this follows immediately from Proposition 1.3(3).

If \star_1 and \star_2 are two semistar operations on D , we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\mathbf{F}}(D)$. This relation induces a partial order on the set of all semistar operations.

Proposition 1.5. [65, Lemma 16] *Let \star_1, \star_2 be two semistar operations on an integral domain D . The following are equivalent:*

$$(i) \star_1 \leq \star_2.$$

$$(ii) (E^{\star_1})^{\star_2} = E^{\star_2}, \text{ for each } E \in \overline{\mathbf{F}}(D).$$

$$(iii) (E^{\star_2})^{\star_1} = E^{\star_2}, \text{ for each } E \in \overline{\mathbf{F}}(D).$$

$$(iv) \overline{\mathbf{F}}^{\star_2}(D) \subseteq \overline{\mathbf{F}}^{\star_1}(D).$$

Proof. (i) \Rightarrow (ii) Since $E^{\star_1} \subseteq E^{\star_2}$, we have $E^{\star_2} \subseteq (E^{\star_1})^{\star_2} \subseteq (E^{\star_2})^{\star_2} = E^{\star_2}$. Thus, $(E^{\star_1})^{\star_2} = E^{\star_2}$.

(ii) \Rightarrow (iii) Since $E^{\star_2} \in \overline{\mathbf{F}}(D)$, from (ii) we have that $((E^{\star_2})^{\star_1})^{\star_2} = (E^{\star_2})^{\star_2} = E^{\star_2}$. So, $E^{\star_2} \subseteq (E^{\star_2})^{\star_1} \subseteq ((E^{\star_2})^{\star_1})^{\star_2} = (E^{\star_2})^{\star_2} = E^{\star_2}$. Hence, $(E^{\star_2})^{\star_1} = E^{\star_2}$.

(iii) \Rightarrow (iv) Let $F \in \overline{\mathbf{F}}^{\star_2}(D)$. Then, there exists $E \in \overline{\mathbf{F}}(D)$ such that $E^{\star_2} = F$. Then, by (iii), $F^{\star_1} = (E^{\star_2})^{\star_1} = E^{\star_2} = F$ and $F \in \overline{\mathbf{F}}^{\star_1}(D)$.
(iv) \Rightarrow (iii) Let $E^{\star_2} \in \overline{\mathbf{F}}^{\star_2}(D) \subseteq \overline{\mathbf{F}}^{\star_1}(D)$. Then, there exists $F \in \overline{\mathbf{F}}(D)$ such that $F^{\star_1} = E^{\star_2}$ for some $F \in \overline{\mathbf{F}}(D)$. Then, $(E^{\star_2})^{\star_1} = (F^{\star_1})^{\star_1} = F^{\star_1} = E^{\star_2}$.
(iii) \Rightarrow (i) $E^{\star_1} \subseteq (E^{\star_1})^{\star_2} = E^{\star_2}$. \square

If \star is a semistar operation on D , then we can consider a map $\star_f : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ defined for each $E \in \overline{\mathbf{F}}(D)$ as follows: $E^{\star_f} := \bigcup \{F^{\star} \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\}$. It is easy to see that \star_f is a semistar operation on D , called *the semistar operation of finite type associated to \star* . Note that, for each $F \in \mathbf{f}(D)$, $F^{\star} = F^{\star_f}$. A semistar operation \star is called a *semistar operation of finite type* if $\star = \star_f$. It is easy to see that $(\star_f)_f = \star_f$ (that is, \star_f is of finite type).

Remark 1.6. We note here that a semistar operation of finite type \star is completely determined by the image of the elements of $\mathbf{f}(D)$, that is, if \star_1, \star_2 are two semistar operations of finite type, such that $\star_1|_{\mathbf{f}(D)} = \star_2|_{\mathbf{f}(D)}$, then $\star_1 = \star_2$. In particular, since $\mathbf{f}(D) \subseteq \mathbf{F}(D)$, if two semistar operations of finite type coincide on $\mathbf{F}(D)$, then they coincide on $\overline{\mathbf{F}}(D)$.

Proposition 1.7. [23, Proposition 1.6(2),(3)] *Let D be an integral domain and \star, \star_1 and \star_2 semistar operations on D . Then:*

- (1) $\star_f \leq \star$.
- (2) $\star_1 \leq \star_2$ implies $(\star_1)_f \leq (\star_2)_f$.

Proof. (1) It is straightforward from the definition of \star_f .

(2) Let $E \in \overline{\mathbf{F}}(D)$ and $x \in E^{(\star_1)_f}$. Then, there exists $F \in \mathbf{f}(D)$, $F \subseteq E$, such that $x \in F^{\star_1} \subseteq F^{\star_2} \subseteq E^{(\star_2)_f}$. \square

We say that a nonzero integral ideal I of D is a *quasi- \star -ideal* if $I^{\star} \cap D = I$. If I is a prime ideal, we say that I is a *quasi- \star -prime*. If M is a maximal element in the set of all quasi- \star -ideals of D , we will say that M is a *quasi- \star -maximal ideal*. We denote by $\mathcal{M}(\star)$ the set of all quasi- \star -maximal ideals.

If I is a nonzero ideal such that $I^{\star} \subsetneq D^{\star}$, then it is easy to see that $I^{\star} \cap D$ is a quasi- \star -ideal. In fact, $(I^{\star} \cap D)^{\star} \cap D \subseteq (I^{\star})^{\star} \cap D^{\star} \cap D = I^{\star} \cap D$ (we have used the straightforward fact that, for each semistar operation \star on D and for each $E, F \in \overline{\mathbf{F}}(D)$, $(E \cap F)^{\star} \subseteq E^{\star} \cap F^{\star}$).

We note that if \star_1, \star_2 are two semistar operations on D such that $\star_1 \leq \star_2$ and I is a quasi- \star_2 -ideal of D , then I is a quasi- \star_1 -ideal of D . In fact, we have $I \subseteq I^{\star_1} \subseteq I^{\star_2}$ and then $I \subseteq I^{\star_1} \cap D \subseteq I^{\star_2} \cap D = I$.

We give two important results about quasi- \star -maximal ideals:

Proposition 1.8. *Let D be an integral domain and \star a semistar operation on D .*

(1) If $M \in \mathcal{M}(\star)$ then M is a prime ideal.

(2) If \star is a semistar operation of finite type, then each quasi- \star -ideal is contained in a quasi- \star -maximal.

Proof. (1) Let $x, y \in D$, with $xy \in M$ and $x \notin M$. Consider the ideal $M + xD$. We have that $(M + xD)^\star = D^\star$, otherwise $(M + xD)^\star \cap D$ would be a quasi- \star -ideal larger than M . Consider the ideal $y(M + xD) = yM + yxD \subseteq M$. We have that $y \in yD^\star \cap D = y(M + xD)^\star \cap D = (yM + yxD)^\star \cap D \subseteq M^\star \cap D = M$. Then, $y \in M$ and M is prime.

(2) It is in [23, Lemma 4.20]. \square

We have noted in Lemma 1.2 that, for each $E, F \in \overline{\mathbf{F}}(D)$, $(E + F)^\star = (E^\star + F^\star)^\star$ and $(EF)^\star = (E^\star F^\star)^\star$. It is natural to ask if something similar holds for the intersection of two D -modules, that is, if $(E \cap F)^\star = (E^\star \cap F^\star)^\star$. It is easy to see that this is true if and only if $(E \cap F)^\star = (E^\star \cap F^\star)$. Indeed, by Lemma 1.2(3), it follows that $(E^\star \cap F^\star)^\star = (E^\star \cap F^\star)$. So, if $(E \cap F)^\star = (E^\star \cap F^\star)^\star$, we have $(E \cap F)^\star = (E^\star \cap F^\star)$. Conversely, if $(E \cap F)^\star = (E^\star \cap F^\star)$, then, by applying again \star , we obtain $(E \cap F)^\star = (E^\star \cap F^\star)^\star$.

We say that a semistar operation is *stable* if $(E \cap F)^\star = (E^\star \cap F^\star)$, for each $E, F \in \overline{\mathbf{F}}(D)$.

We will exhibit in Section 1.2 examples of semistar operations that are stable and examples of semistar operations that are not stable. More results about stable semistar operations will be given in Section 1.3.

We say that a semistar operation \star is *cancellative* (or that D has the \star -*cancellation law*) if, for each $E, F, G \in \overline{\mathbf{F}}(D)$, $(EF)^\star = (EG)^\star$ implies $F^\star = G^\star$. We say that \star is *a.b.* if the same holds for each $E \in \mathbf{f}(D)$, $F, G \in \overline{\mathbf{F}}(D)$ and that \star is *e.a.b.* if the same holds for each $E, F, G \in \mathbf{f}(D)$. Clearly, a cancellative semistar operation is a.b. and an a.b. semistar operation is e.a.b.

1.2 Examples of semistar operations

We give some examples of semistar operations. In the following D is always an integral domain with quotient field K and we will denote by $\mathbf{Star}(D)$ the set of the star operations on D , by $\mathbf{SStar}(D)$ the set of the semistar operations on D and by $(\mathbf{S})\mathbf{Star}(D)$ the set of the (semi)star operations on D .

1.2.1 The d -, e - semistar operations and the trivial extension of a star operation

The first example of semistar operation is the *identity semistar operation*, denoted by d_D (or simply d), defined by $E \mapsto E^d := E$, for each $E \in \overline{\mathbf{F}}(D)$. Another trivial semistar operation on D is the e -operation, given by $E \mapsto$

$E^e := K$, for each $E \in \overline{\mathbf{F}}(D)$. It is clear that $d \leq \star \leq e$, for each semistar operation \star on D .

A star operation \star on D induces canonically a (semi)star operation \star_e on D (the *trivial extension* of \star) defined by $E^{\star_e} := E^\star$, if $E \in \mathbf{F}(D)$, and $E^{\star_e} := K$ otherwise. So, the set $\mathbf{Star}(D)$ of the star operations on D is canonically embedded in the set $\mathbf{SStar}(D)$ of the semistar operations on D .

We notice that a star operation can have different extensions to a semistar operation (in fact, a (semi)star operation). For example, let \star be the identity star operation on D (i.e. $F^\star = F$, for each $F \in \mathbf{F}(D)$), and suppose that D is not conducive (we recall that D is a *conductive domain* if $\overline{\mathbf{F}}(D) \setminus \mathbf{F}(D) = \{K\}$, see [19]). Then, the identity semistar operation d_D and the trivial extension \star_e are distinct and both extend \star (since they coincide with \star on the set $\mathbf{F}(D)$).

More precisely, we have the following straightforward lemma:

Lemma 1.9. *Let D be an integral domain. The following are equivalent:*

- (1) D is conducive.
- (2) The trivial extension of the identity star operation coincides with the identity semistar operation d_D on D .
- (3) Each star operation has only one extension to a semistar operation. \square

We notice that from Remark 1.6 it follows that a finite type semistar operation has only one extension to a finite type semistar operation.

Remark 1.10. Let \star be a semistar operation on an integral domain D with quotient field K . More generally, we associate with \star a map \star_e , defined by $E \mapsto E^{\star_e} := E^\star$, if $E \in \mathbf{F}(D)$ and $E \mapsto E^{\star_e} := K$, if $E \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$. We note that \star_e is a semistar operation if and only if $(D : D^\star) \neq (0)$. Indeed, if $(D : D^\star) = (0)$, we have $(D^{\star_e})^{\star_e} = (D^\star)^{\star_e} = K$, since $D^\star \notin \mathbf{F}(D)$. So, the condition (\star_3) of Definition 1.1 does not hold. Conversely, suppose $(D : D^\star) \neq (0)$. It is easy to see that the conditions (\star_1) and (\star_2) of Definition 1.1 are satisfied. For (\star_3) it is enough to observe that, if $E \in \mathbf{F}(D)$ then $E^\star \in \mathbf{F}(D^\star)$ (Proposition 1.3) and then $E^\star \in \mathbf{F}(D)$, since $(D : D^\star) \neq 0$.

We note that $\star|_{\mathbf{F}(D)} = (\star_e)|_{\mathbf{F}(D)}$ and that if \star_1 and \star_2 are two semistar operations on D , then $(\star_1)_e = (\star_2)_e$ if and only if $(\star_1)|_{\mathbf{F}(D)} = \star_2|_{\mathbf{F}(D)}$. If \star is a (semi)star operation, then \star_e coincides with the trivial extension of the star operation $\star|_{\mathbf{F}(D)}$. For example, the semistar operation d_e coincides with the trivial extension of the identity star operation.

1.2.2 The v and the t semistar operations

Consider the map v_D (or, simply, v) defined by $E^v := (E^{-1})^{-1}$, for each $E \in \overline{\mathbf{F}}(D)$, with $E^{-1} := (D : E) := \{z \in K \mid zE \subseteq D\}$. This map

defines a (semi)star operation on D and it is easy to see that this is the trivial extension of the “classical” v -star operation as studied in [38, Section 34]. In fact, it is clear that the v -semistar operation and the v -star operation coincide on the set $\mathbf{F}(D)$. Moreover, if $E \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, $E^{-1} = (0)$ and so $(E^{-1})^{-1} = K$.

It is well known that the v -star operation is the largest star operation on an integral domain D [38, Theorem 34.1(4)]. The v -semistar operation preserves this property, if we restrict to the set of (semi)star operations.

Lemma 1.11. *Let D be an integral domain and let \star be a (semi)star operation. Then, $\star \leq v$.*

Proof. Since \star restricted to $\mathbf{F}(D)$ is a star operation (1.4), it is clear that $E^\star \subseteq E^v$, for each $E \in \mathbf{F}(D)$. If $E \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, it is clear that $E^\star \subseteq K = E^v$. Then, $E^\star \subseteq E^v$, for each $E \in \overline{\mathbf{F}}(D)$ and $\star \leq v$. \square

As in the case of star operations, we denote by t_D (or, simply, t) the semistar operation of finite type v_f associated to v (that is, $E^t := \bigcup\{F^v \mid F \in \mathbf{f}(D), F \subseteq E\}$). This is the only semistar operation of finite type that coincides with the classical t -star operation on the set $\mathbf{F}(D)$ (see Remark 1.6).

The following lemma is an immediate consequence of Lemma 1.11 and Proposition 1.7.

Lemma 1.12. *Let D be an integral domain and let \star be a (semi)star operation on D . Then $\star_f \leq t$.* \square

1.2.3 Semistar operations defined by overrings

Let T be a proper overring of D . The map $\star_{\{T\}}$, given by $E \mapsto E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\mathbf{F}}(D)$ is a semistar operation (called *extension to the overring T*). This is the first non trivial example of a semistar operation that is not a (semi)star operation, that is, that is not extended from a star operation.

It is easy to see that $\star_{\{T\}}$ is a semistar operation of finite type.

The following proposition concerns the stability of the semistar operation defined by the extension to an overring.

Proposition 1.13. *Let D be an integral domain and T an overring of D . The following are equivalent:*

- (i) T is flat over D .
- (ii) The semistar operation $\star_{\{T\}}$ on D is stable.

Proof. (i) \Leftrightarrow (ii) It follows from [62, Theorem 7.4(i)] and [75, Proposition 1.7]. \square

We have the following result about some cancellation properties for the extension to an overring:

Proposition 1.14. *Let D be an integral domain and T an overring of D . The following are equivalent:*

- (i) T is a Prüfer domain.
- (ii) $\star_{\{T\}}$ is a.b.
- (iii) $\star_{\{T\}}$ is e.a.b.

Proof. (i) \Rightarrow (ii) Let $E \in \mathbf{f}(D)$, $F, G \in \overline{\mathbf{F}}(D)$, such that $(EF)^{\star_{\{T\}}} = (EG)^{\star_{\{T\}}}$, that is, $EFT = EGT$. Since $ET \in \mathbf{f}(T)$ and T is a Prüfer domain, ET is invertible. It follows that $FT = GT$, that is, $G^{\star_{\{T\}}} = F^{\star_{\{T\}}}$.

(ii) \Rightarrow (iii) It is straightforward.

(iii) \Rightarrow (i) Let E, F, G be finitely generated ideals of T such that $EF = EG$. Let E_0, F_0, G_0 be the D -modules generated by the generators respectively of E, F and G . Then, $E_0, F_0, G_0 \in \mathbf{f}(D)$ and $E_0T = E$, $F_0T = F$ and $G_0T = G$. So, $(E_0F_0)^{\star_{\{T\}}} = (E_0G_0)^{\star_{\{T\}}}$. Since $\star_{\{T\}}$ is e.a.b. it follows that $F_0^{\star_{\{T\}}} = G_0^{\star_{\{T\}}}$, that is, $F = G$. Then, T is a Prüfer domain by [38, Theorem 24.3 (3) \Rightarrow (1)]. \square

More generally, if $\mathcal{R} := \{D_\alpha\}_{\alpha \in A}$ is a set of overrings of D , and, for each $\alpha \in A$, \star_α is a semistar operation on D_α , the map $\star_{\mathcal{R}} : E \mapsto E^{\star_{\mathcal{R}}} := \bigcap \{(ED_\alpha)^{\star_\alpha} \mid \alpha \in A\}$, for each $E \in \overline{\mathbf{F}}(D)$, is a semistar operation.

It is easy to see that, if each D_α is flat over D and each \star_α is stable, then $\star_{\mathcal{R}}$ is stable.

A particular case of this construction is when \star_α is the identity semistar operation of D_α , for each $\alpha \in A$. In this case, $E^{\star_{\mathcal{R}}} = \bigcap_\alpha ED_\alpha$, for each $E \in \overline{\mathbf{F}}(D)$. We note that in this case, we have $E^{\star_{\mathcal{R}}}D_\beta = ED_\beta$, for each $E \in \overline{\mathbf{F}}(D)$ and for each $\beta \in A$. Indeed, $E^{\star_{\mathcal{R}}}D_\beta = (\bigcap_\alpha ED_\alpha)D_\beta \subseteq ED_\beta \subseteq E^{\star_{\mathcal{R}}}D_\beta$.

Particularly interesting is the case in which $\mathcal{R} := \{V_\alpha\}$ is the set of all valuation overrings of D and \star_α is the identity semistar operation of V_α for each α . In this case, the semistar operation $\star_{\mathcal{R}}$ on D is called *b_D -semistar operation* (or simply *b -operation*). Clearly, b is a (semi)star operation if and only if D is integrally closed. We notice that, in this case, the restriction of the b -semistar operation to the set $\mathbf{F}(D)$ is the classical b -star operation (see [38, Page 398]).

Another interesting particular case of a semistar operation defined by overrings is obtained if we let all $D_\alpha = D$. In this case the semistar operation $E \mapsto \bigcap \{E^{\star_\alpha} \mid \alpha \in A\}$ is denoted by $\wedge \star_\alpha$ and it is the largest semistar operation \star on D such that $\star \leq \star_\alpha$ for each α . Moreover, if we have a family $\{\star_\beta\}_{\beta \in B}$, we define a new semistar operation as $\vee \star_\beta := \wedge \{\star \mid \star_\beta \leq \star, \beta \in B\}$.

This is the smallest semistar operation \star on D such that $\star_\beta \leq \star$ for each $\beta \in B$. In the case of star operations, these constructions are investigated in [4].

1.2.4 Spectral semistar operations

Among the semistar operations in the class defined in Section 1.2.3, particularly interesting are the semistar operations induced by overrings that are localizations of D at prime ideals. More precisely, if $\Delta \subseteq \text{Spec}(D)$, we denote by \star_Δ the semistar operation defined by $E \mapsto E^{\star_\Delta} := \bigcap \{ED_P \mid P \in \Delta\}$. We refer to these semistar operations as *spectral semistar operations*. If $\Delta = \{P\}$, where $P \in \text{Spec}(D)$, we denote $\star_\Delta (= \star_{\{D_P\}})$ simply by $\star_{\{P\}}$.

As a consequence of what we have proven in Section 1.2.3 in the general case, we have that, for each $E \in \overline{\mathbf{F}}(D)$ and for each $P \in \Delta$, $ED_P = E^{\star_\Delta}D_P$. Moreover, since each localization of D is a flat overring of D , we have that the semistar operation \star_Δ is stable.

If we let $\Delta^\downarrow := \{Q \in \text{Spec}(D) \mid Q \subseteq P \text{ for some } P \in \Delta\}$, it is easy to see that $\star_{\Delta'} = \star_\Delta = \star_{\Delta^\downarrow}$, for each $\Delta \subseteq \Delta' \subseteq \Delta^\downarrow$. Moreover, for each $P \in \Delta^\downarrow$, P is a quasi- \star -prime, that is, $P^{\star_\Delta} \cap D = P$.

1.2.5 The semistar operation $v(I)$

As an other example of semistar operation, we want to introduce a semistar operation that generalizes the v -operation.

Consider $I \in \overline{\mathbf{F}}(D)$ and the map $v(I)$ defined by $E \mapsto E^{v(I)} := (I : (I : E))$, for each $E \in \overline{\mathbf{F}}(D)$. Clearly, $v(D)$ coincides with the v -operation. This map, restricted to $\mathbf{F}(D)$, when I is an ideal of D such that $(I : I) = D$ has been studied in [48] and [12]. In this particular case, it has been proven [48, Proposition 3.2] that it is a star operation. We want to prove that, in general, this map is a semistar operation.

We need two lemmas:

Lemma 1.15. *Let $I, J \in \overline{\mathbf{F}}(D)$, L an invertible fractional ideal of D and $0 \neq u \in K$. Then,*

- (1) $(uI : J) = u(I : J)$;
- (2) $(I : uJ) = u^{-1}(I : J)$;
- (3) $(LI : J) = L(I : J)$;
- (4) $(I : LJ) = L^{-1}(I : J)$.

Proof. It is exactly as in [48, Lemma 2.1]. □

Lemma 1.16. *Let D be an integral domain and $I, J \in \overline{\mathbf{F}}(D)$. Then $(I : (I : J)) = \bigcap \{Iu \mid u \in K, J \subseteq Iu\}$.*

Proof. It is exactly as in [48, Lemma 3.1]. \square

Proposition 1.17. *Let D be an integral domain and $I \in \overline{\mathbf{F}}(D)$:*

- (1) $(I : (I : I)) = I$.
- (2) *The map $v(I) : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ defined by $E^{v(I)} = (I : (I : E))$, for each $E \in \overline{\mathbf{F}}(D)$, is a semistar operation.*

Proof. (1) It follows immediately from the fact that I is an ideal in $(I : I)$.
(2) We have to prove (\star_1) , (\star_2) and (\star_3) of Definition 1.1. Let $E, F \in \overline{\mathbf{F}}(D)$, $a \in K$, $a \neq 0$.

(\star_1) From Lemma 1.15, we have $(aE)^{v(I)} = (I : (I : aE)) = a(I : (I : E)) = aE^{v(I)}$.

(\star_2) Since $E(I : E) \subseteq I$, then $E \subseteq (I : (I : E)) = E^{v(I)}$. And, if $E \subseteq F$, we have $(I : F) \subseteq (I : E)$ and $E^{v(I)} = (I : (I : E)) \subseteq (I : (I : F)) = F^{v(I)}$.

(\star_3) We have to prove that $(E^{v(I)})^{v(I)} = E^{v(I)}$. By Lemma 1.16, $(E^{v(I)})^{v(I)} = \bigcap \{Iu \mid u \in K, E^{v(I)} \subseteq Iu\}$. It is clear from (\star_2) that $E^{v(I)} \subseteq (E^{v(I)})^{v(I)}$. On the other hand, if $E \subseteq Iu$, then $E^{v(I)} \subseteq (Iu)^{v(I)} = I^{v(I)}u = Iu$, since, from (a), $I^{v(I)} = I$. Then $(E^{v(I)})^{v(I)} = \bigcap \{Iu \mid u \in K, E^{v(I)} \subseteq Iu\} \subseteq \bigcap \{Iu \mid u \in K, E \subseteq Iu\} = E^{v(I)}$. \square

We note that, if $I \in \mathbf{F}(D)$, then the semistar operation $v(I)$ is trivial on $\overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, that is, if $E \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, then $(I : (I : E)) = K$. Indeed, take $y \in (D : I)$, $y \neq 0$. If $x \in (I : E)$ and $x \neq 0$, then $xy \in (D : E)$ and $E \in \mathbf{F}(D)$. Then $(I : E) = 0$ and this implies immediately $(I : (I : E)) = K$.

Next lemma gives some basic properties of the semistar operation $v(I)$.

Lemma 1.18. *Let D be an integral domain. Let $I, J, L, E, F \in \overline{\mathbf{F}}(D)$. Then:*

- (1) $D^{v(I)} = (I : I)$.
- (2) $v(I)$ is a (semi)star operation if and only if $(I : I) = D$.
- (3) If I is invertible, then $v(I) = v$.
- (4) If J is invertible, then $J^{v(I)} = J(I : I)$.
- (5) If J is invertible, then $v(IJ) = v(I)$.
- (6) If $J^{v(I)} = (I : (I : J)) = L$, then $(I : J) = (I : L)$.
- (7) $E^{v(I)} = F^{v(I)}$ if and only if $(I : E) = (I : F)$.

Proof. (1) It is straightforward, since $(I : D) = I$

(2) It is an immediate consequence of (1).

(3),(4) and (5) follow immediately from Lemma 1.15.

(6) Since $J \subseteq (I : (I : J)) = L$, we have $(I : J) \subseteq (I : J)^{v(I)} = (I : (I : (I : J))) = (I : L) \subseteq (I : J)$, and then $(I : J) = (I : L)$.

(7) The ‘if’ part is clear. To prove the ‘only if’ we observe that, if $E^{v(I)} = F^{v(I)} = L$, then, from (5), $(I : E) = (I : F) = (I : L)$. □

In [48] the operation $v(I)$ is studied in the case $(I : I) = D$, that is, when, restricted to $\mathbf{F}(D)$, it is a star operation. In that paper an ideal I is defined to be *m-canonical* if the star operation $v(I)$ is the identity, that is, when any ideal of D is a $v(I)$ -ideal (in particular, $(I : I) = D$). Here, we do not require $(I : I) = D$, and so we work in the context of semistar operations. The role played by \star -ideals in the context of star operation, in our context is played by quasi- \star -ideals. Then, it appears natural to ask for which nonzero ideals I , the semistar operation $v(I)$ is such that every nonzero ideal of D is a quasi- $v(I)$ -ideal. Next proposition answers this question: such ideals are exactly the *m-canonical* ideals, that is, if all nonzero ideals are quasi- $v(I)$ -ideals then all ideals are, in fact, $v(I)$ -ideals.

Proposition 1.19. *Let D be an integral domain and let \star be a semistar operation on D such that every nonzero ideal of D is a quasi- \star -ideal. Suppose that there exists a nonzero ideal I of D such that $I^\star = I$. Then $D^\star = D$. In particular, if I is a nonzero ideal ideal such that every ideal of D is a quasi- $v(I)$ -ideal, then $(I : I) = D$ (and I is *m-canonical*).*

Proof. Let I such that $I^\star = I$ and let $x \in I$, $x \neq 0$. We have $(xD)^\star \subseteq I^\star = I \subseteq D$. But, by the hypothesis, $(xD)^\star \cap D = xD$ and then $xD = (xD)^\star = xD^\star$, that implies $D = D^\star$. The statement about $v(I)$ follows immediately from the first part and the fact that $I^{v(I)} = I$ (Proposition 1.17 (1)). □

We conclude this section proving a result that characterizes the semistar operation $v(I)$ as the largest semistar operation such that $I^\star = I$.

Proposition 1.20. *Let D be an integral domain and \star a semistar operation on D . Let $I \in \mathbf{F}(D)$. Then $\star \leq v(I)$ if and only if $I^\star = I$.*

Proof. If $\star \leq v(I)$, then $I^\star \subseteq I^{v(I)} = I$ and then $I^\star = I$. On the other hand, if $I^\star = I$, then $(J^\star)^{v(I)} = (I : (I : J^\star)) = (I : (I : J)) = J^{v(I)}$, by using that $(E^\star : F^\star) = (E^\star : F)$, for every semistar operation. Then, $\star \leq v(I)$, by Proposition 1.5(ii) \Rightarrow (i). □

Corollary 1.21. *Let D be an integral domain and let I, J be ideals of D . Then $v(I) = v(J)$ if and only if $I^{v(J)} = I$ and $J^{v(I)} = J$. □*

1.3 Localizing systems and stable semistar operations

The relation between semistar operations (in particular, stable semistar operations) and localizing systems has been deeply investigated by M. Fontana

and J. Huckaba in [23] and by F. Halter-Koch in the context of module systems [43]. In this section we recall some of the results concerning this relation.

First, we recall the definition of localizing system .

Definition 1.22. Let D be an integral domain. A localizing system of D is a family \mathcal{F} of ideals of D such that:

- (LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$.
- (LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$, for each $i \in I$, then $J \in \mathcal{F}$.

We recall [24, Proposition 5.1.1] that a localizing system is a multiplicative system of ideals (i.e., if \mathcal{F} is a localizing system and $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$).

A localizing system \mathcal{F} is *finitely generated* if, for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.

Example 1.23. (1) Let T be an overring of D . The set $\mathcal{F}(T) := \{I \mid I \text{ ideal of } D, IT = T\}$ is a finitely generated localizing system.

(2) Let P be a prime ideal of D . The set $\mathcal{F}(P) := \{I \mid I \text{ ideal of } D, I \not\subseteq P\}$ is a localizing system of finite type (it is, in fact, a particular case of (1)).

(3) It is easy to see that the intersection of localizing systems of D is a localizing system of D . So, let $\Delta \subseteq \text{Spec}(D)$. Then, $\mathcal{F}(\Delta) := \bigcap_{P \in \Delta} \mathcal{F}(P)$ is a localizing system of D . A localizing system \mathcal{F} of D such that there exists $\Delta \subseteq \text{Spec}(D)$ with $\mathcal{F} = \mathcal{F}(\Delta)$ is called a *spectral localizing system*. We note that, if $\Delta \subseteq \Delta' \subseteq \text{Spec}(D)$ then $\mathcal{F}(\Delta') \subseteq \mathcal{F}(\Delta)$.

We recall an important result about localizing systems [24, Proposition 5.1.8]:

Proposition 1.24. Let D be an integral domain and \mathcal{F} a localizing system of D . The following are equivalent:

- (i) \mathcal{F} is a finitely generated localizing system.
- (ii) $\mathcal{F} = \mathcal{F}(\Delta)$ for some quasi-compact (in the Zariski topology) subspace Δ of $\text{Spec}(D)$. □

To each localizing system \mathcal{F} of D , it is associated a map $\star_{\mathcal{F}} : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, defined by $E \mapsto E^{\star_{\mathcal{F}}} := E_{\mathcal{F}} = \bigcup \{(E : J) \mid J \in \mathcal{F}\}$. This map is a stable semistar operation [23, Proposition 2.4]. If $\mathcal{F} \subseteq \mathcal{F}'$ are two localizing systems of D , it is easy to see that $\star_{\mathcal{F}} \leq \star_{\mathcal{F}'}$. Moreover, if \mathcal{F} is a finitely generated localizing system, then $\star_{\mathcal{F}}$ is a semistar operation of finite type, [23, Proposition 3.2(1)].

Example 1.25. Let $P \in \text{Spec}(D)$ and let $\mathcal{F}(P)$ as in Example 1.23. Then, $\star_{\mathcal{F}(P)} = \star_{\{P\}}$. Indeed, let $E \in \overline{\mathbf{F}}(D)$ and let $x \in E^{\star_{\mathcal{F}(P)}} = \bigcup_{I \in \mathcal{F}(P)} (E : I)$. Then, there exists $I \not\subseteq P$ such that $xI \subseteq E$. It follows that $xy \in E$ for some $y \in I \setminus P$. Hence, $x \in ED_P = E^{\star_{\{P\}}}$. On the other hand, if $x \in ED_P$, there exists $y \in D \setminus P$ such that $xy \in E$. Clearly $I := yD \in \mathcal{F}(P)$. Then, $xI \subseteq E$ and $x \in E^{\star_{\mathcal{F}(P)}}$.

Conversely, to each semistar operation \star on D , it is possible to associate a localizing system \mathcal{F}^\star defined by $\mathcal{F}^\star := \{I \text{ ideal of } D \mid I^\star = D^\star\} = \{I \text{ ideal of } D \mid I^\star \cap D = D\}$. If $\star_1 \leq \star_2$ are two semistar operations on D , it is easy to see that $\mathcal{F}^{\star_1} \subseteq \mathcal{F}^{\star_2}$. Moreover, if \star is a finite type semistar operation, then the localizing system \mathcal{F}^\star is finitely generated.

Example 1.26. The localizing system $\mathcal{F}^{\star_{\{P\}}}$ coincides with the localizing system $\mathcal{F}(P)$. This is straightforward since $I \in \mathcal{F}^{\star_{\{P\}}}$ if and only if $I^{\star_{\{P\}}} = ID_P = D_P$ and this is equivalent to $I \in \mathcal{F}(P)$.

Theorem 1.27. [23, Theorem 2.10] *Let D be an integral domain with quotient field K .*

(1) *Let \mathcal{F} be a localizing system of D . Then $\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}}$.*

(2) *Let \star be a semistar operation on D and let \mathcal{F}^\star be the localizing system associated to \star . Then $\star_{\mathcal{F}^\star} \leq \star$. Moreover, the following are equivalent:*

(i) $\star_{\mathcal{F}^\star} = \star$.

(ii) \star is a stable semistar operation.

(iii) $(E :_D F)^\star = (E^\star :_{D^\star} F^\star)$, for each $E \in \overline{\mathbf{F}}(D)$ and for each $F \in \mathbf{f}(D)$.

(iv) $(E :_D xD)^\star = (E^\star :_{D^\star} xD^\star)$, for each $E \in \overline{\mathbf{F}}(D)$ and for each $0 \neq x \in K$. \square

Next proposition generalizes Example 1.25 and Example 1.26.

Proposition 1.28. [23, Lemma 4.2] *Let D be an integral domain and let $\Delta \subseteq \text{Spec}(D)$. Then $\star_\Delta = \star_{\mathcal{F}(\Delta)}$ and $\mathcal{F}^{\star_\Delta} = \mathcal{F}(\Delta)$.*

Proof. It is clear that $\mathcal{F}^{\star_\Delta} = \mathcal{F}(\Delta)$, since if I is an ideal of D , then $I \in \mathcal{F}^{\star_\Delta}$ if and only if $\bigcap_{P \in \Delta} ID_P = \bigcap_{P \in \Delta} D_P$, that is, if and only if $I \not\subseteq P$ for each $P \in \Delta$. Now, that $\star_\Delta = \star_{\mathcal{F}(\Delta)}$ is a straightforward consequence of the fact that a spectral semistar operation is stable and of Theorem 1.27(2). \square

Next proposition, that is a consequence of Proposition 1.24 and Proposition 1.28, characterizes the spectral semistar operations that are of finite type:

Proposition 1.29. [23, Corollary 4.6(2)] *Let D be an integral domain and $\Delta \subseteq \text{Spec}(D)$. The following are equivalent :*

- (1) \star_Δ is a semistar operation of finite type.
- (2) There exists a quasi-compact subspace F of $\text{Spec}(D)$ such that $\star_\Delta = \star_F$.
□

We finish summarizing the main results of this section in the following theorem.

Theorem 1.30. *Let D be an integral domain. Then, the map given by $\star \mapsto \mathcal{F}^\star$ establishes an order preserving bijection between the set of all stable semistar operations on D and the set of all localizing systems of D , with inverse map given by $\mathcal{F} \mapsto \star_{\mathcal{F}}$. Moreover, the restriction of this map to the set of the semistar operations of finite type (resp., spectral) establishes an order preserving bijection with the set of the finitely generated (resp., spectral) localizing systems.*

1.4 The semistar operation $\tilde{\star}$

Let \star be a semistar operation on D . We can consider the localizing system \mathcal{F}^\star associated with \star and the stable semistar operation $\star_{\mathcal{F}^\star}$ associated with \mathcal{F}^\star . We say that $\star_{\mathcal{F}^\star}$ is the *stable semistar operation associated to \star* . We have already observed (Theorem 1.27) that in general $\star_{\mathcal{F}^\star} \leq \star$ and that $\star_{\mathcal{F}^\star} = \star$ if and only if \star is a stable semistar operation. It is easy to see [23, Proposition 3.7] that $\star_{\mathcal{F}^\star}$ is the largest stable semistar operation smaller than \star .

More interesting it is the construction of the *stable semistar operation of finite type associated to \star* . We proceed in this way: we consider the localizing system \mathcal{F}^\star associated to \star . Then, we consider the set $(\mathcal{F}^\star)_f$ of the ideals J of D such that $J \supseteq I$ for some finitely generated ideal $I \in \mathcal{F}^\star$. It is easy to see that $(\mathcal{F}^\star)_f$ is a finitely generated localizing system and that $(\mathcal{F}^\star)_f = \mathcal{F}^{\star_f}$. We define $\tilde{\star} := \star_{(\mathcal{F}^\star)_f} = \star_{\mathcal{F}^{\star_f}}$. This is a semistar operation stable and of finite type (since it is associated to a finitely generated localizing system) and it is the largest semistar operation stable and of finite type smaller than \star . Clearly, $\tilde{\star} = \widetilde{\star_f}$ and then $\tilde{\star} \leq \star_f$. Moreover $\star = \tilde{\star}$ if and only if \star is stable and of finite type.

Remark 1.31. In [46], J. Hedstrom and E. Houston introduce a star operation under the name F_∞ . The same operation, under the name w -operation, has been considered in [79] by Fanggui Wang and R.L. Mc Casland. This star operation is defined in this way: first, consider the set $GV(D)$ of the *Glaz-Vasconcelos ideals*, that is, the finitely generated ideals J of D such that $J^{-1} = D$. Then, let $I^w := \bigcup_{J \in GV(D)} (I : J)$. We notice that $w = \tilde{v}$. In fact, $\mathcal{F}^v = \{I \text{ ideal of } D \mid I^v = D\} = \{I \text{ ideal of } D \mid I^{-1} = D\}$ and $(\mathcal{F}^v)_f = \{I \text{ ideal of } D \mid I \supseteq J \text{ for some } J \in GV(D)\}$. It is clear now that $I^w = \bigcup_{J \in GV(D)} (I : J) = \bigcup_{J \in (\mathcal{F}^v)_f} (I : J) = I^{\star(\mathcal{F}^v)_f} = I^{\tilde{v}}$.

Lemma 1.32. *Let D be an integral domain and \star a semistar operation on D .*

(1) *If \star is of finite type then \star is stable if and only if it is spectral.*

(2) *The following are equivalent:*

(i) $\star = \tilde{\star}$.

(ii) \star is stable and of finite type.

(iii) \star is spectral and of finite type.

Proof. (1) We have already observed that spectral semistar operations are stable. Conversely, if \star is a semistar operation stable of finite type, the localizing system \mathcal{F}^\star associated to \star is finitely generated, then it is spectral, say $\mathcal{F}^\star = \mathcal{F}(\Delta)$, $\Delta \subseteq \text{Spec}(D)$, by Proposition 1.24. Since \star is stable, $\star = \star_{\mathcal{F}^\star} = \star_{\mathcal{F}(\Delta)} = \star_\Delta$, by Theorem 1.27 and Proposition 1.28. Hence, \star is a spectral semistar operation.

(2) We have already noticed that (i) \Rightarrow (ii). The equivalence between (ii) and (iii) is given by (1). So, we have only to prove that (ii) \Rightarrow (i). Since \star is stable, $\star = \star_{\mathcal{F}^\star}$. Moreover, since \star is of finite type, \mathcal{F}^\star is a finitely generated localizing system. It follows that $\mathcal{F}^\star = (\mathcal{F}^{\star f})$. Hence, $\star = \star_{\mathcal{F}^\star} = \star_{(\mathcal{F}^{\star f})} = \tilde{\star}$. \square

So, $\tilde{\star}$ is spectral. More precisely, if \star is a semistar operation on D , we have that $\tilde{\star} = \star_{\mathcal{M}(\star_f)}$, that is, $\tilde{\star}$ is the spectral semistar operation induced by the set of all quasi- \star_f -maximal ideals. To prove this, we show that $\mathcal{F}(\mathcal{M}(\star_f)) = \mathcal{F}^{\star f}$. Indeed, $I \in \mathcal{F}(\mathcal{M}(\star_f))$ if and only if $I \not\subseteq M$, for each $M \in \mathcal{M}(\star_f)$. But this is equivalent to $I^{\star f} = D^{\star f} (= D^\star)$, that is, $I \in \mathcal{F}^{\star f}$. In fact, if $I \subseteq M$, for some $M \in \mathcal{M}(\star_f)$, clearly $I^{\star f} \subseteq M^{\star f} \subsetneq D^\star$. Conversely, if $I \not\subseteq M$ for each $M \in \mathcal{M}(\star_f)$, then $I^{\star f} = D^\star$, otherwise $I^{\star f} \cap D$ would be a quasi- \star_f -ideal not contained in M , for each $M \in \mathcal{M}(\star_f)$, and this contradicts Proposition 1.8. So, $\mathcal{F}(\mathcal{M}(\star_f)) = \mathcal{F}^{\star f}$ and $\tilde{\star} = \star_{\mathcal{F}(\mathcal{M}(\star_f))} = \star_{\mathcal{M}(\star_f)}$.

Example 1.33. Let D be an integral domain and T an overring of D . Let $\star := \star_{\{T\}}$ the semistar operation of finite type given by the extension to T . We have that $\tilde{\star} = \star$ if and only if \star is stable, that is, if and only if T is flat over D (by Proposition 1.13). In general, $\tilde{\star} \neq \star$. For example, let (D, M) be a pseudo-valuation domain and $V = M^{-1}$. Let $\star := \star_{\{V\}}$. Note that $\star = \star_f$. Since $MV = M$, we have that M is a quasi- \star -prime ideal and clearly $\mathcal{M}(\star_f) = \{M\}$. It follows that $\tilde{\star} = \star_{\mathcal{M}(\star_f)} = \star_{\{M\}} = d$. Then, in this case, $\tilde{\star} = d \leq \star$. Moreover, $D^{\tilde{\star}} \subsetneq D^\star$.

Finally we recall the following proposition about the set of quasi- $\tilde{\star}$ -maximal ideals:

Proposition 1.34. [29, Corollary 3.5(2)] *Let D be an integral domain and \star a semistar operation on D . Then, $\mathcal{M}(\tilde{\star}) = \mathcal{M}(\star_f)$. \square*

1.5 The semistar operations \star_ι and \star^ι

In this section, we show how a semistar operation on a domain D induces canonically a semistar operation on an overring T of D and how a semistar operation on T induces canonically a semistar operation on D .

Proposition 1.35. ([65, Lemma 45] and [30]) *Let D be an integral domain, T an overring of D , $\iota : D \rightarrow T$ the canonical embedding of D in T , \star a semistar operation on D and $*$ a semistar operation on T . Then:*

- (1) *The map $\star_\iota : \overline{\mathbf{F}}(T) \rightarrow \overline{\mathbf{F}}(T)$, $E \mapsto E^{\star_\iota} := E^\star$ is a semistar operation on T .*
- (2) *The map $\star^\iota : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^{\star^\iota} := (ET)^\star$ is a semistar operation on D . \square*

We will denote by $(-)_\iota$ the map $\mathbf{SStar}(D) \rightarrow \mathbf{SStar}(T)$, $\star \mapsto \star_\iota$, and by $(-)^\iota$ the map $\mathbf{SStar}(T) \rightarrow \mathbf{SStar}(D)$, $*$ $\mapsto \star^\iota$. We will refer to this map respectively as the ‘‘ascent’’ and the ‘‘descent’’ maps, as in [65]. In next Lemma we observe that these two maps are order preserving.

Lemma 1.36. *Let D be an integral domain, T an overring of D , $\iota : D \rightarrow T$ the canonical embedding of D in T , \star_1, \star_2 semistar operations on D and $*_1, *_2$ semistar operations on T . Then:*

- (1) $\star_1 \leq \star_2$ implies $(\star_1)_\iota \leq (\star_2)_\iota$.
- (2) $*_1 \leq *_2$ implies $(*_1)^\iota \leq (*_2)^\iota$.

Proof. (1) Let $E \in \overline{\mathbf{F}}(T)$. Then, $E^{(\star_1)_\iota} = E^{\star_1} \subseteq E^{\star_2} = E^{(\star_2)_\iota}$. Hence, $(\star_1)_\iota \leq (\star_2)_\iota$
(2) Let $F \in \overline{\mathbf{F}}(D)$, then $F^{(*_1)^\iota} = (FT)^{*_1} \subseteq (FT)^{*_2} = F^{(*_2)^\iota}$. Hence, $(*_1)^\iota \leq (*_2)^\iota$. \square

Example 1.37. (1) Let the notation be as in Proposition 1.35. If $\star_{\{T\}}$ is the semistar operation given by the extension to the overring T and d_T is the identity (semi)star operation on T , it is easy to see that $(\star_{\{T\}})_\iota = d_T$ and $(d_T)^\iota = \star_{\{T\}}$.

(2) Let D be an integral domain. Consider the b -semistar operation, as defined in Section 1.2.3. It is easy to see that, if D is not integrally closed, the b -semistar operation, is exactly the ‘‘descent’’ of the $b_{D'}$ -(semi)star operation of the integral closure D' of D , i.e. $b_D = (b_{D'})^\iota$, where ι is the canonical embedding of D in D' .

Here we recall only two properties of the semistar operation \star_ι when the overring T coincides with D^\star , that we will need in next section (we will show the proofs and study in details these constructions in Section 2.2).

Proposition 1.38. *Let D be an integral domain and ι the canonical embedding of D in D^* . Then:*

(1) \star_ι is a (semi)star operation on D .

(2) If \star is e.a.b. then \star_ι is e.a.b. □

1.6 Kronecker function rings and Nagata rings

In the context of star operations, the classical Kronecker function ring (defined as in [38, Section 32]) allows to embed “nicely” an integrally closed domain in a Bezout domain. More precisely, to an integrally closed integral domain D and to an e.a.b. star operation \star , it is possible to associate the integral domain $\text{Kr}(D, \star) := \{f/g \mid f, g \in D[X] \setminus \{0\}, \mathbf{c}(f)^\star \subseteq \mathbf{c}(g)^\star\} \cup \{0\}$, where $\mathbf{c}(f)$ is the content of a polynomial $f \in D[X]$.

In [27], M. Fontana and K.A. Loper generalize this construction to the context of semistar operations. The approach in [27] allows to define a Kronecker function ring associated to any semistar operation \star defined on an integral domain D without assuming that D is integrally closed or that \star satisfies some cancellation property. Moreover, it is possible to associate to \star an e.a.b. semistar operation \star_a , such that the Kronecker function ring $\text{Kr}(D, \star)$ coincides with the classical Kronecker function rings of D^{\star_a} with respect to an e.a.b. star operation on D^{\star_a} canonically induced by \star . So, it is clear that $\text{Kr}(D, \star)$ has all the properties of the classical Kronecker functions ring (for example, it is a Bezout domain).

So, let D be an integral domain with quotient field K and \star a semistar operation on D . Consider the set:

$$\text{Kr}(D, \star) := \{f/g \mid f, g \in D[X] \setminus \{0\} \text{ and there exists } h \in D[X] \setminus \{0\} \\ \text{such that } (\mathbf{c}(f)\mathbf{c}(h))^\star \subseteq (\mathbf{c}(g)\mathbf{c}(h))^\star\} \cup \{0\}.$$

We have [27, Theorem 5.1] that $\text{Kr}(D, \star)$ is an integral domain with quotient field $K(X)$, called *the Kronecker function ring of D with respect to \star* .

It is easy to see that $\text{Kr}(D, \star) = \text{Kr}(D, \star_f)$ and that, if $\star_1 \leq \star_2$, then $\text{Kr}(D, \star_1) \leq \text{Kr}(D, \star_2)$.

Now, let \star be a semistar operation on D and consider the map $\star_a : \overline{\mathbf{f}}(D) \rightarrow \overline{\mathbf{f}}(D)$, defined by $F \mapsto F^{\star_a} := \bigcup \{((FH)^\star : H) \mid H \in \mathbf{f}(D)\}$, for each $F \in \mathbf{f}(D)$, and $E^{\star_a} := \bigcup \{F^{\star_a} \mid F \in \mathbf{f}(D), F \subseteq E\}$. The map \star_a is an e.a.b. semistar operation of finite type, and it is called *the e.a.b. semistar operation on D associated to \star* .

We note that $\star = \star_a$ if and only if \star is e.a.b. and of finite type [27, Proposition 4.5(5)] and that D^{\star_a} is an integrally closed domain ([27, Proposition 4.5(9) and Proposition 4.3(2)]). We note that, by Proposition 1.38,

$* := (\star_a)_\iota|_{\mathbf{F}(D)}$ is an e.a.b. star operation on the integrally closed domain D^{\star_a} (then, it is possible to construct the classical Kronecker function ring of D^{\star_a} with respect to $*$).

Let \star be a semistar operation on D and let V be a valuation overring of D . We say that V is a \star -valuation overring of D if, for each $F \in \mathbf{f}(D)$, $F^\star \subseteq FV$ (or equivalently, $\star_f \leq \star_{\{V\}}$).

Note that a valuation overring V of D is a \star -valuation overring of D if and only if $V^{\star_f} = V$. (The “only if” part is obvious; for the “if” part recall that, for each $F \in \mathbf{f}(D)$, there exists a nonzero element $x \in K$ such that $FV = xV$, thus $F^\star \subseteq (FV)^{\star_f} = (xV)^{\star_f} = xV^{\star_f} = xV = FV$).

We now recall some main results about Kronecker function rings.

Proposition 1.39. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star .*

Then we have:

- (1) $\text{Kr}(D, \star)$ is a Bézout domain.
- (2) $\text{Kr}(D, \star) = \text{Kr}(D, \star_a) = \text{Kr}(D^{\star_a}, (\star_a)_\iota)$.
- (3) $E^{\star_a} = E\text{Kr}(D, \star) \cap K$, for each $E \in \overline{\mathbf{F}}(D)$.
- (4) $\text{Kr}(D, \star) = \bigcap \{V(X) \mid V \text{ is a } \star\text{-valuation overring of } D\}$.
- (5) If $F := (a_0, a_1, \dots, a_n) \in \mathbf{f}(D)$ and $f(X) := a_0 + a_1X + \dots + a_nX^n \in K[X]$, then:

$$F\text{Kr}(D, \star) = f(X)\text{Kr}(D, \star) = \mathbf{c}(f)\text{Kr}(D, \star).$$

□

In particular, we have that the Kronecker function ring $\text{Kr}(D, \star)$ of an integral domain D coincides with the classical Kronecker function rings of the integrally closed domain D^{\star_a} with respect to the e.a.b. star operation $*$ given by the restriction of the (semi)star operation $(\star_a)_\iota$ to the set of fractional ideals of D^{\star_a} , i.e. $\text{Kr}(D, \star) = \text{Kr}(D^{\star_a}, *)$.

If R is a ring and X an indeterminate over R , then the ring $R(X) := \{f/g \mid f, g \in R[X] \text{ and } \mathbf{c}(g) = R\}$ (where $\mathbf{c}(g)$ is the content of the polynomial g) is called the *Nagata ring* of R [38, Proposition 33.1]. A more general construction of a Nagata ring associated to a semistar operation defined on an integral domain D was considered in [29] (cf. also [53], for the star case). We present briefly this construction.

Let D be an integral domain and \star a semistar operation on D . Consider the set $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0 \text{ and } (\mathbf{c}(h))^\star = D^\star\}$. It can be proven that $N(\star)$ is a saturated multiplicative subset of $D[X]$ (more precisely, $N(\star) = N(\star_f) = D[X] \setminus \bigcup \{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$). So, let $\text{Na}(D, \star) :=$

$D[X]_{N(\star)} = \{f/g \mid f, g \in D[X], g \neq 0, \mathbf{c}(g)^\star = D^\star\}$ be the Nagata ring of D with respect to the semistar operation \star .

We notice that, if $\star_1 \leq \star_2$, then $N(\star_1) \subseteq N(\star_2)$ and so $\text{Na}(D, \star_1) \subseteq \text{Na}(D, \star_2)$. We observe also that $\text{Na}(D, \star) \subseteq \text{Kr}(D, \star)$.

We summarize in next proposition, the main results about the Nagata ring.

Proposition 1.40. *Let D be an integral domain and \star be a semistar operation on D . Then:*

- (1) $\text{Max}(\text{Na}(D, \star)) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$ and $\mathcal{M}(\star_f)$ coincides with the canonical image in $\text{Spec}(D)$ of $\text{Max}(\text{Na}(D, \star))$.
- (2) $\text{Na}(D, \star) = \bigcap \{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\} = \bigcap \{D[X]_{QD[X]} \mid Q \in \mathcal{M}(\star_f)\}$.
- (3) $E^\bar{\star} = E\text{Na}(D, \star) \cap K$, for each $E \in \overline{\mathbf{F}}(D)$.
- (4) $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star}) = \text{Na}(D^{\bar{\star}}, (\tilde{\star})_\iota) \supseteq D^{\bar{\star}}(X)$, where ι is the canonical embedding of D in $D^{\bar{\star}}$ □

Example 1.41. (1) Let (D, M) be a pseudo-valuation domain and $V = M^{-1}$. Suppose that V is the integral closure of D . Consider the semistar operation $\star_{\{V\}}$ (it coincides with the b -operation defined in Section 1.2.3). It is clear that V is a \star -valuation overring (since $V^{\star_{\{V\}}} = V$). So, $\text{Kr}(D, \star_{\{V\}}) = V(X)$, by Proposition 1.39(4). We notice that in this case $\text{Kr}(D, \star_{\{V\}}) \neq \text{Na}(D, \star_{\{V\}})$. Indeed, by Proposition 1.40(4), $\text{Na}(D, \star_{\{V\}}) = \text{Na}(D, \widetilde{\star_{\{V\}}})$ and $\widetilde{\star_{\{V\}}} = d$ (Example 1.33). Then, $\text{Na}(D, \star_{\{V\}}) = D(X) \subsetneq V(X)$, since $V(X) \cap K = V$ and $D(X) \cap K = D$ (this is a consequence of Proposition 1.40(3)).

(2) [27, Example 3.3(1)] Let D be an integral domain and $P \in \text{Spec}(D)$. Consider the semistar operation of finite type $\star := \star_{\{P\}}$. It is clear that $\mathcal{M}_{\star_f} = \{P\}$. Then, by Proposition 1.40(2), $\text{Na}(D, \star) = D_P(X) = \{f/g \mid f, g \in D[x], g \neq 0, \mathbf{c}(g) \not\subseteq P\}$.

Chapter 2

Star operations on overrings and semistar operations

2.1 Composition of semistar operations

Semistar operations are, in fact, maps. Then, it is possible to define, in some cases and in a “natural” way, a composition of semistar operations. Let \star_1 be a semistar operation on an integral domain D and let \star_2 be a semistar operation on an integral domain T , with $D \subseteq T \subseteq D^{\star_1}$. It follows that $\overline{\mathbf{F}}(D^{\star_1}) \subseteq \overline{\mathbf{F}}(T) \subseteq \overline{\mathbf{F}}(D)$. We can define the map $\star_1\star_2 : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, by $E \mapsto (E^{\star_1})^{\star_2}$, for each $E \in \overline{\mathbf{F}}(D)$. This map is well defined, since $E^{\star_1} \in \overline{\mathbf{F}}(D^{\star_1}) \subseteq \overline{\mathbf{F}}(T)$.

The map $\star_1\star_2$ can be, but in general is not, a semistar operation on D . The properties (\star_1) and (\star_2) of the Definition 1.1 are easily checked, while (\star_3) is not always satisfied.

Example 2.1. (1) Let D be an integral domain with quotient field K and $R (\neq K)$ an overring of D , such that $(D : R) = 0$. Let $\star_1 = v$, the v -operation of D , and let $\star_2 = \star_{\{R\}}$, the semistar operation on D given by the extension to R (that is, $E^{\star_2} = ER$, for each $E \in \overline{\mathbf{F}}(D)$). Let $\star := \star_1\star_2$. This is a map, defined on the set $\overline{\mathbf{F}}(D)$. We prove that it is not a semistar operation, by showing that, in general, if $I \in \overline{\mathbf{F}}(D)$, $I^\star \neq (I^\star)^\star$. Then, let $I \in \overline{\mathbf{F}}(D)$. We have $I^\star = I^v R$ and $(I^\star)^\star = (I^v R)^v R$. We notice that $(I^v R)^v = (D : (D : I^v R)) = (D : ((D : R) : I^v)) = (D : (0 : I^v)) = (D : 0) = K$. Then, if, for instance, I is a principal ideal (or, more generally, a divisorial ideal) of D , we have $I^\star = IR \subsetneq (I^\star)^\star = K$.

(2) Let D be an integral domain, T an overring of D , let ι be the canonical embedding of D in T and \ast a semistar operation on T . Then the semistar operation \ast^ι on D defined in Proposition 1.35(2) is exactly the composition of $\ast_{\{T\}}$ and \ast , i.e. $\ast_{\{T\}}\ast = \ast^\iota$.

(3) As a consequence of Proposition 1.5, we have that, if \star_1, \star_2 are semistar operations on an integral domain D , such that $\star_1 \leq \star_2$, then $\star_1\star_2 = \star_2$

(in particular, $\star_1\star_2$ is a semistar operation).

(4) Let D be an integral domain and A and B two overrings of D . Then, the map $\star := \star_{\{A\}}\star_{\{B\}}$ is a semistar operation. It is sufficient to prove property (\star_3) of Definition 1.1, thus, if $E \in \overline{\mathbf{F}}(D)$, we have $E^{\star\star} = EABAB = EAABB = EAB = E^\star$. More precisely, $\star = \star_{\{R\}}$, where $R = AB$ is the overring of D given by the product of A and B . We notice that, if A and B are not comparable, then \star is different from both $\star_{\{A\}}$ and $\star_{\{B\}}$.

It is natural to ask when the composition of two semistar operations is a semistar operation, that is, under which assumptions, condition (\star_3) of Definition 1.1 holds.

Lemma 2.2. *Let D be an integral domain, \star_1 a semistar operation on D , T an overring of D , $T \subseteq D^{\star_1}$, $\iota : D \hookrightarrow T$ the canonical embedding of D in T . Let \star_2 be a semistar operation on T and $\star := \star_1\star_2$. Then:*

- (1) $E^{\star_1} \subseteq E^\star$, for each $E \in \overline{\mathbf{F}}(D)$. (Then, when \star is a semistar operation, $\star_1 \leq \star$.)
- (2) $\star = \star_1$ if and only if $\star_2 \leq (\star_1)_\iota$. In this case, \star is a semistar operation.
- (3) $E^{(\star_2)^\iota} \subseteq E^\star$, for each $E \in \overline{\mathbf{F}}(D)$. (Then, when \star is a semistar operation, $(\star_2)^\iota \leq \star$.)
- (4) $\star = (\star_2)^\iota$ if and only if $(\star_1)_\iota \leq \star_2$. In this case, \star is a semistar operation.

Proof. (1) and (3) are straightforward.

(2) Suppose $\star = \star_1$. Let $E \in \overline{\mathbf{F}}(T) \subseteq \overline{\mathbf{F}}(D)$. We have $(E^{(\star_1)_\iota})^{\star_2} = (E^{\star_1})^{\star_2} = E^{\star_1} = E^{(\star_1)_\iota}$ and then $\star_2 \leq (\star_1)_\iota$. Conversely, let $\star_2 \leq (\star_1)_\iota$. Then, $(E^{\star_1})^{\star_2} \subseteq ((ET)^{\star_1})^{\star_2} = ((ET)^{(\star_1)_\iota})^{\star_2} = (ET)^{(\star_1)_\iota} \subseteq (ED^{\star_1})^{\star_1} = E^{\star_1}$.

(4) Suppose $\star = (\star_2)^\iota$. Let $E \in \overline{\mathbf{F}}(T)$. Then, $E^{(\star_1)_\iota} \subseteq (E^{(\star_1)_\iota})^{\star_2} = (E^{\star_1})^{\star_2} = E^{\star_2^\iota} = (ET)^{\star_2} = E^{\star_2}$. Hence, $(\star_1)_\iota \leq \star_2$. Conversely, let $(\star_1)_\iota \leq \star_2$. Then, $E^{(\star_2)^\iota} = (ET)^{\star_2} \subseteq (E^{\star_1})^{\star_2} \subseteq ((ET)^{\star_1})^{\star_2} = (ET)^{\star_2} = E^{\star_2^\iota}$. \square

Example 2.3. Let D be an integral domain, \star_1 a semistar operation on D and ι the canonical embedding of D in D^{\star_1} . Let $\star_2 := v_{D^{\star_1}}$ be the v -operation on D^{\star_1} . Consider the composition $\star := \star_1\star_2$. This map, defined by $E^\star = (D^{\star_1} : (D^{\star_1} : E^{\star_1}))$, for each $E \in \overline{\mathbf{F}}(D)$, is a semistar operation and it coincides with the semistar operation $(v_{D^{\star_1}})^\iota$, by Lemma 2.2(4), since $(\star_1)_\iota$ is a (semi)star operation on D^{\star_1} and so $(\star_1)_\iota \leq \star_2$ by Lemma 1.11. As in Section 1.2.5, we will denote this semistar operation on D by $v(D^{\star_1})$; we note that in general, if T is an overring of D and $\iota : D \hookrightarrow T$ is the canonical embedding, the semistar operation $v(T)$ defined in Section 1.2.5 coincides with the descent to D of the v -operation of T , i.e.

$$v(T) = (v_T)^\iota.$$

We will denote by $t(T)$ the descent of the t -operation of T , i.e.

$$t(T) := (t_T)^t.$$

Note that $t(T)$ coincides with the semistar operation of finite type $(v(T))_f$ associated to $v(T)$ (cf. Proposition 2.13(1)).

Remark 2.4. Let D be an integral domain and let T be an overring of D . Let \star_1 be a semistar operation on D and \star_2 be a semistar operation on T . We have shown in Lemma 2.2 that, if $(\star_1)^t$ and \star_2 are comparable in T , then $\star_1\star_2$ is a semistar operation. We notice that this is not a necessary condition. For instance, take A and B two not comparable overrings of D and let $T := A \cap B$. Let $\star_1 := \star_{\{A\}}$ be the semistar operation on D given by the extension to A and let \star_2 be the semistar operation defined on T given by the extension to B . It is easy to see that $\star_1\star_2$ is a semistar operation (with an argument similar to the one used in Example 2.1(4)), but it is clear that \star_1 and \star_2 are not comparable on T , since A and B are not comparable. This is not a necessary condition even if $T = D^{\star_1}$. For example, let D be an integral domain, that is not conducive and that is not a Prüfer domain (for example, the domain $K[X, Y]$, where K is a field and X, Y two indeterminates on K , is clearly not a Prüfer domain and it is not conducive, since a Noetherian conducive domain is local and one dimensional, by [19, Corollary 2.7]). Let $\star_1 = d_e$ be the trivial extension of the identity star operation on D and let $\star_2 = b$, the b semistar operation of D as defined in Section 1.2.3. In this case, $T = D = D^{\star_1}$. We have $b \not\leq d_e$ since there exists a nonzero ideal I of D such that $I^b \neq I$ (otherwise D would be a Prüfer domain, [38, Theorem 24.7]). On the other hand, $d_e \not\leq b$. Indeed, since D is not conducive, there exists a valuation overring V of D that is not a fractional ideal, [19, Lemma 2.0(ii)]. Then, $V^{d_e} = K$, by the definition of d_e , but clearly $V^b = V$. Thus, $d_e \not\leq b$. So, these two semistar operations are not comparable on $D = D^{\star_1}$, but it is easy to see that the composition $\star_1\star_2$ is a semistar operation. More precisely, $\star_1\star_2 = b_e$, the trivial extension of the b star operation of D .

Proposition 2.5. *Let \star_1 be a semistar operation on an integral domain D and let \star_2 be a semistar operation on an integral domain T , with $D \subseteq T \subseteq D^{\star_1}$ and ι the canonical embedding of D in T . Let $\star := \star_1\star_2$. The following are equivalent:*

- (i) \star is a semistar operation.
- (ii) $((E^{\star_1})^{\star_2})^{\star_1} = (E^{\star_1})^{\star_2}$, for each E in $\overline{\mathbf{F}}(D)$.
- (iii) $(F^{\star_2})^{\star_1} = F^{\star_2}$, for each $F \in \overline{\mathbf{F}}^{\star_1}(D)$.
- (iv) $(E^{\star_2})^{\star_1} \subseteq (E^{\star_1})^{\star_2}$, for each $E \in \overline{\mathbf{F}}(T)$.
- (v) $\star_1\star_2 = \star_1 \vee (\star_2)^t$ (Section 1.2.3).

Proof. (ii) \Leftrightarrow (iii) It is straightforward.

(i) \Rightarrow (ii) It is clear, since if $(E^{\star_1})^{\star_2} \subsetneq ((E^{\star_1})^{\star_2})^{\star_1}$, then $E^{\star} \subsetneq (E^{\star})^{\star}$, and \star is not a semistar operation.

(ii) \Rightarrow (i) We have only to prove (\star_3) of Definition 1.1, that is $((((E^{\star_1})^{\star_2})^{\star_1})^{\star_2}) = (E^{\star_1})^{\star_2}$. But this follows immediately from the hypothesis, since we have $((((E^{\star_1})^{\star_2})^{\star_1})^{\star_2}) = ((E^{\star_1})^{\star_2})^{\star_2} = (E^{\star_1})^{\star_2}$.

(ii) \Rightarrow (iv) Let $E \in \overline{\mathbf{F}}(T)$. Then $(E^{\star_2})^{\star_1} \subseteq ((E^{\star_1})^{\star_2})^{\star_1} = (E^{\star_1})^{\star_2}$.

(iv) \Rightarrow (ii) Let $E \in \overline{\mathbf{F}}(D)$. Since $E^{\star_1} \in \overline{\mathbf{F}}(D^{\star_1}) \subseteq \overline{\mathbf{F}}(T)$, by the hypothesis we have $((E^{\star_1})^{\star_2})^{\star_1} \subseteq ((E^{\star_1})^{\star_1})^{\star_2} = (E^{\star_1})^{\star_2} \subseteq ((E^{\star_1})^{\star_2})^{\star_1}$. Hence, $(E^{\star_1})^{\star_2} = ((E^{\star_1})^{\star_2})^{\star_1}$.

(i) \Rightarrow (v) It is enough to show that, if \star is a semistar operation on D such that $\star_1 \leq \star$ and $\star_2^l \leq \star$ then $E^{\star_1 \star_2} \subseteq E^{\star}$ for each $E \in \overline{\mathbf{F}}(D)$. So, let $E \in \overline{\mathbf{F}}(D)$. Note that $E^{\star} \in \overline{\mathbf{F}}(D^{\star}) \subseteq \overline{\mathbf{F}}(T)$, since $T \subseteq D^{\star_1} \subseteq D^{\star}$. Then $(E^{\star})^{\star_2}$ is defined and $(E^{\star})^{\star_2} = E^{\star(\star_2)^l} = E^{\star}$, by Proposition 1.5, since $(\star_2)^l \leq \star$. So, $\star_1 \leq \star$ implies $E^{\star_1} \subseteq E^{\star}$ and then $E^{\star_1 \star_2} \subseteq E^{\star \star_2} = E^{\star}$.

(v) \Rightarrow (i) It is obvious, since $\star_1 \vee (\star_2)^l$ is a semistar operation. \square

Example 2.6. Let the notation be like in Example 2.1(1). In this case, \star_1 and \star_2 are both defined on D . We show again that $\star_1 \star_2$ is not a semistar operation by exhibiting an ideal E of D that does not satisfy condition (iv) of Proposition 2.5. Indeed, let $E := xD$, for some nonzero $x \in D$. Then $(xD)^{\star_2 \star_1} = x(D^{\star_{\{R\}}})^{\star_1} = x(D : (D : R)) = x(D : 0) = K$ that clearly is not contained in $(xD)^{\star_1 \star_2} = xD^{\star_{\{R\}}} = xR$.

Proposition 2.7. *Let D be an integral domain and \star_1 a semistar operation on D . Let T be an overring of D , with $T \subseteq D^{\star_1}$, and \star_2 a semistar operation on T . Suppose that $\star_1 \star_2$ is a semistar operation on D .*

(1) *If \star_1, \star_2 are of finite type, then $\star_1 \star_2$ is of finite type.*

(2) *If \star_1, \star_2 are stable, then $\star_1 \star_2$ is stable.*

(3) *If $\star_1 = \widetilde{\star}_1$ and $\star_2 = \widetilde{\star}_2$ then $\star_1 \star_2 = \widetilde{\star}_1 \widetilde{\star}_2$.*

(4) *If \star_1 and \star_2 are spectral and of finite type, then $\star_1 \star_2$ is spectral of finite type.*

Proof. (1) Let $E \in \overline{\mathbf{F}}(D)$, $x \in (E^{\star_1})^{\star_2}$. Since \star_2 is of finite type, there exists $F \in \mathbf{f}(D^{\star_1})$, $F \subseteq E^{\star_1}$, such that $x \in F^{\star_2}$. Let $F = x_1 D^{\star_1} + \dots + x_n D^{\star_1}$. Since $F \subseteq E^{\star_1}$ and \star_1 is of finite type, there exists $G_1, \dots, G_n \subseteq E$, $G_i \in \mathbf{f}(D)$, $i = 1, \dots, n$, such that $x_1 \in G_1^{\star_1}, \dots, x_n \in G_n^{\star_1}$. It follows that $F \subseteq G_1^{\star_1} + \dots + G_n^{\star_1} \subseteq (G_1^{\star_1} + \dots + G_n^{\star_1})^{\star_1} = (G_1 + \dots + G_n)^{\star_1}$. Let $G = G_1 + \dots + G_n$. Then, $F \subseteq G^{\star_1}$ implies $F^{\star_2} \subseteq (G^{\star_1})^{\star_2}$. It follows that $x \in (G^{\star_1})^{\star_2}$ with $G \in \mathbf{f}(D)$, $G \subseteq E$. Hence, $\star_1 \star_2$ is of finite type.

(2) It is straightforward.

(3) It follows from (1) and (2), since a semistar operation \star coincides with $\widetilde{\star}$

if and only if \star is stable and of finite type (Lemma 1.32(2)(i) \Leftrightarrow (ii)).

(4) It follows immediately from (3), since a semistar operation \star is spectral and of finite type if and only if $\star = \tilde{\star}$ (Lemma 1.32(2)(i) \Leftrightarrow (iii)). \square

Remark 2.8. (1) The converse of each statement in Proposition 2.7 is not true in general. That is, for $\star_1\star_2$ to be of finite type (resp. stable, stable of finite type) it is not necessary that \star_1, \star_2 are of finite type (stable, stable of finite type). For example, if \star_1 and \star_2 are both defined on D and $\star_1 \leq \star_2$, if \star_2 is of finite type (stable, stable of finite type) then $\star_1\star_2$ and $\star_2\star_1$ are of finite type (stable, stable of finite type) without further conditions on \star_1 (since $\star_1\star_2 = \star_2\star_1 = \star_2$).

(2) In the proof of Proposition 2.7, we do not use the fact that $\star_1\star_2$ is a semistar operation. So, even if $\star_1\star_2$ is not a semistar operation, we have that $E^{\star_1\star_2} = \bigcup\{F^{\star_1\star_2} \mid F \in \mathbf{f}(D)\}$, for each $E \in \overline{\mathbf{F}}(D)$, when \star_1 and \star_2 are of finite type, and $(E \cap F)^{\star_1\star_2} = E^{\star_1\star_2} \cap F^{\star_1\star_2}$, for each $E, F \in \overline{\mathbf{F}}(D)$, when \star_1 and \star_2 are stable.

Example 2.9. Let D be an integral domain, P and Q incomparable prime ideals of D . Let $\star_1 := \star_{\{P\}}$ and $\star_2 := \star_{\{Q\}}$. Consider $\star := \star_{\{P\}}\star_{\{Q\}}$. From Example 2.1, it follows that \star is a semistar operation. Since both \star_1 and \star_2 are spectral and of finite type, \star must be spectral and of finite type (Proposition 2.7(4)). Indeed, it is easy to check that $D_P D_Q = D_S$, the localization of D at the multiplicative set $S := \{ab \mid a \in D \setminus P, b \in D \setminus Q\}$. Then, $\star = \star_{\{D_S\}}$, that is a semistar operation of finite type. Moreover, since \star is of finite type and D_S is flat over D , it follows by Proposition 1.13 and Lemma 1.32 that \star is spectral (defined by the set of the primes H of D such that $H \cap S = \emptyset$).

2.2 Star operations on overrings and semistar operations

In the following, we recall some properties and prove new ones of the semistar operations defined in Section 1.5.

First we state a result for stable semistar operations that is a consequence of a well-known fact about localizing systems:

Proposition 2.10. *Let D be an integral domain and \star a stable semistar operation on D . Then, P^\star is a prime ideal of D^\star , for each prime ideal P of D such that $P^\star \neq D^\star$. Moreover, $D_{P^\star}^\star = D_P$.*

Proof. It follows from Theorem 1.27(2) and [9, Theorem 1.1]. \square

Proposition 2.11. *Let D be an integral domain, T an overring of D , $\iota : D \hookrightarrow T$ the canonical embedding of D in T , \star a semistar operation on D and \star_ι the semistar operation on T defined as in Section 1.5.*

- (1) If \star is of finite type, then \star_ι is of finite type.
- (2) If \star is stable then \star_ι is stable.
- (3) If \star is cancellative on D , then \star_ι is cancellative on T .
- (4) If \star is a.b. then \star_ι is a.b.
- (5) Assume $T = D^\star$ or $T \in \mathbf{f}(D)$. If \star is e.a.b. then \star_ι is e.a.b.
- (6) Assume $T = D^\star$. If \star is spectral, then \star_ι is spectral.
- (7) If $T = D^\star$ then \star_ι is a (semi)star operation on T .

Proof. (1) and (7) are in [27, Proposition 2.8]

(2) is straightforward.

(3) is straightforward, since, on T -modules, \star and \star_ι coincide.

(4) Let $E \in \mathbf{f}(T)$ and $F, G \in \overline{\mathbf{f}}(T)$ with $(EF)^{\star_\iota} = (EG)^{\star_\iota}$. There exists $E_0 \in \mathbf{f}(D)$ such that $E_0T = E$. Then, $(E_0TF)^\star = (E_0TG)^\star$ and, since \star is a.b. and $E_0 \in \mathbf{f}(D)$, $F^{\star_\iota} = F^\star = (FT)^\star = (GT)^\star = G^\star = G^{\star_\iota}$. Hence, \star_ι is an a.b. semistar operation.

(5) Let $E, F, G \in \mathbf{f}(T)$ such that $(EF)^{\star_\iota} = (EG)^{\star_\iota}$. Then, there exists $E_0, F_0, G_0 \in \mathbf{f}(D)$ with $E = E_0T, F = F_0T$ and $G = G_0T$. It follows $(E_0F_0T)^{\star_\iota} = (E_0G_0T)^{\star_\iota}$, that is, $(E_0F_0T)^\star = (E_0G_0T)^\star$. If $T \in \mathbf{f}(D)$, then $F_0T, G_0T \in \mathbf{f}(D)$, thus, since \star is e.a.b., we obtain $(F_0T)^\star = (G_0T)^\star$. Hence, $F^{\star_\iota} = G^{\star_\iota}$. On the other hand, if $T = D^\star$, then $(E_0F_0D^\star)^\star = (E_0G_0D^\star)^\star$ implies $(E_0F_0)^\star = (E_0G_0)^\star$ and thus $F_0^\star = G_0^\star$. So, $F^{\star_\iota} = (F_0D^\star)^\star = F_0^\star = G_0^\star = (G_0D^\star)^\star = G^{\star_\iota}$. Hence, in any case, \star_ι is an e.a.b. semistar operation.

(6) Let $\star = \star_\Delta$, for some $\Delta \subseteq \text{Spec}(D)$. We want to prove that $\star_\iota = \star_{\Delta^\star}$, where $\Delta^\star := \{P^\star : P \in \Delta\}$. It is clear that, for $P \in \Delta$, $P^\star \neq D^\star$, so, from Proposition 2.10, it follows that P^\star is prime and that $(D^\star)_{P^\star} = D_P$. Then, if $E \in \overline{\mathbf{f}}(D^\star)$, we have $E^{\star_\iota} = \bigcap \{ED_P \mid P \in \Delta\} = \bigcap \{ED_{P^\star} \mid P \in \Delta\} = E^{\star_{\Delta^\star}}$. Hence, \star_ι is spectral. \square

Proposition 2.12. *Let D be an integral domain, \star a semistar operation on D and T an overring of D . Let ι be the canonical embedding of D in T . Then:*

- (1) $(\star_f)_\iota \leq (\star_\iota)_f$.
- (2) If $T = D^\star$ then $(\star_f)_\iota = (\star_\iota)_f$

Proof. (1) Since $\star_f \leq \star$, we have $(\star_f)_\iota \leq \star_\iota$ (Lemma 1.36). Moreover, $(\star_f)_\iota$ is of finite type (Proposition 2.11(1)), then $(\star_f)_\iota \leq (\star_\iota)_f$ (Proposition 1.7(2)).

(2) Let $E \in \overline{\mathbf{f}}(T)$ and let $x \in E^{(\star_f)_\iota}$. Then, there exists $F \in \mathbf{f}(T)$, say $F = a_0T + a_1T + \dots + a_nT$, with $a_0, a_1, \dots, a_n \in K$, such that $F \subseteq E$ and

$x \in F^{\star\iota} = F^{\star}$. Consider $F_0 := a_0D + a_1D + \dots + a_nD$. We have $F_0 \in \mathbf{f}(D)$, $F_0 \subseteq E$ and $F_0^{\star} = (F_0D)^{\star} = (F_0D^{\star})^{\star} = (F_0T)^{\star} = F^{\star}$. So, $x \in F_0^{\star}$. It follows that $x \in E^{\star_f} = E^{(\star_f)\iota}$. Hence, $(\star_{\iota})_f \leq (\star_f)_{\iota}$. Now, (1) gives the thesis. \square

We study now the semistar operation \star^{ι} defined in Proposition 1.35(2), when \star is a semistar operation on an overring T of an integral domain D . We recall here (Example 2.1(2)) that this semistar operation (defined on D) is exactly the composition of two semistar operations, the extension $\star_{\{T\}}$ to T and the semistar operation \star .

Proposition 2.13. *Let D be an integral domain, T an overring of D , $\iota : D \hookrightarrow T$ the canonical embedding of D in T , \star a semistar operation on T and \star^{ι} the semistar operation on D defined as in Proposition 1.35(2). Then:*

- (1) $(\star^{\iota})_f = (\star_f)^{\iota}$ (in particular, if \star is of finite type, then \star^{ι} is of finite type).
- (2) If \star is cancellative, then \star^{ι} is cancellative.
- (3) If \star is e.a.b. (resp., a.b.) then \star^{ι} is e.a.b. (resp., a.b.).

Proof. (1) See [26, Lemma 3.1].

(2) Let $E, F, G \in \overline{\mathbf{F}}(D)$ such that $(EF)^{\star^{\iota}} = (EG)^{\star^{\iota}}$. Then, $(ETFT)^{\star} = (ETGT)^{\star}$, and, since \star is cancellative, we obtain $(FT)^{\star} = (GT)^{\star}$, that is, $F^{\star^{\iota}} = G^{\star^{\iota}}$.

(3) See [27, Proposition 2.9(2)] \square

Remark 2.14. (1) Note that the fact that, with the notation of Proposition 2.13, if \star is a semistar operation of finite type, then \star^{ι} is a semistar operation of finite type can be proven also considering that, by Example 2.1(2), \star^{ι} is the composition of the semistar operation of finite type $\star_{\{T\}}$ and of \star and thus it is of finite type by Proposition 2.7(1).

(2) In Proposition 2.11 we have shown that the map $(-)^{\iota} : \mathbf{SStar}(D) \rightarrow \mathbf{SStar}(T)$ preserves all the main properties of a semistar operation, that is, the finite character, the stability, the property of being spectral, a.b. and, under some conditions, e.a.b. The map $(-)^{\iota} : \mathbf{SStar}(T) \rightarrow \mathbf{SStar}(D)$ does not behave so well: in fact, while, as we have seen in Proposition 2.13, the finite character and the properties of being a.b. and e.a.b. are preserved, the stability and the properties of being spectral are not preserved. For instance, with the notation of Proposition 2.13, take T not flat over D , and $\star = d_T$, the identity semistar operation of T . Clearly \star is spectral (defined by the set of all maximal ideals of T) and then it is stable, but $\star^{\iota} = \star_{\{T\}}$ is not stable (and then not spectral), by Proposition 1.13.

For the next Proposition, see for example [65, Lemma 45] or [26, Example 2.1 (e)].

Proposition 2.15. *Let D be an integral domain, T an overring of D , $\iota : D \hookrightarrow T$ the canonical embedding of D in T . Then:*

- (1) *For each semistar operation $*$ on T , $(\star)_\iota = *$ (that is, $(-)_\iota \circ (-)^\iota = \text{id}_{\mathbf{SStar}(T)}$).*
- (2) *For each semistar operation \star on D , $\star \leq (\star)_\iota$.* □

Proof. (1) Let $E \in \overline{\mathbf{F}}(T)$. Then, $E^{(\star)_\iota} = E^{\star_\iota} = (ET)^* = E^*$.

(2) Let $E \in \overline{\mathbf{F}}(D)$. Then $E^{(\star)_\iota} = (ET)^{\star_\iota} = (ET)^* \supseteq E^*$. □

It follows that the map $(-)_\iota : \mathbf{SStar}(D) \rightarrow \mathbf{SStar}(T)$ is surjective (that is, each semistar operation on T is an ‘‘ascent’’ of a semistar operation on D) and the map $(-)_\iota : \mathbf{SStar}(T) \rightarrow \mathbf{SStar}(D)$ is injective.

Denote by $\mathbf{SStar}(D, T)$ the set of all semistar operations \star on D such that $D^\star = T$. From Proposition 2.11(7) it follows that $\{\star_\iota | \star \in \mathbf{SStar}(D, T)\}$ is a subset of $(\mathbf{S})\mathbf{Star}(T)$. We will denote by $(-)_\iota^T$ the map $(-)_\iota$ restricted to $\mathbf{SStar}(D, T)$, i.e. $(-)_\iota^T : \mathbf{SStar}(D, T) \rightarrow (\mathbf{S})\mathbf{Star}(T)$. Analogously, we denote by $(-)_T^\iota$ the map $(-)_\iota$ restricted to $(\mathbf{S})\mathbf{Star}(T)$, i.e. $(-)_T^\iota : (\mathbf{S})\mathbf{Star}(T) \rightarrow \mathbf{SStar}(D, T)$. We prove that these maps are one the inverse of the other.

Proposition 2.16. *Let D be an integral domain, T an overring of D , $\iota : D \hookrightarrow T$ the canonical embedding of D in T . Let $(-)_\iota^T$ and $(-)_T^\iota$ be defined as above. Then:*

- (1) *For each $\star \in \mathbf{SStar}(D, T)$, $(\star)_\iota = \star$ (i.e. $(-)_T^\iota \circ (-)_\iota^T = \text{id}_{\mathbf{SStar}(D, T)}$).*
- (2) *For each $*$ in $(\mathbf{S})\mathbf{Star}(T)$, $(\star)_\iota = *$ (i.e. $(-)_\iota^T \circ (-)_T^\iota = \text{id}_{(\mathbf{S})\mathbf{Star}(T)}$).*
- (3) *The maps $(-)_\iota^T$ and $(-)_T^\iota$ are bijective.*

Proof. (1) Suppose $\star \in \mathbf{SStar}(D, T)$ and let $E \in \overline{\mathbf{F}}(D)$. We have $E^{(\star)_\iota} = (ED^\star)^{\star_\iota} = (ED^\star)^* = (ED)^\star = E^\star$, that is, $(\star)_\iota = \star$.

(2) It is immediate by Proposition 2.15.

(3) Straightforward from (1) and (2). □

It follows that each semistar operation on a domain D can be decomposed, in a canonical way, as the composition of two semistar operations, more precisely, the first semistar operation is $\star_{\{T\}}$ for some overring T of D and the second one is a (semi)star operation on T .

Corollary 2.17. *Let D be an integral domain.*

- (1) *Let \star be a semistar operation on D , let $T = D^\star$ and ι the canonical embedding of D in T . Then \star is the composition of the semistar operation $\star_{\{T\}}$ and of a (semi)star operation $*$ on T , i.e. $\star = \star_{\{T\}}*$ (equivalently, $\star = \star^\iota$, for some (semi)star operation $*$ on T).*

- (2) Let T be an overring of D and ι the canonical embedding of D in T .
Then $\mathbf{SStar}(D, T) = \{\star^\iota \mid \star \in (\mathbf{S})\mathbf{Star}(T)\}$.

Proof. (1) Take $\star := \star_\iota$ and apply Proposition 2.16(1).

(2) follows immediately from Proposition 2.16(3). \square

Remark 2.18. Consider an overring R of D and let ι be the canonical embedding of D in R . Let T be an overring of R . Let $\star \in \mathbf{SStar}(R, T)$. Then, the descent \star^ι of \star to D is in $\mathbf{SStar}(D, T)$. Then $\{\star^\iota \mid \star \in \mathbf{SStar}(R, T)\} \subseteq \mathbf{SStar}(D, T)$. We want to show the other inclusion, that is, that each semistar operation in $\mathbf{SStar}(D, T)$ is the descent of a semistar operation in $\mathbf{SStar}(R, T)$. To do this, denote by λ the canonical embedding of D in T and by μ the canonical embedding of R in T . Let $\star \in \mathbf{SStar}(D, T)$. Consider the semistar operation $(\star_\lambda)^\mu \in \mathbf{SStar}(R, T)$ (it is the descent to R of the ascent of \star to T). We show that the descent of $(\star_\lambda)^\mu$ to D coincides with \star . Let $E \in \overline{\mathbf{F}}(D)$. Then $E^{((\star_\lambda)^\mu)^\iota} = (ER)^{(\star_\lambda)^\mu} = (ERT)^{\star_\lambda} = (ET)^{\star_\lambda} = E^{(\star_\lambda)^\lambda} = E^\star$ (the last equality follows from Proposition 2.16(1), since $\star \in \mathbf{SStar}(D, T)$). It follows that \star is the descent of a semistar operation in $\mathbf{SStar}(R, T)$. Hence, $\mathbf{SStar}(D, T) = \{\star^\iota \mid \star \in \mathbf{SStar}(R, T)\}$.

We deduce that $\bigcup_{R \subseteq T} \mathbf{SStar}(D, T) = \{\star^\iota \mid \star \in \mathbf{SStar}(R)\}$. In particular, if D has a unique minimal overring R , then $\mathbf{SStar}(D) = (\mathbf{S})\mathbf{Star}(D) \cup \{\star^\iota \mid \star \in \mathbf{SStar}(R)\}$.

Moreover, consider $\star \in \mathbf{SStar}(R, T)$. We have seen that $\star^\iota \in \mathbf{SStar}(D, T)$. By Proposition 2.15(1), we have that $\star = (\star^\iota)_\iota$, that is, \star is the ascent to R of the semistar operation $\star^\iota \in \mathbf{SStar}(D, T)$. It follows that $\mathbf{SStar}(R, T) = \{\star_\iota \mid \star \in \mathbf{SStar}(D, T)\}$.

Then, there is a bijection between $\mathbf{SStar}(D, T)$ and $\mathbf{SStar}(R, T)$. Thus, $\text{Card}(\mathbf{SStar}(R, T)) = \text{Card}(\mathbf{SStar}(D, T))$. Moreover, $\text{Card}(\mathbf{SStar}(R)) \leq \text{Card}(\mathbf{SStar}(D))$ and, since, for example, $d_D \in (\mathbf{S})\mathbf{Star}(D)$ is not a descent from a semistar on R , we have that, if $\text{Card}(\mathbf{SStar}(R)) < \infty$, then $\text{Card}(\mathbf{SStar}(R)) < \text{Card}(\mathbf{SStar}(D))$.

Since by Proposition 2.11(1) and Proposition 2.13(1), the finite type property is preserved by the maps $(-)_\iota$ and $(-)^\iota$, we obtain a result similar to Corollary 2.17 for finite type semistar operations.

Corollary 2.19. Let D be an integral domain, T an overring of D .

- (1) Let \star be a semistar operation of finite type on D , let $T = D^\star$ and ι the canonical embedding of D in T . Then \star is the composition of the semistar operation $\star_{\{T\}}$ and of a (semi)star operation of finite type \ast on T , i.e. $\star = \star_{\{T\}}\ast$ (equivalently, $\star = \star^\iota$, for some (semi)star operation of finite type \ast on T).
- (2) Let T be an overring of D and ι the canonical embedding of D in T .
Then $\mathbf{SStar}_f(D, T) = \{\star^\iota \mid \star \in (\mathbf{S})\mathbf{Star}_f(T)\}$. \square

Corollary 2.20. *Let D be an integral domain, T an overring of D . Let \star be a semistar operation on D , such that $D^\star = T$ (that is, $\star \in \mathbf{SStar}(D, T)$). Then:*

$$(1) \star_{\{T\}} \leq \star \leq v(T).$$

$$(2) \star_{\{T\}} \leq \star_f \leq t(T).$$

Proof. (1) By Corollary 2.17(1), there exists a (semi)star operation $*$ on T such that $\star = *^t$. Since $*$ is a (semi)star operation on T , $d_T \leq * \leq v_T$ (Lemma 1.11). It follows (Lemma 1.36) that $\star_{\{T\}} = (d_T)^t \leq *^t = \star \leq (v_T)^t = v(T)$.

(2) Use the same argument of (1), applying Corollary 2.19(1) and Example 1.12. \square

In the following, we will denote by $\mathcal{O}(D)$ the set of all overrings of an integral domain D . Since it is clear that $\mathbf{SStar}(D) = \bigcup \{ \mathbf{SStar}(D, T) \mid T \in \mathcal{O}(D) \}$ and $\mathbf{SStar}_f(D) = \bigcup \{ \mathbf{SStar}_f(D, T) \mid T \in \mathcal{O}(D) \}$, and these are unions of disjoint sets, we have the following theorem.

Theorem 2.21. *Let D be an integral domain. For each $T \in \mathcal{O}(D)$, let ι_T be the canonical embedding of D in T .*

(1) *The map $\star \mapsto \star_{\iota_D^*}$ establishes a bijection between the set $\mathbf{SStar}(D)$ and the set $\bigcup \{ (\mathbf{S})\mathbf{Star}(T) \mid T \in \mathcal{O}(D) \}$.*

(2) $\mathbf{SStar}(D) = \bigcup \{ \{ \star^t \mid \star \in (\mathbf{S})\mathbf{Star}(T) \} \mid T \in \mathcal{O}(D) \}$.

(3) *The restriction of the map in (1) establishes a bijection between the set $\mathbf{SStar}_f(D)$ and the set $\bigcup \{ (\mathbf{S})\mathbf{Star}_f(T) \mid T \in \mathcal{O}(D) \}$.*

(4) $\mathbf{SStar}_f(D) = \bigcup \left\{ \{ \star^t \mid \star \in (\mathbf{S})\mathbf{Star}_f(T) \} \mid T \in \mathcal{O}(D) \right\}$. \square

Note that the bijection defined in Theorem 2.21 holds between the set of semistar operations on D and the set of (semi)star operations on the overrings of D (which, in general, contains properly the set of star operations on the overrings under the canonical embedding $\mathbf{Star}(T) \hookrightarrow (\mathbf{S})\mathbf{Star}(T)$, $\star \mapsto \star_e$ (see Section 1.2.1). We have already noted in Section 1.2.1 that, in general, a star operation can be extended to a semistar operation (clearly a (semi)star operation) in different ways. We have also remarked that the extension is unique for conducive domains, or for the case of star operations of finite type (which extend uniquely to semistar operations of finite type). More precisely, we have the following proposition.

Proposition 2.22. *Let D be an integral domain.*

- (1) The map $\star \mapsto \star|_{\mathbf{F}(D)}$ (where $\star|_{\mathbf{F}(D)}$ is the star operation given by the restriction of \star to $\mathbf{F}(D)$, Remark 1.4) establishes a bijection between the set $(\mathbf{S})\mathbf{Star}_f(D)$ of all (semi)star operations of finite type on D and the set $\mathbf{Star}_f(D)$ of all star operations of finite type on D .
- (2) If D is conducive, the map $\star \mapsto \star|_{\mathbf{F}(D)}$ establishes a bijection between the set $(\mathbf{S})\mathbf{Star}(D)$ of all (semi)star operations on D and the set $\mathbf{Star}(D)$ of all star operations on D . \square

Corollary 2.23. *Let D be an integral domain. For each $T \in \mathcal{O}(D)$, let ι_T be the canonical embedding of D in T .*

- (1) The map $\star \mapsto (\star|_{D^*})|_{\mathbf{F}(D)}$ establishes a bijection between the set $\mathbf{SStar}_f(D)$ and the set $\bigcup\{\mathbf{Star}_f(T) \mid T \in \mathcal{O}(D)\}$.
- (2) If D is conducive, the map $\star \mapsto (\star|_{D^*})|_{\mathbf{F}(D)}$ establishes a bijection between the set $\mathbf{SStar}(D)$ and the set $\bigcup\{\mathbf{Star}(T) \mid T \in \mathcal{O}(D)\}$.

Proof. (1) Apply Theorem 2.21(3) and Proposition 2.22(1).

(2) Apply Theorem 2.21(1) and Proposition 2.22(2). \square

As a direct consequence of Theorem 2.21 and Proposition 2.11, we have the following corollary concerning the properties preserved by the map $(-)_\iota$.

Corollary 2.24. *Let D be an integral domain. Let \star be a stable (resp. cancellative, a.b., e.a.b., spectral) semistar operation on D , let $T = D^*$ and ι the canonical embedding of D in T . Then \star is the composition of the semistar operation $\star|_{\{T\}}$ and of a stable (resp., cancellative; a.b.; e.a.b.; spectral) (semi)star operation $*$ on T , i.e. $\star = \star|_{\{T\}} \circ *$ [equivalently, $\star = \star^\iota$, for some stable (resp., cancellative; a.b.; e.a.b.; spectral) (semi)star operation $*$ on T]. \square*

From the previous corollary, we deduce only that, for each semistar operation \star on an integral domain D with a certain property, there exists a particular (semi)star operation $*$ induced by \star on an overring T of D (more precisely $*$ $= \star|_{\{T\}}$, where ι is the canonical embedding of D in T), with the same property, such that \star is the composition of the extension to T and this (semi)star operation. We cannot deduce that, taking a (semi)star operation $*$ on an overring T of D with a certain property, the composition of the extension to T and $*$ has the same property. This is true for the properties preserved by the map $(-)_\iota$ (see Proposition 2.13). In these cases, we have a bijection between the semistar operations on D with a certain property and the (semi)star operations on the overrings of D with the same property. This is the content of the following theorem.

Theorem 2.25. *There is a canonical bijection between the set of all cancellative (resp. a.b., e.a.b.) semistar operations on D and the set of all cancellative (resp. a.b., e.a.b.) (semi)star operations on the overrings of D .*

Proof. It follows from Proposition 2.11 (3),(4),(5), Proposition 2.13(2),(3) and Proposition 2.16. \square

By the previous considerations the study of semistar operations (of finite type, cancellative, a.b., e.a.b.) on an integral domain D is equivalent to the study of all (semi)star operations (of finite type, cancellative, a.b., e.a.b.) on D and on each of its overrings.

We have shown that the properties for which there is a bijection between the semistar operations with that property and the (semi)star operations with that properties on overrings are exactly the properties preserved by the map $(-)^{\iota}$. For example, we have already shown (Remark 2.14(2)) that if \star is a semistar operation on an integral domain D , if T is an overring of D , and ι the canonical embedding of D in T , then, it is not true, in general, that \star^{ι} is stable (resp. spectral), when \star is a stable (resp. spectral) semistar operation on D .

Next goal is to study when a canonical bijection also holds for stable and for spectral semistar operations.

Proposition 2.26. *Let D be an integral domain, T an overring of D and ι the canonical embedding of D in T . Let \star be a semistar operation on T .*

- (1) *If \star is stable and T is a flat overring of D , then \star^{ι} is stable.*
- (2) *If T is a flat overring of D , then the map $(-)^{\iota}_T$ induces a bijection between the set of all stable semistar operations \star on D such that $T = D^{\star}$ and the set of all stable (semi)star operations on T .*

Proof. (1) Since \star is stable by the hypothesis and $\star_{\{T\}}$ is stable because T is flat (Proposition 1.13), the composition \star^{ι} (Example 2.1(2)) is stable by Proposition 2.7(2).

(2) It follows from (1) and Proposition 2.11(2). \square

Remark 2.27. We have shown in Proposition 2.26(1) that if T is an overring of D , and \star is a stable semistar operation on T , the flatness of T over D is a sufficient condition for \star^{ι} to be stable. This condition is not necessary: in fact, let \mathcal{F} be a localizing system on D , such that $D_{\mathcal{F}}$ is not flat over D . Let ι be the canonical embedding of D in $D_{\mathcal{F}}$. Take the semistar operation $\star_{\mathcal{F}}$ on D (as defined in Section 1.3) and consider the (semi)star operation $\star := (\star_{\mathcal{F}})_{\iota}$ on $D_{\mathcal{F}} (= D^{\star_{\mathcal{F}}})$. By Proposition 2.11, \star is stable, and, by Proposition 2.16(1), $\star^{\iota} = \star_{\mathcal{F}}$. Then, \star^{ι} is stable, but $D_{\mathcal{F}}$, by the choice of \mathcal{F} , is not flat over D .

We recall [20, Section 4] that if D is an integral domain, T an overring of T , $\star \in \mathbf{SStar}(D)$ and $\star' \in \mathbf{SStar}(T)$. We say that T is (\star, \star') -flat over D if $(Q \cap D)^{\star_{\mathcal{F}}} \neq D^{\star}$ and $D_{Q \cap D} = T_Q$, for each quasi- $\star'_{\mathcal{F}}$ -prime ideal Q of T .

Proposition 2.28. *Let D be an integral domain, T an overring of D and ι the canonical embedding of D in T . Let $\star = \star_\Delta$ be a spectral semistar operation on T , defined by $\Delta \subseteq \text{Spec}(T)$.*

- (1) *If $T_P = D_{P \cap D}$ (in particular if T is flat over D , [73, Theorem 2]), for each $P \in \Delta$, then \star^ι is spectral.*
- (2) *If T is (\star^ι, \star) -flat over D , then \star^ι is spectral.*
- (3) *If one of the hypotheses in (1) or in (2) holds, then the map $(-)_\iota^T$ induces a bijection between the set of all spectral semistar operations on D such that $T = D^\star$ and the set of all spectral (semi)star operations on T .*

Proof. (1) Let $E \in \overline{\mathbf{F}}(D)$. Then, $E^{\star^\iota} = (ET)^\star = \bigcap \{(ET)T_P (= ET_P) \mid P \in \Delta\} = \bigcap \{ED_{P \cap D} \mid P \in \Delta\}$, that is, $\star^\iota = \star_{\Delta'}$, where $\Delta' = \{P \cap D \mid P \in \Delta\}$.
(2) For each $P \in \Delta$, P is a quasi- \star -prime ideal (Section 1.2.4). So, by definition of (\star^ι, \star) -flatness, $T_P = D_{P \cap D}$. Then, apply (1).
(3) It is clear, because in these cases, the map $(-)_\iota^T$ preserves the spectral property. \square

Corollary 2.29. *Let D be an integral domain. For each $T \in \mathcal{O}(D)$, let ι_T be the canonical embedding of D in T . Then the following are equivalent:*

- (i) *D is a Prüfer domain.*
- (ii) *The map $\star \mapsto \star_{\iota_{D^\star}}$ establishes a bijection between the set of all stable semistar operations on D and the set of all stable (semi)star operations on the overrings of D .*
- (iii) *The map $\star \mapsto \star_{\iota_{D^\star}}$ establishes a bijection between the set of all spectral semistar operations on D and the set of all spectral (semi)star operations on the overrings of D .*

Proof. (i) \Rightarrow (ii),(iii) It follows from the fact that each overring of a Prüfer domain is flat, Proposition 2.26(2) and Proposition 2.28(3).
(ii),(iii) \Rightarrow (i) In both cases, the fact that the bijection holds implies that the semistar operation given by the extension to an overring of D is stable for each overring of D and therefore each overring of D is flat, by Proposition 1.13. Then, D is a Prüfer domain by [24, Theorem 1.1.1]. \square

2.3 Some applications

In this section, we give some applications of the results proven in Section 2.2.

2.3.1 Semistar operations on valuation domains

We start discussing some properties of semistar operations on valuation domains. Some of the results we obtain have already been proven, but only for finite dimensional domains, in [61], [57], [64] and [63]. We generalize several statements without restrictions on the dimension, as corollaries of the results proven in the previous section.

First we recall a result about star operations [4, Proposition 12] (see also [41, Theorem 15.3] for an analogous result in the context of ideal systems):

Proposition 2.30. *Let V be a valuation domain, with maximal ideal M .*

- (1) *If $M^2 \neq M$, then each ideal of V is divisorial, that is, $\mathbf{Star}(V) = \{d\}$.*
- (2) *If $M^2 = M$, then $\mathbf{Star}(V) = \{d, v\}$. □*

Since a valuation domain is conducive [19, Proposition 2.1], by applying this result and Corollary 2.23, we have the following Proposition:

Proposition 2.31. *Let P be a prime ideal of a valuation domain V .*

- (1) *If $P \neq P^2$, then $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}}\}$.*
- (2) *If $P = P^2$, then $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}}, v(V_P)\}$ (where $v(V_P)$ is defined as in Example 2.3). More precisely, if $\star \in \mathbf{SStar}(V, V_P)$ is a semistar operation such that $P^\star = P$, then $\star = \star_{\{P\}}$, otherwise $\star = v(V_P)$ (and in this case $P^\star = V^\star = V_P$).*
- (3) *$\mathbf{SStar}(V)$ coincides with*

$$\left(\bigcup \{ \star_{\{P\}} \mid P \in \text{Spec}(V) \} \right) \cup \left(\bigcup \{ v(V_Q) \mid Q \in \text{Spec}(V), Q \neq (0), Q^2 = Q \} \right).$$

Proof. (1) Since $(PV_P) \neq (PV_P)^2$, we have $\mathbf{Star}(V_P) = \{d_{V_P}\}$ by Proposition 2.30(1). Then, by Corollary 2.17, $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}}\}$.

(2) Apply the same argument, using Proposition 2.30(2). The final statement follows from the fact that $P^{v(V_P)} = (V_P : (V_P : P)) = V_P$ (since a maximal ideal of a Prüfer domain is divisorial if and only if it is invertible, [24, Lemma 4.1.8]).

(3) It is immediate, since $\mathbf{SStar}(D) = \bigcup \{ \mathbf{SStar}(D, T) \mid T \in \mathcal{O}(D) \}$, as we have already observed. □

In particular, for finite dimensional valuation domains, we reobtain the following results (cf. [57, Theorem 4] and [61, Theorem 4]).

Corollary 2.32. *Let (V, M) be an n -dimensional valuation domain. Then*

- (1) *$\text{Card}(\mathbf{SStar}(V)) = n + 1 + \text{Card}(\{P \in \text{Spec}(V) \mid P^2 \neq P\})$.*
- (2) *V is discrete if and only if $\text{Card}(\mathbf{SStar}(V)) = n + 1$. □*

Remark 2.33. Note that, if V is a valuation domain and $P \in \text{Spec}(V)$, the semistar operations $v(P)$ and $v(V_P)$ in general do not coincide. More precisely, Proposition 2.31 shows that $v(P) = v(V_P)(= \star_{\{V_P\}})$ if and only if $P^2 \neq P$. We will show in Lemma 2.53 that, in general, if D is an integral domain and I a nonzero ideal of D then $v(I) = v((I : I))$ if and only if I is divisorial in $(I : I)$.

In the following we study the semistar operations of type $v(I)$ (as introduced in Section 1.2.5) for some ideal I of a valuation domain V . In particular, we prove that all semistar operations on valuation domains are of this type and we characterize the ideals I of V such that $v(I)$ is the extension to an overring of V .

First we prove an easy lemma after recalling by [48] that an ideal I of an integral domain D is *m-canonical* if $(I : (I : J)) = J$ for each nonzero ideal J of D . Note that I is m-canonical if and only if $v(I) = d_e$.

Lemma 2.34. *Let D be a conducive domain and I a nonzero ideal of D . The following are equivalent:*

- (i) I is m-canonical in $(I : I)$.
- (ii) $v(I) = \star_{\{(I:I)\}}$.

Proof. (i) \Rightarrow (ii) Let ι be the canonical embedding of D in $(I : I)$. From the hypothesis in (i), $v(I)_\iota$ coincides with the identity on integral ideals of $(I : I)$. Since D is conducive, it follows that $v(I)_\iota = d_{(I:I)}$. So, $\star_{\{(I:I)\}} = (d_{(I:I)})^\iota = ((v(I))_\iota)^\iota = v(I)$, using the fact that $D^{v(I)} = (I : I)$ and Proposition 2.16(1). (ii) \Rightarrow (i) If ι is the canonical embedding of D in $(I : I)$, we have $(v(I))_\iota = (\star_{\{(I:I)\}})_\iota = d_{(I:I)}$ and so, for each ideal J of $(I : I)$, we have $(I : (I : J)) = J$, that is, I is m-canonical in $(I : I)$. \square

Proposition 2.35. *Let V be a valuation domain with quotient field K .*

- (1) $\star_{\{P\}} = v(P)$ for each $P \in \text{Spec}(V)$.
- (2) If $\star \in \mathbf{SStar}(V)$, there exists a nonzero ideal I of D such that $\star = v(I)$.
- (3) Let I be a nonzero ideal of V . Then $v(I) = \star_{\{P\}}$ if and only if $I = xP$ for some $x \in K$.

Proof. (1) It is easy to see that $(P : P) = V_P$. So, let $P \neq P^2$. Then, by Proposition 2.31(1), it follows immediately that $v(P) = \star_{\{P\}}$. If $P = P^2$, since $P^{v(P)} = P$, it follows by Proposition 2.31(2) that $v(P) = \star_{\{P\}}$. (2) If $\star = \star_{\{P\}}$ for some prime ideal P of V , then $\star = v(P)$, by (1). So, let $\star = v(V_P)$. Then $V^\star = V_P$ is a fractional ideal of V (since V is conducive), that is, there exist an ideal I of V and $x \in V$ such that $V_P = x^{-1}I$. It follows by Lemma 1.18(5) that $\star = v(I)$.

(3) The ‘if’ part follows from (1) and Lemma 1.18(5). So, suppose $v(I) = \star_{\{P\}} (= \star_{\{V_P\}})$. Then, $(I : I) = V_P$. Thus, by Lemma 2.34, I is m -canonical in V_P . Moreover, also P is m -canonical in V_P (the maximal ideal of a valuation domain is m -canonical, [12, Proposition 3.1]), then $I = xP$ for some nonzero $x \in K$, by [48, Proposition 4.2]. \square

As a straightforward consequence, we reobtain this well-known fact [24, Proposition 5.3.8] about strongly discrete valuation domains, that is, valuation domain with no idempotent prime ideals.

Corollary 2.36. *Let V be a valuation domain. The following are equivalent:*

- (i) V is strongly discrete.
- (ii) For each ideal I of V there exists $P \in \text{Spec}(V)$ and $x \in K$ such that $I = xP$.

Proof. By Proposition 2.31, $P \neq P^2$ for each $P \in \text{Spec}(D)$ if and only if the only semistar operations on V are the extensions to the overrings of V . Since each semistar operation on V is of the type $v(I)$ for a nonzero ideal I of V (Proposition 2.35(2)), this is equivalent to the fact that for each nonzero ideal I of V , the semistar operation $v(I)$ coincides with the semistar operation $\star_{\{P\}}$ for some prime ideal P of V . By Proposition 2.35(3), this is equivalent to the fact that for each ideal I of V there exists $P \in \text{Spec}(V)$ and $x \in K$ such that $I = xP$. \square

Remark 2.37. We have seen in Proposition 2.35 that, if $P = P^2$ is an idempotent prime ideal of a valuation domain V , there exists an integral ideal $I \subseteq V$ such that $v(V_P) = v(I)$. We want to determine “explicitly” this ideal I , when P is a branched prime ideal. By Proposition 2.31(2), it is enough to find an ideal I such that $(I : I) = V_P$ and $P^{v(I)} = V_P$.

First we recall the following result, [24, Lemma 3.1.9]:

Let I be a nonzero ideal of a valuation domain V . Then $(I : I) = V_P$ where P is the prime ideal of all the zero divisors on V/I .

As a consequence we have the following:

If I is a P -primary ideal of a valuation domain V then $(I : I) = V_P$.

Indeed, let I be a P -primary ideal. Let Q be the prime ideal of all the zero-divisors on V/I . We want to show that $P = Q$. Clearly $P \subseteq Q$, since $I \subseteq Q$ and P is minimal over I . Conversely, let $x \in Q$. Then, by definition of Q , there exists $y \in V \setminus I$ such that $xy \in I$. Since I is P -primary, it follows that $x \in P$. Thus, $Q \subseteq P$. Hence, P is the set of all the zero-divisors on V/I and, by the previous result, $(I : I) = V_P$.

This result suggest to look for a P -primary ideal I . We begin observing that since P is branched, there exists a principal ideal $J = (x)$ such that

P is minimal on J , [38, Theorem 17.3]. Let $I = JV_P \cap V$. Then I is P -primary, since JV_P is PV_P -primary, having maximal radical. Then, as the result above shows, $(I : I) = V_P$. Let $v := v_{V_P}$ be the v -operation of V_P . Since $P^2 = P$ we have $(PV_P)^v = V_P$. Then, $(I : P) \subseteq (I : P)V_P \subseteq (IV_P : PV_P)_v \subseteq ((IV_P)_v : (PV_P)_v) = ((IV_P)_v : V_P) = (IV_P)_v$. Since $IV_P = xV_P$, we have $(IV_P)_v = IV_P$ and $IV_P = I$, since I is primary. Then $(I : P) \subseteq I$, that is $(I : P) = I$, since the other inclusion is straightforward. Then $P^{v(I)} = (I : (I : P)) = (I : I) = V_P$. From the considerations above, it follows that $v(I) = v(V_P)$.

Now, we want to determine which of the semistar operations on a valuation domain are spectral. Clearly, we have only to discuss the case of the semistar $v(V_P)$ when P is a nonzero idempotent prime ideal of the valuation domain V . In fact, the other semistar operations are spectral since they are the extensions to a localization of V . We recall that a prime ideal P of a valuation domain V is *branched* if there exists a P -primary ideal of V distinct from P , and it is *unbranched* otherwise. From [38, Theorem 17.3(e)], P is branched if and only if the set of the prime ideals of V properly contained in P has a maximal element.

We have the following proposition.

Proposition 2.38. *Let V be a valuation domain and $P \in \text{Spec}(V)$, $P \neq 0$, such that $P^2 = P$. Then, the following are equivalent:*

- (i) *The semistar operation $v(V_P)$ is spectral.*
- (ii) *P is unbranched and $v(V_P) = \star_\Delta$, where $\Delta = \{Q \in \text{Spec}(V) \mid Q \subsetneq P\}$.*

Proof. (i) \Rightarrow (ii) It is easy to see that the only spectral semistar operation \star such that $V^\star = V_P$ for a branched prime ideal P is the extension to V_P . This follows from the fact that, to obtain $V^\star = V_P$, the prime P must be in the set of primes defining \star . (Otherwise, $V^\star \supseteq V_Q \supsetneq V_P$, if Q is the maximal element in the set of prime ideals properly contained in P .)

(ii) \Rightarrow (i) We have only to prove that $V^{\star_\Delta} = V_P$ and that $\star_\Delta \neq \star_{\{P\}}$. That $V^{\star_\Delta} = V_P$ follows from the fact that P is unbranched. Indeed, suppose $V^{\star_\Delta} = V_Q \supsetneq V_P$. Since P is unbranched, there exists $H \in \text{Spec}(V)$, $Q \subsetneq H \subsetneq P$. Then $H \in \Delta$ and $V^{\star_\Delta} \subseteq V_H \subsetneq V_Q$, a contradiction. Hence $V^{\star_\Delta} = V_P$. That $\star_\Delta \neq \star_{\{P\}}$ is straightforward since P is not contained in any $Q \in \Delta$ and then $P^{\star_\Delta} = \bigcap_{Q \in \Delta} PV_Q = \bigcap_{Q \in \Delta} V_Q = V_P \neq P = P^{\star_{\{P\}}}$. \square

Remark 2.39. (1) A different approach to the study of semistar operations on valuation domains is suggested by the following fact, which is a straightforward consequence of the fact that the set of ideals of a valuation domain is totally ordered:

Each semistar operation on a valuation domain is stable.

It follows that each semistar operation on a valuation domain is induced by a localizing system (Theorem 1.27(2)). It follows that the study of semistar operations on valuation domains is equivalent to the study of localizing systems on valuation domains. For localizing systems on valuation domains the following result holds (see [15, Theorem 3.3]):

Let V be a valuation domain. A set \mathcal{F} of ideals of V is a localizing system if and only if

- (1) $\mathcal{F} = \mathcal{F}(P)$ for some $P \in \text{Spec}(V)$, or
- (2) $\mathcal{F} = \hat{\mathcal{F}}(P) := \{I \mid I \supseteq P\} (= \mathcal{F}(P) \cup \{P\})$ for some $P \in \text{Spec}(V)$, with $P^2 = P$.

So, the localizing systems $\mathcal{F}(P)$ induce for each $P \in \text{Spec}(V)$ the semistar operation $\star_{\{P\}}$ (Example 1.25). Moreover, if $P = P^2$, there is also the semistar operation induced by the localizing system $\hat{\mathcal{F}}(P)$. It is easy to see that $V^{\star_{\hat{\mathcal{F}}(P)}} = (P : P) = V_P$. Comparing this result with Proposition 2.31, we have that $v(V_P) = \star_{\hat{\mathcal{F}}(P)}$.

(2) We remark that semistar operations on a valuation domain are stable but not necessarily spectral (we recall that a spectral semistar operation is stable, but the converse is not true in general). Proposition 2.38 shows that, if P is a branched idempotent prime ideal of a valuation domain V then the semistar operation $v(V_P)$ is not spectral (but it is stable). For example, if V is a one dimensional valuation domain, with idempotent maximal ideal M , the v -operation of V is not the identity (Proposition 2.30) and it is clearly not spectral, since M is the only nonzero prime ideal of V and $\star_{\{M\}} = d$.

2.3.2 Semistar operations on Prüfer domains

In this section, we study more generally semistar operations on Prüfer domains. We start with semistar operations of finite type:

Lemma 2.40. *Let D be a Prüfer domain, \star a semistar operation of finite type on D . Then, $\star = \star_{\{D^\star\}}$.*

Proof. Let \star be a semistar operation of finite type on D and let ι be the canonical embedding of D in D^\star . Then, \star_ι is a finite type (semi)star operation on the Prüfer domain D^\star , that is, $\star_\iota = d_{D^\star}$, the identity semistar operation (by Lemma 1.12 and [38, Proposition 34.12]). By Proposition 2.16(1) and Example 1.37(1), we have that $\star = \star_{\{D^\star\}}$. \square

Remark 2.41. (1) Applying this result to a valuation domain V , we obtain that $\mathbf{SStar}_f(V) = \{\star_{\{P\}} \mid P \in \text{Spec}(V)\}$.

(2) The result of Lemma 2.40 can be also proven directly using the semistar analogue of [38, Lemma 32.17], that is:

Let \star be a semistar operation on an integral domain D , let I be an invertible ideal of D and $E \in \overline{\mathbf{F}}(D)$. Then $(IE)^\star = IE^\star$.

The proof of this result is exactly the same as the proof in the case of star operations.

From this, we reobtain Lemma 2.40. As a matter of fact, by the previous statement, it follows that, if \star is a finite type semistar operation on a Prüfer domain D and I is a finitely generated (then, invertible) ideal of D , we have $I^\star = (ID)^\star = ID^\star$. So, since \star coincides with the extension to D^\star on the set of finitely generated ideals of D , it is clear that, if \star is of finite type, it is the extension to D^\star (Remark 1.6).

We can use this result to characterize Prüfer domains such that each semistar operation is of finite type (cf. [63]). To do this, we need another lemma:

Lemma 2.42. *Let D be a conducive Prüfer domain such that each nonzero prime ideal is contained in only one maximal ideal. Then, D is a valuation domain.*

Proof. Since D is a Prüfer, conducive domain, $\text{Spec}(D)$ is pinched, by [19, Corollary 3.4]. That is, there exists a nonzero prime ideal P comparable under inclusion to each prime of D . Suppose that D has two distinct maximal ideal M and N . Then, both must contain P , a contradiction. Hence, D is a local Prüfer domain, that is, a valuation domain. \square

We recall that an integral domain D is *divisorial* if each nonzero fractional ideal of D is divisorial, that is, if $I^v = I$, for each $I \in \mathbf{F}(D)$. With the notation of Remark 1.10, this is equivalent to the fact that $v = d_e$. The domain D is *totally divisorial* if each overring of D is divisorial. (For results on divisorial domains see for example [47], [14]; for totally divisorial domains see [13], [66].)

Theorem 2.43. *Let D be a Prüfer domain. Then, the following are equivalent:*

- (i) *Each semistar operation on D is of finite type.*
- (ii) *Each semistar operation on each overring of D is of finite type.*
- (iii) *Each semistar operation on D is an extension to an overring of D .*
- (iv) *D is conducive and totally divisorial.*
- (v) *D is a strongly discrete valuation domain.*

Proof. (i) \Leftrightarrow (ii) It is a consequence of Proposition 2.15(1) and Proposition 2.11(1).

(i) \Leftrightarrow (iii) It follows immediately by Lemma 2.40 and the fact that the

extensions to overrings are semistar operations of finite type (Section 1.2.3).
(iii) \Rightarrow (iv) Let d_e be the trivial extension of the identity star operation. From the hypothesis, it is the extension to D , that is, the identity semistar operation. Then, D is conducive (Lemma 1.9). To see that D is totally divisorial, take an overring T of D and let ι be the canonical embedding of D in T . Consider v_T the v -(semi)star operation on T . Note that by the hypothesis, $\mathbf{SStar}(D, T) = \{\star_{\{T\}}\}$. Thus, $(v_T)^\iota = \star_{\{T\}}$, since $(v_T)^\iota \in \mathbf{SStar}(D, T)$. So, by Proposition 2.15(1), $v_T = ((v_T)^\iota)_\iota = (\star_{\{T\}})_\iota = d_T$, and T is a divisorial domain. Hence, D is totally divisorial.

(iv) \Rightarrow (v) Since D is divisorial, each nonzero prime of D is contained only in one maximal ideal [47, Theorem 2.4]. Then, by Lemma 2.42, D is a valuation domain. Let $P \in \text{Spec}(D)$. By the hypothesis, $\mathbf{Star}(D_P) = \{d\}$. Then, by Corollary 2.17(2), $\mathbf{SStar}(D, D_P) = \{\star_{\{P\}}\}$ and $P \neq P^2$, by Proposition 2.31. Hence, D is strongly discrete.

(v) \Rightarrow (i) It is a straightforward consequence of Proposition 2.31. \square

Remark 2.44. We note that in the proof of (iii) \Rightarrow (iv) we have not used the hypothesis that D is a Prüfer domain. We will see in Proposition 2.51 that also (iv) \Rightarrow (iii) is always true.

We recall that a Prüfer domain D is called a *generalized Dedekind domain* (GDD for short) if each localizing system of D is of finite type, [24, Theorem 5.2.1.]. Then, by Theorem 1.30, D is a GDD if and only if each stable semistar operation is of finite type. A domain D is an *H-domain* if, for each ideal I such that $I^{-1} = D$, there exists a finitely generated ideal $J \subseteq I$ such that $J^{-1} = D$ [39]. To finish this section, we show another example on how the techniques developed in Section 2.2 can be used, giving a characterization of generalized Dedekind domains in terms of H-domains. First, we need a characterization of H-domains in terms of semistar operations.

Proposition 2.45. *Let D be an integral domain. The following are equivalent:*

- (i) D is an H-domain.
- (ii) The localizing system \mathcal{F}^v (associated to the v -operation of D , i.e. $\mathcal{F}^v := \{I \mid I \text{ ideal of } D, I^v = D\}$) is finitely generated.
- (iii) $\star_{\mathcal{F}^v} = \tilde{v}(= w)$.

Moreover, if D is a Prüfer domain, these conditions are equivalent to:

- (iv) $\star_{\mathcal{F}^v} = d$.

Proof. (i) \Leftrightarrow (ii) It is straightforward, since, for each ideal I of D , we have $I^v = D$ (that is, $I \in \mathcal{F}^v$) if and only if $I^{-1} = D$.

(ii) \Leftrightarrow (iii) It follows easily from the results in Section 1.3 and Section 1.4.

Indeed, \mathcal{F}^v is finitely generated if and only if $\mathcal{F}^v = (\mathcal{F}^v)_f$ if and only if $\star_{\mathcal{F}^v} = \star_{(\mathcal{F}^v)_f} = \tilde{v}$.

(iii) \Leftrightarrow (iv) It follows from the fact that, in a Prüfer domain, $\tilde{v} = d$ (since \tilde{v} is a finite type (semi)star operation and so $\tilde{v} \leq t$, by Lemma 1.12, and in a Prüfer domain $t = d$ [38, Proposition 34.23]). \square

Now, we conclude with the following:

Theorem 2.46. *Let D be a Prüfer domain. The following are equivalent:*

- (i) D is a GDD.
- (ii) D is an H -domain and each overring of D is an H -domain.
- (iii) Every stable semistar operation on D is an extension to an overring of D .

Proof. (i) \Rightarrow (ii) The localizing system \mathcal{F}^v , where v is the v -(semi)star operation of D is finitely generated by the hypothesis and so D is an H -domain. So, we have proved that a GDD is an H -domain. Since each overring of a GDD is a GDD [24, Theorem 5.4.1], then each overring of D is an H -domain.

(ii) \Rightarrow (iii) Let \star be a stable semistar operation on D . Let ι be the canonical embedding of D in D^\star . Then \star_ι is a stable (semi)star operation on D^\star , by Proposition 2.11(2) and (7). Then, by the fact that the bijection between stable semistar operations and localizing systems is order preserving (Theorem 1.30), we have $\star_\iota \leq \star_{\mathcal{F}^v_{D^\star}}$. Moreover, since D^\star is a Prüfer H -domain, Proposition 2.45 implies $\star_{\mathcal{F}^v_{D^\star}} = d_{D^\star}$. Thus $\star_\iota = d_{D^\star}$. Hence $\star = \star_{\{D^\star\}}$, by Proposition 2.16(1) and Example 1.37(1).

(iii) \Rightarrow (i) Since the semistar operation given by the extension to an overring is of finite type (Section 1.2.3), we have that every stable semistar operation is of finite type. Hence, D is a GDD. \square

2.3.3 Integral domains with all semistar operations spectral

Next question is: when are all semistar operations on an integral domain spectral? We give a complete characterization of such domains in the local case.

To begin, we give an easy consequence of Proposition 1.13.

Corollary 2.47. *Let D be an integral domain such that each semistar operation on D is stable. Then:*

- (1) D is a Prüfer domain.
- (2) Each semistar operation on each overring of D is stable.

Proof. (1) If each semistar operation is stable, in particular each extension to an overring is stable. Then, by Proposition 1.13, each overring of D is flat. It follows that D is a Prüfer domain [24, Theorem 1.1.1].

(2) By (1), it follows that each overring of D is flat over D . Now, the result is a consequence of Proposition 2.26(1), Proposition 2.11(2) and Proposition 2.15(1). \square

A similar result holds for spectral semistar operations.

Proposition 2.48. *Let D be an integral domain such that each semistar operation on D is spectral. Then:*

(1) D is a Prüfer domain.

(2) Each semistar operation on each overring of D is spectral.

Proof. (1) Note that spectral semistar operations are stable and apply Corollary 2.47(1).

(2) By (1), it follows that each overring of D is flat over D . Now, the result is a consequence of Proposition 2.28(1), Proposition 2.11(6) and Proposition 2.15(1). \square

We have the following characterization of local domains such that each semistar operation is spectral.

Corollary 2.49. *Let D be a local domain. The following are equivalent:*

(i) Every semistar operation on D is spectral.

(ii) D is a discrete valuation domain (that is, a valuation domain with all idempotent prime ideals unbranched).

Proof. (i) \Rightarrow (ii) By Proposition 2.48(1), D is a valuation domain. Let $P \in \text{Spec}(D)$. If $P^2 \neq P$, the only semistar operation in $\mathbf{SStar}(D, D_P)$ is $\star_{\{P\}}$, which is spectral. If $P^2 = P$, $P \neq 0$, with the notations of Proposition 2.31, $\mathbf{SStar}(D, D_P) = \{\star_{\{P\}}, v(V_P)\}$. We have already shown in Proposition 2.38, that, if P is branched, only $\star_{\{P\}}$ is spectral. Thus, $v(V_P)$ is not spectral, a contradiction, since each semistar operation on D is spectral. Thus, D has not branched idempotent nonzero prime ideals and D is a discrete valuation domain.

(ii) \Rightarrow (i) It is an immediate consequence of Proposition 2.38. \square

As a consequence, we have the following:

Proposition 2.50. *Let D be an integral domain, such that each semistar operation on D is spectral. Then D is a Prüfer domain, such that D_P is a discrete valuation domain for each $P \in \text{Spec}(D)$.*

Proof. It follows by Proposition 2.48(1) and (2), and Corollary 2.49. \square

2.3.4 Totally divisorial domains and the semistar operation $v(I)$

In this section, we want to improve the following result (for a different proof see [63]):

Proposition 2.51. *Let D be an integral domain. The following are equivalent:*

- (i) D is totally divisorial and conducive.
- (ii) Each semistar operation on D is an extension to an overring.

Proof. (i) \Rightarrow (ii) Let \star be a semistar operation on D and ι the canonical embedding of D in D^\star . Then, \star_ι is a (semi)star operation on the divisorial (conductive) domain D^\star . It follows that $\star_\iota = d_{D^\star}$. Hence, $\star = \star_{\{D^\star\}}$, by Proposition 2.16(1) and Example 1.37(1).

(ii) \Rightarrow (i) See Remark 2.44. □

To do this, we will use the notion of “stable” domain. We recall that an ideal I of an integral domain D is *stable* (or *SV-stable*) if I is invertible in the domain $(I : I)$. A domain D is *stable* if each nonzero ideal of D is stable. This concept was introduced by J. Sally and W. Vasconcelos [74], and developed in particular by B. Olberding in a series of papers (see for example [69], [67], [68], [70]).

We will use the following characterization of totally divisorial domains, due to Olberding [68, Theorem 3.12]:

Theorem 2.52. *An integral domain D is totally divisorial if and only if D is a stable divisorial domain.* □

We start with an easy lemma.

Lemma 2.53. *Let D be an integral domain, I a nonzero ideal of D . Then $v(I) = v((I : I))$ if and only if I is divisorial in $(I : I)$.*

Proof. Since $D^{v(I)} = (I : I)$, we have $v(I) \leq v((I : I))$ by Corollary 2.20. On the other hand, Proposition 1.20 implies that $v((I : I)) \leq v(I)$ if and only if $I^{v((I:I))} = I$, that is, if and only if I is divisorial in $(I : I)$. □

We have the following characterization of stable domains (the equivalence (i) \Leftrightarrow (iv) is due to B. Olberding, [70, Theorem 3.5]).

Theorem 2.54. *Let D be an integral domain. Then, the following are equivalent:*

- (i) D is stable.
- (ii) $v(I) = v((I : I))$, for each nonzero ideal I of D .

(iii) If I and J are nonzero ideals of D such that $(I : I) = (J : J)$, then $v(I) = v(J)$.

(iv) I is divisorial in $(I : I)$, for each nonzero ideal I of D .

Proof. (i) \Leftrightarrow (iv) [70, Theorem 3.5].

(ii) \Rightarrow (iii) It is straightforward.

(iii) \Rightarrow (ii) Since $(I : I)$ is a fractional ideal of D , there exists a nonzero ideal J of D and a nonzero element $x \in D$ such that $(I : I) = x^{-1}J$. Clearly, $(I : I) = ((I : I) : (I : I)) = (x^{-1}J : x^{-1}J) = (J : J)$ (this is a consequence of Lemma 1.15(1) and (2)). By the hypothesis, $v(I) = v(J)$. Moreover, $v(J) = v((I : I))$, by Lemma 1.18(5). Hence, $v((I : I)) = v(I)$.

(ii) \Leftrightarrow (iv) is the equivalence proven in Lemma 2.53. \square

We define an ideal I to be *quasi-m-canonical* if I is m-canonical as an ideal of $(I : I)$.

Example 2.55. Let D be a pseudo-valuation domain (that is not a valuation domain) with maximal ideal M and let $V := M^{-1} = (M : M)$ the valuation domain associated to D . Since in a local ring R the maximal ideal is m-canonical if and only if R is a valuation domain [12, Proposition 3.1], M is not m-canonical as an ideal of D , but it is m-canonical as an ideal of V . So, M is a quasi-m-canonical ideal of D that is not m-canonical.

We have shown in Lemma 2.34 that, in a conducive domain, an ideal I is quasi-m-canonical if and only if $v(I) = \star_{\{(I:I)\}}$. In general we have the following:

Lemma 2.56. *Let D be an integral domain and I an ideal of D . The following are equivalent:*

(i) I is quasi-m-canonical.

(ii) $(v(I))|_{\mathbf{F}(D)} = (\star_{\{(I:I)\}})|_{\mathbf{F}(D)}$.

(iii) $v(I) = (\star_{\{(I:I)\}})_e$ (where $(\star_{\{(I:I)\}})_e$ is defined as in Remark 1.10).

Proof. (i) \Rightarrow (ii) Let J be a nonzero ideal of D . Then $J(I : I)$ is an ideal of $(I : I)$. It is easy to see that $(I : J) = (I : J(I : I))$ (this follows from the fact that $I(I : I) = I$). Moreover, since I is m-canonical in $(I : I)$ it follows that $J^{v(I)} = (I : (I : J)) = (I : (I : J(I : I))) = J(I : I) = J^{\star_{\{(I:I)\}}}$.

(ii) \Rightarrow (i) Let J be a nonzero ideal of $(I : I)$. It is clear that $J \in \mathbf{F}(D)$, because $J \in \overline{\mathbf{F}}(D)$ and for each element $x \in I \subseteq D$, $xJ \subseteq x(I : I) \subseteq I \subseteq D$. Then, by the hypothesis, $(I : (I : J)) = J^{v(I)} = J^{\star_{\{(I:I)\}}} = J(I : I) = J$, and I is m-canonical as an ideal of $(I : I)$.

(ii) \Rightarrow (iii) First, we notice that $(\star_{\{(I:I)\}})_e$ is a semistar operation on D , since $(D : (I : I)) \supseteq I \neq (0)$ (Remark 1.10). Moreover, since $(\star_{\{(I:I)\}})_e$

and $\star_{\{(I:I)\}}$ coincide on $\mathbf{F}(D)$ (Remark 1.10), by the hypothesis, $v(I)$ and $(\star_{\{(I:I)\}})_e$ coincide on $\mathbf{F}(D)$. Since both $v(I)$ and $(\star_{\{(I:I)\}})_e$ are trivial on $\overline{\mathbf{F}}(D)$ (Section 1.2.5 and Remark 1.10), they coincide on all $\overline{\mathbf{F}}(D)$. Hence $v(I) = (\star_{\{(I:I)\}})_e$.

(iii) \Rightarrow (ii) $v(I)|_{\mathbf{F}(D)} = ((\star_{\{(I:I)\}})_e)|_{\mathbf{F}(D)} = (\star_{\{(I:I)\}})|_{\mathbf{F}(D)}$. \square

We can now prove a characterization of totally divisorial domains without the conducive assumption.

Theorem 2.57. *Let D be an integral domain. The following are equivalent:*

(i) D is totally divisorial.

(ii) Each nonzero ideal of D is quasi- m -canonical.

(iii) $(I : I)$ is a divisorial domain for each nonzero ideal I of D .

Proof. (i) \Rightarrow (ii) Let I be an ideal of D and let ι be the embedding of D in $(I : I)$. Since the overring $(I : I)$ is divisorial, it follows that $v_{(I:I)} = (d_{(I:I)})_e$. Since $D^{v(I)} = (I : I)$, we have that $v(I) \leq v((I : I)) = (v_{(I:I)})^\iota$, by Corollary 2.20(1) and Example 2.3. Let $J \in \mathbf{F}(D)$. It follows that $J^{v(I)} \subseteq J^{(v_{(I:I)})^\iota} = (J(I : I))^{(v_{(I:I)})^\iota} = (J(I : I))^{(d_{(I:I)})_e} = J(I : I) = J^{\star_{\{(I:I)\}}}$. The opposite inclusion is obvious, since $\star_{\{(I:I)\}} \leq v(I)$ by Corollary 2.20(1). Therefore, by Lemma 2.56, I is quasi- m -canonical.

(ii) \Rightarrow (i) We use the characterization given in Theorem 2.52. Note that D is divisorial since, by the hypothesis, D is quasi- m -canonical and then m -canonical, since $(D : D) = D$. This is equivalent to the fact that D is a divisorial domain. We want to prove that D is a stable domain using Theorem 2.54(i) \Leftrightarrow (iii). Let I, J be nonzero ideals of D such that $(I : I) = (J : J) =: T$. Since both I and J are quasi- m -canonical, by Lemma 2.56(i) \Leftrightarrow (iii), $v(I) = v(J) = (\star_{\{T\}})_e$, thus D is stable. Since D is divisorial and stable, then it is totally divisorial by Theorem 2.52.

(i) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (ii) Let I be a nonzero ideal of D and let ι be the embedding of D in $(I : I)$. The semistar operation $v(I)_\iota$ is a (semi)star operation on $(I : I)$. It follows by Lemma 1.11 that $v(I)_\iota \leq v_{(I:I)}$. Since $(I : I)$ is divisorial, $v((I : I)) = (d_{(I:I)})_e$. Then, $v(I) = (v((I : I)))^\iota = ((d_{(I:I)})_e)^\iota = (\star_{\{(I:I)\}})_e$. Hence, I is quasi- m -canonical by Lemma 2.56(iii) \Rightarrow (i). \square

So, it is enough that the overrings of the form $(I : I)$ are divisorial for all nonzero ideals I of D , to have that D is totally divisorial. We notice that the set $\{(I : I) \mid I \text{ nonzero ideal of } D\}$ in general is properly contained in the set $\mathcal{O}(D)$ of the overrings of D . It is easy to see that $\mathcal{O}(D) = \{(I : I) \mid I \text{ nonzero ideal of } D\}$ if and only if D is a conducive domain.

Remark 2.58. We have seen that domains with each nonzero ideal quasi-m-canonical are exactly the totally divisorial domain. It is natural to ask when all nonzero ideals of an integral domain D are m-canonical. So, let D be an integral domain such that each nonzero ideal is m-canonical. We notice that, if I is an m-canonical ideal of D , then $(I : I) = D$. Thus, $(I : I) = D$ for each nonzero ideal I of D and then D is completely integrally closed by [38, Theorem 34.3]. In addition, D is m-canonical and then every nonzero ideal of D is divisorial. Then, by [24, Proposition 4.3.5], D is a Dedekind domain. On the other hand it is clear that every ideal of a Dedekind domain is m-canonical, since a nonzero ideal I of a Dedekind domain is invertible and so $v(I) = v_D = d_D$ (note that in this case $\overline{F}(D) = F(D) = f(D)$).

2.3.5 Integral domains with at most $\dim(D) + 2$ semistar operations

The study of the number of semistar operations on integral domain of finite Krull dimension has been object recently of a series of papers by R. Matsuda ([61] with T. Sugatani, [56], [58], [59], [60]). In this section we study when a domain D has at most or exactly $\dim(D) + 2$ semistar operations.

Let D be an integral domain of Krull dimension n . We give a new proof of the following result about integral domains with $n + 1$ semistar operations (cf. [57, Theorem 4]).

Theorem 2.59. *Let D be an integral domain of Krull dimension n , with quotient field K . The following are equivalent:*

- (1) D is a strongly discrete valuation domain.
- (2) $\text{Card}(\mathbf{SStar}(D)) = n + 1$. □

Proof. (1) \Rightarrow (2) It is Corollary 2.32

(2) \Rightarrow (1) Let M be a maximal ideal of height n . Since each overring of D induces a semistar operation and since D has at least $n + 1$ overrings given by the localizations to the prime ideals contained in M (including (0)), the semistar operations of D are exactly the extensions to the localizations of D at these prime ideals. In particular, D is local (another maximal ideal would give another overring and then another semistar operation). Moreover, each semistar operation is obviously spectral (and of finite type). Then D is a discrete (strongly discrete since D is of finite dimension) valuation domain by Corollary 2.49. □

In the following we want to characterize the domains D of Krull dimension n such that $\text{Card}(\mathbf{SStar}(D)) = n + 2$. So, let D be an integral domain of Krull dimension n with quotient field K and $n + 2$ semistar operations.

First, we notice that D has a chain of primes $M = P_n \supsetneq P_{n-1} \supsetneq P_{n-2} \supsetneq \dots \supsetneq P_1 \supsetneq P_0 = (0)$, where M is a maximal ideal of D . Each of this

primes induces a semistar operation $\star_{\{P_i\}}$, $i = 0, 1, \dots, n$. So, $n + 1$ semistar operations are given by the extension to a localization of D (we notice that $\star_{\{P_0\}}$ is the e -operation as defined in Section 1.2.1). Then, there is only one semistar operation not given by an extension to one of these primes.

The first consequence is that $\text{Spec}(D)$ is totally ordered.

Lemma 2.60. *Let D be an integral domain of dimension n such that $\text{Card}(\mathbf{SStar}(D)) = n + 2$. Then, $\text{Spec}(D)$ is totally ordered.*

Proof. Suppose that M is a maximal ideal of D with $\text{ht}(M) = n$ and assume that D has another maximal ideal N . Then, we have the $n + 1$ semistar operations given by the extension to the primes contained in M , the extension to D_N and the identity semistar operation, that is, we have (at least) $n + 3$ semistar operations, a contradiction. So, D is local.

Suppose now that $\text{Spec}(D)$ is not totally ordered. It means that there are the prime ideals $M = P_n \supsetneq P_{n-1} \supsetneq P_{n-2} \supsetneq \dots \supsetneq P_1 \supsetneq P_0 = (0)$ and also another prime ideal $Q \neq P_i$, $i = 0, 1, \dots, n$. Now, D has (at least) $n + 2$ primes. It follows that the $n + 2$ semistar operations are exactly the spectral semistar operations induced by these prime ideals. So, D is a local domain such that each semistar operation is spectral, and so, by Corollary 2.49, D is a (strongly discrete) valuation domain, a contradiction since we have supposed that $\text{Spec}(D)$ is not totally ordered. \square

We study now the case in which D is integrally closed.

Theorem 2.61. *Let D be an integrally closed integral domain of dimension n . The following are equivalent:*

- (i) $\text{Card}(\mathbf{SStar}(D)) = n + 2$.
- (ii) D is a valuation domain with exactly one idempotent prime ideal.

Proof. (i) \Rightarrow (ii) By Lemma 2.60, we have that the prime ideals of D are exactly $M = P_n \supsetneq P_{n-1} \supsetneq P_{n-2} \supsetneq \dots \supsetneq P_1 \supsetneq P_0 = (0)$ that give $n + 1$ semistar operations. Consider the v -operation of D . Note that $v \neq \star_{\{P_k\}}$, for each $k = 0, 1, \dots, n - 1$, since v is a (semi)star operation. If $v = d$, in particular $t = d$. Then, since D is a local integrally closed domain, by [38, Proposition 34.12], it follows that D is a valuation domain. Suppose that $v \neq d$. In this case, the $n + 2$ semistar operations are the $n + 1$ spectral ones and the v -operation. It follows that the only overrings of D are its localizations at the primes (another overring would give another semistar operation) and D is a Prüfer domain (and then a valuation domain) by [24, Theorem 1.1.1]. So, in any case, D is a valuation domain and it has exactly one idempotent prime ideal by Corollary 2.32(1).

(ii) \Rightarrow (i) It is a consequence of Corollary 2.32(1). \square

We examine now the case in which D is not integrally closed.

Theorem 2.62. *Let D be a not integrally closed integral domain of dimension n . The following are equivalent:*

- (i) $\text{Card}(\mathbf{SStar}(D)) = n + 2$.
- (ii) D is a divisorial pseudo-valuation domain with maximal ideal M , such that the valuation domain $V = M^{-1}$ is strongly discrete and there are no proper overrings of D properly contained in V .
- (iii) D is a totally divisorial pseudo-valuation domain with maximal ideal M , such that the valuation domain $V = M^{-1}$ is strongly discrete and there are no proper overrings of D properly contained in V .

Proof. (i) \Rightarrow (iii) By Lemma 2.60, $\text{Spec}(D)$ is totally ordered. Let $M = P_n \supsetneq P_{n-1} \supsetneq P_{n-2} \supsetneq \dots \supsetneq P_1 \supsetneq P_0 = (0)$ be the prime ideals of D . Let D' be the integral closure of D . Since D' has dimension n it has exactly $n + 1$ semistar operations. Indeed, it has at least $n + 1$ semistar operations, the extension to the localizations at prime ideals. Suppose that $\text{Card}(\mathbf{SStar}(D')) \geq n + 2$. By Remark 2.18, since D has a finite number of semistar operations, it follows that $n + 2 \leq \text{Card}(\mathbf{SStar}(D')) < \text{Card}(\mathbf{SStar}(D))$, a contradiction. Then, D' has exactly $n + 1$ semistar operations, that is, D' is a strongly discrete valuation domain (Theorem 2.59). Moreover, $D' \subseteq D_{P_{n-1}}$. In fact, if $D' \not\subseteq D_{P_{n-1}}$, we would have on D the $n + 1$ semistar operations descending from D' , the extension to $D_{P_{n-1}}$ and the identity semistar operation. Then, at least $n + 3$ semistar operations, a contradiction. It follows that $D \subsetneq D' \subsetneq D_{P_{n-1}} \subsetneq D_{P_{n-2}} \subsetneq \dots \subsetneq D_{P_1} \subsetneq K$ are the overrings of D and the $n + 2$ semistar operations are exactly the extensions to these overrings. Hence, D is totally divisorial (and conducive) by Proposition 2.51. In particular, the v -operation of D coincides with the d -operation. Then, each (semi)star operation is the identity, by Lemma 1.11. From this, we can deduce that $D' = (M : M)$. Indeed, consider the semistar operation $v(M)$. If $(M : M) = D$, then $v(M)$ is a (semi)star operation, thus $v(M) = d$ and so M is m -canonical. By [12, Proposition 3.1], a local domain with the maximal ideal m -canonical is a valuation domain. So, D is a valuation domain, a contradiction. Then, $D \subsetneq (M : M)$ and $(M : M) = D'$ (clearly $(M : M)$ cannot be one of the D_{P_i} , $i \neq n$, since M is an ideal of $(M : M)$). It follows that $(M : M)$ is a valuation domain with maximal ideal M (since the extension $D \subset (M : M)$ is integral). Hence D is a pseudo-valuation domain by [45, Theorem 2.7].

(ii) \Rightarrow (iii) Note that there are no proper overrings of D not containing V . Indeed, let T be a proper overring of D not containing V and let $x \in V \setminus T$. By the hypothesis, $T \not\subseteq V$. Then, there exists $t \in T$ such that $t \notin V$. It follows that $t^{-1} \in M \subseteq D \subseteq T$. Now, $x = xtt^{-1}$, with $xt^{-1} \in M \subseteq T$ and $t \in T$. It follows that $x \in T$, a contradiction. Hence, the proper overrings of D are exactly the overrings of V . Now, D is divisorial, and each

overring of V is divisorial, since V is strongly discrete and so totally divisorial (this is a straightforward consequence of Proposition 2.30(1), cf. also [13, Proposition 7.6]). Hence, D and all its proper overrings are divisorial, i.e., D is totally divisorial.

(iii) \Rightarrow (ii) is straightforward.

(iii) \Rightarrow (i) D is totally divisorial (by the hypothesis) and conducive (by [19, Proposition 2.1]). Hence, Proposition 2.51 implies that the semistar operations on D are as many as the overrings of D . As we have shown in the proof of (ii) \Rightarrow (iii), the proper overrings of D are also overrings of V . Note that V is the integral closure of D , so $\dim(V) = n$ and V has exactly $n + 1$ overrings. Hence, D has $n + 2$ overrings and so $n + 2$ semistar operations (the identity and the $n + 1$ operations given by the extensions to the proper overrings of D). \square

Chapter 3

Semistar invertibility

3.1 Semistar invertibility

Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$, we say that I is \star -invertible if $(II^{-1})^\star = D^\star$ [26, page 30]. In particular when $\star = d$ [respectively, v , t ($:= v_f$), w ($:= \tilde{v}$)] is the identity (semi)star operation [respectively, the v -operation, the t -operation, the w -operation] we reobtain the classical notion of *invertibility* [respectively, *v -invertibility*, *t -invertibility*, *w -invertibility*] of a fractional ideal.

Lemma 3.1. *Let \star, \star_1, \star_2 be semistar operations on an integral domain D . Let $\text{Inv}(D, \star)$ be the set of all \star -invertible fractional ideals of D and $\text{Inv}(D)$ (instead of $\text{Inv}(D, d)$) the set of all invertible fractional ideals of D . Then:*

- (0) $D \in \text{Inv}(D, \star)$.
- (1) If $\star_1 \leq \star_2$, then $\text{Inv}(D, \star_1) \subseteq \text{Inv}(D, \star_2)$. In particular, $\text{Inv}(D) \subseteq \text{Inv}(D, \tilde{\star}) \subseteq \text{Inv}(D, \star_f) \subseteq \text{Inv}(D, \star)$.
- (2) $I, J \in \text{Inv}(D, \star)$ if and only if $IJ \in \text{Inv}(D, \star)$.
- (3) If $I \in \text{Inv}(D, \star)$ then $I^{-1} \in \text{Inv}(D, \star)$.
- (4) If $I \in \text{Inv}(D, \star)$ then $I^v \in \text{Inv}(D, \star)$.

Proof. (0) and (1) are obvious.

(2) If $I, J \in \text{Inv}(D, \star)$, then $D^\star = (II^{-1})^\star (JJ^{-1})^\star \subseteq (II^{-1}JJ^{-1})^\star \subseteq (IJ(IJ)^{-1})^\star \subseteq D^\star$. Thus, $IJ \in \text{Inv}(D, \star)$. Conversely, if $IJ \in \text{Inv}(D, \star)$, then $D^\star = ((IJ)(D : IJ))^\star = (I(J(D : IJ)))^\star$. Since $(J(D : IJ)) \subseteq (D : I)$, it follows $(I(D : I))^\star = D^\star$. Similarly, $(J(D : J))^\star = D^\star$.

(3) $D^\star = (II^{-1})^\star \subseteq ((I^{-1})^{-1}I^{-1})^\star \subseteq D^\star$.

(4) follows from (3). □

Remark 3.2. (a) Note that D is the unit element of $\text{Inv}(D, \star)$ with respect to the *standard multiplication* of fractional ideals of D . Nevertheless, $\text{Inv}(D, \star)$ is *not* a group in general (under the standard multiplication), because for $I \in \text{Inv}(D, \star)$, then $I^{-1} \in \text{Inv}(D, \star)$, but $II^{-1} \neq D$, if $I \notin \text{Inv}(D)$. For instance, let k be a field, X and Y two indeterminates over k , and let $D := k[X, Y]_{(X, Y)}$. Then D is a local Krull domain, with maximal ideal $M := (X, Y)D$. Let $\star = v$, then clearly $M^v = D$, since $\text{ht}(M) = 2$, thus M is v -invertible but M is not invertible in D , since it is not principal. Therefore $(MM^{-1})^v = D$, but $M = MM^{-1} \subsetneq D$. We will discuss later what happens if we consider the semistar (fractional) ideals semistar invertible with the “semistar product”.

(b) Let $I \in \mathbf{F}(D)$. Assume that $I \in \text{Inv}(D, \star)$ and $(D^\star : I) \in \mathbf{F}(D)$, then we will see later that $(D^\star : I) = (D : I)^\star$ (Lemma 3.11, Remark 3.14(d1) and Proposition 3.17), more precisely that:

$$(I^{-1})^\star = (D : I^v)^\star = (D^\star : I)^\star = (D^\star : I) = (I^\star)^{-1}.$$

However, in this situation, we may not conclude that $(D^\star : I)$ (or, $(D : I)^\star$) belongs to $\text{Inv}(D, \star)$ (even if $(D : I) \in \text{Inv}(D, \star)$, by Lemma 3.1(3)). As a matter of fact, more generally, if $J \in \text{Inv}(D, \star)$ and $J^\star \in \mathbf{F}(D)$, then J^\star does not belong necessarily to $\text{Inv}(D, \star)$.

For instance, let K be a field and X, Y two indeterminates over K , set $T := K[X, Y]$ and $D := K + YK[X, Y]$. Let $\star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\mathbf{F}}(D)$. Then $J := YD$ is obviously invertible (hence $\star_{\{T\}}$ -invertible) in D and $J^{\star_{\{T\}}} = JT = YT = YK[X, Y] = (D : T)$ is a nonzero (maximal) ideal of D (and, at the same time, a (prime) ideal of T), but $J^{\star_{\{T\}}}$ is not $\star_{\{T\}}$ -invertible in D , because $(J^{\star_{\{T\}}}(D : J^{\star_{\{T\}}}))^{\star_{\{T\}}} = (JT(D : JT))T = (YT(D : YT))T = (YTY^{-1}(D : T))T = (T(YT))T = YT \subsetneq T = D^{\star_{\{T\}}}$.

(c) Note that the converses of (3) and (4) of Lemma 3.1 are not true in general. For instance, take an integral domain D that is not an H-domain (recall that an *H-domain* is an integral domain D such that, if I is an ideal of D with $I^{-1} = D$, then there exists a finitely generated $J \subseteq I$, such that $J^{-1} = D$ [39, Section 3]). Then, there exists an ideal I of D such that $I^v = I^{-1} = D$ and $I^t \subsetneq D$. It follows that $(I^{-1}I^v)^t = D$ (and so, I^{-1} and I^v are t -invertibles), but $(II^{-1})^t = I^t \subsetneq D$, that is, I is not t -invertible. On the other hand, note that, trivially, I is v -invertible.

An explicit example is given by a 1-dimensional non discrete valuation domain V with maximal ideal M . Clearly, V is not an H-domain [39, (3.2d)], $M^{-1} = M^v = V$ [38, Exercise 12 p.431] and $M^t = \bigcup \{J^v \mid J \subseteq M, J \text{ finitely generated}\} = \bigcup \{J \mid J \subseteq M, J \text{ finitely generated}\} = M \subsetneq V$. In this case, M^{-1} and M^v are obviously t -invertibles, but M is not t -invertible.

Corollary 3.3. *Let \star be a semistar operation on an integral domain D and let $I \in \overline{\mathbf{F}}(D)$.*

(1) *If I is \star -invertible, then I is $v(D^\star)$ -invertible and $I^\star = I^{v(D^\star)}$.*

(2) If I is \star_f -invertible, then I is $t(D^\star)$ -invertible and $I^\star = I^{t(D^\star)}$.

Proof. (1) Since $\star \leq v(D^\star)$ (Corollary 2.20(1)), it follows from Lemma 3.1(1) that if I is \star -invertible then I is $v(D^\star)$ -invertible. Now, observe that $I^{v(D^\star)} = (D^\star : (D^\star : I)) \subseteq (D^\star : (D : I))$, so $I^{-1}I^{v(D^\star)} \subseteq D^\star$. Thus we have that $I^\star \subseteq I^{v(D^\star)} \subseteq ((II^{-1})^\star I^{v(D^\star)})^\star = (I(I^{-1}I^{v(D^\star)}))^\star \subseteq (ID^\star)^\star = I^\star$. It follows that $I^\star = I^{v(D^\star)}$.

(2) It follows, using a similar argument, by the fact that $\star \leq t(D^\star)$ (Corollary 2.20(2)). \square

If $I \in \overline{\mathbf{F}}(D)$, we say that I is \star -finite if there exists $J \in \mathbf{f}(D)$ such that $J^\star = I^\star$. It is immediate to see that if $\star_1 \leq \star_2$ are semistar operation and I is \star_1 -finite, then I is \star_2 -finite. In particular, if I is \star_f -finite, then it is \star -finite.

We notice that, in the previous definition of \star -finite, we do not require that $J \subseteq I$. Next result shows that, with this “extra” assumption, \star -finite is equivalent to \star_f -finite.

Lemma 3.4. *Let \star be a semistar operation on an integral domain D with quotient field K . Let $I \in \overline{\mathbf{F}}(D)$. Then, the following are equivalent:*

(i) I is \star_f -finite.

(ii) There exists $J \subseteq I$, $J \in \mathbf{f}(D)$ such that $J^\star = I^\star$.

Proof. It is clear that (ii) implies (i), since $J^\star = J^{\star_f}$, if J is finitely generated. On the other hand, suppose I \star_f -finite. Then, $I^{\star_f} = J_0^{\star_f}$, with $J_0 := (a_1, a_2, \dots, a_n)D$, for some family $\{a_1, a_2, \dots, a_n\} \subseteq K$. Since $J_0 \subseteq I^{\star_f}$, there exists a finite family of finitely generated fractional ideals of D , $J_1, J_2, \dots, J_n \subseteq I$, such that $a_i \in J_i^\star$, for $i = 1, 2, \dots, n$. It follows that $I^{\star_f} = J_0^{\star_f} \subseteq (J_1^{\star_f} + J_2^{\star_f} + \dots + J_n^{\star_f})^{\star_f} = (J_1 + J_2 + \dots + J_n)^{\star_f} \subseteq I^{\star_f}$. Set $J := J_1 + J_2 + \dots + J_n$. Then, J is finitely generated, $J \subseteq I$ and $J^{\star_f} = I^{\star_f}$, thus $J^\star = I^\star$. \square

Remark 3.5. Extending the terminology introduced by Zafrullah in the star setting [81] (cf. also [82, p. 433]), given a semistar operation on an integral domain D , we can say that $I \in \overline{\mathbf{F}}(D)$ is *strictly \star -finite* if $I^\star = J^\star$, for some $J \in \mathbf{f}(D)$, with $J \subseteq I$. With this terminology, Lemma 3.4 shows that \star_f -finite coincides with strictly \star_f -finite. This result was already proven, in the star setting, by Zafrullah [81, Theorem 1.1]. Note that Querré studied the strictly v -finite ideals [71], using often the terminology of *quasi-finite ideals*.

For examples of \star -finite ideals that are not \star_f -finite (when \star is the v -operation), see [34, Section (4c)], where domains with all the ideals v -finite

(called DVF-domains), that are not Mori domains (that is, such that not all the ideals are t -finite) are described.

Lemma 3.6. *Let \star be a semistar operation on an integral domain D and let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if, and only if, $(I'I'')^\star = D^\star$, for some $I' \subseteq I, I'' \subseteq I^{-1}$, and $I', I'' \in \mathbf{f}(D)$. Moreover, $I'^\star = I^\star$ and $I''^\star = (I^{-1})^\star$.*

Proof. The “if” part is trivial. For the “only if”: if $(II^{-1})^{\star_f} = D^{\star_f}$, then $H^\star = D^\star$ for some $H \subseteq II^{-1}$, $H \in \mathbf{f}(D)$. Therefore, $H = (h_1, h_2, \dots, h_n)D$, with $h_i = x_{1,i}y_{1,i} + x_{2,i}y_{2,i} + \dots + x_{k_i,i}y_{k_i,i}$, with the x 's in I and the y 's in I^{-1} . Let I' be the (fractional) ideal of D generated by the x 's and let I'' be the (fractional) ideal of D generated by the y 's. Then, $H \subseteq I'I'' \subseteq II^{-1}$ and so $D^\star = (I'I'')^\star$, and, thus, also $D^\star = (I'I^{-1})^\star = (II'')^\star$. Moreover, $I^\star = (ID^\star)^\star = (I(I'I^{-1})^\star)^\star = ((II^{-1})^\star I')^\star = (D^\star I')^\star = I'^\star$. In a similar way, we obtain also that $I''^\star = (I^{-1})^\star$. \square

A classical result due to Krull [51, Théorème 8, Ch. I, §4] shows that, for a star operation of finite type, star-invertibility implies star-finiteness. The following result gives a more complete picture of the situation in the general semistar setting.

Proposition 3.7. *Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if and only if I and I^{-1} are \star_f -finite (hence, in particular, \star -finite) and I is \star -invertible.*

Proof. The “only if” part follows from Lemma 3.6 and from the fact that $\star_f \leq \star$.

For the “if” part, note that by assumption $I^{\star_f} = J'^{\star_f} = J'^\star$ and $(I^{-1})^{\star_f} = J''^{\star_f} = J''^\star$, with $J', J'' \in \mathbf{f}(D)$. Therefore:

$$(II^{-1})^{\star_f} = (J'J'')^{\star_f} = (J'J'')^\star = (J'^\star J''^\star)^\star = (I^\star (I^{-1})^\star)^\star = (II^{-1})^\star = D^\star.$$

\square

Next goal is to investigate when the \star -invertibility coincides with the \star_f -invertibility.

Let \star be a semistar operation on an integral domain D , we say that D is an $H(\star)$ -domain if, for each nonzero integral ideal I of D such that $I^\star = D^\star$, there exists $J \in \mathbf{f}(D)$ with $J \subseteq I$ and $J^\star = D^\star$. It is easy to see that, for $\star = v$, the $H(v)$ -domains coincide with the H -domains introduced by Glaz and Vasconcelos.

Lemma 3.8. *Let \star be a semistar operation on an integral domain D . Then D is an $H(\star)$ -domain if and only if each quasi- \star_f -maximal ideal of D is a quasi- \star -ideal of D .*

Proof. Assume that D is an $H(\star)$ -domain. Let $Q = Q^{\star_f} \cap D$ be a quasi- \star_f -maximal ideal of D . Assume that $Q^{\star} = D^{\star}$. Then, for some $J \in \mathbf{f}(D)$, with $J \subseteq Q$, we have $J^{\star} = D^{\star}$, thus $Q^{\star_f} = D^{\star}$, which leads to a contradiction. Therefore $Q^{\star_f} \cap D \subseteq Q^{\star} \cap D \subsetneq D$ and, hence, there exists a quasi- \star_f -maximal ideal of D containing $Q^{\star} \cap D$. This is possible only if $Q^{\star_f} \cap D = Q^{\star} \cap D$.

Conversely, let I be a nonzero ideal of D with the property $I^{\star} = D^{\star}$. Then, necessarily $I \not\subseteq Q$ for each quasi- \star_f -maximal ideal of D (because, otherwise, by assumption $I \subseteq Q = Q^{\star_f} \cap D = Q^{\star} \cap D$, and so $I^{\star} \subseteq Q^{\star} \subsetneq D^{\star}$). Therefore $I^{\star_f} = D^{\star}$. \square

Next result provides several characterizations of the $H(\star)$ -domains; note that, in the particular case that $\star = v$, the equivalence (i) \Leftrightarrow (iii) was already known [50, Proposition 2.4], the equivalence (i) \Leftrightarrow (iv) was considered in [79, Proposition 5.7] and we have stated the equivalence (i) \Leftrightarrow (v) \Leftrightarrow (vi) in Proposition 2.45.

Proposition 3.9. *Let \star be a semistar operation on an integral domain D . The following are equivalent:*

- (i) D is an $H(\star)$ -domain.
- (ii) For each $I \in \mathbf{F}(D)$, I is \star -invertible if and only if I is \star_f -invertible.
- (iii) $\mathcal{M}(\star_f) = \mathcal{M}(\star)$.
- (iv) $\mathcal{M}(\tilde{\star}) = \mathcal{M}(\star)$.
- (v) The localizing system \mathcal{F}^{\star} is finitely generated.
- (vi) $\star_{\mathcal{F}^{\star}} = \tilde{\star}$.

Proof. Obviously, (iii) \Leftrightarrow (iv) by Proposition 1.34 and (iii) \Leftrightarrow (i) by Lemma 3.8, recalling that a quasi- \star -ideal is also a quasi- \star_f -ideal.

(iii) \Rightarrow (ii). Let I be a \star -invertible ideal of D . Assume that I is not \star_f -invertible. Then, there exists a quasi- \star_f -maximal ideal M such that $II^{-1} \subseteq M$. But M is also quasi- \star -maximal, since $\mathcal{M}(\star_f) = \mathcal{M}(\star)$. Thus $M^{\star} \subsetneq D^{\star}$. It follows that $(II^{-1})^{\star} \subseteq M^{\star} \subsetneq D^{\star}$, a contradiction. Hence I is \star_f -invertible.

(ii) \Rightarrow (i) Let I be a nonzero integral ideal I of D such that $I^{\star} = D^{\star}$. Then, $I \subseteq II^{-1} \subseteq D$ implies that $(II^{-1})^{\star} = D^{\star}$, that is I is \star -invertible. By assumption, it follows that I is \star_f -invertible, and so I is \star_f -finite (Proposition 3.7). By Lemma 3.4, we conclude that there exists $J \in \mathbf{f}(D)$ with $J \subseteq I$ and $J^{\star} = I^{\star} = D^{\star}$.

(i) \Leftrightarrow (v) The localizing system \mathcal{F}^{\star} is the set of the ideals I of D such that $I^{\star} = D^{\star}$ (Section 1.3). So, the thesis is exactly the definition of $H(\star)$ domain in the languages of localizing systems, in fact, D is an $H(\star)$ -domain if and only if for each $I \in \mathcal{F}^{\star}$ there exists $J \in \mathbf{f}(D)$, $J \subseteq I$ such that $J \in \mathcal{F}^{\star}$, that is, if and only if \mathcal{F}^{\star} is finitely generated (Section 1.3).

(v) \Rightarrow (vi) It is immediate by the definition of $\tilde{\star}$. Indeed, $\tilde{\star} := \star_{(\mathcal{F}^\star)_f}$. It is clear that, if \mathcal{F}^\star is finitely generated, then $\mathcal{F}^\star = (\mathcal{F}^\star)_f$. Then, by the hypothesis, $\tilde{\star} = \star_{\mathcal{F}^\star}$.

(vi) \Rightarrow (v) $\mathcal{F}^\star = \mathcal{F}^{\star_{\mathcal{F}^\star}} = \mathcal{F}^{\tilde{\star}}$, by Theorem 1.27(1). Since $\tilde{\star}$ is of finite type, then $\mathcal{F}^{\tilde{\star}}$ is finitely generated (Theorem 1.30). Hence \mathcal{F}^\star is finitely generated. \square

3.2 Quasi- \star -invertibility

In this section, we introduce a new notion of invertibility with respect to a semistar operation, the *quasi- \star -invertibility*. The reason for introducing this notion is the following observation: if \star is a star operation on an integral domain D , it is easy to see that, for an ideal I of D , the fact that $(II^{-1})^\star = D$ is equivalent to the fact that there exists a fractional ideal H such that $(IH)^\star = D$, that is, I is \star -invertible if and only if I is invertible with respect to the “star composition” \times given by $I \times J := (IJ)^\star$, for $I, J \in \mathbf{F}(D)$. This is not true in general for semistar invertibility, as the following example shows.

Example 3.10. Let D be an almost Dedekind domain, that is not Dedekind (cf. for instance [37, Section 2 and the references]). Then, in D there exists a prime (= maximal) ideal P , such that P is not invertible (otherwise, D would be a Dedekind domain). Then, $P^{-1} = D$ [24, Corollary 3.1.3], since D is a Prüfer domain. Consider the semistar operation $\star := \star_{\{P\}}$. Then, $(P(D_P : P))^\star = (PD_P(D_P : PD_P))D_P = D_P = D^\star$, since D_P is a DVR and so PD_P is invertible with inverse $(D_P : PD_P)$. Then, P is invertible with respect to the “semistar composition” (with inverse $(D_P : P)$), but $(PP^{-1})^\star = (PD)^\star = P^\star = PD_P \subsetneq D_P = D^\star$, thus P is not \star -invertible.

We show in next lemma that the fact that an ideal I of an integral domain D is invertible with respect to the “semistar” product is equivalent to the fact that I^\star is \star_ι -invertible in D^\star (where ι is the canonical embedding of D in D^\star).

Lemma 3.11. *Let \star be a semistar operation on an integral domain D and ι the canonical embedding of D in D^\star . Let $I \in \overline{\mathbf{F}}(D)$. Then, I^\star is \star_ι -invertible if and only if there exists $H \in \overline{\mathbf{F}}(D)$ such that $(IH)^\star = D^\star$.*

Proof. If I^\star is \star_ι -invertible, then $(I^\star(D^\star : I^\star))^{\star_\iota} = (D^\star)^{\star_\iota} = D^\star$. It follows that $(I(D^\star : I))^\star = D^\star$. So, take $H = (D^\star : I)$. Conversely, if $(IH)^\star = D^\star$, then $H \subseteq (D^\star : I)$. It follows that $D^\star = (IH)^\star \subseteq (I^\star(D^\star : I^\star))^\star \subseteq D^\star$. Hence, $(I^\star(D^\star : I^\star))^{\star_\iota} = D^\star$ and I^\star is \star_ι -invertible. \square

Let D be an integral domain and \star a semistar operation on D . We say that an ideal I of D is *quasi- \star -invertible* if it satisfies one of the equivalent conditions of Lemma 3.11.

Note that $I^\star \in \text{Inv}(D^\star, \star_\iota)$ implies that $I^\star \in \mathbf{F}(D^\star)$. We denote by $\text{QInv}(D, \star)$ the set of all quasi- \star -invertible D -submodules of K and, when $\star = d$, we set $\text{QInv}(D)$, instead of $\text{QInv}(D, d)$. We note that $\text{Inv}(D, \star) \subseteq \text{QInv}(D, \star)$. As a matter of fact, we have: $D^\star = (II^{-1})^\star = (I^\star(D : I)^\star)^\star \subseteq (I^\star(D^\star : I^\star))^\star = (I^\star(D^\star : I^\star))^{\star_\iota} \subseteq (D^\star)^\star = D^\star$. We have already shown in Example 3.10 that the inclusion can be proper. Moreover, it is obvious that $\text{QInv}(D) = \text{Inv}(D)$.

Next we prove an analogue of Lemma 3.1 for quasi- \star -invertible ideals.

Lemma 3.12. *Let \star, \star_1, \star_2 be semistar operations on an integral domain D . Then:*

- (0) $D^\star \in \text{QInv}(D, \star)$.
- (1) If $\star_1 \leq \star_2$, then $\text{QInv}(D, \star_1) \subseteq \text{QInv}(D, \star_2)$. In particular, we have $\text{QInv}(D) \subseteq \text{QInv}(D, \tilde{\star}) \subseteq \text{QInv}(D, \star_f) \subseteq \text{QInv}(D, \star)$.
- (2) $I, J \in \text{QInv}(D, \star)$ if and only if $IJ \in \text{QInv}(D, \star)$.
- (3) If $I \in \text{QInv}(D, \star)$, then $(D^\star : I) \in \text{QInv}(D, \star)$.
- (4) If $I \in \text{QInv}(D, \star)$, then $I^{v(D^\star)} \in \text{QInv}(D, \star)$.

Proof. (0) and (1) are straightforward.

(2) we notice that $I, J \in \text{QInv}(D, \star)$ if and only if $I^\star, J^\star \in \text{Inv}(D^\star, \star_\iota)$, where \star_ι is defined as above. It follows (from Lemma 3.1(2)) that $I, J \in \text{QInv}(D, \star)$ if and only if $I^\star J^\star \in \text{Inv}(D^\star, \star_\iota)$. It is easy to see that this happens if and only if $(IJ)^\star \in \text{Inv}(D^\star, \star_\iota)$, that is, if and only if $IJ \in \text{QInv}(D, \star)$.

(3) It is clear.

(4) It is an immediate consequence of Lemma 3.1(4) and of the fact that $(v(D^\star))_\iota = v_{D^\star}$ (Example 2.3 and Proposition 2.15(1)), where ι is the canonical embedding of D in D^\star and, as usual, v_{D^\star} is the v -operation of D^\star . \square

Corollary 3.13. *Let \star be a semistar operation on an integral domain D and let $I \in \overline{\mathbf{F}}(D)$.*

- (1) *If I is quasi- \star -invertible, then I is quasi- $v(D^\star)$ -invertible and $I^\star = I^{v(D^\star)}$.*
- (2) *If I is quasi- \star_f -invertible, then I is quasi- $t(D^\star)$ -invertible and $I^\star = I^{t(D^\star)}$.*

Proof. (1) First, we notice that, since $\star \leq v(D^\star)$ (Corollary 2.20(1)), I is quasi- $v(D^\star)$ -invertible (Lemma 3.12(1)) and $(I^\star)^{v(D^\star)} = I^{v(D^\star)}$ (Proposition 1.5). Now, by definition of quasi- \star -invertibility, I^\star is \star_ι -invertible as an ideal of D^\star . So, by Corollary 3.3, $I^\star = (I^\star)^{\star_\iota} = (I^\star)^{v(D^\star)^{\star_\iota}} = (I^\star)^{v(D^\star)} = I^{v(D^\star)}$.

(2) It follows, using a similar argument, by the fact that $\star \leq t(D^\star)$ (Corollary 2.20(2)), applying Corollary 3.3(2). \square

Remark 3.14. (a) Note that if I is a quasi- \star -invertible ideal of D , then every ideal J of D , with $I \subseteq J \subseteq I^* \cap D$, is also quasi- \star -invertible.

More precisely, let $I, J \in \mathbf{F}(D)$ [respectively, $I, J \in \overline{\mathbf{F}}(D)$], assume that $J \subseteq I$, $J^* = I^*$ and that I is \star -invertible [respectively, quasi- \star -invertible] then J is \star -invertible [respectively, quasi- \star -invertible].

Conversely, let $I, J \in \overline{\mathbf{F}}(D)$, assume that $J \subseteq I$, $J^* = I^*$ and that J is quasi- \star -invertible then I is quasi- \star -invertible (but not necessarily \star -invertible, even if J is \star -invertible).

As a matter of fact, if I is \star -invertible, then $D^* = (I(D : I))^* = (J(D : I))^* \subseteq (J(D : J))^* \subseteq D^*$. The quasi- \star -invertible case is similar. Conversely, if J is quasi- \star -invertible then $D^* = (J(D^* : J))^* = (I(D^* : J))^*$, thus I is quasi- \star -invertible and $(D^* : J) = (D^* : J)^* = (D^* : I)^* = (D^* : I)$ (cf. also (d1)).

Example 3.10 shows the parenthetical part of the statement. Let D , P and \star be as in Example 3.10. Note that P^* is principal in (the DVR) $D^* = D_P$, thus $P^* = PD_P = tD_P$, for some nonzero $t \in PD_P$. Therefore, if $J := tD$, then $J^* = P^*$, i.e. P is \star -finite. We already observed that P is quasi- \star -invertible but not \star -invertible, even if obviously J is (\star -)invertible.

(b) Let $I, H', H'', J, L \in \overline{\mathbf{F}}(D)$. The following properties are straightforward:

$$(b1) \quad (IH')^* = D^* = (IH'')^* \Rightarrow H'^* = H''^* = (D^* : I)^* = (D^* : I).$$

$$(b2) \quad I \in \text{QInv}(D, \star), \quad IJ \subseteq IL \Rightarrow J^* \subseteq L^*.$$

$$(b3) \quad I \in \text{QInv}(D, \star), \quad J \subseteq I^* \Rightarrow \exists L \in \overline{\mathbf{F}}(D), (IL)^* = J^*.$$

[Take $L := (D^* : I)J$.]

$$(b4) \quad I, J \in \text{QInv}(D, \star), \quad (IL)^* = J^* \Rightarrow L \in \text{QInv}(D, \star).$$

[Set $H := I(D^* : J)$, and note that $(LH)^* = D^*$.]

$$(b5) \quad I, J \in \text{QInv}(D, \star) \Rightarrow (D^* : IJ) = (D^* : IJ)^* = ((D^* : I)(D^* : J))^*.$$

$$(b6) \quad I, J \in \text{QInv}(D, \star) \Rightarrow \exists L \in \text{QInv}(D, \star), L \subseteq I^*, L \subseteq J^*.$$

[Take $z \in K$, $z \neq 0$, such that $zI \subseteq D^*$, $zJ \subseteq D^*$, and set $L := zIJ$.]

$$(b7) \quad I, J \in \text{QInv}(D, \star), \quad I+J \in \text{QInv}(D, \star) \Rightarrow I^{v(D^*)} \cap J^{v(D^*)} \in \text{QInv}(D, \star).$$

[Recall that $\star \leq v(D^*)$ and note that:

$$\begin{aligned} & ((D^* : I)(D^* : J)(I+J))^* = (((D^* : I)I)^*(D^* : J) + (D^* : I)((D^* : J)J))^* \\ & = ((D^* : J) + (D^* : I))^* = ((D^* : J^{v(D^*)}) + (D^* : I^{v(D^*)}))^* \Rightarrow \\ & ((D^* : I)(D^* : J)(I+J))^{v(D^*)} = ((D^* : I^{v(D^*)}) + (D^* : J^{v(D^*)}))^{v(D^*)} \\ & \Rightarrow \\ & (D^* : ((D^* : I)(D^* : J)(I+J))) = (D^* : ((D^* : I^{v(D^*)}) + (D^* : J^{v(D^*)}))) \\ & = (D^* : (D^* : I^{v(D^*)})) \cap (D^* : (D^* : J^{v(D^*)})) = I^{v(D^*)} \cap J^{v(D^*)}. \end{aligned}$$

.]

(b8) $I, J \in \mathbf{QInv}(D, \star)$, $I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathbf{QInv}(D, \star) \Rightarrow I+J \in \mathbf{QInv}(D, v(D^\star))$.
 [Since $I^{v(D^\star)} \cap J^{v(D^\star)} = (D^\star : ((D^\star : I)(D^\star : J)(I+J)))$ and hence
 $(D^\star : (I^{v(D^\star)} \cap J^{v(D^\star)})) = ((D^\star : I)(D^\star : J)(I+J))^{v(D^\star)}$, then apply
 (b4) to conclude that $I+J \in \mathbf{QInv}(D, v(D^\star))$.]

(c) *Mutatis mutandis*, the statements in (b) hold for \star -invertibles. More precisely: Let \star be a semistar operation on an integral domain D and let $I, H', H'', J, L \in \mathbf{F}(D)$, then:

$$(c1) \quad I \in \mathbf{Inv}(D, \star), (IH')^\star = D^\star = (IH'')^\star \Rightarrow H'^\star = H''^\star = (I^{-1})^\star.$$

$$(c2) \quad I \in \mathbf{Inv}(D, \star), IJ \subseteq IL \Rightarrow J^\star \subseteq L^\star.$$

$$(c3) \quad I \in \mathbf{Inv}(D, \star), J \subseteq I^\star \Rightarrow \exists L \in \mathbf{F}(D), (IL)^\star = J^\star.$$

$$(c4) \quad I, J \in \mathbf{Inv}(D, \star), (IL)^\star = J^\star \Rightarrow L \in \mathbf{QInv}(D, \star), (D^\star : L) = (I(D : J))^\star.$$

Note that, under the present hypotheses, $L \in \mathbf{Inv}(D, \star)$ if and only if $(D : L)^\star = (I(D : J))^\star$.

$$(c5) \quad I, J \in \mathbf{Inv}(D, \star) \Rightarrow (D : IJ)^\star = ((D : I)(D : J))^\star.$$

$$(c6) \quad I, J \in \mathbf{Inv}(D, \star) \Rightarrow \exists L \in \mathbf{Inv}(D, \star), L \subseteq I, L \subseteq J.$$

$$(c7) \quad I, J \in \mathbf{Inv}(D, \star), I+J \in \mathbf{Inv}(D, \star) \Rightarrow I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathbf{Inv}(D, \star).$$

$$(c8) \quad I, J \in \mathbf{Inv}(D, \star), I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathbf{Inv}(D, \star) \Rightarrow I+J \in \mathbf{Inv}(D, v(D^\star)).$$

Our next goal is to extend Proposition 3.7 to the case of quasi- \star_f -invertibles. We need the following:

Lemma 3.15. *Let \star be a semistar operation on an integral domain D with quotient field K and let ι be the canonical embedding of D in D^\star . Let $I \in \overline{\mathbf{F}}(D)$. Then, I is \star -finite if and only if I^\star is \star_ι -finite.*

Proof. If I is \star -finite, then there exists $J \in \mathbf{f}(D)$ such that $I^\star = J^\star$. It is clear that $(JD^\star)^{\star_\iota} = I^\star$, with $JD^\star \in \mathbf{f}(D^\star)$. Thus, I^\star is \star_ι -finite. Conversely, let I^\star be \star_ι -finite. Then, there exists a finitely generated fractional ideal J_0 of D^\star , $J_0 = (a_1, a_2, \dots, a_n)D^\star$, with $\{a_1, a_2, \dots, a_n\} \subseteq K$, such that $J_0^\star = J_0^{\star_\iota} = I^{\star_\iota} = I^\star$. Set $J = (a_1, a_2, \dots, a_n)D \in \mathbf{f}(D)$. Then, $J^\star = (a_1D + a_2D + \dots + a_nD)^\star = (a_1D^\star + a_2D^\star + \dots + a_nD^\star)^\star = J_0^\star = I^\star$, and so I is \star -finite. \square

Proposition 3.16. *Let \star be a semistar operation on an integral domain D and let $I \in \overline{\mathbf{F}}(D)$. Then I is quasi- \star_f -invertible if and only if I and $(D^\star : I)$ are \star_f -finite (hence, \star -finite) and I is quasi- \star -invertible.*

Proof. Let ι be the canonical embedding of D in D^* .

For the “if” part, use the same argument of the proof of the “if” part of Proposition 3.7.

The “only if” part. Since I is quasi- \star_f -invertible, then $(D^* : I)$ is also quasi- \star_f -invertible, thus we have that I^{\star_f} and $(D^* : I)^{\star_f} = (D^* : I)$ are $(\star_f)_\iota$ -invertibles. Then, I^{\star_f} and $(D^* : I)$ are $(\star_f)_\iota$ -finite (Corollary 3.7) and then I and $(D^* : I)$ are \star_f -finite, by Lemma 3.15. Clearly I is quasi- \star -invertible, since $\star_f \leq \star$ (Lemma 3.12 (1)). \square

It is natural to ask under which conditions a quasi- \star -invertible fractional ideal is \star -invertible. Let $I \in \mathbf{F}(D)$ be quasi- \star -invertible. Then $(I(D^* : I))^* = D^*$. Suppose that I is also \star -invertible, that is, $(I(D : I))^* = D^*$. Then, $(D : I)^* = ((D : I)(I(D^* : I))^*)^* = (((D : I)I)^*(D^* : I))^* = (D^* : I)^* = (D^* : I) = (D^* : I^*) \supseteq (D : I)^*$. Therefore we have the following (cf. also Remark 3.2(b)):

Proposition 3.17. *Let \star be a semistar operation on an integral domain D . Let I be a quasi- \star -invertible fractional ideal of D . Then, I is \star -invertible if and only if $(D : I)^* = (D^* : I)$ (i.e. $(I^{-1})^* = (I^*)^{-1}$). \square*

The following corollary is straightforward (in particular, part (2) follows immediately from the fact that if \star is a stable semistar operation on an integral domain D , then $(E : F)^* = (E^* : F^*)$, for each $E \in \overline{\mathbf{F}}(D)$, $F \in \mathbf{f}(D)$).

Corollary 3.18. *Let \star be a semistar operation on an integral domain D , and let $I \in \mathbf{F}(D)$.*

- (1) *If \star is a (semi)star operation then I is quasi- \star -invertible if and only if I is \star -invertible.*
- (2) *If \star is stable and $I \in \mathbf{f}(D)$ then I is quasi- \star -invertible if and only if I is \star -invertible. \square*

We notice that if \star is a semistar operation of finite type, \star -invertibility depends only on the set of quasi- \star -maximal ideals of D . Indeed, it is clear that $I \in \mathbf{F}(D)$ is \star -invertible if and only if $(II^{-1})^* \cap D$ is not contained in any quasi- \star -maximal ideal. Then, from Proposition 1.8(2), we deduce immediately the following general result (cf. [23, Proposition 4.25]):

Proposition 3.19. *Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if and only if I is $\tilde{\star}$ -invertible. \square*

A classical example due to Heinzer can be used for describing the content of the previous proposition.

Example 3.20. Let K be a field and X an indeterminate over K . Set $D := K[[X^3, X^4, X^5]]$ and $M := (X^3, X^4, X^5)D$. It is easy to see that D is a one-dimensional Noetherian local integral domain, with maximal ideal M . Let $\star := v$, note that in this case $v = \star = \star_f = t$ and $\mathcal{M}(v) = \{M\}$, since $M = (D : K[[X]])$. Therefore, $w = \tilde{v} = d$. In this situation $\text{Inv}(D, v) = \text{Inv}(D, t) = \text{Inv}(D, w) = \text{Inv}(D) = \{zD \mid z \in K, z \neq 0\}$. But $v = t \neq w = d$, because in general $(I \cap J)^t$ is different from $I^t \cap J^t$ in D , since D is not a Gorenstein domain [2, Theorem 5, Corollary 5.1] and [54, Theorem 222].

A result “analogous” to Proposition 3.19 does not hold, in general, for quasi-semistar-invertibility, as we show in the following:

Example 3.21. Let D be a pseudo-valuation domain, with maximal ideal M , such that $V := M^{-1}$ is a DVR (for instance, take two fields $k \subsetneq K$ and let $V := K[[X]]$, $M := XK[[X]]$ and $D := k + M$). Consider the semistar operation of finite type $\star := \star_{\{V\}}$, defined by $E^{\star_{\{V\}}} := EV$, for each $E \in \overline{\mathbf{F}}(D)$. It is clear that M is the only quasi- \star -maximal ideal of D . Thus, $\tilde{\star} = \star_{\{M\}} = d$, the identity (semi)star operation of D . We have $(M(V : M))^{\star} = (M(V : M))V = V$, since V is a DVR. Hence, M is quasi- \star -invertible. On the other side, M is not invertible (i.e., not quasi- $\tilde{\star}$ -invertible), since $MM^{-1} = MV = M$.

Under the assumption $D^{\star} = D^{\tilde{\star}}$ we have the following extension of Proposition 3.19 to the case of quasi-semistar-invertibility:

Proposition 3.22. *Let \star be a semistar operation on an integral domain D . Suppose that $D^{\star} = D^{\tilde{\star}}$. Let $I \in \overline{\mathbf{F}}(D)$. Then I is quasi- \star_f -invertible if and only if I is quasi- $\tilde{\star}$ -invertible*

Proof. If I is quasi- $\tilde{\star}$ -invertible, then there exists $J \in \overline{\mathbf{F}}(D)$ with $(IJ)^{\tilde{\star}} = D^{\tilde{\star}}$. This implies $(IJ)^{\star_f} = D^{\star_f}$, since $\tilde{\star} \leq \star_f$. Conversely, suppose that there exists $J \in \overline{\mathbf{F}}(D)$ such that $(IJ)^{\star_f} = D^{\star_f}$. Then $IJ \subseteq D^{\star_f} = D^{\star} = D^{\tilde{\star}}$. Thus, $(IJ)^{\tilde{\star}} \subseteq D^{\tilde{\star}}$. If $(IJ)^{\tilde{\star}} \subsetneq D^{\tilde{\star}}$, then $(IJ)^{\tilde{\star}} \cap D \subsetneq D$ is a quasi- $\tilde{\star}$ -ideal of D . It follows that $(IJ)^{\tilde{\star}} \cap D$ is contained in a quasi- $\tilde{\star}$ -maximal P of D . From Proposition 1.8, P is also a quasi- \star_f -maximal. Then, $(IJ)^{\star_f} \cap D \subseteq ((IJ)^{\tilde{\star}} \cap D)^{\star_f} \subseteq P^{\star_f} \subsetneq D^{\star_f}$, a contradiction. Then, I is quasi- $\tilde{\star}$ -invertible. \square

Remark 3.23. (a) If \star is a semistar operation on an integral domain D , we already observed (Remark 3.2(a)) that $\text{Inv}(D, \star)$ is not a group with respect to the standard multiplication of fractional ideals. In the set of the \star -invertible \star -fractional ideals, i.e. in the set $\text{Inv}^{\star}(D) := \{I \in \text{Inv}(D, \star) \mid I = I^{\star}\}$, we can introduce a semistar composition “ \times ” in the following way $I \times J := (IJ)^{\star}$. Note that $(\text{Inv}^{\star}(D), \times)$ is still not a group, in general, because for instance it does not possede an identity element (e.g. when $D^{\star} \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$).

On the other hand, $\text{QInv}^*(D) := \{I \in \text{QInv}(D, \star) \mid I = I^*\}$, with the semistar composition “ \times ” introduced above, is always a group, having as identity D^* and unique inverse of $I \in \text{QInv}^*(D)$ the D -module $(D^* : I) \in \overline{\mathbf{F}}(D)$, which belongs to $\text{QInv}^*(D)$. This fact provides also one of the motivations for considering $\text{QInv}(D, \star)$ and $\text{QInv}^*(D)$ (and not only $\text{Inv}(D, \star)$ and $\text{Inv}^*(D)$, as in the “classical” star case).

It is not difficult to prove that: *let \star be a semistar operation on an integral domain D , then:*

$$(\text{Inv}^*(D), \times) \text{ is a group} \Leftrightarrow (D : D^*) \neq (0).$$

As a matter of fact, (\Rightarrow) holds because $D^* \in \text{Inv}^*(D) \subseteq \mathbf{F}(D)$ and so $(D : D^*) \neq (0)$. (\Leftarrow) holds because $(D : D^*) \neq (0)$ implies that $D^* \in \text{Inv}^*(D)$ and, for each $I \in \text{Inv}^*(D)$, we have $(D^* : I) \in \mathbf{F}(D)$, thus $(D : I)^* = (D^* : I)$ (Remark 3.14(d1)) and so the inverse of each element $I \in \text{Inv}^*(D)$ exists and is uniquely determined in $\text{Inv}^*(D)$.

Note that, even if $(\text{Inv}^*(D), \times)$ is a group, $\text{Inv}^*(D)$ could be a proper subset of $\text{QInv}^*(D)$. For this purpose, take D, V, M as in Example 3.21, in this case $D^* = V$ and $(D : V) = M \neq (0)$, hence $(\text{Inv}^*(D), \times)$ is a group, but $M \in \text{QInv}^*(D) \setminus \text{Inv}^*(D)$.

(b) Note that, if \star is a semistar operation on an integral domain D , the group $(\text{QInv}^*(D), \times)$ can be identified with a more classic group of star-invertible star-ideals. As a matter of fact, it is easy to see that:

$$(\text{QInv}^*(D), \times) = (\text{Inv}^{\star_\iota}(D^*), \times')$$

where $\iota : D \rightarrow D^*$ is the canonical embedding and the (semi)star composition “ \times' ” in $\text{Inv}^{\star_\iota}(D^*)$ is defined by $E \times' F := (EF)^{\star_\iota}$.

(c) Let \star_1, \star_2 be two semistar operations on an integral domain D . If $\star_1 \leq \star_2$ then $\text{Inv}(D, \star_1) \subseteq \text{Inv}(D, \star_2)$ and $\text{QInv}(D, \star_1) \subseteq \text{QInv}(D, \star_2)$. Note that it is not true in general that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$ or that $\text{QInv}^{\star_1}(D) \subseteq \text{QInv}^{\star_2}(D)$, because there is no reason for a \star_1 -ideal (or $-$ module) to be a \star_2 -ideal (or $-$ module). For instance, let T be a proper overring of an integral domain D , let $\star_1 := d$ be the identity (semi)star operation on D and let $\star_1 := \star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\mathbf{F}}(D)$. If I is a nonzero principal ideal of D , then obviously $I \in \text{Inv}^{\star_1}(D)$ ($= \text{Inv}(D) = \text{QInv}^{\star_1}(D)$) but I does not belong to $\text{QInv}^{\star_2}(D)$ (and, in particular, it does not belong to $\text{Inv}^{\star_2}(D)$), because $I^{\star_2} = IT \neq I$.

Note that, even if $\text{Inv}(D, \star_1) = \text{Inv}(D, \star_2)$, for some pair of semistar operations $\star_1 \leq \star_2$, it is not true in general that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$. Take D, V, M as in Example 3.21. Let $\star_1 := d$ be the identity (semi)star operations on D and let $\star_2 := \star_{\{V\}}$. In this case, $\text{Inv}(D, \star_1) = \text{Inv}(D, \star_2)$, because $\star_1 = \widetilde{\star_2}$ and $\star_2 = (\star_2)_f$ (Proposition 3.19). But, $\text{Inv}^{\star_2}(D) \subsetneq \text{Inv}^{\star_1}(D) = \text{Inv}(D)$, because $\text{Inv}^{\star_2}(D) \subseteq \text{Inv}^{\star_1}(D) = \text{Inv}(D)$ since each \star_2 -ideal is obviously a \star_1 -ideal, and moreover the proper inclusion holds because, as above, a nonzero principal ideal of D belongs to $\text{Inv}(D)$ but not to $\text{Inv}^{\star_2}(D)$.

On the other hand, if $\star_1 \leq \star_2$ are two *star operations* on D , then it is known that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$, essentially because, in this case, $I \in \text{Inv}^{\star_1}(D)$ implies that $I = I^{\star_1} = I^v$ and so $I = I^{\star_2}$ [6, Proposition 3.3].

(d) Let \star be a semistar operation on an integral domain D and let $I, J \in \mathbf{F}(D)$ [respectively, $I, J \in \overline{\mathbf{F}}(D)$]. Assume that I is a \star -invertible [respectively, quasi- \star -invertible] \star -ideal of D , then:

$$(IJ^v)^\star = (I : (D : J)) \text{ [respectively, } (IJ^{v(D^\star)})^\star = (I : (D^\star : J))].$$

Recall that, since $I = I^\star$, then $(I : (D : J)) = (I : (D : J))^\star$. It is obvious that $IJ^v \subseteq (I : (D : J^v)) = (I : (D : J))$ and thus $(IJ^v)^\star \subseteq (I : (D : J))$. Conversely, if $z \in (I : (D : J))$ then $z(D : J) \subseteq I$ and so $z(D : I) \subseteq J^v$. Therefore $z \in zD^\star = z((D : I)I)^\star \subseteq (IJ^v)^\star$.

For the quasi- \star -invertible case, if $I = I^\star$, then $(I : (D^\star : J)) = (I : (D^\star : J))^\star$ and $I = ID^\star$. It is obvious that $IJ^{v(D^\star)} \subseteq (I : (D^\star : J^{v(D^\star)})) = (I : (D^\star : J))$ and thus $(IJ^{v(D^\star)})^\star \subseteq (I : (D^\star : J))$. Conversely, if $z \in (I : (D^\star : J))$ then $z(D^\star : J) \subseteq I$ and so $z(D^\star : I) \subseteq J^{v(D^\star)}$. Therefore $z \in zD^\star = z((D^\star : I)I)^\star \subseteq (IJ^{v(D^\star)})^\star$.

3.3 Semistar invertibility and the Nagata ring

In this section, we investigate the behavior of a \star -invertible ideal (when \star is a semistar operation) with respect to the localizations at quasi- \star -maximal ideals and in the passage to semistar Nagata ring.

In the next theorem, in the spirit of Kaplansky's theorem on (d -)invertibility [54, Theorem 62], we extend a characterization of t -invertibility proven in [55, Corollary 3.2] and two Kang's results proven in the star setting [53, Theorem 2.4 and Proposition 2.6].

Theorem 3.24. *Let \star be a semistar operation on an integral domain D . Assume that $\star = \star_f$. Let $I \in \mathbf{f}(D)$, then the following are equivalent:*

- (i) I is \star -invertible.
- (ii) $ID_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star)$ (and then ID_Q is principal in D_Q).
- (iii) $I \text{Na}(D, \star) \in \text{Inv}(\text{Na}(D, \star))$.

Proof. (i) \Rightarrow (ii). If $(II^{-1})^\star = D^\star$, then $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. Since $I \in \mathbf{f}(D)$, by flatness we have:

$$I^{-1}D_Q = (D : I)D_Q = (D_Q : ID_Q) = (ID_Q)^{-1}.$$

Therefore, for each $Q \in \mathcal{M}(\star)$, since $II^{-1} \not\subseteq Q$, we have:

$$D_Q = (II^{-1})D_Q = ID_Q I^{-1}D_Q = ID_Q (ID_Q)^{-1}.$$

(ii) \Rightarrow (iii). From the assumption and from the proof of (i) \Rightarrow (ii), we have that $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. Since $I \in \mathbf{f}(D)$, by the flatness of the canonical homomorphism $D \rightarrow D[X]_{N(\star)} = \text{Na}(D, \star)$, we have:

$$(I[X]_{N(\star)})^{-1} = (D[X]_{N(\star)} : I[X]_{N(\star)}) = (D : I)[X]_{N(\star)} = I^{-1}[X]_{N(\star)}.$$

Since $II^{-1} \not\subseteq Q$, then $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$, for each $Q \in \mathcal{M}(\star)$. From [29, Proposition 3.1(3)], we deduce that:

$$D[X]_{N(\star)} = (II^{-1})[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1},$$

where $I\text{Na}(D, \star) = I[X]_{N(\star)}$.

(iii) \Rightarrow (i). From the assumption and from the previous considerations, we have:

$$D[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1} = (II^{-1})[X]_{N(\star)},$$

and thus $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$, for each $Q \in \mathcal{M}(\star)$. This fact implies that $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. From [29, Lemma 2.4 (1)], we deduce immediately that $(II^{-1})^\star = D^\star$. \square

Corollary 3.25. *Let \star be a stable semistar operation of finite type on D , and let $I \in \mathbf{f}(D)$. Then, the conditions (i)–(iii) of Theorem 3.24 are equivalent to:*

(iv) *I is quasi- \star -invertible.*

Proof. Apply Corollary 3.18. \square

Remark 3.26. It is known [53, Proposition 2.6] (cf. also [5, Section 1] and [17, Section 1]) that, *if \star is a star operation of finite type on an integral domain D , an ideal I of D is \star -invertible if and only if it is \star -finite and locally principal (when localized at the \star -maximal ideals).* As a matter of fact, by Corollary 3.7, we have that, if I is \star -invertible, then I is \star -finite. Moreover, $(II^{-1})^\star = D$ implies $II^{-1} \not\subseteq Q$, for each \star -maximal ideal Q of D . It follows that $ID_Q I^{-1} D_Q = D_Q$. Thus, ID_Q is invertible (hence, principal) in D_Q . Conversely, assume that $I^\star = J^\star$, with $J \in \mathbf{f}(D)$, $J \subseteq I$. It is clear that $I^{-1} = J^{-1}$, since $I^v = (I^\star)^v = (J^\star)^v = J^v$, being $\star \leq v$ [38, Theorem 34.1(4)]. Suppose that I is not \star -invertible, that is, $(II^{-1})^\star \subsetneq D$. Then, there exists a \star -maximal ideal Q of D , such that $II^{-1} \subseteq Q$. It follows $QD_Q \supseteq ID_Q I^{-1} D_Q = ID_Q J^{-1} D_Q = ID_Q (JD_Q)^{-1} \supseteq ID_Q (ID_Q)^{-1}$, a contradiction, since ID_Q is principal.

We will see in a moment that the “if” part of a similar result for semistar operations does not hold, even if $I = I^\star$. More precisely, we can extend partially [32, Proposition 1.1] in the following way:

Let $I \in \mathbf{F}(D)$ and let \star be a semistar operation on D , the following properties are equivalent:

(i) I is \star_f -invertible;

(ii) $(Q : I) \subsetneq (D : I)$, for each $Q \in \mathcal{M}(\star_f)$;

(iii) $(Q : I) \subsetneq (D : I)$, for each $Q \in \mathcal{M}(\star_f)$ such that $Q \supseteq I(D : I)$.

Moreover, each of the previous properties implies the following:

(iv) I is \star_f -finite and $ID_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star_f)$ (and so ID_Q is principal in D_Q).

As a matter of fact, (i) \Rightarrow (ii) because $D^\star = (I(D : I))^\star$ and if $(Q : I) = (D : I)$, for some $Q \in \mathcal{M}(\star_f)$, then $I(D : I) = I(Q : I) \subseteq Q$, thus $(I(D : I))^{\star_f} \subseteq Q^{\star_f} \subsetneq D^\star$, hence we reach a contradiction. (ii) \Rightarrow (iii) is trivial. (iii) \Rightarrow (i): if not, $I(D : I) \subseteq Q$, for some $Q \in \mathcal{M}(\star_f)$, thus $(D : I) \subseteq (Q : I)$ and hence $(D : I) = (Q : I)$, which contradicts (iii).

Finally (ii) \Rightarrow (iv), because of Proposition 3.7 and because for $z_Q \in (D : I) \setminus (Q : I)$, we have $z_Q I \subset D \setminus Q$, and so $z_Q ID_Q = D_Q$, i.e. $ID_Q = (z_Q)^{-1} D_Q$, for each $Q \in \mathcal{M}(\star_f)$.

But note that, in the semistar setting, (iv) $\not\Rightarrow$ (i), even in case I is a \star_f -ideal, \star_f -finite, as the following example will show. However, we can re-establish a characterization in the quasi- \star -invertibility setting in the following way: if \star is a semistar operation of finite type on an integral domain D and if $I \in \overline{\mathbf{F}}(D)$, then $I \in \text{QInv}(D, \star)$ if and only if I^\star is \star -finite and $I^\star D_M^\star$ is principal, for each \star_l -maximal ideal M of D^\star .

Example 3.27. Let D be a valuation domain, P a nonzero nonmaximal noninvertible prime ideal of D such that D_P is a discrete valuation domain. (For instance, if K is a field and X, Y are two indeterminates over K , let $D := K + XK[X]_{(X)} + YK(X)[Y]_{(Y)}$ and $P := YK(X)[Y]_{(Y)}$; in this case D is a two-dimensional valuation domain, $D_P = K(X)[Y]_{(Y)}$ and $P = PD_P = YD_P \supsetneq YD$.) Set $\star := \star_{\{P\}}$. In this situation, $\star = \star_f$ and $\mathcal{M}(\star) = \{P\}$, thus $\star = \tilde{\star}$, i.e. \star is a stable semistar operation of finite type on D . Note that P is in fact a \star -ideal of D , since $P^\star = PD_P = P$. Moreover, $P^\star = PD_P = tD_P = (tD)^\star$ for some nonzero $t \in D_P$, i.e. P is a non zero principal ideal in $D^\star = D_P$, since D_P is a DVR, by assumption. Thus, P is a \star -ideal, \star -finite and locally principal, when localized at the quasi- \star -maximal ideal(s) of D . But P is not \star -invertible, since in this situation $(D : P) = (P : P) = D_P$ [24, Proposition 3.1.5] and hence $(P(D : P))^\star = (P(P : P))^\star = (PD_P)^\star = P^\star = P$. Note also that, in this situation, P is quasi- \star -invertible (because $(P(D^\star : P))^\star = (tD_P t^{-1} D_P)^\star = D_P = D^\star$) and $D^\star = D_P = (PD_P : PD_P) = (P : P)D_P = (P : P)^\star$.

Next two results generalize to the semistar setting [53, Theorem 2.12 and Theorem 2.14].

Corollary 3.28. *Let \star be a semistar operation on an integral domain D . Assume that $\star = \star_f$. Let $h \in D[X], h \neq 0$, then:*

$$\mathbf{c}(h) \in \text{Inv}(D, \star) \quad \Leftrightarrow \quad h \text{Na}(D, \star) = \mathbf{c}(h) \text{Na}(D, \star).$$

In particular, $\mathbf{c}(h) \in \text{Inv}(D, \star)$ if and only if $\mathbf{c}(h) \in \text{QInv}(D, \star)$.

Proof. The proof of the first part of the statement is based on the following result by D.D. Anderson [1, Theorem 1]: If R is a ring and $h \in R[X], h \neq 0$, then $hR(X) \subseteq \mathbf{c}(h)R(X)$ and, moreover, the following are equivalent:

- (1) $hR(X) = \mathbf{c}(h)R(X)$.
- (2) $\mathbf{c}(h)$ is locally principal (in R).
- (3) $\mathbf{c}(h)R(X)$ is principal (in $R(X)$).

(\Rightarrow) By Theorem 3.24 ((i) \Rightarrow (ii)), we have that $\mathbf{c}(h)D_Q$ is principal, for each $Q \in \mathcal{M}(\star)$. Hence,

$$\mathbf{c}(h)D_Q[X]_{N(\star)} = \mathbf{c}(h)(D[X]_{N(\star)})_{QD[X]_{N(\star)}} = \mathbf{c}(h)D_Q(X)$$

is principal, for each $Q \in \mathcal{M}(\star)$. By applying Anderson's result to the local ring $R = D_Q$, we deduce that $hD_Q(X) = \mathbf{c}(h)D_Q(X)$, for each $Q \in \mathcal{M}(\star)$. The conclusion follows from Proposition 1.40, (2) and (3)

(\Leftarrow) If $h \text{Na}(D, \star) = \mathbf{c}(h) \text{Na}(D, \star)$, then by localization we obtain that $hD_Q(X) = \mathbf{c}(h)D_Q(X)$, for each $Q \in \mathcal{M}(\star)$ (Proposition 1.40 and [38, Corollary 5.3]). By Anderson's result, we deduce that $\mathbf{c}(h)D_Q$ is principal, i.e. $\mathbf{c}(h)D_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star)$. The conclusion follows from Theorem 3.24 ((ii) \Rightarrow (i)).

The last part of the statement follows from the fact that $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star})$ (Proposition 1.40(4)) and from Corollary 3.18 and Proposition 3.19 or, directly, from Corollary 3.25. \square

Proposition 3.29. *Let \star be a semistar operation on an integral domain D . If H is an invertible ideal of $\text{Na}(D, \star)$, then H is principal in $\text{Na}(D, \star)$.*

Proof. We can assume that $H \in \text{Inv}(\text{Na}(D, \star))$ and $H \subseteq \text{Na}(D, \star)$, then, in particular, $H = (h_1, h_2, \dots, h_n) \text{Na}(D, \star)$, with $h_i \in D[X], 1 \leq i \leq n$. For each $Q \in \mathcal{M}(\star_f)$, by localization, $HD_Q(X) = (h_1, h_2, \dots, h_n)D_Q(X)$ is a nonzero principal ideal (Theorem 3.24 ((iii) \Rightarrow (ii))). By a standard argument, if $d_i := \deg(h_i)$, for $1 \leq i \leq n$, and if

$$h := h_1 + h_2X^{d_1+1} + h_3X^{d_1+d_2+2} + \dots + h_nX^{d_1+d_2+\dots+d_{n-1}+n-1} \in D[X],$$

then it is not difficult to see that $HD_Q(X) = hD_Q(X)$, for each $Q \in \mathcal{M}(\star_f)$. From Proposition 1.40(3), we deduce that $H \text{Na}(D, \star) = h \text{Na}(D, \star)$. \square

Chapter 4

Semistar Dedekind domains

4.1 Prüfer semistar multiplication domains

In this section we recall some results and prove new ones about Prüfer semistar multiplication domains. This class of domains was introduced by M. Fontana, P. Jara and E. Santos [26], to generalize the concept of Prüfer domains and PvMD to the context of semistar operations.

Let D be an integral domain and \star a semistar operation on D . We say that D is a *Prüfer \star -multiplication domain* ($P\star$ MD for short) if each finitely generated ideal of D is \star_f -invertible.

Since by Proposition 3.19, \star_f -invertibility and $\tilde{\star}$ -invertibility coincide, we have that D is a $P\star$ MD if and only if each finitely generated ideal of D is $\tilde{\star}$ -invertible.

Clearly D is a $P\star$ MD if and only if D is a $P\star_f$ MD if and only if D is a $P\tilde{\star}$ MD.

Let $\star_1 \leq \star_2$ be two semistar operations on D . If D is a $P\star_1$ MD then D is a $P\star_2$ MD. Indeed, by Proposition 1.7(2), $(\star_1)_f \leq (\star_2)_f$, and by Lemma 3.1(1), if an ideal I of D is $(\star_1)_f$ -invertible then it is also $(\star_2)_f$ -invertible.

The following characterization of $P\star$ MDs is due to M. Fontana, P. Jara and E. Santos [26, Theorem 3.1, Remark 3.1] and generalize several known results about PvMDs (cf. M. Griffin [40, Theorem 5], R. Gilmer [36, Theorem 2.5], J. Arnold and J. Brewer [8, Theorem 3], J. Querré [72, Théorème 3, page 279] and B. G. Kang [53, Theorem 3.5, Theorem 3.7]).

Proposition 4.1. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is a $P\star$ MD.
- (ii) D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$.
- (iii) $\text{Na}(D, \star)$ is a Prüfer domain.
- (iv) $\text{Na}(D, \star) = \text{Kr}(D, \star)$.

(v) $\tilde{\star}$ is an e.a.b. semistar operation.

(vi) \star_f is stable and e.a.b.

In particular, in a P \star MD, $\tilde{\star} = \star_f$. □

We can add three other equivalent conditions. The first two follow immediately from Corollary 3.18(2) and Proposition 3.22.

Proposition 4.2. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

(i) D is a P \star MD

(ii) Each finitely generated ideal of D is quasi- $\tilde{\star}$ -invertible.

(iii) $D^\star = D^{\tilde{\star}}$ and each finitely generated ideal of D is quasi- \star_f -invertible.

(iv) \star_f is stable and a.b.

Proof. (i) \Leftrightarrow (ii) We have noticed that D is a P \star MD if and only if each finitely generated ideal of D is $\tilde{\star}$ -invertible. By Corollary 3.18(2) we deduce immediately that, since $\tilde{\star}$ is a stable semistar operation, for finitely generated ideals $\tilde{\star}$ -invertibility and quasi- $\tilde{\star}$ -invertibility coincide. Hence we have the thesis. (i) \Rightarrow (iii) By definition of P \star MD, each finitely generated ideal of D is \star_f -invertible and then quasi- \star_f -invertible. Moreover, $\star_f = \tilde{\star}$ (Proposition 4.1). Then, $D^{\tilde{\star}} = D^{\star_f} = D^\star$.

(iii) \Rightarrow (ii) Since $D^\star = D^{\tilde{\star}}$, a quasi- \star_f -invertible ideal is also quasi- $\tilde{\star}$ -invertible (Proposition 3.22).

(i) \Rightarrow (iv) We know already (Proposition 4.1(i) \Rightarrow (vi)) that \star_f is stable. Let $E \in \mathbf{f}(D)$, $F, G \in \overline{\mathbf{f}}(D)$, such that $(EF)^{\star_f} = (EG)^{\star_f}$. From the definition of P \star MD, $(EE^{-1})^{\star_f} = D^\star$. We have that $(E^{-1}(EF)^{\star_f})^{\star_f} = (E^{-1}(EG)^{\star_f})^{\star_f}$. Thus, $F^{\star_f} = ((E^{-1}E)^{\star_f}F)^{\star_f} = ((E^{-1}E)^{\star_f}G)^{\star_f} = G^{\star_f}$.

(iv) \Rightarrow (i) It is a straightforward consequence of Proposition 4.1(vi) \Rightarrow (i) since an a.b. semistar operation is e.a.b. □

Next proposition is a consequence of Corollary 3.3(2) and gives a stronger condition than $\star_f = \tilde{\star}$ in a P \star MD (Proposition 4.1).

Proposition 4.3. *Let D be an integral domain and \star a semistar operation on D . If D is a P \star MD then $\tilde{\star} = \star_f = t(D^\star)$.*

Proof. Let $F \in \mathbf{f}(D)$. So, F is \star_f -invertible, since D is a P \star MD. Then, by Corollary 3.3(2), $F^\star = F^{t(D^\star)}$. It follows that \star coincides with $t(D^\star)$ on the finitely generated ideals and so $\star_f = t(D^\star)$. That $\tilde{\star} = \star_f$ is in Proposition 4.1. □

We give a new short proof of a result about the ascent of the P \star MD property, [26, Proposition 3.1].

Proposition 4.4. *Let D be an integral domain, T an overring of D and \star a semistar operation on D . Let ι be the canonical embedding of D in T . If D is a $P\star MD$ then T is a $P\star_\iota MD$. \square*

Proof. By Proposition 4.2(i) \Rightarrow (iv), \star_f is stable and a.b. By Proposition 2.11(2) and (4), the semistar operation $(\star_f)_\iota$ on T is stable and a.b. Then, T is a $P(\star_f)_\iota MD$ by Proposition 4.2(iv) \Rightarrow (i). Now, $(\star_f)_\iota \leq (\star_\iota)_f$, and so T is a $P(\star_\iota)_f MD$, that is, a $P\star_\iota MD$. \square

We prove a corollary of Proposition 4.3 and Proposition 4.4.

Corollary 4.5. *Let D be an integral domain and \star a semistar operation on D . If D is a $P\star MD$ then D^\star is a $Pv MD$.*

Proof. Let ι be the canonical embedding of D in D^\star . Since $\star_f = t(D^\star)$ (Proposition 4.3), we have that $(\star_f)_\iota = (t(D^\star))_\iota$. By Example 2.3 and Proposition 2.15(1), we have that $(\star_f)_\iota = t_{D^\star}$, the t -(semi)star operation of D^\star . Then, Proposition 4.4 implies that D^\star is a $Pt_{D^\star} MD$ or, equivalently, a $Pv_{D^\star} MD$. \square

Next example shows that, if D^\star is a $P\star_\iota MD$ (where ι is the canonical embedding of D in D^\star), not necessarily D is a $P\star MD$.

Example 4.6. Let (D, M) be a pseudo-valuation domain, and $V = M^{-1}$. Consider the semistar operation $\star := \star_{\{V\}}$ on D . Let ι be the canonical embedding of D in $D^\star = V$. Clearly, $\star_\iota = d_V$ (Example 1.37(1)) and V is a $Pd_V MD$ (that is, a Prüfer domain). On the contrary, D is not a $P\star MD$ (by Proposition 4.1(i) \Rightarrow (ii)), since M is the unique quasi- \star -maximal ideal and D_M is not a valuation domain.

More in general, the example shows that, if D is an integral domain, T is an overring of D , ι the embedding of D in T and \star a semistar operation such that T is a $P\star MD$, then D is not always a $P\star' MD$. In [26, Proposition 3.2], the authors prove that the flatness of T over D is a sufficient condition for D to be a $P\star' MD$.

We want to improve this result, giving necessary and sufficient conditions.

To do this we need to recall some results on quasi- \star -ideals and on semistar flatness.

Lemma 4.7. [29, Lemma 2.3(3)(4)] *Let D be an integral domain, \star a semistar operation on D and ι the embedding of D in D^\star .*

- (1) *If Q is a quasi- \star_f -maximal ideal of D , then there exists a quasi- $(\star_f)_\iota$ -maximal ideal M of D^\star such that $Q = M \cap D$.*
- (2) *If N is a quasi- $(\star_f)_\iota$ -prime ideal of D^\star then $N \cap D$ is a quasi- \star_f -prime ideal of D \square*

Let D be an integral domain and T an overring of D . Let \star be a semistar operation on D and let \star' be a semistar operation on T . We say [20, Section 3] that T is (\star, \star') -linked to D if, for each finitely generated integral ideal F of D , we have that $F^\star = D^\star$ implies $(FT)^{\star'} = T^{\star'}$ (equivalently, if $F \in \mathcal{F}^\star$ then $FT \in \mathcal{F}^{\star'}$). We say [20, Section 4] that T is (\star, \star') -flat over D if T is (\star, \star') -linked to D and for each $Q \in \mathcal{M}(\star'_f)$, $D_{Q \cap D} = T_Q$ (we have given earlier an equivalent definition of semistar flatness in Section 2.2 before Proposition 2.28: the equivalence of the two definitions follows from Proposition 4.8(2)).

We recall some results on semistar linkedness and semistar flatness that we will need in this and in the following sections.

Proposition 4.8. [20, Lemma 3.1(e), Proposition 3.2, Corollary 5.4, Theorem 4.5 and 5.7] *Let D be an integral domain and \star a semistar operation on D . Let T be an overring of D and \star' a semistar operation on T .*

- (1) T is (\star, \star_ι) -linked to D (where ι is the canonical embedding of D in T).
- (2) T is (\star, \star') -linked over D if and only if $(N \cap D)^{\star_f} \neq D^\star$, for each quasi- \star'_f -maximal ideal N .
- (3) T is (\star, \star') -flat over D if and only if $\text{Na}(T, \star')$ is a flat overring of $\text{Na}(D, \star)$.
- (4) If D is a $P^\star MD$ and T is (\star, \star') -linked to D then T is a $P^\star MD$.
- (5) D is a $P^\star MD$ if and only if for each overring R of D and for each semistar operation \star' on R such that R is (\star, \star') -linked to D then R is (\star, \star') -flat over D . \square

We can now prove a result about the relation between semistar flatness and the Nagata ring.

Proposition 4.9. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star . Then, the following are equivalent:*

- (i) $\text{Na}(D, \star) = \text{Na}(D^\star, \star_\iota)$.
- (ii) D^\star is (\star, \star_ι) -flat over D .
- (iii) $(D^\star)_P = D_{P \cap D}$ for each $P \in \mathcal{M}((\star_\iota)_f)$.
- (iv) $D^\star = D^{\tilde{\star}}$ and $\widetilde{(\star_\iota)} = (\tilde{\star})_\iota$.

Proof. (i) \Rightarrow (ii) It follows immediately from Proposition 4.8(3).

(ii) \Rightarrow (iii) It is trivial.

(iii) \Rightarrow (iv) First we note that, in this case, $(\star_f)_\iota = (\star_\iota)_f$ (Proposition 2.12). Now, let $E \in \overline{\mathbf{F}}(D^\star)$. Then, $E^{\tilde{\star}} = \bigcap_{Q \in \mathcal{M}(\star_f)} ED_Q$ and $E^{\tilde{\star}_\iota} =$

$\bigcap_{M \in \mathcal{M}((\star_f)_\iota)} E(D^\star)_M$. For each $Q \in \mathcal{M}(\star_f)$ there exists $M \in \mathcal{M}((\star_f)_\iota)$ such that $Q = M \cap D$ (Lemma 4.7(1)). Since, by the hypothesis, $D_Q = (D^\star)_M$, we have that $E^{\tilde{\star}_\iota} \subseteq E^{\tilde{\star}}$. Conversely, if M is a quasi- $(\star_f)_\iota$ -maximal ideal, then $M \cap D$ is a quasi- \star_f -prime ideal of D (Lemma 4.7(2)). So, $M \cap D \subseteq Q$, for some $Q \in \mathcal{M}(\star_f)$. It follows that $D_Q \subseteq D_{M \cap D} = (D^\star)_M$. Thus $E^{\tilde{\star}} \subseteq E^{\tilde{\star}_\iota}$ and so $E^{\tilde{\star}} = E^{\tilde{\star}_\iota}$. Moreover $D^\star = D^{\tilde{\star}}$. Indeed $D^\star \subseteq \bigcap_{M \in \mathcal{M}((\star_f)_\iota)} (D^\star)_M = \bigcap_{Q \in \mathcal{M}(\star_f)} D_Q = D^{\tilde{\star}} \subseteq D^\star$.

(iv) \Rightarrow (i) By Proposition 1.40(4), observing that, by the hypothesis, $D^\star = D^{\tilde{\star}}$ (and so, ι is also the canonical embedding of D in $D^{\tilde{\star}}$) we have $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star}) = \text{Na}(D^{\tilde{\star}}, (\tilde{\star})_\iota) = \text{Na}(D^{\tilde{\star}}, (\tilde{\star}_\iota)) = \text{Na}(D^\star, \star_\iota)$. \square

Corollary 4.10. *Let D be an integral domain and T an overring of D . Let ι be the canonical embedding of D in T and let \star be a semistar operation on T . Then, the following are equivalent:*

- (i) $\text{Na}(D, \star^\iota) = \text{Na}(T, \star)$
- (ii) T is (\star^ι, \star) -flat over D .
- (iii) $T_P = D_{P \cap D}$ for each $P \in \mathcal{M}(\star_f)$.
- (iv) $T = D^{\tilde{\star}^\iota}$ and $\tilde{\star} = (\tilde{\star}^\iota)_\iota$.

Proof. Apply Proposition 4.9 to $\star := \star^\iota$, recalling that by Proposition 2.15(1), $\star = (\star^\iota)_\iota$. \square

We have the following result for $P\star$ MDs:

Proposition 4.11. *Let D be an integral domain and \star a semistar operation on D . Assume that D^\star is a $P\star_\iota$ MD. Then, D is a $P\star$ MD if and only if one of the equivalent conditions of Proposition 4.9 holds.*

Proof. If D is a $P\star$ MD then \star_f is e.a.b. (Proposition 4.1(i) \Leftrightarrow (iv)), so $\star_f = (\star_f)_a$. So, by Proposition 4.1(i) \Leftrightarrow (iv), Proposition 1.39(ii) and Proposition 2.12, $\text{Na}(D, \star) = \text{Kr}(D, \star) = \text{Kr}(D, \star_f) = \text{Kr}(D^{(\star_f)^a}, ((\star_f)_a)_\iota) = \text{Kr}(D^{\star_f}, (\star_f)_\iota) = \text{Na}(D^{\star_f}, (\star_f)_\iota) = \text{Na}(D^{\star_f}, (\star_\iota)_f) = \text{Na}(D^\star, \star_\iota)$. The converse is straightforward by Proposition 4.1(i) \Leftrightarrow (iii). \square

With a similar argument we have the following corollary.

Corollary 4.12. *Let D be an integral domain, T an overring of D . Let ι be the canonical embedding of D in T and \star a semistar operation on T . Assume that T is a $P\star$ MD. Then, D is a $P\star^\iota$ MD if and only if one of the equivalent conditions of Corollary 4.10 holds.* \square

As a corollary we have the results proven in [26, Proposition 3.2 and Proposition 3.3]. First we prove an easy lemma.

Lemma 4.13. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star . Then D^\star is $(\tilde{\star}, \tilde{\star}_\iota)$ -flat over D .*

Proof. It is immediate by Proposition 4.8, since $\text{Na}(D, \star) = \text{Na}(D^\star, \tilde{\star}_\iota)$ (Proposition 1.40(4)). \square

Corollary 4.14. cf. [26, Proposition 3.2 and Proposition 3.3] *Let D be an integral domain.*

- (1) *Let T be a flat overring of D and let \ast be a semistar operation on T . Let $\iota := \iota_T$ be the canonical embedding of D in T . Assume that T is a $P\ast MD$. Then D is a $P\ast^\iota MD$.*
- (2) *Let \star be a semistar operation on D . Let $\iota := \iota_{D^\star}$ be the embedding of D in D^\star . Then D is a $P\star MD$ if and only if D^\star is a $P\star_\iota MD$.*

Proof. (1) It is immediate by Corollary 4.12.

(2) It is immediate by Lemma 4.13 and Proposition 4.11. \square

By using Corollary 4.5 and Corollary 4.12, we can characterize $P\star MD$ as a particular class of subrings of PvMDs.

Theorem 4.15. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) *D is a $P\star MD$;*
- (ii) *There exists an overring T of D such that T is a PvMD, $\star_f = t(T)$ and for each t_T -maximal ideal Q of T , $T_Q = D_{Q \cap D}$.*
- (iii) *There exists an overring T of D such that T is a PvMD, $\star_f = t(T)$ and T is $(t(T), t_T)$ -flat over D .*

Proof. (i) \Rightarrow (ii) Let $T := D^\star$. By Proposition 4.3 $\star_f = t(T)$ and by Corollary 4.5 T is a PvMD. Moreover, by Proposition 4.1 $\star_f = \tilde{\star}$. Thus, by Lemma 4.13 D^\star is $(\star_f, (\star_f)_\iota)$ -flat over D (where ι is the canonical embedding of D in T). Since $t_T = (t(T))_\iota = (\star_f)_\iota$ (Example 2.3 and Proposition 2.15(2)) we have the thesis.

(ii) \Rightarrow (i) Recall that D is a $P\star MD$ if and only if it is a $P\star_f MD$. Let ι be the canonical embedding of D in T . Since T is a $Pv_T MD$, it is a $Pt_T MD$. By the hypothesis on the t_T -maximal ideals, applying Corollary 4.12, we have that D is a $P(t_T)^\iota MD$, that is, a $P\star_f MD$ (recalling that, by definition, $(t_T)^\iota = t(T)$).

(ii) \Leftrightarrow (iii) follows immediately by Corollary 4.10(ii) \Leftrightarrow (iii), since by definition $t(T) = (t_T)^\iota$. \square

4.2 Semistar Noetherian domains

Domains having the ascending chain condition (a.c.c. for short) on classes of ideals play an important role in multiplicative ideal theory. Examples of this kind of domains are Noetherian domains, Mori domains (domains having the a.c.c. on divisorial ideals, see [10]) and strong Mori domains (domains having the a.c.c. on w -ideals, where w is the star operation described in Remark 1.31, [79]) have been intensively studied. We want to introduce a class of domains containing all these classes.

Let D be an integral domain and \star a semistar operation on D . We say that D is a \star -Noetherian domain if D has the ascending chain condition on quasi- \star -ideals.

Note that, if $d (= d_D)$ is the identity (semi)star operation on D , the d -Noetherian domains are just the usual Noetherian domains and the notions of v -Noetherian [respectively, w -Noetherian] domain and Mori [respectively, strong Mori] domain coincide.

Recall that the concept of *star* Noetherian domain has already been introduced, see for instance [3], [81] and [35]. Using ideal systems on commutative monoids, a similar general notion of noetherianity was considered in [41, Chapter 3].

Lemma 4.16. *Let D be an integral domain.*

- (1) *Let $\star \leq \star'$ be two semistar operations on D , then D is \star -Noetherian implies D is \star' -Noetherian.*

In particular:

- (1a) *A Noetherian domain is a \star -Noetherian domain, for any semistar operation \star on D .*

- (1b) *If \star is a (semi)star operation and if D is a \star -Noetherian domain, then D is a Mori domain.*

- (2) *Let T be an overring of D and $*$ a semistar operation on T . Let ι be the canonical embedding of D in T . If T is $*$ -Noetherian, then D is \star' -Noetherian. In particular, if \star is a semistar operation on D , such that D^\star is a \star_ι -Noetherian domain, then D is a \star -Noetherian domain.*

Proof. (1) The first statement holds because each quasi- \star' -ideal is a quasi- \star -ideal. (1a) and (1b) follow from (1) since, for each semistar operation \star , $d \leq \star$ and, if \star is a (semi)star operation, then $\star \leq v$.

(2) If we have a chain of quasi- \star' -ideals $\{I_n\}_{n \geq 1}$ of D that does not stop then, by considering $\{(I_n T)^\star\}_{n \geq 1}$, we get a chain of quasi- $*$ -ideals of T that does not stop, since two distinct quasi- \star' -ideals $I \neq I'$ of D are such that $(IT)^\star \neq (I'T)^\star$. The second part of the statement follows immediately applying the first part to the semistar operation $*$:= \star_ι , since $\star' = \star$ (Proposition 2.16(1)). \square

Remark 4.17. The converse of (2) in Lemma 4.16 does not hold in general. For instance, take $D \subset T$, where D is a Noetherian domain and T is a non-Noetherian overring of D and let ι be the canonical embedding of D in T . Let $\star := d_T$ and $\star := \star_{\{T\}}$. Note that $\star^\iota = \star$. Then, D is \star -Noetherian, by (1a) of Lemma 4.16, but $D^\star = T^\star = T$ is not \star -Noetherian (or, equivalently, \star_ι -Noetherian), because $\star = d_T = \star_\iota$ and T is not Noetherian. However, if $\star = \tilde{\star}$, the last statement of (2) in Lemma 4.16 can be reversed, as we will see in Proposition 4.23.

Lemma 4.18. *Let D be an integral domain and let \star be a semistar operation on D . Then, D is a \star -Noetherian domain if and only if each nonzero ideal I of D is \star_f -finite. In particular, if \star is a star operation on D and D is \star -Noetherian then \star is a star operation of finite type on D .*

Proof. For the “only if” part, let $x_1 \in I$, $x_1 \neq 0$, and set $I_1 := x_1D$. If $I^\star = I_1^\star$ we are done. Otherwise, it is easy to see that $I \not\subseteq I_1^\star \cap D$. Let $x_2 \in I \setminus (I_1^\star \cap D)$ and set $I_2 := (x_1, x_2)D$. By iterating this process, we construct a chain $\{I_n^\star \cap D\}_{n \geq 1}$ of quasi- \star -ideals of D . By assumption this chain must stop, i.e., for some $k \geq 1$, $I_k^\star \cap D = I_{k+1}^\star \cap D$, and so $I_k^\star = (I_k^\star \cap D)^\star = I^\star$. So, we conclude by taking $J := I_k$. Conversely, let $\{I_n\}_{n \geq 1}$ be a chain of quasi- \star -ideals in D and set $I := \bigcup_{n \geq 1} I_n$. Let $J \subseteq I$ be a finitely generated ideal of D such that $J^\star = I^\star$, so there exists $k \geq 1$ such that $J \subseteq I_k$ and $J^\star = I_k^\star = I^\star$. This implies that the chain of quasi- \star -ideals $\{I_n\}_{n \geq 1}$ stops (in fact, $I_n = I_k = I^\star \cap D$, for each $n \geq k$). \square

Corollary 4.19. *Let D be an integral domain and \star a semistar operation on D . Then, D is \star -Noetherian if and only if D is \star_f -Noetherian.*

Proof. The “if” part follows from Lemma 4.16(1), since $\star_f \leq \star$. The converse follows immediately from Lemma 4.18. \square

Remark 4.20. Note that:

$$\tilde{\star}\text{-Noetherian} \Rightarrow \star\text{-Noetherian},$$

because $\tilde{\star} \leq \star$ (Lemma 4.16(1)). The converse is not true in general. Indeed, if $\star := v$, then $\star_f = t$ and $\tilde{\star} = w$ and we know that v -Noetherian (= t -Noetherian) is Mori and that w -Noetherian is strong Mori [79, Section 4]. Since it is possible to give examples of Mori domains that are not strong Mori [79, page 1295], we deduce that \star -Noetherian does not imply $\tilde{\star}$ -Noetherian.

We want to prove an analogue in the semistar case of the Cohen’s theorem for Noetherian domains, that says that a domain is Noetherian if and only if each prime ideal is finitely generated [54, Theorem 8]. A similar result has been already proven for strong Mori domains [79, Theorem 4.3].

We notice that a Cohen-type theorem cannot be proven for an arbitrary semistar operation: for example [33, Examples 2.8] consider the domain $A = \mathbb{Z} + X\mathbb{R}[[X]]$. Each nonzero prime ideal of A is t -finite but A is not a

Mori domain. In fact, the maximal ideals of A are principal (and so t -finite) and the only non maximal prime ideal of A is $Q := XK[[X]]$ and $Q = J_v$, where $J := \sqrt{2}XA + \sqrt{3}XA$, but A is not Mori, since for example, the chain of principal ideals $\{(1/2^n)xA\}_{n \geq 0}$ is not stationary.

So, we restrict to the case of semistar operation stable and of finite type.

We need a lemma (cf. [54, Theorem 7] and [79, Theorem 4.2] for similar results for Noetherian and strong Mori domains).

Lemma 4.21. *Let D be an integral domain and \star a semistar operation on D . Let P be an ideal that is not $\tilde{\star}$ -finite, such that each ideal of D properly containing P is $\tilde{\star}$ -finite. Then, P is prime.*

Proof. Let $ab \in P$, with $a, b \in D \setminus P$. Then, $P + aD$ contains properly P , so, it is $\tilde{\star}$ -finite, that is, there exists a finitely generated ideal $A \subseteq P + aD$ such that $A^{\tilde{\star}} = (P + aD)^{\tilde{\star}}$ (the condition that $A \subseteq P + aD$ follows from the fact that $\tilde{\star}$ is a semistar operation of finite type). Take $p_1, p_2, \dots, p_n \in P$ and $x_1, x_2, \dots, x_n \in D$ such that $A = (p_1 + x_1a, p_2 + x_2a, \dots, p_n + x_na)$. Let $P_0 := (p_1, p_2, \dots, p_n) \subseteq P$ and $J := (P :_D aD)$. Since $b \in J \setminus P$ we have $J \supsetneq P$ and so there exists $(c_1, c_2, \dots, c_k) \subseteq J$ such that $(c_1, c_2, \dots, c_k)^{\tilde{\star}} = J^{\tilde{\star}}$. Let $z \in P^{\tilde{\star}} \subseteq (P + aD)^{\tilde{\star}} = A^{\tilde{\star}}$. Since $z \in P^{\tilde{\star}}$, there exists a finitely generated ideal $H \in \mathcal{F}^{\star f}$ such that $zH \subseteq P$. But z is also an element of $A^{\tilde{\star}}$, so there exists a finitely generated ideal $L \in \mathcal{F}^{\star f}$ such that $zL \subseteq A$. Now, let $B := HL$. We note that $B \in \mathcal{F}^{\star f}$ (since a localizing system is a multiplicative system of ideals, see Section 1.3) and it is finitely generated. Moreover $zB \subseteq P \cap A$. Let $B = (b_1, b_2, \dots, b_l)$. So, for each $t = 1, 2, \dots, l$, there exist $s_{1t}, s_{2t}, \dots, s_{nt} \in D$ such that $zb_t = s_{1t}(p_1 + x_1a) + s_{2t}(p_2 + x_2a) + \dots + s_{nt}(p_n + x_na) = (s_{1t}p_1 + s_{2t}p_2 + \dots + s_{nt}p_n) + (s_{1t}x_1 + s_{2t}x_2 + \dots + s_{nt}x_n)a = (s_{1t}p_1 + s_{2t}p_2 + \dots + s_{nt}p_n) + y_t a$, where $y_t = s_{1t}x_1 + s_{2t}x_2 + \dots + s_{nt}x_n$. We have $y_t a = zb_t - (s_{1t}p_1 + s_{2t}p_2 + \dots + s_{nt}p_n)$, with $zb_t, (s_{1t}p_1 + s_{2t}p_2 + \dots + s_{nt}p_n) \in P$. Then, $ay_t \in P$ and $y_t \in J \subseteq J^{\tilde{\star}} = (c_1, c_2, \dots, c_k)^{\tilde{\star}}$. Then, for $t = 1, 2, \dots, l$, there exists an ideal $B_t \in \mathcal{F}^{\star f}$ such that $y_t B_t \in (c_1, c_2, \dots, c_k)$. For each $t = 1, 2, \dots, l$, $B_1 B_2 \cdots B_l a y_t \in (ac_1, ac_2, \dots, ac_k) \subseteq P$, since $c_1, c_2, \dots, c_k \in J$. Since $B_1 B_2 \cdots B_l \in \mathcal{F}^{\star f}$ (using again the fact that a localizing system is a multiplicative system of ideals, cf. Section 1.3), then, $(ay_1, ay_2, \dots, ay_l) \subseteq (ac_1, ac_2, \dots, ac_k)^{\tilde{\star}}$. Then, $zB \subseteq P_0 + (ay_1, ay_2, \dots, ay_l) \subseteq P_0 + (ac_1, ac_2, \dots, ac_k)^{\tilde{\star}}$. It follows $z \in (P_0 + (ac_1, ac_2, \dots, ac_k)^{\tilde{\star}})^{\tilde{\star}} = (p_1, p_2, \dots, p_n, ac_1, ac_2, \dots, ac_k)^{\tilde{\star}} \subseteq P^{\tilde{\star}}$. Since z is an arbitrary element of $P^{\tilde{\star}}$, it follows $P^{\tilde{\star}} = (p_1, p_2, \dots, p_n, ac_1, ac_2, \dots, ac_k)^{\tilde{\star}}$, with $p_1, p_2, \dots, p_n, ac_1, ac_2, \dots, ac_k \in P$, a contradiction, since P is not $\tilde{\star}$ -finite. \square

Theorem 4.22. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is $\tilde{\star}$ -Noetherian.

(ii) Each prime ideal of D is $\tilde{\star}$ -finite.

Proof. (i) \Rightarrow (ii) It is obvious by Lemma 4.18.

(ii) \Rightarrow (i) Let S be the set of all ideal of D which are not $\tilde{\star}$ -finite. If D is not $\tilde{\star}$ -Noetherian, then S is not empty by Lemma 4.18. Consider a chain $\{I_\alpha \mid \alpha \in A\}$ of elements of S . Let $I := \bigcup_{\alpha \in A} I_\alpha$. We want to show that I is not $\tilde{\star}$ -finite (and so, $I \in S$). Suppose that I is $\tilde{\star}$ -finite. Then, by Lemma 3.4, there exists $J \in \mathbf{f}(D)$, $J \subseteq I$ such that $J^{\tilde{\star}} = I^{\tilde{\star}}$ (using the fact that $\tilde{\star}$ is a semistar operation of finite type). Since J is finitely generated, there exists $\alpha_0 \in A$ such that $J \subseteq I_{\alpha_0}$. Then, $J^{\tilde{\star}} \subseteq I_{\alpha_0}^{\tilde{\star}} \subseteq I^{\tilde{\star}} = J^{\tilde{\star}}$ and so $J^{\tilde{\star}} = I_{\alpha_0}^{\tilde{\star}}$, that is, I_{α_0} is $\tilde{\star}$ -finite, a contradiction. It follows that I is not $\tilde{\star}$ -finite, so we can apply the Zorn's Lemma to the set S . Then, there exists a maximal element in S , which is a prime ideal by Lemma 4.21. But no prime ideals are in S by the hypothesis, so S is empty and D is $\tilde{\star}$ -Noetherian by Lemma 4.18. \square

In the following two propositions, we study the problem of the transfer of the semistar Noetherianity to overrings.

Proposition 4.23. *Let D be an integral domain and let \star be a semistar operation on D .*

- (1) *Assume that \star is stable. Then D is \star -Noetherian if and only if D^\star is \star_ι -Noetherian (where ι is the canonical embedding of D in D^\star).*
- (2) *D is $\tilde{\star}$ -Noetherian if and only if $D^{\tilde{\star}}$ is $(\tilde{\star})_\iota$ -Noetherian (where ι is the canonical embedding of D in $D^{\tilde{\star}}$).*

Proof. (1) The "if" part follows from Lemma 4.16(2) and Proposition 2.16(1) (without using the hypothesis of stability). Conversely, let I be a nonzero ideal of D^\star and set $J := I \cap D$. Then, $J^\star = (I \cap D)^\star = I^\star \cap D^\star = I^\star$. Therefore, by Lemma 4.18 (applied to D), we can find $F \in \mathbf{f}(D)$ such that $F \subseteq J$ and $F^\star = J^\star$. Hence, $(FD^\star)^{\star_\iota} = F^\star = J^\star = I^\star = I^{\star_\iota}$. The conclusion follows from Lemma 4.18 (applied to D^\star , since $FD^\star \subseteq I$ and $FD^\star \in \mathbf{f}(D^\star)$). (2) is a straightforward consequence of (1), since $\tilde{\star}$ is a stable semistar operation. \square

Proposition 4.24. *Let D be an integral domain and let T be an overring of D . Let \star be a semistar operation on D and \star' a semistar operation on T . Assume that T is (\star, \star') -flat over D . If D is $\tilde{\star}$ -Noetherian, then T is $\tilde{\star}'$ -Noetherian.*

Proof. Let A be a nonzero ideal of T . Let $N \in \mathcal{M}(\tilde{\star}') = \mathcal{M}(\star'_f)$ (Proposition 1.40(5)). From the (\star, \star') -flatness, it follows that $T_N = D_{N \cap D}$. Then, $A^{\tilde{\star}'} = \bigcap \{AT_N \mid N \in \mathcal{M}(\star'_f)\} = \bigcap \{AD_{N \cap D} \mid N \in \mathcal{M}(\star'_f)\}$. Now, $N \cap D$ is a prime of D such that $(N \cap D)^{\tilde{\star}} \neq D^{\tilde{\star}}$ (by [20, Proposition 3.2], since T is (\star, \star') -linked to D , by definition of (\star, \star') -flatness). Hence, $N \cap D$ is a quasi- $\tilde{\star}$ -ideal. Consider the ideal $A \cap D$ of D . Since D is $\tilde{\star}$ -Noetherian,

it follows by Lemma 4.18 that there exists a finitely generated ideal C of D , such that $C \subseteq A \cap D$ and $C^{\tilde{\star}} = (A \cap D)^{\tilde{\star}}$. Then, $AT_N = AD_{N \cap D} = (A \cap D)D_{N \cap D} = (A \cap D)^{\tilde{\star}}D_{N \cap D} = C^{\tilde{\star}}D_{N \cap D} = CD_{N \cap D} = (CT)T_N$. Thus, $A^{\tilde{\star}'} = (CT)^{\tilde{\star}'}$, with CT finitely generated ideal of T , such that $CT \subseteq A$. Hence, T is $\tilde{\star}'$ -Noetherian. \square

In particular, when $\star = d$ is the identity semistar operation, Proposition 4.24 gives the following result about flat overrings of Noetherian domains, cf. [73, Corollary of Theorem 3]:

If D is a Noetherian domain and T is a flat overring of D , then T is Noetherian.

Let D be an integral domain and \star a semistar operation on D . We say that D has the \star -finite character property (for short, \star -FC property) if each nonzero element x of D belongs to only finitely many quasi- \star -maximal ideals of D . Note that the \star_f -FC property coincides with the $\tilde{\star}$ -FC property, because $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$ (Proposition 1.8(1)).

Proposition 4.25. *Let D be an integral domain and \star a semistar operation on D . If D is $\tilde{\star}$ -Noetherian, then D_M is Noetherian, for each $M \in \mathcal{M}(\star_f)$. Moreover, if D has the \star_f -FC property, then the converse holds.*

Proof. Let $M \in \mathcal{M}(\star_f)$, A an ideal of D_M and $I := A \cap D$. Since D is $\tilde{\star}$ -Noetherian, there exists a finitely generated ideal $J \subseteq I$ of D with $J^{\tilde{\star}} = I^{\tilde{\star}}$ (Lemma 4.18). Then, $A = ID_M = I^{\tilde{\star}}D_M = J^{\tilde{\star}}D_M = JD_M$ (we used twice the fact that $\tilde{\star}$ is spectral, defined by $\mathcal{M}(\star_f)$). Then A is a finitely generated ideal of D_M and so D_M is Noetherian. For the converse, assume that the \star_f -FC property holds on D . Let I be a nonzero ideal of D and let $0 \neq x \in I$. Let $M_1, M_2, \dots, M_n \in \mathcal{M}(\star_f)$ be the quasi- \star_f -maximal ideals containing x . Since D_{M_i} is Noetherian for each $i = 1, 2, \dots, n$, then $ID_{M_i} = J_i D_{M_i}$, for some finitely generated ideal $J_i \subseteq I$ of D . The ideal $B := xD + J_1 + J_2 + \dots + J_n$ of D is finitely generated and contained in I . It is clear that, for each $i = 1, 2, \dots, n$, $ID_{M_i} = BD_{M_i}$. Moreover, if $M \in \mathcal{M}(\star_f)$ and $M \neq M_i$, for each $i = 1, 2, \dots, n$, then $x \notin M$ and this fact implies $ID_M = BD_M = D_M$. Then, $I^{\tilde{\star}} = \bigcap \{ID_M \mid M \in \mathcal{M}(\star_f)\} = \bigcap \{BD_M \mid M \in \mathcal{M}(\star_f)\} = B^{\tilde{\star}}$. Thus, by Lemma 4.18, D is $\tilde{\star}$ -Noetherian. \square

Remark 4.26. (1) Note that Proposition 4.25, in case of star operations, can be deduced from [42, Proposition 4.6], proven in the context of weak ideal systems on commutative monoids.

(2) Note that strong Mori domains (that is, w -Noetherian domains, where $w := \tilde{v}$) or, more generally, Mori domains satisfy always the t -FC property (= w -FC property, since $\mathcal{M}(w) = \mathcal{M}(t)$, for every integral domain) by [11, Proposition 2.2(b)]. But it is not true in general that the $\tilde{\star}$ -Noetherian domains satisfy the \star_f -FC property (take, for instance, $D := \mathbb{Z}[X]$, $\star := d$,

and observe that X is contained in infinitely many maximal ideals of $\mathbb{Z}[X]$. Note that, from Proposition 4.25 and from the previous considerations, we obtain in particular that *an integral domain D is strong Mori if and only if D_M is Noetherian, for each $M \in \mathcal{M}(t)$, and D has the w -FC property (cf. also [80, Theorem 1.9]).*

The result in Proposition 4.25 allows us to prove, in the context of semi-star Noetherian domains, the generalization of two of the main theorems that hold for Noetherian domains.

We start with the Krull intersection Theorem ([54, Theorem 77] and [80, Theorem 1.8]).

Theorem 4.27. *Let D be an integral domain and \star a semistar operation on D . Assume that D is $\tilde{\star}$ -Noetherian. Let $I \subseteq D$ be an ideal such that $I^{\tilde{\star}} \subsetneq D^{\tilde{\star}}$. Then $\bigcap_{n \in \mathbb{N}} I^n = (0)$.*

Proof. Since $I^{\tilde{\star}} \neq D^{\tilde{\star}}$, I is contained in a quasi- $\tilde{\star}$ -maximal ideal M (Proposition 1.8(2), considering that $I^{\tilde{\star}} \cap D$ is a proper quasi- $\tilde{\star}$ -ideal). By Proposition 4.25, D_M is a Noetherian domain. So, by the Krull intersection Theorem for Noetherian domains [54, Theorem 77], $\bigcap_{n \in \mathbb{N}} (ID_M)^n = (0)$. Since $I \subseteq ID_M$ we have the thesis. \square

Next result is the generalization of the Principal Ideal Theorem for Noetherian domain (PIT, cf. [54, Theorem 142], see also [80, Corollary 1.11] for an analogous result for strong Mori domains).

First we need a lemma.

Lemma 4.28. [20, Lemma 2.3(d)] *Let D be an integral domain and \star a semistar operation on D . Then each prime ideal of D minimal over a quasi- \star_f -ideal of D is a quasi- \star_f -prime.*

Theorem 4.29. *Let D be an integral domain and \star a semistar operation on D . Assume that D is $\tilde{\star}$ -Noetherian. Let $x \in D$, $x \neq 0$, such that $xD^{\tilde{\star}} \neq D^{\tilde{\star}}$. Let P be a prime minimal over $xD^{\tilde{\star}} \cap D$. Then P has height 1.*

Proof. Since P is minimal over the quasi- $\tilde{\star}$ -ideal $xD^{\tilde{\star}} \cap D$, Lemma 4.28 implies that P is a quasi- \star -prime. Then, there exists a quasi- $\tilde{\star}$ -maximal ideal M of D such that $P \subseteq M$. So, PD_M is a minimal prime over $(xD^{\tilde{\star}} \cap D)D_M = xD_M \cap D_M = xD_M$. Thus, PD_M is minimal over the principal ideal xD_M of the Noetherian domain D_M (Proposition 4.25). It follows by the PIT for Noetherian domains [54, Theorem 142] that PD_M (and so P) has height 1. \square

4.3 The Nagata ring of a semistar Noetherian domain

In this section, we want to prove that the Nagata ring of a domain D with respect to a semistar operation stable and of finite type \star is Noetherian if and only if the D is \star -Noetherian.

This result generalizes (and gives a converse of) the fact that the “classical” Nagata ring of a Noetherian domain is Noetherian [38, Section 33, Exercise 1], and that the Nagata ring with respect to the w -operation of a strong Mori domain is Noetherian [78, Theorem 2.8]. The result in the case of Noetherian domain is a straightforward consequence of the Hilbert basis Theorem, since the Nagata ring is a localization of the polynomial ring. The result for strong Mori domain is a consequence of a generalization of the Hilbert Basis Theorem for strong Mori domain (the polynomial ring over a strong Mori domain is a strong Mori domain, [80, Theorem 1.13]). So, to prove our result in the general case, we look for a generalization of the Hilbert Basis Theorem to our context.

There is an obstruction to doing this: in the classical case, the theorem states that the polynomial ring $D[X]$ over a (d_D) -Noetherian domain D is $d_{D[X]}$ -Noetherian. The theorem in the “strong Mori” case states that the polynomial ring $D[X]$ over a w_D -Noetherian domain (that is, a strong Mori domain) D is $w_{D[X]}$ -Noetherian (that is, a strong Mori domain). Consider the general case, that is, an integral domain D is \star -Noetherian, for an arbitrary semistar operation D . It is not clear what is an appropriate generalization of this theorem, or, more precisely, with respect to which semistar operation the polynomial ring $D[X]$ should be semistar Noetherian?

So, we have to determine this semistar operation on the polynomial ring in a way that allows us to reach our goal: a proof that the Nagata ring of a semistar Noetherian domain with respect to a finite type, stable semistar operation is Noetherian.

We can do this for stable semistar operations, by using the fact that a localizing system on an integral domain D induces in a canonical way a localizing system on the polynomial ring. This is the content of the following proposition.

Proposition 4.30. *Let D be an integral domain and \mathcal{F} be a localizing system of D . Let X be an indeterminate on D . Then $\mathcal{F}[X] := \{A \text{ ideal of } D[X] \mid A \supseteq JD[X], \text{ for some } J \in \mathcal{F}\}$ is a localizing system of $D[X]$.*

Proof. First, we prove that $A \in \mathcal{F}[X]$ if and only if $A \cap D \in \mathcal{F}$. Indeed, let $A \in \mathcal{F}[X]$. Then $A \supseteq JD[X]$, $J \in \mathcal{F}$. It follows that $A \cap D \supseteq JD[X] \cap D = J \in \mathcal{F}$. Then, $A \cap D \in \mathcal{F}$. For the converse: let $(A \cap D) \in \mathcal{F}$. Since $A \supseteq (A \cap D)D[X]$, then $A \in \mathcal{F}[X]$ by the definition of $\mathcal{F}[X]$.

Now we prove that for this set of ideals of $D[X]$ the properties **(LS1)** and **(LS2)** of Definition 1.22 are satisfied.

(LS1): Let $A \in \mathcal{F}[X]$ and $B \supseteq A$. Then $A \cap D \in \mathcal{F}$ and $B \cap D \supseteq A \cap D$. It follows that $B \cap D \in \mathcal{F}$ and then $B \in \mathcal{F}[X]$.

(LS2): We have to prove that if $I \in \mathcal{F}[X]$ and A is an ideal of $D[X]$ such that $(A :_{D[X]} iD[X]) \in \mathcal{F}[X]$, for each $i \in I$, then $A \in \mathcal{F}[X]$.

We want to prove that $A \cap D \in \mathcal{F}$. We know that $I \cap D \in \mathcal{F}$. Since \mathcal{F} is a localizing system, it is enough to prove that $(A \cap D :_D iD) \in \mathcal{F}$, for each $i \in I \cap D$. If we prove this, from the condition **(LS2)** for \mathcal{F} , it follows that $A \cap D \in \mathcal{F}$. So, let $i \in I \cap D$. The hypothesis is that $(A :_{D[X]} iD[X]) \in \mathcal{F}[X]$. So, $(A :_{D[X]} iD[X]) \cap D = (A :_D iD[X]) \in \mathcal{F}$. Then, to conclude, it is enough to prove that $(A :_D iD[X]) \subseteq (A \cap D :_D iD)$ and apply the property **(LS1)** of \mathcal{F} . Let $r \in (A :_D iD[X])$. In particular, $r \in D$. Since $riD[X] \subseteq A$, we have $riD \subseteq A$. But, since both r and i are in D , it follows that $riD \subseteq A \cap D$. Thus, $r \in (A \cap D :_D iD)$. Hence, $(A :_D iD[X]) \subseteq (A \cap D :_D iD)$, for each $i \in I \cap D \in \mathcal{F}$. It follows that $A \cap D \in \mathcal{F}$ and $A \in \mathcal{F}[X]$. \square

Proposition 4.31. *Let D be an integral domain. If \mathcal{F} is a finitely generated localizing system of D then $\mathcal{F}[X]$ is a finitely generated localizing system of $D[X]$.*

Proof. Let $A \in \mathcal{F}[X]$. Then, $A \cap D \in \mathcal{F}$. Since \mathcal{F} is finitely generated, there exists a finitely generated $J \in \mathcal{F}$ such that $J \subseteq A \cap D$. It follows that $JD[X] \subseteq (A \cap D)D[X] \subseteq A$ and $JD[X] \in \mathcal{F}[X]$ by definition and it is finitely generated. \square

So, let \star be a stable semistar operation on an integral domain D , and let $\mathcal{F} := \mathcal{F}^\star$ be the localizing system of D associated to \star . Consider the localizing system $\mathcal{F}[X]$ of $D[X]$ defined as in Proposition 4.30. We denote by \star' the stable semistar operation $\star_{\mathcal{F}[X]}$ on $D[X]$ induced by $\mathcal{F}[X]$. If \star is of finite type, then \star' is of finite type, by Proposition 4.31.

Theorem 4.32. *Let D be an integral domain, \star a stable semistar operation on D and \star' the semistar operation on $D[X]$ defined above (i.e., if $\star = \star_{\mathcal{F}}$ for some localizing system \mathcal{F} of D , then $\star' = \star_{\mathcal{F}[X]}$). If D is \star -Noetherian, then $D[X]$ is \star' -Noetherian.*

Proof. Let A be an ideal of $D[X]$. We want to show that there exists a finitely generated ideal $B \subseteq A$ of $D[X]$ such that $B^{\star'} = A^{\star'}$. For each $h \in \mathbb{N}$, let I_h be the set of the leading coefficients of the polynomials in A of degree less or equal than h . Since D is \star -Noetherian, each I_h is \star_f -finite (Lemma 4.18), that is, for each $h \in \mathbb{N}$, there exists a finitely generated ideal $J_h \subseteq I_h$ of D such that $J_h^\star = I_h^\star$. Now, since $I_0 = A \cap D \subseteq I_1 \subseteq I_2 \subseteq \dots$, $I := \bigcup_{h \in \mathbb{N}} I_h$ is an ideal of D . It follows that there exists a finitely generated ideal $J \subseteq I$ of D such that $J^\star = I^\star$. Since J is finitely generated, there exists $m \in \mathbb{N}$ such that $J \subseteq I_m$. Let $J = (b_1, b_2, \dots, b_k)$ and let f_1, f_2, \dots, f_k polynomials in A having respectively b_1, b_2, \dots, b_k as leading coefficients. Clearly, we can choose f_1, f_2, \dots, f_n of degree m . For each $h < m$, let $b_{1,h}, b_{2,h}, \dots, b_{k,h}$

the generators of J_h , and let $g_{1,h}, g_{2,h}, \dots, g_{k_h,h}$ polynomials in A having $b_{1,h}, b_{2,h}, \dots, b_{k_h,h}$ as leading coefficients (again, we can choose them of degree h). Let $B = (\{f_1, f_2, \dots, f_k\} \cup \{g_{1,h}, g_{2,h}, \dots, g_{k_h,h}\}_{h=0,1,\dots,m-1})$, the ideal generated by the f 's and the g 's. We want to prove that $B^{\star'} = A^{\star'}$. Since $B \subseteq A$, it is clear that $B^{\star'} \subseteq A^{\star'}$. For the converse, it is enough to prove that $A \subseteq B^{\star'}$ (it implies immediately that $A^{\star'} \subseteq (B^{\star'})^{\star'} = B^{\star'}$). So let $f \in A$. We prove by induction on the degree of f that $f \in B^{\star'}$. First, consider the case in which f has degree 0, that is, $f \in A \cap D = I_0$. It follows that $f \in I_0^* = J_0^*$. We show that $J_0^* \subseteq B^{\star'}$. Let $x \in J_0^*$. Then, there exists $E \in \mathcal{F}$ such that $xE \subseteq J_0$. Since the generators of J_0 are in B (by the construction of B), we have $xED[X] \subseteq J_0D[X] \subseteq BD[X] = B$. Since $ED[X] \in \mathcal{F}[X]$ by definition of $\mathcal{F}[X]$, we obtain that $x \in B^{\star'}$. Then $J_0^* \subseteq B^{\star'}$ and in particular $f \in B^{\star'}$. Let now $\deg f = n$. By the inductive hypothesis we have that if $g \in A$ and $\deg g < n$, then $g \in B^{\star'}$. First, we suppose $n \geq m$. Let a be the leading coefficient of f . Then, $a \in I$. Since $I^* = J^*$, we have that $a \in J^*$, that is, there exists $H \in \mathcal{F}$ such that $aH \subseteq J$ ($J := (b_1, b_2, \dots, b_k)$). Let $\lambda \in H$. Then, $\lambda a = r_1 b_1 + r_2 b_2 + \dots + r_k b_k \in J$, for some $r_1, r_2, \dots, r_k \in D$. Let $g := \lambda f - (r_1 f_1 + r_2 f_2 + \dots + r_k f_k) x^{n-m}$ (we recall that the f 's are polynomials in B having the b 's as leading coefficients). Now, $(r_1 f_1 + r_2 f_2 + \dots + r_k f_k) x^{n-m}$ is a polynomial of B (and so, of A) of degree n having λa as leading coefficient. Since also $\lambda f \in A$ and has leading coefficient λa and degree n , we have that g is a polynomial in A of degree strictly smaller than n . By induction, $g \in B^{\star'}$. Then, $\lambda f = g + (r_1 f_1 + r_2 f_2 + \dots + r_k f_k) x^{n-m} \in B^{\star'}$, since we have already observed that $(r_1 f_1 + r_2 f_2 + \dots + r_k f_k) x^{n-m} \in B \subseteq B^{\star'}$. It follows that, for each $\lambda \in H$, $\lambda f \in B^{\star'}$. That is, $fH \subseteq B^{\star'}$. It follows that $fHD[X] \subseteq B^{\star'}D[X] = B^{\star'}$, and by definition $HD[X] \in \mathcal{F}[X]$. It follows that $f \in (B^{\star'})^{\star'} = B^{\star'}$. Suppose now that $n < m$. Again, let a be the leading coefficient of f . Then, $a \in I_n \subseteq J_n^*$. Let $c_i := b_{i,n}$, $i = 1, 2, \dots, k_n$ the generators of J_n and let $g_i := g_{i,n}$. Since $a \in J_n^*$, there exists $H \in \mathcal{F}$ such that $aH \subseteq J_n$. Let $\lambda \in H$, then again $\lambda a = r_1 c_1 + r_2 c_2 + \dots + r_{k_n} c_{k_n}$. Let $g := \lambda f - (r_1 g_1 + r_2 g_2 + \dots + r_{k_n} g_{k_n})$. Again, $\deg g < n$, since $(r_1 g_1 + r_2 g_2 + \dots + r_{k_n} g_{k_n})$ is a polynomial of degree n having λa as leading coefficient. Moreover, $g \in A$, since λf and the g_i 's are in A . Then, by induction, $g \in B^{\star'}$. Now, $\lambda f = g + (r_1 g_1 + r_2 g_2 + \dots + r_{k_n} g_{k_n})$, with both g and $(r_1 g_1 + r_2 g_2 + \dots + r_{k_n} g_{k_n})$ elements of $B^{\star'}$. It follows that $\lambda f \in B^{\star'}$. Then, $fH \subseteq B^{\star'}$ and again $fHD[X] \subseteq B^{\star'}D[X] = B^{\star'}$. Since $HD[X] \in \mathcal{F}[X]$, we have that $f \in (B^{\star'})^{\star'} = B^{\star'}$. So, independently from the degree, if f is a polynomial of A , then $f \in B^{\star'}$. Thus, $A \subseteq B^{\star'}$ and so, $A^{\star'} \subseteq B^{\star'}$. Hence, $A^{\star'} = B^{\star'}$, and $D[X]$ is \star' -Noetherian. \square

Remark 4.33. We observe that both the ‘‘classical’’ Hilbert Basis Theorem and its generalization for strong Mori domains are consequences of Theorem 4.32. For the classical case, note that the localizing system \mathcal{F}^{dD} associated with the identity semistar operation of D is the set $\{D\}$. So, let $A \in \mathcal{F}^{dD}[X]$.

By definition of $\mathcal{F}^{d_D}[X]$, $A \cap D$ contains as a subset an element of \mathcal{F}^{d_D} , so, $D \subseteq A \cap D$. It follows that $A = D[X]$. Hence, $\mathcal{F}^{d_D}[X] = \{D[X]\} = \mathcal{F}^{d_{D[X]}}$, and $(d_D)' = d_{D[X]}$. Then, Theorem 4.32 gives exactly the classical Hilbert Basis Theorem.

For the strong Mori case, recall [46, Proposition 4.3] that, for a fractional ideal I of D , $(ID[X])^{t_{D[X]}} = I^{t_D}D[X]$. It follows that, if $\mathcal{F} := \mathcal{F}^{t_D}$, then $\mathcal{F}[X] \subseteq \mathcal{F}^{t_{D[X]}}$. Indeed, if $A \in \mathcal{F}[X]$ then $A \cap D \supseteq I$ for some ideal I such that $I^{t_D} = D$. It follows that $A^{t_{D[X]}} \supseteq ((A \cap D)D[X])^{t_{D[X]}} \supseteq (ID[X])^{t_{D[X]}} = I^{t_D}D[X] = D[X]$. Hence, $A \in \mathcal{F}^{t_{D[X]}}(D[X])$. It follows that $(w_D)' \leq (w_{D[X]})$ (we recall that, for an integral domain R , $w_R := \star_{\mathcal{F}t(R)}$), since the bijection between stable semistar operations and localizing systems is order preserving. So, by Theorem 4.32 and Lemma 4.16(1), if D is w_D -Noetherian (that is, strong Mori) then $D[X]$ is $w_{D[X]}$ -Noetherian (that is, strong Mori).

Let \star be stable and of finite type (that is, $\star = \tilde{\star}$, Lemma 1.32). Then \star is the spectral semistar operation on D associated to the set $\mathcal{M}(\star_f)$ (see Section 1.4, after Lemma 1.32). In this case $\mathcal{F}^\star = \mathcal{F}(\mathcal{M}(\star_f)) := \{I \subseteq D \mid I \not\subseteq P, \text{ for each } P \in \mathcal{M}(\star_f)\}$, by Proposition 1.28.

Proposition 4.34. *Let D be an integral domain and $\star = \tilde{\star}$ a semistar operation stable of finite type on D . Let $\ast := \star_\Delta$ be the spectral semistar operation on $D[X]$ defined by the set $\Delta := \{PD[X] \mid P \in \mathcal{M}(\star_f)\}$ and let \star' defined as before Theorem 4.32. Then:*

- (1) $\star' \leq \ast$.
- (2) If $D[X]$ is \star' -Noetherian then $D[X]$ is \ast -Noetherian.
- (3) If D is \star -Noetherian then $D[X]$ is \ast -Noetherian.

Proof. (1) We have noticed that $\mathcal{F} := \mathcal{F}^\star = \mathcal{F}(\mathcal{M}(\star_f))$. It is enough to prove that the localizing system $\mathcal{F}[X]$ is contained in the localizing system $\mathcal{F}(\Delta) := \{A \subseteq D[X] \mid A \not\subseteq PD[X], \text{ for each } P \in \Delta\}$, since the bijection between stable semistar operation and localizing systems is order preserving (Theorem 1.30). Let $A \in \mathcal{F}[X]$. Then, $A \supseteq JD[X]$, with $J \not\subseteq P$, for each $P \in \mathcal{M}(\star_f)$. Then, $JD[X] \not\subseteq PD[X]$, for each $P \in \mathcal{M}(\star_f)$, that is, $JD[X] \in \mathcal{F}(\Delta)$. Since $A \supseteq JD[X]$, it follows that $A \in \mathcal{F}(\Delta)$.

(2) It is straightforward from (1) and Lemma 4.16(1).

(3) If D is \star -Noetherian, then $D[X]$ is \star' -Noetherian, by Theorem 4.32. Hence, $D[X]$ is \ast -Noetherian by (2). \square

Lemma 4.35. *Let D be an integral domain, \star a semistar operation on D and let ι be the canonical embedding of $D[X]$ in $\text{Na}(D, \star)$. Let $\ast := \star_\Delta$ be the spectral semistar operation on $D[X]$ defined by the set $\Delta := \{PD[X] \mid P \in \mathcal{M}(\star_f)\}$. Then, $\ast_\iota = d_{\text{Na}(D, \star)}$.*

Proof. Let $E \in \overline{\mathbf{F}}(\text{Na}(D, \star))$. Then, $E^{*\iota} = \bigcap_{P \in \mathcal{M}(\star_f)} ED[X]_{PD[X]} = \bigcap_{P \in \mathcal{M}(\star_f)} E\text{Na}(D, \star)_{P\text{Na}(D, \star)} = E$, by Proposition 1.40(1). \square

Theorem 4.36. *Let D be an integral domain and let \star be a semistar operation on D . The following are equivalent:*

- (i) D is $\tilde{\star}$ -Noetherian.
- (ii) $\text{Na}(D, \star)$ is Noetherian.

Proof. (i) \Rightarrow (ii) Let $\ast := \star_\Delta$ be the spectral semistar operation on $D[X]$ defined by the set $\Delta := \{PD[X] \mid P \in \mathcal{M}(\star_f)\}$. Since D is $\tilde{\star}$ -Noetherian, then $D[X]$ is \ast -Noetherian, by Proposition 4.34(3). We note that, since \ast is spectral, then it is stable (Section 1.2.4). Moreover, $(D[X])^\ast = \text{Na}(D, \star)$, by Proposition 1.40(2). Thus, by Proposition 4.23(1), $\text{Na}(D, \star)$ is \ast_ι -Noetherian (where ι is the canonical embedding of $D[X]$ in $\text{Na}(D, \star)$). Hence, $\text{Na}(D, \star)$ is Noetherian, since, by Lemma 4.35, $\ast_\iota = d_{\text{Na}(D, \star)}$.

(ii) \Rightarrow (i) Let $I_0 \subseteq I_1 \subseteq \dots$ be a chain of quasi- $\tilde{\star}$ -ideals of D . Then, $I_0\text{Na}(D, \star) \subseteq I_1\text{Na}(D, \star) \subseteq \dots$ is a chain of ideals in $\text{Na}(D, \star)$. Since $\text{Na}(D, \star)$ is a Noetherian domain, it follows that there exists $n \in \mathbb{N}$ such that $I_k\text{Na}(D, \star) = I_n\text{Na}(D, \star)$, for each $k \geq n$. Since the I_i s are quasi- $\tilde{\star}$ -ideals, $I_k = (I_k)^\tilde{\star} \cap D = I_k\text{Na}(D, \star) \cap D = I_n\text{Na}(D, \star) \cap D = (I_n)^\tilde{\star} \cap D = I_n$, for each $k \geq n$, by Proposition 1.40(3). So, the chain of quasi- $\tilde{\star}$ -ideals ends, and D is $\tilde{\star}$ -Noetherian. \square

Remark 4.37. (1) If we let $\star = d$ in Theorem 4.36, we have the following result about the ‘‘classical’’ Nagata ring $D(X)(= \text{Na}(D, d))$ of a domain D :

Let D be an integral domain. The following are equivalent:

- (i) D is Noetherian.
- (ii) $D(X)$ is Noetherian.

(2) We note that Theorem 4.36 implies that:

if $\text{Na}(D, \star)$ is Noetherian, then D is \star -Noetherian,

since a $\tilde{\star}$ -Noetherian domain is \star -Noetherian (Remark 4.20). The converse of this result is not true in general. Indeed, consider a Mori domain D that is not strong Mori (Remark 4.20) and suppose that $\text{Na}(D, t)$ is a Noetherian domain. Then, by Theorem 4.36(ii) \Rightarrow (i), D is a w -Noetherian domain, i.e. a strong Mori domain, a contradiction.

(3) We remark that a result similar to Theorem 4.36, using the language of hereditary torsion theories, has been presented by P. Jara during the Conference ‘‘Commutative Rings and their Modules’’ held in Cortona, Italy, from May 31 to June 4, 2004.

4.4 Semistar almost Dedekind domains

Let D be an integral domain and \star a semistar operation on D . We say that D is a *semistar almost Dedekind domain* (for short, a \star -ADD) if D_M is a rank-one discrete valuation domain (for short, DVR), for each quasi- \star_f -maximal ideal M of D .

Note that, by definition, \star -ADD = \star_f -ADD and that, if $\star = d$, we obtain the classical notion of “almost Dedekind domain” (for short, ADD) as in [38, Section 36]. Note that, if $\star = v$, the v -ADDs coincide with the t -almost Dedekind domains studied by Kang [53, Section 4]; more generally, if \star is a star operation, then D is a \star -ADD if and only if D is a \star -almost Dedekind domain in the sense of [41, Chapter 23]. Note also that, a field has only the identity (semi)star operation and thus a field is, by convention, a trivial example of a (d -)ADD (since, in this case, $\mathcal{M}(d) = \emptyset$).

Remark 4.38. Let \star_1, \star_2 be two semistar operations on D such that $(\star_1)_f \leq (\star_2)_f$. If D is a \star_1 -ADD, then D is a \star_2 -ADD. In particular:

- D is a ADD $\Rightarrow D$ is a \star -ADD, for each semistar operation \star on D ;
- if \star is a (semi)star operation on D (so, $\star \leq v$), then:

D is a \star -ADD $\Rightarrow D$ is a v -ADD (and, hence, D is integrally closed).

Note that, in general, for a semistar operation \star , a \star -ADD may be not integrally closed. For instance, let K be a field and $T := K[[X]] = K + M$, where $M := XT$ is the maximal ideal of the discrete valuation domain T . Set $D := R + M$, where R is a non integrally closed integral domain with quotient field K (hence, D is not integrally closed [22, Proposition 2.2(10)]). Let ι be the canonical embedding of D in T and let $\star := \star_{\{T\}}$ on D . Then, we have $\star = \star_f$, $\star_\iota = d_T$ is the identity (semi)star operation on T (Example 1.37(1)) and $\mathcal{M}(\star_f) = \{M\}$ (by Lemma 4.7) and $D_M = T$ [22, Proposition 1.9]. So D is a \star -ADD which is not integrally closed (hence, in particular, D is not an ADD).

Proposition 4.39. *Let D be an integral domain, which is not a field, and \star a semistar operation on D . Then:*

- (1) D is a \star -ADD if and only if D_P is a DVR, for each quasi- \star_f -prime ideal P of D .
- (2) If D is a \star -ADD, then D is a $P\star MD$ and each quasi- \star_f -prime of D is a quasi- \star_f -maximal of D .
- (3) Let T be an overring of D and \star' a semistar operation on T . Assume that $D \subseteq T$ is a (\star, \star') -linked extension. If D is a \star -ADD, then T is a \star' -ADD.

(4) If D is a \star -ADD, then D^\star is a \star_ι -ADD (where ι is the canonical embedding of D in D^\star).

Proof. (1) It follows easily from the fact that each quasi- \star_f -prime is contained in a quasi- \star_f -maximal Proposition 1.8.

(2) It is a straightforward consequence of (1) and of Proposition 4.1 ((i) \Leftrightarrow (ii)).

(3) Let $N \in \mathcal{M}(\star'_f)$, then $(N \cap D)^{\star'_f} \neq D^\star$ by Proposition 4.8(2). Let $M \supseteq N \cap D$ be a quasi- \star_f -maximal ideal of D . We have $D_M \subseteq D_{N \cap D} \subseteq T_N$. So $T_N = D_{N \cap D} = D_M$, because D_M is a DVR (by assumption D is a \star -ADD). From this proof we deduce also that $N \cap D (= M)$ is a quasi- \star_f -maximal ideal of D , for each quasi- \star'_f -maximal ideal N of T .

(4) It follows from Proposition 4.8(1) and (3). \square

Remark 4.40. (1) We will show that, for a converse of Proposition 4.39(2), we will need additional conditions (cf. Theorem 4.55((1) \Leftrightarrow (3), (4))).

(2) The converse of Proposition 4.39(4) is not true in general. Indeed, let K be a field and $k \subset K$ a proper subfield of K . Let $T := K[[X]]$ and $D := k + M$, where $M := XT$ is the maximal ideal of T . Take $\star := \star_{\{T\}}$ on D . Note that $\star = \star_f$ and that $\star_\iota = d_T$ is the identity (semi)star operation on T (if ι is the canonical embedding of D in T). We have that $T = D^\star$ is a \star_ι -ADD (= ADD since T is a DVR), but D is not a \star -ADD, since M is a quasi- \star_f -maximal ideal of D and (by [22, Proposition 1.9]) $D_M = D$ is not a valuation domain.

Proposition 4.41. *Let D be an integral domain and \star a semistar operation on D . Then the following are equivalent:*

(i) D is a \star -ADD.

(ii) D is a $t(D^\star)$ -ADD and $\star_f = t(D^\star)$.

In particular, if \star is a (semi)star operation, D is a \star -ADD if and only if D is a t -ADD and $\star_f = t$.

Proof. (i) \Rightarrow (ii) By Corollary 2.20(2) and Remark 4.38, if D is a \star -ADD, then D is a $t(D^\star)$ -ADD. Moreover, by Proposition 4.39(2) D is a $P\star MD$, so $\star_f = t(D^\star)$, by Proposition 4.3.

(ii) \Rightarrow (i) It is clear. \square

Theorem 4.42. *Let D be an integral domain, which is not a field, and \star a semistar operation on D . The following are equivalent:*

(1) D is a \star -ADD.

(2) $\text{Na}(D, \star)$ is an ADD (i.e. $\text{Na}(D, \star)$ is a 1-dimensional Prüfer domain and contains no idempotent maximal ideals).

(3) $\text{Na}(D, \star) = \text{Kr}(D, \star)$ is an ADD and a Bézout domain.

Proof. (1) \Leftrightarrow (2) By Proposition 1.40(1), the maximal ideals of $\text{Na}(D, \star)$ are of the form $M\text{Na}(D, \star)$, where $M \in \mathcal{M}(\star_f)$. Also, for each $M \in \mathcal{M}(\star_f)$, we have $\text{Na}(D, \star)_{M\text{Na}(D, \star)} = D_M(X)$. Moreover, it is well-known that, for $M \in \mathcal{M}(\star_f)$, D_M is a DVR if and only if $D_M(X)$ is a DVR [38, Theorem 19.16 (c), Proposition 33.1 and Theorem 33.4 ((1) \Leftrightarrow (3))]. From these facts we conclude easily.

(1) \Rightarrow (3) If D is a \star -ADD, in particular D is a P \star MD (Proposition 4.39(2)), then $\text{Na}(D, \star) = \text{Kr}(D, \star)$, by Proposition 4.1 ((i) \Leftrightarrow (iv)). Therefore, we deduce that $\text{Na}(D, \star)$ is a Bézout domain (Proposition 1.39(1)) and an ADD by (1) \Rightarrow (2).

(3) \Rightarrow (2) It is trivial. \square

Corollary 4.43. *Let D be an integral domain and \star a semistar operation on D . Let ι be the canonical embedding of D in D^\star . The following are equivalent:*

- (1) D is a \star -ADD.
- (2) D is a $\tilde{\star}$ -ADD.
- (3) D^\star is a $\tilde{\star}_\iota$ -ADD.
- (4) D^\star is a t -ADD and $\tilde{\star}_\iota = t_{D^\star}$.

Proof. Note that $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star}) = \text{Na}(D^\star, \tilde{\star}_\iota)$ (Proposition 1.40(4)), then apply Theorem 4.42((1) \Leftrightarrow (2)) to obtain the equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3). The equivalence (3) \Leftrightarrow (4) follows from Proposition 4.41, since $\tilde{\star}_\iota$ is a (semi)star operation on D^\star . \square

More in general, since the property of being a \star -ADD depends only on the Nagata ring, recalling Proposition 4.9, we can give necessary and sufficient conditions for the converse of Proposition 4.39(4).

Proposition 4.44. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star . Assume that D^\star is a \star_ι -DD. The following are equivalent:*

- (i) D is a \star -ADD.
- (ii) $\text{Na}(D, \star) = \text{Na}(D^\star, \star_\iota)$
- (iii) D^\star is (\star, \star_ι) -flat over D .
- (iv) $(D^\star)_P = D_{P \cap D}$ for each $P \in \mathcal{M}((\star_\iota)_f)$.
- (v) $D^\star = D^{\tilde{\star}}$ and $\widetilde{(\star_\iota)} = (\tilde{\star})_\iota$.

Proof. (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) are in Proposition 4.9.

(i) \Rightarrow (ii) It follows from the fact that a \star -ADD is a P \star MD (Proposition 4.39(2)) and from Proposition 4.11.

(ii) \Rightarrow (i) It is immediate from Theorem 4.42. \square

Corollary 4.45. *Let D be an integral domain, T an overring of D . Let ι be the canonical embedding of D in T and \star a semistar operation on T . Assume that T is a \star -ADD. Then, the following are equivalent:*

(i) D is a \star^ι -ADD.

(ii) $\text{Na}(D, \star^\iota) = \text{Na}(T, \star)$

(iii) T is (\star^ι, \star) -flat over D .

(iv) $T_P = D_{P \cap D}$ for each $P \in \mathcal{M}(\star_f)$.

(v) $T = D^{\tilde{\star}^\iota}$ and $\tilde{\star} = (\tilde{\star}^\iota)_\iota$. \square

Next theorem characterizes \star -ADDs as a particular type of subrings of t -ADDs.

Theorem 4.46. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

(i) D is a \star -ADD.

(ii) There exists an overring T of D such that T is a t -ADD, $\star_f = t(T)$ and for each t_T -maximal ideal Q of T , $T_Q = D_{Q \cap D}$.

(iii) There exists an overring T of D such that T is a t -ADD, $\star_f = t(T)$ and T is $(t(T), t_T)$ -flat over D .

Proof. (i) \Rightarrow (ii) follows immediately from Theorem 4.15 and Proposition 4.39.

(ii) \Rightarrow (i) is a consequence of Corollary 4.45(iv) \Rightarrow (i).

(ii) \Leftrightarrow (iii) is a straightforward consequence of Corollary 4.10(ii) \Rightarrow (iii). \square

Next goal is a characterization of \star -ADD's in terms of valuation overrings, in the style of [38, Theorem 36.2]. We recall that a valuation overring V of D is a \star -valuation overring if $\star_f \leq \star_{\{V\}}$ (we have introduced this notion in Section 1.6). For this purpose, we prove preliminarily the following:

Lemma 4.47. *Let D be an integral domain and \star a semistar operation on D . Let V be a valuation overring of D and ι the canonical embedding of D in V . Then the following are equivalent:*

(1) V is a $\tilde{\star}$ -valuation overring of D .

(2) V is $(\tilde{\star}, d_V)$ -linked to D .

Proof. (1) \Rightarrow (2) Since V is a $\tilde{\star}$ -valuation overring, then $\tilde{\star} \leq \star_{\{V\}}$. Thus, the present implication follows from the fact that $(\star_{\{V\}})_l = d_V$ (Example 1.37) (so $\tilde{\star}_l = d_V$) and from Proposition 4.8(1).

(2) \Rightarrow (1): Let N be the maximal ideal of V (which is $(\tilde{\star}, d_V)$ -linked to D). Then $(N \cap D)^{\tilde{\star}} \neq D^{\tilde{\star}}$ by Proposition 4.8(2). Thus, there exists $M \in \mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$ (Proposition 1.34) such that $N \cap D \subseteq M$. Hence $D_M \subseteq D_{N \cap D} \subseteq V$. So, if $F \in \mathbf{f}(D)$, then $F^{\tilde{\star}} \subseteq FD_M \subseteq FV$. Therefore, V is a $\tilde{\star}$ -valuation overring of D . \square

We recall a result about \star -valuation overrings:

Theorem 4.48. [29, Theorem 3.9] *Let D be an integral domain and let \star be a semistar operation on D . A valuation overring V of D is a $\tilde{\star}$ -valuation overring if and only if V is an overring of D_P for some $P \in \mathcal{M}(\star_f)$.* \square

Theorem 4.49. *Let D be an integral domain, which is not a field, and \star a semistar operation on D . The following are equivalent:*

- (1) D is \star -ADD.
- (2) $D^{\tilde{\star}}$ is integrally closed and each $\tilde{\star}$ -valuation overring of D is a DVR.
- (3) $D^{\tilde{\star}}$ is integrally closed and each valuation overring V of D , which is $(\tilde{\star}, d_V)$ -linked to D , is a DVR.
- (4) $D^{\tilde{\star}}$ is integrally closed and each valuation overring V of D , which is (\star, d_V) -linked to D , is a DVR.

Proof. (1) \Rightarrow (2). Since $D^{\tilde{\star}} = \bigcap \{D_M \mid M \in \mathcal{M}(\star_f)\}$ and D_M is a DVR, for each $M \in \mathcal{M}(\star_f)$, then $D^{\tilde{\star}}$ is integrally closed. Now, let V be a $\tilde{\star}$ -valuation overring of D , then $V \supseteq D_M$ for some $M \in \mathcal{M}(\star_f)$ by Theorem 4.48. Since D_M is a DVR, then $V = D_M$ (is a DVR).

(2) \Leftrightarrow (3). Follows immediately from Lemma 4.47.

(3) \Rightarrow (4) Let V be a valuation overring, (\star, d_V) -linked to D . Suppose that it is not $(\tilde{\star}, d_V)$ -linked to D . Then, there exists a (quasi- d_V -)maximal ideal N of D , such that $(N \cap D)^{\tilde{\star}} = D^{\tilde{\star}}$ (by Proposition 4.8(2)). Then, $(N \cap D)^{\star_f} = D^{\star_f}$, since $\tilde{\star} \leq \star_f$. It follows that V is not (\star, d_V) -linked to D , a contradiction. So, the thesis follows immediately from the fact that overrings V which are (\star, d_V) -linked to D are also $(\tilde{\star}, d_V)$ -linked to D .

(4) \Rightarrow (1). Let $M \in \mathcal{M}(\star_f)$ and V be valuation overring of D_M . Then $V = V_{D \setminus M}$ is (\star, d_V) -linked to D (cf. [20, Example 3.4(1)]). Hence, by assumption, V is a DVR. Furthermore, D_M is integrally closed, since $D^{\tilde{\star}} \subseteq D_M$ and thus $D_M = D^{\tilde{\star}}_{MD_M \cap D^{\tilde{\star}}}$. So D_M is an ADD, by [38, Theorem 36.2], that is, D_M is a DVR. Therefore D is a \star -ADD. \square

Corollary 4.50. *Let D be an integral domain, which is not a field. Then the following are equivalent:*

- (1) D is t -almost Dedekind domain.
- (2) D is integrally closed and each w -valuation overring of D is a DVR.
- (3) D is integrally closed and each t -linked valuation overring of D is a DVR.

Proof. This is an immediate consequence of Theorem 4.49 and of the well-known fact that for a valuation domain V , $d_V = w_V = t_V$. \square

Remark 4.51. If D is a \star -ADD, which is not a field, then, by Theorem 4.49 and by the fact that a \star -valuation overring is a $\tilde{\star}$ -valuation overring, each \star -valuation overring of D is a DVR. Note that the converse is not true, even if $D^{\tilde{\star}}$ is integrally closed. Let D and T be as in Remark 4.40(2). Assume that k is algebraically closed in K . Since $\star = \star_{\{T\}}$, then $\star = \star_f$, $\mathcal{M}(\star_f) = \{M\}$ and $D = D_M = D^{\tilde{\star}}$ is integrally closed, where $\tilde{\star} = d_D$. Moreover, each \star -valuation overring of D is necessarily a valuation overring of T (since $T = D^{\star_f} = D^{\star} \subseteq V = V^{\star_f} = V^{\star}$). This implies that each \star -valuation overring of D is a DVR (since the only non trivial valuation overring of T is T , which is a DVR). Therefore, by Proposition 1.40(4) and 1.39(4), $\text{Na}(D, \star) = \text{Na}(D^{\tilde{\star}}, \tilde{\star}_\iota) = \text{Na}(D, d_D) = D(Z) \subsetneq \text{Kr}(D, \star) = \text{Kr}(T, d_T) = T(Z)$ (where ι is the canonical embedding of D in $D^{\tilde{\star}}$ and Z is an indeterminate over T and D). On the other hand, since $\text{t.deg}_k(K) \geq 1$, it is possible to find $(\tilde{\star}$ -) valuation overrings of D (of rank ≥ 2) contained in T [38, Theorem 20.7].

Let D be an integral domain and \star a semistar operation on D . For each quasi- \star -prime P of D , we define the \star -height of P (for short, $\star\text{-ht}(P)$) the supremum of the lengths of the chains of quasi- \star -prime ideals of D , between (0) and P (included). Obviously, if $\star = d$ is the identity (semi)star operation on D , then $d\text{-ht}(P) = \text{ht}(P)$, for each prime ideal P of D . If the set of quasi- \star -primes of D is not empty, the \star -dimension of D is defined as follows:

$$\star\text{-dim}(D) := \text{Sup}\{\star\text{-ht}(P) \mid P \text{ is a quasi-}\star\text{-prime of } D\}.$$

If the set of quasi- \star -primes of D is empty, then we pose $\star\text{-dim}(D) := 0$.

Note that, if $\star_1 \leq \star_2$, then $\star_2\text{-dim}(D) \leq \star_1\text{-dim}(D)$. In particular, $\star\text{-dim}(D) \leq d\text{-dim}(D) = \text{dim}(D)$ (= Krull dimension of D), for each semistar operation \star on D . Note that, recently, the notions of t -dimension and of w -dimension have been received a considerable interest by several authors (cf. for instance, [49], [76] and [77]).

Lemma 4.52. *Let D be an integral domain and \star a semistar operation on D , then*

$$\begin{aligned} \tilde{\star}\text{-dim}(D) &= \text{Sup}\{\text{ht}(M) \mid M \in \mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})\} = \\ &= \text{Sup}\{\text{ht}(P) \mid P \text{ is a quasi-}\tilde{\star}\text{-prime of } D\}. \end{aligned}$$

Proof. Let $M \in \mathcal{M}(\star_f)$ and $P \subseteq M$ be a nonzero prime ideal of D . Since $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$ (Proposition 1.34) we have $P \subseteq P^{\tilde{\star}} \cap D \subseteq PD_M \cap D = P$. So P is a quasi- $\tilde{\star}$ -prime ideal of D . Hence $\text{ht}(M) = \tilde{\star}\text{-ht}(M)$, so we get the Lemma. \square

Remark 4.53. Note that, in general,

$$\star_f\text{-dim}(D) \leq \text{Sup}\{\text{ht}(P) \mid P \text{ is a quasi-}\star_f\text{-prime of } D\}.$$

Moreover, it can happen that $\star_f\text{-dim}(D) \not\leq \text{Sup}\{\text{ht}(P) \mid P \text{ is a quasi-}\star_f\text{-prime of } D\}$, as the following example shows.

Let T be a DVR, with maximal ideal N , dominating a two-dimensional local Noetherian domain D , with maximal ideal M [18] (or [16, Theorem]), and let $\star := \star_{\{T\}}$. Then, clearly, $\star = \star_f$ and the only quasi- \star_f -prime ideal of D is M , since if P is a nonzero prime ideal of D , then $P^{\star} = PT = N^k$, for some integer $k \geq 1$. Therefore, if we assume that P is quasi- \star_f -ideal of D , then we would have $P = PT \cap D = N^k \cap D \supseteq M^k$, which implies that $P = M$. Therefore, in this case, $1 = \star_f\text{-dim}(D) = \star_f\text{-ht}(M) \leq \text{Sup}\{\text{ht}(P) \mid P \text{ is a quasi-}\star_f\text{-prime of } D\} = \text{ht}(M) = \dim(D) = 2$. Note that, in the present example, $\tilde{\star}$ coincides with the identity (semi)star operation on D . It is already known that, when $\star = v$, it may happen that $t\text{-dim}(D) < w\text{-dim}(D)$, [77, Remark 2].

The following lemma generalizes [38, Theorem 23.3, the first statement in (a)].

Lemma 4.54. *Let D be a $P\star MD$. Let Q be a nonzero P -primary ideal of D , for some prime ideal P of D , and let $x \in D \setminus P$. Then $Q^{\tilde{\star}} = (Q(Q + xD))^{\tilde{\star}}$.*

Proof. Let $M \in \mathcal{M}(\star_f)$. If $Q \not\subseteq M$, then $QD_M = Q^2D_M = Q(Q + xD)D_M (= D_M)$. If $Q \subseteq M$, then QD_M is PD_M -primary and $x \in D_M \setminus PD_M$; so $QD_M = QxD_M$, by [38, Theorem 17.3(a)], since D_M is a valuation domain. Thus $QD_M = (Q^2 + Qx)D_M$, hence $Q^{\tilde{\star}} = (Q(Q + xD))^{\tilde{\star}}$. \square

The following theorem provides several characterizations of the semistar almost Dedekind domains and, in particular, it generalizes [38, Theorem 36.5] and [53, Theorem 4.5].

Theorem 4.55. *Let D be an integral domain which is not a field and let \star be a semistar operation on D . The following are equivalent:*

- (1) D is \star -ADD.
- (2) D has the $\tilde{\star}$ -cancellation law.
- (3) D is a $P\star MD$, $\star_f\text{-dim}(D) = 1$ and $(M^2)^{\star_f} \neq M^{\star_f}$, for each $M \in \mathcal{M}(\star_f) (= \mathcal{M}(\tilde{\star}))$.
- (4) D is a $P\star MD$ and $\bigcap_{n \geq 1} (I^n)^{\star_f} = 0$ for each proper quasi- \star_f -ideal I of D .

(5) D is a $P\star MD$ and it has the \star_f -cancellation law.

Proof. (1) \Rightarrow (2). Let A, B, C be three nonzero (fractional) ideals of D such that $(AB)^{\tilde{\star}} = (AC)^{\tilde{\star}}$. Let $M \in \mathcal{M}(\star_f)$. Then, we have $ABD_M = (AB)^{\tilde{\star}}D_M = (AC)^{\tilde{\star}}D_M = ACD_M$ (we used twice the fact that $\tilde{\star}$ is spectral, defined by $\mathcal{M}(\star_f)$). Moreover, since D_M is a DVR then, in particular, AD_M is principal, thus $BD_M = CD_M$. Hence $B^{\tilde{\star}} = C^{\tilde{\star}}$.

(2) \Rightarrow (3). If D has $\tilde{\star}$ -CL, then in particular, $\tilde{\star}$ is an e.a.b. semistar operation on D , thus D is a $P\star MD$ (Proposition 4.1 ((v) \Rightarrow (i))). Let $M \in \mathcal{M}(\star_f)$. Clearly, by $\tilde{\star}$ -CL, $(M^2)^{\tilde{\star}} \neq M^{\tilde{\star}}$, and hence $(M^2)^{\star_f} \neq M^{\star_f}$ (since $\tilde{\star} = \star_f$ by Proposition 4.1). Next we show that $\text{ht}(M) = 1$, for each $M \in \mathcal{M}(\star_f)$. Deny, let $P \subset M$ be a nonzero prime ideal of D and let $x \in M \setminus P$. By Lemma 4.54, $P^{\tilde{\star}} = (P(P + xD))^{\tilde{\star}}$. Hence $D^{\tilde{\star}} = (P + xD)^{\tilde{\star}}$, by $\tilde{\star}$ -CL. So $P + xD \not\subseteq M$, which is impossible. Hence $\text{ht}(M) = 1$, for each $M \in \mathcal{M}(\star_f)$. Therefore, we conclude that $\star_f\text{-dim}(D) = \tilde{\star}\text{-dim}(D) = 1$ (Lemma 4.52).

(3) \Rightarrow (4). Recall that each proper quasi- \star_f -ideal is contained in a quasi- \star_f -maximal ideal, then it suffices to show that $\bigcap_{n \geq 1} (M^n)^{\star_f} = 0$, for each $M \in \mathcal{M}(\star_f)$. Since, by assumption $(M^2)^{\star_f} \neq M^{\star_f}$, then in particular $(M^2)^{\tilde{\star}} \neq M^{\tilde{\star}}$, and so $M^2D_M \neq MD_M$. Henceforth $\{M^nD_M\}_{n \geq 1}$ is the set of MD_M -primary ideals of D_M [38, Theorem 17.3(b)]. From the assumption we deduce that $\dim(D_M) = 1$ (because $\star_f = \tilde{\star}$ by Proposition 4.1), then $\bigcap_{n \geq 1} M^nD_M = 0$ [38, Theorem 17.3 (c) and (d)]. In particular, we have $\bigcap_{n \geq 1} (M^n)^{\tilde{\star}} \subseteq \bigcap_{n \geq 1} ((M^n)^{\tilde{\star}}D_M) = \bigcap_{n \geq 1} (M^nD_M) = 0$, therefore $\bigcap_{n \geq 1} (M^n)^{\star_f} = 0$.

(4) \Rightarrow (1). Let $M \in \mathcal{M}(\star_f)$. It is easy to see that $(M^n)^{\tilde{\star}} = M^nD_M \cap D^{\tilde{\star}}$, for each $n \geq 1$. So, $(\bigcap_{n \geq 1} M^nD_M) \cap D^{\tilde{\star}} \subseteq \bigcap_{n \geq 1} (M^nD_M \cap D^{\tilde{\star}}) = \bigcap_{n \geq 1} (M^n)^{\tilde{\star}} \subseteq \bigcap_{n \geq 1} (M^n)^{\star_f} = 0$ (the last equality holds by assumption). Hence $\bigcap_{n \geq 1} M^nD_M = 0$, since D_M is an essential valuation overring of $D^{\tilde{\star}}$. It follows that D_M is a DVR [38, p. 192 and Theorem 17.3(b)].

(2) \Leftrightarrow (5) is a consequence of the fact that in a $P\star MD$, $\tilde{\star} = \star_f$ and that the $\tilde{\star}$ -CL implies $P\star MD$. \square

Remark 4.56. As a comment to Theorem 4.55 ((1) \Leftrightarrow (5)), note that D may have the \star_f -CL without being a \star -ADD. It is sufficient to consider the example in Remark 4.40(2). In that case, $\star = \star_f$ and $\tilde{\star} = d_D$, since $\mathcal{M}(\star_f) = \{M\}$. Clearly, D has the \star -cancellation law (because T is a DVR), but, as we have already remarked, D is not a \star -ADD, hence, equivalently, D has not the $(\tilde{\star})$ -cancellation law.

Next result provides a generalization to the semistar case of [38, Theorem 36.4 and Proposition 36.6].

Proposition 4.57. *Let D be an integral domain, which is not a field, and \star a semistar operation on D . The following are equivalent:*

(1) D is a \star -ADD.

(2) For each nonzero ideal I of D , such that $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} =: P$ is a prime ideal of D , then $I^{\tilde{\star}} = (P^n)^{\tilde{\star}}$, for some $n \geq 1$.

(3) $\tilde{\star}\text{-dim}(D) = 1$ and, for each primary quasi- $\tilde{\star}$ -ideal Q of D , then $Q^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, for some $M \in \mathcal{M}(\star_f)$ and for some $n \geq 1$.

Proof. (1) \Rightarrow (2) and (3). Let I be a nonzero ideal of D with $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} = P$ is prime. Let M be a quasi- \star_f -maximal ideal of D such that $I \subseteq M$. So $\sqrt{I} = P \subseteq M$, and hence $P = M$, since D_M is a DVR. Thus $ID_M = M^n D_M$ for some $n \geq 1$. On the other hand, if $N \in \mathcal{M}(\star_f)$ and $N \neq M$, then $ID_N = D_N = M^n D_N$. Hence $I^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, i.e. $I^{\tilde{\star}} = (P^n)^{\tilde{\star}}$. The fact that $\tilde{\star}\text{-dim}(D) = 1$ follows from Theorem 4.55((1) \Rightarrow (3)) (since, in the present situation, $\star_f = \tilde{\star}$).

(2) \Rightarrow (1). Let $M \in \mathcal{M}(\star_f)$. Let A be an ideal of D_M and assume that $\sqrt{A} = PD_M$, for some prime ideal P of D , $P \subseteq M$. Set $B := A \cap D$. We have $\sqrt{B} = P$ and hence $B^{\star_f} \subseteq M^{\star_f} \subset D^{\star}$. By assumption, $B^{\tilde{\star}} = (P^n)^{\tilde{\star}}$, for some $n \geq 1$, hence $A = (A \cap D)D_M = BD_M = B^{\tilde{\star}}D_M = (P^n)^{\tilde{\star}}D_M = P^n D_M$. It follows from [38, Proposition 36.6] that D_M is an ADD. Hence D_M is a DVR.

(3) \Rightarrow (1). We can assume $\star = \star_f$, since $\star\text{-ADD}$ and $\star_f\text{-ADD}$ coincide. Let $M \in \mathcal{M}(\star_f)$ ($= \mathcal{M}(\tilde{\star})$ (Proposition 1.34)). Since $\tilde{\star}\text{-dim}(D) = 1$, then $\text{ht}(M) = \dim(D_M) = 1$ (Lemma 4.52). We can now proceed and conclude as in the proof of (2) \Rightarrow (1). (In this case, we have $\sqrt{A} = MD_M$ and so B is a M -primary quasi- $\tilde{\star}$ -ideal of D . Therefore, by assumption, $B^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, for some $n \geq 1$.) \square

Remark 4.58. Note that, if D is a $\star\text{-ADD}$, which is not a field, then necessarily D satisfies the following conditions (obtained from the statements (2) and (3) of Proposition 4.57; recall that, in this case, $\star_f = \tilde{\star}$, by Proposition 4.39(2) and Proposition 4.1):

(2_f) For each nonzero ideal I of D , such that $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} =: P$ is a prime ideal of D , then $I^{\star_f} = (P^n)^{\star_f}$, for some $n \geq 1$.

(3_f) $\star_f\text{-dim}(D) = 1$ and, for each primary quasi- \star_f -ideal Q of D , then $Q^{\star_f} = (M^n)^{\star_f}$, for some $M \in \mathcal{M}(\star_f)$ and for some $n \geq 1$.

On the other hand, D may satisfy either (2_f) or (3_f) without being a $\star\text{-ADD}$. It is sufficient to consider the example in Remark 4.40(2). In that case, $\star = \star_f$ and $\mathcal{M}(\star_f) = \{M\}$. Clearly, since D is a local one-dimensional domain (in fact, $\tilde{\star}\text{-dim}(D) = \star_f\text{-dim}(D) = \dim(D) = 1$), for each nonzero ideal I of D , with $I^{\star_f} \neq D^{\star}$, then $\sqrt{I} = M$ and $I^{\star_f} = (M^n)^{\star_f}$, for some $n \geq 1$, since T is a DVR. But, as we have already remarked, D is not a $\star\text{-ADD}$.

4.5 Semistar Dedekind domains

In this section we want to generalize the concept of a Dedekind domain, that is, a Noetherian Prüfer domain. So, given a semistar operation \star on an integral domain D , it is natural to define a \star -Dedekind domain (\star -DD for short) to be a \star -Noetherian $P\star MD$.

Next proposition generalizes the well-known fact that an integral domain D is a Dedekind domain if and only each fractional ideal of D is invertible. We notice that, to have an analogue of this result in the semistar case, we need the notion of quasi- \star -invertibility. This is one of the main reasons that led to introducing this concept.

Proposition 4.59. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (1) D is a \star -Noetherian domain and a $P\star MD$ (i.e., a \star -DD);
- (1_f) D is a \star_f -Noetherian domain and a $P\star_f MD$;
- (2) $\mathbf{F}^{\tilde{\star}}(D) := \{F^{\tilde{\star}} \mid F \in \mathbf{F}(D)\}$ is a group under the multiplication “ \times ”, defined by $F^{\tilde{\star}} \times G^{\tilde{\star}} := (F^{\tilde{\star}}G^{\tilde{\star}})^{\tilde{\star}} = (FG)^{\tilde{\star}}$, for all $F, G \in \mathbf{F}(D)$;
- (3) Each nonzero fractional ideal of D is quasi- $\tilde{\star}$ -invertible;
- (4) Each nonzero (integral) ideal of D is quasi- $\tilde{\star}$ -invertible.

Proof. (1) \Leftrightarrow (1_f) is obvious (Proposition 4.19 and Proposition 4.1 ((i) \Leftrightarrow (vi))).

(1) \Rightarrow (2). One can easily check that $\mathbf{F}^{\tilde{\star}}(D)$ is a monoid, with $D^{\tilde{\star}}$ as the identity element (with respect to “ \times ”). We next show that each element of $\mathbf{F}^{\tilde{\star}}(D)$ is invertible for the monoid structure. Let $F \in \mathbf{F}(D)$, then there exists $0 \neq d \in D$ such that $I := dF \subseteq D$. Write $I^{\star_f} = J^{\star_f}$, where $J \subseteq I$ is a finitely generated ideal of D (Lemma 4.18 and Proposition 4.19). Since D is a $P\star MD$, then $\star_f = \tilde{\star}$ (Proposition 4.1). So, $I^{\tilde{\star}} = J^{\tilde{\star}}$. We have $(JJ^{-1})^{\tilde{\star}} = D^{\tilde{\star}}$, since D is a $P\tilde{\star}MD$ (Proposition 4.1). Then, $D^{\tilde{\star}} = (J^{\tilde{\star}}J^{-1})^{\tilde{\star}} = (IJ^{-1})^{\tilde{\star}} = (dFJ^{-1})^{\tilde{\star}} = (F^{\tilde{\star}}(dJ^{-1})^{\tilde{\star}})^{\tilde{\star}}$. Thus $F^{\tilde{\star}}$ is invertible in $(\mathbf{F}^{\tilde{\star}}(D), \times)$.

(2) \Rightarrow (3). Let $F \in \mathbf{F}(D)$. By assumption, there exists $G \in \mathbf{F}(D)$ such that $(FG)^{\tilde{\star}} = D^{\tilde{\star}}$. We have $FG \subseteq D^{\tilde{\star}}$, so $G \subseteq (D^{\tilde{\star}} : F)$. Thus $D^{\tilde{\star}} = (FG)^{\tilde{\star}} \subseteq (F(D^{\tilde{\star}} : F))^{\tilde{\star}} \subseteq D^{\tilde{\star}}$. Hence $(F(D^{\tilde{\star}} : F))^{\tilde{\star}} = D^{\tilde{\star}}$, that is, F is quasi- $\tilde{\star}$ -invertible.

(3) \Rightarrow (4) is straightforward.

(4) \Rightarrow (1) From the previous comments on quasi semistar invertibility for nonzero finitely generated ideals in the stable case, it is clear that the assumption implies that D is a $P\tilde{\star}MD$ and hence D is a $P\star MD$ (Proposition 4.1). To prove that D is a \star -Noetherian domain, since $\tilde{\star} = \star_f$ (Proposition 4.1), it is enough to show, by using Proposition 4.19, that D is $\tilde{\star}$ -Noetherian.

Let I be a nonzero ideal of D . By assumption, I is quasi- $\tilde{\star}$ -invertible, then I is $\tilde{\star}$ -finite, by Proposition 3.16 applied to $\tilde{\star}$. Thus, from Lemma 4.18, we deduce that D is $\tilde{\star}$ -Noetherian. \square

Note that, by definition, the notions of \star -DD and \star_f -DD coincide.

Remark 4.60. (1) By Proposition 4.59(1), if $\star = d$ we obtain that a d -DD coincides with a classical Dedekind domain [38, Theorem 37.1]; if $\star = v$, we have that a v -DD coincides with a Krull domain (since a Mori PvMD is a Krull domain [52, Theorem 3.2 ((1) \Leftrightarrow (3))]; note that a Mori domain satisfies the t -FC property by [11, Proposition 2.2(b)]). More generally, if \star is a star operation, then D is a \star -DD if and only if D is \star -Dedekind in the sense of [41, Chapter 23].

(2) If D is \star -DD then D is \star -ADD (for a converse, see the following Theorem 4.69). Indeed, a \star -DD is a P \star MD and so $\tilde{\star} = \star_f$ (Proposition 4.1). This equality implies also that D is $\tilde{\star}$ -Noetherian (Proposition 4.19 and Proposition 4.59(1)). Therefore D_M is Noetherian (by Proposition 4.25) and, hence, we conclude that D_M is a DVR, for each $M \in \mathcal{M}(\star_f)$.

Corollary 4.61. *Let D be an integral domain and \star a semistar operation on D . Then D is a \star -DD if and only if D is a $\tilde{\star}$ -DD.*

Proof. It follows from Proposition 4.59(4) and from the fact that $\tilde{\tilde{\star}} = \tilde{\star}$, since $\mathcal{M}(\tilde{\star}) = \mathcal{M}(\star_f)$ (Proposition 1.34). \square

Theorem 4.62. *Let D be an integral domain.*

- (1) *Let $\star \leq \star'$ be two semistar operations on D . Then:*
 D is a \star -DD $\Rightarrow D$ is a \star' -DD.

In particular:

- (1a) *If D is a Dedekind domain, then D is a \star -DD, for any semistar operation \star on D .*
(1b) *Assume that \star is a (semi)star operation on D . Then a \star -DD is a Krull domain.*

- (2) *Let T be an overring of D . Let \star be a semistar operation on D and \star' a semistar operation on T . Assume that T is a (\star, \star') -linked overring of D . If D is a \star -DD, then T is a \star' -DD. In particular, If D is a \star -DD, then D^\star is a \star_ι -DD (where ι is the canonical embedding of D in D^\star).*

Proof. (1) It follows from the comments before Proposition 4.1 and Lemma 4.16(1). (1a) and (1b) are consequence of (1), Remark 4.60(1) and of the fact that $d \leq \star$, for each semistar operation \star , and if \star is a (semi)star operation, then $\star \leq v$.

(2) Note that if T is a (\star, \star') -linked overring of D and if D is a P \star MD, then

T is a (\star, \star') -flat over D (Proposition 4.8(5)). By Proposition 4.59(1) and Corollary 4.61, we know that D is $\tilde{\star}$ -Noetherian and a $P\star$ MD (or, equivalently, a $P\tilde{\star}$ MD). Hence, T is $\tilde{\star}'$ -Noetherian (Proposition 4.24) and T is a $P\star'$ MD (or, equivalently, a $P\tilde{\star}'$ MD) by Proposition 4.8(4). The first statement follows from Proposition 4.59(1) and Corollary 4.61. The last statement is a consequence of Proposition 4.8(1). \square

Proposition 4.63. *Let D be an integral domain and \star a semistar operation on D . Then the following are equivalent:*

- (i) D is a \star -DD
- (ii) D is a $t(D^\star)$ -DD and $\star_f = t(D^\star)$.

In particular, if \star is a (semi)star operation, D is a \star -DD if and only if D is a Krull domain and $\star_f = t_D$.

Proof. (i) \Rightarrow (ii). Since D is a $P\star$ MD, $\star_f = t(D^\star)$, by Proposition 4.3. (ii) \Rightarrow (i). It is clear. \square

Note that Proposition 4.63 has already been proven in [41, Theorem 23.3((a) \Leftrightarrow (d))], by using the language of monoids and ideal systems.

Remark 4.64. Note that if D is \star -DD, then by Theorem 4.62(2) D^\star is \star_ι -DD (where ι is the canonical embedding of D in D^\star), that is D^\star is a Krull domain and $(\star_\iota)_f = t_{D^\star}$ (and so, $\star_f = t(D^\star)$) (Proposition 4.63). However, the converse does not hold in general as the example in Remark 4.40(2) shows. We will study the converse of this result in Proposition 4.78 and following.

Next result is a ‘‘Cohen-type’’ Theorem for quasi- \star -invertible ideals.

Lemma 4.65. *Let D be an integral domain and \star a semistar operation of finite type on D . The following are equivalent:*

- (1) Each nonzero quasi- \star -prime of D is a quasi- \star -invertible ideal of D .
- (2) Each nonzero quasi- \star -ideal of D is a quasi- \star -invertible ideal of D .
- (3) Each nonzero ideal of D is a quasi- \star -invertible ideal of D .

Proof. (1) \Rightarrow (2). Let S be the set of the quasi- \star -ideals of D that are not quasi- \star -invertible. Assume that $S \neq \emptyset$. Since $\star = \star_f$ by assumption, then Zorn’s Lemma can be applied, thus we deduce that S has maximal elements. We next show that a maximal element of S is prime. Let P be a maximal element of S and let $r, s \in D$, with $rs \in P$. Suppose $s \notin P$. Let $J := (P :_D rD)$. We claim that $J^\star \cap D = J$. Indeed, since $(P :_D rD)^\star \subseteq (P^\star :_{D^\star} rD)$, then $J^\star \cap D \subseteq (P^\star :_{D^\star} rD) \cap D = (P^\star :_D rD)$. Moreover, if $x \in (P^\star :_D rD)$,

then $xr \in P^* \cap D = P$, and hence $(P^* :_D rD) \subseteq (P :_D rD) = J$. Thus $J = J^* \cap D$, i.e. J is a quasi- \star -ideal of D . Clearly, J contains properly P (since $s \in J \setminus P$). By the maximality of P in S , it follows that J is quasi- \star -invertible, that is $(J(D^* : J))^* = D^*$. We notice that, by Lemma 3.12(2), $P(D^* : J) \in \overline{\mathbf{F}}(D)$ is not quasi- \star -invertible, since P is not quasi- \star -invertible. We deduce that $(P(D^* : J))^* \cap D$ is a proper quasi- \star -ideal, that is not quasi- \star -invertible (Remark 3.14(a)) and, obviously, it contains P . From the maximality of P in S , we have $(P(D^* : J))^* \cap D = P$. Now, $rJ \subseteq P$ implies $(rJ)^* \subseteq P^*$. Then $r \in (rD)^* = (rJ(D^* : J))^* \subseteq (P(D^* : J))^*$. Therefore, $r \in (P(D^* : J))^* \cap D = P$ and so we have proven that P is a prime ideal of D .

(2) \Rightarrow (3) is a consequence of Remark 3.14(a), after remarking that, for each nonzero ideal J of D , then $J \subseteq I := J^* \cap D$, where I is a quasi- \star -ideal of D and $J^* = I^*$.

(3) \Rightarrow (2) \Rightarrow (1) are trivial. \square

Remark 4.66. Note that, in the situation of Lemma 4.65, the statement:

- (0) *each nonzero quasi- \star -maximal ideal of D is a quasi- \star -invertible ideal of D ,*

is, in general, strictly weaker than (1). Take, for instance, D equal to a discrete valuation domain of rank ≥ 2 , and $\star = d_D$.

The next two theorems generalize [38, Theorem 37.8 ((1) \Leftrightarrow (4)), Theorem 37.2]. Similar results are proven in [41, Theorem 23.3((a) \Leftrightarrow (c), (h))].

Theorem 4.67. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (1) D is a \star -DD.
- (2) Each nonzero quasi- $\tilde{\star}$ -prime ideal of D is quasi- $\tilde{\star}$ -invertible.

Proof. Easy consequence of Lemma 4.65 ((1) \Leftrightarrow (3)) and Proposition 4.59 (4). \square

From the previous theorem, we deduce the following characterization of Krull domains (cf. [50, Theorem 2.3 ((1) \Leftrightarrow (3))], [52, Theorem 3.6 ((1) \Leftrightarrow (4))] and [79, Theorem 5.4 ((i) \Leftrightarrow (vi))]).

Corollary 4.68. *Let D be an integral domain. The following are equivalent:*

- (1) D is a Krull domain.
- (2) Each nonzero w -prime ideal of D is w -invertible.
- (3) Each nonzero t -prime ideal of D is t -invertible.

Proof. (1) \Leftrightarrow (2) is a direct consequence of Theorem 4.67.

(1) \Rightarrow (3) is a straightforward consequence of (1) \Rightarrow (2) and of the fact that, in a Krull domain (which is a particular PvMD), $t = \tilde{t} = w$ (Proposition 4.1).

(3) \Rightarrow (2). Note that, by assumption, and by Lemma 4.65 ((1) \Leftrightarrow (3)), every nonzero ideal of D is t -invertible. Let Q be a nonzero w -prime. If $(QQ^{-1})^w \neq D$, then $Q \subseteq (QQ^{-1})^w \subseteq M$, for some $M \in \mathcal{M}(w) = \mathcal{M}(t)$ (Proposition 1.34(5)), thus $(QQ^{-1})^t = ((QQ^{-1})^w)^t \subseteq M^t = M$, which is a contradiction. \square

Theorem 4.69. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (1) D is a \star -DD.
- (2) D is a \star -ADD and each nonzero element of D is contained in only finitely many quasi- \star_f -maximal ideals (i.e. D has the \star_f -FC property).
- (3) D is a \star -Noetherian \star -ADD.

Proof. (1) \Rightarrow (2). Clearly D is a \star -ADD, by Remark 4.60(2). Since by Lemma 4.7(1), each quasi- \star_f -maximal ideal of D is a contraction of a $(\star_\iota)_f$ -maximal ideal of D^\star (where ι is the canonical embedding of D in D^\star), in order to show that D has \star_f -FC property, it is enough to check that D^\star satisfies the $(\star_\iota)_f$ -FC property. On the other hand, since (1) implies that D^\star is a \star_ι -DD (Theorem 4.62(2)), without loss of generality, we can assume that \star is a (semi)star operation on D and D is a \star -DD. By Proposition 4.63, D is a Krull domain and $\star_f = t$. Thus, each nonzero element is contained in only finitely many t -maximal ideals (= \star_f -maximal ideals) of D .

(2) \Rightarrow (1). We need to show that D is \star_f -DD. First, note that D is a $P\star_f$ MD and D_M is Noetherian, for each $M \in \mathcal{M}(\star_f)$ (Proposition 4.39 (1) and (2)). The conclusion now follows from Proposition 4.25 and Proposition 4.59(1), after recalling that, in a $P\star_f$ MD, $\star_f = \tilde{\star}$ (Proposition 4.1).

(1) \Leftrightarrow (3) is a consequence of Proposition 4.39(2), Proposition 4.59 and Remark 4.60(2). \square

From the previous theorem, we deduce a restatement of a well-known characterization of Krull domains:

Corollary 4.70. *Let D be an integral domain, then the following are equivalent:*

- (1) D is a Krull domain.
- (2) D is a t -almost Dedekind domain and each nonzero element of D is contained in only finitely many t -maximal ideals (= t -FC property). \square

Let D be an integral domain and \star a semistar operation on D . We recall that the \star -integral closure $D^{[\star]}$ of D (or, the *semistar integral closure with respect to the semistar operation \star of D*) is the integrally closed overring of D^\star defined by $D^{[\star]} := \bigcup \{(F^\star : F^\star) \mid F \in \mathbf{f}(D)\}$ [27, Definition 4.1]. We say that D is *quasi- \star -integrally closed* (respectively, *\star -integrally closed*) if $D^\star = D^{[\star]}$ (respectively, $D = D^{[\star]}$). It is clear that:

– D is quasi- \star -integrally closed if and only if D is quasi- \star_f -integrally closed (respectively, D is \star -integrally closed if and only if D is \star_f -integrally closed);

– D is \star -integrally closed if and only if D is quasi- \star -integrally closed and \star is a (semi)star operation on D .

We recall also [26, Example 2.1(c2)] that $D^{[\tilde{\star}]} = (D')^{\tilde{\star}}$ (where D' is the integral closure of D).

Note that when $\star = v$, then the overring $D^{[v]} = D^{[t]}$ was studied in [7] under the name of *pseudo-integral closure of D* .

Lemma 4.71. *Let D be an integral domain and \star a semistar operation on D .*

(1) *If \star is e.a.b., then $D^\star = D^{[\star]}$ (i.e. D is quasi- \star -integrally closed).*

(2) *D is quasi- $\tilde{\star}$ -integrally closed if and only if $D^{\tilde{\star}}$ is integrally closed.*

Proof. (1) Note that, in general, $D^\star \subseteq D^{[\star]}$. For the converse, let $F \in \mathbf{f}(D)$ and let $x \in (F^\star : F^\star)$. Then, $xF^\star \subseteq F^\star$ and $F^\star = F^\star + F^\star(xD)$. Therefore we have $(F(D + xD))^\star = (F^\star(D + xD))^\star = (F^\star + F^\star(xD))^\star = F^\star$. From the fact that F is finitely generated and that \star is e.a.b., we obtain $(D + xD)^\star = D^\star$. It follows that $x \in D^\star$ and so $(F^\star : F^\star) \subseteq D^\star$. Hence, $D^\star = D^{[\star]}$.

(2) The “only if” part is clear, since we have mentioned earlier that $D^{[\star]}$ is integrally closed for each semistar operation \star . For the “if” part, let D' be the integral closure of D , since $D^{\tilde{\star}}$ is integrally closed, then $(D')^{\tilde{\star}} \subseteq D^{\tilde{\star}} \subseteq D^{[\tilde{\star}]}$. Since $(D')^{\tilde{\star}} = D^{[\tilde{\star}]}$, hence, $(D')^{\tilde{\star}} = D^{\tilde{\star}} = D^{[\tilde{\star}]}$ (by the comments above about $D^{[\tilde{\star}]}$). Therefore, D is quasi- $\tilde{\star}$ -integrally closed. \square

Corollary 4.72. *Let \star be a semistar operation on an integral domain D . If D is a $P\star MD$ (in particular, a \star -DD) then D is quasi- \star -integrally closed.*

Proof. It follows from Lemma 4.71(1) and from the fact that, in a $P\star MD$, $\tilde{\star} = \star_f$ is an e.a.b. semistar operation (Proposition 4.1 ((i) \Rightarrow (v), (vi))). \square

The following result shows that a semistar version of the “Noether’s Axioms” provides a characterization of the semistar Dedekind domains.

Theorem 4.73. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

(1) *D is a \star -DD.*

(2) D is $\tilde{\star}$ -Noetherian, $\tilde{\star}\text{-dim}(D) = 1$ and D is quasi- $\tilde{\star}$ -integrally closed.

(3) D is $\tilde{\star}$ -Noetherian, $\tilde{\star}\text{-dim}(D) = 1$ and $D^{\tilde{\star}}$ is integrally closed.

Proof. The equivalence (2) \Leftrightarrow (3) follows from Lemma 4.71 (2).

(1) \Rightarrow (2). Since D is a \star -DD, then D is \star -ADD (Remark 4.60(2)). Hence $\tilde{\star}\text{-dim}(D) = 1$ (Proposition 4.57). Moreover, recall that a \star -DD is a $\tilde{\star}$ -DD (Corollary 4.61). Then D is $\tilde{\star}$ -Noetherian and a P $\tilde{\star}$ MD (Proposition 4.59), and so D is quasi- $\tilde{\star}$ -integrally closed by Corollary 4.72.

(3) \Rightarrow (1) For each $M \in \mathcal{M}(\star_f)$, it is well-known that $D^{\tilde{\star}} \subseteq D_M$ and $D^{\tilde{\star}}_{MD_M \cap D^{\tilde{\star}}} = D_M$. Since $D^{\tilde{\star}}$ is integrally closed, this implies that D_M is also integrally closed. Therefore D_M is a local, Noetherian (by Proposition 4.25), integrally closed, one dimensional (by Lemma 4.52) domain, that is, a DVR [38, Theorem 37.8]. Hence D is a P \star MD. In particular, we have $\tilde{\star} = \star_f$ (Proposition 4.1), thus D is \star_f -Noetherian, by the assumption, and so D is \star -Noetherian (Proposition 4.19). We conclude that D is a \star -DD. \square

By taking $\star = v$ in Theorem 4.73, we obtain the following characterization of Krull domains:

Corollary 4.74. *Let D be an integral domain. The following are equivalent:*

(1) D is a Krull domain.

(2) D is a strong Mori domain, $w\text{-dim}(D) = 1$ and $D = D^{[w]}$.

(3) D is a strong Mori domain, $w\text{-dim}(D) = 1$ and D is integrally closed.

(4) D is a strong Mori domain, $t\text{-dim}(D) = 1$ and D is integrally closed.

Proof. The only part which needs a justification is the statement on t -dimension and w -dimension (in the equivalence (3) \Leftrightarrow (4)). This follows from the fact that, in every integral domain, $w \leq t$ and $\mathcal{M}(t) = \mathcal{M}(w)$. \square

Remark 4.75. Note that, if D is a \star -DD, then we know that $\tilde{\star} = \star_f$, and so D satisfies the properties:

(2_f) D is \star_f -Noetherian, $\star_f\text{-dim}(D) = 1$ and D is quasi- \star_f -integrally closed;

(3_f) D is \star_f -Noetherian, $\star_f\text{-dim}(D) = 1$ and D^{\star_f} ($= D^{\star}$) is integrally closed

obtained from (2) and (3) of Theorem 4.73, replacing $\tilde{\star}$ with \star_f . But, conversely, if D satisfies either (2_f) or (3_f) then D is not necessarily a \star -DD. Indeed, let D, T and \star be as in the example of Remark 4.40(2). Then we have already observed that $\star = \star_f$ and $\tilde{\star} = d_D$. Moreover, D is not a \star -DD (because it is not a \star -ADD), but $D^{\star_f} = T = D^{[\star_f]}$ is integrally closed (since T is a DVR), $\star_f\text{-dim}(D) = 1$ (since $\mathcal{M}(\star_f) = \{M\}$ and $\star_f\text{-dim}(D) \leq \dim(D) = 1$) and D is \star_f -Noetherian (Lemma 4.18, since T is Noetherian).

Note that (3_f) does not imply that D is a \star -DD, even if \star is a *(semi)star* operation on D . Take T and D as in the example described in Remark 4.40(2) and, moreover, assume that k is algebraically closed in K . It is wellknown that, in this situation, D is integrally closed. Let $\star := v$ on D . It is easy to see that $\mathcal{M}(v) = \mathcal{M}(t) = \{M\}$, thus $w = d$ is the identity (semi)star operation on D (hence, $D^{[w]} = D^{[d]} = D$) and $t\text{-dim}(D) = 1$ ($= v\text{-dim}(D) = w\text{-dim}(D) = \dim(D)$). Moreover, it is known that D is a Mori domain [34, Theorem 4.18] and thus D is a t -Noetherian domain. However, D is not a Krull domain, since D is not completely integrally closed (being T the complete integral closure of D). Note that, in this situation, D is even not a strong Mori domain (by Corollary 4.74).

Note also that, in the previous example, $D \subsetneq D^{[t]}$ (i.e. D is not t -integrally closed, hence does not satisfies condition (2_f) for $\star = v$), since $D^{[t]} = T$ by [7, Theorem 1.8(ii)].

On the other hand, if \star is a *(semi)star* operation on D , then we know that D is a \star -DD if and only if D is a v -DD (i.e. a Krull domain) and $\star_f = t$ (Proposition 4.63). It is interesting to observe that, for $\star = v$, condition (1) of Theorem 4.73 is equivalent to (2_f) . More precisely we have the following variation of the equivalence (1) \Leftrightarrow (4) of Corollary 4.74:

D is a Krull domain if and only if D is t -Noetherian, $t\text{-dim}(D) = 1$ and D is t -integrally closed (i.e. $D = D^{[t]}$).

As a matter of fact, let $F \in \mathbf{f}(D)$, then $D = D^{[t]} = D^{[v]}$ implies that $D = (F^v : F^v) = (F^{-1} : F^{-1}) = (FF^{-1})^{-1}$ and so $(FF^{-1})^v = D$. Moreover, since t -Noetherian is equivalent to v -Noetherian (Proposition 4.19) and v -Noetherian implies that $v = t$ (Lemma 4.18), then $(FF^{-1})^t = D$. Thus D is a PvMD and so D is a v -DD (Proposition 4.59).

Finally, from the previous considerations we deduce that D is a \star -DD if and only if

$(\bar{2}_f)$ D is \star_f -Noetherian, $\star_f\text{-dim}(D) = 1$, D is quasi- \star_f -integrally closed and $\star_f = t$.

Next result generalizes [38, Proposition 38.7].

Theorem 4.76. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is a \star -DD.
- (ii) $\text{Na}(D, \star)$ ($= \text{Kr}(D, \star)$) is a PID.
- (iii) $\text{Na}(D, \star)$ ($= \text{Kr}(D, \star)$) is a Dedekind domain.

Proof. (i) \Rightarrow (ii) Note that D is a $\tilde{\star}$ -DD (Corollary 4.61), and $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star})$ (Proposition 1.40(4)). Since D is $\tilde{\star}$ -Noetherian, we have that $\text{Na}(D, \tilde{\star}) = \text{Na}(D, \star)$ is a Noetherian domain (Theorem 4.36). Moreover, D

is a P $\tilde{\star}$ MD, so $\text{Na}(D, \tilde{\star}) = \text{Na}(D, \star) = \text{Kr}(D, \star)$ is a Bezout domain (Proposition 4.1(i) \Rightarrow (iv) and Proposition 1.39(1)). Hence, $\text{Na}(D, \star)$ is a Noetherian Bezout domain, that is, a PID.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Assume that $\text{Na}(D, \star)$ is a Dedekind domain. Then, in particular, $\text{Na}(D, \star)$ is a Prüfer domain, and so D is a P \star MD, by Proposition 4.1(iii) \Rightarrow (i). Moreover, $\text{Na}(D, \star)$ is Noetherian, and so D is a $\tilde{\star}$ -Noetherian domain by Theorem 4.36(ii) \Rightarrow (i), and so D is \star -Noetherian by Lemma 4.16(1), since $\tilde{\star} \leq \star$. Hence, D is a \star -DD. □

From the previous result, we deduce immediately:

Corollary 4.77. *Let D be an integral domain. The following are equivalent:*

(1) D is a Krull domain.

(2) $\text{Na}(D, v)$ ($= \text{Kr}(D, v)$) is a PID.

(3) $\text{Na}(D, v)$ ($= \text{Kr}(D, v)$) is a Dedekind domain. □

We go back to the problem mentioned in Remark 4.64: the problem of the descent of the property of being a \star -DD. Theorem 4.76 shows that this property depends only on the Nagata ring: this allows us to use Proposition 4.9 and Corollary 4.10.

Proposition 4.78. *Let D be an integral domain, \star a semistar operation on D . Assume that D^\star is a \star_ι -DD (where ι is the canonical embedding of D in D^\star). Then the following are equivalent:*

(i) D is a \star -DD.

(ii) $\text{Na}(D, \star) = \text{Na}(D^\star, \star_\iota)$

(iii) D^\star is (\star, \star_ι) -flat over D .

(iv) $(D^\star)_P = D_{P \cap D}$ for each $P \in \mathcal{M}((\star_\iota)_f)$.

(v) $D^\star = D^{\tilde{\star}}$ and $(\star_\iota) = (\tilde{\star})_\iota$.

Proof. (i) \Rightarrow (ii) It is a consequence of Proposition 4.11, since a \star -DD is a P \star MD.

(ii) \Rightarrow (i) It is straightforward by Theorem 4.76.

The other equivalences are in Proposition 4.9. □

It follows, by using Corollary 4.12, the general result for the descent from an arbitrary overring.

Proposition 4.79. *Let D be an integral domain, T an overring of D . Let ι be the canonical embedding of D in T and $*$ a semistar operation on T . Assume that T is a $*$ -DD. Then, the following are equivalent:*

- (i) D is a $*^\iota$ -DD.
- (ii) $\text{Na}(D, *^\iota) = \text{Na}(T, *)$
- (iii) T is $(*^\iota, *)$ -flat over D .
- (iv) $T_P = D_{P \cap D}$ for each $P \in \mathcal{M}(*_f)$.
- (v) $T = D^{\widetilde{*}^\iota}$ and $\tilde{*} = (*^\iota)_\iota$. □

Corollary 4.80. *Let D be an integral domain and \star a semistar operation on D . Let ι be the canonical embedding of D in D^\star . The following are equivalent:*

- (i) D is a \star -DD.
- (ii) D is a $\tilde{\star}$ -DD.
- (iii) $D^{\tilde{\star}}$ is a $\tilde{\star}_\iota$ -DD.
- (iv) $D^{\tilde{\star}}$ is a Krull domain and $\tilde{\star}_\iota = t_{D^{\tilde{\star}}}$.

Proof. (i) \Leftrightarrow (ii) It is Corollary 4.61.

(ii) \Rightarrow (iii) It follows from Lemma 4.62

(iii) \Rightarrow (iv) It follows from Proposition 4.63, since $\tilde{\star}_\iota$ is a (semi)star operation on $D^{\tilde{\star}}$.

(iv) \Rightarrow (ii) It follows by Proposition 4.78 and Lemma 4.13, since $(\tilde{\star}_\iota)^\iota = \tilde{\star}$ (Proposition 2.16(1)). □

Next result characterize \star -DD domains as a particular class of subrings of Krull domains (the proof is straightforward, cf. Theorem 4.15).

Theorem 4.81. *If D is an integral domain and \star is a semistar operation on D , then the following are equivalent:*

- (1) D is a \star -DD.
- (2) *There exists an overring T of D such that T is a Krull domain, $\star_f = (t_T)^\iota$ (where ι is the canonical embedding of D in T) and, for each t_T -maximal ideal Q of T , $D_{Q \cap D} = T_Q$. □*

Example 4.82. *Let D be a Mori domain, let Θ be the set of all the maximal t -ideals of D which are t -invertible and let \star_Θ be the spectral semistar operation on D associated to Θ (Section 1.2.4). Assume that $\Theta \neq \emptyset$ (i.e. that D is a Mori non strongly Mori domain, accordingly to the terminology introduced by Barucci and Gabelli [11, page 105]), then D is a \star_Θ -DD.*

Note that by [11, Proposition 3.1 and Theorem 3.3 (a)], $D^{\star\ominus}$ is a Krull domain such that the map $P \mapsto P^{\star\ominus}$ defines a bijection between Θ and the set $\mathcal{M}(t_{D^{\star\ominus}})$ of all the t -maximal ideals of $D^{\star\ominus}$ and $D_P = (D^{\star\ominus})_{P^{\star\ominus}}$. Therefore the (semi)star operation $(\star_{\Theta})_{\iota}$ (ι is the canonical embedding of D in $D^{\star\ominus}$) on $D^{\star\ominus}$ coincides with the t -operation, $t_{D^{\star\ominus}}$, on $D^{\star\ominus}$. Then, D is a \star_{Θ} -DD, by Theorem 4.81.

Let D be an integral domain and \star a semistar operation on D . We say that two nonzero ideals A and B are \star -comaximal if $(A + B)^{\star} = D^{\star}$. Note that, if \star is a semistar operation of finite type, then A and B are \star -comaximal if and only if A and B are not contained in a common quasi- \star -maximal ideal.

Lemma 4.83. *Let D be an integral domain and \star a semistar operation on D . Let A and B be two nonzero \star -comaximal ideals of D . Then $(A \cap B)^{\star} = (AB)^{\star}$.*

Proof. In general $(A+B)(A \cap B) \subseteq AB$. Then, $((A+B)(A \cap B))^{\star} \subseteq (AB)^{\star} \subseteq (A \cap B)^{\star}$. But $((A+B)(A \cap B))^{\star} = ((A+B)^{\star}(A \cap B))^{\star} = (D^{\star}(A \cap B))^{\star} = (A \cap B)^{\star}$. Hence, $(A \cap B)^{\star} = (AB)^{\star}$. \square

Corollary 4.84. *Let D be an integral domain and \star a semistar operation of finite type. Let $n \geq 2$ and let A_1, A_2, \dots, A_n be nonzero ideals of D , such that $(A_i + A_j)^{\star} = D^{\star}$, for $i \neq j$. Then, $(A_1 \cap A_2 \cap \dots \cap A_n)^{\star} = (A_1 A_2 \cdot \dots \cdot A_n)^{\star}$.*

Proof. We prove it by induction on $n \geq 2$, using Lemma 4.83 for the case $n = 2$. Set $A := A_1 \cap A_2 \cap \dots \cap A_{n-1}$ and $B := A_n$. Then, A and B are not contained in a common quasi- \star -maximal ideal, otherwise, A_n and A_j (for some $1 \leq j \leq n-1$) would be contained in a common quasi- \star -maximal ideal. Hence $(A_1 \cap A_2 \cap \dots \cap A_{n-1} \cap A_n)^{\star} = (A \cap B)^{\star} = (AB)^{\star} = (A^{\star} B)^{\star} = (A_1 A_2 \cdot \dots \cdot A_n)^{\star}$. \square

Theorem 4.85. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (1) D is a \star -DD.
- (2) For each nonzero ideal I of D , there exists a finite family of quasi- \star_f -prime ideals P_1, P_2, \dots, P_n of D , pairwise \star_f -comaximals, and a finite family of non negative integers e_1, e_2, \dots, e_n such that $I^{\bar{\star}} = (P_1^{e_1} P_2^{e_2} \cdot \dots \cdot P_n^{e_n})^{\bar{\star}}$.

Moreover, if (2) holds and if $I^{\bar{\star}} \neq D^{\bar{\star}}$, then we can assume that $P_i^{\bar{\star}} \neq D^{\bar{\star}}$, for each $i = 1, 2, \dots, n$. In this case, the integers e_1, e_2, \dots, e_n are positive and the factorization is unique.

Proof. (1) \Rightarrow (2). Let I be a nonzero ideal of D . To avoid the trivial case, we can assume that $I^{\tilde{\star}} \neq D^{\tilde{\star}}$. Let P_1, P_2, \dots, P_n be the finite (non empty) set of quasi- \star_f -maximal ideals such that $I \subseteq P_i$, for $1 \leq i \leq n$ (Theorem 4.69). We have $I^{\tilde{\star}} = \cap\{ID_P \mid P \in \mathcal{M}(\star_f)\} = \cap_{i=1}^n (ID_{P_i} \cap D^{\tilde{\star}})$. Since D_{P_i} is a DVR, then $ID_{P_i} = P_i^{e_i} D_{P_i}$, for some integers $e_i \geq 1$, $i = 1, 2, \dots, n$. Therefore, we have $ID_{P_i} \cap D^{\tilde{\star}} = P_i^{e_i} D_{P_i} \cap D^{\tilde{\star}} = (P_i^{e_i})^{\tilde{\star}}$. Hence $I^{\tilde{\star}} = (P_1^{e_1})^{\tilde{\star}} \cap (P_2^{e_2})^{\tilde{\star}} \cap \dots \cap (P_n^{e_n})^{\tilde{\star}} = (P_1^{e_1} \cap P_2^{e_2} \cap \dots \cap P_n^{e_n})^{\tilde{\star}} = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}}$, by Corollary 4.84.

For the last statement, let $I^{\tilde{\star}} = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}}$, if $P_i^{\tilde{\star}} = D^{\tilde{\star}}$, for some i , then obviously we can cancel P_i from the factorization of $I^{\tilde{\star}}$.

We prove next the uniqueness of the representation of $I^{\tilde{\star}}$. From (Proposition 1.40(4)), we deduce that $INa(D, \star) = P_1^{e_1} P_2^{e_2} \dots P_n^{e_n} Na(D, \star) = (P_1 Na(D, \star))^{e_1} (P_2 Na(D, \star))^{e_2} \dots (P_n Na(D, \star))^{e_n}$ is the unique factorization into primes of the ideal $INa(D, \star)$ in the PID $Na(D, \star)$ (Theorem 4.76). Since $P_i = P_i Na(D, \star) \cap D$ (because each P_i is a quasi- $\tilde{\star}$ -maximal ideal of D), the factorization of $I^{\tilde{\star}}$ is unique.

(2) \Rightarrow (1) Without loss of generality, we can assume that D is not a field. First, we prove that each localization to a quasi- \star_f -maximal ideal of D is a DVR. Let $M \in \mathcal{M}(\star_f)$ and let J be a nonzero proper ideal of D_M . Set $I := J \cap D (\subseteq M)$. Then, it is easy to see that $I^{\tilde{\star}} \neq D^{\tilde{\star}}$ thus, by assumption, $I^{\tilde{\star}} = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}}$, for some family of quasi- \star_f -prime ideals P_i , with $P_i^{\tilde{\star}} \neq D^{\tilde{\star}}$ and for some family of integers $e_i \geq 1$, $i = 1, 2, \dots, n$. It follows that $J = ID_M = I^{\tilde{\star}} D_M = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}} D_M = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n}) D_M$ (since $\tilde{\star}$ is a spectral semistar operation defined by the set $\mathcal{M}(\star_f)$). Hence J is a finite product of primes of D_M . Therefore D_M is a local Dedekind domain [38, Theorem 37.8 ((1) \Leftrightarrow (3))], that is, D_M is a DVR.

Now we show that each quasi- $\tilde{\star}$ -prime ideal of D is quasi- $\tilde{\star}$ -invertible. Let Q be a quasi- $\tilde{\star}$ -prime of D and let $0 \neq x \in Q$. Then, by assumption, $(xD)^{\tilde{\star}} = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}}$, with P_1, P_2, \dots, P_n nonzero prime ideals of D and $e_i \geq 1$, $i = 1, 2, \dots, n$. Since xD is obviously invertible (and thus, clearly, quasi- $\tilde{\star}$ -invertible), then each P_i is quasi- $\tilde{\star}$ -invertible (Lemma 3.12(2)). Moreover, since Q is a quasi- $\tilde{\star}$ -ideal of D , then $P_1^{e_1} P_2^{e_2} \dots P_n^{e_n} \subseteq (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\tilde{\star}} \cap D \subseteq Q$. Therefore, $P_j \subseteq Q$ for some j , with $1 \leq j \leq n$, and since D_Q is a DVR, we have $Q = P_j$. Hence Q is a quasi- $\tilde{\star}$ -invertible ideal of D . Therefore, by Theorem 4.67, we conclude that D is $\tilde{\star}$ -Dedekind. \square

Remark 4.86. It is clear that, if D is a \star -DD then, for each nonzero ideal I of D , such that $I^{\star_f} \neq D^{\star_f}$, we have a unique factorization $I^{\star_f} = (P_1^{e_1} P_2^{e_2} \dots P_n^{e_n})^{\star_f}$, for some family of quasi- \star_f -prime ideals P_i , with $P_i^{\star_f} \neq D^{\star_f}$, and for some family of positive integers e_i , $i = 1, 2, \dots, n$, since $\tilde{\star} = \star_f$ (Proposition 4.1). The converse is not true. For instance, take D, T and \star as in Remark 4.40(2). For each nonzero proper ideal I of D , we have $I^{\star_f} = IT = M^e = (M^e)^{\star_f}$, for some positive integer e , since T is a DVR.

Note that this representation is unique, since D is local with maximal ideal M and $\dim(D) = 1$. But we have already observed that D is not a \star -DD.

Next result generalizes [38, Theorem 38.5 ((1) \Leftrightarrow (3))] to the semistar setting .

Theorem 4.87. *Let D be an integral domain which is not a field and \star a semistar operation on D . The following are equivalent:*

- (1) D is a \star -DD.
- (2) For each nonzero ideal I and for each $a \in I$, $a \neq 0$, there exists $b \in I^{\tilde{\star}}$ such that $I^{\tilde{\star}} = ((a, b)D)^{\tilde{\star}}$.

Proof. (1) \Rightarrow (2). We start by proving the following:

Claim. *If D is a \star -DD, then the map $M \mapsto M^{\tilde{\star}}$ establishes a bijection between the set $\mathcal{M}(\star_f)$ ($= \mathcal{M}(\tilde{\star})$ by Proposition 1.40 (5)) of the quasi- \star_f -maximal ideals of D and the set $\mathcal{M}(t_{D^{\tilde{\star}}})$ of the $t_{D^{\tilde{\star}}}$ -maximal ideals of (the Krull domain) $D^{\tilde{\star}}$.*

Let ι be the canonical embedding of D in $D^{\tilde{\star}}$. For each $M \in \mathcal{M}(\star_f)$, $M^{\tilde{\star}}$ is a prime ideal of $D^{\tilde{\star}}$ (Proposition 2.10) and so it is a $\tilde{\star}_\iota$ -prime ideal. Furthermore, by Corollary 4.80, we know that $D^{\tilde{\star}}$ is a Krull domain and $\tilde{\star}_\iota = t_{D^{\tilde{\star}}}$. On the other hand, for each $\tilde{\star}_\iota$ -prime ideal N of $D^{\tilde{\star}}$, we know that $N \cap D$ is a quasi- $\tilde{\star}$ -prime of D (Lemma 4.7). Since D is a \star -DD (or, equivalently, a $\tilde{\star}$ -DD), we have that each quasi- $\tilde{\star}$ -prime is a quasi- $\tilde{\star}$ -maximal (Proposition 4.39 (2)), thus we easily conclude.

Let $a \in I$, $a \neq 0$, and $\{M_1, M_2, \dots, M_n\}$ the (finite) set of quasi- \star_f -maximal ideals such that $a \in M_i$. Since D_{M_i} is a DVR, then $ID_{M_i} = x_i D_{M_i}$, for some $x_i \in I$, for each $i = 1, 2, \dots, n$. We use the fact that $D^{\tilde{\star}}$ is a Krull domain and, by the Claim, that $\{D_{M^{\tilde{\star}}}^{\tilde{\star}} = D_M \mid M \in \mathcal{M}(\star_f)\}$ is the defining family of the rank-one discrete valuation overrings of $D^{\tilde{\star}}$, in order to apply the approximation theorem to $D^{\tilde{\star}}$. Let v_1, v_2, \dots, v_n be the valuations associated respectively to $D_{M_1}, D_{M_2}, \dots, D_{M_n}$ and let $v_{M'}$ be the valuation associated to $D_{M'} = D_{M^{\tilde{\star}}}$, for $M' \in \mathcal{M}' := \mathcal{M}(\star_f) \setminus \{M_1, M_2, \dots, M_n\}$. Set $k_1 := v_1(x_1), k_2 := v_2(x_2), \dots, k_n := v_n(x_n)$. Then there exists $b \in K$ such that $v_i(b) = k_i$, for each $i = 1, 2, \dots, n$, and $v_{M'}(b) \geq 0$, for each $M' \in \mathcal{M}'$ [38, Theorem 44.1]. We have $I^{\tilde{\star}} = ((a, b)D)^{\tilde{\star}}$. Indeed, let $M \in \mathcal{M}(\star_f)$. If $M = M_i$, for some i , then $ID_M = ID_{M_i} = x_i D_{M_i} = b D_{M_i} = (a, b)D_{M_i}$. If $M \neq M_i$ for each i , then $ID_M = D_M = (a, b)D_M$.

(2) \Rightarrow (1). Let $M \in \mathcal{M}(\star_f)$ and J a nonzero ideal of D_M . Let $a \in J$, $a \neq 0$, there exists $s \in D$, $s \notin M$, such that $sa \in I := J \cap D$. Then, by assumption, there exists $b \in I^{\tilde{\star}}$ such that $I^{\tilde{\star}} = ((sa, b)D)^{\tilde{\star}}$. Therefore, we have $J = ID_M = I^{\tilde{\star}}D_M = ((sa, b)D)^{\tilde{\star}}D_M = (sa, b)D_M = (a, b)D_M$. By [38, Theorem 38.5], D_M is a Dedekind domain, and hence a DVR. Thus, D is a \star -ADD, hence, in particular, is a $P\tilde{\star}MD$ (Corollary 4.43 and Proposition

4.39(2)). In addition, from the assumption each ideal of D is $\tilde{\star}$ -finite, then D is $\tilde{\star}$ -Noetherian (Lemma 4.18), hence D is a \star -DD (Corollary 4.61 and Proposition 4.59(1)). \square

Remark 4.88. Note that, if D is a \star -DD (and hence $\tilde{\star} = \star_f$), then D satisfies also a statement concerning \star_f , analogous to the statement (2) in Theorem 4.87:

(2_f) for each nonzero ideal I of D and for each $0 \neq a \in I$, there exists $b \in I^{\star_f}$ such that $((a, b)D)^{\star_f} = I^{\star_f}$.

But (2_f) does not imply that D is a \star -DD. For instance, let D, T and \star be as in Remark 4.40. Obviously, for each nonzero proper ideal I of D and for each nonzero $a \in I \subseteq D$ we have $I^{\star_f} = IT = X^n T = (a, X^n)T = ((a, X^n)D)^{\star_f}$, for some $n \geq 1$, (where $X^n \in I^{\star_f} \cap D$), but D is not a \star -DD.

Bibliography

- [1] D. D. Anderson, *Some remarks on the ring $R(X)$* , Comment. Math. Univ. St. Paul. **26** (1977/78), no. 2, 137–140.
- [2] ———, *Star-operations induced by overrings*, Comm. Algebra **16** (1988), no. 12, 2535–2553.
- [3] D. D. Anderson and David F. Anderson, *Some remarks on star operations and the class group*, J. Pure Appl. Algebra **51** (1988), no. 1-2, 27–33.
- [4] ———, *Examples of star operations on integral domains*, Comm. Algebra **18** (1990), no. 5, 1621–1643.
- [5] D. D. Anderson and Muhammad Zafrullah, *On t -invertibility. III*, Comm. Algebra **21** (1993), no. 4, 1189–1201.
- [6] David F. Anderson, *A general theory of class groups*, Comm. Algebra **16** (1988), no. 4, 805–847.
- [7] David F. Anderson, Evan G. Houston, and Muhammad Zafrullah, *Pseudo-integrality*, Canad. Math. Bull. **34** (1991), no. 1, 15–22.
- [8] J. T. Arnold and J. W. Brewer, *Kronecker function rings and flat $D[X]$ -modules*, Proc. Amer. Math. Soc. **27** (1971), 483–485.
- [9] ———, *On flat overrings, ideal transforms and generalized transforms of a commutative ring*, J. Algebra **18** (1971), 254–263.
- [10] Valentina Barucci, *Mori domains*, Non-Noetherian commutative ring theory, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000, pp. 57–73.
- [11] Valentina Barucci and Stefania Gabelli, *How far is a Mori domain from being a Krull domain?*, J. Pure Appl. Algebra **45** (1987), no. 2, 101–112.
- [12] Valentina Barucci, Evan Houston, Thomas G. Lucas, and Ira Papick, *m -canonical ideals in integral domains. II*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), Dekker, New York, 2001, pp. 89–108.

- [13] S. Bazzoni and L. Salce, *Warfield domains*, J. Algebra **185** (1996), no. 3, 836–868.
- [14] Silvana Bazzoni, *Divisorial domains*, Forum Math. **12** (2000), no. 4, 397–419.
- [15] Willy Brandal and Erol Barbut, *Localizations of torsion theories*, Pacific J. Math. **107** (1983), no. 1, 27–37.
- [16] Paul-Jean Cahen, Evan G. Houston, and Thomas G. Lucas, *Discrete valuation overrings of Noetherian domains*, Proc. Amer. Math. Soc. **124** (1996), no. 6, 1719–1721.
- [17] Gyu Whan Chang and Jeanam Park, *Star-invertible ideals of integral domains*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **6** (2003), no. 1, 141–150.
- [18] C. Chevalley, *La notion d’anneau de décomposition*, Nagoya Math. J. **7** (1954), 21–33.
- [19] David E. Dobbs and Richard Fedder, *Conducive integral domains*, J. Algebra **86** (1984), no. 2, 494–510.
- [20] Said El Baghdadi and Marco Fontana, *Semistar linkedness and flatness, Prüfer semistar multiplication domains*, Comm.Algebra **32** (2004), no. 3, 1101–1126.
- [21] Said El Baghdadi, Marco Fontana, and Giampaolo Picozza, *Semistar Dedekind domains*, to appear in J. Pure Appl. Algebra.
- [22] Marco Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. (4) **123** (1980), 331–355.
- [23] Marco Fontana and James A. Huckaba, *Localizing systems and semistar operations*, Non-Noetherian Commutative Ring Theory (Scott T. Chapman and Sarah Glaz, eds.), Kluwer Academic Publishers, 2000, pp. 169–198.
- [24] Marco Fontana, James A. Huckaba, and Ira J. Papick, *Prüfer domains*, Marcel Dekker Inc., New York, 1997.
- [25] Marco Fontana, Pascual Jara, and Eva Santos, *Local-global properties of semistar operations*, to appear in Comm.Alg.
- [26] ———, *Prüfer \star -multiplication domains and semistar operations*, J. Algebra Appl. **2** (2003), no. 1, 21–50.

- [27] Marco Fontana and K. Alan Loper, *Kronecker function rings: a general approach*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, New York, 2001, pp. 189–205.
- [28] ———, *A Krull-type theorem for the semistar integral closure of an integral domain*, Arab. J. Sci. Eng. Sect. C Theme Issues **26** (2001), no. 1, 89–95.
- [29] ———, *Nagata rings, Kronecker function rings, and related semistar operations*, Comm. Algebra **31** (2003), no. 10, 4775–4805.
- [30] Marco Fontana and Mi Hee Park, *Star operations and pullbacks*, J.Algebra **274** (2004), no. 1, 387–421.
- [31] Marco Fontana and Giampaolo Picozza, *Semistar invertibility on integral domains*, Preprint (2003).
- [32] Stefania Gabelli, *Completely integrally closed domains and t -ideals*, Boll. Un. Mat. Ital. B (7) **3** (1989), no. 2, 327–342.
- [33] ———, *Domains with the radical trace property and their complete integral closure*, Comm. Algebra **20** (1992), no. 3, 829–845.
- [34] Stefania Gabelli and Evan Houston, *Coherentlike conditions in pullbacks*, Michigan Math. J. **44** (1997), no. 1, 99–123.
- [35] Josefa M. García, Pascual Jara, and Eva Santos, *Prüfer $*$ -multiplication domains and torsion theories*, Comm. Algebra **27** (1999), no. 3, 1275–1295.
- [36] Robert Gilmer, *An embedding theorem for HCF-rings*, Proc. Cambridge Philos. Soc. **68** (1970), 583–587.
- [37] ———, *Prüfer domains and rings of integer-valued polynomials*, J. Algebra **129** (1990), no. 2, 502–517.
- [38] ———, *Multiplicative ideal theory*, Queen’s University, Kingston, ON, 1992.
- [39] Sarah Glaz and Wolmer V. Vasconcelos, *Flat ideals. II*, Manuscripta Math. **22** (1977), no. 4, 325–341.
- [40] Malcolm Griffin, *Some results on v -multiplication rings*, Canad. J. Math. **19** (1967), 710–722.
- [41] Franz Halter-Koch, *Ideal systems*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 211, Marcel Dekker Inc., New York, 1998.

- [42] ———, *Construction of ideal systems with nice Noetherian properties*, Non-Noetherian commutative ring theory, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000, pp. 271–285.
- [43] ———, *Localizing systems, module systems, and semistar operations*, J. Algebra **238** (2001), no. 2, 723–761.
- [44] ———, *Kronecker function rings and generalized integral closures*, Comm. Algebra **31** (2003), no. 1, 45–59.
- [45] John R. Hedstrom and Evan G. Houston, *Pseudo-valuation domains*, Pacific J. Math. **75** (1978), no. 1, 137–147.
- [46] ———, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.
- [47] William Heinzer, *Integral domains in which each non-zero ideal is divisorial*, Mathematika **15** (1968), 164–170.
- [48] William J. Heinzer, James A. Huckaba, and Ira J. Papick, *m -canonical ideals in integral domains*, Comm. Algebra **26** (1998), no. 9, 3021–3043.
- [49] Evan Houston, *Prime t -ideals in $R[X]$* , Commutative ring theory (Fès, 1992), Lecture Notes in Pure and Appl. Math., vol. 153, Dekker, New York, 1994, pp. 163–170.
- [50] Evan Houston and Muhammad Zafrullah, *Integral domains in which each t -ideal is divisorial*, Michigan Math. J. **35** (1988), no. 2, 291–300.
- [51] Paul Jaffard, *Les systèmes d'idéaux*, Travaux et Recherches Mathématiques, IV, Dunod, Paris, 1960.
- [52] B. G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra **124** (1989), no. 2, 284–299.
- [53] ———, *Prüfer v -multiplication domains and the ring $R[X]_{N_v}$* , J. Algebra **123** (1989), no. 1, 151–170.
- [54] Irving Kaplansky, *Commutative rings*, The University of Chicago Press, Chicago, Ill.-London, 1974.
- [55] Saroj Malik, Joe L. Mott, and Muhammad Zafrullah, *On t -invertibility*, Comm. Algebra **16** (1988), no. 1, 149–170.
- [56] Ryūki Matsuda, *A note on the number of semistar-operations. II*, Far East J. Math. Sci. (FJMS) **2** (2000), no. 1, 159–172.
- [57] ———, *Note on valuation rings and semistar-operations*, Comm. Algebra **28** (2000), no. 5, 2515–2519.

- [58] ———, *Note on the number of semistar-operations. III*, Commutative rings, Nova Sci. Publ., Hauppauge, NY, 2002, pp. 77–81.
- [59] ———, *Note on the number of semistar-operations. IV*, Sci. Math. Jpn. **55** (2002), no. 2, 345–347.
- [60] ———, *Note on the number of semistar-operations. V*, Sci. Math. Jpn. **57** (2003), no. 1, 57–62.
- [61] Ryūki Matsuda and Takasi Sugatani, *Semistar-operations on integral domains. II*, Math. J. Toyama Univ. **18** (1995), 155–161.
- [62] Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
- [63] A. Mimouni, *Semistar operations of finite character on integral domains*, Preprint.
- [64] A. Mimouni and M. Samman, *Semistar operations on valuation domains*, IJCR **2**, no. 3.
- [65] Akira Okabe and Ryūki Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.
- [66] Bruce Olberding, *Globalizing local properties of Prüfer domains*, J. Algebra **205** (1998), no. 2, 480–504.
- [67] ———, *On the classification of stable domains*, J. Algebra **243** (2001), no. 1, 177–197.
- [68] ———, *Stability, duality, 2-generated ideals and a canonical decomposition of modules*, Rend. Sem. Mat. Univ. Padova **106** (2001), 261–290.
- [69] ———, *Stability of ideals and its applications*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, New York, 2001, pp. 319–341.
- [70] ———, *On the structure of stable domains*, Comm. Algebra **30** (2002), no. 2, 877–895.
- [71] Julien Querré, *Sur une propriété des anneaux de Krull*, Bull. Sci. Math. (2) **95** (1971), 341–354.
- [72] ———, *Idéaux divisoriels d'un anneau de polynômes*, J. Algebra **64** (1980), no. 1, 270–284.
- [73] Fred Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 794–799.

- [74] Judith D. Sally and Wolmer V. Vasconcelos, *Stable rings*, J. Pure Appl. Algebra **4** (1974), 319–336.
- [75] Hirohumi Uda, *LCM-stableness in ring extensions*, Hiroshima Math. J. **13** (1983), no. 2, 357–377.
- [76] Fanggui Wang, *w-dimension of domains*, Comm. Algebra **27** (1999), no. 5, 2267–2276.
- [77] ———, *w-dimension of domains. II*, Comm. Algebra **29** (2001), no. 6, 2419–2428.
- [78] ———, *Modules over R and $R\{X\}$* , Sichuan Shifan Daxue Xuebao Ziran Kexue Ban **25** (2002), no. 6, 557–562.
- [79] Fanggui Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.
- [80] ———, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.
- [81] Muhammad Zafrullah, *Ascending chain conditions and star operations*, Comm. Algebra **17** (1989), no. 6, 1523–1533.
- [82] ———, *Putting t-invertibility to use*, Non-Noetherian commutative ring theory, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000, pp. 429–457.
- [83] Oscar Zariski and Pierre Samuel, *Commutative algebra. Vol. II*, Springer-Verlag, New York, 1975.

List of notation

$\overline{F}(D)$, 8	$\star_{\mathcal{F}}$, 18
$F(D)$, 8	\mathcal{F}^* , 19
$f(D)$, 8	$\tilde{\star}$, 20
$\star _{F(D)}$, 9	\star_l , 22
\star_f , 10	\star^t , 22
$\mathcal{M}(\star)$, 10	$\text{Kr}(D, \star)$, 23
Star (D), 11	\star_a , 23
SStar (D), 11	$\text{Na}(D, \star)$, 24
(S)Star (D), 11	$v(T)$, 27
\star_e , 12	$t(T)$, 28
$\star_{\{T\}}$, 13	$(-)_l$, 33
$\wedge \star_\alpha$, 14	$(-)^t$, 33
$\vee \star_\beta$, 14	$(-)_l^T$, 33
\star_Δ , 15	$(-)_T^t$, 33
$\star_{\{P\}}$, 15	SStar (D, T), 33
$v(I)$, 15	$\mathcal{O}(D)$, 35
$\mathcal{F}(P)$, 18	$\text{Inv}(D, \star)$, 55
$\mathcal{F}(\Delta)$, 18	$\text{QInv}(D, \star)$, 61

$\text{Inv}^*(D)$, 65

$\text{QInv}^*(D)$, 66

$\mathcal{F}[X]$, 83

$\star\text{-dim}$, 93

$\star\text{-ht}$, 93

$[\star]$, 101

Index

- ★-ADD, 88
- ★-DD, 97
- ★-finite character property, 81
- ★-dimension, 93
- ★-height, 93

- Ascent of a semistar operation, 22

- Descent of a semistar operation, 22
- Domain
 - H -domain, 45
 - $H(\star)$ -domain, 58
 - conducive , 12
 - divisorial, 44
 - generalized Dedekind domain, 45
 - Prüfer ★-multiplication domain, 71
 - quasi-★-integrally closed, 102
 - semistar almost Dedekind, 88
 - semistar Dedekind, 97
 - semistar Noetherian, 77
 - stable, 48
 - totally divisorial, 44

- Ideal
 - ★-finite , 57
 - ★-invertible, 55
 - m-canonical, 17
 - quasi-★-invertible, 60
 - quasi-★-ideal, 10
 - quasi-★-maximal, 10
 - quasi-★-prime, 10
 - quasi-m-canonical, 49
 - stable, 48

- Kronecker function ring, 23

- Localizing system, 18
 - associated to a semistar operation, 19
 - finitely generated, 18
 - spectral, 18

- Nagata ring, 25

- Operation
 - b , 14
 - d , 11
 - e , 11
 - t , 13
 - $t(T)$, 28
 - v , 12
 - $v(I)$, 15
 - “extension to an overring”, 13
 - trivial extension (of a star operation), 12

- Overring
 - (★, ★′)-linked, 74
 - (★, ★′)-flat, 37
 - ★-valuation, 24

- P★MD, 71

- Semistar almost Dedekind domain, 88
- Semistar Dedekind domain, 97
- Semistar integral closure, 102
- Semistar Noetherian domain, 77
- Semistar operation, 8
 - (semi)star operation, 9
 - of finite type, 10
 - a.b., 11
 - associated to a localizing system, 18
 - cancellative, 11
 - e.a.b., 11

spectral, 15

stable, 11

Trivial extension (of a star operation),

12

Acknowledgments

I express my sincere gratitude to my advisor, Prof. Marco Fontana, for his guidance and patience and for suggesting the topic which I really enjoyed, and to Dr. Francesca Tartarone, for her useful advice and suggestions.

I thank also Prof. Stefania Gabelli and Dr. Florida Girolami for their continuous encouragement and Prof. Said El Baghdadi for his collaboration.

I acknowledge the Dipartimento di Matematica di “Roma Tre” for the support given during the preparation of my thesis.

I wish to express my thanks to Prof. E. Houston and Prof. K. Alan Loper for carefully reading my thesis and for their helpful comments and remarks.

I feel also grateful to Prof. Valentina Barucci, Prof. Paul-Jean Cahen and Prof. Stefania Gabelli for accepting to be members of my thesis committee.

Ringrazio infine i miei compagni di stanza Luigi, Isabella, Riccardo e Tiziana, e i dottorandi e gli studenti di Matematica di Roma Tre.