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Asymptotic analysis for a singularly perturbed Dirichlet problem

PHD THESIS

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Contents

Introduction and statement of the Main Results

We are concerned with the study of solutions for the semilinear elliptic Dirichlet problem:

$$\begin{cases} -\Delta u + \lambda V(x) u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where $\lambda > 0$ is a large parameter, $p > 1$, Ω is bounded domain of $\mathbb{R}^N$, $N \geq 2$, $V : \overline{\Omega} \to \mathbb{R}$ is a positive potential. Solutions of (0.1) necessarily blow-up as $\lambda \to +\infty$ in the sense that they are not uniformly bounded (see Proposition 0.1). The aim is to obtain an accurate description of their asymptotic behavior as $\lambda \to +\infty$ through an energy or a Morse index information. We are interested in describing situations where blow-up occurs at finitely as well as infinitely many points. The objective is to give an asymptotic counter-part to several existence results available in literature and based on a constructive approach.

The asymptotic analysis in problems with critical and sub-critical polynomial nonlinearities, with Dirichlet or Neumann boundary condition, has been largely considered in case of pointwise blow-up. Under the transformation $v(x) = \frac{u(x)}{\lambda^{\frac{1}{p-1}}}$, with $\frac{1}{\lambda} = \varepsilon^2$ and $V \equiv 1$ for simplicity, note that problem (0.1) reads equivalently as a singularly perturbed Dirichlet problem:

$$\begin{cases} -\varepsilon^2 \Delta v + v = v^p & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

Problem (0.2) and related ones have been widely considered in literature, as they arise as steady state equation in several biological and physical models, such as dynamic population, pattern formation theories and chemical reactor theory. In order to investigate the long-time behavior of the dynamical solutions, it is very important to understand the properties of the steady-state ones.

The most interesting features of problem (0.1) concern the existence and multiplicity of solutions, and their asymptotic behavior as $\lambda \to +\infty$. When $V \equiv 1$, multiplicity results of solutions and their asymptotic behavior, were first obtained by Ljusternik-Schnirelman category in [**12, 14**]: they prove that equation (0.1) has a family of solutions exhibiting a spike-layer pattern as $\lambda \to +\infty$. Subsequently, in [**89**] Ni and Wei studied the behavior as $\lambda \to +\infty$ of a least energy solution to problem (0.1), characterized variationally as a mountain pass of the associated energy functional, by an asymptotic expansion of the critical value associated to the least energy solution. They proved that, for λ sufficiently large, a least-energy solution possesses a single spike-layer with its unique peak in the interior of Ω. Moreover, the peak

must be situated near the most centered part of Ω, i. e. where the distance function $d(P, \partial\Omega)$, $P \in \Omega$, assumes its global maximum. Intuitively, the location of the blow-up points should depend on the geometric properties of the domain, a natural problem being that of determining the role of the distance function in the location of blow-up points (when $V \equiv 1$).
More recently in [**75, 99**] it was proved that, for any local maximum P_0 of the distance function, there exists a family of solutions with a maximum point that approaches P_0.
In particular in [**75**] it is also shown that, if the Brower degree $\deg(\nabla d(\cdot, \partial\Omega), M, 0) \neq 0$ on a suitable subset M of Ω, then there exists a family of solutions with a unique local maximum point which converges to a critical point of the distance function in M.
In [**45**] the authors proved the existence of single k-peaks solutions at any topologically nontrivial critical point of the distance function, which satisfies a suitable non-degeneracy condition. In [**40**] Dancer and Wei proved the existence of $2-$peaks solutions. Concerning the effect of the domain topology on the existence of multi-peaks solution Dancer and Yan in [**36, 37**] proved that if the homology of the domain is nontrivial, then, for any positive integer k, problem (0.2) has at least one $k-$peaks solution. They assume that the distance function has k isolated compact connected critical sets $T_1, \ldots, T_k$ with suitable properties. They construct a solution which has exactly one local maximum point in a small neighborhood of T_i for $i = 1, \ldots, k$. Moreover they proved that if Ω is strictly convex, problem (0.2) does not have $k-$peaks solutions.
Other papers that deal with this problem ($V \equiv 1$) are [**64, 65**]. In [**64**] it is proved that any "topologically non trivial" critical point of the distance function generates a family of single peak interior spike solutions. Moreover they proved that the peak of any single solution must converge to a critical point of the distance function, and treated also $k-$peaks solutions in the Neumann case. This method is based on an approach of Bahri (see [**11**]), and the new idea is to evaluate, in terms of the generalized gradient of Clarke, the energy of the solutions. In [**23, 24**] Cao, Dancer, Noussair and Yan constructed $k-$peaks solutions with the peaks near local maximum points or saddle points of $d(\,\cdot\,, \partial\Omega)$.

Let us fix some notations and terminology. Now for any solution u of (0.1), one can introduce the linearized operator at u defined as

$$L_{u,\lambda} = -\Delta + \lambda V - p u^{p-1}, \tag{0.3}$$

and its corresponding eigenvalues $\{\, \mu_{k,\lambda};\ k = 1, 2, \ldots \}$. Note that the first eigenvalue is given by

$$\inf \Big\{ < L_{u,\lambda}\phi, \phi >_{H_0^1(\Omega)};\ \phi \in C_0^\infty(\Omega),\ \int_\Omega \phi^2(x) \mathrm{d}x = 1 \Big\} \tag{0.4}$$

with the infimum being attained at a first eigenfunction ϕ_1, while the second eigenvalue is given by the formula

$$\mu_{2,\lambda} = \inf \Big\{ < L_{u,\lambda}\phi, \phi >_{H_0^1(\Omega)};\ \phi \in C_0^\infty(\Omega),\ \int_\Omega \phi^2(x) \mathrm{d}x = 1\,,\ \int_\Omega \phi(x)\phi_1(x) \mathrm{d}\, x = 0 \Big\}. \tag{0.5}$$

This construction can be iterated to obtain the $k-$th eigenvalue $\mu_{k,\lambda}(u)$ with the convention that eigenvalues are repeated according to their multiplicities.

Let u_n be a solution of

$$(0.6)\qquad \begin{cases} -\Delta u_n + \lambda_n V(x) u_n = u_n^p & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

for a sequence $\lambda_n \to +\infty$ as $n \to +\infty$.
We will assume u_n to have uniformly bounded Morse indices, i.e.

$$(0.7)\qquad \exists\, \bar{k} \in \mathbb{N} \ \text{ s. t. } \ \mu_{\bar{k}+1,\lambda_n}(u_n) > 0 \ \ \forall\, n \in \mathbb{N}.$$

We first show that $\|u_n\|_\infty \to \infty$ as $n \to +\infty$ so as to justify the blow-up analysis we will perform later.

PROPOSITION 0.1. *Let $p > 1$ and u_n be a solution of (0.6). Then $\|u_n\|_\infty \to \infty$ as $n \to +\infty$.*

PROOF. We suppose by contradiction that $\|u_n\|_\infty \leq C$. Multiply the equation (0.6) by u_n and integrate to get

$$\int_\Omega |\nabla u_n|^2 + \lambda_n \int_\Omega u_n^2 = \int_\Omega u_n^{p+1} \leq C^{p+1}|\Omega|.$$

We have that u_n is bounded in $H_0^1(\Omega)$ and then, up to a subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $\int_\Omega u_n^2 \to \int_\Omega u^2$ as $n \to +\infty$.
Since

$$\int_\Omega u_n^2 \leq \frac{C^{p+1}|\Omega|}{\lambda_n} \quad \text{and} \quad \lambda_n \to +\infty,$$

we must have $\int_\Omega u^2 = 0$, and then $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$ as $n \to +\infty$. Since $u_n \to 0$ in $L^2(\Omega)$, we have that

$$\int_\Omega |\nabla u_n|^2 \leq \int_\Omega u_n^{p+1} \leq C^{p-1} \int_\Omega u_n^2 \to 0.$$

By Sobolev embedding we have that for $p \geq \frac{N+2}{N-2}$:

$$\int_\Omega |\nabla u_n|^2 \leq \int_\Omega u_n^{p+1} \leq C^{p+1-\frac{2N}{N-2}} S_N^{-\frac{N}{N-2}} \Big(\int_\Omega |\nabla u_n|^2 \Big)^{\frac{N}{N-2}}$$

where S_N is the Sobolev constant. A contradiction with $u_n \to 0$ in $H_0^1(\Omega)$.
By the Hölder's inequality and the Sobolev embedding, we have that for $1 < p < \frac{N+2}{N-2}$

$$\int_\Omega |\nabla u_n|^2 \leq \tilde{C} \Big(\int_\Omega u_n^{\frac{2N}{N-2}} \Big)^{\frac{N-2}{2N}(p+1)} \leq \tilde{C}\, S_N^{-\frac{p+1}{2}} \Big(\int_\Omega |\nabla u_n|^2 \Big)^{\frac{p+1}{2}},$$

and a contradiction still arise. □

In the first part of this thesis, we consider pointwise blow-up and obtain results, already known for solutions sequences with uniformly bounded energy, under an hypothesis of boundedness for their Morse indices. For the asymptotic analysis we need to give a sort of classification of solutions of (0.1) which are stable, or stable outside a compact set. For this classification we use techniques first used by Farina [**54**] and by Esposito et al. [**49, 48**].

In particular, given u_n a solutions sequence of (0.1) with $\lambda_n \to +\infty$ as $n \to +\infty$, we have to consider the description of the blow-up behavior of u_n. Observe that it is always true that

$$\max_{\Omega} u_n(x) \to +\infty \quad \text{as} \quad n \to +\infty,$$

as we have shown in Proposition 0.1.
As in the usual asymptotic techniques, we want to describe, asymptotically, the shape of u_n around any blow-up point. To this aim we rescale u_n around a blow-up sequence and try to identify a limiting problem. We denote by $m(u)$ the Morse index of u, as a solution of (1.1.1). To carry out our analysis, the crucial assumption is that $\sup_n m(u_n) < +\infty$. Given $\varepsilon_n = \|u_n\|_\infty^{-\frac{p-1}{2}}$, one naturally scales u_n in the form $U_n(y) = \frac{u_n(\varepsilon_n\, y+P_n)}{\|u_n\|_\infty} = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n\, y + P_n)$ where P_n is an absolute maximum point of u_n, defined in the rescaled domain $\Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$. We study the asymptotic behavior of U_n and prove that $U_n \to U$ locally uniformly with U a solution of a suitable problem in $\tilde{\Omega} = \lim_{n\to+\infty} \Omega_n$ (an hyperspace or $\mathbb{R}^N$). The limiting domain $\tilde{\Omega}$ depends on how fast P_n possibly approaches $\partial\Omega$.

We recall the main results.
First in the sub-critical case we have:

THEOREM 0.2 (Local profile). *Let (λ_n, u_n) be a positive solution of*

$$\left\{ \begin{array}{ll} -\Delta u_n + \lambda_n V u_n = u_n^p, & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{array} \right. \tag{0.8}$$

with $1 < p < \frac{N+2}{N-2}$.
Assume either

$$\sup_n m(u_n) < +\infty,$$

or

$$\sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1} - \frac{N}{2}}} \int_\Omega u_n^{p+1} < +\infty.$$

Let $P_n \in \Omega$ s.t. $u_n(P_n) = \max_{\Omega \cap B_{R_n \varepsilon_n}(P_n)} u_n$ for some $R_n \to +\infty$ where $\varepsilon_n = u_n(P_n)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$.
Setting $U_n(y) = \frac{u_n(\varepsilon_n y+P_n)}{\|u_n\|_\infty} = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ for $y \in \Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$, then for a subsequence we have that, as $n \to +\infty$:

- *$\lambda_n \varepsilon_n^2 V(P_n) \to \tilde{\lambda} \in (0, 1]$ for some universal constant $\tilde{\lambda}$;*
- *$\frac{\varepsilon_n}{d(P_n, \partial\Omega)} \to 0$;*
- *$U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$ where U is a solution of*

$$\left\{ \begin{array}{ll} -\Delta U + \tilde{\lambda} U = U^p, & \text{in } \mathbb{R}^N \\ 0 < U \le U(0) = 1 & \text{in } \mathbb{R}^N. \end{array} \right.$$

Moreover, $\exists \phi_n \in C_0^\infty(\Omega)$ with supp $\phi_n \subset B_{R\varepsilon_n}(P_n)$, for some $R > 0$, so that

$$\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1}) \phi_n^2 \, dx < 0, \quad \forall\, n \text{ large},$$

and

$$\lim_{n\to+\infty}\frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}}\int_{B_{R\,\varepsilon_n(P_n)}}u_n^{p+1}=\tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}}\Big(\lim_{n\to+\infty}V(P_n)\Big)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{B_R(0)}U^{p+1}.$$

After the limiting problem has been identified and the local behavior around a blow up sequence P_n has been described, we can prove global estimates. We will show in such way that the sequence u_n decays exponentially away from the blow-up points.

THEOREM 0.3 (Global behavior). *Let $1 < p < \frac{N+2}{N-2}$. Let $\lambda_n \to +\infty$, u_n be solution of (0.8), so that either*

$$\overline{k}=\limsup_{n\to+\infty}m(u_n)<+\infty$$

or

$$\overline{k}=\tilde{\lambda}^{\frac{p+1}{p-1}-\frac{N}{2}}(\min_{\Omega}V)^{\frac{N}{2}-\frac{p+1}{p-1}}\Big(\int_{\mathbb{R}^N}U^{p+1}\Big)^{-1}\lim_{n\to+\infty}\frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}}\int_{\Omega}u_n^{p+1}$$

with u_n satisfying (0.7). Up to a subsequence, there exist $P_n^1,\dots,P_n^k$, $k\le\overline{k}$ with $\varepsilon_n^i = u_n(P_n^i)^{-\frac{p-1}{2}}\to 0$ as $n\to+\infty$ s. t.

$$\varepsilon_n^1\le\varepsilon_n^i\le C_0\varepsilon_n^1,\quad \textit{for all } i=1,\dots,k$$

$$\frac{\varepsilon_n^i+\varepsilon_n^j}{|P_n^i-P_n^j|}\to 0\quad \textit{as } n\to+\infty,\quad \textit{for all } i,j=1,\dots,k,\ i\neq j$$

$$\frac{\varepsilon_n^i}{d(P_n^i,\partial\Omega)}\to 0\quad \textit{as } n\to+\infty,\quad \textit{for all } i=1,\dots,k,\ i\neq j$$

$$u_n(P_n^i)=\max_{\Omega\cap B_{R_n\varepsilon_n^i}(P_n^i)}u_n,$$

for some $R_n\to+\infty$ as $n\to+\infty$.
Moreover, there holds

$$u_n(x)\le C(\varepsilon_n^1)^{-\frac{2}{p-1}}\sum_{i=1}^{k}e^{-\gamma\frac{|x-P_n^i|}{\varepsilon_n^1}}\quad\forall\, x\in\Omega,\ n\in\mathbb{N}$$

with $C>0$.

A first goal concerns the investigation of the link between the Morse index and the energy in case of pointwise blow-up. In the context of the Schrödinger operator, they are related in terms of the so-called Rozenbljum-Lieb-Cwikel inequality [**29, 71, 93**]- an estimate of the number of negative eigenvalues of the Schrödinger operator $-\Delta+V$ in terms of a suitable Lebesgue norm of the negative part V_- of V - a one side bound, where the universal constants are however not explicit.
A-posteriori, we show that Morse index information and an energy one are equivalent. Indeed, taking advantage of the special structure of our equation, by an asymptotic analysis approach,

we establish a double-side bound between these two quantities with explicit and essentially sharp constants.

THEOREM 0.4 (Rozenbljum-Lieb-Cwikel type estimate). *Let u_n be a solution of (0.8). The following are equivalent*

(1) $\sup_n m(u_n) < +\infty$;

(2) $\sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty$.

Moreover, when (1) *or* (2) *holds we have*

$$\begin{aligned}\frac{\tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}}}{N+1}(\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U^{p+1} &\leq \liminf_{n\to+\infty}\frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}}{m(u_n)} \leq \limsup_{n\to+\infty}\frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}}{m(u_n)}\\ &\leq \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}}(\max_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U^{p+1}.\end{aligned}$$

Furthermore, the complete knowledge of the limiting problem allows us to establish strong pointwise estimates to get an expansion of the Pohozaev identities and to localize the position of the blow-up points in terms of the potential V or of the distance to the boundary if $V \equiv 1$. Note that the profile around each blow-up point should resemble, in many situations, to the unique radial solution given by M.K. Kwong [**70**], which has given energy and exactly $N+1$ nonnegative eigenvalues for the linearized operator counted with multiplicities.
The case of the critical nonlinearity is quite different. Solutions of (0.8) with uniformly bounded energy never have pointwise blow-up [**25**]. We will extend this analysis to solutions with uniformly bounded Morse indices. In such way we show that problem (0.8) doesn't have such solutions. In the supercritical case a similar result is in order.

For blow-up on manifolds of positive dimension, few results are known from the asymptotical point of view, while many existence results are available through perturbative methods.
The basic result is due to A. Ambrosetti, A. Malchiodi and W.-M. Ni [**5**] for radial solutions in an annulus $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$. They identify a modified potential $M(r) = r^{N-1}V^\theta(r)$, with $\vartheta = \frac{p+1}{p-1} - \frac{1}{2}$ and show that there exist families of solutions which blow-up on spheres whose radii are critical points of M.
It has been conjectured that if $N \geq 3$ there could exist also solutions blowing-up onto some manifold of dimension $\bar{h}$ with $1 \leq \bar{h} \leq N-2$.
Actually, we have two papers where some asymptotic analysis is carried over. The asymptotic analysis has been firstly performed by E.N. Dancer [**30**] by means of ODE techniques. Dancer shows that, for λ large, $V \equiv 1$ and p sub-critical, the only positive radial solution is the radial ground state and it takes its unique maximum on a sphere whose radius goes to 1. In general an energy information seems useless. Let us notice that, for example, the radial ground state solution in the annulus has both energy and Morse index very large, and the usual asymptotic techniques, based on the energy, do not work.
Just these difficulties Esposito, Mancini, Santra and Srikanth [**50**] have then developed an alternative asymptotic approach for radial solutions with uniformly bounded radial Morse indices and general V's in an annulus. They rigorously establish the correspondence between

c.p.'s of M and the blow-up radii. For example, the radial ground state solution has radial Morse index one, but "unbounded" energy and blows-up at the inner radius when $V \equiv 1$.
In a recent paper [**94**] the authors consider the problem on an annulus in $\mathbb{R}^4$, and look for solutions which are invariant under a $1-$parameter group action without fixed points.

In the second part of the thesis, our aim is to consider the asymptotic analysis when blow-up occurs on manifolds of positive dimension, with the purpose of proving a variational construction for such type of solutions, alternative to the perturbative approach [**5**].
We will restrict our attention to the case of $3-$dimensional annulus and to solutions with partial symmetries and uniformly bounded invariant Morse indices, as it has been done in [**50**] for radial solutions. The symmetry group G will be simply the one of rotations around the $z-$axis.
The first objective is to investigate the role of a (full or reduced) Morse index information in the study of the asymptotic behavior. When consider solutions which are invariant under a proper subgroup $G \subset O(N)$ of symmetries, we need to develop an asymptotic approach based on a $G-$invariant Morse index information. In this way we try to carefully localize the blow-up $G-$orbits still in terms of a c.p.'s of suitable modified potential.
Our aim is to exhibit potentials for which the orbits of maximum points for the corresponding solutions do not degenerate on the fixed points set. If G has no fixed points, one can provide, in this way, solutions (for example the $G-$invariant ground state) which blow-up on an orbit with dimension as G. Unfortunately, in general fixed points always arise and just higher order conditions on the blow-up set might prevent, for suitable potential, the blow-up set to degenerate onto the fixed points set.

First we have:

THEOREM 0.5. *Let* (λ_n, u_n) *be a positive,* $G-$*invariant, solutions of (0.8) with* $\sup_n m_G(u_n) < +\infty$, $1 < p \neq 5$.
Let $Q_n = (0, y_n, z_n) \in \Omega$, $y_n \geq 0$ *be so that* $u_n(Q_n) = \max_{\Omega \cap B_{R_n \varepsilon_n}(Q_n)} u_n \to 0$ *as* $n \to +\infty$, *for some* $R_n \to +\infty$.
Setting

$$U_n(X,Y,Z) = \frac{u_n(\varepsilon_n\,(X,Y,Z) + P_n)}{u_n(P_n)}, \quad \varepsilon_n = u_n(P_n)^{-\frac{p-1}{2}},$$

where

$$P_n := \begin{cases} (0,0,z_n) & \text{if } \frac{|y_n|}{\varepsilon_n} \leq C \\ Q_n & \text{if } \frac{|y_n|}{\varepsilon_n} \to +\infty; \end{cases}$$

up to a subsequence we have that

(1) *when* $u_n(Q_n)^{\frac{p-1}{2}}\, y_n \leq C$, *then there hold* $1 < p < 5$ *and*
- $\frac{y_n}{\varepsilon_n} \to 0$
- $\lambda_n \varepsilon_n^2 V(P_n) \to \tilde{\lambda}_3 \in (0,1]$ *for some universal constant* $\tilde{\lambda}_3$
- $\frac{\varepsilon_n}{d(P_n, \partial\Omega)} \to 0$
- $U_n \to U$ *in* $C^1_{loc}(\mathbb{R}^3)$, *where* U *is a positive* $G-$ *invariant solution of*

$$\begin{cases} -\Delta U + \tilde\lambda_3 U = U^p & in \quad \mathbb{R}^3 \\ U \le U(0) = 1 & in \quad \mathbb{R}^3, \end{cases}$$

with $m_G(U) < +\infty$;

(2) *when $u_n(Q_n)^{\frac{p-1}{2}}\, y_n \to +\infty$, then there hold*
- *$\lambda_n \varepsilon_n^2 V(P_n) \to \tilde\lambda_2 \in (0,1]$ for some universal constant $\tilde\lambda_2$*
- *$\frac{\varepsilon_n}{d(P_n,\partial\Omega)} \to 0$*
- *$U_n \to U$ in $C^1_{loc}(\mathbb{R}^3)$, where $U(X,Y,Z) = U(Y,Z)$ is a positive solution of*

$$\begin{cases} -\Delta U + \tilde\lambda_2 U = U^p & in \quad \mathbb{R}^2 \\ U \le U(0) = 1 & in \quad \mathbb{R}^2, \end{cases}$$

with $m(U) < +\infty$ (two dimensional Morse index).

Moreover, there exists a $G-$invariant $\phi_n \in C^1_0(\Omega)$, with

$$supp\ \phi_n \subset A_R(Q_n) := \{\, (x,y,z) \in \mathbb{R}^3 \,:\, (\sqrt{x^2+y^2} - y_n)^2 + (z - z_n)^2 \le R^2 \varepsilon_n^2 \,\},$$

$R > 0$, so that

$$\int_\Omega |\phi_n|^2 + (\lambda_n V - p u_n^{p-1})\phi_n^2 < 0, \tag{0.9}$$

for all n large.

From this local analysis we then deduce the global picture.

THEOREM 0.6. *Let (λ_n, u_n) be a positive, $G-$invariant solution of (0.8) so that $\sup_n m_G(u_n) < +\infty$ and $1 < p \ne 5$. Up to a subsequence, there exist $P_n^1 = (0, y_n^1, z_n^1), \ldots, P_n^h = (0, y_n^h, z_n^h)$, $h \le \sup_n m_G(u_n)$, with $y_n^i \ge 0$ and $\varepsilon_n^i = u_n(P_n^i)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$ s. t.*

$$\varepsilon_n^1 \le \varepsilon_n^i \le C_0 \varepsilon_n^1, \quad for\ all\ \ i = 1, \ldots, h$$

$$\frac{\varepsilon_n^i + \varepsilon_n^j}{|P_n^i - P_n^j|} \to 0 \quad as\ \ n \to +\infty, \quad for\ all\ \ i,j = 1,\ldots,h,\ \ i \ne j$$

$$\frac{\varepsilon_n^i}{d(P_n^i, \partial\Omega)} \to 0 \quad as\ \ n \to +\infty, \quad for\ all\ \ i = 1, \ldots, h,$$

$$u_n(P_n^i) = (1 + o(1)) \max_{B_{R_n \varepsilon_n^i}(P_n^i)} u_n,$$

for some $R_n \to +\infty$ as $n \to +\infty$.
Moreover, there holds

$$u_n(0,y,z) \le C(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^h e^{-\gamma \frac{|(0,y,z) - P_n^i|}{\varepsilon_n^1}} \quad \forall\, (0,y,z) \in \Omega, \ \ n \in \mathbb{N}$$

with $C > 0$.

The main results for invariant solutions are the following:

THEOREM 0.7 (Classification of blow-up points). *Let (λ_n, u_n) be a positive, $G-$invariant solution of (0.8) with* $\sup_n m_G(u_n) < +\infty$ *and* $1 < p \neq 5$ *and* $\lambda_n \to +\infty$ *as* $n \to +\infty$.
Let P_n^i, $i = 1, \ldots, h$ be the points given by Theorem 0.6 and $P^i = \lim_{n\to+\infty} P_n^i = (0, y^i, z^i)$. *Let us assume* $J_2^i = \{\, j = 1, \ldots, h \,:\, P_n^j \to P^i\,,\quad \frac{y_n^j}{\varepsilon_n^j} \to +\infty \,\} = \emptyset$ *whenever* $P^i \in G_0 = \{\, z-axis \,\}$.
Setting

$$\partial\Omega^{\pm} = \{\, x^2 + y^2 + z^2 = b^2,\ \pm z > 0 \,\} \cup \{\, x^2 + y^2 + z^2 = a^2,\ \pm z < 0 \,\},$$
$$\partial\Omega_a = \{\, x^2 + y^2 + z^2 = a^2\},\quad \partial\Omega_b = \{\, x^2 + y^2 + z^2 = b^2 \,\},$$

we have that

(1) *if* $P^i \in \partial\Omega^{\pm}$, *then* $\pm\partial_s V(P^i) \leq 0$, *and if* $P^i = (0, y^i, 0) \in \partial\Omega$, *then* $\partial_s V(P^i) = 0$;
(2) *if* $P^i \in \partial\Omega \setminus G_0$, *we also have* $\partial_r \tilde{V}(P^i) \leq 0$ *if* $P^i \in \partial\Omega_b$ *and* $\partial_r \tilde{V}(P^i) \geq 0$ *if* $P^i \in \partial\Omega_a$;
(3) *if* $P^i \in \Omega \cap G_0$, *then* $\partial_s V(P^i) = \partial_{r\,r} V(P^i) = 0$;
(4) *if* $P^i \in \Omega \setminus G_0$, *then* $\partial_s V(P^i) = \partial_r \tilde{V}(P^i) = 0$.

Here, $r = \sqrt{x^2 + y^2}$ *and* $\tilde{V}(r, s) = r^{\frac{p-1}{2}}\, V(r, s)$.

COROLLARY 0.8. *Let $(u_n \lambda_n)$ be a positive, $G-$invariant solution of (0.8) with u_n satisfying* $m_G(u_n) = 1$ *and* $1 < p \neq 5$, $\lambda_n \to +\infty$ *as* $n \to +\infty$.
Suppose that

$$\partial_s V = 0 \quad in \quad G_0 \implies \partial_{r\,r} V \neq 0,$$

and

$$\partial_s V \begin{cases} > 0 & in\ G_0 \cap \partial\Omega^+, \\ < 0 & in\ G_0 \cap \partial\Omega^-. \end{cases}$$

Then u_n blows-up on a suitable $G-$invariant, one dimensional curve, i.e. a circle with a suitable radius r_n such that $r_n\, \|u_n\|_\infty^2 \to \infty$.

The thesis is organized as follows.
In Chapter 1 we recall some results and also classification Theorems, about solutions which are stable or stable outside a compact set. Preliminary results and this classification, that we will give under more general hypothesis, are used to have some information of the limiting problem.
In Chapter 2 we introduce a blow-up approach to identify the limiting problem, and we give a complete description of the blow-up behavior for u_n.
We then deduce an asymptotic estimate on u_n in terms of its local maximum points.
In this chapter the most important and new result is the equivalence between an energy information and Morse index information (the Rozenblyum-Lieb-Cwikel type estimate).
In Chapter 3 we proceed further in the asymptotic analysis in order to localize the blow-up points in the case $V \neq 1$ and $V \equiv 1$. We show that a Morse Index information, in the sub-critical case, provides a complete description of the blow-up behavior, in the sense that we obtain some crucial global estimates to localize the blow-up set. In this section we show

exactly how the geometry of the domain affects the existence of such solutions. When in the equation (0.1) the potential $V \equiv 1$, we re-derive some results which are already known. For a generic potential V the geometry of the domain does not influence the location of the peak, which must be just a critical point of V.
In Chapter 4, we work with solutions of (0.1), which are invariant under rotations around the $z-$axis in an annulus of $\mathbb{R}^3$. We discuss the asymptotic analysis of these solutions, and in some cases we are able to show that blow-up occurs on circles. The main difficulty is to discuss what happens on this axis, which is fixed under the action of this symmetry group.

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CHAPTER 1

Notations, definitions and preliminary results

In this Chapter we collect some preliminary theorems that will be used frequently in the sequel.
It explains the interest for qualitative results on semilinear elliptic problems on $\mathbb{R}^N$ or hyperplanes with polynomial nonlinearity. In particular we will focus on solutions with finite Morse indices.

1.1. Some classification results

We focus on the study of the asymptotic behavior as $\lambda \to +\infty$ of solutions of

$$\begin{cases} -\Delta u + \lambda V(x) u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.1}$$

where $p > 1$, Ω is bounded domain of $\mathbb{R}^N$, $N \geq 2$, $V : \overline{\Omega} \to \mathbb{R}$ is a potential so that:

- $V \in C^1(\overline{\Omega}, \mathbb{R})$;
- V is a bounded away from zero: $\inf_\Omega V > 0$.

Consider the related problem

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p, & \text{in } \tilde{\Omega} \\ 0 < U \leq U(0) = 1 \end{cases} \tag{1.1.2}$$

with $p > 1$ and $\tilde{\Omega}$ an hyperspace or $\mathbb{R}^N$. We study the existence of solutions of (1.1.2) and related properties in the following section.
We observe that all the results are expressed, in a more general form, for invariant solutions, to achieve a unified approach.
In order to state the results, let us introduce the following definition of stability, stability outside a compact set and of Morse index k.

Definition 1.1. Let G be a subgroup of $O(N)$ and Ω be a $G-$invariant domain (i.e. $g\,x \in \Omega$, $\forall\, x \in \Omega$, $\forall\, g \in G$).
We say that a positive $G-$invariant solution $U \in H^1_0(\Omega)$ (i.e. $U(g\,x) = U(x)$ $\forall\, x \in \Omega$, $\forall\, g \in G$) of

$$-\Delta U + \tilde{\lambda} U = U^p \quad \text{in } \Omega \tag{1.1.3}$$

- is $G-$stable if

$$\forall \varphi \in C^1_0(\Omega) \;\; G-\text{invariant} \qquad Q_U(\varphi) := \int_\Omega |\nabla \varphi|^2 + \tilde{\lambda}\varphi^2 - p\, U^{p-1}\, \varphi^2 \geq 0;$$

- is $G-$stable outside a $G-$invariant compact set K if $Q_U(\varphi) \geq 0$ for any $G-$invariant $\varphi \in C^1_0(\Omega \setminus K)$;
- has $G-$invariant Morse index $m_G(U)$ equal to $k \geq 1$ if k is the maximal dimension of a subspace W_k of $G-$invariant functions in $C^1_0(\Omega)$ s. t. $Q_u(\varphi) < 0$ for any $\varphi \in W_k \setminus \{ 0 \}$.

For $G = \{ Id \}$ we have the classical Morse index $m_{\{Id\}}(u) = m(u)$.
Observe that if U is a solution of (1.1.2), the corresponding linearized operator is

$$L = -\Delta + \tilde{\lambda} - p\, U^{p-1}$$

and the Morse index of U is the number of negativity directions of the quadratic form associated to the linearized operator.

REMARK 1.2. Any finite Morse index solution U is stable outside a compact set $K \subset \Omega$. Indeed, there exist a maximal dimension $k \geq 1$ and $W_k := \text{span}\{ \varphi_1, \ldots, \varphi_k \} \subset C^1_0(\Omega)$ s. t. $Q_U(\varphi) < 0$ for any $\varphi \in W_k \setminus \{ 0 \}$. So $Q_U(\varphi) \geq 0$ for every $\varphi \in C^1_0(\Omega \setminus K)$, where $K := \cup_{i=1}^k \text{supp}(\varphi_i)$.

Following the techniques used first by Farina [**54**], and by Esposito et al. [**49, 48**], we obtain some classification results. The theorems in the sequel are based on the following crucial result:

PROPOSITION 1.3. *Let be a subgroup of $O(N)$, Ω be a $G-$invariant domain (bounded or not) of $\mathbb{R}^N$. Let $U \in C^2(\Omega)$ be a $G-$stable solution of*

$$-\Delta U + \tilde{\lambda} U = |U|^{p-1} U \ \text{in}\, \Omega \tag{1.1.4}$$

with $\tilde{\lambda} \geq 0$, $p > 1$. Then, for any $\gamma \in [1, 2\,p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geq \frac{p+\gamma}{p-1}$ there exists a constant $C_{p,m,\gamma} > 0$, such that

$$\int_\Omega (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\psi^{2\,m} \leq C_{p,m,\gamma} \int_\Omega (|\nabla \psi|^2 + |\psi||\Delta\psi|)^{\frac{p+\gamma}{p-1}} \tag{1.1.5}$$

for all $G-$invariant test functions $\psi \in C^2_0(\Omega)$ satisfying $|\psi| \leq 1$ in Ω.
Moreover if $\tilde{\lambda} > 0$ and $\gamma > 1$, there holds

$$\int_\Omega |U|^{\gamma+1} \psi^{2\,m} \leq C_{p,\gamma} \int_\Omega |U|^{\gamma+1}(|\nabla\psi|^2 + |\psi|\Delta\psi) \tag{1.1.6}$$

for all $G-$invariants test functions $\psi \in C^2_0(\Omega)$ satisfying $|\psi| \leq 1$ in Ω.

PROOF. We prove this proposition with $G = \{ Id \}$, but the same proof works also in the general case.
We divide the proof in four steps:

1^{st} step *For any $\varphi \in C^2_0(\Omega)$ we have:*

$$\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 \varphi^2 = \frac{(\gamma+1)^2}{4\,\gamma} \int_\Omega (|U|^{p+\gamma} - \tilde{\lambda}|U|^{\gamma+1})\varphi^2 + \frac{\gamma+1}{4\,\gamma} \int_\Omega |U|^{\gamma+1} \Delta(\varphi^2). \tag{1.1.7}$$

Multiply the equation (1.1.4) by $U^\gamma\,\varphi^2$ and integrate by parts, to have

$$\int_\Omega \gamma|\nabla U|^2|U|^{\gamma-1}\varphi^2+\int_\Omega \nabla U\,\nabla(\varphi^2)|U|^{\gamma-1}U+\tilde\lambda\int_\Omega |U|^{\gamma+1}\varphi^2=\int_\Omega |U|^{p+\gamma}\varphi^2.$$

Therefore we have that

$$\begin{aligned}&\frac{\gamma}{(\frac{\gamma+1}{2})^2}\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\varphi^2+\int_\Omega \nabla\Big(\frac{|U|^{\gamma+1}}{\gamma+1}\Big)\nabla(\varphi^2)\\ =\;&\frac{\gamma}{(\frac{\gamma+1}{2})^2}\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\varphi^2-\int_\Omega \frac{|U|^{\gamma+1}}{\gamma+1}\Delta(\varphi^2)=\int_\Omega(|U|^{p+\gamma}-\tilde\lambda|U|^{\gamma+1})\varphi^2,\end{aligned}$$

and then (1.1.7) follows.

2^{nd} step *For any $\varphi\in C^2_0(\Omega)$ we have:*

$$\begin{aligned}\Big(p-\frac{(\gamma+1)^2}{4\,\gamma}\Big)\int_\Omega |U|^{p+\gamma}\varphi^2\;&-\;\tilde\lambda\Big(1-\frac{(\gamma+1)^2}{4\,\gamma}\Big)\int_\Omega |U|^{\gamma+1}\varphi^2\\ &\le\;\frac{1-\gamma}{4\,\gamma}\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2)+\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2.\end{aligned}\tag{1.1.8}$$

We observe that the function $\psi=|U|^{\frac{\gamma-1}{2}}U\,\varphi$ belongs to $C^1_0(\Omega)$, for all $\varphi\in C^2_0(\Omega)$ and so it can be used as a test function in the quadratic form Q_U. Then the stability assumption on U gives:

$$\begin{aligned}&p\int_\Omega |U|^{p+\gamma}\varphi^2-\tilde\lambda\int_\Omega |U|^{\gamma+1}\varphi^2\le\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\varphi^2+\int_\Omega(|U|^{\frac{\gamma-1}{2}}U)^2|\nabla\varphi|^2\\ +\;&\int_\Omega 2\,\nabla(|U|^{\frac{\gamma-1}{2}}U)\nabla\varphi\,|U|^{\frac{\gamma-1}{2}}U\,\varphi=\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\varphi^2+\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2\\ -\;&\frac12\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2)=\frac{(\gamma+1)^2}{4\,\gamma}\int_\Omega(|U|^{p+\gamma}-\tilde\lambda|U|^{\gamma+1})\varphi^2+\frac{\gamma+1}{4\,\gamma}\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2)\\ +\;&\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2-\int_\Omega \frac12|U|^{\gamma+1}\Delta(\varphi^2)\\ =\;&\frac{(\gamma+1)^2}{4\,\gamma}\int_\Omega(|U|^{p+\gamma}-\tilde\lambda|U|^{\gamma+1})\varphi^2+\frac{1-\gamma}{4\,\gamma}\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2)+\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2.\end{aligned}$$

3^{rd} step *For any $\gamma\in[1,2\,p+2\sqrt{p(p-1)}-1)$ and any integer $m\ge\frac{p+\gamma}{p-1}$ there exists a constant $C(p,m,\gamma)$ such that*

$$\int_\Omega |U|^{p+\gamma}\,\psi^{2\,m}\le C(p,m,\gamma)\int_\Omega\Big[|\nabla\psi|^2+|\psi|\,|\Delta\psi|\Big]^{\frac{p+\gamma}{p-1}};\tag{1.1.9}$$

$$\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\,\psi^{2\,m}\le C(p,m,\gamma)\Big(\int_\Omega |\nabla\psi|^2+|\psi|\,|\Delta\psi|\Big)^{\frac{p+\gamma}{p-1}};\tag{1.1.10}$$

for all test functions $\psi \in C_0^2(\Omega)$ satisfying $|\psi| \le 1$ in Ω (the constant can be explicitly computed).

Let $\alpha_1 = p - \frac{(\gamma+1)^2}{4\gamma}$ and $\alpha_2 = 1 - \frac{(\gamma+1)^2}{4\gamma} = -\frac{(\gamma-1)^2}{4\gamma}$.
Observe that $\alpha_1 = \frac{-\gamma^2+2(2p-1)\gamma-1}{4\gamma} > 0 \iff \gamma_- = 2p-1-\sqrt{4p^2-4p} < \gamma < \gamma_+ = 2p-1+\sqrt{4p^2-4p}$. Observe that $\gamma_- < 1 < \gamma_+$. So we have that $\alpha_1 > 0$ and $\alpha_2 \le 0$ and the following inequality does hold:

$$\alpha_1 \int_\Omega |U|^{p+\gamma}\varphi^2 - \tilde{\lambda}\alpha_2 \int_\Omega |U|^{\gamma+1}\varphi^2 \le \beta \int_\Omega |U|^{\gamma+1}\Delta(\varphi^2) + \int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2 \tag{1.1.11}$$

with $\beta = \frac{1-\gamma}{4\gamma} \le 0$. For any $\psi \in C_0^2(\Omega)$, with $|\psi| \le 1$ in Ω, we set $\varphi = \psi^m$. Observe that the function $\varphi \in C_0^2(\Omega)$, since $m \ge 1$ is an integer, so we can apply (1.1.11) to get

$$\begin{aligned} \alpha_1 \int_\Omega |U|^{p+\gamma}\psi^{2m} &- \tilde{\lambda}\alpha_2 \int_\Omega |U|^{\gamma+1}\psi^{2m} \\ &\le \int_\Omega |U|^{\gamma+1}\psi^{2m-2}[m^2|\nabla\psi|^2 + 2\beta m(2m-1)|\nabla\psi|^2 + 2\beta m\psi\Delta\psi]. \end{aligned} \tag{1.1.12}$$

Since $\alpha_2 \le 0$, we have that

$$\begin{aligned} \alpha_1 \int_\Omega |U|^{p+\gamma}\psi^{2m} &\le \alpha_1 \int_\Omega |U|^{p+\gamma}\psi^{2m} - \tilde{\lambda}\alpha_2 \int_\Omega |U|^{\gamma+1}\psi^{2m} \\ &\le \int_\Omega |U|^{\gamma+1}\psi^{2m-2}[m^2|\nabla\psi|^2 + 2\beta m(2m-1)|\nabla\psi|^2 + 2\beta m\psi\Delta\psi], \end{aligned} \tag{1.1.13}$$

and we deduce that

$$\int_\Omega |U|^{p+\gamma}|\psi|^{2m} \le C_1 \int_\Omega |U|^{\gamma+1}|\psi|^{2m-2}[|\nabla\psi|^2 + |\psi\Delta\psi|], \tag{1.1.14}$$

with $C_1 = \frac{m^2+2\beta m(2m-1)}{\alpha_1}$. Apply Hölder's inequality to get:

$$\int_\Omega |U|^{p+\gamma}|\psi|^{2m} \le C_1\Big(\int_\Omega [|U|^{\gamma+1}|\psi|^{2m-2}]^{\frac{p+\gamma}{1+\gamma}}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega [|\nabla\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}. \tag{1.1.15}$$

Notice that $m \ge \frac{p+\gamma}{p-1}$, implies $(2m-2)\frac{p+\gamma}{1+\gamma} \ge 2m$, and thus $|\psi|^{(2m-2)\frac{p+\gamma}{1+\gamma}} \le |\psi|^{2m}$ in Ω, in view of $|\psi| \le 1$.
We obtain

$$\int_\Omega |U|^{p+\gamma}|\psi|^{2m} \le C_1\Big(\int_\Omega |U|^{p+\gamma}|\psi|^{2m}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega [|\nabla\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}, \tag{1.1.16}$$

and then

$$\int_\Omega |U|^{p+\gamma}|\psi|^{2m} \le C_1^{\frac{p+\gamma}{p-1}} \int_\Omega [|\nabla\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}, \tag{1.1.17}$$

Inequality (1.1.9) does hold with $C = C_1^{\frac{p+\gamma}{p-1}}$. Observe that

$$\begin{aligned}\int_\Omega |U|^{p+\gamma}\varphi^2 - \tilde\lambda\int_\Omega |U|^{\gamma+1}\varphi^2 &\leq \int_\Omega |U|^{p+\gamma}\varphi^2 - \tilde\lambda\frac{\alpha_2}{\alpha_1}\int_\Omega |U|^{\gamma+1}\varphi^2 \\ &\leq \frac{\beta}{\alpha_1}\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2) + \frac{1}{\alpha_1}\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2 \end{aligned} \tag{1.1.18}$$

in view of $\frac{\alpha_2}{\alpha_1} < 0$. By a combination of (1.1.7) and (1.1.18) we have that

$$\begin{aligned}&\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\varphi^2 \\ \leq\ & \frac{(\gamma+1)^2}{4\gamma}\Big[\frac{\beta}{\alpha_1}\int_\Omega |U|^{\gamma+1}\varphi\,\Delta\varphi + \frac{1}{\alpha_1}\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2\Big] + \frac{\gamma+1}{4\gamma}\int_\Omega |U|^{\gamma+1}\Delta(\varphi^2) \\ =\ & \frac{(\gamma+1)^2}{4\gamma}\Big[\frac{1}{\alpha_1}\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2 + \frac{\beta}{\alpha_1}\int_\Omega |U|^{\gamma+1}\varphi\Delta\varphi\Big] + \frac{\gamma+1}{2\gamma}\Big[\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2 \\ +\ & \int_\Omega |U|^{\gamma+1}\varphi\,\Delta\varphi\Big] = A\int_\Omega |U|^{\gamma+1}|\nabla\varphi|^2 + B\int_\Omega |U|^{\gamma+1}\varphi\,\Delta\varphi\end{aligned}$$

where $A = \frac{(\gamma+1)^2\,(2\beta+1)}{4\gamma\,\alpha_1} + \frac{\gamma+1}{2\gamma} > 0$ and $B = \frac{(\gamma+1)^2}{2\gamma\,\alpha_1}\beta + \frac{\gamma+1}{2\gamma} \in \mathbb{R}$. Now we insert the test function $\varphi = \psi^m$ in the latter inequality to find

$$\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\psi^{2m} \leq \int_\Omega |U|^{\gamma+1}\psi^{2m-2}[Am^2|\nabla\psi|^2 + B\,m\,(m-1)\,|\nabla\psi|^2 + B\,m\,\psi\Delta\psi], \tag{1.1.19}$$

and then

$$\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\psi^{2m} \leq C_2\int_\Omega |U|^{\gamma+1}\psi^{2m-2}[|\nabla\psi|^2 + |\,\psi\Delta\psi|], \tag{1.1.20}$$

with $C_2 = \max\{\,|A\,m^2 + B\,m\,(m-1)|, |B\,m|\,\} > 0$. Applying Hölder inequality in (1.1.20) we get

$$\begin{aligned}\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\psi^{2m} &\leq C_2\Big(\int_\Omega [|U|^{\gamma+1}\psi^{2m-2}]^{\frac{p+\gamma}{1+\gamma}}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega [|\nabla\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}, \\ &\leq C_2\Big(\int_\Omega |U|^{p+\gamma}\psi^{2m}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega [|\nabla\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}.\end{aligned}$$

Inserting (1.1.17) into the latter we obtain

$$\int_\Omega |\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2\psi^{2m} \leq C_2\,C_1^{\frac{1+\gamma}{p-1}}\int_\Omega [|\psi|^2 + |\psi\Delta\psi|]^{\frac{p+\gamma}{p-1}}, \tag{1.1.21}$$

ans so the proof of 3^{rd} step is complete.

4^{th} step *End of proof*

Formula (1.1.5) follows adding inequalities (1.1.9) and (1.1.10), (1.1.6) is an easy consequence of (1.1.18), in view of $\alpha_1 \geq 0$ and $\alpha_2 < 0$.

□

If we supply the equation with the boundary condition $U = 0$ on $\partial\Omega$, we obtain the following generalization of Proposition 1.3 above:

PROPOSITION 1.4. *Let $p > 1$, $0 < \alpha < 1$ and let Ω be a $G-$invariant proper $C^{2,\alpha}$ domain (bounded or not) of $\mathbb{R}^N$. Let $U \in C^2(\Omega)$ be a $G-$invariant solution of*

$$\left\{ \begin{array}{ll} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \Omega, \\ U = 0 & \text{in } \partial\Omega, \end{array} \right. \tag{1.1.22}$$

with $\tilde{\lambda} \geq 0$, which is $G-$stable outside a $G-$stable compact set K. Then, for any $\gamma \in [1, 2\,p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geq \frac{p+\gamma}{p-1}$, there exists a constant $C_{p,m,\gamma} > 0$, such that

$$\int_\Omega (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\psi^{2\,m} \leq C_{p,m,\gamma} \int_\Omega (|\nabla\psi|^2 + |\psi||\nabla\psi|)^{\frac{p+\gamma}{p-1}} \tag{1.1.23}$$

for all $G-$invariant test functions $\psi \in C_0^2(\mathbb{R}^N \setminus K)$ satisfying $|\psi| \leq 1$ in $\mathbb{R}^N \setminus K$.
Moreover if $\tilde{\lambda} > 0$ and $\gamma > 1$

$$\int_\Omega |U|^{\gamma+1}\psi^2 \leq C_{p,\gamma} \int_\Omega |U|^{\gamma+1}(|\nabla\psi|^2 + |\psi||\Delta\psi|) \tag{1.1.24}$$

for all $G-$invariant test functions $\psi \in C_0^2(\mathbb{R}^N \setminus K)$ satisfying $|\psi| \leq 1$ in $\Omega \setminus K$.

PROOF. We prove this proposition with $G = \{ Id \}$, but the same proof is true also in the generic case.
Since Ω is smooth, $U \in C^2(\overline{\Omega})$ and $U = 0$ on $\partial\Omega$ allows to proceed as in the proof op Proposition 1.3. Observe that 1^{st} step goes without any change if we remark that for any $\varphi \in C_0^2(\mathbb{R}^N \setminus K)$, the function $|U|^{\gamma-1}U\,\varphi^2 \in H_0^1(\Omega \setminus K)$ and integration by parts does hold.
In the same way, 2^{nd} step can be carried over since for the function $|U|^{\frac{\gamma-1}{2}}\,U\,\varphi$ the quadratic form Q_U is non negative. The rest of the proof is unchanged.

□

REMARK 1.5. The crucial fact is that we can use test functions supported in $\mathbb{R}^N \setminus K$ and not only in $\Omega \setminus K$, in view of the zero Dirichlet boundary condition of U.

We prove now that the stability outside a compact set implies strong integrability properties.

PROPOSITION 1.6. *Let G a subgroup of $O(N)$. Let U be a $G-$invariant solution of*

$$\left\{ \begin{array}{ll} -\Delta U + \tilde{\lambda} U = |U|^{p-1}U & \text{in } \tilde{\Omega} \\ U = 0 & \text{on } \partial\tilde{\Omega} \end{array} \right. \tag{1.1.25}$$

with $p > 1$, $\tilde{\lambda} > 0$ and $\tilde{\Omega}$ is either $\mathbb{R}^N$ or a $G-$invariant half-space. Assume that U is $G-$stable outside a ball $B_{R_0}(0)$, $R_0 > 0$.
Then, for any $q \in (0, 2\,p + 2\sqrt{p(p-1)})$ we have that :

$$\int_{\mathbb{R}^N} |U|^q(y)(1 + |y|^2)^\alpha < \infty$$

and $\alpha \in \mathbb{R}$.

Proof. Let $R > R_0 + 2$ and $\eta \in C_0^\infty(\mathbb{R}^N)$ radial so that

$$0 \le \eta \le 1, \quad \eta \equiv 0 \ \text{ in } \ B_{R_0+1}(0) \cup B_{2R}^C(0), \ \ \eta \equiv 1, \ \text{ in } \ B_R(0) \setminus B_{R_0+2}(0), \\ R|\nabla \eta| + R^2 |\Delta \eta| \le 2, \ \text{ in } \ B_{2R}(0) \setminus B_R(0). \tag{1.1.26}$$

By (1.1.23) we get that

$$\int_{\tilde{\Omega}} |U|^{p+\gamma} \le C_{p,m,\gamma} \int_{\tilde{\Omega}} \frac{1}{(1+|y|^2)^{\beta \frac{p+\gamma}{p-1}}} \tag{1.1.27}$$

for every $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$, $m \ge \frac{p+\gamma}{p-1}$ integer, $\beta \ge 1$, in view of

$$\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}} \left| \Delta \left(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}} \right) \right| + \left| \nabla \left(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}} \right) \right|^2 \le \frac{C_1}{(1+|y|^2)^\beta} \tag{1.1.28}$$

for some constant C_1 independent on $R > 0$.
Hence, letting $R \to +\infty$ we get that $\int_{\mathbb{R}^N} \frac{|U|^{p+\gamma}}{(1+|y|^2)^{(\beta-1)\,m}} < +\infty$ for $\beta > \frac{N}{2} \frac{p-1}{p+\gamma}$. Resuming, for every $\gamma \in [p, 3p + 2\sqrt{p(p-1)} - 2)$ there exists $\beta_\gamma > 1$ so that

$$\int_{\mathbb{R}^N} \frac{|U|^{\gamma+1}}{(1+|y|^2)^\beta} < +\infty \ \ \forall \beta \ge \beta_\gamma. \tag{1.1.29}$$

Now, use (1.1.24) with $\psi = \frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}$ and by (1.1.28) we obtain

$$\int_{\tilde{\Omega}} |U|^{\gamma+1} \frac{\eta^2}{(1+|y|^2)^{\beta-1}} \le C'_{p,\gamma} \int_{\mathbb{R}^N} \frac{|U|^{\gamma+1}}{(1+|y|^2)^\beta}, \tag{1.1.30}$$

for $\gamma \in (1, 2p + 2\sqrt{p\,(p-1)} - 1)$. Letting $R \to +\infty$, we then get

$$\int_{\mathbb{R}^N} \frac{|U|^{\gamma+1}}{(1+|y|^2)^{\beta-1}} \le C_U + C_{p,\gamma} \int_{\mathbb{R}^N} \frac{|U|^{\gamma+1}}{(1+|y|^2)^\beta} \tag{1.1.31}$$

for $\gamma \in (1, 2p + 2\sqrt{p(p-1)} - 1)$, where $C_U = \int_{B_{R_0+2}(0)} \frac{|U|^{\gamma+1}}{(1+|y|^2)^{\beta-1}}$. Starting from $\beta = \beta_\gamma$ in (1.1.28), we can iterate (1.1.31) to obtain

$$\int_{\mathbb{R}^N} |U|^{\gamma+1} (1+|y|^2)^\alpha < +\infty \tag{1.1.32}$$

for every $\alpha \in \mathbb{R}$ and $\gamma \in [p, 2p + 2\sqrt{p(p-1)} - 1)$. By (1.1.32) for every $\alpha \in \mathbb{R}$ and the Hölder inequality we easily show that

$$\int_{\mathbb{R}^N} |U|^q \, (1+|y|^2)^\alpha < +\infty$$

for every $\alpha \in \mathbb{R}$ and $q \in (0, 2p + 2\sqrt{p)(p-1)})$, as claimed.

□

1.1.1. The whole-space. In this subsection we consider existence and non existence results of the problem (1.1.2) in $\mathbb{R}^N$, $\tilde{\lambda} \geq 0$.

For $\tilde{\lambda} = 0$ we use some results contained in [**52**] concerning the Lane-Emden equation $-\Delta U = |U|^{p-1}U$ on unbounded domain of $\mathbb{R}^N$. Several classification theorems and Liouville-type results are obtained for different classes of solutions. We are interested in $G-$invariant solutions in $\mathbb{R}^N$, which are stable outside a compact set.
Let

$$p_c(N) := \begin{cases} +\infty & \text{if } N \leq 10 \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11 \end{cases}$$

be the Joseph-Lundgren exponent [**68**]. Note that $p_c(N)$ is larger then $p_S(N)$, where $p_S(N)+1$ is the critical exponent in Sobolev embedding $H_0^1(\tilde{\Omega}) \subset L^q(\tilde{\Omega})$, and $p_c(N) < p_c(N-1)$.

THEOREM 1.7. *Let $U \in C^2(\mathbb{R}^N)$ be a $G-$stable solution of*

$$-\Delta U = |U|^{p-1}U \tag{1.1.33}$$

with

$$\begin{cases} 1 < p < +\infty & \text{if } \; N \leq 10 \\ 1 < p < p_c(N) := \frac{(N-2)^2-4\,N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } \; N \geq 11. \end{cases} \tag{1.1.34}$$

Then $U \equiv 0$.

PROOF. For every $R > 0$, we consider the function $\psi_R(x) = \varphi(\frac{|x|}{R})$, where $\varphi \in C_0^2(\mathbb{R})$, satisfies $0 \leq \varphi \leq 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{if } \; |t| \leq 1 \\ 0 & \text{if } \; |t| \geq 2. \end{cases} \tag{1.1.35}$$

Let us fix $p > 1$. Observe that for any $\gamma \in [1, 2\,p + 2\,\sqrt{p(p-1)} - 1)$ and $m \geq \frac{p+\gamma}{p-1}$, Proposition 1.3 yields to

$$\begin{aligned} \int_{B_R(0)} (|\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2 + |U|^{p+\gamma}) &\leq C_{p,m,\gamma} \int_{\mathbb{R}^N} (|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|)^{\frac{p+\gamma}{p-1}} \\ &\leq C(p,\gamma,m,N,\varphi) R^{N-2\frac{p+\gamma}{p-1}}, \end{aligned} \tag{1.1.36}$$

for every $R > 0$.
We claim that we can always choose $\gamma \in [1, 2\,p + 2\,\sqrt{p(p-1)} - 1)$ such that

$$N - 2\Big(\frac{p+\gamma}{p-1}\Big) < 0. \tag{1.1.37}$$

We set $\gamma_M(p) = 2\,p + 2\,\sqrt{p(p-1)} - 1$ and consider separately the case $N \leq 10$ and $N \geq 11$. When $N \leq 10$, we have that

$$p + \gamma_M(p) = 3\,p + 2\,\sqrt{p(p-1)} - 1 \geq 3\,p + 2(p-1) - 1 > 5(p-1)$$

and therefore

$$N - 2\Big(\frac{p+\gamma_M(p)}{p-1}\Big) < N - 10 \leq 0.$$

The latter inequality and the continuity of the function $t \to N - 2\frac{p+t}{p-1}$ imply the existence of $\gamma \in [1, 2p + 2\sqrt{p(p-1) - 1})$ satisfying (1.1.37).
Consider $N \geq 11$ and $1 < p < p_c(N)$. In this case we consider the real-valued function in $(1, +\infty) \to f(t) := 2(\frac{t+\gamma_M(t)}{t-1})$. Since f is a strictly decreasing function satisfying $\lim_{t\to 1^+} f(t) = +\infty$ and $\lim_{t\to +\infty} f(t) = 10$, there exists a unique $p_0 > 1$ such that $N = 2(\frac{p_0+\gamma_M(p_0)}{p_0-1})$. We claim that $p_0 = p_c(N)$. Indeed there holds

$$N = 2\Big(\frac{p_0 + \gamma_M(p_0)}{p_0 - 1}\Big) \iff (N-2)(p_0-1) - 4\,p_0 = 4\sqrt{p_0(p_0-1)},$$

which implies that p_0 satisfies:

$$|(N-2)(N-10)|p_0^2 + [-2(N-2)^2 + 8N]p_0 + (N-2)^2 = 0, \tag{1.1.38}$$

whose roots are

$$p_c(N) = \frac{(N-2)^2 - 4\,N + 8\sqrt{N-1}}{(N-2)(N-10)},\ \frac{(N-2)^2 - 4\,N - 8\sqrt{N-1}}{(N-2)(N-10)} < p_c(N).$$

Since $(N-2)(p_c(N)-1) - 4\,p_c(N) > 0$, we have that $p_0 = p_c(N)$
Since f is a strictly decreasing function, it follows that

$$\forall\ 1 < p < p_c(N)\ \ f(p) = 2\Big(\frac{p + \gamma_M(p)}{p-1}\Big) > f(p_c(N)) = N. \tag{1.1.39}$$

Now the continuity of $t \to N - 2\frac{p+t}{p-1}$ implies the existence of this γ.
Therefore, by letting $R \to +\infty$ in (1.1.36), we have

$$\int_{\mathbb{R}^N} (|\nabla(|U|^{\frac{\gamma-1}{2}}U)|^2 + |U|^{p+\gamma}) = 0,$$

and so $U \equiv 0$. □

Next we can improve the argument to obtain the following result:

THEOREM 1.8. *Let $U \in C^2(\mathbb{R}^N)$ be a $G-$invariant solution of (1.1.33) which is $G-$stable outside a compact $G-$invariant set K of $\mathbb{R}^N$. Suppose*

$$\begin{cases} 1 < p < +\infty & \text{if } N = 2, \\ 1 < p < +\infty, p \neq \frac{N+2}{N-2} & \text{if } 3 \leq N \leq 10, \\ 1 < p < p_c(N) & \text{if } N \geq 11, \end{cases} \tag{1.1.40}$$

then $U \equiv 0$.

PROOF. Let $\varphi \in C_0^2(\mathbb{R})$ be as in the previous proof, and $\vartheta_s \in C_0^2(\mathbb{R})$, so that $0 \leq \vartheta_s \leq 1$, and

$$\vartheta(t) = \begin{cases} 0 & \text{if } |t| \leq s+1, \\ 1 & \text{if } |t| \geq s+2, \end{cases} \tag{1.1.41}$$

where $s > 0$. We divide the proof in several steps.

1^{st} step. *Let $p > 1$. There exists $R_0 = R_0(U) > 0$ such that*

(1) *for every* $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ *and every* $r > R_0 + 3$, *we have*

$$\int_{\{ R_0+2<|x|<r \}} (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\, dx \le A + B\, r^{N-2\frac{p+\gamma}{p-1}},$$

where A *and* B *are positive constants depending on* p, γ, N, R_0 *but not on* r.

(2) *for every* $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$ *and every open ball* $B_R(y)$ *such that* $B_{2R}(y) \subset \{ x \in \mathbb{R}^N : |x| > R_0 \}$, *we have*

$$\int_{B_R(y)} (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\, dx \le C R^{N-2\frac{p+\gamma}{p-1}},$$

R_0 *so that* $K \subset B_{R_0}(0)$, *where* C *is a positive constant depending on* p, γ, N, R_0 *but neither on* R *nor on* y.

For every $r > R_0 + 3$, we consider the function

$$\xi_r(t) = \begin{cases} \vartheta_{R_0}(|x|) & \text{if } |x| \le R_0 + 3, \\ \varphi(\frac{|x|}{r}) & \text{if } |x| \ge R_0 + 3. \end{cases} \tag{1.1.42}$$

We choose an integer $m \ge \frac{p+\gamma}{p-1}$. Notice that the function ξ_r belongs to $C_0^2(\mathbb{R}^N \setminus \overline{B(0,R_0)})$ and satisfies $0 \le \xi_r \le 1$ everywhere on $\mathbb{R}^N$. Therefore an application of Proposition 1.3 with $\Omega := \mathbb{R}^N \setminus \overline{B(0,R_0)}$ yields to

$$\begin{aligned}
&\int_{\{ R_0+2<|x|<r \}} (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\mathrm{d}x \le C_{p,m,\gamma} \int_{\mathbb{R}^N} (|\nabla\xi_r|^2 + |\xi_r||\Delta\xi_r|)^{\frac{p+\gamma}{p-1}} \mathrm{d}x \\
\le\ & C_{p,m,\gamma}\Big[\int_{\{ |x| \le R_0+3 \}} (|\nabla\vartheta_{R_0}|^2 + |\vartheta_{R_0}||\Delta\vartheta_{R_0}|)^{\frac{p+\gamma}{p-1}} \mathrm{d}x + \int_{\{ r \le |x| \le 2r \}} (|\nabla\xi_r|^2 + |\xi_r||\Delta\xi_r|)^{\frac{p+\gamma}{p-1}} \Big] \\
\le\ & A + B\, r^{N - \frac{2\,p+\gamma}{p-1}},
\end{aligned}$$

for all $r > R_0 + 3$ and all $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$, and the first estimate follows. To prove the other estimate, let $\nu_{R,y} \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function so that $0 \le \nu_{R,y} \le 1$, $\nu_{R,y} \equiv 1$ in $||x| - |y|| \le R$, $\nu_{R,y} \equiv 0$ in $||x| - |y|| \ge 2R$ and $|\nabla\nu_{R,y}|^2 + |\Delta\nu_{R,y}| \le \frac{C}{R^2}$ in $\mathbb{R}^2$ uniformly in x and y. Using the Proposition 1.3 we have

$$\begin{aligned}
&\int_{B_R(y)} (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\mathrm{d}x \le \int_{||x|-|y|| \le R} (|\nabla(|U|^{\frac{\gamma-1}{2}} U)|^2 + |U|^{p+\gamma})\mathrm{d}x \\
\le\ & C_{p,m,\gamma} \int_{\mathbb{R}^N} (|\nabla\nu_{R,y}|^2 + |\nu_{R,y}||\Delta\nu_{R,y}|)^{\frac{p+\gamma}{p-1}} \mathrm{d}x \\
\le\ & C(p,m,\gamma) R^{N-2\frac{p+\gamma}{p-1}}.
\end{aligned}$$

2^{nd} step. *The sub-critical case.*

We assume, either $N = 2$ and $1 < p < +\infty$ or $N \ge 3$ and $1 < p < \frac{N+2}{N-2}$. By choosing $\gamma = 1$ in

1^{st} step we get $\nabla U \in L^2(\mathbb{R}^N)$ and $U \in L^{p+1}(\mathbb{R}^N)$, and therefore we can obtain the classical Pohozaev identity, see [**92, 95**],

$$\Big(\frac{N}{2}-1\Big)\int_{\mathbb{R}^N}|\nabla U|^2=\frac{N}{p+1}\int_{\mathbb{R}^N}|U|^{p+1}. \tag{1.1.43}$$

Multiply the equation (1.1.33) by $U\,\psi_{R,0}$ and integrate by parts to get:

$$\int_{\mathbb{R}^N}|\nabla U|^2\psi_{R,0}-\int_{\mathbb{R}^N}|U|^{p+1}\psi_{R,0}=\frac{1}{2}\int_{\mathbb{R}^N}U^2\Delta(\psi_{R,0}).$$

Observing that

$$\Big|\int_{\mathbb{R}^N}U^2\Delta(\psi_{R,0})\Big|\le\Big(\int_{R<|x|<2\,R}|U|^{p+1}\Big)^{\frac{2}{p+1}}\Big(\int_{R<|x|<2\,R}[\Delta(\psi_{R,0})]^{\frac{p+1}{p-1}}\Big)^{\frac{p-1}{p+1}}\le o(1)\,(R^{N-2\frac{p+1}{p-1}})^{\frac{p-1}{p+1}}\to 0$$

as $R\to+\infty$, we have that

$$\int_{\mathbb{R}^N}|\nabla U|^2=\int_{\mathbb{R}^N}|U|^{p+1}. \tag{1.1.44}$$

We combine (1.1.43) and (1.1.44) to get

$$\Big(\frac{N}{2}-1-\frac{N}{p+1}\Big)\int_{\mathbb{R}^N}|U|^{p+1}=0$$

where $\frac{N}{2}-1-\frac{N}{p+1}\le 0$. Hence U must be identically zero, as claimed.

3^{rd} step. *Let $\eta>0$. Assume either*

$$N\ge 11\quad\textit{and}\quad\frac{N+2}{N-2}\le p<p_c(N)\ \ \textit{or}\ \ 3\le N\le 10\quad\textit{and}\quad\frac{N+2}{N-2}\le p<+\infty.$$

Then

$$\exists\,\gamma_1=\gamma_1(p,N)\in(1,2\,p+2\sqrt{p(p-1)}-1):\ (p-1)\frac{N}{2}=p+\gamma_1, \tag{1.1.45}$$

$$\exists\,R_1=R_1(p,N,\eta,U)>R_0:\ \int_{|x|\ge R_1}|U|^{(p-1)\frac{N}{2}}\,dx<\eta, \tag{1.1.46}$$

$$\exists\varepsilon=\varepsilon(p,N)\in(0,1]:\ 1\le(p-1)\frac{N}{2-\varepsilon}-p<2\,p+2\sqrt{p(p-1)}-1. \tag{1.1.47}$$

We observe that $p\ge\frac{N+2}{N-2}\ \Rightarrow\ p+1\le(p-1)\frac{N}{2}$, and by (1.1.39) we have

$$p+(2\,p+2\sqrt{p(p-1)}-1)=p+\gamma_M(p)>(p-1)\frac{N}{2}.$$

These facts and the continuity of the function $(\gamma,\varepsilon) \to (p-1)\frac{N}{2-\varepsilon} - p - \gamma$ imply the existence of γ_1 and ε. By the existence of such γ_1 and 1^{st} step we have

$$\begin{aligned}\int_{\{R_0+2<|x|<r\}} |U|^{(p-1)\frac{N}{2}} \mathrm{d}x &= \int_{\{R_0+2<|x|<r\}} |U|^{p+\gamma_1} \mathrm{d}x \\ &\leq A + Br^{N-2\frac{p+\gamma_1}{p-1}} = A + B \quad \forall\, r > R_0+3.\end{aligned}$$

Then

$$\int_{|x|\geq R_0+2} |U|^{(p-1)\frac{N}{2}} \mathrm{d}x < +\infty,$$

and the thesis follows for a suitable R_0 depending on η.

4^{th} step. *Assume that*

$$3 \leq N \leq 10 \quad \textit{and} \quad \frac{N+2}{N-2} \leq p < +\infty, \quad \textit{or} \quad N \geq 11 \quad \textit{and} \quad \frac{N+2}{N-2} \leq p < p_c(N).$$

Then

$$\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} U(x) = 0, \qquad \lim_{|x|\to+\infty} |x|^{1+\frac{2}{p+1}} |\nabla U(x)| = 0.$$

We omit the proof of this step, see [**70**] or [**52**].

5^{th} step. *Assume*

$$3 \leq N \leq 10 \quad \textit{and} \quad \frac{N+2}{N-2} < p < +\infty, \quad \textit{or} \quad N \geq 11 \quad \textit{and} \quad \frac{N+2}{N-2} < p < p_c(N).$$

Then $U \equiv 0$.

As in [**21**] we use the change of variable

$$U(r,\sigma) = r^{-\frac{2}{p-1}} v(t,\sigma), \quad t = \ln(r).$$

Then v satisfies the equation:

$$v_{tt} + A\, v_t + \Delta_{S^{N-1}} v + B\, v + |v|^{p-1} v = 0, \quad \text{in } \mathbb{R} \times S^{N-1},$$

with $A = (N-2-\frac{4}{p-1})$, $B = -(\frac{2}{p-1}(N-2-\frac{2}{p-1}))$. Here S^{N-1} is the unit sphere of $\mathbb{R}^N$ and $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on S^{N-1}.
Setting

$$E(w) := \int_{S^{N-1}} \Big(\frac{1}{2}|\nabla_{S^{N-1}} w|^2 - \frac{B}{2} w^2 - \frac{1}{p-1}|w|^{p+1}\Big) \mathrm{d}\sigma,$$

we have for $v(t) := w(t,\cdot)$

$$A \int_{S^{N-1}} v_t^2 \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big[E(v)(t) - \frac{1}{2}\int_{S^{N-1}} v_t^2 \mathrm{d}\sigma\Big]. \tag{1.1.48}$$

Observe that $A \neq 0$, since $p \neq \frac{N+2}{N-2}$, therefore, after integrating, we find

(1.1.49)
$$\forall s > 0 \quad \int_{-s}^{s} A \int_{S^{N-1}} v_t^2 \mathrm{d}\sigma \mathrm{d}t = E(v)(s) - E(v)(-s) - \frac{1}{2} \int_{S^{N-1}} v_t^2(s,\sigma)\mathrm{d}\sigma + \frac{1}{2} \int_{S^{N-1}} v_t^2(-s,\sigma)\mathrm{d}\sigma.$$

We use the crucial decay estimates in 4^{th} step to have

(1.1.50)
$$\lim_{t\to+\infty} v(t,\sigma) = 0, \quad \lim_{t\to+\infty} |v_t(t,\sigma)| = \lim_{t\to+\infty} |\nabla_{S^{N-1}} v(t,\sigma)| = 0,$$

the limits being uniform respect to $\sigma \in S^{N-1}$. Observe that U is regular at the origin and then (1.1.50) holds true when $t \to -\infty$. Letting $s \to \infty$ in (1.1.48) we have $A \int_{\mathbb{R}} \int_{S^{N-1}} v_t^2 \mathrm{d}\sigma \mathrm{d}t = 0$. Hence $v = v(\sigma)$ and $\lim_{t\to+\infty} v(t,\sigma) = 0$. This implies $v \equiv 0$, and so $U \equiv 0$.

□

REMARK 1.9.
(1) Theorem 1.8 is sharp. Indeed, on one hand, for $N \geq 3$ the set of functions
$$u_\lambda(x) := \Big(\frac{\lambda \sqrt{N(N-2)}}{\lambda^2 + |x|^2} \Big)^{\frac{N-2}{2}}, \quad \lambda > 0,$$
is a one-parameter family of positive solutions of equation (1.1.33), with $\Omega = \mathbb{R}^N$ and $p = \frac{N+2}{N-2}$, and all these solutions are shown to be stable outside a large ball centered at the origin by using Hardy's inequality. On the other hand, for $N \geq 11$ and $p \geq p_c(N)$ equation (1.1.33) admits a positive, bounded, stable and radial solution in $\mathbb{R}^N$ (see [**52, 53**]).
(2) Theorem 1.8 improves upon a Liouville-type result proved by A.Bahri and P.L.Lions [**17**], where solutions are assumed to be both bounded and with finite Morse index, and p is sub-critical.
(3) In case $\tilde{\lambda} = 0$ and $p = \frac{N+2}{N-2}$ there is a complete classification of the solutions, see [**22, 58, 90**].

We give a Liouville-type result for $p \geq \frac{N+2}{N-2}$:

THEOREM 1.10. *Let $\tilde{\lambda} \in (0,1]$ and $U \in C^2(\mathbb{R}^N)$ be a $G-$invariant solution of $-\Delta U + \tilde{\lambda} U = |U|^{p-1}U$ which is $G-$stable outside a compact set of $\mathbb{R}^N$. Suppose $p \geq \frac{N+2}{N-2}$. Then $U \equiv 0$.*

PROOF. We have that the 1^{st} step in the proof of Theorem 1.8 is valid also in this case because Proposition 1.3 holds for $\tilde{\lambda} > 0$. We have that $U \in L^{p+1}(\mathbb{R}^N)$ and $\nabla U \in L^2(\mathbb{R}^N)$.

Therefore we can obtain the classical Pohozaev identity, see [**92, 95**],

(1.1.51)
$$\Big(\frac{N}{2} - 1\Big) \int_{\mathbb{R}^N} |\nabla U|^2 = \frac{N}{p+1} \int_{\mathbb{R}^N} |U|^{p+1} - \tilde{\lambda} \frac{N}{2} \int_{\mathbb{R}^N} |U|^2.$$

Arguing as for (1.1.44), we get that

(1.1.52)
$$\int_{\mathbb{R}^N} |\nabla U|^2 = \int_{\mathbb{R}^N} |U|^{p+1} - \tilde{\lambda} \int_{\mathbb{R}^N} |U|^2.$$

We combine (1.1.51) and (1.1.52) we have

$$\Big(\frac{N}{2}-1-\frac{N}{p+1}\Big)\int_{\mathbb{R}^N}|U|^{p+1}+\tilde\lambda\int_{\mathbb{R}^N}|U|^2=0$$

where $\frac{N}{2}-1-\frac{N}{p+1}\geq 0$ whenever $p>\frac{N+2}{N-2}$. Hence U must be identically zero, as claimed. $\square$

1.1.2. A new deal with $\tilde\lambda>0$ and p subcritical.

We have that

THEOREM 1.11. *Let U be a $G-$invariant solution of*

$$-\Delta U+\tilde\lambda U=|U|^{p-1}U,\quad in\ \ \mathbb{R}^N \tag{1.1.53}$$

for $1<p<\frac{N+2}{N-2}$ and $\tilde\lambda>0$. Assume that either

- *U is $G-$stable outside a ball $B_{R_0}(0)$;*
  or
- *$\int_{\mathbb{R}^N}U^{p+1}<+\infty$.*

Then $U\to 0$ as $|x|\to+\infty$.

PROOF. If U is $G-$stable outside $B_{R_0}(0)$, by Proposition 1.6 we get that $\int_{\mathbb{R}^N}|U|^{p+1}<+\infty$ for $q=p+1$ and $\alpha=0$. The two different assumptions can be re-formulated as $U\in L^{p+1}(\mathbb{R}^N)$. Let η a smooth cut-off function so that $0\leq\eta\leq 1$, $\eta=1$ in $B_1(0)$, $\eta=0$ in $B_2(0)$. Given $x_0\in\mathbb{R}^N$ multiply (1.1.53) by $\eta^2(x-x_0)U$ to obtain

$$\begin{aligned}\int_{B_1(x_0)}|\nabla U|^2 &\leq \int_{\mathbb{R}^N}\eta^2(x-x_0)|\nabla U|^2=\int_{\mathbb{R}^N}\eta^2(x-x_0)|U|^{p+1}-\tilde\lambda\int_{\mathbb{R}^N}\eta^2(x-x_0)U^2\\ &+ \frac12\int_{\mathbb{R}^N}U^2\Delta\eta^2(x-x_0)\leq\|U\|^{p+1}_{L^{p+1}(B_2(x_0))}.\end{aligned}$$

In view of $p>1$, we then have that

$$\|U\|^2_{H^1(B_1(x_0))}\leq C\|U\|^{p+1}_{L^{p+1}(B_2(x_0))}+C\|U\|_{L^{p+1}(B_2(x_0))}, \tag{1.1.54}$$

for some $C>0$ independent on x_0.
By Sobolev embedding, (1.1.54) leads to

$$\|U\|_{L^{\frac{2N}{N-2}}(B_1(x_0))}\leq C(\|U\|^{\frac{p+1}{2}}_{L^{p+1}(B_2(x_0))}+C\|U\|_{L^{p+1}(B_2(x_0))}). \tag{1.1.55}$$

Set $q_0=\frac{2N}{N-2}$. By (1.1.55) we get that

$$\begin{aligned}\||U|^{p-1}U-\tilde\lambda U\|_{L^{\frac{q_0}{p}}(B_1(x_0))} &\leq C\Big[\|U\|^{p\frac{p+1}{2}}_{L^{p+1}(B_2(x_0))}+\|U\|^p_{L^{p+1}(B_2(x_0))}\\ &+ \|U\|^{\frac{p+1}{2}}_{L^{p+1}(B_2(x_0))}+\|U\|_{L^{p+1}(B_1(x_0))}\Big].\end{aligned} \tag{1.1.56}$$

Decompose U as U_1+U_2, where

$$\begin{cases}-\Delta U_1=U^p-\tilde\lambda U & \text{in } \ B_1(x_0)\\ U=0 & \text{on } \ \partial B_1(x_0).\end{cases} \tag{1.1.57}$$

By regularity theory, we get that

$$\|U_1\|_{W^{2,\frac{q_0}{p}}(B_1(x_0))} \le C \| |U|^{p-1}U - \tilde\lambda\, U\|_{L^{\frac{q_0}{p}}(B_1(x_0))}.$$

Assume $\frac{q_0}{p} < \frac{N}{2}$. Then, by Sobolev embedding

$$\|U_1\|_{L^{\frac{N\,q_0}{N\,p-2\,q_0}}(B_1(x_0))} \le C \| |U|^{p-1}U - \tilde\lambda U\|_{L^{\frac{q_0}{p}}(B_1(x_0))}.$$

Since U_2 is harmonic in $B_1(x_0)$ with $U_2 = U$ on $\partial B_1(x_0)$, by the mean-value theorem we get that

$$\begin{aligned}\|U_2\|_{L^\infty(B_{\frac12}(x_0))} &\le C\Big[\|U\|^{p\frac{p+1}{2}}_{L^{p+1}(B_2(x_0))} + \|U\|^p_{L^{p+1}(B_2(x_0))} + \|U\|^{\frac{p+1}{2}}_{L^{p+1}(B_2(x_0))} + \|U\|_{L^{p+1}(B_1(x_0))}\Big]\\ &\le C\|U_2\|_{L^1(B_2(x_0))} \le C(\|U\|_{L^1(B_2(x_0))} + \|U_1\|_{L^1(B_2(x_0))})\end{aligned}$$

in view of $\frac{N\,q_0}{N\,p-2\,q_0} > 1$ (valid for $q_0 > \frac{N}{N+2}p$) and (1.1.56).
In conclusion, setting $q_1 = \frac{N\,q_0}{N\,p-2\,q_0}$, and $f_0(s) = s^{p\frac{p+1}{2}} + s^p + s^{\frac{p+1}{2}} + s$ we have that

$$\|U\|_{L^{q_1}(B_{\frac12}(x_0))} \le C f_0\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big) \tag{1.1.58}$$

for some $C > 0$ independent on x_0, provided $\frac{q_0}{p} < \frac{N}{2}$.
Now, we replace condition (1.1.55) with (1.1.58) and the same argument leads to

$$\|U\|_{L^{q_2}(B_{\frac14}(x_0))} \le C f_1\Big(|U\|_{L^{p+1}(B_2(x_0))}\Big), \tag{1.1.59}$$

where $q_2 = \frac{N\,q_1}{N\,p-2\,q_1}$, and $f_1(s)$ is a suitable function, for some C independent on x_0, (in view of $q_1 > q_0 > \frac{N}{N+2}p$ for $p < \frac{N+2}{N-2}$), provided $\frac{q_1}{p} < \frac{N}{2}$.
Defining $q_k = \frac{N\,q_{k-1}}{N\,p-2q_{k-1}}$ in an inductive way (whenever $\frac{q_{k-1}}{p} < \frac{N}{2}$), we have that q_k is strictly increasing in k (so as to have always $q_k > \frac{N}{N+2}p$), and, for some finite $k \ge 1$, $\frac{q_k}{p} \ge \frac{N}{2}$. Indeed, assume by induction $q_k > \cdots > q_0 = \frac{2\,N}{N-2}$ (for $k = 1$ it's already true). If $q_k \ge \frac{N}{2}p$, nothing to prove. If $q_k < \frac{N}{2}p$, we need to show that $q_{k+1} = \frac{N\,q_k}{N\,p-2\,q_k} > q_k$. This is equivalent to $p < 1 + \frac{2}{N}q_k$, which is true in view of $1 + \frac{2}{N}q_k > 1 + \frac{2}{N}\frac{2\,N}{N-2} = \frac{N+2}{N-2}$.
Hence, whenever it is defined, q_k is strictly increasing. If $q_k < \frac{N}{2}p\ \forall\, k$, the sequence q_k is defined $\forall\, k$ and $q_k \to q_\infty \in (0, \frac{N}{2}p]$. Since $q_\infty = \frac{N\,q_\infty}{N\,p-2\,q_\infty}$, we get that $p = 1 + \frac{2}{N}q_\infty$, a contradiction with $q_\infty > q_0 = \frac{2\,N}{N-2}$ and $p < \frac{N+2}{N-2}$.
Let k be so that $q_{k-1} < \frac{N}{2}p$ and $q_k \ge \frac{N}{2}p$. Iterating the argument to obtain (1.1.59), we get that

$$\|U\|_{L^{q_{k-1}}(B_{\frac{1}{2^{k-1}}}(x_0))} \le C f_{k-2}\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big)$$

for a suitable f_{k-2} and some $C > 0$ independent on x_0.
If $q_k > \frac{N}{2}p$, iterate once more to get

$$\|U\|_{L^\infty(B_{\frac{1}{2^{k-1}}}(x_0))} \le C f_{k-1}\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big)$$

for some $C > 0$ independent on x_0.
If $q_k = \frac{N}{2}p$, we have only that

$$\|U\|_{L^q(B_{\frac{1}{2^k}}(x_0))} \le C f_{k-1,q}\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big) \quad \forall \; q < +\infty$$

and then

$$\|U\|_{L^\infty(B_{\frac{1}{2^{k+1}}}(x_0))} \le C f_k\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big).$$

In conclusion, there exist $r \in (0,1)$ and $C < +\infty$ (possibly depending on U) so that

$$\|U\|_{L^\infty(B_r(x_0))} \le C f\Big(\|U\|_{L^{p+1}(B_2(x_0))}\Big) \tag{1.1.60}$$

for all $x_0 \in \mathbb{R}^N$, where $f : [0,+\infty) \to [0,+\infty)$ is continuous so that $f(0) = 0$.
Since $\|U\|_{L^{p+1}(B_2(x_0))} \to 0$ as $|x_0| \to +\infty$ in view of $\int_{\mathbb{R}^N} |U|^{p+1} < +\infty$, by (1.1.60) we get that

$$\lim_{|x|\to\infty} |U(x)| = 0$$

as claimed. □

Now, let us recall some results of Gidas, Ni, Nirenberg [**58**], about symmetry and related properties of positive solutions of second order elliptic equations.

THEOREM 1.12. *Let U be a positive solution of*

$$-\Delta U + \tilde\lambda U = g(U) \quad in \quad \mathbb{R}^N,$$

with $N \ge 2$, $\tilde\lambda > 0$, $U(x) \to 0$ as $|x| \to +\infty$, and g continuous so that $g(U) = O(U^\alpha)$, $\alpha > 1$, near $U = 0$. On the interval $0 \le s \le U_0 = \max_{\mathbb{R}^N} U(x)$, assume $g(s)$ nondecreasing.
Then $U(x)$ is spherically symmetric about some point in $\mathbb{R}^N$ and $U_r < 0$ for $r > 0$, where r is the distance from that point.
Furthermore

$$\lim_{r\to\infty} r^{-\frac{N-1}{2}} e^r U(r) = \mu > 0. \tag{1.1.61}$$

In this paper Gidas, Ni and Nirenberg [**58**] first prove that the solutions in Theorem 1.12 decay exponentially at infinity.

PROPOSITION 1.13. *Let $U(x) > 0$ be a solution of*

$$-\Delta U + \tilde\lambda U = g(U) \;\; in \;\; \mathbb{R}^N, \tag{1.1.62}$$

with $N \ge 2$, $\tilde\lambda > 0$, $U \to 0$ at infinity. Assume $g(U) = O(U^\alpha)$, for some $\alpha > 1$, near $U = 0$. Then

$$U(x) + |\nabla U(x)| = O\left(\frac{e^{-|x|}}{|x|^{\frac{N-1}{2}}}\right) \;\; as \;\; |x| \to +\infty.$$

REMARK 1.14. Thanks to the exponential decay and the monotonicity of g, Theorem 1.12 then follows by the moving plane method.

A particular equation covered by Theorem 1.12 is

$$-\Delta U + \tilde{\lambda} U = U^p \text{ in } \mathbb{R}^N$$

with $\tilde{\lambda} > 0$, and p sub-critical. Combining Theorems 1.11 and 1.12 we get that

THEOREM 1.15. *Let U be a $G-$invariant solution of*

$$-\Delta U + \tilde{\lambda} U = U^p \quad \textit{in } \mathbb{R}^N$$

with $\tilde{\lambda} > 0$, and p sub-critical. Assume that either

- *U is $G-$stable outside $B_{R_0}(0)$;*
  or
- *$\int_{\mathbb{R}^N} |U|^{p+1} < +\infty$.*

Then $U(x)$ is is spherically symmetric about some point in $\mathbb{R}^N$ and $U_r < 0$ for $r > 0$, where r is the distance from that point.
Furthermore

$$\lim_{r\to\infty} r^{-\frac{N-1}{2}} e^r U(r) = \mu > 0. \tag{1.1.63}$$

Kwong [**70**] establishes, for p sub-critical, the uniqueness of the positive, radially symmetric solution to the differential equation $\Delta\tilde{U} - \tilde{U} + \tilde{U}^p = 0$ in a bounded or unbounded annular region in $\mathbb{R}^N$ for all $N \geq 2$ with suitable boundary condition.
We recall the following result (see, for example, [**17**], [**57**], [**70**]).

THEOREM 1.16. *For $p < \frac{N+2}{N-2}$, with $N \geq 2$, the equation*

$$-\Delta\tilde{U} + \tilde{U} = \tilde{U}^p \quad \textit{in } \mathbb{R}^N, \quad \tilde{U}(x) \to 0 \quad \textit{for} \quad |x| \to +\infty$$

possesses a unique positive radial solution U_k.

REMARK 1.17. Let $\tilde{U}$ be a radial solution of

$$\begin{cases} -\Delta\tilde{U} + \tilde{U} = \tilde{U}^p & \text{in } \mathbb{R}^N \\ 0 < \tilde{U} \leq \tilde{U}(0) = \tilde{\lambda}^{-\frac{1}{p-1}}. \end{cases} \tag{1.1.64}$$

If we consider $U(y) = \tilde{\lambda}^{\frac{1}{p-1}} \tilde{U}(\tilde{\lambda}^{\frac{1}{2}}\, y)$ then U is solution of

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \mathbb{R}^N \\ 0 < U \leq U(0) = 1. \end{cases} \tag{1.1.65}$$

Then we are led to study: U solution of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^N \\ U(0) = \lambda^{-\frac{1}{p-1}} = U_0 \end{cases} \tag{1.1.66}$$

with $p > 1$.

We collect now Theorems 1.15 and 1.16 and Remark 1.17 to get

THEOREM 1.18. *Let U be a $G-$invariant solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & in\ \mathbb{R}^N \\ 0 < U \le U(0) = 1 & in\ \mathbb{R}^N \end{cases} \tag{1.1.67}$$

with $\tilde{\lambda} > 0$ and $1 < p < \frac{N+2}{N-2}$. If U is $G-$stable outside $B_{R_0}(0)$ or $\int_{\mathbb{R}^N} U^{p+1} < +\infty$, then $U(y)$ coincides with $\tilde{\lambda}^{\frac{1}{p-1}} U_k(\tilde{\lambda}^{\frac{1}{2}}\, y)$ and is unstable .
Furthermore, U has the first negative eigenvalue $\mu_1 < 0$ (that is simple) with eigenfunction φ_1 in $H^1(\mathbb{R}^N)$; the second eigenvalue is 0 (with multiplicity N) and the eigenspace in $H^1(\mathbb{R}^N)$ is given by

$$span\{\, \partial_{x_1} U, \dots, \partial_{x_N} U \,\}.$$

REMARK 1.19. Note that U_k can be obtained as a mountain-pass solution for the corresponding energy functional in $H^1(\mathbb{R}^N)$. We have that U_k is unstable in view of the exponential decay and

$$\int_{\mathbb{R}^N} |\nabla U|^2 + \tilde{\lambda} \int_{\mathbb{R}^N} U^2 - p \int_{\mathbb{R}^N} U^{p-1}\, U^2 = -(p-1) \int_{\mathbb{R}^N} U^{p+1} < 0$$

by the equation.
As far as the zero eigenvalue, it is know (see [**64**]) that

$$\text{kernel}\,(-\Delta + 1 + pU_k^{p-1}) = \text{span}\{\, \partial_{x_1} U_k, \dots, \partial_{x_N} U_k \,\}$$

in $H^1(\mathbb{R}^N)$.
Observe that we can find the first eigenfunction as the minimum of the quadratic form associated to $-\Delta + 1 - p\, U_k^{p-1}$ on $\{\, \phi \in H^1(\mathbb{R}^N)\ : \int_{\mathbb{R}^N} \phi^2 = 1 \,\}$.
By [**62**] we know that U_k has Morse index at most 1. Then, U_k has exactly Morse index 1.

1.1.3. The half-space. Assume $\tilde{\Omega} = \{\, x = (x_1, \dots, x_N) \in \mathbb{R}^N\ :\ x_N > 0 \,\}$. By the moving plane method (see [**54**]), it is possible to show that for a solution of U_0 of

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p \ \text{ in } \ \tilde{\Omega} \\ U = 0 \ \text{ in } \ \{\, x_N = 0 \,\}, \end{cases} \tag{1.1.68}$$

there holds $\partial_{x_N} U > 0$ in $\tilde{\Omega}$, and then U is semi-stable on $\tilde{\Omega}$.

We study non-negative solutions, the first Theorem extends the celebrated results of Gidas and Spruck (see [**59, 60**]) to the case where the unbounded domain $\tilde{\Omega}$ is the half-space.

THEOREM 1.20. *Let $\tilde{\Omega}$ be a $G-$invariant half-space and let $U \in C^2(\overline{\tilde{\Omega}})$ be a positive $G-$invariant solution of*

$$\begin{cases} -\Delta U = U^p & in\ \tilde{\Omega} \\ U = 0 & on\ \partial\tilde{\Omega}, \end{cases} \tag{1.1.69}$$

which is stable outside $B_{R_0}(0)$. Assume that

$$\begin{cases} 1 < p < +\infty & if\ N \le 10, \\ 1 < p < p_c(N) & if\ N \ge 11. \end{cases} \tag{1.1.70}$$

Then $U \equiv 0$.

PROOF. Let us consider the odd extension of U,

$$v(x) = v(x', x_N) := \begin{cases} U(x', x_N), & x_N \geq 0 \\ -U(x', -x_N), & x_N < 0. \end{cases} \tag{1.1.71}$$

Clearly v belongs to $C^2(\mathbb{R}^2)$ and solves the equation in $\mathbb{R}^N$. We can find $\tilde{R}_0 > 0$ large so that $B_{R_0}(0) \subset B'_{R_0}(0) \subset B_{\tilde{R}_0}(0)$, where $B'_{R_0}(0) = \{ (x', x_N) : (x', -x_N) \in B_{R_0}(0) \}$.
We claim that

For every $\gamma \in [1, 2\,p + 2\,\sqrt{p(p-1)} - 1)$ *and every open ball* $B_{2\,R}(y) \subset \{ x \in \mathbb{R}^N \,:\, |x| > R_0 \}$, *we have:*

$$\int_{B_R(y)} (|\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |v|^{p+\gamma})\, dx \leq C\, R^{N-2\frac{p+\gamma}{p-1}},$$

where C *is a positive constant depending on* p, γ, N, R_0 *and neither on* R *nor on* y.

We fix $m \geq \frac{p+\gamma}{p-1}$ and consider the test functions $\psi_{R,y}(x) := \varphi\Big(\frac{|x-y|}{R}\Big)$, where $\varphi \in C^2_0(\mathbb{R})$ satisfies $0 \leq \varphi \leq 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2. \end{cases} \tag{1.1.72}$$

An application of Proposition 1.4 to $v_{|\Omega} = U$ in $\tilde{\Omega}$ gives

$$\begin{aligned} &\int_{\tilde{\Omega} \cap B_R(y)} (|\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |v|^{p+\gamma}) \mathrm{d}x \leq C_{p,m,\gamma} \int_{\mathbb{R}^N} (|\nabla \psi_{R,y}|^2 + |\psi_{R,y}||\Delta \psi_{R,y}|)^{\frac{p+\gamma}{p-1}} \mathrm{d}x \\ = \;& C_{p,m,\gamma} \int_{B_{2\,R}(0)} (|\nabla \psi_{R,0}|^2 + |\psi_{R,0}||\Delta \psi_{R,0}|)^{\frac{p+\gamma}{p-1}} \mathrm{d}x \leq C(p, m, \gamma, N, \varphi) R^{N-2\frac{p+\gamma}{p-1}}. \end{aligned}$$

Since v is the odd extension of U, we observe that also the following holds true:

$$\int_{\tilde{\Omega}' \cap B_R(y)} (|\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |v|^{p+\gamma}) \mathrm{d}x \leq C_{p,m,\gamma,N,\varphi} R^{N-2\frac{p+\gamma}{p-1}},$$

where $\tilde{\Omega}' := \{ x = (x', x_N) \in \mathbb{R}^N \,: x_N < 0 \}$. The conclusion follows by adding the last two estimates. Then we conclude the proof as in $3^{rd}, 4^{th}$, and 5^{th} step of the proof of Theorem 1.8, and conclude that $v \equiv 0$. Then $U \equiv 0$.

□

Now we consider a bounded no negative solution of our problem

THEOREM 1.21. *Assume* $N \geq 2$ *and* $\tilde{\Omega}$ *be the half space* $\{ x \in \mathbb{R}^N \,:\, x_N > 0 \}$. *Let* $U \in C^2(\overline{\tilde{\Omega}})$ *be a bounded non negative solution of*

$$\begin{cases} -\Delta U = U^p & \text{in } \tilde{\Omega} \\ U = 0 & \text{on } \partial\tilde{\Omega}. \end{cases} \tag{1.1.73}$$

$$\begin{cases} 1 < p < +\infty & \text{if } N \leq 11, \\ 1 < p < p_c(N-1) & \text{if } N \geq 12. \end{cases} \tag{1.1.74}$$

Then $U \equiv 0$.

Proof. We claim that U is a stable solution. Indeed, by the strong minimum principle either $U \equiv 0$, and then U is stable, or $U > 0$ in $\tilde{\Omega}$. Let us prove that the second possibility does not happen. Suppose, to the contrary, that $U > 0$ in $\tilde{\Omega}$, then, by moving plane method, a result of N. Dancer [**32**], implies that $\frac{\partial U}{\partial x_N} > 0$ everywhere in $\tilde{\Omega}$. Therefore $\frac{\partial U}{\partial x_N}$ is a positive solution of the linearized equation:

$$-\Delta s + \tilde{\lambda} s - p\, U^{p-1}\, s = 0 \quad \text{in } \tilde{\Omega},$$

and thus U is a stable solution of (1.1.73), indeed this is a well-known fact in the theory of the linear PDEs, see [**53, 84, 9**]. The boundedness of U, standard elliptic estimates [**61**] and the monotonicity of U with respect to the variable x_N, imply that the function

$$v(x_1, \ldots, x_{N-1}) := \lim_{x_N \to +\infty} U(x)$$

is a positive solution of the equation in $\mathbb{R}^{N-1}$. Furthermore v is stable in $\mathbb{R}^{N-1}$, see [**16, 9**]. At this point an application of Theorem 1.7, to the solution v in $\mathbb{R}^{N-1}$, gives $v \equiv 0$ in $\mathbb{R}^{N-1}$. This result contradicts $v > 0$ in $\mathbb{R}^{N-1}$. Hence $U \equiv 0$. □

Remark 1.22. This Theorem improves upon a results proved in [**32**] where the exponent p was assumed to satisfy $1 < p < +\infty$ if $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ if $N > 3$.

In this passage of the section we adapt the existence and non existence results of Esteban-Lions [**51**] to our equation. They prove that there exist no solution distinct from 0 of $-\Delta U = f(U)$ in unbounded domain, with Dirichlet condition and $U \to 0$ as $|x| \to +\infty$, for any smooth f satisfying $f(0) = 0$.

Theorem 1.23 (Esteban-Lions). *Let f be locally Lipschitz continuous on $\mathbb{R}$ such that $f(0) = 0$ and let Ω be a smooth unbounded connected domain with the following condition*

$$\textit{there exists} \quad X \in \mathbb{R}^N,\ |X| = 1 \ \textit{s.t.}\ \nu(x) \cdot X \geq 0,\ \nu(x) \cdot X \neq 0 \ \textit{on}\ \partial\Omega$$

where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at the point x. Under these assumptions, a solution U in $C^2(\overline{\Omega})$ of

$$\begin{cases} -\Delta U = f(U) & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.75}$$

satisfying $\nabla U \in L^2(\Omega)$, $F(U) \in L^1(\Omega)$ with $F(t) = \int_0^t f(s)ds$, is necessarily trivial.

Remark 1.24. The half-space is a typical domain which satisfies the assumptions of Theorem 1.23.

Theorem 1.25. *Assume $N \geq 3$, $p > 1$ and $\tilde{\Omega}$ be the half space $\{ x \in \mathbb{R}^N : x_N > 0 \}$. Let $U \in C^2(\overline{\tilde{\Omega}})$ be a bounded non negative solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \tilde{\Omega} \\ U = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \tag{1.1.76}$$

with $\tilde{\lambda} \in (0,1]$. *Then* $U \equiv 0$.

Proof. We claim that U is stable solution. Indeed, by the strong minimum principle either $U \equiv 0$, and then U is stable, or $U > 0$ in $\tilde{\Omega}$. In the latter case, since $\tilde{\Omega}$ is an half space, a result of Dancer [**32**], that use the moving plane method, implies that $\frac{\partial U}{\partial x_N} > 0$ in $\tilde{\Omega}$. It is known that $\frac{\partial U}{\partial x_N}$ is a positive solution of the linearized equation:

$$-\Delta s + \tilde{\lambda} U - pU^{p-1}\, s = 0 \ \text{ in } \ \tilde{\Omega},$$

and thus U is stable solution of (1.1.76), see [**53, 83, 9**].
We have that U is stable solution, then we can apply Proposition 1.3 with $\gamma = 1$ to have

$$\frac{U^{p+1}}{p+1} - \frac{\tilde{\lambda}}{2} U^2 \in L^1(\tilde{\Omega}), \quad \nabla U \in L^2(\tilde{\Omega}). \tag{1.1.77}$$

Therefore we can apply Theorem 1.23 to have the desired conclusion $U \equiv 0$. □

1.2. Pohozaev-type identity

In this thesis we will use some Pohozaev-Type Identities, in order to localize the blow-up set. We want to point out that, these are fundamental in our analysis.
To explain the integral identities we are going to derive, we argue as follows: notice that the equation

$$\begin{cases} -\Delta U = f(U) & \text{in } \ \Omega \\ U = 0 & \text{on } \ \partial\Omega \end{cases} \tag{1.2.1}$$

where $f \in C(\mathbb{R})$ and Ω is any smooth domain, has a "certain" invariance by multiplicative group of dilatations $(T_t U)(x) = U(t\,x)$ for $t \in (0, +\infty)$. By the use of this multiplier, Pohozaev [**92**] obtained a well known identity when Ω is bounded. For $\Omega = \mathbb{R}^N$ the same identity is proved under optimal conditions in [**18**]. We recall this so-called Pohozaev identity and we prove it by a simple adaptation of the method in [**18**].

Proposition 1.26. *Let* U *be a solution of*

$$-\Delta U = U^p - \tilde{\lambda} U \ \text{ in } \Omega \tag{1.2.2}$$

where $1 < p < +\infty$*, and* Ω *is a smooth domain. Assume* $\nabla U \in L^2(\Omega)$, $\frac{U^{p+1}}{p+1} - \tilde{\lambda} V \frac{U^2}{2} \in L^1(\Omega)$. *Then*

$$\int_\Omega \Big\{ \frac{N}{p+1} U^{p+1} - \frac{N}{2} \tilde{\lambda} U^2 + \Big(1 - \frac{N}{2}\Big) |\nabla U|^2 \Big\}\, dx = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) |\nabla U|^2 ds. \tag{1.2.3}$$

Remark 1.27. The exact meaning of the boundary integral in (1.2.3), if Ω is unbounded, is that there exists a sequence $R_n \to +\infty$ as $n \to +\infty$ such that we have

$$\int_\Omega \Big\{ \frac{N}{p+1} U^{p+1} - \frac{N}{2} \tilde{\lambda} U^2 + \Big(1 - \frac{N}{2}\Big) |\nabla U|^2 \Big\} \mathrm{d}x = \lim_{n \to +\infty} \frac{1}{2} \int_{\partial\Omega \cap B_{R_n}} (x \cdot \nu) |\nabla U|^2 \mathrm{d}s. \tag{1.2.4}$$

Proof of Proposition 1.26. We proceed as in [**18**]. Define $F(U) = \frac{U^{p+1}}{p+1} - \tilde{\lambda} V \frac{U^2}{2}$. Multiplying the equation $-\Delta U = U^p - \lambda U$ by $\sum_i x_i \frac{\partial U}{\partial x_i}$ and integrating by parts over $\Omega \cap B_R$, one gets (here and in all that follows, we use the implicit summation convention)

$$-\int_{\partial(\Omega\cap B_R)} \frac{\partial U}{\partial \nu} x_i \frac{\partial U}{\partial x_i} + \int_{\Omega\cap B_R} \left\{ \frac{\partial U}{\partial x_j} \delta_{i\,j} \frac{\partial U}{\partial x_i} + x_i \frac{\partial U}{\partial x_j} \frac{\partial^2 U}{\partial x_i \partial x_j} \right\} =$$

$$\int_{\Omega\cap B_R} x_i \frac{\partial F(U)}{\partial x_i} + \tilde{\lambda} \int_{\Omega\cap B_R} U\, x_i\, \frac{\partial V}{\partial x_i}$$

where ν denotes the unit outward normal to $\partial(\Omega \cap B_R)$. This implies, since $\nabla u = \frac{\partial u}{\partial \nu} \cdot \nu$ on $\partial\Omega$,

$$\begin{aligned} N \int_{\Omega\cap B_R} (F(U)) \quad &- \quad \int_{\partial\Omega\cap B_R} (x\cdot\nu)|\nabla U|^2 \mathrm{d}s + \int_{\Omega\cap B_R} |\nabla U|^2 + \int_{\Omega\cap B_R} x_i \frac{1}{2} \frac{\partial}{\partial x_i} (|\nabla U|^2) \\ &= \quad \int_{\Omega\cap\partial B_R} \left\{ \frac{\partial U}{\partial \nu} x_i \frac{\partial U}{\partial x_i} + |x| F(U) \right\}, \end{aligned}$$

or

$$\begin{aligned} &\int_{\Omega\cap B_R} \left\{ NF(U) + \left(1 - \frac{N}{2}\right) |\nabla U|^2 \right\} \mathrm{d}x - \frac{1}{2} \int_{\partial\Omega\cap B_R} (x\cdot\nu)|\nabla U|^2 \mathrm{d}s \\ = \quad &\int_{\partial B_R\cap\Omega} \left\{ \frac{\partial U}{\partial \nu} x_i \frac{\partial U}{\partial x_i} + |x| F(U) - \frac{1}{2} |x| |\nabla U|^2 \right\} \mathrm{d}s. \end{aligned}$$

Now the right hand member is bounded by

$$M(R) = R \int_{\Omega\cap B_R} \left\{ \frac{1}{2} |\nabla U|^2 + |F(U)| \right\} \mathrm{d}s.$$

If we assume that $\nabla U \in L^2(\Omega)$, $F(U) \in L^1(\Omega)$, we have

$$\int_0^{+\infty} \mathrm{d}r \int_{\Omega\cap\partial B_R} \left\{ \frac{1}{2} |\nabla U|^2 + |F(U)| \right\} \mathrm{d}s < +\infty;$$

therefore there exists a sequence R_n such that $M(R_n) \to 0$ as $n \to +\infty$. This proves the proposition. □

Proposition 1.28. *Let U be a solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \Omega \\ U > 0 & \text{in } \Omega \\ U = 0 & \text{in } \partial\Omega \end{cases} \tag{1.2.5}$$

where $1 < p < +\infty$, and Ω is a smooth domain. Let $\delta > 0$ small. Assume $\nabla U \in L^2(\Omega)$, $U^p - \tilde{\lambda} U \in L^1(\Omega)$.

Then

(1.2.6)
$$\int_{B_\delta}\Big\{\frac{N}{p+1}U^{p+1}-\frac{N}{2}\tilde\lambda U^2+\Big(1-\frac{N}{2}\Big)|\nabla U|^2\Big\}\,dx$$
$$=\ \delta\int_{\partial B_\delta}\Big(\frac{U^{p+1}}{p+1}-\frac{\tilde\lambda}{2}U^2\Big)\,ds-\frac{\delta}{2}\int_{\partial B_\delta}|\nabla U|^2\,ds+\delta\int_{\partial B_\delta}\left(\frac{\partial U}{\partial\nu}\right)^2 ds.$$

Proof. Let δ small, and define $F(U)=\frac{U^{p+1}}{p+1}-\tilde\lambda\frac{U^2}{2}$. Multiply the equation $-\Delta U=U^p-\lambda V\,U$ by $\sum_i x_i\frac{\partial U}{\partial x_i}$ and integrate by parts on $B_\delta:=B_\delta(P)$:

$$-\int_{\partial B_\delta}\frac{\partial U}{\partial\nu}\sum_i x_i\frac{\partial U}{\partial x_i}+\sum_i\int_{B_\delta}\Big\{\frac{\partial U}{\partial x_i}\delta_{ij}\frac{\partial U}{\partial x_i}+\frac{\partial U}{\partial x_i}\frac{\partial^2 U}{\partial x_i\partial x_j}\Big\}=\int_{B_\delta}\sum_i x_i\frac{\partial F(U)}{\partial x_i}.$$

Hence
$$\int_{B_\delta}|\nabla U|^2+\sum_i\int_{B_\delta}x_i\frac{1}{2}\frac{\partial}{\partial x_i}(|\nabla U|^2)-\int_{\partial B_\delta}\frac{\partial U}{\partial\nu}\sum_i x_i\frac{\partial U}{\partial x_i}$$
$$=-\int_{B_\delta}N\Big(\frac{U^{p+1}}{p+1}-\tilde\lambda\frac{U^2}{2}\Big)+\int_{\partial B_\delta}|x|\Big(\frac{U^{p+1}}{p+1}-\lambda\frac{U^2}{2}\Big)\mathrm{d}s$$
$$=\int_{B_\delta}\Big(1-\frac{N}{2}\Big)|\nabla U|^2\mathrm{d}x+\frac{\delta}{2}\int_{\partial B_\delta}|\nabla U|^2\mathrm{d}s-\int_{\partial B_\delta}\frac{\partial U}{\partial\nu}\sum_i x_i\frac{\partial U}{\partial x_i}\mathrm{d}s$$
$$=\int_{B_\delta}\Big(1-\frac{N}{2}\Big)|\nabla U|^2\mathrm{d}x+\frac{\delta}{2}\int_{\partial B_\delta}|\nabla U|^2\mathrm{d}s-\delta\int_{\partial B_\delta}\left(\frac{\partial U}{\partial\nu}\right)^2\mathrm{d}s.$$

Then
$$-\quad\int_{B_\delta}N\Big(\frac{U^{p+1}}{p+1}-\tilde\lambda\frac{U^2}{2}\Big)+\delta\int_{\partial B_\delta}\Big(\frac{U^{p+1}}{p+1}-\lambda\frac{U^2}{2}\Big)\mathrm{d}s$$
$$=\quad\int_{B_\delta}\Big(1-\frac{N}{2}\Big)|\nabla U|^2\mathrm{d}x+\frac{\delta}{2}\int_{\partial B_\delta}|\nabla U|^2\mathrm{d}s-\delta\int_{\partial B_\delta}\left(\frac{\partial U}{\partial\nu}\right)^2\mathrm{d}s,$$

so the thesis follows.

□

By the decay and the Pohozaev identity we have that if $\tilde\lambda>0$ and $p\geq\frac{N+2}{N-2}$ there are no solutions of our problem in $\mathbb{R}^N$, which is stable outside a compact set.

1.3. Conclusions

We resume the results obtained about the limiting problem:

Theorem 1.29. *Let U be a $G-$stable outside a compact set solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \tilde{\Omega} \\ 0 \leq U \leq U(0) = 1. \end{cases} \tag{1.3.1}$$

Consider $\tilde{\Omega}$ the whole-space, we have that

1. *if $\tilde{\lambda} > 0$, $1 < p < \frac{N+2}{N-2}$ or $\tilde{\lambda} = 0, p = \frac{N+2}{N-2}$ then there exists non trivial solution;*
2. *if $\tilde{\lambda} > 0$, $p \geq \frac{N+2}{N-2}$ or $\tilde{\lambda} = 0$, $1 < p < p_c(N)$, $p \neq \frac{N+2}{N-2}$ then $U \equiv 0$.*

Consider $\tilde{\Omega}$ the half-space, we have that

1. *if $\tilde{\lambda} = 0$, $1p > p_c(N-1)$ then there exists non trivial solution;*
2. *if $\tilde{\lambda} \in (0,1]$, $p > 1$ or $\tilde{\lambda} = 0$, $1 < p < p_c(N-1)$ then $U \equiv 0$.*

Therefore, the possible limiting problems are:

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \mathbb{R}^N \\ 0 \leq U \leq U(0) = 1, \quad \tilde{\lambda} \in (0,1], \ 1 < p < \frac{N+2}{N-2} \end{cases} \tag{1.3.2}$$

$$\begin{cases} -\Delta U = U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N \\ 0 \leq U \leq U(0) = 1. \end{cases} \tag{1.3.3}$$

CHAPTER 2

Asymptotic analysis and blow-up profile

It is known that solutions u of problem (1.1.1), must blow up as $\lambda \to +\infty$, and we address here in this chapter the asymptotic description of such a blow up behavior. When the "energy" is uniformly bounded, the behavior is well understood and the solutions can develop just a finite number of sharp peaks. When V is not constant, the blow up points must be c.p.'s of the potential V. The situation is more involved when $V \equiv 1$, and the crucial role is played by the mutual distances between the blow-up points as well as the boundary distances. The construction of these blowing-up solutions has also been addressed.
In this chapter we give a complete asymptotic analysis, also under the new hypothesis of bounded Morse index solutions. In the next we localize the blow-up set, in the different case (V generic potential, or $V \equiv 1$).
The most new and interesting feature is that, a-posteriori Morse index information and energy information are equivalent. Then we are able to find with an asymptotic approach, a Rozemljum-Lieb-Cwikel type estimate.

2.1. Blow up profile: $1 < p < \frac{N+2}{N-2}$

In the sequel we will work with $1 < p < \frac{N+2}{N-2}$ and give a complete description of the blow-up profile. Let u_n be a solution of (1.1.1), which satisfies (0.7), $P_n \in \Omega$ be a point of maximum of u_n, $u_n(P_n) = \|u_n\|_\infty$. Let us introduce $\varepsilon_n = \|u_n\|_\infty^{-\frac{p-1}{2}}$ and the change of variable:

$$U_n(y) = \frac{u_n(\varepsilon_n y + P_n)}{\|u_n\|_\infty} = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n).$$

It is easily seen that U_n is a solution of

$$\begin{cases} -\Delta U_n + \lambda_n \varepsilon_n^2 V(\varepsilon_n y + P_n) U_n = U_n^p & \text{in } \Omega_n \\ 0 < U_n(y) \le U_n(0) = 1 & \text{in } \Omega_n \end{cases} \tag{2.1.1}$$

with $\Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$. In this section we prove that, up to a subsequence, there exists $\tilde\lambda \in (0,1]$ s. t.

$$\lambda_n \varepsilon_n^2 V(P_n) \to \tilde\lambda, \text{ as } n \to +\infty.$$

So, up to a subsequence, U_n is uniformly bounded in $C^{1,1}_{loc}(\Omega_n)$. Then we assume that $U_n \to U$ in C^1_{loc} as $n \to +\infty$, where U satisfies

$$\begin{cases} -\Delta U + \tilde\lambda U = U^p, & \text{in } \tilde\Omega \\ 0 < U(y) \le U(0) = 1 \end{cases} \tag{2.1.2}$$

with $\tilde{\Omega} = \mathbb{R}^N$, $\tilde{\lambda} \in (0,1]$. The new main features of the previous results are the role of the potential V and the fact that we consider solutions with bounded Morse Indices.
We observe that if u_n has bounded Morse index, than U_n and U too. If φ_1 is a test function for u_n s. t. the linearized operator $L = -\Delta + 1 - pu_n^{p-1}$ satisfies

$$< L\varphi_1, \varphi_1 >< 0 \Leftrightarrow \int |\nabla\varphi|^2 + \lambda_n\, V(x)\varphi^2 - p\, u_n^{p-1}\varphi^2 < 0,$$

then the test function be $\varphi_{\varepsilon_n,1}(y) = \varphi_1\left(\frac{y-P_n}{\varepsilon_n}\right)$ satisfies

$$\int |\nabla\varphi_{\varepsilon,1}|^2 + \lambda_n\,\varepsilon_n^2\, V(\varepsilon_n\, y + P_n)\varphi_{\varepsilon,1}^2 - p\, U_n^{p-1}\varphi_{\varepsilon,1}^2 < 0.$$

So the Morse index of U_n is less or equal to $\bar{k}$ (see (0.7)).

2.1.1. Local profile.

We can prove this first result:

THEOREM 2.1. *Let (λ_n, u_n) be a positive solution of*

$$\begin{cases} -\Delta u_n + \lambda_n V u_n = u_n^p, & in\ \ \Omega \\ u_n = 0 & on\ \ \partial\Omega \end{cases} \tag{2.1.3}$$

with $1 < p < \frac{N+2}{N-2}$.
Assume either

$$\sup_n m(u_n) < +\infty,$$

or

$$\sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1} < +\infty.$$

Let $P_n \in \Omega$ s.t. $u_n(P_n) = \max_{\Omega\cap B_{R_n\varepsilon_n}(P_n)} u_n$ for some $R_n \to +\infty$ where $\varepsilon_n = u_n(P_n)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$.
Setting $U_n(y) = \frac{u_n(\varepsilon_n y + P_n)}{\|u_n\|_\infty} = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ for $y \in \Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$, then for a subsequence we have that, as $n \to +\infty$:

- $$\lambda_n \varepsilon_n^2 V(P_n) \to \tilde{\lambda} \in (0,1] \tag{2.1.4}$$

  for some universal constant $\tilde{\lambda}$;
- $\frac{\varepsilon_n}{d(P_n,\partial\Omega)} \to 0$;
- *$U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$ where U is a solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p, & in\ \ \mathbb{R}^N \\ 0 < U \le U(0) = 1 & in\ \ \mathbb{R}^N. \end{cases} \tag{2.1.5}$$

Moreover, $\exists \phi_n \in C_0^\infty(\Omega)$ with supp $\phi_n \subset B_{R\varepsilon_n}(P_n)$, for some $R > 0$, so that

$$\int_\Omega |\nabla\phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 dx < 0, \quad \forall\, n\ large, \tag{2.1.6}$$

and

$$\lim_{n\to+\infty} \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_{B_{R\,\varepsilon_n(P_n)}} u_n^{p+1} = \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \Big(\lim_{n\to+\infty} V(P_n)\Big)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{B_R(0)} U^{p+1}.$$

PROOF. Let d_n be $d(P_n, \partial\Omega)$. Suppose that $\frac{\varepsilon_n}{d_n} \to L \in [0, +\infty]$, up to a subsequence. Then $\Omega_n \to H$, H halfspace s. t. $0 \in \overline{H}$ and $d(0, \partial H) = \frac{1}{L}$. The function U_n satisfies

$$\begin{cases} -\Delta U_n + \lambda_n \varepsilon_n^2 V(\varepsilon_n y + P_n) U_n = U_n^p, & \text{in } \Omega_n \\ 0 < U_n \leq U_n(0) = 1, & \text{in } \Omega_n \cap B_{R_n}(0) \\ U_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{2.1.7}$$

Since P_n is a point of local maximum of u_n, we have

$$0 \leq -\Delta U_n(0) = 1 - \lambda_n \varepsilon_n^2 V(P_n) \leq 1 \;\Rightarrow\; 0 \leq \lambda_n \varepsilon_n^2 V(P_n) \leq 1.$$

Setting $\omega(V) := [\max_{\overline{\Omega}} V][\min_{\overline{\Omega}} V]^{-1}$, it follows that

$$\lambda_n\, \varepsilon_n^2 V(x) \leq \omega(V),$$

and up to a subsequence,

$$\varepsilon_n^2 \lambda_n V(P_n) \to \tilde{\lambda} \text{ as } n \to +\infty,$$

for some $\tilde{\lambda} \in [0, 1]$. By regularity theory we have that $U_n \to U$ in $C^1_{loc}(\overline{H})$, as $n \to +\infty$, where U satisfies

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } H \\ 0 < U(y) \leq U(0) = 1 & \text{in } H \\ U = 0 & \text{on } \partial H \end{cases} \tag{2.1.8}$$

where $\tilde{\lambda} \in [0, 1]$. In particular U is stable outside a compact set and by Theorems 1.8 and 1.21, we have $\tilde{\lambda} > 0$. Since $\tilde{\lambda} > 0$ by Theorem 1.1.76 we have that $H = \mathbb{R}^N$.
By Theorem 1.18 U is unstable. There exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi \subset B_R(0)$ and

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + (\tilde{\lambda} - p\, U^{p-1})\phi^2 < 0.$$

Then, the function $\phi_n(x) := \frac{1}{\varepsilon_n^{\frac{N+2}{2}}} \phi(\frac{x-P_n}{\varepsilon_n})$, ϕ_n satisfies

$$\int_\Omega |\nabla\phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 \mathrm{d}x \to \int_\Omega |\nabla\phi|^2 + (\tilde{\lambda} - p\, U^{p-1})\phi^2 \mathrm{d}x < 0$$

and $\operatorname{supp}\phi_n \subset B_{R\varepsilon_n}(P_n)$. Assume now

$$\sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1} < +\infty.$$

By Theorem 1.21 and [**59**] we have $\tilde{\lambda} > 0$ whenever $1 < p < \frac{N+2}{N-2}$. By Theorem 1.1.76 we also get that $H = \mathbb{R}^N$. Since

$$\int_{B_R(0)} U_n^{p+1} = (\lambda_n\, \varepsilon_n^2)^{\frac{p+1}{p-1}-\frac{N}{2}} \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_{B_{R\,\varepsilon_n(P_n)}} u_n^{p+1} \leq \frac{1}{(\min V)^{\frac{p+1}{p-1}-\frac{N}{2}}} \sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1} < +\infty,$$

we get that $\int_{\mathbb{R}^N} U^{p+1} < +\infty$ and by Theorem 1.18 U is the unique radial solution of (2.1.5). Moreover

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}}\int_{B_{R\varepsilon_n(P_n)}} u_n^{p+1} = \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}}\Big(\lim_{n\to+\infty}V(P_n)\Big)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U^{p+1}.$$

□

REMARK 2.2. The argument above works also for $\frac{N+2}{N-2} < p < p_c(N)$ whenever $\sup_n m(u_n) < +\infty$. Indeed, Theorem 1.21 holds for $1 < p < p_c(N)$, $p \neq \frac{N+2}{N-2}$. So, we still get in this case that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of (2.1.5) with $m(U) < +\infty$.
For $\tilde{\lambda} > 0$ and $p > \frac{N+2}{N-2}$, such solution U cannot exists as it follows by

THEOREM 2.3. *Let U be a nonnegative solution of*

$$-\Delta U + \tilde{\lambda} U = U^p, \quad in\ \ \mathbb{R}^N, \tilde{\lambda} > 0, \tag{2.1.9}$$

which is stable outside a compact set.
If $p > \frac{N+2}{N-2}$, then $U \equiv 0$.
This means that, whenever u_n blows-up in $L^\infty(\Omega)$: $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$, we have

$$m(u_n) \to +\infty \quad as\, n \to +\infty$$

for all $p > \frac{N+2}{N-2}$.

2.1.2. Global behavior. After the limiting problem has been identified and the local behavior around a blow up sequence P_n has been described, we can prove global estimates. We will show in such way that the sequence u_n decays exponentially away from the blow-up points.

THEOREM 2.4. *Let $1 < p < \frac{N+2}{N-2}$. Let $\lambda_n \to +\infty$, u_n be solution of (2.1.3), so that either*

$$\overline{k} = \limsup_{n\to+\infty} m(u_n) < +\infty$$

or

$$\overline{k} = \tilde{\lambda}^{\frac{p+1}{p-1}-\frac{N}{2}}(\min_{\Omega} V)^{\frac{N}{2}-\frac{p+1}{p-1}}\Big(\int_{\mathbb{R}^N} U^{p+1}\Big)^{-1}\sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}}\int_\Omega u_n^{p+1}$$

with u_n satisfying (0.7). Up to a subsequence, there exist $P_n^1, \dots, P_n^k$, $k \le \overline{k}$ with $\varepsilon_n^i = u_n(P_n^i)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$ s. t.

$$\varepsilon_n^1 \le \varepsilon_n^i \le C_0\varepsilon_n^1, \quad for\ all\ \ i = 1, \dots, k \tag{2.1.10}$$

$$\frac{\varepsilon_n^i + \varepsilon_n^j}{|P_n^i - P_n^j|} \to 0 \quad as\ \ n \to +\infty, \quad for\ all\ \ i, j = 1, \dots, k,\ \ i \neq j \tag{2.1.11}$$

$$\frac{\varepsilon_n^i}{d(P_n^i, \partial\Omega)} \to 0 \quad as\ \ n \to +\infty, \quad for\ all\ \ i = 1, \dots, k,\ \ i \neq j \tag{2.1.12}$$

$$u_n(P_n^i) = \max_{\Omega \cap B_{R_n \varepsilon_n^i}(P_n^i)} u_n, \tag{2.1.13}$$

for some $R_n \to +\infty$ as $n \to +\infty$.
Moreover, there holds

$$u_n(x) \leq C(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^{k} e^{-\gamma \frac{|x-P_n^i|}{\varepsilon_n^1}} \quad \forall\, x \in \Omega, \;\; n \in \mathbb{N} \tag{2.1.14}$$

with $C > 0$.

Proof. The proof is divided in two steps (see also [**50**]).

1[st] step *There exist $k \leq \overline{k}$ sequences $P_n^1, \dots, P_n^k$ satisfying (2.1.10) - (2.1.13) such that:*

$$\lim_{R\to+\infty} \left(\limsup_{n\to+\infty} \left[(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{d_n(x) \geq R\varepsilon_n^1\}} u_n(x) \right] \right) = 0 \tag{2.1.15}$$

where $d_n(x) = \min\{|x - P_n^i| \,:\, i = 1, \dots, k\}$ is the distance function in Ω from $\{P_n^1, \dots, P_n^k\}$.

Let P_n^1 be a point of global maximum of u_n: $u_n(P_n^1) = \max_{\Omega} u_n(x)$. Since (2.1.13) holds, if (2.1.15) holds for P_n^1, then we take $k = 1$ and the claim is proved. Otherwise, we suppose by contradiction that

$$\limsup_{R\to+\infty} \limsup_{n\to+\infty} (\varepsilon_n^1)^{\frac{2}{p-1}} \max_{|x-P_n^1| \geq R\varepsilon_n^1} u_n = 4\delta > 0.$$

Applying Theorem 2.1, up to a subsequence we have

$$(\varepsilon_n^1)^{\frac{2}{p-1}} u_n(\varepsilon_n^1 y + P_n^1) = U_n^1(y) \to U(y) \quad \text{in } C^1_{loc}(\mathbb{R}^N), \tag{2.1.16}$$

where U is the unique radial solution of (2.1.5) in $\mathbb{R}^N$. Since $U \to 0$ as $|x| \to +\infty$, we can find R large such that :

$$U(y) \leq \delta, \quad \forall\, |y| \geq R. \tag{2.1.17}$$

Up to take R larger, we can assume

$$\limsup_{n\to+\infty} (\varepsilon_n^1)^{\frac{2}{p-1}} \max_{|x-P_n^1| \geq R\varepsilon_n^1} u_n \geq 3\delta > 0. \tag{2.1.18}$$

Up to a subsequence, we can also assume that

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{|x-P_n^1| \geq R\varepsilon_n^1,} u_n \geq 2\,\delta. \tag{2.1.19}$$

Since $u_n = 0$ on $\partial\Omega$, we have that

$$\exists\, P_n^2 \in \Omega \setminus B_{R\,\varepsilon_n^1}(P_n^1) \quad \text{s.t.} \quad u_n(P_n^2) = \max_{B_{R\,\varepsilon_n^1}(P_n^1)} u_n.$$

By (2.1.17) and (2.1.16) we have $\frac{|P_n^2-P_n^1|}{\varepsilon_n^1}\to+\infty$. Indeed, if $\frac{|P_1^2-P_n^1|}{\varepsilon_n^1}\to R'\geq R$

$$u_n(P_n^2)=U_n^1\left(\frac{P_n^2-P_n^1}{\varepsilon_n^1}\right)\to U(R')\leq\delta$$

contradicting (2.1.19).

We take $R_n^2=\frac{1}{2}\frac{|P_n^2-P_n^1|}{\varepsilon_n^2}$ $(R_n^1=\frac{1}{2}\frac{d(P_n^1,\partial\Omega)}{\varepsilon_n^1})$. By (2.1.19) we get $\varepsilon_n^2:=u_n(P_n^2)^{-\frac{p-1}{2}}\leq\varepsilon_n^1(2\,\delta)^{-\frac{p-1}{2}}$, and since $\varepsilon_n^1\leq\varepsilon_n^2$, we see that (2.1.10) and (2.1.11) are fulfilled. So this implies (2.1.13):

$$u_n(P_n^2)=\max_{|x-P_n^1|\geq R\varepsilon_n^1}u_n=\max_{B_{R_n^2\,\varepsilon_n^2}(P_n^2)\cap\Omega}u_n.$$

Indeed $R_n^2\,\varepsilon_n^2=\frac{1}{2}|P_n^2-P_n^1|$, and $R\,\varepsilon_n^1<<\frac{1}{2}|P_n^2-P_n^1|$ imply $\forall x\in B_{R_n^2\,\varepsilon_n^2}(P_n^2)$,

$$|x-P_n^1|\geq|P_n^2-P_n^1|-|x-P_n^2|\geq\frac{1}{2}|P_n^2-P_n^1|\geq R\,\varepsilon_n^1,$$

i. e. $\Omega\cap B_{R_n^2\,\varepsilon_n^2}(P_n^2)\subset\Omega\cap B_{R\,\varepsilon_n^1}(P_n^1)$. Since $R_n^2\to+\infty$ as $n\to+\infty$. By Theorem 2.1 we get that (2.1.10)-(2.1.13) hold true for $\{\,P_n^1,P_n^2\,\}$. If (2.1.15) holds for $\{\,P_n^1,P_n^2\,\}$, we are done.

Otherwise, we iterate the above argument: let $P_n^1,\dots,P_n^s$ s sequences so that (2.1.10)-(2.1.13) hold true, but (2.1.15) is not satisfied. We have

$$\limsup_{R\to+\infty}\limsup_{n\to+\infty}(\varepsilon_n^1)^{\frac{2}{p-1}}\max_{d_n(x)\geq R\,\varepsilon_n^1}u_n=4\,\delta>0$$

with $d_n(x)=\min\{\,|x-P_n^i|\,:i=1,\dots,s\,\}$. There exists $R>0$ large s. t.

$$(\varepsilon_n^1)^{\frac{2}{p-1}}\max_{\{\,d_n(x)\geq R\varepsilon_n^1\,\}}u_n(x)\geq2\,\delta$$

holds for a subsequence. By (2.1.10) and Theorem 2.1:

$$\exists\vartheta_i\in\left[\frac{1}{C},1\right]:\ \frac{\varepsilon_n^1}{\varepsilon_n^i}\to\vartheta_i,$$

$$(\varepsilon_n^1)^{\frac{2}{p-1}}u_n(\,\varepsilon_n^1\,y+P_n^1)=\left(\frac{\varepsilon_n^1}{\varepsilon_n^i}\right)^{\frac{2}{p-1}}U_n^i\left(\frac{\varepsilon_n^1}{\varepsilon_n^i}y\right)\to\vartheta_i^{\frac{2}{p-1}}U(\vartheta_i\,y)\tag{2.1.20}$$

in $C^1_{loc}(\mathbb{R}^N)$.

Since $U\to0$ as $|x|\to+\infty$ we can find R large so that $\vartheta_i^{\frac{2}{p-1}}U(\vartheta_i\,y)<\delta$ for $|y|\geq R_\delta$. We repeat the argument above, replacing $|x-P_n^1|$ with $d_n(x)$. Let P_n^{s+1} be s. t. $u_n(P_n^{s+1})=\max_{d_n\geq R\,\varepsilon_n^1}u_n\geq2\,\delta\,(\varepsilon_n^1)^{-\frac{2}{p-1}}$. As above we have that $\frac{d_n(P_n^{s+1})}{\varepsilon_n^1}\to+\infty$, and (2.1.10) holds for $\{\,P_n^1,\dots,P_n^{s+1}\,\}$. For $R_n^{s+1}=\frac{1}{2}\frac{d_n(x)}{\varepsilon_n^{s+1}}$ we get the validity of (2.1.10) for P_n^{s+1} so by Theorem 2.1 we get that (2.1.10)-(2.1.13) holds for $\{\,P_n^1,\dots,P_n^{s+1}\,\}$ with $R_n=\min_k R_n^k$. We can use Theorem 2.1 for any sequence P_n^i, $i=1,\dots,s+1$, for n large, we can find functions $\phi_n^i\in C_0^\infty(\Omega)$ with $\operatorname{supp}\phi_n^i\subset B_{R\,\varepsilon_n^i}(P_n^i)$, for some $R>0$, which satisfy (2.1.6). By (2.1.11) $\phi_n^1,\dots,\phi_n^{s+1}$

have disjoint compact supports for n large and then $s+1 \le \sup_n m(u_n)$. The argument must stop for some $k \le \overline{k}$. By Theorem 2.1 and (2.1.11), we get that $\forall R > 0$ (up to a subsequence):

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1} &\ge \lim_{n\to+\infty} \sum_{i=1}^{s+1} \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_{B_{R\,\varepsilon_n^i}(P_n^i)} u_n^{p+1} \\ &\ge (s+1)\tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} (\min_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{B_R(0)} U^{p+1},\end{aligned}$$

and then

$$s+1 \le \tilde{\lambda}^{\frac{p+1}{p-1}-\frac{N}{2}} (\min_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \Big(\int_{\mathbb{R}^N} U^{p+1}\Big)^{-1} \sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1}.$$

We want to show now the validity of (2.1.14) and (2.1.15) does hold.

2nd step *Let $P_n^1, \dots, P_n^k$ be as in the 1^{st} step. Then there are γ, $C > 0$ such that:*

(2.1.21) $$u_n(x) \le C\,(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^k e^{-\gamma \frac{|x-P_n^i|}{\varepsilon_n^1}}, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}.$$

By (2.1.15), for $R > 0$ large and $n \ge n(R)$, there holds

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{\, d_n(x) \ge R\varepsilon_n^1 \,\}} u_n(x) \le \Big(\frac{\tilde{\lambda}}{2\omega(V)}\Big)^{\frac{1}{p-1}}.$$

Hence, in $\{d_n(x) \ge R\,\varepsilon_n^1\,\}$ we have

$$(\varepsilon_n^1)^2\, u_n^{p-1}(x) \le \frac{\tilde{\lambda}}{2\omega(V)},$$

where $\omega(V) := [\max_{\overline{\Omega}} V][\min_{\overline{\Omega}} V]^{-1}$. Moreover, by Theorem 2.1 we have the validity of (2.1.4), and there we get

$$\lambda_n(\varepsilon_n^1)^2\, V(x) \ge [\omega(V)]^{-1} \lambda_n (\varepsilon_n^1)^2\, V(P_n^1) \to_n \frac{\tilde{\lambda}}{\omega(V)}.$$

Therefore, for $n \ge n(R)$ we have that

$$(\varepsilon_n^1)^2 [\lambda_n\, V(x) - u_n^{p-1}(x)] \ge \frac{\tilde{\lambda}}{2\,\omega(V)} > 0, \quad \text{if } d_n(x) \ge R\,\varepsilon_n^1.$$

Now consider the following linear operator:

$$L_n := -\Delta + (\lambda_n\, V(x) - u_n^{p-1}(x)).$$

Since u_n is a positive solution in Ω of L_n, L_n satisfies the minimum principle in any $\widehat{\Omega} \subset \Omega$: $L_n\,\phi > 0$ in $\widehat{\Omega}$, $\phi > 0$ on $\partial\widehat{\Omega}$ implies $\phi \ge 0$ in $\widehat{\Omega}$.

Let $\phi_n^i(x) = e^{-\gamma(\varepsilon_n^1)^{-1}|x-P_n^i|}$. We have that in $d_n(x) \geq R\,\varepsilon_n^1$:

$$L_n(\phi_n^i) = (\varepsilon_n^1)^{-2}\phi_n^i\Big[-\gamma^2 + (N-1)\frac{\varepsilon_n^1}{|x-P_n^i|}\gamma + (\varepsilon_n^1)^2(\lambda_n\,V(x) - u_n^{p-1}(x))\Big] > 0$$

for n large, provided $\gamma^2 \leq \frac{\tilde\lambda}{4\,\omega(V)}$. Observe that

$$\Big(e^{\gamma R}\phi_n^i(x) - (\varepsilon_n^1)^{\frac{2}{p-1}}u_n(x)\Big)|_{\partial B_{R\,\varepsilon_n^1}(P_n^i)} \to 1 - \vartheta_i^{\frac{2}{p-1}}U(\vartheta_i R) > 0$$

for R large, where ϑ_i as in (2.1.20).
Then, if we define $\phi_n := e^{\gamma\,R}(\varepsilon_n^1)^{-\frac{2}{p-1}}\sum_{i=1}^k \phi_n^i$, we have

$$L_n(\phi_n - u_n) > 0 \quad \text{in} \quad \{d_n(x) > R\,\varepsilon_n^1\}$$

and $\phi_n - u_n > 0$ on $\{d_n(x) = R\,\varepsilon_n^1\} \cup \partial\Omega$. Note that by (2.1.10)-(2.1.13))

$$\{d_n(x) = R\,\varepsilon_n^1\} = \cup_{i=1}^k \partial B_{R\,\varepsilon_n^1}(P_n^i) \subset \Omega,$$

for $n \geq n(R)$. Then, by the minimum principle

$$u_n \leq \phi_n = e^{\gamma\,R}(\varepsilon_n^1)^{-\frac{2}{p-1}}\sum_{i=1}^k e^{-\gamma\frac{|x-P_n^i|}{\varepsilon_n^1}}$$

in $\{d_n(x) > R\ \varepsilon_n^1\}$, if R is large and $n \geq n(R)$. Since

$$u_n(x) \leq \max_\Omega u_n = (\varepsilon_n^1)^{-\frac{2}{p-1}} \leq e^{\gamma\,R}(\varepsilon_n^1)^{-\frac{2}{p-1}}\sum_{i=1}^k e^{-\gamma\frac{|x-P_n^i|}{\varepsilon_n^1}}$$

if $d_n(x) \leq R\,\varepsilon_n^1$. We have that (2.1.14) holds true in Ω, for $C = e^{\gamma\,R}$ and $n \geq n(R)$.
Up to take a larger constant C, we have the validity of (2.1.14) in Ω for every $n \in \mathbb{N}$.

□

2.2. Morse Index information and Energy information

It is know that any solution u of (1.1.1) is a critical point of an energy functional and vice versa. It can be proved that for any family of solutions u_n of (0.1), with finite energy we can obtain Morse index information.
In this section, we obtain an equivalence in the form of a double-side bound between Morse index and "energy" with essentially optimal constants. This result can be seen as a sort of Rozenblyum-Lieb-Cwikel inequality, where the number of negative eigenvalues of a Schrödinger operator $-\Delta+V$ can be estimated in terms of a suitable Lebesgue norm of the negative part V_- of V. Thanks to the specificity of our problem, we improve it by getting the correct Lebesgue exponent (in view of the double-side bound) as well as the sharp constants.

Let u_n be a positive solutions sequence of

$$\begin{cases} -\Delta u_n + \lambda_n V u_n = u_n^p & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2.1}$$

where $\lambda_n \to +\infty$ as $n \to +\infty$, $1 < p < \frac{N+2}{N-2}$ and $\inf_\Omega V > 0$.
Assume

$$\sup_n \frac{1}{\lambda_n^{\frac{p+1}{p-1}-\frac{N}{2}}} \int_\Omega u_n^{p+1} < +\infty. \tag{2.2.2}$$

Consider the problem

$$\begin{cases} -\Delta U + \tilde{\lambda} U = U^p & \text{in } \mathbb{R}^N \\ 0 < U \le U(0) = 1 & \text{in } \mathbb{R}^N \end{cases} \tag{2.2.3}$$

for suitable $\tilde{\lambda} > 0$,. We remember that Kwong [**70**] proved that this problem has a unique solution.

THEOREM 2.5. *Let u_n be a solution of (2.2.1) so that $\sup_n m(u_n) < +\infty$. Then*

$$\limsup_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} \le \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} (\max_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U^{p+1} \tag{2.2.4}$$

PROOF. Take a subsequence so that

$$\frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} \to \beta \quad \text{as} \quad n \to +\infty.$$

By Theorem 2.4, up to a further sub-sequence, we can assume that $m(u_n) \to \overline{k}$ and there exist $\{ P_n^1, \dots, P_n^k \}$, $k \le \overline{k}$, so that (2.1.10)-(2.1.14) hold.
We can then write $\forall\, R > 0$:

$$\begin{aligned} &\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} \\ = \;& \sum_{i=1}^k (\lambda_n\, (\varepsilon_n^i)^2)^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{B_R(0)} (U_n^i)^{p+1} + O\Big((\lambda_n\, (\varepsilon_n^i)^2)^{\frac{N}{2}-\frac{p+1}{p-1}} \sum_{j=1}^k \int_{\mathbb{R}^N \setminus B_{R\frac{\varepsilon_n^j}{\varepsilon_n^1}}(0)} e^{-\gamma(p+1)|y|} \mathrm{d}y \Big), \end{aligned}$$

and in view of (2.1.4) we get that

$$\begin{aligned} &\lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} \\ = \;& \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{B_R(0)} U^{p+1} \sum_{i=1}^k \lim_{n\to+\infty} V(P_n^i)^{\frac{p+1}{p-1}-\frac{N}{2}} + O\Big(\int_{\mathbb{R}^N \setminus B_{\varepsilon\, R}(0)} e^{-\gamma(p+1)|y|} \mathrm{d}y \Big). \end{aligned}$$

Since this is true for every $R > 0$ we get as $R \to +\infty$:

$$\begin{aligned} \lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} &= \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{\mathbb{R}^N} U^{p+1} \sum_{i=1}^k \lim_{n\to+\infty} V(P_n^i)^{\frac{p+1}{p-1}-\frac{N}{2}} \\ &\le k\, \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{\mathbb{R}^N} U^{p+1} (\max_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \\ &\le \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{\mathbb{R}^N} U^{p+1} (\max_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \lim_{n\to+\infty} m(u_n). \end{aligned}$$

We then deduce (2.2.4) on the value β. □

Now we want to show the lower bound:

THEOREM 2.6. *Let u_n be a solution of (2.2.1) so that*

$$\sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty.$$

Then

$$\liminf_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} \geq \frac{1}{N+1} \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{\mathbb{R}^N} U^{p+1} \, (\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}. \tag{2.2.5}$$

REMARK 2.7. In this context the so-called Rozenblyum-Lieb-Cwikel inequality [**93, 71, 29**], can be considered for a solution U as a relation between the Morse index and a bound on U in same Lebesgue space.
Let consider the operator $-\Delta + V$ in $\mathbb{R}^N$, the number of negative eigenvalue of this operator is estimated by this Rozemblyum-Lieb-Cwikel inequality, in the following way:

$$\#\{\, \lambda < 0 \ \text{ eigenvalue of } \ -\Delta + V\} \leq C_2 \|V_-\|_{L^{\frac{N}{2}}}^{\frac{N}{2}}.$$

In our specific case, we obtain better estimates, in the sense that we obtain a double bound inequality and with optimal constants. Indeed let u_n be a solution of

$$\begin{cases} -\Delta u_n + \lambda_n u_n = u_n^p & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2.6}$$

with $1 < p < \frac{N+2}{N-2}$ (for simplicity $V \equiv 1$), and after a rescaling we can consider this problem like $\tilde{U_n}$ solution of

$$\begin{cases} -\Delta \tilde{U_n} + \tilde{U_n} = \tilde{U_n}^p & \text{in } \Omega_n \\ \tilde{U_n} = 0 & \text{on } \partial\Omega_n \\ \tilde{U_n} \leq \tilde{U_n}(0) \leq (\lambda_n \varepsilon_n^2)^{-\frac{1}{p-1}}. \end{cases} \tag{2.2.7}$$

In the sequel of this section we obtain a double bound of the energy:

$$C_0 \, \lambda_n^\gamma \int_\Omega u_n^{p+1} \leq m(u_n) \leq C_1 \, \lambda_n^\gamma \int_\Omega u_n^{p+1}, \tag{2.2.8}$$

with a suitable γ. If we consider the Morse index $m(\tilde{U_n})$ of $\tilde{U_n}$ in $H_0^1(\Omega_n)$, where the domain Ω_n is expanding to $\mathbb{R}^N$, it coincides with $m(u_n)$. Now, rescaling the norms in (2.2.8), we have

$$C_0 \int_{\Omega_n} \tilde{U_n}^{p+1} \leq m(U_n) \leq C_1 \int_{\Omega_n} \tilde{U_n}^{p+1}. \tag{2.2.9}$$

Observe that

$$m(\tilde{U_n}) = \#\{\, \lambda < 0 \text{ eigenvalue of } \ -\Delta + (1 - p\,\tilde{U_n}^{p-1})\,\}.$$

A Rozenblyum-Lieb-Cwikel type formula would read on Ω_n as

$$m(\tilde{U_n}) \leq C_2 \|(1 - p\,\tilde{U_n}^{\,p-1})_-\|_{L^{\frac{N}{2}}}^{\frac{N}{2}}. \tag{2.2.10}$$

While in (2.2.9) we have a better estimate.
Resuming, we are able to exhibit a double bound estimate, based on techniques of asymptotic analysis and a characterization of the unique radial stable outside a compact set or with bounded energy solution.

As a consequence of Theorems 2.5 and 2.6 we get

THEOREM 2.8 (Rozenblyum-Lieb-Cwikel type estimate). *Let u_n be a solution of (2.2.1). The following are equivalent*

(1) $\sup_n m(u_n) < +\infty$;
(2) $\sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty$.

Moreover, when (1) *or* (2) *holds we have*

$$\begin{aligned}\frac{\tilde{\lambda}^{\frac{p+1}{p-1}-\frac{N}{2}}}{N+1}(\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U^{p+1} \quad &\leq \quad \liminf_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}}{m(u_n)} \leq \limsup_{n\to+\infty}\frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}}{m(u_n)} \\ &\leq \quad \tilde{\lambda}^{\frac{N}{2}-\frac{p+1}{p-1}}(\max_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U^{p+1}.\end{aligned}$$

PROOF OF THEOREM 2.6. Take a subsequence so that

$$\frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}}{m(u_n)} \to \beta \quad \text{as} \quad n\to+\infty.$$

Assume, up to a further subsequence, that

$$\lim_{n\to+\infty} m(u_n) = \hat{N} \in [0,+\infty].$$

By Theorem 2.4 we know that there exists $\{\,P_n^1,\dots,P_n^k\,\}$, $k \leq \sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}$, so that (2.1.10)-(2.1.14) hold.
Let φ_n^m be the $m-^{th}$ eigenfunction of $-\Delta + \lambda\,V - p\,u_n^{p-1}$ in $H_0^1(\Omega)$ corresponding to the eigenvalue μ_n^m and assume φ_n^m be normalized to have $\max_\Omega|\varphi_n^m| = \max_\Omega \varphi_n^m = 1$ (considered with multiplicities):

$$\begin{cases} -\Delta\varphi_n^m + \lambda_n\,V\,\varphi_n^m - p\,u_n^{p-1}\,\varphi_n^m = \mu_n^m\,\varphi_n^m & \text{in } \Omega \\ |\varphi_n^m| \leq \max_\Omega \varphi_n^m = 1,\ \varphi_n^m = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2.11}$$

Fix now m such that $\mu_n^m \leq 0$ for n large. We have
1^{st} claim

–

$$\mu_n^m \geq \lambda_n \inf_\Omega V - \frac{p}{(\varepsilon_n^1)^2};$$

– $\exists\, M > 0$ *s. t.*

$$Q_n^m \in \bigcup_{j=1}^{k} B_{M\,\varepsilon_n^j}(P_n^j)$$

where Q_n^m *is so that* $\varphi_n^m(Q_n^m) = 1$.

Proof of 1^{st} claim: Let Q_n^m be a maximum point of φ_n^m. We have that

$$\mu_n^m\, \varphi_n^m(Q_n^m) \geq \lambda_n\, V(Q_n^m)\varphi_n^m(Q_n^m) - p\, u_n^{p-1}(Q_n^m)\varphi_n^m(Q_n^m).$$

Since $\varphi_n^m(Q_n^m) = 1$, we get that

$$\mu_n^m \geq \lambda_n\, V(Q_n^m) - p\, u_n^{p-1}(Q_n^m) \geq \lambda_n\, \inf_{\Omega} V - \frac{p}{(\varepsilon_n^1)}.$$

Further, observe that by (2.1.14) we have

$$u_n^{p-1}(x) \leq C(\varepsilon_n^1)^{-2} \sum_{j=1}^{k} e^{-(p-1)\gamma \frac{|x-P_n^j|}{\varepsilon_n^j}} \leq C(\varepsilon_n^1)^{-2} k e^{-\gamma\, M(p-1)}$$

in $\Omega \setminus \bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)$. By (2.1.4) we get

$$\lambda_n\, V(Q_n^m) - p\, u_n^{p-1}(Q_n^m) \geq (\varepsilon_n^1)^{-2}\Big[\frac{\tilde{\lambda}}{2}\frac{\inf_\Omega V}{\max_\Omega V} - p\, C\, k\, e^{-(p-1)\gamma\, M}\Big] > 0$$

for n large, whenever $Q_n^m \in \Omega \setminus \bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)$, for some $M > 0$ large. A contradiction to $\mu_n^m \leq 0$. Hence,

$$Q_n^m \in \bigcup_{j=1}^{k} B_{M\,\varepsilon_n^j}(P_n^j) \quad \text{for some } M > 0 \text{ large.}$$

□

Set $\phi_n^{m,j}(y) = \varphi_n^m(\varepsilon_n^j\, y + P_n^j)$. The function $\phi_n^{m,j}$ solves

(2.2.12)
$$\begin{cases} -\Delta\phi_n^{m,j} + \lambda_n(\varepsilon_n^j)^2 V(\varepsilon_n^j\, y + P_n^j)\phi_n^{m,j} - p(U_n^j)^{p-1}\phi_n^{m,j} = (\varepsilon_n^j)^2\mu_n^m\, \phi_n^{m,j} & \text{in } \frac{\Omega - P_n^j}{\varepsilon_n^j} \\ |\phi_n^{m,j}| \leq \phi_n^{m,j}\Big(\frac{Q_n^m - P_n^j}{\varepsilon_n^j}\Big) = 1 & \text{in } \frac{\Omega - P_n^j}{\varepsilon_n^j} \\ \phi_n^{m,j} = 0 & \text{on } \partial(\frac{\Omega - P_n^j}{\varepsilon_n^j}). \end{cases}$$

By the 1^{st} claim and (2.1.4), (2.1.10) we get

$$0 \geq (\varepsilon_n^j)^2\mu_n^m \geq \lambda_n(\varepsilon_n^j)^2 \inf_\Omega V - p\Big(\frac{\varepsilon_n^j}{\varepsilon_n^1}\Big)^2 \geq \frac{\tilde{\lambda}}{2}\frac{\inf_\Omega V}{\max_\Omega V} - p\, U_0^2.$$

Up to a subsequence, we can assume that

$$(\varepsilon_n^j)^2\mu_n^m \to \mu^{m,j} \leq 0 \quad \text{as } n \to +\infty.$$

Multiply (2.2.12) by $\phi_n^{m,j}$ and integrate on $\frac{\Omega-P_n^j}{\varepsilon_n^j}$ to get

$$\begin{aligned}
&\int_{\frac{\Omega-P_n^j}{\varepsilon_n^j}} |\nabla\phi_n^{m,j}|^2 + \lambda_n(\varepsilon_n^j)^2\, V(\varepsilon_n^j\, y + P_n^j)(\phi_n^{m,j})^2 \le \int_{\frac{\Omega-P_n^j}{\varepsilon_n^j}} (U_n^j)^{p-1} \\
\le\ & \frac{C(\varepsilon_n^j)^2}{(\varepsilon_n^1)^2} \sum_{i=1}^k \int_{\frac{\Omega-P_n^j}{\varepsilon_n^j}} e^{-(p-1)\gamma\left|\frac{\varepsilon_n^j}{\varepsilon_n^1} y + \frac{P_n^j - P_n^i}{\varepsilon_n^1}\right|} \\
=\ & \frac{C(\varepsilon_n^j)^2}{(\varepsilon_n^1)^2} \frac{(\varepsilon_n^i)^N}{(\varepsilon_n^j)^N} \sum_{i=1}^k \int_{\frac{\Omega-P_n^i}{\varepsilon_n^i}} e^{-(p-1)\gamma|y|} \\
\le\ & C^{N+2}\, k \int_{\mathbb{R}^N} e^{-(p-1)\gamma\,|y|} < +\infty
\end{aligned}$$

in view of (2.1.10) and (2.1.14).
In particular, $\phi_n^{m,j}{}_{H^1(B_M(0))} \le C_M\ \forall\, M$ and, up to a subsequence and a diagonal process, $\phi_n^{m,j} \rightharpoonup \phi^{m,j}$ in $H^1_{loc}(\mathbb{R}^N)$ as $n \to +\infty$. Moreover $\phi^{m,j}$ solves

$$\left\{ \begin{array}{ll} -\Delta\phi^{m,j} + \tilde{\lambda}\,\phi^{m,j} - p\,U^{p-1}\,\phi^{m,j} = \mu^{m,j}\,\phi^{m,j} & \text{in} \quad \mathbb{R}^N \\ |\phi^{m,j}| \le 1 & \end{array} \right. \tag{2.2.13}$$

in view of (2.1.4). We have that $\phi^{m,j} \ne 0$ for at least one $j \in \{1, \dots, k\}$.
This follows by:

2^{nd} claim *Let $j \in \{1, \dots, k\}$ so that (up to a subsequence) $Q_n^m \in B_{M\,\varepsilon_n^j}(P_n^j)$ for some $M > 0$ large. Then $\phi^{m,j} \ne 0$.*

Proof of 2^{nd} claim: Decompose $\phi_n^{m,j}$ as $h_n + t_n$, where h_n satisfies

$$\left\{ \begin{array}{ll} \Delta h_n = 0 & \text{in } B_{M+1}(0) \\ h_n = \phi_n^{m,j} & \text{on } \partial B_{M+1}(0). \end{array} \right. \tag{2.2.14}$$

If $\phi^{m,j} \equiv 0$, then $\phi_n^{m,j} \rightharpoonup 0$ in $H^1(B_{M+1}(0))$, and by the trace Sobolev embedding Theorem $\phi_n^{m,j} \to 0$ in $L^1(\partial B_{M+1}(0))$. By the mean value Theorem, then $h_n \to 0$ uniformly in $B_M(0)$. Since

$$\left\{ \begin{array}{ll} -\Delta t_n = -\Delta\phi_n^{m,j} = O(1) & \text{in } B_{M+1}(0) \\ t_n = 0 & \text{on } \partial B_{M+1}(0), \end{array} \right. \tag{2.2.15}$$

by regularity theory t_n is uniformly bounded in $C^{0,\alpha}(\overline{B_{M+1}(0)})$. In particular, by Ascoli-Arzelá Theorem $t_n \to t$ uniformly in $\overline{B_{M+1}(0)}$. Hence, $\phi_n^{m,j} = h_n + t_n \to t$ uniformly in $B_M(0)$, where $t = \phi^{m,j} \equiv 0$. In particular,

$$\phi_n^{m,j}\Big(\frac{Q_n^m - P_n^j}{\varepsilon_n^j}\Big) = \varphi_n^m(Q_n^m) = 1 \to 0 \text{ as } n \to +\infty.$$

A contradiction and then $\phi^{m,j} \ne 0$. □

By Theorem 1.18, recall that $-\Delta+\tilde{\lambda}-p\,U^{p-1}$ has a first negative eigenvalue $\mu_1<0$ (with corresponding eigenfunction $\psi_1\in L^2(\mathbb{R}^N)$), $\mu_2=\mu_3=\cdots=\mu_{N+1}=0$ vanish (with corresponding eigenfunctions $\psi_2=\partial_{x_1}U,\dots,\psi_{N+1}=\partial_{x_N}U\in L^2(\mathbb{R}^N)$), and all the other eigenvalues are positive.
We have that necessarily $\mu^{m,j}=\mu_1\ \forall\, j$ or $\mu^{m,j}=0\ \forall j$. Assume that for $m=1,\dots,M$ $\mu^{m,j}=\mu_1\ \forall\, j$ and we will try to estimate M.

3^{rd} claim

$$|\varphi_n^m|\le C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\quad in\ \Omega,\ \forall\ n$$

for some C, $\gamma>0$.

Proof of 3^{rd} claim: Let $L_n=-\Delta+a_n(x)$, where $a_n=\lambda_n\,V-p\,u_n^{p-1}-\mu_n^m$. As in the proof of 1^{st} claim, we have that $a_n(x)\ge\lambda_n\,V(x)-p\,u_n^{p-1}(x)\ge\delta\,(\varepsilon_n^1)^{-2}>0$, $\delta>0$, for $x\in\Omega\setminus\bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)$ and M large.
In view of (2.1.10) we have that

$$L_n\Big(\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\Big)=\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\Big[-\frac{\gamma^2}{(\varepsilon_n^j)^2}+\frac{N-1}{|x-P_n^j|}\frac{\gamma}{\varepsilon_n^j}+a_n\Big]$$
$$\ge\ \sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}(\varepsilon_n^j)^{-2}\Big[-\gamma^2+\delta\Big]>0$$

in $\Omega\setminus\bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)$, for $0<\gamma<\sqrt{\delta}$. Since for $C\ge e^{\gamma\,M}$

$$(2.2.16)\qquad \begin{aligned}|\varphi_n^m|\ &\le\ 1\le C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\quad\text{on}\ \bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)\\ |\varphi_n^m|\ &=\ 0\le C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\quad\text{on}\ \partial\Omega,\end{aligned}$$

by the maximum principle

$$L_n(\pm\varphi_n^m)=0\le L_n\Big(C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\Big)\ \text{ in }\ \Omega\setminus\bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j)$$

implies that

$$|\varphi_n^m|\le C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\quad\text{in}\ \Omega\setminus\bigcup_{j=1}^k B_{M\,\varepsilon_n^j}(P_n^j).$$

Hence, by (2.2.16) we get that

$$|\varphi_n^m|\le C\sum_{j=1}^k e^{-\gamma\frac{|x-P_n^j|}{\varepsilon_n^j}}\quad\text{in}\ \Omega$$

for n large. Up to take C larger, the estimate is true $\forall\, n$. $\square$

For $m, l \in \{1, \dots, M\}$, $m \neq l$, we want to take the limit of the orthogonality condition:

$$\begin{aligned} 0 &= \int_\Omega \varphi_n^m \varphi_n^l = \sum_{j=1}^k \int_{B_{M\varepsilon_n^j}(P_n^j)} \varphi_n^m \varphi_n^l + \int_{\Omega \setminus \bigcup_{j=1}^k B_{M\varepsilon_n^j}(P_n^j)} \varphi_n^m \varphi_n^l \\ &= \sum_{j=1}^k (\varepsilon_n^j)^N \int_{B_M(0)} \phi_n^{m,j} \phi_n^{l,j} + \int_{\Omega \setminus \bigcup_{j=1}^k B_{M\varepsilon_n^j}(P_n^j)} \varphi_n^m \varphi_n^l. \end{aligned}$$

We have that

$$\begin{aligned} \Big| \int_{\Omega \setminus \bigcup_{j=1}^k B_{M\varepsilon_n^j}(P_n^j)} \varphi_n^m \varphi_n^l \Big| &\leq C' \sum_{i=1}^k \int_{\Omega \setminus \bigcup_{j=1}^k B_{M\varepsilon_n^j}(P_n^j)} e^{-2\gamma \frac{|x-P_n^i|}{\varepsilon_n^i}} \\ &\leq C' \sum_{i=1}^k (\varepsilon_n^i)^N \int_{\mathbb{R}^N \setminus B_M(0)} e^{-2\gamma |y|} \mathrm{d}y. \end{aligned}$$

Since $\phi_n^{m,j} \rightharpoonup \phi^{m,j}$ and $\phi_n^{l,j} \rightharpoonup \phi^{l,j}$ in $H^1(B_M(0))$, we have that $\phi_n^{m,j} \to \phi^{m,j}$ and $\phi_n^{l,j} \to \phi^{l,j}$ in $L^2(B_M(0))\ \forall\, M$. By $(\varepsilon_n^j)^2 \mu_n^m \to \mu^{m,j} = \mu_1 \quad \forall\, j$, we get $\frac{\varepsilon_n^j}{\varepsilon_n^i} \to 1 \quad \forall\, i \neq j$ as $n \to +\infty$. Finally, by

$$0 = \frac{1}{(\varepsilon_n^1)^N} \int_\Omega \varphi_n^m \varphi_n^l = \sum_{j=1}^k \Big(\frac{\varepsilon_n^j}{\varepsilon_n^1}\Big)^N \int_{B_M(0)} \phi_n^{m,j} \phi_n^{l,j} + O\Big(\int_{\mathbb{R}^N \setminus B_M(0)} e^{-2\gamma|y|} \mathrm{d}y \Big)$$

we get that as $n \to +\infty$:

$$0 = \sum_{j=1}^k \int_{B_M(0)} \phi^{m,j} \phi^{l,j} + O\Big(\int_{\mathbb{R}^N \setminus B_M(0)} e^{-2\gamma|y|} \mathrm{d}y \Big) \quad \forall\, M.$$

As $M \to +\infty$ we get

$$0 = \sum_{j=1}^k \int_{\mathbb{R}^N} \phi^{m,j} \phi^{l,j}.$$

Since $\phi^{m,j}$ and $\phi^{l,j}$ are eigenfunctions of $-\Delta + \tilde{\lambda} - p\, U^{p-1}$ with eigenvalue $\mu^{m,j} = \mu^{l,j} = \mu_1 < 0$, we can write

$$\phi^{m,j} = \frac{\lambda_{m,j}}{(\int_{\mathbb{R}^N} \psi_i^2)^{\frac{1}{2}}} \psi_i, \quad \phi^{l,j} = \frac{\lambda_{l,j}}{(\int_{\mathbb{R}^N} \psi_i^2)^{\frac{1}{2}}} \psi_i.$$

We prove that $\phi^{m,j}$ and $\phi^{l,j} \in L^2$, i.e. that the eigenfunctions of eigenvalues less or equal to zero of the operator $-\Delta + 1 - pU^{p-1}$ are in L^2. If $\phi^{m,j}$ are eigenfunctions of eigenvalue zero, we say that U is a unique solution which is non-degenerate, i.e.

$$\mathrm{kernel}\,(-\Delta + 2 + pU^{p-1}) = \mathrm{span}\{\partial_1 U, \dots, \partial_N U\},$$

with $\partial_i U = \partial_{x_i} U$.
By the well-known result of Gidas, Ni and Nirenberg [**57**], U is radially symmetric: $U(y) =$

$U(|y|)$ and strictly decreasing: $U'(r) < 0$ for $r > 0$, $r = |y|$.
Moreover, we have the following asymptotic behavior of U:

$$U(r) = A_N\, r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})),$$

$$U'(r) = -A_N\, r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})),$$

for r large , where $A_N > 0$ is a generic constant.
We have that $U, |\nabla U|$ have an exponential decay, so also $\partial_i U$ for all $i = 1, \dots, N$. Observe that, in this case, $\phi^{m,j}, \phi^{l,j}$ are $\partial_i U$, so have an exponential decay and then they are in L^2.
Consider the case of $\mu < 0$, the case of negative eigenvalue. First we have to prove that there exists a unique eigenfunction associated to μ. Observe that U, the unique solution obtained by Kwong, must be obtained also like Mountain Pass solution.
By [**62**] we state that the Morse Index of a ground state solution is always less than 1. We have that there exists a negativity direction of the linearized operator, then we have that the first eigenvalue is negative.
Moreover we have that $m(U)$ is at least one, with one negative eigenvalue and so there is one and only one eigenvalue strictly negative μ and the others are less or equal to zero. There is one and only one eigenfunction associated to $\mu < 0$ (1-dimensional space) by the property of the first eigenvalue.
Then we can prove that this eigenfunction is in L^2. Observe that we can find the first eigenfunction like the minimum of a functional. Observe that the minimum of this functional is attained in L^2, then the eigenfunction is in L^2. In this way we get

$$0 = \sum_{j=1}^{k} \lambda_{m,j}\, \lambda_{l,j}.$$

Set $\lambda_m = (\lambda_{m,1}, \dots, \lambda_{m,k})\ \forall\ m = 1, \dots, M$. By the 2^{nd} claim we have that $\lambda_m \neq 0\ \forall m = 1, \dots, M$ and $< \lambda_m, \lambda_l >= 0\ \forall m \neq l,\ m, l = 1, \dots, M$. Hence $M \leq k$. Assume that $\mu_n^{m,j} \to 0$ (for every $j = 1, \dots, k$) for $m = M+1, \dots, M+s$. By (2.1.10) let us assume that $\frac{\varepsilon_n^j}{\varepsilon_n^1} \to D_j^{\frac{2}{N}} > 0$ $\forall\, j = 1, \dots, k$. We want to show that $s \leq N\, k$.
Indeed, we have always that

$$\sum_{j=1}^{k} D_j^2 \int_{\mathbb{R}^N} \phi^{m,j}\, \phi^{l,j} = 0,\ \ \forall\, m \neq l,\ \ m, l = M+1, \dots, M+S.$$

Since $\phi^{m,j}$ and $\phi^{l,j}$ are eigenfunctions of $-\Delta + \tilde{\lambda} - p\, U^{p-1}$ with eigenvalues $\mu^{m,j} = \mu^{l,j} = 0$, we can write

$$\begin{aligned} \phi^{m,j} &= \sum_{i=2}^{N+1} \frac{\beta_{m,j}^i\, \psi_i}{D_j(\int_{\mathbb{R}^N} \psi_i^2)^{\frac{1}{2}}} \\ \phi^{l,j} &= \sum_{i=2}^{N+1} \frac{\beta_{l,j}^i\, \psi_i}{D_j(\int_{\mathbb{R}^N} \psi_i^2)^{\frac{1}{2}}}. \end{aligned}$$

In this way, the orthogonality condition rewrites as

$$\sum_{j=1}^{k}\sum_{i=2}^{N+1}\beta^i_{m,j}\,\beta^i_{l,j}=0$$

in view of $\int_{\mathbb{R}^N}\psi_i\,\psi_j=0$ for $i\neq j$. We consider $\beta_m=(\beta^2_{m,1},\beta^2_{m,2},\ldots,\beta^2_{m,k},\ldots,\beta^{N+1}_{m,1},\ldots,\beta^{N+1}_{m,k})$. We have that $\beta_m\neq 0\ \forall m=M+1,\ldots,M+S$ and $<\beta_m,\beta_l>=0\ \forall m\neq l,\ m,l=M+1,\ldots,M+S$. Hence $S\leq Nk$.
Observe that $\hat N<+\infty$. Indeed, for $\hat N=+\infty$ we would have that $\mu^m_n\leq 0$ for $m=1,\ldots,(N+1)\,k+1$ and n large. The corresponding limits $\mu^{m,j}$ have to satisfy:

$$\mu^{m,j}=\mu_j\quad(\forall\ j=1,\ldots,k)\quad\text{for every } m=1,\ldots,M,$$

$$\mu^{m,j}=0\quad(\forall\ j=1,\ldots,k)\quad\text{for every } m=M+1,\ldots,(N+1)\,k+1,,$$

for some $M\in[1,(N+1)\,k+1]$. Hence, we should have $M\leq k$, $(N+1)\,k+1-M\leq N\,k$ and then $M+s=(N+1)\,k+1\leq(N+1)\,k$. A contradiction.
Also we have that

$$\lim_{n\to+\infty}(u_n)=\hat N\leq(N+1)\,k,$$

where by Theorem 2.4

$$k\leq\tilde\lambda^{\frac{p+1}{p-1}-\frac{N}{2}}(\min V)^{\frac{N}{2}-\frac{p+1}{p-1}}\Big(\int_{\mathbb{R}^N}U^{p+1}\Big)^{-1}\lim_{n\to+\infty}\tilde\lambda^{\frac{N}{2}-\frac{p+1}{p-1}}\int_\Omega u_n^{p+1}.$$

The Theorem is estabilished. □

2.3. Blow up profile: $p=\frac{N+2}{N-2}$

In this section we adapt some results of Druet, Hebey and Robert (see [**47**]) for the case of $p=\frac{N+2}{N-2}$ to have a blow-up profile. For simplicity we can take $V\equiv 1$ (everything holds with $V\neq 1$ too).
Let u_n be a sequence of positive solutions of

$$\begin{cases}-\Delta u_n+\lambda_n u_n=u_n^{\frac{N+2}{N-2}} & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega\end{cases}\tag{2.3.1}$$

with $\limsup_n m(u_n)=\overline{k}$.

2.3.1. Exhaustion of blow-up points.

Given $k\in\mathbb{N}^*$, let (P^i_n), $i=1,\ldots,k$, be k converging sequences of points in Ω, and (ε^i_n), $i=1,\ldots,k$, be k sequences of positive numbers converging to 0. We set

$$S:=\Big\{\lim_{n\to+\infty}P^i_n,\ i=1,\ldots,k\Big\}\tag{2.3.2}$$

and when $k\geq 2$ we set

$$S_i:=\Big\{\lim_{n\to+\infty}\frac{1}{\varepsilon^i_n}(P^j_n-P^i_n)\ j\neq i\Big\}\tag{2.3.3}$$

for $i = 1, \dots k$, where the limits, up to a subsequence, are assumed to exist.

$$U_n^i(x) = (\varepsilon_n^i)^{\frac{N}{2}-1} u_n(\varepsilon_n^i x + P_n^i). \tag{2.3.4}$$

Let U_0^1 be the function defined in $\mathbb{R}^N$ as

$$U_0^1(x) = \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{1-\frac{N}{2}}. \tag{2.3.5}$$

It is known [**22, 57, 90**] that U_0^1 is the only positive solution of the equation $-\Delta u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$ satisfying $U_0^1(0) = \max_{\mathbb{R}^N} U_0^1 = 1$.
When $k \geq 2$ we consider the following statement:

$$\frac{|P_n^j - P_n^i|}{\min(\varepsilon_n^i, \varepsilon_n^j)} \to +\infty \tag{2.3.6}$$

for all $i \neq j$ as $n \to +\infty$. We also define

$$d_n^k(x) = \min_{i=1,\dots,k} |x - P_n^i|. \tag{2.3.7}$$

By Proposition 0.1 that

$$\lim_{n \to +\infty} \max_{\Omega} u_n = +\infty. \tag{2.3.8}$$

Then the following theorem holds:

THEOREM 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$ and u_n be a sequence of positive solutions to (2.3.1). Assume that $\sup_n m(u_n) < +\infty$. Then there exist $k \in \mathbb{N}^*$, converging sequences (P_n^i) in $\overline{\Omega}$ and sequences (ε_n^i) of positive real numbers converging to 0, $i = 1, \dots, k$, such that (2.3.6) holds and such that, up to a subsequence, the following properties hold:*

(1) *For any $x \in \Omega$ and any n*

$$d_n^k(x)^{\frac{N}{2}-1} u_n(x) \leq C \tag{2.3.9}$$

for some $C > 0$ where $d_n^k(x)$ is given by (2.3.7). Moreover

$$\lim_{R \to +\infty} \lim_{n \to +\infty} \sup_{x \in \Omega \setminus \cup_{i=1}^k B_{R\varepsilon_n^i}(P_n^i)} d_n^k(x)^{\frac{N}{2}-1} u_n(x) = 0. \tag{2.3.10}$$

(2) *$u_n \to 0$ strongly in $C^0_{loc}(\overline{\Omega} \setminus S)$ as $n \to +\infty$, where S is given by (2.3.2).*

PROOF. The proof of (1) proceeds in several two steps. The claims below are all up to a subsequence.

1st step *There exists a converging sequence (P_n^1) of points in Ω and a sequence $\varepsilon_n^1 \to 0$ such that*

$$U_n^1 \to U_0^1 \ \text{ in } \ C^2_{loc}(\mathbb{R}^N) \tag{2.3.11}$$

holds as $n \to +\infty$, where $U_n^1(x) = (\varepsilon_n^1)^{\frac{N}{2}-1} u_n(\varepsilon_n^1 x + P_n^1)$.

Let P_n^1 be a point in Ω where u_n achieves its maximum and set

$$u_n(P_n^1) = (\varepsilon_n^1)^{1-\frac{N}{2}} = \max_{\Omega} u_n. \tag{2.3.12}$$

We have

$$U_n^1(0) = (\varepsilon_n^1)^{\frac{N}{2}-1} u_n(P_n^1) = 1. \tag{2.3.13}$$

From (2.3.8) we have that $\varepsilon_n^1 \to 0$ as $n \to +\infty$. Moreover U_n^1 satisfies

$$\begin{cases} -\Delta U_n^1 + \lambda_n (\varepsilon_n^1)^2 U_n^1 = (U_n^1)^{\frac{N+2}{N-2}} & \text{in } \Omega_n := \frac{\Omega - P_n^1}{\varepsilon_n^1} \\ 0 < U_n^1 \le U_n^1(0) = 1 & \text{in}\, \Omega_n \\ U_n^1 = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{2.3.14}$$

Since P_n^1 is a point of local maximum of u_n we have that

$$0 \le \lambda_n (\varepsilon_n^1)^2 \le 1,$$

and, up to a subsequence, we can assume

$$\lambda_n (\varepsilon_n^1)^2 \to \tilde\lambda \in [0,1] \text{ as } n \to +\infty.$$

Since U_n^1 is bounded, by regularity theory we have

$$U_n^1 \to U \quad \text{in} \quad C^2_{loc}(\overline{T}) \tag{2.3.15}$$

where U satisfies

$$\begin{cases} -\Delta U + \tilde\lambda U = U^{\frac{N+2}{N-2}} & \text{in } T \\ 0 < U \le U(0) = 1 & \text{in}\, T \\ U = 0 & \text{on}\, \partial T \end{cases} \tag{2.3.16}$$

with $m(U) < +\infty$ and $T = \lim_{n\to+\infty} \Omega_n$. Note that T is an hyperplane (without loss of generality $\mathbb{R}^N_+$) or $\mathbb{R}^N$. Observe that, if $\tilde\lambda > 0$ by Theorems 1.10 and 1.1.76 we have no solutions of the problem neither in $\mathbb{R}^N$ and in $\mathbb{R}^N_+$. If $\tilde\lambda = 0$ then by Theorem 1.21 there are no solutions in $\mathbb{R}^N_+$. By [**22, 57, 90**] $U = U_0^1$ solves the limiting problem

$$\begin{cases} -\Delta U = U^{\frac{N+2}{N-2}} & \text{in}\, \mathbb{R}^N \\ 0 < U \le 1. \end{cases} \tag{2.3.17}$$

In particular $\lambda_n\, \varepsilon_n^2 \to 0$ and $\frac{\text{dist}(P_n^1, \partial\Omega)}{\varepsilon_n^1} \to +\infty$ as $n \to +\infty$.

Let $m \in \mathbb{N} \setminus \{0\}$ and for $i = 1, \dots, m$, let (P_n^i) be m converging sequences of points in Ω and (ε_n^i) be m sequences of positive real numbers converging to 0. Note that U_0^1 is unstable, so there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ so that supp$\phi \subset B_R(0)$, $R > 0$ and

$$\int |\nabla\phi|^2 - \frac{N+2}{N-2} (U_0^1)^{\frac{4}{N-2}}\, \phi^2 < 0.$$

Then, we define $\phi_n^1(x) := \frac{1}{(\varepsilon_n^1)^{\frac{N-2}{2}}}\,\phi\Big(\frac{x-P_n^1}{\varepsilon_n^1}\Big)$, so that $\operatorname{supp}\phi_n^1 \subset B_{R\,\varepsilon_n^1}(P_n^1)$ and

$$\begin{aligned}\int_\Omega |\nabla\phi_n|^2 + (\lambda_n - \frac{N+2}{N-2}\,u_n^{\frac{4}{N-2}})\phi_n^2 \quad &= \quad \int |\nabla\phi|^2 + (\lambda_n\,(\varepsilon_n^1)^2 - \frac{N+2}{N-2}\,U_n^{\frac{4}{N-2}})\phi^2 \\ &\to \quad \int |\nabla\phi|^2 - \frac{N+2}{N-2}\,(U_0^1)^{\frac{4}{N-2}}\phi^2 < 0.\end{aligned}$$

We say that (H_m^1) holds if there exist m converging sequences (P_n^i) of points in Ω and m sequences (ε_n^i) of positive real numbers converging to 0 such that, up to a subsequence, the following assertion holds:

(1) when $m \geq 2$, (P_n^i) and (ε_n^i) satisfy

$$\frac{|P_n^j - P_n^i|}{\varepsilon_n^i} \to +\infty \quad \text{as} \quad n \to +\infty, \quad \text{for any} \quad j \neq i, \tag{2.3.18}$$

(2) for any $i \in \{\,1,\dots,m\,\}$, $U_n^i \to U_0^1$ in $C^2_{loc}(\mathbb{R}^N)$ as $n \to +\infty$.

By $1^{\mathbf{st}}$ step we know that (H_1^1) holds.
We claim:

$2^{\mathbf{nd}}$ step *Assume that (H_m^1) holds, then (H_{m+1}^1) holds or $d_n^m(x)^{\frac{N}{2}-1}u_n(x) \leq C$ for all $x \in \Omega$ and all n, where d_n^m is given by (2.3.7) and $C > 0$ is independent of x and n.*

We prove this $2^{\mathbf{nd}}$ step. Suppose that (H_m^1) holds. We assume that

$$\max_{x\in\Omega} d_n^m(x)^{\frac{N}{2}-1}u_n(x) \to +\infty \tag{2.3.19}$$

as $n \to +\infty$, so we aim to prove that (H_{m+1}^1) holds. We consider $P_n^{m+1} \in \Omega$ such that

$$\max_{x\in\Omega} d_n^m(x)^{\frac{N}{2}-1}u_n(x) = d_n^m(P_n^{m+1})^{\frac{N}{2}-1}u_n(P_n^{m+1}) = (\min_{i=1,\dots,m} |P_n^{m+1} - P_n^i|)^{\frac{N}{2}-1}u_n(P_n^{m+1}) \tag{2.3.20}$$

and we set $u_n(P_n^{m+1}) = (\varepsilon_n^{m+1})^{1-\frac{N}{2}}$. We have that $\varepsilon_n^{m+1} \to 0$ as $n \to +\infty$. It follows from (2.3.19) that for any $i \in \{\,1,\dots,m\,\}$

$$\frac{|P_n^{m+1} - P_n^i|}{\varepsilon_n^{m+1}} \to +\infty \quad \text{as} \;\; n \to +\infty.$$

We have also that $\frac{|P_n^{m+1}-P_n^i|}{\varepsilon_n^i} \to +\infty$ as $n \to +\infty$ $\forall\, i = 1,\dots,m$. Otherwise, we find $i \in \{\,1,\dots,M\,\}$ and M s.t. $P_n^{m+1} \in B_{M\,\varepsilon_n^i}(P_n^i)$. Then

$$d_n^m(P_n^{m+1})^{\frac{N}{2}-1}u_n(P_n^{m+1}) \leq M^{\frac{N}{2}-1}\,U_n^i\Big(\frac{P_n^{m+1} - P_n^i}{\varepsilon_n^i}\Big) \leq 2\,M^{\frac{N}{2}-1}\sup_{|y|\leq M} U_0^1(y)$$

contradicting (2.3.19). Letting $0 < \delta < 1$, for $y \in B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0)$ define

$$U_n^{m+1}(y) = (\varepsilon_n^{m+1})^{\frac{N}{2}-1}u_n(\varepsilon_n^{m+1}\,y + P_n^{m+1}).$$

We have that $U_n^{m+1}(0) = 1$ and U_n^{m+1} satisfies

$$-\Delta U_n^{m+1} + \lambda_n\, (\varepsilon_n^{m+1})^2 U_n^{m+1} = (U_n^{m+1})^{\frac{N+2}{N-2}} \quad \text{in} \ \ B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0).$$

For $y \in B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0)$, and $i \in \{\, 1, \ldots, m\,\}$, by the estimate

$$|P_n^{m+1} - P_n^i| - \, \varepsilon_n^{m+1}\, |y| \le |(\varepsilon_n^{m+1}\, y + P_n^{m+1}) - P_n^i| \le |P_n^{m+1} - P_n^i| + \, \varepsilon_n^{m+1}\, |y|$$

and by (2.3.19) we deduce

$$\lim_{n\to+\infty} \frac{|(\, \varepsilon_n^{m+1}\, y + P_n^{m+1}) - P_n^i|}{|P_n^{m+1} - P_n^i, |} = 1.$$

This implies that on $B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0)$, then holds:

$$U_n^{m+1}(y) = \frac{u_n(\varepsilon_n^{m+1}\ y + \ P_n^{m+1})}{u_n(P_n^{m+1})} \le \Big(\frac{d_n^m(P_n^{m+1})}{d_n^m(\varepsilon_n^{m+1}\ y + \ P_n^{m+1})} u_n(P_n^{m+1})\Big)^{\frac{N}{2}-1} \to 1 \ \ \text{as } n \to +\infty.$$

Arguing as before, up to a subsequence we have:

$$\lim_{n\to+\infty} U_n^{m+1} = U^{m+1} \quad \text{in} \ \ C^2_{loc}(\mathbb{R}^N),$$

where $U^{m+1}(y) \le U^{m+1}(0) = 1$ and $-\Delta U^{m+1} = (U^{m+1})^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$. So we have that $U^{m+1} = U_0^1$ proving (H^1_{m+1}). Set now $m_0 = \sup\{\, m \in \mathbb{N} \, : (H^1_m)$ holds for a subsequence$\,\}$ and we have $m_0 < +\infty$. Indeed, whenever (H^1_m) holds we can find a function $\phi_n^i(x) = \frac{1}{\varepsilon_n^i}^{\frac{N-2}{2}} \phi\Big(\frac{x-P_n^i}{\varepsilon_n^i}\Big)$ so that $\operatorname{supp} \phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$ and

$$\int |\phi_n^i|^2 + (\lambda_n - \frac{N+2}{N-2} u_n^{\frac{4}{N-2}})(\phi_n^i)^2 < 0,$$

for every $i = 1, \ldots, m$. By (2.3.6) the functions ϕ_n^i have disjoint supports, and are therefore orthogonal in $L^2(\Omega)$. This implies $m_0 \le \sup_n \, m(u_n) < +\infty$.

By the **2$^{\text{nd}}$** step, we get that there exist $m \ge 1$, converging sequences (P_n^i) of points in Ω and m sequences (ε_n^i) of positive real numbers converging to 0 such that (H^1_m) holds and such that

$$d_n^m(x)^{\frac{N}{2}-1} u_n(x) \le C \quad \forall\, x \in \Omega, \ \forall\, n \in \mathbb{N}, \tag{2.3.21}$$

for some $C > 0$. The proof of (1) is complete.
As far as (2), let $s = \{\, \lim_{n\to+\infty} P_n^i \, : \, i = 1, \ldots, m\,\}$. We have that $u_n \to 0$ uniformly in $\overline{\Omega} \setminus S$. If not, up to a subsequence, we can find $P_n \in \Omega$ s.t. $P_n \to P \in \overline{\Omega} \setminus S$ and $u_n(P_n) \ge \delta_0 > 0$. Let $r > 0$ small so that $\overline{B_{3\,r}(P)} \subset S^C$.
We consider a cut-off function χ so that $\chi = 1$ in $B_{2\,r}(P)$, $\chi = 0$ in $B_{3\,r}(P)^C$ and $0 \le \chi \le 1$. We multiply (2.3.1) by $\chi^2\, u_n$ and integrate in Ω to get:

$$\begin{aligned} \int_\Omega |\nabla(\chi\, u_n)|^2 + \lambda_n \int_\Omega (\chi\, u_n)^2 \ &= \ \int_\Omega [\nabla\, u_n \ \nabla(\chi^2\, u_n) + \lambda_n\, u_n(\chi^2\, u_n)] + \int_\Omega u_n^2\, |\nabla\chi|^2 \\ &= \ \int_\Omega (\chi^2\, u_n^{\frac{2\,N}{N-2}} + |\nabla\chi|^2\, u_n^2). \end{aligned}$$

By (2.3.9) we get that $u_n \le C$ in $\overline{B_{3r}(P)\cap\Omega}$, and then $\chi\, u_n$ is uniformly bounded in $H^1_0(\Omega)$. Then u_n is uniformly bounded in $H^1(B_{2r}(P)\cap\Omega)$ with $\lambda_n \int_{B_{2r}(P)\cap\Omega} u_n^2 \le C$. Up to a subsequence, by the Sobolev embedding Theorem $u_n \rightharpoonup u$ in $H^1(B_{2r}(P)\cap\Omega)$ and strongly in $L^2(B_{2r}(P)\cap\Omega)$. Since $\lambda_n \int_{B_{2r}(P)\cap\Omega} u_n^2 \le C$ and $\lambda_n \to +\infty$, we get that $u \equiv 0$. Up to a subsequence, we have that $u_n \to 0$ a. e. in $B_{2r}(P)\cap\Omega$. By the Lebesgue Theorem and $u_n \le C$ in $B_{2r}(P)\cap\Omega$, we get that $u_n \to 0$ in $L^p(B_{2r}(P)\cap\Omega)$, $\forall\, p \ge 1$, as $n \to +\infty$ and

$$u_n \to 0 \;\text{ in } L^p(\partial B_{2r}(P)\cap\overline{\Omega}),\;\; \forall\, p\ge 1,\;\text{ as } n\to+\infty. \tag{2.3.22}$$

Let v_n be the solution of

$$\begin{cases} -\Delta v_n = u_n^{\frac{N+2}{N-2}} & \text{in } B_{2r}(P)\cap\Omega \\ v_n = u_n & \text{on } \partial(B_{2r}(P)\cap\Omega). \end{cases} \tag{2.3.23}$$

Letting $G(x,y)$ be the Green function of $-\Delta$ in $H^1_0(B_{2r}(P)\cap\Omega)$, by the representation formula: $\forall\, x \in B_{2r}(P)\cap\Omega$ there holds

$$v_n(x) = \int_{B_{2r}(P)\cap\Omega} u_n^{\frac{N+2}{N-2}}(y)\, G(x,y)\mathrm{d}y - \int_{\partial B_{2r}(P)\cap\Omega} u_n^{\frac{N+2}{N-2}}(y)\, \partial_\nu\, G(x,y)\mathrm{d}y,$$

in view of $u_n = 0$ on $\partial\Omega \cap B_{2r}(P)$.
By (1) and (2) we get that $v_n \to 0$ uniformly in $B_r(P)\cap\Omega$. Since $-\Delta u_n = u_n^{\frac{N+2}{N-2}} - \lambda_n\, u_n \le u_n^{\frac{N+2}{N-2}} = -\Delta v_n$ in $B_{2r}(P)\cap\Omega$, by the maximum principle we get that $0 \le u_n \le v_n$ in $B_{2r}(P)\cap\Omega$. Hence, $u_n \to 0$ uniformly in $B_r(P)\cap\Omega$ and in particular $u_n(P_n) \to 0$ as $n \to +\infty$.
This contradicts $u_n(P_n) \ge \delta_0 > 0$.

We say that (H^2_m) holds if, up to a subsequence there exist m converging sequences (P^i_n) of points in Ω and m sequences (ε^i_n) of positive real numbers converging to 0 such that

(1) when $m \ge 2$, (P^i_n) and (ε^i_n) satisfy

$$\frac{|P^i_n - P^j_n|}{\min\{\,\varepsilon^i_n, \varepsilon^j_n\,\}} \to +\infty \quad \text{as} \quad n \to +\infty, \quad \text{for any} \quad j \neq i; \tag{2.3.24}$$

(2) for any $x \in \Omega$ and any n, $d^m_n(x)^{\frac{N}{2}-1} u_n(x) \le C$ for some $C > 0$;
(3) there exists $\varepsilon_0 > 0$ and $P^i \in \mathbb{R}^N$, $i = 1, \dots, m$ such that for any $i = 1, \dots, m$ $U^i_n \to U^1_0(\cdot\, - P^i)$ in $C^2_{loc}(\mathbb{R}^N \setminus S_i)$.

We have seen that (H^2_m) holds for a suitable $m_0 \in \mathbb{N}\setminus\{\,0\,\}$. Note that (2) of $(H^1_{m_0})$ implies (3) of $(H^2_{m_0})$ by simply taking $P^i = 0\;\forall\, i$. We now prove $3^{\textbf{rd}}$ step:

$3^{\textbf{rd}}$ step *Assume that (H^2_m) holds, then (H^2_{m+1}) holds or*

$$\lim_{R\to+\infty}\limsup_{n\to+\infty} \sup_{x\in\Omega\setminus\cup_{i=1}^m B_{R\,\varepsilon^i_n}(P^i_m)} d^m_n(x)^{\frac{N}{2}-1} u_n(x) = 0. \tag{2.3.25}$$

Assume that (2.3.25) is false. Then, up to a subsequence, there exists a sequence P_n^{m+1} of points in Ω such that for any $i = 1, \dots, m$

$$\frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^i} \to_{n\to+\infty} +\infty \tag{2.3.26}$$

and such that

$$d_n^m(P_n^{m+1})^{\frac{N}{2}-1} u_n(P_n^{m+1})) \geq \delta_0 \tag{2.3.27}$$

for some $\delta_0 > 0$. So we need to prove that (H_{m+1}^2) holds. We let $u_n(P_n^{m+1}) = (\varepsilon_n^{m+1})^{1-\frac{N}{2}}$ and claim that (H_{m+1}^2) holds when adding (P_n^{m+1}) and (ε_n^{m+1}) to the (P_n^i)'s and (ε_n^i)'s $i = 1, \dots, m$. We observe that (1) of (H_{m+1}^2) is a consequence of (2.3.26). We observe that $d_n^m(P_n^{m+1}) \to 0$ as $n \to +\infty$. Otherwise, for a subsequence, we can assume that $P_n^{m+1} \to P$ with $d_n^m(P) \geq \delta_0 > 0$. By (2) of (H_m^2) we get that $u_n \leq C$ in $B_\varepsilon(P)$, for some $\varepsilon > 0$ small, and then $B_\varepsilon(P) \subset \Omega \setminus S$. By (1) of Theorem (2.9) $u_n \to 0$ uniformly in $B_\varepsilon(P)$, and in particular $u_n(P_n^{m+1}) \to 0$, contradicting (2.3.27). Since $d_n(P_n^{m+1}) \to 0$, by (2.3.27) we have that $\varepsilon_n^{m+1} \to 0$ as $n \to +\infty$. Given $0 < \delta < 1$, for $y \in B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0)$ we set $U_n^{m+1}(y) = (\varepsilon_n^{m+1})^{\frac{N}{2}-1} u_n(\varepsilon_n^{m+1}\, x + P_n^{m+1})$. We have that $U_n^{m+1}(0) = 1$ and

$$-\Delta U_n^{m+1} + \lambda_n\, (\varepsilon_n^{m+1})^2 U_n^{m+1} = (U_n^{m+1})^{\frac{N+2}{N-2}} \quad \text{in} \quad B_{\frac{\delta}{\varepsilon_n^{m+1}}}(0). \tag{2.3.28}$$

We let

$$S_{m+1} = \left\{ \lim_{n\to+\infty} \frac{P_n^i - P_n^{m+1}}{\varepsilon_n^{m+1}}, \quad 1 \leq i \leq m \right\}$$

where, up to a subsequence, the limits are ok. Note that by (2.3.27) $0 \notin S_{m+1}$.
Let $R > 0$ and (P_n) a sequence in $B_R(0)$ such that $d(P_n, S_{m+1}) \geq \frac{1}{R}$, then

$$d_n^m(\varepsilon_n^{m+1}\, P_n +\, P_n^{m+1}) \geq \frac{\varepsilon_n^{m+1}}{2\,R}$$

for n sufficiently large. Letting $y_n = \varepsilon_n^{m+1}\, P_n +\, P_n^{m+1}$ it follows from (2) of (H_m^2) that

$$\begin{aligned} U_n^{m+1}(P_n) &= (\varepsilon_n^{m+1})^{\frac{N}{2}-1}\, u_n(\varepsilon_n^{m+1}\, P_n +\, P_n^{m+1}) \\ &\leq (2\,R)^{\frac{N}{2}-1} d_n^m(y_n)^{\frac{N}{2}-1} u_n(y_n) \leq (2\,R)^{\frac{N}{2}-1}\, C. \end{aligned}$$

Then, for any $K \subset\subset \mathbb{R}^N \backslash S_{m+1}$ there exists $C_K > 0$, independent of n such that $\|U_n^{m+1}\|_{L^\infty(K)} \leq C_K$.
As in $1^{\mathbf{st}}$ step and $2^{\mathbf{nd}}$ step we have, up to a subsequence,

$$\lim_{n\to+\infty} U_n^{m+1} = U^{m+1} \quad \text{in} \quad C_{loc}^2(\mathbb{R}^N \setminus S_{m+1})$$

where U_{m+1} is such that $U_{m+1}(0) = 1$ and $-\Delta U^{m+1} = (U^{m+1})^{\frac{N+2}{N-2}}$, in $\mathbb{R}^N \setminus S_{m+1}$. Due to **[22]**

$$U^{m+1}(y) = \gamma_{m+1}^{\frac{N-2}{2}}\, U_0^1(\gamma_{m+1}(y - P^{m+1}))$$

where $\gamma_{m+1} > 0$ and $P^{m+1} \in \mathbb{R}^N$ are such that $\gamma_{m+1} = 1 + \frac{\gamma_{m+1}^2\, |P^{m+1}|^2}{N(N-2)}$. Since S_i with respect to (H_m^2) coincides with S_i with respect to (H_{m+1}^2) in view of (2.3.26) Up to changing

ε_n^{m+1} into $(\gamma_{m+1})^{-1}\varepsilon_n^{m+1}$ and P_{m+1} into $(\gamma_{m+1})^{-1}P_{m+1}$, property (3) of (H^2_{m+1}) is proved. Since $\min_{i=1,\dots,m+1}|x-P_n^i| \le \min_{i=1,\dots,m}|x-P_n^i|$ property (2) of (H^2_m) is still true. Since $d_n^m(P_n^{m+1}) \to 0$ note that $\lim_{n\to+\infty} P_n^{m+1} \in S$, where S is the set composed by $\lim_{n\to+\infty} P_n^i$ for $i=1,\dots,m$.

To complete the proof of Theorem 2.9, we need only to show that $\exists\, k\in\mathbb{N}$, $k\ge m_0$, so that (2.3.10) holds. This will follow by $\sup_n m(u_n) < +\infty$ and the following construction. Let $\phi\in C_0^\infty(\mathbb{R}^N)$ so that

$$\int |\nabla\phi|^2 - \frac{N+2}{N-2}(U_0^1)^{\frac{4}{N-2}}\phi^2 < 0. \tag{2.3.29}$$

We now want to show that we can remove a point P: $\exists\tilde\phi \in C_0^\infty(\mathbb{R}^N\setminus\{\,P\,\})$ so that (2.3.29) holds. Given $0<\delta<1$, let χ_δ a cut-off function defined as:

$$\chi_\delta(y) = \begin{cases} 0 & \text{if } |y-P|\le\delta^2 \\ 2-\frac{\log|y-P|}{\log\delta} & \text{if } \delta^2<|y-P|<\delta \\ 1 & \text{if } |y-P|\ge\delta. \end{cases} \tag{2.3.30}$$

The function $\phi_\delta := \chi_\delta\,\phi$ satisfies:

$$\int|\nabla\phi_\delta|^2 - \int|\nabla\phi|^2 = \int|\nabla\chi_\delta|^2\phi^2 + (\chi_\delta^2-1)\phi^2 + O\Big(\int \chi_\delta|\nabla\chi_\delta|\,|\phi|\,|\nabla\phi|\Big)$$
$$= O\Big(\frac{1}{\log^2\delta}\int_{B_\delta(P)\setminus B_{\delta^2}(P)}\frac{\mathrm{d}y}{|y-P|^2} + \delta^N + \frac{1}{|\log\delta|}\int_{B_\delta(P)\setminus B_{\delta^2}(P)}\frac{\mathrm{d}y}{|y-P|^2}\Big) \to 0,$$

and

$$\int(U_0^1)^{\frac{4}{N-2}}\phi_\delta^2 - \int(U_0^1)^{\frac{4}{N-2}}\phi^2 = O\Big(\delta^N\Big)\to 0$$

as $\delta\to0^+$. Then for $\delta>0$ small the function ϕ_δ still satisfies (2.3.29). This the function $\tilde\phi$ we were searching for. The construction can be clearly repeated for many points $P^1,\dots,P^j$: there exists $\phi\in C_0^\infty(\mathbb{R}^N\setminus\{\,P^1,\dots,P^j\,\})$ so that (2.3.29) is valid for ϕ.
Now we proceed by induction. Assume that (H^2_m) holds and we have found $\phi_n^1,\dots,\phi_n^m \in C_n^\infty(\Omega)$ with disjoint supports so that $\mathrm{supp}\phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$ and

$$\int_\Omega |\nabla\phi_n^i|^2 + (\lambda_n - \frac{N+2}{N-2}\,u_n^{\frac{4}{N-2}})(\phi_n^i)^2 < 0,$$

for every $i=1,\dots,m$ and for some $R>0$.
If (2.3.18) holds, we set $k=m$ and we are done. If (2.3.18) doesn't hold, up to a subsequence we can find a point P_n^{m+1} as in 3^{rd} step.
Letting $S_{m+1}\{\,P^1,\dots,P^j\,\}$, we can find $\phi\in C_0^\infty(\mathbb{R}^N\setminus S_{m+1})$ so that (2.3.18) is valid with U_0^1 replaced by $U_0^1(\,\cdot\,,P^{m+1})$. So, there exists $\delta>0$ small so that

$$\mathrm{supp}\,\phi \subset \left(\bigcup_{i=1}^m B_\delta\Big(\frac{P_n^i-P_n^{m+1}}{\varepsilon_n^{m+1}}\Big)\right)^C \cap B_{\frac1\delta}(0)$$

(We have to treat separately the case $\frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^{m+1}} \to S_{m+1}$ and $\frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^{m+1}} \to +\infty$). Define now $\phi_n^{m+1}(x) = \frac{1}{(\varepsilon_n^{m+1})^{\frac{N-2}{2}}}\psi\Big(\frac{x-P_n^{m+1}}{\varepsilon_n^{m+1}}\Big) \in C_0^\infty(\Omega)$. Then $\operatorname{supp}\phi_n^{m+1} \subset B_{\frac{1}{\delta}\varepsilon_n^{m+1}}(P_n^{m+1}) \setminus \bigcup_{i=1}^m B_{\delta\varepsilon_n^{m+1}}(P_n^i)$. If $\frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^{m+1}} \to S_{m+1}$, by (2.3.18) we have that $\varepsilon_n^{m+1} \geq \varepsilon_n^i$ and

$$\frac{\varepsilon_n^i}{\varepsilon_n^{m+1}} = \frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^{m+1}} \frac{\min\{\,\varepsilon_n^i, \varepsilon_n^{m+1}\,\}}{|P_n^i - P_n^{m+1}|} \to 0,$$

which implies $\operatorname{supp}\phi_n^{m+1} \cap \operatorname{supp}\phi_n^i = \phi$ in view of $\operatorname{supp}\phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$.
When $\frac{|P_n^i - P_n^{m+1}|}{\varepsilon_n^{m+1}} \to +\infty$, by (2.3.26) we get $B_{\frac{1}{\delta}\varepsilon_n^{m+1}}(P_n^{m+1}) \cap B_{R\,\varepsilon_n^i}(P_n^i)$ which implies $\operatorname{supp}\phi_n^{m+1} \cap \operatorname{supp}\phi_n^i = \phi$ also in this case.
It is easy to see that

$$\begin{aligned} \int_\Omega |\phi_n^{m+1}|^2 + (\lambda_n - \frac{N+2}{N-2} u_n^{\frac{4}{N-2}})(\phi_n^{m+1})^2 &= \int |\nabla\phi|^2 + (\lambda_n - \frac{N+2}{N-2}(U_n^{m+1})^{\frac{4}{N-2}})(\phi)^2 \\ &\to \int [|\nabla\phi|^2 - \frac{N+2}{N-2}(U_0^1)^{\frac{4}{N-2}}(y - P^{m+1})\phi^2] < 0 \end{aligned}$$

as $n \to +\infty$, in view of $U_n^{m+1} \to_{n\to+\infty} U_0^1(\,\cdot\, - P^{m+1})$ in $C^2_{loc}(\mathbb{R}^N \setminus S_{m+1})$ and $\phi \in C_0^\infty(\mathbb{R}^N \setminus S_{m+1})$. By 3^{rd} step, we know that (H^2_{m+1}) holds for $\{\,P_n^1, \ldots, P_n^{m+1}\,\}$ and we have found $\phi_n^1, \ldots, \phi_n^{m+1} \in C_0^\infty(\Omega)$ with disjoint supports so that $\operatorname{supp}\phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$ and

$$\int_\Omega |\nabla\phi_n^i|^2 + (\lambda_n - \frac{N+2}{N-2} u_n^{\frac{4}{N-2}})(\phi_n^i)^2 < 0$$

for every $i = 1, \ldots, m+1$ and some $R > 0$, unless (2.3.18) holds for m. Since this last property holds for every $m \geq m_0$, and then we get a contradiction:

$$m \leq \sup_n m(u_n) \quad \forall\, m \geq m_0.$$

This show that $\exists\, k \in \mathbb{N}$, $k \geq m_0$, so that (2.3.10) holds and Theorem 2.9 is established. □

2.3.2. Sharp pointwise estimates. We define $P_n^i \in \Omega$ and $\varepsilon_n^i > 0$ by the relations

$$u_n(P_n^i) = \max_\Omega u_n = (\varepsilon^i)_n^{1-\frac{N}{2}}$$

$i = 1\ldots, k$, clearly $P_n^i \to P_0^i$, (P_0^i is a geometrical point of blow-up) and $\varepsilon_n^i \to 0$ as $n \to +\infty$. We know, by Theorem 2.9, that there exists $C > 0$, independent of n, s. t. for any n and any $x \in \Omega$

$$|x - P_n^i|^{\frac{N-2}{2}} u_n(x) \leq C \tag{2.3.31}$$

and that

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \sup_{\Omega \setminus B_{R\varepsilon_n^i}(P_n^i)} |x - P_n^i|^{\frac{N-2}{2}} u_n(x) = 0. \tag{2.3.32}$$

The Standard Bubble $(\varepsilon_n^i)^{-\frac{N-2}{2}}U(\frac{x-P_n^i}{\varepsilon_n^i})$ is given by the expression:

$$U_{P_n^i,\varepsilon_n^i}(x) = \left(\frac{\varepsilon_n^i}{\varepsilon{^i}^2_n + \frac{|x-P_n^i|^2}{N(N-2)}}\right)^{\frac{N-2}{2}}. \tag{2.3.33}$$

We claim that the following sharp estimate holds.

THEOREM 2.10. *Under the above assumption, up to a subsequences, there exists $C > 1$ independent of n, such that for any n and any $x \in \Omega$*

$$u_n(x) \le C \sum_{i=1}^{k} U_{P_n^i + \varepsilon_n^i\, P^i, \varepsilon_n^i}(x) \tag{2.3.34}$$

where P_n and ε_n are as above, i.e. the u_n's are C^0 controlled, by the standard Bubble. Moreover, $\int_\Omega u_n^{\frac{2N}{N-2}} < +\infty$.

REMARK 2.11. Using the analysis done by Druet, Hebey, Robert [**47**], we could be able to obtain the estimate (2.3.34), in our case, for solutions with uniformly bounded Morse indices and $k-$peaks.

PROOF. For simplicity, we explain only the proof for $1-$peak case.
Rather, we proof a weakest estimate i.e. that:

$$u_n \le C \frac{(\varepsilon_n)^{\frac{N-2}{2}-\tilde{\varepsilon}}}{|x-P_n|^{N-2-\tilde{\varepsilon}}}, \tag{2.3.35}$$

in $\Omega \setminus B_{M\,\varepsilon_n}(P_n)$, with $M > 0$, $\tilde{\varepsilon} > 0$ small.

Let u_n be positive solution of (2.3.1) with $sup_n m(u_n) < +\infty$. By Theorem 2.9 we have that

$$u_n \to 0 \quad \text{in} \quad C^0_{loc}(\Omega \setminus \{\, S\}) \tag{2.3.36}$$

as $n \to +\infty$.
We define ε_n by the relation

$$u_n(P_n) = (\varepsilon_n)^{1-\frac{N}{2}}$$

where P_n are local maximum for u_n. We have that

$$\varepsilon_n^i = (u_n(P_n))^{\frac{2}{2-N}}$$

and $\varepsilon_n \to 0$ as $n \to +\infty$. We say that (2.3.31) and (2.3.32) hold, then we have that there exists $C > 0$ independent of n, such that for any n and $x \in \Omega$

$$|P_n - x|^{\frac{N-2}{2}} u_n(x) \le C$$

and

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \sup_{\Omega\setminus B_{R\varepsilon_n}(P_n)} |x-P_n|^{\frac{N-2}{2}} u_n(x) = 0. \tag{2.3.37}$$

Therefore the theorem reduces to the existence of $C > 1$, independent of n, such that for any n and any x:

$$u_n(x) \le C\left(\frac{\varepsilon_n}{(\varepsilon_n)^2 + \frac{|x-P_n|^2}{N(N-2)}}\right)^{\frac{N-2}{2}}, \tag{2.3.38}$$

and for any $\varepsilon > 0$, to the existence of $\delta_\varepsilon > 0$ independent of n such that for any n and any $x \in B_{\delta_\varepsilon}(P_0)$

$$u_n(x) \le C_\varepsilon\left(\frac{\varepsilon_n}{(\varepsilon_n)^2 + \frac{|x-P_n|^2}{N(N-2)}}\right)^{\frac{N-2}{2}}, \tag{2.3.39}$$

where $C_\varepsilon = 1+\varepsilon$. We prove (2.3.38) and (2.3.39) in several steps.
We claim first that there exists $C > 0$ such that, up to a subsequence,

$$u_n(x) \le C\left(\frac{\varepsilon_n}{(\varepsilon_n)^2 + \frac{|x-P_n|^2}{N(N-2)}}\right)^{\frac{N-2}{2}}$$

for all n and $x \in \Omega$, the proof of this claim reduces to the prove that there exists $C > 0$ such that for all n and $x \in \Omega$

$$|P_n - x|^{N-2} u_n(P_n) u_n(x) \le C. \tag{2.3.40}$$

As a first step we prove that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any n and $x \in \Omega$,

$$|P_n - x|^{N-2-\varepsilon} (\varepsilon_n)^{-\frac{N-2}{2}+\varepsilon} u_n(x) \le C_\varepsilon. \tag{2.3.41}$$

We prove (2.3.41), it is suffices to prove this for $\varepsilon > 0$ small. Indeed, let $0 < \varepsilon^1 < \varepsilon^2$, (2.3.41) with respect $\varepsilon = \varepsilon^1, \varepsilon^2$ is true if $|P_n - x| \le \varepsilon_n$. If (2.3.41) is true with respect to ε^1 and $|x - P_n| \ge \varepsilon_n$, then

$$|x-P_n|^{N-2-\varepsilon^2} (\varepsilon_n)^{-\frac{N-2}{2}+\varepsilon^2} u_n(x) \le (|x-P_n|^{-1}\varepsilon_n)^{\varepsilon^2-\varepsilon^1} |P_n - x|^{N-2-\varepsilon^1} (\varepsilon_n)^{-\frac{N-2}{2}+\varepsilon^1} u_n(x)$$

then (2.3.41) with respect ε^2 is true. There exists $\varepsilon > 0$ such that, fix $\varepsilon_0 > 0$ small, we can consider the Green's function of the operator $-\Delta + \frac{1-\varepsilon_0}{1+\varepsilon}$. By the maximum principle, G_ε is positive. We let L_n be the operator

$$L_n u = -\Delta u + \lambda_n u - u_n^{\frac{4}{N-2}} u.$$

Since $L_n u_n = 0$ with $u_n > 0$ on Ω, we get from [**20**] that the maximum principle holds for L_n. We have

$$\frac{L_n {G_\varepsilon}^{1-\varepsilon}(P_n^i, x)}{G_\varepsilon^{1-\varepsilon}(P_n^i, x)} = \varepsilon_0 + \lambda_n + 1 - u_n^{\frac{4}{N-2}}(x) + \varepsilon(1-\varepsilon)\frac{|\nabla G_\varepsilon|^2(P_n, x)}{G_\varepsilon^2(P_n, x)}$$

so that

$$\frac{L_n {G_\varepsilon}^{1-\varepsilon}(P_n, x)}{G_\varepsilon^{1-\varepsilon}(P_n, x)} \ge \varepsilon_0 - u_n^{\frac{4}{N-2}}(x) + \varepsilon(1-\varepsilon)\frac{|\nabla G_\varepsilon|^2(P_n, x)}{G_\varepsilon^2(P_n, x)}. \tag{2.3.42}$$

We use a standard property of the Green's function, i. e. there exists $C>0$ and $\rho>0$ such that, for any n and any $x\in B_\rho(P_n)\setminus\{\,P_n\,\}$,

$$\frac{|\nabla G_\varepsilon|(P_n^i,x)}{G_\varepsilon(P_n,x)}\geq\frac{C}{|x-P_n|}. \tag{2.3.43}$$

Let $R>0$, to be fixed later on. By (2.3.36) and (2.3.42), we get that for n sufficiently large

$$L_nG_\varepsilon^{1-\varepsilon}(P_n,x)\geq 0$$

in $\Omega\setminus B_\rho(P_n)$. On the other hand, if $x\in B_\rho(P_n)\setminus B_{R\,\varepsilon_n}(P_n)$, then by (2.3.37)

$$|x-P_n|^2u_n^{\frac{4}{N-2}}(x)\leq\varepsilon_R,$$

where $\varepsilon_R\to 0$ as $R\to+\infty$. This, (2.3.42) and (2.3.43) gives that

$$\frac{L_nG_\varepsilon^{1-\varepsilon}(P_n,x)}{G_\varepsilon^{1-\varepsilon}(P_n,x)}\geq\frac{C\varepsilon(1-\varepsilon)-B_n\varepsilon_R}{|x-P_n|^2}.$$

We choose $R>0$, sufficiently large such that $C\varepsilon(1-\varepsilon)-\,\varepsilon_R\geq 0$. Then $L_n\,G_\varepsilon^{1-\varepsilon}(P_n,x)\geq 0$ in $B_\rho(P_n)\setminus B_{R\,\varepsilon_n}(P_n)$, and we have proved that for n sufficiently large, $L_nG_\varepsilon^{1-\varepsilon}(P_n,x)\geq 0$ in $\Omega\setminus B_{R\,\varepsilon_n}(P_n^i)$. We recall another standar property of the Green's function, i. e. there exist $C>0$ such that for any n, and any $x\in\partial B_{R\,\varepsilon_n}(P_n)$,

$$G_\varepsilon^{1-\varepsilon}(P_n^i,x)\geq C\,(\varepsilon_n^i)^{-(1-\varepsilon)(N-2)}.$$

If we let $C_n=C^{-1}(\varepsilon_n)^{(1-\varepsilon)(N-2)}$, then we have

$$C_nG_\varepsilon^{1-\varepsilon}(P_n,x)\geq u_n(x)$$

for all n and all $x\in\partial B_{R\,\varepsilon_n}(P_n)$. We can use the maximum principle, then

$$C_nG_\varepsilon^{1-\varepsilon}(P_n,x)\geq u_n(x)$$

for all n and all $x\in\Omega\setminus B_{R\,\varepsilon_n}(P_n)$. Noting that there exists $C>0$ such that for any n, and any $x\in\Omega\setminus\{\,P_n\,\}$,

$$|P_n-x|^{N-2}G_\varepsilon(P_n,x)\leq C$$

it follows that for any $\varepsilon>0$, any n, and any $x\in\Omega\setminus\{\,P_0\,\}$,

$$|P_n-x|^{N-2-\tilde\varepsilon}(\varepsilon_n)^{-\frac{N-2}{2}+\tilde\varepsilon}u_n(x)\leq C_\varepsilon$$

where $\tilde\varepsilon=(N-2)\varepsilon$, and $C_\varepsilon>0$ is independent of n. (2.3.41) is true in $\Omega\setminus B_{R\,\varepsilon_n^i}(P_n)$, and it is obviously satisfied in $B_{R\,\varepsilon_n}(P_n)$. This proves (2.3.41).
Now we have that

$$u_n\leq C\frac{(\varepsilon_n)^{\frac{N-2}{2}-\tilde\varepsilon}}{|x-P_n|^{N-2-\tilde\varepsilon}}, \tag{2.3.44}$$

in $\Omega\setminus B_{M\,\varepsilon_n}(P_n)$, with $M>0$. Observe that

$$\int_{B_{M\,\varepsilon_n(P_n)}}u_n^{p+1}\leq C_1. \tag{2.3.45}$$

Integrate (2.3.44) and by a change of variable ($x - P_n = \varepsilon_n\, y$)

$$\int_{\Omega\setminus B_{M\,\varepsilon_n}(P_n)} u_n^{\frac{2N}{N-2}} \le C\int_{\Omega\setminus B_{M\,\varepsilon_n}(P_n)} \frac{(\varepsilon_n)^{N-\tilde\varepsilon\frac{2N}{N-2}}}{|x-P_n|^{2N-\tilde\varepsilon\frac{2N}{N-2}}}\mathrm{d}x \tag{2.3.46}$$

$$\le C\int_{\mathbb{R}^N\setminus B_1(0)} \frac{(\varepsilon_n)^{2N-\tilde\varepsilon\frac{2N}{N-2}}}{|\varepsilon_n\, y|^{2N-\tilde\varepsilon\frac{2N}{N-2}}}\mathrm{d}y \tag{2.3.47}$$

$$\le \tilde C\int_{\mathbb{R}^N\setminus B_1(0)} \frac{1}{|y|^{2N-\tilde\varepsilon\frac{2N}{N-2}}}\mathrm{d}y < +\infty. \tag{2.3.48}$$

This proves that $\int_\Omega u_n^{\frac{2N}{N-2}} < +\infty$. □

2.3.3. A non existence result for $\frac{N+2}{N-2} \le p < p_c(N)$. When $\frac{N+2}{N-2} \le p < p_c(N)$, we have that $m(u_n) \to +\infty$ whenever $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. For a solutions sequence u_n of (2.1.3) with $\lambda_n \to +\infty$ we always have that $\|u_n\|_\infty \to +\infty$ as shown in Proposition 0.1.
In conclusion, we have that

THEOREM 2.12. *Let $\frac{N+2}{N-2} \le p < p_c(N)$. Problem (1.1.1) doesn't have, for λ large, solutions u_n with uniformly bounded Morse indices or with uniformly bounded energy.*

Indeed, with $p = \frac{N+2}{N-2}$, if holds (2.3.34) then u_n have bounded energy (i.e. $\int_\Omega u_n^{\frac{2N}{N-2}} < +\infty$). Castorina and Mancini [**25**] were able to prove that, in the critical case for all $\lambda_n \to +\infty$, any possible blowing-up solutions sequence have the property that $\int_\Omega u^{p+1} \to +\infty$, so there aren't solutions of this problem with uniformly bounded Morse Indices.

CHAPTER 3

Location of the blow-up set

This chapter deals with the location of the blow-up set of our problem. We show that Morse Index information on solutions of (1.1.1), with $1 < p < \frac{N+2}{N-2}$, provide a complete description of the blow-up behavior, in the sense that we obtain some crucial global estimates to localize the blow-up set. If u_n is a a family of solutions of the problem (1.1.1), u_n has exactly a finite number of maximum points $P_n^1, \dots, P_n^k$. The question is to find where are P_n^i $i = 1, \dots, k$. Intuitively, the location of should depend on the geometric properties of the domain. We show exactly how the geometry of the domain affects the solutions. When in the equation (0.1) the potential $V \equiv 1$, we derive some results are already known, having information on the Morse index, i.e. we obtain that a solution which posses a single peak has its peak in the interior of Ω and this peak must be situated near the most centered part of Ω, that is where the distance function assumed its maximum. Otherwise with a generic potential V, the geometry of the domain does not influence the location of the peaks, that must be critical points of V.

3.1. Case $V \neq 1$

Our aim is to localize the blow-up set as critical points of the potential V. In the radial case a modified potential $M(r) := r^{N-1}V^{\theta}(r)$, ($\theta = \frac{p+1}{p-1} - \frac{1}{2}$), introduced by Ambrosetti, Malchiodi and Ni [**5**], has an important role in concentration phenomena. In our case the presence of V is fundamental for the location of blow-up points.

3.1.1. Preliminary estimates. In this section we prove some estimates that we will need in the sequel. First of all, a lemma is in order. We anticipate that the pointwise estimate (3.1.1) below is the key ingredient that will allows us to localize the blow-up set.
Let us start with some asymptotic estimates for u_n solution of (1.1.1).
We say that u_n has, up to a subsequence, at least k points of local maximum $P_n^1, \dots, P_n^k \in \Omega$, with $P_n^i \to P^i \in \Omega$, $i = 1, \dots, k$.
Let

$$J_i = \{j = 1, \dots, k \ : \ P_n^j \to P^i\}, \qquad I_\delta^i := B_\delta(P^i) \cap \Omega,$$

$\delta > 0$ fixed small that $I_\delta^i \cap \{P^1, \dots, P^k\} = \{P^i\}$. We have the following estimates:

Lemma 3.1. *Let g be some smooth functions on Ω. Let $1 < p < \frac{N+2}{N-2}$, $q > 1$. Fix $i \in \{1, \dots, k\}$. Then*

$$\int_{I_\delta^i} g\, u_n^q = g(P^i)\Big(\sum_{j \in J^i} (\varepsilon_n^j)^{-\frac{2q}{p-1}+N}\Big)\Big(\int_{\mathbb{R}^N} U^q + o_n(1)\Big) \tag{3.1.1}$$

where $o_n(1) \to 0$ *as* $n \to +\infty$.
In particular

$$\int_\Omega u_n^{p+1} = \Big(\sum_{i=1}^k (\varepsilon_n^i)^{-\frac{2\,(p+1)}{p-1}+N}\Big)\Big(\int_{\mathbb{R}^N} U^{p+1} + o_n(1)\Big) \tag{3.1.2}$$

PROOF. Let $d_n(x) := \min\{|x - P_n^i| \ : i = 1, \dots, k\}$. Given $R > 0$, for $n > n(R)$,

$$\{d_n(x) \le R\,\varepsilon_n^1\} \subset \Omega \text{ and } I_\delta^i \cap \{d_n(x) \le R\,\varepsilon_n^1\} = \cup_{j\in J_i}\{|x - P_n^j| \le R\,\varepsilon_n^1\}.$$

We know by Theorem 2.4 that

$$u_n^q \le C(\varepsilon_n^1)^{-\frac{2\,q}{p-1}} \sum_{j=1}^k e^{-q\,\gamma\frac{|x-P_n^i|}{\varepsilon_n^1}}. \tag{3.1.3}$$

We have

$$\begin{aligned}
\int_{I_\delta^i} g\,u_n^q &= \int_{I_\delta^i \cap \{\,d_n(x)\le R\,\varepsilon_n^1\,\}} g(x)u_n^q + \int_{I_\delta^i \cap \{\,d_n(x)\ge R\,\varepsilon_n^1\,\}} g(x)u_n^q \\
&= \sum_{j\in J_i} \int_{\{\,|x-P_n^j|\le R\,\varepsilon_n^1\,\}} g(x)u_n^q \\
&+ O\Big((\varepsilon_n^1)^{-\frac{2\,q}{p-1}} \sum_{j=1}^k \int_{\{\,|x-P_n^j|\ge R\,\varepsilon_n^1\,\}} e^{-q\,\gamma\frac{|x-P_n^j|}{\varepsilon_n^1}}\Big) \\
&= \sum_{j\in J_i} (\varepsilon_n^j)^{-\frac{2\,q}{p-1}+N} \int_{\{|y|\le \frac{R\,\varepsilon_n^1}{\varepsilon_n^j}\}} g(\varepsilon_n^j\,y + P_n^j)u_n^q(\varepsilon_n^j\,y + P_n^j)(\varepsilon_n^j)^{\frac{2\,q}{p-1}} \\
&+ O\Big((\varepsilon_n^1)^{-\frac{2\,q}{p-1}} \sum_{j=1}^k (\varepsilon_n^j)^N \int_{\{\,|y|\ge R\frac{\varepsilon_n^1}{\varepsilon_n^j}\,\}} e^{-q\,\gamma|y|\frac{\varepsilon_n^j}{\varepsilon_n^1}}\Big) \\
&= \sum_{j\in J_i} (\varepsilon_n^j)^{-\frac{2\,q}{p-1}+N} \int_{\{|y|\le \frac{R\,\varepsilon_n^1}{\varepsilon_n^j}\}} g(\varepsilon_n^j\,x + P_n^j)(U_n^j)^q \\
&+ O\Big((\varepsilon_n^1)^{-\frac{2\,q}{p-1}} \sum_{j=1}^k (\varepsilon_n^j)^N \int_{\{\,|y|\ge R\frac{\varepsilon_n^1}{\varepsilon_n^j}\,\}} e^{-q\,\gamma|y|\frac{\varepsilon_n^j}{\varepsilon_n^1}}\Big).
\end{aligned}$$

Up to a subsequence, $\varepsilon_n^1 \le \varepsilon_n^j \le C\varepsilon_n^1$, so we can assume that $\frac{\varepsilon_n^1}{\varepsilon_n^j} \to \theta_j \in [\frac{1}{C}, 1]$. Since $U_n^j \to_n U$ in $C^1_{loc}(\mathbb{R}^N)$ for any $j = 1, \dots, k$, we find, along some subsequences:

$$\lim_{n\to+\infty} (\varepsilon_n^1)^{\frac{2\,q}{p-1}-N} \int_{I_\delta^i} g\,u_n^q$$

$$= g(P^i) \sum_{j\in J_i} (\theta_j)^{\frac{2\,q}{p-1}-N} \int_{\{\,|y|\le R\,\theta_j\,\}} U^q + O\Big(\sum_{j=1}^k \int_{|y|\ge R\,\theta_j} e^{-q\,\gamma\frac{|y|}{\theta_j}}\Big);$$

sending R to infinity, we get, along the same sequence

$$\lim_{n\to+\infty} (\varepsilon_n^1)^{\frac{2\,q-N(p-1)}{p-1}} \int_{I_\delta^i} g(x)u_n^q = g(P^i)\Big(\sum_{j\in J_i} \theta_j^{\frac{2\,q-N(p-1)}{p-1}}\Big)\int_{\mathbb{R}^N} U^q.$$

Rescaling the definition of θ_j, the proof is complete. Moreover, by (3.1.3) we have that $u_n \to 0$ as $n \to +\infty$ uniformly away from $\{\, P^1, \dots, P^k \,\}$. So, if we take $q = p+1$ and $g \equiv 1$ in (3.1.1), we have (3.1.2) by summing over i. □

3.1.2. Blow-up set. Therefore, we combine the asymptotic expansions in Lemma 3.1 with a Pohozaev identity to obtain the location of blow-up points $P^1, \dots, P^k$.

THEOREM 3.2. *For any $i = 1, \dots, k$, P^i is a critical point of the potential V:*

$$\nabla V(P^i) = 0.$$

PROOF. Let's get the right Pohozaev-Type Identity to work. Let $\delta > 0$ small, and $P_n^i \to P^i \in \Omega$. We multiply the equation $-\Delta u_n = u_n^p - \lambda_n V\, u_n$ by $\partial_h u_n$, so we have the Pohozaev identity on $B_\delta(P^i) \cap \Omega = I_\delta^i$,

$$\begin{aligned}
\int_{I_\delta^i} (u_n^p - \lambda_n V\, u_n)\partial_h u_n &= \int_{I_\delta^i} \partial_i\Big(\frac{u_n^{p+1}}{p+1} - \lambda_n V\,\frac{u_n^2}{2}\Big) + \int_{I_\delta^i} \lambda_n \frac{u_n^2}{2}\partial_h V \\
= \quad & \int_{\partial I_\delta^i} \Big(\frac{u_n^{p+1}}{p+1} - \lambda_n V\,\frac{u_n^2}{2}\Big)\nu_h + \int_{I_\delta^i} \frac{\lambda_n}{2} u_n^2\,\partial_h V \\
= \quad & \int_{I_\delta^i} -\sum_i \partial_{i\,i} u_n\; \partial_h u_n = \int_{\partial I_\delta^i} -\sum_i \partial_i u_n\,\nu_i\,\partial_h u_n + \int_\Omega \sum_i \partial_i\, u_n \partial_{h\,i} u_n \\
= \quad & -\int_{\partial I_\delta^i} \partial_\nu u_n\,\partial_h u_n + \int_{I_\delta^i} \partial_h\Big(\frac{1}{2}|\nabla u_n|^2\Big) \\
= \quad & -\int_{\partial I_\delta^i} \partial_\nu u_n\,\partial_h u_n + \int_{\partial I_\delta^i} \frac{1}{2}|\nabla u_n|^2\,\nu_h
\end{aligned}$$

$\lambda_n V(x)u_n \to 0$ uniformly in Ω away from $P^1, \dots, P^k$. Observe that by the exponential decay in Theorem 2.4 we have that $u_n, |\nabla u_n| \to 0$ as $n \to +\infty$ on $\partial B_\delta(P^i)$, and we say that $u_n = 0$ on $\partial\Omega$, so the integrals on ∂I_δ^i are zero.
We have

$$\lambda_n \int_{I_\delta^i} \partial_h V\, u_n^2 \to_{n\to+\infty} 0 \quad \forall\, h,\ \forall i.$$

By (3.1.1) we get that

$$\lambda_n \int_{I_\delta^i} \partial_h V\, u_n^2 = \lambda_n \partial_h V(P^i)\Big(\sum_{j\in J^j} (\varepsilon_n^j)^{-\frac{4}{p-1}+N}\Big)\Big(\int_{\mathbb{R}^N} U^2 + o_n(1)\Big) \quad \forall\, h, \forall\, i,$$

by Theorem 2.4 we say that $\lambda_n\ (\varepsilon_n^j)^2 V \to \tilde{\lambda} \in (0,1]$ so

$$\lambda_n \int_{I_\delta^i} \partial_h V\, u_n^2 = \frac{\tilde{\lambda}}{V(P^i)} \partial_h V(P^i) \Big(\sum_{j \in J^j} (\varepsilon_n^j)^{-\frac{4}{p-1}+N-2} \Big) \Big(\int_{\mathbb{R}^N} U^2 + o_n(1) \Big) \quad \forall\, h,\, \forall\, i,$$

and divided by $\Big(\sum_{j\in J^j} (\varepsilon_n^j)^{-\frac{4}{p-1}+N-2} \Big)$ and $\int_{\mathbb{R}^N} U^2 < +\infty$

$$\Rightarrow\ \frac{\partial_h V(P^i)}{V(P^i)} = 0 \quad \forall\, h,\, \forall\, i \quad \Rightarrow\ \partial_h V(P^i) = 0\ \ \forall\, h,\, \forall\, i.$$

Therefore

$$\nabla V(P^i) = 0, \quad \forall\, i.$$

□

3.2. Case $V \equiv 1$

When the potential $V \equiv 1$, the approach used in the previous section is useless, because we must have a more precise expansion of the Pohozaev-Type Identity. In this case the idea is to use some techniques of [**89**] and [**104, 105**], that consider the projection of U in Ω_n and also a vanishing viscosity method to derive some properties of this projection.
We consider the case when the blow-up occurs in one peak and would be possible to generalize the idea for the case of multi-peaks, using the appoach in [**65, 64, 91**].
We obtain that the blow-up occurs in critical points of the distance function.

3.2.1. Some properties of the distance function. Let Ω be a smooth bounded domain of $\mathbb{R}^N$. Let $f : \Omega \to \mathbb{R}$ be a Lipschitz continuous function. We recall the following definition due to Clarke [**28**].

Definition 3.3 (The generalized gradient). The generalized gradient of f at $x \in \Omega$ is the set:

$$\partial f(x) = \{\, \alpha \in \mathbb{R}^N \mid f^0(x,v) \geq \alpha \cdot v \ \ \forall\, v \in \mathbb{R}^N \,\}$$

where the generalized directional derivative $f^0(x,v)$ is defined by

$$f^0(x,v) = \limsup_{h \to 0,\, \lambda \to 0^+} \frac{f(x+h+\lambda v) - f(x+h)}{\lambda}.$$

If f is continuously differentiable at x then $\partial f(x) = \{\nabla f(x)\}$.
If f is only differentiable at x, $\partial f(x)$ can contain points other than $\nabla f(x)$.
For example, for $f(x) = x^2 \sin\frac{1}{x}$ it is easy to show that $f^0(0,v) = |v|$. So $\partial f(0) = [-1,1]$ which contains the derivative $f'(0) = 0$.

Definition 3.4. The function f is said to be regular at $x \in \Omega$ provided that for any $v \in \mathbb{R}^N$ $\partial_v f(x) = f^0(x,v)$, where $\partial_v f(x)$ is the usual directional derivative.

In [**28**] we deduce (in Proposition 2.2.4 and (b) of Proposition 2.3.6):

Proposition 3.5. *If $\partial f(x)$ reduces to a singleton $\{\alpha\}$, then f is differentiable at x and $\nabla f(x) = \alpha$. Conversely, if f is differentiable and regular at x then $\partial f(x) = \{\nabla f(x)\}$.*

We recall the following property in [**28**] (in Proposition 2.1.5).

Proposition 3.6. *Let x_n and α_n be sequences in $\mathbb{R}^N$ such that $x_n \in \Omega$ and $\alpha_n \in \partial f(x_n)$. Suppose that $x_n \to x$ and $\alpha_n \to \alpha$ as $n \to +\infty$. Then $\alpha \in \partial f(x)$.*

Let $x = (x_1, x_2)$. Denote by $\partial_1 f(x_1, x_2)$ and $\partial_2 f(x_1, x_2)$ the partial generalized gradient of $f(\cdot, x_2)$ at x_1 of $f(x_1, \cdot)$ at x_2 respectively. The following result holds (see [**28**], Proposition 2.1.5).

Remark 3.7. If f is regular at (x_1, x_2) then

$$\partial f(x_1, x_2) = \partial_1 f(x_1, x_2) \times \partial_2 f(x_1, x_2).$$

Assume that $\{f_i\}_{i \in I}$ is a finite collection of Lipschitz continuous function defined on D. The function $f(x) = \min\{f_i(x) \,|\, i \in I\}$ is easily seen to be a Lipschitz continuous function. For any $x \in D$ we let $I(x)$ denote the set of indices i for which $f(x) = f_i(x)$. Then the following holds (see [**28**], Proposition 2.3.12).

Proposition 3.8. *If f_i is regular at x for any $i \in I(x)$ then f is regular at x and*

$$\partial f(x) = co\{\, \partial f_i(x) \,|\, i \in I(x)\,\},$$

where co is the convex hull of the set.

Definition 3.9. A point x_0 is said to be a critical point of f if $0 \in \partial f(x_0)$. A real number c is said to be a critical value of f if there exists a critical point x_0 of f such that $f(x_0) = c$.

By Definition 3.3 we easily deduce that if x_0 is a minimum point or a maximum point for a Lipschitz continuous function f then $0 \in \partial f(x_0)$.

Definition 3.10. Let $d_{\partial\Omega} : \Omega \to \mathbb{R}$ be the distance function defined by

$$d_{\partial\Omega}(x) = \mathrm{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x - y|.$$

The function $d_{\partial\Omega}$ is a Lipschitz continuous function. In [**28**] (by Corollary 2 p.87), we can compute the generalized gradient of the distance function.

Remark 3.11. For any $x \in \Omega$ we have

$$\begin{aligned} \partial d_{\partial\Omega}(x) \quad = \quad & \Big\{ \int_{\partial\Omega} \nu^{(i)}(y) \mathrm{d}_{\mu_x}(y) \,:\, \mathrm{d}_{\mu_x}(y) \text{ is a bounded Borel measure on } \partial\Omega, \\ & \int_{\partial\Omega} \mathrm{d}_{\mu_x} = 1, \mathrm{supp}(\mathrm{d}_{\mu_x}) \subset \Pi_{\partial\Omega}(x) \Big\}, \end{aligned} \tag{3.2.1}$$

where

$$\Pi_{\partial\Omega}(x) = \{\, y \in \partial\Omega \,|\, |y-x| = \mathrm{d}_{\partial\Omega}(x)\,\} \tag{3.2.2}$$

and $\nu^{(i)}(y)$ denotes the unit inward normal at the point y of $\partial\Omega$.

In [**28**] (by Corollary 2, p. 87), we deduce that the distance function is regular at any $x \in \Omega$. Therefore by Proposition 3.5 we get:

REMARK 3.12. $\mathrm{d}_{\partial\Omega}$ is differentiable at x if and only if $\Pi_{\partial\Omega}(x)$ reduces to a singleton $\{\,\pi(x)\,\}$. In this case, $\nabla \mathrm{d}_{\partial\Omega}(x) = \nu^{(i)}(\pi(x))$.

Finally, we have:

PROPOSITION 3.13. *There exists a neighborhood J of the boundary of Ω such that $\pi_{\partial\Omega}(x) = \{\,\pi(x)\,\}$ for any $x \in J\cap\Omega$. In particular, $|\nabla d_{\partial\Omega}(x)| = 1$, $\forall\, x \in J\cap\Omega$.*

3.2.2. Blow-up in one peak: notations and preliminaries. There are many papers concerning the effect of the domain topology on the solutions of problems related to our (see [**12, 13**] and [**14**]) and concerning the importance of the shape of the domain on the solutions (see [**33, 34**]). In [**35**], the uniqueness of solutions was proved under a very strong symmetry hypothesis of the domain. The first precise result on the effect of the domain geometry on the generic solutions of $-\varepsilon\Delta u - u + u^p = 0$ with Dirichlet boundary condition is [**104**]. It seems extremely difficult and interesting to see how the geometry of the domain affects the locations of multi-spikes solutions. Due to the special structure of the problem, it is necessary to estimate the exponentially small error terms. Thus traditional techniques in singular perturbations do not seem to apply; it is necessary a detailed study of a vanishing viscosity solution.

Let us introduce some auxiliary problems that will be used in the sequel. Let u_n be solutions of

$$\begin{cases} -\Delta u_n + \lambda_n u_n = u_n^p & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{in } \partial\Omega. \end{cases} \tag{3.2.3}$$

with $\sup_n m(u_n) < +\infty$, for $1 < p < \frac{N+2}{N-2}$. Let P_n be a local maximum of u_n, and consider the usual change of variable $U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ that satisfies

$$\begin{cases} -\Delta U_n + \tilde\lambda_n U_n = U_n^p & \text{in } \Omega_n = \frac{\Omega - P_n}{\varepsilon_n} \\ U_n = 0 & \text{on } \partial\Omega_n \end{cases} \tag{3.2.4}$$

with $\tilde\lambda_n = \lambda_n\varepsilon_n^2$. By previuos Theorem 2.1 we say that $\tilde\lambda_n \to \tilde\lambda \in (0,1]$, and $U_n \to_n U$ in $C^1_{loc}(\mathbb{R}^N)$, with U solution in the whole space of

$$\begin{cases} -\Delta U + \tilde\lambda U = U^p & \text{in } \mathbb{R}^N \\ 0 < U(x) \le U(0) = 1. \end{cases} \tag{3.2.5}$$

Consider some related problems that permit to obtain an expansion of U_n.

We define the projection $P_{\Omega_n}U$ to be the unique solution of

$$\begin{cases} -\Delta P_{\Omega_n}U + \tilde\lambda P_{\Omega_n}U = U^p & \text{in } \Omega_n \\ P_{\Omega_n}U(x) = 0 & \text{in } \partial\Omega_n. \end{cases} \tag{3.2.6}$$

Let

$$\varphi_n := U - P_{\Omega_n}U, \;\; \psi_n := -\varepsilon_n \log\Big(\varphi_n\Big(\frac{x-P_n}{\varepsilon_n}\Big)\Big), \quad V_n := e^{\frac{1}{\varepsilon_n}\psi_n(P_n)}\varphi_n = \frac{\varphi_n}{\|\,\varphi_n\,\|},$$

which satisfy respectively

$$\begin{cases} -\Delta\varphi_n + \tilde\lambda\varphi_n = 0 & \text{in } \Omega_n \\ 0 < \varphi_n(x) \le 1 & \text{in } \partial\Omega_n \\ \varphi(x) = U\Big(\frac{x-P_n}{\varepsilon_n}\Big) & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.7}$$

$$\begin{cases} \varepsilon_n\Delta\psi_n - |\nabla\psi_n|^2 + \tilde\lambda = 0 & \text{in } \Omega \\ \psi_n(x) = -\varepsilon_n \log U\big(\frac{x-P_n}{\varepsilon_n}\big) & \text{in } \partial\Omega, \end{cases} \tag{3.2.8}$$

$$\begin{cases} -\Delta V_n + \tilde\lambda\, V_n = 0 & \text{in } \Omega_n \\ V_n(0) = 1. \end{cases} \tag{3.2.9}$$

Then, following the ideas in [**89**], we can prove that, up to a subsequence, for every sequence $\varepsilon_n \to 0$

$$U_n(y) = P_{\Omega_n}U(y) + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\phi_n(y)$$

so

$$P_{\Omega_n}U = U_n - e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\phi_n,$$

and

$$\varphi_n = U - U_n - e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\phi_n,$$

with ϕ_n solution of

$$\begin{cases} -\Delta\phi_n + \tilde\lambda_n\phi_n + e^{\frac{1}{\varepsilon_n}\varphi_n(P_n)}[U_n^p - U^p] = 0 & \text{in } \Omega_n \\ \phi_n(x) > 0 & \text{in } \partial\Omega_n, \end{cases} \tag{3.2.10}$$

such that $\|\phi_n - \phi_0\|_{L^\infty(\Omega_n)} \to 0$, where ϕ_0 is a solution of

$$-\Delta\phi_0 + \tilde\lambda\phi_0 = pU^{p-1}(\phi_0 - V_0), \quad \text{in } \mathbb{R}^N \tag{3.2.11}$$

and V_0 is a solution of

$$\begin{cases} \Delta u - \tilde\lambda u = 0 & \text{in } \mathbb{R}^N \\ u > 0. \quad u(0) = 1. \end{cases} \tag{3.2.12}$$

Remark 3.14. We observe that, by the Maximum Principle, $P_{\Omega_n}U < U$ on Ω, in fact

$$\begin{cases} -\Delta P_{\Omega_n}U + \tilde\lambda P_{\Omega_n}U = -\Delta U + \tilde\lambda U & \text{in}\,\Omega_n, \\ P_{\Omega_n}U = 0 \le U & \text{on } \partial\Omega_n. \end{cases} \tag{3.2.13}$$

We have the following results for the profile of U_n.

PROPOSITION 3.15. *Let $U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n\, y + P_n)$, then the following statements hold.*

(1) *For any $\eta > 0$, there exist positive constants ε_0 and k_0, such that, for all $0 < \varepsilon_n < \varepsilon_0$, we have $B_{2\,k_0\,\varepsilon_n}(P_n) \subset \Omega$ and $\|U_n - U\|_{C^1(B_{k_0}(0))} < \eta$, where U is the unique solution of (3.2.5).*

(2) *For any $0 < \delta < 1$ there is a constant C such that*

$$U_n(y) \le C\, e^{-(\tilde\lambda - \delta)\,|y|} \quad \text{for } y \in \Omega_n. \tag{3.2.14}$$

(3) *$\|U_n - U\|_{L^s(\Omega_n)} \to 0$ for all $1 \le s \le \infty$ as $\varepsilon_n \to 0$.*

PROOF. (1) We omit the detail of this proof. Indeed repeating the proof of Theorem 2.1 in [**86**] we see that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$ as $\varepsilon_n \to 0$, where U is the unique solution of (3.2.5). This proves the thesis.

(2) By a result in [**57**]

$$U(r) \le C_0\, e^{-r} \quad \text{for } r \ge 0. \tag{3.2.15}$$

For any $\eta > 0$, set

$$R := \log\left(\frac{C_0}{\eta}\right)$$

so that $\eta = C\, e^{-R}$. Then we say that there is a $\varepsilon_0 > 0$ such that

$$\|U_n - U\|_{C^1(\overline{B}_{2\,R}(0))} \le \eta$$

if $0 < \varepsilon_n < \varepsilon_0$. Thus

$$U_n(y) \le U(y) + \eta \le C_0\, e^{-R} + \eta = 2\,\eta$$

for $|y| = R$. We have $u_n(x) \le 2\,\eta$ for $x \in \partial B_{R\,\varepsilon_n}(P_n))$ and $\varepsilon_n \le \varepsilon_0$. By the fact that there exists only one peak, the set $\{\, x \in \Omega \mid u_n(x) > 2\,\eta \,\}$ has only one connected component. Consequently

$$u_n(x) \le 2\,\eta \quad \text{in} \quad \Omega \setminus B_{R\,\varepsilon_n}(P_n).$$

We choose η such that $\lambda_n\, \varepsilon_n^2 - \alpha^{p-1} > \lambda_n\, \varepsilon_n^2 - \delta$ for $\alpha < 2\,\eta$. Then U_n satisfies

$$\begin{cases} \Delta U_n - (\lambda_n\, \varepsilon_n^2 - U_n^{p-1}) U_n = 0 & \text{in } \Omega_n \setminus B_R(0), \\ U_n|_{\partial B_R(0)} \le 2\,\eta, & \\ U_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.2.16}$$

Observe that $\tilde\lambda - \alpha^{p-1} > \tilde\lambda - \delta$ for $\alpha < 2\,\eta$.

Let $G(y, z)$ be the Green's function for $-\Delta + \tilde\lambda$ on $\mathbb{R}^N$, i. e.

$$G(y, z) = C_n |y - z|^{-\frac{(N-2)}{2}} K_{\frac{N-2}{2}}(|y - z|) \tag{3.2.17}$$

where C_n is a positive constant depending only on N and $K_m(z)$ is the modified Bessel function of order m; see Appendix C in [**57**].

Let $G_0(|y|) = G(|y|) = G(y, 0)$ and $\overline{U}(y) = \frac{2\,\eta\, G_0(\sqrt{\tilde\lambda - \delta}\,|y|)}{G_0(\sqrt{\tilde\lambda - \delta}\, R)}$. Then $\overline{U}$ satisfies

$$\begin{cases} \Delta \overline{U} - (\tilde\lambda - \delta)\overline{U} = 0 & \\ \overline{U} = 2\,\eta & \text{on } \partial B_R(0). \end{cases} \tag{3.2.18}$$

By the Maximum Principle on $\Omega_n \setminus B_R(0)$, we have

$$U_n(y) \le \overline{U}(y) \quad \text{on } \Omega_n \setminus B_R(0).$$

But on $B_{R\varepsilon_n}(P_n)$, $U_n(y) \le C$. Hence

$$U_n(y) \le C\, e^{-(\tilde{\lambda}-\delta)|y|}, \quad \text{for all } y \in \Omega_n.$$

(3) For any $\eta > 0$, by the exponential decay of U_n and U, there is an R large such that $\|U_n - U\|_{L^s(\Omega \setminus B_R(0))} \le \frac{\eta}{2}$. On the other hand, we say that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$ as $n \to \infty$, so there is $\varepsilon_R > 0$ such that $\varepsilon_n < \varepsilon_0$, $\|U_n - U\|_{L^s(\Omega_n \cap B_R(0))} \le \frac{\eta}{2}$. Then $\|U_n - U\|_{L^s(\Omega_n)} \le \eta$.

□

We recall the following results about U (see [**17**], [**57**], [**70**]).

Theorem 3.16. *The equation*

$$-\Delta U + U = U^p, \quad \text{in } \mathbb{R}^N, \quad U(x) for |x| \to +\infty$$

possesses a unique non trivial regular solution U with the following properties:

(1) $U(x) > 0\ \forall\, x \in \mathbb{R}^N$,
(2) *U is spherically symmetric, i. e., $U(x) = U(r)$ where $r = |x|$, and U decreases with respect to r,*
(3) $U \in C^2(\mathbb{R}^N)$,
(4) *U together with its derivatives up to order 2 have exponential decay at infinity; that is, there exists $C > 0$ and $\delta > 0$ such that $|D^\alpha U(x)| \le C\, e^{-\delta|x|}\ \forall\, x \in \mathbb{R}^N$ and $|\alpha| \le 2$.*
(5) *there exists $\lambda_0 > 0$ such that $\lim_{r\to+\infty} r^{\frac{N-1}{2}} e^r U(r) = \lambda_0$; moreover*

$$\lim_{r\to\infty} \frac{U'(r)}{U(r)} = -1.$$

By this properties of U, we immediately have:

$$|x - P_n|^{\frac{N-1}{2}} e^{|x-P_n|} U(|x - P_n|) = \lambda_0 + o(1),$$

that implies

$$\psi_n(x) = -\varepsilon_n \log \varphi_n\Big(\frac{x - P_n}{\varepsilon_n}\Big) = |x - P_n| + \frac{N-1}{2}\varepsilon_n \log \frac{|x - P_n|}{\varepsilon_n} - \varepsilon_n \log(\lambda_0 + o(1)), \tag{3.2.19}$$

uniformly for $x \in \partial\Omega$.
Note that on $\partial\Omega$, $|x - P_n| \ge d(P_n, \partial\Omega)$, i. e. $\frac{|x-P_n|}{\varepsilon_n} \ge \rho_n \to_n \infty$.

We have the following statement:

Lemma 3.17. *It holds that*

$$\lim_{n\to\infty} \frac{\psi_n(x)}{|x - P_n|} = 1 \quad \textit{uniformly for} \quad x \in \partial\Omega.$$

PROOF. By (3.2.19) we have

$$\frac{\psi_n(x)}{|x-P_n|} = 1 + \frac{N-1}{2}\frac{\log\frac{|x-P_n|}{\varepsilon_n}}{\frac{|x-P_n|}{\varepsilon_n}} - \frac{\log(\lambda_0+o(1))}{\frac{|x-P_n|}{\varepsilon^n}} \to 1$$

uniformly for $x \in \partial\Omega$ as $n \to \infty$.

□

In order to study the properties of ψ_n, we first consider a closely related problem which is simpler.

LEMMA 3.18. *Let ψ^n, for ε_n sufficiently small, be the unique solution of the equation*

$$\begin{cases} \varepsilon_n \Delta\psi - |\nabla\psi|^2 + \tilde{\lambda} = 0 & \text{in } \Omega \\ \psi(x) = |x-P_n| & \text{on } \partial\Omega. \end{cases} \tag{3.2.20}$$

Furthermore, there exist two positive constants C_1, C_2 s. t.

$$\|\psi^n\|_{L^\infty(\Omega)} \le C_1, \qquad \|\nabla\psi^n\|_{L^\infty(\Omega)} \le C_2.$$

PROOF. We observe that 0 is a subsolution of (3.2.20) in Ω, on the other hand, for ε_n sufficiently small ψ_n is a supersolution to (3.2.20) and $\psi_n > 0$ in Ω, by the Maximum Principle. By Theorem 1 in [**1**], there is a solution ψ^n to (3.2.20) s. t. $0 < \psi^n < \psi_n$.
We want to obtain an upper bound of ψ^n. We choose a vector X_0 such that $|X_0| > 1$ and a number b large such that $g(x) =< x\,, X_0 > +b > |x-P_n|$ on $\partial\Omega$. Then by comparison, we have that $g(x) > \psi^n(x)$ on $\partial\Omega$, which proves that $\|\psi^n\|_{L^\infty(\Omega)} \le C_1$.
By computation

$$\begin{cases} -\varepsilon_n \Delta g - |\nabla g|^2 + \tilde{\lambda} = 1 - |X_0|^2 < 0 \\ g(x) > |x-P_n|. \end{cases} \tag{3.2.21}$$

The uniqueness of ψ^n follows from the usual Maximum Principle.
Now prove that $\|\nabla\psi^n\|_{L^\infty(\Omega)} \le C_2$. We first show that $\|\nabla\psi^n\|_{L^\infty(\partial\Omega)} \le C_2$. The idea of the proof is to use a barrier method.
We choose $\delta > 0$ small and $\rho > 0$ large such that the distance function $d(x) := d(x, \partial\Omega)$ is C^2 in $\Omega_\delta := \{\, x \in \Omega \,|\, d(x) < \delta \,\}$ and $\rho\,\delta > C_1$. Considering the functions

$$\psi_-^n = |x-P_n|, \quad \psi_+^n = |x-P_n| + \rho\, d(x),$$

and observe that

$$\varepsilon_n \Delta\psi_-^n - |\nabla\psi_-^n|^2 + \tilde{\lambda} = \frac{\varepsilon_n\,(N-1)}{|x-P_n|} \tag{3.2.22}$$

for $x \ne P_n$ and that $\psi^n \ge C(\varepsilon_n) > 0$. Hence if we take $\varepsilon_n \le \varepsilon_0$ and $\delta(\varepsilon_n)$ small it is easy to see that ψ_-^n is a subsolution on $\Omega \setminus B_{\delta(\varepsilon_n)}(P_n)$. Therefore it is a subsolution on Ω_δ and $\psi_-^n \le \psi^n$ on $\overline{\Omega_\delta}$.
We have that

$$\begin{aligned} \varepsilon_n \Delta\psi_+^n - |\nabla\psi_+^n|^2 + \tilde{\lambda} &= \varepsilon_n(\Delta\psi_-^n + \rho\Delta d) - |\nabla\psi_-^n + \rho\nabla d|^2 + \tilde{\lambda} \\ &= -\rho^2|\nabla\, d|^2 - 2\,\rho\,\nabla\psi_-^n \cdot \nabla d + \frac{\varepsilon_n\,(N-1)}{|x-P_n|} + \varepsilon_n\,\rho\Delta d. \end{aligned}$$

We observe that

$$|\nabla d| = 1 \text{ on } \partial\Omega, \quad |\Delta d| \le 1 \text{ in } \Omega_\delta.$$

Then, if we choose ρ large, we have

$$\begin{cases} \varepsilon_n \Delta\psi_+^n - |\nabla\psi_+^n|^2 + \tilde{\lambda} < 0 & \text{in } \Omega_\delta \\ \psi_+^n > \psi^n & \text{on } \partial\Omega_\delta. \end{cases} \tag{3.2.23}$$

By comparison, we conclude $\psi_+^n > \psi^n$ on Ω_δ, so $\psi_-^n < \psi^n < \psi_+^n$ on Ω_δ. Thus $|\psi^n - \psi_-^n| < \rho\, d(x)$ on Ω_δ. Since $\psi^n = \psi_-^n$ on $\partial\Omega$, it follows that $\|\nabla\psi^n\|_{L^\infty(\partial\Omega)} \le C_2$. By a simply computation we have

$$\Delta(|\nabla\psi^n|^2) - \frac{2}{\varepsilon_n}\nabla\psi^n \cdot \nabla(|\nabla\psi^n|^2) \ge 0 \text{ in } \Omega.$$

Hence by the Maximum Principle, $|\nabla\psi^n|^2 \le C$ in Ω. □

We need to analyze the limit of ψ^n as $n \to +\infty$. We obtain that this limit is a viscosity solution.

Lemma 3.19. *Let ψ^n be the solution of 3.2.20, then ψ^n converges, as $n \to +\infty$, uniformly to a function $\psi_0 \in W^{1,\infty}(\Omega)$ which can be explicitly written as*

$$\psi_0(x) = \inf_{P\in\partial\Omega}(|P - P_n| + L(P, x)) \tag{3.2.24}$$

where $L(x, y)$ denotes the infimum of T such that there exists $\xi(s) \in C^{0,1}([0, T], \overline{\Omega})$, with $\xi(0) = x$, $\xi(T) = y$, and $|\frac{d\xi}{ds}| \le \sqrt{\tilde{\lambda}}$ almost everywhere in $[0, T]$.

Proof. Divided the proof in two-step.

1^{st} step: *Let*

$$S := \{\, v \in W^{1,\infty}(\Omega) \,:\, v(x) \le |x - P_n| \text{ on } \partial\Omega,\ |\nabla v| \le \sqrt{\tilde{\lambda}} \text{ a.e. in } \Omega \,\},$$

we have that ψ_0 is the maximum element of S.

First we show that $\psi_0 \in S$. Indeed, since $L(x, y)$ is the length of the shortest path in $\overline{\Omega}$ connecting x and y, we see that $|L(x, y) - L(\overline{x}, y)| \le L(x, \overline{x})$ for all $\overline{x} \in \Omega$. Therefore $|\psi_0(x) - \psi_0(\overline{x})| \le L(x, \overline{x})$. So we have that, when $x \in \Omega$ and $\overline{x} \in \Omega$ are closed, it is easy to see that $L(x, \overline{x}) = |x - \overline{x}|$ and $|\psi_0(x) - \psi_0(\overline{x})| \le |x - \overline{x}|$. Hence $\psi_0 \in W^{1,\infty}(\Omega)$ and $|\nabla\psi_0| \le \tilde{\lambda} \le 1$ almost everywhere in Ω. It is also easy to see that $\psi_0(x) = |x - P_n|$ on $\partial\Omega$ since $|x - P_n| - |y - P_n| \le L(x, y)$ for $x,\ y \in \partial\Omega$.

We next prove that ψ_0 is the maximum element of S. Indeed, let $v \in S$, since Ω is smooth we can extend v in the following way:

for h_0 small enough there exists $\tilde{v} \in W^{1,\infty}(\Omega^{h_0})$ such that $\tilde{v} = v$ in Ω and $|\nabla\tilde{v}| \le \tilde{k}$ a. e. in Ω^{h_0}, where

$$\Omega^{h_0} := \overline{\Omega} \cup \{\, x \in \mathbb{R}^N \setminus \overline{\Omega} \mid d(x, \partial\Omega) < h_0 \,\},$$

$\tilde{k} \in C(\overline{\Omega^{h_0}})$ and $\tilde{k} \equiv \sqrt{\tilde{\lambda}}$ on $\overline{\Omega}$. In fact, if h_0 is small enough, each point x in $\overline{\Omega^{h_0}} \setminus \overline{\Omega}$ is uniquely determined by the equation:

$$x = z + h\ \nu(z)$$

where $z \in \partial\Omega$, $h > 0$ and $\nu(z)$ is the unit outer normal to $\partial\Omega$ at the point z. Moreover, the map $x \to (z, h)$ is C^1 diffeomorphism on $\overline{\Omega}^{h_0} \setminus \overline{\Omega}$. Then we set $\tilde{v}(x) = \nu(z)$ and $\tilde{k}(x) = \tilde{\lambda} + C\, h$ for a large constant $C > 0$ (independent on h). We regularize v in a classical way, for α small enough, we may define $v_\alpha = \tilde{v} * \rho_\alpha$ where $\rho_\alpha = \alpha^{-N}\rho(\frac{\cdot}{\alpha})$, $\rho \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\rho \subset B_1$, $\int_{\mathbb{R}^N} \rho(y)\,\mathrm{d}y = 1$.
Then we have

$$|\nabla v_\alpha|^2 \leq (|\nabla \tilde{v}|^2) * \rho_\alpha \leq \tilde{k}^2 * \rho_\alpha \leq \tilde{\lambda} + C\,\alpha$$

on $\overline{\Omega}$ and $v_\alpha \to v$ in $C(\overline{\Omega})$ as $\alpha \to 0$.
Let $x, y \in \overline{\Omega}$ and for every $\eta > 0$, let ξ, T_0 be such that $\xi(0) = x$, $\xi(T_0) = y$, $\left|\frac{\mathrm{d}\xi}{\mathrm{d}t}\right| \leq \tilde{\lambda}$ a. e. in $[0, T_0]$, $\xi(t) \in \overline{\Omega}$ for all $t \in [0, T_0]$ and $T_0 \leq L(x, y) + \eta$.
Since Ω is smooth, there exist x_α, y_α, ξ_α, T_α such that $\xi_\alpha(0) = x_\alpha$, $\xi(T_\alpha) = y_\alpha$, $\left|\frac{\mathrm{d}\xi_\alpha}{\mathrm{d}t}\right| \leq \sqrt{\tilde{\lambda}} + C\,\alpha$ in $[0, T_\alpha]$, $\xi_\alpha \in C^1([0, T_\alpha], \overline{\Omega^{h_0}})$ and $\xi_\alpha \to \xi$, $T_\alpha \to T_0$ as $\alpha \to 0$, for example we can take ξ_α to be the regularization of ξ.
We have

$$\begin{aligned} |v_\alpha(y_\alpha) - v_\alpha(x_\alpha)| &= \Big| \int_0^{T_\alpha} \nabla v_\alpha(\xi_\alpha(t)) \cdot \frac{\mathrm{d}\xi_\alpha}{\mathrm{d}t}(t)\,\mathrm{d}t \Big| \\ &\leq \int_0^{T_\alpha} (\sqrt{\tilde{\lambda}} + C\,\alpha)^2\,\mathrm{d}t. \end{aligned}$$

Letting $\alpha \to 0$ and then $\eta > 0$, we obtain $|v(y) - v(x)| \leq L(x, y)$. Hence $v(x) \leq v(y) + L(x, y) \leq |y - P_n| + L(x, y)$ for all $y \in \partial\Omega$. Then we have $v \leq \psi_0$.

2^{nd} step: *For any sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ such that $\psi^{\varepsilon_{n_k}} \to \psi_0$ uniformly in $\overline{\Omega}$ as $\varepsilon_{n_k} \to 0$. Then it follows that $\psi^{\varepsilon_n} \to \psi^0$ uniformly in $\overline{\Omega}$ as $n \to \infty$.*

By Lemma 3.18 and the Ascoli-Arzela theorem, for any sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ such that $\psi^{\varepsilon_{n_k}} \to \psi^0$ uniformly in $\overline{\Omega}$ as $\varepsilon_{n_k} \to 0$. We have to prove that $\psi^0 = \psi_0$. We observe that $\psi^0 \in S$, in fact by taking limits in the sense of distributions in the equation satisfied by ψ^{n_k}, we obtain $|\nabla\psi^0|^2 \leq \tilde{\lambda} \leq 1$ in $D'(\Omega)$. Thus $\psi^0 \in W^{1,\infty}(\Omega)$, $|\nabla\psi^0| \leq \sqrt{\tilde{\lambda}} \leq 1$ a. e. in Ω and $\psi^0(x) = |x - P_n|$ on $\partial\Omega$. Hence $\psi^0 \leq \psi_0$. On the other hand, let $v \in S$, like in 1^{st} step we extend v to $\tilde{v}$ in Ω^{h_0} and regularize $\tilde{v}$ to v_α, in such a way that $\|v - v_\alpha\|_{L^\infty(\overline{\Omega})} \leq C\,\alpha$ and $|\nabla\tilde{v}| \leq \tilde{k}$. We have that

$$|\nabla v_\alpha|^2 \leq (|\nabla \tilde{v}|^2) * \rho_\alpha \leq \tilde{k}^2 * \rho_\alpha \leq \tilde{\lambda} + C\,\alpha$$

and $v_\alpha \to v$ in $C(\overline{\Omega})$ as $\alpha \to 0$.
Moreover we have

$$\begin{cases} \varepsilon_n \nabla v_\alpha - |\nabla v_\alpha|^2 + \tilde{\lambda} + C\,\alpha + A_\alpha\,\varepsilon_n \geq 0 & \text{in } \Omega \\ v_\alpha(x) \leq |x - P_n| + C\,\alpha & \text{on } \partial\Omega, \end{cases} \tag{3.2.25}$$

where $A_\alpha \geq 0$.
Let $\tilde{v}_\alpha := \frac{v_\alpha}{\sqrt{\tilde\lambda + C\,\alpha + A_\alpha\,\varepsilon_n}}$, then by comparison we deduce that

$$\tilde{v} \leq \psi^{\frac{\varepsilon_n}{\sqrt{\tilde\lambda + C\,\alpha + A_\alpha\,\varepsilon_n}}} + C\,\alpha. \tag{3.2.26}$$

Choosing $\varepsilon_n = \varepsilon'_{n_k}$ such that $\varepsilon_{n_k} = \frac{\varepsilon'_{n_k}}{\sqrt{\tilde\lambda + C\,\alpha + A_\alpha\,\varepsilon'_{n_k}}}$, we see that $\frac{v_\alpha}{\sqrt{\tilde\lambda + C\,\alpha}} \leq \psi^0 + C\,\alpha$ as $\varepsilon'_{n_k} \to 0$. Then, letting $\alpha \to 0$, we obtain $v \leq \psi^0$. In particular, we have $\psi_0 \leq \psi^0$ and then $\psi_0(x) = \psi^0(x)$. □

REMARK 3.20. Note that ψ_0 is a viscosity solution of the Hamilton-Jacobi equation: $|\nabla u| = \sqrt{\tilde\lambda}$ in Ω, see ([**72**]). Observe that $\psi_0(P_n) \to 2d(P_0, \partial\Omega)$ as $n \to +\infty$, for the proof see ([**89**]).

We give an estimate for $\psi_n(x)$.

PROPOSITION 3.21. (1) *There exists a positive constant C s. t. $\|\psi_n\|_{L^\infty(\Omega)} \leq C$.*

(2) *For any $\sigma_0 > 0$, there is an ε_0 s. t. for any $\varepsilon_n < \varepsilon_0$,*

$$\psi_n(P_n) \leq (1 + \tilde\lambda + \sigma_0) d(P_n, \partial\Omega).$$

(3) *Up to a subsequence, for every sequence $\varepsilon_n \to 0$, $V_n \to V_0$ uniformly on every compact set of $\mathbb{R}^N$, where V_0 is a positive solution of (3.2.12).*
Moreover, for any $\sigma_1 > 0$,

$$\sup_{y \in \overline{\Omega_n}} e^{-((\tilde\lambda + \sigma_1)|y|)} |V_n(y) - V_0(y)| \to 0, \quad \textit{as} \;\; n \to \infty.$$

PROOF. (1) The proof of this is almost identical to that of its counterpart in Lemma 3.2.20 and is thus omitted.

(2) Assume that $d(P_n, \partial\Omega) = |P_n - \overline{P_n}|$ where $\overline{P_n} \in \partial\Omega$. Let y_n be the point on the ray $\overline{P_n\overline{P_n}}$ s. t. $|P_n - y_n| = (\tilde\lambda + \eta)|P_n - \overline{P_n}|$ where $\eta < \min\{1, \sigma_0/10\}$ so small that $B_{r_0}(y_n) \subset \Omega^C$ and $\overline{B_{r_0}(y_n)} \cup \overline{\Omega} = \{\overline{P_n}\}$ for $r_0 = \eta|P_n - \overline{P_n}|$. Consider $w_n(x) = (\tilde\lambda + 2\,\eta)(|P_n - \overline{P_n}| + |y_n - x|)$, we have , on $\partial\Omega$

$$\begin{aligned} \psi_n(x) &\leq \Big(\tilde\lambda + \frac{\eta}{2}\Big)|x - P_n| \\ &< \Big(\tilde\lambda + \frac{\eta}{2}\Big)(|P_n - y_n| + |y_n - x|) \\ &= \Big(\tilde\lambda + \frac{\eta}{2}\Big)((\tilde\lambda + \eta)|P_n - \overline{P_n}| + |y_n - x|) \\ &< w_n(x) \end{aligned}$$

for ε_n sufficiently small, since $(\tilde\lambda + \frac{\eta}{2})(\tilde\lambda + \eta) < \tilde\lambda + 2\,\eta$. Moreover, $w_n(x) \in C^2(\overline{\Omega})$ (observe that $P_n \notin \partial\Omega$) and

$$\nabla w_n(x) = (\tilde\lambda + 2\,\eta)\frac{x - y_n}{|x - y_n|},$$

$$|\Delta w_n| \leq \frac{C}{|y_n - x|} \leq \frac{C}{(\eta|P_n - \overline{P_n}|)}.$$

Then, for ε_n sufficiently small,

$$\varepsilon_n \Delta w_n - |w_n|^2 + \tilde{\lambda} \leq \frac{C\ \varepsilon_n}{(\eta |P_n - \overline{P_n}|)} - (\tilde{\lambda} + \eta)^2 + \tilde{\lambda} < 0,$$

since $\frac{\varepsilon_n}{|P_n - \overline{P_n}|} = \frac{1}{\rho_n} \to 0$ as $\varepsilon_n \to 0$. Then, by the Maximum Principle, we have

$$\psi_n(x) \leq w_n(x), \quad \text{for} \quad x \in \Omega.$$

Therefore

$$\begin{aligned} \psi_n(P_n) &\leq w_n(P_n) = (\tilde{\lambda} + 2\,\eta)(|P_n - \overline{P_n}| + |y_n - P_n|) \\ &= (\tilde{\lambda} + 2\ \eta)(1 + \tilde{\lambda} + \eta)|P_n - \overline{P_n}| \\ &< (1 + \tilde{\lambda} + \sigma_0)|P_n - \overline{P_n}|. \end{aligned}$$

(3) We just need to proof that

$$V_n(y) \leq C\, e^{(\tilde{\lambda} + \frac{\sigma_1}{2})|y|}$$

for $\varepsilon_n < \varepsilon_0$. Observe that $\frac{\varepsilon_n}{d(P_n, \partial\Omega)} \to 0$ as $n \to +\infty$. Now, for any given $\sigma_1 > 0$, we let $\delta_1 = \frac{\sigma_1}{10}$ and χ_1 be a cut-off function such that $\chi_1(r) \equiv 1$ for $r \leq \tilde{\lambda} - \delta_1$ and $\chi_1(r) \equiv 0$ for $r > \tilde{\lambda} - \frac{\delta_1}{2}$. Setting $\tau_n^1(x) = \nabla\psi_n(x)$ and $\tau_n(x) = \nabla\psi_n(x)\chi_1\left(\frac{|x - P_n|}{d(P_n, \partial\Omega)}\right)$, we prove that

$$\max_{x \in \Omega} \tau_n(x) \leq \frac{C}{d(P_n, \partial\Omega)} \tag{3.2.27}$$

where $C = C(\delta_1)$, but is independent of ε_n.
Suppose that (3.2.27) is proved, it implies that

$$\Delta\psi_n(x) \leq \frac{C}{d(P_n, \partial\Omega)} \quad \text{for } x \in B_{(\tilde{\lambda} - \delta_1)\, d(P_n, \partial\Omega)}(P_n).$$

Hence

$$|\nabla\psi_n(x)|^2 = \tilde{\lambda} + \varepsilon_n \Delta\psi_n(x) \leq \tilde{\lambda} + C\frac{C}{d(P_n, \partial\Omega)} \leq \tilde{\lambda} + \frac{\sigma_1}{2}$$

for $x \in B_{(\tilde{\lambda} - \delta_1)\, d(P_n, \partial\Omega)}(P_n)$ and $\varepsilon_n < \varepsilon_0$. Then

$$\begin{aligned} V_n(z) &= e^{\frac{1}{\varepsilon_n}(\psi_n(P_n) - \psi_n(x))} = e^{-\frac{1}{\varepsilon_n}\nabla\psi_n(\overline{x})(x - P_n)} \\ &= e^{-\nabla\psi_n(\overline{x})\, z} \leq e^{(\tilde{\lambda} + \frac{\sigma_1}{2})|z|} \end{aligned}$$

for $z \in B_{(\tilde{\lambda} - \delta_1)\frac{d(P_n, \partial\Omega)}{\varepsilon_n}}(0)$.
If $\frac{(\tilde{\lambda} - \delta_1)\, d(P_n, \partial\Omega)}{\varepsilon_n} \leq |z|$, i.e. $|x - P_n| \geq (1 - \delta_1)\, d(P_n, \partial\Omega)$, we observe that by (3.2.19) $\psi_n(x) \geq |x - P_n|$ on $\partial\Omega$ and thus function $|x - P_n|$ is a subsolution of (3.2.8) on $\Omega \setminus B_{(\tilde{\lambda} - \delta_1)\, d(P_n, \partial\Omega)}(P_n)$ for some $\delta_1(\varepsilon_n) > 0$ sufficiently small, by (3.2.22) and the arguments following it in the proof of Lemma 3.2.20. Hence, we have $\psi_n(x) \geq |x - P_n|$

for $x \in \overline{\Omega} \setminus B_{(\tilde{\lambda}-\delta_1)\, d(P_n,\partial\Omega)}(P_n)$. Then $\varphi_n(z) \le e^{-|z|}$ for $z \in \overline{\Omega}_n \setminus B_{(\tilde{\lambda}-\delta_1)\, \frac{d(P_n,\partial\Omega)}{\varepsilon_n}}(0)$. By the first part of this Lemma, it follows that

$$\begin{aligned} V_n(z) &= e^{\frac{1}{\varepsilon_n}\,\psi_n(P_n)}\,\varphi(z) \le e^{\frac{1}{\varepsilon_n}\,\psi_n(P_n)-|z|} \\ &\le e^{(1+\tilde{\lambda}+\sigma_0)\,\frac{d(P_n,\partial\Omega)}{\varepsilon_n}-|z|} \le e^{(1+\sigma_0+\delta_1)\,\frac{1}{\varepsilon_n}\,d(P_n,\partial\Omega)} \\ &\le e^{\frac{(1+\sigma_0+\delta_1)}{(\tilde{\lambda}-\delta_1)}\,|z|} \le C\,e^{(\tilde{\lambda}+\frac{\sigma_1}{2})\,|z|}. \end{aligned}$$

So we have for all $z \in \Omega_n$,

$$V_n(z) \le C\,e^{(\tilde{\lambda}+\frac{\sigma_1}{2})\,|z|}.$$

Taking a diagonal process and passing to a subsequence $\varepsilon_{n_k} \to 0$, we have that $V_{n_k}(z) \to V_0(z)$ uniformly on any compact set of $\mathbb{R}^N$ and $V_0(0)$ is a solution of (3.2.12). Moreover

$$\sup_{z\in\overline{\Omega}_{n_k}} e^{-(\tilde{\lambda}+\sigma_1)\,|z|}|V_{n_k}(z)-V_0(z)| \to 0 \ \text{ as } \ \varepsilon_n \to 0.$$

We have to prove (3.2.27). Indeed τ_n^1 satisfies

$$-\varepsilon_n\,\Delta\tau_n^1 + 2\,\nabla\,\psi_n\,\cdot\,\nabla\tau_n^1 + 2\,|\nabla^2\psi_n|^2 = 0 \quad \text{in } \ \Omega.$$

Since $|\nabla^2\psi_n|^2 = \sum_{i,j=1}^N(\frac{\partial^2\,\psi_n}{\partial x_i\,\partial\,x_j})^2 \ge C_1\,|\tau_n^1|^2$ for some constant C_1, we see that

$$C_1\,|\tau_n^1|^2 - \varepsilon_n\,\Delta\tau_n^1 + 2\,\nabla\,\psi_n\,\cdot\,\nabla\tau_n^1 \le 0. \tag{3.2.28}$$

Multiplying (3.2.28) by $\chi_1^2\Big(\frac{|x-P_n|}{d(P_n,\partial\Omega)}\Big)$, we obtain

$$\begin{aligned} C_1|\tau_n|^2 \ &- \ \varepsilon_n\,\chi_1\,\Delta\tau_n + 2\,\chi_1\nabla\psi_n\,\cdot\,\tau_n + 2\,\varepsilon_n\,\nabla\tau_n\,\cdot\,\nabla\chi_1 \\ &- \ 2(\nabla\psi_n\,\cdot\,\nabla\chi_n)\,\tau_n + \varepsilon_n(\delta\chi_1 - 2\,\frac{|\nabla\chi_1|^2}{\chi_1})\,\tau_n \le 0. \end{aligned} \tag{3.2.29}$$

Observe that in (3.2.28) and the rest of the proof for simplicity, we always write χ_1 for $\chi_1\Big(\frac{|x-P_n|}{d(P_n,\partial\Omega)}\Big)$, while the argument in the other functions is x and the differentiations are taken with respect to x.
Let $\tau_n(x_0) = \max_{x\in\Omega}\tau_n(x)$. If $\tau_n(x_0) \le 0$, (3.2.27) holds.
If $\tau_n(x_0) > 0$, then we have

$$C_1\,|\tau_n|^2 \le 2\,(\nabla\psi_n\,\cdot\,\nabla\chi_1)\,\tau_n - \varepsilon_n\left(\nabla\chi_1 - 2\,\frac{|\nabla\chi_1|^2}{\chi_1}\right)\tau_n \ \text{ as } x_0.$$

Observe that

$$\begin{aligned}\nabla\psi_n \cdot \nabla\chi_1 &\leq |\nabla\psi_n|\,|\nabla\chi_1| = \sqrt{|\nabla\psi_n|^2\,|\chi_1|^2}\\ &= \sqrt{(1+\varepsilon_n\,\Delta\psi_n)\,|\nabla\chi_1|^2} = \sqrt{|\nabla\chi_1|^2 + \varepsilon_n \frac{|\nabla\chi_1|^2}{\chi_1}\,\tau_n}\\ &\leq C\frac{\sqrt{1+\varepsilon_n\,\tau_n}}{d(P_n,\partial\Omega)} \leq C\,\frac{(1+\varepsilon_n\,\tau_n)}{d(P_n,\partial\Omega)}\\ &\leq \frac{C}{d(P_n,\partial\Omega)} + \frac{C\,\varepsilon_n\,\tau_n}{d(P_n,\partial\Omega)}.\end{aligned}$$

Note that $|\Delta\chi_1 - 2\,\frac{|\nabla\chi_1|^2}{\chi_1}| \leq \frac{C}{d^2(P_n,\partial\Omega)}$. Therefore, by (3.2.29), we have at x_0

$$\begin{aligned}C_1\,\tau_n^2 &\leq \Big(\frac{C\,\varepsilon_n}{d^2(P_n,\partial\Omega)}\Big)\,\tau_n + \Big(\frac{C\,\varepsilon_n}{d(P_n,\partial\Omega)}\Big)\,\tau_n + \Big(\frac{C\,\varepsilon_n}{d(P_n,\partial\Omega)}\Big)\,\tau_n^2\\ &\leq \Big(\frac{C\,\varepsilon_n}{d(P_n,\partial\Omega)}\Big)\,\tau_n + \Big(\frac{C\,\varepsilon_n}{d(P_n,\partial\Omega)}\Big)\,\tau_n^2.\end{aligned}$$

Hence if we choose $\varepsilon_n < \varepsilon_0$ such that $\frac{C\,\varepsilon_n}{d(P_n,\partial\Omega)} \leq \frac{1}{2}\,C_1$, then we have

$$\tau_n(x_0) \leq \frac{C}{d(P_n,\partial\Omega)}$$

so (3.2.27) is established.

□

We want to derive an asymptotic formula. To this end, we define ϕ_n by

$$u_n(P_n+\varepsilon_n\,y) = U_n(y) = P_{\Omega_n}U(y) + e^{-\frac{1}{\varepsilon_n}\,\psi_n(P_n)}\phi_n(y),$$

for all $y\in\Omega_n$, where ϕ_n satisfies (3.2.10).

Remark 3.22. Observe that:

$$\Big|e^{\frac{1}{\varepsilon_n}\,\psi_n(P_n)}\Big[U_n^p - U^p - e^{-\frac{1}{\varepsilon_n}\,\psi_n(P_n)}p\,U^{p-1}\phi_n\Big] + p\,U^{p-1}V_n\Big|$$

$$= \Big|e^{\frac{1}{\varepsilon_n}\,\psi_n(P_n)}(U_n-U) - p\,U^{p-1}(\phi_n - V_n)\Big| \leq C|U_n-U|^\sigma|V_n-\phi_n|,$$

in fact if we consider $f(s)=s^p$, $f'(s) = p\,s^{p-1}$

$$|f'(t)-f'(s)| = f(s)(t-s) + o((t-s)^2) = p(t^{p-1}-s^{p-1}) \leq pC|t-s|^\sigma, \quad \sigma>1;$$

then, by the mean value theorem , if $u_n \leq t \leq U$

$$|f(u_n) - f(U) + e^{-\frac{1}{\varepsilon_n}\,\psi_n(P_n)}f'(U)(V_n-\phi_n)|$$

$$= \Big||f'(t)|\Big[-e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}(V_n-\phi_n)\Big] + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}f'(U)(V_n-\phi_n)\Big|$$

$$\leq |f'(U) - f'(t)||V_n - \phi_n||e^{-\frac{1}{\varepsilon_n}\psi_n}| \leq C\,|U-t|^\sigma |V_n - \phi_n| \leq |u_n - U|^\sigma |V_n - \phi_n|.$$

Next result is crucial in deriving the asymptotic expansion.

PROPOSITION 3.23. *We define ϕ_n by*

$$U_n(y) = P_{\Omega_n} U(y) + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\phi_n(y) \tag{3.2.30}$$

where $y \in \Omega_n$. Then ϕ_n satisfies (3.2.10). Moreover, let $\sigma > 1$, we have the following properties

1. *For $s > N$, $\| \phi_n \|_{W^{2,s}(\Omega_n)} \leq C(s)$.*
2. *For every sequence $\varepsilon_n \to 0$, there is a subsequence ε_{n_k} and a solution V_0 of (3.2.12) s. t. $\|\phi_{\varepsilon_k} - \phi_0\|_{L^\infty(B_{(1-\delta_2)\rho_{\varepsilon_k}})} \to 0$ as $n \to \infty$ where $\delta_2 = \frac{\sigma_1}{10}$ and ϕ_0 is a solution of (3.2.11). Furthermore, $\phi_0 \in W^{2,s}(\mathbb{R}^N)$ for $s > 1$.*

We need some preliminary results before the proof. The following two Lemmas play a basic role in our estimates.

LEMMA 3.24. (1) *Let $s > 1$ and u be a solution of*

$$\begin{cases} \Delta u - u + \bar{f} = 0 & \text{in } \Omega_n \\ u = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.2.31}$$

Then

$$\|u\|_{W^{2,s}(\Omega_n)} \leq C\,(\|\bar{f}\|_{L^s(\Omega_n)} + \|\bar{f}\|_{L^2(\Omega_n)}), \tag{3.2.32}$$

where C is a constant independent of $\varepsilon_n \leq \varepsilon_0$.

(2) *Let u be a solution of*

$$\Delta u - u + \tilde{f} = 0 \quad \text{in } \mathbb{R}^N$$

with $\|\tilde{f}\|_{L^s(\mathbb{R}^N)} < \infty$ and $\|\tilde{f}\|_{L^2(\mathbb{R}^N)} < \infty$. Then

$$\|u\|_{W^{2,s}(\mathbb{R}^N)} \leq C\,(\|\tilde{f}\|_{L^s(\mathbb{R}^N)} + \|\tilde{f}\|_{L^2(\mathbb{R}^N)}). \tag{3.2.33}$$

(3) *For every function $\tilde{K} \in C^2(\overline{\Omega_n})$, there exists an extension $K \in C_0^2(\mathbb{R}^N)$ with*

$$\|K\|_{W^{2,s}(\mathbb{R}^N)} \leq C\|\tilde{K}\|_{W^{2,s}(\Omega_n)}, \tag{3.2.34}$$

where $s > 1$ and C is independent of $\tilde{K}$ and $\varepsilon_n \leq 1$

PROOF. We carry off the proof in [**89**].
The second item follows from the first one by truncation. For the proof of third item see Lemma 4.2 (2) of [**102**]. We just prove the first item. We use the same idea of Lemma 1.1 in [**55**].
We claim that there exists constants δ_0 and C^* (independent of $\varepsilon_n \leq \varepsilon_0$), such that for each $y \in \partial\Omega_n$ the set $\partial\Omega_n \cup \{\, x : |x-y| < \delta_0 \,\}$ can be represented in the form

$$x_i - y_i = \Phi(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$$

for some $i,\ 1 \le i \le N$ and

$$|\Phi| + \sum \Big|\frac{\partial \Phi}{\partial x_j}\Big| + \sum \Big|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\Big| \le C^*,$$

for the proof of this claim see paragraph 3 in [**89**].

Then we introduce a mesh in $\mathbb{R}^N$ made up of cubes with sides parallel to the coordinate axes and having length $\eta = \frac{\delta_0}{3\sqrt{N}}$. Denote $\Gamma_1, \dots, \Gamma_{h_0}$ those cubes whose closure intersects $\partial\Omega_n$. Denote the center of Γ_j by y_j. Let Γ'_j, Γ''_j be cubes with center y_j and with sides parallel to the coordinate axes, having length $2\,\eta$ and $3\,\eta$ respectively. Then $\Gamma'_1, \dots, \Gamma'_{h_0}$ form an open covering of $\partial\Omega_n$. We note that for any $y \in \partial\Omega_n$ there is a cube Γ'_j such that $y \in \Gamma_j$ and $d(y, \partial\Gamma'_j) \ge \frac{\delta_0}{2}$.

Let Ψ be a C^∞ function such that

$$\begin{array}{lll} \Psi(x) = 1 & \text{if } |x_i| < \eta & \text{for all } i = 1, 2, \dots, N, \\ \Psi(x) = 0 & \text{if } |x_i| > \frac{5}{4}\eta & \text{for some } i, \\ 0 \le \Psi(x) \le 1 & & \text{elsewhere,} \end{array} \tag{3.2.35}$$

and set $\Psi_j(x) = \Psi(y_j + x)$. Then $\Psi_j = 1$ in Γ'_j and $\Psi_j = 0$ in a small neighborhood of $\partial\Gamma''_j$ and outside Γ''_j.

Denote by $\Omega_{n,\eta} : \{\, x \in \Omega_n \ : \ d(x, \partial\Omega_n) > \frac{\eta}{2}\,\}$. We introduce a mesh made up of cubes with sides parallel to the coordinate axes and having length $\eta_0 = \frac{\delta_0}{8\sqrt{N}}$.

Denote by $\Delta_1, \dots, \Delta_{h_1}$ those cubes whose closure intersects Ω_n. Let Δ'_j, Δ''_j be the cubes with the same center z_j as Δ_j and with sides parallel to the coordinate axes, having length $2\,\eta_0$ and $3\,\eta_0$ respectively.

The cubes $\Delta'_1, \dots, \Delta'_{h_1}$ form an opening covering of $\overline{\Omega}_{n,\eta}$ and the cubes $\Delta''_1, \dots, \Delta''_{h_1}$ lie entirely in Ω_n.

Let χ be the C^∞ function

$$\chi(x) = \Psi\Big(\frac{\eta}{\eta_0} x\Big)$$

and let $\chi_j(x) = \chi(z_j + x)$. Let

$$\begin{array}{ll} \varphi_j = \frac{\Psi_j}{\sum \Psi_k + \sum \chi_k} & \text{if } 1 \le j \le h_0, \\ \varphi_{j+h_0} = \frac{\chi_j}{\sum \Psi_k + \sum \chi_k} & \text{if } 1 \le j \le h_1, \\ G_j = \Gamma''_j,\ G'_j = \Gamma'_j & \text{if } 1 \le j \le h_0, \\ G_{j+h_0} = \Delta''_j,\ G'_{j+h_0} = \Delta'_j & \text{if } 1 \le j \le h_1, \end{array} \tag{3.2.36}$$

and let $h = h_0 + h_1$. Then $G_1, \dots, G_h$ form an open covering of $\overline{\Omega}_n$ and $\varphi_1, \dots, \varphi_h$ form a partition of unity subordinate to this converging, such that

1. $G_1, \dots, G_{h_0}$ intersect $\partial\Omega_n$ and $G_{h_0+1}, \dots, G_h$ lie entirely in Ω_n;
2. $\varphi_k \in C_0^\infty(G_k)$;
3. each $x \in \overline{\Omega}_n$ belongs to at most N_1 sets G_k, where N_1 is a positive integer independent of $\varepsilon_n \le \varepsilon_0$;
4. $\varphi_k \ge \frac{1}{N_1}$ on the set $G'_1, \dots, G'_h$ form an opening covering of $\overline{\Omega}_n$;
5. there is a constant N_2 independent of k, ε_n, such that

$$|D^\alpha \varphi_k| \le N_2 \ \text{ if } |\alpha| \le 2, \quad x \in G_k,\ \ 1 \le k \le h.$$

Let $G_{k,\varepsilon_n} = G_k \cap \Omega_n$ and $Au = -\Delta u + u$. Note that φ_k has compact support in G_k. By standard regularity theorem, we have

$$\|u\,\varphi_k\|_{W^{2,s}(G_{k,n})} \le C(\|A(u\,\varphi_k)\|_{L^s(G_{k,n})} + \|u\,\varphi_k\|_{L^s(G_{k,n})}) \tag{3.2.37}$$

where C is a constant independent of k, $\varepsilon_n \le \varepsilon_0$. Now we note that

$$A(u\,\varphi_k) = \bar{f}\,\varphi_k - 2\,\nabla u\,\nabla\varphi_k - u\,\Delta\varphi_k.$$

Hence

$$|A(u\,\varphi_k)|^s \le C|\bar{f}|^s + C|Du|^s + C|u|^s. \tag{3.2.38}$$

By (3.2.37), we have

$$\begin{aligned}\int_{G_{k,n}} (|D^2u|\varphi_k)^s \mathrm{d}x \;\; &\le \;\; C\int_{G_{k,n}} (|Du|\;|D\varphi_k|)^s \mathrm{d}x + C\int_{G_{k,n}} (|u|\;|D^2\varphi_k|)^s \mathrm{d}x \\ &+ \;\; C\int_{G_{k,n}} (A(u\,\varphi_k))^s \mathrm{d}x + C\int_{G_{k,n}} (|u\varphi_k|)^s \mathrm{d}x.\end{aligned}$$

By using (3.2.37) and (3.2.38) we have

$$\begin{aligned}\int_{G_{k,n}} (|D^2u|\varphi_k)^s \mathrm{d}x \;\; &\le \;\; C\int_{G_{k,n}} |\bar{f}|^s \mathrm{d}x \\ &+ \;\; \int_{G_{k,n}} |Du|^s \mathrm{d}x + \int_{G_{k,n}} |u|^s \mathrm{d}x.\end{aligned}$$

Summing for $k = 1, \ldots, h$ we obtain

$$\begin{aligned}\int_{\Omega_n} (|D^2u|)^s \mathrm{d}x \;\; &\le \;\; C\int_{\Omega_n} |\bar{f}|^s \mathrm{d}x \\ &+ \;\; \int_{\Omega_n} |Du|^s \mathrm{d}x + \int_{G_{k,n}} |u|^s \mathrm{d}x.\end{aligned} \tag{3.2.39}$$

We have thus proved that

$$\int_{\Omega_n} A\,u(x)\;u(x)\mathrm{d}x \ge C\|u\|_{W^{1,2}(\Omega_n)}. \tag{3.2.40}$$

Therefore

$$\|u\|_{W^{1,2}(\Omega_n)} \le C\|\bar{f}\|_{L^2(\Omega_n)}, \tag{3.2.41}$$

for $s = 2$ the thesis follows from (3.2.39) and (3.2.41). By using (3.2.41) and a Sobolev's inequality, the rest of the proof is exactly the same as that of Lemma 1.1 in [**55**] □

Now we give a characterization of the kernel of the operator associated to the equation $-\Delta U + \tilde{\lambda}U - U^p = 0$.

LEMMA 3.25. *If the domain of the operator $L = -\Delta + \tilde{\lambda} - U^{p-1}$ is $W^{2,s}(\mathbb{R}^N)$ where $s > \frac{N}{2}$, then ker$(L) = X =$ Span$\{\,\partial_{x_1}U, \ldots, \partial_{x_N}U\,\}$.*

PROOF. Let $\varphi \in Ker(L) \cap W^{2,s}(\mathbb{R}^N)$, then $\|\varphi\|_{W^{2,s}} \le C$. Hence $\varphi(y) \le C$. Moreover, by regularity Theorem, $\varphi \in C^\infty(\mathbb{R}^N)$. By third part of Lemma 3.24, we have

$$\varphi(y) = p\int_{\mathbb{R}^N} G(y-z)U^{p-1}(z)\,\varphi(z)\mathrm{d}z \tag{3.2.42}$$

where $G(y-z)$ is the Green's function for $-\Delta+\tilde{\lambda}$ and $0 < G(y-z) < \frac{C_N}{|y-z|^{N-2}}(1+|y-z|)^{\frac{N-3}{2}}e^{-|y-z|}$. We have

$$\varphi(y) \le \int_{\mathbb{R}^N} \frac{(1+|y-z|)^{\frac{N-3}{2}}}{|y-z|^{N-2}} e^{-|y-z|}e^{-\sigma|z|}\mathrm{d}z \le C.$$

Let $\sigma = \min\{1, p-1\}$. Substituting this into (3.2.42) we obtain

$$\begin{aligned} e^{\sigma'|y|}\varphi(y) &\le \int_{\mathbb{R}^N} \frac{(1+|y-z|)^{\frac{N-3}{2}}}{|y-z|^{N-2}} e^{-|y-z|}e^{-\sigma|z|}e^{\sigma'|y|}\mathrm{d}z \\ &\le C\int_{\mathbb{R}^N} \frac{(1+|y-z|)^{\frac{N-3}{2}}}{|y-z|^{N-2}} e^{-(1-\sigma')|y-z|}e^{\sigma'(|y|-|z|-|y-z|)}e^{(\sigma'-\sigma)|z|}\mathrm{d}z \\ &\le C \end{aligned}$$

for $y \in \mathbb{R}^N$ and $0 < \sigma' < \sigma$, i.e., φ decays exponentially. Once φ decays exponentially, standard elliptic regularity estimates guarantee that $\varphi(x) \in W^{2,s}(\mathbb{R}^N)$, for all $s > 1$. By Lemma 4.2 in [**87**], we finish the proof. □

PROOF OF PROPOSITION 3.23. This proof will follow the idea in [**87, 102**]. We first prove that $\|\phi_n\|_{L^s(\Omega_n)}$ is bounded for $s > N$. Then by Proposition 3.21, for every sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ and a solution V_0 of (3.2.12) such that $V_{\varepsilon_{n_k}} \to V_0$. Letting $\overline{\phi}_{n_k} = \chi(\frac{|y|}{\rho_{\varepsilon_{n_k}}})\phi_0(y)$ where $\chi(r) = 1$ when $r \le \tilde{\lambda}-\sigma_2$ and $\chi(r) = 0$ when $r > \tilde{\lambda}-\frac{\sigma_2}{2}$, we show that $\|\phi_{n_k} - \overline{\phi}_{n_k}\|_{W^{2,s}(\Omega_{n_k})} = o(1)$, which by the Sobolev Imbedding Theorem, proves Proposition.

We begin with the following estimates.

LEMMA 3.26. *For every sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ and a solution V_0 of (3.2.12) such that for $2 < s \le \infty$*

$$\|e^{\frac{1}{\varepsilon_{n_k}}}\{U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}}p\,U^{p-1}\phi_n\} + p\,U^{p-1}V_0\|_{L^s(\Omega_{\varepsilon_{n_k}})} \le C(o(1)\|\phi_{n_k}\|_{L^s(\Omega_{n_k})} + o(1)), \tag{3.2.43}$$

$$\|e^{\frac{1}{\varepsilon_{n_k}}}\{U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}}p\,U^{p-1}\phi_n\} + p\,U^{p-1}V_0\|_{L^2(\Omega_{\varepsilon_{n_k}})} \le C(o(1)\|\phi_{n_k}\|_{L^s(\Omega_{n_k})} + o(1)). \tag{3.2.44}$$

PROOF. By Proposition 3.21, for every sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ and a solution V_0 of (3.2.12) such that $V_{n_k} \to V_0$. By remark 3.22, we have

$$\begin{aligned}\left|e^{\frac{1}{\varepsilon_{n_k}}}\{\, U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}}} p\, U^{p-1}\phi_n \,\} + p\, U^{p-1}\, V_0\right| &\leq C|p\, U^{p-1}|\,|V_0 - V_{n_k}| \\ &+ C|U_{n_k} - U|^{\sigma}\,|\phi_{n_k} - V_{n_k}|.\end{aligned} \tag{3.2.45}$$

Since $|V_{n_k} - V_0| = o(1)\, e^{(\tilde\lambda+\sigma_1)|y|}$ by Proposition 3.21, it follows that

$$\begin{aligned}&\left|e^{\frac{1}{\varepsilon_{n_k}}}\{\, U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}}} p\, U^{p-1}\phi_n \,\} + p\, U^{p-1}\, V_0\right| \\ \leq\ & o(1)\ e^{(-\sigma+(\tilde\lambda+\sigma_1))|y|} + C\,|U_{n_k} - U|^{\sigma}\,|\phi_{n_k} - V_{n_k}|.\end{aligned}$$

Then, from Proposition 3.15 we conclude, for $2 < s < \infty$,

$$\begin{aligned}&\left\|e^{\frac{1}{\varepsilon_{n_k}}}\{\, U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}}} p\, U^{p-1}\phi_n \,\} + p\, U^{p-1}\, V_0\right\|_{L^s(\Omega_{n_k})} \\ \leq\ & o(1) + C\|\phi_{n_k}\|_{L^s(\Omega_{n_k})}\|U_{n_k} - U\|_{L^\infty(\Omega_{n_k})} + C\|V_{n_k}(U_{n_k} - U)^{\sigma}\|_{L^s(\Omega_{n_k})} \\ =\ & o(1) + o(1)\|\phi_{n_k}\|_{L^s(\Omega_{n_k})},\end{aligned}$$

since $\|V_{n_k}(U_{n_k} - U)^{\sigma}\|_{L^s(\Omega_{n_k})} = o(1)$, by Lebesgue's Dominated Convergence Theorem. In the same way we have

$$\begin{aligned}&\left\|e^{\frac{1}{\varepsilon_{n_k}}}\{\, U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}}} p\, U^{p-1}\phi_n \,\} + p\, U^{p-1}\, V_0\right\|_{L^2(\Omega_{n_k})} \\ \leq\ & o(1) + C\|\phi_{n_k}\|_{L^s(\Omega_{n_k})}\|(U_{n_k} - U)^{\sigma}\|_{L^{\frac{2s}{s-2}}(\Omega_{n_k})} \\ +\ & C\|V_{n_k}(U_{n_k} - U)^{\sigma}\|_{L^2(\Omega_{n_k})} = o(1) + o(1)\|\phi_{n_k}\|_{L^s(\Omega_{n_k})}.\end{aligned}$$

For the case $s = \infty$ we can proceed in a similar manner. □

LEMMA 3.27. *Let $N < s < \infty$. Then $\|\phi_n\|_{L^s(\Omega_n)} \leq C(s)$.*

PROOF. We prove this Lemma by contradiction. Suppose that there exists a sequence of $\varepsilon_{n_k} \to 0$ such that $\|\phi_{n_k}\|_{L^s(\Omega_{n_k})} \to \infty$.
Let $M_k = \|\phi_{n_k}\|_{L^s(\Omega_{n_k})}$, $g_k = \frac{\phi_{n_k}}{M_k}$, that satisfies

$$\begin{cases} \Delta g_k - \tilde\lambda_{n_k} g_k + pU^{p-1} g_k + \dfrac{e^{\frac{1}{\varepsilon_{n_k}}\psi_{n_k}(P_n)}}{M_k}(U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}}} pU^{p-1}\psi_{n_k}) & \text{in } \Omega_{n_k} \\ g_k = 0 & \text{on } \partial\Omega_{n_k} \end{cases} \tag{3.2.46}$$

We show that $\|g_k\|_{W^{2,s}(\Omega_{n_k})}$ is bounded. By Lemma 3.24

$$\begin{aligned}\|g_k\|_{W^{2,s}(\Omega_{n_k})} &\leq C\Bigg(\|pU^{p-1}\,g_j\|_{L^s(\Omega_{n_k})}+\|pU^{p-1}\,g_k\|_{L^2(\Omega_{n_k})}\\ &+\left\|\frac{e^{\frac{1}{\varepsilon_{n_k}}\psi_{n_k}(P_n)}}{M_j}(U^p_{n_k}-U^p-e^{-\frac{1}{\varepsilon_{n_k}}}pU^{p-1}\psi_{n_k})\right\|_{L^s(\Omega_{n_k})}\\ &+\left\|\frac{e^{\frac{1}{\varepsilon_{n_k}}\psi_{n_k}(P_n)}}{M_j}(U^p_{n_k}-U^p-e^{-\frac{1}{\varepsilon_{n_k}}}pU^{p-1}\psi_{n_k})\right\|_{L^2(\Omega_{n_k})}\Bigg).\end{aligned} \tag{3.2.47}$$

Since

$$\begin{aligned}\|pU^{p-1}\,g_k\|_{L^s(\Omega_{n_k})} &\leq C\|g_k\|_{L^s(\Omega_{n_k})},\\ \|pU^{p-1}\,g_k\|_{L^2(\Omega_{n_k})} &\leq C\|U^\sigma\|_{L^{\frac{2s}{s-2}}(\Omega_{n_k})}\|g_k\|_{L^s(\Omega_{n_k})}\leq C\|g_k\|_{L^s(\Omega_{n_k})},\end{aligned}$$

and by Lemma 3.26

$$\begin{aligned}&\left\|\frac{e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k}-U^p-e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k})}{M_k}\right\|_{L^s(\Omega_{n_k})}\\ \leq\;& C\frac{\|pU^{p-1}V_0\|_{L^s(\Omega_{n_k})}}{M_k}+o(1)\|g_k\|_{L^s(\Omega_{n_k})}+o(1)=o(1),\\ &\left\|\frac{e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k}-U^p-e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k})}{M_k}\right\|_{L^2(\Omega_{n_k})}\\ \leq\;& C\frac{\|pU^{p-1}V_0\|_{L^2(\Omega_{n_k})}}{M_k}+o(1)\|g_k\|_{L^s(\Omega_{n_k})}+o(1)=o(1)\end{aligned}$$

we obtain $\|g_k\|_{W^{2,s}(\Omega_{n_k})}\leq C$.
Now from Lemma 3.24, we can extend g_k to a C^2 function with compact support in $\mathbb{R}^N$, still denoted by g_k, in such a way that $\|g_k\|_{W^{2,s}(\mathbb{R}^N)}\leq C$, where the constant C is independent of k. We can conclude that

$$\|g_k\|_{L^\infty(\mathbb{R}^N)}\leq C \tag{3.2.48}$$

by Sobolev Imbedding Theorem, and that there exists a function $g_0\in W^{2,s}(\mathbb{R}^N)$ such that, by passing to a subsequence if necessary, $g_k\to g_0$ weakly in $W^{2,s}(\mathbb{R}^N)$ and $g_k\to g_0$ in $C^1_{loc}(\mathbb{R}^N)$. We have to show that $g_k=0$. We estimate

$$\left\|\frac{e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k}-U^p-e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k})}{M_k}\right\|_{L^\infty(\Omega_{n_k})}.$$

Note that in Lemma 3.26, we now take $s = \infty$. Then, as before, we have

$$\left\| \frac{e^{\frac{1}{\varepsilon_{n_k}} \psi_{n_k}(P_n)}(U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}} p\, U^{p-1}\phi_{n_k})}{M_k} \right\|_{L^\infty(\Omega_{n_k})} \to 0$$

as $k \to +\infty$ by (3.2.48). Therefore

$$\left\| \frac{e^{\frac{1}{\varepsilon_{n_k}} \psi_{n_k}(P_n)}(U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}} p\, U^{p-1}\phi_{n_k})}{M_k} \right\|_{L^\infty(\Omega_{n_k})} \to 0$$

on every compact set of $\mathbb{R}^N$.
Hence g_0 is a weak, thus classical, solution of the following equation

$$\text{(3.2.49)} \qquad \begin{cases} L\, g_0 = \Delta g_0 - \tilde{\lambda}\, g_0 + pU^{p-1}\, g_0 = 0 & \text{in } \mathbb{R}^N, \\ g_0 \in W^{2,s}(\mathbb{R}^N) & N < s. \end{cases}$$

By Lemma 3.25, $g_0 \in X$. That is $g_0 = \sum_{i=1}^N a_i\, e_i$, with $e_i = \partial_{x_i} U$, for some constants a_i, $i = 1, \ldots, N$.
By definition $U_{n_k}(y) = U(y) + e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}(P_{n_k})}(\phi_{n_k} - V_{n_k})$. Hence

$$0 = \nabla U_{n_k}(0) = \nabla U(0) + e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}(P_{n_k})}(\nabla \phi_{n_k}(0) - \nabla V_{n_k}(0))$$

which implies that $\nabla \phi_{n_k}(0) = \nabla V_{n_k}(0)$. Thus

$$\nabla g_k(0) = \frac{\nabla V_{n_k}(0)}{M_k} \to 0 \;\text{ as } k \to +\infty$$

since V_{n_k} is bounded in $C^2_{loc}(\mathbb{R}^N)$ and standard elliptic regularity estimates. Therefore

$$\nabla g_0(0) = \sum_{i=1}^N a_i\, \nabla e_i(0) = 0.$$

Observing that $\nabla e_1(0), \ldots, \nabla e_N(0)$ are linearly independent, we conclude that $a_i = 0$, $i = 1, \ldots, N$. Hence $g_0 = 0$ and $g_k \to 0$ weakly in $W^{2,s}(\mathbb{R}^N)$, which completes the proof.
Now we prove that $\|g_j\|_{W^{2,s}(\mathbb{R}^N)} = o(1)$, which gives a contradiction, in fact $\|g_k\|_{L^s(\Omega_{n_k})} = 1$. As in the previous calculation, we have (3.2.47) and

$$\begin{aligned} \left\| \frac{e^{\frac{1}{\varepsilon_{n_k}} \psi_{n_k}(P_n)}(U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}} p\, U^{p-1}\phi_{n_k})}{M_k} \right\|_{L^s(\Omega_{n_k})} &= o(1), \\ \left\| \frac{e^{\frac{1}{\varepsilon_{n_k}} \psi_{n_n}(P_n)}(U_{n_k}^p - U^p - e^{-\frac{1}{\varepsilon_{n_k}} \psi_{n_k}} p\, U^{p-1}\phi_{n_k})}{M_k} \right\|_{L^2(\Omega_{n_k})} &= o(1). \end{aligned}$$

Let $\sigma = \min\{1, p-1\}$. We have the following estimates:

$$\begin{aligned}\|p\,U^{p-1}\,g_k\|^s_{L^s(\Omega_{n_k})} &\leq C\int_{\Omega_{n_k}\cap B_R^C} U^{s\,\sigma}\,g_k^s + C\int_{\Omega_{n_k}\cap B_R} U^{s\,\sigma}\,g_k^s \\ &\leq Ce^{-s\,\sigma\,R}\|g_k\|_{L^s(\Omega_{n_k})} + C\int_{\Omega_{n_k}\cap B_R} U^{s\,\sigma}\,g_k^s \\ &\leq Ce^{-s\,\sigma\,R}\|g_k\|_{L^s(\Omega_{n_k})} + C\int_{\Omega_{n_k}\cap B_R} g_k^s, \qquad (3.2.50)\end{aligned}$$

$$\begin{aligned}\|p\,U^{p-1}\,g_j\|^2_{L^2(\Omega_{n_k})} &\leq C\int_{\Omega_{n_k}\cap B_R^C} U^{2\,\sigma}\,g_k^2 + C\int_{\Omega_{n_k}\cap B_R} U^{2\,\sigma}\,g_k^2 \\ &\leq C\|U^{2\,\sigma}\|_{L^{\frac{s}{s-2}}(B_R^C)}\|g_k\|^2_{L^2(\Omega_{n_k})} + C\int_{\Omega_{n_k}\cap B_R} g_k^2 \\ &\leq CR^{\frac{N(s-2)}{s}}\,e^{-2\,\sigma\,R}\|g_k\|_{L^s(\Omega_n)} + C\int_{\Omega_n\cap B_R} g_k^2, \qquad (3.2.51)\end{aligned}$$

where $R \geq 1$ is an arbitrary number and C is independent of R.
Since $g_k \to 0$ in $C^1_{loc}(\mathbb{R}^N)$, we have

$$\begin{aligned}\limsup_{k\to+\infty}\|p\,U^{p-1}\,g_k\|_{L^s(\Omega_{n_k})} &\leq C\,e^{-\sigma\,R} \\ \limsup_{k\to+\infty}\|p\,U^{p-1}\,g_k\|_{L^2(\Omega_{n_k})} &\leq C\,R^{\frac{N(s-2)}{2\,s}}\,e^{-\sigma\,R}.\end{aligned}$$

Letting $R \to +\infty$ from (3.2.47) we conclude that

$$\|g_k\|_{W^{2,s}(\Omega_{n_k})} = o(1).$$

□

Corollary 3.28. $\|\phi_{n_k}\|_{W^{2,s}(\Omega_n)} \leq C(s)$ *for* $s > N$.

Proof. Indeed, we note that $\|\phi_{n_k}\|_{W^{2,s}(\Omega_{n_k})} \leq C(s)$, and so our conclusion follows from the same argument in the first part of the proof of Lemma 3.27.

□

By Lemma 3.24, we can extend ϕ_{n_k} to a function, still denoted by ϕ_{n_k}, such that $\|\phi_{n_k}\|_{W^{2,s}(\mathbb{R}^N)} \leq C(s)$ for $N < s < \infty$. We fix $s > N$. For any subsequence ε_{n_k}, we can take a further sequence, still denoted by ε_{n_k}, such that $\phi_{n_k} \to \phi_0$ weakly in $W^{2,s}(\mathbb{R}^N)$ and $\phi_{n_k} \to \phi_0$ in $C^1_{loc}(\mathbb{R}^N)$.

Now we continue with the proof of Proposition 3.23.
The first part of the proof is just Corollary 3.28. Next we show that ϕ_0 is a solution of (3.2.11). We need to prove that

$$e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}\,p\,U^{p-1}\phi_{n_k}) \to -p\,U^{p-1}V_0$$

in $L^1_{loc}(\mathbb{R}^N)$, but this can be easily deduced from (3.2.45). Hence ϕ_0 is a solution of the equation (3.2.11).

We have to derive the property of ϕ_0. We prove that $\|\phi_0\|_{W^{2,s}(\mathbb{R}^N)} < \infty$ for all $s > 1$. Indeed by (3.2.11), ϕ_0 satisfies

$$-\Delta\phi_0 + \tilde{\lambda}\phi_0 = p\,U^{p-1}(\phi_0 - V_0),$$

and we have

$$\begin{aligned} |p\,U^{p-1}(y)(\phi_0(y) - V_0(y))| &\leq C\,e^{-\sigma\,|y|} + C\,e^{-\sigma\,|y|}e^{(1+\sigma_1)\,|y|} \\ &\leq C\,e^{(1-\sigma+\sigma_1)\,|y|}. \end{aligned}$$

Then $\|p\,U^{p-1}(\phi_0 - V_0)\|_{L^s(\mathbb{R}^N)} < \infty$ for all $s > 1$. Therefore, by Lemma 3.24, our assertion is established.

It remains to prove that $\phi_{n_k} \to \phi_0$ in some sense. Let $\chi(r) = 1$ for $r \leq \tilde{\lambda} - \delta_1$, $\chi(r) = 0$ for $r \geq \tilde{\lambda} - \frac{\delta_1}{2}$ where $\delta_1 = \frac{\sigma_1}{10}$. Setting $\chi_k(y) = \chi\Big(\frac{|y|}{\rho_k}\Big)$ and $\overline{\phi}_{n_k} = \chi_k(y)\phi_0$, we see that $\phi_{n_k} - \overline{\phi}_{n_k}$ satisfies the following equation

$$\begin{aligned} &\Delta(\phi_{n_k} - \overline{\phi}_{n_k}) - \lambda_n(\phi_{n_k} - \overline{\phi}_{n_k}) + p\,U^{p-1}(\phi_{n_k} - \overline{\phi}_{n_k}) \\ = \;& -[e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k}) + p\,U^{p-1}V_0)] \\ + \;& (1-\chi_k)p\,U^{p-1}V_0 - 2\,\nabla\chi_k\,\nabla\phi_0 - (\Delta\,\chi_k)\,\phi_0. \end{aligned}$$

From Lemma 3.26 we have that

$$\begin{aligned} &\|e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k}) + p\,U^{p-1}V_0)\|_{L^s(\Omega_{n_k})} \\ + \;& \|e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k}) + p\,U^{p-1}V_0)\|_{L^2(\Omega_{n_k})} = o(1). \end{aligned} \tag{3.2.52}$$

Then

$$\begin{aligned} &\|(1-\chi_k)p\,U^{p-1}V_0\|_{L^s(\Omega_{n_k})} + \|(1-\chi_k)p\,U^{p-1}V_0\|_{L^2(\Omega_{n_k})} \\ \leq \;& \|p\,U^{p-1}\,V_0\|_{L^s(\Omega_{n_k}\cap B^C_{(1-\delta_1)\rho_k})} + C\|p\,U^{p-1}\,V_0\|_{L^2(\Omega_{n_k}\cap B^C_{(\tilde{\lambda}-\delta_1)\rho_k})} \\ \leq \;& C\,\rho_k^{\frac{N}{2}}\,e^{((\tilde{\lambda}-\sigma)+\sigma_1)\rho_k} = o(1), \end{aligned} \tag{3.2.53}$$

$$\begin{aligned} \|2\,\nabla\chi_k\,\nabla\phi_0\|_{L^s(\Omega_{n_k})} &+ \|2\,\nabla\chi_k\,\nabla\phi_0\|_{L^2(\Omega_{n_k})} + \|(\Delta\,\chi_k)\,\phi_0\|_{L^s(\Omega_{n_k})} + \|(\Delta\,\chi_k)\,\phi_0\|_{L^2(\Omega_{n_k})} \\ &\leq \frac{C}{\rho_k}(\|\phi_0\|_{W^{2,s}(\mathbb{R}^N)} + \|\phi_0\|_{W^{2,2}(\mathbb{R}^N)}). \end{aligned} \tag{3.2.54}$$

The same argument leading to (3.2.50) and (3.2.51) yields

$$\|p\,U^{p-1}\,(\phi_{n_k} - \overline{\phi}_{n_k})\|^s_{L^s(\Omega_{n_k})} \leq C\,e^{-s\,\sigma\,R} + C\int_{\Omega_{n_k}\cap B_R} |\phi_{n_k} - \overline{\phi}_{n_k}|^s, \tag{3.2.55}$$

$$\|p\,U^{p-1}\,(\phi_{n_k} - \overline{\phi}_{n_k})\|^2_{L^2(\Omega_{n_k})} \leq C\,R^{\frac{N(s-2)}{s}}\,e^{-2\,\sigma\,R} + C\int_{\Omega_{n_k}\cap B_R} |\phi_{n_k} - \overline{\phi}_{n_k}|^2 \tag{3.2.56}$$

where $R \geq 1$ is an arbitrary number and C is independent of R.
By Lemma 3.24 we have

$$\begin{aligned} &\|\phi_{n_k} - \overline{\phi}_{n_k}\|_{W^{2,s}(\Omega_{n_k})} \leq C\|p\,U^{p-1}\,(\phi_{n_k} - \overline{\phi}_{n_k})\|_{L^s(\Omega_{n_k})} + C\|p\,U^{p-1}\,(\phi_{n_k} - \overline{\phi}_{n_k})\|_{L^2(\Omega_{n_k})} \\ + \quad & C\Big(\|e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k}) + p\,U^{p-1}V_0)\|_{L^s(\Omega_{n_k})} \\ + \quad & \|e^{\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}(P_n)}(U^p_{n_k} - U^p - e^{-\frac{1}{\varepsilon_{n_k}}\,\psi_{n_k}}p\,U^{p-1}\phi_{n_k}) + p\,U^{p-1}V_0)\|_{L^2(\Omega_{n_k})}\Big) \\ + \quad & C\Big(\|(1-\chi_k)p\,U^{p-1}V_0\|_{L^s(\Omega_{n_k})} + \|\|(1-\chi_k)p\,U^{p-1}V_0\|_{L^2(\Omega_{n_k})}) \\ + \quad & C\Big(\| - 2\,\nabla\chi_k\,\nabla\phi_0\|_{L^s(\Omega_{n_k})} + \| - 2\,\nabla\chi_k\,\nabla\phi_0\|_{L^2(\Omega_{n_k})}\Big) \\ + \quad & C\Big(\| - (\Delta\,\chi_k)\,\phi_0\|_{L^s(\Omega_{n_k})} + \| - (\Delta\,\chi_k)\,\phi_0\|_{L^2(\Omega_{n_k})}\Big). \end{aligned}$$

By the previous estimates we have

$$\limsup_{k\to+\infty} \|\phi_{n_k} - \overline{\phi}_{n_k}\|_{W^{2,s}(\Omega_{n_k})} \leq C\,e^{-\sigma\,R} + C\,R^{\frac{N(s-2)}{2\,s}}\,e^{-\sigma\,R},$$

that gives $\phi_{n_k} \to \phi_0$ in $C^1_{loc}(\mathbb{R}^N)$. Our thesis follows by letting $R \to +\infty$.

□

The following lemma holds (see [**105**], Lemma 2.5 or [**101**], Lemma 2.4) and plays a fundamental role.

Lemma 3.29. *Assume that P_n in Ω is such that $\lim_{\varepsilon_n\to 0} P_n = P_0 \in \Omega$. Then there exists a bounded Borel measure $d\mu_{P_0}(z)$ on $\partial\Omega$ with $\int_{\partial\Omega} d\mu_{P_0}(\xi) = 1$ and supp$(d\mu_{P_0}(\xi)) \subset \Pi_{\partial\Omega}(P_0)$ such that, up to a subsequence,*

$$\lim_{n\to\infty} V_n(y) = \int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|},y>_{\mathbb{R}^N}}\,d\mu_{P_0}(z).$$

If $P_0 \in \partial\Omega$ then $d\mu_P = \delta_P$ and $\lim_{n\to+\infty} V_n(y) = e^{<b\,,\,y>}$ for some $|b| = 1$, $b \in \mathbb{R}^N$.

Proof. Let $G_n(x,y)$ be the Green's function of $-\Delta + \tilde{\lambda}_n$ on $W^{1,2}_0(\Omega)$. Then we have by standard representation formula:

$$\varphi_n(x) = \int_{\partial\Omega} U\Big(\frac{z-P_n}{\varepsilon_n}\Big)\frac{\partial G_n}{\partial\nu}(z,x)\mathrm{d}\,z, \tag{3.2.57}$$

see Lemma 2.1 in [**104**].
By the Theorems 1.15 and 1.18, and the estimate in Section 3 of [**103**], we calculate

$$\begin{aligned} \varphi_n(x) \quad &= \quad \frac{C_N + O(\varepsilon_n)}{\varepsilon_n^N} \\ \times \quad & \int_{\partial\Omega}\Big\{e^{-\frac{(|z-P_0|+|z-x|)}{\varepsilon_n}}|z-P_0|^{-\frac{(N-1)}{2}}|z-x|^{-\frac{N-1}{2}}\frac{<\,z-P_0,\,\nu\,>}{|z-x|}\Big\}\mathrm{d}\,z, \end{aligned} \tag{3.2.58}$$

as $n \to +\infty$. We have

$$\varphi_n(P_0) = \frac{C_N + O(\varepsilon_n)}{\varepsilon_n^N}\int_{\partial\Omega}\Big\{e^{-\frac{2|z-P_0|}{\varepsilon_n}}|z-P_0|^{-N+1}\frac{<\,z-P_0,\,\nu\,>}{|z-x|}\Big\}\mathrm{d}\,z. \tag{3.2.59}$$

Let $\varepsilon_n\, y + P_n = x$ and $|y| \le K$; then

$$\begin{aligned} |z-x| &= |z - P_0 - \varepsilon_n\, y| = \varepsilon_n \Big| y - \frac{z-P_0}{\varepsilon_n} \Big| \\ &= |z-P_0| - <\, y,\ \frac{z-P_0}{|z-P_0|}\ > + O((\varepsilon_n)^2). \end{aligned}$$

By (3.2.58) and (3.2.59), we have

$$V_n(y) = (1 + O(\varepsilon_n)) \frac{\int_{\partial\Omega} \Big\{ e^{-\frac{2|z-P_n|}{\varepsilon_n}} e^{<\frac{z-P_n}{|z-P_n|}, y>} |z-P_n|^{-N+1} \frac{<z-P_n, \nu>}{|z-P_n|} \Big\} \mathrm{d}\, z}{\int_{\partial\Omega} \Big\{ e^{-\frac{2|z-P_n|}{\varepsilon_n}} |z-P_n|^{-N+1} \frac{<z-P_n, \nu>}{|z-P_n|} \Big\} \mathrm{d}\, z}.$$

If $P_n \to P_0$, then it is easy to see that

$$V_n(y) \to \int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|}, y>_{\mathbb{R}^N}} \mathrm{d}\mu_{P_0}(z)$$

for some $\mathrm{d}\mu_{P_0}(z)$. This proves Lemma. □

REMARK 3.30. For any $b \in \mathbb{R}^N$ with $|b| = 1$, we have

$$\gamma = \int_{\mathbb{R}^N} U^p(y) e^{<b,y>_{\mathbb{R}^N}} \mathrm{d}y$$

(see Lemma 4.7 of [**89**]).

Using Proposition 3.23, we obtain an asymptotic expansion that will be used in the sequel.

PROPOSITION 3.31. *For ε_n sufficiently small, we have*

$$\int_{\Omega_n} U_n^p = \int_{\mathbb{R}^N} U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} \int_{\mathbb{R}^N} p\, U^{p-1}(V_0 - \phi_0) + o\Big(e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} \Big).$$

PROOF. By the mean value theorem, we have

$$U_n f(U_n) = U_n U_n^{p-1} = U_n^p = U f(U) + e^{-\frac{1}{\varepsilon_n}\psi_n(P_\varepsilon)} (f(t) + t\, f'(t))(\phi_n - V_n),$$

where $f(s) = s^{p-1}$, $U_n < t < U$, i. e.

$$U_n^p = U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} (t^{p-1} + (p-1)\ t^{p-1})(\phi_n - V_n) = U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} p\, t^{p-1}(\phi_n - V_n).$$

Therefore

$$\int_{\Omega_n} U_n^p = \int_{\Omega_n} U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\, p\, t^{p-1}(\phi_n - V_n). \tag{3.2.60}$$

We have

$$\begin{aligned} \int_{\Omega_n} |\, p\, t^{p-1}(\phi_n - V_n)| &\le C \int_{\Omega_n} t^{p-1} |\phi_n - V_n| \\ &\le C \int_{\Omega_n} e^{-(1-\delta)(1+\sigma)|y|} (e^{(1+\sigma_1)|y|} + e^{\mu|y|}) \le C \end{aligned}$$

for $\varepsilon_n \leq \varepsilon_0$. By Proposition 3.21, there is a subsequence $\varepsilon_{n_k} \to 0$ and a solution V_0 of (3.2.12) s. t. $V_n \to_n V_0$. So, by Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\Omega_n} p\, t^{p-1}(\phi_n - V_n) \to_{n\to\infty} \int_{\mathbb{R}^N} p\, U^{p-1}(\phi_0 - V_0).$$

Moreover, by Proposition 3.21 (2) with $\sigma_0 < \sigma$

$$\int_{\mathbb{R}^N\setminus\Omega_n} U|U^{p-1}| \leq Ce^{-\frac{1}{\varepsilon_n}(2+\sigma_0)d(P_n,\partial\Omega)} \int_{\mathbb{R}^N\setminus\Omega_n} e^{-(\sigma-\sigma_0)|y|} = o(e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}).$$

Hence, up to a subsequence,

$$\int_{\Omega_n} U_n^p = \int_{\Omega_n} U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\, p\, t^{p-1}(\phi_n - V_n)$$

$$= \int_{\mathbb{R}^N} U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} \int_{\mathbb{R}^N} p\, U^{p-1}(V_0 - \phi_0) + o\Big(e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}\Big)$$

□

3.2.3. Blow-up set. We obtain the location of blow-up set:

THEOREM 3.32. *P_0 is a critical point of distance function $d_{\partial\Omega}$.*

PROOF. Let denote $\partial_i U_n = \partial_{x_i} U_n$. Let U_n, $P(\partial_i U_n)$, PU and $P(\partial_i U)$ be, respectively, solutions of

$$\begin{cases} -\Delta U_n + \tilde{\lambda}_n U_n = U_n^p & \text{in } \Omega_n \\ U_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.61}$$

$$\begin{cases} -\Delta P(\partial_i U_n) + \tilde{\lambda}_n P(\partial_i U_n) = -\Delta \partial_i U_n + \tilde{\lambda}_n \partial_i U_n = \partial_i U_n^p & \text{in } \Omega_n \\ P(\partial_i U_n) = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.62}$$

$$\begin{cases} -\Delta P U + \tilde{\lambda} P U = U^p & \text{in } \Omega_n \\ P U = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.63}$$

$$\begin{cases} -\Delta P(\partial_i U) + \tilde{\lambda} P(\partial_i U) = -\Delta \partial_i U + \tilde{\lambda} \partial_i U = \partial_i U^p & \text{in } \Omega_n \\ P(\partial_i U) = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.2.64}$$

Multiply (3.2.61) by $P(\partial_i U)$ and integrate by parts, omitting for simplicity the integrals on the boundary that are zero, we have:

$$\int_{\Omega_n} U_n^p P(\partial_i U) = \int_{\Omega_n} U_n(-\Delta P(\partial_i U) + \tilde{\lambda} P(\partial_i U)) = \int_{\Omega_n} U_n \partial_i (U^p) = -\int_{\Omega_n} U^p \partial_i U_n \tag{3.2.65}$$

in view of (3.2.64).
Multiply (3.2.63) by $P(\partial_i U)$ and integrate by parts, and as previously, we omit the zero integral on the boundary:

(3.2.66)
$$\int_{\Omega_n} U^p P(\partial_i U) = \int_{\Omega_n} PU(-\Delta P(\partial_i U) + \tilde{\lambda} P(\partial_i U)) = \int_{\Omega_n} PU\partial_i(U^p) = -\int_{\Omega_n} U^p \partial_i PU$$

in view of (3.2.64).

Summing up (3.2.65) and (3.2.66), we get

$$\int_{\Omega_n} U^p \partial_i U_n + \int_{\Omega_n} (U_n^p - U^p) P(\partial_i U) - \int_{\Omega_n} U^p \partial_i PU = 0.$$

Since

$$\int_{\Omega_n} U_n^p \partial_i U_n = \frac{1}{p+1} \int_{\Omega_n} \partial_i(U_n^{p+1}) = 0 = \int_{\Omega_n} (PU)^p \partial_i PU,$$

finally we get that

$$0 = \int_{\Omega_n} (U^p - U_n^p)(\partial_i U_n - P(\partial_i U)) + \int_{\Omega_n} ((PU)^p - U^p) \partial_i PU.$$

We can prove that

$$\int_{\Omega_n} (U^p - U_n^p)(\partial_i U_n - P(\partial_i U))$$

is quadratically (exponentially) small, indeed we say that the following expansions hold

$$U_n(y) = PU(y) + e^{-\frac{1}{\varepsilon_n}\phi_n(P_n)} \psi_n(y)$$

$$\partial_i U_n(y) = \partial_i PU(y) + e^{-\frac{1}{\varepsilon_n}\partial_i \psi_n(P_n)} \phi_n(y)$$

where ϕ_n is a solution of (3.2.10);

$$U_n^p = U^p + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} p U^{p-1}(\phi_n - V_n).$$

Then we have, by the expansion in the Proposition 3.31,

$$\int_{\Omega_n} U^p - U_n^p = \int_{\Omega_n} -e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} p U^{p-1}(\phi_n - V_n) \sim -e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)}$$

because U is such that there exist $C > 0$ and $\delta > 0$ such that

$$|D^\alpha U(x)| \leq C\, e^{-\delta|x|}, \quad \forall x \in \mathbb{R}^N \text{ and } |\alpha| \leq 2$$

and ϕ_n and V_n bounded (see Proposition 3.21). Therefore $P(\partial_i U) \sim \partial_i U$ and so

$$\partial_i U_n - P(\partial_i U) \sim \partial_i U_n - \partial_i U = \partial_i PU - \partial_i U + e^{-\frac{1}{\varepsilon_n}\psi_n(P_n)} \partial_i \phi_n.$$

Observe that the function $W(y) = U\left(\frac{y-x}{\varepsilon_n}\right) - PU\left(\frac{y-x}{\varepsilon_n}\right)$ solves the problem

$$-\Delta W + \tilde{\lambda} W = 0, \text{ in } \Omega, \quad W = U\left(\frac{\cdot - x}{\varepsilon_n}\right) \text{ on } \partial_i \Omega$$

and by Theorem 2.3.3.6 of [**63**] it follows that exist $\varepsilon_0 > 0$ and a constant $C > 0$ such that for any $\varepsilon_n \in (0, \varepsilon_0)$

$$\|W\|_{H^2(\Omega)} \leq C \left\| U\left(\frac{\cdot - x}{\varepsilon_n}\right) \right\|_{W^{3/2,2}(\partial\Omega)}.$$

Then we have to estimate the term

$$\begin{aligned}\Big\|U\Big(\frac{\cdot - x}{\varepsilon_n}\Big)\Big\|^2_{W^{3/2,2}(\partial\Omega)} &= \Big\|U\Big(\frac{\cdot - x}{\varepsilon_n}\Big)\Big\|_{L^2(\partial\Omega)} \\ &+ \int\int_{\partial\Omega\times\partial\Omega}\sum_{|\alpha|=1}\frac{|D^\alpha U\Big(\frac{y-x}{\varepsilon_n}\Big) - D^\alpha\Big(\frac{z-x}{\varepsilon_n}\Big)|^2}{|y-z|^N}\mathrm{d}\sigma(y)\mathrm{d}\sigma(z)\end{aligned}$$

to obtain that

$$\Big\|U\Big(\frac{\cdot - x}{\varepsilon_n}\Big) - P\Big(\frac{\cdot - x}{\varepsilon_n}\Big)\Big\|_{H^2(\Omega)} \le C\,\varepsilon^{-2}e^{-\delta\frac{\mathrm{dist}(x,\partial\Omega)}{\varepsilon_n}}.$$

So we have that

$$\int_{\Omega_n}((P\,U)^p - U^p)\partial_i P\,U = 0.$$

Finally we have to estimate the term (by the mean value theorem)

$$\begin{aligned}\int_{\Omega_n}((P\,U)^p - U^p)\partial_i P\,U &= \int_{\Omega_n}\Big(p\int_0^1\Big[t\,U + (1-t)P\,U\Big]^{p-1}\mathrm{d}\,t\Big)\partial_i P\,U(U - P\,U) \\ &= \varphi_n(P_n)\int_{\Omega_n}\Big(p\int_0^1\Big[t\,U + (1-t)P\,U\Big]^{p-1}\mathrm{d}\,t\Big)\partial_i U\quad V_n \\ &+ \varphi_n(P_n)\int_{\Omega_n}\Big(p\int_0^1\Big[t\,U + (1-t)P\,U\Big]^{p-1}\mathrm{d}\,t\Big)V_n(\partial_i U - \partial_i U) \\ &= \frac{\varphi_n(P_n)}{\varepsilon}\int_{\Omega_n}\partial_i(U^p)V_n + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big) \\ &= \frac{\varphi_n(P_n)}{\varepsilon}\int_{\mathbb{R}^N}\partial_i(U^p)\int_{\partial\Omega}e^{<\frac{\xi-P_0}{|\xi-P_0|},y>_{\mathbb{R}^N}}\mathrm{d}\mu_{P_0}(\xi) + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big)\end{aligned}$$

integrating by parts and by Lemma 3.29

$$\begin{aligned}&= -\frac{\varphi_n(P_n)}{\varepsilon}\int_{\mathbb{R}^N}U^p\partial_i\Big(\int_{\partial\Omega}e^{<\frac{\xi-P_0}{|\xi-P_0|},y>_{\mathbb{R}^N}}\mathrm{d}\mu_{P_0}(\xi)\Big) + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big) \\ &= -\frac{\varphi_n(P_n)}{\varepsilon}\int_{\mathbb{R}^N}U^p\int_{\partial\Omega}\frac{\xi_i - P_{0_i}}{|\xi - P_0|}e^{<\frac{\xi-P_0}{|\xi-P_0|},y>_{\mathbb{R}^N}}\mathrm{d}\mu_{P_0}(\xi) + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big)\end{aligned}$$

by Fubini's Theorem

$$= -\frac{\varphi_n(P_n)}{\varepsilon}\Big(\int_{\partial\Omega}\frac{\xi_i - P_{0_i}}{|\xi - P_0|}\Big(\int_{\partial\Omega}U^p\,e^{<\frac{\xi-P_0}{|\xi-P_0|},y>_{\mathbb{R}^N}}\Big)\mathrm{d}\mu_{P_0}(\xi)\Big) + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big)$$

by Remark 3.30

$$\begin{aligned}&= -\frac{\varphi_n(P_n)}{\varepsilon}\gamma\Big(\int_{\partial\Omega}\frac{\xi_i - P_{0_i}}{|\xi - P_0|}\mathrm{d}\mu_{P_0}(\xi)\Big) + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big) \\ &= -\frac{\varphi_n(P_n)}{\varepsilon}\gamma\Big(\alpha(P_0)\Big)_i + o\Big(\frac{\varphi_n(P_n)}{\varepsilon}\Big)\end{aligned}$$

where $\alpha(P_0) = \int_{\partial\Omega}\frac{\xi-P_0}{|\xi-P_0|}\mathrm{d}\mu_{P_0}(\xi) \in \partial d_{\partial\Omega}(P_0)$.

□

REMARK 3.33. Observe that, in the previous section, we have obtained easily the location of the blow-up points, in the case of $V \neq 1$. This approach can't be used with $V \equiv 1$. For this reason we have introduced a different approach [**89**]. Vice versa the techniques used in the case of $V \equiv 1$ can by applied with a generic potential $V \neq 1$ in the equation. So we can obtain a more precisely expansion for U_n.
For the location of the blow-up set, we proceed as in Theorem 3.32. Consider U_n, $P(\partial_i U_n)$, PU and $P(\partial_i U)$ respectively, solutions of

$$\begin{cases} -\Delta U_n + \tilde{\lambda}_n V(\varepsilon_n y + P_n) U_n = U_n^p & \text{in } \Omega_n \\ U_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.67}$$

$$\begin{cases} -\Delta \partial_i U_n + \tilde{\lambda}_n [\partial_i V(\varepsilon_n y + P_n) U_n + V(\varepsilon_n y + P_n)\partial_i U_n] = \partial_i U_n^p & \text{in } \Omega_n \\ P(\partial_i U_n) = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.68}$$

$$\begin{cases} -\Delta PU + \tilde{\lambda} PU = U^p & \text{in } \Omega_n \\ PU = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{3.2.69}$$

$$\begin{cases} -\Delta P(\partial_i U) + \tilde{\lambda} P(\partial_i U) = -\Delta \partial_i U + \tilde{\lambda}\partial_i U = \partial_i U^p & \text{in } \Omega_n \\ P(\partial_i U) = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{3.2.70}$$

Multiply (3.2.67) by $P(\partial_i U)$ and integrate by parts, omitting for simplicity the integrals on the boundary that are zero, we have:

$$\begin{aligned} \int_{\Omega_n} U_n^p P(\partial_i U) &= \int_{\Omega_n} U_n(-\Delta P(\partial_i U) + \tilde{\lambda}_n V(\varepsilon_n y + P_n) P(\partial_i U)) \\ &= \int_{\Omega_n} U_n \partial_i (U^p) + \int_{\Omega_n} [\tilde{\lambda}_n V(\varepsilon_n y + P_n) - \tilde{\lambda}] U_n \, P(\partial_i U) \quad (3.2.71) \\ &= -\int_{\Omega_n} U^p \partial_i U_n + \int_{\Omega_n} [\tilde{\lambda}_n V(\varepsilon_n y + P_n) - \tilde{\lambda}] U_n \, P(\partial_i U) \quad (3.2.72) \end{aligned}$$

in view of (3.2.70).
Multiply (3.2.69) by $P(\partial_i U)$ and integrate by parts, and as previously, we omit the zero integral on the boundary:

(3.2.73)
$$\int_{\Omega_n} U^p P(\partial_i U) = \int_{\Omega_n} PU(-\Delta P(\partial_i U) + \tilde{\lambda} P(\partial_i U)) = \int_{\Omega_n} PU \partial_i (U^p) = -\int_{\Omega_n} U^p \partial_i PU$$

in view of (3.2.70).
Summing up (3.2.71) and (3.2.73), we get

$$\int_{\Omega_n} U^p \partial_i U_n + \int_{\Omega_n} (U_n^p - U^p) P(\partial_i U) - \int_{\Omega_n} U^p \partial_i PU = \int_{\Omega_n} [\tilde{\lambda}_n V(\varepsilon_n y + P_n) - \tilde{\lambda}] U_n \, P(\partial_i U).$$

Therefore, we obtain

$$\begin{aligned} &\int_{\Omega_n}(-\Delta U+\tilde\lambda U)\partial_i U_n - U^p\partial_i P\,U + \int_{\Omega_n}(U_n^p-U^p)P(\partial_i U)\\ - &\int_{\Omega_n}[\tilde\lambda_n V(\varepsilon_n y+P_n)-\tilde\lambda]U_n\,P(\partial_i U) = \int_{\Omega_n} -\Delta\partial_i U_n\,U + \tilde\lambda U\,\partial_i U_n - U^p\partial_i P\,U\\ + &\int_{\Omega_n}(U_n^p-U^p)P(\partial_i U) - \int_{\Omega_n}[\tilde\lambda_n V(\varepsilon_n y+P_n)-\tilde\lambda]U_n\,P(\partial_i U)\\ = &\int_{\Omega_n}[-\tilde\lambda_n(V\,\partial_i U_n+\partial_i V\ U_n)U+\tilde\lambda U\,\partial_i U_n - U^p\,\partial_i P\,U]\\ + &\int_{\Omega_n}(U_n^p-U^p)P(\partial_i U) - \int_{\Omega_n}[\tilde\lambda_n V(\varepsilon_n y+P_n)-\tilde\lambda]U_n\,P(\partial_i U)=0. \end{aligned}$$

so

$$\begin{aligned} \int_{\Omega_n}(U_n^p-U^p)P(\partial_i U) \quad - &\int_{\Omega_n}[\tilde\lambda_n V(\varepsilon_n y+P_n)-\tilde\lambda]U_n\,P(\partial_i U)+\int_{\Omega_n}[\tilde\lambda-\tilde\lambda_n V(\varepsilon_n y+P_n)]U\,\partial_i U_n\\ - &\int_{\Omega_n}\tilde\lambda_n\,\partial_i V\ U_n\ U - \int_{\Omega_n}U^p\ \partial_i P\,U = 0. \end{aligned}$$

We can prove that some terms in the expansion are exponentially small and finally we have the expected result.

3.2.4. Approaching blow up in multi-peaks. Following the idea used in the previous section, we try to give an approach of the possible global analysis of the blow-up in multi-peaks. The first that considers the case of 2−peaks is Wei in [**104**].
Let us define the function D_k which will play a crucial role in the sequel, introduced in [**64, 65, 91**].

Definition 3.34. Let $k \geq 1$ be an integer. Set $\Omega^k = \Omega\times\cdots\times\Omega$. Let $D_k : \Omega^k \to \mathbb{R}$ be defined by

$$D_k(X) = \min_{\left\{ i,j,l=1,\dots,k\ j\neq l\right\}} \left\{ d(x^i,\partial\Omega),\ \frac{|x^j-x^l|}{2}\right\}.$$

Set $M_k(\Omega) = \left\{ X=(x^1,\dots,x^k)\in\Omega^k \mid x^i\neq x^j,\ i\neq j,\ i,j=1,\dots,k \right\}.$

By the regularity of the distance function and Proposition 3.8 we can compute the generalized gradient of D_k.

Lemma 3.35. *For any $X\in M_k(\Omega)$ we have that $\beta(X)\in\partial D_k(X)$ if and only if*

$$\beta(X) = \left(a_1\,\alpha(x^1)+\frac12\sum_{j=1,j\neq1}^k b_{1\,j}\frac{x^1-x^j}{|x^1-x^j|},\dots,a_k\,\alpha(x^k)+\frac12\sum_{j=1,j\neq k}^k b_{1\,j}\frac{x^k-x^j}{|x^k-x^j|}\right),$$

with $\alpha(x^i)\in\partial d_{\partial\Omega}(x^i)$, a_j, $b_{j\,l}\geq 0$, $b_{j\,l}=b_{l\,j}$, $\sum_{i=1}^k a_i+\frac12\sum_{j,l=1,\,l\neq j}^k b_{j\,l}=1$.

In particular, by this Lemma, we deduce that if $x^1, \ldots, x^k$ are k different critical points of the distance function, then $X = (x^1, \ldots, x^k)$ is a critical point of D_k.
Observe that there is not any critical point of D_k close to the boundary of $M_k(\Omega)$, indeed there holds:

PROPOSITION 3.36. *There exists a neighborhood U of the boundary of $M_k(\Omega)$ such that $0 \notin \partial D_k(X)$ for any $X \in U \cap M_k(\Omega)$.*

PROOF. We prove that if X_ε is a sequence in $M_k(\Omega)$ such that $\lim_{\varepsilon\to 0} X_\varepsilon = X_0$ and $X_0 \in \partial M_k(\Omega)$, then there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$|\beta_\varepsilon(X_\varepsilon)| \geq C > 0 \quad \forall\, \beta_\varepsilon(X_\varepsilon) \in \partial D_k(X_\varepsilon).$$

We proceed by induction on the number k.
Let $k = 1$ and let x_ε be a sequence in Ω such that $x_0 = \lim_\varepsilon x_\varepsilon \in \partial\Omega$. If follows that for ε small enough $\partial D_1(x_\varepsilon) = \{\, \nu^{(i)}(\pi(x_\varepsilon)) \,\}$ and the claim follows.
Suppose the claim to be true for any integer $1 \leq h \leq k-1$. Let us prove that the claim is true for k.
Let X_ε be a sequence in $M_k(\Omega)$ such that $\lim_{\varepsilon\to 0} X_\varepsilon = X_0$ and $X_0 \in \partial M_k(\Omega)$.
Then we have either

- $\exists\, i, j \in \{\, 1, \ldots, k \,\}$ such that $x_0^i \neq x_0^j$,
- $x_0^1 = \cdots = x_0^k \in \partial\Omega$,
- $x_0^1 = \cdots = x_0^k \in \Omega$.

Using Lemma 3.35 and the inductive assumptions the claim easily follows. □

PROPOSITION 3.37. *Let $(x^1, \ldots, x^k) \in M_k(\Omega)$ be a critical point of D_k. Assume that for any integer $1 \leq h \leq k-1$ and for any set of indices $\{\, i_1, \ldots, i_h \,\} \subset \{\, 1, \ldots, k \,\}$, $(x^{i_1}, \ldots, x^{i_h})$ is not a critical point of D_k. Then $d_{\partial\Omega}(x^i) = \frac{|x^l - x^h|}{2}$ for any i, j, h and $0 \in co\{\, \alpha(x^i) \,|\, \alpha(x^i) \in \partial d_{\partial\Omega}(x^i,\ i = 1, \ldots, k) \,\}$.*

PROOF. We prove the thesis by contradiction. We have either

(1) $\exists\, i, j \in \{\, 1, \ldots, k \,\}$ such that $D_k(X) < \frac{|x^i - x^j|}{2}$,
(2) $\forall\, l, h \in \{\, 1, \ldots, k \,\} \in D_k(X) = \frac{|x^l - x^h|}{2}$ and $\exists\, i \in \{\, 1, \ldots, k \,\}$ such that $D_k(X) < d_{\partial\Omega}(x^i)$.

A contradiction arise in both cases, using Lemma 3.35. □

We recall the following characterization of the critical points of D_2.

COROLLARY 3.38. *Let $(x^1, x^2) \in M_2(\Omega)$ be a critical point of D_2 such that the distance function is differentiable at x^1 and x^2. Then $d_{\partial\Omega}(x^1) = d_{\partial\Omega}(x^2) = \frac{|x^1 - x^2|}{2}$ and $\nu^{(i)}(\pi(x^1)) = -\nu^{(i)}(\pi(x^2)) = \frac{x^2 - x^1}{|x^2 - x^1|}$.*

Let $X = (x^1, \ldots, x^k) \in M_k(\Omega)$ and $P_n^1, \ldots, P_n^k$, local maximum of u_n and ε_n^i, $i = 1, \ldots, k$, as in Theorem 2.4 with $k \leq \bar{k}$. Consider the usual change of variable $U_n^i(y) = (\varepsilon_n^i)^{\frac{2}{p-1}} u_n(\varepsilon_n^i y + P_n^i)$,

$i = 1, \ldots, k$ that satisfies

$$\begin{cases} -\Delta U_n^i + \lambda_n (\varepsilon_n^i)^2 = (U_n^i)^p & \text{in } \Omega_n^i \\ U_n = 0 & \text{on } \partial\Omega_n^i \end{cases} \tag{3.2.74}$$

with $\Omega_n^i = \frac{\Omega - P_n^i}{\varepsilon_n^i}$.

Let introduce the projection $P_{\Omega_n^i} U$ of U (solution of (3.2.5) in the whole space), in Ω_n^i, $i = 1, \ldots, k$, as the unique solution of

$$\begin{cases} -\Delta P_{\Omega_n^i} U + \tilde{\lambda} P_{\Omega_n^i} U = U^p & \text{in } \Omega_n^i \\ P_{\Omega_n^i} U = 0 & \text{on } \partial\Omega_n^i. \end{cases} \tag{3.2.75}$$

Let $\varphi_n^i := U - P_{\Omega_n^i} U$ and, as a generalization of ϕ_n for $1-$peak, we define for $X \in \mathbf{M}_k(\Omega)$

$$\Psi_n(X) = -\varepsilon_n^1 \log \Big[\sum_{i=1}^{k} \varphi_n^i(x^i) \Big].$$

Moreover, it is possible to show that Ψ_n is C^1 in X and $\|\Psi_n\|_{W^{2,s}(\Omega)} \leq C\, e^{-(1+\sigma)\frac{D_k(X)}{\varepsilon_n^1}}$, where $\sigma = \min\{1,\, p-1\}$. In this way, as previously seen for $1-$peak, we would obtain that the blow-up occurs in critical points of $D_k(X)$.

CHAPTER 4

Solutions with symmetries

This chapter deals with invariant solutions. Much work has been devoted to our problem, with Dirichlet or Neumann boundary conditions, in order to understand where concentration occurs and how the profile of solutions looks like.
The structure of solutions blown-up at points, called spike-layers, has been shown to be very rich, and solutions that blow-up at $k-$points, the so called $k-$peaks solutions, too.
In addition to solutions blowing at points, it is natural to ask whether there exist other ones which scale only in some of variables, and which therefore blow-up at higher dimensional sets (dimension k), like curves, surfaces, etc.
Just recently, existence of solutions blowing-up at different sets has been proved. Indeed under generic assumptions, for example in the case of Neumann boundary condition, see [**85**], if $\Omega \subset \mathbb{R}^N$ and $k = 1, ..., N-1$, it was conjectured the existence of solutions that concentrate at suitable $k-$ dimensional sets. The phenomenon was known for particular domains with some symmetries. For these and related issues see [**5, 6, 10, 15, 39, 42, 83, 81**]. This conjecture has been recently proved in [**76**] for the general case, while the result has been shown in [**79, 80**] for $k = N-1$ and in [**78**] for $N = 3$ and $k = 1$.
We study blow-up on manifolds in the case of an annulus and consider solutions with partial symmetry assumptions and bounded invariant Morse index, as it has been done in [**50**] for radial solution. We are interested in solutions which are invariant under a proper subgroup $G \subset O(N)$ of symmetries. By an asymptotic approach based on $G-$invariant Morse index information, we try to carefully localize the blow-up $G-$orbits in terms of a modified potential.
Let us notice that the ground state solution in the space of invariant solutions under a proper subgroup $G \subset O(N)$ has Morse index 1 in this space, while its full Morse index or its energy is very large. Thanks to this information, we perform an asymptotic analysis and localize the concentration set. Our aim is to exhibit potential in which the orbit of the maximum doesn't degenerate on the fixed points set of this symmetry group. If G has not fixed points, one can provide, in this way, solutions (for example the $G-$invariant ground state) which concentrate on a whole orbit with dimension as G.
Let G simply the rotations around the $z-$ axis group.

Let u_n be a positive, $G-$invariant solution of

$$\begin{cases} -\Delta u_n + \lambda_n V(x) u_n = u_n^p & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega := \{ (x, y, z) \in \mathbb{R}^3 \, : \, a \leq |(x, y, z)| \leq b \}$ is an annulus and $V : \overline{\Omega} \to \mathbb{R}$ is a smooth potential bounded away from zero, a, $b > 0$. Let G_0 be the set of fixed points under the action group, i. e. in this case the z-axis.

We want to study the asymptotic behavior of such solutions when $\lambda_n \to +\infty$ under an uniform bound of the $G-$invariant Morse indices: $\sup_n m_G(u_n) < +\infty$.

4.1. Blow-up profile

In this section we discuss the blow-up profile for solutions of the invariant problem.
A question arises concerning whether the limiting profile keeps the invariances of u_n.
We have that in general the limiting function U is no longer $G-$invariant.
Let assume that $P_n \to P_0$ as $n \to +\infty$, where P_n is a local maximum point of u_n. We have different situations depending on the location of P_0 and the rate $\frac{d(P_n,G_0)}{\varepsilon_n}$, where $G_0 = \{\, z - axis \,\}$ is kept fixed by the action of G. Accordingly, we discuss now each one of these situations.

4.1.1. Some preliminary results and local analysis. Let $m_G(u_n)$ be the Morse Index of u_n with respect to $G-$invariant test functions. We say that a positive solution $u_n \in H^1_0(\Omega)$ has Morse index $m_G(u_n) = k \geq 1$, if k is the maximal dimension of a subspace W_k of

$$H^1_{0,G}(\Omega)_G := \{\; \phi \in H^1_0(\Omega),\; \phi \text{ is } G\text{−invariant a. e. in } \Omega \,\}$$

s.t.

$$Q_{u_n}(\phi) = \int_\Omega |\nabla \phi|^2 + \tilde{\lambda}\, u_n\, \phi^2 - p\, u_n^{p-1}\, \phi^2 < 0$$

for all $\phi \in W_k \setminus \{\, 0 \,\}$.

We have that:

Theorem 4.1. *Let (λ_n, u_n) be a positive, $G-$invariant, solutions of*

$$\begin{cases} -\Delta u_n + \lambda_n V\, u_n = u_n^p & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1.1}$$

with $\sup_n m_G(u_n) < +\infty$, $1 < p \neq 5$.
Let $Q_n = (0, y_n, z_n) \in \Omega$, $y_n \geq 0$ be so that $u_n(Q_n) = \max_{\Omega \cap B_{R_n\, \varepsilon_n}(Q_n)} u_n \to 0$ as $n \to +\infty$, for some $R_n \to +\infty$.
Setting

$$U_n(X,Y,Z) = \frac{u_n(\varepsilon_n\,(X,Y,Z) + P_n)}{u_n(P_n)}, \quad \varepsilon_n = u_n(P_n)^{-\frac{p-1}{2}},$$

where

$$P_n := \begin{cases} (0,0,z_n) & \text{if } \frac{|y_n|}{\varepsilon_n} \leq C \\ Q_n & \text{if } \frac{|y_n|}{\varepsilon_n} \to +\infty; \end{cases} \tag{4.1.2}$$

up to a subsequence we have that

(1) *when $u_n(Q_n)^{\frac{p-1}{2}}\; y_n \leq C$, then there hold $1 < p < 5$ and*
- $\frac{y_n}{\varepsilon_n} \to 0$
- $\lambda_n \varepsilon_n^2 V(P_n) \to \tilde{\lambda}_3 \in (0,1]$ *for some universal constant $\tilde{\lambda}_3$*
- $\frac{\varepsilon_n}{d(P_n,\partial\Omega)} \to 0$

- *$U_n \to U$ in $C^1_{loc}(\mathbb{R}^3)$, where U is a positive $G-$invariant solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda}_3 U = U^p & \text{in } \mathbb{R}^3 \\ U \le U(0) = 1 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.1.3}$$

with $m_G(U) < +\infty$;

(2) *when $u_n(Q_n)^{\frac{p-1}{2}}\ y_n \to +\infty$, then there hold*
- *$\lambda_n \varepsilon_n^2 V(P_n) \to \tilde{\lambda}_2 \in (0,1]$ for some universal constant $\tilde{\lambda}_2$*
- *$\frac{\varepsilon_n}{d(P_n,\partial\Omega)} \to 0$*
- *$U_n \to U$ in $C^1_{loc}(\mathbb{R}^3)$, where $U(X,Y,Z) = U(Y,Z)$ is a positive solution of*

$$\begin{cases} -\Delta U + \tilde{\lambda}_2 U = U^p & \text{in } \mathbb{R}^2 \\ U \le U(0) = 1 & \text{in } \mathbb{R}^2, \end{cases} \tag{4.1.4}$$

with $m(U) < +\infty$ (two dimensional Morse index).

Moreover, there exists a $G-$invariant $\phi_n \in C^1_0(\Omega)$, with

$$supp\, \phi_n \subset A_R(Q_n) := \{ (x,y,z) \in \mathbb{R}^3 \ : \ (\sqrt{x^2+y^2} - y_n)^2 + (z - z_n)^2 \le R^2 \varepsilon_n^2 \},$$

$R > 0$, so that

$$\int_\Omega |\phi_n|^2 + (\lambda_n V - p u_n^{p-1}) \phi_n^2 < 0, \tag{4.1.5}$$

for all n large.

Proof. Let d_n simply denote $d(Q_n, \partial\Omega)$ and suppose $\frac{\mu_n}{d_n} \to L \in [0,+\infty]$, where $\mu_n = u_n(Q_n)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$. Then $\Omega_n = \frac{\Omega - Q_n}{\mu_n} \to H$, when H is an halfspace with $0 \in \overline{H}$ and $d(0, \partial H) = \frac{1}{L}$. The function $W_n(X,Y,Z) = \mu_n^{\frac{p-1}{2}} u_n(\mu_n(X,Y,Z) + Q_n)$ solves

$$\begin{cases} -\Delta W_n + \lambda_n \mu_n^2 V(\mu_n(X,Y,Z) + Q_n) W_n = W_n^p, & \text{in } \Omega_n \\ 0 < W_n \le W_n(0) = 1, & \text{in } \Omega_n \cap B_{R_n}(0) \\ W_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{4.1.6}$$

Since Q_n is a point of local maximum of u_n, we have

$$0 \le -\Delta W_n(0) = 1 - \lambda_n \mu_n^2\ V(Q_n) \ \Rightarrow \ 0 \le \lambda_n \mu_n^2\ V(Q_n) \le 1.$$

Denoting $\omega(V) := [\max_{\overline{\Omega}} V][\min_{\overline{\Omega}} V]^{-1}$ it follows that

$$\lambda_n \mu_n^2 V(\mu_n(X,Y,Z) + Q_n) \le \omega(V),$$

and, up to a subsequence,

$$\lambda_n \mu_n^2 V(Q_n) \to \tilde{\lambda} \in [0,1]$$

as $n \to +\infty$ (up to a subsequence). Since $W_n^p - \lambda_n \mu_n^2 V(\mu_n(X,Y,Z) + Q_n) W_n$ is uniformly bounded in $\Omega_n \cap B_{R_n}(0)$, by regularity theory we have that $W_n \to W$ in $C^1_{loc}(\overline{H})$, as $n \to +\infty$, where W solves

$$\begin{cases} -\Delta W + \tilde{\lambda} W = W^p, & \text{in } H \\ 0 < W \le U(0) = 1 & \text{in } H \\ W = 0 & \text{on } \partial H. \end{cases} \tag{4.1.7}$$

Since $W(0) = 1$ and $W = 0$ on ∂H, we deduce that $0 \in H$ and $L < +\infty$.
Consider the two cases above

(1) when $\frac{y_n}{\mu_n} \le C$, we can assume that $-\frac{y_n}{\mu_n} \to y_0$.
Since
$$b - |z_n| \ge b - \sqrt{y_n^2 + z_n^2} \ge d(P_n, \partial\Omega) = \min\{\, b - \sqrt{y_n^2 + z_n^2}, \sqrt{y_n^2 + z_n^2} - a \,\}$$
and
$$|z_n| - a = \sqrt{y_n^2 + z_n^2} - a - \frac{y_n^2}{|z_n| + \sqrt{y_n^2 + z_n^2}} \ge d(P_n, \partial\Omega) - \frac{y_n^2}{a},$$
we get that
$$d(P_n, \partial\Omega) = \min\{\, b - |z_n|, |z_n| - a \,\} \ge d(P_n, \partial\Omega) - \frac{y_n^2}{a}.$$
Since
$$d\Big(\frac{P_n - Q_n}{\mu_n}, \partial\Omega\Big) = \frac{d(P_n, \partial\Omega)}{\mu_n} \ge \frac{d(Q_n, \partial\Omega)}{\mu_n} + o(1),$$
we finally get that
$$d((0, y_0, 0), \partial H) \ge \frac{1}{L} > 0.$$
Computing now
$$\frac{u_n(P_n)}{u_n(Q_n)} = W_n\Big(\frac{Q_n - P_n}{\mu_n}\Big) = W_n\Big(0, -\frac{y_n}{\mu_n}, 0\Big) \to W(0, y_0, 0)$$
as $n \to +\infty$, we get that
$$\frac{u_n(P_n)}{u_n(Q_n)} \ge \delta_0^{\frac{2}{p-1}} > 0$$
for n large in view of $(0, y_0, 0) \in H$ and $W(0, y_0, 0) > 0$. In particular, we have that

(4.1.8)
$$\delta_0 \le \frac{\mu_n}{\varepsilon_n} \le 1.$$

We consider now U_n as the scaling of u_n w.r.t ε_n and P_n. Since by (4.1.8)
$$U_n \le U_n\Big(\frac{Q_n - P_n}{\varepsilon_n}\Big) = \Big(\frac{\varepsilon_n}{\mu}\Big)^{\frac{2}{p-1}} \le \delta_0^{-\frac{2}{p-1}}$$
in $\Omega_n \cap B_{\frac{R_n}{2}}(0)$, $\Omega_n := \frac{\Omega - P_n}{\varepsilon_n}$, we get that U_n converges in $C^1_{loc}(\overline{H'})$ to a solution U of

(4.1.9)
$$\begin{cases} -\Delta U + \tilde\lambda' U = U^p, & \text{in } H' \\ 0 < U \le U(0, \tilde y_0, 0) = 1 & \text{in } H \\ U = 0 & \text{on } \partial H'. \end{cases}$$

where

(4.1.10)
$$H' = \begin{cases} \Big\{ (X, Y, Z) \ : Z < \lim \frac{d(P_n, \partial\Omega)}{\varepsilon_n} \Big\} & \text{if } z_n \to b^- \text{ or } z_n \to -a^- \\ \Big\{ (X, Y, Z) \ : Z > \lim \frac{d(P_n, \partial\Omega)}{\varepsilon_n} \Big\} & \text{if } z_n \to a^+ \text{ or } z_n \to -b^+ \end{cases}$$

and $\tilde y_0 = \lim \frac{y_n}{\varepsilon_n}$, $\tilde\lambda' = \lim \lambda_n \varepsilon_n^2 V(P_n)$. Note that the existence of $\tilde y_0$ and $\tilde\lambda'$ follows by (4.1.8), up to a subsequence.

Observe also that U is still $G-$invariant and $m_G(U) < +\infty$. Since $p \neq 5$, by Theorems 1.8 and 1.21, we deduce that $\tilde{\lambda} > 0$. By Theorem 1.1.76 we can show that $H' = \mathbb{R}^3$. By Theorems 1.16 and 1.18, we know that U is radial and radially decreasing.
In particular, $\tilde{y}_0 = 0$ and then $\frac{y_n}{\varepsilon_n} \to 0$. Since $U(0) = 1$ by definition, we get that

$$U_n\Big(\frac{Q_n - P_n}{\varepsilon_n}\Big) = \Big(\frac{\varepsilon_n}{\mu_n}\Big)^{\frac{2}{p-1}} \to U(0, \tilde{y}_0, 0) = U(0) = 1.$$

By Theorems 1.18, we know that $U = (\tilde{\lambda}')^{\frac{1}{p-1}} U_k((\tilde{\lambda}')^{\frac{1}{2}} y)$ where U_k is given in Theorem 1.16, and then

$$\tilde{\lambda}' = U_k(0)^{-(p-1)} := \tilde{\lambda}_3$$

is an universal constant.

(2) when $\frac{y_n}{\mu_n} \to +\infty$, we have that $Q_n = P_n$ and $W_n = U_n$. The analysis follows the same lines of the previous case: show that $W = \lim U_n$ solves

$$\begin{cases} -\Delta W + \tilde{\lambda} W = W^p, & \text{in } \mathbb{R}^3 \\ 0 < W \le W(0) = 1 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.1.11}$$

for some $\tilde{\lambda} = \lim \lambda_n \varepsilon_n^2 V(P_n) \in [0, 1]$. The case $H \neq \mathbb{R}^3$ can be excluded thanks to $W > 0$.
The crucial point is that $W(x, y, z) = W(y, z)$ does really solve (4.1.11) in $\mathbb{R}^2$ with $m(W) < +\infty$ so as to use the same classification result in one dimensional less.
Indeed, notice that u_n is constant on

$$\{ (x, y, z) \ : \ x^2 + y^2 = y_n^2 + 2\, \varepsilon_n\, y_n\, r,\ z = \varepsilon_n\, s + z_n \}$$

for $(r, s) \in \mathbb{R}^2$ and n large.
Observe that on this set

$$x^2 + y^2 + z^2 = y_n^2 + z_n^2 + O(\varepsilon_n),$$

and then $(x, y, z) \in \Omega$:

$$a^2 < a^2 + (y_n^2 + z_n^2 - a^2) + O(\varepsilon_n) < x^2 + y^2 + z^2 < b^2 - (b^2 - y_n^2 - z_n^2) + O(\varepsilon_n) < b^2$$

in view of $\varepsilon_n = o(d(Q_n, \partial\Omega))$.
Moreover, $y_n^2 + 2\, \varepsilon_n\, y_n\, r = y_n^2(1 + \frac{2\, \varepsilon_n}{y_n}\, r) > 0$ in view of $\frac{\varepsilon_n}{y_n} \to 0$. Then U_n is constant on

$$\Big\{ (X, Y, Z) \ : \ \frac{\varepsilon_n}{2\, y_n}(X^2 + Y^2) + Y - r = 0,\ Z = s \Big\}.$$

Namely, $W(X, Y, Z) = W(Y, Z)$ and $m(W) < +\infty$ easily follows by $\sup_n m_G(u_n) < +\infty$.
The constant $\tilde{\lambda}$ satisfies

$$\tilde{\lambda} = \tilde{\lambda}_2 = U_k(0)^{-(p-1)},$$

where U_k now is given in Theorem 1.16 with $N = 2$.

$\square$

The last part of Theorem 4.1 follows by the next two Propositions.

PROPOSITION 4.2. *In case* (1) *of Theorem 4.1, we have that*

$$\exists \phi_n \in C^1_0(\Omega) \ \ G-\textit{invariant so that } \ supp\, \phi_n \subset B_{R\,\varepsilon_n}(P_n) \subset A_{R+1}(P_n)$$

and

$$\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V \ - p\, u_n^{p-1}) \phi_n^2\, dx < 0, \tag{4.1.12}$$

for all n *large and for some fixed* $R > 0$.

PROOF. By Theorem 4.1 we know that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^3)$ with $P_n = (0,0,z_n)$ and U is a $G-$invariant solution of (4.1.3).
By Theorem 2.1 let $\phi \in C^1_0(\mathbb{R}^3)$ be a radial function so that $\int |\nabla \phi|^2 + (\tilde\lambda_3 - p\, U^{p-1})\phi^2 < 0$. Set $\phi_n(x,y,z) = \frac{1}{\varepsilon_n^{\frac{1}{2}}} \phi\Big(\frac{(x,y,z)-P_n}{\varepsilon_n}\Big)$. Then ϕ_n is a $G-$invariant function in Ω so that $\mathrm{supp}\, \phi_n \subset B_{R\,\varepsilon_n}(P_n)$ for some $R > 0$ and

$$\begin{aligned}\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1}) \phi_n^2 \quad &= \quad \int_{\frac{\Omega - P_n}{\varepsilon_n}} |\nabla \phi|^2 + (\lambda_n\, \varepsilon_n^2\, V(\varepsilon_n\,(x,y,z) + P_n) - p\, U_n^{p-1})\phi^2 \\ &\to \quad \int |\nabla \phi|^2 + (\tilde\lambda_3 - p\, U^{p-1})\phi^2 < 0,\end{aligned}$$

for n large. Note that

$$\frac{\Omega - P_n}{\varepsilon_n} \to \mathbb{R}^3, \quad \frac{V(\varepsilon_n\,(x,y,z)+P_n)}{V(P_n)} \to 1 \ \text{ in}\, C_{loc}(\mathbb{R}^3). \tag{4.1.13}$$

Notice that, if $(x,y,z) \in B_{R\,\varepsilon_n}(P_n)$, then

$$(\sqrt{x^2+y^2} - y_n)^2 + (z - z_n)^2 \le \varepsilon_n^2 \Big(R^2 + 2\, R \frac{|y_n|}{\varepsilon_n} + \frac{y_n^2}{\varepsilon_n^2} \Big) \le (R+1)^2\, \varepsilon_n^2$$

in view of $\frac{|y_n|}{\varepsilon_n} \to 0$. It means that $\mathrm{supp}\phi_n \subset B_{R\,\varepsilon_n}(P_n) \subset A_{(R+1)\varepsilon_n}(P_n)$.
Observe that, if $\psi \in C^1_0(\mathbb{R}^3)$ is a $G-$invariant function so that

$$\int |\nabla \psi|^2 + (\tilde\lambda_3 - p\, U^{p-1})\psi^2 < 0$$

and $\int \phi\, \psi = 0$, one can correspondingly define ψ_n and property

$$\int_\Omega \psi_n\, \phi_n = \varepsilon_n^2 \int \psi\, \phi = 0$$

is still true. In this way, we deduce that

$$m_G(U) \le \liminf_{n\to+\infty} m_G(u_n) < +\infty. \tag{4.1.14}$$

□

PROPOSITION 4.3. *In case* (2) *of Theorem 4.1, we have that*

$$\exists \phi_n \in C^1_0(\Omega) \quad G-\textit{invariant so that } \ supp\, \phi_n \subset A_{R\varepsilon_n}(P_n)$$

and

$$\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2\, dx < 0, \tag{4.1.15}$$

for all n large and for some fixed $R > 0$.

PROOF. By Theorem 4.1 we know that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^3)$, where $U(X,Y,Z) = U(Y,Z)$ depends only on the $YZ-$ variables. By Theorem 2.1 let $\phi(Y,Z) \in C^1_0(\mathbb{R}^2)$ so that

$$\int |\nabla \phi|^2 + (\tilde\lambda_2 - p\, U^{p-1})\phi^2 < 0.$$

Set $\phi_n(x,y,z) = \phi\Big(\frac{\sqrt{x^2+y^2}-y_n}{\varepsilon_n}, \frac{z-z_n}{\varepsilon_n}\Big)\, y_n^{-\frac12}$. Then, ϕ_n is a $G-$invariant function so that

$$\begin{aligned}
&\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 \\
= \;& \frac{1}{y_n\,\varepsilon_n^2}\int_\Omega \Big[|\nabla\phi|^2\Big(\frac{\sqrt{x^2+y^2}-y_n}{\varepsilon_n}, \frac{z-z_n}{\varepsilon_n}\Big) + (\lambda_n V - p\, u_n^{p-1})\,\varepsilon_n^2\,\phi^2\Big(\frac{\sqrt{x^2+y^2}-y_n}{\varepsilon_n}, \frac{z-z_n}{\varepsilon_n}\Big)\Big] \\
= \;& \frac{2\pi}{y_n\,\varepsilon_n^2}\int_{\tilde B} \Big[|\nabla\phi|^2\Big(\frac{s-y_n}{\varepsilon_n}, \frac{t-z_n}{\varepsilon_n}\Big) + (\lambda_n V\,\varepsilon_n^2 - p\, u_n^{p-1}\,\varepsilon_n^2)\,\phi^2\Big(\frac{s-y_n}{\varepsilon_n}, \frac{t-z_n}{\varepsilon_n}\Big)\Big] s\mathrm{d}s\,\mathrm{d}t \\
= \;& 2\pi\int_B [|\nabla\phi|^2 + (\lambda_n\,\varepsilon_n^2\, V(0,\varepsilon_n S + y_n, \varepsilon_n T + z_n) - p\, U_n^{p-1}(0,S,T))\phi^2]\Big(\frac{\varepsilon_n S}{y_n} + 1\Big)\mathrm{d}S\,\mathrm{d}T
\end{aligned}$$

where $\tilde B = \{\, a^2 \le s^2+t^2 \le b^2,\ s \ge 0 \,\}$ and $B = \{\, a^2 \le (\varepsilon_n S + y_n)^2 + (\varepsilon_n T + z_n)^2 \le b^2,\ S \ge \frac{-y_n}{\varepsilon_n} \,\}$. Since ϕ has compact support, there holds

$$\mathrm{supp}\,\phi \subset \{\, a^2 \le (\varepsilon_n S + y_n)^2 + (\varepsilon_n T + z_n)^2 \le b^2,\ S \ge \frac{-y_n}{\varepsilon_n} \,\}.$$

Indeed, for S bounded from below it is true that $S \ge -\frac{y_n}{\varepsilon_n}$ for n large, in view of $\frac{y_n}{\varepsilon_n} \to +\infty$. Similarly, for S, T bounded we have that

$$(\varepsilon_n S + y_n)^2 + (\varepsilon_n T + z_n)^2 = y_n^2 + z_n^2 + O(\varepsilon_n) \in (a^2, b^2)$$

for n large in view of $\frac{\mathrm{d}\,((0,y_n,z_n),\partial\Omega)}{\varepsilon_n} \to +\infty$ as $n \to +\infty$.
Then for n large

$$\begin{aligned}
&\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 \\
= \;& \int_{\mathrm{supp}\,\phi} [|\nabla\phi|^2 + \lambda_n\,\varepsilon_n^2\, V(0,\varepsilon_n S + y_n, \varepsilon_n T + z_n)\phi^2 - p\, U_n^{p-1}(0,S,T)\,\phi^2]\Big(\frac{\varepsilon_n}{y_n} S + 1\Big)\mathrm{d}S\,\mathrm{d}T \\
\to \;& \int |\nabla\phi|^2 + (\tilde\lambda_2 - p\, U^{p-1})\phi^2 < +\infty
\end{aligned}$$

in view of $\frac{\varepsilon_n}{y_n} \to 0$, as $n \to +\infty$.
If $\mathrm{supp}\,\phi \subset B_R(0)$, then $\mathrm{supp}\,\phi_n \subset \{\, (\sqrt{x^2+y^2} - y_n)^2 + (z-z_n)^2 \le R^2\,\varepsilon_n^2 \,\} = A_{R\varepsilon_n}(P_n)$.

$\square$

REMARK 4.4. Observe that in Proposition 4.3 we fix one direction of "negativity" ϕ in $\mathbb{R}^2$ and bring it back to the original problem in $\Omega \subset \mathbb{R}^3$. Notice that U has just finite two-dimensional Morse index, while U has infinite Morse index as a solution in $\mathbb{R}^3$.

4.1.2. Global analysis. After the limiting problem has been identified and the local behavior has been described, we can control the global behavior.

THEOREM 4.5. *Let (λ_n, u_n) be a positive, $G-$invariant solution of (4.1.1) so that $\sup_n m_G(u_n) < +\infty$ and $1 < p \neq 5$. Up to a subsequence, there exist $P_n^1 = (0, y_n^1, z_n^1), \ldots, P_n^h = (0, y_n^h, z_n^h)$, $h \leq \sup_n m_G(u_n)$, with $y_n^i \geq 0$ and $\varepsilon_n^i = u_n(P_n^i)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$ s. t.*

$$\varepsilon_n^1 \leq \varepsilon_n^i \leq C_0 \varepsilon_n^1, \quad \text{for all } i = 1, \ldots, h \tag{4.1.16}$$

$$\frac{\varepsilon_n^i + \varepsilon_n^j}{|P_n^i - P_n^j|} \to 0 \quad \text{as } n \to +\infty, \quad \text{for all } i, j = 1, \ldots, h, \ i \neq j \tag{4.1.17}$$

$$\frac{\varepsilon_n^i}{d(P_n^i, \partial\Omega)} \to 0 \quad \text{as } n \to +\infty, \quad \text{for all } i = 1, \ldots, h, \tag{4.1.18}$$

$$u_n(P_n^i) = (1 + o(1)) \max_{B_{R_n \varepsilon_n^i}(P_n^i)} u_n, \tag{4.1.19}$$

for some $R_n \to +\infty$ as $n \to +\infty$.
Moreover, there holds

$$u_n(0, y, z) \leq C(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^{h} e^{-\gamma \frac{|(0,y,z) - P_n^i|}{\varepsilon_n^1}} \quad \forall\, (0, y, z) \in \Omega, \ \ n \in \mathbb{N} \tag{4.1.20}$$

with $C > 0$.

PROOF. We follow the proof of Theorem 2.4.

1^{st} step *There exist $k \leq \sup_n m_G(u_n)$ sequences $P_n^1, \ldots, P_n^k$ satisfying (4.1.16)-(4.1.19) such that:*

$$\lim_{R \to +\infty} \left(\limsup_{n \to +\infty} \left[(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{d_n(0,y,z) \geq R\varepsilon_n^1\}} u_n(0, y, z) \right] \right) = 0 \tag{4.1.21}$$

where $d_n(0, y, z) = \min\{|(0, y, z) - P_n^i| \, : \, i = 1, \ldots, k\}$ is the distance function in Ω from $\{P_n^1, \ldots, P_n^k\}$.

Let Q_n^1 be a point of global maximum of u_n: $u_n(Q_n^1) = \max\limits_{\Omega} u_n$. By Theorem 4.1 we have that there hold (4.1.18), (4.1.19) and $\lambda_n (\varepsilon_n^i)^2 V(P_n^i) \to \tilde{\lambda} \in (0, 1]$ as $n \to +\infty$. By Propositions 4.2 and 4.3 we can deduce the existence of a $G-$invariant $\phi_n^1 \in C_0^1(\Omega)$ so that (4.1.5) holds and $\operatorname{supp} \phi_n^1 \subset \{ \, (\sqrt{x^2 + y^2} - y_n^1)^2 + (z - z_n^1)^2 < R^2 \, (\varepsilon_n^1)^2 \, \}$.

If (4.1.21) holds for P_n^1, then we take $k=1$ and the claim is proved. Otherwise, we have that

$$\limsup_{R\to+\infty}\ \limsup_{n\to+\infty}(\varepsilon_n^1)^{\frac{2}{p-1}}\max_{|(0,y,z)-P_n^1|\geq R\varepsilon_n^1} u_n(0,y,z)=4\delta>0.$$

Applying Theorem 4.1, up to a subsequence, we have

$$(\varepsilon_n^1)^{\frac{2}{p-1}}u_n(\varepsilon_n^1(0,Y,Z)+P_n^1)=U_n^1(0,Y,Z)\to U(0,Y,Z)\quad\text{in } C^1_{loc}(\mathbb{R}^3), \tag{4.1.22}$$

where U is solution of (4.1.3) or (4.1.4). By Theorem 1.12 we have that $U\to 0$ as $|(X,Y,Z)|\to\infty$ and then $\exists\, R$ so that:

(1)

$$U(0,Y,Z)\leq\delta\quad \text{if } |(0,Y,Z)|\geq R; \tag{4.1.23}$$

(2) up to take R large, we have the following property

$$\limsup_{n\to+\infty}(\varepsilon_n^1)^{\frac{2}{p-1}}\max_{|(0,y,z)-P_n^1|\geq R\varepsilon_n^1} u_n\geq 3\delta>0. \tag{4.1.24}$$

Up to a subsequence, we can also assume that

$$(\varepsilon_n^1)^{\frac{2}{p-1}}\max_{|(0,y,z)-P_n^1|\geq R\varepsilon_n^1} u_n\geq 2\delta. \tag{4.1.25}$$

Since $u_n=0$ on $\partial\Omega$, then we have that

$$\exists\, Q_n^2=(0,y_n^2,z_n^2)\in\Omega\setminus\{\,|(0,y,z)-P_n^1|>R\,\varepsilon_n^1\,\}\quad\text{s.t.}\quad u_n(Q_n^2)=\max_{|(0,y,z)-P_n^1|\geq R\,\varepsilon_n^1}u_n.$$

By (4.1.22) and (4.1.23) we get that $\frac{|Q_n^2-P_n^1|}{\varepsilon_n^1}\to+\infty$. Indeed, if $\frac{|Q_n^2-P_n^1|}{\varepsilon_n^1}\to R'\geq R$

$$u_n(Q_n^2)=U_n^1\Big(\frac{|Q_n^2-P_n^1|}{\varepsilon_n^1}\Big)\to U(R')\leq\delta$$

contradicting (4.1.25). We take $R_n^2=\frac{1}{2}\frac{|Q_n^2-P_n^1|}{\varepsilon_n^1}$, $(R_n^1=\frac{1}{2}\frac{d(P_n^1,\partial\Omega)}{\varepsilon_n^1})$, $C_1=(2\,\delta)^{-\frac{p-1}{2}}$. By this and (4.1.25) we get $\varepsilon_{n_j}^2:=u_{n_j}(Q_{n_j}^2)^{-\frac{p-1}{2}}\leq\varepsilon_{n_j}^1(2\,\delta)^{-\frac{p-1}{2}}$, and since $\varepsilon_{n_j}^1\leq\varepsilon_{n_j}^2$ we see that (4.1.16) and (4.1.17) are fulfilled because $|P_{n_j}^2-P_{n_j}^1|\geq R\,\varepsilon_{n_j}^1$. So this implies (4.1.19)

$$u_n(Q_n^2)=\max_{|(0,y,z)-P_n^1|\geq R\varepsilon_n^1}u_n=\max_{B_{R_n^2\,\varepsilon_n^2}(Q_n^2)\cap\Omega}u_n.$$

Indeed $R_n^2\,\varepsilon_n^2=\frac{1}{2}|Q_n^2-P_n^1|$, and $R\,\varepsilon_n^1<<\frac{1}{2}|Q_n^2-P_n^1|$ imply $\forall\,(0,y,z)\in B_{R_n^2\,\varepsilon_n^2}(Q_n^2)$,

$$|(0,y,z)-P_n^1|\geq|Q_n^2-P_n^1|-|(0,y,z)-Q_n^2|\geq\frac{1}{2}|Q_n^2-P_n^1|\geq R\,\varepsilon_n^1,$$

i. e. $\Omega\cap B_{R_n^2\,\varepsilon_n^2}(Q_n^2)\subset\Omega\cap B_{R\,\varepsilon_n^1}(P_n^1)$. Since $R_n^2\to+\infty$ as $n\to+\infty$. By Theorem 4.1 we get that, up to a subsequence, (4.1.16)- (4.1.19) hold true for $\{\,P_n^1,Q_n^2\,\}$. If (4.1.21) holds for $\{\,P_n^1,Q_n^2\,\}$ we are done.

Otherwise, we iterate the above argument: let $P_n^1,Q_n^2\dots,Q_n^s$ s sequences, so that (4.1.16)-(4.1.19) hold true, but (4.1.21) is not satisfied. We have

$$\limsup_{R\to+\infty}\limsup_{n\to+\infty}((\varepsilon_n^1)^{\frac{2}{p-1}}\max_{d_n(0,y,z)\geq R\,\varepsilon_n^1}u_n)=4\,\delta>0$$

with $d_n(0,y,z) = \min\{\,|(0,y,z) - P_n^i|\,: i = 1,2\,\}$. There exists $R > 0$ large s.t.

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{\,d_n(0,y,z)\geq R\varepsilon_n^1\,\}} u_n((0,y,z)) \geq 2\,\delta$$

holds for a subsequence. By (4.1.16) and Theorem 2.1:

$$\exists \vartheta_i \in \Big[\frac{1}{C}, 1\Big] : \ \frac{\varepsilon_n^1}{\varepsilon_n^i} \to \vartheta_i, \tag{4.1.26}$$

$$(\varepsilon_n^1)^{\frac{2}{p-1}} u_n(\,\varepsilon_n^1\,(0,Y,Z) + P_n^1) = \Big(\frac{\varepsilon_n^1}{\varepsilon_n^i}\Big)^{\frac{2}{p-1}} U_n^i\Big(\frac{\varepsilon_n^1}{\varepsilon_n^i}(0,Y,Z)\Big) \to \vartheta_i^{\frac{2}{p-1}} U(\vartheta_i\,(0,Y,Z))$$

in $C^1_{loc}(\mathbb{R}^3)$.

Since $U \to 0$ as $|x| \to +\infty$ we can find R large so that $\vartheta_i^{\frac{2}{p-1}} U(\vartheta_i\,(0,Y,Z)) < \delta$ for $|(0,Y,Z)| \geq R_\delta$. We repeat the argument above, replacing $|(0,y,z) - P_n^1|$ with $d_n(0,y,z)$.

Let Q_n^{s+1} be s. t. $u_n(Q_n^{s+1}) = \max_{d_n(0,y,z)\geq R\,\varepsilon_n^1} u_n \geq 2\,\delta$. As above we have that $\frac{d_n(Q_n^{s+1})}{\varepsilon_n^1} \to +\infty$ and (4.1.16) holds for $\{\,P_n^1, Q_n^2, \ldots, Q_n^{s+1}\,\}$. For $R_n^{s+1} = \frac{1}{2}\frac{d_n(Q_n^{s+1})}{\varepsilon_n^{s+1}}$ we get the validity of (4.1.16) for Q_n^{s+1} so by Theorem 4.1 we get that (4.1.16)-(4.1.19) hold for $\{\,P_n^1, Q_n^2, \ldots, Q_n^{s+1}\,\}$ with $R_n = \min_k R_n^k$.

We can use Theorem 4.1 for any sequence Q_n^i, $i = 1, \ldots, s+1$, for n large. If $P_n^i \to P^i \in z-$axis and $\frac{d(P^i, G_0)}{\varepsilon_n^i} \leq C < +\infty$, we can find functions $\phi_n^i \in C_0^\infty(\Omega)$ with $\operatorname{supp}\phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$, for some $R > 0$, which satisfy (4.1.12). If $P_n^i \to P^i \notin G_0$, we can find functions $\tilde\phi_n^i \in C_0^\infty(\Omega \cap \mathbb{R}^2)$, with $\operatorname{supp}\tilde\phi_n^i \subset B_{R\,\varepsilon_n^i}(P^i) \cap \mathbb{R}^2$, which satisfy (4.1.15). By (4.1.17) $\phi_n^i, \tilde\phi_n^j$ $i,j \in 1, \ldots s+1$, $i \neq j$, have disjoint compact supports for n large and then $s + 1 \leq \sup_n m(u_n)$. The argument must stop for some $k \leq \overline{k}$.

We want to show now the validity of (4.1.20) and this prove the Theorem. this is the contribute of the following $2^{\mathbf{nd}}$ step.

$2^{\mathbf{nd}}$ step *Let $P_n^1, \ldots, P_n^k$ be as in the first step, $1 < p < \infty$. Then there are γ, $C > 0$ such that:*

$$u_n(0,y,z) \leq C\,(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^{k} e^{-\gamma\frac{|(0,y,z)-P_n^i|}{\varepsilon_n^1}}, \quad \forall (0,y,z) \in \Omega, \quad \forall n \in \mathbb{N}. \tag{4.1.27}$$

By (4.1.21), for $R > 0$ large and $n \geq n(R)$, it results

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{\,d_n(x)\geq R\varepsilon_n^1\,\}} u_n(0,y,z) \leq \Big(\frac{\tilde\lambda}{2\omega(V)}\Big)^{\frac{1}{p-1}}.$$

Hence in $\{d_n(0,y,z) \geq R\,\varepsilon_n^1\,\}$ we have

$$(\varepsilon_n^1)^2\, u_n^{p-1}(0,y,z) \leq \frac{\tilde\lambda}{2\omega(V)},$$

where $\omega(V) := [\max_{\overline{\Omega}} V][\min_{\overline{\Omega}} V]^{-1}$. Moreover, by Theorem 4.1 we get

$$\lambda_n(\varepsilon_n^1)^2\, V(0,y,z) \geq [\omega(V)]^{-1}\lambda_n(\varepsilon_n^1)^2\, V(P_n^1) \to_n \frac{\tilde{\lambda}}{\omega(V)}.$$

Therefore for $n \geq n(R)$, we have that

$$(\varepsilon_n^1)^2[\lambda_n\, V(0,y,z) - u_n^{p-1}(0,y,z)] \geq \frac{\tilde{\lambda}}{2\,\omega(V)} > 0, \quad \text{if } d_n(0,y,z) \geq R\,\varepsilon_n^1.$$

Now consider the following linear operator:

$$L_n := -\Delta + (\lambda_n\, V(0,y,z) - u_n^{p-1}(0,y,z)).$$

Since u_n is a positive solution in Ω of L_n, L_n satisfies the minimum principle in any $\tilde{\Omega} \subset \Omega$: $L_n\phi > 0$ in $\tilde{\Omega}$, $\phi > 0$ on $\partial\tilde{\Omega}$ implies $\phi > 0$ in $\tilde{\Omega}$. Let $\phi_n^i((0,y,z)) = e^{-\gamma(\varepsilon_n^1)^{-1}|(0,y,z)-P_n^i|}$. We have that in $d_n(0,y,z) \geq R\,\varepsilon_n^1$:

$$L_n(\phi_n^i) = (\varepsilon_n^1)^{-2}\phi_n^i\Big[-\gamma^2 + (N-1)\frac{\varepsilon_n^1}{|(0,y,z)-P_n^i|}\gamma + (\varepsilon_n^1)^2(\lambda_n\, V(0,y,z) - u_n^{p-1}(0,y,z))\Big] > 0$$

for n large, provided $\gamma^2 \leq \frac{\tilde{\lambda}\, V(P^i)}{4\,\omega(V)}$.
Observe that

$$\Big(e^{\gamma R}\phi_n^i(0,y,z) - (\varepsilon_n^1)^{\frac{2}{p-1}} u_n(0,y,z)\Big)|_{\partial B_{R\,\varepsilon_n^1}(P_n^i)} \to 1 - \theta_i^{\frac{2}{p-1}} U(\theta_i R) > 0$$

for R large, where θ_i are given by (4.1.26).
Then if we define $\phi_n := e^{\gamma\, R}\sum_{i=1}^k \phi_n^i$, we have

$$L_n(\phi_n - u_n) > 0 \quad \text{in} \quad \{d_n(0,y,z) > R\,\varepsilon_n^1\}$$

and $\phi_n - u_n \geq 0$ on $\{d_n(0,y,z) = R\,\varepsilon_n^1\} \cup \partial\Omega$. Note that by (4.1.16)-(4.1.19),

$$\{d_n(0,y,z) = R\,\varepsilon_n^1\} = \bigcup_{j=1}^k \partial B_{R\,\varepsilon_n^1}(P_n^j) \subset \Omega,$$

for $n \geq n(R)$. Then by the minimum principle

$$u_n \leq \phi_n = e^{\gamma\, R}(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^k e^{-\gamma\frac{|(0,y,z)-P_n^i|}{\varepsilon_n^1}}$$

in $\{d_n(0,y,z) > R\ \varepsilon_n^1\}$, if R is large and $n \geq n(R)$. Since

$$u_n(0,y,z) \leq \max_{\Omega} u_n = (\varepsilon_n^1)^{-\frac{2}{p-1}} \leq e^{\gamma\, R}(\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^k e^{-\gamma\frac{|(0,y,z)-P_n^i|}{\varepsilon_n^1}}$$

if $d_n(0,y,z) \leq R\,\varepsilon_n^1$.
We have that (4.1.20) holds true in Ω, for $C = e^{\gamma\, R}$ and $n \geq n(R)$. Up to take a larger constant C, we have the validity of (4.1.20) for every $n \in \mathbb{N}$.

□

4.2. Location of the blow-up set

Our aim is to obtain solutions which blow-up on suitable circles. There is a great literature of this kind based on a constructive approach of perturbative type.
With the notations of the Theorem 4.5, let us set $P^i = \lim_{n\to+\infty} P_n^i$ and

$$J_i = \{j = 1, \dots, h \ : \ P_n^j \to P^i\},$$

for every $i = 1, \dots, h$.
For a point $P_0 = (0, y_0, z_0)$ let us define

$$A_\delta(P_0) = \{\, (x,y,z) \in \mathbb{R}^3 \ : \ \big(\sqrt{x^2+y^2} - y_0\big)^2 + (z-z_0)^2 \le \delta^2 \,\}$$

for $\delta > 0$. Define

$$J^1 = \{\, j = 1, \dots, h \ : \ \frac{y_n^j}{\varepsilon_n^j} \le C \,\}, \quad J^2 = \{\, j = 1, \dots, h \ : \ \frac{y_n^j}{\varepsilon_n^j} \to +\infty \,\}$$

and

$$J_1^i = J_i \cap J^1, \quad J_2^i = J_i \cap J^2.$$

Fix $\delta > 0$ small so that $I_\delta^i := A_\delta(P^i) \cap \Omega$, satisfies

$$I_\delta^i \cap \{P^1, \dots, P^h\} = \{P^i\}.$$

We have the following expansions:

LEMMA 4.6. *Let g be a continuous $G-$invariant function in $\bar\Omega$. Let $1 < p \neq 5$ and $q > 1$ and $i \in \{1, \dots, h\}$. Then, there hold:*

- *If $J_2^i = \emptyset$*

$$\int_{I_\delta^i} g\, u_n^q = g(P^i)\Big(\int_{\mathbb{R}^3} U^q\Big)\Big(\sum_{j \in J^i} (\varepsilon_n^j)^{-\frac{2q}{p-1}+3}\Big)(1 + o_n(1)). \tag{4.2.1}$$

- *If $J_2^i \neq \emptyset$*

$$\int_{I_\delta^i} g\, u_n^q = 2\pi\, g(P^i)\Big(\int_{\mathbb{R}^2} V^q\Big)\Big(\sum_{j \in J^i_{2,max}} (\varepsilon_n^j)^{-\frac{2q}{p-1}+2}\, y_n^j\Big)(1 + o_n(1)) \tag{4.2.2}$$

where $o_n(1) \to 0$ as $n \to +\infty$, U and V are the solutions of (4.1.3) and (4.1.4) and

$$J^i_{2,max} = \Big\{\, j = 1, \dots, h \ : \ \lim_{n\to+\infty} \frac{y_n^j}{\max_{j \in J_2^i}\{\, y_n^j \,\}} \Big\}.$$

PROOF. Let us define $A^j_{R\,\varepsilon_n^1}(P_n^j)$, for every $R > 0$ and $j = 1, \dots, h$. In view of (4.1.16) - (4.1.18), we have that

$$A^j_{R\,\varepsilon_n^1} \subset \Omega \quad \text{and} \quad A^j_{R\,\varepsilon_n^1} \cap A^k_{R\,\varepsilon_n^1} = \emptyset, \quad j \neq k,$$

for $n \geq n(R)$ large and $j \in J^2$. By Theorem 4.5, we know that

$$u_n^q(0,y,z) \le C(\varepsilon_n^1)^{-\frac{2q}{p-1}} \sum_{j=1}^h e^{-q\gamma \frac{|(0,y,z)-P_n^j|}{\varepsilon_n^1}} \quad \forall (0,y,z) \in \Omega \tag{4.2.3}$$

and, by the $G-$invariance of u_n,

$$u_n^q(x,y,z) \le C(\varepsilon_n^1)^{-\frac{2q}{p-1}} \sum_{j=1}^h e^{-q\gamma \frac{|(0,\sqrt{x^2+y^2},z)-P_n^j|}{\varepsilon_n^1}} \quad \forall (x,y,z) \in \Omega, \tag{4.2.4}$$

for some $C > 0$.
For $j \in J^1$, we have that

$$\begin{aligned} \int_{B_{R\varepsilon_n^1}(P_n^j)} g\, u_n^q &= (\varepsilon_n^j)^{-\frac{2q}{p-1}+3} \int_{B_{R\frac{\varepsilon_n^1}{\varepsilon_n^j}}(0)} g(\varepsilon_n^j(X,Y,Z)+P_n^j)\,(U_n^j)^q \\ &= (\varepsilon_n^j)^{-\frac{2q}{p-1}+3}\Big(g(P^j)\int_{B_{R\theta_j}(0)} U^q + o_n(1)\Big), \end{aligned} \tag{4.2.5}$$

where $\frac{\varepsilon_n^1}{\varepsilon_n^j} \to \theta_j \in [\frac{1}{C_0}, 1]$ in view of (4.1.16) and U is the solution of (4.1.3). Since $P_n^j = (0,0,z_n^j)$ in this case, we have that

$$|(0,\sqrt{x^2+y^2},z) - P_n^j| = |(x,y,z) - P_n^j|$$

so as to get

$$(\varepsilon_n^1)^{-\frac{2q}{p-1}} \int_{\mathbb{R}^3\setminus B_{R\varepsilon_n^1}(P_n^j)} e^{-q\gamma\frac{|(0,\sqrt{x^2+y^2},z)-P_n^j|}{\varepsilon_n^1}} = (\varepsilon_n^1)^{-\frac{2q}{p-1}+3}\int_{\mathbb{R}^3\setminus B_R(0)} e^{-q\gamma|(X,Y,Z)|}. \tag{4.2.6}$$

For $j \in J^2$, we have that

$$\begin{aligned} &\int_{A^j_{R\varepsilon_n^1}} g\, u_n^q \\ = \;& 2\pi \int_{\{(s,t)\,:\, s\ge 0,\,(s-y_n^j)^2+(t-z_n^j)^2 \le R^2(\varepsilon_n^1)^2\}} g(0,s,t)\; u_n^q(0,s,t)\, s \,\mathrm{d}\,s\,\mathrm{d}\,t = 2\pi\,(\varepsilon_n^j)^{-\frac{2q}{p-1}+2}\, y_n^j \\ & \int_{\{(S,T)\,:\,S^2+T^2\le R^2\left(\frac{\varepsilon_n^1}{\varepsilon_n^j}\right)^2\}} g(0,\varepsilon_n^j S + y_n^j, \varepsilon_n^j T + z_n^j)\, U_n^q(0,S,T)\Big(\frac{\varepsilon_n^j}{y_n^j} S + 1\Big)\mathrm{d}\,S\,\mathrm{d}\,T \end{aligned}$$

in view of $\frac{y_n^j}{\varepsilon_n^j} \to +\infty$.
Hence, we get that

$$\int_{A^j_{R\varepsilon_n^1}} g\, u_n^q = 2\pi\,(\varepsilon_n^j)^{-\frac{2q}{p-1}+2}\, y_n^j\Big(g(P^j)\int_{B_{R\theta_j}(0)} W^q(S,T)\mathrm{d}\,S\,\mathrm{d}\,T + o_n(1)\Big) \tag{4.2.7}$$

where W is solution of (4.1.4).
Similarly, we have that

$$
\begin{aligned}
&(\varepsilon_n^1)^{-\frac{2q}{p-1}} \int_{\mathbb{R}^3\setminus A^j_{R\,\varepsilon_n^1}} e^{-q\,\gamma\,\frac{|(0,\sqrt{x^2+y^2},z)-P_n^j|}{\varepsilon_n^1}} \\
&\leq \; 2\,\pi\,(\varepsilon_n^1)^{-\frac{2q}{p-1}+2}\, y_n^j \int_{\mathbb{R}^2\setminus B_R(0)} e^{-q\,\gamma\,|(S,T)|}\Big(\frac{\varepsilon_n^1}{y_n^j}\,S+1\Big)\mathrm{d}\,S\,\mathrm{d}\,T \\
&\leq \; 2\,\pi\,(\varepsilon_n^1)^{-\frac{2q}{p-1}+2}\, y_n^j \int_{\mathbb{R}^2\setminus B_R(0)} e^{-q\,\gamma\,|(S,T)|}\Big(|S|+1\Big)\mathrm{d}\,S\,\mathrm{d}\,T. \qquad (4.2.8)
\end{aligned}
$$

In conclusion, by (4.2.5)- (4.2.8) we get that

$$
\begin{aligned}
&\int_{I_\delta^i} g\,u_n^1 = \sum_{j\in J_1^i}\int_{B_{R\,\varepsilon_n^1}(P_n^j)} g\,u_n^q + \sum_{j\in J_2^i}\int_{A^j_{R\,\varepsilon_n^1}} g\,u_n^q \\
&+\; O\Big((\varepsilon_n^1)^{-\frac{2q}{p-1}} \sum_{j\in J_1^i}\int_{\mathbb{R}^3\setminus B_{R\,\varepsilon_n^1}(P_n^j)} e^{-q\,\gamma\,\frac{|(0,\sqrt{x^2+y^2},z)-P_n^j|}{\varepsilon_n^1}} \\
&+\; (\varepsilon_n^1)^{-\frac{2q}{p-1}} \sum_{j\in J_2^i}\int_{\mathbb{R}^3\setminus A^j_{R\,\varepsilon_n^1}} e^{-q\,\gamma\,\frac{|(0,\sqrt{x^2+y^2},z)-P_n^j|}{\varepsilon_n^1}}\Big) \\
&=\; \Big[\sum_{j\in J_1^i}(\varepsilon_n^1)^{-\frac{2q}{p-1}+3}\int_{B_{R\,\theta_j}(0)} U^q + \sum_{j\in J_2^i}(\varepsilon_n^1)^{-\frac{2q}{p-1}+2}\, y_n^j\, 2\,\pi \int_{B_{R\,\theta_j}(0)} W^q\Big]\, g(P^i)\,(1+o_n(1)) \\
&+\; O\Big((\varepsilon_n^1)^{-\frac{2q}{p-1}+3}\Big)\int_{\mathbb{R}^3\setminus B_R(0)} e^{-q\,\gamma\,|(X,Y,Z)|} \\
&+\; (\varepsilon_n^1)^{-\frac{2q}{p-1}+2}\sum_{j\in J_2^i} y_n^j \int_{\mathbb{R}^2\setminus B_R(0)} e^{-q\,\gamma\,|(S,T)|}(|S|+1)\mathrm{d}\,S\,\mathrm{d}\,T\Big),
\end{aligned}
$$

in view of $B_{R\,\varepsilon_n^1}(P_n^j)\subset\Omega$ and $B_{R\,\varepsilon_n^1}(P_n^1)\cap B_{R\,\varepsilon_n^1}(P_n^k)=\varnothing$, $j\neq k$, and $B_{R\,\varepsilon_n^1}(P_n^j)\cap A^m_{R\,\varepsilon_n^1}$ $\forall\, j,\ k$ and $m\in J^2$, due to (4.1.16) - (4.1.18).
If $J_2^i=\varnothing$

$$
\lim_{n\to+\infty}(\varepsilon_n^1)^{\frac{2q}{p-1}+3}\int_{I_\delta^i} g\,u_n^q = \Big(\sum_{j\in J_1^i}\theta_j^{\frac{2q}{p-1}+3}\int_{B_{R\,\theta_j}(0)} U^q\Big)g(P^i) + O\Big(\int_{\mathbb{R}^3\setminus B_R(0)} e^{-q\,\gamma|(X,Y,Z)|}\Big)
$$

for all $R>0$, and then as $R\to+\infty$

$$
\lim_{n\to+\infty}(\varepsilon_n^1)^{\frac{2q}{p-1}}\int_{I_\delta^i} g\,u_n^q = g(P^i)\int_{\mathbb{R}^3} U^q \sum_{j\in J_1^i}\theta_j^{\frac{2q}{p-1}+3}.
$$

Therefore

$$
\int_{I_\delta^i} g\,u_n^q = g(P^i)\Big(\int_{\mathbb{R}^3} U^q\Big)\Big(\sum_{j\in J_1^i}(\varepsilon_n^1)^{-\frac{2q}{p-1}+3}\Big)(1+o(1)). \qquad (4.2.9)
$$

If $J_2^i \neq \emptyset$ we define the subset

$$J^i_{2,max} = \Big\{ j = 1, \dots, h \, : \, \frac{y_n^j}{\max\{ y_n^j \, : \, j \in J_2^i \}} \to \mu_j > 0 \Big\}$$

(up to a subsequence). Then

$$\begin{aligned} \lim_{n\to+\infty} (\varepsilon_n^1)^{\frac{2q}{p-1}+2} \Big(\max_{j \in J_2^i} \{ y_n^j \} \Big)^{-1} \int_{I^i_\delta} g\, u_n^q &= \Big(\sum_{j \in J^i_{2,max}} 2\,\pi\, \theta_j^{\frac{2q}{p-1}+2} \mu_j \int_{B_{R\,\theta_j}(0)} W^q \Big) g(P^i) \\ &+ O\Big(\int_{\mathbb{R}^2 \setminus B_R(0)} e^{-q\,\gamma\, |(S,T)|} (|S|+1) \mathrm{d}\, S \,\mathrm{d}\, T \Big) \end{aligned}$$

in view of

$$\frac{\varepsilon_n^j}{\max_{j \in J_2^i} \{ y_n^j \}} = \frac{\varepsilon_n^j}{\varepsilon_n^1} \frac{\varepsilon_n^{j_n}}{y_n^{j_n}} \frac{\varepsilon_n^1}{\varepsilon_n^{j_n}} \le C_0 \frac{\varepsilon_n^{j_n}}{y_n^{j_n}} \to 0$$

for a suitable $j_n \in J^i_{2,max}$. We have used (4.1.16) and $\frac{\varepsilon_n^j}{y_n^j} \to 0 \;\forall\, j \in J_2^i$.
Letting $R \to +\infty$, finally we get

$$\lim_{n\to+\infty} (\varepsilon_n^1)^{\frac{2q}{p-1}+2} \Big(\max_{j \in J_2^i} \{ y_n^j \} \Big)^{-1} \int_{I^i_\delta} g\, u_n^q = 2\,\pi\, g(P^i) \Big(\int_{\mathbb{R}^2} W^q \Big) \Big(\sum_{j \in J^i_{2,max}} \theta_j^{\frac{2q}{p-1}+2} \mu_j \Big),$$

and therefore

$$\int_{I^i_\delta} g\, u_n^q = 2\,\pi\, g(P^i) \Big(\int_{\mathbb{R}^2} W^q \Big) \Big(\sum_{j \in J^i_{2,max}} (\varepsilon_n^j)^{-\frac{2q}{p-1}+2} y_n^j \Big) (1 + o(1)).$$

□

In this part of section we want to localize the blow-up set of $G-$invariant solutions. For their symmetry properties, we may guess that the blow-up set is of positive dimension.
We are searching for a modified potential that localizes the blow-up set.

First we have that:

THEOREM 4.7. *Let u_n be a positive solution of (4.1.1). Then we have that for all $k \in (0, +\infty)$*

$$\lambda_n\, u_n^2 + |\nabla u_n| = O((\varepsilon_n^1)^k), \quad on \;\; \partial\, B_\delta(P_0),$$

where $\delta > 0$ is small so that P_0 is the only geometrical blow-up point in $\overline{B_{2\,\delta}(P_0)}$.

PROOF. Consider the general case $P_0 \in \overline{\Omega}$ and obtain the estimate on $\partial A_\delta(P_0) \cap \Omega$, for $\delta > 0$ small. By 4.1.20 we obtain that

$$u_n(x,y,z) \le C (\varepsilon_n^1)^{-\frac{2}{p-1}} \sum_{i=1}^h e^{-\gamma \frac{|(0, \sqrt{x^2+y^2}, z) - P_n^i|}{\varepsilon_n^1}} \quad \forall\, (x,y,z) \in \Omega, \;\; n \in \mathbb{N} \tag{4.2.10}$$

with $C > 0$. In particular we have that $u_n = O((\varepsilon_n^1)^k)$ uniformly in $B_{2\,\delta}(P_0) \setminus B_\delta(P_0)$

We decompose u_n as $u_n = u_n^1 + u_n^2$ where u_n^1 and u_n^2 satisfy

$$\begin{cases} -\Delta u_n^1 = u_n^p - \lambda_n u_n & \text{in } B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0) \\ u_n^1 = 0 & \text{on } \partial(B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0)); \end{cases} \tag{4.2.11}$$

$$\begin{cases} \Delta u_n^2 = 0 & \text{in } B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0) \\ u_n^2 = u_n = O((\varepsilon_n^1)^k) & \text{on } \partial(B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0)). \end{cases} \tag{4.2.12}$$

By the mean value theorem we have that $u_n^2 + |\nabla u_n^2| = O((\varepsilon_n^1)^k)$ on $B_{\frac{3}{2}\delta}(P_0) \setminus B_{\frac{5}{4}\delta}(P_0)$. We have that in $B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0)$

$$-\Delta u_n^1 = O((\varepsilon_n^1)^{k-2})$$

in view of (4.1.16) and $\lambda_n\,(\varepsilon_n^1)^2 V(P_n^1) \leq C$. Then we have $u_n^1 + |\nabla u_n^1| = O((\varepsilon_n^1)^{k-2})$ in $B_{2\,\delta}(P_0) \setminus B_{\frac{\delta}{2}}(P_0)$. The result then follows. □

We consider the main result of this chapter, in which we obtain a complete asymptotic analysis:

THEOREM 4.8 (Classification of blow-up points). *Let (λ_n, u_n) be a positive, $G-$invariant solution of (4.1.1) with $\sup_n m_G(u_n) < +\infty$ and $1 < p \neq 5$ and $\lambda_n \to +\infty$ as $n \to +\infty$. Let P_n^i, $i = 1, \ldots, h$ be the points given by Theorem 4.5 and $P^i = \lim_{n\to+\infty} P_n^i = (0, y^i, z^i)$. According to the notations of Lemma 4.6, let us assume $J_2^i = \emptyset$ whenever $P^i \in G_0$. Setting*

$$\partial\Omega^{\pm} = \{\, x^2 + y^2 + z^2 = b^2,\ \pm z > 0 \,\} \cup \{\, x^2 + y^2 + z^2 = a^2,\ \pm z < 0 \,\},$$
$$\partial\Omega_a = \{\, x^2 + y^2 + z^2 = a^2 \},\quad \partial\Omega_b = \{\, x^2 + y^2 + z^2 = b^2 \,\}$$

we have that

1. *if $P^i \in \partial\Omega^{\pm}$, then $\pm\partial_s V(P^i) \leq 0$, if $P^i = (0, y^i, 0) \in \partial\Omega$, then $\partial_s V(P^i) = 0$;*
2. *if $P^i \partial\Omega \setminus G_0$, we also have $\partial_r \tilde{V}(P^i) \leq 0$ if $P^i \in \partial\Omega_b$ and $\partial_r \tilde{V}(P^i) \geq 0$ if $P^i \in \partial\Omega_a$;*
3. *if $P^i \in \Omega \cap G_0$, then $\partial_s V(P^i) = \partial_{r\,r} V(P^i) = 0$;*
4. *if $P^i \in \Omega \setminus G_0$, then $\partial_s V(P^i) = \partial_r \tilde{V}(P^i) = 0$.*

Here, $r = \sqrt{x^2 + y^2}$ and $\tilde{V}(r, s) = r^{\frac{p-1}{2}}\, V(r, s)$.

PROOF. Fix $k \in \mathbb{R}$. By Theorem 4.7 we have that

$$\lambda_n\, u_n^2 + u_n^2 + |\nabla u_n|^2 = O((\varepsilon_n^1)^k) \tag{4.2.13}$$

uniformly on $\partial I_\delta^i \cap \Omega$, where $I_\delta^i = A_\delta(P^i) \cap \Omega$.
Multiply the equation (4.1.1) by $\partial_s u_n$ and integrate by parts in I_δ^i so as to get:

$$\begin{aligned} \frac{\lambda_n}{2} \int_{I_\delta^i} \partial_s V\, u_n^2 &= \int_{I_\delta^i} \Big[-\Delta u_n\, \partial_s u_n + \frac{\lambda_n}{2}\, \partial_s (V\, u_n^2) - \frac{1}{p+1} \partial_s (u_n^{p+1}) \Big] \\ &= \int_{\partial I_\delta^i \cap \Omega} \Big[\frac{\lambda_n}{2} V\, u_n^2 - \frac{1}{p+1} u_n^{p+1} \Big] - \int_{\partial I_\delta^i} \partial_\nu u_n \partial_s u_n + \int_{I_\delta^i} \frac{1}{2} \partial_s (|\nabla u_n|^2) \\ &= O((\varepsilon_n^1)^k) - \frac{1}{2} \int_{A_\delta(P^i) \cap \partial\Omega} (\partial_\nu u_n)^2\, \nu_s \end{aligned} \tag{4.2.14}$$

in view of (4.2.13). Observe that would not have sense to multiply by $\partial_x u_n$ or $\partial_y u_n$, having no $G-$invariance. Indeed the contributions of these Pohozaev would cancel each other.

If $P^i = (0, y_i, z_i) \in \Omega$ with $y_i > 0$, by Lemma 4.6 we get that

$$\frac{\lambda_n}{2}\int_{I^i_\delta} \partial_s V\, u_n^2 = \pi\,\lambda_n\,\partial_s V(P^i)\Big(\int_{\mathbb{R}^2} W^2\Big)\Big(\sum_{j\in J_2}(\varepsilon_n^j)^{-\frac{4}{p-1}+2}\Big) y_i(1+o_n(1))$$

in view of $J_1^i = \emptyset$ and $J^i_{2,\max} = J^i_2$.
Since $\frac{\varepsilon_n^1}{\varepsilon_n^j} \to \vartheta_j \in [\frac{1}{C_0}, 1]$ and $\lambda_n(\varepsilon_n^j)^2\, V(P_n^j) \to \tilde\lambda_2 \in (0,1]\ \ \forall\ j \in J_2$, we get that

$$\frac{\lambda_n}{2}\int_{I^i_\delta} \partial_s V\, u_n^2 = \pi\,\tilde\lambda_2\,\frac{\partial_s V(P^i)}{V(P^i)}\Big(\int_{\mathbb{R}^2} W^2\Big)\Big(\sum_{j\in J_2}\vartheta_j^{\frac{4}{p-1}}\Big)\, y_i\,(\varepsilon_n^1)^{-\frac{4}{p-1}}(1+o_n(1)). \tag{4.2.15}$$

For $k > -\frac{4}{p-1}$, by (4.2.14) we get that

$$\partial_s V(P^i) = 0$$

in view of $B_\delta(P^i) \subset \Omega$, for some $\delta > 0$ small.
Similarly, if $P^i = (0,0,z_i) \in \Omega$ with $J_2^i = \emptyset$, by Lemma 4.6 we get that

$$\frac{\tilde\lambda_3}{2}\frac{\partial_s V(P^i)}{V(P^i)}\Big(\int_{\mathbb{R}^2} U^2\Big)\Big(\sum_{j\in J^i}\vartheta_j^{-1+\frac{4}{p-1}}\Big)(\varepsilon_n^1)^{1-\frac{4}{p-1}}(1+o_n(1)) = O((\varepsilon_n^1)^k), \tag{4.2.16}$$

and then $\partial_s V(P^i) = 0$ for $k > 1 - \frac{4}{p-1}$.

In case $P^i \in \partial\Omega$ with $z_i \neq 0$, we have that $\pm\nu_s \geq 0$ in $A_\delta(P^i) \cap \partial\Omega$ according to whether $P^i \in \partial\Omega^\pm$, and (4.2.14) leads to

$$\pm\frac{\lambda_n}{2}\int_{I^i_\delta} \partial_s V\, u_n^2 \leq O((\varepsilon_n^1)^k),$$

respectively. By the same computation as above, we get that

$$\pm\partial_s V(P^i) \leq 0$$

according to whether $P^i \in \partial\Omega^\pm$.

If $P^i \notin G_0$, we choose $Q = (0,0,z_0)$, we multiply the equation (4.1.1) by $[(x,y,z)-Q]\cdot\nabla u_n$ and integrate by parts in $B_\delta(P^i)\cap\Omega = I^i_\delta$

$$\begin{aligned}
&\int_{I^i_\delta}[(x,y,z)-Q]\cdot\nabla u_n(u_n^p - \lambda_n V\,u_n)\\
=\ & -\int_{I^i_\delta} 3\frac{u_n^{p+1}}{p+1} - \frac{\lambda_n}{2}\int_{I^i_\delta} V\ \cdot\ [(x,y,z)-Q]\cdot\nabla(u_n^2) + O((\varepsilon_n^1)^k)\\
=\ & -\int_{I^i_\delta} 3\Big(\frac{u_n^{p+1}}{p+1} - \lambda_n\,V\,\frac{u_n^2}{2} - \lambda_n\,\nabla V\ \cdot\ [(x,y,z)-Q]\,\frac{u_n^2}{6}\Big) + O((\varepsilon_n^1)^k)
\end{aligned}$$

in view of $u_n = 0$ on $\partial\Omega$ and by Theorem 4.7. On the other hand, we have that

$$\begin{aligned}\int_{I^i_\delta} -\Delta u_n\ [(x,y,z)-Q]\cdot\nabla u_n &= \int_{I^i_\delta} |\nabla u_n|^2 + [(x,y,z)-Q]\cdot\nabla\Big(\frac{1}{2}|\nabla u_n|^2\Big)\\ &- \int_{\partial\Omega\cap B_\delta(P^i)} (\partial_\nu u_n)^2\,[(x,y,z)-Q]\,\nu + O((\varepsilon^i_n)^k)\\ &= -\frac{1}{2}\int_{I^i_\delta} |\nabla u_n|^2 - \frac{1}{2}(\partial_\nu u_n)^2\ [(x,y,z)-Q]\cdot\nu + O((\varepsilon^1_n)^k),\end{aligned}$$

and then

$$\begin{aligned}\frac{1}{2}\int_{I^i_\delta}|\nabla u_n|^2 &= \int_{I^i_\delta} 3\Big(\frac{u_n^{p+1}}{p+1} - \lambda_n V\frac{u_n^2}{2} - \lambda_n [(x,y,z)-Q]\cdot\nabla V\frac{u_n^2}{6}\Big)\\ &- \frac{1}{2}\int_{\partial\Omega\cap B_\delta(P^i)}(\partial_\nu u_n)^2\ [(x,y,z)-Q]\ \cdot\nu + O((\varepsilon^1_n)^k).\end{aligned}\tag{4.2.17}$$

Then we multiply the equation (4.1.1) by u_n and integrate on I^i_δ

$$\int_{I^i_\delta}|\nabla u_n|^2 = \int_{I^i_\delta}\Big(u_n^{p+1} - \lambda_n V u_n^2\Big) + O((\varepsilon^1_n)^k).\tag{4.2.18}$$

Substituting (4.2.18) in (4.2.17) we have

$$\begin{aligned}&-\frac{1}{2}\int_{I^i_\delta}\Big(u_n^{p+1}-\lambda_n V u_n^2\Big) - \frac{1}{2}\int_{\partial\Omega\cap B_\delta(P^i)}(\partial_\nu u_n)^2\ [(x,y,z)-Q]\ \cdot\nu + O((\varepsilon^1_n)^k)\\ &= -\int_{I^i_\delta}3\Big(\frac{u_n^{p+1}}{p+1} - \lambda_n V\frac{u_n^2}{2}\Big) + \frac{\lambda_n}{2}\int_{I^i_\delta}[(x,y,z)-Q]\cdot\nabla V\ u_n^2\end{aligned}$$

that is

$$\begin{aligned}&\Big(\frac{1}{2}-\frac{3}{p+1}\Big)\int_{I^i_\delta}u_n^{p+1} + \lambda_n\int_{I^i_\delta}V\,u_n^2\\ &+\ \frac{\lambda_n}{2}\int_{I^i_\delta}[(x,y,z)-Q]\cdot\nabla V\,u_n^2 + \frac{1}{2}\int_{\partial\Omega\cap B_\delta(P^i)}(\partial_\nu u_n)^2\ [(x,y,z)-Q]\ \cdot\nu = O((\varepsilon^1_n)^k).\end{aligned}\tag{4.2.19}$$

Observe that the choice of the form $Q=(0,0,z_0)$ is to ensure that the term $[(x,y,z)-Q]\cdot\nabla V$ is $G-$invariant. As previously observed for the (4.2.14), the contributions in $\partial_x V$ and $\partial_y V$ are not invariant and therefore can not apply the Lemma 4.6, or we may observe that the two terms do not give contribution because they cancel each other.
We consider the different case.

If $P^i\in\Omega$, $\frac{1}{2}\int_{\partial\Omega\cap B_\delta(P^i)}(\partial_\nu u_n)^2\ [(x,y,z)-Q]\ \cdot\nu = 0$ and using Lemma 4.6, we have

$$\begin{aligned}&\sum_{i\in J^2} 2\pi y_i\Big[\Big(\frac{1}{2}-\frac{3}{p+1}\Big)\Big(\int_{\mathbb{R}^2}W^{p+1}\Big)(\varepsilon^i_n)^{-2\frac{p+1}{p-1}+2} + \tilde\lambda_2\Big(\int_{\mathbb{R}^2}V^2\Big)(\varepsilon^i_n)^{-\frac{4}{p-1}}\\ &+\ \frac{\tilde\lambda_2}{2}\frac{[P^i-Q]\,\nabla V(P^i)}{V(P^i)}\Big(\int_{\mathbb{R}^2}W^2\Big)(\varepsilon^i_n)^{-\frac{4}{p-1}}\Big] = O((\varepsilon^1_n)^k),\end{aligned}\tag{4.2.20}$$

where W satisfies (4.1.4). Multiply the equation (4.1.4) by $(r,s)\cdot\nabla W$ and integrate by part

$$\int_{\mathbb{R}^2} -\Delta W(r,s)\cdot\nabla W = 0 = -2\int_{\mathbb{R}^2}\Big(\frac{W^{p+1}}{p+1}-\frac{\tilde\lambda_2}{2}W^2\Big) \tag{4.2.21}$$

and then we derive

$$\int_{\mathbb{R}^2} W^{p+1} = \tilde\lambda_2\,\frac{p+1}{2}\int_{\mathbb{R}^2} W^2. \tag{4.2.22}$$

We substitute (4.2.22) in (4.2.20)

$$\sum_{i\in J^2} 2\pi\, y_i\,(\varepsilon_n^i)^{-\frac{4}{p-1}}\Big(\int_{\mathbb{R}^2}W^2\Big)\tilde\lambda_2\Big[\frac{p-5}{4}+1+\frac{1}{2}\frac{[P^i-Q]\,\nabla V(P^i)}{V(P^i)}\Big] = O((\varepsilon_n^1)^k), \tag{4.2.23}$$

we can divide by $\sum_{i\in J^2} 2\pi\, y^i\,(\varepsilon_n^i)^{-\frac{4}{p-1}}\;\tilde\lambda_2\int_{\mathbb{R}^2}V^2$, therefore

$$\frac{p-1}{2}+\frac{[P^i-Q]\,\nabla V(P^i)}{V(P^i)} = 0. \tag{4.2.24}$$

By previous Pohozaev we have $\partial_s V(P^i)=0$ so in condition (4.2.24) we have

$$\frac{p-1}{2}+y^i\,\frac{\partial_r V(P^i)}{V(P^i)} = 0 \implies \frac{p-1}{2}V(P^i)+y^i\,\partial_r V(P^i)=0 \tag{4.2.25}$$

and this implies that P^i is a critical point of a modified potential $\tilde V(r,s)=r^{\frac{p-1}{2}}\,V(r,s)$.

At this point, if $z^i\neq 0$, we choose $Q=(0,0,z^i)$. Under this choice we have that

$$[P^i-Q]\cdot\nu(P^i) = (0,y^i,0)\cdot\nu(P^i) = \begin{cases}(y^i)^2>0 & \text{if } P^i\in\partial\Omega_b\\ -(y^i)^2<0 & \text{if } P^i\in\partial\Omega_a\end{cases} \tag{4.2.26}$$

where $\partial\Omega_a=\{\,x^2+y^2+z^2=a^2\,\}$ and $\partial\Omega_b=\{\,x^2+y^2+z^2=b^2\,\}$, so in (4.2.19) we have that

$$-\frac{1}{2}\int_{\partial\Omega\cap B_\delta(P^i)}(\partial_\nu u_n)^2\;[(x,y,z)-Q]\;\cdot\nu\begin{cases}<0 & \text{if } P^i\in\partial\Omega_b\\ >0 & \text{if } P^i\in\partial\Omega_a.\end{cases}$$

Therefore we obtain the following estimate

$$\Big(\frac12-\frac{3}{p+1}\Big)\int_{I_\delta^i}u_n^{p+1}+\lambda_n\int_{I_\delta^i}V\,u_n^2+\frac{\lambda_n}{2}\int_{I_\delta^i}[(x,y,z)-Q]\cdot\nabla V\,u_n^2\begin{cases}<0 & \text{if } P^i\in\partial\Omega_b\\ >0 & \text{if } P^i\in\Omega_a.\end{cases} \tag{4.2.27}$$

Using Lemma 4.6 and (4.2.22) we have

$$\sum_{i\in J^2} 2\pi\, y_i\,(\varepsilon_n^i)^{-\frac{4}{p-1}}\Big(\int_{\mathbb{R}^2}W^2\Big)\tilde\lambda_2\Big[\frac{p-5}{4}+1+\frac12\frac{(0,y^i,0)\cdot\nabla V(P^i)}{V(P^i)}\Big)\begin{cases}\le 0 & \text{if } P^i\in\partial\Omega_b\\ \ge 0 & \text{if } P^i\in\Omega_a,\end{cases} \tag{4.2.28}$$

therefore

$$\frac{p-1}{2}+\frac{(0,y^i,0)\cdot\nabla V(P^i)}{V(P^i)} = \Big(\frac{p-1}{2}V+r\partial_r V\Big)_{|_{P^i}}\begin{cases}\le 0 & \text{if } P^i\in\partial\Omega_b\\ \ge 0 & \text{if } P^i\in\partial\Omega_a,\end{cases} \tag{4.2.29}$$

that is

$$\partial\tilde V(P^i)\begin{cases}\le 0 & \text{if } P^i\in\partial\Omega_b\\ \ge 0 & \text{if } P^i\in\partial\Omega_a.\end{cases} \tag{4.2.30}$$

If $z^i = 0$, i. e. P^i is a point of the equator, we choose $Q = (0, 0, z_0)$ and by the previous Pohozaev we have

$$\frac{p-1}{2} + \frac{(0, y^i, z_0) \cdot \nabla V(P^i)}{V(P^i)} = \frac{p-1}{2} V(P^i) + z_0 \partial_s V(P^i) \lesseqgtr 0 \quad \forall z_0 \tag{4.2.31}$$

and then $\partial_s V(P^i) = 0$.

We consider the case in which $P^i = (0, 0, z^i) = (0, s^i)$ with $s^i \in (-b, -a) \cup (a, b)$, get the condition $\partial_{r\,r} V(P^i) = 0$. Multiplying the equation (4.1.1) by $\partial_r u_n$ where $r = \sqrt{x^2 + y^2}$ and integrating on $B_\delta(P^i)$ (in $\mathbb{R}^3$) we have

$$\int_{B_\delta(P^i)} u_n^p \partial_r u_n - \lambda_n V\, u_n\, \partial_r u_n = \int_{B_\delta(P^i)} -\Delta u_n \partial_r u_n$$

and integrating by parts

$$\int_{B_\delta(P^i)} u_n^p \partial_r u_n - \lambda_n V\, u_n\, \partial_r u_n = \int_{B_\delta(P^i)} \partial_r \Big(\frac{u_n^{p+1}}{p+1}\Big) - \lambda_n \partial_r \Big(\frac{V\, u_n^2}{2}\Big) + \frac{\lambda_n}{2} \partial_r V\, u_n^2$$

$$\int_{B_\delta(P^i)} -\Delta u_n \partial_r u_n = \int_{B_\delta(P^i)} \nabla u_n \nabla(\partial_r u_n) - \int_{\partial B_\delta(P^i)} .\partial_\nu u_n\, \partial_r u_n$$

We make a change of variable $r = \sqrt{x^2 + y^2}$, $z = s$ and indicating with $\tilde{B}_\delta(P^i) = \{\, (r, s) : \sqrt{r^2 + s^2} \le \delta \,\}$ we have

$$\begin{aligned} & \int_{B_\delta(P^i)} \nabla u_n \nabla(\partial_r u_n) - \int_{\partial B_\delta(P^i)} \partial_\nu u_n\, \partial_r u_n \\ = \; & \int_{B_\delta(P^i)} \partial_r \Big(\frac{1}{2} |\nabla u_n|^2\Big) - \int_{\partial B_\delta(P^i)} \partial_\nu u_n\, \partial_r u_n \\ = \; & \int_{\tilde{B}_\delta(P^i)} r \partial_r \Big(\frac{1}{2} |\nabla u_n|^2\Big) \mathrm{d}r\, \mathrm{d}s + o(\varepsilon_n^k) = - \int_{\tilde{B}_\delta(P^i)} \frac{1}{2} |\nabla u_n|^2 \mathrm{d}r\, \mathrm{d}s + O((\varepsilon_n^1)^k) \end{aligned}$$

and

$$\begin{aligned} & \int_{B_\delta(P^i)} \partial_r \Big(\frac{u_n^{p+1}}{p+1}\Big) - \lambda_n \partial_r \Big(\frac{V\, u_n^2}{2}\Big) + \frac{\lambda_n}{2} \int_{B_\delta(P^i)} \partial_r V\, u_n^2 \\ = \; & \int_{\tilde{B}_\delta(P^i)} \Big[\partial_r \Big(\frac{u_n^{p+1}}{p+1}\Big) - \lambda_n\, V \partial_r \Big(\frac{u_n^2}{2}\Big)\Big] r \mathrm{d}r\, \mathrm{d}s \\ = \; & - \int_{\tilde{B}_\delta(P^i)} \frac{u_n^{p+1}}{p+1} \mathrm{d}r\, \mathrm{d}s + \lambda_n \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2}{2} \partial_r (r\, V) + O((\varepsilon_n^1)^k), \end{aligned}$$

that give

$$- \int_{\tilde{B}_\delta(P^i)} \frac{1}{2} |\nabla u_n|^2 \mathrm{d}r\, \mathrm{d}s = - \int_{\tilde{B}_\delta(P^i)} \frac{u_n^{p+1}}{p+1} \mathrm{d}r\, \mathrm{d}s + \lambda_n \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2}{2} \partial_r (r\, V) + O((\varepsilon_n^1)^k) \tag{4.2.32}$$

Observe that $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz} = \partial_x(\partial_r \frac{\partial r}{\partial x}) + \partial_y(\partial_r \frac{\partial r}{\partial y}) + \partial_{ss} = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{ss}$. Since $\frac{u_n}{r}$ has the same behavior of $\frac{1}{r}$ and $\frac{1}{r} \in L^1(\Omega)$, we can multiply the equation (4.1.1) by $\frac{u_n}{r}$ and integrating by parts, we have

$$\begin{aligned}
&\int_{B_\delta(P^i)} -\Delta u_n \frac{u_n}{r} = \int_{B_\delta(P^i)} \frac{u_n^{p+1}}{r} - \lambda_n V \frac{u_n^2}{r} = \int_{\tilde{B}_\delta(P^i)} u_n^{p+1} - \lambda_n V u_n^2 \mathrm{d}r\, \mathrm{d}s \\
= &\; -\int_{\tilde{B}_\delta(P^i)} (\partial_{rr} u_n + \frac{1}{r}\partial_r u_n + \partial_{ss} u_n) u_n \mathrm{d}r\, \mathrm{d}s = \int_{\tilde{B}_\delta(P^i)} (\partial_r u_n)^2 + (\partial_s u_n)^2 \\
- &\int_{\tilde{B}_\delta(P^i)} \frac{1}{2\, r}\partial_r |u_n^2|\, \mathrm{d}r\, \mathrm{d}s \\
= &\int_{\tilde{B}_\delta(P^i)} (\partial_r u_n)^2 + (\partial_s u_n)^2 - \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2(r,s) - u_n^2(0,s)}{2\, r^2} \mathrm{d}r\, \mathrm{d}s.
\end{aligned}$$

So we have that

$$\int_{\tilde{B}_\delta(P^i)} u_n^{p+1} - \lambda_n V u_n^2 \mathrm{d}r\, \mathrm{d}s = \int_{\tilde{B}_\delta(P^i)} |\nabla u_n|^2 - \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2(r,s) - u_n^2(0,s)}{2\, r^2} \mathrm{d}r\, \mathrm{d}s + O((\varepsilon_n^1)^k)$$

and then we obtain

$$\int_{\tilde{B}_\delta(P^i)} |\nabla u_n|^2 \mathrm{d}r\, \mathrm{d}s = \int_{\tilde{B}_\delta(P^i)} u_n^{p+1} - \lambda_n V u_n^2 + \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2(r,s) - u_n^2(0,s)}{2\, r^2} \mathrm{d}r\, \mathrm{d}s + = ((\varepsilon_n^1)^k).$$

Using this results in (4.2.32), we reduce to

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{\tilde{B}_\delta(P^i)} u_n^{p+1} + \frac{\lambda_n}{2} \int_{\tilde{B}_\delta(P^i)} u_n^2\, r\, \partial_r V + \frac{1}{4} \int_{\tilde{B}_\delta(P^i)} \frac{u_n^2(r,s) - u_n^2(0,s)}{r^2} = O((\varepsilon_n^j)^k).$$

By Lemma 4.6

$$\int_{B_\delta(P^i)} \frac{u_n^2(x,y,z) - u_n^2(0,0,z)}{4\, r^3} = \sum_{j \in J^i} (\varepsilon_n^j)^{-\frac{4}{p-1}} \int_{\mathbb{R}^2} \frac{U^2(r,s) - U^2(0,s)}{4\, r^2} \mathrm{d}r\, \mathrm{d}s.$$

Let U be a positive solution of $-\Delta U + \tilde{\lambda} U = U^p$ in $\mathbb{R}^3$. Multiplying this equation by $\partial_r U$, integrating by parts and changing variables we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} -\Delta U\, \partial_r U = \int_{\mathbb{R}^3} (U^p - \tilde{\lambda} U) \partial_r U \\
= &\int_{\mathbb{R}^2} \Big(\partial_r \Big(\frac{U^{p+1}}{p+1}\Big) - \tilde{\lambda} \partial_r \Big(\frac{U^2}{2}\Big)\Big) r\, \mathrm{d}r\, \mathrm{d}s = -\int_{\mathbb{R}^2} \frac{U^{p+1}}{p+1} - \tilde{\lambda} \frac{U^2}{2} \\
&\int_{\mathbb{R}^3} -\Delta U\, \partial_r U = \int_{\mathbb{R}^2} \nabla U\, \nabla(\partial_r U) = \int_{\mathbb{R}^3} \partial_r \Big(\frac{1}{2} |\nabla U|^2\Big) r\, \mathrm{d}r\, \mathrm{d}s = -\frac{1}{2} \int_{\mathbb{R}^2} |\nabla U|^2 \mathrm{d}r \mathrm{d}s \\
\Longrightarrow &\; \frac{1}{2} \int_{\mathbb{R}^2} |\nabla U|^2 \mathrm{d}r \mathrm{d}s = \int_{\mathbb{R}^2} \frac{U^{p+1}}{p+1} - \tilde{\lambda} \frac{U^2}{2}.
\end{aligned}$$

Moreover, multiplying the equation of U by $\frac{U}{r}$ and integrating by parts we obtain

$$\int_{\mathbb{R}^2} U^{p+1} - \tilde{\lambda} U^2 = \int_{\mathbb{R}^3} -\Delta U \, \frac{U}{r} = \int_{\mathbb{R}^2} -(\partial_{rr} U + \frac{1}{r}\partial_r U + \partial_{ss} U) U$$

$$= \int_{\mathbb{R}^2} |\nabla U|^2 - \frac{1}{r}\partial_r \Big(\frac{U^2}{2}\Big) = \int_{\mathbb{R}^2} |\nabla U|^2 - \frac{1}{r^2} \frac{U^2(r,s) - U^2(0,s)}{2}.$$

By these two Pohozaev we obtain that

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{\mathbb{R}^2} U^{p+1} = -\frac{1}{4} \int_{\mathbb{R}^2} \frac{U^2(r,s) - U^2(0,s)}{r^2}.$$

We have that $U_n^j \to U$ as $n \to +\infty$ exponentially, by the previous chapter, so

$$\sum_{j \in J^i} (\varepsilon_n^j)^{-\frac{4}{p-1}} \Big(\int_{B_{\frac{\delta}{\varepsilon_n^j}}(0, z_i - z_n^i)} \frac{(U_n^j)^2(r,s) - (U_n^j)^2(0,s)}{r^2} \mathrm{d}\, r \, \mathrm{d}\, s$$

$$+ \quad \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{B_{\frac{\delta}{\varepsilon_n^j}}(0, z_i - z_n^i)} (U_n^j)^{p+1} \Big) = O((\varepsilon_n^1)^k).$$

So we make an expansion of:

$$\frac{\lambda_n}{2} \int_{\tilde{B}_\delta(P^i)} u_n^2 \, r \, \partial_r V$$

$$= \sum_{j \in J^i} (\varepsilon_n^j)^{-\frac{4}{p-1}+3} \; \frac{\lambda_n}{2} \int_{B_{\frac{\delta}{\varepsilon_n^j}}(0, z_i - z_n^j)} \frac{(U_n^j)^2(r,s) \, r \, \partial_r V(\varepsilon_n^j r, \varepsilon_n^j s + z_n^i)}{V(\varepsilon_n^j r + y_n^j, \varepsilon_n^j s + z_n^j)} \mathrm{d}r \, \mathrm{d}s = O((\varepsilon_n^1)^k)$$

$$\partial_r V(\varepsilon_n^j r, \varepsilon_n^j s + z_n^j) = \partial_r V(0, \varepsilon_n^j s + z_n^j) + \varepsilon_n^j \partial_{rr} V(0, \varepsilon_n^j s + z_n^j)\, r + O((\varepsilon_n^j)^2 r^2)$$

therefore for $k > 2\,\frac{p-3}{p-1}$

$$\frac{1}{V(P^i)} \frac{\tilde{\lambda}_2}{2} \sum_{j \in J^i} (\varepsilon_n^j)^{\frac{p-5}{p-1}} \int_{\tilde{B}_{\frac{\delta}{\varepsilon_n^j}}(0, z_i - z_n^i)} (U_n^j)^2 \, r^2 \varepsilon_n^j \, \partial_{rr} V(0, \varepsilon_n^j s + z_n^j) \mathrm{d}r \, \mathrm{d}s = O((\varepsilon_n^1)^k)$$

$$\Longrightarrow \quad \partial_{rr} V(P^i) \int_{\mathbb{R}^2} U^2 r^2 \mathrm{d}r \, \mathrm{d}s = 0 \Longrightarrow \partial_{rr} V(P^i) = 0.$$

□

COROLLARY 4.9. *Let $(u_n \lambda_n)$ be a positive, $G-$invariant solution of (4.1.1) with u_n satisfying $m_G(u_n) = 1$ and $1 < p \neq 5$, $\lambda_n \to +\infty$ as $n \to +\infty$.*
Suppose that

$$\partial_s V = 0 \;\; in \;\; G_0 \;\; \Longrightarrow \partial_{rr} V \neq 0,$$

and

$$\partial_s V \begin{cases} > 0 & in \;\; G_0 \cap \partial\Omega^+ \\ < 0 & in \;\; G_0 \cap \partial\Omega^-. \end{cases}$$

Then u_n *blows-up on a suitable* $G-$*invariant, one dimensional curve, i.e. a circle with a suitable radius* r_n *such that* $\frac{r_n}{(\varepsilon_n^1)^{\frac{4}{p-1}}} \to \infty$.

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