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# A double-copy perspective on asymptotic symmetries

Titolo della tesi


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# Chapter 1

# Introduction and general outlook

Is it possible to describe gravity by means of simpler spin-one gauge theories? This question is undeniably relevant to our understanding of the Universe, as gravity continues to underlie the most challenging problems in fundamental physics. At first glance, however, a positive answer may seem unlikely, given the inherent differences between gravity and gauge theories. This is further supported by the Weinberg-Witten theorem [1], which asserts that the graviton is not a composite object, but rather a fundamental spin-two particle.

Nevertheless, evidence that has emerged over the past fifteen years in the study of scattering amplitudes suggests that the initial question may not be as trivial as it seems. Specifically, it was discovered that two tree-level Yang-Mills amplitudes, when expressed in a certain symmetric form, can be combined to yield gravitational tree-level amplitudes [2,3]. This observation led to the rapid development of the **double-copy** framework[1]. Important results have been achieved by employing the double-copy or Bern-Carrasco-Johansonn (BCJ) relations for amplitudes, such as the proof about $\mathcal{N} = 8$ supergravity being finite at five loops [7].

One might wonder whether the double-copy construction finds applications in other contexts. A natural arena is provided by solutions to classical field equations, being closely related to the semiclassical limit of tree-level scattering amplitudes. Indeed, with some subtleties, it is possible to arrange some non-perturbative, geometric solutions in general relativity as a double-copy of classical solutions in suitable Yang-Mills theories [8–10].

How is it possible that gauge theories, which are generally unaware of spacetime geometry, can provide information about it? To what extent does

[1] See [4–6] for reviews.

double-copy know about geometry? To address this, we must establish a connection between gravity and Yang-Mills (YM) theories at a fundamental, field-theoretical level. This was partially achieved in the framework of the **convolutional double-copy** [11–13], where the DC gravitational field is expressed as a particular convolution of Yang-Mills fields. The convolutional double-copy dictionary is able to connect gravity and gauge theories at the level of symmetries, although both theories are considered in the linearised limit. In fact, the "input" of the double-copy in this particular setup is not the full non-Abelian structure of Yang-Mills fields but only its abelianised limit, meaning that the Yang-Mills fields are Maxwell fields with a global color index. At the same time, the "output" of the convolutional double copy is not a full gravitational field, but rather its linearised version, describing a spin-two particle on a flat background. As of now, a complete field-theoretical framework that connects non-Abelian Yang-Mills theories to full gravity has not been developed, leaving the geometrical interpretation of the double-copy still unclear.

Nonetheless, the convolutional approach appears to be a promising step towards answering this fundamental question. In particular, it seems well-suited to addressing an intermediate, yet nontrivial, problem: can gauge theories provide information on the asymptotic geometry of spacetime? This is the central question guiding this thesis, and based on our understanding, we believe the convolutional dictionary allows us to provide a positive answer in the case of asimptotically flat configurations. By means of the convolutional DC, we are able to show that the boundary conditions, studied at future null infinity $\mathcal{I}$, in (linearised) gravity can be expressed as a double copy of the boundary conditions in electromagnetism. Moreover, we are able to express the non-trivial class of gauge symmetries preserving the boundary conditions in gravity as a double-copy of those in electromagnetism, mapping the Maxwell **asymptotic symmetries** to the gravitational ones. Previous analysis on double-copy and asymptotic symmetries can be found in [14, 15], with reference to the self-dual sectors of gauge and gravitational theories. Further related works are [16, 17].

An asymptotic symmetry is a local gauge symmetry which preserves the boundary condition while mapping different physical configurations[2]. Therefore, gauge symmetries can have a physical meaning, and this is encoded in the presence of (infinitely many) conserved charges which do not die off at the boundary under investigation. Furthermore, asymptotic symmetries are associated to important observable effects [19], which are **soft theorems** for scattering amplitudes and **memory effects**. This connection is elegantly described by the so-called **infrared triangle**, depicted in Figure 1.1.

[2]See [18] for a review.

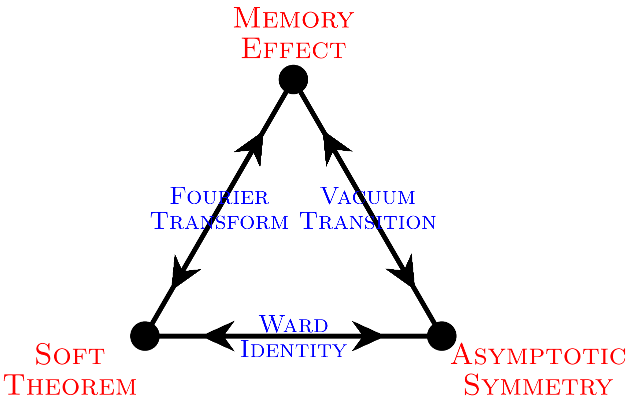


Figure 1.1: The infrared triangle, connecting asymptotic symmetries with soft theorems and memory effects. *Image from the ArXiv version of [18].*

The concept of asymptotic symmetry finds applications in both gravity and gauge theories. The very first example of it can be found in the context of general relativity and the study of asymptotically flat spacetime at null infinity $\mathcal{I}$, as discussed in 1962 by Bondi, van der Burg, Metzner and Sachs [20–22]. Specifically, they identified the class of nontrivial diffeomorphisms that preserve the asymptotic flatness condition to be an infinite-dimensional enhancement of the Poincaré group, encompassing the Lorentz group and the supertranslations, an infinite-dimensional generalisation of the ordinary translations. Surprisingly, the first discussion of asymptotic symmetries in gauge theories is far more recent, with the first study on Yang-Mills theories in 2013 by Strominger [23]. The fundamental idea of this project is to study the connection between these two examples of asymptotic symmetries from the point of view of the double-copy.

Basically, the plan of asymptotic symmetries consists in identifying those gauge parameters compatible with the boundary conditions and a nonvanishing **asymptotic charge** at the boundary. Typically, these charges are further constrained by the condition of being non-divergent. This request limits the possible classes of gauge parameters allowed and, for the theories considered in this thesis, in $D = 4$ it implies that the Minkowskian components of the gauge parameters have to approach $\mathcal{I}$ with an $\mathcal{O}(r^0)$ falloff condition. However, recent developments on the possibility to renormalize the divergent terms of asymptotic charges [24] allow one to explore for **higher-order asymptotic symmetries**, parametrised by in principle arbitrary divergent, $\mathcal{O}(r^N)$, gauge parameters. The idea at the basis of infrared triangle is that a higher-order asymptotic symmetries is associated to a certain sub$^n$-leading soft theorem.

Within this framework, in this thesis we also study higher-order asymptotic symmetries in $D = 4$ for electromagnetism, refining the existing analyses [25, 26, 135, 136], and for the two-form theory, for which previous investigations only concerned the finite sector of asymptotic charges [27, 28]. This discussion is strictly related to the double-copy viewpoint. In fact, the convolutional double-copy describes an extended graviton, which propagates a spin-two graviton, a two-form Kalb-Ramond field and a scalar dilaton. This is the $\mathcal{N} = 0$ supergravity multiplet, which emerge as the particle content of the low-energy limit of bosonic string theory. In particular, we are able to employ the DC dictionary in order to map the ordinary Maxwell asymptotic symmetries to the corresponding (extended) gravitational ones, where with "ordinary" we refer to those standard asymptotic symmetries which do not require a renormalization procedure. Surprisingly, this analysis unveils some aspects about two-form asymptotic symmetries that were not fully clarified in the existing literature [27, 28]. A natural outlook stemming from the DC analysis would be that of generalising our study to higher order. In this sense, the two-form higher order symmetries that we explored in [29] and illustrate in this thesis, can provide a benchmark for testing deeper investigations from the double-copy point of view. Furthermore, in $D = 4$ a two-form is dual, on-shell, to a scalar field. This duality makes the two-form analysis appealing since it could allow us to better explore the relation between asymptotic symmetries and physical observables, taking advantage of the simplicity of scalar field theories.

We divide this thesis in two parts, the first on asymptotic symmetries and the second on double-copy. We start with a review on asymptotic symmetries and their connection with physical observables, respectively analysed in section 2.1 and section 2.2. We thus focus in section 2.3 on the peculiarities that emerge in Lorenz gauge, for which gauge parameters require logarithms in their expansion near $\mathcal{I}$. This is instrumental to present the work [29] in chapter 3, which discusses higher-order asymptotic symmetries for electromagnetism and for the two-form. For the electromagnetic part, the work can be partially understood as a review of the already existing analyses [25, 26], but in a slightly different setup. Focusing on $\mathcal{I}^+$, reached by the limit $r \to \infty$ at fixed retarded time $u = t - r$, we consider different choices for the expansions of quantities near the boundary and, for what concerns the renormalization, we discuss more general expansions for the physical Cauchy data of the theory. Specifically, while the gauge parameter expansion require logarithmic terms, we use a power expansion for the field strength, thus implying that the gauge field components admit logarithms exclusively within pure gauge sectors. At the same time, we consider the Cauchy data, encoded in the angular components of the gauge field $A_i$ ($i = 1, 2$), to admit a power expansion in the retarded time $u$ near $\mathcal{I}^+_-$, which is reached in the limit $u \to -\infty$. The symplectic renormalization allow us to obtain $N$ independent asymptotic charges

$$\begin{aligned}
Q_e^{(0)} &= \int_{\mathcal{I}^+_-} dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon_c^{(0)} \mathcal{D}^i A_i^{(0,0)}\,, \\
Q_e^{(n)} &= -\int_{\mathcal{I}^+_-} dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon_c^{(-n)} \mathcal{D}^i A_i^{(n,0)} \qquad (1 \leq n \leq N)\,,
\end{aligned} \tag{1.1}$$

where we parametrised the angular variables by means of complex stereographic coordinates $z, \bar{z}$. In particular, $\mathcal{D}^i A_i^{(n,0)}$ is the two-sphere divergence of $A_i^{(n,0)}$, which is the $u-$independent part of the $\mathcal{O}(r^{-n})$ Cauchy data component near $\mathcal{I}^+_-$, while $\epsilon_c^{(-n)}$ is the $u-$independent part of the (divergent) $\mathcal{O}(r^n)$ gauge parameter.

The same approach is then applied to the two-form case. In particular, the Cauchy data is encoded in the $B_{z\bar{z}}$ component, which is of order $\mathcal{O}(r)$, for which we assume the same expansions used for the electromagnetic case. Focusing on arbitrary $\mathcal{O}(r^N)$ parameters, we identify higher-order asymptotic symmetries and $N$ independents asymptotic charges:

$$\begin{aligned}
Q_B^{(0)} &= -\int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-1)} \mathcal{D}^2 B^{(-1,0)}.\,, \\
Q_B^{(-1)} &= -\int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-2)} \frac{(\mathcal{D}^2)^2}{2} B_r'^{(1,0)}\,, \\
Q_B^{(-n)} &= \frac{n-2}{2} \int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-n)} \mathcal{D}^2 B'^{(n-1,0)} \qquad (3 \leq n \leq N+1)\,,
\end{aligned} \tag{1.2}$$

where $\mathcal{D}^2$ is the two-sphere Laplace operator, $\lambda_c^{(-n)}$ is the $u-$independent part of the $\mathcal{O}(r^n)$ gauge parameter, while the $B^{(n,0)}$'s encode the $u-$independent $\mathcal{O}(r^n)$ term of the Cauchy data. We thus find that the result of electromagnetism can be generalised to the two-form. Furthermore, the structural similarities between electromagnetic and two-form charges suggest also an interpretation in terms of double-copy.

These results suggest various possible directions to be explored in the future. For instance, it would be interesting to generalise our approach to arbitrary spacetime dimension $D$ and to study $p$-forms, with the aim of exploring the on-shell duality between $p$- and $(D-p-2)$-forms from the perspective of asymptotic symmetries [30]. Furthermore, our analysis could be applied to gravity, for which only $\mathcal{O}(r^2)$ asymptotic symmetries were formulated in $D = 4$ [31] and studied to arbitrary order only in the self-dual sector [15], and to higher-spin theories [32, 33].

The usual prescription used for symplectic renormalization is to cancel all the divergences while leaving the finite parts unaltered, which are interpreted as physical. The divergences are cancelled exploiting ambiguities intrinsic to the definition of the presymplectic potential. However, there are additional residual ambiguities whose role has not been clarified yet. It could be possible that these residual terms are able to cancel some finite parts, whose physical meaning should be reconsidered.

In the second part of this thesis, we focus instead on double-copy asymptotic symmetries. We start with a review of the various incarnations of the double-copy in chapter 4, with a particular emphasis on the field-theoretical aspects in section 4.3 and section 4.4. We then review the work [34] in chapter 5, employing the convolutional dictionary to map the boundary conditions of electrodynamics in Lorenz gauge to the boundary conditions of a graviton in de-Donder gauge, a two-form in Lorenz gauge and a dilaton. We further find that ordinary Maxwell asymptotic symmetries are mapped to the (extended) gravitational ones, deriving asymptotic charges for all the fields in the $\mathcal{N}=0$ supergravity multiplet. In particular, for the symmetric part of the DC field, encoding a graviton and a scalar, we obtain BMS supertranslations charge [19, 31, 35, 36]

$$Q^h = \frac{1}{2}\int dz d\bar{z}\gamma_{z\bar{z}}T(z,\bar{z})\mathcal{D}^i\mathcal{D}^j h_{ij}^{(-1)}\,, \tag{1.3}$$

with $T$ being the function parametrising supertranslations and $h_{ij}^{(-1)}$ the $\mathcal{O}(r)$ components of $h_{ij}$ encoding the two graviton polarisations, and the asymptotic scalar charge

$$Q^\varphi = -\frac{1}{2}\int dz d\bar{z}\gamma_{z\bar{z}}(\mathcal{D}^2+1)T(z,\bar{z})\varphi^{(1)}\,, \tag{1.4}$$

which provide an explanation to the existence of infinitely-many conserved charges for the scalar field proposed in [37]. For the anti-symmetric part of the DC field, describing a two-form, we obtain indeed the charge $Q_B^{(0)}$ presented in (1.2). In particular, we find that the DC predict the presence of logarithmic terms in the two-form gauge parameter, which are required in order to obtain a non-vanishing asymptotic charge and were not considered in previous analyses of the two-form asymptotic symmetries. Finally, in the last section of chapter 5 we also mention some preliminary results of the work in progress [38], where the convolutional DC dictionary is applied to the first higher-order asymptotic symmetries.

In future investigations we seek to extend our double-copy results to arbitrary higher-order $\mathcal{O}(r^N)$ asymptotic symmetries. As mentioned, however, gravitational asymptotic symmetries so far have been discussed only up to $\mathcal{O}(r^2)$ parameters [31]. In this respect, a DC analysis could provide also novel results at the level of gravity. In fact, a better understanding of the relation between asymptotic symmetries and the double-copy could be useful in simplifying computations for asymptotic symmetries, which become particularly challenging when dealing with gravity, in arbitrary spacetime dimension or for theories involving $p-$forms or higher-spin fields.

At the same time, the problem of symplectic renormalization and residual ambiguities could receive clarifying inputs from the DC perspective. Generalising the double-copy results to higher-order asymptotic symmetries might provide insights on the role of the residual terms. For instance, it would be interesting to understand whether we are able to find some DC relations at the level of renormalized asymptotic charges or if the DC dictionary connects quantities only before the renormalization procedure. Furthermore, since the symplectic renormalization involve the addition of some boundary terms to the Lagrangian, it might be interesting to study whether some double-copy relations can be established at the level of these boundary terms.

The double copy can be viewed, in a sense, as a duality between gravity and gauge theories. Unlike holography, which connects gravity to lower-dimensional gauge theories defined on the boundary, the double copy is a map between theories defined in the same spacetime dimension. Currently, there are no analyses about the possible interplay between double copy and holography, but we hope that the results on asymptotic symmetries could help in unveiling a possible relation between double-copy and, for instance, flat holography, introduced in subsection 2.2.4. In this respect, let us observe that the convolutional DC dictionary has been recently extended to the AdS background [39] and this could provide a starting point for confrontation with the holographic viewpoint.

Finally, let us remark that asymptotic symmetries could be a non-trivial arena to test improved DC definition, since the convolutional dictionary employed in this thesis has the limitation of connecting linearised theories. We expect that an improved definition of the DC field, involving the non-Abelian structure of Yang-Mills and aiming to fully reproduce geometrical structure of gravity, should help to implement the connection also at the level of asymptotic symmetries.

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The research carried out during this three-years PhD is contained in the two works [29,34] and a work in progress [38]. These works are presented in Chapter 3, based on [29], and in Chapter 5, based on [34, 38].

Furthermore, during the PhD, I was also involved in a different project on string theory, whose content is not detailed in this thesis:

- M. Romoli and O. Zanusso, "Different kind of four-dimensional brane for string theory", *Phys.Rev.D* 105 (2022) 12, 126009

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# Part I

# Asymptotic symmetries

# Chapter 2

# Topics on asymptotic symmetries and infrared effects

Over the past decade, there has been a renewed interest in the subject of asymptotic symmetries in gravity and gauge theories, mostly due to the established connection with soft theorems and memory effects (see [18] for a review), elegantly described by the infrared triangle. In fact, the concept of asymptotic symmetry has been extended to other theories, including electrodynamics [40–42], Yang-Mills theory [23, 43], $p$-forms theories [27, 28, 44] and higher-spin theories [32, 45]. It has also been generalized to arbitrary spacetime dimension [33, 46–51], different backgrounds [52–54] and to more general contexts [37, 55–57].

In its most basic incarnation, the program of asymptotic symmetries consists in finding, among the residual gauge symmetries, the ones that preserve the boundary conditions, *i.e.* the chosen (field) falloffs, and still act nontrivially on asymptotic field configurations. In the case of gauge fields, we focus on boundary conditions at null infinity, particularly $\mathcal{I}^+$, characterized by $r \to \infty$ with fixed retarded time $u = t - r$. Falloffs are typically determined by ensuring the finiteness of physical quantities and the inclusion of all physically reasonable solutions. For instance, in electromagnetism, the long-range electric field falls off as $\frac{1}{r^2}$, while in gravity, the characterization of the boundary is captured by the concept of asymptotic flatness at null infinity [20, 21, 58–60].

In this chapter we review the subject of asymptotic symmetries starting from a discussion about conformal infinity and the derivation of the BMS group. Then, we present the concept of asymptotic symmetry in the case of electromagnetism in the simplest setup: the radial gauge. Subsequently, we derive the asymptotic charges for the theories analysed and explore the connection with physical observable quantities. Finally, in the last section we focus on the issues that occur when working in Lorenz gauge, such as the requirement of different expansions for the gauge parameters. This is instrumental to the discussion of higher order symmetries which is the subject of Chapter 2.

## 2.1 Basics on asymptotic symmetries

### 2.1.1 Conformal infinity

We start by reviewing the conformal compactification of Minkowski spacetime, which allows one to endow flat spacetime with a boundary. The idea at the basis of conformal compactification is to bring points at "infinity" on a non-compact pseudo-Riemannian manifold to a finite distance by a conformal rescaling of the metric. This was formally introduced by Penrose [58, 59]. We focus on the four-dimensional case, even if most of the concepts are generalisable to arbitrary spacetime dimension $D$. The Minkowski metric reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 d\gamma^2, \tag{2.1}$$

where $d\gamma^2 = \gamma_{ij} dz^i dz^j$ $(i, j = 1, 2)$ is the line element on the Euclidean 2-sphere, which can be either described through standard angular variables $(\theta, \phi)$ or, for instance, by means of a stereographic projection, described either in terms of complex coordinates $(z, \bar{z})$ or real variables $(z_1 = \mathrm{Re} z,\ z_2 = \mathrm{Im} z)$.

To study causal motion on spacetime we employ retarded and advanced null coordinates

$$u = t - r, \qquad v = t + r, \tag{2.2}$$

with $-\infty < v \leq u < \infty$, so that the metric reads

$$ds^2 = -du\, dv + \left(\frac{u - v}{2}\right)^2 d\gamma^2. \tag{2.3}$$

These coordinates represent affine parameters along outgoing and incoming null geodesics, respectively. To make the range compact we introduce "rescaled" coordinates

$$u = \tan U, \qquad v = \tan V, \tag{2.4}$$

with $U, V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the metric reads

$$ds^2 = \Omega^{-2} \left(-4dUdV + \sin^2(U - V) d\gamma^2\right), \tag{2.5}$$

with $\Omega = 4\cos^2 U \cos^2 V$ being the conformal factor. This metric diverges for $U, V \to \pm\frac{\pi}{2}$, which corresponds to the limit $u, v \to \infty$. We perform the conformal rescaling

$$ds^2 \to d\tilde{s}^2 = \Omega^2 ds^2 \tag{2.6}$$

in order to add the "points at infinity" $U, V = \pm\frac{\pi}{2}$ and make the metric finite. Let us remark that after the rescaling distances get warped, but the causal structure is preserved since conformal transformations preserve the angles.

We now introduce new time and radial coordinates $T$ and $R$, defined analogously to (2.2)

$$U = T - R, \qquad V = T + R, \tag{2.7}$$

with $0 \leq R < \pi$ and $|T| + R < \pi$. We have (conformally) mapped the Minkowski spacetime into the triangle of vertices $i^0(\pi, 0)$, $i^+(0, \pi)$ and $i^-(0, -\pi)$, which goes under the name of **Penrose diagram**, depicted in Figure 2.1. Every point in this diagram except the $R = 0$ line corresponds to a two-sphere $\mathcal{S}^2$ and the boundaries are defined as

- **Future timelike infinity** $i^+$, corresponding to $t \to +\infty$ with $r$ fixed, equivalent to $R = 0, T = \pi$.
- **Spacelike infinity** $i^0$, which corresponds to the limit $r \to \infty$ at constant $t$, equivalent to $R = 0, T = \pi$.
- **Past timelike infinity** $i^-$, corresponding to $t \to -\infty$ with $r$ fixed, equivalent to $R = 0, T = -\pi$.
- **Future null infinity** $\mathcal{I}^+$, which connects $i^+$ and $i^0$ and correspond to the limit $r \to \infty$ at fixed $u = t - r$, equivalent to $R + T = \pi$.
- **Past null infinity** $\mathcal{I}^-$, which connects $i^+$ and $i^0$ and correspond to the limit $r \to \infty$ at fixed $v = t + r$, equivalent to $R - T = \pi$.

### 2.1.2 The antipodal matching condition

In Figure 2.1 we specified also 4 other points:

- $\mathcal{I}^+_+$ which corresponds to the limit $u \to \infty$ of $\mathcal{I}^+$. This point, which is associated to a $\mathcal{S}^2$, does not coincide with $i^+$.
- $\mathcal{I}^+_-$ is defined as the limit $u \to -\infty$ of $\mathcal{I}^+$ and does not coincide with $i^0$.
- $\mathcal{I}^-_+$ consists in the limit $v \to +\infty$ of $\mathcal{I}^-$, distinct from $i^0$.
- $\mathcal{I}^-_-$ is reached by the limit $v \to -\infty$ of $\mathcal{I}^-$, and it is different from $i^-$.

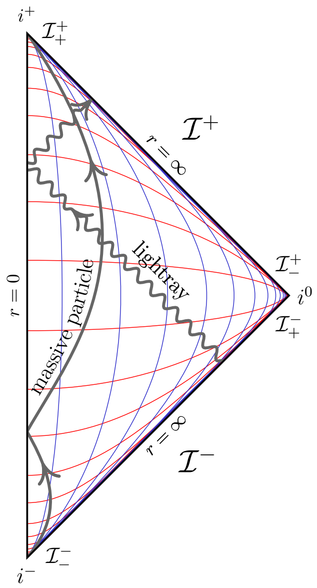



Figure 2.1: Penrose diagram of Minkowski space in which every point except for $r = 0$ is a $\mathcal{S}^2$. Red lines represent surfaces of constant $t$, while blue lines represent surfaces of constant $r$. The thick gray line is the worldline of a massive particle moving at constant velocity, and the thick wavy gray line is a light ray, propagating on constant $u$ surfaces. *Image from the ArXiv version of [18].*

Likewise, $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ do not coincide as well. In fact, $i^0$ is in particular a special "point" and several considerations are required in order to discuss physics around spacelike infinity, which are summarised in the **antipodal matching condition**, first introduced in [61]. To present it, it is convenient to depict the Penrose diagram as is Figure 2.2 taking into account the $\mathcal{S}^2$, so that every two sphere is represented by two points which, as we shall see, are exchanged by the antipodal map. Furthermore, we describe the 2−sphere through complex stereographic coordinates $(z, \bar{z})$, which are related to the standard polar and azimuthal angles as

$$z = \tan\frac{\theta}{2}e^{i\phi}, \qquad \bar{z} = \tan\frac{\theta}{2}e^{-i\phi}, \tag{2.8}$$

so that $z^* = \bar{z}$. The variable $z$ runs over the entire complex plane: $z = 0$ is the north pole ($\theta = 0$), $z = \infty$ is the south pole ($\theta = \pi$) and $z\bar{z} = 1$ corresponds to the equator. The two-sphere metric in these coordinates is anti-diagonal with

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}. \tag{2.9}$$

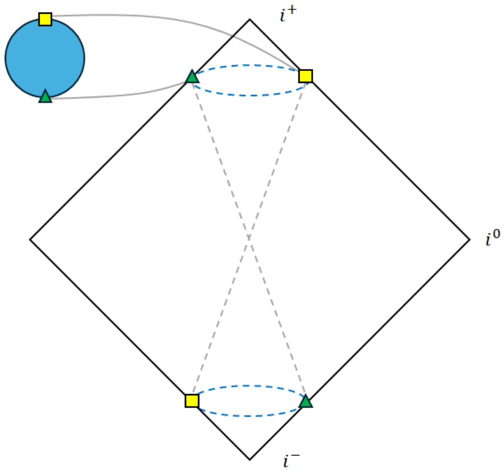



Figure 2.2: Alternative Penrose diagram of Minkowski space and the antipodal matching condition.

The antipodal matching condition specifies the relation between fields valued at $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ in a CPT and Lorentz-invariant way stating that

$$\Phi(\theta,\phi)\big|_{\mathcal{I}^+_-} = \Phi(-\theta,-\phi)\big|_{\mathcal{I}^-_+}, \tag{2.10}$$

which in stereographic coordinates reads

$$\Phi(z,\bar{z})\big|_{\mathcal{I}^+_-} = \Phi(-1/\bar{z},-1/z)\big|_{\mathcal{I}^-_+}, \tag{2.11}$$

Even if surprising at first, this condition is fundamental in order to define conserved quantities and to connect them with soft theorems and memory effects and it is instrumental in the definition of scattering problem from $\mathcal{I}^-$ to $\mathcal{I}^+$ [40, 61]. In fact, considering a trivial scattering process, the out states are antipodally related to the in states, as one can understand by studying the trajectory of a free massless particle in Minkowski space. Hence, the antipodal map provides a natural CPT and Lorentz-invariant definition of "identity" between in and out Fock spaces [62]. A convenient way to understand the antipodal matching condition is to write explicitly the metric in the $T, R$ coordinates, which is

$$ds^2 = \Omega^{-2} d\tilde{s}^2 = \Omega^{-2}(T,R)\left(-dT^2 + dR^2 + \sin^2 R\, d\gamma^2\right), \tag{2.12}$$

and note that $d\tilde{s}^2$ is the natural Lorentz metric on $R_T \times S^3$, known as the Einstein static universe. In fact, one can recognize the $S^3$ metric being $dR^2 + \sin^2 R\, d\gamma^2$, with the restriction on $R \in (0, \pi)$. The entire $R_T \times S^3$ can be drawn as a cylinder in which each circle of constant $T$ represents a three-sphere. From the figure Figure 2.3, we can observe that null infinity $\mathcal{I}$ is the lightcone

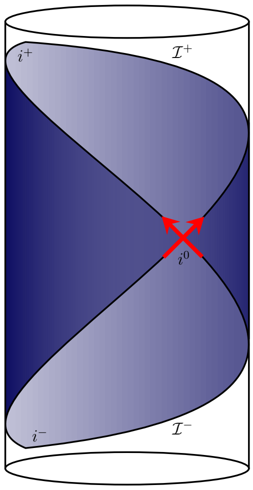



Figure 2.3: Conformal compactification of Minkowski spacetime. Unrolling the shaded region gives the Penrose diagram for Minkowski space. The red arrows indicate how the generators of null infinity pass through spatial infinity. *Image from the ArXiv version of [18].*

of spatial infinity $i^0$. The future lightcone wraps around the cylinder and converges at $i^+$, while the past lightcone at $i^-$. The matching condition (2.11) states that the fields are continuous along the generators of $\mathcal{I}$ when crossing $i^0$, even if it is a singular point.

We are interested in describing physics on the null boundaries of spacetime $\mathcal{I}^\pm$. To this end, we employ coordinates suited to describe those regions of spacetime, which are retarded Bondi coordinates $(u, r, z, \bar{z})$ for $\mathcal{I}^+$ and advanced Bondi coordinates $(v, r, z, \bar{z})$ for $\mathcal{I}^-$. However, the definition of $(z, \bar{z})$ in the two cases is not the same, but we choose them in order to satisfy the antipodal match in the simplest possible way. In particular, respect to the Minkowski coordinates $X^\mu = (t, X^I)$ with $I = 1, 2, 3$, we write

$$\begin{aligned} \mathcal{I}^+ : \qquad & u = t - r, \quad r = |\vec{X}|, \quad z = \frac{X^1 + iX^2}{X^3 + r}, \\ \mathcal{I}^- : \qquad & v = t + r, \quad r = |\vec{X}|, \quad z = \frac{X^1 + iX^2}{X^3 - r}. \end{aligned} \tag{2.13}$$

This choice is perfectly suited to account for the antipodal matching condition, because we have

$$z\Big|_{\mathcal{I}^+} = -\frac{1}{\bar{z}}\Big|_{\mathcal{I}^-}, \tag{2.14}$$

so that the condition (2.11) is now

$$\Phi(z,\bar{z})\big|_{\mathcal{I}^+_-} = \Phi(z,\bar{z})\big|_{\mathcal{I}^-_+}. \tag{2.15}$$

The Minkowski metric in retarded Bondi coordinates reads

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{ij}dz^i dz^j, \tag{2.16}$$

while when working in advanced coordinate we have

$$ds^2 = -dv^2 + 2dvdr + r^2\gamma_{ij}dz^i dz^j. \tag{2.17}$$

### 2.1.3 Massive particles

So far, we introduced a setup which is naturally suited to discuss massless particles, since they are the primary focus of this thesis. However, it is worth to mention that it is possible to take into account massive particles, which follow timelike trajectories that start from $i^-$ and end at $i^+$. To discuss physics around these points we need to consider a hyperbolic slicing of Minkowski space [63, 64], with slices labelled by the coordinate

$$\tau^2 = t^2 - r^2 = u\, v. \tag{2.18}$$

Hypersurfaces at constant $\tau$ are are hyperbolic spaces and for $\tau^2 > 0$ the hypersurfaces have the topology of $\mathbb{H}_3$ (or Euclidean $AdS_3$), while for $\tau^2 < 0$ the topology is $dS_3$. In particular, $i^\pm$ have obviously $\tau^2 > 0$ and therefore around those points we consider a $\mathbb{H}_3$ slicing. On the other hand, to take into account also $i^0$, which is characterized by $\tau^2 < 0$, we consider the $dS_3$ case.

On $\mathbb{H}_3$, we introduce the coordinate $\rho$ defined as

$$\rho = \frac{r}{\sqrt{t^2 - r^2}} = \frac{r}{\tau}, \tag{2.19}$$

so that the Minkowski metric reads

$$ds^2 = -d\tau^2 + \tau^2\left[\frac{d\rho^2}{1+\rho^2} + \rho^2 d\gamma^2\right]. \tag{2.20}$$

The slicing is depicted in figure Figure 2.4 and it allow us to study the action of asymptotic symmetries around $i^\pm$ and $i^0$, as for instance is done in [46, 65, 66].

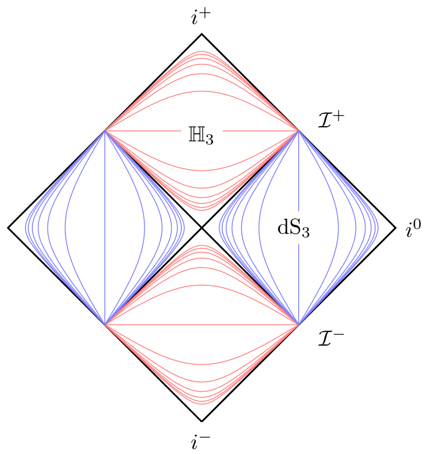



Figure 2.4: Hyperbolic slicing of Minkowski spacetime. *Image from the ArXiv version of [18].*

### 2.1.4 Asymptotic flatness

Having discussed the boundary structure of Minkowski spacetime and the meaning of approaching infinity, we are now ready to take into account gravity and the concept of **asymptotically flat spacetimes**. This was first discussed by Bondi, van der Burg, Metzner and Sachs [20, 21] and later reformulated in a more geometrical fashion by Penrose and Carter [58, 60, 67]. We start by observing that in any four-dimensional spacetime, it is always possible [22] to write the spacetime metric in the following form

$$ds^2 = e^{2\beta}\left(\frac{V}{r}du^2 - 2du\,dr\right) + g_{ij}\left(dz^i - \mathcal{U}^i du\right)\left(dz^j - \mathcal{U}^j du\right) , \tag{2.21}$$

with

$$\partial_r \det\left(\frac{g_{ij}}{r^2}\right) = 0. \tag{2.22}$$

Equation (2.22) implies that $r$ is the luminosity distance. The conditions (2.21) and (2.22) are referred as Bondi gauge, which is commonly used in most of the literature on asymptotically flat spacetimes. Some peculiarities emerge when working in other gauges, as we discuss later.

In (2.21), $\beta$, $V$, $\mathcal{U}^i$ and $g_{ij}$ have arbitrary dependence on the coordinates, except for the condition (2.22). Indeed, any geometry can be described locally by that metric. For instance, the Minkowski spacetime corresponds to

$$\beta = 0, \qquad V = -r, \qquad g_{ij} = \gamma_{ij}, \qquad \mathcal{U}^i = 0. \tag{2.23}$$

Imposing asymptotic flatness at large $r$ with fixed $(u, z, \bar{z})$ leads to **falloff conditions** on the metric components. There is no a priori preferred method of determining what these falloffs must be and the literature discusses various options. Typically they are chosen in order to allow for all physically relevant solutions, including gravitational radiation, while excluding the unphysical ones, such as those with infinite energy.

The most restrictive way to assign falloffs consistent with asymptotic flatness and the presence of gravitational radiation is

$$\begin{aligned} \frac{V}{r} &= -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2})\,, \\ \beta &= \frac{\bar{\beta}}{r^2} + \mathcal{O}(r^{-3})\,, \\ g_{ij} &= r^2\gamma_{ij} + r\,C_{ij} + \mathcal{O}(r^{-2})\,, \\ \mathcal{U}^i &= \frac{U^i}{r^2} + \frac{4}{3}\frac{N^i}{r^3} + \mathcal{O}(r^{-4})\,, \end{aligned} \tag{2.24}$$

where all the introduced functions depend on $u$, $z$ and $\bar{z}$. In addition, we have $\gamma^{ij}C_{ij} = 0$ because of the requirement (2.22). The function $m_B(u, z, \bar{z})$ goes under the name of **Bondi mass aspect**, since it is related to the notion of energy at a given retarded time, while the definition of angular momentum is related to $N^i(u, z, \bar{z})$, which is called **angular momentum aspect**

The Einstein equations provide constraints between these components, namely

$$U^i = -\frac{1}{2}\mathcal{D}_j C^{ij}\,, \qquad \partial_u m_B = -\frac{1}{8}N_{ij}N^{ij} + \frac{1}{4}\mathcal{D}_i\mathcal{D}_j N^{ij}\,, \qquad \bar{\beta} = -\frac{1}{32}C_{ij}C^{ij}\,, \tag{2.25}$$

where $N_{ij} = \partial_u C_{ij}$ is the **Bondi news tensor**, which identifies the two independent components of $C_{ij}$ as the **free radiative boundary data**, namely as the two independent graviton polarizations that propagate to infinity. In fact, the time dependent part of $C_{ij}$ gives rise to the energy flux carried by radiation. Indeed, considering a linearised perturbation on Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ satisfying the falloff conditions, the energy flux across a sphere of retarded time $u$ and radius $r$ is proportional to [20]

$$\oint N_{ij}N^{ij}d\Omega\,, \tag{2.26}$$

where $d\Omega$ is the measure element on the Euclidean unit sphere.

### 2.1.5 The BMS group

BMS asymptotics symmetries are defined as the residual diffeomorphisms compatible with the Bondi gauge and that preserve the falloffs (2.24), mapping an asymptotically flat geometry to an inequivalent one, acting nontrivially on the leading field components $C_{ij}$, $U^i$, $m_B$. They are also called **large diffeomorphism**, while a "small diffeomorphism" acts only at subleading level, inducing an identity transformation on $\mathcal{I}^+$.

As we shall see in the next section, we can distinguish a large diffeomorphism from a small one because the first is associated to a finite and non-vanishing asymptotic charge in the limit $r \to \infty$, while the second leads to a vanishing charge in that limit.

We can determine asymptotic symmetries by solving the asymptotic Killing equation, imposing that the infinitesimal transformations

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \,, \tag{2.27}$$

preserve the falloffs and the gauge. We start imposing that the vectors preserve the gauge condition, namely

$$\begin{aligned} &\mathcal{L}_\xi g_{rr} = g_{\mu r}\partial_r \xi^\mu = 0, \\ &\mathcal{L}_\xi g_{ri} = g_{\mu r}\partial_i \xi^\mu + g_{\mu i}\partial_r \xi^\mu = 0, \\ &\mathcal{L}_\xi \left(g_{ij} g^{ij}\right) = g^{ij}\left(\xi^\mu \partial_\mu g_{ij} + g_{\mu(i}\partial_{j)}\xi^\mu\right) = 0 \,, \end{aligned} \tag{2.28}$$

where round brackets denote symmetrization of the indices. From the first equation we obtain that $\xi^u$ must be $r-$independent, namely

$$\xi^u = f(u, z, \bar{z}) \,. \tag{2.29}$$

Inserting this solution in the second of (2.28), we find

$$\xi^i = -\partial_j f \int_r^\infty e^{2\beta} g^{ij} dr' + Y^i \,, \tag{2.30}$$

with $Y^i = Y^i(u, z, \bar{z})$ being an $r-$integration function. By means of the identities $g^{ij}\partial_r g_{ij} = 4r^{-1}$, $g^{ij}\partial_u g_{ij} = 0$ and $g^{ij}\partial_k g_{ij} = \gamma^{ij}\partial_k \gamma_{ij}$, which can be derived from the condition $\det(g_{ij}) = r^4 \det(\gamma_{ij})$, the third equation in (2.28) leads to

$$\xi^r = \frac{r}{2}\left(\mathcal{U}^i \partial_i \xi^u - \mathcal{D}_i \xi^i\right) \,, \tag{2.31}$$

where we recall that $\mathcal{D}_i$ is the covariant derivative on the two-sphere respect to the metric $\gamma_{ij}$.

Substituting the falloffs (2.24) into (2.29), (2.30) and (2.31) we obtain

$$\begin{aligned}
\xi^u &= f(u, z, \bar{z}) \,, \\
\xi^r &= -\frac{r}{2}\mathcal{D}_i Y^i(u, z, \bar{z}) + \frac{1}{2}\mathcal{D}^2 f(u, z, \bar{z}) + \mathcal{O}(r^{-1}) \,, \\
\xi^i &= Y^i(u, z, \bar{z}) - \frac{1}{r}\mathcal{D}^i f(u, z, \bar{z}) + \mathcal{O}(r^{-2}) \,,
\end{aligned} \tag{2.32}$$

where $\mathcal{D}^2 = \mathcal{D}^i \mathcal{D}_i$ and $\mathcal{D}^i = \gamma^{ij}\mathcal{D}_j$. Imposing that $\mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2})$ and $\mathcal{L}_\xi g_{ui} = \mathcal{O}(1)$ gives

$$\partial_u f = \frac{1}{2}\mathcal{D}_i Y^i \,, \qquad \partial_u Y^i = 0 \,, \tag{2.33}$$

so that $Y^i(u, z, \bar{z}) = Y^i(z, \bar{z})$ is actually a vector on the sphere, while

$$\xi^u = T(z, \bar{z}) + \frac{u}{2}\mathcal{D}_i Y^i(z, \bar{z}) + \mathcal{O}(r^{-1}) \,. \tag{2.34}$$

Finally, the condition $\mathcal{L}_\xi g_{ij} = \mathcal{O}(r)$ implies that $Y^i$ satisfies the **conformal Killing equation** on the two-sphere

$$\mathcal{D}_{(i} Y_{j)} = \gamma_{ij} \mathcal{D} \cdot Y \,, \tag{2.35}$$

where the dot denotes contraction with $\gamma_{ij}$. The last remaining equation $\mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1})$ gives

$$(\mathcal{D}^2 + 2)\mathcal{D} \cdot Y = 0 \,, \tag{2.36}$$

which is identically true if (2.35) holds.

In the $(z, \bar{z})$ coordinates, the conformal Killing equation (2.35) reads simply

$$\partial_z Y^{\bar{z}} = 0 \,, \qquad \partial_{\bar{z}} Y^z = 0 \,, \tag{2.37}$$

so that for any holomorphic (antiholomorphic) function local solutions are given by $Y^z(z, \bar{z}) = Y^z(z)$ and $Y^{\bar{z}}(z, \bar{z}) = Y^{\bar{z}}(\bar{z})$. Globally well-defined solutions are selected by requiring that the Laurent expansion of $Y^z$ $(Y^{\bar{z}})$ be nonsingular for $z = 0$ and $z \to \infty$, and similarly for $Y^{\bar{z}}(\bar{z})$. This leaves us with

$$Y^z = c_0 + c_1 z + c_2 z^2 \,, \qquad Y^{\bar{z}} = \tilde{c}_0 + \tilde{c}_1 \bar{z} + \tilde{c}_2 \bar{z}^2 \,. \tag{2.38}$$

To summarise, BMS asymptotic symmetries are parametrised by

$$\begin{aligned}\xi^u &= T(z,\bar z) + \frac{u}{2}\mathcal{D}\cdot Y(z,\bar z)\,,\\ \xi^r &= \frac{1}{2}\mathcal{D}^2 T(z,\bar z) - \frac{u+r}{2}\mathcal{D}\cdot Y(z,\bar z) + \mathcal{O}(r^{-1})\,,\\ \xi^i &= Y^i(z,\bar z) - \frac{1}{r}\mathcal{D}^i T(z,\bar z) - \frac{u}{2r}\mathcal{D}^i\mathcal{D}\cdot Y(z,\bar z) + \mathcal{O}(r^{-2})\,,\end{aligned} \tag{2.39}$$

where $T(z,\bar z)$ is an arbitrary function of the angles, while $Y^i$ satisfies the conformal Killing equation (2.35) on the Euclidean unit sphere and assume the general form (2.38).

As a first comment, it is possible to check that the 6 global conformal Killing vectors found $Y^i$ are indeed the generators of the Lorentz transformations. In fact, an infinitesimal rotation/boost generator is given by

$$l_\mu = \omega_{\mu\nu} x^\nu \tag{2.40}$$

with $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ antisymmetric and $x^\nu$ denoting Minkowski coordinates, which are conveniently related to the Bondi coordinates through

$$x^\mu = u t^\mu + r\hat q^\mu(z) \tag{2.41}$$

with $t^\mu = (1,0,0,0)$ and $\hat q^\mu$ is a null vector expressed as

$$\hat q^\mu(z) = \left(1, \frac{z-\bar z}{1+z\bar z}, -i\frac{z-\bar z}{1+z\bar z}, \frac{1-z\bar z}{1+z\bar z}\right) =: (1,\hat n^I)\,. \tag{2.42}$$

Hence, the Bondi components read

$$l^u = u n^I \omega_{0I}\,,\quad l^r = -(u+r)n^I\omega_{0I}\,,\quad l^i = -\partial^i n^I \omega_{0I} + \partial^i n^I \omega_{IJ} n^J - \frac{u}{r}\partial^i n^I \omega_{0I}\,, \tag{2.43}$$

which correspond to the case

$$T = 0\,,\qquad Y^i = -\partial^i n^I\omega_{0I} + \partial^i n^I \omega_{IJ} n^J\,. \tag{2.44}$$

Using the explicit expression for $n^I$, we identify

$$\begin{aligned}c_0 &= \frac{1}{2}\big(\omega_{01} - \omega_{13} + i(\omega_{02}+\omega_{23})\big)\,,\\ c_1 &= \omega_{03} + i\omega_{12}\,,\\ c_2 &= \frac{1}{2}\big(\omega_{13} - \omega_{01} + i(\omega_{02}-\omega_{23})\big),\end{aligned} \tag{2.45}$$

thus providing the correspondence between the Lorentz generator and the global conformal Killing vector.

At the same time, we find the 4 spacetime translations when

$$T(z,\bar{z}) = T_{0,0} + \sum_{m=-1}^{1} T_{1,m} Y_{1,m}(z,\bar{z}), \tag{2.46}$$

where $Y_{1,m}$ are the spherical harmonics functions with $l=1$ and $T_{0,0}, T_{1,m}$ are constants coefficients. In fact if we move back to Minkowski components, the choice (2.46) correspond to a constant $\xi^\mu$.

Hence, the set asymptotic symmetries (2.39) contains the Poincaré transformations. However, it is much bigger because the function $T(z,\bar{z})$ is indeed arbitrary, corresponding to an infinite-dimensional enhancement of ordinary translations (2.46). These are indeed called **supertranslations** and were introduced by Bondi, van der Burg, Metzner and Sachs in 1962 [20–22].

Supertranslations map an asymptotically flat geometry to a physically different one despite the fact that they are diffeomorphisms. To understand this, we can consider a solution [18] describing an outgoing pulse of gravitational waves crossing the south pole of $\mathcal{I}^+$ and another pulse crosses the north pole of $\mathcal{I}^+$ , both at retarded time $u = 100$. We now on act with a supertranslation parametrised by a function $T(z,\bar{z})$ that has the property that $T$(south pole)=100 and $T$(north pole)=0. The new solution now has one outgoing pulse at the north pole only at $u = 100$ and one at the south pole only at $u = 200$. The outgoing data are measurably changed by a supertranslation. The action of a supertranslation is indeed an independent shift of the retarded time at every angle, as depicted in Figure 2.5

Projecting (2.39) on a surface of fixed $r$ and taking the limit $r \to \infty$, for fixed $u$, we find

$$\xi^u = T + \frac{u}{2}\mathcal{D}\cdot Y\,, \qquad \xi^i = Y^i\,. \tag{2.47}$$

Evaluating the commutator between two vectors $\xi_1, \xi_2$ we can observe that they give rise to the following algebra

$$[\xi_1,\xi_2]^u = Y_{[1}\cdot \mathcal{D}T_{2]} + \frac{1}{2}T_{[1}\mathcal{D}\cdot Y_{2]} + \frac{u}{2}\mathcal{D}\cdot [Y_1,Y_2]\,, \qquad [\xi_1,\xi_2]^i = [Y_1,Y_2]^i\,, \tag{2.48}$$

which goes under the name of $\mathfrak{bms}_4$−algebra. If $s = (T,Y) \in \mathfrak{bms}_4$ denotes a generic element of the algebra, we can schematically write [68]

$$[(T_1,Y_1),(T_2,Y_2)] = \Big(Y_{[1}\cdot \mathcal{D}T_{2]} + \frac{1}{2}T_{[1}\mathcal{D}\cdot Y_{2]}\,,[Y_1,Y_2]\Big)\,. \tag{2.49}$$

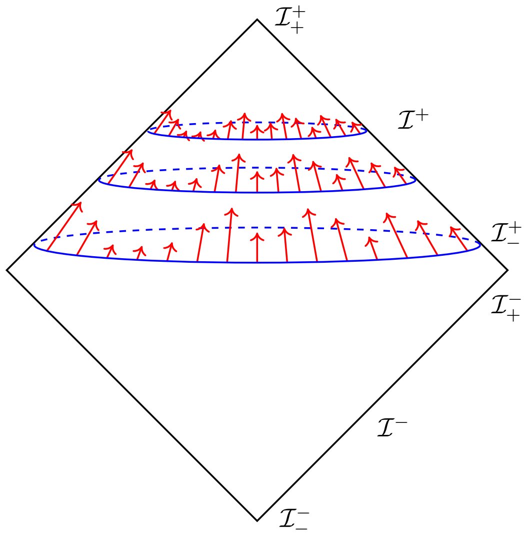



Figure 2.5: Action of a supertranslation. *Image from the ArXiv version of [18].*

A key structural feature of the $\mathfrak{bms}_4$−algebra is the existence of an infinite-dimensional Abelian subalgebra given by supertranslations $\xi = T(z,\bar{z})\partial_u$, which is in fact a Lie ideal since by (2.49) the commutator of a supertranslation with any transformation is again a supertranslation.

At the group level, supertranslations give rise to an infinite-dimensional Abelian normal subgroup $ST$ of the full $BMS$ group. Supertranslations contain ordinary translations (2.46) as a four-dimensional subgroup and enter the full asymptotic symmetry group as follows:

$$\text{BMS} = \text{ST} \rtimes \text{SO(3,1)}\,. \tag{2.50}$$

### 2.1.6 Generalisations of BMS

Up to this point, we have discussed the BMS group largely in the context of its original formulation [20, 21]. However, over the past fifteen years the BMS group has undergone significant extensions. These generalisations are among the motivations of the renewed interest towards the subject of asymptotic symmetries.

#### Superrotations and extended BMS

As discussed previously, there are only six global conformal Killing vector on the sphere, consisting in the non-singular solutions of the CKV equations

$$\partial_{\bar{z}} Y^z = 0, \qquad \partial_z Y^{\bar{z}} = 0. \tag{2.51}$$

However, it has been proposed to generalize the choice of $Y^i$ by considering also two-sphere vector that satisfy the CKV equations everywhere except for isolated points. This was first suggested by Banks in a footnote of [69] but formally developed later by Barnich and Troessart [55, 70, 71]. This proposal is analogous to the work of Belavin, Polyakov, and Zamolodchikov, who found the same equation in analysing the symmetries of two-dimensional CFTs [72].

This generalisation goes under the name of **superrotations**, which extend the six global CKV to the local solutions of equation (2.51). For instance, we now consider all the functions $Y^z = z^n$ with arbitrary $n$, admitting isolated poles, as well as singular solutions as $Y^z = \frac{1}{z-w}$, for which

$$\partial_{\bar{z}} \frac{1}{z-w} = 2\pi\delta^2(w-z) \neq 0, \tag{2.52}$$

and that violate the falloff only in an isolated point $w$ (this violation in particular has been interpreted in terms of cosmic strings [73]). At the local level, the subalgebra given by superrotations is also infinite-dimensional and is in fact given by the direct sum of two Virasoro algebras, one for the holomorphic sector and one for the antiholomorphic sector (2.37). This result is at the basis of the **celestial holography** program, whose goal is to recast the bulk physics in asymptotically flat spacetimes in terms of 2D CFTs. The asymptotic symmetries comprising supertranslations and superrotations go under the name of **extended BMS**, that are

$$\mathrm{eBMS} = \mathrm{ST} \rtimes (\mathrm{Vir} \times \mathrm{Vir})\,. \tag{2.53}$$

On the asymptotic shear $C_{ij}$, which encodes the graviton polarisations, the asymptotic Killing vectors (2.39) act as

$$\delta_\xi C_{zz} = \delta_\xi^H C_{zz} + \delta_\xi^S C_{zz}\,, \tag{2.54}$$

where we split the contribution in hard and soft parts. In particular, we have

$$\begin{aligned} \delta_\xi^H C_{zz} &= \left[ \left( T + \frac{u}{2}\mathcal{D}_i Y^i \right) \partial_u + \frac{3}{2} Y \mathcal{D}_z - \frac{1}{2} \bar{Y} \mathcal{D}_{\bar{z}} \right] C_{zz} \\ \delta_\xi^S C_{zz} &= -\, 2\mathcal{D}_z^2 T - u \mathcal{D}_z^3 Y\,, \end{aligned} \tag{2.55}$$

and analogous expressions for $\delta_\xi C_{\bar{z}\bar{z}}$.

### Generalized BMS

More recently, it has been proposed a further generalisation of the superrotations sector. In particular, while considering local CKV imply that we find falloffs violations in isolated point, we can allow two-sphere vectors that simply violate some falloffs. This generalisation has been suggested by Campiglia and Laddha in 2014 [74] and goes under the name of **generalised BMS**. The idea is to extend the local CKV to an arbitrary two-sphere vector $Y^i$. The generalised BMS group is

$$\text{gBMS} = \text{ST} \rtimes \text{Diff}(S^2)\,. \tag{2.56}$$

This result has allowed to connect the gBMS to the subleading soft graviton theorem, formulated by Cachazo and Strominger in [75].

In the logic of the original presentation of gBMS the idea was to substitute the preservation of the falloffs, written as

$$\nabla_{(\mu}\xi_{\nu)} = 0\big|_{\mathcal{I}^+} \tag{2.57}$$

with the idea of having asymptotically divergence-free Killing vectors

$$\nabla_\mu \xi^\mu = 0\big|_{\mathcal{I}^+}, \tag{2.58}$$

thus allowing for specific violations of the falloffs. However, as we discuss in the next section, a more general attitude concerning falloff preservation can be assumed, as long the asymptotic charge can be renormalized.

The action on the asymptotic shear is the same of (2.55), but the meaning of $Y^i$ is different.

## 2.1.7 Electromagnetic asymptotic symmetries

Up to this point in the discussion we have been concerned with the symmetries of weakly radiating gravitational systems in the limit $r \to \infty$. However, the concept of asymptotic symmetry finds applications also in the context of gauge theories. In this subsection we discuss the simplest case: electromagnetism, originally presented in [23].

We start by considering the classical action for electromagnetism coupled to a conserved current $\mathcal{J}^\mu$,

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} - \int d^4x\sqrt{-g}\, A_\mu \mathcal{J}^\mu, \tag{2.59}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the field strength which is invariant under the gauge transformation

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon. \tag{2.60}$$

The action (2.59) is invariant under a gauge transformation (2.60) up to a boundary term, because the current is conserved, $\partial_\mu \mathcal{J}^\mu = 0$.

In retarded Bondi coordinates, we choose the retarded radial gauge condition

$$A_r = 0, \qquad A_u\big|_{\mathcal{I}^+} = 0, \tag{2.61}$$

which can be thought as the spin-one analogue of the Bondi gauge. The first condition leaves as residual gauge freedom an $r-$independent scalar function $\epsilon(u, z, \bar{z})$, while the second condition implies that $\epsilon$ has to be $u-$independent. We are left with a residual gauge parameter

$$\epsilon(z, \bar{z}), \tag{2.62}$$

resembling the spin-one analogue of the supertranslation parameter $T$.

The falloff conditions are chosen in order to take into account all physically reasonable solutions, while avoiding all the unphysical ones, characterized by the divergence of physical quantities like the energy. This constraints fix the falloffs of the field strength to be

$$F_{ur} = \mathcal{O}(r^{-2}), \qquad F_{uA} = \mathcal{O}(1), \qquad F_{rA} = \mathcal{O}(r^{-2}), \qquad F_{AB} = \mathcal{O}(1). \tag{2.63}$$

Fields falloffs compatible with the gauge choice and the field strength are

$$A_u = A_u(u, z, \bar{z})/r + O(r^{-2})\,, \qquad A_i = A_i(u, z, \bar{z}) + O(r^{-1})\,, \tag{2.64}$$

where it is worth to note that these falloffs can be in principle generalised in pure gauge sectors, as we do in chapter 3. Furthermore, we can assign falloff conditions to the current and, in particular, accounting for the possible presence of massless charged particles implies that $\mathcal{J}^\mu$ behaves like

$$\mathcal{J}_u = \frac{J_u}{r^2} + \mathcal{O}(r^{-3})\,, \qquad \mathcal{J}_r = \frac{J_r}{r^2} + \mathcal{O}(r^{-3})\,, \qquad \mathcal{J}_i = \frac{J_i}{r} + \mathcal{O}(r^{-2})\,. \tag{2.65}$$

Note that not all the field components are independent since they satisfy the relation

$$\partial_u A_u = \partial_u \mathcal{D}^i A_i + J_u\,, \tag{2.66}$$

which can be derived from the equations of motion. From this condition, we see that the $A_i$'s have the role of the free Cauchy data, corresponding to the two physical polarisations of the electromagnetic field.

As one can check by direct computation, the parameter (2.62) preserve the falloffs chosen and, by construction, the gauge. This analysis can be generalised in several ways as for the BMS case discussed in the previous subsection. We shall discuss in detail these generalisation in the next chapter.

## 2.2 Physics of asymptotic symmetries

### 2.2.1 Asymptotic charges

#### Covariant phase space

The most important tool in order to understand whether an asymptotic symmetry has a physical interpretation is the **asymptotic charge**. We introduce this notion employing the **covariant phase space** formalism, whose main idea is to combine the calculus in both spacetime and field space. This technique was introduced in [76–78] and later refined in [79–81]. Discussions with a particular focus on asymptotic symmetries can be found in [82,83]. Reviews can be found, for instance, in [84–86].

On a four-dimensional differentiable manifold $M$, the space of forms induce the de Rham cohomology, where we denote with $d$ and $i$ the exterior derivative and interior product on this complex, respectively. The exterior derivative is assumed to be nilpotent, *i.e.* $d^2 = 0$. The Lie derivative along $\xi \in TM$ is given by

$$\mathcal{L}_\xi = d\, i_\xi + i_\xi d. \tag{2.67}$$

These notions can be introduced also in the space of all possible field configurations $\Gamma$. In this case, we denote with $\delta$ the exterior derivative, assumed to be nilpotent as well, while we use $I_V$ for the interior product, so that the Lie derivative reads

$$\mathfrak{L}_V := \delta\, I_V + I_V \delta\,. \tag{2.68}$$

Putting together spacetime and field-space forms to form the variational bicomplex, defined on $(M, \Gamma)$. We refer to "$(p,q)-$forms" as a form on the bicomplex which is a spacetime $p-$form and a field-space $q-$form.

Given a Lagrangian theory, whose action reads

$$S = \int_M L, \tag{2.69}$$

we can write the way $L$, which is a $(4,0)-$form, transforms under a generic field variation $\varphi \to \varphi + \delta\varphi$ as

$$\delta L = \mathrm{EOM}\delta\varphi + d\theta, \tag{2.70}$$

with $\theta$ being the presymplectic potential, which is a $(3,1)$-form. We define the presymplectic two-form, which is precisely a $(3,2)-$form, as

$$\omega = \delta\theta, \tag{2.71}$$

and the symplectic two-form

$$\Omega = \int_\Sigma \omega, \tag{2.72}$$

with $\Sigma$ being an arbitrary Cauchy surface. Therefore, $\Omega$ is a $(0,2)-$form.

Assuming trivial cohomology in the space of 1-forms on $\Gamma$, a vector $V \in T\Gamma$ is called a symplectomorphism or Hamiltonian vector field if $\mathfrak{L}_V\omega = 0$. For a gauge symmetry, we can define the current as

$$J_V = I_V\theta \tag{2.73}$$

and the charge

$$Q = \int_\Sigma J_V. \tag{2.74}$$

Furthermore, for a gauge symmetry the current is in general an on-shell total derivative and the charge can be written as a integral over $\partial\Sigma$ by means of the Stokes theorem. At the same time, for a spacetime symmetry we have an on-shell identity for the Noether current, which is

$$J_\xi \approx I_\xi\theta - i_\xi L\,. \tag{2.75}$$

The presymplectic potential admits two types of ambiguities

$$\theta \to \theta + \delta\Xi + d\Upsilon. \tag{2.76}$$

The first does not change $\omega$, since $\delta$ is nilpotent, and corresponds to the addition of a boundary term to the Lagrangian $L \to L + d\Xi$. The second modifies $\omega \to \omega + d\delta\Upsilon$ but does not affect $d\theta$ in (2.70), since $d$ is nilpotent.

The existence of these ambiguities is fundamental in order to discuss asymptotic charges. In fact, as we discuss later, some divergences occur when dealing with higher order asymptotic symmetries. Therefore, it is not possible to perform the limit to define asymptotic charges. However, we can absorb the divergent terms in ambiguities of the presymplectic potential, while leaving the finite and non-vanishing parts of the asymptotic charges unchanged. The latter provide the physical content of the asymptotic symmetries.

### Asymptotic electric charges

We start by determining the asymptotic charges in the case of electromagnetism. Given the action (2.59), the presymplectic potential reads

$$\theta = \delta A \wedge *F, \tag{2.77}$$

with $*$ denoting the Hodge dual. We define the presymplectic potential current by means of the Hodge dual $*\theta$, for which we use the same letter

$$\theta^{\mu}_{\text{em}} = \sqrt{-g}\, F^{\mu\nu}\delta A_{\nu}. \tag{2.78}$$

The Noether current is therefore

$$J^{\mu} = \sqrt{-g}\, F^{\mu\nu}\partial_{\nu}\epsilon \approx \partial_{\nu}(\sqrt{-g}\, F^{\mu\nu}\epsilon). \tag{2.79}$$

A Cauchy slice is a hypersurface of constant $t$, therefore we are interested in

$$\theta^{t} = \theta^{u} + \theta^{r}. \tag{2.80}$$

The Noether charge is then

$$Q_t = \int_{\Sigma_t} dzd\bar{z}du\gamma_{z\bar{z}}(\partial_u - \partial_r)(r^2 F^{ur}\epsilon), \tag{2.81}$$

where we neglected a total two-sphere divergence. The asymptotic charge consists in evaluating $Q_t$ in the limit $t \to \infty$ and taking the finite and nonvanishing part [25, 31, 74]. In this respect, we observe that in other analyses of asymptotic charges a fixed $r$ surface $\Sigma_r$ is considered in the limit $r \to \infty$, which is not however a Cauchy surface. In our understanding, these two definitions are identical for parameters of order $\mathcal{O}(r^0)$ and therefore, for the cases under examination, studying the limit $t \to \infty$ for the integral over $\Sigma_t$ coincides with studying the limit $r \to \infty$. However, as we discuss in Chapter 3, these two definitions are not equivalent when one consider higher-order asymptotic symmetries and in particular, the definition with $\Sigma_t$ is the one allowing to recover the charge of the subleading soft photon theorem [25, 87].

Reminding that we have $F^{ur} = \mathcal{O}(r^{-2})$ and $\epsilon = \mathcal{O}(r^0)$, we observe that only the $\partial_u$ part survives in the limit $r \to \infty$ and, using the eom , we find

$$Q = \int_{\mathcal{I}^+} dz d\bar{z} du \gamma_{z\bar{z}} \epsilon(z, \bar{z}) \partial_u (\mathcal{D} \cdot A^{(0)} - J_u). \tag{2.82}$$

which is the asymptotic electric charge.

**eBMS charges**

Starting from Einstein-Hilbert action

$$S = \frac{1}{2k^2} \int d^4 x \sqrt{-g}\, R, \tag{2.83}$$

with $R$ being the Ricci scalar, we can determine the presymplectic potential, which is

$$\theta^\mu = \nabla_\nu \delta g^{\mu\nu} - g_{\nu\rho} \nabla^\mu \delta g^{\nu\rho}. \tag{2.84}$$

Employing the definition, the Noether current is

$$j^\mu = \frac{1}{2k^2} \sqrt{-g}\, (\theta^\mu - \xi^\mu R) \tag{2.85}$$

and, substituting $\theta^\mu$, we find

$$j^\mu = \frac{1}{2k^2} \sqrt{-g} \left( \nabla_\nu \nabla^{[\nu} \xi^{\mu]} + 2 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \right) \xi_\nu \right) \approx \frac{1}{2k^2} \sqrt{-g} \left( \nabla_\nu \nabla^{[\nu} \xi^{\mu]} \right) \tag{2.86}$$

Substituting the form of asymptotic Killing vectors found before and the falloffs conditions (2.24), we find the asymptotic charge of BMS is

$$Q_T = \frac{1}{4\pi G} \int_{\mathcal{I}^+} dz d\bar{z} du \gamma_{z\bar{z}} T(z, \bar{z}) \left( T_{uu} - \frac{1}{4} \mathcal{D}_i \mathcal{D}_j N^{ij} \right) , \tag{2.87}$$

with $T_{\mu\nu}$ being the stress-energy tensor which enter in the gravitational asymptotic charge exactly as the current density $J_u$ in the electromagnetic case.

At the same time, we find the contribution from the superrotation part to be

$$Q_Y = \frac{1}{4\pi G} \int_{\mathcal{I}^+} dz d\bar{z} du \gamma_{z\bar{z}} Y^i \left( u \partial_u \partial_i m_B + T_{ui} - \frac{1}{4} \partial_i (\mathcal{D}_z^2 C^{zz} - \mathcal{D}_{\bar{z}}^2 C^{\bar{z}\bar{z}}) \right) , \tag{2.88}$$

which, by means of the equations of motion

$$\begin{aligned}\partial_u N_z &= \frac{1}{4}\partial_z(\mathcal{D}_z^2 C^{zz} - \mathcal{D}_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_z m_B - T_{uz}\,,\\ \partial_u N_{\bar{z}} &= \frac{1}{4}\partial_z(\mathcal{D}_z^2 C^{zz} - \mathcal{D}_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_{\bar{z}} m_B - T_{u\bar{z}}\,,\end{aligned} \tag{2.89}$$

can be written as an integral over $\mathcal{I}^+_-$

$$Q_Y = \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} dz d\bar{z}\,\gamma_{z\bar{z}} Y^i N_i\,. \tag{2.90}$$

### 2.2.2 Soft theorems as Ward identities

Soft theorems are identities relating two scattering amplitudes whereas one differs from the other for the emission or absorption of low-energy ("soft") particles. They were first presented in the case of QED, where the soft particle is a photon, by Bloch and Nordsieck in 1937 [88] and further extended by Low et al [89–93]. It has been generalized to the gravitational case by Weinberg in 1965 [94] and extended by Cachazo and Strominger in 2013 [75]. More recently, there have been included the case of the spin 1/2 photino [95] and the spin 3/2 gravitino [96, 97]. Furthermore, it has been generalized to scalar theories [37, 98].

In this section, we discuss the leading soft photon theorem and the leading soft graviton theorem, with the soft particles being photons and gravitons, respectively. For simplicity, we present the scenario with only one soft emission or absorption, though the generalization is straightforward. Additionally, we discuss only the tree level structure of these theorems, as loop corrections have been studied but are not relevant for the purposes of this thesis.

#### Soft theorems in a nutshell

**Soft photon theorem.** We begin by evaluating an $(n+1)$-particle scattering amplitude that describes a process involving a soft photon with four-momentum $\omega q^\mu$ and polarization $\varepsilon^\pm_\mu$ (with $\pm$ denoting the two helicities) along with $n$ "hard" massive scalar particles, each with momentum $p^\mu_k$; $(k = 1, \ldots, n)$, which is large compared to the soft momentum, and charge $e_k$. The amplitude can be depicted diagrammatically by the l.h.s. of Figure 2.6. This process can be divided into two scenarios: in the first, the soft photon attaches to an internal line; in the second, it is emitted (or absorbed) by an external particle. As we shall see, only the latter contribution is relevant at leading order.

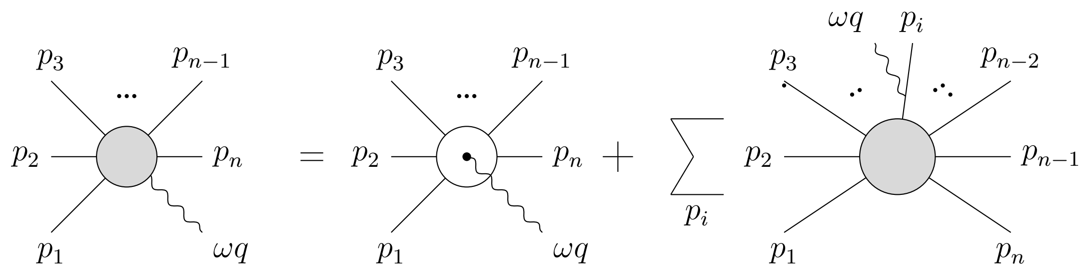



Figure 2.6: Soft theorem

We analyse the second case, denoted with an apex "(2)", where the soft photon is attached to an external line with momentum $p_k$. The diagram is equivalent to a diagram with $n$ hard particle of momentum $p_1, \dots, p_k + \omega q^\mu, \dots, p_n$, times a propagator

$$\frac{1}{(p_k + \omega q)^2 + m^2} = \frac{1}{2\omega p_k \cdot q}, \tag{2.91}$$

where we used the fact that $p_k$ is on-shell, and a vertex $e_k\left(2p_k^\mu + \omega q^\mu\right)$ that has to be contracted with the polarization vector $\varepsilon_\mu^\pm$. Putting all together, we find

$$\begin{aligned} A_{n+1}^{\pm\,(2)}(p_1, \dots, p_n;\ \omega q) =& \frac{1}{\omega} \sum_k \eta_k e_k \frac{p_k \cdot \varepsilon^\pm}{p_k \cdot q} A_n(p_1, \dots, p_k + \omega q, \dots, p_n) \\ =& \frac{1}{\omega} \sum_k \eta_k e_k \frac{p_k \cdot \varepsilon^\pm}{p_k \cdot q} \left[A_n(p_1, \dots, p_n)\right] + \mathcal{O}(\omega^0)\,. \end{aligned} \tag{2.92}$$

When the soft photon is inserted in an internal line of momentum $k^\mu$, the logic is similar but we encounter a crucial difference: the momentum $k^\mu$ is the sum of on-shell momenta but $k^2 + m^2 \neq 0$, so that

$$\frac{1}{(k_k + \omega q)^2 + m^2} = \frac{1}{k_k^2 + 2\omega k_k \cdot q + m^2} = \mathcal{O}(\omega^0), \tag{2.93}$$

thus implying that the contribution from the first of the r.h.s. of Figure 2.6 is of order $\mathcal{O}(\omega^0)$. The contribution to the amplitude, denoted with the apex "(1)" is

$$A_{n+1}^{\pm\,(1)}(p_1, \dots, p_n;\ \omega q) = \sum_k \eta_k e_k \frac{k_k \cdot \varepsilon^\pm}{k_k^2 + m^2} \left[A_n(p_1, \dots, p_n) + \mathcal{O}(\omega)\right], \tag{2.94}$$

where the $k$-sum runs over all the inequivalent way of attaching a soft photon to an internal line.

Therefore, to leading order the soft photon theorem is

$$\lim_{\omega\to 0}[\omega\, A^{\pm}_{n+1}(p_1,\dots,p_n;\ \omega q)] = \mathbf{S}^{\pm}_{lead} A_n(p_1,\dots,p_n;\,) \tag{2.95}$$

where $\mathbf{S}^{\pm}_{lead}$ is the leading soft factor, which can be read directly from (2.92). While we referred to a specific theory involving scalar matter, the result is independent from the theory considered, at least if we focus on the leading contribution. Therefore, the structure of the soft factor is the same if we consider QED as well as complex scalar QED. This property, which is valid also when considering quantum corrections, goes under the name of **universality** of leading soft theorems.

The subleading corrections to the soft factor, which are of order $\mathcal{O}(\omega^0)$, receive contributions from both cases. However, it is highly non-trivial to evaluate the contribution of the emission/absorption from an internal line using Feynman diagrams. However, using other arguments such as the request of gauge invariance, it is possible to determine the form of the subleading soft factor

$$\mathbf{S}^{\pm}_{subl} = -ie\sum_k Q_k \frac{q_\mu \varepsilon^{\pm}_n J^{\mu\nu}_k}{p_k\cdot q}\,, \tag{2.96}$$

where $J_{\mu\nu}$ is the angular momentum of the $k-$th hard particle. This result was derived in 1958 by Low [91] and later extended in [92, 93].

**Soft graviton theorem.** Similar results hold in the gravitational case. Indeed, considering the analogous process involving $n$ hard particle of momentum $p_i$ and the emission/absorption of a soft graviton of momentum $\omega q^\mu$ and polarization $\varepsilon^{\pm}_{\mu\nu}$ we can write

$$\mathcal{M}^{\pm}_{n+1}(p_1,\dots,p_n;\ \omega q) = \left(\frac{\mathcal{S}^{\pm}_{lead}}{\omega} + \mathcal{S}^{\pm}_{subl} + \mathcal{O}(\omega)\right)\mathcal{M}_n(p_1,\dots,p_n), \tag{2.97}$$

where the leading soft factor was derived by Weinberg in 1965 [94] and it has the form

$$\mathbf{S}^{\pm}_{lead} = \sqrt{8\pi G}\sum_k \frac{p_k\cdot p_k\cdot\varepsilon^{\pm}}{p_k\cdot q}, \tag{2.98}$$

while the derivation of the subleading soft graviton theorem is more recent, from Cachazo and Strominger [75], which implies

$$\mathbf{S}^{\pm}_{subl} = -i\sqrt{8\pi G}\sum_k Q_k \frac{p^\mu_k \varepsilon^{\pm}_{\mu\nu} q_\lambda J^{\lambda\nu}_k}{p_k\cdot q}\,, \tag{2.99}$$

with $J_{k\,\mu\nu}$ is the angular momentum of the $k-$th hard particle, being the some of the orbital angular momentum and the helicity of the internal spin.

**Leading soft photon theorem as a Ward identity**

We start from writing explicitly the soft theorems in term of retarded momentum coordinates, parametrised as

$$\begin{aligned}
p^\mu =&\omega q^\mu(w,\bar{w}) = \omega\frac{1}{1+w\bar{w}}\left(1+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-w\bar{w}\right)\,,\\
p_k^\mu =&\omega_k q^\mu(w_k,\bar{w}_k)\,,\\
\varepsilon_i^\mu =&\frac{1}{\sqrt{2}}\partial_i\left[(1+w\bar{w})q^\mu(w,\bar{w})\right]\,,
\end{aligned} \tag{2.100}$$

where we are considering the case of massless hard particle of momentum $p_i$ and we identify $\varepsilon_+^\mu := \varepsilon_w^\mu = \frac{1}{\sqrt{2}}(\bar{w},1,-i,-\bar{w})$, $\varepsilon_-^\mu := \varepsilon_{\bar{w}}^\mu = \frac{1}{\sqrt{2}}(w,1,i,-w)$. With these choices, the soft theorem for a positive helicity outgoing photon at leading order reads

$$\lim_{\omega\to 0}\omega\, A_{n+1}^+ = (1+w\bar{w})\sum_{in,out}\left(\frac{e_k^{out}}{w-w_k^{out}} - \frac{e_k^{in}}{w-w_k^{in}}\right)A_n \tag{2.101}$$

We write $A_{n+1}^+$ as

$$\langle out|a_+^{out}\mathcal{S}|in\rangle, \tag{2.102}$$

with $\mathcal{S}$ being the S-matrix, and we note that, to leading order in $\omega$, it is equivalent to

$$-\langle out|\mathcal{S}a_-^{in}|in\rangle. \tag{2.103}$$

due to CPT invariance. This can be proved by observing that mapping outgoing hard particles to incoming hard particles changes the overall sign of the soft factor and, at the same time, having a negative-helicity incoming photon leads to the replacement $\varepsilon_+^\mu \to (\varepsilon_-^\mu)^* = \varepsilon_+^\mu$. A convenient way to write the soft photon theorem is therefore

$$\lim_{\omega\to 0}\left[\omega\,\langle out|a_+^{out}\mathcal{S} - \mathcal{S}a_-^{in}|in\rangle\right] = 2\mathbf{S}_{lead}^+\langle out|\mathcal{S}|in\rangle \tag{2.104}$$

We now want to prove that this is equivalent to the Ward identity for the asymptotic symmetry parametrized by $\epsilon(z,\bar{z})$. Having a conserved charge that obeys to the antipodal matching condition, its conservation can be written as

$$\langle out|Q_\epsilon^+\mathcal{S} - \mathcal{S}Q_\epsilon^-|in\rangle = 0, \tag{2.105}$$

where we have

$$
\begin{aligned}
Q_\epsilon^+ &= \int_{\mathcal{I}^+} dzd\bar{z}du\gamma_{z\bar{z}}\epsilon(z,\bar{z})(\partial_u(D^z A_z + D^{\bar{z}} A_{\bar{z}}) + j_u)\,,\\
Q_\epsilon^- &= \int_{\mathcal{I}^-} dzd\bar{z}dv\gamma_{z\bar{z}}\epsilon(z,\bar{z})(\partial_v(D^z A_z + D^{\bar{z}} A_{\bar{z}}) + j_v)\,,
\end{aligned}
\tag{2.106}
$$

where the second charge refers to past null infinity, for which we did not performed the derivation since there are no crucial differences. Note that we use the same $\epsilon(z,\bar{z})$ in force of the antipodal matching.

Now we decompose the two-sphere vector by means of the Helmoltz decomposition:

$$
A_i = \mathcal{D}_i A + \varepsilon_{ij}\mathcal{D}^j A', \tag{2.107}
$$

so that

$$
D^z A_z + D^{\bar{z}} A_{\bar{z}} = 2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}} A. \tag{2.108}
$$

Hence, the charge can be written as

$$
Q_\epsilon^+ = 2\int dzd\bar{z}A\,\partial_z\partial_{\bar{z}}\epsilon(z,\bar{z}) + \int_{\mathcal{I}^+} dzd\bar{z}du\gamma_{z\bar{z}}\epsilon(z,\bar{z})j_u. \tag{2.109}
$$

In most of the literature the term $A$ is denoted as $N$, but here we adapt to the notation we use in the next chapter.

Integrating by parts, the charge acts on the states as

$$
\begin{aligned}
Q_\epsilon^-|in\rangle &= -2\int dzd\bar{z}\partial_{\bar{z}}\epsilon(z,\bar{z})\partial_z A^-|in\rangle + \sum_{i=1}^m q_i^{in}\epsilon(z_i^{in},\bar{z}_i^{in})|in\rangle\,;\\
\langle out|Q_\epsilon^+ &= -2\int dzd\bar{z}\partial_z\partial_{\bar{z}}\epsilon(z,\bar{z})\langle out|A^+ + \sum_{i=1}^m q_i^{out}\epsilon(z_i^{out},\bar{z}_i^{out})\langle out|.
\end{aligned}
\tag{2.110}
$$

Using the auxiliary boundary condition $\partial_z A_{\bar{z}} = \partial_{\bar{z}} A_z$ at $\mathcal{I}^+_\pm$, which is equivalent to demand the absence of long-range magnetic fields on $\mathcal{I}^+$, we can integrate by parts and choose $\epsilon(z,\bar{z}) = \frac{1}{z-w}$. Using $\partial_{\bar{z}}\frac{1}{z-w} = 2\pi\delta^{(2)}(w-z)$, we find

$$
4\pi\langle out|\partial_w\mathcal{A}^+\mathcal{S} - \mathcal{S}\partial_w\mathcal{A}^-|in\rangle = \left[\sum_{i=1}^m \frac{q_i^{in}}{w-w_i^{in}} - \sum_{i=1}^n \frac{q_i^{out}}{w-w_i^{out}}\right]\langle out|\mathcal{S}|in\rangle\,. \tag{2.111}
$$

As we discuss in detail in Chapter 5, we can prove, using a stationary phase argument that

$$
\partial_w\mathcal{A}^+ = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+w\bar{w}}\lim_{\omega\to 0}\left[\omega a_+ + \omega a_-\right] \tag{2.112}
$$

and substituting it in the l.h.s. of (2.111) it gives the soft theorem (2.101), thus proving that the Ward identity for (leading) electromagnetic asymptotic symmetry is equivalent to the leading soft photon theorem.

### Supertranslations soft theorem and a hint of Double Copy

The derivation of the gravitational counterpart of this result passes through the same steps, therefore we highlight only the main differences. Interestingly, this analysis leads to the first hints of double-copy, which we explore in the forthcoming parts of this thesis. The BMS charges read

$$\begin{aligned} Q^+ &= \frac{1}{4\pi G}\int_{I^+_-} dud^2z\gamma_{z\bar z}\,T(z,\bar z)\left[T_{uu} - \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar z}^2 N^{\bar z\bar z}\right)\right], \\ Q^- &= \frac{1}{4\pi G}\int_{I^-_+} dvd^2z\gamma_{z\bar z}\,T(z,\bar z)\left[T_{vv} + \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar z}^2 N^{\bar z\bar z}\right)\right]. \end{aligned} \tag{2.113}$$

The polarization vectors can be written in a double-copy formulation, as originally presented by Weinberg [94]

$$\varepsilon^{\pm}_{\mu\nu} = \varepsilon^{\pm}_{\mu}\varepsilon^{\pm}_{\nu}, \tag{2.114}$$

with $\varepsilon^{\pm}_{\mu}$ being the ones used for electromagnetism. With this decomposition the leading gravitational soft factor reads

$$\mathbf{S}^{\pm}_{grav} = \sqrt{8\pi G}\sum_k \frac{(p_k\cdot\varepsilon^{\pm})^2}{p_k\cdot q} := \sqrt{8\pi G}\sum \mathbf{S}^{k\,\pm}_{\text{grav}}. \tag{2.115}$$

Setting all the couplings to one, we can rearrange the soft factor in this form

$$\mathbf{S}^{\pm}_{grav} = \sum_k \frac{(p_k\cdot\epsilon^{\pm})}{p_k\cdot q}p_k\cdot q\frac{(p_k\cdot\epsilon^{\pm})}{p_k\cdot q} = \sum_k \mathbf{S}^{k\pm}_{\text{em}}(\mathbf{S}^{k\,\pm}_{\text{scal}})^{-1}\mathbf{S}^{k\pm}_{\text{em}}. \tag{2.116}$$

where we defined $\mathbf{S}^{k\pm}_{\text{em}}$ in the same spirit as $\mathbf{S}^{k\,\pm}_{\text{grav}}$ in (2.115) and introduced the scalar soft factor $\mathbf{S}^{k\,\pm}_{\text{scal}}$. This rearrangement is similar to the double-copy discussion of soft theorems [99].

With the choices (2.100), the soft factor assumes the form

$$\mathbf{S}^{\pm}_{grav} = -\sum \frac{\omega_i}{(1+w_i\bar w_i)}\left(\frac{\bar w - \bar w_i}{w - w_i}\right)^{\pm 1}. \tag{2.117}$$

With the definition (2.114), the analysis of the stationary phase leads to

$$h_{z\bar{z}} = -\frac{\sqrt{8\pi G}}{(1+w\bar{w})^2} \lim_{\omega\to 0} [\omega a_+ + \omega a_-] \tag{2.118}$$

and substituting in (2.101), we find the soft graviton theorem.

### 2.2.3 Memory effects

Memory effects are observable phenomena that occur when radiation interacts with a test charge, leaving a permanent imprint on the charge even after the radiation has died out. For instance, we can observe that two test masses invested by gravitational radiation exhibit a permanent relative displacement. Analogously, a small electric charge, initially at rest, might exhibit a non-zero velocity after being invested by electromagnetic radiation. Memory effects on a particle located near null infinity can be induced either by radiation emitted from the movement of charged sources within the interior of spacetime or by the outflow of charged massless matter traveling along null rays. The first scenario, known as linear or ordinary memory, reflects the movement of bulk charges, which is captured in the properties of test particles. The second scenario, called nonlinear or null memory, indicates the passage of charged radiation, which can occur in both nonlinear field theories involving self-interacting massless particles and linearized theories with charged massless sources.

In order to present the gravitational memory effect we start from the **geodesic deviation** equation. We start by considering a family of closely spaced geodesics indexed by a continuous variable $\alpha$ and parametrized by an affine parameter $\tau$. For each $\alpha$, the curve swept out by $\gamma_\alpha(\tau)$ is a geodesic. If $x^\mu(\alpha,\tau)$ are the coordinates of the geodesic $\gamma_\alpha(\tau)$, then the tangent vector to the geodesic is

$$t^\mu = \partial_\tau x^\mu. \tag{2.119}$$

If $\tau$ is the proper time, then $t^\mu$ is the four velocity of the object travelling along the geodesic. The deviation vector, on the other hand, is

$$s^\mu = \partial_\alpha x^\mu. \tag{2.120}$$

The relative acceleration is defined as

$$a^\mu = D_\tau^2 s^\mu, \tag{2.121}$$

with $D_\tau$ being the covariant derivative along $\tau$. The geodesic deviation equation reads

$$D_\tau^2 s^\mu = R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho s^\sigma . \tag{2.122}$$

At null infinity, the deviation is encoded in $(s^z, s^{\bar z})$ and we have $t^\mu = \delta_u^\mu$. Hence, the geodesic deviation equation reads

$$\partial_u^2 s^{\bar z} = R^{\bar z}{}_{uuz} s^z . \tag{2.123}$$

From the asymptotically flat metric in Bondi gauge (2.21) with the falloff conditions (2.24), we have

$$R^{\bar z}{}_{uuz} = \frac{\gamma^{z\bar z}}{2r} \partial_u^2 C_{zz} \tag{2.124}$$

and, upon integration between a $u_1$ and $u_2$, denoting the interval of radiation, we find

$$\mathcal{D}^2 s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \mathcal{D}^2 C_{zz} s^z , \tag{2.125}$$

thus showing a permanent separation between the initial and final state [100–104]. Proposals to observe and measure the gravitational memory effect are given in Lasky et al. [105] and van Haasteren and Levin [106].

Interestingly, the connection between the memory effect and the soft theorems is easy to be understood [107]. In fact, the shift of the outgoing metric sourced by black hole or neutron star collisions of momentum $p^A$ was presented by Braginsky and Thorne [100]

$$\mathcal{D}^2 h_{ij} = \mathcal{D}^2 \left( \sum_A \frac{p_i^A p_j^A}{p^A \cdot k} \right) \tag{2.126}$$

where we recognize the soft theorem if we substitute the momentum of the colliding stars or black holes with the momentum of the hard particles and write the graviton momentum as $k = \omega q$. Then the two result match up to a Fourier transform $\int dt e^{i\omega t}$. Note that, at very long distances, astrophysical black holes can be consistently approximated to point-like object like elementary particles.

Let us now understand the connection with asymptotic symmetries. In the assumption that there is no energy flux or retarded time dependence of the asymptotic data at late and early times, $C_{zz}$ must vanish up to a supertranslation:

$$C_{zz}^{\text{early}} = -2D_z^2 C^{\text{early}} , \quad C_{zz}^{\text{late}} = -2D_z^2 C^{\text{late}} ,$$

where $\partial_u C^{\text{early}} = \partial_u C^{\text{late}} = 0$. The early and late geometries are related by a

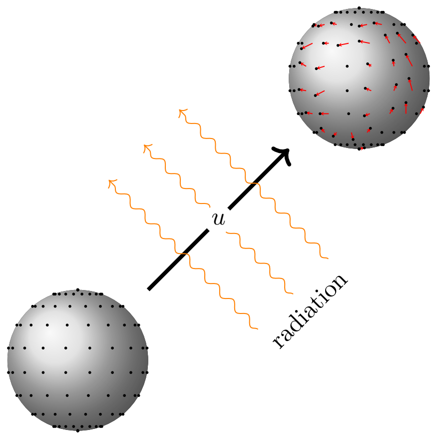



Figure 2.7: Memory effect: an array of evenly spaced inertial detectors (black dots) on the sphere near $\mathcal{I}^+$ will be permanently displaced (red arrows) by the passage of gravitational radiation. *Image from the ArXiv version of [18].*

supertranslation

$$\mathcal{D}^2 C = C^{\text{late}} - C^{\text{early}} . \tag{2.127}$$

Hence we can think of a pulse of radiation passing through $\mathcal{I}^+$ as a domain wall separating diffeomorphic but BMS-inequivalent vacua.

$$D_z^2 \mathcal{D}^2 C^{zz} = 2\mathcal{D}^2 m_B + 2 \int du T_{uu}, \tag{2.128}$$

where the $u$-integral extends from the early to the late regions. Solving this differential equation, it follows that

$$\mathcal{D}^2 C(z,\bar{z}) = -\int d^2 w \gamma_{w\bar{w}} G(z,\bar{z};w,\bar{w}) \left( \int du T_{uu}(w,\bar{w}) + \mathcal{D}^2 m_B \right), \tag{2.129}$$

where the Green's function is

$$G\left(z,\bar{z};z',\bar{z}'\right) = \frac{1}{\pi} \sin^2 \frac{\mathcal{D}^2\Theta}{2} \log \sin^2 \frac{\mathcal{D}^2\Theta}{2}, \tag{2.130}$$

with $\mathcal{D}^2\Theta$ the angle on $\mathcal{C}S^2$ between $(z,\bar{z})$ and $(z',\bar{z}')$. This is an explicit formula for the supertranslation induced by waves passing through $\mathcal{I}^+$, as depicted in Figure 2.7

The memory effect finds application in gauge theories as well and the discussion is quite similar. However, it is worth to note that the electromagnetic memory effect, for instance, if harder to observe respect to the gravitational one which, although being extremely difficult to see, has decent chances of being measured in the next years.

### 2.2.4 Flat holography

The holographic principle [108, 109] states that it is possible to encode the description of gravity in a given spacetime region on the boundary of that region. This paradigm is realised mostly in the AdS/CFT correspondence [110–112], a duality between a gravitational theory in a $d-$dimensional Anti-de Sitter (AdS) space and a conformal field theory (CFT) on its boundary, originating from the dynamics of $D-$branes in the bulk and which can be constructed, for instance, through geometrical engineering of quantum field theory [113–118].

Extending the correspondence beyond AdS spacetime is one of the most challenging problems in theoretical physics. Researchers are currently making significant efforts to apply the holographic principle to the case of asymptotically flat spacetimes, which represent a model more similar to our Universe. This attempt is at the basis of the **flat holography** program. Specifically, two main approaches have emerged within this program: the **Carrollian holography** and the **Celestial holography**.

**Carrollian holography.** In the first approach, the dual theory is a BMS field theory defined on the $3d$ null boundary of the spacetime [119–126], called also a conformal Carrollian field theory CCarrFT. Carriollan holography involves a codimension-one holographic duality, compatible with the AdS/CFT perspective. At the same time, establishing a holographic dictionary in the flat case introduces new challenges not encountered in AdS, due to the null nature of the conformal boundary and the radiative flux passing through it. Taking a flat-space limit in the bulk, specifically by letting the cosmological constant approach zero, induces an ultra-relativistic limit $c \to 0$ in the boundary theory, which contracts the conformal symmetries to BMS symmetries. In order to understand this approach, let us consider eBMS Killing vectors

$$\begin{aligned}
\xi^u &= T + \frac{u}{2}\mathcal{D}_i Y^i + \mathcal{O}(r^{-1})\,, \\
\xi^r &= -\frac{r}{2}\mathcal{D}_i Y^i + \frac{1}{2}\mathcal{D}^2 Y - \frac{u}{2}\mathcal{D}_i Y^i + \mathcal{O}(r^{-1})\,, \\
\xi^i &= Y^i - \frac{1}{r}\left(\mathcal{D}^i T + \frac{u}{2}\mathcal{D}^i\mathcal{D}_j Y^j\right) + \mathcal{O}(r^{-2})\,,
\end{aligned} \tag{2.131}$$

with $T = T(z,\bar{z})$ parametrising supertranslations and $Y^i = (Y(z), \bar{Y}(\bar{z}))$ parametrising superrotations, where $Y$ satisfies the conformal Killing equation. Geometrically, eBMS symmetries are the conformal symmetries of a Carrollian manifold $C$ on $\mathcal{I}^+$ with coordinates $x^a = (u, z, \bar{z})$ [121, 122]. This structure is given by a degenerate metric

$$q_{ab} = 0du^2 + \gamma_{ij} dz^i dz^j \,, \tag{2.132}$$

whose kernel is generated by a nowhere vanishing vector Carrollian field vector field $n^a$, such that $q_{ab}n^b = 0$. In particular, we have

$$n^a \partial_a = \partial_u \,. \tag{2.133}$$

In [127] it was added the further requirement for a Carroll manifold to carry, in addition, a symmetric affine connection[1], compatible with both $q$ and $n$.

In [121], it was introduced a notion of conformal Carroll group of level $N$ $\mathrm{CCarr}_N(C, q, n)$ of those transformations which act as

$$q \to \Omega^2 q \qquad n \to \Omega^{-2/N} n \,. \tag{2.134}$$

The Lie algebra of infinitesimal conformal Carrollian transformations on $C$, $\mathfrak{ccarr}_N(C, q, n)$, is generated by vector fields $\bar{\xi} = \bar{\xi}^a \partial_a$ on $\mathcal{I}^+$ satisfying

$$\mathcal{L}_{\bar{\xi}} q_{ab} = \alpha q_{ab} \,, \qquad \mathcal{L}_{\bar{\xi}} n^a = -\frac{\alpha}{N} n^a \,, \tag{2.135}$$

with $\alpha$ being a function on $\mathcal{I}^+$. As pointed out in equation (III.5) of [121], the most general solution $\bar{\xi}$ to (2.135) with $N = 2$ is precisely the restriction to $\mathcal{I}^+$ of the asymptotic Killing vector (2.131)

$$\bar{\xi} = \left( T + \frac{u}{2} D \cdot Y \right) \partial_u + Y^i \partial_i \,, \tag{2.136}$$

with

$$\alpha = D \cdot Y \,. \tag{2.137}$$

The standard Lie bracket on $\mathcal{I}^+$ of these vector fields reproduces the BMS algebra, as discussed in equation (2.49). In particular, this results is valid in any spacetime dimension $D$ with

$$\mathrm{CCarr}_2(C_D, q, n) \equiv \mathrm{BMS}_{D+1} \,. \tag{2.138}$$

Inspired by this equivalence, the idea of Carrollian holography is that three-dimensional quantum CCarrFT coupled with external sources describe gravity in four-dimensional asymptotically flat spacetimes. Specifically, in [128] a holographic dictionary is proposed, connecting Carrollian momenta and grav-

[1] Since $q$ is degenerate this connection might not be uniquely defined.

itational data at $\mathcal{I}^+$.

**Celestial holography.** In the second approach to flat holography, the holographic dual of asymptotically flat gravity is a two-dimensional celestial conformal field theory CCFT, living on the conformal sphere at infinity.

To discuss soft theorems, we used the standard framework of S-matrix that maps an incoming to an outgoing Hilbert space, for which multiparticle states in the in- and out- Hilbert spaces are described as asymptotic, non-interacting momentum eigenstates. However, it is possible to formulate an alternate description of scatterings as a type of correlation function on the celestial sphere [129], as depicted in Figure 2.8. In this case, in and out states are labelled by operator defined on the celestial sphere $O_k(z_k, \bar{z}_k)$. In this

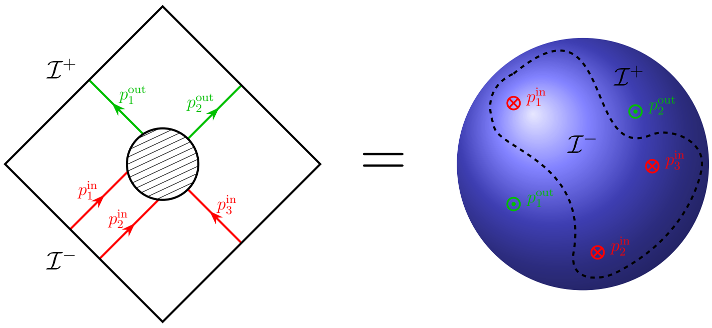



Figure 2.8: Asymptotic momentum eigenstates are mapped to points on the celestial sphere $\mathcal{CS}^2$.

language, scattering amplitudes are mapped to two-dimensional celestial correlators [64, 130, 131]

$$\langle \text{out}|\, \mathcal{S}\, |\text{in}\rangle \to \langle O_{\Delta_1}(w_1, \bar{w}_1), \ldots, O_{\Delta_n}(w_n, \bar{w}_n) \rangle\,. \tag{2.139}$$

Specifically, instead of using energy-momentum eigenstates, the S-matrix is written in boost eigenstates basis. This change of basis is realised through the Mellin integral transform, defined as

$$\mathcal{O}_\Delta(w, \bar{w}) = \int_0^\infty d\omega\, \omega^{\Delta-1} \mathcal{O}(\omega, w, \bar{w})\,. \tag{2.140}$$

Working in this basis, the operators

$$\hat{h} = \frac{1}{2}(l - \omega\partial_\omega), \qquad \hat{\bar{h}} = \frac{1}{2}(-l - \omega\partial_\omega)\,, \tag{2.141}$$

through which the subleading soft graviton mode couples to an external particle, as explained in equation (2.96), are diagonalised. In particular, $l$ and $\omega$ are the helicity and the energy of the external particle. The scaling dimension $\Delta_i$ of the operator $\mathcal{O}_{\Delta_i}(w_i, \bar{w}_i)$ on the celestial sphere is defined as

$$\Delta_i = h_i + \bar{h}_i. \tag{2.142}$$

In particular, we have

$$\langle O_{\Delta_1}(w_1, \bar{w}_1), \ldots, O_{\Delta_n}(w_n, \bar{w}_n) \rangle = \prod_{i=1}^{n} \int d\omega_i \, \omega_i^{\Delta_i - 1} \langle \text{out} | \, \mathcal{S} \, | \text{in} \rangle \, . \tag{2.143}$$

Hence, in celestial holography the duality is realised between a four-dimensional bulk and a two-dimensional boundary, providing a crucial difference with the standard AdS/CFT paradigm and the Carrollian approach. However, in [128, 132] was argued that Celestial holography and Carrollian holography are complementary to each other. Specifically, Carrollian source operators living at null infinity can be mapped to CCFT operators living on the celestial sphere after using an appropriate integral transform, thus providing a relation between the correlation functions between the sourced CCarrFT and those of the CCFT. In particular, the Ward identities of the (sourced) Carrollian CFT are equivalent to those of the CCFT.

## 2.3 Peculiarities of Lorenz gauge

In this thesis, we work in Lorenz gauge. We motivate this choice noting that, in this gauge, we can generalise to arbitrary higher order the falloff of the gauge parameter, thus providing a systematic way to analyse higher-order asymptotic symmetries. Furthermore, the Lorenz gauge is natural from the point of view of the double copy, as will become clear in the second part of this thesis. However, there are some technical issues that required to be discussed. It turns out that, in order to find asymptotic symmetries in Lorenz gauge, we need to generalise the gauge parameter expansion by means of logarithmic terms. In this section, we review these issues for spin-one and spin-two theories. The two-form analysis is left for the next chapter.

### 2.3.1 Electromagnetism

The Lorenz gauge consists in

$$\nabla^\mu A_\mu = 0, \tag{2.144}$$

so that the Maxwell equations reduce to

$$\Box A_\mu = 0 \tag{2.145}$$

and the residual gauge parameter obeys to the d'Alembert equation as well

$$\Box \epsilon = 0. \tag{2.146}$$

Let us forget for the moment about the falloff preservation, since this is addressed in the next chapter. We focus instead on the electric asymptotic charge, defined as

$$Q = \int_{\mathcal{I}^+_-} dz d\bar{z}\, r^2 \gamma_{z\bar{z}} \epsilon\, F_{ur} \,. \tag{2.147}$$

Since we have $F_{ur} = \mathcal{O}(r^{-2})$ we observe that, in order to have non-vanishing and non-divergent[2] charge in the limit $r \to \infty$, we need $\epsilon = \mathcal{O}(1)$. In force of the other attempts in the context electromagnetism, one might be tempted to write

$$\epsilon = \sum_{n=0}^{\infty} \frac{\epsilon^{(n)}(u, z, \bar{z})}{r^n} \,. \tag{2.148}$$

Inserting this expansion in the wave equation, we find

$$-2(n-1)\partial_u \epsilon^{(n)} = [\mathcal{D}^2 + (n-1)(n-2)]\epsilon^{(n-1)} \,. \tag{2.149}$$

To leading order, this reads

$$\partial_u \epsilon^{(0)}(u, z, \bar{z}) = 0 \,, \tag{2.150}$$

thus implying that $\epsilon^{(0)}$ is a $u-$independent term, consistently with what expected from analysis in other gauges. However, going at the first subleading order, the equation implies

$$\mathcal{D}^2 \epsilon^{(0)}(z, \bar{z}) = 0, \tag{2.151}$$

[2]In the next section we discuss divergent charges by means of symplectic renormalization, but in this section we stick to the standard approach.

which means that only constants parameters are allowed to leading order. Specifically, this means we have only space for global symmetries, as can be understood by substituting a constant $\epsilon^{(0)}$ in the asymptotic charge, which would be the standard electric charge.

One might be tempted to interpret this result as a signal that Lorenz gauge is not suited for asymptotic symmetries, meaning that we are gauging away some physical degrees of freedom. This is partially true: the absence of a nontrivial leading parameter is indeed the evidence that our setup is not enough to encompass asymptotic symmetries. However, our setup is not only the gauge choice and the falloffs, but also the series (2.148). Crucially, it was noted [25, 50, 133] that generalising the expansion by means of logarithmic (subleading) terms

$$\epsilon = \sum_{n=0}^{\infty} \frac{\epsilon^{(n)}(u, z, \bar{z})}{r^n} + \sum_{n=1}^{\infty} \hat{\epsilon}^{(n)}(u, z, \bar{z}) \frac{\log r}{r^n} \,, \tag{2.152}$$

we are able to "restore" asymptotic symmetries. The equation (2.151) is indeed modified as

$$2\partial_u \hat{\epsilon}^{(1)} = \mathcal{D}^2 \epsilon^{(0)} \,, \tag{2.153}$$

and since $\hat{\epsilon}^{(1)}$ is unconstrained by other equations, we can find an arbitrary $\epsilon^{(0)}(z, \bar{z})$ and therefore asymptotic symmetries. We refer to the series (2.152) as a polyhomogeneous expansion.

Generalising the expansion modifies the result about asymptotic symmetries. Our interpretation of this fact is the following: in principle, the Lorenz gauge allows for an arbitrary $\epsilon(u, r, z, \bar{z})$ obeying the d'Alembert equation. By writing a power expansion, we have further restricted our residual gauge, adding an extra condition to the gauge parameter. Crucially, we (unknowingly) gauged away some "physical" parts of $\epsilon$, which are recovered when we choose a more general expansion. This is an accident of Lorenz gauge, which was not noted in the analyses in radial gauge and temporal gauge.

However, the generalisation of the series leads to some side effects: the standard falloffs are violated. This is addressed in the next chapter with great detail, but this issue is at the basis of some attempts to generalise the falloffs for both the field components that the gauge parameter, which led to higher-order asymptotic symmetries.

### 2.3.2 Spin two

Now let us move to the case of supertranslations, where the leading order of the residual gauge symmetries are parametrised by

$$\xi^{u\,(0)} = T(z,\bar{z})\,, \qquad \xi^{r\,(0)} = \frac{1}{2}\mathcal{D}^2 T(z,\bar{z})\,, \qquad \xi^{i\,(1)} = -\mathcal{D}^i T(z,\bar{z})\,. \tag{2.154}$$

In the case of spin two, the gauge condition we consider is the de-Donder gauge

$$\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h = 0. \tag{2.155}$$

Note that if we work in terms of the trace-reversed graviton, namely

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,, \tag{2.156}$$

the gauge condition becomes equivalent to the Lorenz gauge, since

$$\nabla^\mu \bar{h}_{\mu\nu} = 0\,. \tag{2.157}$$

The residual gauge vector obeys again to the d'Alembert equation

$$\Box \xi^\mu = 0\,. \tag{2.158}$$

Assuming the power expansions

$$\xi^u = \sum_{n=0}^{\infty} \frac{\xi^{u\,(n)}}{r^n}, \quad \xi^r = \sum_{n=0}^{\infty} \frac{\xi^{r\,(n)}}{r^n}, \quad \xi^i = \sum_{n=1}^{\infty} \frac{\xi^{i\,(n)}}{r^n}\,, \tag{2.159}$$

the order-by-order wave equation reads

$$\begin{aligned}
&-2(n-1)\partial_u \xi^{r\,(n)} = [\mathcal{D}^2 + (n-1)(n-2) - 2]\xi^{r\,(n-1)} - 2\mathcal{D}_i \xi^{i\,(n)} \\
&-2(n-1)\partial_u \xi^{u\,(n)} = [\mathcal{D}^2 + (n-1)(n-2)]\xi^{u\,(n-1)} + 2\xi^{r\,(n-1)} + 2\mathcal{D}_i \xi^{i\,(n)} \\
&-2(n-1)\partial_u \xi^{i\,(n+1)} = [\mathcal{D}^2 + (n-1)(n-2) - 1]\xi^{i\,(n)} + 2\mathcal{D}^i \xi^{r\,(n-1)}\,.
\end{aligned} \tag{2.160}$$

For $n=0$ we find the leading orders to be $u-$independent, while for $n=1$ we have

$$\begin{aligned}
&[\mathcal{D}^2 - 2]\xi^{r\,(0)} - 2\mathcal{D}_i \xi^{i\,(1)} = 0 \\
&\mathcal{D}^2 \xi^{u\,(0)} + 2\xi^{r\,(0)} + 2\mathcal{D}_i \xi^{i\,(1)} = 0 \\
&[\mathcal{D}^2 - 1]\xi^{i\,(1)} + 2\mathcal{D}^i \xi^{r\,(0)} = 0\,.
\end{aligned} \tag{2.161}$$

Summing the first two, we already obtain

$$\mathcal{D}^2(\xi^{r\,(0)} + \xi^{u\,(0)}) = 0, \tag{2.162}$$

meaning

$$\xi^{r\,(0)} = -\xi^{u\,(0)} + c_1\,, \tag{2.163}$$

with $c_1$, constant, implying already we cannot find supertranslations in the form (2.154). Evaluating the divergence of the third equation, we also find

$$\mathcal{D}_i\xi^{i\,(1)} = 2\xi^{u\,(0)} + c_2. \tag{2.164}$$

Substituting everything in the second equation, we find

$$[\mathcal{D}^2 - 2]\xi^{u\,(0)} = 0\,. \tag{2.165}$$

The solutions to this equations are ordinary four-dimensional translations, parametrised by the spherical harmonics $\{Y_{0,0}, Y_{1,m}\}$. We thus find the same result of the spin-one case.

Modifying the expansions by means of subleading logarithmic series as

$$\begin{aligned}
\xi^u &= \sum_{n=0}^{\infty}\frac{\xi^{u\,(n)}}{r^n} + \sum_{n=1}^{\infty}\hat{\xi}^{u\,(n)}\frac{\log r}{r^n},\\
\xi^r &= \sum_{n=0}^{\infty}\frac{\xi^{r\,(n)}}{r^n} + \sum_{n=1}^{\infty}\hat{\xi}^{r\,(n)}\frac{\log r}{r^n},\\
\xi^i &= \sum_{n=1}^{\infty}\frac{\xi^{i\,(n)}}{r^n} + \sum_{n=2}^{\infty}\hat{\xi}^{i\,(n)}\frac{\log r}{r^n}\,,
\end{aligned} \tag{2.166}$$

the equations (2.161) are modified analogously to the spin-one case, namely

$$\begin{aligned}
&2\partial_u\hat{\xi}^{r\,(1)} = (\mathcal{D}^2 - 2)\xi^{r\,(0)} - 2\mathcal{D}_i\xi^{i\,(1)};\\
&2\partial_u\hat{\xi}^{u\,(1)} = \mathcal{D}^2\xi^{u\,(0)} + 2\hat{\xi}^{r\,(0)} + 2\mathcal{D}_i\xi^{i\,(1)} = 0;\\
&2\partial_u\hat{\xi}^{i\,(2)} = (\mathcal{D}^2 - 1)\xi^{i\,(1)} + 2\mathcal{D}^i\xi^{r\,(0)} = 0;
\end{aligned} \tag{2.167}$$

thus proving we can find standard supertranslations (2.154), since $\hat{\xi}^{r\,(1)}, \hat{\xi}^{u\,(1)}, \hat{\xi}^{i\,(2)}$ are unconstrained by other equations.

# Chapter 3

# Two-form higher-order asymptotic symmetries

In this chapter, we review the work [29], with the aim of exploring higher-order asymptotic symmetries for electromagnetism and two-form theory. In fact, building on the ideas behind the formulation of superrotations [70, 71], asymptotic symmetries have been generalised to higher orders. The order of an asymptotic symmetry is defined with respect to the radial power of the corresponding asymptotic parameter in Minkowskian components[1]. Supertranslations are, for instance, $\mathcal{O}(1)$ (or $\mathcal{O}(r^0)$) asymptotic symmetries, while superrotations are $\mathcal{O}(r)$. The key concept of the infrared triangle is that the semiclassical Ward Identity for an $\mathcal{O}(r^n)$ asymptotic symmetry corresponds to the sub$^n$-leading soft theorem. We can find analyses of $\mathcal{O}(r)$ asymptotic symmetries in electromagnetism [25], recently generalised to arbitrary $\mathcal{O}(r^N)$ in [26]. The non-abelian case was studied at $\mathcal{O}(r)$ in [134], and recently extended to arbitrary higher order in [135, 136]. For gravity, $\mathcal{O}(r^2)$ asymptotic symmetries were studied in [31], exploring the connection with sub-subleading soft graviton theorem. In this work, we present the $\mathcal{O}(r^N)$ generalisation of two-form asymptotic symmetries, whose $\mathcal{O}(1)$ formulation was studied in the context of the scalar duality in [27, 28] and recently refined in [34].

One of the issues about higher-order asymptotic symmetries regards the preservation of the boundary conditions, defined by assigning falloff conditions to the gauge fields at the boundary under investigation, which in our case is $\mathscr{I}^+$. Typically, one starts by determining the falloffs of the field strength of the theory under investigation and then deduces the gauge fields falloffs. This process, however, is not entirely unambiguous, as field falloffs can include

[1] In fact, this is not equivalent to study the radial power of Bondi components, since when focusing on angular components we have an extra power of $r$ from the change of coordinates.

overleading pure gauge terms, allowing for the extension of the order of the asymptotic parameter that preserves these falloffs. An exception is the radial gauge in electromagnetism, where this flexibility is not present for $\mathcal{O}(r^N)$; $(N > 0)$ parameters.

Another key point is the choice of field and parameter expansions. While early work focused on $\frac{1}{r}$ expansions, recent analysis have introduced logarithmic terms [47,48,50,137], which seem unavoidable in certain setups, such as in the Lorenz gauge, which we adopt. In this gauge, the asymptotic symmetries require logarithmic terms in their parameters, though not in the field components. Therefore, we allow for logarithms in the gauge fields only in pure gauge sectors. As a consequence, the field strength admits a simple power expansion, as detailed in section 3.2 and section 3.3.

A challenge with higher-order asymptotic symmetries is the divergence of the corresponding asymptotic charge. These divergences are interpreted as ambiguities in the presymplectic potential and can be renormalized in the manner of [24]. We apply this procedure to eliminate divergences while preserving the finite parts.

There are several motivations for this work. To begin with, $\mathcal{O}(r^N)$ two-form asymptotic symmetries are interesting because of the existence of the scalar duality, since in $D = 4$ a two-form is dual (on-shell) to a scalar field. Hence, understanding these higher order asymptotic symmetries could be the key to understand asymptotic symmetries for scalars. And since scalar theories represent the simplest field theories, these results could help towards a deeper understanding of the connection between asymptotic symmetries and physical effects. Furthermore, as discussed in the second part of this thesis, $\mathcal{O}(1)$ asymptotic symmetries of electromagnetism are linked via double copy to BMS supertranslations and $\mathcal{O}(1)$ two-form symmetries [34]. This suggests a potential generalization to higher order and specifically to $\mathcal{O}(r)$, providing a two-form counterpart to superrotations.

Additionally, let us observe that the two-form provides the simplest example of a theory possessing a gauge-for-gauge redundancy, namely the gauge parameter has itself a gauge symmetry. We thus might expect some peculiarities that are not present in the electromagnetic case. From our analysis, however, it seems that no particular role is played by the gauge-for-gauge parameter, except that of simplifying some computations.

We begin with an analysis of electromagnetism, which can be partially understood as a review of [26], and then extend our approach to the two-form theory, first reviewing the existing results in literature about $\mathcal{O}(1)$ asymptotic symmetries [27,28,34] and then generalising the discussion to arbitrary $\mathcal{O}(r^N)$.

The symplectic renormalization is presented in section 3.4.

# 3.1 Preliminaries

## 3.1.1 Review of notation and conventions

We focus on $D = 4$ and adopt the mostly plus convention for the metric. For the discussion of asymptotic parameter, we employ retarded Bondi coordinates $(u, r, z^i)$, with $r$ being the radial coordinate, $u = t - r$ the retarded time and $z^i$ $(i = 1, 2)$ the angular coordinates, which we parameterise by means of stereographic projection in terms of $(z, \bar{z})$, related to the standard angles as

$$z = e^{i\phi} \cot \frac{\theta}{2}, \qquad \bar{z} = e^{-i\phi} \cot \frac{\theta}{2}, \tag{3.1}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

The Minkowski metric reads

$$ds^2 = -du^2 - 2du\, dr + r^2 \gamma_{ij} dz^i\, dz^j, \tag{3.2}$$

where $\gamma_{ij}$ is the unit metric on the two-sphere, which is anti-diagonal in terms of $(z, \bar{z})$, with

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}. \tag{3.3}$$

The covariant derivative with respect to $\gamma_{ij}$ is denoted with $\mathcal{D}_i$ and the two-sphere Laplace operator is $\mathcal{D}^2 = \mathcal{D}_i \mathcal{D}^i$. At the same time, we use $\nabla_\mu$ for the covariant derivative respect to the metric (3.2) and $\Box = \nabla_\mu \nabla^\mu$ for the d'Alembert operator. Divergences are often denoted as $\mathcal{D} \cdot V := \mathcal{D}^i V_i$ and $\nabla \cdot V := \nabla^\mu V_\mu$.

The non-vanishing Christoffel symbols are

$$\Gamma^u_{ij} = -\Gamma^r_{ij} = r\gamma_{ij}, \qquad \Gamma^i_{rj} = \frac{1}{r}\delta^i_j, \qquad \Gamma^z_{zz} = \bar{\Gamma}^{\bar{z}}_{\bar{z}\bar{z}} = -\frac{2z}{1 + z\bar{z}}. \tag{3.4}$$

We often make use of the following commutation relation, which we write here explicitly:

$$\mathcal{D}_i \mathcal{D}^2 V_j = (\mathcal{D}^2 - 3)\mathcal{D}_i V_j + 2\gamma_{ij} D_k V^k. \tag{3.5}$$

Future null infinity $\mathcal{I}^+$ is reached by sending $r \to \infty$ (or $t \to \infty$) with fixed $u$. It is a null hypersurface with $\mathbb{R} \times S^2$ topology. Sending $u \to \pm\infty$ we reach $\mathcal{I}^+_\pm$. Fields and parameter are expanded near $\mathcal{I}^+$. The most general radial expansion we write is

$$\varphi = \sum_{n_1} \frac{\varphi^{(n_1)}}{r^{n_1}} + \sum_{n_2} \hat{\varphi}^{(n_2)} \frac{\log r}{r^{n_2}}, \tag{3.6}$$

where the coefficients of the radial expansion are always denoted with a superscript "$(n)$" and, in particular, we use the hat for the logarithmic coefficients.

When discussing charges and renormalization we employ $(t, u, z, \bar{z})$ coordinates, since we are interested in the large $t-$behaviour around $\mathcal{I}^+$. The expansion coefficients are denoted with a subscript "$(n)$" and

$$\varphi = \sum_{n_1} t^{n_1} \varphi_{(n_1)} + \log t \sum_{n_2} t^{n_2}\, \hat{\varphi}_{(n_2)}. \tag{3.7}$$

To describe the behaviour near $\mathcal{I}^+_-$ we consider only power expansions that we denote as

$$\varphi^{(n)} = \sum_m u^m \varphi^{(n,m)}. \tag{3.8}$$

In some cases this expansion is valid for all values of $u$ and not only for $u \to -\infty$, but we use the same notation, specifying always the meaning.

### 3.1.2 Asymptotic charges

**Electromagnetism.** Considering the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{3.9}$$

the presymplectic potential current reads

$$\theta^\mu_e = -\sqrt{-g}\, F^{\mu\nu} \delta \mathcal{A}_\nu. \tag{3.10}$$

Given a gauge transformation $\delta\mathcal{A}_\nu = \partial_\nu \epsilon$ the current is an on-shell total derivative

$$\mathcal{J}^\mu_e = -\partial_\nu (\sqrt{-g}\, F^{\mu\nu} \epsilon)\,. \tag{3.11}$$

We consider a surface $\Sigma_t$ at constant $t = u + r$ and then study the limit $t \to \infty$. This defines the asymptotic charge, evaluated in $(u, t, z, \bar{z})-$coordinates, which is

$$
\begin{aligned}
Q_e &= \lim_{t\to\infty} - \int_{\Sigma_t} dudzd\bar{z}\,\partial_\mu(\sqrt{-g}\,F^{t\mu}\epsilon) \\
&= \lim_{t\to\infty} - \int_{\Sigma_t} dudzd\bar{z}\gamma_{z\bar{z}}(\partial_u - \partial_r)(r^2 F_{ur}\epsilon),
\end{aligned}
\tag{3.12}
$$

where in the last step we substituted the expression in terms of Bondi components and neglected a total two-sphere divergence.

At the same time, we can write a dual asymptotic magnetic charge, which is

$$
\tilde{Q}_m = \int_{\Sigma_t} dudzd\bar{z}(\partial_u - \partial_r)(\sqrt{-g}\,\tilde{F}_{ur}\tilde{\epsilon})\,, \tag{3.13}
$$

where $\tilde{F}$ is the dual field strength, defined as through the Hodge dual $\tilde{F} = \star F$, and $\tilde{\epsilon}$ is the gauge parameter of the dual gauge field $\tilde{A}$, with $\tilde{F} = d\tilde{A}$.

However, it should be noted that the definition (3.13) is not entirely correct. As studied in detail in [138], the charge (3.13) is not derived from a well-defined canonical analysis. To obtain its correct form, one should extend the covariant phase space by means of edge modes degrees of freedom. In particular, the correct magnetic charge is different from the one (3.13) when the dual gauge parameter has singularities.

We do not perform such an analysis here, as our primary interest lies in the two-form and the comparison with the electric charges. The correct form of the charge is used in [26] for a $\mathcal{O}(r^n)$ dual gauge parameter. We leave this analysis for future investigations.

**2-form.** The Lagrangian is

$$
L = -\frac{1}{6}H_{\mu\nu\rho}H^{\mu\nu\rho}\,, \tag{3.14}
$$

so that the presymplectic potential current reads

$$
\theta^\mu_B = -\sqrt{-g}\,H^{\mu\nu\rho}\delta\mathcal{B}_{\nu\rho}\,. \tag{3.15}
$$

Given a gauge transformation $\delta\mathcal{B}_{\nu\rho} = \frac{1}{2}(\partial_\nu\lambda_\rho - \partial_\rho\lambda_\nu)$ the current takes the form

$$
\mathcal{J}^\mu_B = -\partial_\nu(\sqrt{-g}\,H^{\mu\nu\rho}\lambda_\rho). \tag{3.16}
$$

Considering again a constant time surface $\Sigma_t$ , the asymptotic charge is defined as the limit $t \to \infty$ of the quantity

$$Q_B = \lim_{t\to\infty} \int_{\Sigma_t} dudzd\bar{z}\gamma_{z\bar{z}}(\partial_u - \partial_r)\left(r^2 H^{uri}\lambda_i\right) \tag{3.17}$$

where we are neglecting again a total two-sphere divergence.

For higher-order asymptotic symmetries, there are $t-$divergences which imply that the both $Q_e$ and $Q_B$ are not well-defined in the limit. However, we can employ symplectic renormalization in order to neglect these divergences and perform the limit.

## 3.2 Electromagnetism

We start by deriving $\mathcal{O}(r^N)$ asymptotic symmetries for electromagnetism. We do not consider sources in our analysis and focus only on the soft part of the charges, with the aim of comparing the spin-one and the two-form results. This is motivated by the fact that the double-copy results [34], which partially inspired this work, are based on a dictionary that treats differently fields and sources [11]. In addition to this, we observe that the source of [25, 26], namely the current of scalar electrodynamics

$$j_\mu = ie\phi\mathcal{D}_\mu\phi^* + c.c \tag{3.18}$$

with $\mathcal{D}_\mu\phi = \partial_\mu\phi - ie\mathcal{A}_\mu\phi$, with the standard assumption $\phi = \mathcal{O}(\frac{1}{r})$ and $\mathcal{A}_u = \mathcal{O}(\frac{1}{r})$, leads to a nontrivial constraint on the $u-$dependence of $\phi^{(1)}$, which comes from the condition $j_u^{(2)} = 0$.

Let us also remark that we cannot treat $A_\mu$ and $j_\mu$ independently in an order-by-order analysis of the equations of motion, crucial to determine the correct falloffs. Indeed, some leading components of the current $j_\mu$ already involve $A_\mu$, making it nontrivial to assume that the results align with those derived from the free equations of motion. For instance, in the case of $\phi^3$ theory in $D = 4$, as noted in [98], the standard falloff of scalar field $\phi = \mathcal{O}(\frac{1}{r})$, is not consistent with the equations of motion in the presence of the interaction, while being fully compatible with the vacuum equations of motion.

A full analysis is left for future investigation, while in the present setup we restrict our study to the scenario where all quantities satisfy the vacuum equations of motion, analysed in Appendix A. Let us observe that this setup applies also to the case of solutions far from localised sources, to which the vacuum equations of motion provide a good approximation [139].

The falloffs analysis starts by asking compatibility with some physical requests about the field strength:

$$F_{ur} = \mathcal{O}(r^{-2}), \qquad F_{ui} = \mathcal{O}(1), \qquad F_{ri} = \mathcal{O}(r^{-2}), \qquad F_{ij} = \mathcal{O}(1). \tag{3.19}$$

Moreover, we propose a key ansatz regards the expansion of the field strength components:

$$F_{\mu\nu} = \sum_n \frac{F^{(n)}_{\mu\nu}(u, z, \bar{z})}{r^n}, \tag{3.20}$$

where we add details on the $u-$dependence in the next section. Hence, logarithms are not included in the field strength expansion, unlike the gauge parameter and the gauge field components. This choice is motivated by observing how logarithms in (3.20) are unnecessary. In fact, they are required in the gauge parameter because otherwise the Lorenz gauge condition does not allow for a non-trivial $\mathcal{O}(1)$ term. Consequently, logarithms are added in the field components or otherwise the falloffs cannot be preserved (see, for instance, section 5 of [50]). However, this issue is specific to Lorenz gauge but does not arise, for instance, in retarded radial gauge or in radiation gauge. In these well-studied and somehow simpler cases, the parameter does not require logarithms, therefore there is no need to consider these terms in the gauge field components and, accordingly, no reason to add them in the field strength. And since the field strength is a gauge invariant quantity, we do not expect the expansion considered to be gauge-dependent. This choice is one of the main differences with [26]. Note that this does not imply that including logarithms in the field strength does not make sense, but only that the confrontation with other analyses in other gauges must be performed cautiously. To our best understanding, our setup is not modified when considering sources, as they are gauge invariant.

We work in Lorenz gauge

$$\nabla^\mu \mathcal{A}_\mu = 0, \tag{3.21}$$

so that the residual gauge parameter $\epsilon$ obeys to

$$\Box \epsilon = 0 \tag{3.22}$$

and the Maxwell equations reduces to

$$\Box \mathcal{A}_\mu = 0. \tag{3.23}$$

### 3.2.1 Falloffs

The choice (3.20) has a crucial consequence on the logarithmic terms of the gauge field components: they are pure gauge terms. Therefore, the gauge field admits the general expansion

$$\mathcal{A}_\mu = A_\mu + \partial_\mu a = \sum_{n,n_1,n_2} \left[ \frac{A_\mu^{(n)}}{r^n} + \partial_\mu \left( \frac{a^{(n_1)}}{r^{n_1}} + \hat{a}^{(n_2)} \frac{\log r}{r^{n_2}} \right) \right], \tag{3.24}$$

where we factored out all the pure gauge sector. The Lorenz gauge condition implies that the $a$-series satisfies the wave equation.

Only the first series in (3.24) contributes to the field strength, allowing us to determine the falloffs of $A_\mu$ from (3.19), which are :

$$A_u = \mathcal{O}(r^{-1}), \qquad A_r = \mathcal{O}(r^{-2}) \qquad A_i = \mathcal{O}(1), \tag{3.25}$$

where constant terms are neglected[2]. It is straightforward to verify that these falloffs are consistent with the equations of motion. In fact, since the logarithms appear in pure gauge sectors, it is sufficient to study the equations of motion for a $\frac{1}{r}$ expansion. Specifically, $A_i$ has the role of the free Cauchy data, as it remains unconstrained by the equations of motion, meaning that $A_i^{(0)} = A_i^{(0)}(u, z, \bar{z})$ is arbitrary on $\mathcal{I}^+$. We assume that, as we approach $\mathcal{I}^+_-$, $A_i^{(0)}$ tends to a well-defined function of $(z, \bar{z})$ as

$$A_i^{(0)}(u, z, \bar{z}) = \sum_{m=-\infty}^{0} u^m A_i^{(0,m)}(z, \bar{z}). \tag{3.26}$$

Although other expansions[3] are possible, the structure in (3.26) simplifies the discussion of renormalization. In particular, this assumption has consequences on the expansion of the presymplectic potential in (3.127) and the consequent renormalization procedure.

---

[2] See section 3.2.4 for further comments.

[3] In [26] it is assumed that the term $A_i^{(0)}$ satisfies the so-called tree-level assumption, namely

$$A_i^{(0)} = A_i^{(0,0)} + \mathcal{O}(|u|^{-\infty}) \tag{3.27}$$

in the limit $u \to \pm\infty$, meaning that $\lim_{u \to \pm\infty} u^n \partial_u A_i = 0 \;\; \forall\, n > 0$. However, this assumption is not required for the purposes of this work and therefore we relax it. See the conclusions for further comments.

We can also rearrange the expansion (3.24) as

$$\mathcal{A}_\mu = \sum_n \frac{\mathcal{A}_\mu^{(n)}}{r^n} + \sum_{n'} \hat{\mathcal{A}}_\mu^{(n')} \frac{\log r}{r^{n'}}, \tag{3.28}$$

where it is important to remark that the second series is entirely pure gauge, while the first series receives contribution from both $A_\mu$ and $a$.

Let us notice that, by allowing for overleading pure gauge terms, we are extending the space of asymptotic field configuration $\Gamma$ and therefore we are generalising the presymplectic potential, since the variation $\delta\mathcal{A}_\mu$ is within the space $\Gamma$. This turns out to be crucial for the discussion of charge renormalization.

## 3.2.2 Field strength

As shown in (3.12), the component $F_{ur} = \mathcal{O}(r^{-2})$ enters in the asymptotic electric charge definition. Specifically, the assumption

$$F_{ur} = \sum_{n=2}^{\infty} \frac{F_{ur}^{(n)}}{r^n} \tag{3.29}$$

implies that only the first series of (3.24) contributes to it. Using the equations of motion, we find

$$\begin{aligned} F_{ur}^{(2)} &= \mathcal{D} \cdot A^{(0)} \\ F_{ur}^{(n)} &= -\frac{\mathcal{D}^2}{n-2} A_r^{(n-1)} - \mathcal{D} \cdot A^{(n-2)} \qquad (n > 2), \end{aligned} \tag{3.30}$$

as well as

$$\begin{aligned} \partial_u F_{ur}^{(2)} &= \partial_u \mathcal{D} \cdot A^{(0)} \\ \partial_u F_{ur}^{(3)} &= -\frac{1}{2} \mathcal{D}^2 \mathcal{D} \cdot A^{(0)} \\ \partial_u F_{ur}^{(n)} &= \frac{\mathcal{D}^2}{2} \left( \frac{\mathcal{D}^2}{(n-2)(n-3)} + 1 \right) A_r^{(n-2)} + \left( \frac{\mathcal{D}^2 + (n-2)(n-3)}{2(n-2)} \right) \mathcal{D} \cdot A^{(n-3)} \qquad (n > 3). \end{aligned} \tag{3.31}$$

We can perform a further gauge fixing and express $F_{ur}^{(n)}$ only in terms of $\mathcal{D} \cdot A$. This is discussed in the rest of the section and in detail in section A.2.

These equations, together with the assumption (3.26), fixes the $u-$dependence around $u \to -\infty$ to be

$$F_{ur} = \sum_{n=2}^{\infty} \frac{1}{r^n} \left[ \sum_{m=0}^{n-2} u^m F_{ur}^{(n,m)}(z,\bar{z}) + f_{ur}^{(n)}(u,z,\bar{z}) \right] \tag{3.32}$$

where $f_{ur}^{(n)}(u,z,\bar{z}) \to 0$ for $u \to -\infty$ and, consistently with (3.26), has a $u-$power expansion. On $\mathcal{I}_+^+$ we assume that the field strength goes to zero because we are not considering massive charged particles.

The dual tensor $\tilde{F}_{ur} := (\star F)_{ur}$ is defined as

$$\tilde{F}_{ur} = \varepsilon_{urij} F^{ij} = \frac{\varepsilon^{ij}}{r^2} F_{ij}, \tag{3.33}$$

where $\varepsilon^{ij}$ is the antisymmetric Levi-Civita tensor on the two-sphere.

In order to analyse it, let us introduce a particularly useful splitting which generally valid for every two-sphere vector

$$A_i = \mathcal{D}_i A + \varepsilon_{ij} \mathcal{D}^j A'. \tag{3.34}$$

This is the Helmholtz decomposition on the two-sphere which, in stereographic coordinates, takes the simple form

$$\begin{aligned} A_z &= D_z(A + A'), \\ A_{\bar{z}} &= D_{\bar{z}}(A - A'). \end{aligned} \tag{3.35}$$

This splitting is particularly well-suited for analysing asymptotic symmetries since these two components decouple in the asymptotic charges. Indeed, the tensor $F_{ur}$, that appear in the asymptotic electric charge definition, depends on $\mathcal{D} \cdot A = \mathcal{D}^2 A$, while the dual tensor, which enters in the magnetic charge, depends exclusively on $\varepsilon^{ij} F_{ij} = 2\mathcal{D}^2 A'$.

### 3.2.3 Asymptotic symmetries

#### $\mathcal{O}(1)$ asymptotic symmetries

Working in Lorenz gauge, the residual gauge parameter satisfies the wave equation

$$\Box \epsilon = 0. \tag{3.36}$$

In order to find a $\mathcal{O}(1)$ term with arbitrary dependence from angular variables, a standard power expansion is not sufficient[4]. A solution is to use a polyhomogeneous expansion

$$\epsilon = \sum_{n=0}^{\infty} \frac{\epsilon^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\epsilon}^{(n)} \frac{\log r}{r^n}. \tag{3.37}$$

This parameter does not preserve the falloffs (3.25). We thus modify them to be

$$\mathcal{A}_u = \mathcal{O}(\tfrac{\log r}{r}), \qquad \mathcal{A}_r = \mathcal{O}(\tfrac{\log r}{r^2}), \qquad \mathcal{A}_i = \mathcal{O}(1), \tag{3.38}$$

where the overleading terms respect to (3.25) are understood as pure gauge terms. Note that, if this were not the case, we would have found $F_{ri} = \mathcal{O}(\frac{\log r}{r^2})$, overleading respect to (3.19), or should have imposed a constraint[5] on $\hat{\mathcal{A}}_r^{(2)}$ and $\hat{\mathcal{A}}_i^{(1)}$, namely $\hat{\mathcal{A}}_i^{(1)} = -\mathcal{D}_i \hat{\mathcal{A}}_r^{(2)}$. According to the notation (3.24), the falloffs (3.38) consist in

$$A_u = \mathcal{O}(r^{-1}), \qquad A_r = \mathcal{O}(r^{-2}) \qquad A_i = \mathcal{O}(1), \qquad a = \mathcal{O}\left(1\right), \tag{3.39}$$

with $a$ polyhomogeneously expanded as in (3.24).

The equation (3.36) for a parameter of the form (3.37) implies for the first orders that

$$\begin{aligned} \epsilon^{(0)} &= \epsilon_c^{(0)}(z,\bar{z}), \\ \hat{\epsilon}^{(1)} &= \frac{1}{2} u \mathcal{D}^2 \epsilon^{(0)} + \hat{\epsilon}_c^{(1)}(z,\bar{z}), \\ \epsilon^{(1)} &= \epsilon^{(1)}(u,z,\bar{z}), \end{aligned} \tag{3.40}$$

where the subscript "$c$ " denotes a $u-$integration constant, that appears in general at every order, while $\epsilon^{(1)}$ is completely free. All the functions that appear with a subscript "$c$ " depend only on $z$ and $\bar{z}$. In particular, the $u-$dependent part of $\epsilon^{(1)}$ can be used to fix $A_u^{(1)} = 0$, leaving only $\epsilon_c^{(1)}(z,\bar{z})$ and all the $\epsilon_c^{(n)}(z,\bar{z})$ for $n \geq 1$ can be used to gauge away $u-$independent components of $A_r^{(n)}$. In particular, we use the various $\epsilon_c^{(n)}$ to reach the condition

$$A_r^{(n,0)}(z,\bar{z}) = 0, \tag{3.41}$$

[4]In fact, the wave equation for a parameter $\epsilon = \sum_{n=0}^{\infty} \frac{\epsilon^{(n)}}{r^n}$ implies $\mathcal{D}^2 \epsilon^{(0)}(z,\bar{z}) = 0$, thus showing that only constants are $\epsilon^{(0)}$ are allowed. In $D > 4$ we do not even find constants [50].

[5]To be precise, this constraint should be imposed only on the $u-$independent part of $\hat{\mathcal{A}}_r^{(2)}$ and $\hat{\mathcal{A}}_i^{(1)}$, because for the $u-$dependent part it comes from the equations of motion.

at $u \to -\infty$, meaning that we are cancelling the $u-$independent part at $\mathcal{I}^+_-$. Comments are added in section A.2. Having used all the subleading terms in $\epsilon$, we have reached a complete gauge fixing.

The parameter (3.40) preserve the falloffs (3.38) and, as we discuss below, it corresponds to the standard $\mathcal{O}(1)$ asymptotic charge.

### $\mathcal{O}(r)$ asymptotic symmetries

We are now interested in a $\mathcal{O}(r)$ asymptotic parameter which, however, would violate the falloffs (3.38). Therefore, we consider

$$\mathcal{A}_r = \mathcal{O}(1), \quad \mathcal{A}_u = \mathcal{O}(1), \quad \mathcal{A}_i = \mathcal{O}(r), \tag{3.42}$$

where all the leading terms are now in pure gauge sectors, meaning that we consider $a = \mathcal{O}(r)$ respect to (3.24). The parameter takes the form

$$\epsilon = \sum_{n=-1}^{\infty} \frac{\epsilon^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\epsilon}^{(n)} \frac{\log r}{r^n}, \tag{3.43}$$

where we emphasize that the logarithmic series starts again from $n = 1$. In fact, the second series is necessary to find a $\mathcal{O}(r^0)$ term with arbitrary angular dependence, but then the $\mathcal{O}(r)$ term can have it without requiring additional terms[6]

For what concerns the gauge parameter, the wave equation implies

$$\begin{aligned}
\epsilon^{(-1)} &= \epsilon_c^{(-1)}(z,\bar{z}), \\
\epsilon^{(0)} &= u\frac{\mathcal{D}^2+2}{2}\epsilon_c^{(-1)} + \epsilon_c^{(0)}(z,\bar{z}), \\
\hat{\epsilon}^{(1)} &= u^2\frac{\mathcal{D}^2-2}{8}\mathcal{D}^2\epsilon_c^{(-1)} + \frac{1}{2}u\,\mathcal{D}^2\epsilon_c^{(0)} + \hat{\epsilon}_c^{(1)}(z,\bar{z}), \\
\epsilon^{(1)} &= \epsilon^{(1)}(u,z,\bar{z}).
\end{aligned} \tag{3.44}$$

The same comments as before on the subleading integrations $\epsilon^{(n)}, \hat{\epsilon}^{(n)}$ with $n \geq 1$ can be repeated here. However, in this case we find two distinct arbitrary functions of the angular variables, $\epsilon_c^{(-1)}(z,\bar{z})$ and $\epsilon_c^{(0)}(z,\bar{z})$, to which we

---

[6]We could, in principle, consider a $\log r$ term in the series (3.43), but in order for it to have a non-trivial angular dependence we should add a third series

$$\sum_{n=1}^{\infty} \tilde{\epsilon}^{(n)} \frac{\log^2 r}{r^n}$$

.

can associate two independent contributions to the asymptotic charge, which correspond to the leading and the sub-leading soft photon theorem, as was derived in [25].

#### $\mathcal{O}\left(r^N\right)$ asymptotic symmetries

As understood, we can in principle generalize the falloffs to an arbitrary $r^N$ order ($N > 0$) by taking $a = \mathcal{O}(r^N)$. Consequently, we consider $\mathcal{O}(r^N)$ gauge parameter, with expansion

$$\epsilon = \sum_{n=-N}^{\infty} \frac{\epsilon^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\epsilon}^{(n)} \frac{\log r}{r^n}, \tag{3.45}$$

where the second series starts again from $n = 1$.

The wave equation implies that, for every $0 \leq n \leq N$

$$\epsilon^{(-n)} = \sum_{m=0}^{N-n} u^m \epsilon^{(-n,m)} \tag{3.46}$$

where the $u$-independent part $\epsilon^{(-n,0)}$ is unconstrained at every order $n$, since it comes from a $u-$integration constant. Note that, although we used the same notation of (3.32), this expansion is valid for every value of $u$.

Consistently with the previous notation, we define

$$\epsilon_c^{(-n)} := \epsilon^{(-n,0)}. \tag{3.47}$$

Given all the $\epsilon_c^{(-n)}$ for $0 \leq n \leq N$, the wave equation completely determines the structure of the $u-$dependent terms to be

$$\epsilon^{(-n,m)} = \frac{1}{2^m} \frac{n!}{m!(n+m)!} \left[ \prod_{i=1}^{m} \left( \mathcal{D}^2 + (i+n)(i+n+1) \right) \right] \epsilon_c^{(-n-m)}, \tag{3.48}$$

where $\epsilon_c^{(-n-m)} = 0$ if $n+m > N$. Hence, there are $N$ independent free functions of the angular variables which give contributions to the asymptotic charges.

### 3.2.4 Charges

#### Leading and subleading electric charges

The asymptotic electric charge is formally determined by

$$Q_e = \lim_{t\to\infty} -\int_{\Sigma_t} dudzd\bar{z}\,\gamma_{z\bar{z}}(\partial_u - \partial_r)(r^2 F_{ur}\epsilon) \tag{3.49}$$

and taking the finite part. The integral is ill-defined because there are both $t-$ and $u-$divergences. However, these divergences are understood as ambiguities of the presymplectic potential and are analysed in detail in section 3.4.

We start defining

$$\rho := r^2 F_{ur}\,\epsilon. \tag{3.50}$$

Note that studying the $\mathcal{O}(t^0)$ term is not equivalent to studying the $\mathcal{O}(r^0)$ term in general, except when considering a $\mathcal{O}(r^0)$ asymptotic parameter. Indeed, for a $\mathcal{O}(r)$ parameter, we have

$$\rho = r\left[F^{(2)}_{ur}\epsilon^{(-1)}\right] + \left[F^{(3)}_{ur}\epsilon^{(-1)} + F^{(2)}_{ur}\epsilon^{(0)}\right] + \dots \tag{3.51}$$

which implies

$$(\partial_u - \partial_r)\rho = r\left[\epsilon^{(-1)}\partial_u F^{(2)}_{ur}\right] + \left[\epsilon^{(-1)}\partial_u F^{(3)}_{ur} + \partial_u(F^{(2)}_{ur}\epsilon^{(0)}) - F^{(2)}_{ur}\epsilon^{(-1)}\right] + \dots \tag{3.52}$$

while, substituting $r = t - u$, we find

$$(\partial_u - \partial_r)\rho = t\left[\epsilon^{(-1)}\partial_u F^{(2)}_{ur}\right] + \left[\epsilon^{(-1)}\partial_u F^{(3)}_{ur} + \partial_u(F^{(2)}_{ur}\epsilon^{(0)}) - \epsilon^{(-1)}\partial_u\left(u\,F^{(2)}_{ur}\right)\right] + \dots\,. \tag{3.53}$$

Using the expression for $\epsilon^{(0)}$ in (3.44) as well as those for $F_{ur}$ (3.30)-(3.31) and, substituting $r = t - u$, we find up to some integration by parts at $\mathcal{O}(t^0)$

$$Q_e = -\int dudzd\bar{z}\gamma_{z\bar{z}}\left(\frac{1}{2}u\mathcal{D}^2\epsilon^{(-1)}_c\partial_u\mathcal{D}\cdot A^{(0)} + \epsilon^{(0)}_c\partial_u\mathcal{D}\cdot A^{(0)}\right) = Q^{(-1)}_e + Q^{(0)}_e \tag{3.54}$$

which matches the soft part of [25, 26]. In particular, the charge $Q^{(-1)}$ is the one derived in the context of the subleading soft photon theorem in [87], thus indicating that the $\mathcal{O}(t^0)$ term is the correct one to consider. This approach differs, for instance, from the one proposed in [140] where the charge corresponding to the subleading soft photon theorem is not overleading as our $Q^{(-1)}$ but the first subleading.

### $\mathcal{O}\left(r^N\right)$ charges

Next, we determine the $\mathcal{O}(r^N)$ charge. For a $\mathcal{O}(r^N)$ parameter, $\rho$, defined in (3.50), has the following expansion

$$\rho = \sum_{n=-N}^{\infty} \frac{\rho^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\rho}^{(n)} \frac{\log r}{r^n}. \tag{3.55}$$

We focus on the part containing positive powers of $r$, namely the first series with $-N \leq n \leq 0$, which we rewrite as

$$\sum_{n=0}^{N} r^n \rho^{(-n)} \tag{3.56}$$

with

$$\rho^{(-n)} = \sum_{k=0}^{N-n} F_{ur}^{(2+k)} \epsilon^{(-n-k)}. \tag{3.57}$$

To evaluate the charge we need to integrate

$$(\partial_u - \partial_r)(r^n \rho^{(-n)}) = r^n \partial_u \rho^{(-n)} - n r^{(n-1)} \rho^{(-n)} \tag{3.58}$$

so that the $\mathcal{O}(t^0)$ terms reads

$$\sum_{n=0}^{N} \Big( (-u)^n \partial_u \rho^{(-n)} - n(-u)^{(n-1)} \rho^{(-n)} \Big) = \sum_{n=0}^{N} \left( \partial_u ((-u)^n \rho^{(-n)}) \right) \tag{3.59}$$

Hence, the charge at $\mathcal{O}(t^0)$ reads

$$Q_e = -\sum_{n=0}^{N} \int_{\Sigma_t} du dz d\bar{z} \big( \partial_u ((-u)^n \rho^{(-n)}) \big). \tag{3.60}$$

As one can observe, there are divergences from the $du$ integral, coming from powers in $u$ of both $\epsilon$ and $F_{ur}$. These divergences can be renormalized as well, leaving us with the $\mathcal{O}(u^0)$ term. We can now perform the limit for $t \to \infty$ and find

$$Q_{e,\,\text{ren}} = \sum_{n=0}^{N} Q_{e,\,\text{ren}}^{(-n)} = \sum_{n=0}^{N} \int_{\mathcal{I}^+_-} dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon_c^{(-n)} F_{ur}^{(2+n,0)}\,, \tag{3.61}$$

where the meaning of $F_{ur}^{(2+n,0)}$ is explained in (3.32). The renormalization discussion is presented in detail in section 3.4 and the results match the ones discussed in [26].

Finally, we can express everything as a function of the free data $A^{(0)}$ by means of the residual gauge fixing and, using the expressions derived in section A.2, we find

$$
\begin{aligned}
Q^{(0)}_{e,\,\mathrm{ren}} &= \int_{\mathcal{I}^+_-} dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon^{(0)}_c \mathcal{D}^2 A^{(0,0)}\,. \\
Q^{(n)}_{e,\,\mathrm{ren}} &= -\int_{\mathcal{I}^+_-} dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon^{(-n)}_c \mathcal{D}^2 A^{(n,0)}\,. \qquad (n \geq 1)
\end{aligned}
\tag{3.62}
$$

### Global charges in Lorenz gauge

Even though it is not strictly related to our purposes, let us add a few comments to the subject of the global charges, which should appear in

$$
Q^{(0)}_e = \int dz d\bar{z} \gamma_{z\bar{z}} \epsilon^{(0)}_c F^{(2)}_{ur} \tag{3.63}
$$

as the case of $\epsilon^{(0)}_c$ being a constant parameter. However, in Lorenz gauge we have $F^{(2)}_{ur} = \mathcal{D} \cdot A^{(0)}$ (also in the presence of sources[7]), and therefore we find $Q^{(0)}_{\mathrm{const}} = 0$, which would imply the absence of the standard electric charge.

This problem is solved by noting that the field strength falloffs, consistently with the equations of motion, allow for a

$$
A_r = \mathcal{O}\left(\frac{1}{r}\right) \tag{3.64}
$$

with a constant $A^{(1)}_r$. The correct expression for $F^{(2)}_{ur}$, also in the presence of sources, would be

$$
F^{(2)}_{ur} = \mathcal{D} \cdot A^{(0)} + A^{(1)}_r, \tag{3.65}
$$

thus providing a term in (3.63) that survives in the constant $\epsilon^{(0)}_c$ case.

Neglecting the constant $A^{(1)}_r$ has no consequence in the analysis of asymptotic symmetries, characterized by non-trivial $\epsilon^{(n)}_c$, but it is crucial to find consistent global charges. To our knowledge, this detail has been first pointed out in [34].

[7] The condition $F^{(2)}_{ur} = \mathcal{D} \cdot A^{(0)}$ can be derived by the Lorenz gauge condition and therefore it is unaltered in the presence of sources. See the Appendix A for the order-by-order equations.

## 3.3 Two-form

In $D = 4$, a two-form gauge field $\mathcal{B}_{\mu\nu}$ is dual, on-shell, to a scalar field $\phi$ through

$$H = *d\phi, \tag{3.66}$$

where we introduced the field strength $H = d\mathcal{B}$. By assuming the standard behaviour of a scalar field approaching null infinity

$$\phi = \frac{\phi^{(1)}(u, z, \bar{z})}{r} + \dots\,, \tag{3.67}$$

the duality induces the field strength falloffs, namely

$$H_{uri} = \mathcal{O}(r^{-1}), \qquad H_{uij} = \mathcal{O}(r), \qquad H_{rij} = \mathcal{O}(1)\,. \tag{3.68}$$

Accordingly to the setup chosen for the spin-one case, the field strength admits a power expansion, thus allowing for logarithms in the gauge field components only in pure gauge sectors. We work again in Lorenz gauge, namely

$$\nabla^{\mu}\mathcal{B}_{\mu\nu} = 0, \tag{3.69}$$

so that the residual gauge parameter obeys to the Maxwell equations

$$\Box\lambda_{\mu} - \nabla_{\mu}\nabla\cdot\lambda = 0. \tag{3.70}$$

Furthermore, $\lambda_{\mu}$ itself possesses a so-called gauge-for-gauge redundancy, namely

$$\delta_{\epsilon}\lambda_{\mu} = \partial_{\mu}\epsilon\,, \tag{3.71}$$

with $\epsilon$ being a scalar function. We use $\epsilon$ to reach the Lorenz gauge for the parameter $\lambda_{\mu}$, so that it obeys to

$$\Box\lambda_{\mu} = 0, \qquad \nabla\cdot\lambda = 0, \tag{3.72}$$

and the residual gauge-for-gauge parameter obeys to

$$\Box\epsilon = 0. \tag{3.73}$$

This equaton is analysed in detail in Appendix A.

### 3.3.1 Falloffs and field strength

The two-form analogue of (3.24) is

$$\mathcal{B}_{\mu\nu} = B_{\mu\nu} + \partial_{[\mu} b_{\nu]} = \sum_{n,n_1,n_2} \left[ \frac{B^{(n)}_{\mu\nu}}{r^n} + \partial_{[\mu} \left( b^{(n_1)}_{\nu]} \frac{1}{r^{n_1}} + \hat{b}^{(n_2)}_{\nu]} \frac{\log r}{r^{n_2}} \right) \right] , \tag{3.74}$$

where the pure gauge part has been factored out, so that only the first series is relevant to the field strength.

The field strength falloffs (3.68), together with the ansatz about the power expansion, the equations of motion and the Lorenz gauge condition, determine the falloffs of the first series to be

$$B_{ij} = \mathcal{O}(r), \qquad B_{ui} = \mathcal{O}(1), \qquad B_{ri} = \mathcal{O}(r^{-1}), \qquad B_{ur} = \mathcal{O}(r^{-2})\,. \tag{3.75}$$

In particular, $B^{(-1)}_{ij}(u, z, \bar{z})$ has the role of the free data and we assume again that it approaches a well-defined function in the limit $u \to -\infty$ and admits a $u-$power expansion near $\mathcal{I}^+_-$ as in (3.26).

We can also write (3.74) as

$$\mathcal{B}_{\mu\nu} = \sum \frac{\mathcal{B}^{(n)}_{\mu\nu}}{r^n} + \sum \hat{\mathcal{B}}^{(n)}_{\mu\nu} \frac{\log r}{r^n}\,. \tag{3.76}$$

As a consequence of the previous discussion and the equations of motion, we find that the field strength approaches $\mathcal{I}^+_-$ as

$$H_{uri} = \sum_{n=1}^{\infty} \frac{1}{r^n} \left[ \sum_{m=0}^{n-1} u^m H^{(n,m)}_{uri} + h^{(n)}_{uri}(u, z, \bar{z}) \right] , \tag{3.77}$$

where $h^{(n)}_{uri} \to 0$ for $u \to -\infty$.

In order to discuss asymptotic symmetries we apply the splitting previously used for $A_i$ to the parameter component $\lambda_i$, with components

$$\lambda_i = \mathcal{D}_i \lambda + \varepsilon_{ij} \mathcal{D}^j \lambda', \tag{3.78}$$

and expansion coefficients $\lambda^{(n)}, \hat{\lambda}^{(n)}$ and $\lambda'^{(n)}, \hat{\lambda}'^{(n)}$ . By looking at (3.17), we observe that only the second component can contribute to the asymptotic charge, since

$$\mathcal{D}^i H_{uri} = 0 \tag{3.79}$$

as a consequence of the Bianchi identity $\nabla_{[i} H_{urj]} = 0$. This can be understood by noting how the term $\mathcal{D}_i \lambda$ in $\lambda_i$ resembles a gauge-for-gauge redundancy.

Similar expansions can be proposed for the components

$$\begin{aligned} B_{ri} &= \mathcal{D}_i B_r + \varepsilon_{ij}\mathcal{D}^j B'_r \\ B_{ui} &= \mathcal{D}_i B_u + \varepsilon_{ij}\mathcal{D}^j B'_u, \end{aligned} \tag{3.80}$$

while, for the free data $B_{ij}$ we write

$$B_{ij} = \varepsilon_{ij} B. \tag{3.81}$$

Under a gauge transformation parametrised by $\lambda, \lambda'$, we have

$$\delta_\lambda B = 0, \qquad \delta_{\lambda'} B = \mathcal{D}^2 \lambda', \tag{3.82}$$

which further clarifies the different roles played by the two components of the splitting (3.78).

### 3.3.2 Asymptotic symmetries

Given the expression (3.17), we observe that to find a non-vanishing charge, $\lambda_i$ has to fall off at most as $\mathcal{O}(r)$, because of (3.68). To find a non-trivial $\mathcal{O}(r)$ term in $\lambda'$, it must admit a polyhomogeneous expansion such as

$$\lambda' = \sum_{n=-(N+1)}^{\infty} \frac{\lambda'^{(n)}}{r^n} + \sum_{n=0}^{\infty} \hat{\lambda}'^{(n)} \frac{\log r}{r^n}, \tag{3.83}$$

with $N \geq 0$. In particular, the second series is necessary to have a nontrivial $\lambda'^{(-1)}$.

Furthermore, we observe how the splitting (3.78) is particularly useful when working in Lorenz gauge. Indeed, the parameter, obeying to (3.72), completely decouples after the splitting. As detailed in Appendix A, the equations $\Box\lambda_u = 0$, $\Box\lambda_r = 0$ and $\nabla \cdot \lambda = 0$ involve only the divergence of $\lambda_i$ and therefore only $\lambda$. At the same time, the equation for $\Box\lambda_i$ decouples as well and, in particular, the order-by-order equations for $\lambda'$ are

$$\begin{cases} -2n\partial_u \lambda'^{(n)} + 2\partial_u \hat{\lambda}'^{(n)} = [\mathcal{D}^2 + n(n-1)]\lambda'^{(n-1)} - 2(n-1)\hat{\lambda}'^{(n-1)}, \\ -2n\partial_u \hat{\lambda}'^{(n)} = [\mathcal{D}^2 + n(n-1)]\hat{\lambda}'^{(n-1)} \end{cases} \tag{3.84}$$

where $\hat{\lambda}^{(n)} = 0$ for $n < 0$ in our expansion. This system is quite similar to the one for a scalar obeying the wave equation, up to a redefinition of the orders due to the fact that we are studying angular components $\lambda_i$[8]. In particular, it is easy to check that, without a $\hat{\lambda}'^{(0)}$, the system (3.84) leads to

$$\mathcal{D}^2 \lambda'^{(-1)} = 0\,, \tag{3.85}$$

thus proving we cannot find asymptotic symmetries in Lorenz gauge without a polyhomogeneous expansion.

### $\mathcal{O}(1)$ asymptotic symmetries

While it would be sufficient to consider only $\lambda'$, as clarified earlier, we start by briefly reviewing the result [34], where the $\mathcal{O}(1)$ of a two-form asymptotic symmetries are understood as a component of the double-copy supertranslations. To this end, we consider

$$\mathcal{B}_{ur} = \mathcal{O}(r^{-1}), \qquad \mathcal{B}_{ui} = \mathcal{O}(\log r), \qquad \mathcal{B}_{ri} = \mathcal{O}(1), \qquad \mathcal{B}_{ij} = \mathcal{O}(r), \tag{3.86}$$

where the overleading terms respect to (3.75) are understood as pure gauge. Relative to the writing (3.74), the falloffs (3.86) mean

$$\begin{aligned} &B_{ij} = \mathcal{O}(r), \qquad &&B_{ui} = \mathcal{O}(1), \qquad &&B_{ri} = \mathcal{O}(r^{-1}), \qquad &&B_{ur} = \mathcal{O}(r^{-2}), \\ &b_u = \mathcal{O}(1), &&b_r = \mathcal{O}(1), &&b_i = \mathcal{O}(r)\,. \end{aligned} \tag{3.87}$$

Note in fact that it is not possible to have a non-trivial $\mathcal{B}^{(1)}_{ur}$ and $\mathcal{B}^{(0)}_{ri}$ without having $\mathcal{B}_{ui} = \mathcal{O}(\log r)$, as observed in [34].

To find leading orders with a non-trivial angular dependence we require the gauge parameter to admit a polyhomogeneous expansion as well, in particular

$$\begin{aligned} \lambda_u &= \sum_{n=0}^{\infty} \frac{\lambda_u^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\lambda}_u^{(n)} \frac{\log r}{r^n}, \\ \lambda_r &= \sum_{n=0}^{\infty} \frac{\lambda_r^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\lambda}_r^{(n)} \frac{\log r}{r^n}, \\ \lambda_i &= \sum_{n=-1}^{\infty} \frac{\lambda_i^{(n)}}{r^n} + \sum_{n=0}^{\infty} \hat{\lambda}_i^{(n)} \frac{\log r}{r^n}\,. \end{aligned} \tag{3.88}$$

[8]Sometimes a different parametrization is chosen, so that a vector reads

$$\lambda = \lambda^u \partial_u + \lambda^r \partial_r + \frac{1}{r} \lambda^i \partial_i\,,$$

with the extra $\frac{1}{r}$ in front of the angular component. With this parametrization, the system would have been identical to the scalar one.

Equations (3.72) imply on the first terms that

$$\lambda_u^{(0)} = \lambda_u^{(0)}(z,\bar z), \quad \lambda_r^{(0)} = \lambda_r^{(0)}(z,\bar z), \quad \lambda_i^{(-1)} = \lambda_i^{(-1)}(z,\bar z), \quad \hat\lambda_r^{(1)} = \hat\lambda_r^{(1)}(z,\bar z) \tag{3.89}$$

and fix a relation between the leading orders, namely

$$\mathcal{D}^2\lambda_r^{(0)} + 2(\lambda_u^{(0)} - \lambda_r^{(0)}) - 2\mathcal{D}^i\lambda_i^{(-1)} = 0. \tag{3.90}$$

We parametrise the solutions to this equation as "supertranslations+something else", namely[9]

$$\lambda_r^{(0)} = -\mathcal{T}(z,\bar z), \quad \lambda_i^{(-1)} = -\mathcal{D}_i\mathcal{T} + \mathcal{R}_i(z,\bar z), \quad \lambda_u^{(0)} = -\frac{\mathcal{D}^2+2}{2}\mathcal{T} - \mathcal{D}^i\mathcal{R}_i. \tag{3.91}$$

The supertranslations part here resembles a gauge-for-gauge redundancy. Indeed it can be fixed away by means of a gauge-for-gauge symmetry of a scalar parameter with $\epsilon^{(-1)} = \mathcal{T}(z,\bar z)$. The other part, parametrised by $\mathcal{R}_i$, can be decomposed again in

$$\mathcal{R}_i = \mathcal{D}_i\mathcal{R} + \varepsilon_{ij}\mathcal{D}^j\mathcal{R}' \tag{3.92}$$

and only $\mathcal{R}'$ contributes to the charge, while the existence of $\mathcal{R}$ signals the possibility of performing a further residual gauge fixing. For more comments see section 5.5.2 .

### $O\left(r^N\right)$ asymptotic symmetries

Let us now discuss a $\mathcal{O}(r^N)$ asymptotic parameter, with the expansion (3.83). Using the splitting (3.78), the only contribution to the charge comes from the term $\lambda'$. In fact, to study the asymptotic symmetries of a two-form we can consistently choose

$$\lambda_u = 0, \qquad \lambda_r = 0, \qquad \lambda = 0. \tag{3.93}$$

With this choice we cannot appreciate the possible double-copy structure as in the previous discussion, but it is the simplest way to analyse the two-form asymptotic symmetries. This setup implies, for instance, that the falloffs of $B_{ur}, B_r, B_u$, where we used the notation introduced in (3.80), can be always

[9] Let us remind that indices are raised as

$$\lambda^{u\,(n)} = -\lambda_r^{(n)}, \quad \lambda^{r\,(n)} = \lambda_r^{(n)} - \lambda_u^{(n)}, \quad \lambda^{A\,(n)} = \gamma^{ij}\lambda_j^{(n+2)}$$

so that we recognise $\mathcal{T}(z,\bar z)$ as the conterparto of the function that parametrises supertranslations in the gravitational case.

preserved. The falloffs we need to generalise by means of pure gauge terms are only $B, B'_r, B'_u$. In particular, $\lambda'$ acts on these components as

$$[\delta B]^{(n)} = \mathcal{D}^2\lambda'^{(n)}, \quad [\delta B'_u]^{(n)} = \partial_u\lambda'^{(n)}, \quad [\delta B'_r]^{(n)} = -(n-1)\lambda'^{(n-1)} + \hat{\lambda}'^{(n-1)}$$
$$\widehat{[\delta B]}^{(n)} = \mathcal{D}^2\hat{\lambda}'^{(n)}, \quad \widehat{[\delta B'_u]}^{(n)} = \partial_u\hat{\lambda}'^{(n)}, \quad \widehat{[\delta B'_r]}^{(n)} = -(n-1)\hat{\lambda}'^{(n-1)} \tag{3.94}$$

The wave equation for $\lambda'$, which is reported order-by-order in (3.84), completely fixes the $u-$dependence on $\mathcal{I}^+$ being again of the form

$$\lambda'^{(-n)} = \sum_{m=0}^{N-n} u^m \lambda'^{(-n,m)}\,, \tag{3.95}$$

where we are now focusing on the $n \geq 1$ part, which gives non-vanishing contributions to the charge, and in particular

$$\lambda'^{(-n,m)} = \frac{1}{2^m}\frac{(n-1)!}{m!(n+m-1)!}\left[\prod_{i=1}^{m}\left(\mathcal{D}^2 + (i+n-1)(i+n)\right)\right]\lambda_c'^{(-n-m)}\,, \tag{3.96}$$

where the result is identical the case of the scalar wave equation except for a shift $n \to n-1$ due to the fact that the equations of motion of the angular component change the $r$-order (see Equation 8).

### 3.3.3 Charges

**Leading and subleading charges.**

We study the limit $t \to \infty$ of

$$-\int_{\Sigma_t} dudzd\bar{z}\gamma_{z\bar{z}}(\partial_u - \partial_r)(\gamma^{ij}H_{uri}\lambda_j)\,. \tag{3.97}$$

and consider the $\mathcal{O}(t^0)$ term. We rewrite the integral using the splitting (3.78) and integrating by parts as

$$\int_{\Sigma_t} dudzd\bar{z}\gamma_{z\bar{z}}(\partial_u - \partial_r)\rho_H\,, \tag{3.98}$$

where we define

$$\rho_H := \lambda'\varepsilon^{ij}\mathcal{D}_j H_{uri}\,. \tag{3.99}$$

We start considering the leading and subleading contributions to the asymptotic charge, with a $\lambda' = \mathcal{O}(r^2)$. In particular, we have

$$\begin{aligned}(\partial_u - \partial_r)\rho_H =& r\left[\lambda'^{(-2)}\varepsilon^{ij}\partial_u\mathcal{D}_j H^{(1)}_{uri}\right] + r^0\left[-\lambda'^{(-2)}\varepsilon^{ij}\mathcal{D}_j H^{(1)}_{uri}\right.\\ &\left.+\partial_u(\lambda'^{(-1)}\varepsilon^{ij}\mathcal{D}_j H^{(1)}_{uri}) + \lambda'^{(-2)}\varepsilon^{ij}\partial_u\mathcal{D}_j H^{(2)}_{uri}\right] + \dots\end{aligned} \tag{3.100}$$

and the $\mathcal{O}(t^0)$ term is given by

$$\partial_u\left(-u\lambda'^{(-2)}\varepsilon^{ij}\mathcal{D}_j H^{(1)}_{uri} + \lambda'^{(-1)}\varepsilon^{ij}\mathcal{D}_j H^{(1)}_{uri} + \lambda'^{(-2)}\varepsilon^{ij}\mathcal{D}_j H^{(2)}_{uri}\right), \tag{3.101}$$

while the term linear in $t$ is neglected by means of symplectic renormalization.

To evaluate this, we need the expressions

$$\begin{aligned}H^{(1)}_{uri} &= -\mathcal{D}^j B^{(-1)}_{ij}\\ \partial_u H^{(2)}_{uri} &= \frac{\mathcal{D}^2-1}{2}\mathcal{D}^j B^{(-1)}_{ij},\end{aligned} \tag{3.102}$$

and, by means of (3.81), we find

$$\begin{aligned}\varepsilon^{ij}\mathcal{D}_j H^{(1)}_{uri} &= \mathcal{D}^2 B^{(-1)},\\ \varepsilon^{ij}\partial_u\mathcal{D}_j H^{(1)}_{uri} &= -\frac{(\mathcal{D}^2)^2}{2}B^{(-1)}.\end{aligned} \tag{3.103}$$

Finally, using

$$\begin{aligned}\lambda^{(-2)} &= \lambda^{(-2)}_c(z,\bar z),\\ \lambda^{(-1)} &= u\frac{\mathcal{D}^2+2}{2}\lambda'^{(-2)}_c(z,\bar z) + \lambda'^{(-1)}_c(z,\bar z),\end{aligned} \tag{3.104}$$

we arrive to

$$Q_B = Q^{(0)}_B + Q^{(-1)}_B\,, \tag{3.105}$$

with

$$Q^{(0)}_B = \int dudzd\bar z\gamma_{z\bar z}\,\lambda'^{(-1)}_c\partial_u\mathcal{D}^2 B^{(-1)}, \tag{3.106a}$$

$$Q^{(-1)}_B = \int dudzd\bar z\gamma_{z\bar z}\frac{u}{2}\lambda'^{(-2)}_c\partial_u(\mathcal{D}^2)^2 B^{(-1)}. \tag{3.106b}$$

The charge $Q^{(0)}_B$ matches the one presented in [34]. It corresponds to the leading soft scalar theorem, originally derived in [37], as studied in [27,28]. The charge $Q^{(-1)}_B$, to which we refer as subleading, is presented here for the first time. Interestingly, it has a form identical to its electromagnetic counterpart

$Q_e^{(-1)}$ derived in (3.54) which, up to an integration by parts, reads

$$Q_e^{(-1)} = \int dudzd\bar{z}\gamma_{z\bar{z}}\frac{u}{2}\epsilon_c^{(-1)}\partial_u(\mathcal{D}^2)^2 A^{(0)} \tag{3.107}$$

where we used the splitting (3.34) to highlight the similarity between the two charges. Our guess is that the conservation of the charge (3.106b), up to the identification of $B^{(-1)}$ with the scalar mode $\phi^{(1)}$, might correspond to a subleading version of the soft scalar theorem presented in [37].

To our knowledge, however, no subleading formulation of the soft scalar theorem of [37] has been derived yet. Nonetheless, we expect the subleading soft scalar factor to have a similar structure to the photon and graviton analogues. This is motivated, for instance, by the structure of subleading soft factors for dilatons, derived in [141, 142].

Furthermore, we notice the proposal in (3.43) of [37] for the smeared version of the scalar charges is also motivated by the analogy with electromagnetism and gravity. In this sense, the natural proposal for a subleading version of the charge would be our $Q_B^{(-1)}$.

### $\boldsymbol{O\left(r^N\right)}$ charges.

Let us now consider the $\mathcal{O}(r^N)$ charges with arbitrary $N$ by repeating the same step of the previous case. We have

$$\rho_H = \sum_{n=-N}^{\infty} \frac{\rho_H^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\rho}^{(n)}\frac{\log r}{r^n}\,. \tag{3.108}$$

We focus on the part with $-N \leq n \leq 0$ with

$$\rho_H^{(-n)} = \sum_{k=0}^{N-n} \lambda'^{(-n-k-1)}\varepsilon^{ij}\mathcal{D}_j H_{uri}^{(1+k)}\,, \tag{3.109}$$

so that the $\mathcal{O}(t^0)$ term reads

$$\sum_{n=0}^{N} \partial_u\left((-u)^n\rho^{(-n)}\right), \tag{3.110}$$

with the charge

$$Q_B = \sum_{n=0}^{N}\int_{\Sigma_t} dudzd\bar{z}\gamma_{z\bar{z}}\,\partial_u\left((-u)^n\rho^{(-n)}\right). \tag{3.111}$$

There are again $u-$divergences, but they are understood in terms of ambiguities of the presymplectic potential. We are left only with the $u-$independent terms and a well-defined integral, so that we can perform the limit $t \to \infty$ and obtain

$$Q_{B,\,\text{ren}} = \sum_{n=0}^{N} Q_{B,\,\text{ren}}^{(-n)} = -\int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}}\, \lambda_c^{(-(n+1))} \varepsilon^{ij} \mathcal{D}_j H_{uri}^{(n+1,0)}. \tag{3.112}$$

The confrontation with the spin-one counterpart is straightforward: we find $N$ independent asymptotic charges, parametrised by $N$ arbitrary scalar functions of the angular variables, arising from $u-$integrations. Using the expressions obtained in section A.2 we find

$$\begin{aligned} Q_B^{(0)} &= -\int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-1)} \mathcal{D}^2 B^{(-1,0)}.\,, \\ Q_B^{(-1)} &= -\int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-2)} \frac{(\mathcal{D}^2)^2}{2} B_r^{\prime(1,0)}\,, \\ Q_B^{(-n)} &= \frac{n-2}{2} \int_{\mathcal{I}^+_-} dz d\bar{z} \gamma_{z\bar{z}} \lambda_c^{(-n)} \mathcal{D}^2 B^{\prime(n-1,0)} \qquad (n \geq 2)\,. \end{aligned} \tag{3.113}$$

## 3.4 Renormalization of the charge

Our study of asymptotic charges encounters two types of divergences: those arising from positive powers of $t$ and those resulting from the $du$ integration over positive powers of $u$. To address these issues, we employ a symplectic renormalization procedure, as proposed in [24].

The presymplectic potential admits two types of ambiguities that allow us to define an equivalence relation

$$\theta^\mu \sim \theta^\mu + \partial_\nu \Upsilon^{\mu\nu} + \delta \Xi^\mu, \tag{3.114}$$

with $\Upsilon^{\mu\nu}$ being antisymmetric. Our prescription is to use these ambiguities to cancel the divergences while leaving the rest unchanged.

We do not need to specify the theory to which we are referring, since all the expansions and assumptions apply to both one- and two-form cases, as we prove. This is a consequence of using the same setup for both theories. In $(u, t, z, \bar{z})$ coordinates, we have

$$\partial_t \theta^t \approx \delta \mathcal{L} - \partial_u \theta^u - \partial_i \theta^i, \tag{3.115}$$

where "$\approx$" denotes an on-shell relation. As $t \to +\infty$, $\theta^t$ has the general structure

$$\theta^t = \sum_{n=1}^{N} t^n \theta^t_{(n)}(u,z,\bar{z}) + \theta^t_{(0)}(u,t,z,\bar{z}), \tag{3.116}$$

where $\theta^t_{(0)}(u,t,z,\bar{z})$ is such that $\lim_{t\to\infty} \theta^t_{(0)}(u,t,z,\bar{z}) = \theta^t_{(0)}(u,z,\bar{z})$. In particular, logarithmic terms such as $t^{-n}\log t$ are inside $\theta^t_{(0)}(u,z,\bar{z})$ and, therefore, not relevant. Similar expansions are introduced for $\mathcal{L}, \theta^u, \theta^i$ with respective coefficients $\mathcal{L}_{(n)}, \theta^u_{(n)}, \theta^i_{(n)}$ .

Let us verify that the expansion (3.116) holds in both one- and two-form cases. For the spin-one case, we write

$$\theta^t_e = \theta^u_e + \theta^r_e = r^2\gamma_{z\bar{z}}(F^{u\nu} + F^{r\nu})\delta\mathcal{A}_\nu \,. \tag{3.117}$$

In our setup, the field strength admits a power expansion, while for the gauge field variation we have

$$\delta\mathcal{A}_\nu = \delta A_\nu - \partial_\nu \delta a, \tag{3.118}$$

where $A_\nu$ admits a power expansion as well, while for $a$ we chose to consider a polyhomogeneous expansion. In particular, we used

$$a = \sum_{n=-N}^{\infty} \frac{a^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{a}^{(n)} \frac{\log r}{r^n}, \tag{3.119}$$

with the logarithmic series starting at order $\mathcal{O}\left(\frac{\log r}{r}\right)$. Summing over indices we find

$$\theta^t_e = r^2\gamma_{z\bar{z}} \left[ F_{ur}(\delta\mathcal{A}_u - \delta\mathcal{A}_r) - \frac{\gamma^{ij}}{r^2} F_{uj}\delta\mathcal{A}_i \right] \tag{3.120}$$

and, using the field strength falloffs (3.19), we deduce that the first logarithmic term in $\theta^t_e$ is of order $\mathcal{O}\left(\frac{\log r}{r}\right)$ and therefore of order $\mathcal{O}\left(\frac{\log t}{t}\right)$. The highest order, instead, is of order $\mathcal{O}(r^N)$ and therefore $\mathcal{O}(t^N)$.

Let us repeat the analysis for the two-form case. We have

$$\begin{aligned} \theta^t_B &= r^2\gamma_{z\bar{z}}(H^{u\mu\nu} + H^{r\mu\nu})\delta\mathcal{B}_{\mu\nu} \\ &= \gamma_{z\bar{z}} \left[ \gamma^{ij} H_{uri}(\delta\mathcal{B}_{uj} - \delta\mathcal{B}_{rj}) - \frac{\gamma^{ik}\gamma^{jl}}{r^2} H_{uij}\delta\mathcal{B}_{kl} \right]. \end{aligned} \tag{3.121}$$

Field strength components admit a power expansion with falloffs (3.68). The two-form gauge field variation is

$$\delta\mathcal{B}_{\mu\nu} = \delta B_{\mu\nu} - \partial_{[\mu}\delta b_{\nu]} \tag{3.122}$$

with $b$ admitting a polyhomogeneous expansion of the form

$$\begin{aligned} b_u &= \sum_{n=-N}^{\infty} \frac{b_u^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{b}_u^{(n)} \frac{\log r}{r^n}, \\ b_r &= \sum_{n=-N}^{\infty} \frac{b_r^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{b}_r^{(n)} \frac{\log r}{r^n}, \\ b_i &= \sum_{n=-N-1}^{\infty} \frac{b_i^{(n)}}{r^n} + \sum_{n=0}^{\infty} \hat{b}_i^{(n)} \frac{\log r}{r^n}. \end{aligned} \tag{3.123}$$

Substituting these in the equation(3.121), we find exactly the expansion (3.116).

The equation (3.115) implies that every $\theta^t_{(n)}$ with $n > 0$ corresponds to an ambiguity, namely

$$\theta^t_{(n)} = \frac{1}{n}\left(\delta\mathcal{L}_{(n-1)} - \partial_u \theta^u_{(n-1)} - \partial_i \theta^i_{(n-1)}\right) . \tag{3.124}$$

Therefore, we can identify

$$\begin{aligned} \Upsilon^{tu} &= \sum_{n=1}^{N} \frac{1}{n} t^n \theta^u_{(n-1)} + \Upsilon^{tu}_{(0)} , \\ \Upsilon^{ti} &= \sum_{n=1}^{N} \frac{1}{n} t^n \theta^i_{(n-1)} + \Upsilon^{ti}_{(0)} , \\ \Xi^t &= \sum_{n=1}^{N} -\frac{1}{n} t^n \mathcal{L}_{(n-1)} + \Xi^t_{(0)} , \end{aligned} \tag{3.125}$$

where $\Upsilon^{tu}_{(0)}, \Upsilon^{tA}_{(0)}, \Xi^t_{(0)}$ are (free) functions with a well-defined limit for $t \to \infty$. We thus find that

$$\theta^t \sim \theta^t_{(0)} + \partial_u \Upsilon^{tu}_{(0)} + \partial_i \Upsilon^{ti}_{(0)} + \delta \Xi^t_{(0)}, \tag{3.126}$$

implying that only the $t^0$ order contributes to the charge in the limit $t \to \infty$.

Next, we address the $u-$divergences, which arise from the $du$ integration. This is managed using $\Upsilon^{t\mu}_{(0)}$. Near $u \to -\infty$, $\theta^t_{(0)}$ reads

$$\theta^t_{(0)} = \sum_{n=1}^{N} \theta^t_{(0,n)} u^{n-2} + \Theta^t_{(0)}(u, z, \bar{z}), \tag{3.127}$$

where we factored out the part that is not $u-$integrable, absent for $N = 0$, while $\Theta^t_{(0)}(u, z, \bar{z})$ admits a $u-$power expansion with leading order $\mathcal{O}(u^{-2})$.

To prove this expansion, we start from the setup used for $\mathcal{O}(1)$ asymptotic symmetries. The $\mathcal{O}(r^N)$ case would differ from the $\mathcal{O}(1)$ only for overleading powers of $u$. Let us begin with electromagnetism and determine the $\mathcal{O}(t^0)$ term. The only contribution is from the last term in (3.120) and reads

$$\theta^t_{e(0)} = \gamma_{z\bar{z}}\gamma^{ij}\partial_u A^{(0)}_i \delta\mathcal{A}^{(0)}_j , \tag{3.128}$$

where we substituted $F^{(0)}_{ui} = \partial_u A^{(0)}_i$. Near $\mathcal{I}^+_-$ we use the ansatz (3.26) for $A^{(0)}_i$ which, for $u \to -\infty$, means

$$\partial_u A^{(0)}_i = \mathcal{O}(u^{-2}) . \tag{3.129}$$

Consistently with this assumption, we find that $\delta\mathcal{A}^{(0)}_j$ tends to a well-defined function of the angles for $u \to -\infty$. Combining these two results, we find a $u-$integrable quantity for $\theta^t_{(0)}$ in the $N = 0$ case. Considering higher order terms with arbitrary $N$, the setup differs only for $a = \mathcal{O}(r^N)$, therefore we are adding overleading powers of $u$, finding non-integrable quantities as in the first series of the expansion (3.127).

Let us repeat the same computation for the two-form case in the $N = 0$ setup. We have

$$\theta^t_{B(0)} = \gamma_{z\bar{z}}\gamma^{ik}\gamma^{jl}\partial_u B^{(-1)}_{ij} \delta\mathcal{B}^{(-1)}_{kl} , \tag{3.130}$$

and near $\mathcal{I}^+_-$

$$\begin{aligned} B^{(-1)}_{ij} &= B^{(-1,0)}_{ij}(z,\bar{z}) + \mathcal{O}\left(u^{-1}\right) \\ \partial_u B^{(-1)}_{ij} &= \mathcal{O}\left(u^{-2}\right) . \end{aligned} \tag{3.131}$$

Hence, the presymplectic potential in the $N = 0$ case is integrable and order $\mathcal{O}(u^{-2})$. Modifying the setup to arbitrary $N$ adds positive powers of $u$.

We now use $\Upsilon^{t\mu}_{(0)}$ to cancel the first series. This can be achieved, for instance, employing these expansions

$$\Upsilon^{tu}_{(0)} = \sum_{n=3}^{N+1} u^{n-2}\Upsilon^{tu}_{(0,n)} , \qquad \Upsilon^{ti}_{(0)} = \frac{1}{u}\Upsilon^{ti}_{(0,-1)} \tag{3.132}$$

with

$$\Upsilon^{tu}_{(0,n)} = -\frac{\theta^t_{(0,n+1)}}{n-2} , \qquad \partial_i \Upsilon^{ti}_{(0,-1)} = -\theta^t_{(0,1)} . \tag{3.133}$$

This solution is not unique and depends on the specific structure of the renormalizing terms. Here we are assuming that $\Upsilon^{tu}_{(0)}$ has a $u-$power expansion, making it impossible to cancel a $\frac{1}{u}$ term with a $u-$derivative, which is why we used $\partial_i \Upsilon^{ti}_{(0,-1)}$.

Let us comment on the fact that the expansion (3.127) is related to the ansatz (3.26), specifically to the assumption that the free data approaches $\mathcal{I}^+_-$ with a $u-$power expansion. As mentioned in Equation 3, other expansions can be considered. This is the case of [26], for which the free data approaches a well defined function $A_i^{(0,0)}$ faster that any $u^{-n}$. This choice, however, would not alter equation (3.127), but only the falloff of the integrable quantity, preserving the validity of the renormalization process. A different expansion is proposed in [143], namely

$$A_i^{(0)} = A_i^{(0,0)} + \sum_{n=1}^{\infty} \hat{A}_i^{(0,n)} \frac{(\log u)^{n-1}}{u^n} + \dots\,, \tag{3.134}$$

which instead would modify the equation (3.127), and therefore would lead to some differences in the renormalization procedure, whose analysis we leave to a future work.

Ultimately, this process leaves us with well-defined quantities, thus showing that it is possible to renormalize the presymplectic potential to make the asymptotic charge finite while accommodating arbitrary powers of the asymptotic parameter.

The symplectic renormalization procedure is a powerful tool that allow us to study higher order asymptotic symmetries and deal with divergences. This approach suggests the possibility of discovering an infinite set of asymptotic charges, broadly meaning an infinity of infinitely-many conserved charges. It is crucial, however, to emphasize that we used a specific prescription: to cancel the divergences by means of ambiguities while leaving unaltered the finite parts, which enter in the definition of asymptotic charges. However, it is clear that there are residual ambiguities that might further constraint the number of physical asymptotic charges. We could find more general forms for $\Upsilon$ and $\Xi$ to cancel some finite parts of $\theta^t$. In particular, we did not use the residual term $\delta\Xi^t_{(0)}$.

## 3.5 Outlook

In this chapter, we analysed $\mathcal{O}(r^N)$ one- and two-form asymptotic symmetries. We began by revisiting electromagnetism, which was studied in detail in [26]. We focused only on the soft part of the charges, with the aim of confronting it with the two-form ones, and we chose a slightly different setup w.r.t. [26], mostly concerning to the expansion of the quantities involved. At any rate, our results are in agreement with those of [26], in all places where comparisons were possible.

We then moved to $\mathcal{O}(r^N)$ two-form asymptotic symmetries. Up to the symplectic renormalization procedure, we found $N$ independent asymptotic charges, with each charge parametrised by an arbitrary function of the angles. The result for the $\mathcal{O}(r)$ case is particularly interesting, since it is the charge one would naively expect for the subleading soft scalar theorem in the spirit of [37].

These results leave space for several future investigations. First, it would be interesting to repeat our analysis considering sources of various form, thus providing the hard part of the asymptotic charges. As mentioned, we think that this would require particular attention in the falloffs analysis.

Another possible generalisation involves exploring arbitrary spacetime dimension $D$ and generic $p-$forms, with the aim of exploring the duality between $p-$ and $(D-p-2)-$forms from the perspective of asymptotic symmetries.

Additionally, we seek to better understand the potential connection between the $\mathcal{O}(r^n)$ asymptotic charge and the sub$^n$-leading soft theorem. In this respect, the existing duality between the two-form and a scalar in $D = 4$ is particularly appealing, given the relative simplicity of scalar theories.

Finally, we aim to explore the limitations of the symplectic renormalization and the role of residual ambiguities. In our view, this issue is one of the most challenging aspects regarding asymptotic symmetries. A deeper understanding of this problem is crucial, as it would help to determine the physical significance of asymptotic symmetries at every order.

# Part II

# Double-copy asymptotic symmetries

# Chapter 4

# Basics on double copy

## 4.1 Double copy for amplitudes

### 4.1.1 Properties of Yang-Mills tree level amplitudes

We start by analysing the structure of tree level Feynman diagrams describing $n$ external $SU(N)$ gauge bosons, to which we refer as gluons, to show that it is possible to rearrange every Feynman diagram in terms of a new set of diagrams with only trivalent vertices. We explicitly show this result for a four point amplitude, but it can easily generalized to arbitrary $n$ by means of induction.

Given a Yang-Mills theory, described by the gauge-fixed Lagrangian

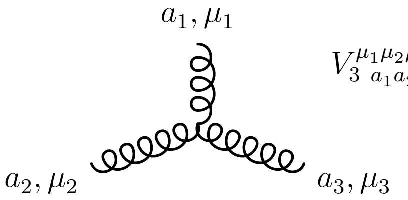
$$L = \frac{1}{2} A^a_\mu \Box A^{a\mu} - g f^{abc} \left( \partial_\mu A^a_\nu \right) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu}, \tag{4.1}$$

we have the following Feynman rules:

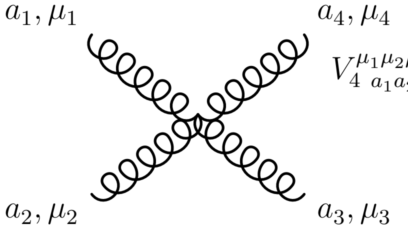
- propagator:


$$S^{\mu\nu}_{ab} = -i \frac{\delta_{ab} \eta^{\mu\nu}}{p^2}, \tag{4.2}$$

- interacting vertices



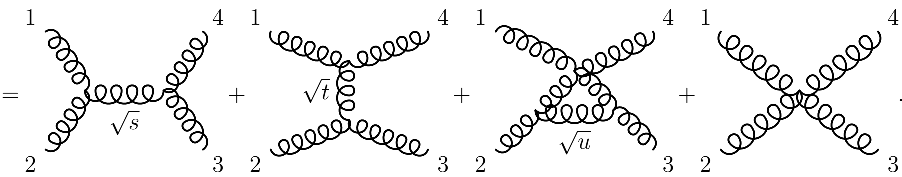

$$
\begin{aligned}
V_{3\ a_1a_2a_3}^{\mu_1\mu_2\mu_3} = {} & g f^{a_1a_2a_3}\big\{\eta^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + \eta^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + \\
& + \eta^{\mu_3\mu_1}(p_1-p_1)^{\mu_2}\big\},
\end{aligned}
\tag{4.3}
$$




$$
\begin{aligned}
V_{4\ a_1a_2a_3a_4}^{\mu_1\mu_2\mu_3\mu_4} = {} & -ig^2\big\{f^{a_1a_2e}f^{a_3a_4e}(\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}) + \\
& + f^{a_1a_4e}f^{a_2a_3e}(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}) + \\
& + f^{a_1a_3e}f^{a_4a_2e}(\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4})\big\}.
\end{aligned}
\tag{4.4}
$$

At tree level, the four gluons amplitude receives contributions from

$\mathcal{A}_4^{TL} =$




. (4.5)

While the first three diagrams have only trivalent vertices and describe the usual $s-$channel, $t-$channel and $u-$channel, after the relative Mandelstam variables, the last one corresponds to the contact term. Our goal is to prove that it is possible to decompose the contact term into contributions to the various channels. In our conventions, Mandelstam variables are are defined as

$$
\begin{cases}
s = (p_1+p_2)^2 = 2p_1\cdot p_2 = 2p_3\cdot p_4, \\
t = (p_1+p_4)^2 = 2p_1\cdot p_4 = 2p_2\cdot p_3, \\
u = (p_1+p_3)^2 = 2p_1\cdot p_3 = 2p_2\cdot p_4,
\end{cases}
\tag{4.6}
$$

since we parametrize all the momenta as incoming, so that the conservation reads $\sum_i p_i = 0$.

We evaluate the contributions of these diagrams as

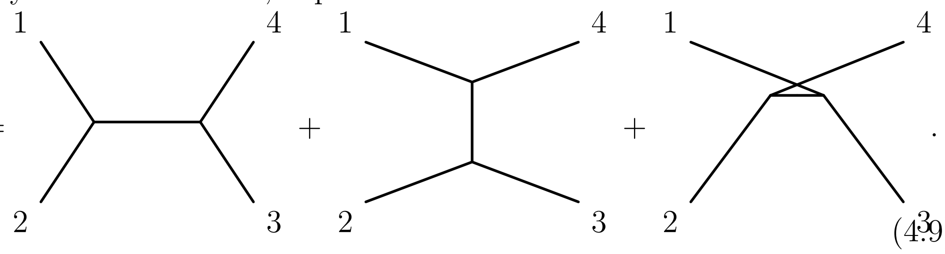
$$
\begin{aligned}
\text{s-channel:}\quad & -i\frac{g^2}{s} f^{a_1a_2e} f^{a_3a_4e} \left\{(\varepsilon^1\cdot\varepsilon^2)(p_1-p_2)^\alpha + 2\varepsilon^{2\alpha}(\varepsilon^1\cdot p_2) - 2\varepsilon^{1\alpha}(\varepsilon^2\cdot p_1)\right\} \\
& \times \left\{(\varepsilon^3\cdot\varepsilon^4)(p_3-p_4)^\alpha + 2\varepsilon^{4\alpha}(\varepsilon^3\cdot p_4) - 2\varepsilon^{3\alpha}(\varepsilon^4\cdot p_3)\right\}, \\
\text{t-channel:}\quad & -i\frac{g^2}{t} f^{a_1a_4e} f^{a_2a_3e} \left\{(\varepsilon^1\cdot\varepsilon^4)(p_1-p_4)^\alpha + 2\varepsilon^{4\alpha}(\varepsilon^1\cdot p_4) - 2\varepsilon^{1\alpha}(\varepsilon^4\cdot p_1)\right\} \\
& \times \left\{(\varepsilon^2\cdot\varepsilon^3)(p_2-p_3)^\alpha + 2\varepsilon^{3\alpha}(\varepsilon^2\cdot p_3) - 2\varepsilon^{2\alpha}(\varepsilon^3\cdot p_2)\right\}, \\
\text{u-channel:}\quad & -i\frac{g^2}{u} f^{a_1a_3e} f^{a_4a_2e} \left\{(\varepsilon^1\cdot\varepsilon^3)(p_1-p_3)^\alpha + 2\varepsilon^{3\alpha}(\varepsilon^1\cdot p_3) - 2\varepsilon^{1\alpha}(\varepsilon^3\cdot p_1)\right\} \\
& \times \left\{(\varepsilon^4\cdot\varepsilon^2)(p_4-p_2)^\alpha + 2\varepsilon^{2\alpha}(\varepsilon^4\cdot p_2) - 2\varepsilon^{4\alpha}(\varepsilon^2\cdot p_4)\right\},
\end{aligned}
\tag{4.7}
$$

while the contact term reads

$$
\begin{aligned}
\text{Contact:}\quad & -ig^2 f^{a_1a_2e} f^{a_3a_4e} \left[(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^2\cdot\varepsilon^4) - (\varepsilon^1\cdot\varepsilon^4)(\varepsilon^2\cdot\varepsilon^3)\right] \\
& -ig^2 f^{a_1a_4e} f^{a_2a_3e} \left[(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^3\cdot\varepsilon^4) - (\varepsilon^1\cdot\varepsilon^3)(\varepsilon^2\cdot\varepsilon^4)\right] \\
& -ig^2 f^{a_1a_3e} f^{a_4a_2e} \left[(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^2\cdot\varepsilon^3) - (\varepsilon^1\cdot\varepsilon^2)(\varepsilon^3\cdot\varepsilon^4)\right].
\end{aligned}
\tag{4.8}
$$

Interestingly, the term in the first line contains $g^2 f^{a_1a_2e} f^{a_3a_4e}$, which is exactly the same combination of structure constants that enters in the $s-$channel contribution. The same result holds for the other lines, corresponding to the other channels. Hence, we can split these contribution as "corrections" to the $s-, t-, u-$channels, rearranging the Feynman diagrams into new diagrams with only trivalent vertices, depicted as

$\mathcal{A}_4^{TL} =$




. (4.9)

This way, the 4-points amplitude at tree-level takes the form

$$\mathcal{A}_4^{TL} = g^2 \left\{\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\right\}, \tag{4.10}$$

where the $c_i$'s are

$$c_s = f^{a_1a_2e} f^{a_3a_4e}, \quad c_t = f^{a_1a_4e} f^{a_2a_3e}, \quad c_u = f^{a_1a_3e} f^{a_4a_2e}, \tag{4.11}$$

and are called **color factors**, since they contain only the structure constants, while the $n_i$'s are the **kinematics factors**:

$$\begin{aligned}
n_s =& - i\Big\{(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^3\cdot\varepsilon^4)(u-t) + 2(\varepsilon^1\cdot\varepsilon^2)\big[(\varepsilon^4\cdot p_3)\varepsilon^3\cdot(p_2-p_1)\\
&+(\varepsilon^3\cdot p_4)\varepsilon^4\cdot(p_1-p_2)\big] + 2(\varepsilon^3\cdot\varepsilon^4)\big[(\varepsilon^1\cdot p_2)\varepsilon^2\cdot(p_3-p_4)\\
&+(\varepsilon^2\cdot p_1)\varepsilon^1\cdot(p_4-p_3)\big] - 4(\varepsilon^2\cdot\varepsilon^3)(\varepsilon^1\cdot p_2)(\varepsilon^4\cdot p_3)\\
&+4(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^2\cdot p_1)(\varepsilon^4 p_3) - 4(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^2\cdot p_1)(\varepsilon^3\cdot p_4)\\
&+\underbrace{s(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^2\cdot\varepsilon^4) - s(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^2\cdot\varepsilon^3)}_{\text{contact term}}\Big\},\\
n_t =& - i\Big\{(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^3\cdot\varepsilon^2)(s-u) + 2(\varepsilon^1\cdot\varepsilon^4)\big[(\varepsilon^3\cdot p_2)\varepsilon^2\cdot(p_4-p_1)\\
&+(\varepsilon^2\cdot p_3)\varepsilon^3\cdot(p_1-p_4)\big] + 2(\varepsilon^2\cdot\varepsilon^3)\big[(\varepsilon^1\cdot p_4)\varepsilon^4\cdot(p_2-p_3)\\
&+(\varepsilon^4\cdot p_1)\varepsilon^1\cdot(p_3-p_2)\big] + 4(\varepsilon^4\cdot\varepsilon^3)(\varepsilon^1\cdot p_4)(\varepsilon^2\cdot p_3)\\
&+4(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^4\cdot p_1)(\varepsilon^3\cdot p_2) - 4(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^4\cdot p_1)(\varepsilon^2\cdot p_3)\\
&+\underbrace{t(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^4\cdot\varepsilon^3) - t(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^4\cdot\varepsilon^2)}_{\text{contact term}}\Big\}\\
n_u =& - i\Big\{(\varepsilon^1\cdot\varepsilon^3)(\varepsilon^2\cdot\varepsilon^4)(t-s) + 2(\varepsilon^1\cdot\varepsilon^3)\big[(\varepsilon^2\cdot p_4)\varepsilon^4\cdot(p_3-p_1)\\
&+(\varepsilon^4\cdot p_2)\varepsilon^2\cdot(p_1-p_3)\big] + 2(\varepsilon^2\cdot\varepsilon^4)\big[(\varepsilon^1\cdot p_3)\varepsilon^3\cdot(p_4-p_2)\\
&+(\varepsilon^3\cdot p_1)\varepsilon^1\cdot(p_2-p_4)\big] + 4(\varepsilon^3\cdot\varepsilon^2)(\varepsilon^1\cdot p_3)(\varepsilon^4\cdot p_2)\\
&+4(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^3\cdot p_1)(\varepsilon^2\cdot p_4) - 4(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^3\cdot p_1)(\varepsilon^4\cdot p_2)\\
&+\underbrace{u(\varepsilon^1\cdot\varepsilon^4)(\varepsilon^3\cdot\varepsilon^2) - u(\varepsilon^1\cdot\varepsilon^2)(\varepsilon^3\cdot\varepsilon^4)}_{\text{contact term}}\Big\}.
\end{aligned} \tag{4.12}$$

This **splitting** procedure, namely the multiplication and division by appropriate propagators in order to reproduce the same color and propagator structures of the Feynman diagrams with only trivalent vertices, can be implemented for amplitudes with more than four particles as well. Therefore, it is possible to rearrange the amplitude in terms of a new set of diagrams (which are not Feynman diagrams) with only trivalent vertices, generalizing the result (4.10) to

$$A_n^{(TL)} = g^{n-2}\sum_{i\in\Gamma_3}\frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} \tag{4.13}$$

where the sum runs over thi new set of diagrams $\Gamma_3$ and the variables $s_{\alpha_i}$ are a $n-$point generalization of the Mandelstam variables.

### 4.1.2 Color-Kinematics duality

The factorization (4.10) is important in order to discuss a symmetry of amplitudes, namely the **color-kinematics duality** [2, 144]. Let us observe that the color factors in (4.10) obey to the Jacobi identity

$$c_s + c_t + c_u = 0, \tag{4.14}$$

whose origin can be traced back to the existence of a $SU(N)$ group structure. On the other hand, as one can verify from a direct computation, we also have

$$n_s + n_t + n_u = 0, \tag{4.15}$$

whose origin is a priori nontrivial. This is the first example of the color-kinematics (CK) duality, which is a symmetry of every $n-$point tree level amplitude.

The generalization to arbitrary $n$ requires a further observation: the $n-$point amplitude (4.13) is invariant under

$$n_i \to n_i + \Delta_i \tag{4.16}$$

if

$$\sum_{i\in\Gamma_3} \frac{\Delta_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0. \tag{4.17}$$

This property goes under the name **generalized gauge invariance**, where the use of "generalized" is due to the fact that the transformations considered with $\Delta_i$ correspond, at the field-theoretical level, to field redefinition more general than the ones usually considered for standard gauge transformation comprising, for instance, non-local terms.

The CK duality is a symmetry of a $n-$points YM amplitude that states that it is always possible to find a generalized gauge transformation such that, for every Jacobi identity satisfied by the color factors of a triplet of diagrams in the set $\Gamma_3$, the transformed kinematic factors obey to an analogous identity

$$c_i + c_j + c_k = 0 \implies n'_i + n'_j + n'_k = 0. \tag{4.18}$$

Furthermore, the new kinematic factors possess the same antisymmetry of the color factors, namely

$$c_i = -c_j \implies n'_i = -n'_j \tag{4.19}$$

where $c_j$ is obtained from the diagram of $c_i$ by exchange of two external legs attached to the same vertices.

While for the 4-points case we did not require a generalized gauge transformation, it is already necessary at 5 points. At tree level, it is always possible to arrange the (transformed) kinematic factors such that they behave as the color factors. Indeed, the CK duality is a **proved theorem** at tree level and strongly supported at various loop, although a general proof is still missing.

Given an $L$-loop amplitude, it is possible to apply the same splitting procedure we discussed for $\mathcal{A}_n^{TL}$ and obtain

$$\mathcal{A}_n^{L-loop} = i^L g^{n-2+2L} \sum_{i\in\Gamma_3} \int \left( \prod_{k=1}^{L} \frac{d^D l_k}{(2\pi)^D} \right) \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{4.20}$$

where we introduced the loop momenta $l_k$ and the symmetry factors $S_i$. The CK-duality has been successfully tested in various specific cases, such as

- 4 gluons, 4 loops in $N = 4$ super Yang-Mills (in every dimension) [3,145];
- 5 gluons, 2 loops in $N = 4$ SYM (in every dimension) [146];
- 4 gluons, 2 loops in pure YM (in dimension $D = 4$) [3];
- 4 gluons, 1 loop in pure YM (in every dimension) [147];
- 4 gluons, 1 loop in pure YM with matter (in every dimension) [148];
- 4 gluons, 1 loop with less than maximal supersymmetry (in dimension $D = 4$) [149].

### 4.1.3 Double copy relations

The CK duality suggests a sort of interchangeability between color and kinematic factors. This observation, together with the existence of Kawai-Lewellen-Tye relations [150] in string theory, which relate closed string amplitudes to products of open ones, is at the basis of the double copy construction, also known as Bern-Carrasco-Johansson (BCJ) relations [3,151]. In its essence, the DC states that, given a tree-level $n$-gluon scattering amplitude $\mathcal{A}_n^{TL}$ written in the CK-dual form, it is possible to obtain an $n$-point tree-level amplitude of a gravitational theory $\mathcal{M}_n^{TL}$ by replacing the color factors with a second copy of kinematic factors $\tilde{n}_i$, possibly taken from a different YM theory.

$$\mathcal{A}_n^{TL} = g^{n-2} \sum_{i\in\Gamma_3} \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} \;\Rightarrow\; \mathcal{M}_n^{TL} = i(2k)^{n-2} \sum_{i\in\Gamma_3} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{4.21}$$

where we contextually redefined the couplings. Let us highlight however that it is not necessary that both $n_i$ and $\tilde{n}_i$ are in the CK-dual form, but it is sufficient to have it for only one of them.

While being not particularly relevant for this thesis, one of the main feature of the DC construction is that it holds at tree-level for numerous supersymmetric YM models [3, 151], with the resulting amplitudes corresponding to supergravity theories. In particular, we can take $n_i, \tilde{n}_i$ being the kinematic factors of an $n-$gluons amplitudes of two different super Yang-Mills and the DC relations provide the map

$$[\mathcal{N}_L \quad SYM] \times [\mathcal{N}_R \quad SYM] = [\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R \quad SUGRA]. \tag{4.22}$$

### 4.1.4 Particle content of the DC

Having discussed the inputs of the DC relations, let us now focus on understanding the outputs. Hence, we now explore the particle content of the corresponding gravitational theory.

The DC amplitude (4.21) contains, among other terms, products of spin-one polarization vectors $\varepsilon_\mu, \tilde{\varepsilon}_\mu$. In the case of a single massless particle, we can always choose a frame where its momentum is

$$p_\mu = E(1, 0, \ldots, 0, 1), \tag{4.23}$$

so that its little group (LG), i.e. the subgroup of the Poincaré group which preserve this vector, is $ISO(D-2)$. Particle states with finite spin are defined as reducible but not decomposable representations of the LG such that, in order to avoid continuous spin representations, an equivalence relation between polarization tensors is assumed such to trivialize the action of the commuting generators of $ISO(D-2)$:

$$\varepsilon_\mu \sim \varepsilon_\mu + \lambda(p) p_\mu. \tag{4.24}$$

These equivalence relation allow us to use (discrete) representations of $SO(D-2)$.

Given our frame (4.23) and the equivalence relation (4.24), we can always find $\lambda(p)$ such that $\varepsilon^i_0 = \varepsilon^i_D = 0$, where $i$ is the polarization. Therefore, the product $\varepsilon^i_\mu \tilde{\varepsilon}^i_\nu$ can be arranged as

$$\varepsilon^i \tilde{\varepsilon}^i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon^i_1 \tilde{\varepsilon}^i_1 & \cdots & \varepsilon^i_1 \tilde{\varepsilon}^i_{D-2} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \varepsilon^i_{D-2} \tilde{\varepsilon}^i_1 & \cdots & \varepsilon^i_{D-2} \tilde{\varepsilon}^i_{D-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.25}$$

Hence, $\varepsilon_{\mu\nu}$ is a reducible rank-two $SO(D-2)$ tensor whose particle content can be easily understood in terms of Young Tableaux

$$\square \otimes \square = \square\!\square\Big|_T \oplus \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} \oplus \bullet, \tag{4.26}$$

where:

1. $\square\!\square\Big|_T$ symmetric traceless tensor with $\frac{D(D-3)}{2}$ d.o.f. thus describing a (massless) graviton $h_{ij}$
2. $\begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array}$ antisymmetric rank 2 tensor with $\frac{(D-2)(D-3)}{2}$ polarisations, corresponding to a two-form $B_{ij}$.
3. $\bullet$ represents the trace part of (4.25) and describes a scalar particle $\varphi$.

This clarifies what is the output of the DC relations when the inputs are non-supersymmetric YM amplitudes: a graviton, a Kalb-Ramond two form and a dilaton. This matches the particle content of the low-energy limit of closed bosonic string theory, which is sometimes referred as $\mathcal{N} = 0$ supergravity. When the inputs are SYM theories we have a wider particle content as output, compatibly with (4.22).

To write the outer product covariantly, let us introduce the identity matrix for the LG as

$$\mathcal{I}^{LG}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_{ij}^{(D-2)\times(D-2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \eta_{\mu\nu} + \frac{1}{2} p_{(\mu} q_{\nu)}, \tag{4.27}$$

with $q_\mu$ being an auxiliary vector defined as

$$q_\mu = E^{-1}(1, 0, \cdots, 0, -1). \tag{4.28}$$

We thus obtain the following covariant decomposition of $\varepsilon^i_{\mu\nu}$ [152]:

$$\begin{aligned} \varepsilon^i_{\mu\nu} = \varepsilon^i_\mu \tilde{\varepsilon}^i_\nu = & \underbrace{\frac{1}{2}\varepsilon^i_{(\mu}\tilde{\varepsilon}^i_{\nu)} - \frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}\left(\eta_{\mu\nu} + \frac{1}{2}p_{(\mu}q_{\nu)}\right)}_{\textit{graviton}} \\ & + \underbrace{\frac{1}{2}\varepsilon^i_{[\mu}\tilde{\varepsilon}^i_{\nu]}}_{\textit{two-form}} + \underbrace{\frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}\left(\eta_{\mu\nu} + \frac{1}{2}p_{(\mu}q_{\nu)}\right)}_{\textit{scalar}}. \end{aligned} \tag{4.29}$$

An equivalent decomposition can be obtained by neglecting the terms with the auxiliary vector $q_\mu$, which can be eliminated by means of an on-shell gauge transformation

$$\varepsilon^i_{\mu\nu} = \underbrace{\frac{1}{2}\varepsilon^i_{(\mu}\tilde{\varepsilon}^i_{\nu)} - \frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}\eta_{\mu\nu}}_{graviton} + \underbrace{\frac{1}{2}\varepsilon^i_{[\mu}\tilde{\varepsilon}^i_{\nu]}}_{two\text{-}form} + \underbrace{\frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}\eta_{\mu\nu}}_{scalar}, \tag{4.30}$$

so that we can define the polarisations of the graviton, the two-form and the scalar respectively as:

$$G^i_{\mu\nu} = \frac{1}{2}\varepsilon^i_{(\mu}\tilde{\varepsilon}^i_{\nu)} - \frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}\eta_{\mu\nu}, \quad A^i_{\mu\nu} = \frac{1}{2}\varepsilon^i_{[\mu}\tilde{\varepsilon}^i_{\nu]}, \quad S = \frac{\varepsilon^i \cdot \tilde{\varepsilon}^i}{D-2}. \tag{4.31}$$

Let us observe that the first of (4.31) is such that $G^i_{\mu\nu}$ satisfies the de Donder gauge condition:

$$p \cdot G_\nu = \frac{1}{2}p_\nu G^\alpha{}_\alpha, \tag{4.32}$$

which results from the gauge-fixing necessary to change the decomposition (4.29) into (4.30).

### 4.1.5 The zeroth copy

As was observed in [153], the replacement (4.21) is not the only possible one. Indeed, one can replace the kinematic factor with another copy of color factors, coming from a YM theory with a possibly different gauge group. This operation builds what is called the **zeroth copy** and reads

$$\mathcal{A}^{TL}_n = g^{n-2}\sum_{i\in\Gamma_3}\frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} \;\Rightarrow\; \mathcal{T}^{TL}_n = (y)^{n-2}\sum_{i\in\Gamma_3}\frac{c_i\tilde{c}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{4.33}$$

where the latter expression is expected to describe a scalar amplitude. At the same time, it contains two color factors. Hence, $\mathcal{T}_n$ is a scattering amplitude in a theory of scalar fields $\Phi_{aa'}$, transforming in the adjoint representation of two (possibly different) Lie algebras. The Lagrangian of this theory, which correctly reproduces the structure $\mathcal{T}_n$, is

$$\mathcal{L}_\Phi = -\frac{1}{2}\partial_\mu\Phi_{aa'}\partial^\mu\Phi^{aa'} - \frac{y}{3}f^{abc}f^{a'b'c'}\Phi_{aa'}\Phi_{bb'}\Phi_{cc'} \tag{4.34}$$

where primed and unprimed indices refer to two possibly different Lie algebras. The field $\Phi_{aa'}$ goes under the name of **biadjoint scalar field** and it is fundamental in order to derive a field theoretical version of the double copy.

## 4.2 Classical double copy

Given the BCJ relations for tree-level amplitudes, one might expect to finds relations at the level of classical solutions, as they are related to tree level scattering amplitudes. This is indeed the case, even if with subtleties. In fact, it is possible to express geometric, non-perturbative solutions in general relativity as the double-copy of classical solutions in gauge theories. This framework goes under the name of **classical double copy.**

### 4.2.1 The Kerr-Schild solutions

The Kerr-Schild (KS) class of solutions to Einstein equations is well-suited to have an interpretation in terms of DC. For instance, working in this framework it is possible to write the Schwarzschild solution as a double copy of the Coulomb potential [8] or the (A)dS solutions as double copies of uniform distribution of (color) charge solutions [154]. The KS double-copy finds applications also in more complicated solutions, including the Taub-NUT spacetime as an example of double Kerr-Schild metric [154], Kerr-Schild metrics over curved backgrounds [9], perturbative solutions [155], accelerating particles and radiative solutions [156–161], GR and YM solutions over maximally symmetric spacetimes [10].

Kerr-Schild solutions are characterised by the existence of a coordinate system where the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi k_\mu k_\nu, \tag{4.35}$$

with $\eta_{\mu\nu}$ being the Minkowski metric, $\varphi$ a scalar field an $k_\mu$ a vector that satisfies

$$\begin{cases} \eta^{\mu\nu} k_\mu k_\nu = g^{\mu\nu} k_\mu k_\nu = 0, \\ k^\mu \partial_\mu k_\nu = 0. \end{cases} \tag{4.36}$$

Equations (4.36) imply several useful properties:

- $g^{\mu\nu} = \eta^{\mu\nu} - \varphi k^\mu k^\nu$
- $\eta^{\mu\nu} k_\mu \partial_\rho k_\nu = 0, \;\; g^{\mu\nu} k_\mu \partial_\rho k_\nu = 0.$
- $\sqrt{-g} = 1$
- $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\left\{\partial_\beta(\varphi k^\alpha k_\gamma) + \partial_\gamma(\varphi k^\alpha k_\beta) - \partial^\alpha(\varphi k_\beta k_\gamma) + \varphi k^\alpha k_\beta k_\gamma k \cdot \partial\varphi\right\}$
- $\Gamma^\alpha_{\beta\gamma} k^\beta k^\gamma = 0, \;\; \Gamma^\alpha_{\beta\gamma} k^\beta k_\alpha = 0$

- $(\nabla_\mu k_\nu)k^\nu = (\partial_\mu k_\nu)k^\nu = 0,\ \ (\nabla_\mu k_\nu)k^\mu = (\partial_\mu k_\nu)k^\mu = 0,$

- $R^\lambda{}_\nu = \frac{1}{2}\{\partial_\alpha\partial^\lambda(\varphi k^\alpha k_\nu) + \partial_\alpha\partial_\nu(\varphi k^\alpha k^\lambda) - \partial^\alpha\partial_\alpha(\varphi k^\lambda k_\nu)\}$, where we use the mixed convention $R^\lambda{}_\nu = g^{\lambda\mu}R_{\mu\nu}$, $\partial^\lambda = \eta^{\lambda\mu}\partial_\mu$.

Let us highlight that this last property shows that, for a solution in the Kerr-Schild form, the Ricci tensor is linear in the (non-perturbative) graviton field $\varphi k_\mu k_\nu$, implying that the Einstein-Hilbert equations are also linear. This fact will be of great use for the DC.

### The Double Copy for Kerr-Schild solutions

The authors of [8] focused on the particular case of vacuum, stationary Kerr-Schild solutions. Stationarity means

$$\begin{aligned} \partial_0\varphi &= 0, \\ \partial_0 k_\mu &= 0, \end{aligned} \tag{4.37}$$

while the vacuum condition reads

$$R^\lambda{}_\nu = 0, \tag{4.38}$$

which are further simplified by the stationarity condition (4.37):

$$\begin{aligned} &\bullet\ R^0{}_0 = -\frac{1}{2}\Box(\varphi k^0 k_0) = -\frac{1}{2}\nabla^2(\varphi k^0 k_0) = 0, \\ &\bullet\ R^i{}_0 = \frac{1}{2}\{\partial_j\partial^i(\varphi k^j k_0) - \nabla^2(\varphi k_i k_0)\} = 0, \\ &\bullet\ R^i{}_j = \frac{1}{2}\{\partial_l\partial^i(\varphi k^l k_j) + \partial_l\partial_j(\varphi k^l k^i) - \partial_l\partial^l(\varphi k^i k_j)\} = 0. \end{aligned} \tag{4.39}$$

The Kerr-Schild metric is invariant under

$$k_\mu \to \alpha(x)k_\mu, \quad \varphi(x) \to \frac{\varphi(x)}{\alpha^2(x)}, \tag{4.40}$$

and we can use this invariance to fix $k_0 = 1$. Therefore, the first two of equations (4.39) are

$$\begin{aligned} &\bullet\ -\frac{1}{2}\nabla^2\varphi = 0, \\ &\bullet\ \frac{1}{2}\{\partial_j\partial^i(\varphi k^j) - \nabla^2(\varphi k_i)\} = 0. \end{aligned} \tag{4.41}$$

Starting from this setup, the DC mapping works in this case at reverse: we can identify a **single copy field** $A^a_\mu$ for the Kerr-Schild solutions and show that the equations of motion for the metric imply the YM equations for $A^a_\mu$.

The particular structure of the Kerr-Schild solution is especially suited to this procedure, thus suggesting that the single copy field is related to $k_\mu$. In particular, it is identified as

$$A^a_\mu = \varphi^a k_\mu, \tag{4.42}$$

where the set of scalars $\varphi^a$ satisfy the same equations of the scalar $\varphi$ .

The key observation to show that $A^a_\mu$ satisfies the YM equations of motion is that its field strength must be linear due to the factorization of the colour and Lorentz indices in the definition (4.42):

$$F^a_{\mu\nu} = \partial_\mu(\varphi^a k_\nu) - \partial_\nu(\varphi^a k_\mu), \tag{4.43}$$

so that equations (4.41) lead to:

$$\begin{aligned} &\bullet\ \partial^\mu F^a_{\mu 0} = \nabla^2\varphi = 0, \\ &\bullet\ \partial^\mu F^a_{\mu i} = \nabla^2(\varphi^a k_i) - \partial_j\partial_i(\varphi^a k_j) = 0. \end{aligned} \tag{4.44}$$

### Double Copy for the Schwarzschild solution

We now apply the same procedure to the Schwarzschild metric in $D = 4$, which reads

$$ds^2 = -\mathcal{S}(r)dt^2 + \frac{1}{\mathcal{S}(r)}dr^2 + r^2 d\Omega^2, \tag{4.45}$$

with

$$\mathcal{S}(r) = 1 - \frac{2GM}{r}, \tag{4.46}$$

and $M$ is the mass of a point-like source. Let us write the metric in the Kerr-Schild form by changing the coordinates

$$dt \to du = dt + \frac{dr}{\mathcal{S}(r)}, \tag{4.47}$$

and redefining

$$u \to T = u - r\,. \tag{4.48}$$

The metric reduces to

$$\begin{aligned} ds^2 &= (1-\mathcal{S}(r))\,(dT + dr)^2 - dT^2 + dr^2 + r^2 d\Omega^2 \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + \varphi k_\mu k_\nu dx^\mu dx^\nu, \end{aligned} \tag{4.49}$$

where

$$\begin{aligned} \varphi &= 1 - \mathcal{S}(r) = \frac{2GM}{r}, \\ k_\mu &= \left(1, \frac{x_i}{r}\right). \end{aligned} \tag{4.50}$$

We can observe that this $k_\mu$ satisfies the properties that define a Kerr-Schild metric.

To extract the single copy construction [8], we factor out from $\varphi$ the gravitational coupling constant $\mathrm{k}^2 = 8\pi G$ and the mass $M$

$$\varphi = \frac{1}{4\pi r}. \tag{4.51}$$

Thus, writing the metric tensor as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}, \tag{4.52}$$

we have

$$2\kappa h_{\mu\nu} = \frac{2GM}{r} k_\mu k_\nu = \frac{\kappa^2 M}{4\pi r} k_\mu k_\nu = \mathrm{k}^2 M \varphi k_\mu k_\nu \;\Rightarrow\; h_{\mu\nu} = \frac{\kappa}{2} M \varphi k_\mu k_\nu. \tag{4.53}$$

The single-copy is constructed by mapping

$$\kappa \to 2g, \quad M \to c^a, \quad k_\mu k_\nu \to k_\mu, \tag{4.54}$$

where $g$ is the Yang-Mills coupling and $c^a$ is a colour charge. Therefore, our single-copy field reads

$$A^a_\mu = g \frac{c^a}{4\pi r} k_\mu. \tag{4.55}$$

A physical interpretation of this result can be found by choosing a gauge-fixing such that the spatial part of the YM field is set to zero so that $A^a_\mu$ assumes the form of the Coulomb solution:

$$A^a = g \frac{c^a}{4\pi r} (1, 0, 0, 0). \tag{4.56}$$

### 4.2.2 Weyl Double Copy

Another approach to the realm of classical solutions is that of the Weyl double copy, first proposed in [162]. In the Weyl DC, the connection is at the level of the Weyl tensor $C$ which, written in spinorial language, is the product of two Maxwell tensors $F$. This is motivated by the structure of linear waves in gauge theory and in gravity:

$$\begin{aligned} A_\mu = \varepsilon_\mu\, e^{ik\cdot x} \quad &\Longrightarrow \quad F_{\mu\nu} = i(k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu)\, e^{ik\cdot x}\,, \\ h_{\mu\nu} = \varepsilon_{\mu\nu}\, e^{ik\cdot x} \quad &\Longrightarrow \quad R_{\mu\nu\rho\lambda} = \frac{1}{2}(k_\mu\varepsilon_\nu - k_\nu\varepsilon_\mu)(k_\rho\varepsilon_\lambda - k_\lambda\varepsilon_\rho)\, e^{ik\cdot x}\,, \end{aligned} \tag{4.57}$$

where we used $\varepsilon_{\mu\nu} = \varepsilon_\mu\varepsilon_\nu$. Therefore, at linearised level we have

$$e^{ik\cdot x} R_{\mu\nu\rho\lambda} \sim F_{\mu\nu}F_{\rho\lambda}. \tag{4.58}$$

Then, working with the spinorial formulation of general relativity [163], the idea at the basis of the Weyl DC is to connect the Weyl tensor and the field strength in position space through the relation

$$C_{ABCD} = \frac{1}{S} F_{(AB}F_{CD)}\,, \tag{4.59}$$

with $S$ being a scalar field, which is required for dimensional reasons, and $A, B, C, D$ denoting (undotted) spinorial indices. The Weyl DC approach is an extension of the KS DC. In [162] the focus is on four-dimensional vacuum metrics of algebraic type D, a class that includes several important solutions. In particular, in the vanishing cosmological constant case the metric takes the form [164]

$$\begin{aligned} ds^2 = \frac{1}{(1-pq)^2}\Bigg[ &\frac{p^2+q^2}{P(p)}\, dp^2 + \frac{P(p)}{p^2+q^2}\,(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Q(q)}\, dq^2 \\ &- \frac{Q(q)}{p^2+q^2}\,(d\tau - p^2 d\sigma)^2 \Bigg]\,, \end{aligned} \tag{4.60}$$

with $x^\mu = (\tau, \sigma, p, q)$ real coordinates (in $(-,+,+,+)$ signature) and $P, Q$ functions of the form

$$\begin{aligned} P(p) &= \gamma(1-p^4) + 2np - \epsilon p^2 + 2mp^3\,, \\ Q(q) &= \gamma(1-q^4) - 2mq + \epsilon q^2 - 2nq^3\,, \end{aligned} \tag{4.61}$$

where $m, n, \gamma, \epsilon$ are free parameters that, in appropriate combinations, can be associated to mass, NUT charge, angular momentum and acceleration. Working in the spinorial formalism it is thus possible to identify the electromagnetic and scalar fields associated with this family. In particular, the single-copy fields still depend on the free parameters.

The Weyl DC finds application in several other contexts, such as gravitational waves [165], twistor theory [166–168] and asymptotic symmetries

[14–16, 169].

## 4.3 Convolutional double copy

The framework of classical DC seems to connect the geometry of spacetime to YM theories, which are generally unaware of spacetime geometry. In order to understand to what extent it is possible to describe general relativity by means of YM theories, one would like to relate gravity and YM at a field-theoretical level. This is one of the reasons driving the efforts to understand the off-shell origins of the DC relations. In addition to this, let us mention that achieving a Lagrangian understanding of the DC could assist in proving (or disproving) the DC correspondence at the loop level.

### 4.3.1 The Double-Copy field

We want to define a DC field which describe the particle content of the DC and that can be written as specific product of YM fields. Starting from the construction at the level of scattering amplitudes, where the double copy is realised through a product in momentum space, it has been proposed to define a DC field in configuration space in terms of a specific **convolution** product [11–13].

The convolution, defined as

$$(f \circ g)(x) = \int d^D y f(y) g(x-y)\,, \tag{4.62}$$

is a commutative and associative linear product which behaves in a peculiar way under differentiation, namely

$$\partial_\mu (f \circ g)(x) = (\partial_\mu f \circ g)(x) = (f \circ \partial_\mu g)(x). \tag{4.63}$$

This property is crucial in order to connect YM theories and gravity at the level of symmetries.

In order to consistently define a DC field that describes gravity through a convolution we require that:

- The DC field is a rank two tensor which propagates the $\mathcal{N} = 0$ gravitational degrees of freedom.
- The DC field is a color scalar, since gravity is a non-colored theory.
- The DC field has the correct mass dimensions.

- Gauge transformations of spin-one fields are mapped to the corresponding transformations of the DC field.

The definition of the DC field [11–13] is

$$H_{\mu\nu}(x) = (A^a_\mu \circ \Phi^{-1}_{aa'} \circ \tilde{A}^{a'}_\nu)(x) := A_\mu \star \tilde{A}_\nu, \tag{4.64}$$

where $A^a_\mu, \tilde{A}^{a'}_\nu$ are the YM fields and $\Phi^{-1}_{aa'}$ is the convolution inverse of the biadjoint scalar field $\Phi_{aa'}$, introduced in (4.33). The presence of the biadjoint scalar is necessary in order to have a color scalar DC field, while the use of the convolution inverse, defined as

$$\Phi^{-1}_{aa'} \circ \Phi_{bb'} = \delta^D(x)\delta_{ab}\delta_{a'b'}, \tag{4.65}$$

is necessary to obtain the correct mass dimensions:

$$[A^a_\mu], [\tilde{A}^{a'}_\nu], [\Phi_{aa'}] = \frac{D-2}{2} \implies [H_{\mu\nu}] = \frac{D-2}{2}. \tag{4.66}$$

However, in order to realize the DC map at the level of symmetries, a crucial constraint is in order: $A^a_\mu, \tilde{A}^{a'}_\nu$ are not full-fledged YM fields but an "abelianised" version. The color index is thus understood as global index, while gauge transformations are parametrised as

$$\delta A^a_\mu = \partial_u \epsilon^a, \quad \delta \tilde{A}^{a'}_\mu = \partial_u \tilde{\epsilon}^{a'}. \tag{4.67}$$

Therefore, for what concerns gauge symmetries, $A^a_\mu, \tilde{A}^{a'}_\nu$ are formally $N^2 - 1$ and $(N')^2 - 1$ copies of a Maxwell field. This simplification results that the DC field describes gravity at a linearised level.

### 4.3.2 DC field content

A straightforward approach might be to define the graviton and the two-form as the symmetric and antisymmetric parts of $H_{\mu\nu}$, respectively. This setup would yield the correct linearized gauge transformations, as we discuss later. However, this identification overlooks some important aspects.

For massless particles, the physical states lie in the irreducible representations of $SO(D-2)$, as discussed in the scattering amplitudes construction. On the other hand, covariant fields correspond to representations of $GL(D, R)$. We can understand the product in terms of Young tableaux but, when referring to $GL(D,R)$, the product reads simply

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \tag{4.68}$$

where there are no constraints on the trace of the symmetric part. As a consequence, in the covariant construction of the graviton $h_{\mu\nu}$ there is some degree of arbitrariness in the definition of its trace, and this leads to possible mixing with the scalar field of the $\mathcal{N}=0$ Supergravity multiplet.

As mentioned, the naive identification $h_{\mu\nu} = H_{(\mu\nu)}$ would work in reproducing linearised diffeomorphism. Postponing this proof to the next subsection, we now observe that the more general identication

$$h_{\mu\nu} = H_{(\mu\nu)} - \gamma\eta_{\mu\nu}\varphi, \tag{4.69}$$

where $\gamma \in \mathbb{R}$ is a parameter and $\varphi$ is a gauge invariant scalar field, works as well by construction.

Let us discuss how we can obtain a gauge invariant scalar field. Postponing the proof, a notion of DC gauge symmetry is induced on the DC field. Its symmetric part transforms as

$$\delta H_{(\mu\nu)} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.70}$$

and therefore the trace is not generally gauge invariant since

$$\delta H = 2\partial \cdot \xi\,. \tag{4.71}$$

In order to define a gauge-invariant quantity, we need compensate that transformation and this can be achieved by adding non-localities in the definition of the scalar field, which turns out to be

$$\varphi = H - \frac{\partial \cdot \partial \cdot H}{\square}. \tag{4.72}$$

To summarize, the off-shell decomposition of the DC field is

$$\begin{aligned} h_{\mu\nu} &= H_{(\mu\nu)} - \gamma\eta_{\mu\nu}\varphi \\ B_{\mu\nu} &= H_{[\mu\nu]} \\ \varphi &= H - \frac{\partial \cdot \partial \cdot H}{\square}\,. \end{aligned} \tag{4.73}$$

Let us observe that, given the differentiation rule of the convolution, we have that for

$$\gamma = \frac{1}{D-2} \tag{4.74}$$

the Lorenz gauge conditions on the single copy fields implies the Lorenz gauge condition of the two-form and the de-Donder condition on the graviton. From now on, we specialize to this value of $\gamma$.

### 4.3.3 Linearised symmetries and field strength

In the abelianized limit $g \to 0$, a gauge field transformation, together with a global one, reduces to:

$$\delta A^a_\mu = \partial_\mu \epsilon^a + f^{abc} A^b_\mu \theta^c \,, \tag{4.75}$$

where $\epsilon = \epsilon(x)$ is the function parametrising the gauge freedom, while $\theta$ is a constant parameter used for global transformations.

The biadjoint scalar field is chosen to transform only globally as

$$\delta \Phi^{-1}_{aa'} = f^{abc} \Phi^{-1}_{ba'} \, \theta^c + f^{a'b'c'} \Phi^{-1}_{ab'} \, \tilde{\theta}^{c'} . \tag{4.76}$$

Thus, when the inputs undergo a gauge transformation, a variation is induced in the DC field, which reads

$$\begin{aligned} \delta H_{\mu\nu} &= \delta(A^a_\mu \circ \Phi^{-1}_{aa'} \circ \tilde{A}^{a'}_\nu) \\ &= (\delta A^a_\mu) \circ \Phi^{-1}_{aa'} \circ \tilde{A}^{a'}_\nu + A^a_\mu \circ \delta(\Phi^{-1}_{aa'}) \circ \tilde{A}^{a'}_\nu + A^a_\mu \circ \Phi^{-1}_{aa'} \circ (\delta \tilde{A}^{a'}_\nu) \,. \end{aligned} \tag{4.77}$$

Substituting the transformation laws and using the differentiation rule of convolution, we find

$$\delta H_{\mu\nu} = \partial_\mu \alpha_\nu + \partial_\nu \tilde{\alpha}_\mu \tag{4.78}$$

where

$$\begin{aligned} \alpha_\mu &= \epsilon^a \circ \Phi^{-1}_{aa'} \circ \tilde{A}^{a'}_\mu \,, \\ \tilde{\alpha}_\mu &= A^a_\mu \circ \Phi^{-1}_{aa'} \circ \tilde{\epsilon}^{a'} \,. \end{aligned} \tag{4.79}$$

By means of the decomposition in terms of the DC multiplet, we identify the linearised diffeomorphism parameter to be

$$\xi_\mu := \frac{1}{2}(\alpha_\mu + \tilde{\alpha}_\mu) \,, \tag{4.80}$$

while the two-from gauge parameter reads

$$\Lambda_\mu := \frac{1}{2}(\alpha_\mu - \tilde{\alpha}_\mu) \,. \tag{4.81}$$

An important observation is in order: gauge-invariant quantities are mapped, through double copy, to gauge invariant quantities. Indeed, it is possible to define a Riemann tensor in terms of the abelianised field strength $F^a_{\mu\nu}$ as

$$R_{\mu\nu\rho\sigma} = -\frac{1}{2} F_{\mu\nu} \star \tilde{F}_{\rho\sigma}, \tag{4.82}$$

which as element of $GL(D)$, decomposes as follows

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2,2} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} \tag{4.83}$$

suggesting that the geometry associated to the DC field be that of a manifold with torsion $T_{\mu\nu\rho}$, encoded in the field strength of the two-form $T = dB$.

### 4.3.4 Equations of motions

We now want to see that, whenever the single copy fields are on-shell, the DC field is on-shell as well. The vacuum equations of motions reads

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = 0, \\ &\Box B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} = 0, \\ &\Box \varphi = 0, \end{aligned} \tag{4.84}$$

and we want to show that they are induced by the equations of motion of the single copy fields

$$\Box A^a_\mu - \partial_\mu \partial \cdot A^a = 0. \tag{4.85}$$

In order to prove it, we work in Lorenz gauge, so that the DC multiplet reads

$$\begin{aligned} &h_{\mu\nu} = H_{(\mu\nu)} - \frac{1}{D-2}\eta_{\mu\nu} H\,, \\ &B_{\mu\nu} = H_{[\mu\nu]}\,, \\ &\varphi = H\,. \end{aligned} \tag{4.86}$$

The vacuum equations of motion for a spin two in de-Donder gauge read

$$\Box \bar{h}_{\mu\nu} = 0, \tag{4.87}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ being the trace-reversed graviton. Let us prove that they are indeed induced by the on-shell condition of the spin one field

$$\Box\bar{h}_{\mu\nu} = \Box[A_{(\mu} \star \tilde{A}_{\nu)}] = [\Box A_{(\mu}] \star \tilde{A}_{\nu)} \approx 0, \tag{4.88}$$

where we used the differentiation rule of the convolution product and that, in Lorenz gauge, the equations of motion reads $\Box A_\mu = 0$. Identical results hold for the two form and the dilaton. Starting from the equations of motion (4.84), the derivation can be performed in a gauge independent manner.

Furthermore, we can include sources in our discussion. In particular, let us introduce the linearised equations of motion for YM fields

$$\begin{cases} \partial^\mu F^a_{\mu\nu} = \Box A^a_\mu - \partial_\mu \partial \cdot A^a = j^a_\nu, \\ \partial^\mu \tilde{F}^{a'}_{\mu\nu} = \Box \tilde{A}^{a'}_\mu - \partial_\mu \partial \cdot \tilde{A}^{a'} = \tilde{j}^{a'}_\nu. \end{cases} \tag{4.89}$$

in terms of the abelianized sources $j_\mu, \tilde{j}_\mu$.

The discussion for the convolution require some steps. For the biadjoint scalar (not the inverse), we have

$$\Box\Phi_{aa'} = j^\Phi_{aa'}. \tag{4.90}$$

We can define the convolution inverse of the current $[j^\Phi_{aa'}]^{-1}$ and, convoluting it with both sides of the previous equation, we find

$$[j^\Phi_{bb'}]^{-1} \circ \Box\Phi_{aa'} = [j^\Phi_{bb'}]^{-1} \circ j^\Phi_{aa'} = \delta_{ab}\delta_{a'b'}\delta^{(D)}(x) \tag{4.91}$$

And since we have$[j^\Phi_{bb'}]^{-1} \circ \Box\Phi_{aa'} = (\Box[j^\Phi_{bb'}]^{-1}) \circ \Phi_{aa'}$ we can identify

$$\Box[j^\Phi_{aa'}]^{-1} = \Phi^{-1}_{aa'}. \tag{4.92}$$

We can thus define the DC source as

$$j_{\mu\nu} = j^a_\mu \circ [j^\Phi_{aa'}]^{-1} \circ \tilde{j}^{a'}_\nu := j_\mu \circledast \tilde{j}_\nu \tag{4.93}$$

and identify its components in terms of the DC multiplet sources

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -\frac{1}{4} j_{(\mu} \circledast \tilde{j}_{\nu)} := -T_{\mu\nu}, \\ \Box B_{\mu\nu} - \partial_\mu\partial^\rho B_{\rho\nu} + \partial_\nu\partial^\rho B_{\rho\mu} = \frac{1}{2} j_{[\mu} \circledast \tilde{j}_{\nu]} := j^{(B)}_{\mu\nu}, \\ \Box\varphi = j^\alpha \circledast \tilde{j}_\alpha := j^{(\varphi)}. \end{cases} \tag{4.94}$$

# 4.4 Lagrangians

## 4.4.1 Attempts towards a Lagrangian DC

There were several attempts to provide a Lagrangian descriptions ot the double-copy. The first approach was proposed by Bern, Dennen, Huang and Kiermaier in [151] and it was based on the introduction of several auxiliary fields. Their logic was similar to the BCJ algorithm for the scattering amplitudes. In fact, the aim of the BDHK approach is to modify the YM Lagrangian in order to have CK-duality encoded into it. Such modification consisted in the inclusion of additional vertices which are all identically vanishing due to the Jacobi identity, but produce new Feynman rules which make every $n$-gluon tree-level amplitude manifestly CK-dual. All the new vertices are non-local, but it is possible to eliminate the non-localities by introducing, order by order, a set of auxiliary fields. This procedure has been carried out explicitly up to the fifth order. For instance, the additional 5-gluon vertex reads

$$\begin{aligned}\mathcal{L'}_5 = -\frac{g^3}{2} f^{a_1a_2b} f^{ba_3c} f^{ca_4a_5} \big(&\partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3\mu} \\ &+ \partial_{[\mu} A^{a_2}_{\nu]} A^{a_3}_{\rho} A^{a_1\mu} + \partial_{[\mu} A^{a_3}_{\nu]} A^{a_1}_{\rho} A^{a_2\mu}\big) \frac{1}{\Box} \big(A^{a_4\nu} A^{a_5\rho}\big),\end{aligned} \tag{4.95}$$

which manifestly vanishes if rearranged

$$\begin{aligned}\mathcal{L'}_5 = -\frac{g^3}{2} &\big(f^{a_1a_2b} f^{ba_3c} + f^{a_3a_1b} f^{ba_2c} + f^{a_2a_3b} f^{ba_1c}\big) \\ &\times \big(\partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3\mu}\big) \frac{1}{\Box} \big(A^{a_4\nu} A^{a_5\rho}\big) = 0.\end{aligned} \tag{4.96}$$

Adding the auxiliary fields, this 5-point Lagrangian reads

$$\begin{aligned}\mathcal{L}'_5 \to & Y^{a\mu\nu} \Box X^a_{\mu\nu} + D^{a\mu\nu\rho}_{(3)} \Box C^a_{(3)\mu\nu\rho} + D^{a\mu\nu\rho\sigma}_{(4)} \Box C^a_{(4)\mu\nu\rho\sigma} \\ &+ g f^{abc} \left( Y^{a\mu\nu} A^b_\mu A^c_\nu + \partial_\mu D^{a\mu\nu\rho}_{(3)} A^b_\nu A^c_\rho - \frac{1}{2} \partial_\mu D^{a\mu\nu\rho\sigma}_{(4)} \partial_{[\nu} A^b_{\rho]} A^c_\sigma \right) \\ &+ g f^{abc} X^{a\mu\nu} \left( \frac{1}{2} \partial_\rho C^{b\rho\sigma}_{(3)} \partial_{[\sigma} A^c_{\nu]} + \partial_\rho C^{b\rho\sigma}_{(4)\nu[\mu} A^c_{\sigma]} \right).\end{aligned} \tag{4.97}$$

Squaring the Lagrangian, we can identify the gravitational field as

$$A^\mu \tilde{A}^\nu \to H^{\mu\nu}\,, \tag{4.98}$$

where $H^{\mu\nu}$ is the physical field, which includes the graviton, anti-symmetric tensor, and dilaton. All the other combinations of fields are squared through

DC to gravitational auxiliary fields. In 2013, this approach was generalised up to six points [170].

### 4.4.2 Lagrangian Convolutional DC

The first attempts on a Lagrangian derivation based on the convolutional approach were proposed in [12, 13], with an extension to the purely gravitational cubic vertex in [171]. However, from an off-shell point of view, there are two issues: on the one hand, the number of degrees of freedom of the DC field does not seem to match that of the extended gravitational multiplet, upon discarding those gauge components that can be fixed without making use of the equations of motion, one would be left with $3 \times 3$ components for the DC field, against $6 + 3 + 1$ for the off-shell system given by graviton, Kalb-Ramond field and dilaton. On the other hand, when interpreted in DC terms, the source for the scalar field naturally mixes with the trace of the stress-energy tensor sourcing the graviton equations of motion, thus posing an issue of relative dependence of the two types of coupling. This problem was partially solved by the inclusion of an auxiliary scalar field in [13].

The solution considered in [172] is the Maxwell-like Lagrangian [173, 174]

$$\mathcal{L}_{ML} \,=\, \frac{1}{2}\, H^{\mu\nu}\left(\eta_{\rho\mu}\,\eta_{\nu\sigma}\,\Box - \eta_{\rho\mu}\,\partial_\sigma\,\partial_\nu - \eta_{\sigma\nu}\,\partial_\rho\,\partial_\mu\right) H^{\rho\sigma}\,, \tag{4.99}$$

with $H$ being the DC field. This Lagrangian correctly reproduce the equations of motion

$$\Box\, H^{\mu\nu} \,-\, \partial_\mu\,\partial^\rho\, H_\rho{}^\nu \,-\, \partial_\nu\,\partial^\sigma\, H^\mu{}_\sigma \,=\, 0\,. \tag{4.100}$$

The Lagrangian $\mathcal{L}_{ML}$ is not invariant under a full DC gauge transformation, since

$$\delta_0\,\mathcal{L}_{ML} \,=\, -\,\partial^\mu(\alpha_\mu + \tilde{\alpha}_\mu)\,\partial\cdot\partial\cdot H\,. \tag{4.101}$$

unless

$$\partial^\mu\,(\alpha_\mu \,+\, \tilde{\alpha}_\mu) \,=\, 0\,. \tag{4.102}$$

This condition can be understood as a generalisation of the one used in unimodular gravity (see *e.g.* [175–177]). Under (4.102) the trace and the double divergence of $H^{\mu\nu}$ are separately gauge invariant.

The constraint (4.102) implies a linear dependence between the two gauge sectors that in particular reduces by one scalar component the possibility of performing independent gauge transformations off shell, thus restoring the matching between the expected DC components and their vector seeds. Moreover, it is relevant to recall that for unimodular gravity the actual source of the gravitational field effectively is the traceless part of the stress-energy tensor, since the variational derivative with respect to metrics with fixed determinant selects the traceless component of the Einstein tensor. There is no missing information with respect to the full sourced Einstein-Hilbert equations, though, since the two setups are known to be classically equivalent (barring the different role played by the cosmological constant). In this respect, for the symmetric part of $H_{\mu\nu}$, (4.99) is tantamount to a version of unimodular gravity where the trace of the symmetric tensor is not just frozen, as it could consistently chosen to be, nor is it pure gauge as for the Fierz-Pauli case, rather it represents an independent physical scalar field. It is then clear that, if one were to source (4.100) with a traceful tensor, the trace of the latter would unambiguously source the scalar field alone encoded in the trace of $H_{\mu\nu}$.

It is also possible to provide a formulation in order to obtain a full gauge-invariant free Lagrangian. This can be achieved by admitting non-localities. The result is still rewarding, however, as it makes the DC structure apparent. The non-local DC Lagrangian is [172]

$$\mathcal{L}_{NL} = \frac{1}{2} H^{\alpha\beta} \{ \Box \eta_{\alpha\mu}\eta_{\beta\nu} - \partial_\alpha \partial_\mu \eta_{\beta\nu} - \partial_\beta \partial_\nu \eta_{\alpha\mu} + \partial_\alpha \partial_\beta \frac{1}{\Box} \partial_\mu \partial_\nu \} H^{\mu\nu} \,, \tag{4.103}$$

and, integrating by parts and bearing in mind the definition of the DC curvature (4.82), one finds that the free Lagrangian (4.103) admits the rewriting

$$\begin{aligned} \mathcal{L}_{NL} &= \frac{1}{2} R_{\mu\nu\rho\sigma} \frac{1}{\Box} R^{\mu\nu\rho\sigma} \\ &= \frac{1}{8} (F_{\mu\alpha} \star \tilde{F}_{\nu\beta}) \frac{1}{\Box} (F^{\mu\alpha} \star \tilde{F}^{\nu\beta}) \,. \end{aligned} \tag{4.104}$$

Splitting the curvature in terms of $H^S_{\mu\nu}$ and $B_{\mu\nu}$:

$$\begin{aligned} R_{\mu\nu\rho\sigma} = &\frac{1}{2} \{ \partial_\nu \partial_\rho H^S_{\mu\sigma} + \partial_\mu \partial_\sigma H^S_{\nu\rho} - \partial_\nu \partial_\sigma H^S_{\mu\rho} - \partial_\mu \partial_\rho H^S_{\nu\sigma} \} \\ &+ \frac{1}{2} \{ \partial_\nu \partial_\rho B_{\mu\sigma} + \partial_\mu \partial_\sigma B_{\nu\rho} - \partial_\nu \partial_\sigma B_{\mu\rho} - \partial_\mu \partial_\rho B_{\nu\sigma} \}, \end{aligned} \tag{4.105}$$

where in the first row one can recognise the linearised Riemann tensor $R^S_{\mu\nu\rho\sigma}$ of a manifold with Levi-Civita connection and metric $g_{\mu\nu} = \eta_{\mu\nu} + H^S_{\mu\nu}$, while in the second row the field strength of the two-form,

$$\mathcal{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \,, \tag{4.106}$$

appears in the combination $\partial_\mu \mathcal{H}_{\nu\rho\sigma} - \partial_\nu \mathcal{H}_{\mu\rho\sigma}$.

Substituting (4.105) in (4.104) one obtains

$$\mathcal{L}_{NL} \,=\, \frac{1}{2}\, R^S_{\mu\nu\rho\sigma}\, \frac{1}{\Box}\, R^{S\mu\nu\rho\sigma} \,-\, \frac{1}{6}\, \mathcal{H}_{\mu\nu\rho}\, \mathcal{H}^{\mu\nu\rho}\,, \tag{4.107}$$

where the first term is the spin-two representative of the Lagrangians found in [173] from the first Regge trajectory of tensionless limit of string theory, while the the second term is justified by the presence of a torsion, related to the two-form field strength as

$$T_{\mu\rho\sigma} \,=\, -\,\mathcal{H}_{\,\mu\rho\sigma}\,. \tag{4.108}$$

The nonlocal equations of motion obtained from (4.103)

$$\Box\, H^{\mu\nu} \,-\, \partial^\mu \partial_\alpha\, H^{\alpha\nu} \,-\, \partial^\nu \partial_\beta\, H^{\mu\beta} \,+\, \partial^\mu\, \partial^\nu \frac{1}{\Box} \partial_\alpha\, \partial_\beta\, H^{\alpha\beta} \,=\, 0\,, \tag{4.109}$$

are equivalent to (4.100) up to a partial gauge fixing.

Furthermore, the massless DC Lagrangian (4.104) admits a simple extension to the massive case of the Proca-like form [172, 178]

$$\begin{aligned}\mathcal{L}_{NL} \,&=\, \frac{1}{2}\, R_{\mu\nu\rho\sigma}\, \frac{1}{\Box}\, R^{\mu\nu\rho\sigma} \,-\, \frac{1}{2}\, m^2\, H_{\mu\nu}\, H^{\mu\nu} \\ &=\, \frac{1}{8}\, (F_{\mu\alpha} \,\star\, \tilde{F}_{\nu\beta})\, \frac{1}{\Box}\, (F^{\mu\alpha} \,\star\, \tilde{F}^{\nu\beta}) \,-\, \frac{1}{2}\, m^2\, (A_\mu \,\star\, \tilde{A}_\nu)\, (A^\mu \,\star\, \tilde{A}^\nu)\,.\end{aligned} \tag{4.110}$$

Indeed, the corresponding equations of motion

$$\Box\, H^{\mu\nu} - \partial^\mu \partial_\alpha\, H^{\alpha\nu} - \partial^\nu \partial_\beta\, H^{\mu\beta} + \partial^\mu\, \partial^\nu \frac{1}{\Box} \partial_\alpha\, \partial_\beta\, H^{\alpha\beta} - m^2\, H^{\mu\nu} \,=\, 0\,, \tag{4.111}$$

imply

$$\partial_\alpha\, H^{\alpha\nu} \,=\, 0 \,=\, \partial_\beta\, H^{\mu\beta}\,, \tag{4.112}$$

and thus reduce to

$$(\Box \,-\, m^2)\, H^{\mu\nu} \,=\, 0\,, \tag{4.113}$$

which describe the (free) propagation of massive graviton, two-form and scalar.

# Chapter 5

# Double-copy supertranslations

The convolutional framework is not enough to understand the geometrical meaning of the double-copy, since it works between abelianised Yang-Mills and linearised gravity. Nonetheless, it appears to be well-suited to addressing an intermediate, yet nontrivial, problem: how much DC knows about the asymptotic geometry of spacetime? In this chapter, we review the analysis performed in [34] to answer this question and, at the end, we also present some preliminary results of [38].

## 5.1 Introduction and setup

We aim to understand whether the convolutional DC dictionary works in mapping Maxwell asymptotic symmetries to the $\mathcal{N} = 0$ counterparts. This chapter is based the results of [34, 38] and partially [29].

In the discussion we often refer to the Maxwell asymptotic fields and parameters as "inputs", while we call "outputs" the double-copied quantities. In particular, our inputs are the falloffs of the spin-one field components $A_\mu, \tilde{A}_\mu$ at $\mathcal{I}^+$ and the properties of the corresponding asymptotic gauge parameters $\epsilon, \tilde{\epsilon}$. Clearly, the outputs we want to analyse are the falloffs of the DC field multiplet $h_{\mu\nu}, B_{\mu\nu}, \varphi$ and the parameters $\xi_\mu, \lambda_\mu$.

This approach already raises a conceptual issue regarding the convolution. In fact, this operation involves an integral that evaluate quantities in all the spacetime points, while we are only specifying their asymptotic behaviour. However, it is known that the asymptotic properties of a convolution can be determined by the asymptotic properties of the convoluting quantities [179]. Specifically, we address this point by means of the method of regions [180,181], whose detail we clarify below, generalising the approach used in [182].

There are technical issues as well, which we address in this chapter. First, it is important to note how the convolution does not transform covariantly. This can be understood by considering the convolution product of two Maxwell fields, without taking into account the convolution inverse $\Phi^{-1}$ for the moment. We define

$$\mathfrak{h}_{\mu\nu}(x) = [A_\mu \circ A_\nu](x) = \int d^4y\, A_\mu(y) A_\nu(x-y)\,. \tag{5.1}$$

Under a change of coordinates $x^\mu \to x'^\mu$, the l.h.s. transforms as

$$\mathfrak{h}'_{\mu\nu}(x') = \partial_\mu x'^\rho \partial_\nu x'^\sigma [A_\rho \circ A_\sigma](x')\,. \tag{5.2}$$

However, this operation is not equivalent to

$$[A'_\mu \circ A'_\nu](x') = [(\partial_\mu x'^\rho A_\rho) \circ (\partial_\nu x'^\sigma A_\sigma)](x')\,, \tag{5.3}$$

thus showing we do not have a covariant definition. We thus choose the convolutional dictionary to work with Minkowskian components. For instance, this implies

$$H_{12} = A_1 \star \tilde{A}_2, \tag{5.4}$$

with $1,2$ denoting Minkowskian components and where we are now considering the full DC convolution. At the same time, we have

$$H_{ur} \neq A_u \star \tilde{A}_r, \tag{5.5}$$

but rather

$$H_{ur} = \partial_u x^\mu \partial_r x^\nu H_{\mu\nu} = \partial_u x^\mu \partial_r x^\nu (A_\mu \star \tilde{A}_\nu) \tag{5.6}$$

where $x^\mu$ are the Minkowski coordinates. For the rest of this thesis, we always assume that the convolutional DC relations apply only in this case. One might wonder whether it is possible to generalise the convolutional dictionaries in a covariant manner. To our knowledge there is not such a definition, but this is undoubtedly a technical issue and not a conceptual one. However, let us remark that, whatever generalisation of the convolution is proposed, the results have to be compatible with those obtained working with Minkowskian components. Another possibility is to use a modified dictionary with a convolution defined at null infinity. For instance, a dictionary involving a two-sphere convolution has been proposed in [183]. This generalisation in particular could be useful when working with the celestial sphere.

Having clarified how to use DC relations, leads to another important issue: the gauge choice. Indeed, it is straightforward to verify that not all the gauge are related in a convolutional DC approach. For instance, when the input is in radial gauge, we see from (5.5) that the output is not in the corresponding gravitational one, namely the Bondi gauge (or the Newman-Unti gauge).

Therefore, while the radial gauge is the simplest choice to discuss asymptotic symmetries, we opt for the Lorenz gauge, for which the DC dictionary provide a one-to-one correspondence. Indeed, the derivation rules of convolution imply that

$$\begin{cases} \nabla^\mu A_\mu = 0 \\ \nabla^\mu \tilde{A}_\mu = 0 \end{cases} \implies \nabla^\mu H_{\mu\nu} = 0, \tag{5.7}$$

so that

$$\begin{aligned} &\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h = 0\,, \\ &\nabla^\mu B_{\mu\nu} = 0\,, \end{aligned} \tag{5.8}$$

meaning the spin-two field $h_{\mu\nu}$ satisfies the de-Donder gauge condition, whereas the two-form is in Lorenz gauge as well.

The gauge parameters satisfy

$$\begin{cases} \Box\epsilon = 0 \\ \Box\tilde{\epsilon} = 0 \end{cases} \implies \begin{cases} \Box\alpha_\mu = 0 \\ \Box\tilde{\alpha}_\mu = 0 \end{cases} \tag{5.9}$$

Furthermore, the Lorenz gauge condition enforces the transversality of the DC parameters, namely

$$\nabla^\mu \alpha_\mu = 0, \qquad \nabla^\mu \tilde{\alpha}_\mu = 0. \tag{5.10}$$

Therefore the spin-two parameter satisfies

$$\nabla \cdot \xi = 0\,, \qquad \Box\xi^\mu = 0\,, \tag{5.11}$$

which provides the condition constraining diffeomorphisms in the various incarnations of unimodular gravity [177].

At the same time, the two-form parameter obeys to

$$\nabla \cdot \lambda = 0\,, \qquad \Box\lambda_\mu = 0\,, \tag{5.12}$$

and this condition does not exactly coincide with Lorenz gauge. In fact, the parameter should obey the Maxwell equations. However, it is possible to employ the gauge-for-gauge redudancy in order to enforce the Lorenz gauge condition on the two-form parameter, leading to (5.12). Consistently, the gauge-for-gauge parameter satisfies the wave equation.

The falloff conditions and the properties of the gauge parameters in Lorenz gauge are presented in Section 2.3 and in chapter 3.

Finally, let us comment on the approach we are going to follow in order to determine the properties of the DC field. Let us recall that the formal properties of the convolution allow us to work in Fourier space, where the convolutional DC dictionary connects the Maxwell fields with the gravitational ones through a product.

In particular, denoting the Fourier transform of a field as

$$a^{(\varphi)}(p) := \mathcal{F}\{\varphi(x)\}(p), \tag{5.13}$$

and given the formal definition of the biadjoint inverse

$$a^{(\Phi^{-1})} = \left(a^{(\Phi)}\right)^{-1}, \tag{5.14}$$

we find

$$a^{(H)}_{\mu\nu} = \frac{a^{(A)a}_{\mu}\ \tilde{a}^{(\tilde{A})a'}_{\nu}}{a^{(\Phi)}_{aa'}}, \tag{5.15}$$

for the DC field, while the DC parameters can be expressed as

$$a^{(\alpha)}_{\mu} = \frac{a^{(\epsilon)}\ \tilde{a}^{(\tilde{A})}_{\mu}}{a^{(\Phi)}}, \qquad \tilde{a}^{(\tilde{\alpha})}_{\mu} = \frac{\tilde{a}^{(\tilde{\epsilon})}\ a^{(\tilde{A})}_{\mu}}{a^{(\Phi)}} \tag{5.16}$$

where we omitted the color indices, as we do for most of this thesis.

Our logic is the following: first, we want to study the single-copy quantities in Fourier space. In fact, knowing the falloff conditions and the formal properties of the spin-one field components and parameters, we can deduce their behaviour in Fourier space. Then, we employ the relations (5.15) and (5.16) to determine the properties of the Fourier transform of the DC field and parameters. From these results, we can deduce the falloffs of the fields $h_{\mu\nu}$, $B_{\mu\nu}$, $\varphi$ and the parameters $\xi_\mu$, $\lambda_\mu$ and study the outputs from an asymptotic symmetry point of view.

## 5.2 Single-copy

In this section, we aim to classify the properties of $a^{(A)}_{\mu}, \tilde{a}^{(\tilde{A})}_{\mu}, a^{(\epsilon)}, \tilde{a}^{(\tilde{\epsilon})}$ that lead to the correct falloff conditions of $A_\mu, \tilde{A}_\mu, \epsilon, \tilde{\epsilon}$ at $\mathcal{I}^+$ and the expected arbitrary dependence on the angular variables for the parameter that lead to Maxwell asymptotic symmetries. Let us recall the falloffs here

$$A_u = \mathcal{O}(r^{-1}), \qquad A_r = \mathcal{O}(r^{-2}), \qquad A_i = \mathcal{O}(1), \qquad \epsilon = \mathcal{O}(1)\,, \tag{5.17}$$

where the same conditions hold for $\tilde{A}_\mu$ and $\tilde{\epsilon}$.

As discussed, all these quantities satisfy the wave equation, therefore we start by studying it in Fourier space for a scalar field.

### 5.2.1 Scalar field at null infinity

Let us consider a scalar field satisfying the d'Alembert equation

$$\Box\varphi(x) = 0\,. \tag{5.18}$$

In Fourier space, the most general solution is

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a^{(\varphi)}(k)\,, \tag{5.19}$$

and, splitting into positive- and negative- frequency parts, we find the usual expression in terms of creation and annihilation operators

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}e^{i|\vec{k}|t - i\vec{k}\cdot\vec{x}}a^{(\varphi)}(\vec{k})\ +\ \text{h.c.}\,. \tag{5.20}$$

We are interested in the behaviour of this field near $\mathcal{I}^+$. In this analysis, retarded Bondi coordinates are described with $(u, r, z_1, z_2)$, where $\vec{z} = (z_1, z_2)$ is a $\mathbb{R}^2$-vector used to parametrise the two-sphere. It is related to the stereographic coordinates as

$$z_1 = \operatorname{Re} z, \qquad z_2 = \operatorname{Im} z\,. \tag{5.21}$$

In terms of these $\mathbb{R}^2$ real coordinates, the two-sphere metric is diagonal, with

$$\gamma_{ij} = \frac{2}{(1+|\vec{z}|^2)^2}\delta_{ij}. \tag{5.22}$$

We denote $\mathcal{D}_i$ the covariant derivative respect to this metric and with $\mathcal{D}^2 = D_iD^i$ the Laplace operator.

Cartesian and Bondi coordinates are related as

$$x^\mu = ut^\mu + \frac{2r}{1+|\vec{z}|^2}q^\mu(\vec{z}) \tag{5.23}$$

with $t^\mu = (1, 0, 0, 0)$ and $q^\mu$ is a null vector expressed as

$$q^\mu(\vec{z}) = \left( \frac{1 + |\vec{z}|^2}{2}, z_1, z_2, \frac{1 - |\vec{z}|^2}{2} \right) . \tag{5.24}$$

At the same time, we introduce Bondi-like coordinates for the momenta, but with a crucial difference:

$$k^\mu = \mu t^\mu + \omega q^\mu(\vec{w}). \tag{5.25}$$

This change of coordinates is particular because we have that $\omega$ is not the standard frequency $\omega \neq |\vec{k}|$, but a convenient rescaling of it which is

$$\omega = \frac{2}{1 + |\vec{w}|^2} |\vec{k}|, \tag{5.26}$$

while $\mu, w_1, w_2$ are related to the Cartesian components $k^\mu$ in the same way of $u, z_1, z_2$ are to $x^\mu$.

This rescaling, while appearing strange at first, provides a great simplification in computations. In fact, in $(\mu, \omega, \vec{w})$ coordinates the metric reads

$$dk^2 = -d\mu^2 - (1 + |\vec{w}|^2)\, d\mu d\omega - 2\omega w_i \, d\mu dw^i + \omega^2 \delta_{ij} dw^i dw^j \tag{5.27}$$

and therefore the unit metric on the two sphere is now simply $\gamma_{ij}(\vec{w}) = \delta_{ij}$. Hence, the covariant derivative w.r.t. $\gamma_{ij}(\vec{w})$ is an ordinary partial derivative $\partial_i$, providing great simplifications when performing integration by parts. Correspondingly, the Laplace operator is denoted with $\Delta = \partial_i \partial^i$.

Employing these coordinates and focusing on the positive-frequency part, we find

$$\varphi_+(x) = \int_0^\infty \omega^2 \frac{d\omega}{2\omega} \int d^2\vec{w} e^{-i\omega q^0(\vec{w})u - \frac{i\omega r |\vec{w} - \vec{z}|^2}{1 + |\vec{z}|^2}} a^{(\varphi)}(\omega q(\vec{w})) \,. \tag{5.28}$$

In the large-$r$ limit, the leading behaviour of $\varphi$ is determined by those regions where

$$\frac{\omega r |\vec{w} - \vec{z}\,|^2}{1 + |\vec{z}\,|^2} \sim \mathcal{O}(r^0) \,. \tag{5.29}$$

This condition, which is equivalent to

$$\omega |\vec{w} - \vec{z}\,|^2 \sim \mathcal{O}\left( \frac{1}{r} \right) , \tag{5.30}$$

is solved by

$$\omega \sim r^{-\alpha}, \qquad |\vec{w}-\vec{z}|^2 \sim r^{-1+\alpha}\,, \tag{5.31}$$

with $\alpha$ arbitrary.

However, the only two relevant regions that are those for which $\alpha = 0, 1$, whereas the contributions from other regions are either scale-less or lead to rapidly oscillating integrals. The first case goes under the name of **collinear region**, while we refer to the second as **regular region**.

### Collinear region

Let us start studying the region for which

$$|\vec{w}-\vec{z}|^2 \sim \frac{1}{r}, \qquad \omega \sim 1\,. \tag{5.32}$$

In order to analyse it, we write

$$\vec{w} = \vec{z} + \frac{1}{\sqrt{r}}\vec{s}\,, \tag{5.33}$$

with $\vec{s}$ formally of $\mathcal{O}(1)$. To leading order, we find

$$\varphi_+^{(c)}(x) \sim \frac{1}{r}\int_0^\infty d\omega\,\frac{\omega}{2}e^{-i\omega q^0(\vec{z})u}a^{(\varphi)}(\omega q(\vec{z}))\int_{\mathbb{R}^2} d^2\vec{s}\, e^{-\frac{i\omega|\vec{s}|^2}{1+|\vec{z}|^2}}\,. \tag{5.34}$$

Performing the resulting Gaussian $\vec{s}$-integral, we find

$$\varphi_+^{(c)}(x) \sim \frac{\pi\left(1+|\vec{z}|^2\right)}{2ir}\int_0^\infty d\omega\, e^{-i\omega q^0(\vec{z})u}a^{(\varphi)}(\omega q(\vec{z}))\,. \tag{5.35}$$

This is the radiation falloffs and it is easy to check that, in arbitrary spacetime dimension $D$, this gives a contribution of order $\mathcal{O}(r^{-\frac{D-2}{2}})$ with a $u$-dependence arising from the exponent in the $d\omega$ integrand.

### Regular region

The other region consists in

$$\omega \sim \frac{1}{r}, \qquad |\vec{w}-\vec{z}|^2 \sim \mathcal{O}(r^0)\,. \tag{5.36}$$

In this case, we assume the small $\omega$ expansion[1] for

$$a^{(\varphi)}(\omega q(\vec{w})) \sim \omega^{\beta-2} a_\beta(q(\vec{w})). \tag{5.37}$$

For fields, the form of the leading coefficient is determined by the soft factors[2]. Substituting this expansion in the integral, we find

$$\varphi_+^{(r)}(x) \sim \frac{1}{2}\int_{\mathbb{R}^2} d^2\vec{w}\, a_\beta^{(\varphi)}(\vec{w}) \int_0^\infty d\omega\, \omega^{\beta-1} e^{-\frac{i\omega r|\vec{w}-\vec{z}|^2}{1+|\vec{z}|^2}}, \tag{5.38}$$

and performing the $\omega$ integral, which is convergent for $\beta > 0$, we find

$$\varphi_+^{(r)}(x) \sim \frac{\Gamma(\beta)}{(ir)^\beta}\left(1+|\vec{z}|^2\right)^\beta \int_{\mathbb{R}^2} \frac{a_\beta^{(\varphi)}(\vec{w})}{2|\vec{w}-\vec{z}|^{2\beta}} d^d\vec{w}. \tag{5.39}$$

Including the corresponding analysis for negative-frequency modes, we obtain

$$\varphi_\pm(x) \sim \frac{\Gamma(\beta)}{(\pm ir)^\beta}\left(1+|\vec{z}|^2\right)^\beta \int_{\mathbb{R}^2} \frac{a_\beta(w)}{2|\vec{w}-\vec{z}|^{2\beta}} d^2w + \frac{\pi(1+|\vec{z}|^2)}{\pm 2ir}\int_0^\infty e^{\mp i\omega u} a\left(\omega q(\vec{z})\right) d\omega, \tag{5.40}$$

The collinear region is thus responsible for the radiative order, while the regular region can, in principle, lead to an arbitrary falloff depending on the value of $\beta$. The structure of the soft theorems, however, tell us that, for fields in $D = 4$, $\beta = 1$. This is studied in [184–186].

Interestingly, when generalising the spacetime dimension, the collinear region leads to the standard radiation falloff $\mathcal{O}\left(r^{-\frac{D-2}{2}}\right)$, while the regular region, compatibly with the structure of soft theorems, leads to a $\mathcal{O}\left(r^{-(D-3)}\right)$, namely the Coulombic falloffs.

### Example: O(1/r) leading order

Let us study the $\beta \to 1$ case, which is relevant for the falloffs. In particular, we consider the case studied in [182], for which

$$a^{(\varphi)} = a_1(\vec{w})\omega^{\varepsilon-1}. \tag{5.41}$$

[1] We can also consider more general low-frequency scaling. For instance, we assume a $\omega^{\beta-2}\log\omega$ term for the gauge parameter.

[2] This will be particularly relevant in the asymptotic symmetries discussion.

with $\varepsilon$ being a small parameter we study for $\varepsilon \to 0$. To simplify the notation, we remove the superscript $(\varphi)$ on $a^{(\varphi)}$. We have

$$\varphi(x) = \int \left[e^{ik\cdot x}a(k) + e^{-ik\cdot x}a^*(k)\right] \theta\left(k^0\right) 2\pi\delta\left(k^2\right) \frac{d^D k}{(2\pi)^D}. \tag{5.42}$$

In this case, the collinear region gives

$$\begin{aligned} \varphi_+^{(c)}(x) &\sim a_1(\vec{z})\frac{\pi\left(1+|\vec{z}|^2\right)}{2ir}\int_0^\infty d\omega\,\omega^{\varepsilon-1}\,e^{-i\omega q^0(\vec{z})u}\,, \\ \varphi_-^{(c)}(x) &\sim a_1^*(\vec{z})\frac{\pi\left(1+|\vec{z}|^2\right)}{-2ir}\int_0^\infty d\omega\,\omega^{\varepsilon-1}\,e^{i\omega q^0(\vec{z})u}\,. \end{aligned} \tag{5.43}$$

The integrals on the right hand sides can be performed by means of an analytic continuation, the result of which is

$$\varphi^{(c)} = \varphi_+^{(c)} + \varphi_-^{(c)} \sim \frac{\pi\left(1+|\vec{z}|^2\right)\Gamma(\varepsilon)}{2ir}\left[a_1(\vec{z})(iu)^{-\varepsilon} - a_1^*(\vec{z})(-iu)^{-\varepsilon}\right]. \tag{5.44}$$

We observe that there is a possible divergence for $\varepsilon \to 0$. However, we can impose $a_1(\vec{z}) = a_1^*(\vec{z})$ and, using

$$\begin{aligned} &(iu)^{\varepsilon} - (-iu)^{-\varepsilon} = -i\varepsilon\pi\mathrm{sng}(u) + \mathcal{O}(\varepsilon^2), \\ &\Gamma(\varepsilon) = \frac{1}{\varepsilon} + \mathcal{O}(1)\,, \end{aligned} \tag{5.45}$$

we find for $\varepsilon \to 0$

$$\varphi^{(c)}(u, r, \vec{z}) \sim -\mathrm{sgn}(u)\frac{\pi^2\left(1+|\vec{z}|^2\right)}{2ir}a_1(\vec{z})\,. \tag{5.46}$$

We now move to the regular region. The integrals give

$$\varphi^{(r)} \sim \frac{\Gamma(1-\epsilon)\left(1+|\vec{z}|^2\right)^{1-\epsilon}}{(2\pi)^{3-2\epsilon}r^{1-\epsilon}}\int_{\mathbb{R}^d}\frac{\sin(\pi\epsilon)}{|\vec{w}-\vec{z}|^{2-2\epsilon}}a_1(\vec{w})d^2\vec{w}\,. \tag{5.47}$$

We now use the property

$$\lim_{\epsilon\to 0}\frac{\sin(\pi\epsilon)}{|\vec{w}-\vec{z}|^{2-4\epsilon}} = \lim_{\epsilon\to 0}\frac{\pi\epsilon}{2|\vec{w}-\vec{z}|^{2-4\epsilon}} = \frac{\pi^2}{2}\delta^{(2)}(\vec{w}-\vec{z}), \tag{5.48}$$

for which we find

$$\varphi(x) = \left(1+|\vec{z}|^2\right)\left[\frac{a_1(\vec{z})}{8\pi r} - \frac{a_1(\vec{z})}{8\pi r}\,\mathrm{sgn}(u)\right] = \left(1+|\vec{z}|^2\right)\theta(-u)\frac{a_1(\vec{z})}{4\pi r}. \tag{5.49}$$

Let us comment that, in principle, the choice (5.41) allow us to determine the structure of $\varphi$ at $\mathcal{I}^+$. In more general case, for which (5.41) is not exact but only the leading order of a more complex expansion, the scalar field can show an arbitrary dependence on the retarded time. This is compatible with the fact that, when analysing the wave equation for a $\mathcal{O}(r^{-1})$ field, the leading order is indeed free.

### 5.2.2 Single-copy vector

Let us now apply our analysis to spin-one vectors. In particular we focus on $A_\mu$, which satisfies

$$\Box A_\mu = 0, \qquad \nabla \cdot A = 0. \tag{5.50}$$

Most of the discussion is similar to the scalar one, with the main difference given by the presence of polarization vectors. Indeed, the Fourier transform $a^A_\mu$ can be written as

$$a^A_\mu = a^{(A)}_i \varepsilon^i_\mu \,, \tag{5.51}$$

where the index $i$ refers to the sum over physical polarizations. Let us note that the use of the same letter employed for denoting the angular variables is not a mistake. In fact, polarisations in momentum space [182] can be defined as

$$\varepsilon^\mu_i(\vec{w}) = \partial_i \left( \frac{q^\mu(\vec{w})}{y \cdot q(\vec{w})} \right) , \tag{5.52}$$

with $y$ being a null reference vector. Let us observe that this choice automatically satisfies the Lorenz gauge condition, which in Fourier space is $k^\mu a^{(A)}_\mu = 0$. And since the on-shell momentum is $k^\mu = \omega q^\mu(\vec{w})$, we have

$$q \cdot \varepsilon^i = 0. \tag{5.53}$$

Hence, the solution to the equations (5.50) can be written as

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} \delta(k^2) e^{ik\cdot x} a^{(A)}_i \varepsilon^i_\mu \,. \tag{5.54}$$

Let us recall that polarisation vectors depend only on the angles. Therefore, it is straightforward to observe that nothing change from the point of view of the collinear region, since the peculiarity of that region is to evaluate, at leading order, the functions of $\vec{w}$ in $\vec{z}$. The result is thus simply modified as

$$A^{(c)}_{\mu+}(x) \sim \frac{\pi\left(1+|\vec{z}|^2\right)}{2ir} \varepsilon^i_\mu(\vec{z}) \int_0^\infty d\omega \, e^{-i\omega q^0(\vec{z})u} a^{(A)}_i(\omega q(\vec{z})) \,. \tag{5.55}$$

At the same time, assuming $\beta^{(A)} = 1$ and $a_1^{(A)}$ real again, the regular region analysis encounters a $\delta^{(2)}(\vec{w}-\vec{z})$, solving trivially the presence of an extra function of the angles.

Altogether, for the leading order from both regions one has

$$A_\mu \sim \frac{1}{r}\varepsilon_\mu(\vec{z})\left(\eta^{\text{reg}}(\vec{z})+\eta^{\text{coll}}(u,\vec{z})\right), \tag{5.56}$$

where the sum over polarisations is left implicit and $\eta^{\text{reg}}(\vec{z})$, $\eta^{\text{coll}}(u,\vec{z})$ are scalar functions encoding contributions for the regular and collinear regions, respectively. In particular, the regular region does not bring any dependence on $u$, while the collinear does.

These are the falloffs of the Minkowskian components. We are now interested in the discussion of the Bondi components. Changing the coordinates, we find

$$A_i = \mathcal{O}(1), \qquad A_u = \mathcal{O}(r^{-1}), \qquad A_r = \mathcal{O}(r^{-2}). \tag{5.57}$$

In particular, the fact that $A_r^{(1)} = 0$ is a consequence of the transversality of the polarisation vectors, since

$$A_r^{(1)} = \partial_r x^\mu A_\mu = \frac{q^\mu}{q^0}\varepsilon_\mu\left(\eta^{\text{reg}}(\vec{z})+\eta^{\text{coll}}(u,\vec{z})\right) = 0\,. \tag{5.58}$$

### 5.2.3 Single-copy parameter

To include in the discussion the gauge parameters, we need to find the set of properties that lead to a $\mathcal{O}(1)$ term with an arbitrary angular dependence. Consistently, they have to admit a polyhomogeneous expansion, namely the parameter has to be of the form

$$\epsilon = \sum_{n=0}^{\infty}\frac{\epsilon_n}{r^n} + \sum_{n=1}^{\infty}\hat{\epsilon}_n\frac{\log r}{r^n} \tag{5.59}$$

As shown, the collinear region always brings about a $\sim \frac{1}{r}$ behaviour, therefore the regular region is the only one that can be responsible for a $\mathcal{O}(r^0)$ term, which is the leading order of the gauge parameter expansion. The most general expansion for small $\omega$, compatible with the series (5.59), is

$$(a^{(\epsilon)})^a(\omega q(\vec{w})) \sim \frac{i}{\pi}\big(\omega^{\beta-2}(b_1^a(\vec{w}) - i\,b_2^a(\vec{w})) + \log\omega\,\omega^{\beta-2}\big(a_0^{(\epsilon)}\big)^a(\vec{w})\big), \tag{5.60}$$

where all the function introduced are real and $a$ is the colour index, which we leave implicit from now on.

As we see below, the first term is associated to global symmetries, while the last two are responsible of asymptotic symmetries. In particular, they both need to be total derivatives as

$$a_0 = \Delta f; \qquad b_2 = \Delta g, \tag{5.61}$$

where we recall that $\Delta$ is the $\mathbb{R}^2-$ Laplacian[3].

The terms $b_1(\vec{w}), b_2(\vec{w})$ can be easily discussed, since the result of the $\omega$ integration has already been presented in (5.39). Hence, we just need to discuss the limit $\beta \to 0$ considering the explicit form of (5.60). For $b_1$, we find to leading order

$$\epsilon_{glob} = \frac{\Gamma(\beta)}{\pi\, r^{\beta}} \sin\left(\frac{\pi}{2}\beta\right) \int d^2\vec{w}\, b_1(\vec{w})\, |\vec{w}-\vec{z}|^{-2\beta}, \tag{5.62}$$

which, in the limit $\beta \to 0$, implies

$$\epsilon_{glob} = \frac{1}{2} \int d^2\vec{w}\, b_1(\vec{w})\,. \tag{5.63}$$

This result is well-defined as long as $b_1(\vec{w})$ is integrable and, as one can see, it does no longer depends on $\vec{z}$, meaning that it corresponds to global symmetries. Note that, in principle, $b_1$ can be associated to non-trivial subleading terms. This is equivalent to solve the wave equation for a $\mathcal{O}(1)$ scalar field **without** a polyhomogeneous expansion: we cannot find a leading order with arbitrary angular dependence, but only a constant. However, the subleading terms are still there, but do not represent asymptotic symmetries.

For $b_2$, one finds

$$\epsilon(x)_{asy,1} \sim 2\left(\log r + \gamma_E - \frac{1}{\beta}\right) \int d^2\vec{w}\; b_2 + 2 \int d^2\vec{w}\; b_2\; \log\left(\frac{|\vec{w}-\vec{z}|^2}{1+|\vec{z}|^2}\right), \tag{5.64}$$

and, given $b_2 = \Delta g$ as written in (5.61), the integral is convergent and compatible with the expansion (5.59). In fact, the first term vanishes since it is a total derivate. At the same time, by integrating by parts the second term, the $\Delta \log(|\vec{w}-\vec{z}|)$ gives a $\delta^{(2)}(\vec{w}-\vec{z})$ and we find

$$\epsilon^a_{as,1} = \#_1 g(\vec{z}) \tag{5.65}$$

[3] As we discuss in the DC chapter, we need to assume $a_0 = \Delta^2 f$ for the DC parameter to have an arbitrary angular dependence. However, for this section we maintain this simpler ansatz.

thus showing how one can obtain asymptotic gauge parameters of $\mathcal{O}(1)$ in the radial coordinate, while still retaining an arbitrary dependence on the angular coordinates as well as global transformations.

Note that, without the ansatz (5.61), we would have encountered two problems: a divergence and a $\log r$ term, not compatible with the polyhomogeneous expansion chosen.

For the last term, $a_0$, we need to perform the $d\omega$ integral with extra $\log \omega$ term considered in the expansion. The result is quite similar to the previous one, but we do not find any possible divergence, since

$$\epsilon_{as,2} \sim (\log r + \gamma_E) \int d^2\vec{w}\; a_0 + \int d^2\vec{w}\; a_0 \;\log\left(\frac{|\vec{w}-\vec{z}|^2}{1+|\vec{z}|^2}\right), \tag{5.66}$$

and, asking again $a_0 = \Delta f$, we find as well

$$\epsilon_{as,2} = \#_2 f(\vec{z}) \tag{5.67}$$

Being the two contributions practically identical, we consider only the one coming from $a_0$, since it does not bring any divergence but only a $\log r$ term which is not as problematic.

The result for the leading order parameter is thus

$$\epsilon^{(0)} = \epsilon^{(0)}_{glob} + \epsilon_{as}, \tag{5.68}$$

where we defined $\epsilon_{as} := \epsilon_{as,2}$.

Let us observe that the first subleading term resulting from the ansatz (5.60), is indeed of order $\frac{\log r}{r}$ with coefficient

$$\hat{\epsilon}^{(1)} = \frac{1}{2}\, u\, \mathcal{D}^2 \epsilon_{as} \tag{5.69}$$

linear in $u$, consistently with [50].

Summarising, we write for the Cartesian components of the fields

$$\begin{aligned} A^a_\mu(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a^{(A)}_{ai}(k)\,\varepsilon^i_\mu(k) \\ \tilde{A}^{a'}_\nu(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}\tilde{a}^{(\tilde{A})}_{a'j}(k)\,\tilde{\varepsilon}^j_\nu(k) \\ \Phi_{aa'}(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a^{(\Phi)}_{aa'}(k) \end{aligned} \tag{5.70}$$

and the parameters

$$\begin{aligned}\epsilon^a(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a_a^{(\epsilon)}(k)\,,\\ \tilde{\epsilon}^{a'}(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}\tilde{a}_a^{(\tilde{\epsilon})}(k)\,.\end{aligned} \tag{5.71}$$

The structure of spin-one asymptotic symmetries fix the properties of $a_{ai}^{(A)}, \tilde{a}_{a'i}^{(\tilde{A})}, a_{aa'}^{(\Phi)}$ to be real and with a low $\omega$ behaviour as

$$a_{ai}^{(A)}, \tilde{a}_{a'i}^{(\tilde{A})}, a_{aa'}^{(\Phi)} = \mathcal{O}(\omega^{-1}), \tag{5.72}$$

with $\beta^{(A)} = \beta^{(\tilde{A})} = \beta^{(\Phi)} = 1$.

The Fourier transforms of the parameters need a double description. The asymptotic symmetries part is associated to a low-$\omega$ scaling

$$a_a^{(\epsilon)}, \tilde{a}_{a'}^{(\tilde{\epsilon})} = \mathcal{O}(\omega^{-2}\,\log\omega). \tag{5.73}$$

with $a_a^{(\epsilon)}, \tilde{a}_{a'}^{(\tilde{\epsilon})}$ imaginary[4]. At the same time, the discussion of global symmetries require

$$a_{a,glob}^{(\epsilon)}, \tilde{a}_{a',glob}^{(\tilde{\epsilon})} = \mathcal{O}(\omega^{-2}). \tag{5.74}$$

with $a_{a,glob}^{(\epsilon)}, \tilde{a}_{a',glob}^{(\tilde{\epsilon})}$ imaginary as well.

Having classified the inputs, we are ready to employ the DC relations and study the output.

## 5.3 DC field

The DC field satisfies the wave equation as well. We can write the solution as

$$H_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a_{ij}^{(H)}(k)\,\varepsilon_{\mu\nu}^{ij}(k) \tag{5.75}$$

and, using the formal properties of the convolution, we can identify

$$a_{ij}^{(H)} = \frac{a_i^{(A)}\tilde{a}_j^{(A)}}{a^{(\Phi)}}\,, \qquad \varepsilon_{\mu\nu}^{ij} = \varepsilon_\mu^i\tilde{\varepsilon}_\nu^j\,. \tag{5.76}$$

The falloffs of Cartesian components of $H_{\mu\nu}$ are therefore determined by

[4] Note that we collected a factor $i$ in (5.60)

the same discussion as before. The collinear region provides again a $\mathcal{O}(r^{-1})$ falloff, while the regular region is determined by the value of $\beta^{(H)}$. We can use (5.76) and find

$$\beta^{(H)} = \beta^{(A)} + \beta^{(\tilde{A})} - \beta^{(\Phi)} = 1, \tag{5.77}$$

and therefore we find that the regular region leads again to a $\mathcal{O}(r^{-1})$ falloff. We now study this result for the Bondi components.

### 5.3.1 Falloffs of Bondi components

The single-copy discussion can be easily generalised to the DC field. The corresponding analysis of the collinear region gives

$$H_{\mu\nu}^{\text{coll}} \sim \frac{i\pi}{r} \int_0^\infty \frac{d\omega}{2\omega} e^{-i\omega q^0(\vec{z})u} a^{(H)}(\omega q(\vec{z}))\varepsilon_{\mu\nu}(q(\vec{z})) + \text{c.c.}\,, \tag{5.78}$$

while for the regular, we find

$$H_{\mu\nu}^{\text{reg}} \sim \frac{\Gamma(\beta)}{(ir)^\beta}(1+|\vec{z}|^2) \int d^2\vec{w}\, \frac{a_0^{(H)}(q(\vec{w}))\varepsilon_{\mu\nu}(q(\vec{w}))}{2|\vec{w}-\vec{z}|^{2\beta}} + \text{c.c.}\,. \tag{5.79}$$

where we are interested in the $\beta \to 1$ limit. In this case, we can write

$$\cos\left(\frac{\pi}{2}\beta\right) |\vec{w}-\vec{z}|^{-2\beta} = \frac{\pi^2}{2}\delta^{(2)}(\vec{w}-\vec{z}) + \mathcal{O}(\beta-1), \tag{5.80}$$

so that the result for the Cartesian components from both regions is

$$H_{\mu\nu} \sim \frac{1}{r}\varepsilon_{\mu\nu}(\vec{z})\left(\eta^{\text{reg}}(\vec{z}) + \eta^{\text{coll}}(u,\vec{z})\right), \tag{5.81}$$

with $\eta^{\text{reg}}(\vec{z})$, $\eta^{\text{coll}}(u,\vec{z})$ collecting the results for the collinear and regular regions.

Moving to Bondi coordinates and using the transversality of $\varepsilon_{\mu\nu}$, we obtain

$$H_{r\mu}^{(0)} = 0 \tag{5.82}$$

so that, performing the change of coordinates, we obtain the $H-$falloffs

$$H_{ij} = \mathcal{O}(r), \qquad H_{ui} = \mathcal{O}(1), \qquad H_{uu}, H_{ri} = \mathcal{O}(r^{-1}), \qquad H_{ur}, H_{rr} = \mathcal{O}(r^{-2})\,. \tag{5.83}$$

These falloffs imply the following scalings on the graviton

$$
\begin{aligned}
h_{uu}, h_{ur}, h_{ri} &= \mathcal{O}\left(\tfrac{1}{r}\right), \qquad h_{rr} = \mathcal{O}\left(\tfrac{1}{r^2}\right), \\
h_{ui}, h_{z\bar{z}} &= \mathcal{O}\left(1\right), \qquad h_{zz}, h_{\bar{z}\bar{z}} = \mathcal{O}\left(r\right),
\end{aligned}
\tag{5.84}
$$

while the two form we find

$$
\begin{aligned}
B_{ur} &= \mathcal{O}\left(\tfrac{1}{r^2}\right), \qquad B_{ui} = \mathcal{O}\left(1\right), \\
B_{ri} &= \mathcal{O}\left(\tfrac{1}{r}\right), \qquad B_{ij} = \mathcal{O}\left(r\right).
\end{aligned}
\tag{5.85}
$$

At the same time, we can determine the falloff of the dilaton, which is

$$
\varphi = \mathcal{O}\left(\tfrac{1}{r}\right). \tag{5.86}
$$

These falloffs are compatible with those considered for the corresponding asymptotic symmetries analysis. This is our first original result: the DC dictionary works in mapping the spin-one falloffs to the corresponding DC ones. Our aim for the next chapter is to understand whether this works also for the asymptotic parameters.

## 5.4 Asymptotic symmetries

### 5.4.1 Falloffs of $\alpha_\mu$

The DC parameters satisfy the wave equation and are divergnece-free. We thus can write

$$
\begin{aligned}
\alpha_\mu(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}a^{(\alpha)}\tilde{\varepsilon}_\mu, \\
\tilde{\alpha}_\mu(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2)e^{ik\cdot x}\tilde{a}^{(\tilde{\alpha})}\varepsilon_\mu,
\end{aligned}
\tag{5.87}
$$

where we are leaving implicit the polarisations indices. The DC dictionary implies that

$$
a^{(\alpha)} = \frac{a^{(\epsilon)}\tilde{a}^{(A)}}{a^{(\Phi)}}, \qquad \tilde{a}^{(\tilde{\alpha})} = \frac{a^{(A)}\tilde{a}^{(\epsilon)}}{a^{(\Phi)}}. \tag{5.88}
$$

As discussed previously, $a^{(A)}, \tilde{a}^{(\tilde{A})}$ and $a^{(\Phi)}$ are real functions with a small $\omega$ scaling as $a(\omega q(\vec{w})) \sim \omega^{-1}a_1(q(\vec{w}))$, i.e. $\beta^{(A)} = \beta^{(\tilde{A})} = \beta^{(\Phi)} = 1$. At the same time, $a^{(\epsilon)}, a^{(\tilde{\epsilon})}$ are imaginary functions for which we consider two different contributions, involving a $\omega^{-2}\log\omega$ scaling for the part responsible for asymptotic symmetries, while $\omega^{-2}$ scaling for the global part.

Focusing only on $\alpha_\mu$, these considerations lead to a small $\omega$ expansion of the form

$$a^{(\alpha)}(\omega q(\vec{w})) = \frac{a^{(\epsilon)}\tilde{a}^{(A)}}{a^{(\Phi)}} \sim \log\omega\,\omega^{\beta-2}\,a_0^{(\alpha)} + \omega^{\beta-2}a_1^{(\alpha)}, \tag{5.89}$$

with both $a_0^{(\alpha)}, a_1^{(\alpha)}$ both being imaginary. We expect $a_0$ to be associated to asymptotic symmetries, while $a_1$ to the global ones. As we shall see, the discussion about asymptotic symmetries require some extra considerations.

## 5.4.2 DC parameters

As a reminder, we considered

$$a^{(\epsilon)} \sim \frac{i}{\pi}\left(\log\omega\,\omega^{\beta-2}a_0(\vec{w}) + \omega^{\beta-2}b_1(\vec{w})\right), \tag{5.90}$$

with the first term, obeying to $a_0 = \Delta f$, corresponding to asymptotic symmetries, while the second one leads to global symmetries.

### Global symmetries

First of all, let us discuss global symmetries, which means we will consider $b_1(\vec{w})$ as an input. The associate DC Fourier coefficient is

$$a_1^{(\alpha)} \sim \frac{i}{\pi}\omega^{\beta-2}\frac{a_1^{(A)}}{a_1^{(\Phi)}}b_1\,. \tag{5.91}$$

This leads to

$$\alpha_\mu^{glob} \sim \frac{\Gamma(\beta)}{\pi r^\beta}\sin\left(\frac{\pi}{2}\beta\right)\int d^2\vec{w}\,\frac{i}{\pi}\omega^{\beta-2}\frac{a_1^{(A)}}{a_1^{(\Phi)}}b_1\,\varepsilon_\mu|\vec{w}-\vec{z}|^{-2\beta}, \tag{5.92}$$

which, asking only for integrability conditions, implies a $\mathcal{O}(r^0)$ constant term in the limit $\beta \to 0$, meaning global symmetries.

### Asymptotic symmetries

As we shall see, the ansatz $a_0^{(\epsilon)} = \Delta f$ is not enough to find DC asymptotic symmetries. In particular, we obtain a $\mathcal{O}(\log r)$ term, which would violate the falloffs and it is not considered in the discussions of $\mathcal{O}(1)$ asymptotic symmetries in Lorenz gauge.

In order cancel the $\mathcal{O}(\log r)$ term, we require a stronger condition on the single-copy parameter, namely

$$a_0^{(\epsilon)} = \Delta^2 f. \tag{5.93}$$

The associate DC Fourier coefficient to asymptotic symmetries would be

$$a^{(\alpha)} \sim \log \omega\, \omega^{\beta-2}\, a_0^{(\alpha)} \tag{5.94}$$

with

$$a_0^{(\alpha)} = \frac{a_1^{(A)}}{a_1^{(\Phi)}} \Delta^2 f\,. \tag{5.95}$$

We now make use of the explicit form of $a_1^{(A)}, a_1^{(\Phi)}$, which is fixed by soft theorems. In fact, let us recall that, for asymptotic fields generated by sources moving in the bulk, the form of the leading contribution for soft momenta $q$ is fixed by soft theorems to be the sum of terms of the following type [107, 184–186]

$$a_1^{(A)} = e\,\frac{\varepsilon \cdot p}{p \cdot q}\,, \qquad a_1^{(\tilde{A})} = \tilde{e}\,\frac{\tilde{\varepsilon} \cdot p}{p \cdot q}\,, \qquad a_1^{(\Phi)} = \frac{g^{\Phi}}{p \cdot q}\,, \tag{5.96}$$

each associated to the motion of a given background particle of hard momentum $p$, with $e$, $\tilde{e}$ and $g^{\Phi}$ the corresponding couplings and where we choose $\varepsilon_i^{\mu}(\vec{z}) = (z_i, \delta_{ij}, -z_i) = \tilde{\varepsilon}_i^{\mu}(\vec{z})$, which is equivalent to set the reference vector $y$ introduced in (5.52) to $y^{\mu} = (-1, 0, 0, 1)$. From the DC perspective [99, 187], where at each vertex the particles in the gravitational multiplets are interpreted as products of single-copy constituents, upon further choosing $g^{\Phi} = \frac{e\tilde{e}}{\sqrt{8\pi G}}$, with $G$ denoting the gravitational constant, one finds

$$a_1^{(H)} = \sqrt{8\pi G}\,\frac{\varepsilon \cdot p\, \tilde{\varepsilon} \cdot p}{p \cdot q}\,. \tag{5.97}$$

This observation has an immediate implication on

$$\frac{\tilde{a}_1^{(A)}}{a_1^{(\Phi)}} = \frac{\sqrt{8\pi G}}{e}\,\tilde{\varepsilon} \cdot p\,. \tag{5.98}$$

Making use of (5.98) and (5.93) one finds that the dangerous term eventually cancels, since

$$\frac{\log r + \gamma_E}{2\pi} \int d^2\vec{w}\, (\Delta^2 f)\, \tilde{\varepsilon} \cdot p\, \tilde{\varepsilon}_{\mu} = 0 \tag{5.99}$$

because $\Delta^2(\tilde{\varepsilon} \cdot p\, \tilde{\varepsilon}_{\mu}) = 0$, which motivates our choice (5.93).

By integrating by parts, the DC parameters is

$$\alpha_\mu \sim f\Delta\Psi_\mu + \Psi_\mu \Delta f - 2\partial_A f \partial^A \Psi_\mu, \tag{5.100}$$

where we defined

$$\tilde{\Psi}_\mu := \frac{\tilde{a}_1^{(\tilde{A})}}{a_1^{(\Phi)}} \tilde{\varepsilon}^i_\mu := \frac{\tilde{a}_{1\,\mu}^{(\tilde{A})}}{a_1^{(\Phi)}} . \tag{5.101}$$

.

In fact, the connection with soft theorems, explored in (5.96), implies that

$$\Psi_\mu = g(\varepsilon \cdot p)\varepsilon_\mu \qquad \tilde{\Psi}_\mu = \tilde{g}(t\varepsilon \cdot \tilde{p})\tilde{\varepsilon}_\mu .. \tag{5.102}$$

Putting this inside the integral, we find

$$\alpha_\mu \sim (\log r + \gamma_E) g \int d^2\vec{w}\ \Delta^2 f(\vec{w})\ (\varepsilon \cdot p)\varepsilon_\mu + g \int d^2\vec{w}\ \Delta^2 f(\vec{w})\ (\varepsilon \cdot p)\varepsilon_\mu\ \log\left(\frac{|\vec{w}-\vec{z}|^2}{1+|\vec{z}|^2}\right) . \tag{5.103}$$

Integrating by parts the first term, the integral goes to zero, therefore we do not have any $\log r$ term. This would not be the case if we decided to use a single Laplacian.

Integrating by parts the second term, we find the result (5.100), which is a well-defined $\mathcal{O}(1)$ term with an arbitrary angular dependence.

Let us highlight that the so-derived DC parameters keep a nontrivial angular dependence at $\mathcal{O}(1)$ for large $r$ with

$$\begin{aligned} \alpha^{(0)\mu}(\vec{z}) &= \alpha^\mu_{glob} + \Delta f\,\tilde{\Psi}^\mu + f\Delta\tilde{\Psi}^\mu - 2\partial_i f \partial^i \tilde{\Psi}^\mu , \\ \tilde{\alpha}^{(0)\mu}(\vec{z}) &= \tilde{\alpha}^\mu_{glob} + \Delta \tilde{f}\,\Psi^\mu + \tilde{f}\Delta\Psi^\mu - 2\partial_i \tilde{f} \partial^i \Psi^\mu, \end{aligned} \tag{5.104}$$

and inherit from their single-copy constituents a polyhomogeneous expansion

$$\alpha^\mu = \alpha^{(0)\mu}(\vec{z}) + \mathcal{O}\left(\frac{\log r}{r}\right) . \tag{5.105}$$

The Bondi components of $\alpha^\mu$ are related to the Minkowski ones, here denoted as $\alpha^\mu_M$, by

$$\alpha^u = -\hat{q} \cdot \alpha_M, \quad \alpha^r = \hat{n}^I \alpha^I_M, \quad \alpha^i = \frac{1}{r}(\mathcal{D}^i \hat{n}^I)\alpha^I_M , \tag{5.106}$$

with $I = 1, 2, 3$ and

$$\hat{q}^\mu(\vec{z}) = \frac{q^\mu(\vec{z})}{q^0(\vec{z})} = (1, \hat{n}^I(\vec{z})), \tag{5.107}$$

where we recall that $\mathcal{D}_i$ denotes the covariant derivative on the sphere. They satisfy

$$\mathcal{D}^2\,\alpha^{(0)u} + 2\alpha^{(0)r} + 2\mathcal{D}_i\alpha^{(1)i} = -\hat{q}\cdot\mathcal{D}^2\,\alpha_M^{(0)} = 0\,, \tag{5.108}$$

where $\mathcal{D}^2 = \mathcal{D}_i\mathcal{D}^i$ and we used $\mathcal{D}^2\hat{n}^I = -2\hat{n}^I$. For the ensuing discussion it is useful to parameterise the solutions of (5.108) in terms of a scalar function $\Theta(\vec{z})$ and vector on the sphere $\Sigma^i(\vec{z})$ as follows:

$$\begin{aligned}
\alpha^{(0)u}(\vec{z}) &= \Theta(\vec{z}),\\
\alpha^{(1)i}(\vec{z}) &= -\mathcal{D}^i\Theta(\vec{z}) + \Sigma^i(\vec{z}),\\
\alpha^{(0)r}(\vec{z}) &= \frac{1}{2}\mathcal{D}^2\,\Theta(\vec{z}) - \mathcal{D}_i\Sigma^i(\vec{z})\,.
\end{aligned} \tag{5.109}$$

Finally, we collected all the results needed to analyse the asymptotic symmetries of the multiplet subsumed in the DC field $H_{\mu\nu}$. In particular, since the symmetric part $H^S_{\mu\nu}$, encoding the d.o.f. of $h_{\mu\nu}$ and $\varphi$, and the antisymmetric part $H^A_{\mu\nu}$ decouple from a Lagrangian point of view [172], we can analyse their asymptotic properties separately.

In particular, since the asymptotic charges involve specific components of the curvatures, it is useful to recall the definition of the DC curvature [11]:

$$R_{\mu\nu\rho\sigma} = -\frac{1}{2}F_{\mu\nu}\star\tilde{F}_{\rho\sigma} = R^S_{\mu\nu\rho\sigma} + R^A_{\mu\nu\rho\sigma}\,, \tag{5.110}$$

where we can recognise the linearised Riemann tensor for $H^S_{\mu\nu}$ together with a gauge invariant combination of derivatives of $H^A_{\mu\nu}$, related to the torsion part of the Riemann tensor. In particular,the Minkowski components are

$$R^I_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu\partial_\sigma H^I_{\nu\rho} - \partial_\nu\partial_\sigma H^I_{\mu\rho} + \partial_\nu\partial_\rho H^I_{\mu\sigma} - \partial_\mu\partial_\rho H^I_{\nu\sigma})\,, \tag{5.111}$$

with $I = A, S$. For the antisymmetric part, we can write

$$R^A_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\mu\mathcal{H}_{\nu\rho\sigma} - \partial_\nu\mathcal{H}_{\mu\rho\sigma},\right) \tag{5.112}$$

with

$$\mathcal{H}_{\alpha\beta\gamma} = \partial_\alpha H^A_{\beta\gamma} + \partial_\gamma H^A_{\alpha\beta} + \partial_\beta H^A_{\gamma\alpha} \tag{5.113}$$

being the two-form field strength.

## 5.5 Asymptotic symmetries

### 5.5.1 Supertranslations and scalar charges

The DC dictionary implies that the gauge parameter of $H^S_{\mu\nu}$ is

$$\xi^\mu = \frac{1}{2}(\alpha^\mu + \tilde{\alpha}^\mu)\,. \tag{5.114}$$

The asymptotic surface charges correspond to the fixed$-u$, large$-r$ limit of

$$Q^S = -r^3 \int dz d\bar{z} \gamma_{z\bar{z}} \left( \xi^u R^S_{urur} + \xi^i R^S_{irur} \right)\,. \tag{5.115}$$

where one can identify two contributions to the curvature

$$R^S_{\mu\nu\rho\sigma} = R^h_{\mu\nu\rho\sigma} + R^\varphi_{\mu\nu\rho\sigma}, \tag{5.116}$$

associated to the graviton $h_{\mu\nu}$ and to the scalar $\varphi$, respectively.

In particular, the graviton $h_{\mu\nu}$ is in de Donder gauge, while the condition $\nabla \cdot \xi = 0$ is consistent with the scalar degree of freedom being encoded in the equation for the trace of $H^S_{\mu\nu}$ [172–174]. We aim to show that the results obtained from the DC coincide, up to some considerations, with supertranslations, which in de Donder gauge were discussed in [31, 36, 133].

Let us recall the falloffs we derived from the DC analysis

$$\begin{aligned} \varphi, h_{uu}, h_{ur}, h_{ri} &= \mathcal{O}\left(\tfrac{1}{r}\right), & h_{rr} &= \mathcal{O}\left(\tfrac{1}{r^2}\right), \\ h_{ui}, h_{z\bar{z}} &= \mathcal{O}\left(1\right), & h_{zz}, h_{\bar{z}\bar{z}} &= \mathcal{O}\left(r\right), \end{aligned} \tag{5.117}$$

while the leading Bondi components of $\xi^\mu$ satisfy both (5.108) and (5.109). We define

$$\begin{aligned} \Theta(\vec{z}) \quad &\to \quad T(\vec{z}) \;= \frac{1}{2}\left(\Theta(\vec{z}) + \tilde{\Theta}(\vec{z})\right), \\ \Sigma(\vec{z}) \quad &\to \quad S^i(\vec{z}) = \frac{1}{2}\left(\Sigma^i(\vec{z}) + \tilde{\Sigma}^i(\vec{z})\right). \end{aligned} \tag{5.118}$$

In particular, it is possible to perform small gauge transformations employing $S^i$ setting to zero $h^{(1)}_{ur}$ and $h^{(2)}_{rr}$.

The residual parameters are then further constrained by $\hat{q} \cdot \mathcal{D}_i \xi = 0$, which suffices to fix the correct cross-dependences among the components of $\xi^\mu$, namely

$$\xi^{(1)i} = -\mathcal{D}^i T(\vec{z}), \qquad \xi^{(0)r} = \frac{1}{2}\mathcal{D}^2\, T(\vec{z})\,. \tag{5.119}$$

This identifies the leading-order asymptotic symmetries of the DC graviton as BMS supertranslations. The corresponding charge receives contribution only from the first term in (5.115), on account of (5.84), and has the form

$$Q^h = -\int dz d\bar{z} \gamma_{z\bar{z}} T(z,\bar{z}) R^{h(3)}_{urur}, \tag{5.120}$$

with

$$R^{h(3)}_{urur} = -\frac{1}{2}\mathcal{D}^i\mathcal{D}^j h^{(-1)}_{ij}\,. \tag{5.121}$$

In addition, from both terms in (5.115) one finds that the DC builds infinitely-many asymptotic charges for the scalar field of the form

$$Q^\varphi = -\frac{1}{2}\int dz d\bar{z} \gamma_{z\bar{z}} (\mathcal{D}^2+1) T(z,\bar{z}) \varphi^{(1)}\,, \tag{5.122}$$

which reproduce the asymptotic charges proposed for scalar fields in [37], with the identification of the corresponding smearing function with $(\mathcal{D}^2+1)T(z,\bar{z})$.

### 5.5.2 Two-form asymptotic symmetries

We now focus on the two-form part $B_{\mu\nu}$. The parameter is defined as

$$\lambda^\mu = \frac{1}{2}(\alpha^\mu - \tilde{\alpha}^\mu) \tag{5.123}$$

and satisfies (5.12). The asymptotic charge corresponding to (5.115) emerges from the $r \to +\infty$ limit of

$$Q^B = -r^3 \int dz d\bar{z} \gamma_{z\bar{z}} \Big( \lambda^u R^A_{urur} + \lambda^i R^A_{irur} \Big)\,, \tag{5.124}$$

where in this case the first contribution vanishes identically, while the latter gives a non-vanishing contribution originating from $\lambda^{i(1)} R^{A(2)}_{irur}$ with

$$R^{A(2)}_{irur} = \mathcal{H}^{(1)}_{iur} = \partial_u B^{(1)}_{ri} = -\mathcal{D}^j B^{(-1)}_{ij}\,, \tag{5.125}$$

where in the last equality we used the equations of motion. Altogether, the corresponding asymptotic charge is

$$Q^B = \int dz d\bar{z} \gamma_{z\bar{z}}\, \lambda^{i(1)}\, \mathcal{D}^j B^{(-1)}_{ij}\,, \tag{5.126}$$

and we need to determine whether in our DC setup there is room for a nonvanishing coefficient $\lambda^{i(1)}$. From (5.105) we obtain

$$\begin{aligned}
\lambda^u &= \sum_{n=0}^{\infty} \frac{\lambda^{(n)u}}{r^n} + \sum_{n=1}^{\infty} \hat{\lambda}^{(n)u} \frac{\log r}{r^n}, \\
\lambda^r &= \sum_{n=0}^{\infty} \frac{\lambda^{(n)r}}{r^n} + \sum_{n=1}^{\infty} \hat{\lambda}^{(n)r} \frac{\log r}{r^n}, \\
\lambda^i &= \sum_{n=1}^{\infty} \frac{\lambda^{(n)i}}{r^n} + \sum_{n=2}^{\infty} \hat{\lambda}^{(n)i} \frac{\log r}{r^n},
\end{aligned} \tag{5.127}$$

where in particular the leading coefficients are related by

$$\begin{aligned}
&\mathcal{D}^2 \lambda^{(0)u} + 2\lambda^{(0)r} + 2\mathcal{D} \cdot \lambda^{(1)} = 0\,, \\
&2\partial_u \hat{\lambda}^{(1)r} = (\mathcal{D}^2 - 2)\lambda^{(0)r} - 2\mathcal{D} \cdot \lambda^{(1)}\,, \\
&2\partial_u \hat{\lambda}^{(2)i} = (\mathcal{D}^2 - 1)\lambda^{(1)i} + 2\mathcal{D}^i \lambda^{(0)r}\,,
\end{aligned} \tag{5.128}$$

as a consequence of the Lorenz gauge condition. We parameterise the corresponding solutions as follows

$$\begin{aligned}
\Lambda^{(0)u} &= \tau(\vec{z}\,)\,, \\
\Lambda^{(1)i} &= -\mathcal{D}^i \tau(\vec{z}\,) + \sigma^i(\vec{z}\,)\,, \\
\Lambda^{(0)r} &= \frac{1}{2}\mathcal{D}^2 \tau(\vec{z}\,) - \mathcal{D} \cdot \sigma(\vec{z}\,)\,.
\end{aligned} \tag{5.129}$$

having defined

$$\begin{aligned}
\tau(\vec{z}\,) &= \frac{1}{2}\left(\Theta(\vec{z}\,) - \tilde{\Theta}(\vec{z}\,)\right), \\
\sigma^i(\vec{z}\,) &= \frac{1}{2}\left(\Sigma^i(\vec{z}\,) - \tilde{\Sigma}^i(\vec{z}\,)\right).
\end{aligned} \tag{5.130}$$

with $\tau(\vec{z}\,)$ and $\sigma^i(\vec{z}\,)$ providing the counterparts of (5.118). From the second of (5.129) we see that in principle there are two types of contributions to $Q^B$, depending on $\tau(\vec{z}\,)$ and on $\sigma^i(\vec{z}\,)$, respectively. The contribution to the charge involving $\tau(\vec{z}\,)$, however, actually vanishes:

$$Q^B_\tau = \int dz d\bar{z}\, \gamma^{z\bar{z}} (\mathcal{D}_{\bar{z}} \mathcal{D}_z \tau - \mathcal{D}_z \mathcal{D}_{\bar{z}} \tau) B^{(-1)}_{z\bar{z}} = 0\,. \tag{5.131}$$

This outcome can be understood upon recognising that variations involving $\Lambda^{(1)i} = -\mathcal{D}^i \tau$ represent gauge-for-gauge transformations that do not affect the free data encoded in $B_{ij}^{(-1)}$. Thus, it is the presence of a non-trivial $\sigma^i(\vec{z})$ that allows one to identify the asymptotic charge of the two-form:

$$Q_\sigma^B = \int dz d\bar{z}\, \gamma^{z\bar{z}} (\mathcal{D}_{\bar{z}}\sigma_z - \mathcal{D}_z\sigma_{\bar{z}}) B_{z\bar{z}}^{(-1)} \,. \tag{5.132}$$

Let us observe that the parameters $\tau(\vec{z})$ and $\sigma^i(\vec{z})$ play a role which is somehow reversed w.r.t. their gravitational counterparts $T(\vec{z})$ and $S^i(\vec{z})$ defined in (5.118), with the supertranslation parameter $T(\vec{z})$ providing the relevant asymptotic symmetry and $S^i(\vec{z})$ encoding small gauge transformations.

Finally, let us recall that the polyhomogeneous expansion (5.127) induces partial violations of the falloffs (5.85), consistently with the analysis of chapter 3. Indeed, if $\sigma^i \neq 0$ then

$$\begin{aligned} \delta B_{ur} &= \mathcal{O}\left(\tfrac{1}{r}\right), & \delta B_{ui} &= \mathcal{O}(\log r), \\ \delta B_{ri} &= \mathcal{O}(1), & \delta B_{ij} &= \mathcal{O}(r)\,, \end{aligned} \tag{5.133}$$

and therefore the falloffs are generically not preserved except for $B_{ij}$, which encodes the free data. The same mechanism is at work for the spin-one and spin-two cases in Lorenz and de Donder gauges, respectively, where polyhomogeneous scalings of the parameters were found to be necessary in order for asymptotic charges to exist. One possible attitude is to intepret the violations of (5.85) as physically innocuous, since they affect only pure-gauge components of the field and as such do not alter any physical observables [50]. Alternatively, one may also envisage the possibility to generalise the falloffs of the fields from the very start, as suggested in [26,36], possibly in order to take the presence of matter into account.

The charges (5.132) should provide the dual counterparts of the asymptotic charges for scalar fields found in [37].

## 5.6 Double-copy superrotations

Here, we present some preliminary results which will be discussed in detail in [38].

### 5.6.1 Single-copy parameter

#### O(r) Maxwell asymptotic symmetries

As an input, it seems natural to consider the $\mathcal{O}(r)$ Maxwell asymptotic parameter, first presented in [25].

The parameter admits the expansion

$$\epsilon = \sum_{n=-1}^{\infty} \frac{\epsilon^{(n)}}{r^n} + \sum_{n=1}^{\infty} \hat{\epsilon}^{(n)} \frac{\log r}{r^n}, \tag{5.134}$$

where it is worth to remark how the logarithmic series starts again from $n = 1$. In fact, we remind that the second series is required to have a $\mathcal{O}(r^0)$ term with arbitrary angular dependence, but then the $\mathcal{O}(r)$ term has it without asking extra terms[5].

The wave equation, discussed in detail in the first part of the thesis and in the appendix, implies

$$\begin{aligned}
\epsilon^{(-1)} &= \epsilon^{(-1)}(z,\bar{z}) \\
\epsilon^{(0)} &= u\frac{\Delta+2}{2}\epsilon^{(-1)} + \epsilon_c^{(0)}(z,\bar{z}) \\
\hat{\epsilon}^{(1)} &= u^2\frac{\Delta-2}{8}\Delta\epsilon^{(-1)} + \frac{1}{2}u\,\Delta\epsilon_c^{(0)} + \hat{\epsilon}_c^{(1)}(z,\bar{z}) \\
\epsilon^{(1)} &= \epsilon^{(1)}(u,z,\bar{z})
\end{aligned} \tag{5.135}$$

where $\epsilon_c^{(n)}$ denotes a $u-$integration function that enters at every order and $\epsilon^{(1)}$ is completely free. Let us observe that this parameter does not preserve the falloffs considered in the DC discussion, but this also happens for DC $\mathcal{O}(1)$ asymptotic symmetries. We are again admitting violations in pure gauge sectors. This is compatible with the approach used in Chapter 3, for which we considered generalised falloffs in pure gauge sectors from the beginning. Using the notation of Chapter 3, in this case, we can assume we are starting from a gauge orbit with $a = 0$. However, from the point of view of the DC analysis, to our best understanding this would not add more to the results. Furthermore, as discussed in the first part of the thesis, the parameter leads to a divergent but renormalizable charge.

In order to apply the same procedure used for supertranslations, we need to understand what are the properties of the Fourier transform $a^{(\epsilon)}$ needed to find a $\mathcal{O}(r)$ parameter. We start with the assumption

$$a_{\pm}^{(\epsilon)} \sim \omega^{\beta-2} a_{\pm}^{(\epsilon)}(\vec{w}) \qquad \omega \to 0\,. \tag{5.136}$$

[5] We could consider a $\log r$ term in the series (5.134), but in order for it to have a non-trivial angular dependence we should add a third series

$$\sum_{n=1}^{\infty} \tilde{\epsilon}^{(n)} \frac{\log^2 r}{r^n}$$

.

and note that the integral

$$\epsilon_\pm \sim \frac{1}{2}\int d^2\vec{w}\, a_\pm \int d\omega\, \omega^{\beta-1} e^{\mp i\omega r x} \tag{5.137}$$

is divergent for $\beta < 0$ and therefore the limit $\beta \to -1$ is not in principle a definite operation. However, we can perform the integral in the case $\beta > 0$ and then extend the result to all values of $\beta$ by means of an analytic continuation.

**Asymptotic symmetries.** We start analysing the case $a_+^{(\epsilon)} = -a_-^{(\epsilon)}$. In particular, in this case the integral of the sum $\epsilon_+ + \epsilon_-$ is defined for $\beta > -1$ and therefore the $\beta \to -1$ limit is a well-defined operation. The integral gives

$$\epsilon \sim \frac{r^{-\beta}}{2} \sin\left(\frac{\pi\beta}{2}\right) \Gamma(\beta) \int d^2\vec{w}\, a_+^{(\epsilon)}(\vec{w}) x^{-\beta}\,. \tag{5.138}$$

In order to study the $\beta \to -1$ limit, we define $\beta = \varepsilon - 1$ so that

$$\begin{aligned}
&\sin\left(\frac{\pi\beta}{2}\right) = -\cos\left(\frac{\pi\varepsilon}{2}\right) = -1 + \mathcal{O}(\varepsilon^2),\\
&r^{-\beta} = r(1 - \varepsilon \log r + \mathcal{O}(\varepsilon^2)),\\
&\Gamma(\beta) = -\frac{1}{\varepsilon} + (\gamma - 1) + \mathcal{O}(\varepsilon),\\
&x^{-\beta} \sim |\vec{w} - \vec{z}|^{2-2\varepsilon}.
\end{aligned} \tag{5.139}$$

We have a divergence coming from $\frac{1}{\varepsilon}$, but we still might ask conditions on $a_+^{(\epsilon)}$ so that the integral results finite. Using the same logic of supertranslations discussion, we ask $a_+^{(\epsilon)}$ to be a total derivative. In particular, we have

$$\begin{aligned}
&\Delta|\vec{w} - \vec{z}|^{2-2\varepsilon} = (2 - 2\varepsilon)^2 |\vec{w} - \vec{z}|^{-2\varepsilon}\,,\\
&\Delta^2|\vec{w} - \vec{z}|^{2-2\varepsilon} \sim \varepsilon\, \delta^{(2)}(\vec{w} - \vec{z})\,,
\end{aligned} \tag{5.140}$$

therefore we find that $a_+ = \Delta^2 f(\vec{w})$ implies

$$\epsilon = r\, f(\vec{w}) + \ldots, \tag{5.141}$$

thus allowing for a leading order with arbitrary angular variables.

**Global symmetries.** We now focus on the case $a_+^{(\epsilon)} = a_-^{(\epsilon)}$ which, in principle, leads to a divergent integral for $\beta \leq 0$. However, we can perform the integral considering $\beta > 0$ and then generalise the result to every value of

$\beta$ by means of an analytic continuation. The result is

$$\epsilon \sim -\frac{\pi}{1+|\vec{z}|^2} r \int d^2w \, a_+^{(\epsilon)} |\vec{w}-\vec{z}|^2, \tag{5.142}$$

which is convergent if the integral is. For instance, we can ask a falloff condition on $a_+^{(\epsilon)}$, namely

$$a_+^{(\epsilon)} \sim \frac{1}{|\vec{w}|^a} \tag{5.143}$$

with $a > 4$.

Let us observe that the integral does not have an arbitrary angular dependence on $\vec{z}$, but the structure

$$\epsilon \sim \frac{a + z^i b_i + c|\vec{z}|^2}{1+|\vec{z}|^2}, \tag{5.144}$$

with $a, b_i, c$ being (not all independent) constants. This, to our understanding, might be a candidate to discuss the Lorentz group at the level of DC.

The last possibility is to consider an expansion

$$a^{(\epsilon)}(\omega, \vec{w}) \sim \omega^{\beta-2} \log\omega \, a^{(\epsilon)}(\vec{w}), \tag{5.145}$$

but in the case $\beta \to -1$ ($\varepsilon \to 0$) we have a $\frac{1}{\varepsilon^2}$ divergence for $a_+^{(\epsilon)} = -a_-^{(\epsilon)}$ and the unavoidable presence of a $r\log r$ for $a_+^{(\epsilon)} = a_-^{(\epsilon)}$. Hence, this expansion does not seem useful to discuss a $\mathcal{O}(r)$ parameter.

## 5.6.2 DC parameter

We now want to study the DC parameter. By means of the DC relations, the regular region for the DC parameter gives

$$\alpha_\mu \sim \frac{r^{-\beta}}{2} \sin\left(\frac{\pi\beta}{2}\right) \Gamma(\beta) \int d^2\vec{w} \, a_+^{(\epsilon)}(\vec{w}) \Psi_\mu x^{-\beta} \tag{5.146}$$

where $\Psi_\mu = \frac{a_\mu^{(A)}}{a^{(\Phi)}}$. In [34], we use $\Psi^\mu = (\varepsilon^i \cdot p_\mu)\varepsilon_i^\mu$, where $\varepsilon_i^\mu$ is the polarisation vector and $p_\mu$ is the momentum of a hard particle. For the standard choice of polarization vectors, $\varepsilon_i^\mu = D_i q^\mu$, $\Psi_\mu$ is quadratic in $\vec{w}$.

As a consequence of $\Psi_\mu \sim \mathcal{O}(|\vec{w}|^2)$, a mechanism similar to the supertranslations case occurs: to ask $a_+^{(\epsilon)} = \Delta^2 f$ is not enough, but we need

$$a_+^{(\epsilon)} = \Delta^3 f. \tag{5.147}$$

Integrating by parts, we find a well-defined $\mathcal{O}(r)$ DC parameter, candidate to describe superrotations, $\mathcal{O}(r)$ two-form asymptotic symmetries [29] and the subleading asymptotic charge for the scalar dilaton. Since it maintains the angular dependence, we expect to find generalised BMS instead of extended BMS.

At the same time, the global symmetries seem to be mapped to the Lorentz group, described by the six global conformal Killing vectors, with a structure analogous to (5.144).

## 5.7 Outlook

In this chapter, we reviewed the derivation of DC supertranslations presented in [34], whereby electromagnetic large gauge transformations "square" to BMS supertranslations for the graviton and to asymptotic symmetries for the Kalb-Ramond two-form, while also providing an explanation to the existence of infinitely-many asymptotic charges for the scalar field. Our approach consisted in performing the asymptotic expansions of the on-shell fields by means of the method of regions, which allowed us to deduce the falloffs of the double-copy fields from those of their single copy constituents under suitable assumptions. A similar mechanism works for the corresponding asymptotic symmetry parameters. The appearance of terms involving $\log r$ at subleading orders, which is a common feature of asymptotic expansions performed in Lorenz and de Donder gauges, motivated us to revisit the study of the asymptotic symmetries and charges for the double-copy fields, in particular for the two-form. We found that the logarithms "predicted" by the convolutional double copy in this way are precisely those needed to allow for the existence of nontrivial asymptotic symmetries and charges.

As a first generalisation, we proposed some preliminary results on DC superrotations. We find that, in order to obtain DC parameter of $\mathcal{O}(r)$, we need to consider $\mathcal{O}(r)$ gauge parameter at the single-copy level. This is different from the results of [14], where a subset of superrotations seems to emerge from a particular subset of $\mathcal{O}(1)$ YM symmetries. However, in [14] is also observed that there is potentially another class of asymptotic symmetries if one allows for $\mathcal{O}(r)$ single-copy gauge transformations. Therefore, it might be very interesting to apply our approach to the self-dual sector and to confront the results with those of [14] and to understand which results are a peculiarity of the self-dual sector.

There are several other possible generalisations of our work that could be considered in the future. First of all, we would like to study the DC of arbitrary higher-order asymptotic symmetries. The structural similarities between spin-one and two-form found in [29] seem to indicate that there might be some DC relations. However, it is worth to remark that gravity is understood only up to $\mathcal{O}(r^2)$, as the computations become very challenging to higher orders. Therefore, it is not clear what to expect on the gravitational side.

Furthermore, as observed, the study of the DC for asymptotic symmetries in the convolutional approach is tightly linked to soft theorems and it would be interesting to better investigate this connection. In this respect, the generalisation to $\mathcal{O}(r^n)$ asymptotic symmetries would help in exploring the connection with the sub$^n$-leading soft theorem.

Further potential developments include the generalisation of our approach to different, yet related, contexts. The study of asymptotic symmetries and the associated soft theorems has been generalised to (Anti) de Sitter backgrounds [52–54, 188], for which the convolutional DC dictionary has recently been investigated in [39]. Another enticing perspective is to investigate asymptotic symmetries of higher spin gauge fields [32,33] formulated as convolutional double copies of lower-spin building blocks[6]. Moreover, since the convolutional DC can be equivalently formulated in any number of spacetime dimensions, it is certainly possible to imagine that our work and the generalisations thereof could be also explored beyond $D = 4$.

[6]See [189] for a chiral higher-spin DC and [190] for a Kerr-Schild-type one.

# Appendix A

# Equations of motion

## A.1 Order-by-order expressions

Let us remind that, for radial expansions, we employ the following notation

$$\varphi = \sum_{n_1} \frac{\varphi^{(n_1)}}{r^{n_1}} + \sum_{n_2} \hat{\varphi}^{(n_2)} \frac{\log r}{r^{n_2}}, \tag{A.1}$$

so that $\varphi^{(n_1)}$ is the coefficient for $r^{-n_1}$.

### Order by order tricks

In evaluating order-by-order equations, these are the relevant terms that often occur:

$$\begin{aligned} \frac{1}{r^\alpha}\partial_r\varphi : \quad & \begin{cases} \left[\frac{1}{r^\alpha}\partial_r\varphi\right]^{(n)} = \hat{\varphi}^{(n-(\alpha+1))} - (n-(\alpha+1))\varphi^{(n-(\alpha+1))} \\ \left[\widehat{\frac{1}{r^\alpha}\partial_r\varphi}\right]^{(n)} = -(n-(\alpha+1))\hat{\varphi}^{(n-(\alpha+1))} \end{cases} \\ \partial_r^2\varphi \ : \quad & \begin{cases} \left[\partial_r^2\varphi\right]^{(n)} = (n-1)(n-2)\varphi^{(n-2)} - (2n-3)\hat{\varphi}^{(n-2)} \\ \left[\widehat{\partial_r^2\varphi}\right]^{(n)} = (n-1)(n-2)\hat{\varphi}^{(n-2)} \end{cases} \end{aligned} \tag{A.2}$$

### Scalar

The scalar wave equation reads

$$\Box\epsilon = \left(\partial_r^2 - 2\partial_u\partial_r - \frac{2}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right)\epsilon = 0 \tag{A.3}$$

which, order-by-order, implies the system

$$\begin{cases} (2-2n)\partial_u \epsilon^{(n)} + 2\partial_u \hat{\epsilon}^{(n)} = [\Delta + (n-1)(n-2)]\epsilon^{(n-1)} + (3-2n)\hat{\epsilon}^{(n-1)} \,, \\ (2-2n)\partial_u \hat{\epsilon}^{(n)} = [\Delta + (n-1)(n-2)]\hat{\epsilon}^{(n-1)} \,. \end{cases} \tag{A.4}$$

## Vector

A vector in Lorenz gauge obeys to

$$\Box \mathcal{A}_u = \left(\partial_r^2 - 2\partial_u\partial_r - \frac{2}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right)\mathcal{A}_u = 0 \,, \tag{A.5a}$$

$$\Box \mathcal{A}_r = \left(\partial_r^2 - 2\partial_u\partial_r - \frac{2}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right)\mathcal{A}_r + \frac{2}{r^2}(\mathcal{A}_u - \mathcal{A}_r) - \frac{2}{r^3}D\cdot A = 0 \,, \tag{A.5b}$$

$$\Box \mathcal{A}_i = \left(\partial_r^2 - 2\partial_u\partial_r + \frac{\Delta - 1}{r^2}\right)\mathcal{A}_i - \frac{2}{r}D_i(\mathcal{A}_u - \mathcal{A}_r) = 0 \,, \tag{A.5c}$$

$$\nabla \cdot \mathcal{A} = -\partial_u \mathcal{A}_r + \left(\partial_r + \frac{2}{r}\right)(\mathcal{A}_r - \mathcal{A}_u) + \frac{1}{r^2}D\cdot \mathcal{A} = 0 \,. \tag{A.5d}$$

The gauge condition reads

$$\begin{cases} \partial_u \mathcal{A}_r^{(n)} - (n-3)\left(\mathcal{A}_u^{(n-1)} - \mathcal{A}_r^{(n-1)}\right) - D\cdot\mathcal{A}^{(n-2)} + \hat{\mathcal{A}}_u^{(n-1)} - \hat{\mathcal{A}}_r^{(n-1)} = 0 \,, \\ \partial_u \hat{\mathcal{A}}_r^{(n)} - (n-3)\left(\hat{\mathcal{A}}_u^{(n-1)} - \hat{\mathcal{A}}_r^{(n-1)}\right) - D\cdot\hat{\mathcal{A}}^{(n-2)} = 0 \,, \end{cases} \tag{A.6}$$

while the equations of motions are[1]

$$\begin{aligned} &\begin{cases} (2-2n)\partial_u \mathcal{A}_u^{(n)} + 2\partial_u \hat{\mathcal{A}}_u^{(n)} = [\Delta + (n-1)(n-2)]\mathcal{A}_u^{(n-1)} - (2n-3)\hat{\mathcal{A}}_u^{(n-1)} \,, \\ (2-2n)\partial_u \hat{\mathcal{A}}_u^{(n)} = [\Delta + (n-1)(n-2)]\hat{\mathcal{A}}_u^{(n-1)} \,, \end{cases} \\ &\begin{cases} (2-2n)\partial_u \mathcal{A}_r^{(n)} + 2\partial_u \hat{\mathcal{A}}_r^{(n)} = [\Delta + n(n-3)]\mathcal{A}_r^{(n-1)} + 2\mathcal{A}_u^{(n-1)} - 2D\cdot\mathcal{A}^{(n-2)} - (2n-3)\hat{\mathcal{A}}_r^{(n-1)}, \\ (2-2n)\partial_u \hat{\mathcal{A}}_r^{(n)} = [\Delta + n(n-3)]\hat{\mathcal{A}}_r^{(n-1)} + 2\,\hat{\mathcal{A}}_u^{(n-1)} - 2D\cdot\hat{\mathcal{A}}^{(n-2)} \,, \end{cases} \\ &\begin{cases} -2n\,\partial_u \mathcal{A}_i^{(n)} + 2\partial_u \hat{\mathcal{A}}_i^{(n)} = [\Delta + n(n-1) - 1]\mathcal{A}_i^{(n-1)} - 2D_i(\mathcal{A}_u^{(n)} - \mathcal{A}_r^{(n)}) - (2n-1)\hat{\mathcal{A}}_i^{(n-1)} \,, \\ -2n\,\partial_u \hat{\mathcal{A}}_i^{(n)} = [\Delta + n(n-1) - 1]\hat{\mathcal{A}}_i^{(n-1)} - 2D_i(\hat{\mathcal{A}}_u^{(n)} - \hat{\mathcal{A}}_r^{(n)}) \,. \end{cases} \end{aligned} \tag{A.7}$$

These equations, containing the logarithmic part, are important for the two-form parameter. However, in order to establish the explicit relation between

[1] Moving to the right the l.h.s. of each equation we recover the order-by-order expression of $\left[\Box \mathcal{A}_\mu\right]^{(n+1)}$.

the field strength and the spin-one field, we need the equations only for the $A-$part of $\mathcal{A}$, which admits a power expansion. These equations are identical to the equations of the pure logarithmic part, but here we report them for simplicity

$$\partial_u A_r^{(n)} = (n-3)\big(A_u^{(n-1)} - A_r^{(n-1)}\big) + D\cdot A^{(n-2)}\,, \tag{A.8a}$$
$$(2-2n)\partial_u A_u^{(n)} = [\Delta + (n-1)(n-2)]A_u^{(n-1)}\,, \tag{A.8b}$$
$$(2-2n)\partial_u A_r^{(n)} = [\Delta + n(n-3)]A_r^{(n-1)} + 2\,A_u^{(n-1)} - 2D\cdot A^{(n-2)}\,, \tag{A.8c}$$
$$-2n\,\partial_u A_i^{(n)} = [\Delta + n(n-1)-1]A_i^{(n-1)} - 2D_i(A_u^{(n)} - A_r^{(n)})\,. \tag{A.8d}$$

The last equation can be split by means of (3.34) and we find

$$-2n\,\partial_u A^{(n)} = [\Delta + n(n-1)]A^{(n-1)} - 2(A_u^{(n)} - A_r^{(n)}) \tag{A.9a}$$
$$-2n\,\partial_u A'^{(n)} = [\Delta + n(n-1)]A_i^{('n-1)}\,. \tag{A.9b}$$

where we used that $(\Delta-1)D_iA = D_i\Delta A$.

## Spin two

We work with De-donder gauge, namely

$$G_\mu := \nabla\cdot h_\mu - \frac{1}{2}\nabla_\mu h = 0\,. \tag{A.10}$$

The equations of motion are

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = 0 \implies \Box h_{\mu\nu} = 0\,, \tag{A.11}$$

where we used that, in absence of sources, $\Box h = 0$ and

$$h = h_{rr} - 2h_{ur} + \frac{\gamma^{ij}}{r^2}h_{ij}\,. \tag{A.12}$$

We have

$$\begin{aligned}
\Box h_{rr} =& \partial_r^2 h_{rr} - 2\partial_u\partial_r h_{rr} + \frac{\Delta}{r^2}h_{rr} - \frac{4}{r^3}D^i h_{ir} - \frac{2}{r}(\partial_u - \partial_r)h_{rr} \\
&+ \frac{4}{r^2}(h_{ur} - h_{rr}) + \frac{2}{r^4}\gamma^{ij}h_{ij} \\
\Box h_{ur} =& \partial_r^2 h_{ur} - 2\partial_u\partial_r h_{ur} + \frac{\gamma^{ij}}{r^2}\big(D_iD_j h_{ur} - \frac{1}{r}D_i h_{uj} - r\gamma_{ij}(\partial_u - \partial_r)h_{ur} \\
&- \frac{1}{r}D_j h_{iu} + \gamma_{ij}(h_{uu} - h_{ur})\big) \\
\Box h_{uu} =& \partial_r^2 h_{uu} - 2\partial_u\partial_r h_{uu} + \frac{\Delta}{r^2}h_{uu} - \frac{2}{r}(\partial_u - \partial_r)h_{uu} \\
\Box h \;\; =& \partial_r^2 h - 2\partial_u\partial_r h + \frac{\Delta}{r^2}h - \frac{2}{r}(\partial_u - \partial_r)h
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\Box h_{ui} =& \partial_r^2 h_{ui} - \frac{1}{r^2}h_{ui} - 2\partial_u\partial_r h_{ui} + \frac{1}{r^2}(\Delta h_{ui} - 2rD_i(h_{uu} - h_{ur})) \\
\Box h_{ri} =& \partial_r^2 h_{ri} - \frac{5}{r^2}h_{rA} - 2\partial_u\partial_r h_{rA} + \frac{1}{r^2}(\Delta h_{ri} - \frac{2}{r}D^j h_{ij} - 2rD_i(h_{ru} - h_{rr}) + 4h_{ui}) \\
\Box h_{ij} =& \partial_r^2 h_{ij} - \frac{2}{r}\partial_r h_{ij} - 2\partial_u\partial_r h_{ij} + \frac{2}{r}\partial_u h_{ij} \\
&+ \frac{1}{r^2}\left[\Delta h_{ij} - 2rD_i(h_{uj} - h_{rj}) - 2rD_j(h_{ui} - h_{ri})\right] \\
&+ 2\gamma_{ij}(h_{uu} - 2h_{ur} + h_{rr})
\end{aligned} \tag{A.14}$$

The gauge condition is instead

$$\begin{aligned}
G_u =& \partial_r h_{ru} - \partial_r h_{uu} - \partial_u h_{ur} + \frac{\gamma^{ij}}{r^2}(D_i h_{ju} - r\gamma_{ij}(h_{uu} - h_{ur})) \\
&- \frac{1}{2}\partial_u(h_{rr} - 2h_{ur} + \frac{\gamma^{ij}}{r^2}h_{ij}) \\
G_r =& \partial_r h_{rr} - \partial_r h_{ur} - \partial_u h_{rr} + \frac{\gamma^{ij}}{r^2}(D_i h_{rj} - r\gamma_{ij}(h_{ur} - h_{rr}) - \frac{1}{r}h_{ij}) \\
&- \frac{1}{2}\partial_r(h_{rr} - 2h_{ur} + \frac{\gamma^{ij}}{r^2}h_{ij}) \\
G_i =& \partial_r h_{ri} + \frac{2}{r}h_{ri} - \partial_u h_{ri} - \partial_r h_{ui} - \frac{2}{r}h_{ui} \\
&+ \frac{\gamma^{jk}}{r^2}(D_j h_{ik}) - \frac{1}{2}D_i(h_{rr} - 2h_{ur} + \frac{\gamma^{jk}}{r^2}h_{jk})
\end{aligned} \tag{A.15}$$

Order by order, these are

$$
\begin{aligned}
\Box h_{rr}^{(n)} :\ & (\Delta + (n-2)(n-3) - 4)h_{rr}^{(n-2)} + 2(n-2)\partial_u h_{rr}^{(n-1)} + 4h_{ur}^{(n-2)} \\
& - 4D^i h_{ri}^{(n-3)} + 2\gamma^{ij} h_{ij}^{(n-4)} \\
\Box h_{ur}^{(n)} :\ & (\Delta + (n-1)(n-4))h_{ur}^{(n-2)} + 2(n-2)\partial_u h_{ur}^{(n-1)} + 2h_{uu}^{(n-2)} - 2D^i h_{ui}^{(n-3)} \\
\Box h_{uu}^{(n)} :\ & (\Delta + (n-2)(n-3))h_{uu}^{(n-2)} + 2(n-2)\partial_u h_{uu}^{(n-1)} \\
\Box h_{ui}^{(n)} :\ & (\Delta + (n-1)(n-2) - 1)h_{ui}^{(n-2)} + 2(n-1)\partial_u h_{ui}^{(n-1)} - 2D_A h_{uu}^{(n-1)} \\
& + 2D_i h_{ur}^{(n-1)} \\
\Box h_{ri}^{(n)} :\ & (\Delta + (n-1)(n-2) - 5)h_{ri}^{(n-2)} + 4h_{uA}^{(n-2)} + 2(n-1)\partial_u h_{ri}^{(n-1)} \\
& - 2D_i h_{ru}^{(n-1)} + 2D_i h_{rr}^{(n-1)} - 2D^j h_{ij}^{(n-3)} \\
\Box h_{ij}^{(n)} :\ & (\Delta + (n+1)(n-2))h_{ij}^{(n-2)} + 2n\partial_u h_{ij}^{(n-1)} + 2\gamma_{ij}(h_{uu}^{(n)} - 2h_{ur}^{(n)} \\
& + h_{rr}^{(n)}) - 2D_i(h_{uj}^{(n-1)} - h_{rj}^{(n-1)}) - 2D_j(h_{ui}^{(n-1)} - h_{ri}^{(n-1)}) \\
\Box h^{(n)} :\ & (\Delta + (n-2)(n-3))(h_{rr}^{(n-2)} - 2h_{ur}^{(n-2)} + \gamma^{ij} h_{ij}^{(n-4)}) \\
& + 2(n-2)\partial_u(h_{rr}^{(n-1)} - 2h_{ur}^{(n-1)} + \gamma^{ij} h_{ij}^{(n-3)}) \\
2G_u^{(n)} :\ & -2(n-3)h_{ur}^{(n-1)} + 2(n-3)h_{uu}^{(n-1)} - \partial_u h_{rr}^{(n)} + 2D^i h_{ui}^{(n-2)} - \gamma^{ij}\partial_u h_{ij}^{(n-2)} \\
2G_r^{(n)} :\ & -(n-5)h_{rr}^{(n-1)} - 2\partial_u h_{rr}^{(n)} + 2D^i h_{ir}^{(n-2)} - 4h_{ur}^{(n-1)} + (n-3)\gamma^{ij} h_{ij}^{(n-3)} \\
2G_i^{(n)} :\ & -2(n-3)h_{ri}^{(n-1)} + 2(n-3)h_{ui}^{(n-1)} - 2\partial_u h_{ri}^{(n)} + 2D^j h_{ij}^{(n-2)} - D_i h_{rr}^{(n)} \\
& + 2D_i h_{ur}^{(n)} - \gamma^{jk} D_i h_{jk}^{(n-2)}
\end{aligned}
\tag{A.16}
$$

## Two-form

The wave equation is

$$\Box \mathcal{B}_{ur} = \left(\partial_r^2 - 2\partial_u\partial_r + \frac{\Delta}{r^2} - \frac{2}{r}(\partial_u - \partial_r) - \frac{2}{r^2}\right)\mathcal{B}_{ur} - \frac{2}{r^3}D^i\mathcal{B}_{ui}\,, \tag{A.17a}$$

$$\Box \mathcal{B}_{ui} = \left(\partial_r^2 - 2\partial_u\partial_r + \frac{\Delta}{r^2} - \frac{1}{r^2}\right)\mathcal{B}_{ui} + \frac{2}{r}D_i\mathcal{B}_{ur}\,, \tag{A.17b}$$

$$\Box \mathcal{B}_{ri} = \left(\partial_r^2 - 2\partial_u\partial_r + \frac{\Delta}{r^2} - \frac{1}{r^2}\right)\mathcal{B}_{ri} + \frac{2}{r^3}D^j\mathcal{B}_{ij} + \frac{2}{r}D_i\mathcal{B}_{ur}\,, \tag{A.17c}$$

$$\Box \mathcal{B}_{ij} = \left(\partial_r^2 - 2\partial_u\partial_r + \frac{\Delta}{r^2} + \frac{2}{r}(\partial_u - \partial_r)\right)\mathcal{B}_{ij} - \frac{2}{r}\big(D_i(\mathcal{B}_{uj} - \mathcal{B}_{rj}) - D_j(\mathcal{B}_{ui} - \mathcal{B}_{ri})\big)\,, \tag{A.17d}$$

while the Lorenz gauge condition $G_\nu := \nabla^\mu \mathcal{B}_{\mu\nu} = 0$ implies

$$G_u = \left(\partial_u - \partial_r - \frac{2}{r}\right)\mathcal{B}_{ur} - \frac{1}{r^2}D^i\mathcal{B}_{ui}\,, \tag{A.18a}$$

$$G_r = -\left(\partial_r + \frac{2}{r}\right)\mathcal{B}_{ur} - \frac{1}{r^2}D^i\mathcal{B}_{ri}\,, \tag{A.18b}$$

$$G_i = (\partial_r - \partial_u)\mathcal{B}_{ri} - \partial_r\mathcal{B}_{ui} - \frac{1}{r^2}D^j\mathcal{B}_{ij}\,. \tag{A.18c}$$

In our setup the logarithms appear only in pure gauge sectors and therefore trivially solve the equations of motion, implying non-trivial conditions only in the $B-$part of the $\mathcal{B}$ field. However, here we analyse the order-by-order equations for a two form gauge field polyhomogeneously expanded.

In particular, the order-by-order wave equation for the logarithmic part is[2]

$$-2(n-2)\partial_u\hat{\mathcal{B}}_{ur}^{(n-1)} = (\Delta + (n-1)(n-4))\hat{\mathcal{B}}_{ur}^{(n-2)} - 2D^i\hat{\mathcal{B}}_{ui}^{(n-3)} \tag{A.19a}$$

$$-2(n-1)\partial_u\hat{\mathcal{B}}_{ui}^{(n-1)} = (\Delta + (n-1)(n-2) - 1)\hat{\mathcal{B}}_{ui}^{(n-2)} + 2D_i\hat{\mathcal{B}}_{ur}^{(n-1)} \tag{A.19b}$$

$$-2(n-1)\partial_u\hat{\mathcal{B}}_{ri}^{(n-1)} = (\Delta + (n-1)(n-2) - 1)\hat{\mathcal{B}}_{ri}^{(n-2)} + 2D_i\hat{\mathcal{B}}_{ur}^{(n-1)} + 2D^j\hat{\mathcal{B}}_{ij}^{(n-3)} \tag{A.19c}$$

$$\begin{aligned}-2n\partial_u\hat{\mathcal{B}}_{ij}^{(n-1)} =& (\Delta + (n+1)(n-2))\hat{\mathcal{B}}_{ij}^{(n-2)} - 2D_i(\hat{\mathcal{B}}_{uj}^{(n-1)} - \hat{\mathcal{B}}_{rj}^{(n-1)}) \\ &+ 2D_j(\hat{\mathcal{B}}_{ui}^{(n-1)} - \hat{\mathcal{B}}_{ri}^{(n-1)})\end{aligned} \tag{A.19d}$$

while for the rest we find

$$\begin{aligned}-2(n-2)\partial_u\mathcal{B}_{ur}^{(n-1)} + 2\partial_u\hat{\mathcal{B}}_{ur}^{(n-1)} =& (\Delta + (n-1)(n-4))\mathcal{B}_{ur}^{(n-2)} - (2n-5)\hat{\mathcal{B}}_{ur}^{(n-2)} \\ &- 2D^i\mathcal{B}_{ui}^{(n-3)}\end{aligned} \tag{A.20a}$$

$$\begin{aligned}-2(n-1)\partial_u\mathcal{B}_{ui}^{(n-1)} + 2\partial_u\hat{\mathcal{B}}_{ui}^{(n-1)} =& (\Delta + (n-1)(n-2) - 1)\mathcal{B}_{ui}^{(n-2)} - (2n-3)\hat{\mathcal{B}}_{ui}^{(n-2)} \\ &+ 2D_i\mathcal{B}_{ur}^{(n-1)}\end{aligned} \tag{A.20b}$$

$$\begin{aligned}-2(n-1)\partial_u\mathcal{B}_{ri}^{(n-1)} + 2\partial_u\hat{\mathcal{B}}_{ri}^{(n-1)} =& (\Delta + (n-1)(n-2) - 1)\mathcal{B}_{ri}^{(n-2)} \\ &- (2n-3)\hat{\mathcal{B}}_{ri}^{(n-2)} + 2D_i\mathcal{B}_{ur}^{(n-1)} + 2D^j\mathcal{B}_{ij}^{(n-3)}\end{aligned} \tag{A.20c}$$

$$\begin{aligned}-2n\partial_u\mathcal{B}_{ij}^{(n-1)} + 2\partial_u\hat{\mathcal{B}}_{ij}^{(n-1)} =& (\Delta + (n+1)(n-2))\mathcal{B}_{ij}^{(n-2)} - (2n-1)\hat{\mathcal{B}}_{ij}^{(n-2)} \\ &- 2D_i(\mathcal{B}_{uj}^{(n-1)} - \mathcal{B}_{rj}^{(n-1)}) + 2D_j(\mathcal{B}_{ui}^{(n-1)} - \mathcal{B}_{ri}^{(n-1)}).\end{aligned} \tag{A.20d}$$

[2] In this case, moving to the right the l.h.s. of each equation we recover the order-by-order expression of $\left[\Box\mathcal{B}_{\mu\nu}\right]^{(n)}$.

The order-by-order gauge condition for the logarithms is

$$-\partial_u \hat{\mathcal{B}}^{(n)}_{ur} = (n-3)\hat{\mathcal{B}}^{(n-1)}_{ur} - D^i \hat{\mathcal{B}}^{(n-2)}_{ui} \tag{A.21a}$$

$$(n-3)\hat{\mathcal{B}}^{(n-1)}_{ur} - D^i \hat{\mathcal{B}}^{(n-2)}_{ri} = 0 \tag{A.21b}$$

$$\partial_u \hat{\mathcal{B}}^{(n)}_{ri} = (n-1)(\hat{\mathcal{B}}^{(n-1)}_{ui} - \hat{\mathcal{B}}^{(n-1)}_{ri}) - D^j \hat{\mathcal{B}}^{(n-2)}_{ij} \tag{A.21c}$$

while

$$-\partial_u \mathcal{B}^{(n)}_{ur} = -\hat{\mathcal{B}}^{(n-1)}_{ur} + (n-3)\mathcal{B}^{(n-1)}_{ur} - D^i \mathcal{B}^{(n-2)}_{ui} \tag{A.22a}$$

$$(n-3)\mathcal{B}^{(n-1)}_{ur} - \hat{\mathcal{B}}^{(n-1)}_{ur} - D^i \mathcal{B}^{(n-2)}_{ri} = 0 \tag{A.22b}$$

$$\partial_u \mathcal{B}^{(n)}_{ri} = \hat{\mathcal{B}}^{(n-1)}_{ri} - \hat{\mathcal{B}}^{(n-1)}_{ui} + (n-1)(\mathcal{B}^{(n-1)}_{ui} - \mathcal{B}^{(n-1)}_{ri}) - D^j \mathcal{B}^{(n-2)}_{ij} \tag{A.22c}$$

For completeness, the following are the equations for the part that admits a $\frac{1}{r}$ expansion, which are identical to the previous ones for the logarithmic part. The gauge condition is

$$-\partial_u B^{(n)}_{ur} = (n-3)B^{(n-1)}_{ur} - D^i B^{(n-2)}_{ui} \tag{A.23a}$$

$$(n-3)B^{(n-1)}_{ur} - D^i B^{(n-2)}_{ri} = 0 \tag{A.23b}$$

$$\partial_u B^{(n)}_{ri} = (n-1)(B^{(n-1)}_{ui} - B^{(n-1)}_{ri}) - D^j B^{(n-2)}_{ij} \tag{A.23c}$$

and the equations of motion are

$$-2(n-2)\partial_u B^{(n-1)}_{ur} = (\Delta + (n-1)(n-4))B^{(n-2)}_{ur} - 2D^i B^{(n-3)}_{ui} \tag{A.24a}$$

$$-2(n-1)\partial_u B^{(n-1)}_{ui} = (\Delta + (n-1)(n-2) - 1)B^{(n-2)}_{ui} + 2D_i B^{(n-1)}_{ur} \tag{A.24b}$$

$$-2(n-1)\partial_u B^{(n-1)}_{ri} = (\Delta + (n-1)(n-2) - 1)B^{(n-2)}_{ri} + 2D_i B^{(n-1)}_{ur} + 2D^j B^{(n-3)}_{ij} \tag{A.24c}$$

$$\begin{aligned}-2n\partial_u B^{(n-1)}_{ij} =& (\Delta + (n+1)(n-2))B^{(n-2)}_{ij} - 2D_i(B^{(n-1)}_{uj} - B^{(n-1)}_{rj}) \\ &+ 2D_j(B^{(n-1)}_{ui} - B^{(n-1)}_{ri})\end{aligned} \tag{A.24d}$$

By means of the splitting (3.80) and the writing (3.81), we can write the equations for $B'_r, B'_r$ and $B$, which are relevant for the field strength. In particular, we have

$$\partial_u B_r'^{(n)} = (n-1)(B_u'^{(n-1)} - B_r'^{(n-1)}) - B^{(n-2)}, \tag{A.25a}$$

$$-2(n-1)\partial_u B_u'^{(n-1)} = (\Delta + (n-1)(n-2))B_u'^{(n-2)} \tag{A.25b}$$

$$-2(n-1)\partial_u B_r'^{(n-1)} = (\Delta + (n-1)(n-2))B_r'^{(n-2)} + 2B^{(n-3)} \tag{A.25c}$$

$$-2n\partial_u B^{(n-1)} = (\Delta + (n+1)(n-2))B^{(n-2)} + 4\Delta(B_u'^{(n-1)} - B_r'^{(n-1)}) \tag{A.25d}$$

## A.2 Field strength expressions

### A.2.1 $F_{ur}$.

Let us start employing the complete gauge fixing mentioned in the text. Using the subleading integration functions we reach a setup with

$$\nabla^\mu A_\mu = 0, \qquad \Box A_\mu = 0, \qquad A_u^{(1)} = 0, \qquad A_r^{(n,0)} = 0, \tag{A.26}$$

where $A_r^{(n,0)}$ is the $u-$independent part of $A_r^{(n)}$ near $\mathcal{I}^+_-$. We do not have further free subleading terms in the residual gauge parameter $\epsilon$.

First, we solve the order-by-order equation $\Box A_u$, which reads

$$(2-2n)\partial_u A_u^{(n)} = [\Delta + (n-1)(n-2)]A_u^{(n-1)}\,, \tag{A.27}$$

and use $A_u^{(1)} = 0$. We start with a $A_u^{(2)} = A_{u,c}^{(2)}(z,\bar{z})$ and the equation gives

$$A_u^{(n)} = \sum_{m=0}^{n-2} (-u)^m A_u^{(n,m)}, \tag{A.28}$$

with

$$A_u^{(n,m)} = \frac{1}{2^m}\frac{(n-m-1)!}{m!(n-1)!}\prod_{i=1}^{m}\left[\Delta + i(i+1)\right] A_{u,c}^{(n-m)}\,. \tag{A.29}$$

This result is exact at $\mathcal{I}^+$. Now we can use the other gauge conditions to determine a relation between $A_{u,c}^{(n-m)}$ and $A$.

The equations $\Box A_r$ and $\nabla^\mu A_\mu$ give a order-by-order constraint

$$[\Delta - (n-2)(n-3)]A_r^{(n-1)} + 2(n-2)^2 A_u^{(n-1)} + 2(n-2)D\cdot A^{(n-2)} = 0 \tag{A.30}$$

Let us focus on the limit $u\to-\infty$ and, in particular, to the $u-$independent part. The condition $A_r^{(n,0)} = 0$ implies that

$$A_u^{(n,0)} = -\frac{1}{n-1} D^i A_i^{(n-1,0)}, \tag{A.31}$$

and therefore we are able to find all the $A_u^{(n)}$ in terms of the free Cauchy data

$$A_u^{(n,m)} = -\frac{1}{2^m} \frac{(n-m-2)!}{m!(n-1)!} \prod_{i=1}^{m} \left[\Delta + i(i+1)\right] D^i A_i^{(n-m,0)}. \tag{A.32}$$

Having determined the expression of $A_u$ in terms of the free data, the equation (A.30) allow us to determine the $A_r$ as well. However, we cannot extract an exact form on $\mathcal{I}^+$ because $A_i$ as an arbitrary dependence on $u$.

Still, the gauge condition $A_r^{(n,0)} = 0$ allow us to determine in a compact form the structure of $F_{ur}^{(n,0)}$. In fact, substituting it in (3.30) and using the Helmholtz decomposition, we find

$$\begin{aligned} F_{ur}^{(2,0)} &= \Delta A^{(0,0)} \\ F_{ur}^{(n,0)} &= -\Delta A^{(n-2,0)} \qquad (n > 2)\,. \end{aligned} \tag{A.33}$$

### A.2.2 $\boldsymbol{H_{uri}}$

Let us now discuss the component $H_{uri}$, which enters in the two-form asymptotic charge. We have by definition

$$H_{uri}^{(n)} = \partial_u B_{ri} + (n-1) B_{ui}^{(n-1)} + D_i B_{ur}. \tag{A.34}$$

and substituting the Lorenz gauge condition on $\partial_u B_{ri}$ we find

$$H_{uri}^{(n)} = (n-1)\left(2B_{ui}^{(n-1)} - B_{ri}^{(n-1)}\right) + D_i B_{ur} - D^j B_{ij}^{(n-2)}. \tag{A.35}$$

Since $D^i H_{uri} = 0$, we can write

$$H_{uri}^{(n)} = \varepsilon_{ij} D^j H_{ur}'^{(n)} \tag{A.36}$$

with

$$H_{ur}'^{(n)} = (n-1)\left(2B_u'^{(n-1)} - B_r'^{(n-1)}\right) - B^{(n-2)}\,. \tag{A.37}$$

Combining equations of motion with the gauge condition we find

$$[\Delta - n(n-1)] B_r'^{(n-1)} + 2n(n-1) B_u'^{(n-1)} - 2(n-1) B^{(n-2)} = 0\,, \tag{A.38}$$

which leads to

$$H_{ur}'^{(n)} = -\frac{\Delta}{n} B_r'^{(n-1)} + \frac{n-2}{n} B^{(n-2)}. \tag{A.39}$$

Note that the fact that, for $n = 2$, this object does not depend on $B^{(0)}$ is simply due to the fact that we are left with a free function $\lambda_i^{(0)}(u, z, \bar{z})$ that can be used to fix $B^{(0)} = 0$. Consistently, we can use the subleading integration functions to impose the condition

$$B_r'^{(n,0)} = 0 \qquad (n > 2). \tag{A.40}$$

This leaves us with

$$\begin{aligned} H_{ur}'^{(1,0)} &= - B^{(-1,0)}, \\ H_{ur}'^{(2,0)} &= - \frac{\Delta}{2} B_r'^{(1,0)}, \\ H_{ur}'^{(n,0)} &= \frac{n-2}{n} B^{(n-2,0)} \qquad (n > 2). \end{aligned} \tag{A.41}$$

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