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Numerical Godeaux surfaces with an automorphism of order three

Author
ELEONORA PALMIERI

Advisor
Prof. CIRO CILIBERTO

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Introduction

This thesis is devoted to one of the major open problems about complex surfaces: the classification of surfaces of general type and their automorphisms. We will always work over the complex numbers.

It is well known that complex surfaces have been classified by Enriques and Kodaira in terms of their Kodaira dimension κ, which is defined as follows.

Definition 0.0.1. *Let S be a surface. The* **Kodaira dimension** *of S is the number*

$$\kappa(S) := \max\{\dim \operatorname{Im}(\varphi_{|mK_S|} : S \longrightarrow \mathbb{P}^N), m \in \mathbb{N}\}$$

where $\varphi_{|mK_S|}$ is the rational map defined by the pluricanonical system $|mK_S|$ on S. We will set $\kappa(S) = -\infty$ if $|mK_S| = \emptyset$ for all $m \geq 0$.

While surfaces with $\kappa \leq 1$ are quite well-known, we have much less information about surfaces of general type, i.e. those for which $\kappa = 2$. Their complete classification is still an open problem even though there are important contributions from many mathematicians (for a general reference see [BCP]).

We know that minimal surfaces of general type are subdivided into classes according to the value of three main invariants: the self-intersection of the canonical divisor ${K_S}^2$, the holomorphic Euler characteristic $\chi(S, \mathcal{O}_S)$ and the geometric genus $p_g(S) := h^0(S, \mathcal{O}_S(K_S)) = h^2(S, \mathcal{O}_S)$. In this thesis we are mainly interested in those surfaces with the lowest invariants:

Definition 0.0.2. *A* **numerical Godeaux surface** *is a minimal complex surface of general type S with $p_g(S) = 0, {K_S}^2 = 1, \chi(\mathcal{O}_S) = 1$.*

The first example of such a surface can be found in [G] and it is the quotient of a smooth quintic in $\mathbb{P}^3$ with a free $\mathbb{Z}/5\mathbb{Z}$ action. This example turns out to have non-trivial torsion, and in fact it has $\mathbb{Z}/5\mathbb{Z}$ as a torsion group.

Much information about torsion group of numerical Godeaux surfaces can be obtained by the study of the base points of the tricanonical system $|3K_S|$. This is an important result by Miyaoka [Miy] which is recalled in section 1.5. It is known (see [R] and [Miy]) that the moduli spaces of numerical Godeaux surfaces with torsion group $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ are irreducible and have dimension 8.

As for every surface of general type $Aut(S)$ is a finite group (see also [X1], [X2] and [X3]). It is still a quite difficult problem to determine the group $Aut(S)$.

The simplest case is that of a surfaces S admitting an involution. For Godeaux surfaces in [KL] Keum and Lee study the fixed locus of the involution under the hypothesis that the bicanonical system $|2K_S|$ of the surface has no fixed component.

In their work [CCM2] Calabri, Ciliberto and Mendes Lopes complete the above study by removing this hypothesis. They use intersection theory and the theory of abelian covers to get a classification result, which can be resumed in the following theorem:

Theorem 0.0.3. *A numerical Godeaux surface S with an involution is birationally equivalent to one of the following:*

1. *a double plane of Campedelli type;*
2. *a double plane branched along a reduced curve which is the union of two distinct lines r_1, r_1 and a curve of degree 12 with the following singularities:*
   - *the point $q_0 = r_1 \cap r_2$ of multiplicity 4;*
   - *a point $q_i \in r_i$, $i = 1, 2$ of type $[4, 4]$, where the tangent line is r_i;*
   - *further three points q_3, q_4, q_5 of multiplicity 4 and a point q_6 of type $[3, 3]$, such that there is no conic through $q_1, \ldots, q_6$;*
3. *a double cover of an Enriques surface branched along a curve of arithmetic genus 2.*

In case 3 the torsion group of S is $\mathrm{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$, whilst in case 2 is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

We recall that a double plane of Campedelli type is a double plane branched along a curve of degree 10 with a 4-tuple point and 5 points of type $[3, 3]$, not lying on a conic. An example of such a double plane can be found in [S].

We want to extend the method used in [CCM2] in order to classify such numerical Godeaux surfaces S having an automorphism σ of order three.

Our main result is

Theorem 0.0.4. *A numerical Godeaux surface S cannot have an automorphism of order 3.*

In the first chapter we recall some well-known results about complex surfaces of general type and about fibrations of surfaces over curves. Moreover, in section 1.2 we recall some basic elements of the theory of cyclic triple covers.

In section 1.3 we show how it is possible to construct a minimal smooth resolution of the cover $S \longrightarrow \Sigma \; = S/\sigma$, i.e. a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & S \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\eta} & \Sigma \end{array}$$

where X and Y are smooth surfaces and $X \longrightarrow Y$ is a triple cover induced by σ . The main idea is then to apply the theory of abelian covers following [P].

In section 1.4 we recall the basics about plane quadratic transformations.

In the second chapter we start our analysis, using Hurwitz formula and the topological Euler characteristic e to estimate the number of isolated fixed points of the action of σ on S (which can be mapped either to ordinary triple points or to double points of type A_2). We determine some basic properties of the invariant part Λ of the tricanonical system $|3K_S|$, which can be either a pencil or a net and it is mapped to a system $|N|$ over the quotient surface Y. Moreover we study the adjoint systems to $|N|$ with the help of [CCM1, lemma 2.2]. All their numerical properties are collected in proposition 2.3.12. We also have a subdivision in three major cases according to the intersection number R_0K_S and h_2, where R_0 is the divisorial part of the ramification locus of σ while h_2 is the number of isolated fixed points of σ mapped to A_2-singularities (see the list of page 33).

A numerical analysis of these three cases is worked out in the third chapter, where using some properties of nef divisors and of fibrations it is shown (see theorems 3.1.21 and 3.2.2) that the first two cases cannot occur. In the third case the system $|N|$ on Y (and also Λ on S) is a pencil and its movable part induces a fibration over Y. An analysis of the singular fibres determines the possibilities listed in theorem 3.3.8. It is quite easy to see, although it is a very important information, that Y is a smooth rational surface (proposition 3.3.1).

Chapter four is devoted to a deeper study of the adjoint systems to the pencil $|N|$ and to exclude some of the cases coming from theorem 3.3.8. We also divide the remaining group of cases between Del Pezzo cases and ruled cases (see definitions 4.0.11 and 4.0.12), since either Y is a blow-up of $\mathbb{P}^2$ at a certain number of points, or Y has a rational pencil with self-intersection 0. Moreover we show that the divisorial part R_0 of the ramification locus of the order three automorphism σ on the numerical Godeaux surface S is either 0 or it has only one irreducible component.

Last chapter deals with a more geometric study. The first section is devoted to the ruled cases. We show that Y after contraction of suitable curves can be mapped onto $\mathbb{F}_0, \mathbb{F}_1$ or $\mathbb{F}_2$ and that, by blowing up a point and contracting again, we can always reduce to $\mathbb{F}_1$. Then we can actually see, birationally speaking, our surface S as triple plane.

A computation of the movable part $|A'|$ of the pencil $|N|$ on Y allows us to show that ruled cases cannot actually occur.

The second section is then devoted to the study of Del Pezzo cases where the rational surface Y is mapped to the projective plane blown-up at seven, eight or thirteen points. The computation of the exceptional curves coming from the blow-up of the isolated fixed points on S tells us that also Del Pezzo cases do not occur.

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Chapter 1

Known results about surfaces

In this first chapter we collect some results about complex surfaces, fibrations and cyclic covers and other general theorems which are at the base of our work. For the results which are not proved here we refer to [BPV] or to [H], unless otherwise specified.

1.1 Fibrations and other results about surfaces

Lemma 1.1.1. *Let $f : X \longrightarrow B$ be a fibration, not necessarily connected. Then $h^i(X_b, \mathcal{O}_{X_b})$, $i = 0, 1$ is independent of b. In particular $h^0(X_b, \mathcal{O}_{X_b})$ equals the number of connected components of a nonsingular fibre.*

Lemma 1.1.2 (Zariski's lemma). *Let $X_b = \sum n_i C_i, n_i > 0, C_i \subset X$ irreducible, be a fibre of the fibration $f : X \longrightarrow B$. Then we have*

1. *$C_i X_b = 0$ for all i.*
2. *If $D = \sum m_i C_i$, $m_i \in \mathbb{Z}$, then $D^2 \leq 0$.*
3. *$D^2 = 0$ holds if and only if $D = rX_b$, $r \in \mathbb{Q}$.*

Proposition 1.1.3 (Proposition III.11.4 of [BPV]). *Let $f : X \longrightarrow B$ be a fibration and X_{gen} a nonsingular fibre. Then*

1. *$e(X_b) \geq e(X_{gen})$ for all fibres X_b;*
2. *if X is compact then*
$$e(X) = e(X_{gen})e(B) + \sum_{b \in B} (e(X_b) - e(X_{gen})).$$

Lemma 1.1.4. *An irreducible curve C_1 with $C_1^2 = -n$ in a singular fibre contributes to the Euler number as a curve with at least n nodes.*

Proof. Let us consider a reducible curve of the fibration

$$C = \sum_{i=1}^{l} h_i C_i$$

As shown in [F] (see also [E, section V.1]) C is equivalent to δ_0 curves with a node where

$$\delta_0 \geq \sum_{i=1}^{l} (h_i - 1)(2p_a(C_i) - 2) + \sum_{i \neq j} (h_i + h_j - 1) C_i C_j \tag{1.1}$$

Let us consider one of the curves C_j, say C_1 with $C_1^2 = -n$. Then

$$0 = C_1 C = -nh_1 + C_1 \sum_{i=2}^{l} h_i C_i$$

hence

$$C_1 \sum_{i=2}^{l} h_i C_i = nh_1$$

Since C is connected we also have

$$C_1 \sum_{i=2}^{l} C_i \geq 1$$

Then

$$\begin{aligned} \delta_0 &\geq (h_1 - 1)(2p_a(C_1) - 2) + \sum_{j \geq 2} (h_1 + h_j - 1) C_1 C_j \\ &\geq (h_1 - 1)(-2 + \sum_{j \geq 2} C_1 C_j) + C_1 \sum_{j \geq 2} h_j C_j \\ &\geq (h_1 - 1)(-1) + nh_1 = (n-1)h_1 + 1 \geq n \end{aligned}$$

as wanted. □

The above lemma tells us that a curve with negative self-intersection in a singular fibre can be considered as the sum of a suitable number of curves with a node. Let us now see what the contribution of each node to the Euler number of a fibre is.

Lemma 1.1.5. *Each node in a singular reduced fibre increases the Euler number by one.*

Proof. Assume F is a reduced curve with n nodes. Then if $\nu : \widetilde{F} \longrightarrow F$ is the normalization of F we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \nu_*\mathcal{O}_{\widetilde{F}} \longrightarrow \delta \longrightarrow 0$$

where δ is a sheaf supported on the node and $\chi(\delta) = h^0(\delta) = n$. Then

$$1 - p_g(F) = \chi(\mathcal{O}_{\widetilde{F}}) = \chi(\nu_*\mathcal{O}_{\widetilde{F}}) = \chi(\mathcal{O}_F) + \chi(\delta) = 1 - p_a(F) + h^0(\delta)$$

hence

$$p_g(F) = p_a(F) - h^0(\delta)$$

We also have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_F \arrow[r] \arrow[d] & \nu_*\mathbb{C}_{\widetilde{F}} \arrow[r, "\beta"] \arrow[d, "\gamma"'] & \delta \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathcal{O}_F \arrow[r] & \nu_*\mathcal{O}_{\widetilde{F}} \arrow[r, "\delta"] & \delta \arrow[r] & 0
\end{tikzcd}$$

It follows

$$\begin{aligned} e(F) &= e(\widetilde{F}) - \chi(\delta') = 2 - 2(p_a(F) - h^0(\delta)) - h^0(\delta) = 2 - 2p_a(F) + h^0(\delta) \\ &= e(X_{gen}) + h^0(\delta). \end{aligned}$$

□

Theorem 1.1.6 (Unbranched covering trick). *Let X be a connected complex manifold.*

(i) If $b_1(X) \neq 0$ then X admits unbranched coverings of any order.

(ii) If $H_1(X, \mathbb{Z})$ contains k-torsion, then X has an unbranched coverings of order k.

Proposition 1.1.7. *Let S be a minimal surface of general type with ${K_S}^2 = 1$. Then $q(S) = 0$.*

Proof. Assume $q(S) > 0$. Then $b_1(S) \neq 0$, $H_1(S, \mathbb{Z})$ is infinite and the unbranched covering trick says that S has unramified covers of any order n. Let

$\varphi_n : S' \longrightarrow S$ be such a cover. Then $e(\mathcal{O}_{S'}) = ne(\mathcal{O}_S)$ and $K_{S'} \equiv {\varphi_n}^*(K_S)$ hence

$$\chi(\mathcal{O}_{S'}) = n\chi(\mathcal{O}_S).$$

Since S is of general type we find $p_g(S) \geq q(S)$ and

$$1 + p_g(S') \geq \chi(\mathcal{O}_{S'}) = n\chi(\mathcal{O}_S) = n(1 + p_g(S) - q(S))$$

We remark that, if S is minimal, then also S' is.

On the other hand from the minimality of S' and from Noether's inequality

$$p_g(S') \leq \frac{K_{S'}^2}{2} + 2 = \frac{n{K_S}^2}{2} + 2$$

It follows

$$n - 1 + n(p_g(S) - q(S)) \leq p_g(S') \leq \frac{n{K_S}^2}{2} + 2$$

and then

$$0 \leq p_g(S) - q(S) \leq \frac{1}{n}\left(\frac{n{K_S}^2}{2} + 2 + 1 - n\right) = \frac{1}{n}\left(3 - \frac{n}{2}\right)$$

and we get a contradiction when $n \geq 7$. □

Proposition 1.1.8. *Let S be a surface with $p_g(S) = 0$. Then for any effective divisor D on S we have $h^2(S, \mathcal{O}_S(D)) = 0$.*

Proof. Let D be an effective divisor on the surface S. Then we have the short exact sequence of sheaves (see also [H])

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0 \tag{1.2}$$

By Serre's duality $p_g(S) = h^0(S, \mathcal{O}_S(K_S)) = h^2(S, \mathcal{O}_S)$. Then from the long exact sequence of cohomology associated to (1.2) we have

$$0 \longrightarrow H^2(S, \mathcal{O}_S(D)) \longrightarrow H^2(D, \mathcal{O}_D(D)) \longrightarrow 0$$

Since $\mathcal{O}_D(D)$ is supported on a curve, we find $h^2(D, \mathcal{O}_D(D)) = 0$ and the result is proved. □

Lemma 1.1.9 (Lemma 2.2 of [CCM1]). *Let D be a nef curve on a regular surface X such that $p_a(D) \geq 1$. If $K_X + D$ is not nef, then any irreducible curve Θ such that $\Theta(K_X + D) < 0$ is a (-1)-curve Θ such that $\Theta D = 0$.*

Proof. Let us consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Then from the associated long exact sequence of cohomology we have, since X is a regular surface,

$$0 \longrightarrow H^1(D, \mathcal{O}_D) \longrightarrow H^2(X, \mathcal{O}_X(-D)) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$$

Hence

$$\begin{aligned} h^0(X, \mathcal{O}_X(K_X + D)) &= h^2(X, \mathcal{O}_X(-D)) = h^1(D, \mathcal{O}_D) + h^2(X, \mathcal{O}_X) \\ &= h^0(D, \mathcal{O}_D) - 1 + p_a(D) + p_g(X) \geq p_a(D) \end{aligned}$$

and we have equality if D is 1-connected (e.g. if D is nef and big) and $p_g(X) = 0$. Then, when $p_a(D) \geq 1$, $K_X + D$ is an effective divisor.

If Θ is an irreducible curve such that $\Theta(K_X + D) < 0$ then we obviously have $\Theta K_X < 0$. Moreover since Θ cannot move (i.e. $h^0(X, \mathcal{O}_X(\Theta)) = 1$) from the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\Theta) \longrightarrow \mathcal{O}_\Theta(\Theta) \longrightarrow 0$$

and the regularity of X we find $h^0(X, \mathcal{O}_\Theta(\Theta)) = 0$ and then

$$\begin{aligned} -h^1(X, \mathcal{O}_\Theta(\Theta) &= \chi(\Theta, \mathcal{O}_\Theta(\Theta)) = 1 + \Theta^2 - p_a(\Theta) \\ &= 1 + \Theta^2 - 1 - \frac{\Theta^2 + \Theta K_X}{2} = \frac{\Theta^2 - \Theta K_X}{2} \end{aligned}$$

Thus Θ satisfies the three inequalities

$$\begin{cases} \Theta^2 + \Theta K_X \geq -2 \\ \Theta^2 - \Theta K_X \leq 0 \\ \Theta K_X < 0 \end{cases}$$

which force $\Theta^2 = \Theta K_X = -1$. □

Theorem 1.1.10 (Index Theorem). *Let D, E be divisors with rational coefficients on an algebraic surface S. If $D^2 > 0$ and $DE = 0$, then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is homologous to 0.*

Theorem 1.1.11 (Kawamata-Viehweg). *Let X be a smooth projective variety of dimension n, and let L be an integral divisor (or line bundle) on X. Assume that*

$$L \sim D + \sum a_i A_i$$

where D is a big and nef $\mathbb{Q}$-divisor, and $\Delta = \sum a_i A_i$ is a $\mathbb{Q}$-divisor with simple normal crossing support and fractional coefficients

$$0 \leq a_i < 1 \quad \textit{for all } i.$$

Then

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \textit{for } i > 0.$$

Equivalently

$$H^j(X, \mathcal{O}_X(-L)) = 0 \quad \textit{for } j < n.$$

For the proof of theorem 1.1.11 we refer to [L].

Finally we recall the well-known Castelnuovo's criterion for the rationality of surfaces:

Theorem 1.1.12 (Castelnuovo's criterion). *An algebraic surface X is rational if and only if $q(X) = P_2(X) = 0$.*

1.2 Cyclic triple covers of smooth surfaces

We now briefly recall the theory of Galois abelian covers for surfaces. A more general and detailed description of such covers can be found in [P]. We will focus our attention on the case of covers with Galois group $G = \mathbb{Z}/3\mathbb{Z}$. For the case of triple covers the reader may also refer to [Mir].

Let Y be a smooth surface, G be an abelian group with a faithful action on Y and let X be an abelian cover of Y with group G. This means that there is a finite map $\pi : X \longrightarrow Y$ such that $Y = X/G$ and π is the projection of X to its quotient.

Under these hypotheses there is a splitting

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}$$

where G^* is the group of characters of G, $L_1 = \mathcal{O}_Y$ and G acts on L_χ^{-1} via the character χ.

Definition 1.2.1. *For each component T of the ramification divisor R on X we can define its* **inertia group**

$$H_T := \{h \in G \mid hx = x \, \forall \, x \in T\}$$

Lemma 1.2.2 (Lemma 1.1 of [P]). *If T as above has codimension 1 in X then H_T is a cyclic group.*

Lemma 1.2.3 (Lemma 1.2 of [P]). *If T is as in lemma 1.2.2, then there exists a parameter t for $\mathcal{O}_{X,T}$ such that the action of H_T is given by*

$$ht = \varrho_T(h)t \quad \forall \, h \in H_T$$

where ϱ_T is the representation of H_T on $\mathfrak{M}/\mathfrak{M}^2$ ($\mathfrak{M}$ is the maximal ideal of T in X) induced by the cotangent map.

Let B be the branch locus of the cover $X \longrightarrow Y$. Then to each irreducible component V of B is associated a subgroup of G which is the inertia group $H = H_T$ of all the irreducible components of $\pi^{-1}(V)$ and a character $\psi \in H_T^*$ as in lemma 1.2.3 which is a generator of H_T^*. Hence we can write

$$B = \sum_{H \in \mathcal{C}} \sum_{\psi \in \mathcal{S}_H} B_{H,\psi}$$

where $\mathcal{C}$ is the set of cyclic subgroups of G and $\mathcal{S}_H$ is the set of generators of H^* for any $H \in \mathcal{C}$ and we denote by $B_{H,\psi}$ the sum of the irreducible components of B with inertia group H and character ψ.

Definition 1.2.4. *A G-cover is a* **simple cyclic cover** *if $B = B_{G,\psi}$ for some character ψ that generates G^*.*

Let us now fix a pair (H, ψ). Then for any $\chi, \chi' \in G^*$ we can write

$$\chi_{|H} = \psi^{i_\chi}, \chi'_{|H} = \psi^{i_{\chi'}}$$

with $i_\chi, i_{\chi'} \in \{0, \ldots, m_H - 1\}$ where m_H is the order of H.

Then we can define

$$\varepsilon_{\chi,\chi'}^{H,\psi} = \begin{cases} 0 & \text{if } i_\chi + i_{\chi'} < m_H \\ 1 & \text{otherwise} \end{cases}$$

The invertible sheaves L_χ and the components $B_{H,\psi}$ are called the **building data** of the cover and they define uniquely the Galois cover in the sense of the following

Theorem 1.2.5 (Theorem 2.1 of [P]). *Let G be an abelian group. Let Y be a smooth variety, X a normal one and let $\pi : X \longrightarrow Y$ be an abelian cover with group G. Then the following set of linear equivalences is satisfied by the building data of the cover:*

$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + \sum_{H \in mcC} \sum_{\psi \in \mathcal{S}_H} \varepsilon^{H,\psi}_{\chi,\chi'} B_{H,\psi} \tag{1.3}$$

Conversely, to any set of data $L_\chi, B_{H,\psi}$ satisfying (1.3) *we can associate an abelian cover $\pi : X \longrightarrow Y$ in a natural way. Whenever the cover so constructed is normal, $L_\chi, B_{H,\psi}$ are its building data.*

Moreover, if Y is complete, then the building data determine the cover $\pi : X \longrightarrow Y$ up to isomorphisms of Galois covers.

We now restrict to the case of Galois triple cover. When $G = \mathbb{Z}/3\mathbb{Z}$ the only non-trivial cyclic subgroup of G is G itself and there are two generators of G^*. We also have

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L_1) \oplus \mathcal{O}_Y(-L_2) \tag{1.4}$$

where G acts on $\mathcal{O}_Y(-L_1)$ as the multiplication by $\omega = e^{2\pi i/3}$ while acts on $\mathcal{O}_Y(-L_2)$ as the multiplication by ω^2. In this setting the conditions (1.3) can be rewritten as

$$2L_1 \equiv L_2 + B_{\omega^2}$$

$$L_1 + L_2 \equiv B$$

$$2L_2 \equiv L_1 + B_\omega$$

and can be reduced to the following two:

$$\begin{aligned} 3L_1 &\equiv B + B_{\omega^2} \\ L_1 + L_2 &\equiv B \end{aligned} \tag{1.5}$$

1.3 Resolution for cyclic triple covers

Assume we have a smooth surface S with an action of $G = \mathbb{Z}/3\mathbb{Z}$. Then S has a ramification locus which is composed of a divisorial part R and of some isolated fixed points. We put $\omega = e^{2\pi i/3}$ as in the previous section.

We now look at the G-action near the isolated fixed points. Without loss of generality we can assume the point is $O = (0, 0)$ and, by Cartan's lemma [Car, lemma 2, p.98], the action of G can be linearized near O. Then since O does not belong to any divisorial component of the ramification locus we find two possible actions:

$$(x, y) \longrightarrow (\omega x, \omega y)$$

or

$$(x, y) \longrightarrow (\omega x, \omega^2 y)$$

Then there are two different kinds of isolated fixed points. The quotient surface S/G is a normal surface, but it is singular at the image of the isolated fixed points. Let us consider the first action

$$(x, y) \longrightarrow (\omega x, \omega y)$$

Locally S can be seen as $\mathrm{Spec}(\mathbb{C}[x, y])$ and the coordinate ring of the quotient surface is given by the ring of invariants $\mathbb{C}[x, y]^G$ of $\mathbb{C}[x, y]$. The ring $\mathbb{C}[x, y]^G$ is generated by the monomials $\xi = x^3, \eta = y^3, \zeta = xy^2, \upsilon = x^2y$ satisfying the relations $\zeta^2 = \upsilon\eta$, $\upsilon^2 = \xi\zeta$, $\zeta\upsilon = \xi\eta$.

Then we can locally describe S/G as

$$\mathrm{Spec}(\mathbb{C}[\xi, \eta, \zeta, \upsilon]/(\zeta^2 - \upsilon\eta, \upsilon^2 - \xi\zeta, \zeta\upsilon - \xi\eta)). \tag{1.6}$$

Such a surface has an ordinary triple point singularity at the origin.

We now consider the second action

$$(x, y) \longrightarrow (\omega x, \omega^2 y)$$

In this case the invariant ring is generated by the monomials $\xi = x^3, \eta = y^3, \zeta = xy$ with the relation $\zeta^3 = \xi\eta$.

Then S/G can be locally described as

$$\mathrm{Spec}(\mathbb{C}[\xi, \eta, \zeta]/(\zeta^3 - \xi\eta)) \tag{1.7}$$

and it has a double point A_2 singularity at the origin.

Definition 1.3.1. *Let S be a smooth surface with an action of $G = \mathbb{Z}/3\mathbb{Z}$.*

An isolated fixed point P will be said **of type I** *if the quotient $S \longrightarrow S/G$ maps P to an ordinary triple point.*

P will be said **of type II** *if the quotient $S \longrightarrow S/G$ maps P to a double point of type A_2.*

We now want to show how to find a minimal desingularization of the triple cover $S \longrightarrow \Sigma = S/G$, i.e. we want to find two smooth surfaces X and Y such that X is birational to S, Y is birational to Σ and there is a triple cover $X \longrightarrow Y$ such that the following commutative diagram holds

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Sigma \end{array}$$

A description of the desingularization can be found in [T] and in [Cal]. We now recall the explicit computation.

Since it is a local question, we reduce ourselves to consider separately the two kinds of isolated fixed points. Without loss of generality we can assume that S has local coordinates (x, y) and the isolated fixed point is $(0, 0)$.

Type I:

If $(0, 0)$ is of type I (see definition 1.3.1), the action of $G = \mathbb{Z}/3\mathbb{Z}$ is

$$(x, y) \longrightarrow (\omega x, \omega y)$$

and the quotient surface $\Sigma = S/G$ is described by (1.6).

We now blow up S at the isolated fixed point. Then the blow-up S' will satisfy the condition

$$rk \begin{pmatrix} x & y \\ u_0 & u_1 \end{pmatrix} \leq 1$$

Then we have

$$xu_1 = u_0 y \tag{1.8}$$

We now extend the $\mathbb{Z}/3\mathbb{Z}$-action on S' in the natural way by defining

$$((x, y), [u_0 : u_1]) \longrightarrow ((\omega x, \omega y), [\omega u_0 : \omega u_1]) = ((\omega x, \omega y), [u_0 : u_1])$$

Hence the exceptional divisor $e := \{(0, 0), [u_0 : u_1]\}$ is fixed under the action of $G = \mathbb{Z}/3\mathbb{Z}$ and we have no more isolated fixed points. Then $X = S'$ and the quotient surface $Y = S'/G$ is smooth. Moreover the image e' of e is the (-3)-curve obtained by blowing up the triple point singularity of $\Sigma = S/G$.

Type II:

Let us now consider the case when $(0, 0)$ is an isolated fixed point of type II (see definition 1.3.1). The action of $G = \mathbb{Z}/3\mathbb{Z}$ is now

$$(x, y) \longrightarrow (\omega x, \omega^2 y)$$

and the quotient surface $\Sigma = S/G$ is described by (1.7).

We now blow up S at the isolated fixed point. Then the blow-up S' will satisfy the condition

$$rk \begin{pmatrix} x & y \\ u_0 & u_1 \end{pmatrix} \leq 1$$

Then we have

$$xu_1 = u_0 y \tag{1.9}$$

We now extend the $\mathbb{Z}/3\mathbb{Z}$-action on S' in the natural way by defining

$$((x, y), [u_0 : u_1]) \longrightarrow ((\omega x, \omega^2 y), [u_0 : \omega u_1])$$

Then the exceptional divisor is invariant for the above action but it is not fixed. Hence the action on S' has two more isolated fixed points $P_0 = ((0, 0), [1 : 0])$ and $P_1 = ((0, 0), [0 : 1])$ on the exceptional divisor $e_1 = \{(0, 0), [u_0 : u_1]\}$.

The point P_j is in the open subset $U_j := \{u_j \neq 0\}$ for $j = 0, 1$. Let us define $v_1 := u_1/u_0$ on U_0 and $v_0 := u_0/u_1$ on U_1. Then on U_0 we can write

$$y = v_1 x$$

and we have local coordinates (x, v_1). If we compute the $\mathbb{Z}/3\mathbb{Z}$-action on U_0 we get

$$(x, v_1) \longrightarrow (\omega x, \omega v_1)$$

hence P_0 is an isolated fixed point of type I.

On U_1 instead we have

$$x = v_0 y$$

and the local coordinates are (y, v_0). If we compute the $\mathbb{Z}/3\mathbb{Z}$-action on U_1 we get

$$(y, v_0) \longrightarrow (\omega^2 x, \omega^2 v_1)$$

hence P_1 is again an isolated fixed point of type I.

Therefore a further blow-up of the two points as described in page 18 gives us the surface X which has an action of $G = \mathbb{Z}/3\mathbb{Z}$ with no isolated fixed points. Then the quotient $Y = X/G$ is smooth as wanted. Moreover e_1 on X is a (-3)-curve and its image e_1' satisfies

$$3{e_1'}^2 = \pi^*(e_1')^2 = e_1^2$$

where $\pi : X \longrightarrow Y$ is the quotient map. Hence e_1' is a (-1)-curve on Y.

1.4 Quadratic transformations

In this section we recall some basic definitions and properties of planar Cremona maps and, in particular, of quadratic transformations of $\mathbb{P}^2$. General references for this subject are [D] or the book [AC].

Let S be a nonsingular projective surface and let $B(S)$ be the category of birational morphisms $\pi : S' \longrightarrow S$ of nonsingular projective surfaces.

Definition 1.4.1. *The* **bubble space** S^{bb} *of a nonsingular surface* S *is the factor set*

$$S^{bb} = \left(\bigcup_{(S' \xrightarrow{\pi'} S) \in B(S)} S' \right) / R$$

where R *is the equivalence relation defined as follows:* $x' \in S'$ *is equivalent to* $x'' \in S''$ *if the rational map* ${\pi''}^{-1} \circ \pi' : S' \dashrightarrow S''$ *maps isomorphically an open neighborhood of* x' *onto a neighborhood of* x''*.*

Definition 1.4.2. *If* $\varphi : S'' \longrightarrow S'$ *is isomorphic to the blow-up of a point* $x' \in S'$ *each point* $x'' \in \varphi^{-1}(x')$ *is said to be an* **infinitely near** *point to* x' *of the first order. By induction one can define infinitely near points to* x' *of order* k*. We will write* $x'' \succ_k x'$ *to say that* x'' *is infinitely near of order* k *to* x'*.*

Definition 1.4.3. *A* **bubble cycle** *is an element* $\eta = \sum m(x)x \in \mathbb{Z}^{S^{bb}}$ *such that*

1. η *has a finite support;*
2. $m(x) \geq 0$ *for any* $x \in S^{bb}$*;*
3. *if* $x \succ x'$ *then* $m(x) \leq m(x')$*.*

Definition 1.4.4. *Let η be a bubble cycle and let D be a divisor on a nonsingular surface S. Then the linear system $|D-\eta|$ is* **homaloidal** *if the map associated to $|D-\eta|$ is birational onto its image. When $S=\mathbb{P}^2$ and $D \equiv dl \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ the cycle η is also called a* **homaloidal bubble cycle** *of degree d.*

Then one can show the following theorem

Theorem 1.4.5. *A bubble cycle $\eta = \sum_{i=1}^N m_i x_i$ on $\mathbb{P}^2$ is homaloidal of degree d if and only if $|dl-\eta|$ contains an irreducible divisor and the following numerical conditions are satisfied:*

$$d^2 - \sum_{i=1}^N m_i^2 = 1, \quad 3d - \sum_{i=1}^N m_i = 3.$$

For the proof of the above theorem we refer to [D].

From now on we will assume that the nonsingular surface S is $\mathbb{P}^2$.

Definition 1.4.6. *A* **plane Cremona transformation** *is a birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.*

Each plane Cremona transformation is defined by a homaloidal net $|V|$ on $\mathbb{P}^2$ of polynomial of some degree d and by a choice of a basis in V. We are mainly interested in those maps defined by a net of polynomials of degree 2.

Definition 1.4.7. *A* **quadratic transformation** *is a plane Cremona transformation which is defined by a net of degree 2 homogeneous polynomials.*

Since the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of conics of $\mathbb{P}^2$ has dimension 5, a homaloidal net $|V|$ inside $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is defined, following theorem 1.4.5, by imposing three base points x_1, x_2, x_3 such that $\eta = x_1+x_2+x_3$ is a homaloidal bubble cycle as in definitions 1.4.3 and 1.4.4.

Definition 1.4.8. *Let x_1, x_2 be points in $(\mathbb{P}^2)^{bb}$. Then x_2 is* **proximate** *to x_1 if $x_2 \succ_1 x_1$.*

Definition 1.4.9. *Let x_1, x_2, x_3 be points in $(\mathbb{P}^2)^{bb}$. Then x_3 is* **satellite** *to x_1 if $x_2 \succ_1 x_1$, $x_3 \succ_1 x_2$, $x_3 \succ_1 x_1$. In particular x_3 is the intersection point between the exceptional divisor obtained by blowing up x_2 and the strict transform of the exceptional divisor of x_1.*

When we blow up a certain number of points $x_1, \dots, x_n$ on $\mathbb{P}^2$ each irreducible curve on the blow-up must satisfy the following inequalities (see also section 2.3 of [Cal])

$$\sum_{x_j \succ_1 x_i} m_j \leq m_i \quad \forall\, i = 1, \dots, n \tag{1.10}$$

where m_h is the multiplicity of the curve at the point x_h.

In particular when we blow-up a bubble cycle $\eta = x_1 + x_2 + x_3$ such that x_3 is satellite to x_1 we get

$$m_1 \geq m_2 + m_3 \geq m_3 + m_3 = 2m_3$$

for any irreducible curve passing through the three points with multiplicity m_i.

Let us now fix a bubble cycle $\eta = x_1 + x_2 + x_3$ in $(\mathbb{P}^2)^{bb}$. Then there are no irreducible curves in $|2l - \eta|$ if and only if one of the following cases occurs:

a) all the points in η are on $\mathbb{P}^2$ and they are collinear;

b) $x_2 \succ_1 x_1$, $x_3 \succ_1 x_1$

In the former case all the conics through the three points must contain the line joining them, hence they are reducible. In the latter case either x_3 is satellite to x_1, hence there are no smooth curves in $|dl - \eta|$ for any d, or the conic has two different tangent directions at x_1 hence it is reducible since it is the union of two distinct lines.

Example 1.4.10. *Let us take three non-collinear points x_1, x_2, x_3 on $\mathbb{P}^2$ (with coordinates y_0, y_1, y_2). Then, up to projective transformations, we can assume they are $[1:0:0], [0:1:0].[0:0:1]$. Let us now consider the system of conics through these three points and choose a basis $\{y_1y_2, y_0y_2, y_0y_1\}$ of this system. We now set*

$$T_1 : [y_0 : y_1 : y_2] \longrightarrow [y_1y_2 : y_0y_2 : y_0y_1]$$

It is easy to see that $T_1^2 = \mathrm{id}$ thus T_1 is a birational involution.

Example 1.4.11. *Let us now take three points x_1, x_2, x_3 with $x_3 \succ_1 x_1$ while $x_2 \not\succ x_1, x_3$, $x_1, x_2 \in \mathbb{P}^2$. Then, up to projective transformations, we can assume that x_1, x_2 are $[0:0:1], [1:0:0]$ whereas x_3 is the tangent direction $y_0 = 0$. We now choose the basis $\{y_1^2, y_0y_2, y_0y_1\}$ of the system of conics through x_1, x_2, x_3. We can set*

$$T_2 : [y_0 : y_1 : y_2] \longrightarrow [y_1^2 : y_0y_2 : y_0y_1]$$

Again one has $T_2^2 = \mathrm{id}$ thus also T_2 is a birational involution.

Example 1.4.12. *Let us now take three points x_1, x_2, x_3 with $x_3 \succ_1 x_2 \succ_1 x_1$, $x_1 \in \mathbb{P}^2$. Then, up to projective transformations, we can assume that x_3 is $[0:0:1]$, x_2 is the tangent direction $y_0 = 0$ whereas x_3 lies on the proper transform of the line $y_2 = 0$. We now choose the basis $\{y_0^2, y_0y_1, y_1^2 - y_0y_2\}$ of the system of conics through x_1, x_2, x_3. We can set*

$$T_3 : [y_0 : y_1 : y_2] \longrightarrow [y_0^2 : y_0y_1 : y_1^2 - y_0y_2]$$

Again one has $T_3^2 = \mathrm{id}$ thus also T_3 is a birational involution.

Remark 1.4.13. *Each quadratic transformation on $\mathbb{P}^2$ can be written as*

$$gT_ig'$$

for suitable projective transformations g, g'.

The maps T_i are defined at all points of $\mathbb{P}^2$ except for x_1, x_2, x_3. Let us take a curve of degree d passing through x_1, x_2, x_3 with multiplicities m_1, m_2, m_3 respectively. Let us now understand what the image of such a curve under a quadratic transformation T_1, T_2, T_3 is. In particular we now determine the degree d' and the multiplicities m_1', m_2', m_3' at x_1, x_2, x_3 of the image.

We work out the computation in the case of the transformation T_1 (see example 1.4.10).

Let us take a homogeneous polynomial $F(y_0, y_1, y_2)$ of degree d which has multiplicity m_1, m_2, m_3 at $x_1 = [1:0:0], x_2 = [0:1:0], x_3 = [0:0:1]$. Then

$$F(y_0, y_1, y_2) = \sum_{\substack{i_0+i_1+i_2=d \\ i_0 \le d-m_1 \\ i_1 \le d-m_2 \\ i_2 \le d-m_3}} f_{i_0i_1i_2} y_0^{i_0} y_1^{i_1} y_2^{i_2}$$

The image $F'(y_0, y_1, y_2)$ is then

$$\begin{aligned} F'(y_0, y_1, y_2) &= \sum_{\substack{i_0+i_1+i_2=d \\ i_0 \le d-m_1 \\ i_1 \le d-m_2 \\ i_2 \le d-m_3}} f_{i_0i_1i_2} y_1^{i_0} y_2^{i_0} y_0^{i_1} y_2^{i_1} y_0^{i_2} y_2^{i_2} \\ &= \sum_{\substack{i_0+i_1+i_2=d \\ i_0 \le d-m_1 \\ i_1 \le d-m_2 \\ i_2 \le d-m_3}} f_{i_0i_1i_2} y_0^{i_1+i_2} y_1^{i_0+i_2} y_2^{i_0+i_1} \end{aligned}$$

$$= \sum_{\substack{i_0+i_1+i_2=d \\ i_0 \le d-m_1 \\ i_1 \le d-m_2 \\ i_2 \le d-m_3}} f_{i_0 i_1 i_2} y_0^{d-i_0} y_1^{d-i_1} y_2^{d-i_2} = y_0^{m_1} y_1^{m_2} y_2^{m_3} F''(y_0, y_1, y_2)$$

Thus the strict transform $F'' = 0$ of our initial curve $F = 0$ is of degree

$$d' = 2d - m_1 - m_2 - m_3. \tag{1.11}$$

Moreover each variable y_j appears in F'' with a power $\alpha_j = d - i_j - m_{j+1} \le d - m_{j+1}$ which tells us that the multiplicity of $F'' = 0$ at each point x_h is

$$m'_h = (2d - m_1 - m_2 - m_3) - (d - m_h) = d - \sum_{j \ne h} m_j. \tag{1.12}$$

1.5 Numerical Godeaux surfaces

Definition 1.5.1. *A* **numerical Godeaux surface** *is a minimal complex surface of general type S with $p_g(S) = 0, {K_S}^2 = 1, \chi(\mathcal{O}_S) = 1$.*

Numerical Godeaux surfaces are the minimal surfaces of general type with the lowest invariants. A first example was given by Godeaux in [G] as the quotient of a quintic in $\mathbb{P}^3$ by a free $\mathbb{Z}/5\mathbb{Z}$ action. Later many other examples were constructed, but we still not have a complete classification.

From the definition 1.5.1 one can easily see that the bicanonical map of numerical Godeaux surfaces cannot be a birational map, since $|2K_S|$ is a pencil. Thus they are not covered by the results in [Ci].

In [Miy] Miyaoka studied the properties of the bicanonical and of the tricanonical system of such surfaces. We now recall his results:

Lemma 1.5.2 (Lemma 6 of [Miy]). *Let S be a numerical Godeaux surface and let $|M|$ be the movable part of the bicanonical system $|2K_S|$. Then we can write $|2K_S| = |M| + T$ where T is the fixed part of $|2K_S|$. Then the general $M \in |M|$ is reduced and irreducible. Moreover M and T satisfy one of the following numerical conditions:*

a) $T = 0$;

b) $K_S T = 0, T^2 = -2, M^2 = 2, MT = 2$;

c) $K_S T = 0, T^2 = -4, M^2 = 0, MT = 4$.

Theorem 1.5.3 (Proposition 2, Theorem 2 and Theorem 3 of [Miy]). *Let S be a numerical Godeaux surface. Then the tricanonical system $|3K_S|$ has no fixed part. Every base point of the tricanonical system is simple and the number b of base points is given as follows:*

$$b = |\{t \in H_2(S, \mathbb{Z})_{tor} \mid t \neq -t\}|/2$$

Moreover the map $\varphi_{|3K_S|}$ associated to $|3K_S|$ is birational.

The birationality of the tricanonical map $\varphi_{|3K_S|}$ allows us to see a numerical Godeaux surface S as a surface in $\mathbb{P}^3$ with a finite number of double points as singularities, coming from the contraction of the (-2)-curves.

Catanese and Pignatelli, in their work [CP1], have recently improved the study of the bicanonical system obtaining the following result which excludes case c) of lemma 1.5.2.

Theorem 1.5.4. *Let S be a numerical Godeaux surface and let $f : S \to \mathbb{P}^1$ the fibration induced by the bicanonical pencil of S. Then the genus of the fibre can only be 3 or 4.*

As a consequence of theorem 1.5.3 we also have the following

Theorem 1.5.5 (Lemma 11, Theorem 2' and following remark of [Miy]). *For a numerical Godeaux surface S the order of the torsion group does not exceed 5 and*

$$b = \begin{cases} 0 & \textit{if } H_2(S, \mathbb{Z})_{tor} = 0 \textit{ or } \mathbb{Z}/2\mathbb{Z}; \\ 1 & \textit{if } H_2(S, \mathbb{Z})_{tor} = \mathbb{Z}/3\mathbb{Z} \textit{ or } \mathbb{Z}/4\mathbb{Z}; \\ 2 & \textit{if } H_2(S, \mathbb{Z})_{tor} = \mathbb{Z}/5\mathbb{Z}. \end{cases}$$

The moduli spaces of numerical Godeaux surfaces with torsion group $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z}$ are known to be irreducible of dimension 8 (see [R] and [Miy]). At the best of our knowledge, the analogous question about surfaces with torsion group 0 or $\mathbb{Z}/2\mathbb{Z}$ is still open. Examples with such torsion groups are constructed in [B], [CG] and [W], while a deeper study of the numerical Godeaux surfaces without torsion can be found in [CP2].

Chapter 2

Preliminary results

2.1 Basic properties

Let us consider a numerical Godeaux surface S (cf. section 1.5) with an order 3 automorphism σ and let $p : S \longrightarrow \Sigma$ be the projection of S to its quotient $\Sigma = S/ < \sigma >$. Let also $\pi : X \longrightarrow Y$ be the resolution of the cover $S \longrightarrow \Sigma$ with X and Y smooth as in 1.3. So we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & S \\ \pi\downarrow & & \downarrow p \\ Y & \xrightarrow{\eta} & \Sigma \end{array}$$

Let us fix the notation: R_0 is the ramification divisor of p, h_1 is the number of isolated fixed points p_i of σ which descend to triple point singularities of Σ , whereas h_2 is the number of isolated fixed points q_j of σ which descend to double point singularities of Σ . We also define $E = \sum_{i=1}^{h_1} E_i$ where E_i is the exceptional curve corresponding to the point p_i. We will denote the reducible (-1)-curve which contracts to a point q_j by $F_j + G_i + H_j$ where F_j, H_j are (-1)-curves and G_j is a (-3)-curve with $F_jG_j = H_jG_j = 1$, $F_jH_j = 0$. The sum of the curves F_i, G_i and H_i will be similarly denoted by F, G, H. Let finally $B_0 = \pi(\varepsilon^*(R_0))$ and E_i', F_i' etc. be the images of E_i, F_i,... via π.

So we have $R = Ram(\pi) = \varepsilon^*(R_0) + E + F + H$ and, by Hurwitz formula,

$$K_X = \pi^*(K_Y) + 2R = \pi^*(K_Y) + 2\varepsilon^*(R_0) + 2E + 2F + 2H \tag{2.1}$$

while since X is a blow-up of S

$$K_X = \varepsilon^*(K_S) + E + 2F + G + 2H \tag{2.2}$$

Lemma 2.1.1. *We have*

$$\varepsilon^*(R_0)K_X = R_0K_S, \quad \varepsilon^*(R_0)\pi^*(K_Y) = B_0K_Y \tag{2.3}$$

$$B_0K_Y = R_0K_S - 2{R_0}^2. \tag{2.4}$$

Proof. Let us compute $\varepsilon^*(R_0)K_X$ using the two formulas 2.1 and 2.2. We notice that, since $\pi^*(B_0) = 3R_0$,

$$\varepsilon^*(R_0)\pi^*(K_Y) = \frac{1}{3}\pi^*(B_0)\pi^*(K_Y) = B_0K_Y$$

By (2.1) we obtain

$$\begin{aligned} \varepsilon^*(R_0)K_X &= \varepsilon^*(R_0)(\pi^*(K_Y) + 2\varepsilon^*(R_0) + 2E + 2F + 2H) \\ &= \varepsilon^*(R_0)\pi^*(K_Y) + 2(\varepsilon^*(R_0))^2 = B_0K_Y + 2{R_0}^2 \end{aligned}$$

Instead, by (2.2) we find

$$\varepsilon^*(R_0)K_X = \varepsilon^*(R_0)(\varepsilon^*(K_S) + E + 2F + G + 2H) = R_0K_S$$

The desired result follows. □

Proposition 2.1.2. *Let S, σ , X, Y be as above. Then the number of isolated fixed point of σ satisfies the formula*

$$h_1 + 2h_2 = 6 + \frac{3R_0K_S - {R_0}^2}{2}. \tag{2.5}$$

Moreover we have

$$K_Y^2 = \frac{1}{3}[{K_S}^2 - (h_1 + 3h_2) + 4{R_0}^2 - 4R_0K_S]. \tag{2.6}$$

Proof. Computing the Euler number of X and Y we obtain

$$e(X) = 3e(Y) - 2e(R) \tag{2.7}$$

Now,

$$-e(R) = -e(\varepsilon^*(R_0)) - 2(h_1 + 2h_2) = {R_0}^2 + R_0K_S - 2(h_1 + 2h_2)$$

$$e(X) = 12 - K_X^2, \quad e(Y) = 12 - K_Y^2$$

so from (2.7)

$$12 - K_X^2 = 3(12 - K_Y^2) + 2({R_0}^2 + R_0K_S) - 4(h_1 + 2h_2) \tag{2.8}$$

Again from (2.1), (2.3) and (2.4)

$$\begin{aligned} K_X^2 &= (\pi^*(K_Y) + 2\varepsilon^*(R_0) + 2E + 2F + 2H)^2 \\ &= 3K_Y^2 + 4{R_0}^2 + 4E^2 + 4F^2 + 4H^2 + 4B_0K_Y + 4(E' + F' + H')K_Y \\ &= 3K_Y^2 + 4{R_0}^2 + 4B_0K_Y = 3K_Y^2 + 4{R_0}^2 + 4(R_0K_S - 2{R_0}^2) \\ &= 3K_Y^2 - 4{R_0}^2 + 4R_0K_S \end{aligned} \tag{2.9}$$

hence

$$K_Y^2 = \frac{1}{3}[K_X^2 + 4{R_0}^2 - 4R_0K_S] = \frac{1}{3}[{K_S}^2 - (h_1 + 3h_2) + 4{R_0}^2 - 4R_0K_S]$$

Putting all these together and substituting (2.9) in (2.8) we obtain

$$12 - 3K_Y^2 + 4{R_0}^2 - 4R_0K_S = 36 - 3K_Y^2 + 2({R_0}^2 + R_0K_S) - 4(h_1 + 2h_2)$$

from which we infer

$$h_1 + 2h_2 = 6 + \frac{3R_0K_S - {R_0}^2}{2}$$

as wanted. □

Remark 2.1.3. *Using the above proposition we have*

$$K_Y^2 = \frac{1}{3}[{K_S}^2 - 6 - h_2 + \frac{9}{2}{R_0}^2 - \frac{11}{2}R_0K_S]. \tag{2.10}$$

In fact

$$\begin{aligned} K_Y^2 &= \frac{1}{3}[{K_S}^2 - (h_1 + 3h_2) + 4{R_0}^2 - 4R_0K_S] \\ &= \frac{1}{3}[{K_S}^2 - (h_1 + 2h_2) - h_2 + 4{R_0}^2 - 4R_0K_S] \\ &= \frac{1}{3}[{K_S}^2 - 6 - \frac{3}{2}R_0K_S + \frac{1}{2}{R_0}^2 - h_2 + 4{R_0}^2 - 4R_0K_S] \\ &= \frac{1}{3}[{K_S}^2 - 6 - h_2 + \frac{9}{2}{R_0}^2 - \frac{11}{2}R_0K_S] \end{aligned}$$

Moreover, since

$$2K_X - R = 2(\pi^*(K_Y) + 2R) - R = \pi^*(2K_Y) + 3R = \pi^*(2K_Y + B)$$
$$3K_X = 3(\pi^*(K_Y) + 2R) = \pi^*(3K_Y + 2B)$$

from (1.4) we have

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L_1) \oplus \mathcal{O}_Y(-L_2)$$

and then

$$\begin{aligned}\pi_*(\mathcal{O}_X(2K_X-R)) &= \mathcal{O}_Y(2K_Y+B)\otimes(\mathcal{O}_Y\oplus\mathcal{O}_Y(-L_1)\oplus\mathcal{O}_Y(-L_2))\\ &= \mathcal{O}_Y(2K_Y+B)\oplus\mathcal{O}_Y(2K_Y+L_2)\oplus\mathcal{O}_Y(2K_Y+L_1)\end{aligned} \tag{2.11}$$

$$\begin{aligned}\pi_*(\mathcal{O}_X(3K_X)) &= \mathcal{O}_Y(3K_Y+2B)\otimes(\mathcal{O}_Y\oplus\mathcal{O}_Y(-L_1)\oplus\mathcal{O}_Y(-L_2))\\ &= \mathcal{O}_Y(3K_Y+2B)\oplus\mathcal{O}_Y(3K_Y+B+L_2)\oplus\mathcal{O}_Y(3K_Y+B+L_1)\end{aligned} \tag{2.12}$$

In particular, we have

$$2 = h^0(X,\mathcal{O}_X(2K_X)) \geq h^0(X,\mathcal{O}_X(2K_X-R)) \geq h^0(Y,\mathcal{O}_Y(2K_Y+B)) \geq 0$$

$$4 = h^0(X,\mathcal{O}_X(3K_X)) \geq h^0(Y,\mathcal{O}_Y(3K_Y+2B)) \geq 0$$

Remark 2.1.4. *We note that the case $h^0(Y,\mathcal{O}_Y(3K_Y+2B)) = 4$ cannot occur, because if so, then each curve of the tricanonical system $|3K_X|$ would be invariant under the action of σ , and then the tricanonical map $\varphi_{|3K_X|}$ would be composed with σ : this is not possible since $\varphi_{|3K_X|}$ is a birational map (see [Miy]).*

Lemma 2.1.5. *The divisor $N = 3K_Y + 2B_0 + E' - 3G'$ on Y is nef and big and has the following properties:*

$$N^2 = 3 \tag{2.13}$$

$$NK_Y = 1 - 2R_0K_S. \tag{2.14}$$

Proof. We just observe that

$$\begin{aligned}\pi^*(N) &= 3\pi^*(K_Y) + 6\varepsilon^*(R_0) + 3E - 3G\\ &\overset{(2.1)}{=} 3K_X - 6E - 6F - 6H + 3E - 3G\\ &\overset{(2.6)}{=} 3\varepsilon^*(K_S) + 3E + 6F + 3G + 6H - 6E - 6F - 6H + 3E - 3G\\ &= \varepsilon^*(3K_S)\end{aligned}$$

which is nef and big since S is of general type.

Moreover $9 = (\varepsilon^*(3K_S))^2 = (\pi^*(N))^2 = 3N^2$ and

$$\begin{aligned}NK_Y &= (3K_Y + 2B_0 + E' - 3G')K_Y = 3K_Y^2 + 2B_0K_Y + E'K_Y - 3G'K_Y\\ &\overset{(2.6)}{=} {K_S}^2 - (h_1+3h_2) + 4{R_0}^2 - 4R_0K_S + 2(R_0K_S - 2{R_0}^2) + h_1 + 3h_2\\ &= {K_S}^2 - 2R_0K_S = 1 - 2R_0K_S\end{aligned}$$

completes the proof. □

We now want to apply Kawamata-Viehweg theorem 1.1.11 to compute the dimensions of $H^0(Y, \mathcal{O}_Y(3K_Y+2B))$ and $H^0(Y, \mathcal{O}_Y(2K_Y+B))$ as vector spaces. We obtain the following

Proposition 2.1.6. *In the above setting we have*

(a) $h^0(Y, \mathcal{O}_Y(3K_Y + 2B)) = h^0(Y, \mathcal{O}_Y(N)) = 2 + R_0K_S$

(b) $h^0(Y, \mathcal{O}_Y(2K_Y + B)) = \frac{1}{3}(2h_2 - 2 - R_0K_S)$

Moreover, we have $0 \leq R_0K_S \leq 1$ *and* $R_0K_S = 1$ *if and only if* $h^0(Y, \mathcal{O}_Y(2K_Y + B)) = 1$ *and* $h_2 = 3$*.*

Proof. (a) We determine some curves in the fixed part of $|3K_Y+2B|$: we recall that the curves E'_i, F'_i, H'_i are (-3)-curves, while the G'_i's are (-1)-curves. Then

$$(3K_Y + 2B)E'_i = (3K_Y + 2B)F'_i = (3K_Y + 2B)H'_i = -3$$

$$(3K_Y + 2B - E' - F' - H')G'_i = (3K_Y + 2B_0 + E' + F' + H')G'_i = -1$$

$$\begin{aligned}(3K_Y + 2B_0+&E' + F' + H' - G')F'_i \\ &= (3K_Y + 2B_0 + E' + F' + H' - G')H'_i = -1\end{aligned}$$

$$(3K_Y + 2B_0 + E' - G')G'_i = -2.$$

It follows that we can write $|3K_Y + 2B| = E' + 2F' + 2H' + 3G' + |N|$. So we have

$$\begin{aligned}h^0(Y, \mathcal{O}_Y(3K_Y + 2B)) &= h^0(Y, \mathcal{O}_Y(N)) \\ &= h^0(Y, \mathcal{O}_Y(3K_Y + 2B_0 + E' - 3G')).\end{aligned}$$

Moreover, since $\pi^*(N) = \varepsilon^*(3K_S)$, using the formula

$$\pi_*(\mathcal{O}_X(\varepsilon^*(3K_S))) = \mathcal{O}_Y(N) \oplus \mathcal{O}_Y(N - L_1) \oplus \mathcal{O}_Y(N - L_2)$$

and the fact that $h^i(S, \mathcal{O}_S(3K_S)) = 0$ for all $i > 0$, we find

$$h^i(Y, \mathcal{O}_Y(N)) = 0 \quad \text{for all } i > 0.$$

Then, using lemma 2.1.5 one has

$$\begin{aligned}0 \leq h^0(Y, \mathcal{O}_Y(N)) &= \chi(Y, \mathcal{O}_Y(N)) = 1 + \frac{N(N - K_Y)}{2} \\ = 1 + \frac{3 - 1 + 2R_0K_S}{2} &= 2 + R_0K_S\end{aligned}$$

(b) Again we determine some curves in the fixed part of $|2K_Y + B|$:

$$(2K_Y + B)F'_i = -1$$
$$(2K_Y + B - F')G'_i = -1$$
$$(2K_Y + B_0 + E' + H' - G')H'_i = -2$$

So we have

$$h^0(Y, \mathcal{O}_Y(2K_Y + B)) = h^0(Y, \mathcal{O}_Y(2K_Y + B_0 + E' - G'))$$

But we can also write

$$2K_Y + B_0 + E' - G' = K_Y + (K_Y + B_0 + E' - G') = K_Y + \frac{1}{3}N + \frac{1}{3}B_0 + \frac{2}{3}E'$$

and by Kawamata-Viehweg theorem 1.1.11

$$h^i(Y, \mathcal{O}_Y(2K_Y + B_0 + E' - G')) = 0 \quad \text{for all } i > 0.$$

Then, as in (a),

$$\begin{aligned}
0 &\leq h^0(Y, \mathcal{O}_Y(2K_Y + B_0 + E' - G')) = \chi(Y, \mathcal{O}_Y(2K_Y + B_0 + E' - G')) \\
&= 1 + \frac{(2K_Y + B_0 + E' - G')(K_Y + B_0 + E' - G')}{2} \\
&= 1 + K_Y^2 + \frac{3}{2}B_0K_Y + \frac{3}{2}E'K_Y - \frac{3}{2}G'K_Y + \frac{1}{2}B_0^2 + \frac{1}{2}E'^2 + \frac{1}{2}G'^2 \\
&= 1 + \frac{1}{3}[{K_S}^2 - (h_1 + 3h_2) + 4{R_0}^2 - 4R_0K_S] + \frac{3}{2}(R_0K_S - 2{R_0}^2) + \\
&+ \frac{3}{2}{R_0}^2 + h_2 \\
&= 1 + \frac{1}{3}[{K_S}^2 - 6 - h_2 - \frac{11}{2}R_0K_S + \frac{9}{2}{R_0}^2] + \frac{3}{2}(R_0K_S - {R_0}^2) + h_2 \\
&= \frac{1}{3}[3 + {K_S}^2 - 6 + 2h_2 - R_0K_S] = \frac{1}{3}(2h_2 - 2 - R_0K_S)
\end{aligned}$$

The last assertion follows easily by remark 2.1.4 and the fact that

$$0 \leq h^0(Y, \mathcal{O}_Y(2K_Y + B)) \leq 2.$$

□

So we are left with only three possible cases, according to the values of R_0K_S and of h_2:

(i) $h^0(Y, \mathcal{O}_Y(N)) = 3$, $h^0(Y, \mathcal{O}_Y(2K_Y + B)) = 1$, $R_0K_S = 1$, $h_2 = 3$

(ii) $h^0(Y, \mathcal{O}_Y(N)) = 2$, $h^0(Y, \mathcal{O}_Y(2K_Y + B)) = 2$, $R_0K_S = 0$, $h_2 = 4$

(iii) $h^0(Y, \mathcal{O}_Y(N)) = 2$, $h^0(Y, \mathcal{O}_Y(2K_Y + B)) = 0$, $R_0K_S = 0$, $h_2 = 1$

Lemma 2.1.7. *For any* $1 \leq i \leq 2, 0 \leq j \leq 2$ *we have*

$$h^j(Y, \mathcal{O}_Y(-L_i)) = 0.$$

In particular $L_i^2 + L_iK_Y = -2$ *for* $i = 1, 2$.

Proof. Since X is birational to a numerical Godeaux surface we have $p_g(X) = q(X) = 0$ and $\chi(X, \mathcal{O}_X) = 1$. From (1.4) we find

$$0 = p_g(X) = p_g(Y) + h^2(Y, \mathcal{O}_Y(-L_1)) + h^2(Y, \mathcal{O}_Y(-L_2))$$

$$0 = q(X) = q(Y) + h^1(Y, \mathcal{O}_Y(-L_1)) + h^1(Y, \mathcal{O}_Y(-L_2))$$

$$1 = h^0(X, \mathcal{O}_X) = h^0(Y, \mathcal{O}_Y) + h^0(Y, \mathcal{O}_Y(-L_1)) + h^0(Y, \mathcal{O}_Y(-L_2))$$

Since Y is smooth this implies $p_g(Y) = q(Y) = 0, \chi(Y, \mathcal{O}_Y) = 1$ and

$$h^j(Y, \mathcal{O}_Y(-L_i)) = 0 \quad 1 \leq i \leq 2, 0 \leq j \leq 2$$

as wanted. □

Proposition 2.1.8. *Assume case* (iii) *above holds and* $\ell = 1$. *Then* R_0 *is an irreducible* (-2)*-curve and* $h_1 = 4 + \ell = 5$. *Let* $\omega = e^{\frac{2\pi i}{3}}$ *be a primitive third root of unity and let* h_{11} *and* h_{12} *be the number of curves* E_i *such that the eigenvalue of the action of* $\mathbb{Z}/3\mathbb{Z}$ *on* E_i *is* ω *and* ω^2 *respectively. Then if* ω *is the eigenvalue corresponding to* R_0 *then* $h_{11} = 2, h_{12} = 3$.

Proof. Since case (iii) holds from (2.5) we infer $h_1 = 4 + \ell = 5$.

We now write as $\bar{E}'_+$ and $\bar{E}'_-$ the sum of the curves E'_i associated to the same eigenvalue ω and ω^2 respectively. Since $h_1 = 5 = h_{11} + h_{12}$ from (1.5)

$$3L_1 \equiv B_0 + E'_+ + 2\bar{E}'_- + F' + 2H'$$

and we find

$$L_1K_Y = \frac{1}{3}(B_0 + E'_+ + 2\bar{E}'_- + F' + 2H')K_Y = \frac{1}{3}(4 + h_{11} + 2h_{22} + 3h_2)$$

$$= \frac{4+2h_1-h_{11}}{3} + h_2 = \frac{14-h_{11}}{3} + 1$$

hence $h_{11} \equiv 2 \mod 3$ that forces $h_{11} = 2, h_{12} = 3$ or $h_{11} = 5, h_{12} = 0$. Furthermore

$$L_1^2 = \frac{1}{9}(B_0 + E'_+ + 2\bar{E}'_- + F' + 2H')^2 = \frac{1}{9}(-6 - 3h_{11} - 12h_{22} - 15h_2)$$
$$= \frac{-6-12h_1+9h_{11}-15h_2}{9} = \frac{-81+9h_{11}}{9} = -9 + h_{11}$$

From lemma 2.1.7 we know that $L_1^2 + L_1K_Y = -2$ hence

$$-2 = L_1^2 + L_1K_Y = -9 + h_{11} + \frac{14-h_{11}}{3} + 1$$
$$= \frac{14-24+2h_{11}}{3} = \frac{2h_{11}-10}{3}$$

and $h_{11} = 2$. $\square$

2.2 The invariant part of the tricanonical system

Before going on, we want to better understand the properties of the curves in $|N|$ (which is always non-empty). In particular, in lemma 2.1.5 we have seen that $N^2 = 3$ and $NK_Y = 1 - 2R_0K_S$ so that

$$p_a(N) = 1 + \frac{N^2 + NK_Y}{2} = 1 + \frac{3+1-2R_0K_S}{2} = 3 - R_0K_S$$

Lemma 2.2.1. *Let S be a numerical Godeaux surface and let Λ be a linear subsystem of $|3K_S|$ with $\dim\Lambda \geq 1$ and $\Lambda = \mathcal{A} + \Phi$ where $\mathcal{A}$ is the movable part and Φ is the fixed part of Λ. Then the general member $A \in \mathcal{A}$ is reduced and irreducible and one of the following conditions is satisfied:*

a) $AK_S = 2$, $\Phi K_S = 1$,$A^2 = 0, 2, 4$, $p_a(\Phi) \leq 2$

b) $AK_S = 3$, $\Phi K_S = 0$ *and either* $A^2 = 1, 3, 5, 7$, $p_a(\Phi) \leq 0$ *or* $\Phi = 0$.

Moreover, if $A^2 = 4$ then $A \sim 2K_S$.

Proof. We have

$$3 = 3{K_S}^2 = \Lambda K_S = AK_S + \Phi K_S$$

Moreover, by Miyaoka [Miy] (see also lemma 1.5.2), we know that $AK_S \geq 2$. This implies either $AK_S = 2$, $\Phi K_S = 1$ or $AK_S = 3$, $\Phi K_S = 0$. In the former case by the Index Theorem

$$0 \geq (A - 2K_S)^2 = A^2 + 4 - 8 = A^2 - 4$$

and

$$0 \geq (\Phi - K_S)^2 = \Phi^2 + 1 - 2 = \Phi^2 - 1$$

which proves a). A similar argument shows b). To see the irreducibility of A simply observe that if $A = A_1 + A_2$ was reducible then $A_1 K_S, A_2 K_S \geq 2$ and $AK_S \geq 4$. Contradiction. □

Proposition 2.2.2. *If the linear system $|N|$ has fixed part, then $|N| = |A'| + \Phi'$ with ${A'}^2 = 0, 1, 2$ and the general curve of $|A'|$ is smooth.*

Proof. Since $\pi^*(N) = \varepsilon^*(3K_S)$ there is a linear subsystem Λ of $|3K_S|$ such that $\varepsilon^*(\Lambda) = \pi^*(|N|)$ and $\dim \Lambda = h^0(Y, \mathcal{O}_Y(N)) - 1 = 1 + R_0 K_S$. Thus we can apply lemma 2.2.1 to Λ. Moreover the strict transform $\widetilde{A}$ of A is the movable part of $\pi^*(|N|)$, so $\widetilde{A} = \pi^*(A')$ where $|A'|$ is the movable part of $|N|$. Then

$$9 \geq \varepsilon^*(A)^2 \geq \widetilde{A}^2 = \pi^*(A')^2 = 3{A'}^2$$

This forces $\widetilde{A}^2$ to be 0, 3, 6 or 9. If $\widetilde{A}^2 = 9$ then $\widetilde{A} = A$ and the linear system Λ, hence $|N|$, has no fixed part. □

Lemma 2.2.3. *The curves in $|A'|$ satisfy*

a) $A'N = AK_S$

b) $A'B_0 = AR_0$

Proof. It is an easy computation. In the former case

$$3A'N = \pi^*(A')\pi^*(N) = \widetilde{A}\varepsilon^*(3K_S) = 3AK_S$$

In the latter case

$$3A'B_0 = \pi^*(A')\pi^*(B_0) = 3\widetilde{A}\varepsilon^*(R_0) = 3AR_0$$

□

We now focus our attention on the case $\dim \Lambda = 1 + R_0 K_S = 1$ or equivalently $R_0 K_S = 0$. Then $\mathcal{A}$ is a pencil and A^2 is the number of base points of $\mathcal{A}$.

Remark 2.2.4. *We note that, if $R_0 K_S = 0$, since $\Lambda = A + \Phi \equiv 3K_S$, for each irreducible component R_{0i} of R_0 we have either $AR_{0i} = 0$ or $R_{0i} \leq \Phi$. Then*

$$AR_0 = \sum_{i=1}^{\ell} AR_{0i} \leq A\Phi$$

On the other hand

$$9 = \Lambda^2 = A^2 + 2A\Phi + \Phi^2$$

and

$$3\Phi K_S = \Phi\Lambda = A\Phi + \Phi^2$$

Therefore

$$0 \leq AR_0 \leq A\Phi = 9 - A^2 - 3\Phi K_S$$

Moreover the $A\Phi$ points of intersection between A and Φ form an invariant set for the action of $\mathbb{Z}/3\mathbb{Z}$ on S.

Let us write

$$\varepsilon^*(A) = \widetilde{A} + D$$

with D a sum of exceptional divisors with certain multiplicities.

Remark 2.2.5. *Let us write*

$$\varepsilon^*(\Phi) = \widetilde{\Phi} + D'$$

Then there exists a divisor Φ'' on Y such that $\pi^*(\Phi'') = \widetilde{\Phi}$ and

$$\pi^*(\Phi') = \widetilde{\Phi} + D + D'$$

This implies $(D + D')^2 \equiv 0 \mod 3$. Moreover, the multiplicity of each curve E_k, F or H in $D + D'$ is a multiple of 3, since they appear in the branch locus of the cover $\pi : X \longrightarrow Y$ and $D + D' = \pi^*(\Phi' - \widetilde{\Phi})$ is a pull-back of a divisor on Y.

We also remark that if $\Phi = 0$ we have $\widetilde{\Phi} \equiv D' \equiv 0$ hence $\pi^*(\Phi') \equiv D$.

Lemma 2.2.6. *For each simple base point of A which is an isolated fixed point q_j the self-intersection $\widetilde{A}^2$ of $\widetilde{A}$ drops exactly by 2.*

Proof. The map $\varepsilon : X \longrightarrow S$ blows up each of the points q_j three times. For each of the blow-ups the pre-image of each point is a fixed point under the action of the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Let ε_1 be the blow-up of q_j with exceptional divisor G_{1j} and ε_2 and ε_3 the blow-ups at the fixed point of G_{1j}. Let also $\widetilde{A}_1$ be the strict transform of A under ε_1 and $\widetilde{A}_2$ the strict transform of A under the composition $\varepsilon_2 \circ \varepsilon_1$. The exceptional divisor G_1 of the first blow-up is an invariant (but not fixed) rational curve for this action and it has two fixed points. The other two blow-ups are based exactly on these two points. Therefore each point q_j which is also a base point for A is blown up twice and at each step the self-intersection of the strict transform of A drops by 1. Moreover, we have

$$\begin{aligned}\varepsilon^*(A) &= {\varepsilon_3}^*({\varepsilon_2}^*({\varepsilon_1}^*(A))) = {\varepsilon_3}^*({\varepsilon_2}^*(\widetilde{A}_1 + G_{1j})) \\ &= {\varepsilon_3}^*(\widetilde{A}_2 + F_{2j} + G_{2j} + F_{2j}) = \widetilde{A} + 2F_j + G_j + H_j\end{aligned}$$

or, analogously, $\varepsilon^*(A) = \widetilde{A} + F_j + G_j + 2H_j$ depending on which of the fixed points of G_{1j} is in $\widetilde{A}_1$. $\square$

Lemma 2.2.7. *For each double base point of A which is an isolated fixed point q_j the self-intersection $\widetilde{A}^2$ of $\widetilde{A}$ drops at least by 5. In any case this can only happen when $A^2 \geq 6$.*

Proof. We use the same argument as in the proof of lemma 2.2.6. At a first blow-up the self-intersection of the strict transform of A drops by 4. The pre-image of the point q_j is composed either of two simple points, or of one point (and the curve is tangent to the exceptional divisor G_{1j}) or a double point. In any case we need to blow-up again these points, since they are fixed under the action of $\mathbb{Z}/3\mathbb{Z}$.

When we have two simple points each of the remaining blow-ups ε_2 and ε_3 drops the self-intersection by 1 and then $\widetilde{A}^2 = A^2 - 6$. The points in the pre-image of q_j are no more base points for $\widetilde{A}$ and

$$\begin{aligned}\varepsilon^*(A) &= {\varepsilon_3}^*({\varepsilon_2}^*({\varepsilon_1}^*(A))) = {\varepsilon_3}^*({\varepsilon_2}^*(\widetilde{A}_1 + 2G_{1j})) \\ &= {\varepsilon_3}^*(\widetilde{A}_2 + F_{2j} + 2G_{2j} + 2F_{2j}) = \widetilde{A} + 3F_j + 2G_j + 3H_j\end{aligned}$$

When we have one non-singular point Q without loss of generality we can assume that ε_2 is the blow-up at Q. Then

$$\begin{aligned}\varepsilon^*(A) &= {\varepsilon_3}^*({\varepsilon_2}^*({\varepsilon_1}^*(A))) = {\varepsilon_3}^*({\varepsilon_2}^*(\widetilde{A}_1 + 2G_{1j})) \\ &= {\varepsilon_3}^*(\widetilde{A}_2 + F_{2j} + 2G_{2j} + 2F_{2j}) = \widetilde{A} + 3F_j + 2G_j + 2H_j\end{aligned}$$

hence

$$\widetilde{A}^2 = (\varepsilon^*(A) - 3F_j - 2G_j - 2H_j)^2 = A^2 - 9 - 12 - 4 + 12 + 8 = A^2 - 5$$

The pre-image of q_j is again a base point for $\widetilde{A}$. This can only happen when $A^2 \geq 6$.

Finally when Q is a double point

$$\begin{aligned}\varepsilon^*(A) &= {\varepsilon_3}^*({\varepsilon_2}^*({\varepsilon_1}^*(A))) = {\varepsilon_3}^*({\varepsilon_2}^*(\widetilde{A}_1 + 2G_{1j})) \\ &= {\varepsilon_3}^*(\widetilde{A}_2 + 2F_{2j} + 2G_{2j} + 2F_{2j}) = \widetilde{A} + 4F_j + 2G_j + 2H_j\end{aligned}$$

hence

$$\widetilde{A}^2 = (\varepsilon^*(A) - 4F_j - 2G_j - 2H_j)^2 = A^2 - 16 - 12 - 4 + 16 + 8 = A^2 - 8$$

which is impossible unless $A^2 = 9$. □

Lemma 2.2.8. *If the general $A \in \mathcal{A}$ has a triple point singularity at one of the isolated fixed point q_j we have $A^2 = 9$ and q_j is an ordinary triple point.*

Proof. Assume q_j is a triple point for the general $A \in \mathcal{A}$. Then after a blow-up the pre-image of q_j As in the two previous lemmas we find in the case of an ordinary triple point

$$\begin{aligned}\varepsilon^*(A) &= {\varepsilon_3}^*({\varepsilon_2}^*({\varepsilon_1}^*(A))) = {\varepsilon_3}^*({\varepsilon_2}^*(\widetilde{A}_1 + 3G_{1j})) \\ &= {\varepsilon_3}^*(\widetilde{A}_2 + 3G_{2j} + 3F_{2j}) = \widetilde{A} + 3G_j + 3F_j + 3H_j\end{aligned}$$

and then

$$\widetilde{A}^2 = A^2 + (3G_j + 3F_j + 3H_j)^2 = A^2 + (-27 - 9 - 9 + 18 - 18) = A^2 - 9 = 0$$

If q_j is not an ordinary triple point, its pre-image under the first blow-up is composed either of a flex, or a cusp or another triple point. In all these cases it consists of a single point Q that must be blown up again. Then we have

$$\widetilde{A}^2 < A^2 - 9 = 0$$

which is impossible since $\widetilde{A}$ is nef. □

Remark 2.2.9. *From remark 2.2.5 when $A^2 = 9$ (or equivalently $\Phi = 0$) we have $D' = 0$ and each component of D different from G has multiplicity $m \equiv 0 \mod 3$. In particular if we look at the multiplicities α_j of A at the points q_j we find, using lemmas 2.2.6, 2.2.7 and 2.2.8, the following possibilities:*

1. $\alpha_j = 0$
2. $\alpha_j = 2$ *and* q_j *is a node*
3. $\alpha_j = 3$ *and* q_j *is an ordinary triple point.*

Moreover the multiplicity m_i *of the general curve* A *at any of the points* p_i *can be different from 0 (hence* $m_i = 3$ *since* $m_i \equiv 0 \mod 3$*) only when* $\alpha_j = 0$ *for all the points* q_j*.*

Remark 2.2.10. *Assume* $p_a(A') = g$*. Then*

$$A'K_Y = 2g - 2 - A'^2$$

On the other hand

$$\begin{aligned} 3A'K_Y = \pi^*(A')\pi^*(K_Y) &\overset{(2.1)}{=} \widetilde{A}(K_X - 2\varepsilon^*(R_0) - 2E - 2F - 2H) \\ &\overset{(2.2)}{=} \widetilde{A}(\varepsilon^*(K_S - 2R_0) + E + 2F + G + 2H - 2E - 2F - 2H) \\ &= \widetilde{A}(\varepsilon^*(K_S - 2R_0) + G - E) \\ &= (\varepsilon^*(A) - D)(\varepsilon^*(K_S - 2R_0) + G - E) \\ &= AK_S - 2AR_0 - DG + DE \end{aligned}$$

Therefore

$$AK_S - 2AR_0 - DG + DE = 6g - 6 - 3A'^2 \tag{2.15}$$

Lemma 2.2.11. *In the above setting we have* $DG = 0$ *except when* $A^2 = 9$ *and the general* $A \in \mathcal{A}$ *has an ordinary triple point at* q*. In the latter case* $DG = -3$*. In particular the general* $A \in \mathcal{A}$ *cannot have a cusp at* q*.*

Proof. If $\mathrm{mult}_q A = 0$ then obviously $DG = 0$. Therefore we can assume $\alpha := \mathrm{mult}_q A \geq 1$. We notice that

$$3A'G' = \pi^*(A')\pi^*(G') = \widetilde{A}G = (\varepsilon^*(A) - D)G = -DG \tag{2.16}$$

and then $DG \equiv 0 \mod 3$.
If $\alpha = 1$ then from the proof of lemma 2.2.6 one has $D = 2F + G + H$ and

$$DG = (2F + G + H)G = 2 - 3 + 1 = 0.$$

If $\alpha = 2$ and $D = 3F + 2G + 3H$ then

$$DG = (3F + 2G + 3H)G = 3 - 6 + 3 = 0.$$

If $\alpha = 2$ and $D = 3F + 2G + 2H$ then

$$DG = (3F + 2G + 2H)G = 3 - 6 + 2 = -1$$

which is impossible.

Finally if $\alpha = 3$ then from lemma 2.2.8 $D = 3F + 3G + 3H$ and

$$DG = (3F + 3G + 3H)G = 3 - 9 + 3 = -3.$$

$\square$

As an immediate consequence of the above lemma and of equation (2.16) we have

Corollary 2.2.12. *In the above setting we have* $A'G' = 0$ *unless* $A^2 = 9$ *and the general* A *has an ordinary triple point at* q*. Then* $A'G' = 1$.

We now concentrate our analysis on the case $\dim \Lambda = 1$ and $h_2 = 1$, which is case (iii) of the list at page 34.

Proposition 2.2.13. *Assume* $\dim \Lambda = 1$ *and* $h_2 = 1$*. Then when* $A'^2 = 0$ *one of the following possibilities holds:*

either $\Phi \neq 0$ *and*

$(0a)$ $A^2 = 2$, $AR_0 = 0$, $g = 1$, $A'K_Y = 0$, $D = E_1 + E_2$

$(0b)$ $A^2 = 2$, $AR_0 = 1$, $g = 1$, $A'K_Y = 0$, $D = 2F + G + H$

$(0c)$ $A^2 = 3$, $AR_0 = 0$, $g = 1$, $A'K_Y = 0$, $D = E_1 + E_2 + E_3$

$(0d)$ $A^2 = 3$, $AR_0 = 1$, $g = 1$, $A'K_Y = 0$, $D = E_1 + 2F + G + H$

$(0e)$ $A^2 = 4$, $AR_0 = 0$, $g = 1$, $A'K_Y = 0$, $D = 2E_1$

$(0f)$ $A^2 = 5$, $AR_0 = 0$, $g = 1$, $A'K_Y = 0$, $D = 2E_1 + E_2$

or $\Phi = 0$ *and*

$(0g)$ $A^2 = 9$, $AR_0 = 0$, $g = 2$, $A'K_Y = 2$, $D = 3F + 3G + 3H$

$(0h)$ $A^2 = 9$, $AR_0 = 0$, $g = 1$, $A'K_Y = 0$, $D = 3E_1$

Proof. Let us assume $A'^2 = 0$. We start by considering $A^2 \leq 7$. From lemma 2.2.11 we have $DG = 0$. Then, if $D = uF + vG + wH + \sum_{i=1}^{h_1} a_i E_i$, (2.15) becomes

$$AK_S - 2AR_0 - \sum_{i=1}^{h_1} a_i = 6g - 6 \tag{2.17}$$

Let us begin with $A^2 = 0$. Then $D = 0$ and $\widetilde{A} = \varepsilon^*(A)$. Moreover from lemma 2.2.1 $AK_S = 2$ and from remark 2.2.4 we have $0 \leq AR_0 \leq A\Phi = 6$. Hence by (2.17)

$$2 - 2AR_0 = 6g - 6$$

and then $AR_0 = 1, 4$ since $AR_0 \equiv 1 \mod 3$. The intersection cycle $A \cdot \Phi$ is composed of six points with multiplicities. From remark 2.2.4 (we recall that we are assuming $R_0 K_S = 0$) these points are organized in orbits for the action of $\mathbb{Z}/3\mathbb{Z}$. Each orbit contains either three distinct points or only one fixed point, which can a priori be an isolated fixed point. The latter case cannot actually occur since A and Φ have no isolated fixed point in common. Then we should have $AR_0 \equiv 0 \mod 3$. Contradiction.

When $A^2 = 1$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 8$ and from lemma 2.2.6 $D = E_1$. Then from remark 2.2.5 we have $D' \geq 2E_1$. Since $A \cdot \Phi$ is composed by 8 points with multiplicities and the only isolated fixed point in $A \cap \Phi$ is p_1, which is double for the 0-cycle $A \cdot \Phi$, from remark 2.2.4 we have $AR_0 \equiv 0 \mod 3$. From (2.17) we have

$$2 - 2AR_0 = 3 - 2AR_0 - 1 = 6g - 6$$

Then $AR_0 \equiv 1 \mod 3$ and this is impossible.

When $A^2 = 2$ we have $\widetilde{A}^2 = 0$, $AK_S = 2$, $AR_0 \leq A\Phi = 4$ and from lemma 2.2.6 the following possibilities for D can hold:

a) $D = E_1 + E_2$

b) $D = 2F + G + H$

c) $D = F + G + 2H$.

In case a) from (2.17)

$$-2AR_0 = 2 - 2AR_0 - 2 = 6g - 6$$

hence either $AR_0 = 0, g = 1, A'K_Y = 0$ or $AR_0 = 3, g = 0, A'K_Y = -2$. Moreover by remarks 2.2.4 and 2.2.5 all the four points (counted with their multiplicities) in $A \cap \Phi$ are fixed under the action of $\mathbb{Z}/3\mathbb{Z}$ and $D' \geq 2E_1 + 2E_2$. This forces $AR_0 = 0$.

In cases b) and c)

$$2 - 2AR_0 = 6g - 6$$

hence either $AR_0 = 1, g = 1, A'K_Y = 0$ or $AR_0 = 4, g = 0, A'K_Y = -2$. But now from remarks 2.2.4, 2.2.5 and lemma 2.2.6 $D' \geq F + G + 2H$ in case b) and $D' \geq 2F + G + H$ in case c) and $AR_0 \neq 4$. Then only $AR_0 = 1, g = 1, A'K_Y = 0$ can hold.

When $A^2 = 3$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 6$. Then one of the following is true:
a) $D = E_1 + E_2 + E_3$

$$-2AR_0 = 3 - 2AR_0 - 3 = 6g - 6$$

hence either $AR_0 = 0, g = 1, A'K_Y = 0$ or $AR_0 = 3, g = 0, A'K_Y = -2$. From remark 2.2.5 we have $D' \geq 2E_1 + 2E_2 + 2E_3$ and from remark 2.2.4 $AR_0 = 0$ since p_1, p_2, p_3 are double for $A \cdot \Phi$. Then $AR_0 = 0, g = 1, A'K_Y = 0$ holds;
b) $D = E_1 + 2F + G + H$

$$2 - 2AR_0 = 3 - 2AR_0 - 1 = 6g - 6$$

hence either $AR_0 = 1, g = 1, A'K_Y = 0$ or $AR_0 = 4, g = 0, A'K_Y = -2$. Again from remarks 2.2.4 and 2.2.5 we have $D' \geq 2E_1 + F + G + 2H$ and $AR_0 \neq 4$.

When $A^2 = 4$ by lemma 2.2.1 we have $A \sim 2K_S$ therefore $AR_0 = 0$, $AK_S = 2 = A\Phi$. Then from (2.17) we have

$$\sum_{i=1}^{h_1} a_i = 8 - 6g \equiv 2 \mod 3$$

Then one of the following is true:
a) $D = E_1 + E_2 + 2F + G + H$

$$0 = 2 - 2 = 6g - 6$$

hence $g = 1, A'K_Y = 0$;
b) $D = 2E_1$

$$0 = 2 - 2 = 6g - 6$$

hence $g = 1, A'K_Y = 0$.

In the former case we have $D' \geq 2E_1 + 2E_2 + F + G + 2H$ and then $A\Phi \geq 4$ which is impossible.

When $A^2 = 5$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 4$. Then one of the following is true:
a) $D = E_1 + E_2 + E_3 + E_4 + E_5$

$$-2 - 2AR_0 = 3 - 2AR_0 - 5 = 6g - 6$$

hence $AR_0 = 2, g = 0, A'K_Y = -2$. From remark 2.2.5 we have $D' \geq 2(E_1 + \cdots + E_5)$ and then $A\Phi \geq 10$ which is impossible.
b) $D = E_1 + E_2 + E_3 + 2F + G + H$

$$-2AR_0 = 3 - 2AR_0 - 3 = 6g - 6$$

hence either $AR_0 = 0, g = 1, A'K_Y = 0$ or $AR_0 = 3, g = 0, A'K_Y = -2$. Here $D' \geq 2E_1 + 2E_2 + 2E_3 + F + G + 2H$ and $A\Phi \geq 7$. Contradiction.
c) $D = 2E_1 + E_2$

$$-2AR_0 = 3 - 2AR_0 - 3 = 6g - 6$$

hence either $AR_0 = 0, g = 1, A'K_Y = 0$ or $AR_0 = 3, g = 0, A'K_Y = -2$. From remarks 2.2.5 and 2.2.4 we have $D' \geq E_1 + 2E_2$ and then $AR_0 = 0, g = 1, A'K_Y = 0$ holds.

When $A^2 = 7$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 2$. The general curve A cannot pass through more than one of the isolated fixed points p_i (say p_1). In fact, otherwise we would have $D \geq E_1 + E_2$ and from remark 2.2.5 we should have $D' \geq 2E_1 + E_2$ hence $A\Phi \geq 4$. Contradiction. Then the only possibility for D (cf. lemmas 2.2.6, 2.2.7 and 2.2.8) is:
$D = E_1 + 3F + 2G + 3H$

$$2 - 2AR_0 = 3 - 2AR_0 - 1 = 6g - 6$$

hence $AR_0 = 1, g = 1, A'K_Y = 0$. This is also impossible by remarks 2.2.5 and 2.2.4 since $D' \geq 2E_1$ and $A\Phi = 2$ force $AR_0 = 0$.

Finally we consider $A^2 = 9$. We know from lemma 2.2.1 that $\Phi = 0$. Using remark 2.2.4 we find $AR_0 = 0$. Moreover from remarks 2.2.5 and 2.2.9, $D' = 0$ and the general A has either multiplicity 0 or 3 at each of the isolated fixed points p_j, and it can be 3 only if the multiplicity at q is 0. Then we have the following possibilities for D:
a) $D = 3F + 3G + 3H$:
in this case $DG = -3$ from lemma 2.2.11 and (2.15) becomes

$$6 = 3 - 0 + 3 - 0 = AK_S - 2AR_0 - DG + DE = 6g - 6 - 3A'^2 = 6g - 6$$

which has the only solution $g = 2, A'K_Y = 2$.
b) $D = 3E_1$:
equation (2.15) becomes

$$0 = 3 - 0 + 0 - 3 = AK_S - 2AR_0 - DG + DE = 6g - 6$$

which forces $g = 1, A'K_Y = 0$. □

Proposition 2.2.14. *Assume* $\dim \Lambda = 1$ *and* $h_2 = 1$. *Then when* ${A'}^2 = 1$ *one of the following possibilities holds:*

either $\Phi \neq 0$ *and*

$(1a)$ $A^2 = 3$, $AR_0 = 0$, $g = 2$, $A'K_Y = 1$, $D = 0$

$(1b)$ $A^2 = 3$, $AR_0 = 3$, $g = 1$, $A'K_Y = -1$, $D = 0$

$(1c)$ $A^2 = 3$, $AR_0 = 6$, $g = 0$, $A'K_Y = -3$, $D = 0$

$(1d)$ $A^2 = 5$, $AR_0 = 0$, $g = 2$, $A'K_Y = 1$, $D = 2F + G + H$

$(1e)$ $A^2 = 5$, $AR_0 = 3$, $g = 1$, $A'K_Y = -1$, $D = 2F + G + H$

or $\Phi = 0$ *and*

$(1f)$ $A^2 = 9$, $AR_0 = 0$, $g = 2$, $A'K_Y = 1$, $D = 3F + 2G + 3H$

Proof. Let us assume ${A'}^2 = 1$. Since $\widetilde{A}^2 = 3{A'}^2 = 3$ we have necessarily $A^2 \geq 3$. Again we consider at first $A^2 \leq 7$. From lemma 2.2.11 we have $DG = 0$. Then, if $D = uF + vG + wH + \sum_{i=1}^{h_1} a_i E_i$, for $3 \leq A^2 \leq 7$ from (2.15) equation

$$AK_S - 2AR_0 - \sum_{i=1}^{h_1} a_i = 6g - 9 \tag{2.18}$$

holds. We use the same argument as in the proof of proposition 2.2.13.

When $A^2 = 3$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 6$ (see remark 2.2.4), $D = 0$ and

$$3 - 2AR_0 = 6g - 9$$

hence either $AR_0 = 0, g = 2, A'K_Y = 1$ or $AR_0 = 3, g = 1, A'K_Y = -1$ or $AR_0 = 6, g = 0, A'K_Y = -3$.

When $A^2 = 4$ by lemma 2.2.1 we have $A \sim 2K_S$ therefore $AR_0 = 0$, $AK_S = 2$. Then

$$\sum_{i=1}^{h_1} a_i = 11 - 6g \equiv 2 \mod 3$$

which is impossible since $\widetilde{A}^2 = A^2 - 1 = 3$.

When $A^2 = 5$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 4$. Then the following possibilities for D can hold:

a) $D = E_1 + E_2$

$$1 - 2AR_0 = 3 - 2AR_0 - 2 = 6g - 9$$

hence $AR_0 = 2, g = 1, A'K_Y = -1$. Here from remarks 2.2.4 and 2.2.5 we have $D' \geq 2E_1 + 2E_2$ and this forces $AR_0 = 0$. Contradiction.

b) $D = 2F + G + H$

$$3 - 2AR_0 = 6g - 9$$

hence either $AR_0 = 0, g = 2, A'K_Y = 1$ or $AR_0 = 3, g = 1, A'K_Y = -1$.

When $A^2 = 7$ we have $AK_S = 3$, $AR_0 \leq A\Phi = 2$. Then arguing as in the proof of proposition 2.2.13, we have to consider only the cases when only the curve E_1 appears in D with multiplicity 1 or 2. Hence $D = 2E_1$ and

$$1 - 2AR_0 = 3 - 2AR_0 - 2 = 6g - 9$$

which implies $AR_0 = 2, g = 1, A'K_Y = -1$. But in this case remark 2.2.4 forces $AR_0 = 0$. Contradiction.

When $A^2 = 9$ instead we have $\Phi = 0$ from lemma 2.2.1 and then from remark 2.2.4 $AR_0 = 0$. Since from remark 2.2.5 all the curves E_k, F' and H' must appear with multiplicity $m \equiv 0 \mod 3$ in D, the only possibility for D (see also remark 2.2.9) is $D = 3F + 2G + 3H$. Then (2.15) becomes

$$3 = AK_S - 2AR_0 - DG + DE = 6g - 6 - 3A'^2 = 6g - 6 - 3 = 6g - 9$$

that implies $g = 2, A'K_Y = 1$. □

Proposition 2.2.15. *Assume* $\dim \Lambda = 1$ *and* $h_2 = 1$. *The case* $A'^2 = 2$ *cannot occur.*

Proof. Since $\widetilde{A}^2 = 3A'^2 = 6$, from lemma 2.2.1 we have $A^2 \in \{7, 9\}$. In all the cases from lemma 2.2.11 we have $DG = 0$ and, from (2.15), we find

$$AK_S - 2AR_0 - \sum_{i=1}^{h_1} a_i = 6g - 12 \tag{2.19}$$

When $A^2 = 7$ we have $AK_S = 3$, $AR_0 \leq 2$, $D = E_1$ and

$$2 - 2AR_0 = 3 - 2AR_0 - 1 = 6g - 12$$

hence $AR_0 = 1, g = 2, A'K_Y = 0$. But remarks 2.2.5 and 2.2.4 force $AR_0 = 0$. Contradiction.

From remark 2.2.5 we can easily see that $A^2 = 9$ is impossible. This is because from lemmas 2.2.6, 2.2.7 and 2.2.8 either A has a triple point at q or at the points p_j, and then $A'^2 = 0$ or it has a node at q when $A'^2 = 1$. □

Remark 2.2.16. *There is only one possibility left out by propositions 2.2.13, 2.2.14 and 2.2.15. This is the case $A^2 = 9, D = 0$ or, equivalently, $A' = N$. Then from lemma 2.1.5 we know $g = 3, A'K_Y = NK_Y = 1$.*

Corollary 2.2.17. *In the above setting, when $D = 2F + G + H + \sum_i a_i E_i$ we find $A'H' = 0$.*

Proof. It follows by a very simple observation:

$$\begin{aligned} 3A'H' &= \pi^*(A')\pi^*(H') = \widetilde{A}3H = 3(\varepsilon^*(A) - D)H \\ &= -3DH = -3(2F + G + H + \sum_i a_i E_i)H = 0. \end{aligned}$$

□

2.3 Adjoint systems to the pencil $|N|$

We also state here some properties of the adjoint system $|K_Y + N|$ which will be useful later.

We know that $h^2(Y, \mathcal{O}_Y) = 0$, so Y is a regular surface, and that we have a linear system $|N|$ of nef and big curves on Y. If we look at the adjoint system $|N + K_Y|$ from the short exact sequence

$$0 \to \mathcal{O}_Y(-N) \to \mathcal{O}_Y \to \mathcal{O}_N \to 0$$

using that N is nef and big we have

$$h^0(Y, \mathcal{O}_Y(-N)) = h^1(Y, \mathcal{O}_Y(-N)) = 0$$

and

$$h^0(Y, \mathcal{O}_Y(K_Y + N)) = h^2(Y, \mathcal{O}_Y(-N)) = h^1(N, \mathcal{O}_N) = p_a(N) = 3 - R_0K_S$$

Then $|N+K_Y|$ is a linear system of curves with arithmetic genus given by the formulas (see also lemma 2.1.5)

$$(N+K_Y)^2 = N^2 + K_Y^2 + 2NK_Y = 5 + K_Y^2 - 4R_0K_S \tag{2.20}$$

$$(N+K_Y)K_Y = K_Y^2 + NK_Y = K_Y^2 + 1 - 2R_0K_S$$

$$p_a(N+K_Y) = 1 + \frac{(N+K_Y)(N+2K_Y)}{2} = 4 + K_Y^2 - 3R_0K_S$$

Remark 2.3.1. *We observe that $N+K_Y$ is not nef. In fact*

$$(N+K_Y)G'_i = K_YG'_i = -1$$

From lemma 1.1.9 if $N+K_Y$ is not nef then every irreducible curve Z such that $Z(N+K_Y) < 0$ is a (-1)-curve with $ZN = 0$. By contracting the curves and repeating the above argument we can see that after contracting each (-1)-cycle on Y such that $ZN = 0$ we get a surface on which N and its adjoint are both nef divisors.

Lemma 2.3.2. *The number n of (-1)-cycles Z on Y different from the ones of G' for which $ZN = 0$ is greater or equal than*

$$\frac{35}{6}R_0K_S - \frac{3}{2}{R_0}^2 - \frac{10+2h_2}{3}.$$

Proof. Let Z be such a cycle. Then for any other (-1)-cycle Z' that does not intersect N we have by the Index Theorem 1.1.10

$$(Z+Z')^2 = -1 - 1 + 2ZZ' < 0 \quad \Rightarrow \quad ZZ' = 0$$

In particular Z does not intersect any curve G'_i.

Then

$$0 = ZN = Z(3K_Y + 2B_0 + E' - 3G') = -3 + 2B_0Z + E'Z \tag{2.21}$$

and there is a (-3)-curve E'_i that intersects Z in at least 1 point. Moreover since $E'_iN = 0$ we have

$$(Z+E'_i)^2 = -1 - 3 + 2ZE'_i = 2(ZE'_i - 2) < 0$$

$$(Z-E'_i)^2 = -1 - 3 - 2ZE'_i = -2(ZE'_i + 2) < 0$$

hence $-1 \leq ZE'_i \leq 1$ for all $i = 1, \dots, h_1$.

We know that $N_1 := N + K_Y - \sum_{i=1}^n Z_i - G'$ is nef and, by lemma 2.1.5,

$$\begin{aligned}
0 &\leq (N + K_Y - \sum_{i=1}^n Z_i - G')^2 \\
&= (N + K_Y)^2 - n - h_2 - 2(N + K_Y)(\sum_{i=1}^n Z_i + G') \\
&= 5 + K_Y^2 - 4R_0K_S - n - h_2 + 2n + 2h_2 = 5 - 4R_0K_S + K_Y^2 + n + h_2 \\
&\overset{(2.10)}{=} 5 - 4R_0K_S + \frac{1}{3}[-5 - h_2 + \frac{9}{2}{R_0}^2 - \frac{11}{2}R_0K_S] + n + h_2 \\
&= \frac{10 + 2h_2}{3} + \frac{3}{2}{R_0}^2 - \frac{35}{6}R_0K_S + n
\end{aligned}$$

□

Let us set $N_1 := N + K_Y - G' - \sum_{i=1}^n Z_i$.

2.3.1 The (-1)-cycles Z_i

We now analyse the irreducible components of the above (-1)-cycles Z_i.

Proposition 2.3.3. *In the above setting each irreducible component of the (-1)-cycles Z is a curve C such that $CN = CN_1 = 0$.*

Proof. Since we have $ZN = ZN_1 = 0$ and N and N_1 are nef, for any irreducible component C of Z we find

$$0 \leq CN_1 \leq ZN_1 = 0, \quad 0 \leq CN \leq ZN = 0$$

and the result is proved. □

Corollary 2.3.4. *The curves F'_j and H'_j satisfy $F'_jN_1 = H'_jN_1 = 0$ for any $j = 1, \ldots, h_2$. In particular $F'_j \sum_{i=1}^n Z_i = H'_j \sum_{i=1}^n Z_i = 0$.*

Proof. Since F'_j and H'_j are (-3)-curves such that $F'_jN = 0 = H'_jN$ we find

$$F'_jN_1 = F'_j(N + K_Y - G' - \sum_{i=1}^n Z_i) = 1 - 1 - \sum_{i=1}^n Z_iF'_j = -\sum_{i=1}^n Z_iF'_j \geq 0$$

and the same holds for H'_j.

If the curve F'_j is contained in one of the (-1)-cycles Z_i we have $F'_jN_1 = 0$ from proposition 2.3.3, otherwise we have $F'C \geq 0$ for any irreducible component C of the cycles Z_i. Then

$$0 \leq \sum_{i=1}^{n} Z_i F'_j \leq 0$$

and the result is proved. □

Corollary 2.3.5. *For any irreducible curve E'_k or B_{0k} we find*

$$E'_k \sum_{i=1}^{n} Z_i \geq 0, \quad B_{0k} \sum_{i=1}^{n} Z_i \geq 0.$$

Proof. The statement is obvious if E'_k or B_{0k} are not contained in any of the (-1)-cycles Z_i since then they have non-negative intersection with any irreducible component C of the curves Z_i. On the other hand if E'_k is contained in some (-1)-cycles, from proposition 2.3.3 we find

$$0 = E'_kN_1 = E'_k(N + K_Y - G' - \sum_{i=1}^{n} Z_i) = 1 - E'_k \sum_{i=1}^{n} Z_i$$

hence $E'_k \sum_{i=1}^{n} Z_i = 1$. Analogously if B_{0k} is contained is some cycle Z_{i_0}, then $B_{0k}N = 0$ and then B_{0k} is a (-6)-curve on Y. Hence we find

$$0 = B_{0k}N_1 = B_{0k}(N + K_Y - G' - \sum_{i=1}^{n} Z_i) = 4 - B_{0k} \sum_{i=1}^{n} Z_i$$

as wanted. □

Let us now consider an irreducible (-1)-curve C in a cycle. Recall, from the proof of lemma 2.3.2 and from (2.21), that there is a curve E'_i such that $CE'_i = 1$. On the other hand, $(\pi^*(C))^2 = 3C^2 = -3$ and, for each E'_i with $CE'_i = 1$ and C as above,

$$3 = \pi^*(CE'_i) = 3\pi^*(C)E_i \quad \Rightarrow \quad \pi^*(C)E_i = 1$$

Moreover both C and E'_i are irreducible and $C^2 = -1$ while $E'^2_i = -3$. Thus it cannot be $E_i \leq \pi^*(C)$. In particular $\pi^*(C)$ cannot be singular at the point $\pi^*(C) \cap E_i$ (otherwise we should have $\pi^*(C)E_i \geq 2$).

Lemma 2.3.6. *If C is an irreducible (-1)-curve such that $CN = 0$ and $CF'_j = CH'_j = 0$ ($j = 1, \dots, h_2$) then $\pi^*(C)$ is a rational curve and $\pi^*(C)^2 = -3$.*

Proof. Suppose that $\pi^*(C) = C_1 + C_2 + C_3$ is the union of three distinct curves and consider the curve E_i above. Then $C_iC_j \geq 1$ for $i \neq j$ since the point $\pi^*(Z) \cap C_i$ is fixed for σ . Each component of $\pi^*(C)$ is a rational curve, so

$$-2 = 2p_a(C_i) - 2 = C_i(C_i + K_X)$$

Since the intersection of the components C_i is fixed under the action of σ , we should have, for each i such that E_i' intersect C,

$$1 = \pi^*(C)E_i = (C_1 + C_2 + C_3)E_i = 3C_1E_i$$

Contradiction. It follows that $\pi^*(C)$ is an irreducible curve. We now want to show that $p_g(\pi^*(C)) = 0$. From Hurwitz formula we have

$$2p_g(\pi^*(C)) - 2 = -2 \cdot 3 + 2r$$

where r is the number of ramification points of the triple cover $\pi^*(C)^\nu \longrightarrow C$. We have $2r = 2p_g(\pi^*(C)) + 4 \geq 4$, so $r \geq 2$. On the other hand r is not greater than the number of intersection points of C with $B_0 + E' + F' + H'$. We have $CF' = CH' = 0$ and from (2.21) either

$$CB_0 = CE' = 1$$

or

$$CB_0 = 0 \quad CE' = 3.$$

Furthermore

$$\pi^*(C)K_X = \pi^*(C)(\pi^*(K_Y) + 2R) = -3 + 2CB_0 + 2CE'$$

In the former case, $r = 2$ and $\pi^*(C)^\nu$ is a smooth rational curve. In the latter case $r = 2, 3$. If $r = 2$ then $\pi^*(C)$ has geometric genus 0 and it has a singular point in $\pi^*(C) \cap E$, and this is a contradiction since $\pi^*(C)E_i \leq 1$, $i = 1, \ldots, h_1$.

When $r = 3$, instead, since $p_a(\pi^*(C)) = p_g(\pi^*(C)) = 1$, $\pi^*(C)$ should be a smooth elliptic curve. When we look at the image $\varepsilon(\pi^*(C))$ of this curve on S, since $CE' = 3$ (recall from the proof of lemma 2.3.2 that $0 \leq CE_i' \leq 1$ for any $i = 1, \ldots, h_1$) we would have $\varepsilon(\pi^*(C))^2 = \pi^*(C)^2 - 3 = 0$, and since it is an elliptic curve, $K_S\,\varepsilon(\pi^*(C)) = 0$. This is impossible since S is a minimal surface of general type. □

Corollary 2.3.7. *For any curve C as above $CB_0 = CE' = 1$.*

We now want to determine the composition of the reducible (-1)-cycles.

Lemma 2.3.8. *The curves G'_j cannot be contained in one of the cycles Z_i, $i = 1, \dots, n$.*

Proof. If one of the cycles Z_i, say Z_{i_0} contains a curve G'_j then from corollary 2.3.4 we have

$$0 = F'_j \sum_{i=1}^{n} Z_i = F'_j G'_j + F'_j (Z_{i_0} - G'_j) + F'_j \sum_{i \neq i_0} Z_i = 1 + F'_j (Z_{i_0} - G'_j) + F'_j \sum_{i \neq i_0} Z_i$$

hence F'_j is contained either in Z_{i_0} or in another cycle Z_i with $i \neq i_0$. In this latter case we have

$$0 = G'_j Z_i = G'_j (Z_i - F'_j) + G'_j F'_j = G'_j (Z_i - F'_j) + 1$$

hence G'_j is also contained in Z_i.

Then there exists a cycle containing both G'_j and F'_j. The same argument holds for H'_j. In particular F'_j and H'_j are both contracted to make the adjoint divisor to N a nef divisor.

When we contract the curve G'_j the images of F'_j and H'_j are two (-2)-curves meeting at one point. Since they are both contracted there is a (-1)-cycle C intersecting at least one of them at one point. If C passes through the intersection point of the (-2)-curves, then by contracting C we obtain a cycle which is composed of two (-1)-curves meeting at one point. In particular this cycle is effective with self-intersection 0 and it does not intersect the image $\bar{N}$ of N contradicting the Index theorem 1.1.10.

This implies that C is a (-1)-cycle which intersects at one point only one of the curves F'_j or H'_j. We will assume without loss of generality $CF'_j = 1$.

We show the lemma by reducing ourselves to the case when C is an irreducible (-1)-curve hence $C = Z_1$. This is always possible after the contraction of a suitable number of (-1)-curves. In this case we have the configurations of figure 2.1 hence $n \geq 3$. Moreover we have

$$Z_1 N = Z_1(3K_Y + 2B_0 + E' - 3G') = -3 + 2B_0 Z_1 + E' Z_1$$

Since Z_1 is irreducible then either $B_0 Z_1 = 0, E' Z_1 = 3$ or $B_0 Z_1 = 1, E' Z_1 = 1$. By the Index theorem, since $E'_k N = Z_i N = 0$ for all $k = 1, \dots, h_1, i =$

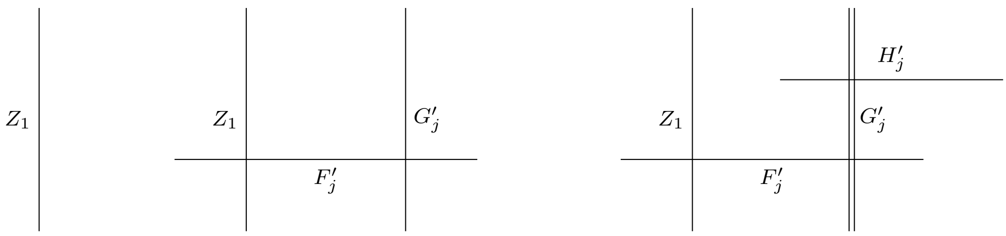


Figure 2.1:

$1, \dots, n$, we have $-1 \leq E'_k Z_i \leq 1$. For any curve E'_k such that $E'_k Z_1 = 1$ we find (see also corollary 2.3.5)

$$0 \leq E'_k \sum_{i=1}^{n} Z_i = 1 - E'_k N_1 \leq 1$$

and

$$E'_k \sum_{i=1}^{n} Z_i = 3E'_k Z_1 + E'_k (\sum_{i \geq 4} Z_i) = 3 + E'_k (\sum_{i \geq 4} Z_i) \leq 1$$

Then E'_k is contained in some (-1)-cycle Z_i $i \geq 4$. Then E'_k is contracted too and one of the cycles has the configuration of figure 2.2.

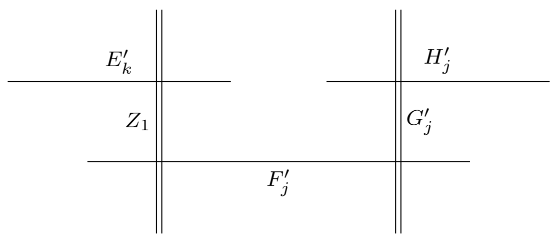


Figure 2.2:

When we contract the curves Z_1 and G'_j the images of E'_k and H'_j are (-2)-curves while the image of F'_j is a (-1)-curve intersecting them at one point. Hence when we contract the (-1)-curve we obtain two (-1)-curve meeting at one point. This new configuration has self-intersection 0 and cannot be contracted to a point. Thus we get a contradiction and the curve G'_j cannot be contained in a cycle. $\square$

Corollary 2.3.9. *The curves F'_j and H'_j are not contained in any of the (-1)-cycles Z_i.*

Proof. If a curve F'_j (or H'_j) is contained in a cycle Z_i then

$$0 = G'_j Z_i = G'_j(Z_i - F'_j) + G'_j F'_j = G'_j(Z_i - F'_j) + 1$$

hence G'_j is also contained in Z_i. This contradicts lemma 2.3.8. $\square$

Lemma 2.3.10. *There is no cycle $Z_i, 1 \leq i \leq n$ containing at least two curves E'_k.*

Proof. Let us assume that two of the curves E'_k, say E'_1 and E'_2 are contained in a reducible cycle Z_{i_0}. Then $E'_k N_1 = 0$ implies $E'_k \sum_i Z_i = 1$ and there are two (-1)-cycles Z_1 and Z_2 such that $E'_1 Z_1 = 1$, $E'_2 Z_2 = 1$.

Then we have the configuration of figure 2.3 where C is a suitable cycle. One

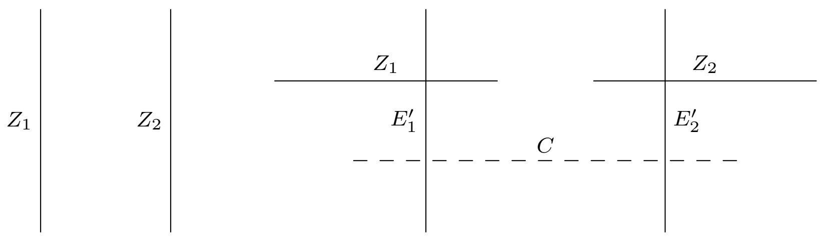


Figure 2.3:

can easily see that, in order to contract E'_1 (and analogously E'_2), the configurations of figure 2.4 are (-1)-cycles. Then we have

$$-1 = (C + E'_2 + Z_2)^2 = C^2 - 3 - 1 + 2CE'_2 + 2 = C^2 + 2CE'_2 - 2$$

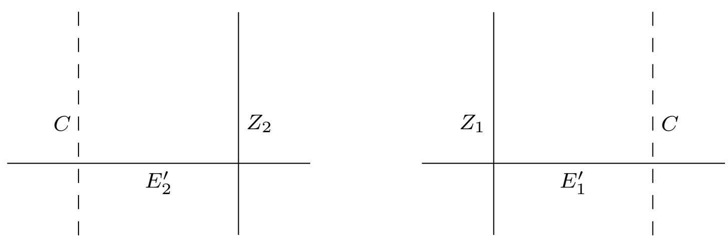


Figure 2.4:

hence

$$C^2 + 2CE_2' = 1 \tag{2.22}$$

and, analogously,

$$C^2 + 2CE_1' = 1 \tag{2.23}$$

Moreover

$$\begin{aligned} -1 = Z_3^2 &= (Z_1 + E_1' + C + E_2' + Z_2)^2 \\ &= -1 - 3 + C^2 - 3 - 1 + 2 + 2CE_1' + 2CE_2' + 2 \\ &= -4 + C^2 + 2CE_1' + 2CE_2' \overset{(2.22)}{=} -3 + 2CE_2' \end{aligned}$$

Thus $CE_2' = 1 = CE_1', C^2 = -1$. Then C is a (-1)-cycle not intersecting N hence by the Index theorem 1.1.10 we should have $CZ_3 = 0$. But

$$CZ_3 = C(Z_1 + E_1' + C + E_2' + Z_2) = 0 + 1 + C^2 + 1 = 2 - 1 = 1$$

and we get a contradiction. □

Corollary 2.3.11. *If there is a reducible cycle Z_{i_0}, then $n \geq 3$ and for $n = 3$ we have one of the following possibilities:*

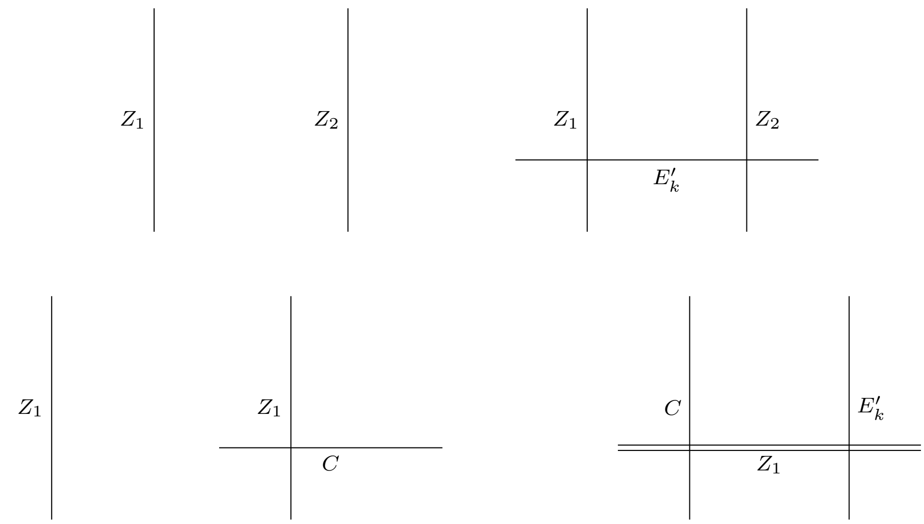
1. *two irreducible (-1)-curves Z_1 and Z_2 and $Z_3 = Z_1 + Z_2 + E_k'$ where $E_k'Z_1 = E_k'Z_2 = 1$;*
2. *only one irreducible (-1)-curve Z_1, $Z_2 = Z_1 + C$, $Z_3 = C + 2Z_1 + E_k'$ where C is a (-2)-curve intersecting Z_1 at one point and E_k' is such that $E_k'Z_1 = 1$.*

Proof. From lemmas 2.3.8 and 2.3.10 if $n \leq 2$ a reducible (-1)-cycle can contain at most one curve E_k' and it does not contain any curve G_j'. Hence there is at least an irreducible curve Z_1. Then for $n = 1$ the result is proved. For $n = 2$ if Z_2 was reducible then $Z_2 \geq Z_1$. Then there exists a curve E_k' which intersects Z_1 at one point and

$$E_k'(Z_1 + Z_2) = E_k'(2Z_1 + (Z_2 - Z_1)) = 2 + E_k'(Z_2 - Z_2) \leq 1$$

This is only possible when E_k' is contained in Z_2. But then there is at least another (-1)-cycle Z_3 which intersects E_k' at one point and which does not intersect N contradicting the assumption $n = 2$.

When $n = 3$ we can apply the above argument and we can see that if Z_3 is a reducible cycle then there is at least an irreducible (-1)-cycle Z_1. Hence, we have one of the following configurations:



where C is an irreducible (-2)-curve. □

2.3.2 The linear systems $|N_i|$

We now compute the arithmetic genus of N_1: from equation (2.20) of page 47 we know that $N_1^2 = 5 - 4R_0K_S + K_Y^2 + n + h_2$ while from lemma 2.1.5

$$\begin{aligned} N_1K_Y &= (N + K_Y - \sum_{i=1}^{n} Z_i - G')K_Y = NK_Y + K_Y^2 + n + h_2 \\ &= 1 - 2R_0K_S + K_Y^2 + n + h_2 \end{aligned}$$

so

$$\begin{aligned} p_a(N_1) &= 1 + \frac{N_1^2 + N_1K_Y}{2} \\ &= 1 + \frac{5 - 4R_0K_S + K_Y^2 + n + h_2 + 1 - 2R_0K_S + K_Y^2 + n + h_2}{2} \\ &= 4 - 3R_0K_S + K_Y^2 + n + h_2 \leq N_1^2 \end{aligned} \tag{2.24}$$

since $0 \leq R_0K_S \leq 1$. $|N_1|$ is again a linear system of nef curves, so when $p_a(N_1) \geq 1$ we can apply the same argument as in page 47 to study the adjoint system $|N_1 + K_Y|$. Under this hypotheses N_1 is nef and big and we find again $h^0(Y, \mathcal{O}_Y(N_1 + K_Y)) = p_a(N_1) \geq 1$. $N_1 + K_Y$ is not nef since the curves Z_i and

G_i' do not intersect N_1, but there could be some other (-1)-cycles Z_i' such that $Z_i'N_1 = 0$ (see lemma 1.1.9). Then

$$N_2 := N_1 + K_Y - \sum_{i=1}^{n} Z_i - G' - \sum_{j=1}^{n'} Z_j'$$

is nef and

$$\begin{aligned} N_2^2 &= (N_1 + K_Y - \sum_{i=1}^{n} Z_i - G' - \sum_{j=1}^{n'} Z_j')^2 \\ &= N_1^2 + K_Y^2 + 2N_1K_Y - n - h_2 - n' + 2n + 2h_2 + 2n' \\ &= 5 - 4R_0K_S + K_Y^2 + n + h_2 + K_Y^2 + 2(1 - 2R_0K_S + K_Y^2 + n + h_2)+ \\ &+ n + h_2 + n' \\ &= 7 - 8R_0K_S + 4K_Y^2 + 4n + 4h_2 + n' \geq 0 \end{aligned}$$

$$\begin{aligned} N_2K_Y &= (N_1 + K_Y - \sum_{i=1}^{n} Z_i - G' - \sum_{j=1}^{n'} Z_j')K_Y = N_1K_Y + K_Y^2 + n + h_2 + n' \\ &= 1 - 2R_0K_S + K_Y^2 + n + h_2 + K_Y^2 + n + h_2 + n' \\ &= 1 - 2R_0K_S + 2K_Y^2 + 2n + 2h_2 + n' \end{aligned}$$

and then

$$\begin{aligned} p_a(N_2) &= 1 + \frac{N_2^2 + N_2K_Y}{2} \\ &= 1 + \frac{7 - 8R_0K_S + 4K_Y^2 + 4n + 4h_2 + n'}{2} + \\ &+ \frac{1 - 2R_0K_S + 2K_Y^2 + 2n + 2h_2 + n'}{2} \\ &= 5 - 5R_0K_S + 3K_Y^2 + 3n + 3h_2 + n' \end{aligned} \tag{2.25}$$

We also remark that

$$\begin{aligned} NN_1 &= N(N + K_Y - \sum_{i=1}^{n} Z_i - G') = N(N + K_Y) \\ &= 2p_a(N) - 2 = 6 - 2R_0K_S - 2 = 4 - 2R_0K_S \end{aligned} \tag{2.26}$$

$$N_1N_2 = N_1(N_1 + K_Y - \sum_{i=1}^{n} Z_i - G' - \sum_{j=1}^{n'} Z_j') = N_1(N_1 + K_Y) \tag{2.27}$$

$$
\begin{aligned}
&= 2p_a(N_1) - 2 = 8 - 6R_0K_S + 2K_Y^2 + 2n + 2h_2 - 2 \\
&= 6 - 6R_0K_S + 2K_Y^2 + 2n + 2h_2
\end{aligned}
$$

Moreover when $p_a(N_2) \geq 1$ and $N_2^2 > 0$ we can repeat the above argument. We find $h^0(Y, \mathcal{O}_Y(N_2 + K_Y)) = p_a(N_2)$ and there are (-1)-cycles Z_i'' such that $Z_i''N_2 = 0$ and

$$N_3 := N_2 + K_Y - G' - \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n'} Z_i' - \sum_{i=1}^{n''} Z_i''$$

is nef. Let us also compute self-intersection and arithmetic genus of N_3:

$$
\begin{aligned}
N_3^2 &= (N_2 + K_Y - G' - \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n'} Z_i' - \sum_{i=1}^{n''} Z_i'')^2 \qquad (2.28)\\
&= N_2^2 + K_Y^2 + 2N_2K_Y + h_2 + n + n' + n'' \\
&= 7 - 8R_0K_S + 4K_Y^2 + 4n + 4h_2 + n' + K_Y^2 + \\
&+ 2(1 - 2R_0K_S + 2K_Y^2 + 2n + 2h_2 + n') + h_2 + n + n' + n'' \\
&= 9 - 12R_0K_S + 9K_Y^2 + 9h_2 + 9n + 4n' + n''
\end{aligned}
$$

$$
\begin{aligned}
N_3K_Y &= (N_2 + K_Y - G' - \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} Z_i' - \sum_{i=1}^{n} Z_i'')K_Y \qquad (2.29)\\
&= N_2K_Y + K_Y^2 + h_2 + n + n' + n'' \\
&= 1 - 2R_0K_S + 2K_Y^2 + 2n + 2h_2 + n' + K_Y^2 + h_2 + n + n' + n'' \\
&= 1 - 2R_0K_S + 3K_Y^2 + 3h_2 + 3n + 2n' + n''
\end{aligned}
$$

$$
\begin{aligned}
p_a(N_3) &= 1 + \frac{N_3^2 + N_3K_Y}{2} \\
&= 1 + \frac{9 - 12R_0K_S + 9K_Y^2 + 9h_2 + 9n + 4n' + n''}{2} + \\
&+ \frac{1 - 2R_0K_S + 3K_Y^2 + 3h_2 + 3n + 2n' + n''}{2} \\
&= 6 - 7R_0K_S + 6K_Y^2 + 6h_2 + 6n + 3n' + n''
\end{aligned}
$$

Moreover

$$N_2N_3 = 2p_a(N_2) - 2 = 2(5 - 5R_0K_S + 3K_Y^2 + 3n + 3h_2 + n') - 2$$

$$= 8 - 10R_0K_S + 6K_Y^2 + 6h_2 + 6n + 2n' \geq 0$$

We can collect all these computations in the following proposition

Proposition 2.3.12. *In the above setting let us define $N_0 := N$. Then the numerical data of the curves N_1, N_2, N_3 are:*

	$i = 1$	$i = 2$
N_i^2	$5 - 4R_0K_S + K_Y^2 + n + h_2$	$7 - 8R_0K_S + 4K_Y^2 + 4n + 4h_2 + n'$
N_iK_Y	$1 - 2R_0K_S + K_Y^2 + n + h_2$	$1 - 2R_0K_S + 2K_Y^2 + 2n + 2h_2 + n'$
$p_a(N_i)$	$4 - 3R_0K_S + K_Y^2 + n + h_2$	$5 - 5R_0K_S + 3K_Y^2 + 3n + 3h_2 + n'$
$N_{i-1}N_i$	$4 - 2R_0K_S$	$6 - 6R_0K_S + 2K_Y^2 + 2n + 2h_2$

	$i = 3$
N_i^2	$9 - 12R_0K_S + 9K_Y^2 + 9h_2 + 9n + 4n' + n''$
N_iK_Y	$1 - 2R_0K_S + 3K_Y^2 + 3h_2 + 3n + 2n' + n''$
$p_a(N_i)$	$6 - 7R_0K_S + 6K_Y^2 + 6h_2 + 6n + 3n' + n''$
$N_{i-1}N_i$	$8 - 10R_0K_S + 6K_Y^2 + 6h_2 + 6n + 2n'$

Chapter 3

Numerical analysis of the possible cases

3.1 Case (i): $R_0K_S = 1$, $h_2 = 3$

Lemma 3.1.1. *In case (i) we find*

(a) $h_1 \geq 1$

(b) no positive multiple of $K_S - 2R_0$ *is an effective divisor.*

Proof. (a) Since $R_0K_S = 1$, there exists a unique irreducible component Γ of R_0 for which $\Gamma K_S = 1$. Using the index theorem we find $\Gamma^2 \leq 1$. The other irreducible components of R_0 are (-2)-curves. Then

$$R_0{}^2 \leq \Gamma^2 \leq 1$$

But now $h_2 = 3$ therefore from (2.5)

$$h_1 = \frac{3R_0K_S - R_0{}^2}{2} \geq 1.$$

(b) Note that $K_S(K_S - 2R_0) = -1$ while K_S is nef.

□

Lemma 3.1.2. *Suppose case (i) holds and* R_0 *is the disjoint union of an irreducible component* Γ *with* $\Gamma K_S = 1$ *and of* ℓ (-2)*-curves. Then*

$$K_Y^2 = -4 - 3\ell + \frac{3\Gamma^2 - 1}{2} \leq -3.$$

Proof. It is an easy computation which uses formula (2.10) of page 29:

$$\begin{aligned} K_Y^2 &= \frac{1}{3}({K_S}^2 - 6 - h_2 - \frac{11}{2}R_0K_S + \frac{9}{2}{R_0}^2) = \frac{1}{3}(-8 - \frac{11}{2} + \frac{9}{2}\Gamma^2 - 9\ell) \\ &= \frac{1}{3}(-12 + \frac{9\Gamma^2 - 3}{2} - 9\ell) = -4 - 3\ell + \frac{3\Gamma^2 - 1}{2}. \end{aligned}$$

□

Lemma 3.1.3. *In the above setting we have* $0 \leq \ell \leq (5 + \Gamma^2)/2$.

Proof. Since $\pi : X \longrightarrow Y$ is a surjective map we have an injection $H^2(Y, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$. In particular we find $e(X) = 12 - K_X^2 \geq e(Y) = 12 - K_Y^2$ hence $K_Y^2 \geq K_X^2$. Thus from lemma 3.1.2 and lemma 3.1.1

$$-4 - 3\ell + \frac{3\Gamma^2 - 1}{2} = K_Y^2 \geq K_X^2 = {K_S}^2 - (h_1 + 3h_2) = 1 - \frac{3 - \Gamma^2}{2} - \ell - 9$$

hence

$$2\ell \leq -4 + 9 - 1 + \frac{3\Gamma^2 - 1}{2} + \frac{3 - \Gamma^2}{2} = 5 + \Gamma^2$$

as wanted. □

As an immediate consequence of the above lemma and of lemma 3.1.2 we find

Corollary 3.1.4. *In the above setting we have* $K_Y^2 \geq -12$.

Corollary 3.1.5. *In the above setting we have* $h_1 \leq 4$.

Proof. We have

$$h_1 = \frac{3R_0K_S - {R_0}^2}{2} = \frac{3 - \Gamma^2}{2} + \ell \leq \frac{3 - \Gamma^2}{2} + \frac{5 + \Gamma^2}{2} = 4.$$

□

Proposition 3.1.6. *Assume case (i) holds. Then Y is a rational surface.*

Proof. By Castelnuovo's criterion 1.1.12, since $q(Y) \leq q(X) = q(S) = 0$, we need to show that $P_2(Y) = h^0(Y, \mathcal{O}_Y(2K_Y)) = 0$. We have

$$\begin{aligned} \pi^*(2K_Y - 2G') &= 2K_X - 4R - 2G = \varepsilon^*(2K_S) + 2E + 4F + 2G + 4H + \\ &\quad - (4\varepsilon^*(R_0) + 4E + 4F + 4H) - 2G = \varepsilon^*(2K_S - 4R_0) - 2E \end{aligned}$$

Since $2K_YG_i' = -2 < 0$, we find

$$0 \leq h^0(Y, \mathcal{O}_Y(2K_Y)) = h^0(Y, \mathcal{O}_Y(2K_Y - 2G')) \leq$$

$$\leq h^0(X, \mathcal{O}_X(\varepsilon^*(2K_S - 4R_0) - 2E))$$

But using lemma 3.1.1 we find

$$h^0(X, \mathcal{O}_X(\varepsilon^*(2K_S - 4R_0) - 2E)) \leq h^0(X, \mathcal{O}_X(\varepsilon^*(2K_S - 4R_0))) = \\ = h^0(S, \mathcal{O}_S(2K_S - 4R_0)) = 0$$

and then

$$h^0(Y, \mathcal{O}_Y(2K_Y)) = h^0(Y, \mathcal{O}_Y(2K_Y - 2G')) = 0.$$

□

Corollary 3.1.7. *In the above setting if* $\Gamma^2 = 1$ *then* $\mathrm{Tors}(S) = \mathbb{Z}/3\mathbb{Z}$*.*

Proof. Since $\Gamma K_S = 1 = K_S{}^2$ from the Index theorem 1.1.10 we find $\Gamma^2 \leq 1$.

If it was $\Gamma^2 = 1$ then we should have $\Gamma \sim K_S$ and then $3\Gamma \sim 3K_S$. Let us set $\Gamma' := \pi(\varepsilon^*(\Gamma))$. Then on Y we have

$$\Gamma' \sim N$$

which implies $\Gamma' \equiv N$ since Y is rational.

The pull-back π^* of $\pi : X \longrightarrow Y$ is injective since π is a surjective map and Y is a rational surface. Using lemma 2.1.5 we find

$$\varepsilon^*(3\Gamma) \equiv \pi^*(\Gamma') \equiv \pi^*(N) \equiv \varepsilon^*(3K_S).$$

Thus S has a non-trivial 3-torsion element and from theorem 1.5.5 we have $\mathrm{Tors}(S) = \mathbb{Z}/3\mathbb{Z}$. □

When $\Gamma^2 \neq 1$ we can show a weaker result.

Proposition 3.1.8. *If case (i) holds the numerical Godeaux surface S has no 2-torsion element.*

Proof. From theorem 1.5.5 we know that $\mathrm{Tors}(S)$ is a cyclic group of order $s \leq 5$. If a 2-torsion element η exists, then it is unique since a cyclic group of order 2 or 4 has one and only one element of order two.

Then S has an unramified double cover S'. Moreover S' has an involution τ such that $S = S'/ < \tau >$. Since η is the only 2-torsion element of S it is fixed under the action of the order three automorphism σ. Then we can extend

σ to an automorphism of order 3 $\widetilde{\sigma}$ on S' such that $\widetilde{\sigma}\tau = \tau\widetilde{\sigma}$. Then we blow up on S' the isolated fixed points for the action of $\widetilde{\sigma}$. The new surface X' is clearly an unramified double cover of our surface X. It is easy to see, since $\gcd(2,3) = 1$, that the quotient $Y' = X'/(\mathbb{Z}/3\mathbb{Z})$ is also an unramified double cover of Y. But then we get a contradiction since Y is a rational surface ad it has no torsion element. □

From now on we use the same notation as in lemma 2.3.2. From proposition 2.3.12 we have

$$p_a(N_1) = 4 - 3R_0K_S + K_Y^2 + n + h_2 = 4 + K_Y^2 + n = N_1^2 \tag{3.1}$$

$$NN_1 = 4 - 2R_0K_S = 2 \tag{3.2}$$

Lemma 3.1.9. *In the above setting we have $N_1^2 = 0, 1$.*

Proof. Since $N^2 = 3$ we have by the Index Theorem 1.1.10

$$0 \geq (3N_1 - 2N)^2 = 9N_1^2 + 4N^2 - 12N_1N \overset{(3.2)}{=} 9N_1^2 + 12 - 24 = 9N_1^2 - 12$$

which implies $N_1^2 = 0, 1$. □

Let us write $|N_1| = |\Delta| + T$, where $|\Delta|$ is the movable part and T is the fixed part of $|N_1|$. Since N_1, Δ, N are nef divisors,

$$0 \leq \Delta N \leq N_1N = 2$$

In particular it cannot be $\Delta N = 0$ otherwise, by the Index Theorem 1.1.10 and the rationality of Y, $\Delta = 0$ whereas $h^0(Y, \mathcal{O}_Y(\Delta)) = h^0(Y, \mathcal{O}_Y(N_1)) = 2$. Thus

$$1 \leq \Delta N \leq N_1N = 2$$

Lemma 3.1.10. *Suppose $N_1^2 = 0$. Then $|N_1|$ has no fixed part.*

Proof. We have $0 = N_1^2 = N_1\Delta + N_1T$ or, equivalently,

$$0 = N_1\Delta = \Delta^2 + \Delta T$$
$$0 = N_1T = \Delta T + T^2$$

which implies $\Delta^2 = \Delta T = T^2 = 0$.

It cannot be $N\Delta = 1 = NT$: we obtain by the Index Theorem

$$0 \geq (\Delta - T)^2 = \Delta^2 + T^2 - 2\Delta T = 0 \quad \Rightarrow \quad \Delta \sim T$$

Since Y is a rational surface, this implies $\Delta \equiv T$ which is impossible.

So $N\Delta = NN_1 = 2$ and then

$$0 \geq (N_1 - \Delta)^2 = T^2 = 0$$

Again, by the rationality of Y we have $T \equiv 0$ and $|N_1|$ has no fixed part. □

Lemma 3.1.11. *Suppose* $N_1^2 = 1$. *Then* $|N_1|$ *has no fixed part unless* $\Delta^2 = 0, p_a(\Delta) = p_a(N_1) = 1$ *and either* $\Delta N = 1, N = N_1 + \Delta$ *or* $\Delta N = 2, N_1 = \Delta + Z_i$ *for some reducible* (-1)*-cycle* Z_i.

Proof. We know that $1 = N_1^2 = N_1\Delta + N_1T$.

It cannot be $N_1\Delta = 0$, otherwise by the Index Theorem 1.1.10 and the rationality of Y it should be $\Delta = 0$, which is impossible.

Then we have $N_1\Delta = 1$, $N_1T = 0$, and this implies $T^2 \leq 0$. When $T^2 = 0$ we see that $|N_1|$ has no fixed part, as wanted, whereas when T^2 is strictly negative, by

$$1 = N_1\Delta = \Delta^2 + \Delta T \tag{3.3}$$

$$0 = N_1T = \Delta T + T^2 \tag{3.4}$$

we find $\Delta T = 1$, $T^2 = -1$, $\Delta^2 = 0$. Then by (3.1)

$$N_1^2 = \Delta^2 + T^2 + 2\Delta T = 1 = p_a(N_1)$$

and $N_1K_Y = -N_1^2 = -1$. We now look at $N\Delta$.

If $N\Delta = 2$ we have

$$1 = N_1\Delta = (K_Y + N - \sum_{i=1}^{n} Z_i - G')\Delta = \Delta K_Y + 2 - (\sum_{i=1}^{n} Z_i + G')\Delta$$

which amounts to say

$$0 \leq (\sum_{i=1}^{n} Z_i + G')\Delta = 1 + \Delta K_Y$$

So the followings are true:

a) $\Delta K_Y \geq 0$

b) there exists a (-1)-curve C (which can be either one of the Z_i's or one of the G_i''s) which intersects Δ positively.

Using b) and the fact that $CN = TN = 0$, the Index theorem 1.1.10 implies

$$0 \geq (T-C)^2 = T^2 + C^2 - 2TC = -2 - 2TC$$

Then $TC \geq -1$, whereas $CN_1 = 0$ implies $CT \leq -1$. This forces $TC = -1$, $T \equiv C$.

But for any irreducible such curve C there is a curve D such that $DN_1 = 0$ which intersects C at one point, i.e. $D = E'_j$ for the Z_i and $D = F'_i$ for G'_i (see corollary 2.3.7), so that

$$0 = DN_1 = D\Delta + DC = D\Delta + 1$$

which contradicts the nefness of Δ. Then $N_1 \equiv \Delta + Z_i$ with Z_i a reducible (-1)-cycle.

We are now left with the case $N\Delta = 1$. Since $NN_1 = 2$ we have $NT = 1$. Moreover $N_1\Delta = 1$ and $\Delta^2 = 0$ by (3.3) and (3.4). From $(N_1 + \Delta)N = 3 = N^2$ we have

$$\begin{aligned} 0 \geq (N_1 + \Delta - N)^2 &= N_1^2 + \Delta^2 + N^2 + 2N_1\Delta - 2N_1N - 2N\Delta \\ &= 1 + 0 + 3 + 2 - 4 - 2 = 0 \end{aligned}$$

then $N \equiv N_1 + \Delta \equiv 2\Delta + T$. Then

$$\Delta K_Y = (N - N_1)K_Y = -1 + 1 = 0 \quad \Rightarrow \quad p_a(\Delta) = 1$$

as wanted. □

Corollary 3.1.12. *We have*

$$n = N_1^2 + 3\ell + \frac{1 - 3\Gamma^2}{2} \leq N_1^2 + 8 \leq 9.$$

Proof. From corollary 3.1.4 we know that $K_Y^2 \geq -12$. From (3.1) and lemma 3.1.9 we find

$$n = N_1^2 - 4 - K_Y^2 \leq N_1^2 - 4 + 12 = N_1^2 + 8 \leq 9$$

Moreover from lemma 3.1.2

$$n = N_1^2 - 4 - K_Y^2 = N_1^2 - 4 - (-4 - 3\ell + \frac{3\Gamma^2 - 1}{2}) = N_1^2 + 3\ell + \frac{1 - 3\Gamma^2}{2}.$$

□

Remark 3.1.13. *We note that from the above corollary*

$$n = N_1^2 + 3\ell + \frac{1-3\Gamma^2}{2} = N_1^2 + 3\ell + 3\frac{1-\Gamma^2}{2} - 1 \equiv N_1^2 - 1 \mod 3$$

In particular when $N_1^2 = 0$ we find $n \equiv 2 \mod 3$ hence $n = 2, 5$ or 8, while when $N_1^2 = 1$ we have $n \equiv 0 \mod 3$ hence $n = 3, 6$ or 9.

Lemma 3.1.14. *In the above setting the pencil $|\Delta|$ determines a fibration $\varphi_{|\Delta|} : Y \dashrightarrow \mathbb{P}^1$. Let us set*

$$\delta := \sum_s (e(\Delta_s) - e(\Delta))$$

where the sum is taken over all the singular curves $\Delta_s \in |\Delta|$. Then δ satisfies

$$18 \le \delta = 12 + 3N_1^2 + n + \Delta^2 \le 16 + n$$

In particular if $N_1^2 = 0$ then $n = 8$.

Proof. Away from its Δ^2 base points, the pencil $|\Delta|$ determines on Y a fibration over $\mathbb{P}^1$ of curves of genus $0 \le p_a(\Delta) = N_1^2 \le 1$. Computing Euler numbers from proposition 1.1.3 we find

$$12 - K_Y^2 + \Delta^2 = e(Y) + \Delta^2 = 2(2 - 2N_1^2) + \sum_s (e(\Delta_s) - e(\Delta))$$

where Δ_s are the singular curves of $|\Delta|$. Let us set

$$\delta := \sum_s (e(\Delta_s) - e(\Delta))$$

Then

$$\begin{aligned} \delta &= 12 - K_Y^2 + \Delta^2 - 4 + 4N_1^2 \\ &= 12 - (N_1^2 - 4 - n) + \Delta^2 - 4 + 4N_1^2 \\ &= 12 + 3N_1^2 + n + \Delta^2 \le 16 + n \end{aligned}$$

Recall that from lemma 1.1.4 a curve with self-intersection $-m$ in a singular fibre contributes m to δ.

Let us first consider the curves F_i' and H_i'. From lemma 2.3.2 we find

$$F_i'N_1 = F_i'(N + K_Y - \sum_{i=1}^{n} Z_i - G') = 0 + 1 + 0 - 1 = 0$$

and the same holds for H_i'.

Moreover if $N_1 \equiv \Delta + T$ with $T \neq 0$ from the above proof, since $N \equiv N_1 + \Delta$ we also find

$$0 = F_i' N = F_i'(N_1 + \Delta) = F_i' \Delta$$

and, again, the same holds for H_i'.

If $N_1 \equiv \Delta + Z_i$ instead we find from corollary 2.3.9

$$0 = F_i' N_1 = F_i'(\Delta + Z_i) \geq F_i' \Delta$$

hence $F_i' \Delta = H_i' \Delta = 0$ in any case.

Then from lemma 1.1.4 each of the curves F_i' and H_i' contributes 3 to δ, hence $\delta \geq 6h_2 = 18$.

If $N_1^2 = 0$ then $\Delta \equiv N_1$

$$18 \leq \delta = 12 + 3N_1^2 + n + \Delta^2 = 12 + n$$

hence $n \geq 6$. In particular from remark 3.1.13 we can deduce $n = 8$. □

Proposition 3.1.15. *Case (i) cannot occur with $N_1^2 = 0$.*

Proof. Let us assume $N_1^2 = 0$. Then from lemma 3.1.14

$$18 \leq \delta = 12 + n = 12 + 8 = 20$$

If all the eight cycles Z_i were irreducible then we should have $Z_i N_1 = 0$ hence from lemma 1.1.4 each of them would contribute 1 to δ. Thus $18 + n = 26 \leq \delta = 20$. Contradiction.

If one of the cycles is reducible, then from lemma 2.3.8, corollary 2.3.9 and lemma 2.3.10 there is a curve E_k' contained in that cycle and from lemma 1.1.4 E_k' increases δ by 3. Then

$$18 + 3 = 21 \leq \delta = 20$$

and, again, we get a contradiction. □

From remark 3.1.13 we know that when $N_1^2 = 1$ we have $n = 3, 6, 9$.

Proposition 3.1.16. *The case $|N_1| = |\Delta| + Z_i$ with Z_i reducible (-1)-cycle cannot occur.*

Proof. From the proof of lemma 3.1.11 we know that $\Delta N = NN_1 = 2, \Delta G' = 0$ and $\Delta^2 = \Delta K_Y = 0$. Hence

$$2 = N\Delta = \Delta(3K_Y + 2B_0 + E' - 3G') = 2B_0\Delta + E'\Delta$$

Since Δ is nef we find either $B_0\Delta = 1, E'\Delta = 0$ or $B_0\Delta = 0, E'\Delta = 2$. We recall that $B_0 = \Gamma' + \sum_{i=1}^{\ell} B_{0i}$ with $B_{0i}^2 = -6$, $B_{0i} \cong \mathbb{P}^1$.

Case I: $B_0\Delta = 1, E'\Delta = 0$.

Since none of the (-3)-curves E'_k intersects Δ, each of them contributes 3 to δ (see lemma 1.1.4). Moreover there is only one irreducible component of B_0 which intersects Δ.

If $\Gamma'\Delta = 1$ then we also have the contribution of ℓ irreducible (-6)-curves of B_0. Then, using lemma 1.1.4 and corollary 3.1.12,

$$\begin{aligned} 18 + 3h_1 + 6\ell = 18 + 3(\frac{3-\Gamma^2}{2} + \ell) + 6\ell \leq \delta = 12 + 3N_1^2 + n + \Delta^2 \\ = 15 + n = 15 + N_1^2 + 3\ell + \frac{1-3\Gamma^2}{2} = 16 + 3\ell + \frac{1-3\Gamma^2}{2} \end{aligned}$$

hence

$$2 + 6\ell \leq \frac{1-3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} = -4$$

and we get a contradiction.

If $\Gamma'\Delta = 0$ we have necessarily $\Gamma'^2 \leq 0$ (hence $\Gamma^2 \leq -1$ on the numerical Godeaux surface S) and there is a (-6)-curve B_{0k} in B_0 such that $B_{0k}\Delta = 1$. In particular $\ell \geq 1$. Then Γ' contributes $-\Gamma'^2 = -3\Gamma^2$ to δ and we can also consider $\ell - 1$ (-6)-curves B_{0i}, $i \neq k$, plus the h_1 curves E'_k. Thus

$$\begin{aligned} 18 - 3\Gamma^2 + 6(\ell - 1) + 3h_1 = 12 + 6\ell - 3\Gamma^2 + \frac{9-3\Gamma^2}{2} + 3\ell \leq \delta \\ = 15 + n = 16 + 3\ell + \frac{1-3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 4 + \frac{1-3\Gamma^2}{2} + \frac{3\Gamma^2-9}{2} + 3\Gamma^2 = 3\Gamma^2 < 0$$

Contradiction.

Case II: $B_0\Delta = 0, E'\Delta = 2$.

Since $\Gamma'\Delta = 0$ we find $\Gamma'^2 \leq 0$ hence $\Gamma^2 \leq -1$ on S. In particular $h_1 = \frac{3-\Gamma^2}{2} + \ell \geq 2$. All the irreducible components of B_0 and $h_1 - 2$ curves E'_k contribute to δ. Thus

$$\begin{aligned} 18 - 3\Gamma^2 + 6\ell + 3(h_1 - 2) &= 12 + 6\ell - 3\Gamma^2 + \frac{9 - 3\Gamma^2}{2} + 3\ell \leq \delta \\ &= 15 + n = 16 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 4 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} - 3\Gamma^2$$

and as before we get a contradiction. □

Proposition 3.1.17. *The case $N \equiv N_1 + \Delta$ cannot occur.*

Proof. From the proof of lemma 3.1.11 we know that $\Delta N = 1, NN_1 = 2, \Delta G' = 0$ and $\Delta^2 = \Delta K_Y = 0$. Hence

$$1 = N\Delta = \Delta(3K_Y + 2B_0 + E' - 3G') = 2B_0\Delta + E'\Delta$$

From the nefness of Δ we find $B_0\Delta = 0, E'\Delta = 1$. Since $\Gamma'\Delta = 0$ we find $\Gamma'^2 \leq 0$ hence $\Gamma^2 \leq -1$ on S. In particular $h_1 = \frac{3-\Gamma^2}{2} + \ell \geq 2$. All the irreducible components of B_0 and $h_1 - 1$ curves E'_k contribute to δ. Thus from lemma 1.1.4 and corollary 3.1.12

$$\begin{aligned} 18 - 3\Gamma^2 + 6\ell + 3(h_1 - 1) &= 15 + 6\ell - 3\Gamma^2 + \frac{9 - 3\Gamma^2}{2} + 3\ell \leq \delta \\ &= 12 + 3N_1^2 + n + \Delta^2 = 15 + n \\ &= 15 + N_1^2 + 3\ell + \frac{1 - 3\Gamma^2}{2} = 16 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 1 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} - 3\Gamma^2 = -3 + 3\Gamma^2 < 0$$

and we get a contradiction. □

Thus from lemma 3.1.11, propositions 3.1.16 and 3.1.17 we immediately find

Corollary 3.1.18. *In the above setting the pencil $|N_1|$ has no fixed part.*

Proposition 3.1.19. *Case (i) with $N_1^2 = 1$ can only occur when $n = 6$ and either $\Gamma^2 = -3$ and $\ell = 0$ or $\Gamma^2 = -1$ and $\ell = 1$. In particular $\Gamma^2 = 1$ cannot occur. Moreover all the curves E'_k intersect N_1 at one point.*

Proof. From the above corollary we have $\Delta \equiv N_1$. Since $p_a(N_1) = N_1^2 = 1$ (cf. (3.1)) we know that $NN_1 = 2, N_1G' = 0$ and $N_1^2 = 1, N_1K_Y = 0$. Hence

$$2 = NN_1 = N_1(3K_Y + 2B_0 + E' - 3G') = -3 + 2B_0N_1 + E'N_1$$

Moreover, from proposition 2.3.3,

$$0 \leq E'_kN_1 = E'_k(N + K_Y + G' - \sum_i Z_i) = 0 + 1 - E'_k \sum_i Z_i \leq 1$$

hence from corollary 3.1.5 $0 \leq E'N_1 \leq h_1 \leq 4$. Then either $B_0N_1 = 2, E'N_1 = 1$ or $B_0N_1 = 1, E'N_1 = 3$.

Case I: $B_0N_1 = 2, E'N_1 = 1$.

All the (-3)-curves E'_k, except for one, have no intersection with N_1 hence each of them contributes (from lemma 1.1.4) 3 to δ. Moreover there are at most two irreducible components of B_0 which intersect N_1.

If $\Gamma'N_1 = 2$ then we have the contribution of ℓ irreducible (-6)-curves of B_0. Then, using lemmas 1.1.4, 3.1.14 and corollary 3.1.12,

$$\begin{aligned} 18 + 3(h_1 - 1) + 6\ell = 18 + 3(\frac{3 - \Gamma^2}{2} + \ell - 1) + 6\ell &\leq \delta \\ &= 12 + 3N_1^2 + n + \Delta^2 = 16 + n \\ &= 16 + N_1^2 + 3\ell + \frac{1 - 3\Gamma^2}{2} = 17 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 2 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} = -2$$

and we get a contradiction.

If $\Gamma'N_1 = 1$ then there is a (-6)-curve B_{0k} which intersects N_1 at one point. Hence we have the contribution of $\ell - 1$ irreducible (-6)-curves of B_0. Then, using lemmas 1.1.4, 3.1.14 and corollary 3.1.12,

$$\begin{aligned} 18 + 3(h_1 - 1) + 6(\ell - 1) = 18 + 3(\frac{3 - \Gamma^2}{2} + \ell - 1) + 6\ell - 6 &\leq \delta \\ &= 16 + n = 17 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 8 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} = 4$$

which forces $\ell = 0$ while we know $\ell \geq 1$. Contradiction.

If $\Gamma' N_1 = 0$ we have necessarily $\Gamma'^2 \leq 0$ (hence $\Gamma^2 \leq -1$ on the numerical Godeaux surface S) and there is at least one (-6)-curve B_{0k} in B_0 such that $B_{0k}N_1 \geq 1$. In particular $\ell \geq 1$. Then Γ' contributes $-\Gamma'^2 = -3\Gamma^2$ to δ. If $\ell \geq 2$ we can also consider $\ell - 2$ (-6)-curves B_{0i}, $i \neq k$, plus the $h_1 - 1$ curves E'_k. Thus

$$\begin{aligned} 18 - 3\Gamma^2 + 6(\ell - 2) + 3(h_1 - 1) = 3 + 6\ell - 3\Gamma^2 + \frac{9 - 3\Gamma^2}{2} + 3\ell \leq \delta \\ = 16 + n = 17 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 14 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} + 3\Gamma^2 = 10 + 3\Gamma^2 \leq 7$$

which forces $\ell \leq 1$ contradicting the assumption $\ell \geq 2$.

If $\ell = 1$ then we only have the contribution of Γ' and of $h_1 - 1$ curves E'_k. Then

$$\begin{aligned} 18 - 3\Gamma^2 + 3h_1 - 3 = 18 - 3\Gamma^2 + \frac{9 - 3\Gamma^2}{2} + 3\ell - 3 \\ = 18 - 3\Gamma^2 + \frac{9 - 3\Gamma^2}{2} \leq \delta = 17 + 3\ell + \frac{1 - 3\Gamma^2}{2} = 20 + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

which forces

$$0 \leq 2 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} + 3\Gamma^2 = -2 + 3\Gamma^2 < -2$$

Contradiction.

Case II: $B_0 N_1 = 1, E' N_1 = 3$.

In this case only one irreducible component of B_0 intersects N_1. Moreover since $0 \leq E'_k N_1 \leq 1$ we infer $h_1 \geq 3$.

If $\Gamma' N_1 = 1$ then we have the contribution of all the (-6)-curves in B_0 and of $h_1 - 3$ curves E'_k. Thus

$$\begin{aligned} 18 + 3(h_1 - 3) + 6\ell = 9 + \frac{9 - 3\Gamma^2}{2} + 3\ell + 6\ell \leq \delta \\ = 16 + n = 17 + 3\ell + \frac{1 - 3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 8 + \frac{1 - 3\Gamma^2}{2} + \frac{3\Gamma^2 - 9}{2} = 4$$

which implies $\ell = 0$. Since

$$h_1 = \frac{3-\Gamma^2}{2} + \ell = \frac{3-\Gamma^2}{2} \geq 3$$

we find $\Gamma^2 \leq -3$. From $\Gamma^2 + \Gamma K_S \geq -2$ it follows $\Gamma^2 = -3$, $h_1 = 3$ and, from corollary 3.1.12, $n = 1 + \frac{1-3\Gamma^2}{2} = 6$.

If $\Gamma' N_1 = 0$ we find $\Gamma'^2 \leq 0$ hence $\Gamma^2 \leq -1$ on S. As before we have $h_1 = \frac{3-\Gamma^2}{2} + \ell \geq 3$. Moreover there is exactly one of the (-6)-curves of B_0 which intersects N_1. Thus

$$\begin{aligned} 18 - 3\Gamma^2 + 6\ell - 6 + 3(h_1 - 3) = 3 + 6\ell - 3\Gamma^2 + \frac{9-3\Gamma^2}{2} + 3\ell \leq \delta \\ = 16 + n = 17 + 3\ell + \frac{1-3\Gamma^2}{2} \end{aligned}$$

hence

$$6\ell \leq 14 + \frac{1-3\Gamma^2}{2} + \frac{3\Gamma^2-9}{2} + 3\Gamma^2 = 10 + 3\Gamma^2 \leq 7$$

forces $\ell = 1$ and $\Gamma^2 = -1$. Then

$$h_1 = \frac{3-\Gamma^2}{2} + \ell = \frac{3-\Gamma^2}{2} + 1 = 3$$

and

$$n = N_1^2 + 3\ell + \frac{1-3\Gamma^2}{2} = 1 + 3 + 2 = 6.$$

$\square$

We now show the following

Proposition 3.1.20. *Case (i) with* $N_1^2 = 1$ *and* $n = 6$ *cannot occur.*

Proof. Let us assume $N_1^2 = 1$ and $n = 6$. Then from lemma 3.1.14

$$18 \leq \delta = 12 + 3N_1^2 + n + \Delta^2 \leq 16 + n = 16 + 6 = 22$$

If all the six cycles Z_i were irreducible then each of them would not intersect N_1 and Δ. Then from lemma 1.1.4 they would contribute $1 \cdot 6 = 6$ to δ hence

$$18 + 6 = 24 \leq \delta \leq 22$$

and we would get a contradiction.

Let us assume there is at least one reducible cycle. Then one of the irreducible (-1)-curves, say Z_1, appears with multiplicity $m_1 \geq 2$ in $\sum_i Z_i$. For any curve E'_k such that $E'_k Z_1 = 1$ we find

$$E'_k \sum_i Z_i = E'_k(m_i Z_1 + (\sum_i Z_i - m_i Z_1)) = m_i + E'_k(\sum_i Z_i - m_i Z_1)$$

It follows that E'_k is contained in some cycle Z_i, $i \geq 2$ and then from proposition 2.3.3 $E'_k N_1 = 0$ contradicting proposition 3.1.19. □

The results of propositions 3.1.15, 3.1.16, 3.1.17, 3.1.19 and 3.1.20 can be summarized in the following theorem.

Theorem 3.1.21. *Case (i) cannot occur.*

3.2 Case (ii): $R_0 K_S = 0$, $h_2 = 4$

Assume case (ii) holds. From proposition 2.1.6 and formula (2.11) we have

$$2 = h^0(Y, \mathcal{O}_Y(2K_Y + B)) \leq h^0(X, \mathcal{O}_Y(2K_X - R)) \leq h^0(X, \mathcal{O}_X(2K_X)) = 2$$

which implies that R_0 is in the fixed part of $|2K_S|$. Then the number ℓ of disjoint (-2)-curves that form R_0 is greater or equal than 2. In fact

$$h_1 + 8 = h_1 + 2h_2 = 6 + \frac{3R_0K_S - {R_0}^2}{2} = 6 + \ell$$

which forces $h_1 = \ell - 2$ and $\ell \geq 2$.

Let M be an effective divisor in the movable part of the pencil $|2K_S - R_0|$. Then M is in the movable part of the bicanonical system $|2K_S| = |M| + T$ and, by lemma 1.5.2 (see also [Miy]), either $M^2 = 0$ or $M^2 = 2$. In any case the general curve of $|M|$ is smooth.

From theorem 1.5.4 we can exclude the case $M^2 = 0$.

The strict transform $\widetilde{M}$ of M satisfies $\widetilde{M} = \pi^*(M')$ for some pencil $|M'|$ on Y. This implies $\widetilde{M}^2 \equiv 0 \mod 3$. Therefore $\widetilde{M}^2 = 0$. We have

$$\varepsilon^*(M) = \widetilde{M} + D$$

where D is a sum of exceptional divisors.

Since $M^2 = 2$ then $D \neq 0$ and the general curve M, see lemma 2.2.6, passes either through one of the $h_2 = 4$ points q_i (without loss of generality we may assume it is q_1) with multiplicity $m_1 = 1$, while $m_2 = m_3 = m_4 = 0$ hence

$$D = 2F + G + H$$

or through two of the points p_j (if $\ell \geq 4$) hence

$$D = E_1 + E_2.$$

In any case

$$p_a(\widetilde{M}) = 1 + \frac{\widetilde{M}^2 + \widetilde{M}K_X}{2} = 3 = p_a(M)$$

and we have $\widetilde{M}K_X = 4$.

Lemma 3.2.1. *$|M'|$ is a pencil of elliptic curves with ${M'}^2 = 0$.*

Proof. We have

$$\begin{aligned} 3M'K_Y = \pi^*(M')\pi^*(K_Y) &= \widetilde{M}(K_X - 2\varepsilon^*(R_0) - 2E - 2F - 2H) \\ &= \begin{cases} 4 - 2MR_0 - 2 = 2 - 2MR_0 & \text{if } D = 2F + G + H \\ 4 - 2MR_0 - 4 = -2MR_0 & \text{if } D = E_1 + E_2 \end{cases} \end{aligned}$$

and then $MR_0 \equiv 0, 1 \mod 3$. Since $0 \leq MR_0 \leq MT = 2$ we get $MR_0 = 0, 1$ hence $M'K_Y = 0$. Since ${M'}^2 = 0$ this proves the lemma. □

Theorem 3.2.2. *Case (ii) cannot occur.*

Proof. Let us consider on Y the fibration over $\mathbb{P}^1$ given by the elliptic pencil $|M'|$. From proposition 1.1.3 we have

$$e(Y) = e(M')e(\mathbb{P}^1) + \sum_s (e(M'_s) - e(M')) = e(M')e(\mathbb{P}^1) + \delta \tag{3.5}$$

where the sum is taken over all the singular curves M'_s in $|M'|$ and we set

$$\delta := \sum_s (e(M'_s) - e(M)).$$

Since $e(M') = 0$ and from (2.6)

$$e(Y) = 12 - K_Y^2 = 12 + 3 + 3\ell = 15 + 3\ell$$

we find $\delta = 15 + 3\ell$.

The general M on S passes only through at most one of the points q_j. Then we have

$$F_2'M' = F_3'M' = F_4'M' = H_2'M' = H_3'M' = H_4'M' = 0$$

and each of these disjoint curves contributes 3 to δ by lemma 1.1.4. Moreover since $0 \leq MR_0 \leq 1$ (see the above proof) we have at least $\ell - 1$ irreducible components B_{0i} of B_0 which do not intersect M'. Each curve B_{0i} contributes 6 more nodes to δ. Therefore

$$6 \cdot 3 + 6(\ell - 1) = 12 + 6\ell \leq \delta = 15 + 3\ell$$

which forces $\ell \leq 1$ and we get a contradiction since we know $\ell \geq 2$. $\square$

3.3 Case (iii): $R_0K_S = 0$, $h_2 = 1$

In this case, from formula (2.5), $h_1 = 4 + \ell$ where ℓ is the number of irreducible components of R_0 .

Proposition 3.3.1. *If case (iii) holds then Y is a rational surface.*

Proof. We know that $q(Y) \leq q(X) = q(S) = 0$. Then from Castelnuovo's theorem 1.1.12 we only need to show that $P_2(Y) = h^0(Y, \mathcal{O}_Y(2K_Y)) = 0$. Then, since we are in case (iii),

$$h^0(Y, \mathcal{O}_Y(2K_Y)) \leq h^0(Y, \mathcal{O}_Y(2K_Y + B)) = 0.$$

$\square$

Then the same argument of proposition 3.1.8 holds and we can show

Proposition 3.3.2. *In the above setting the numerical Godeaux surface S has no 2-torsion element.*

We still have the pencil $|N| = |A'| + \Phi'$ which is composed of curves of arithmetic genus 3. Off the A'^2 base points $\varphi_{|A'|}$ is a fibration over $\mathbb{P}^1$ of curves of genus $0 \leq p_a(A') \leq 2$. Computing Euler numbers we obtain

$$e(Y) + A'^2 = e(A')e(\mathbb{P}^1) + \sum_s (e(A'_s) - e(A')) = e(A')e(\mathbb{P}^1) + \delta \tag{3.6}$$

where the sum is taken over all the singular curves A'_s in $|A'|$ and we set

$$\delta = \sum_s (e(A'_s) - e(A')).$$

Lemma 3.3.3. *In the above setting each of the exceptional curves E'_k, F' and H' which does not intersect A' increases δ by 3. Moreover each component B_{0i} of B_0 for which $B_{0i}A' = 0$ increases δ by 6.*

Proof. For any curve C such that $CA' = 0$ we have

$$(A' - C)C = -C^2$$

and, by lemma 1.1.4, C increases by at least $-C^2$ the Euler number δ associated to $\varphi_{|A'|}$. Therefore when $C = E'_k, F', H'$ we add 3 to δ, whereas when $C = B_{0i}$ we add 6. □

Lemma 3.3.4. *In the above setting we have*

$$\delta = 14 + 3\ell + 3A'^2 + 2A'K_Y.$$

Proof. Let us compute, using (2.10),

$$\begin{aligned} e(Y) + A'^2 &= 12 - K_Y^2 + A'^2 \\ &= 12 - \frac{1}{3}[K_S{}^2 - 6 - h_2 + \frac{9}{2}R_0{}^2 - \frac{11}{2}R_0K_S] + A'^2 \\ &= 12 - \frac{1}{3}[K_S{}^2 - 6 - 1 - 9\ell] + A'^2 = 14 + A'^2 + 3\ell \end{aligned}$$

while

$$e(A')e(\mathbb{P}^1) = (2 - 2 \cdot p_a(A'))2 = 2(-A'^2 - A'K_Y)$$

Therefore we have

$$\delta = \sum_s (e(A'_s) - e(A')) = 14 + 3\ell + 3A'^2 + 2A'K_Y$$

as wanted. □

Proposition 3.3.5. *Assume $A'^2 = 0$. Then $0 \leq \ell \leq 1$ and we have $\ell = 0$ only when $(0a)$, $(0c)$, $(0f)$ or $(0g)$ holds and $\ell = 1$ only when $(0d)$ holds. Moreover cases $(0b)$, $(0e)$ and $(0h)$ of the list of proposition 2.2.13 cannot occur.*

Proof. We refer to the list of proposition 2.2.13. We have from lemma 3.3.4

$$\delta = 14 + 3\ell + 2A'K_Y$$

We have $A'K_Y = 0$ in all cases of the list except for $(0g)$.

In case $(0a)$ we have to consider F', H' and $h_1 - 2 = 2 + \ell$ curves E'_k, plus all the components of B_0. Then

$$6 + 3(2 + \ell) + 6\ell = 12 + 9\ell \leq \delta = 14 + 3\ell$$

which forces $\ell = 0$.

In case $(0b)$ we find $\ell \geq 1$ and we have the contribution of the curves E'_k, H' (see corollary 2.2.17) and of $\ell - 1$ components of B_0. Then

$$3(4 + \ell) + 3 + 6(\ell - 1) = 9 + 9\ell \leq \delta = 14 + 3\ell$$

which implies $6\ell \leq 5$. Impossible.

In case $(0c)$ we have the contribution of F', H' and $h_1 - 3 = 1 + \ell$ of the curves E'_k, plus all the components of B_0. Then

$$6 + 3(1 + \ell) + 6\ell = 9 + 9\ell \leq \delta = 14 + 3\ell$$

which forces $\ell = 0$.

In case $(0d)$ we find $\ell \geq 1$ and we have the contribution of H', of $h_1 - 1 = 3 + \ell$ of the curves E'_k, plus that of $\ell - 1$ components of B_0. Then

$$3(3 + \ell) + 3 + 6(\ell - 1) = 6 + 9\ell \leq \delta = 14 + 3\ell$$

which forces $\ell = 1$.

In case $(0e)$ we have the contribution of F', H' and of $h_1 - 1 = 3 + \ell$ of the curves E'_k, plus that of all the components of B_0. Then

$$6 + 3(3 + \ell) + 6\ell = 15 + 9\ell \leq \delta = 14 + 3\ell$$

which is impossible.

In case $(0f)$ we have the contribution of F', H' and of $h_1 - 2 = 2 + \ell$ of the curves E'_k, plus that of all the components of B_0. Then

$$6 + 3(2 + \ell) + 6\ell = 12 + 9\ell \leq \delta = 14 + 3\ell$$

which forces $\ell = 0$.

In case $(0h)$ we have the contribution of F', H' and of $h_1 - 1 = 3 + \ell$ of the curves E'_k, plus that of all the components of B_0. Then

$$6 + 3(3 + \ell) + 6\ell = 15 + 9\ell \leq \delta = 14 + 3\ell$$

which is impossible.

Finally we consider case $(0g)$. Then

$$\begin{aligned} 3A'F' &= \pi^*(A')\pi^*(F') = 3\widetilde{A}F = 3(\varepsilon^*(A) - D)F = -3DF \\ &= -3(3F + 3G + 3H)F = -3(-3 + 3) = 0 \end{aligned}$$

and analogously $A'H' = 0$. Then we have the contribution of F', H' and of all the curves E'_k and B_{0i}. Consequently

$$6 + 3(4 + 3\ell) + 6\ell = 18 + 9\ell \leq \delta = 14 + 3\ell + 2A'K_Y = 18 + 3\ell$$

which forces $\ell = 0$. $\square$

Proposition 3.3.6. *Assume $A'^2 = 1$. Then $0 \leq \ell \leq 3$. If $(1f)$ holds then $\ell = 0, 1$. If $(1e)$ holds then $\ell = 1, 2, 3$. If $(1a)$ or $(1d)$ holds then $\ell = 0$. Cases $(1b)$ and $(1c)$ of the list of proposition 2.2.14 cannot occur.*

Moreover in case $(1e)$ for each irreducible component B_{0k} of B_0 we find $B_{0k}A' \geq 1$.

Proof. We refer to the list of proposition 2.2.14. From lemma 3.3.4 we have

$$\delta = 14 + 3\ell + 3A'^2 + 2A'K_Y = 17 + 3\ell + 2A'K_Y$$

We have $A'K_Y = 1$ in cases $(1a)$, $(1d)$ and $(1f)$ of the above list.

If $(1a)$ holds all the irreducible components of B_0 and all the exceptional curves F', H', E' contribute to δ. Then

$$6 + 6\ell + 3(4 + \ell) = 18 + 9\ell \leq \delta = 19 + 3\ell$$

This can be satisfied only when $\ell = 0$. If $(1d)$ holds all the components of B_0, H' (see corollary 2.2.17) and all the curves E'_k contribute to δ. Thus

$$6\ell + 3 + 3(4 + \ell) = 15 + 9\ell \leq \delta = 19 + 3\ell$$

hence $\ell = 0$.

If $(1f)$ holds all the components of B_0 and the curves E'_k contribute to δ. Hence

$$6\ell + 3(4+\ell) = 12 + 9\ell \leq \delta = 19 + 3\ell$$

forces $\ell = 0, 1$.

We have $A'K_Y = -1$ in cases $(1b)$ and $(1e)$ of the list.

If $(1b)$ holds, all the exceptional curves F', H', E' contribute to δ and we have also to consider a certain number $0 \leq h \leq \ell - 1$ of irreducible components of B_0 (this can only happen when $\ell \geq 1$). Then

$$6 + 3(4+\ell) + 6h = 18 + 3\ell + 6h \leq \delta = 15 + 3\ell$$

which is impossible.

When $(1e)$ holds, $\ell \geq 1$ and we have to consider the curves E'_k, H' and $0 \leq h \leq \ell - 1$ irreducible components of B_0. Then

$$3(4+\ell) + 3 + 6h = 15 + 3\ell + 6h \leq \delta = 15 + 3\ell$$

hence $h = 0$ and $1 \leq \ell \leq 3$. In particular, for any irreducible component B_{0k} of B_0 we have $B_{0k}A' \geq 1$.

Finally we have $A'K_Y = -3$ only in case $(1c)$ when $\ell \geq 1$. In this case all the exceptional curves F', H', E' contribute to δ and we also have $0 \leq h \leq \ell - 1$ components of B_0. Then

$$6 + 3(4+\ell) + 6h = 18 + 3\ell + 6h \leq \delta = 11 + 3\ell$$

which is impossible. □

From proposition 2.2.15 it cannot be $A'^2 = 2$. Then (see remark 2.2.16) we are left with the case $A' = N$.

Proposition 3.3.7. *Assume* $A' = N$. *Then* $0 \leq \ell \leq 1$.

Proof. When $A' = N$ from lemmas 2.1.5 and 3.3.4 we have

$$\delta = 14 + 3\ell + 3N^2 + 2NK_Y = 14 + 3\ell + 9 + 2 = 25 + 3\ell.$$

All the exceptional curves F', H' and E'_k contribute to δ as all the irreducible components of B_0 do. Then

$$6 + 3(4+\ell) + 6\ell = 18 + 9\ell \leq \delta = 25 + 3\ell$$

which forces $\ell = 0, 1$. □

As a consequence of propositions 2.2.13, 2.2.14, 2.2.15, 3.3.5, 3.3.6, 3.3.7 and of remark 2.2.16 we obtain

Theorem 3.3.8. *Case (iii) of page 34 can only occur when one of the following conditions is satisfied:*

1. $\ell = 0$
   - *Cases* $(0a)$, $(0c)$, $(0f)$, $(0g)$
   - *Cases* $(1a)$, $(1d)$, $(1f)$
   - $A' = N$
2. $\ell = 1$
   - *Case* $(0d)$
   - *Cases* $(1e)$, $(1f)$
   - $A' = N$
3. $\ell = 2$
   - *Case* $(1e)$
4. $\ell = 3$
   - *Case* $(1e)$.

Moreover in cases $(0g)$, $(1f)$ *and* $A' = N$ *we have* $\Phi = 0$*, i.e. the invariant pencil* $\Lambda \leq |3K_S|$ *has no fixed part.*

Chapter 4

More on the case $R_0K_S = 0$, $h_2 = 1$

We recall that from a numerical analysis we are left only with the case $R_0K_S = 0$, $h_2 = 1$ for which the following possibilities can hold:

1. $R_0 = 0$
2. R_0 is a union of $\ell \leq 3$ smooth irreducible (-2)-curves

Moreover there is a pencil $|N|$ on the rational surface Y whose general element is a nef and big curve of genus 3 (see proposition 3.3.1 and lemma 2.1.5).

We also note that from (2.5) $h_1 = 4 + \ell$ and from (2.10)

$$K_Y^2 = \frac{1}{3}[{K_S}^2 - 6 - h_2 + \frac{9}{2}{R_0}^2 - \frac{11}{2}R_0K_S] = \frac{1}{3}[1 - 6 - 1 - 9\ell] = -2 - 3\ell \quad (4.1)$$

From now on we refer to the formulas of proposition 2.3.12 when computing the arithmetic data of the curves in the linear systems $|N|$, $|N_1|$, $|N_2|$, $|N_3|$.

We start by computing N_1^2 and $p_a(N_1)$:

$$N_1^2 = 5 - 4R_0K_S + K_Y^2 + n + h_2 = 5 - 2 - 3\ell + n + 1 = 4 - 3\ell + n \geq 0 \quad (4.2)$$

$$p_a(N_1) = 4 - 3R_0K_S + K_Y^2 + n + h_2 = 4 - 2 - 3\ell + n + 1 = 3 - 3\ell + n = N_1^2 - 1 \quad (4.3)$$

Moreover we have the following

Lemma 4.0.9. *In the above setting we have $3\ell - 4 \leq n \leq 3\ell$.*

Proof. Let us consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_Y(N - N_1) \longrightarrow \mathcal{O}_Y(N) \longrightarrow \mathcal{O}_{N_1}(N) \longrightarrow 0 \quad (4.4)$$

Then, since Y is a rational surface, from the definition of N_1,

$$
\begin{aligned}
h^2(Y, \mathcal{O}_Y(N - N_1)) &= h^0(Y, \mathcal{O}_Y(K_Y - N + N_1)) \\
&= h^0(Y, \mathcal{O}_Y(2K_Y - G' - \sum_{i=1}^{n} Z_i)) \leq \\
&\leq h^0(Y, \mathcal{O}_Y(2K_Y)) = 0
\end{aligned}
$$

Moreover $N - N_1$ cannot be effective, otherwise

$$3 = h^0(Y, \mathcal{O}_Y(N_1)) \leq h^0(Y, \mathcal{O}_Y(N)) = 2$$

The long exact sequence of (4.4) gives therefore

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y(N)) \longrightarrow H^0(N_1, \mathcal{O}_{N_1}(N)) \longrightarrow H^1(Y, \mathcal{O}_Y(N - N_1)) \longrightarrow 0$$

$$0 \longrightarrow H^1(N_1, \mathcal{O}_{N_1}(N)) \longrightarrow 0$$

This forces $H^1(N_1, \mathcal{O}_{N_1}(N)) = 0$. Since N_1 is big and nef, hence 1-connected,

$$h^0(N_1, \mathcal{O}_{N_1}(N)) = \chi(\mathcal{O}_{N_1}(N)) = 1 + NN_1 - p_a(N_1) = 5 - 3 + 3\ell - n = 2 + 3\ell - n$$

Then

$$2 = h^0(Y, \mathcal{O}_Y(N)) \leq h^0(N_1, \mathcal{O}_{N_1}(N)) = 2 + 3\ell - n$$

and $3\ell - 4 \leq n \leq 3\ell$ as wanted (see formula (4.2)). $\square$

Remark 4.0.10. *When $\ell = 0$ one can easily see that $n = 3\ell = 0$ is the only possibility for n.*

We will see in the following sections that a deeper study of the adjoint linear systems $|N_i|$ to the pencil $|N|$ on Y allows us to collect the cases listed in theorem 3.3.8 into two main groups

Definition 4.0.11. *We call* **ruled cases** *those for which one of the linear systems $|N_i|$ induce a morphism $Y \longrightarrow \mathbb{F}_a$ for some $a \geq 0$.*

Definition 4.0.12. *We call* **Del Pezzo cases** *those which are not ruled cases.*

Moreover in section 4.6 we will show that not all the cases listed in theorem 3.3.8 can actually occur.

4.1 $n = 3\ell - 4$

Proposition 4.1.1. *In the case $n = 3\ell - 4$ the net $|N_1|$ has no fixed part and we have $|N_1| = |2\Theta|$ where $|\Theta|$ is a pencil of rational curves with $\Theta^2 = 0$.*

Proof. Assume that $n = 3\ell - 4$. Then $|N_1|$ is a net of curves with $N_1^2 = 0$. Let us write $|N_1| = |\Delta| + T$ where T and $|\Delta|$ are the fixed and the movable part of $|N_1|$ respectively. Then $0 \le \Delta^2 \le \Delta N_1 \le N_1^2$ hence $\Delta N_1 = 0$ and

$$0 = N_1\Delta = \Delta^2 + T\Delta$$

$$0 = N_1 T = \Delta T + T^2$$

It follows $\Delta^2 = \Delta T = T^2 = 0$. Therefore there exists a pencil $|\Theta|$ such that $|\Delta| = |2\Theta|$. Then

$$4 = NN_1 = 2N\Theta + NT$$

and $N\Theta \ge 1$ otherwise by the Index Theorem and the rationality of Y we have $\Theta \equiv 0$. If $N\Theta = 1$, $NT = N\Delta = 2$ then

$$(\Delta - T)^2 = \Delta^2 + T^2 - 2\Delta T = 0$$

hence $\Delta \equiv T$ which is impossible. Thus $N\Theta = 2$ and $NT = 0$ which forces $T = 0$ by the Index Theorem.

Therefore we have $N_1 = \Delta$ and

$$-1 = p_a(N_1) = p_a(\Delta) = p_a(2\Theta) = 1 + \frac{2\Theta(2\Theta + K_Y)}{2} = 1 + \Theta K_Y$$

forces $\Theta K_Y = -2$. Then $|\Theta|$ is a pencil of rational curves. □

Theorem 4.1.2. *The case $n = 3\ell - 4$ cannot occur.*

Proof. If $n = 3\ell - 4 \ge 0$ we have $\ell \ge 2$ and we are in case $(1e)$ of proposition 2.2.14 (see also theorem 3.3.8). Therefore we have a pencil of elliptic curves $|A'|$ for which $A'^2 = 1$. Then, from proposition 4.1.1, lemmas 2.2.1 and 2.2.3 and corollary 2.2.12,

$$\begin{aligned} 2A'\Theta = A'N_1 &= A'(N + K_Y - G' - \sum_{i=1}^{n} Z_i) \\ &= 3 - 1 - A'\sum_{i=1}^{n} Z_i = 2 - A'\sum_{i=1}^{n} Z_i \le 2 \end{aligned}$$

and $A'N_1 \geq 1$ by the Index Theorem 1.1.10, whence $A'N_1 = 2$ and $A'\Theta = 1$.

We have $h^0(A', \mathcal{O}_{A'}(\Theta)) = 2$ since otherwise the point $A' \cap \Theta$ should be a base point for the pencil $|\Theta|$, whereas $\Theta^2 = 0$. Then we get a contradiction since for any divisor D of degree 1 on the smooth elliptic curve A' $h^0(A', \mathcal{O}_{A'}(D)) = 1$. □

4.2 $n = 3\ell - 3$

In this case we have $1 \leq \ell \leq 3$ and from equations (4.2) and (4.3) we find $N_1^2 = 1$ and $p_a(N_1) = 0$.

Lemma 4.2.1. *If $n = 3\ell - 3$ then $|N_1|$ has no fixed part. Then the general element of $|N_1|$ is a smooth rational curve.*

Proof. We can use the same argument as in lemma 3.1.11 and we find that $|N_1|$ has no fixed part unless $|N_1| = |\Delta| + T$ with $\Delta^2 = 0, \Delta N_1 = \Delta T = 1, T^2 = -1$. Since $\Delta^2 = 0$ and $|\Delta|$ is a net, there exists a pencil $|\Theta|$ such that $\Delta \equiv 2\Theta$. But then

$$1 = \Delta T = 2\Theta T$$

and we get a contradiction. □

In this setting $|N_1|$ is base point free and we have a birational morphism $\varphi_{|N_1|} : Y \longrightarrow \mathbb{P}^2$.

Theorem 4.2.2. *The case $n = 3\ell - 3$ cannot occur when $\Phi \neq 0$ and $\ell = 1$.*

Proof. Assume $n = 3\ell - 3$ and $\ell = 1$. Then $n = 0$ and when we contract the curve G' we get a rational surface W such that

$$K_W^2 = K_Y^2 + 1 = -2 - 3\ell + 1 = -4.$$

Let us consider on W the images $\bar{N}$ and $\bar{N}_1$ of N and N_1 respectively. Since, as we have already seen, $|\bar{N}_1|$ gives a birational morphism $\varphi_{|\bar{N}_1|} : W \longrightarrow \mathbb{P}^2$, the plane image of $|\bar{N}_1|$ is the net of lines of $\mathbb{P}^2$.

In this setting, the plane image of $|\bar{N}|$ is the linear system of quartics through 13 points, which has virtual dimension $14 - 13 = 1$.

Let us also assume $\Phi \neq 0$. From theorem 3.3.8 either case $(0d)$ of proposition 2.2.13 or case $(1e)$ of proposition 2.2.14 holds. In any case, see the proofs of propositions 3.3.5 and 3.3.6, we have

$$B_{0k}A' \geq 1$$

for all the irreducible components B_{0k} of B_0. Then

$$B_{0k}\Phi' = B_{0k}(N - A') = -B_{0k}A' \leq -1$$

and we can deduce $B_0 \leq \Phi'$.

We now compute

$$B_0N_1 = B_0(N + K_Y - G') = 0 + 4\ell - 0 = 4$$

while from the nefness of N_1 and of A' we should have

$$B_0N_1 \leq \Phi'N_1 = (N - A')N_1 = NN_1 - A'N_1 = 4 - A'N_1 \leq 3.$$

Contradiction. $\square$

When $n \geq 3\ell - 2$ it makes sense to consider $|N_2| = |N_1 + K_Y - \sum_{i=1}^{n} Z_i - G' - \sum_{j=1}^{n'} Z'_j|$ which is a linear system of dimension $3 - 3\ell + n = p_a(N_1) \leq 3$ and $3 - 3\ell + n \geq 1$ and from proposition 2.3.12

$$\begin{aligned} N_2^2 &= 7 - 8R_0K_S + 4K_Y^2 + 4n + 4h_2 + n' = 7 - 8 - 12\ell + 4n + 4 + n' \\ &= 3 - 12\ell + 4n + n' \\ p_a(N_2) &= 5 - 5R_0K_S + 3K_Y^2 + 3n + 3h_2 + n' = 5 - 6 - 9\ell + 3n + 3 + n' \\ &= 2 - 9\ell + 3n + n' \qquad (4.5) \\ N_1N_2 &= 2p_a(N_1) - 2 = 2(3 - 3\ell + n) - 2 = 4 - 6\ell + 2n \end{aligned}$$

4.3 $n = 3\ell - 2$

In this case $1 \leq \ell \leq 3$ and we have $N_1N_2 = 0$ and by the Index Theorem 1.1.10 we infer $N_2 \equiv 0$. Then from (4.5) $n' = N_2^2 - 3 + 12\ell - 4n = 5$ and

$$\begin{aligned} N_1 &\equiv \textstyle\sum_{i=1}^{n} Z_i + G' + \sum_{j=1}^{5} Z'_j - K_Y \\ N &\equiv 2\textstyle\sum_{i=1}^{n} Z_i + 2G' + \sum_{j=1}^{5} Z'_j - 2K_Y \qquad (4.6) \\ 2B_0 + E' &\equiv 2\textstyle\sum_{i=1}^{n} Z_i + 5G' + \sum_{j=1}^{5} Z'_j - 5K_Y \end{aligned}$$

4.4 $n = 3\ell - 1$

This case can happen for $1 \leq \ell \leq 3$. We have $N_1N_2 = 2$, $N_1^2 = 3$ (cf. equations (4.2), (4.3) and (4.5)) and since $N_1(3N_2 - 2N_1) = 0$ then

$$(3N_2 - 2N_1)^2 = 9N_2^2 + 4N_1^2 - 12N_1N_2 = 9N_2^2 + 12 - 24 = 9N_2^2 - 12 \leq 0$$

Hence from (4.5) $0 \leq N_2^2 = n' - 1 \leq 1$ and $n' = 1, 2$.

Lemma 4.4.1. *If $N_2^2 = 0$ (i.e. $n' = 1$) then $|N_2|$ has no fixed part. In particular the general member of $|N_2|$ is a smooth rational curve with self-intersection 0.*

Proof. Let us write $|N_2| = |\Delta| + T$ with $|\Delta|$ and T the movable and the fixed part of N_2 respectively. We have $0 = N_2^2 = N_2\Delta + N_2T$ or, equivalently,

$$\begin{aligned} 0 &= N_2\Delta = \Delta^2 + \Delta T \\ 0 &= N_2T = \Delta T + T^2 \end{aligned}$$

which implies $\Delta^2 = \Delta T = T^2 = 0$.

We know that $N_1N_2 = 2$. Then $0 \leq N_1\Delta \leq N_1N_2 = 2$. It cannot be $N_1\Delta = 0$ otherwise, by the index theorem and the rationality of Y, $\Delta = 0$.

It cannot be $N_1\Delta = 1 = N_1T$: we obtain by the Index Theorem

$$0 \geq (\Delta - T)^2 = \Delta^2 + T^2 - 2\Delta T = 0 \quad \Rightarrow \quad \Delta \sim T$$

Since Y is a rational surface, this implies $\Delta \equiv T$ which is impossible.

So $N_1\Delta = N_1N_2 = 2$ and then

$$0 \geq (N_2 - \Delta)^2 = T^2 = 0$$

Again, by the rationality of Y we have $T \equiv 0$ and $|N_2|$ has no fixed part. □

Then there exists a morphism $Y \longrightarrow \mathbb{F}_a$ for some $a \geq 0$.

If $n' = 2$ then $|N_2|$ is a pencil of curves with arithmetic genus 1 and therefore $N_3 \equiv 0$. But now from proposition 2.3.12

$$\begin{aligned} N_3^2 &= 9 - 12R_0K_S + 9K_Y^2 + 9h_2 + 9n + 4n' + n'' \\ &= 9 - 18 - 27\ell + 9 + 27\ell - 9 + 8 + n'' = n'' - 1 = 0 \end{aligned}$$

Then

$$N_2 \equiv G' + \textstyle\sum_{i=1}^{n} Z_i + \sum_{i=1}^{2} Z_i' + Z'' - K_Y$$

$$N_1 \equiv 2G' + 2\sum_{i=1}^{n} Z_i + 2\sum_{i=1}^{2} Z_i' + Z'' - 2K_Y$$
$$N \equiv 3G' + 3\sum_{i=1}^{n} Z_i + 2\sum_{i=1}^{2} Z_i' + Z'' - 3K_Y$$
$$2B_0 + E' \equiv N + 3G' - 3K_Y \equiv 6G' + 3\sum_{i=1}^{n} Z_i + 2\sum_{i=1}^{2} Z_i' + Z'' - 6K_Y$$

4.5 $n = 3\ell$

In this case we have $0 \leq \ell \leq 3$. Now $N_1^2 = 4 = N_1 N_2$ and

$$N_2^2 = n' + 3 \geq 3$$

By the Index theorem

$$(N_1 - N_2)^2 = N_1^2 + N_2^2 - 2N_1N_2 = 4 + n' + 3 - 8 = n' - 1 \leq 0$$

Moreover if $n' = 1$ we have $N_1 \equiv N_2$ and then

$$K_Y \equiv G' + \sum_{i=1}^{n} Z_i + Z'$$

which is impossible since K_Y is not effective. This implies $n' = 0$, $N_2^2 = 3$ and $p_a(N_2) = 2$.

If we look at N_3 we have (see also proposition 2.3.12)

$$N_3^2 = 9 - 12R_0K_S + 9K_Y^2 + 9h_2 + 9n + 4n' + n'' = 9 - 18 - 27\ell + 9 + 27\ell + n'' = n''$$

$$p_a(N_3) = 6 - 7R_0K_S + 6K_Y^2 + 6h_2 + 6n + 3n' + n'' = 6 - 12 - 18\ell + 6 + 18\ell + n'' = n''$$

Since $N_2N_3 = 2p_a(N_2) - 2 = 2$ we have

$$(3N_3 - 2N_2)^2 = 9N_3^2 + 4N_2^2 - 12N_2N_3 = 9n'' - 12 \leq 0$$

Then $n'' = 0, 1$. In the former case $|N_3|$ is a pencil of rational curves of self-intersection 0 (see also the formulas of proposition 2.3.12), whereas in the latter case we have a pencil of curves with arithmetical genus one. Again we infer

$$N_4 = N_3 + K_Y - G' - \sum_{i=1}^{n} Z_i - Z'' - \sum_{i=1}^{n'''} Z_i''' \equiv 0.$$

Then

$$N_4^2 = N_3^2 + K_Y^2 + 2N_3K_Y + 1 + n + n' + n'' + n'''$$

$$= 1 - 2 - 3\ell - 2 + 1 + 3\ell + 1 + n''' = n''' - 1 = 0$$

Therefore

$$\begin{aligned}
N_3 &\equiv G' + \textstyle\sum_{i=1}^n Z_i + Z'' + Z''' - K_Y \\
N_2 &\equiv 2G' + 2\textstyle\sum_{i=1}^n Z_i + 2Z'' + Z''' - 2K_Y \\
N_1 &\equiv 3G' + 3\textstyle\sum_{i=1}^n Z_i + 2Z'' + Z''' - 3K_Y \\
N &\equiv 4G' + 4\textstyle\sum_{i=1}^n Z_i + 2Z'' + Z''' - 4K_Y \\
2B_0 + E' &\equiv 7G' + 4\textstyle\sum_{i=1}^n Z_i + 2Z'' + Z''' - 7K_Y
\end{aligned}$$

In case $|N_3|$ is a pencil of rational curves we can show arguing as in lemma 4.4.1 that $|N_3|$ has no fixed part.

Therefore we have a map $Y \longrightarrow \mathbb{F}_a$ for some $a \geq 0$.

4.6 Further results

Proposition 4.6.1. *Case* $(1e)$ *of proposition 2.2.14 cannot occur.*

Proof. Assume case $(1e)$ holds. Then Y has an elliptic pencil $|A'|$ with $A'^2 = 1$. From lemma 1.1.9 if $A' + K_Y$ is not nef then there exist (-1)-cycles D_j such that $D_jA' = 0$. Moreover by the Index theorem 1.1.10 we have

$$A_1 = A' + K_Y - \sum_j D_j \equiv 0 \tag{4.7}$$

We now look at the intersection number $s := A'N_1$: we have (cf. corollary 2.2.12)

$$A'N_1 = A'(N + K_Y - G' - \sum_{i=1}^n Z_i) = 3 - 1 - A'\sum_{i=1}^n Z_i = 2 - A'\sum_{i=1}^n Z_i \geq 1$$

otherwise $N_1 \equiv 0$. Moreover from theorem 4.1.2 we have $1 \leq N_1^2 \leq 4$ (cf. sections 4.2, 4.3, 4.4 and 4.5).

Then by the Index theorem 1.1.10 we have $A'(N_1 - sA') = 0$ and

$$(N_1 - sA')^2 = N_1^2 + s^2A'^2 - 2sN_1A' = N_1^2 - s^2 \leq 0$$

Then if $s = 1$ we should have $N_1^2 = 1$ and, from the rationality of Y, $N_1 \equiv A'$ which is impossible since $|A'|$ is a pencil whereas $|N_1|$ is a net. Hence $A'N_1 = 2$ and $A' \sum_{i=1}^n Z_i = 0$. Moreover, when $N_1^2 = 4$ (or, equivalently, $n = 3\ell$) we get, because of the rationality of Y,

$$N_1 \equiv 2A'$$

which is impossible since then (see lemma 2.1.5 and proposition 2.2.14)

$$4 = NN_1 = 2NA' = 6.$$

Hence in case $(1e)$ we can only have $n = 3\ell - 1$, $n = 3\ell - 2$ or $n = 3\ell - 3$.

Since $A'N_1 = 2$, all the curves Z_i do not intersect A'. Therefore from (4.7)

$$A_1 = A' + K_Y - G' - \sum_{i=1}^n Z_i - \sum_{j=1}^m C_j \equiv 0$$

and

$$\begin{aligned} 0 = A_1^2 &= A'^2 + K_Y^2 + 2A'K_Y + 1 + n + m \\ &= 1 - 2 - 3\ell - 2 + 1 + n + m = n - 2 - 3\ell + m \end{aligned} \tag{4.8}$$

hence $m = 3\ell - n + 2$.

We also know that in case $(1e)$ for any irreducible component B_{0k} of B_0 we have $B_{0k}A' \geq 1$ (see proposition 3.3.6). Moreover $A'N = 3, \Phi'N = 0$. Then

$$0 = B_{0k}N = B_{0k}(A' + \Phi') \geq 1 + B_{0k}\Phi'$$

for any $k = 1, \dots, \ell$ forces $B_0 \leq \Phi'$.

We recall that

$$N_1K_Y = (N + K_Y - G' - \sum_{i=1}^n Z_i)K_Y = 1 - 2 - 3\ell + 1 + n = n - 3\ell$$

Then we find

$$\begin{aligned} 0 = A_1N_1 &= N_1(A' + K_Y - G' - \sum_{i=1}^n Z_i - \sum_{j=1}^m C_j) \\ &= 2 + n - 3\ell - N_1 \sum_{j=1}^m C_j \leq n - (3\ell - 2) \end{aligned}$$

This excludes $n = 3\ell - 3$.

When $n = 3\ell - 2$ all the $m = 3\ell - n + 2 = 4$ curves C_j do not intersect N_1. Hence they are 4 of the $n' = 5$ curves Z'_i. Since $N_2 \equiv A_1 \equiv 0$ we find

$$N_1 \equiv \sum_{i=1}^{n} Z_i + G' + \sum_{j=1}^{5} Z'_j - K_Y \equiv A' + Z'_5.$$

Hence we get a contradiction since

$$0 = F'N_1 = F'A' + F'Z'_5 = 1 + 0 = 1.$$

When $n = 3\ell - 1$ we have $m = 3\ell - n + 2 = 3$ and, using proposition 2.3.12,

$$\begin{aligned} 0 = A_1N_1 &= N_1(A' + K_Y - G' - \sum_{i=1}^{n} Z_i - \sum_{j=1}^{m} C_j) \\ &= 2 - 1 - N_1 \sum_{j=1}^{m} C_j = 1 - N_1 \sum_{j=1}^{m} C_j \end{aligned}$$

Thus there is exactly one curve C_1 with $C_1N_1 = 1$ whereas the remaining two have to be chosen among the $n' \leq 2$ curves Z'_i. This also excludes the case $n' = 1$.

When $n' = 2$ we have

$$A'N_2 = A'(N_1 + K_Y - G' - \sum_{i=1}^{n} Z_i - \sum_{j=1}^{n'} Z'_j) = 2 - 1 = 1$$

and

$$(A' - N_2)^2 = 1 + 1 - 2 = 0.$$

Thus $A' \equiv N_2$ but we get a contradiction since

$$\Phi'N_2 = \Phi'A' = (N - A')A' = 3 - 1 = 2$$

and

$$2 \geq B_0N_2 = B_0A' = 3.$$

□

Proposition 4.6.2. *The case* $(0d)$ *of proposition 2.2.13 cannot occur.*

Proof. Assume case $(0d)$ holds. Then we have $\ell = 1$ (see theorem 3.3.8) and $K_Y^2 = -2 - 3\ell = -5$. Moreover $|A'|$ is a pencil of elliptic curves such that $A'^2 = 0 = A'K_Y$. Then if we look at the adjoint system $A' + K_Y$ we find $A'(A' + K_Y) = 0$. From lemma 1.1.9 there are (-1)-cycles D_j such that

$$A_1 = A' + K_Y - \sum_{j=1}^{m} D_j$$

is nef and $D_j A' = 0$. Since $A'A_1 = A'(A' + K_Y) = 0$ we necessarily have $A_1^2 = 0$. Hence (recall that $G'A' = 0$ from corollary 2.2.12)

$$0 = A_1^2 = (A' + K_Y - G' - \sum_j C_j)^2 = 0 + K_Y^2 + 0 + 1 + m = -4 + m$$

and

$$A_1 K_Y = K_Y(A' + K_Y - G' - \sum_{j=1}^{m} C_j) = 0 - 5 + 1 + m = A_1^2 = 0$$

Since from lemma 2.2.3 $A'N = 3$ we can write

$$0 \leq A_1 N = N(A' + K_Y - G' - \sum_{j=1}^{m} C_j) = 3 + 1 - N \sum_{j=1}^{m} C_j \leq 4$$

Assume now $1 \leq A_1 N = s \leq 4$. Then we have $N(3A_1 - sA') = 0$ and by the Index theorem 1.1.10 and the rationality of Y

$$(3A_1 - sA')^2 = 9A_1^2 + s^2 A'^2 - 6sA'A_1 = 0$$

and $sA' \equiv 3A_1$. Thus

$$1 \leq s = sA'B_0 = 3A_1 B_0$$

forces $s = 3$ and $A' \equiv A_1$ which is impossible since otherwise K_Y would be effective. Thus $A_1 N = 0$ hence $A_1 \equiv 0$ and

$$\begin{aligned} 0 = A_1 B_0 &= B_0(A' + K_Y - G' - \sum_{j=1}^{4} C_j) \\ &= 1 + 4 - B_0 \sum_{j=1}^{4} C_j = 5 - B_0 \sum_{j=1}^{4} C_j \end{aligned} \tag{4.9}$$

We note that B_0 cannot be contained in any singular fibre of $|A'|$ since $B_0 A' = 1 > 0$. In particular it is not contained in any of the (-1)-cycles C_j. Then

$B_0C_j \geq 0$ for any $j = 1, \dots, 4$ and from (4.9) there exists a cycle C_j, say C_1 such that $C_1B_0 \geq 2$. But $C_1A' = 0$ forces $C_1 \leq A'$ and

$$2 \leq C_1B_0 \leq A'B_0 = 1$$

and we get a contradiction. □

As a byproduct of propositions 4.6.1, 4.6.2 and theorem 3.3.8 we obtain

Theorem 4.6.3. *One has* $0 \leq \ell \leq 1$. *We can have* $\ell = 1$ *only in cases* $(1f)$ *and* $A' = N$.

Proposition 4.6.4. *Case* $(0a)$, $(0c)$ *and* $(0f)$ *of proposition 2.2.13 cannot occur.*

Proof. Let us begin with case $(0a)$ of proposition 2.2.13. Then $n = 3\ell = n' = 0$, $A'N = 2$ and

$$A'N_1 = A'(N + K_Y - G') = A'N = 2$$

hence $\Phi'N_1 = (N - A')N_1 = 2$. Moreover

$$A'N_2 = A'(N_1 + K_Y - G') = A'N_1 = 2$$

and

$$NN_2 = N(N_1 + K_Y - G') = 4 + 1 = 5$$

which implies $\Phi'N_2 = 3$. But we know, from proposition 2.2.13,

$$E'_i\Phi' = E'_i(N - A') = -1$$

for $i = 1, 2$ which forces $E'_1 + E'_2 \leq \Phi'$. Moreover

$$E'_kN_2 = E'_k(N + 2K_Y - 2G') = 2$$

for all $k = 1, \dots, h_1$. Then we get a contradiction since

$$4 = (E'_1 + E'_2)N_2 \leq \Phi'N_2 = 3.$$

Assume now that either case $(0c)$ or case $(0f)$ of proposition 2.2.13 holds. Then $n = 3\ell = 0$, $A'N = 3, \Phi'N = 0$ and

$$A'N_1 = A'(N + K_Y - G') = A'N = 3$$

Then

$$\Phi'N_1 = (N - A')N_1 = 4 - 3 = 1$$

Since N_1 is nef there exists exactly one irreducible component D of Φ' such that $DN_1 = 1$.

We also know, from proposition 2.2.13,

$$E_i'\Phi' = E_i'(N - A') \leq -1$$

for $i = 1, 2$ and then since

$$E_k'N_1 = E_k'(N + K_Y - G') = 1$$

for all $k = 1, \dots, h_1$ we get a contradiction. □

From theorems 3.3.8, 4.1.2, 4.2.2 and propositions 4.6.1, 4.6.2 and 4.6.4 we obtain the following result:

Theorem 4.6.5. *Case (iii) of page 34 can only occur when one of the following conditions is satisfied:*

1. $\ell = 0$
   - *Case* $(0g)$
   - *Cases* $(1a)$, $(1d)$, $(1f)$
   - $A' = N$
2. $\ell = 1$
   - *Case* $(1f)$
   - $A' = N$

Moreover in cases $(0g)$, $(1f)$ *and* $A' = N$ *we have* $\Phi = 0$, *i.e. the invariant pencil* $\Lambda \leq |3K_S|$ *has no fixed part.*

4.7 Summary

From a numerical analysis the only possible case is

$$R_0K_S = 0, h_2 = 1$$

and either $R_0 = 0$ or R_0 has $\ell \leq 3$ irreducible components which are smooth (-2)-curves. In all cases we have a pencil $|N|$ of nef and big curves of genus 3 on the rational surface Y which corresponds to the invariant part Λ of the tricanonical system $|3K_S|$ on the numerical Godeaux surface S.

The list of possible cases depends on the number n of curves (different from G') that we have to contract in order to find a surface W' where the adjoint system $|\bar{N}_1|$ of $|\bar{N}|$ is nef. Then we will contract W' to W where the adjoint systems of $|\bar{N}|$ of any index are nef curves (if they are non-empty).

We can actually show that R_0 is composed of at most one irreducible component (see theorem 4.6.3) and we have $3\ell - 3 \leq n \leq 3\ell$.

Let us start by assuming $n = 3\ell$. Then to get W' we need to contract G' and the curves Z_i. We have two possibilities:

- $W = W'$ and
  - $|\bar{N}|$ is a pencil of curves of genus 3 and $\bar{N}^2 = 3$
  - $|\bar{N}_1|$ is a net of curves of genus 3 and $\bar{N}_1^2 = 4$
  - $|\bar{N}_2|$ is a net of curves of genus 2 and $\bar{N}_2^2 = 3$
  - $|\bar{N}_3|$ is a pencil of curves of genus 0 and $\bar{N}_3^2 = 0$

  with $\varphi_{|\bar{N}_3|}$ morphism from W to $\mathbb{P}^1$ and $g : W \longrightarrow \mathbb{F}_a$ birational morphism.

- we have to contract two more curves Z'' and Z''' to get W and
  - $|\bar{N}|$ is a pencil of curves of genus 3 and $\bar{N}^2 = 16$
  - $|\bar{N}_1|$ is a net of curves of genus 3 and $\bar{N}_1^2 = 9$
  - $|\bar{N}_2|$ is a net of curves of genus 2 and $\bar{N}_2^2 = 3$
  - $|\bar{N}_3|$ is a pencil of curves of genus 1 and $\bar{N}_3^2 = 1$
  - $\bar{N}_4 \equiv 0$

  In this case we also find

$$N_3 \equiv G' + \sum_{i=1}^n Z_i + Z'' + Z''' - K_Y$$
$$N_2 \equiv 2G' + 2\sum_{i=1}^n Z_i + 2Z'' + Z''' - 2K_Y$$
$$N_1 \equiv 3G' + 3\sum_{i=1}^n Z_i + 2Z'' + Z''' - 3K_Y$$
$$N \equiv 4G' + 4\sum_{i=1}^n Z_i + 2Z'' + Z''' - 4K_Y$$
$$2B_0 + E' \equiv 7G' + 4\sum_{i=1}^n Z_i + 2Z'' + Z''' - 7K_Y$$

If we assume $n = 3\ell - 1$ we have to add to the above list two more possibilities:

- we have to contract one more cycle Z' to get W and
  - $|\bar{N}|$ is a pencil of curves of genus 3 and $\bar{N}^2 = 4$
  - $|\bar{N}_1|$ is a net of curves of genus 2 and $\bar{N}_1^2 = 3$
  - $|\bar{N}_2|$ is a pencil of curves of genus 0 and $\bar{N}_2^2 = 0$

  with $\varphi_{|\bar{N}_2|}$ morphism from W to $\mathbb{P}^1$ and $g : W \longrightarrow \mathbb{F}_a$ birational morphism.
- we have to contract three more cycles Z_1', Z_2' and Z'' to get W and
  - $|\bar{N}|$ is a pencil of curves of genus 3 and $\bar{N}^2 = 12$
  - $|\bar{N}_1|$ is a net of curves of genus 2 and $\bar{N}_1^2 = 4$
  - $|\bar{N}_2|$ is a pencil of curves of genus 1 and $\bar{N}_2^2 = 1$
  - $\bar{N}_3 \equiv 0$

  In this case we also find

$$\begin{gathered}
N_2 \equiv G' + \textstyle\sum_{i=1}^n Z_i + \sum_{i=1}^2 Z_i' + Z'' - K_Y \\
N_1 \equiv 2G' + 2\textstyle\sum_{i=1}^n Z_i + 2\sum_{i=1}^2 Z_i' + Z'' - 2K_Y \\
N \equiv 3G' + 3\textstyle\sum_{i=1}^n Z_i + 2\sum_{i=1}^2 Z_i' + Z'' - 3K_Y \\
2B_0 + E' \equiv 6G' + 3\textstyle\sum_{i=1}^n Z_i + 2\sum_{i=1}^2 Z_i' + Z'' - 6K_Y
\end{gathered}$$

When $n = 3\ell - 2$ we need to contract five more cycles $Z_1', \ldots, Z_5'$ to get W and

- $|\bar{N}|$ is a pencil of curves of genus 3 and $\bar{N}^2 = 8$
- $|\bar{N}_1|$ is a net of curves of genus 1 and $\bar{N}_1^2 = 1$
- $\bar{N}_2 \equiv 0$

In this case we also find

$$\begin{gathered}
N_1 \equiv G' + \textstyle\sum_{i=1}^n Z_i + \sum_{i=1}^5 Z_i' - K_Y \\
N \equiv 2G' + 2\textstyle\sum_{i=1}^n Z_i + \sum_{i=1}^5 Z_i' - 2K_Y \\
2B_0 + E' \equiv 5G' + 2\textstyle\sum_{i=1}^n Z_i + \sum_{i=1}^2 Z_i' - 5K_Y
\end{gathered}$$

Finally when $n = 3\ell - 3$ we have $\ell = 1$ (then $n = 0$) and the first adjoint $|N_1|$ to $|N|$ is a net of rational curves and it gives a birational morphism to $\mathbb{P}^2$.

Moreover only the cases listed in theorem 4.6.5 can occur.

Chapter 5

Geometric analysis

5.1 Ruled cases

5.1.1 $n = 3\ell$

By definition 4.0.11 and the results of section 4.5 we know that $|\bar{N}_3|$ gives a morphism $g : W \longrightarrow \mathbb{F}_a$ for some $a \geq 0$ with $K_W^2 = K_Y^2 + 1 + 3\ell = -1$. Then we have

$$K_{\mathbb{F}_a} = -2c - (a+2)f, \quad g^*(f) = \bar{N}_3$$

where c is the $(-a)$-section of $\mathbb{F}_a$ and f is a fibre of the ruling $\mathbb{F}_a \longrightarrow \mathbb{P}_1$. Then $K_W = -2g^*(c) - (a+2)\bar{N}_3 + \Delta$ where Δ is the exceptional divisor of g.

Therefore

$$\begin{aligned}
\bar{N}_2 &= \bar{N}_3 - K_W = 2g^*(c) + (a+3)\bar{N}_3 - \Delta \\
\bar{N}_1 &= \bar{N}_2 - K_W = 4g^*(c) + (2a+5)\bar{N}_3 - 2\Delta \\
\bar{N} &= \bar{N}_1 - K_W = 6g^*(c) + (3a+7)\bar{N}_3 - 3\Delta
\end{aligned}$$

Lemma 5.1.1. *In the above setting* $0 \leq a \leq 2$.

Proof. Since $\bar{N}$ is nef we find

$$g^*(c)\bar{N} = 7 - 3a \geq 0$$

and then $0 \leq a \leq 2$. □

We look at $2\bar{B}_0 + \bar{E}'$: from the definition of N on the surface Y one has

$$2\bar{B}_0 + \bar{E}' = \bar{N} - 3K_W = 12g^*(c) + (6a+13)\bar{N}_3 - 6\Delta \tag{5.1}$$

whence $g(2\bar{B}_0 + \bar{E}') = 12c + (6a+13)f$.

Furthermore

$$g^*(c)(2\bar{B}_0+\bar{E}') = g^*(c)(12g^*(c)+(6a+13)\bar{N}_3-6\Delta) = -12a+13+6a = 13-6a$$

From theorem 4.6.3 we have $\ell = 0, 1$.

$\underline{\ell = 0}$

Let us assume $\ell = 0$. Then $2\bar{B}_0 + \bar{E}' = \bar{E}'$ and we can write $\bar{E}'_i = \alpha_i g^*(c) + \beta_i \bar{N}_3 + \sum_j \gamma_{ji}\Delta_j$. We also recall that $h_1 = 4 + \ell = 4$ from (2.5). Then

$$3 = \bar{N}_3\bar{E}'_i = \bar{N}_3[\alpha_i g^*(c) + \beta_i\bar{N}_3 + \sum_j \gamma_{ji}\Delta_j] = \alpha_i$$

This implies

$$g^*(c)\bar{E}'_i = g^*(c)[\alpha_i g^*(c) + \beta_i\bar{N}_3 + \sum_j \gamma_{ji}\Delta_j] = -a\alpha_i + \beta_i = \beta_i - 3a$$

Lemma 5.1.2. *In the above setting we have $g^*(c)\bar{E}'_i \geq 0$. In particular we have $\beta_i \geq 3a$ for all $i = 1, \ldots, 4$.*

Proof. It is obvious since $g^*(c)\bar{E}'_i < 0$ and the irreducibility of $\bar{E}'_i$ would imply $\bar{E}'_i \leq g^*(c)$ and therefore $\bar{E}'_i = \bar{c}$ the strict transform of c. Then we get a contradiction since

$$3 = \bar{E}'_i\bar{N}_3 = \bar{c}\bar{N}_3 = g^*(c)\bar{N}_3 = 1.$$

□

Moreover from equation (5.1) we have $\sum_{i=1}^4 \beta_i = 13 + 6a$.

Lemma 5.1.3. *Each irreducible component C in the singular fibres of $\varphi_{|\bar{N}_3|} : W \longrightarrow \mathbb{P}^1$ is a rational curve with $C^2 = -1, -2$.*

Proof. We know that $\bar{N}_2 = \bar{N}_3 - K_W$ is nef. C satisfies $C\bar{N}_3 = 0$ hence, by Zariski's lemma 1.1.2, $C^2 \leq 0$ and

$$C\bar{N}_2 = C(\bar{N}_3 - K_W) \geq 0$$

which implies $CK_W \leq 0$. By the Index theorem 1.1.10 and the rationality of W if it was $CK_W = 0$ we should have $C^2 < 0$ hence $C^2 = -2$. On the other hand, if $CK_W \leq -1$ then $C^2 \geq -2 - CK_W \geq -1$ forces $C^2 = -1 = CK_W$. □

Remark 5.1.4. *For any (-2)-curve C' which is contained in a singular fibre we have*

$$C'\bar{E}' = C'(\bar{N}_3 - 6K_W) = 0$$

so C' does not intersect any of the curves $\bar{E}'_k$. Therefore the intersection of $\bar{E}'_k$ with the singular fibres is only given by the points of intersection with the (-1)-curves.

Lemma 5.1.5. *Each singular fibre contains two irreducible (-1)-curves with multiplicity 1.*

Proof. Assume that there are m irreducible (-1)-curves appearing with multiplicity $b_i \geq 1$ $i = 1, \ldots, m$. Then from lemma 5.1.3 and the rationality of $|\bar{N}_3|$ we find

$$-2 = \bar{N}_3\bar{K}_W = -\sum_{i=1}^{m} b_i$$

hence either $m = 1, b_1 = 2$ or $m = 2, b_1 = b_2 = 1$.

Since $\bar{E}'_k\bar{N}_3 = 3$ for any $k = 1, \ldots, 4$ and since the curves $\bar{E}'_k$ cannot intersect the (-2)-curves in each singular fibre (see remark 5.1.4) there cannot be a fibre with only one (-1)-curve of multiplicity 2. □

Lemma 5.1.6. *For each singular fibre the curve $\bar{E}'$ intersects the exceptional curves of that fibre.*

Proof. Assume there is a singular fibre ψ of $g : W \longrightarrow \mathbb{F}_a$ such that $\bar{E}'$ does not intersect any of the exceptional curves of that fibre. Then there exists a curve Γ in ψ such that $\Gamma\bar{E}' = \bar{E}'\bar{N}_3 = 12$ and Γ is not contracted by $g : W \longrightarrow \mathbb{F}_a$. Hence

$$12 = \Gamma\bar{E}' = \Gamma(12g^*(c) + (13 + 6a)\bar{N}_3 - 6\Delta) = 12\Gamma g^*(c) - 6\Gamma\Delta$$

Since $\Gamma\Delta \geq 1$ and $\Gamma\Delta \equiv 0 \mod 2$ we find

$$12 \leq 12\Gamma g^*(c) - 12$$

hence $\Gamma g^*(c) \geq 2$. Let $g(\Gamma) = f_1$ be the fibre of the ruling of $\mathbb{F}_a$ obtained by Γ. Then

$$1 = f_1 c = g^*(f_1)g^*(c) = \Gamma g^*(c) \geq 2$$

and we get a contradiction. □

Lemma 5.1.7. *In the above setting we can reduce to the case $a = 1$ unless $a = 2$ and $\varphi_{|\bar{N}_3|} : W \longrightarrow \mathbb{P}^1$ has at most two singular fibres.*

Proof. We know that $\bar{F}'\bar{N}_3 = \bar{H}'\bar{N}_3 = 0$ and the two (-2)-curves are contained in a singular fibre of $\varphi_{|\bar{N}_3|}$. We can choose the map g so that it contracts these curves to a point which is now on a nonsingular fibre f_0 of the map $\mathbb{F}_a \longrightarrow \mathbb{P}^1$.

If $a = 0$ and we blow up the above point and we consider the section c which intersects f_0 at that point, the strict transform of c is a (-1)-curve. By contracting the strict transform of f_0 the exceptional divisor becomes a curve with self-intersection 0. Therefore the surface now obtained is $\mathbb{F}_1$.

We can do the same for $a = 2$ if the point is not the intersection point between f_0 and the (-2)-section c on $\mathbb{F}_2$.

Assume now that $a = 2$ and c passes through the above point P_0. We can reduce to $a = 1$ if we find a singular fibre f_1 such that c does not pass through the point obtained by contracting all the exceptional curves of $g : W \longrightarrow \mathbb{F}_2$ in that fibre.

Let us suppose such a fibre f_1 does not exist. Then for any singular fibre f' the (-2)-section c passes through the point P' which is the contraction of all the exceptional curves in that fibre. From lemma 5.1.6 we can deduce that P must be a point in $\bar{E}'$. Since $g^*(c)\bar{E}' = 13 - 6a = 1$ there can be at most one such fibre. Thus, if the number of singular fibres is at least 3 we are done. □

Lemma 5.1.8. *For any* $i = 1, \ldots, 4$ *and* $j = 1, \ldots, 9$ *we have*

$$\Delta_k \sum_j \gamma_{ji}\Delta_j = 0 \quad \textit{if } \Delta_k^2 = -2$$

$$0 \leq \Delta_k \sum_j \gamma_{ji}\Delta_j \leq 3 \quad \textit{if } \Delta_k^2 = -1.$$

Proof. Since

$$\bar{E}'_i = 3g^*(c) + \beta_i\bar{N}_3 + \sum_j \gamma_{ji}\Delta_j$$

we have

$$0 \leq \bar{E}'_i\bar{\Delta}_k = \Delta_k \sum_j \gamma_{ji}\Delta_j \leq \bar{E}'_i\bar{N}_3 = 3$$

From lemma 5.1.3 the curves Δ_k have self-intersection $-2 \leq \Delta_k^2 \leq -1$.

Furthermore, from remark 5.1.4, if $\Delta_k^2 = -2$

$$\bar{E}'_i\Delta_k = 0$$

hence

$$0 = \bar{E}_i' \Delta_k = \Delta_k \sum_j \gamma_{ji} \Delta_j$$

which proves the lemma. □

Corollary 5.1.9. *In the above setting for any* $i = 1, \ldots, 4$ *we have*

$$\Delta \sum_j \gamma_{ji} \Delta_j \leq 6r$$

where r *is the number of singular fibres of* $g : W \longrightarrow \mathbb{F}_a$.

Proof. Let us set

$$V := \{v \,|\, \Delta_v^2 = -1\}$$

From the above proposition we have

$$\Delta \sum_j \gamma_{ji} \Delta_j = \sum_{v \in V} \Delta_v \gamma_{ji} \Delta_j \leq 3|V|$$

where $|V|$ is the cardinality of the set V. From lemma 5.1.5 there are two simple (-1)-curves in each of the r singular fibres then

$$|V| \leq 2r$$

as wanted. □

We are now ready to show that the reduction to $a = 1$ it is always possible.

Proposition 5.1.10. *The case* $a = 2$ *cannot occur with* $r \leq 2$ *singular fibres.*

Proof. We know from the formulas of page 97 that

$$\bar{N}_2 = 2g^*(c) + (a+3)\bar{N}_3 - \Delta$$

Then

$$\begin{aligned} 2 = E_i' N_2 = \bar{E}_i' \bar{N}_2 &= (3g^*(c) + \beta_i \bar{N}_3 + \sum_j \gamma_{ji} \Delta_j)(2g^*(c) + (a+3)\bar{N}_3 - \Delta) \\ &= -6a + 3a + 9 + 2\beta_i - \Delta \sum_j \gamma_{ji} \Delta_j \end{aligned}$$

hence

$$\Delta \sum_j \gamma_{ji} \Delta_j = 2\beta_i + 7 - 3a$$

Thus, from lemma 5.1.2 and corollary 5.1.9,

$$7 + 3a \leq 2\beta_i + 7 - 3a \leq 6r$$

where r is the number of singular fibres of $g : W \longrightarrow \mathbb{F}_a$. When $a = 2$ we get

$$6r \geq 13$$

and then $r \geq 3$ as wanted. $\square$

From now on we assume $a = 1$. The pencil $|\bar{N}_3|$ is mapped to the pencil of lines of $\mathbb{P}^2$ through a point P. Then $|\bar{N}_2|$ maps to the net of quartics with 1 double point and 9 simple base points, $|\bar{N}_1|$ to the net of curves of degree 7 with 1 triple point and 9 double points (with no other simple base points), and $|\bar{N}|$ to the pencil of curves of degree 10 with one 4-tuple point, 9 triple points and no other base points.

Theorem 5.1.11. *The case $n = 3\ell = 0, n' = n'' = 0$ cannot occur.*

Proof. We compute the plane image of $|\bar{A}'|$. From theorem 4.6.5 we know that $\ell = 0$ can only occur in case $(0g)$ of proposition 2.2.13, in cases $(1a)$, $(1d)$ or $(1f)$ of proposition 2.2.14 and when $A' = N$. Then, using also lemma 2.2.3 and corollary 2.2.12, $A'^2 = 0, A'K_Y = 2, A'N = 3, A'G' = 1$ in the former case while we have $A'K_Y = 1, A'N = 3, A'G' = 0$ in the latter cases with $A'^2 = 1$ unless $A' = N$. Hence we find

$$A'N_3 = A'(N + 3K_Y - 3G') = 3 + 3A'K_Y - 3A'G' = 6$$

in all the above cases. Then A' is mapped onto a plane curve of degree d with a point of multiplicity $d - 6$ at P and, denoting by s_j the number of points of multiplicity j among $P_1, \ldots, P_9$,

$$3 = A'N = \bar{A}'\bar{N} = 10d - 3\sum_j js_j - 4(d-6)$$

hence

$$\sum_j js_j = 2d + 7 \tag{5.2}$$

We also have $\bar{A}'^2 = 1, p_a(\bar{A}') = 2$ in all the above cases except for $A' = N$. Then, if $A' \neq N$,

$$1 = \bar{A}'^2 = d^2 - \sum_j j^2 s_j$$

hence

$$\sum_j j^2 s_j = d^2 - 1 \tag{5.3}$$

and

$$\begin{aligned} 2 = p_a(\bar{A}') &= \frac{(d-1)(d-2)}{2} - \sum_j s_j \frac{j(j-1)}{2} - \frac{(d-6)(d-7)}{2} \\ &= \frac{d^2 - 3d + 2 - d^2 + 13d - 42}{2} - \sum_j s_j \frac{j(j-1)}{2} \end{aligned}$$

hence

$$\sum_j s_j j(j-1) = 10d - 44 \tag{5.4}$$

Then comparing (5.2), (5.3) and (5.4) we get

$$10d - 44 = d^2 - 1 - (2d + 7) = d^2 - 2d - 8$$

hence

$$d^2 - 12d + 36 = (d-6)^2 = 0$$

which forces $d = 6$. In this case (5.2) and (5.3) become

$$\sum_j j s_j = 19$$

$$\sum_j j^2 s_j = 35$$

We now easily infer $j \leq 5$. Subtracting the first equation from the second one we find

$$\begin{aligned} 16 = 35 - 19 = (&25s_5 + 16s_4 + 9s_3 + 4s_2 + s_1) + \\ &- (5s_5 + 4s_4 + 3s_3 + 2s_2 + s_1) = 20s_5 + 12s_4 + 6s_3 + 2s_2 \end{aligned}$$

Then $s_5 = 0, s_4 \leq 1$.

Then we find

$$6s_3 + 2s_2 = 16 - 12s_4$$

or equivalently,

$$3s_3 + s_2 = 8 - 6s_4$$

and substituting in (5.2)

$$s_1 + s_2 = 19 - 4s_4 - (3s_3 + s_2) = 19 - 4s_4 - (8 - 6s_4) = 11 + 2s_4$$

which gives a contradiction since $\sum_j s_j \leq 9$.

We now discuss the case $A' = N$. Then $\bar{A}'^2 = \bar{N}^2 = 3$ and $p_a(\bar{A}') = 3$. Then

$$3 = \bar{A}'^2 = d^2 - \sum_j j^2 s_j$$

forces

$$\sum_j j^2 s_j = d^2 - 3 \tag{5.5}$$

and

$$\begin{aligned} 3 = p_a(\bar{A}') &= \frac{(d-1)(d-2)}{2} - \sum_j s_j \frac{j(j-1)}{2} - \frac{(d-6)(d-7)}{2} \\ &= \frac{d^2 - 3d + 2 - d^2 + 13d - 42}{2} - \sum_j s_j \frac{j(j-1)}{2} \end{aligned}$$

hence

$$\sum_j s_j j(j-1) = 10d - 46 \tag{5.6}$$

Thus comparing (5.2), (5.5) and (5.6) we find

$$10d - 46 = d^2 - 3 - (2d + 7) = d^2 - 2d - 10$$

hence

$$d^2 - 12d + 36 = (d-6)^2 = 0$$

forces $d = 6$ while we know $d = 10$ since $\bar{A}' = \bar{N}$. $\square$

$\underline{\ell = 1}$

Let us now assume $B_0 \neq 0$. For any curve E'_i on the rational surface Y there is at most one of the curves Z_j which intersects E'_i (see corollary 2.3.7).

Assume $n = 3\ell = 3$. If all the cycles Z_j are irreducible then there are exactly 3 of the 5 curves E'_i (we can suppose they are E'_3, E'_4 and E'_5) which are intersected by one (and only one) of the curves Z_j: we have $E'_i N_3 = 0$ for each of them. This implies that they are contained in singular fibres of the map $\varphi_{|N_3|} : Y \longrightarrow \mathbb{P}_1$.

If one of the cycles Z_j is reducible then, from corollary 2.3.11, either Z_1, Z_2 are irreducible and $Z_3 = Z_1 + Z_2 + E'_k$ for some $1 \leq k \leq h_1 = 5$ or Z_1 is irreducible, $Z_2 = Z_1 + C$ (with C a (-2)-curve) and $Z_3 = Z_1 + Z_2 + E'_k$ for some

$k \leq 5$. In any case we have $E'_k N_1 = 0$ (see proposition 2.3.3) hence $E'_k N_3 = 0$. Moreover since $\ell = 1$ from lemma 2.3.6 and corollary 2.3.7

$$B_0 \sum_{i=1}^{n} Z_i = 2B_0 Z_1 + 2B_0 Z_2 = 4$$

hence $B_0 N_1 = 0$ forces $B_0 N_3 = 0$.

We now can show the following

Theorem 5.1.12. *The case $n = 3\ell, n' = n'' = 0$ with $\ell = 1$ cannot occur.*

Proof. Let us now consider the fibration given by the rational pencil $|N_3|$. If we set $\delta := \sum_s (e(N_{3s} - e(N_3))$ from proposition 1.1.3 we have

$$\delta = e(Y) - e(N_3)e(\mathbb{P}_1) = 12 - K_Y^2 - 4 \overset{(4.1)}{=} 12 + 2 + 3\ell - 4 = 13.$$

From lemma 3.3.3 every curve C in a singular fibre contributes $-C^2$ to δ. If Z_1, Z_2, Z_3 are irreducible then we know that the (-3)-curves F', H', E'_3, E'_4, E'_5 are disjoint and they are contained in singular fibres.

If Z_3 is reducible then the (-3)-curves F', H', E'_k and the (-6)-curve B_0 are contained in singular fibres.

In any case we find

$$13 = \delta \geq 15.$$

Contradiction. $\square$

5.1.2 $n = 3\ell - 1$

From section 4.4 we know that $|\bar{N}_2|$ gives a morphism $g : W \longrightarrow \mathbb{F}_a$ for some $a \geq 0$ with $K_W^2 = K_Y^2 + 1 + 3\ell - 1 + 1 = -1$. Then we have

$$K_{\mathbb{F}_a} = -2c - (a+2)f, \quad g^*(f) = \bar{N}_2$$

and $K_W = -2g^*(c) - (a+2)\bar{N}_2 + \Delta$ where Δ is the exceptional divisor of g.

Therefore

$$\bar{N}_1 = \bar{N}_2 - K_W = 2g^*(c) + (a+3)\bar{N}_2 - \Delta$$
$$\bar{N} = \bar{N}_1 - K_W = 4g^*(c) + (2a+5)\bar{N}_2 - 2\Delta$$

Lemma 5.1.13. *In the above setting* $0 \leq a \leq 2$.

Proof. Since $\bar{N}$ is nef we find

$$g^*(c)\bar{N} = 5 - 2a \geq 0$$

and then $0 \leq a \leq 2$. $\square$

Then we look at $2\bar{B}_0 + \bar{E}'$: since

$$2\bar{B}_0 + \bar{E}' = \bar{N} - 3K_W = 10g^*(c) + (5a + 11)\bar{N}_2 - 5\Delta$$

we have $g(2\bar{B}_0 + \bar{E}') = 10c + (5a + 11)f$.

Furthermore

$$g^*(c)(2\bar{B}_0+\bar{E}') = g^*(c)(10g^*(c)+(5a+11)\bar{N}_2-5\Delta) = -10a+11+5a = 11-5a$$

We recall that $n = 3\ell - 1$ can only occur when $\ell \geq 1$. From theorem 4.6.5 and the results of section 4.4 we have $\ell = 1, n = 2, n' = 1$. By definition of N we find

$$1 = Z'N = Z'(2B_0 + E' + 3K_Y - 3G') = 2B_0Z' + E'Z' - 3$$

hence $2B_0Z' + E'Z' = 4$. This implies $0 \leq B_0Z' \leq 2$ and $0 \leq E'Z' = 4 - 2B_0Z' \leq 4$.

We also compute using corollary 2.3.7

$$\begin{aligned} B_0N_2 &= B_0(N + 2K_Y - 2G' - 2Z_1 - 2Z_2 - Z') \\ &= 0 + 8 - 0 - 2 - 2 - B_0Z' = 4 - B_0Z' \end{aligned}$$

$$\begin{aligned} E'_kN_2 &= E'_k(N + 2K_Y - 2G' - 2Z_1 - 2Z_2 - Z') \\ &= 0 + 2 - 0 - 2(Z_1 + Z_2)E'_k - E'_kZ' \leq 2 - E'_kZ' \end{aligned}$$

Since from corollaries 2.3.11 and 2.3.7 we have $Z_1E' = Z_2E' = 1$ there exist two curves E'_k (say E'_4 and E'_5) such that each of them intersects Z_1 or Z_2 at one point. For those curves we necessarily have $E'_4N_2 = E'_5N_2 = 0$ and $E'_4Z' = E'_5Z' = 0$.

Lemma 5.1.14. *In the above setting it cannot be* $B_0Z' = 0$, $E'Z' = 4$. *In particular we have* $0 \leq E'_kZ' \leq 1$ *for all* $k = 1, \ldots, 5$.

Proof. Let us consider the Euler number of the fibration $\varphi_{|N_2|} : Y \longrightarrow \mathbb{P}_1$ according to proposition 1.1.3. If we set

$$\delta := \sum_s (e(N_{2s}) - e(N_2))$$

where the sum is taken over all the singular curves in $|N_2|$, we have

$$\delta = e(Y) - e(N_2)e(\mathbb{P}_1) = 12 - K_Y^2 - 4 \overset{(4.1)}{=} 12 + 5 - 4 = 13.$$

Since every curve C in a singular fibre contributes $-C^2$ to δ (see also lemma 3.3.3) and we know that the (-3)-curves F', H', E_4', E_5' are disjoint and they are contained in singular fibres we find

$$12 \leq \delta = 13.$$

If it was $E_k'Z' = 2$ for some $k = 1, 2, 3$ we would also have $E_k'N_2 = 0$ and E_k' would be disjoint from all the above ones and contained in a singular fibre. Then it would be

$$15 = 3 + 12 \leq \delta = 13.$$

Contradiction. Then $E'Z' = (E_1' + E_2' + E_3')Z' \leq 3$ as wanted. □

Lemma 5.1.14 forces one of the two following options:

1. $B_0Z' = 2, E'Z' = 0, B_0N_2 = 2, E_k'N_2 = 2, k = 1, \dots, 3$
2. $B_0Z' = 1, E'Z' = 2, B_0N_2 = 3, E_1'N_2 = 2, E_k'N_2 = 1, k = 2, 3$

Remark 5.1.15. *It is easy to see that in both cases 1 and 2 there is a curve E_k', say E_1', such that $E_1'Z_i = E_1'Z' = 0$. Then $\bar{E}_1'$ is a (-3)-curve such that $\bar{E}_1'\bar{N} = 0$.*

Let us set $\bar{E}_0' := \bar{B}_0$.

Lemma 5.1.16. *In the above setting for any curve $\bar{E}_i'$ such that $\bar{E}_i'\bar{N}_2 \geq 2$ we have $g^*(c)\bar{E}_i' \geq 0$.*

Proof. It is obvious since $g^*(c)\bar{E}_i' < 0$ and the irreducibility of $\bar{E}_i'$ would imply $\bar{E}_i' \leq g^*(c)$ and therefore $\bar{E}_i' = \bar{c}$ the strict transform of c. Then we get a contradiction as in lemma 5.1.2 since

$$2 \leq \bar{E}_i'\bar{N}_2 = \bar{c}\bar{N}_2 = g^*(c)\bar{N}_2 = 1.$$

□

Lemma 5.1.17. *Each irreducible component C in the singular fibres of $\varphi_{|\bar{N}_2|} : W \longrightarrow \mathbb{P}^1$ is a rational curve with $C^2 = -1, -2$.*

Proof. We know that $\bar{N}_1 = \bar{N}_2 - K_W$ is nef. C satisfies $C\bar{N}_2 = 0$ hence, by Zariski's lemma 1.1.2, $C^2 \leq 0$ and

$$C\bar{N}_1 = C(\bar{N}_2 - K_W) \geq 0$$

which implies $CK_W \leq 0$. By the Index theorem 1.1.10 and the rationality of W if it was $CK_W = 0$ we should have $C^2 < 0$ hence $C^2 = -2$. On the other hand, if $CK_W \leq -1$ then $C^2 \geq -2 - CK_W \geq -1$ forces $C^2 = -1 = CK_W$. □

Remark 5.1.18. *For any (-2)-curve C' different from $\bar{E}'_4$ and $\bar{E}'_5$ which is contained in a singular fibre we have*

$$C'(2\bar{B}_0 + \bar{E}') = C'(\bar{N}_2 - 5K_W) = 0$$

so C' does not intersect the curves $\bar{E}'_k$ and $\bar{B}_0$.

Lemma 5.1.19. *Each singular fibre contains at most two irreducible (-1)-curves.*

Proof. Assume that there are m irreducible (-1)-curves appearing with multiplicity $b_i \geq 1$ $i = 1, \ldots, m$. Then from lemma 5.1.17 and because of the rationality of $|\bar{N}_2|$ we find

$$-2 = \bar{N}_2 \bar{K}_W = -\sum_{i=1}^{m} b_i$$

hence either $m = 1, b_1 = 2$ or $m = 2, b_1 = b_2 = 1$. □

Lemma 5.1.20. *For each singular fibre the curve $\bar{E}'$ intersects the exceptional curves of that fibre.*

Proof. Assume there is a singular fibre ψ of $g : W \longrightarrow \mathbb{F}_a$ such that $\bar{E}'$ does not intersect any of the exceptional curves of that fibre. Then there exists a curve Γ in the singular fibres such that $\Gamma(2\bar{B}_0 + \bar{E}') = (2\bar{B}_0 + \bar{E}')\bar{N}_2 = 10$ and Γ is not contracted by $g : W \longrightarrow \mathbb{F}_a$. Then

$$10 = \Gamma(2\bar{B}_0 + \bar{E}') = \Gamma(10g^*(c) + (11 + 5a)\bar{N}_3 - 5\Delta) = 10\Gamma g^*(c) - 5\Gamma\Delta$$

Since $\Gamma\Delta \geq 1$ and $\Gamma\Delta \equiv 0 \mod 2$ we find

$$10 \leq 10\Gamma g^*(c) - 10$$

hence $\Gamma g^*(c) \geq 2$. Let $g(\Gamma) = f_1$ be the fibre of the ruling of $\mathbb{F}_a$ obtained by Γ. Then

$$1 = f_1 c = g^*(f_1) g^*(c) = \Gamma g^*(c) \geq 2$$

and we get a contradiction. $\square$

Lemma 5.1.21. *In the above setting we can always reduce to the case $a = 1$ unless $a = 2$ and $\varphi_{|\bar{N}_2|} : W \longrightarrow \mathbb{P}^1$ has at most two singular fibres.*

Proof. We know that $\bar{F}'\bar{N}_2 = \bar{H}'\bar{N}_2 = 0$ and the two (-2)-curves are contained in a singular fibre of $\varphi_{|\bar{N}_2|}$. We can choose the map g to contract these curves to a point which is now on a nonsingular fibre f_0 of the map $\mathbb{F}_a \longrightarrow \mathbb{P}^1$.

If $a = 0$ and we blow up the above point and we consider the section c which intersects f_0 at that point, the strict transform of c is a (-1)-curve. By contracting the strict transform of f_0 the exceptional divisor becomes a curve with self-intersection 0. Therefore the surface now obtained is $\mathbb{F}_1$.

We can do the same for $a = 2$ if the point is not the intersection point between f_0 and the (-2)-section c on $\mathbb{F}_2$.

Assume now that $a = 2$ and c passes through the above point P_0. We can reduce to $a = 1$ if we find a singular fibre f_1 such that c does not pass through the point obtained by contracting all the exceptional curves of $g : W \longrightarrow \mathbb{F}_2$ in that fibre.

Let us suppose such a fibre f_1 does not exist. Then for any singular fibre f' the (-2)-section c passes through the point P' which is the contraction of all the exceptional curves in that fibre. From lemma 5.1.6 we can deduce that P must be a point in $\bar{E}'$. Since $g^*(c)(2\bar{B}_0 + \bar{E}') = 11 - 5a = 1$ there can be at most one such fibre. Then, if the number of singular fibres is at least 3 we are done. $\square$

We can write

$$\bar{E}'_0 = \bar{B}_0 = \alpha_0 g^*(c) + \beta_0 \bar{N}_2 + \sum_j \gamma_{j0} \Delta_j.$$

$$\bar{E}'_i = \alpha_i g^*(c) + \beta_i \bar{N}_2 + \sum_j \gamma_{ji} \Delta_j.$$

Then for any $i \geq 0$

$$\bar{N}_2 \bar{E}'_i = \bar{N}_2[\alpha_i g^*(c) + \beta_i \bar{N}_2 + \sum_j \gamma_{ji} \Delta_j] = \alpha_i.$$

This implies

$$g^*(c)\bar{E}'_i = g^*(c)[\alpha_i g^*(c) + \beta_i \bar{N}_3 + \sum_j \gamma_{ji}\Delta_j] = -a\alpha_i + \beta_i \tag{5.7}$$

Lemma 5.1.22. *For any $i = 1, \ldots, 3$ and $j = 1, \ldots, 9$ we have*

$$\Delta_k \sum_j \gamma_{ji}\Delta_j = 0 \quad \textit{if } \Delta_k^2 = -2, \Delta_k \neq \bar{E}'_4, \bar{E}'_5$$

$$0 \leq \Delta_k \sum_j \gamma_{ji}\Delta_j \leq \alpha_i \quad \textit{if } \Delta_k^2 = -1.$$

Proof. Since

$$\bar{E}'_i = \alpha_i g^*(c) + \beta_i \bar{N}_2 + \sum_j \gamma_{ji}\Delta_j$$

if $\Delta_k \neq \bar{E}'_4, \bar{E}'_5$ we have

$$0 \leq \Delta_k \sum_j \gamma_{ji}\Delta_j = \bar{E}'_i\bar{\Delta}_k \leq \bar{E}'_i\bar{N}_2 = \alpha_i \tag{5.8}$$

From lemma 5.1.17 the curves Δ_k have self-intersection $-2 \leq \Delta_k^2 \leq -1$. Furthermore, from remark 5.1.18, if $\Delta_k^2 = -2$, $\Delta_k \neq \bar{E}'_4, \bar{E}'_5$

$$\bar{E}'_i\Delta_k = 0$$

hence

$$0 = \bar{E}'_i\Delta_k = \Delta_k \sum_j \gamma_{ji}\Delta_j$$

which proves the lemma. □

Corollary 5.1.23. *In the above setting for any $i = 1, \ldots, 3$ we have*

$$\Delta \sum_j \gamma_{ji}\Delta_j \leq 2\alpha_i r + \bar{E}'_4\bar{E}'_i + \bar{E}'_5\bar{E}'_i$$

where r is the number of singular fibres of $g : W \longrightarrow \mathbb{F}_a$.

Proof. Let us define

$$V := \{v \mid \Delta_v^2 = -1\}$$

From the above proposition (see (5.8)) we have

$$\Delta \sum_j \gamma_{ji}\Delta_j = \sum_{v \in V} \Delta_v \gamma_{ji}\Delta_j + (\bar{E}'_4 + \bar{E}'_5)\sum_j \gamma_{ji}\Delta_j$$

$$= \sum_{v \in V} \Delta_v \gamma_{ji} \Delta_j + (\bar{E}_4' + \bar{E}_5')\bar{E}_i' \leq \alpha_i |V| + (\bar{E}_4' + \bar{E}_5')\bar{E}_i'$$

where $|V|$ is the cardinality of the set V. From lemma 5.1.19 there are at most two (-1)-curves in each of the r singular fibres then

$$|V| \leq 2r$$

as wanted. □

We are now ready to show that the reduction to $a = 1$ it is always possible.

Proposition 5.1.24. *The case $a = 2$ cannot occur with $r \leq 2$ singular fibres.*

Proof. We know from the formulas of page 105 that

$$\bar{N}_1 = 2g^*(c) + (a+3)\bar{N}_2 - \Delta$$

Then

$$E_i' N_1 = \bar{E}_i' \bar{N}_1 = (\alpha_i g^*(c) + \beta_i \bar{N}_2 + \sum_j \gamma_{ji} \Delta_j)(2g^*(c) + (a+3)\bar{N}_3 - \Delta)$$
$$= -2a\alpha_i + a\alpha_i + 3\alpha_i + 2\beta_i - \Delta \sum_j \gamma_{ji} \Delta_j$$

hence

$$\Delta \sum_j \gamma_{ji} \Delta_j = 2\beta_i + (3-a)\alpha_i - E_i' N_1$$

Let us now fix $i = 1$. From remark 5.1.15 we know that $\bar{E}_1'$ is a (-3)-curve and $E_1' Z' = E_1' Z_1 = E_1' Z_2 = 0$ hence

$$E_1' N_1 = E_1'(N + K_Y - G' - Z_1 - Z_2) = E_1' K_Y = 1$$

$$\alpha_1 = \bar{E}_1' \bar{N}_2 = E_1' N_2 = E_1'(N + 2K_Y - 2Z_1 - 2Z_2 - Z') = 2.$$

Thus, from lemma 5.1.16, equation (5.7) and corollary 5.1.23,

$$2(3+a) - 1 = (3+a)\alpha_1 - 1 \leq 2\beta_1 + (3-a)\alpha_1 - 1$$
$$= \Delta \sum_j \gamma_{j1} \Delta_j \leq 2\alpha_1 r + \bar{E}_1'(\bar{E}_4' + \bar{E}_5') = 2\alpha_1 r = 4r$$

where r is the number of singular fibres of $g : W \longrightarrow \mathbb{F}_a$. When $a = 2$ we get

$$4r \geq 9$$

and then $r \geq 3$ as wanted. □

From now on we assume $a = 1$. The pencil $|\bar{N}_2|$ is mapped to the pencil of lines of $\mathbb{P}^2$ through a point P.

Then $|\bar{N}_1|$ maps to the net of quartics with 1 double point and 9 simple base points, $|\bar{N}|$ to the system of curves of degree 7 with 1 triple point and 9 double points (with no other simple base points).

We now want to show the following

Theorem 5.1.25. *The case $n = 3\ell - 1 = 2, n' = 1$ cannot occur.*

Proof. We compute the plane image of the pencil $|\bar{A}'|$. From theorem 4.6.5 we know that $\ell = 1$ can only occur in case $(1f)$ of proposition 2.2.14 or when $A' = N$. Then, using also lemma 2.2.3 and corollary 2.2.12, $A'K_Y = 1, A'N = 3, A'G' = 0$ and we find

$$\begin{aligned} 1 \le A'N_2 &= A'(N + 2K_Y - 2G' - 2(Z_1 + Z_2) - Z') \\ &= 5 + 0 - 2A'(Z_1 + Z_2) - A'Z' \le 5 \end{aligned} \tag{5.9}$$

Let us first consider the case $A' = N$. Then $A'N_2 = NN_2 = 5 - NZ' = 4$ and $\bar{A}' = \bar{N}$ maps to a plane curve of degree d with a point of multiplicity $d - 4$ at P.

Then

$$4 = \bar{A}'\bar{N} = 7d - 2\sum_j js_j - 3(d-4)$$

and

$$\sum_j js_j = 2d + 4 \tag{5.10}$$

Moreover since $\bar{A}'^2 = A'^2 + 1 = 4$ we have

$$\sum_j j^2 s_j = d^2 - 4 \tag{5.11}$$

hence we find

$$d^2 - 4 \equiv 2d + 4 \equiv 0 \mod 2$$

and d should be an even number. Since $A' = N$ we know $d = 7$ and we get a contradiction.

Let us now study the case $(1f)$. We know that $A'^2 = 1$ and, for any $i = 1, 2$,

$$1 = Z'N = Z'A' + Z'\Phi'$$

Thus if $Z'A' \geq 2$ then Z' and Φ' have some irreducible component in common hence

$$2 \leq Z'A' \leq Z'N + A'\Phi' = 3 \tag{5.12}$$

This implies $0 \leq A'Z' \leq 3$. Let us set $s := A'Z'$.

We notice that since $n = 2$ from corollary 2.3.11 Z_1 and Z_2 are irreducible (-1)-curves hence

$$0 = Z_iN = Z_iA' + Z_i\Phi' = Z_iA' + Z_i(F' + 2G' + H') = Z_iA'$$

hence $A'(Z_1 + Z_2) = 0$.

Then from (5.9) and (5.12) $A'N_2 = 5 - s \geq 2$ and

$$\bar{A}'\bar{N}' = A'N + s = 3 + s = 7d - 2\sum_j js_j - 3(d - 5 + s)$$

hence

$$\sum_j js_j = 2d + 6 - 2s \tag{5.13}$$

Furthermore $\bar{A}'^2 = A'^2 + s^2 = 1 + s^2$ implies

$$\sum_j j^2 s_j = d^2 - 1 - s^2 \tag{5.14}$$

Then

$$\begin{aligned}
p_a(\bar{A}') &= p_a(A') + \frac{s(s-1)}{2} = 2 + \frac{s(s-1)}{2} \\
&= \frac{(d-1)(d-2)}{2} - \sum_j s_j \frac{j(j-1)}{2} - \frac{(d-5+s)(d-6+s)}{2} \\
&= \frac{d^2 - 3d + 2 - (d^2 - 6d + sd - 5d + 30 - 5s + sd - 6s + s^2)}{2} + \\
&\quad - \sum_j s_j \frac{j(j-1)}{2} \\
&= \frac{-3d + 2 + 11d - 2sd - 30 + 11s - s^2}{2} - \sum_j s_j \frac{j(j-1)}{2}
\end{aligned}$$

hence

$$\sum_j s_j j(j-1) = (8 - 2s)d - 32 + 12s - 2s^2 \tag{5.15}$$

Then comparing (5.13), (5.14) and (5.15) we get

$$(8 - 2s)d - 32 + 12s - 2s^2 = d^2 - 1 - s^2 - (2d + 6 - 2s) = d^2 - 2d - 7 + 2s - s^2$$

or equivalently

$$d^2 - (10+2s)d + 25 - 10s + s^2 = (d-5+s)^2 = 0$$

hence $d = 5 - s$.

Since $p_a(\bar{A}') \geq 2$ we have $d \geq 4$ and we can exclude $s = 2, 3$.

When $s = 0, 1$ we can rewrite (5.13) and (5.14) as

$$\sum_j j s_j = 2(5-s) + 6 - 2s = 16 - 4s$$

and

$$\sum_j j^2 s_j = (5-s)^2 - 1 - s^2 = 25 + s^2 - 10s - 1 - s^2 = 24 - 10s$$

In particular one can see that $j \leq 4$ and we can write

$$\begin{cases} 4s_4 + 3s_3 + 2s_2 + s_1 = 16 - 4s \\ 16s_4 + 9s_3 + 4s_2 + s_1 = 24 - 10s \end{cases}$$

Subtracting the second equation from the first one we find

$$12s_4 + 6s_3 + 2s_3 = 8 - 6s$$

which forces $s_4 = 0$ and $3s_3 + s_2 = 4 - 2s$. Then

$$s_1 + s_2 = 16 - 4s - 4s_4 - (3s_3 + s_2) = 16 - 4s - 4 + 2s = 12 - 2s \geq 10$$

while we know $s_1 + s_2 \leq \sum_j s_j \leq 9$. Contradiction. $\square$

5.2 Del Pezzo cases

We now treat separately those cases with $\ell = 0$ from those with $\ell = 1$. We refer then to the list of theorem 4.6.5.

5.2.1 $\ell = 0$

From theorem 4.6.5 we know that we are in cases $(0g)$ of proposition 2.2.13, $(1a)$, $(1d)$ or $(1f)$ of proposition 2.2.14 or $A' = N$. We use the same notation as in section 4.5. Since $N_3^2 = 1$, when we contract the curves G', Z'', Z''' we obtain a rational surface W which is isomorphic to the projective plane $\mathbb{P}^2$ blown up at eight points $P_1, \dots, P_8$. We will denote by $\bar{D}$ the image on W of a divisor $D \in Div(Y)$. We note that since $\bar{N}_4 \equiv 0$ then $|\bar{N}_3|$ is mapped to the system of cubics through the eight points, $|\bar{N}_2|$ to the system of sextics with eight double points, $|\bar{N}_1|$ to that of curves of degree 9 with eight triple points and $|\bar{N}|$ to the system of curves of degree 12 with eight quadruple points.

From the definition of N (since $B_0 = 0$ and $Z''G' = 0$) we have

$$2 = Z''N = Z''(3K_Y + 2B_0 + E' - 3G') = -3 + E'Z''$$

hence $E'Z'' = 5$ and analogously $E'Z''' = 6$.

Lemma 5.2.1. *In the above setting Z'' is an irreducible (-1)-curve.*

Proof. If Z'' was a reducible (-1)-cycle then it should necessarily contain G'. Then

$$0 = F'N_3 = F'(N + 3K_Y - 3G' - Z'') = -F'Z'' = -F'(G' + (Z'' - G'))$$

and $F'G' = 1$ forces $F' \leq Z''$ (and analogously $H' \leq Z''$). Then as in the proof of lemma 2.3.8 there should be another (-1)-curve C not intersecting N_2 such that $C \leq Z''$. Since $n = n' = 0$ it should be $n'' \geq 2$. Contradiction. Hence Z'' is irreducible and $F'Z'' = H'Z'' = 0$. □

From lemma 5.2.1 we have $E'_kZ'' \geq 0$. Since

$$E'_kN_2 = E'_k(N + 2K_Y - 2G') = 2$$

while $Z'''N_2 = 1$ from the nefness of N_2 we deduce that E'_k cannot be contained in Z''' hence $E'_kZ''' \geq 0$ too.

Since $N_4 \equiv 0$ we find

$$0 = E'_kN_4 = E'_k(N + 4K_Y - 4G' - 2Z'' - Z''') = 4 - 2E'_kZ'' - E'_kZ'''$$

hence

$$2E'_kZ'' + E'_kZ''' = 4 \tag{5.16}$$

for any $k = 1, \dots, h_1 = 4$.

Thus $0 \leq E'_kZ'' \leq 2$ and $E'_kZ''' = 4 - 2E'_kZ''$ for any $k = 1, \dots, 4$. We now compute

$$\bar{E}'_k\bar{N}_3 = E'_kN_3 = E'_k(N + 3K_Y - 3G' - Z'') = 3 - E'_kZ'' \geq 1$$

If $E'_k Z'' = 0$ then from (5.16) we have $E'_k Z''' = 4$. Then we get a contradiction since $\bar{E}'_k \bar{N}_3 = 3$ and from the Index theorem 1.1.10

$$(\bar{E}'_k - 3\bar{N}_3)^2 = \bar{E}'^2_k - 9 \leq 0$$

while we know

$$\bar{E}'^2_k = E'^2_k + 16 = 13.$$

If $E'_k Z'' = 1$ then from (5.16) we have $E'_k Z''' = 2$. The plane image of $\bar{E}'_k$ is a curve of degree d with s_j points of multiplicity j. Then

$$\sum_j j^2 s_j = d^2 - \bar{E}'^2_k = d^2 - E'^2_k - 1 - 4 = d^2 - 2$$

and

$$\sum_j j s_j = 3d - \bar{E}'_k \bar{N}_3 = 3d - E'_k N_3 = 3d - 2$$

hence

$$d^2 - 2 - 3d + 2 = d^2 - 3d = d(d-3) \geq 0$$

and $d \geq 3$.

If $E'_k Z'' = 2$ then from (5.16) we have $E'_k Z''' = 0$. Then $\bar{E}'_k \bar{N}_3 = 1$ and from the Index theorem 1.1.10 we find

$$(\bar{E}'_k - \bar{N}_3)^2 = \bar{E}'^2_k - 1 \leq 0$$

Since we know

$$\bar{E}'^2_k = E'^2_k + 4 = 1$$

we find $\bar{E}'_k \equiv \bar{N}_3$ and the plane image of $\bar{E}'_k$ is a cubic through eight points $P_1, \ldots, P_8$.

Furthermore since $E'Z'' = 5$ we have exactly one curve, say E'_1, such that $E'_1 Z'' = 2$ and three curves E'_i ($i = 2, 3, 4$) such that $E'_i Z'' = 1$.

From $\bar{E}' \equiv \bar{N} - 3K_W$ the sum of the curves E'_k has total degree 21. Then we have the following possibilities for the degrees (d_1, d_2, d_3, d_4):

$$(3,3,3,12)$$

$$(3,3,4,11)$$

$$(3,3,5,10)$$

$$(3,3,6,9)$$

$$(3,3,7,8)$$
$$(3,4,6,8)$$
$$(3,4,7,7)$$
$$(3,5,5,8)$$
$$(3,5,6,7)$$
$$(3,6,6,6)$$

As we have already seen $\bar{E}'_1$ is a cubic through 8 points.

Moreover since $E'_i Z'' = 1, E'_i Z''' = 2$ for $i = 2,3,4$ we find $\bar{E}'_i \bar{E}'_1 = 2$ and $\bar{E}'_i \bar{E}'_j = E'_i E'_j + 1 + 4 = 5$ for $i \neq j, 2 \leq i, j \leq 4$.

Let us consider $\bar{E}'_i, i \geq 2$. Then $\bar{E}'^2_i = 2$ and $\bar{E}'_i \bar{N}_2 = 2$ forces $\bar{E}'_i$ to be a solution of the linear system

$$\begin{cases} \sum_j j^2 s_j = d_i^2 - \bar{E}'^2_i = d_i^2 - 2 \\ \sum_j j s_j = 3d_i - \bar{E}'_i \bar{N}_2 = 3d_i - 2 \\ \sum_j s_j \leq 8 \end{cases}$$

By an easy computation we have the following list of solutions:

$$\begin{array}{ll} 1) & d_i = 3, s_1 = 7 \\ 2) & d_i = 4, s_2 = 2, s_1 = 6 \\ 3) & d_i = 5, s_2 = 5, s_1 = 3 \\ 4) & d_i = 6, s_3 = 1, s_2 = 6, s_1 = 1 \\ 5) & d_i = 7, s_3 = 3, s_2 = 5 \\ 6) & d_i = 8, s_3 = 6, s_2 = 2 \\ 7) & d_i = 9, s_4 = 1, s_3 = 7 \end{array}$$

Proposition 5.2.2. *All the above solutions are equivalent up to a finite number of Cremona quadratic transformations of* $\mathbb{P}^2$ *based at* $P_1, \ldots, P_8$.

Proof. Let us consider a curve of degree 9 as in 7) and let us take the 4-tuple point Q_1 and two of the seven triple points Q_2, Q_3. Then they are not collinear otherwise there should be a line meeting the above curve at 10 points. Moreover Q_1 is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point (see also definition

1.4.2), since is the unique point of maximal multiplicity for the curve. Since $\bar{E}_i'$ is an irreducible curve if both Q_2, Q_3 were proximate to P_1 (see definition 1.4.8) from the inequalities (1.10) we should have

$$4 \geq 3 + 3 = 6$$

If Q_2 or Q_3 are not infinitely near to Q_1 we can choose them to be on $\mathbb{P}^2$. Then a quadratic transformation (see section 1.4) based at Q_1, Q_2, Q_3 is well-defined and takes the curve of degree 9 onto an octic as in 6).

Let us consider the octic in 6) and let us take three of the six triple points Q_1, Q_2, Q_3. Then they are not collinear otherwise there should be a line meeting the octic at 9 points. Moreover we can choose the points in such a way that one of them, say Q_1, is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point. Since $\bar{E}_i'$ is an irreducible curve if both Q_2, Q_3 were proximate to P_1 from the inequalities (1.10) we should have

$$3 \geq 3 + 3 = 6$$

If Q_2 or Q_3 are not infinitely near to Q_1 we can choose them to be on $\mathbb{P}^2$. Then a quadratic transformation based at Q_1, Q_2, Q_3 is well-defined and takes the octic onto a septic as in 5).

Let us now take two triple points Q_1, Q_2 and a double point Q_3 for the septic 5). Then they are not collinear otherwise there should be a line meeting the octic at 8 points. Moreover we can choose the points in such a way that one of them, say Q_1, is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point. If both Q_2, Q_3 were proximate to P_1 from the inequalities (1.10) we should have

$$3 \geq 3 + 2 = 5$$

If Q_2 is not infinitely near to Q_1 we can choose it to be on $\mathbb{P}^2$. If Q_3 is not infinitely near to Q_1, Q_2 then either it is a plane double point or it is infinitely near to a plane double point or it is infinitely near to a plane triple point. In the first case we have nothing to do. In the second case we choose as Q_3 the plane double point, while in the third case we choose as Q_1 the plane triple point. Then a quadratic transformation based at Q_1, Q_2, Q_3 is well-defined and takes the septic onto a sextic as in 4).

To get 3) we consider the triple point Q_1 and two double points Q_2, Q_3 of the sextic. They are not collinear and Q_1 is necessarily on $\mathbb{P}^2$. Moreover

$$3 < 2 + 2 = 4$$

hence Q_2, Q_3 cannot be both proximate to Q_1. If Q_2 or Q_3 are not infinitely near to Q_1 we can choose them to be on $\mathbb{P}^2$. Then a quadratic transformation based at Q_1, Q_2, Q_3 takes the sextic onto a quintic.

With a similar argument one can see that we can choose three double points for the quintic such that a quadratic transformation based at those points sends the quintic onto a quartic as in 2). Eventually, if we base a quadratic transformation at the two double points of the quartic and at one of the six simple points, we can take the quartic onto the cubic in 1). The result is then proved. $\square$

Thus we can assume that one of the curves $\bar{E}'_i, i \geq 2$, say $\bar{E}'_2$ is a cubic. Therefore, up to a finite number of quadratic transformations, we have $(d_1, d_2, d_3, d_4) = (3, 3, 7, 8)$ or $(3, 3, 6, 9)$ (we note that $\bar{E}'_1$ is invariant for any quadratic transformation based at three points among $P_1, \ldots, P_8$).

The plane image of $\bar{E}' = \sum_{i=1}^{4} \bar{E}'_i$ is $|-7K_{\mathbb{P}^2}|$. Then the total multiplicity of $\bar{E}'$ at $P_1, \ldots, P_8$ is 7. Since $\bar{E}'_i\bar{E}'_j = 5$ for $2 \leq i, j \leq 4$, we obtain the following configurations

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
3	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	0
7	2	2	2	2	2	3	3	3
8	3	3	3	3	3	2	2	3

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
3	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	0
6	1	2	2	2	2	2	2	3
9	4	3	3	3	3	3	3	3

If we denote by P_9 the point obtained by contracting Z''', by P_{10} the contraction of Z'' and by P_{11} the contraction of G' we can write

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
3	1	1	1	1	1	1	1	1	0	2	0
3	1	1	1	1	1	1	1	0	2	1	0
7	2	2	2	2	2	3	3	3	2	1	0
8	3	3	3	3	3	2	2	3	2	1	0

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
3	1	1	1	1	1	1	1	1	0	2	0
3	1	1	1	1	1	1	1	0	2	1	0
6	1	2	2	2	2	2	2	3	2	1	0
9	4	3	3	3	3	3	3	3	2	1	0

Let us now compute the images of F' and H'. We have $F'G' = H'G' = 1, F'Z'' = G'Z'' = 0$ hence also $F'Z''' = H'Z''' = 0$. In particular the images of F' and H' are curves of degree $d \geq 0$ with multiplicity 0 at P_9, P_{10}.

Theorem 5.2.3. *The case $n = 3\ell, n' = 0, n'' = n''' = 1$ cannot occur with $\ell = 0$.*

Proof. We look at the eigenvalues of the curves $E'_i, 1 \leq i \leq 4$, F' and H' for the action of the automorphism of order 3. We know that F' and H' correspond to different eigenvalues since they come from the blow-up of a singularity of type A_2. Let us set $\omega := e^{2\pi i/3}$. If E'_1 corresponds to the eigenvalue ω then it appears with multiplicity 1 in the branch locus of the simple triple cover associated to $X \longrightarrow Y = X/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that E'_i corresponds to the eigenvalue ω^{ν_i}, F' corresponds to the eigenvalue ω^{ν_F} and H' to $\omega^{2\nu_F}$.

The point P_{10} is double for E'_1 hence it is not infinitely near to any other point. The point P_9 is double for E'_2 hence again it is not infinitely near to any other point.

The total multiplicity at those two points of the branch divisor has to be a multiple of 3. Then we obtain the two equations

$$\begin{cases} 2\nu_2 + 2\nu_3 + 2\nu_4 \equiv 0 \mod 3 \\ 2\nu_1 + \nu_2 + \nu_3 + \nu_4 \equiv 0 \mod 3 \end{cases}$$

which force $\nu_1 \equiv 0 \mod 3$. Contradiction. □

5.2.2 $\ell = 1$

When $\ell = 1$ from theorem 4.6.5 we always have $\Phi = 0$ and either case $(1f)$ of proposition 2.2.14 or $A' = N$ holds. Moreover we have $0 = 3\ell - 3 \leq n \leq 3\ell = 3$.

$\underline{n = 3\ell}$

From the results of section 4.5 when we contract the (-1)-cycles $G', Z_1, Z_2, Z_3, Z'', Z'''$ we get a rational surface W which is isomorphic to the projective plane

$\mathbb{P}^2$ blown up at eight points $P_1, \ldots, P_8$.

Case I: The cycles Z_1, Z_2, Z_3 are irreducible.

In this case

$$B_0 N_1 = B_0(N + K_Y - G' - Z_1 - Z_2 - Z_3) = 0 + 4 - 0 - 1 - 1 - 1 = 1$$

and

$$B_0 N_2 = B_0(N + 2K_Y - 2G' - 2\sum_{i=1}^{3} Z_i) = 0 + 8 - 0 - 6 = 2$$

In particular, since $Z''N_2 = 0$ and $Z'''N_2 = 1$, from the nefness of N_2 one can see that B_0 cannot be an irreducible component of any of these cycles, i.e. B_0 is not contracted on W. Let us compute (recall that $N_4 \equiv 0$)

$$\begin{aligned} 0 = B_0 N_4 &= B_0(N + 4K_Y - 4G' - 4\sum_{i=1}^{3} Z_i - 2Z'' - Z''') \\ &= 0 + 16 - 0 - 12 - B_0(2Z'' + Z'') = 4 - B_0(2\sum_{i=1}^{2} Z_i' + Z'') \end{aligned}$$

hence $0 \le B_0 Z'' \le 2$ and we have

$$\begin{aligned} \bar{B}_0 \bar{N}_3 = B_0 N_3 &= B_0(N + 3K_Y - 3G' - 3\sum_{i=1}^{2} Z_i - Z'') \\ &= 0 + 12 - 0 - 9 - B_0 Z'' = 3 - B_0 Z'' \ge 1 \end{aligned}$$

Thus we can write the following table

	B_0Z_1	B_0Z_2	B_0Z_3	B_0Z''	B_0Z'''	$\bar{B}_0\bar{N}_3$
a)	1	1	1	2	0	1
b)	1	1	1	1	2	2
c)	1	1	1	0	4	3

We now apply the Index theorem 1.1.10. Since $\bar{N}_3^2 = 1$ and $\bar{B}_0\bar{N}_3 = s \ge 1$ we find $\bar{B}_0^2 \le s^2$ which excludes case c) and forces $\bar{B}_0 \equiv \bar{N}_3$ in case a). We also note that in case b) we have $\bar{B}_0^2 = 2$.

Lemma 5.2.4. *Case a) cannot occur.*

Proof. Assume case a) holds. Since $\bar{B}_0 \equiv \bar{N}_3$, $B_0Z_1 = B_0Z_2 = B_0Z_3 = 1$ and $E_5'Z_1 = E_4'Z_2 = 1 = E_3'Z_3$, if E_3', E_4', E_5' were not contracted on W we should have

$$1 \leq \bar{E}'_k \bar{B}_0 = \bar{E}'_k \bar{N}_3 = E'_k N_3 = E'_k(N + 3K_Y - 3G' - 3\sum_{i=1}^{2} Z_i - Z'') = -E'_k Z''$$

for $k = 3, 4, 5$. Then we should have $E'_k \leq Z''$ and $0 \leq E'_k N_3 \leq Z'' N_3 = 0$. Contradiction.

Thus E'_k is contracted on W and, from the nefness of N_3, $E'_k Z'' = E'_k Z''' = 0$ ($k = 3, 4, 5$). From the definition of N and from $Z''' N = 3$ we find

$$6 = 2B_0 Z''' + E' Z''' = E' Z''' = (E'_1 + E'_2) Z''' \tag{5.17}$$

Moreover, for $k = 1, 2$

$$\begin{aligned} 0 = E'_k N_4 &= E'_k(N + 4K_Y - 4G' - 4\sum_{i=1}^{2} Z_i - 2Z'' - Z''') \\ &= 0 + 4 - 0 - 0 - E'_k(2Z'' + Z''') = 4 - E'_k(2Z'' + Z''') \end{aligned}$$

and

$$\begin{aligned} E'_k N_3 &= E'_k(N + 3K_Y - 3G' - 3\sum_{i=1}^{2} Z_i - Z'') \\ &= 0 + 3 - 0 - 0 - E'_k Z'' = 3 - E'_k Z'' \geq 0 \end{aligned}$$

Moreover since $E'_k N_2 = 2$ while $Z'' N_2 = 0, Z''' N_2 = 1$, it cannot be $E'_k \leq Z'', Z'''$ for $k = 1, 2$. In particular $E'_k Z'', E'_k Z''' \geq 0$. Thus

$$2E'_k Z'' + E'_k Z''' = 4 \tag{5.18}$$

forces $0 \leq E'_k Z''' \leq 4$ and from (5.17) there should be at least one of the curves E'_k (say E'_2) such that $E'_k Z''' = 4, \bar{E}'_k \bar{N}_3 = 3$. We get a contradiction since from the Index theorem 1.1.10 we should have

$$(\bar{E}'_2 - 3\bar{N}_2)^2 \leq 0$$

or equivalently $\bar{E}'^2_2 \leq 9$. □

We now study case b). Let us denote by d_0 the degree of the plane image of $\bar{B}_0$. Since

$$2\bar{B}_0 + \bar{E}' = \bar{N} - 3K_W$$

the curve $2\bar{B}_0 + \bar{E}'$ is sent to an element of $|-7K_{\mathbb{P}^2}|$.

We note that a quadratic transformation leaves the plane image of $|2\bar{B}_0 + \bar{E}'|$ unchanged. In particular even after any quadratic transformation the equation

$$2d_0 + \sum_{i=1}^{5} d_i = 21 \tag{5.19}$$

holds, where d_0 is the degree of the image of $\bar{B}_0$ while $d_i, i = 1, \ldots, 5$ are the degrees of the plane images of the curves $\bar{E}_i'$. In particular we have $d_0 \leq 10$.

The curve $\bar{B}_0$ satisfies the linear system

$$\begin{cases} \sum_j j^2 s_j = d_0^2 - \bar{B}_0^2 = d_0^2 - 2 \\ \sum_j j s_j = 3d_0 - \bar{B}_0 \bar{N}_2 = 3d_0 - 2 \\ \sum_j s_j \leq 8 \end{cases}$$

where s_j is the number of points among $P_1, \ldots, P_8$ of multiplicity j for $\bar{B}_0$. By an easy computation one can see that $d_0 \geq 3$ and we have the following list of solutions:

$$\begin{array}{ll} 1) & d_0 = 3, s_1 = 7 \\ 2) & d_0 = 4, s_2 = 2, s_1 = 6 \\ 3) & d_0 = 5, s_2 = 5, s_1 = 3 \\ 4) & d_0 = 6, s_3 = 1, s_2 = 6, s_1 = 1 \\ 5) & d_0 = 7, s_3 = 3, s_2 = 5 \\ 6) & d_0 = 8, s_3 = 6, s_2 = 2 \\ 7) & d_0 = 9, s_4 = 1, s_3 = 7 \end{array}$$

Proposition 5.2.5. *All the above solutions are equivalent up to a finite number of Cremona quadratic transformations of* $\mathbb{P}^2$ *based at* $P_1, \ldots, P_8$.

Proof. See the proof of proposition 5.2.2. □

From the above proposition, up to Cremona transformations, we can set $d_0 = 9$. In particular we can assume that the quadruple point of the curve B_0 is P_8. Then we find

$$\sum_{i=1}^{5} d_i = 21 - 2d_0 = 3$$

and $d_i \leq 3$ for any $i = 1, \ldots, 5$.

Proposition 5.2.6. *In the above setting case I cannot occur.*

Proof. Let us consider the curves E_i', $i = 1, 2$. Then

$$E_i'N_2 = E_i'(N + 2K_Y - 2G' - 2\sum_{i=1}^{3} Z_i) = 2$$

while $Z''N_2 = 0, Z'''N_2 = 1$. In particular, from the nefness of N_2, E_1' and E_2' cannot be contained in Z'' or Z''' and they are not contracted on W. Thus

$$E_i'Z_j = 0, \quad 2E_i'Z'' + E_i'Z''' = 4 \quad (i = 1, 2, j = 1, 2, 3) \tag{5.20}$$

Since (5.20) holds we have a priori three possibilities. In the former case $E_i'Z'' = 2, E_i'Z''' = 0$ and then $\bar{E}'^2_i = 1$ hence $\bar{E}_i'\bar{B}_0 = 2$ and $d_i = 3, s_1 = 8$. In the second case $E_i'Z'' = 1, E_i'Z''' = 2$ and then $\bar{E}'^2_i = 2$ hence $\bar{E}_i'\bar{B}_0 = 5$ and $d_i = 3, s_1 = 7$. In the latter case $E_i'Z'' = 0, E_i'Z'' = 4$ hence $\bar{E}_i'\bar{N}_3 = 3$ and $\bar{E}'^2_i = 13$ contradicting the Index theorem 1.1.10 as in the proof of lemma 5.2.4.

Thus, since $\sum_{i=1}^{5} d_i = 3$, one among E_1' and E_2' is necessarily contracted on W and we get a contradiction. □

Case II: At least one of the cycles Z_i is reducible.

We know from corollary 2.3.11 that either Z_1, Z_2 are irreducible and $Z_3 = Z_1 + Z_2 + E_k'$ for a suitable $1 \leq k \leq 5$ or Z_1 is irreducible $Z_2 = Z_1 + C$, $Z_3 = 2Z_1 + C + E_k'$ for a suitable $1 \leq k \leq 5$ where C is a (-2)-curve.

Let us look at the (-6)-curve B_0. In any case we have $B_0Z_1 = B_0Z_2 = 1, B_0Z_3 = 2$.

Proposition 5.2.7. *In the above setting case II cannot occur.*

Proof. If B_0 was contracted on W, then it should be contained either in Z'' or in Z'''. But when we contract the cycles Z_1, Z_2, Z_2 the self-intersection of the image B_0' of B_0 is

$$B'^2_0 = B_0^2 + 1 + 1 + 4 = -6 + 6 = 0$$

Since B_0, hence B_0', is irreducible, it cannot be a component of a (-1)-cycle.

Computing B_0N_3 and B_0N_4 and recalling that N_3 is nef whereas $N_4 \equiv 0$, one can easily see that $B_0Z'' = B_0Z''' = 0$. Thus the image $\bar{B}_0$ of B_0 on the rational surface W is a curve of self-intersection 0 having a node or a cusp (depending on the structure of the cycles Z_i) at the point obtained by contracting Z_1, Z_2, Z_3.

In particular we note that $\bar{B}_0\bar{N}_3 = B_0N_3 = 0$. Hence by the Index theorem 1.1.10 and the rationality of W we infer $\bar{B}_0 = 0$. Contradiction. □

Hence we obtain

Theorem 5.2.8. *The case $n = 3\ell, n' = 0, n'' = n''' = 1$ cannot occur with $\ell = 1$.*

$\underline{n = 3\ell - 1}$

From the results of section 4.4 when we contract the (-1)-cycles G', Z_1, Z_2, Z_1', Z_2', Z'' we get a rational surface W which is isomorphic to the projective plane $\mathbb{P}^2$ blown up at eight points $P_1, \ldots, P_8$.

We also recall that from corollary 2.3.11 the cycles Z_1 and Z_2 are irreducible (-1)-curves and

$$B_0N_1 = B_0(N + K_Y - G' - Z_1 - Z_2) = 0 + 4 - 0 - 1 - 1 = 2$$

In particular, since $Z_1'N_1 = Z_2'N_1 = 0$ and $Z''N_1 = 1$, from the nefness of N_1 one can see that B_0 cannot be an irreducible component of any of these cycles, i.e. B_0 is not contracted on W.

Let us compute (recall that $N_3 \equiv 0$)

$$\begin{aligned} 0 = B_0N_3 &= B_0(N + 3K_Y - 3G' - 3\sum_{i=1}^{2} Z_i - 2\sum_{i=1}^{2} Z_i' - Z'') \\ &= 0 + 12 - 0 - 6 - B_0(2\sum_{i=1}^{2} Z_i' + Z'') = 6 - B_0(2\sum_{i=1}^{2} Z_i' + Z'') \end{aligned}$$

hence $0 \le B_0 \sum_{i=1}^{2} Z_i' \le 3$ and we have

$$\begin{aligned} \bar{B}_0\bar{N}_2 = B_0N_2 &= B_0(N + 2K_Y - 2G' - 2\sum_{i=1}^{2} Z_i - \sum_{i=1}^{2} Z_i') \\ &= 0 + 8 - 0 - 4 - B_0\sum_{i=1}^{2} Z_i' = 4 - B_0\sum_{i=1}^{2} Z_i' \ge 4 - 3 = 1 \end{aligned}$$

Thus we can write the following table

	B_0Z_1	B_0Z_2	$B_0Z'_1$	$B_0Z'_2$	B_0Z''	$\bar{B}_0\bar{N}_2$
a)	1	1	3	0	0	1
b)	1	1	2	1	0	1
c)	1	1	1	1	2	2
d)	1	1	1	0	4	3
e)	1	1	0	0	6	4

We now apply the Index theorem 1.1.10. Since $\bar{N}_2^2 = 1$ and $\bar{B}_0\bar{N}_2 = s \geq 1$ we find $\bar{B}_0^2 \leq s^2$ which excludes cases a), d), e) and forces $\bar{B}_0 \equiv \bar{N}_2$ in case b). We also note that in case c) we have $\bar{B}_0^2 = 2$.

Lemma 5.2.9. *Case b) cannot occur.*

Proof. Assume case b) holds. Since $\bar{B}_0 \equiv \bar{N}_2$, $B_0Z_1 = B_0Z_2 = 1$ and $E'_5Z_1 = E'_4Z_2 = 1$, if E'_4, E'_5 were not contracted on W we should have

$$1 \leq \bar{E}'_k\bar{B}_0 = \bar{E}'_k\bar{N}_2 = E'_kN_2 = E'_k(N + 2K_Y - 2G' - 2\sum_{i=1}^{2} Z_i - \sum_{i=1}^{2} Z'_i)$$
$$= -E'_k \sum_{i=1}^{2} Z'_i$$

for $k = 4, 5$. Then we should have $E'_k \leq Z'_i$ for some i and $0 \leq E'_kN_2 \leq Z'_iN_2 = 0$. Contradiction.

Thus we have E'_k is contracted on W and, from the nefness of N_2, $E'_k \sum_i Z''_i = E'_kZ'' = 0$ ($k = 4, 5$). From the definition of N and from $Z''N = 2$ we find

$$5 = 2B_0Z'' + E'Z'' = E'Z'' = (E'_1 + E'_2 + E'_3)Z'' \tag{5.21}$$

Moreover, for $k = 1, 2, 3$

$$0 = E'_kN_3 = E'_k(N + 3K_Y - 3G' - 3\sum_{i=1}^{2} Z_i - 2\sum_{i=1}^{2} Z'_i - Z'')$$
$$= 0 + 3 - 0 - 0 - E'_k(2\sum_{i=1}^{2} Z'_i + Z'') = 3 - E'_k(2\sum_{i=1}^{2} Z'_i + Z'')$$

and

$$E'_kN_2 = E'_k(N + 2K_Y - 2G' - 2\sum_{i=1}^{2} Z_i - \sum_{i=1}^{2} Z'_i)$$

$$= 0 + 2 - 0 - 0 - E'_k \sum_{i=1}^{2} Z'_i = 2 - E'_k \sum_{i=1}^{2} Z'_i \geq 0$$

Thus

$$2E'_k \sum_{i=1}^{2} Z'_i + E'_k Z'' = 3 \tag{5.22}$$

and $E'_k \sum_{i=1}^{2} Z'_i \leq 2$ forces $-1 \leq E'_k Z'' \leq 3$.

Thus $E'_k Z''$ is odd and from (5.21) there should be at least one of the curves E'_k (say E'_3) such that $E'_k Z'' = 3, \bar{E}'_k \bar{N}_2 = 2$. We get a contradiction since from the Index theorem 1.1.10 we should have

$$(\bar{E}'_3 - 2\bar{N}_2)^2 \leq 0$$

or equivalently $\bar{E}'^2_3 \leq 4$. □

We now study case c). Let us denote by d_0 the degree of the plane image of $\bar{B}_0$. Since

$$2\bar{B}_0 + \bar{E}' = \bar{N} - 3K_W$$

the curve $2\bar{B}_0 + \bar{E}'$ is sent to an element of $|-6K_{\mathbb{P}^2}|$.

We note that a quadratic transformation leaves the plane image of $|2\bar{B}_0 + \bar{E}'|$ unchanged. In particular even after any quadratic transformation the equation

$$2d_0 + \sum_{i=1}^{5} d_i = 18 \tag{5.23}$$

holds, where d_0 is the degree of the image of $\bar{B}_0$ while $d_i, i = 1, \ldots, 5$ are the degrees of the plane images of the curves $\bar{E}'_i$. In particular we have $d_0 \leq 9$.

The curve $\bar{B}_0$ satisfies the linear system

$$\begin{cases} \sum_j j^2 s_j = d_0^2 - \bar{B}_0^2 = d_0^2 - 2 \\ \sum_j j s_j = 3d_0 - \bar{B}_0 \bar{N}_2 = 3d_0 - 2 \\ \sum_j s_j \leq 8 \end{cases}$$

where s_j is the number of points among $P_1, \ldots, P_8$ of multiplicity j for $\bar{B}_0$. By an easy computation one can see that $d_0 \geq 3$ and we have the following list of solutions:

$$1) \qquad d_0 = 3, s_1 = 7$$

2) $d_0 = 4, s_2 = 2, s_1 = 6$

3) $d_0 = 5, s_2 = 5, s_1 = 3$

4) $d_0 = 6, s_3 = 1, s_2 = 6, s_1 = 1$

5) $d_0 = 7, s_3 = 3, s_2 = 5$

6) $d_0 = 8, s_3 = 6, s_2 = 2$

7) $d_0 = 9, s_4 = 1, s_3 = 7$

Proposition 5.2.10. *All the above solutions are equivalent up to a finite number of Cremona quadratic transformations of $\mathbb{P}^2$ based at $P_1, \ldots, P_8$.*

Proof. See the proof of proposition 5.2.2. □

From the above proposition, up to Cremona transformations, we can set $d_0 = 8$. In particular we can assume that the two double points of the octic are P_7, P_8. Then we find

$$\sum_{i=1}^{5} d_i = 18 - 2d_0 = 2$$

and $d_i \leq 2$ for any $i = 1, \ldots, 5$.

If one of the curves E'_i is contracted on W then it has $d_i = 0$ and the multiplicity at each of the points $P_1, \ldots, P_8$ is 0.

If E'_i is not contracted on W we have two different numerical possibilities:

$$E'_i \sum_{j=1}^{2} Z_j = 1, E'_i Z'_1 = E'_i Z'_2 = 0, E'_i Z'' = 0 \quad (i = 4, 5) \tag{5.24}$$

$$E'_i Z_1 = E'_i Z_2 = 0, 2E'_i \sum_{j=1}^{2} Z'_i + E'_i Z'' = 3 \quad (i = 1, 2, 3) \tag{5.25}$$

When (5.24) holds we find $\bar{E}'^2_i = -2$ hence, since $d_i \leq 2$, either $d_i = 1, s_1 = 3$ or $d_i = 2, s_1 = 6$. Moreover $\bar{E}'_i \bar{B}_0 = 1$.

When (5.25) holds we have a priori two more possibilities. In the former case $E'_i \sum_{j=1}^{2} Z'_j = 1, E'_i Z'' = 1$ and then $\bar{E}'^2_i = -1$ hence $\bar{E}'_i \bar{B}_0 = 3$ and either $d_i = 1, s_1 = 2$ or $d_i = 2, s_1 = 5$. In the latter case $E'_i \sum_{j=1}^{2} Z'_j = 0, E'_i Z'' = 3$ hence $\bar{E}'_i \bar{N}_2 = 2$ and $\bar{E}'^2_i = 6$ contradicting the Index theorem 1.1.10 as in the proof of lemma 5.2.9.

We now fix $i = 4, 5$. Then since $\bar{E}'_i \bar{N}_2 = 0$ we have $\sum_{j=1}^{8} m_j = 3d_i$ where m_j is the multiplicity of $\bar{E}'_i$ at P_j. Moreover

$$\begin{aligned} 1 = \bar{E}'_i \bar{B}_0 &= 8d_i - 3(m_1 + \cdots + m_6) - 2(m_7 + m_8) \\ &= 8d_i - 3\sum_{j=1}^{8} m_j + (m_7 + m_8) = -d_i + m_7 + m_8 \end{aligned}$$

hence

$$d_i + 1 = m_7 + m_8 \leq 2 \tag{5.26}$$

since there are no singular points among $P_1, \ldots, P_8$. Hence $d_i \leq 1$ and when $d_i = 1$ the line $\bar{E}'_i$ must pass through P_7 and P_8.

This excludes the 6-tuple $(d_0, d_1, d_2, d_3, d_4, d_5) = (8, 0, 0, 0, 1, 1)$ since both $\bar{E}'_4$ and $\bar{E}'_5$ would be lines through the points P_7 and P_8.

Then we have the following list of 6-tuples $(d_0, d_1, d_2, d_3, d_4, d_5)$:

$$\begin{gathered} (8, 2, 0, 0, 0, 0) \\ (8, 1, 1, 0, 0, 0) \\ (8, 1, 0, 0, 1, 0) \end{gathered}$$

For $i = 1, 2, 3$, since $\bar{E}'_i \bar{N}_2 = 1$ (hence $\sum_{j=1}^{8} m_j = 3d_i - 1$), we have

$$\begin{aligned} 3 = \bar{E}'_i \bar{B}_0 &= 8d_i - 3(m_1 + \cdots + m_6) - 2(m_7 + m_8) \\ &= 8d_i - 3\sum_{j=1}^{8} m_j + (m_7 + m_8) = -d_i + 3 + m_7 + m_8 \end{aligned}$$

hence

$$m_7 + m_8 = d_i \leq 2 \tag{5.27}$$

We now study the 6-tuple of degrees $(8, 2, 0, 0, 0, 0)$. Using (5.26), (5.27) and the fact that $2\bar{B}_0 + \bar{E}'$ has total multiplicity 6 at each of the points $P_1, \ldots, P_8$ we find the following configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
8	3	3	3	3	3	3	2	2
2	1	1	1	0	0	0	1	1
0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	1	0
0	0	0	-1	0	0	0	0	1

For the 6-tuple $(8,1,1,0,0,0)$ we find

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
8	3	3	3	3	3	3	2	2
1	1	0	0	0	0	0	1	0
1	0	1	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	1	0
0	0	-1	0	0	0	0	0	1

For the 6-tuple $(8,1,0,0,1,0)$ we have

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
8	3	3	3	3	3	3	2	2
1	1	0	0	0	0	0	1	0
0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	1	1
0	0	-1	0	0	0	0	0	1

Remark 5.2.11. *The conditions* (5.26) *and* (5.27)*, the total multiplicity 6 of* $2\bar{B}_0+\bar{E}'$ *at each of the points* $P_1,\dots,P_8$ *and the computation of the intersection numbers*

$$\bar{E}'_i\bar{E}'_j=\begin{cases}0 & i=4,5,\ i\neq j\\ \geq 1 & i\neq j,\ 1\leq i,j\leq 3\\ -1 & 1\leq i=j\leq 3\end{cases}$$

are sufficient to uniquely determine the configuration of points for each 6-tuple $(d_0,d_1,d_2,d_3,d_4,d_5)$.

Lemma 5.2.12. *The three above configurations are equivalent up to a finite number of quadratic transformations.*

Proof. We consider the 6-tuple $(8,1,1,0,0,0)$. Let us apply a quadratic transformation based at P_1,P_2,P_8. Since P_1 is a point of maximal multiplicity for both $\bar{B}_0$ and $\bar{E}'_1$ while P_2 is a point of maximal multiplicity for both $\bar{B}_0$ and $\bar{E}'_2$, they cannot be infinitely near to any other point. Moreover P_8 is proximate to P_2 since the line E'_2 joins the two points and does not pass through any of the other points. Hence a quadratic transformation based at P_1,P_2,P_8 is well-defined and we obtain $(8,1,0,0,0,1)$.

We now show that $(8,1,1,0,0,0)$ is equivalent to $(8,2,0,0,0,0)$. We know that P_8 is proximate to P_2. A similar argument shows that P_7 is proximate to P_1. Let us now consider the points P_3, P_4, P_5, P_6. We claim that none of them can be proximate to P_1 or to P_2. If this was the case, in fact, the octic should satisfy the inequalities (1.10)

$$3 \geq 3 + 2 = 5$$

Contradiction. Hence at least one of them, say P_3, has to be a planar point and we can perform a quadratic transformation based at P_2, P_3, P_8 obtaining the 6-tuple $(8,2,0,0,0,0)$. $\square$

Thus all the 6-tuples are equivalent up to quadratic transformations and we can reduce to one of them, say $(8,2,0,0,0,0)$. We also note that the curve E_3' is contracted on W. In particular we have

$$E_3' \sum_{j=1}^{2} Z_j' = 2, E_3' Z'' = -1.$$

Moreover since $E_3' \leq Z''$ we find $E_3' N_2 = 0$ and then from the Index theorem 1.1.10 we have (recall that $Z_j' N_2 = 0$ for any $j = 1, 2$)

$$(E_3' + Z_j')^2 = -3 - 1 + 2E_3' Z_j' < 0 \tag{5.28}$$

hence $E_3' Z_1' = E_3' Z_2' = 1$.

We now look at the surface Y which is isomorphic to the plane blown up at 14 points. Let us denote by P_9 the point obtained by contracting Z'', P_{10} and P_{11} the points obtained by contracting the cycles Z_j', P_{12} and P_{13} the contractions of Z_1 and Z_2 and, finally, P_{14} the contraction of G'. From section 4.4

$$2B_0 + E' \equiv 6G' + 3\sum_{i=1}^{n} Z_i + 2\sum_{i=1}^{2} Z_i' + Z'' - 6K_Y$$

hence the total multiplicity of $2B_0 + E'$ at P_9 is 5, at P_{10} and P_{11} is 4, at P_{12} and P_{13} is 3 and it is 0 at P_{14}.

Therefore we can write the following table

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0

Let us now see what the images of F' and H' are. We know that $F'G' = H'G' = 1$ hence their images pass through the point P_{14}. We also know they have no intersection with B_0 and with any of the curves E_i'.

Let us now consider F'. The computation for H' is similar. Its plane image is a curve of degree d with multiplicities $m_1, \ldots, m_{14}$ at the points $P_1, \ldots, P_{14}$. From the above remarks we find the following relations

$$\begin{cases} m_{14} = 1 \\ 3\sum_{i=1}^{6} m_i + 2\sum_{i=7}^{9} m_i + \sum_{i=10}^{13} m_i = 8d \\ \sum_{i=1}^{3} m_i + \sum_{i=7}^{10} m_i = 2d \\ -m_1 + m_9 + m_{11} = 0 \\ -m_9 + m_{10} + m_{11} = 0 \\ -m_2 + m_7 + m_{12} = 0 \\ -m_3 + m_8 + m_{13} = 0 \end{cases} \tag{5.29}$$

Let us first assume that F' is contracted on W. Then we immediately find $d = 0, m_i = 0$, $i \leq 8$. The system (5.29) becomes

$$\begin{cases} m_{14} = 1 \\ 2m_9 + \sum_{i=10}^{13} m_i = 0 \\ m_9 + m_{10} = 0 \\ m_9 + m_{11} = 0 \\ -m_9 + m_{10} + m_{11} = 0 \\ m_{12} = 0 \\ m_{13} = 0 \end{cases}$$

which forces $m_9 = m_{10} = m_{11} = m_{12} = m_{13} = 0, m_{14} = 1$. Then we get a contradiction since F' is a (-3)-curve on Y.

Then F' (hence H') is not contracted on W. Since $F'N_1 = F'N_2 = 0$ we have $F'\sum_{i=1}^2 Z_i = F'\sum_{i=1}^2 Z_i' = 0 = F'Z''$. This forces $F'Z_i = F'Z_i' = 0$ for $i = 1, 2$, hence

$$m_i = 0 \quad 9 \leq i \leq 13$$

We can then rewrite (5.29) as

$$\begin{cases} m_{14} = 1 \\ 3\sum_{i=1}^6 m_i + 2\sum_{i=7}^8 m_i = 8d \\ \sum_{i=1}^3 m_i + \sum_{i=7}^8 m_i = 2d \\ -m_1 = 0 \\ 0 = 0 \\ -m_2 + m_7 = 0 \\ -m_3 + m_8 = 0 \end{cases}$$

which is also equivalent to

$$\begin{cases} m_{14} = 1 \\ m_1 = 0 \\ m_2 + m_3 = d \\ m_7 = m_2 \\ m_8 = m_3 \\ m_4 + m_5 + m_6 = d \end{cases} \tag{5.30}$$

Since F' is a (-3)-curve we have

$$\begin{aligned} -3 = d^2 - \sum_{i=1}^{14} m_i^2 &= d^2 - m_2^2 - m_3^2 - m_4^2 - m_5^2 - m_6^2 - m_7^2 - m_8^2 - m_{14}^2 \\ &= d^2 - m_2^2 - (d - m_2)^2 - m_4^2 - m_5^2 - m_6^2 - m_2^2 - (d - m_2)^2 - 1 \\ &= -(d - 2m_2)^2 - m_4^2 - m_5^2 - m_6^2 - 1 \end{aligned}$$

hence

$$(d - 2m_2)^2 + m_4^2 + m_5^2 + m_6^2 = 2$$

First of all we note that

$$2 \geq m_4^2 + m_5^2 + m_6^2 \geq m_4 + m_5 + m_6 = d$$

forces $d \leq 2$. If $d = 2$ we find $m_2 = 1$ and $(m_4, m_5, m_6) = (1,1,0), (1,0,1)$ or $(0,1,1)$. Using (5.30) we find $m_2 = m_3 = m_7 = m_8 = m_{14} = 1$. Hence F' and H' cannot both be sent to conics, since otherwise they should have at least 5 common points while $F'H' = 0$ on Y.

When $d = 1$ we find $-1 \leq 1 - 2m_2 \leq 1$ hence either $m_2 = m_7 = 0, m_3 = m_8 = 1$ or $m_2 = m_7 = 1, m_3 = m_8 = 0$. F' and H' cannot be sent to a conic and a line respectively, since they should have at least 3 common points (P_2, P_7, P_{14} or P_3, P_8, P_{14}). Contradiction.

If $d = 0$ then $-1 \leq -2m_2 \leq 1$ forces $m_2 = 0$. Hence from (5.30) we get $m_2 = m_3 = m_7 = m_8 = 0, m_4 + m_5 + m_6 = 0$. Thus

$$\{m_4, m_5, m_6\} = \{1, 0, -1\}.$$

From the above analysis either F' and H' are both sent to lines or one of them is contracted on $\mathbb{P}^2$. In the former case we have the configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0
1	0	1	0	1	0	0	1	0	0	0	0	0	0	1
1	0	0	1	0	1	0	0	1	0	0	0	0	0	1

In the latter case we can assume that the contracted curve (resp: one of the contracted curves) has $m_4 = 1, m_5 = -1, m_6 = 0$. If the second curve is a conic we find the configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	1
2	0	1	1	0	1	1	1	1	0	0	0	0	0	1

If it is a line we find

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	1
1	0	0	1	0	1	0	0	1	0	0	0	0	0	1

while if they are both contracted we have

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	1	-1	0	0	0	0	0	0	0	1

Lemma 5.2.13. *The configurations* $(8, 2, 0, 0, 0, 0, 1, 1)$ *and* $(8, 2, 0, 0, 0, 0, 0, 1)$ *are equivalent up to quadratic transformations.*

Proof. Let us study the configuration with two lines. One can easily see that P_2 and P_3 are planar points since they are of maximal multiplicity for the octic, the conic and one of the two lines simultaneously. Moreover P_7 is proximate to P_2 while P_3 is proximate to P_8. Thus P_1 cannot be proximate to P_2 or to P_3 since otherwise the octic would contradict the inequalities (1.10)

$$3 \geq 3 + 2 = 5$$

With a similar argument one can show that P_4 and P_5 are planar points too.

If we base a quadratic transformation at P_3, P_4, P_8 we obtain the configuration with a line and a contracted curve. $\square$

Proposition 5.2.14. *The case* $n = 3\ell - 1, n' = 2, n'' = 1$ *cannot occur with degrees* $(d_0, d_1, d_2, d_3, d_4, d_5, d_F, d_H) = (8, 2, 0, 0, 0, 0, 1, 1)$ *and* $(8, 2, 0, 0, 0, 0, 0, 1)$.

Proof. We look at the eigenvalues of the curves B_0, E_i', $1 \leq i \leq 5$, F' and H' for the action of the automorphism of order 3. We know that F' and H' correspond to different eigenvalues since they come from the blow-up of a singularity of type A_2 (see section 1.3). Let us set $\omega := e^{2\pi i/3}$. If B_0 corresponds to the eigenvalue ω then it appears with multiplicity 1 in the branch locus of the simple triple cover associated to $X \longrightarrow Y = X/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that E_i' corresponds to the eigenvalue ω^{ν_i}, F' corresponds to the eigenvalue ω^{ν_F} and H' to $\omega^{2\nu_F}$.

We have already shown that the two configurations $(8,2,0,0,0,0,1,1)$ and $(8,2,0,0,0,0,0,1)$ are equivalent up to quadratic transformations. Let us consider the configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
8	3	3	3	3	3	3	2	2	2	1	1	1	1	0
2	1	1	1	0	0	0	1	1	1	1	0	0	0	0
0	-1	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
0	0	-1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	-1	0	0	0	0	1	0	0	0	0	1	0
1	0	1	0	1	0	0	1	0	0	0	0	0	0	1
1	0	0	1	0	1	0	0	1	0	0	0	0	0	1

Since the total degree of the branch curve on $\mathbb{P}^2$ has to be multiple of 3 and since the two lines correspond to different eigenvalues, the conic appears with multiplicity 2 in the branch divisor, hence $\nu_1 \equiv 2 \mod 3$.

The points P_2,P_3 are not infinitely near to any other point since they are the only points which are triple for the octic and simple for both the conic and one of the two lines.

The total multiplicity at P_3 of the branch divisor has to be a multiple of 3. Then we obtain the equation

$$3 + \nu_1 + \nu_5 + \nu_H \equiv 3 + 2 + \nu_5 + \nu_H \equiv 0 \mod 3$$

which forces $\nu_H + \nu_5 \equiv 2\nu_F + \nu_5 \equiv 1 \mod 3$.

On the other hand the same computation for P_2 gives us

$$3 + \nu_1 + \nu_4 + \nu_F \equiv 3 + 2 + \nu_4 + \nu_F \equiv 0 \mod 3$$

which forces $\nu_F + \nu_4 \equiv 1 \mod 3$.

Then since $\nu_i, \nu_F \equiv 1, 2 \mod 3$ we find

$$\nu_F \equiv \nu_4 \equiv 2 \mod 3$$

hence $\nu_5 \equiv 0 \mod 3$. Contradiction. $\square$

Proposition 5.2.15. *The case* $n = 3\ell - 1, n' = 2, n'' = 1$ *cannot occur with degrees* $(d_0, d_1, d_2, d_3, d_4, d_5, d_F, d_H) = (8, 2, 0, 0, 0, 0, 0, 2)$.

Proof. Let us consider the points P_1, P_2, P_3. They are of maximal multiplicity for both the octic and one of the two conics hence they cannot be infnitely near to any of the points $P_j, j \geq 4$. Since there is an irreducible conic passing through all the three points, we can perform a quadratic transformation based at P_1, P_2, P_3 and we obtain the following configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
7	2	2	2	3	3	3	2	2	2	1	1	1	1	0
1	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	1	1	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
1	1	0	1	0	0	0	1	0	0	0	0	1	0	0
1	1	1	0	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	1
2	0	1	1	0	1	1	1	1	0	0	0	0	0	1

We now show that this new configuration cannot occur.

Let us consider the points P_4, P_5, P_6. Since they are triple points for the septic they cannot be infinitely near to any other point P_j, $j \leq 3$ or $j \geq 7$. Moreover the conic H' passes through P_5 and P_6 but not through P_4. Hence one among P_5 and P_6 has to be a planar point. We look at the eigenvalues of the curves B_0, $E_i', 1 \leq i \leq 5$, F' and H' for the action of the automorphism of order 3. We know that F' and H' correspond to different eigenvalues since they come from the blow-up of a singularity of type A_2 (see section 1.3). Let us set $\omega := e^{2\pi i/3}$. If B_0 corresponds to the eigenvalue ω then it appears with multiplicity 1 in the branch locus of the simple triple cover associated to $X \longrightarrow Y = X/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that E_i' corresponds to the eigenvalue ω^{ν_i}, F' corresponds to the eigenvalue ω^{ν_F} and H' to $\omega^{2\nu_F}$.

If P_6 was planar, then the total multiplicity of P_6 in the branch divisor of the simple triple cover has to be a multiple of 3. Thus

$$3 + \nu_H \equiv 0 \mod 3$$

which forces $\nu_H \equiv 0 \mod 3$. We get a contradiction since H' is an irreducible component of the branch divisor.

Thus P_5 is a planar point and P_6 is proximate to P_5. But then when we blow up P_5 the exceptional divisor F' should pass through P_6. Contradiction. □

Proposition 5.2.16. *The case $n = 3\ell - 1, n' = 2, n'' = 1$ cannot occur with degrees $(d_0, d_1, d_2, d_3, d_4, d_5, d_F, d_H) = (8, 2, 0, 0, 0, 0, 0, 0)$.*

Proof. Let us consider the points P_1, P_2, P_3. They are of maximal multiplicity for both the octic and one of the two conics hence they cannot be infnitely near to any of the points $P_j, j \geq 4$. Since there is an irreducible conic passing through all the three points, we can perform a quadratic transformation based at P_1, P_2, P_3 and we obtain the following configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
7	2	2	2	3	3	3	2	2	2	1	1	1	1	0
1	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	1	1	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	-1	1	1	0	0	0
1	1	0	1	0	0	0	1	0	0	0	0	1	0	0
1	1	1	0	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	1	-1	0	0	0	0	0	0	0	1

We now show that this new configuration cannot occur.

Let us consider the points P_4, P_5, P_6. Since they are triple points for the septic they cannot be infinitely near to any other point P_j, $j \leq 3$ or $j \geq 7$. Moreover we have $P_4 \succ P_5 \succ P_6$ (cf. definition 1.4.2). In particular P_6 is a planar point.

We look at the eigenvalues of the curves B_0, $E_i', 1 \leq i \leq 5$, F' and H' for the action of the automorphism of order 3. We know that F' and H' correspond to different eigenvalues since they come from the blow-up of a singularity of type A_2 (see section 1.3). Let us set $\omega := e^{2\pi i/3}$. If B_0 corresponds to the eigenvalue ω then it appears with multiplicity 1 in the branch locus of the simple triple cover associated to $X \longrightarrow Y = X/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that E_i' corresponds to the eigenvalue ω^{ν_i}, F' corresponds to the eigenvalue ω^{ν_F} and H' to $\omega^{2\nu_F}$.

Since P_6 is planar, the total multiplicity of P_6 in the branch divisor of the simple triple cover has to be a multiple of 3. Thus

$$3 + \nu_H \equiv 0 \mod 3$$

which forces $\nu_H \equiv 0 \mod 3$. We get a contradiction since H' is an irreducible component of the branch divisor. □

Hence we obtain

Theorem 5.2.17. *The case* $n = 3\ell - 1, n' = 2, n'' = 1$ *cannot occur.*

$\underline{n = 3\ell - 2}$

Assume $n = 3\ell - 2 = 1$. Then we contract the (-1)-cycles $G', Z, Z'_i, i = 1, \ldots, 5$ and we obtain a surface W with $K_W^2 = K_Y^2 + 7 = 2$ which is isomorphic to the plane blown-up at seven points $P_1, \ldots, P_7$. Thus $|\bar{N}_1|$ maps to the linear system of cubics through the seven points. Then

$$\bar{B}_0\bar{N}_1 = B_0N_1 = B_0(N + K_Y - G' - Z) = 0 + 4 - 0 - B_0Z$$

Corollary 2.3.11 tells us that Z is an irreducible (-1)-curve. Then from corollary 2.3.7 we find $B_0Z = 1$.

We also notice that from formula (4.6)

$$0 = NB_0 = (2Z + 2G' + \sum_{j=1}^{5} Z'_j - 2K_Y)B_0 = 2 + B_0\sum_{j=1}^{5} Z'_j - 8$$

hence

$$B_0\sum_{j=1}^{5} Z'_j = 6$$

Moreover by definition of $N = 3K_Y + 2B_0 + E' - 3G'$ we have

$$1 = Z'_jN = Z'_j(3K_Y + 2B_0 + E' - 3G') = -3 + (2B_0 + E')Z'_j$$

and then

$$(2B_0 + E')Z'_j = 4.$$

We notice that B_0 cannot be contained in any of the cycles Z'_j otherwise we should have

$$B_0N_1 = 3 \le Z'_jN_1 = 0$$

hence $0 \le B_0Z'_j \le 2$ for all $j = 1 \ldots, 5$.

Therefore we can have one of possibilities of the following table:

	$B_0Z'_1$	$B_0Z'_2$	$B_0Z'_3$	$B_0Z'_4$	$B_0Z'_5$
a)	2	2	2	0	0
b)	2	2	1	1	0
c)	2	1	1	1	1

Proposition 5.2.18. *Cases a) and b) cannot occur.*

Proof. Let us consider on W the image $\bar{B}_0$ of B_0.

Since $\bar{B}_0\bar{N}_1 = B_0N_1 = 3$ and $\bar{N}_1^2 = 2$ from the Index theorem 1.1.10 we find

$$(2\bar{B}_0 - 3\bar{N}_1)^2 = 4\bar{B}_0^2 - 18 \leq 0$$

hence $\bar{B}_0^2 \leq 4$.

If a) holds then we find (recall that in any case we have $B_0Z = 1, B_0G' = 0$)

$$\bar{B}_0^2 = B_0^2 + 1 + 4 + 4 + 4 = B_0^2 + 13 = -6 + 13 = 7$$

If b) holds then we find

$$\bar{B}_0^2 = B_0^2 + 1 + 4 + 4 + 1 + 1 = B_0^2 + 11 = -6 + 11 = 5$$

Finally if c) holds then we find

$$\bar{B}_0^2 = B_0^2 + 1 + 4 + 1 + 1 + 1 + 1 = B_0^2 + 9 = -6 + 9 = 3$$

The result is then proved. □

Let us consider the system $2\bar{B}_0 + \bar{E}' = \bar{N} - 3K_W$. This is mapped to the linear system $|-5K_{\mathbb{P}^2}|$ of curves of degree 15 with 7 quintuple points. In particular the plane image of $\bar{B}_0$ has degree $d_0 \leq 7$.

From proposition 5.2.18 only c) can actually hold. In this case $\bar{B}_0$ satisfies the linear system

$$\begin{cases} \sum_j js_j = 3d_0 - \bar{B}_0\bar{N}_1 = 3d_0 - 3 \\ \sum_j j^2 s_j = d_0^2 - \bar{B}_0^2 = d_0^2 - 3 \\ \sum_j s_j \leq 7 \end{cases}$$

where s_j is the number of points of multiplicity j of $\bar{B}_0$ among $P_1, \ldots, P_7$.

One can easily see that $d_0 \geq 3$ and the system has the following solutions (recall that $d_0 \leq 7$):

$$
\begin{array}{ll}
1) & d_0 = 3, s_1 = 6 \\
2) & d_0 = 4, s_2 = 2, s_1 = 5 \\
3) & d_0 = 5, s_2 = 5, s_1 = 2 \\
4) & d_0 = 6, s_3 = 1, s_2 = 6
\end{array}
$$

Proposition 5.2.19. *All the possibilities for $\bar{B}_0$ are equivalent up to Cremona quadratic transformations based at three of the seven points $P_1, \ldots, P_7$.*

Proof. Let us consider a sextic as in 4) and let us take the triple point Q_1 and two of the six double points Q_2, Q_3. Then they are not collinear otherwise there should be a line meeting the sextic at 7 points. Moreover Q_1 is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point, since is the unique point of maximal multiplicity for the curve. Since $\bar{E}_i'$ is an irreducible curve if both Q_2, Q_3 were proximate to P_1 from the inequalities (1.10) we should have

$$3 \geq 2 + 2 = 4$$

If Q_2 or Q_3 are not infinitely near to Q_1 we can choose them to be on $\mathbb{P}^2$. Then a quadratic transformation (see section 1.4) based at Q_1, Q_2, Q_3 is well-defined and takes the sextic onto a quintic as in 3).

Let us consider the quintic in 3) and let us take three of the five double points Q_1, Q_2, Q_3. Then they are not collinear otherwise there should be a line meeting the quintic at 6 points. Moreover we can choose the points in such a way that one of them, say Q_1, is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point. Since $\bar{E}_i'$ is an irreducible curve if both Q_2, Q_3 were proximate to P_1 from the inequalities (1.10) we should have

$$2 \geq 2 + 2 = 4$$

If Q_2 or Q_3 are not infinitely near to Q_1 we can choose them to be on $\mathbb{P}^2$. Then a quadratic transformation based at Q_1, Q_2, Q_3 is well-defined and takes the quintic onto a quartic as in 2).

Let us now take two double points Q_1, Q_2 and a simple point Q_3 for the quartic 2). Then they are not collinear otherwise there should be a line meeting the quartic at 5 points. Moreover we can choose the points in such a way that one of them, say Q_1, is on $\mathbb{P}^2$, i.e. it is not infinitely near to any other point. If both Q_2, Q_3 were proximate to P_1 from the inequalities (1.10) we should have

$$2 \geq 1 + 2 = 3$$

If Q_2 is not infinitely near to Q_1 we can choose it to be on $\mathbb{P}^2$. If Q_3 is not infinitely near to Q_1, Q_2 then either it is a plane simple point or it is infinitely near to a plane simple point or it is infinitely near to a plane double point. In the first case we have nothing to do. In the second case we choose as Q_3 the plane simple point, while in the third case we choose as Q_1 the plane double point. Then a quadratic transformation based at Q_1, Q_2, Q_3 is well-defined and takes the quartic onto a cubic as in 1). □

From now on we assume that the plane image of $\bar{B}_0$ is a sextic with a triple point and six double points. We also note that a quadratic transformation leaves the plane image of $|2\bar{B}_0 + \bar{E}'|$ unchanged (in fact it is $| - 5K_{\mathbb{P}^2})|$). In particular after any quadratic transformation the equation

$$2d_0 + \sum_{i=1}^{5} d_i = 15 \tag{5.31}$$

holds, where d_0 is the degree of the image of $\bar{B}_0$ while $d_i, i = 1, \dots, 5$ are the degrees of the plane images of the curves $\bar{E}'_i$. When we fix $d_0 = 6$ we obtain

$$\sum_{i=1}^{5} d_i = 3.$$

Remark 5.2.20. *We observe that for any $1 \leq i \leq 4$ we have $E'_i N_1 = 1$. In particular E'_i cannot be contained in a (-1)-cycle Z'_j since otherwise*

$$1 = E'_i N_1 \leq Z'_j N_1 = 0.$$

Moreover since $\bar{E}'_i \bar{N}_1 = E'_i N_1 = 1$ and $\bar{N}_1^2 = 2$ we have

$$(2\bar{E}'_i - \bar{N}_1)^2 = 4\bar{E}'^2_i - 2 \leq 0$$

hence $\bar{E}'^2_i \leq 0$. Computing

$$0 = E'_i N_2 = E'_i(N + 2K_Y - 2G' - 2Z - \sum_{j=1}^{5} Z'_j) = 2 - E'_i \sum_{j=1}^{5} Z'_j$$

the Index theorem 1.1.10 forces $E'_i Z'_{j_1} = E'_i Z'_{j_2} = 1$ for suitable j_1 and j_2.

From remark 5.2.20 we can subdivide the analysis as follows:

Case I: E_5' *is contracted on* W.

Case II: E_5' *is not contracted on* W.

Before studying the two cases we add some useful remarks.

Remark 5.2.21. *From the results of section 4.3 we have*

$$2B_0 + E' \equiv 2Z + 5G' + \sum_{j=1}^{5} Z_j' - 5K_Y$$

hence the total multiplicity of $2B_0 + E'$ *at the points* $P_1, \ldots, P_7$ *is 5, while if we denote by* $P_8, \ldots, P_{12}$ *the points obtained by contracting the cycles* Z_j'*, by* P_{13} *the point obtained by contracting* Z *and by* P_{14} *the point which is the contraction of* G'*,* $2B_0+E'$ *has multiplicity 4 at* $P_8, \ldots, P_{12}$*, multiplicity 3 at* P_{13} *and multiplicity 0 at* P_{14}*.*

Remark 5.2.22. *Since* $\bar{E}_i'\bar{N}_1 = 1$ *the curves* E_i'*,* $1 \leq i \leq 4$ *satisfy* $\sum_{i=1}^{7} m_i = 3d_i - 1$*. Moreover either* $\bar{B}_0\bar{E}_i' = 2$ *or* $\bar{B}_0\bar{E}_i' = 3$*. In the former case we find*

$$2 = \bar{B}_0\bar{E}_i' = 6d_i - 3m_1 - 2\sum_{i=1}^{7} m_i = 6d_i - m_1 - 6d_i + 2$$

hence $m_1 = 0$*.*

In the latter case we find

$$3 = \bar{B}_0\bar{E}_i' = 6d_i - 3m_1 - 2\sum_{i=1}^{7} m_i = 6d_i - m_1 - 6d_i + 2$$

hence $m_1 = -1$ *and* $d_i = 0$*.*

From (5.31) *we have* $\sum_{i=1}^{5} d_i = 3$ *whence* $d_i \leq 3$ *for all* i*. When* $1 \leq i \leq 4$ $\bar{E}_i'$ *is a* (-1)*-curve such that* $\bar{E}_i'\bar{N}_1 = 1$*. Thus it is a solution of the following linear system*

$$\begin{cases} \sum_j j^2 s_j = d_i^2 + 1 \\ \sum_j j s_j = 3d_i - 1 \\ \sum_j s_j \leq 7 \end{cases}$$

Since $d_i \leq 3$ *we find the solutions:*

$$\begin{aligned} &1) \qquad d_i = 1, s_1 = 2 \\ &2) \qquad d_i = 2, s_1 = 5 \end{aligned}$$

$$3) \qquad d_i = 3, s_2 = 1, s_1 = 6$$

From the computation of $\bar{E}_i'\bar{B}_0$ we have $m_1 = 0, -1$ which excludes case 3). In particular $d_i \leq 2$ for $1 \leq i \leq 4$.

Case I: *E_5' is contracted on W.*

If E_5' is contracted on W then there exists a (-1)-cycle Z_j' such that $E_5'Z_j' = -1$. Then $d_5 = 0$ and the multiplicity of E_5' at each point $P_1 \ldots, P_7$ is 0. Then we have the following possibilities for the 6-tuples $(d_0, d_1, d_2, d_3, d_4, d_5)$:

$$(6, 2, 1, 0, 0, 0)$$
$$(6, 1, 1, 1, 0, 0)$$

In any of the two cases using the remarks 5.2.21 and 5.2.22 and the fact that $E_i'E_j' = 0$ for $i \neq j$ and $E_i'B_0 = 0$ on the surface Y we can uniquely determine the multiplicities of each curve at the points $P_1, \ldots, P_{14}$:

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
0	-1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	1	1	1	1	1	0	0	0	1	1	0	0	0
1	0	1	0	0	0	0	1	0	0	1	0	1	0	0
0	0	-1	0	0	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	1	0

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
0	-1	0	0	0	0	0	0	1	1	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	1	1	0	0	0
1	0	0	0	1	1	0	0	0	0	1	0	1	0	0
1	0	0	0	0	0	1	1	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	1	0

Let us consider the second configuration. Then P_4 and P_5 are the only points which are of maximal multiplicity for both the sextic and the line E_3'. Hence at least one of them has to be a planar point, say P_5. Analogously one between P_6

and P_7, say P_7, has to be a planar point. Thus a quadratic transformation based at P_5, P_6, P_7 is well-defined (see section 1.4) and it takes the second configuration to the first one. Hence they are equivalent up to Cremona transformations and we can assume $(6,2,1,0,0,0)$ holds.

We now compute the images of F' and H'. They have no intersection with all the above curves while $F'G' = H'G' = 1$. In particular they pass through P_{14}. Moreover since $F'N_1 = H'N_1 = 0$ we find $F'Z = H'Z = 0$ hence they do not pass through P_{13}. Thus they satisfy the following conditions

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ -m_8 + m_9 + m_{13} = 0 \\ -m_2 + m_{11} + m_{12} = 0 \\ m_2 + m_7 + m_{10} + m_{12} = d \\ m_2 + m_3 + m_4 + m_5 + m_6 + m_{10} + m_{11} = 2d \\ -m_1 + m_8 + m_9 = 0 \\ 3m_1 + 2\sum_{i=2}^{8} m_i + \sum_{i=9}^{13} m_i = 6d \end{cases} \tag{5.32}$$

If one of the two curves, say F', is contracted on W we have $d = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = 0$ and (5.32) becomes

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ -m_8 + m_9 + m_{13} = 0 \\ m_{11} + m_{12} = 0 \\ m_{10} + m_{12} = 0 \\ m_{10} + m_{11} = 0 \\ m_8 + m_9 = 0 \\ 2m_8 + \sum_{i=9}^{13} m_i = 0 \end{cases}$$

which has the only solution $m_8 = m_9 = m_{10} = m_{11} = m_{12} = m_{13} = 0, m_{14} = 1$. Then we get a contradiction since ${F'}^2 = -3$ on Y.

It follows that neither F' nor H' are contracted on W. This forces $m_8 = m_9 = m_{10} = m_{11} = m_{12} = m_{13} = 0$. Then (5.32) becomes

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ 0 = 0 \\ m_2 = 0 \\ m_2 + m_7 = d \\ m_2 + m_3 + m_4 + m_5 + m_6 = 2d \\ m_1 = 0 \\ 3m_1 + 2\sum_{i=2}^{7} m_i = 6d \end{cases}$$

which forces $m_1 = m_2 = 0, m_7 = d, m_3 + m_4 + m_5 + m_6 = 2d$.

Moreover since $F'^2 = -3$ we have

$$-3 = d^2 - m_3^2 - m_4^2 - m_5^2 - m_6^2 - d^2 - 1$$

which forces

$$m_3^2 + m_4^2 + m_5^2 + m_6^2 = 2 \geq m_3 + m_4 + m_5 + m_6 = 2d \tag{5.33}$$

hence $d \leq 1$. If it was $d = 1$ for both F' and H' we would get a contradiction since the two distinct lines would meet at P_7 and P_{14}. Thus for at least one of them we have $d = 0$. Then from (5.33) we have

$$-1 \leq m_i \leq 1 \quad 3 \leq i \leq 6.$$

Without loss of generality we can assume $m_3 = 1, m_4 = -1, m_5 = m_6 = 0$ for one of the two curves, say F'.

If H' is a line we have the configuration

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
0	-1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	1	1	1	1	1	0	0	0	1	1	0	0	0
1	0	1	0	0	0	0	1	0	0	1	0	1	0	0
0	0	-1	0	0	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	1	0
0	0	0	1	-1	0	0	0	0	0	0	0	0	0	1
1	0	0	0	1	1	0	1	0	0	0	0	0	0	1

whereas if it is contracted we find

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
0	-1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	1	1	1	1	1	0	0	0	1	1	0	0	0
1	0	1	0	0	0	0	1	0	0	1	0	1	0	0
0	0	-1	0	0	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	1	0
0	0	0	1	-1	0	0	0	0	0	0	0	0	0	1
0	0	0	-1	0	1	0	0	0	0	0	0	0	0	1

Lemma 5.2.23. *The two configurations are equivalent up to quadratic transformations.*

Proof. We now consider the first configuration. The point P_2 cannot be infinitely near to any other point, since it is of maximal multiplicity for both the conic and the line E_2'. P_7 cannot be infinitely near to P_2 since it is on the line H'. Then we can easily see that P_7 is a planar point too. Then with a similar argument one can see that either P_4 or P_5 is planar. Let us assume P_5 is not planar. Hence P_4 is and we can perform a quadratic transformation based at P_2, P_4, P_7. Then we obtain

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
0	-1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	1	1	1	1	1	0	0	0	1	1	0	0	0
0	0	0	0	-1	0	0	0	0	0	1	0	1	0	0
1	0	0	0	1	0	0	1	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	-1	1	0	0	0	1	0
1	0	1	1	0	0	0	1	0	0	0	0	0	0	1
0	0	-1	0	0	1	0	0	0	0	0	0	0	0	1

Then the line F' has $m_2 + m_7 = 2 \neq 1$ contradicting (5.32). Thus we can apply a quadratic transformation based at P_2, P_5, P_7 and we obtain the second configuration. $\square$

Proposition 5.2.24. *In the above setting case I cannot occur.*

Proof. We look at the eigenvalues of the curves B_0, $E_i', 1 \leq i \leq 5$, F' and H' for the action of the automorphism of order 3. We know that F' and H' correspond to different eigenvalues since they come from the blow-up of a singularity of type A_2 (see section 1.3). Let us set $\omega := e^{2\pi i/3}$. If B_0 corresponds to the eigenvalue ω then it appears with multiplicity 1 in the branch locus of the simple triple cover associated to $X \longrightarrow Y = X/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that E_i' corresponds to the eigenvalue ω^{ν_i}, F' corresponds to the eigenvalue ω^{ν_F} and H' to $\omega^{2\nu_F}$.

The point P_1 is not infinitely near to any other point since it is the only point of maximal multiplicity of the sextic. The total multiplicity at this point of the branch divisor has to be a multiple of 3. Then we obtain the equation

$$3 + \nu_1 \equiv 0 \mod 3$$

which forces $\nu_1 \equiv 0 \mod 3$. Contradiction. □

Case II: *E_5' is not contracted on W.*

Let us now assume E_5' is not contracted on W. Then we have $E_5'Z = 1$, $E_5'Z_j' = 0$ for all $j \leq 5$. Moreover $\bar{E}_5'\bar{N}_1 = 0$ hence $\sum_{i=1}^{7} m_i = 3d_5$. In particular we can compute

$$1 = \bar{B}_0\bar{E}_5' = 6d_5 - 3m_1 - 2\sum_{i=2}^{7} m_i = 6d_5 - m_1 - 6d_5$$

which forces $m_1 = -1$ and $d_5 = 0$.

Then, as before, we have the following possibilities for $(d_0, d_1, d_2, d_3, d_4, d_5)$:

$$(6, 2, 1, 0, 0, 0)$$
$$(6, 1, 1, 1, 0, 0)$$

Using remarks 5.2.20, 5.2.21, 5.2.22 and the fact that $E_i'E_j' = 0, E_i'B_0 = 0$ for any $i \neq j$ on Y each 6-tuple gives us a unique configuration of the points

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
2	0	0	1	1	1	1	1	0	1	1	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	1	1	0	0
0	0	0	0	-1	0	0	0	0	0	1	0	1	0	0
0	0	0	-1	0	0	0	0	0	1	0	1	0	0	0
0	-1	1	0	0	0	0	0	0	0	0	0	0	1	0

and

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
1	0	0	1	1	0	0	0	0	1	1	0	0	0	0
1	0	0	1	0	1	0	0	0	0	0	1	1	0	0
1	0	0	0	0	0	1	1	0	0	1	0	1	0	0
0	0	0	-1	0	0	0	0	0	1	0	1	0	0	0
0	-1	1	0	0	0	0	0	0	0	0	0	0	1	0

Let us consider the second configuration. Then we can base a quadratic transformation (see 1.4) at the points P_3, P_4, P_5. In fact P_3 is a planar point since it is of maximal multiplicity for the lines E_1' and E_2' and double for the sextic.

P_4 and P_5 cannot both be proximate to P_3 otherwise from (1.10) the sextic should satisfy

$$2 = m_3 \leq m_4 + m_5 = 2 + 2 = 4.$$

Hence they are planar points too. Thus we can reduce the second configuration to the first one. Then we can assume that $(d_0, d_1, d_2, d_3, d_4, d_5) = (6, 2, 1, 0, 0, 0)$ holds.

Let us now compute the images of F' and H'. As before they have no intersection with all the above curves and we know that $F'Z = H'Z = 0$. Moreover they both pass through the point P_{14} which is the contraction of G'. Thus we find the conditions

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ -m_1 + m_2 + m_{13} = 0 \\ -m_3 + m_9 + m_{11} = 0 \\ -m_4 + m_{10} + m_{12} = 0 \\ m_3 + m_4 + m_{11} + m_{12} = d \\ m_3 + m_4 + m_5 + m_6 + m_7 + m_9 + m_{10} = 2d \\ 3m_1 + 2\sum_{i=2}^{8} m_i + \sum_{i=9}^{13} m_i = 6d \end{cases} \tag{5.34}$$

If F' or H' are contracted on W then $d = m_1 = m_2 = m_3 = m_4 = m_5 =$

$m_6 = m_7 = 0$ and (5.34) becomes

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ 0 = 0 \\ m_9 + m_{11} = 0 \\ m_{10} + m_{12} = 0 \\ m_{11} + m_{12} = 0 \\ m_9 + m_{10} = 0 \\ 2m_8 + \sum_{i=9}^{12} m_i = 0 \end{cases}$$

which forces $m_8 = 0$, $m_9 = m_{12} = -m_{10} = -m_{11}$. Since $F'^2 = H'^2 = -3$ we obtain a contradiction.

Hence neither F' nor H' are contracted on W. In this case we have $m_8 = m_9 = m_{10} = m_{11} = m_{12} = m_{13} = 0$ and (5.34) becomes

$$\begin{cases} m_{14} = 1 \\ m_{13} = 0 \\ m_1 = m_2 \\ m_3 = 0 \\ m_4 = 0 \\ m_3 + m_4 = d \\ m_3 + m_4 + m_5 + m_6 + m_7 = 2d \\ 3m_1 + 2\sum_{i=2}^{7} m_i = 6d \end{cases}$$

which forces $d = 0, m_1 = m_2 = m_3 = m_4 = 0, m_5 + m_6 + m_7 = 0$. Thus since the self-intersection of F' and H' is -3 we have

$$\{m_5, m_6, m_7\} = \{1, 0, -1\}.$$

Hence we can write the following configuration (recall that $F'H' = 0$ on Y)

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
6	3	2	2	2	2	2	2	2	1	1	1	1	1	0
2	0	0	1	1	1	1	1	0	1	1	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	1	1	0	0
0	0	0	0	-1	0	0	0	0	0	1	0	1	0	0
0	0	0	-1	0	0	0	0	0	1	0	1	0	0	0
0	-1	1	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	-1	0	0	0	0	0	0	0	1
0	0	0	0	0	-1	0	1	0	0	0	0	0	0	1

A similar argument as in proposition 5.2.24 shows

Proposition 5.2.25. *In the above setting case II cannot occur.*

As an immediate consequence we obtain

Theorem 5.2.26. *The case $n = 3\ell - 2, n' = 5$ cannot occur.*

$\underline{n = 3\ell - 3}$

The case $n = 3\ell - 3$ can only occur (see theorem 4.6.5) when we are in cases $(1f)$ of proposition 2.2.14 or when $A' = N$. Then $n = 0$ and when we contract on Y the curve G' we obtain a rational surface W with $K_W^2 = -5 + 1 = -4$ having a net $\bar{N}_1 = \bar{N} + K_W$ of rational curves such that $\bar{N}_1^2 = 1$.

Then as we have already seen in section 4.2 we have a birational morphism $\varphi_{|\bar{N}_1|} : W \longrightarrow \mathbb{P}^2$. The net $|\bar{N}_1|$ is then mapped onto the net of lines of $\mathbb{P}^2$ while $|\bar{N}|$ is mapped to the system of quartics through 13 points. Thus $|2\bar{B}_0 + \bar{E}'| = |\bar{N} - 3K_W|$ is mapped to the system of curves of degree 13 with 13 quadruple points.

The curves F' and H' on Y have no intersection with N_1 (cf. corollary 2.3.4) hence they are contracted by the map $\varphi_{|\bar{N}_1|}$.

We now compute the plane images of B_0 and E'. We have

$$\bar{B}_0\bar{N}_1 = B_0N_1 = B_0(N + K_Y - G') = 0 + 4 - 0 = 4$$

$$\bar{E}_i'\bar{N}_1 = E_i'N_1 = E_i'(N + K_Y - G') = 0 + 1 - 0 = 1$$

$\bar{B}_0$ is then mapped onto a quartic and none of the curves E'_i is contracted. Since the total multiplicity of any of $2\bar{B}_0 + \bar{E}'$ of any of the thirteen points is 4, $\bar{B}_0$ can only have simple or double points at $P_1, \ldots, P_{13}$. Then

$$\sum_j j^2 s_j = d^2 - \bar{B}_0^2 = 16 - B_0^2 + 16 + 6 = 22$$

and

$$\sum_j j s_j = 4d - \bar{B}_0 \bar{N} = 16 - B_0 N = 16$$

with $j \leq 2$. Hence

$$2s_2 = (4s_2 + s_1) - (2s_2 + s_1) = 22 - 16 = 6$$

forces $s_2 = 3, s_1 = 10$.

The (-3)-curves $\bar{E}'_k$ satisfy $\bar{E}'_k \bar{N}_1 = E'_k N_1 = 1$ hence their plane images are lines through four points. Since $\bar{E}'_j \bar{E}'_k = 0$ for any $j \neq k$ and since $2\bar{B}_0 + \bar{E}'$ has multiplicity 4 at the points $P_1, \ldots, P_{13}$ it is easy to see that, up to reordering the points, the only possible configuration for $\bar{B}_0$ and the curves $\bar{E}'_k$ is

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	2	2	2	1	1	1	1	1	1	1	1	1	1	0
1	0	0	0	1	1	1	1	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	1	1	0	0	0	0
1	0	0	0	0	1	0	0	1	0	0	1	1	0	0
1	0	0	0	0	0	1	0	0	1	0	1	0	1	0
1	0	0	0	0	0	0	1	0	0	1	0	1	1	0

We know that F' and H' are (-3)-curves contracted on $\mathbb{P}^2$ since they have no intersection with N_1. Moreover they do not intersect B_0 and the curve E'_i while they intersect G' at one point.

Thus they solve the linear system

$$\begin{cases} m_{14} = 1 \\ 2(m_1 + m_2 + m_3) + \sum_{i=4}^{13} m_i = 0 \\ m_4 + m_5 + m_6 + m_7 = 0 \\ m_4 + m_8 + m_9 + m_{10} = 0 \\ m_5 + m_8 + m_{11} + m_{12} = 0 \\ m_6 + m_9 + m_{11} + m_{13} = 0 \\ m_7 + m_{10} + m_{12} + m_{13} = 0 \end{cases}$$

The sum of the last five equations gives

$$2\sum_{i=4}^{13} m_i = 0$$

hence also $m_1 + m_2 + m_3 = 0$.

Since $F'^2 = H'^2 = -3$ we also have

$$-\sum_{i=1}^{13} m_i^2 = -2 \tag{5.35}$$

and then $-1 \leq m_i \leq 1$ for any $1 \leq i \leq 13$. Each number $m_i, i \geq 4, i \neq 14$ appears in three of the seven equations. Thus if it was $m_i = 1$ there should be at least two values j such that $m_j = -1$. For example when $i = 4$ we have

$$1 + m_5 + m_6 + m_7 = 0$$

$$1 + m_8 + m_9 + m_{10} = 0$$

which force two values, say m_5 and m_8 to be -1. This contradicts (5.35). Hence

$$m_i = 0 \quad i \geq 4$$

and

$$\{m_1, m_2, m_3\} = \{-1, 0, 1\}.$$

Thus the total configuration of the curves B_0, E_i', F', H' is

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
4	2	2	2	1	1	1	1	1	1	1	1	1	1	0
1	0	0	0	1	1	1	1	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	1	1	0	0	0	0
1	0	0	0	0	1	0	0	1	0	0	1	1	0	0
1	0	0	0	0	0	1	0	0	1	0	1	0	1	0
1	0	0	0	0	0	0	1	0	0	1	0	1	1	0
0	-1	1	0	0	0	0	0	0	0	0	0	0	0	1
0	0	-1	1	0	0	0	0	0	0	0	0	0	0	1

Theorem 5.2.27. *The case $n = 3l - 3$ cannot occur.*

Proof. We now want to compute the eigenvalues corresponding to each curve of the branch locus. Let us consider the simple triple cover associated to $X \longrightarrow Y$.

Without loss of generality we can assume that B_0 corresponds to the eigenvalue $\omega = e^{2\pi i/3}$. Each curve E_i' corresponds to ω^{ν_i} hence it will appear with multiplicity ν_i in the branch locus of the simple cover. Moreover F' and H' have different eigenvalues, since they come from the blow-up of a singularity of type A_2 and their multiplicities in the branch locus will be denoted by ν_F and $\nu_H \equiv 2\nu_F \mod 3$ respectively.

Then we look at the points P_4, P_5, P_6, P_7 on $\mathbb{P}^2$. F' and H' do not pass through any of these points and we find the numerical conditions

$$\begin{aligned} 1 + \nu_1 + \nu_2 &\equiv 0 \mod 3 \\ 1 + \nu_1 + \nu_3 &\equiv 0 \mod 3 \\ 1 + \nu_1 + \nu_4 &\equiv 0 \mod 3 \\ 1 + \nu_1 + \nu_5 &\equiv 0 \mod 3 \end{aligned}$$

which force $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5 = 1$ since $\nu_i \equiv 1, 2 \mod 3$. This contradicts proposition 2.1.8. □

5.3 The final statement

Theorem 5.3.1. *A numerical Godeaux surface S cannot have an automorphism of order 3.*

Proof. See the proofs of theorems 3.1.21, 3.2.2, 4.6.5, 5.1.11, 5.1.12, 5.1.25, 5.2.3, 5.2.8, 5.2.17, 5.2.26 and 5.2.27. □

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