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The Chow ring of the classifying space of Spin_8

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Introduction

To algebraic topologists, the cohomology of classifying spaces of linear algebraic groups (or, equivalently, of compact Lie groups) has been an important object of study for a long time. Let G a topological group and X a topological G-space; Armand Borel ([**1**]) defined the equivariant cohomology ring with coefficient in a commutative ring k as

$$\mathrm{H}^*_G(X;k) := \mathrm{H}^*\big((X \times \mathrm{E}\,G)/G;k\big)$$

where $\mathrm{E}\,G \to \mathrm{B}\,G$ is the universal principal G-bundle and the left-hand side is the usual cohomology ring; in particular, $\mathrm{H}^*_G := \mathrm{H}^*_G(pt)$ is identified with the integral cohomology of the classifying space of G (we will write $\mathrm{H}^*(X)$ for the integral cohomology $\mathrm{H}^*(X;\mathbb{Z})$).

Recently, Burt Totaro ([**19**]) has introduced an algebraic analogue of this cohomology, the Chow ring of the classifying space of a linear algebraic group G, denoted by A^*_G. There is a natural ring homomorphism $\mathrm{A}^*_G \to \mathrm{H}^*_G$, which is, in general, neither surjective not injective.

Rationally, the situation is very well understood. If G is a connected algebraic group, then the homomorphism $\mathrm{A}^*_G \otimes \mathbb{Q} \to \mathrm{H}^*_G \otimes \mathbb{Q}$ is an isomorphism, and both rings coincide with the ring of invariants under the Weyl group in the symmetric algebra of the ring of characters of a maximal torus; this is classical, due to Leray and Borel, in the case of cohomology, and to Edidin and Graham ([**6**]) for the Chow ring. Furthermore, this ring of invariants is always a polynomial ring, as was shown by Chevalley. With integral coefficients, the situation is much more subtle.

The Chow ring A^*_G has been computed for the classical groups GL_n, SL_n, Sp_n, O_n or SO_n, but not for the PGL_n series. The results are as follows. Each of the groups above comes with a tautological representation V, of dimension n (or $2n$, in the case of Sp_n). Every representation V of an algebraic group G has Chern classes $\mathrm{c}_i(V) \in \mathrm{A}^i_G$. When G is a classical group, we denote the Chern classes of the tautological representation simply by c_i.

Burt Totaro ([**19**]) and Rahul Pandharipande ([**5**]) described A^*_G when $G = \mathrm{GL}_n$, SL_n, Sp_n, O_n and SO_n when n is odd. We will use the following notation: if R is a ring, t_1, …, t_n are elements of R, f_1, …, f_r are polynomials in $\mathbb{Z}[x_1,\ldots,x_n]$, we write

$$R = \mathbb{Z}[t_1,\ldots,t_n]/\big(f_1(t_1,\ldots,t_n),\ldots,f_r(t_1,\ldots,t_n)\big)$$

to indicate the the ring R is generated by t_1, …, t_n, and the kernel of the evaluation map $\mathbb{Z}[x_1,\ldots,x_n] \to R$ sending x_i to y_i is generated by f_1, …, f_r. When there are no f_i this means that R is a polynomial ring in the t_i.

First the case of the special groups.

THEOREM (B. Totaro).

(1) $\mathrm{A}^*_{\mathrm{GL}_n} = \mathbb{Z}[c_1,\ldots,c_n]$.

(2) $\mathrm{A}^*_{\mathrm{SL}_n} = \mathbb{Z}[c_2, \dots, c_n]$.
(3) $\mathrm{A}^*_{\mathrm{Sp}_n} = \mathbb{Z}[c_2, c_4, \dots, c_{2n}]$.

The first two cases follow very easily from the well known description via generators and relations of the Chow ring of a Grassmannian.

In all three cases, the Chow ring is isomorphic to the cohomology ring.

THEOREM (R. Pandharipande, B. Totaro).
(1) $\mathrm{A}^*_{\mathrm{O}_n} = \mathbb{Z}[c_1, \dots, c_n]/(2c_{\mathrm{odd}})$.
(2) If n is odd, then $\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n]/(2c_{\mathrm{odd}})$.

The notation $2c_{\mathrm{odd}}$ means "all the elements $2c_i$ for i odd"; in a similar way, if $x_{i_1}, \dots, x_{i_n}$ are elements of R, the notation x_{odd} (resp. x_{even}) will mean "all the elements x_i for i odd" (resp. "all the elements x_i for i even").

When n is odd, then $\mathrm{O}_n \simeq \mathrm{SO}_n \times \boldsymbol{\mu}_2$, and this allows to obtain the result for SO_n from that for O_n. When n is even this fails, and the situation is more complicated. Even rationally, the Chern classes of the tautological representation do not generate the Chow ring, or the cohomology. It is well known that when $n = 2m$, the tautological representation has an Euler class $\epsilon_m \in \mathrm{H}^{2m}_{\mathrm{SO}_n}$, whose square is $(-1)^m c_n$: this class, together with the even Chern classes c_2, c_4, $\dots$, c_{n-2} generate $\mathrm{A}^*_{\mathrm{SO}_n} \otimes \mathbb{Q} = \mathrm{H}^*_{\mathrm{SO}_n} \otimes \mathbb{Q}$. Totaro noticed that when $n = 4$ the class ϵ_2 is not in the image of $\mathrm{A}^*_{\mathrm{SO}_n}$; shortly afterwards, Edidin and Graham ([**8**]) constructed a class $y_m \in \mathrm{A}^m_{\mathrm{SO}_n}$, whose image in $\mathrm{H}^*_{\mathrm{SO}_n}$ is, rationally, $2^{m-1}\epsilon_m$.

Subsequently, Pandharipande computed $\mathrm{A}^*_{\mathrm{SO}_4}$: he showed that it is generated by c_2, c_3, c_4 and y_2, and gave the relations (his description of the class y_2 is different, but equivalent to that of Edidin and Graham). Finally, in her Ph.D. thesis Rebecca Field obtained the general result ([**9**]), which is as follows.

THEOREM (R. Field). *When $n = 2m$ is even, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n, y_m]/\big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

The PGL_n series is much harder (this is an example of a universal phenomenon, that of all the classical groups, these are the ones giving rise to the deepest problems). For $n = 2$ we have that $\mathrm{PGL}_2 = \mathrm{SO}_3$, and for this group everything is well understood. For $n = 3$ there is a difficult paper of G. Vezzosi ([**23**]), where he describes $\mathrm{A}^*_{\mathrm{PGL}_3}$ almost completely. Here is his basic idea. The fundamental tool is the equivariant intersection theory that Edidin and Graham ([**7**]) have forged starting from Totaro's idea. Vezzosi stratifies the adjoint representation $\mathfrak{sl}_3$ of PGL_3 by type of Jordan canonical form, compute the Chow ring of each stratum, and then get generators for $\mathrm{A}^*_{\mathrm{PGL}_3}$ using the localization sequence for equivariant Chow groups. To get relations he restricts to appropriate subgroups of PGL_3. His technique has been refined and improved by Angelo Vistoli in [**24**], where he studies the Chow ring and the cohomology of the classifying space of PGL_p, where p is an odd prime.

The purpose of the first part of this thesis, written in collaboration with Angelo Vistoli ([**15**]) is to show how this stratification method provides a unified approach to all the known results on the Chow ring of classical groups over any field. Consider a classical group G with its tautological representation V. Then one stratifies V in strata in which the stabilizers are, up to an extension by a unipotent group, smaller classical group. Using the localization sequence for equivariant Chow groups this gives generators for the Chow rings, with relations that come out naturally. To show that the relations suffice, one restricts to appropriate subgroups of G (a maximal torus first, to show that the relations suffice up to torsion, then to some finite subgroup to handle torsion).

In the second part we determine almost completely the Chow ring of the complex spin group Spin_8. It is well known that for $n \leq 6$ the Spin_n are special and isomorphic to classical groups whose Chow ring is known, so the first interesting spin group is Spin_7. In [**12**] Pierre Guillot determined almost completely the Chow ring of the classifying space of Spin_7 localized at (2). Using the stratification method, he firstly described the Chow ring of the exceptional group G_2; then he found generators and some relations of $\mathrm{A}^*_{\mathrm{Spin}_7}$, and exploited the results on Brown-Peterson cohomology in [**13**] to show that relations sufficies. To state his result, set $c_i := \mathrm{c}_i(\mathbb{A}^7)$, where $\mathbb{A}^7$ is the representation given by the projection $\mathrm{Spin}_7 \to \mathrm{SO}_7$, and $c'_i := \mathrm{c}_i(S)$ where S is the 8-dimensional spin representation. We use the following notation: given a ring R, $R\langle x_1, \ldots, x_n\rangle$ is the free R-module generated by the elements $x_1, \ldots, x_n$. Then Guillot proved the following:

Theorem (P. Guillot). *There is an additive isomorphism*

$$(\mathrm{A}^*_{\mathrm{Spin}_7})_{(2)} \simeq \mathbb{Z}_{(2)}[c_4, c_6, c'_8] \otimes \left(\mathbb{Z}_{(2)}\left\langle 1, c_2, c'_4, c'_6\right\rangle \oplus \mathbb{Z}/2\left\langle \xi_3\right\rangle \oplus \mathbb{Z}/2[c_7]\left\langle c_7\right\rangle\right)$$

where ξ_3 is a class of degree 3 that cannot be expressed in terms of Chern classes; the products in $(\mathrm{A}^*_{\mathrm{Spin}_7})_{(2)}$ are determined by the following relations:

$$\begin{aligned}
&\xi_3^2 = 0\\
&\xi_3 c_7 = 0\\
&\xi_3(c_4 - c'_4) = 0\\
&\xi_3(c_6 - c'_6) = 0\\
&\xi_3 c_2 = 0\\
&c_2^2 - 4c_4 = \frac{8}{3}(c'_3 - c_4)\\
&c_2(c'_4 - c_4) = 6(c'_6 - c_6)\\
&c_2(c'_6 - c_6) = \frac{2}{3}c_4(c'_4 - c_4) + 16c'_8\\
&c'_2 c_7 = \delta_1 c_6 \xi_3\\
&c'_4(c'_4 - c_4) = c_4(c'_4 - c_4) + 36c'_8
\end{aligned}$$

$$\begin{aligned}
c_4'(c_6'-c_6) &= c_4(c_6'-c_6)+6c_2c_8' \\
c_7(c_4'-c_4) &= \delta_2 c_8 \xi_3 \\
c_6'(c_6'-c_6) &= c_6(c_6'-c_6)+c_8(\frac{8}{3}c_4'+\frac{4}{3}c_4) \\
c_7(c_6'-c_6) &= 0.
\end{aligned}$$

Our approach is similar, but, besides the stratification method, we use the triality action of $\mathfrak{S}_3$ on Spin_8. The group Spin_8 has three 8-dimensional representations, the first is the projection $\mathrm{Spin}_8 \to \mathrm{SO}_8$ and the other two are the half-spin representations S^+, S^-. The isomorphism classes of these representations are exchanged by the full symmetric group $\mathfrak{S}_3$. Set $c_i := \mathrm{c}_i(\mathbb{A}^8)$, where $\mathbb{A}^8$ is the representation given by the projection $\mathrm{Spin}_8 \to \mathrm{SO}_8$, and $c_i^{\pm} := \mathrm{c}_i(S^{\pm})$. Here there is our main result, obtained from Corollary 8.7, Proposition 8.5, Lemma 8.6, Proposition 9.7, and Lemma 9.8 of Chapter 2:

Main Theorem.

$$\mathrm{A}^*_{\mathrm{Spin}_8} \simeq \mathbb{Z}[c_2, c_4, c_6, c_8, c_8^+, \zeta_3, \zeta_3^+, \zeta_4, \zeta_4^+, \zeta_5, \zeta_6, \zeta_6^+, \zeta_8, \zeta_{10}]/R$$

where ζ_i, ζ_i^+ are elements of degree i, and R is the ideal generated by the following elements:

$$\begin{aligned}
&2\zeta_{odd} \\
&2c_7 \\
&2\zeta_3^+ \\
&c_2^2 - 4c_4 - 8\zeta_4^+ + 4\zeta_4 \\
&c_2\zeta_4 - 2\zeta_6 \\
&c_2\zeta_4^+ - 2\zeta_6^+ \\
&c_2\zeta_6 - 2c_4\zeta_4 - 8\zeta_8 + 8c_8 \\
&c_2\zeta_6^+ - 2c_4\zeta_4^+ - 16c_8^+ + 4\zeta_8 \\
&c_2\zeta_8 - 2\zeta_{10} \\
&c_2\zeta_{10} - 2c_4\zeta_8 - 8c_8^+\zeta_4 + 4c_8\zeta_4^+ \\
&\zeta_4^2 - 4c_8 \\
&\zeta_4\zeta_4^+ - 2\zeta_8 \\
&\zeta_4\zeta_6 - 2c_2c_8 \\
&\zeta_4\zeta_6^+ - 2\zeta_{10} \\
&\zeta_4\zeta_8 - 2c_8\zeta_4^+ \\
&\zeta_4\zeta_{10} - 2c_8\zeta_6^+ \\
&(\zeta_4^+)^2 - 4c_8^+ \\
&\zeta_4^+\zeta_6 - 2\zeta_{10}
\end{aligned}$$

$$\zeta_4^+\zeta_6^+ - 2c_2c_8^+$$
$$\zeta_4^+\zeta_8 - 2c_8^+\zeta_4$$
$$\zeta_4^+\zeta_{10} - 2c_8^+\zeta_6$$
$$\zeta_6^2 - 4c_8(c_4 + 2\zeta_4^+ - \zeta_4)$$
$$\zeta_6\zeta_6^+ - 2c_4\zeta_8 - 8c_8^+\zeta_4 + 4c_8\zeta_4^+$$
$$\zeta_6\zeta_8 - 2c_8\zeta_6^+$$
$$\zeta_6\zeta_{10} - c_8(c_4\zeta_4^+ + 12c_8^+ - 2\zeta_8 - 8c_8)$$
$$(\zeta_6^+)^2 - 4c_8^+(c_4 + 2\zeta_4^+ - \zeta_4)$$
$$\zeta_6^+\zeta_8 - 2c_8^+\zeta_6$$
$$\zeta_6^+\zeta_{10} - c_8^+(2c_4\zeta_4 + 8\zeta_8 - 8c_8)$$
$$\zeta_8^2 - 4c_8c_8^+$$
$$\zeta_8\zeta_{10} - 2c_2c_8c_8^+$$
$$\zeta_{10}^2 - c_8c_8^+(4c_4 + 8\zeta_4^+ - 4\zeta_4)$$
$$\{\alpha\beta\}_{\alpha\in\{\zeta_{odd},\zeta_3^+\},\beta\in\{c_7,\zeta_{odd},\zeta_3^+\}}$$
$$\{\alpha\beta\}_{\alpha\in\{c_2,\zeta_{even},\zeta_{even}^+\},\beta\in\{\zeta_{odd},\zeta_3^+\}}$$
$$c_2c_7 = \delta_1c_6\zeta_3^+ + \delta_3c_6\zeta_3 + \delta_4c_4\zeta_5$$
$$\zeta_4c_7 + c_8(\delta_5\zeta_3 + \delta_6\zeta_3^+)$$
$$\zeta_4^+c_7 + c_8^+(\delta_2\zeta_3^+ + \delta_7\zeta_3)$$
$$\zeta_6c_7 + \delta_8c_8\zeta_5$$
$$\zeta_6^+c_7 + \delta_9c_8^+\zeta_5$$
$$\zeta_8c_7 + c_8(\delta_{10}c_4\zeta_3 + \delta_{11}c_4\zeta_3^+ + \delta_{12}\zeta_7)$$
$$\zeta_{10}c_7 + c_8(\delta_{13}c_4\zeta_5 + \delta_{14}c_6\zeta_3 + \delta_{15}c_6\zeta_3^+).$$

Here $\delta_i \in \mathbb{Z}/2$ are indeterminate coefficients, and δ_1, δ_2 are the same as in [**12**, Proposition 9.1].

The indeterminate relations are of the same kind of that in [**12**]: they are the products of c_7 with elements of even degree.

Using this result we are able to compute the Chow ring of $\mathrm{B\,Spin}_7$ whitout localizing at (2) (see Proposition 9.10).

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Contents

CHAPTER 1

The Chow ring of the classifying space of classical groups

1. Preliminaries on equivariant intersection theory

In this section we recall some definitions and notations, and state some tecnical results that will be used throughout this paper.

All schemes and algebraic spaces are assumed to be of finite type over an fixed field k. Let G a g-dimensional linear algebraic group over k, and X a smooth scheme over k with a G-action.

Edidin and Graham ([**7**]), expanding on the idea of Totaro, have defined the G-equivariant Chow ring of X, denoted $\mathrm{A}_G^*(X)$, as follows. For each $i \geq 0$, choose a representation V of G with an open susbscheme $U \subset V$ on which G acts freely (in which case we call (V, U) a *good pair for* G), and such that the codimension of $V \setminus U$ is greater than i. The action of G on $X \times U$ is also free, and the quotient $(X \times U)/G$ exists as a smooth algebraic space; then Edidin and Graham define

$$\mathrm{A}_G^i(X) := \mathrm{A}^i\big((X \times U)/G\big),$$

where the right hand term is the Chow group of classes of cycles of codimension i (see [**26**] for the intersection theory on algebraic spaces). This is easily seen to be independent of the good pair (V, U) chosen. Moreover, under mild hypoteses (see [**7**, Proposition 23]) the quotient $(X \times U)/G$ exists as a smooth scheme, so $\mathrm{A}^i\big((X \times U)/G\big)$ is the usual Chow group defined in [**3**]. Then one sets

$$\mathrm{A}_G^*(X) := \bigoplus_{i \geq 0} \mathrm{A}_G^i(X).$$

If G acts freely on X, then there is a quotient X/G as an algebraic space of finite type over k, and the projection $X \to X/G$ makes X into a G-torsor over X/G; in this case the ring $\mathrm{A}_G^*(X)$ is canonically isomorphic to $\mathrm{A}^*(X/G)$.

Totaro's definition of the Chow ring of a classifying space is a particular case of this, as

$$\mathrm{A}_G^* := \mathrm{A}_G^*(\operatorname{Spec} k).$$

The formal properties of ordinary Chow rings extend to equivariant Chow rings. We recall briefly the properties that we need, which will be used without comments in the paper, referring to [**7**] for the details.

If $f\colon X \to Y$ is an equivariant morphism of smooth G-schemes there is an induced ring homomorphism $f^*\colon \mathrm{A}^*_G(Y) \to \mathrm{A}^*_G(X)$, making A^*_G into a contravariant functor from smooth G-schemes to graded commutative rings. Furthermore, if f is proper there is an induced homomorphism of groups $f_*\colon \mathrm{A}^*_G(X) \to \mathrm{A}^*_G(Y)$; the projection formula holds.

There is also a functoriality in the group: if $\phi : H \to G$ is a homomorphism of algebraic groups, the action of G on X induces an action of H on X, and there is homomorphism of graded rings

$$\mathrm{A}^*_G(X) \longrightarrow \mathrm{A}^*_H(X),$$

defined as follows: suppose that (U, V) (resp. (T, W)) is a good pair for X relative to G (resp. relative to H), and let G act on $V \times W$ via $g \cdot (v, w) = (g \cdot v, \phi(g) \cdot w)$ for $g \in G, (v, w) \in V \times W$. Then the projection $X \times U \times T \to X \times T$ induces a map

$$(X \times U \times T)/G \longrightarrow (X \times T)/H$$

and pulling back along this map we obtain the desired ring morphism. When H is a subgroup of G we will refer to this as a *restriction homomorphism*.

If H is a subgroup of G, then there is an H-equivariant embedding X into $X \times G/H$, defined in set-theoretic terms by sending x into $(x, 1)$. Then the composite of the restriction homomorphism $\mathrm{A}^*_G(X \times G/H) \to \mathrm{A}^*_H(X \times G/H)$ with the pullback $\mathrm{A}^*_H(X \times G/H) \to \mathrm{A}^*_H(X)$ is an isomorphism.

Of paramount importance is the localization sequence; if Y is a closed G-invariant subscheme of X, and we denote by $i\colon Y \hookrightarrow X$ and $j\colon X \setminus Y \hookrightarrow X$ the inclusions, then the sequence

$$\mathrm{A}^*_G(Y) \xrightarrow{i_*} \mathrm{A}^*_G(X) \xrightarrow{j^*} \mathrm{A}^*_G(X \setminus Y) \longrightarrow 0$$

is exact.

Furthermore, if E is a G-equivariant vector bundle on X, then by [**7**, Lemma 1] $(E \times U)/G \to (X \times U)/G$ is a vector bundle, so we can define equivariant Chern classes $\mathrm{c}_i(E) \in \mathrm{A}^i_G(X)$ as

$$\mathrm{c}_i(E) := \mathrm{c}_i\big((E \times U)/G\big),$$

enjoying the usual properties. Also, the pullback $\mathrm{A}^*_G(X) \to \mathrm{A}^*_G(E)$ is an isomorphism.

In particular, since the equivariant vector bundles over $\operatorname{Spec} k$ are the representations of G, we get Chern classes $\mathrm{c}_i(V) \in \mathrm{A}^i_G$ for every representation of G; and the pullback $\mathrm{A}^*_G \to \mathrm{A}^*_G(V)$ is an isomorphism.

We also need other easy properties of equivariant Chow rings, for which we do not have a suitable reference.

Lemma 1.1. *Let G a linear algebraic group, X a smooth G-scheme, H a normal algebraic subgroup G. Suppose that the action of H on X is free with quotient X/H. Then there is canonical isomorphism of graded rings*

$$\mathrm{A}^*_G(X) \simeq \mathrm{A}^*_{G/H}(X/H).$$

PROOF. Let (V, U) be a good pair for G, such that the codimension of $V \setminus U$ is greater then i. Then

$$\begin{aligned}\mathrm{A}^i_G(X) &= \mathrm{A}^i\bigl((X \times U)/G\bigr)\\ &= \mathrm{A}^i\bigl(((X \times U)/H)/(G/H)\bigr)\\ &= \mathrm{A}^i_{G/H}\bigl((X \times U)/H\bigr).\end{aligned}$$

Now, the quotient $(X \times V)/H$ is a G/H-equivariant vector bundle over X/H, $(X \times U)/H$ is an open subscheme of $(X \times V)/H$ whose complement has codimension larger than i. This yields isomorphisms

$$\begin{aligned}\mathrm{A}^i_{G/H}\bigl((X \times U)/H\bigr) &\simeq \mathrm{A}^i_{G/H}\bigl((X \times V)/H\bigr)\\ &\simeq \mathrm{A}^i_{G/H}(X/H).\end{aligned}$$

The resulting isomorphisms $\mathrm{A}^i_G(X) \simeq \mathrm{A}^i_{G/H}(X/H)$ yield the desired ring isomorphism $\mathrm{A}^*_G(X) \simeq \mathrm{A}^*_{G/H}(X/H)$. ♪

LEMMA 1.2. *Let G be an affine linear group acting on a smooth scheme X, $E \to X$ an equivariant vector bundle of rank r. Call $E_0 \subseteq E$ the complement of the zero section of E. Then the pullback homomorphism $\mathrm{A}^*_G(X) \to \mathrm{A}^*_G(E_0)$ is surjective, and its kernel is generated by the top Chern class $\mathrm{c}_r(E) \in \mathrm{A}^r_G(X)$.*

PROOF. Call $s \colon X \to E$ the zero-section. Then the statement follows immediately from the exactness of the localization sequence

$$\mathrm{A}^*_G(X) \xrightarrow{s_*} \mathrm{A}^*_G(E) \longrightarrow \mathrm{A}^*_G(E_0) \longrightarrow 0,$$

from the fact that the pullback $s^* \colon \mathrm{A}^*_G(E) \to \mathrm{A}^*_G(X)$ is an isomorphism, and from the self-intersection formula, which implies that the composite $\mathrm{A}^*_G(X) \xrightarrow{s_*} \mathrm{A}^*_G(E) \xrightarrow{s^*} \mathrm{A}^*_G(X)$ is multiplication by $\mathrm{c}_r(E)$. ♪

LEMMA 1.3. *Let H a linear algebraic group with an isomorphism $H \simeq \mathbb{A}^n_k$ of varieties, such that the for any field extension $k \subseteq k'$ and any $h \in H(k')$, the action of h on $H_{k'}$ by multiplication is corresponds to an affine automorphism of $\mathbb{A}^n_{k'}$ under the isomorphism above. Furthermore, let G be a linear algebraic group acting on H via group automorphisms, that corresponds to a linear action of G on $\mathbb{A}^n_k$ under this isomorphism.*

If G acts on a smooth scheme X: form the semidirect product $G \ltimes H$, and let $G \ltimes H$ act on X via the projection $G \ltimes H \to G$. Then the homomorphism

$$\mathrm{A}^*_G(X) \longrightarrow \mathrm{A}^*_{G \ltimes H}(X)$$

induced by the projection $G \ltimes H \to G$ is an isomorphism.

PROOF. Let (V, U) (resp. (V', U')) be a good pair for $G \ltimes H$ (resp. G). Then $G \ltimes H$ acts on U' via the projection $G \ltimes H \to G$: it follows that $G \ltimes H$

acts on $X \times H \times U \times U'$, and since the action of $G \ltimes H$ on H is transitive, and the stabilizer of the origin is H, there is an isomorphism

$$\begin{aligned}(X \times H \times U \times U')/(G \ltimes H) &= (X \times (G \ltimes H)/H \times U \times U')/(G \ltimes H) \\ &\simeq (X \times U \times U')/G.\end{aligned}$$

Look at the following commutative diagram:

$$\begin{array}{ccc}(X \times H \times U \times U')/(G \ltimes H) & \longrightarrow & (X \times U \times U')/G \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ (X \times U \times U')/(G \ltimes H) & \xrightarrow{\ f\ } & (X \times U')/G.\end{array}$$

Note that π_1 is an affine bundle: in fact, it is a fiber bundle with fiber isomorphic to $\mathbb{A}^n$, and structure group $G \ltimes H$ that acts on $\mathbb{A}^n$ by affine transformations, since the action of G on H is affine and the action of H on itself is affine. It follows from [**11**, p. 35] that π_1^* is an isomorphism. On the other hand, since $U \times U'$ is an open set of $V \times V'$ on which G acts freely, π_2^* is the identity on the equivariant Chow ring $\mathrm{A}^*_G(X)$, up to a degree that can be made arbitrarily large: so we have a commutative triangle

$$\begin{array}{ccc} & \mathrm{A}^*_{G \ltimes H}(X \times H) & \\ {\scriptstyle \pi_1^*}\ \nearrow\simeq & & \simeq\nwarrow \\ \mathrm{A}^*_{G \ltimes H}(X) & \xleftarrow{\quad f^* \quad} & \mathrm{A}^*_G(X)\end{array}$$

where the horizontal arrow is exactly the map induced by the projection $G \ltimes H \to G$. ♪

Here is another auxiliary result: it is well known (see for instance [**23**]) that $\mathrm{A}^*_{\boldsymbol{\mu}_n} \simeq \mathbb{Z}[\xi]/(n\xi)$, where ξ is the first Chern class of the character given by the inclusion $\boldsymbol{\mu}_n \hookrightarrow \mathrm{G_m}$. If G is an algebraic group, we will denote by $\xi \in \mathrm{A}^*_{G \times \boldsymbol{\mu}_n}$ the image of ξ under the map $\mathrm{A}^*_{\boldsymbol{\mu}_n} \to \mathrm{A}^*_{G \times \boldsymbol{\mu}_n}$ induced by the projection $G \times \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$. Using the projection $G \times \boldsymbol{\mu}_n \to G$, we can consider $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n}$ as an A^*_G-algebra. Then $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n}$ admits the following description:

LEMMA 1.4. *As an A^*_G-algebra, $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n}$ is generated by the element ξ, and the kernel of the evaluation map $\mathrm{A}^*_G[x] \to \mathrm{A}^*_G[\xi]$ is the ideal (nx). In other words,*

$$\mathrm{A}^*_{G \times \boldsymbol{\mu}_n} = \mathrm{A}^*_G[\xi]/(n\xi).$$

PROOF. The action of $\boldsymbol{\mu}_n$ on $\mathbb{A}^1$ given by the embedding $\boldsymbol{\mu}_n \hookrightarrow \mathrm{G_m}$ can be extended to an action of $G \times \boldsymbol{\mu}_n$ by letting G act trivially on $\mathbb{A}^1$. Then from Lemma 1.2 we have that $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n} \to \mathrm{A}^*_{G \times \boldsymbol{\mu}_n}(\mathrm{G_m})$ is surjective, and its kernel is generated by ξ. Since $\mathrm{G_m}/\boldsymbol{\mu}_n \simeq \mathrm{G_m}$, from Lemma 1.1 we deduce that $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n}(\mathrm{G_m}) \simeq \mathrm{A}^*_G(\mathrm{G_m})$, and since G acts trivially on $\mathrm{G_m}$ and $\mathrm{G_m}$ is an open subset of the affine line, $\mathrm{A}^*_G(\mathrm{G_m}) \simeq \mathrm{A}^*_G$. So we have that $\mathrm{A}^*_{G \times \boldsymbol{\mu}_n} \simeq \mathrm{A}^*_G[\xi]/(n\xi)$, as claimed. ♪

2. The special groups: GL_n, SL_n and Sp_n

Let us fix a field k: we write GL_n, SL_n and Sp_n for the corresponding algebraic groups over k.

These groups are always much easier to study: they are special, in the sense that every principal bundle is Zariski locally trivial. For GL_n and Sp_n the idea works in a very similar way: let us work out Sp_n, that is marginally harder. We proceed by induction on n, the case $n = 0$ being trivial.

Consider $V = \mathbb{A}^{2n}$, the tautological representation of Sp_n, with its symplectic form $h\colon V \times V \to k$ given in coordinates by

$$h(z_1, \dots, z_{2n}, w_1, \dots, w_{2n}) = z_1 w_{n+1} + \cdots + z_n w_{2n} - z_{n+1} w_1 - \cdots - z_{2n} w_n.$$

Denote by $e_1, \dots, e_{2n}$ the canonical basis of V.

The orbit structure of V is very simple: there are two orbits, the origin and its complement $U \stackrel{\text{def}}{=} V \setminus \{0\}$. Consider the subspace

$$V' = \langle e_1, \dots, e_{n-1}, e_{n+1}, \dots, e_{2n-1} \rangle;$$

the restriction of h to V' is a non-degenerate symplectic form, and $V = V' \oplus \langle e_n, e_{2n} \rangle$. This induces an embedding $\mathrm{Sp}_{n-1} \hookrightarrow \mathrm{Sp}_n$, identifying Sp_{n-1} with the stabilizer of the pair (e_n, e_{2n}).

Let G the stabilizer of the element e_n: then we have that $\mathrm{Sp}_{n-1} \subseteq G \subseteq \mathrm{Sp}_n$. The first inclusion admits a splitting: if $A \in G$, then A stabilizes the orthogonal complement $\langle e_n \rangle^{\perp}$. It follows that A induces a linear endomorphism on the quotient $\langle e_n \rangle^{\perp} / \langle e_n \rangle \simeq V'$, and this endomorphism is easily seen to preserve the symplectic form $h|_{V'}$, so it is an element of Sp_{n-1}. Thus we have a projection $G \to \mathrm{Sp}_{n-1}$: let H its kernel, so that $G = \mathrm{Sp}_{n-1} \ltimes H$.

The structure of H is as follows; the matrices in H are exactly those for which there are scalars $a_1, \dots, a_{2n-1}$ such that

$$Ae_i = \begin{cases} e_i - a_{i+n} e_n & \text{if } i = 1, \dots, n-1 \\ e_n & \text{if } i = n \\ e_i + a_{i-n} e_n & \text{if } i = n+1, \dots, 2n-1 \\ a_1 e_2 + \cdots + a_{2n-1} e_{2n-1} + e_{2n} & \text{if } i = 2n. \end{cases}$$

This yields an isomorphism of varieties $H \simeq \mathbb{A}^{2n-1}$. It is not hard to see that the conditions of Lemma 1.3 are satisfied for the action of Sp_{n-1} on H; hence the embedding $\mathrm{Sp}_{n-1} \subseteq G$ induces an isomorphism of rings $\mathrm{A}^*_G \simeq \mathrm{A}^*_{\mathrm{Sp}_{n-1}}$, so the composite

$$\mathrm{A}^*_{\mathrm{Sp}_n}(U) \longrightarrow \mathrm{A}^*_{\mathrm{Sp}_{n-1}}(U) \longrightarrow \mathrm{A}^*_{\mathrm{Sp}_{n-1}}(e_n) = \mathrm{A}^*_{\mathrm{Sp}_{n-1}}$$

is an isomorphism. The restriction of the representation V to Sp_{n-1} is the direct sum of V' and of a trivial 2-dimensional representation: hence the Chern classes $c_i = \mathrm{c}_i(V)$ restrict to the $\mathrm{c}_i(V')$. From the induction hypothesis, we conclude that $\mathrm{A}^*_{\mathrm{Sp}_n}(U)$ is generated by the images of c_2, $\dots$, c_{2n-2}.

From Lemma 1.2 we conclude that every class in $\mathrm{A}^*_{\mathrm{Sp}_n}$ can be written as a polynomial in c_2, …, c_{2n-2}, plus a multiple of c_{2n}. By induction on the degree we conclude that c_2, …, c_{2n} generate $\mathrm{A}^*_{\mathrm{Sp}_n}$.

To prove their algebraic independence, let us restrict to $\mathrm{A}^*_{T_n}$, where $T_n \simeq \mathrm{G}_\mathrm{m}^n$ is the standard maximal torus in Sp_n, consisting of diagonal matrices with entries $(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$. Then $\mathrm{A}^*_{T_n}$ is the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, where x_i is the first Chern class of the 1-dimensional representation given by the i^{th} projection $T_n \to \mathrm{G}_\mathrm{m}$. Then the total Chern class of the restriction of V_n to T_n is

$$(1 + x_1) \dots (1 + x_n)(1 - x_1) \dots (1 - x_n) = (1 - x_1^2) \dots (1 - x_n^2);$$

hence the restriction of c_{2i} is the i^{th} elementary symmetric function of $-x_1^2$, …, $-x_n^2$. This proves the independence of the c_{2i}.

As we mentioned, the argument for GL_n is very similar. For SL_n, one can proceed similarly, but it is easier to use the fact that, if GL_n acts freely on an algebraic variety U, the induced morphism $U/\mathrm{SL}_n \to U/\mathrm{GL}_n$ makes U/SL_n into a principal G_m-bundle on U/GL_n, associated with the determinant homomorphism $\det\colon \mathrm{GL}_n \to \mathrm{G}_\mathrm{m}$. Hence, by Lemma 1.2, we have an isomorphism $\mathrm{A}^*_{\mathrm{SL}_n} \simeq \mathrm{A}^*_{\mathrm{GL}_n}/(c_1)$, which gives us what we want.

Remark 2.1. All these arguments work with cohomology. when $k = \mathbb{C}$. The localization sequence in cohomology does not quite work in the same way, as the restriction homomorphism from the cohomology of the total space to that of an open subset is not necessarily surjective. However, if Y is a smooth closed subvariety of a smooth complex algebraic variety X, of pure codimension d, then there is an exact sequence

$$\cdots \longrightarrow \mathrm{H}_G^{i-2d}(Y) \longrightarrow \mathrm{H}_G^i(X) \longrightarrow \mathrm{H}_G^i(X \setminus Y) \longrightarrow \mathrm{H}_G^{i-2d+1}(Y) \longrightarrow \cdots .$$

Hence if we know that either the pullback $\mathrm{H}^*_G(X) \to \mathrm{H}^*_G(X \setminus Y)$ is surjective, or the pushforward $\mathrm{H}^*_G(Y) \to \mathrm{H}^*_G(X)$ is injective, we can conclude that we have an exact sequence

$$0 \longrightarrow \mathrm{H}^*_G(Y) \longrightarrow \mathrm{H}^*_G(X) \longrightarrow \mathrm{H}^*_G(X \setminus Y) \longrightarrow 0;$$

and this is sufficient to mimic the arguments above and give the result for cohomology.

Remark 2.2. These results can also be proved very simply from a result of Edidin and Graham (see [**6**]): if G is a special algebraic group, T a maximal torus and W the Weyl group, the natural restriction homomorphism $\mathrm{A}^*_G \to \left(\mathrm{A}^*_T\right)^W$ is an isomorphism.

3. The Chow ring of the classifying space of O_n

Let us fix a field k of characteristic different from 2. If $V = k^n$ is an n-dimensional vector space, we define a quadratic form $q\colon V \to k$ in the standard form

$$q(z_1, \dots, z_n) = z_1 z_{m+1} + \cdots + z_m z_{2m}$$

when $n = 2m$, and

$$q(z_1, \dots, z_n) = z_1 z_{m+1} + \dots + z_m z_{2m} + z_{2m+1}^2$$

when $n = 2m + 1$. We will denote by O_n the algebraic group of linear transformations preserving this quadratic form.

THEOREM 3.1 (R. Pandharipande, B. Totaro).

$$\mathrm{A}^*_{\mathrm{O}_n} = \mathbb{Z}[c_1, \dots, c_n]/(2c_{\mathrm{odd}}).$$

REMARK 3.2. Let V' be another n-dimensional vector space over k, with a non-degenerate quadratic form $q' \colon V' \to k$. We can associate with this another algebraic group $\mathrm{O}(q')$, which will not be isomorphic to $\mathrm{O}_n = \mathrm{O}(q)$, in general, unless k is algebraically closed.

However, one can show that there is an isomorphism of Chow rings $\mathrm{A}^*_{\mathrm{O}_n} \simeq \mathrm{A}^*_{\mathrm{O}(q')}$, such that the classes $\mathrm{c}_i(V)$ in the left hand side correspond to the classes $\mathrm{c}_i(V')$ in the right hand side. The principle that allows to prove this has been known for a long time ([**10**]): it is the existence of a bitorsor $I \to \operatorname{Spec} k$. This is the scheme representing the functor of isomorphisms of (V, q) with (V', q'). On I there is a left action of $\mathrm{O}(q')$ and right action of O_n, by composition. These two actions commute, and make I into a torsor under both groups (because (V, q) and (V', q') become isomorphic after a base extensions).

In general, assume that G and G' are algebraic groups over a field k (in fact, any algebraic space will do as a base), and $I \to \operatorname{Spec} k$ is (G', G)-bitorsor: that is, on I there is a right action of G and left action of G', and this makes I into a torsor under both groups. If X is a k-algebraic space on which G' acts on the left, then we can produce a k-algebraic space $I \times^G X$ on which G acts on the left, by dividing the product $I \times_{\operatorname{Spec} k} X$ by the right action of G, defined by the usual formula $(i, x)g = (ig, xg^{-1})$. The left action of G' is by multiplication on the first component: the quotients $G \backslash X$ and $G' \backslash (I \times^G X)$ are canonically isomorphic.

This operation gives an equivalence of the category of G-algebraic spaces with the category of G'-algebraic spaces. When applied to representations, it yields representations, and gives an equivalence of the category of representations of G and of G'. Furthermore, given a representation V of G, with an open subset $U \subseteq V$ on which G acts freely, we get a representation $V' = I \times^G V$ with an open subset $U' = I \times^G U$ on which G' acts freely, so that the quotients $G \backslash U$ and $G' \backslash U'$ are isomorphic. In Totaro's construction this gives an isomorphism of A^*_G with $\mathrm{A}^*_{G'}$.

So, in particular, the result that we have stated for O_n also holds for $\mathrm{O}(q')$ for any other non-degenerate n-dimensional quadratic form q', and we have

$$\mathrm{A}^*_{\mathrm{O}(q')} = \mathbb{Z}[c_1, \dots, c_n]/(2c_{\mathrm{odd}}).$$

The proof of the Theorem will be split into two parts: first we show that the c_i generate $\mathrm{A}^*_{\mathrm{O}_n}$, then that ideal of relations is generated by the given ones.

For the first part we proceed by induction on n.

For $n = 1$, $q(z) = z_1^2$, and $\mathrm{O}_1 = \boldsymbol{\mu}_2$, so

$$\mathrm{A}^*_{\mathrm{O}_1} = \mathrm{A}^*_{\boldsymbol{\mu}_2} \simeq \mathbb{Z}[c_1]/(2c_1).$$

For $n > 1$, let $B = \{v \in \mathbb{A}^n \mid q(v) \neq 0\}$, and set $Q = q^{-1}(1)$. Then $q\colon B \to \mathrm{G_m}$ is a fibration, with fibers isomorphic to Q. This fibration is not trivial, but it becomes trivial after an étale base change. Set

$$\widetilde{B} = \{(t, v) \in \mathrm{G_m} \times B \mid t^2 = q(v)\},$$

and consider the cartesian diagram

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathrm{G_m} & \xrightarrow{(-)^2} & \mathrm{G_m} \end{array}$$

where the first column is projection onto the first factor, and the top row is defined by the formula $(t, v) \mapsto tv$.

There are obvious commuting actions of $\boldsymbol{\mu}_2$ and O_n on $\widetilde{B}$, the first defined by $\epsilon \cdot (t, v) = (\epsilon t, v)$, and the second by $M \cdot (t, v) = (t, Mv)$. The quotient $\widetilde{B}/\boldsymbol{\mu}_2$ is isomorphic to B, and the induced action of O_n on the quotient coincides with the given action on B. From Lemma 1.1, we obtain an isomorphism

$$\mathrm{A}^*_{\mathrm{O}_n}(B) \simeq A_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\widetilde{B}).$$

Then there is an isomorphism of $\mathrm{G_m}$-schemes $\widetilde{B} \simeq \mathrm{G_m} \times Q$ defined by the formula $(t, v) \mapsto (t, v/t)$. The given actions of $\boldsymbol{\mu}_2$ and of O_n on $\widetilde{B}$ induce commuting actions on $\mathrm{G_m} \times Q$ given by $\epsilon \cdot (t, v) = (\epsilon t, \epsilon v)$ for $\epsilon \in \boldsymbol{\mu}_2$ and $M(t, v) = (t, Mv)$ for $M \in \mathrm{O}_n$. These define an action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathrm{G_m} \times Q$, and $\mathrm{A}^*_{\mathrm{O}_n}(B)$ is isomorphic to $\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\mathrm{G_m} \times Q)$.

This action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathrm{G_m} \times Q$ extends uniquely to an action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathbb{A}^1 \times Q$, defined by the same formulae. This action is defined by two separate action on $\mathbb{A}^1$ and Q, and the action on $\mathbb{A}^1$ is linear, defined by the non-trivial character of $\boldsymbol{\mu}_2$ through the projection $\boldsymbol{\mu}_2 \times \mathrm{O}_n \to \boldsymbol{\mu}_2$. Call ξ the first Chern class of this representation. From Lemma 1.2, we have an isomorphism

$$\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\mathrm{G_m} \times Q) \simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)/(\xi). \tag{3.1}$$

To investigate $\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)$ we will also use an orthogonal basis $e'_1, \ldots, e'_n$ of V, in which q has the form

$$q(z_1 e'_1 + \cdots + z_n e'_n) = z_1^2 + \cdots + z_m^2 - z_{m+1}^2 - \cdots - z_n^2$$

when $n = 2m$, and

$$q(z_1 e'_1 + \cdots + z_n e'_n) = z_1^2 + \cdots + z_{m+1}^2 - z_{m+2}^2 - \cdots - z_n^2$$

when $n = 2m + 1$.

Now, the action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on Q is transitive; let H the stabilizer of the point $e'_1 \in Q$. The structure of H is as follows. Set $V' \stackrel{\mathrm{def}}{=} \langle e'_2, \ldots, e'_n \rangle$, so that V is the orthogonal sum $\langle e'_1 \rangle \oplus V'$, and call q' the restriction of q to V'. Then the group $\mathrm{O}_{q'}$ of linear automorphisms of V' preserving q' is naturally embedded into O_n, as the stabilizer of e'_1. Notice that in an appropriate basis q' has the standard form

$$q'(z_1, \ldots, z_{n-1}) = z_1 z_{m+1} + \cdots + z_m z_{2m}$$

when $n = 2m + 1$, and the opposite of the standard form

$$q'(z_1, \ldots, z_{n-1}) = -(z_1 z_m + \cdots + z_{m-1} z_{2m-2} + z_{2m-1}^2)$$

when $n = 2m$; in both cases the orthogonal group $\mathrm{O}(q')$ is isomorphic to O_{n-1}, and we identify it with O_{n-1}.

The stabilizer of e'_1 in $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ is the group $\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}$, embedded into $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ with the injective homomorphism

$$(\epsilon, M) \longmapsto (\epsilon, \epsilon M).$$

It follows that

$$\begin{aligned} \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q) &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}\big((\boldsymbol{\mu}_2 \times \mathrm{O}_n)/(\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1})\big) \\ &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}. \end{aligned}$$

We obtain a chain of isomorphisms

$$\begin{aligned} \mathrm{A}^*_{\mathrm{O}_n}(B) &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)/(\xi) \\ &\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}/(\xi). \end{aligned}$$

Finally, from Lemma 1.1 we get an isomorphism

$$\begin{aligned} \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}/(\xi) &\simeq \mathrm{A}^*_{\mathrm{O}_{n-1}}[\xi]/(\xi) \\ &\simeq \mathrm{A}^*_{\mathrm{O}_{n-1}}. \end{aligned}$$

The composite $\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_n}(U) \to \mathrm{A}^*_{\mathrm{O}_{n-1}}$ is the pullback induced by the embedding $\mathrm{O}_{n-1} \subseteq \mathrm{O}_n$.

The restriction of V to O_{n-1} is the direct sum of V' and a trivial 1-dimensional representation, hence the restriction $\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_{n-1}}$ carries c_i into $\mathrm{c}_i(V')$. Therefore, by induction hypothesis, the images of $c_1, \ldots, c_{n-1}$ generate $\mathrm{A}^*_{\mathrm{O}_n}(B)$.

Next, we claim that the restriction homomorphism $\mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{O}_n}(B)$ is an isomorphism. To see this, set

$$C = \{v \in \mathbb{A}^n \setminus \{0\} \mid q(v) = 0\}$$

with its reduced scheme structure, and consider the fundamental exact sequence

$$\mathrm{A}^*_{\mathrm{O}_n}(C) \xrightarrow{i_*} \mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\}) \longrightarrow \mathrm{A}^*_{\mathrm{O}_n}(U) \longrightarrow 0.$$

We need to show that i_* is the zero map. In fact, $q\colon \mathbb{A}^n\setminus\{0\} \to \mathbb{A}^1$ is smooth, since the characteristic of the base field is not 2, so C is the scheme-theoretic inverse image of $\{0\}$. The map $q\colon \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^1$ is O_n-equivariant, if we let O_n act trivially on $\mathbb{A}^1$; and the fundamental class $[0] \in \mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^1)$ equals zero. Since the inverse image of $[0]$ in $\mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\})$ is $[C]$, we can conclude that

$$[C] = 0 \in \mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\}).$$

Next we show that the pullback $i^*\colon \mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{O}_n}(C)$ is surjective: in this case, for every $\alpha \in \mathrm{A}^*_{\mathrm{O}_n}(C \setminus \{0\})$, we have $\alpha = i^*\beta$ for some $\beta \in \mathrm{A}^*_{\mathrm{O}_n}(\mathbb{A}^n \setminus \{0\})$, so

$$i_*(\alpha) = i_*i^*(\beta) = [C] \cdot \beta = 0$$

by the projection formula, and i_* is the zero map, as claimed.

To show surjectivity, notice that the action of O_n on C is transitive. Let us investigate the stabilizer G of $e_1 \in C$. Set $n = 2m$ or $n = 2m + 1$, as usual. If we define

$$V' = \langle e_2, \dots, e_m, e_{m+2}, \dots, e_n\rangle$$

then the restriction of q to V' has the standard form, and V is the orthogonal sum $V' \oplus \langle e_1, e_{m+1}\rangle$. This gives an embedding $\mathrm{O}_{n-2} \subseteq \mathrm{O}_n$, identifying O_{n-2} with the stabilizer of the pair (e_1, e_{m+1}).

An analysis very similar to that we have carried out for the stabilizer of a vector under Sp_n leads to the conclusion that the stabilizer G of e_1 is a semidirect product $\mathrm{O}_{n-2} \ltimes H$, where H is isomorphic to $\mathbb{A}^{n-1}$ as a variety, the action of an element of H is itself is given by an affine map, and the action of O_{n-2} on H is linear: by Lemma 1.3, the embedding $\mathrm{O}_{n-2} \subseteq G$ induces an isomorphism of rings $\mathrm{A}^*_G \simeq \mathrm{A}^*_{\mathrm{O}_{n-2}}$, so the composite

$$\mathrm{A}^*_{\mathrm{O}_n}(C) \longrightarrow \mathrm{A}^*_{\mathrm{O}_{n-2}}(C) \longrightarrow \mathrm{A}^*_{\mathrm{O}_{n-2}}(e_1) = \mathrm{A}^*_{\mathrm{O}_{n-2}}$$

is an isomorphism. But the c_i restrict in $\mathrm{A}^*_{\mathrm{O}_{n-2}}$ to the Chern classes of V': hence, by inductions hypothesis, they generate $\mathrm{A}^*_{\mathrm{O}_{n-2}}$. Hence the pullback $\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_n}(C)$ is surjective, as claimed. This ends the proof that the c_i generate $\mathrm{A}^*_{\mathrm{O}_n}$. Let us investigate the relations.

The quadratic form q induces an isomorphism $V \simeq V^\vee$ of representations of O_n, hence for each i we have $\mathrm{c}_i(V) = (-1)^i\mathrm{c}_i(V)$. This shows that $2c_i = 0$ when i is odd.

To show that these generate the ideal of relations among the c_i, let $J \subseteq \mathbb{Z}[X_1, \dots, X_n]$ be the ideal generated by $2X_1$, $2X_3$, ... Let $P \in \mathbb{Z}[X_1, \dots, X_n]$ be a homogeneous polynomial such that $P(c_1, \dots, c_n) = 0 \in \mathrm{A}^*_{\mathrm{O}_n}$: we need to check that $P \in J$. By modifying P by an element of J, we may assume that P is of the form $Q + R$, where Q is a polynomial in the even X_i, while R is a polynomial in which every monomial contains some X_i with i odd, and all of whose coefficients are either 0 or 1.

Let $T_m \simeq \mathrm{G}_\mathrm{m}^m$ be the standard torus in O_n: the embedding $T_m \subseteq \mathrm{O}_n$ sends $(t_1, \dots, t_m)$ into the diagonal matrix with entries $(t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1})$ if $n = 2m$, and $(t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}, 1)$ if $n = 2m+1$. Then $\mathrm{A}^*_{T_m} = \mathbb{Z}[x_1, \dots, x_m]$, where x_i is the first Chern class of the i^{th} projection $\chi_i \colon T_n \to \mathrm{G}_\mathrm{m}$. The restriction of V to T_n splits as $\rho \overset{\text{def}}{=} \chi_1 + \dots + \chi_m + \chi_1^{-1} + \dots + \chi_m^{-1}$ when m is even, and $\rho + 1$ when n is odd. Hence the total Chern class of the restriction of V to T_n is

$$(1 + x_1) \dots (1 + x_m)(1 - x_1) \dots (1 - x_m) = (1 - x_1^2) \dots (1 - x_m^2);$$

and this means that the restrictions of the c_i is 0 when i is odd, while c_{2j} restricts to the j^{th} symmetric function of $-x_1^2, \dots, -x_m^2$. Hence the restrictions of even Chern classes are algebraically independent. In the decomposition $0 = P(c_1, \dots, c_n) = Q(c_2, \dots, c_{2m}) + R(c_1, \dots, c_n)$ the summand $R(c_1, \dots, c_n)$ restricts to 0, so $Q(c_2, \dots, c_{2m})$ also restricts to 0. This implies that $Q = 0$. So we have that P has coefficients that are either 0 or 1.

Now take a basis $e_1', \dots, e_n'$ of V in which q has a diagonal form. Consider the subgroup $\boldsymbol{\mu}_2^n \subseteq \mathrm{O}_n$ consisting of linear transformations that take each e_i' into e_i' or $-e_i'$. If we call η_i the first Chern class of the character obtained composing the i^{th} projection $\boldsymbol{\mu}_2^n \to \boldsymbol{\mu}_2$ with the embedding $\boldsymbol{\mu}_2 \hookrightarrow \mathrm{G}_\mathrm{m}$, then by Lemma 1.4 we have

$$\mathrm{A}^*_{\boldsymbol{\mu}_2^n} = \mathbb{Z}[\eta_1, \dots, \eta_n]/(2\eta_1, \dots, 2\eta_n).$$

There is a natural ring homomorphism from $\mathrm{A}^*_{\boldsymbol{\mu}_2^n}$ into the polymomial ring $\mathbb{F}_2[Y_1, \dots, Y_n]$ that sends each η_i into Y_i. The restriction of V to $\boldsymbol{\mu}_2^n$ has total Chern class $(1 + \eta_1) \dots (1 + \eta_n)$; hence the image of c_i in $\mathbb{F}_2[Y_1, \dots, Y_n]$ is the i^{th} elementary symmetric polynomial s_i in the Y_i. The s_i are algebraically independent in $\mathbb{F}_2[Y_1, \dots, Y_n]$, the image of $0 = P(c_1, \dots, c_n)$ is $P(s_1, \dots, s_n)$, and P has coefficients that either 0 or 1. This implies that $P = 0$, and completes the proof of the theorem.

4. The Chow ring of the classifying space of SO_n

Let k be a field of characteristic different from 2, set $V = k^n$, and let $q \colon V \to k$ be the same quadratic form as in the previous section. Consider the subgroup $\mathrm{SO}_n \subseteq \mathrm{O}_n$ of orthogonal linear transformations of determinant 1.

If n is odd, $\mathrm{A}^*_{\mathrm{SO}_n}$ can be easily computed from $\mathrm{A}^*_{\mathrm{O}_n}$, as was noticed in [**5**] and [**19**].

Theorem 4.1 (R. Pandharipande, B. Totaro). *If n is odd, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n]/(2c_{\mathrm{odd}} = 0).$$

Proof. When is n odd there is an isomorphism $\mathrm{O}_n \simeq \boldsymbol{\mu}_2 \times \mathrm{SO}_n$; the determinant character $\det\colon \mathrm{O}_n \to \boldsymbol{\mu}_2$ (whose first Chern class in $\mathrm{A}^*_{\mathrm{O}_n}$ is c_1) corresponds to the projection $\boldsymbol{\mu}_2 \times \mathrm{SO}_n \to \boldsymbol{\mu}_2$. Then from Lemma 1.4 we get that

$$\mathrm{A}^*_{\mathrm{SO}_n} \simeq \mathrm{A}^*_{\mathrm{O}_n}/(c_1)$$

and the conclusion follows. ♪

4.1. The Edidin–Graham construction. From now on we shall assume that n is even, and write $n = 2m$.

In this case, $\mathrm{A}^*_{\mathrm{SO}_n}$ is not generated by the Chern classes of the standard representation, not even rationally. This can be seen easily for $n = 2$. We have that SO_2 consists of matrices of the form

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and so is isomorphic to $\mathrm{G_m}$. Then

$$\mathrm{A}^*_{\mathrm{SO}_2} = \mathrm{A}^*_{\mathrm{G_m}} = \mathbb{Z}[\xi],$$

where ξ is the first Chern class of the tautological representation $L = \mathbb{A}^1$, on which $\mathrm{G_m}$ acts via multiplication. Hence $V = L \oplus L^\vee$, so $\mathrm{c}_2(V) = -\xi^2$.

For general n, the vector space V will still split as the direct sum of two totally isotropic subspaces, one dual to the other: however, when $n > 2$ this splitting is not unique, and the totally isotropic subspaces are not invariant under the action of SO_n, so V is not a direct sum of two nontrivial representations (and V is in fact irreducible). Still, in topology V has an Euler class $\epsilon_m \in \mathrm{H}^{2m}_{\mathrm{SO}_n}$, whose square is $(-1)^m c_m$. Let us recall Edidin and Graham's construction of an algebraic multiple of ϵ_m (see [**8**]).

In what follows we will use the classical conventions for projectivizations and Grassmannians; those seem a little more natural in intersection theory than Grothendieck's. So, if W is a vector space, we denote by $\mathbb{P}(W)$ the vector space of lines in W, and by $\mathbb{G}(r, W)$ the Grassmannian of subspaces of dimension r; and similarly for vector bundles.

Denote by $\mathbb{I}(m, V)$ the smooth subvariety of $\mathbb{G}(m, V)$ consisting of maximal totally isotropic subspaces of V. It is well known that O_n acts transitively on $\mathbb{I}(m, V)$, and that $\mathbb{I}(m, V)$ has two connected components, each of which is an orbit under the action of SO_n. Let us choose one of the orbits, for example, the one containing the subspace $\langle e_1, \dots, e_m\rangle$. Every totally isotropic subspace of dimension $m-1$ of V is contained in exactly two maximal totally isotropic subspaces, one in each connected component.

There is a well known equivalence of categories between O_n-torsors and vector bundles of rank n with a non-degenerate quadratic form. If E is a vector bundle on a scheme X with a non-degenerate quadratic form, this corresponds to a O_n-torsor $\pi\colon P \to X$, the torsor of isometries between E and $V \times X$; with this torsor we can associate a $\boldsymbol{\mu}_2$-torsor (that is, an étale double cover) $P/\mathrm{SO}_n \to X$ via the determinant homomorphism $\det\colon \mathrm{O}_n \to \boldsymbol{\mu}_2$. This cover can be described geometrically as follows.

Consider the subscheme $\mathbb{I}(m,E)$ of totally isotropic subbundles in the relative Grassmannian $\mathbb{G}(m,E) \to X$; the projection $\mathbb{I}(m,E) \to X$ is proper and smooth, and each of its geometric fibers has two connected components. Let $\mathbb{I}(m,E) \to \widetilde{\mathbb{I}}(m,E) \to X$ be the Stein factorization; then $\widetilde{\mathbb{I}}(m,E) \to X$ is an étale double cover, and is precisely the double cover $P/\mathrm{SO}_n \to X$. This can be seen as follows.

On P we have, by definition, an isometry of π^*E with $V \times P$. In $V \times P$ we have a maximal totally isotropic subbundle $\langle e_1, \dots, e_m\rangle \times P$, so we get a maximal totally isotropic subbundle of π^*E. This defines a morphism $P \to \mathbb{I}(m,E)$ over X; the composite $P \to \mathbb{I}(m,E) \to \widetilde{\mathbb{I}}(m,E)$ induces the desired isomorphism $P/\mathrm{SO}_n \simeq \widetilde{\mathbb{I}}(m,E)$.

Hence, to give a reduction of structure group of $P \to X$ to SO_n is equivalent to assigning a section $X \to \widetilde{\mathbb{I}}(m,E)$. This gives an equivalence of the groupoid of SO_n-torsors on X with the groupoid of vector bundles $E \to X$ of rank n with a non-degenerated quadratic form, and a section $X \to \widetilde{\mathbb{I}}(m,E)$. We shall refer to such a structure as an SO_n-*structure on* E.

Furthermore, given an SO_n-structure on E, if $f\colon T \to X$ is a morphism of algebraic varieties, and L is a totally isotropic subbundle of f^*E of rank m, we say that L is *admissible* if the image of T under the morphism $T \to \mathbb{I}(m,X)$ corresponding to L is contained in the inverse image of the given embedding $X \subseteq \widetilde{\mathbb{I}}(m,E)$.

Here is the construction of Edidin and Graham. We will follow their notation. Let E be a vector bundle of rank n with an SO_n-structure on a smooth algebraic variety X. For each $i = 1, \dots, m$ consider the flag variety $f_i\colon Q_i \to X$ of totally isotropic flags $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{m-i} \subseteq E$, with each L_s of rank s. For each i, denote by $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{m-i} \subseteq f_i^*E$ the universal flag on Q_i. The restriction of the quadratic form to $L_{m-i}^{\perp}$ is degenerate, with radical equal to L_{m-i}; hence on Q_i there lives a vector bundle $E_i \overset{\text{def}}{=} L_{m-i}^{\perp}/L_{m-i}$ of rank $2i$ with a non-degenerate quadratic form. For each $i = 1, \dots, m-1$ we have a projection $\pi_i\colon Q_{i-1} \to Q_i$, obtained by dropping the last totally isotropic subbundle in the chain; and Q_{i-1} is canonically isomorphic, as a scheme over Q_i, to the smooth quadric bundle in $\mathbb{P}(E_i)$ defined by the quadratic form on E_i. This means that Q_{i-1} is a family of quadrics of dimension $2(i-1)$ over Q_i. Let us denote by $h_i \in \mathrm{A}^1(Q_{i-1})$ the restriction to Q_{i-1} of the class $\mathrm{c}_1\big(\mathcal{O}_{\mathbb{P}(E_i)}(1)\big) \in \mathrm{A}^1\big(\mathbb{P}(E_i)\big)$.

Each bundle E_i has a canonical SO_{n-2i}-structure. Call $\pi_i\colon L_{m-i}^{\perp} \to E_i$ the projection. From each totally isotropic vector subbundle $L \subseteq E_i$ of rank $m-i$, we get a totally isotropic vector subbundle $\pi_i^*L \subseteq L_{m-i}^{\perp} \subseteq f_i^*E$ of rank m; then L is admissible if and only if π_i^*L is admissible.

The universal flag $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{m-1} \subseteq f_1^*E$ on Q_1 can be completed in a unique way to a maximal totally isotropic flag $L_1 \subseteq \cdots \subseteq L_{m-1} \subseteq L_m \subseteq f_1^*E$ in such a way that L_m is admissible. Then Edidin and Graham define

$$y_m(E) = f_*\big(s \cdot \mathrm{c}_m(L_m)\big) \in \mathrm{A}^m(X)$$

where we have set

$$s = h_2^2 h_3^4 \dots h_m^{2m-2} \in \mathrm{A}^*(Q_1).$$

REMARK 4.2. In this formula each of the classes h_i should be pulled back to Q_1. Here, and in what follows, we use the following convention: when $f\colon Y \to X$ is a morphism of smooth varieties, and $\xi \in \mathrm{A}^*(X)$, we will also write ξ for $f^*\xi \in \mathrm{A}^*(Y)$. Similarly, if $E \to X$ is a vector bundle, we will also write E for f^*E. This has the advantage of considerably simplifying notation, and should not lead to confusion. With this notation, when f is proper the projection formula reads: if $\xi \in \mathrm{A}^*(X)$ and $\eta \in \mathrm{A}^*(Y)$, then

$$f_*(\xi\eta) = \xi f_*\eta.$$

There is also an inductive definition of $y_m(E)$. If $m = 1$ then there is precisely one totally admissible isotropic line subbundle of E, and we have $y_1(E) = \mathrm{c}_1(L)$, by definition.

For $m > 1$ we have a vector bundle E_{m-1} on Q_{m-1} with an SO_{n-2}-structure.

LEMMA 4.3. *The formula*

$$y_m(E) = -f_{m-1*}\bigl(h_m^{2m-1} y_{m-1}(E_{m-1})\bigr)$$

holds.

PROOF. To prove this, call $g\colon Q_1 \to Q_{m-1}$ the projection: on Q_1 we have a flag

$$L_2/L_1 \subseteq L_3/L_1 \subseteq \cdots \subseteq L_{m-1}/L_1 \subseteq g^* E_{m-1}$$

that makes Q_1 into the variety of totally isotropic flags of length $m-2$ in E_{m-1}; we complete this to a maximal totally isotropic flag by adding L_m/L_1. So we get

$$y_{m-1}(E_{m-1}) = g_*\bigl(h_2^2 h_3^4 \dots h_{m-1}^{2m-4} \mathrm{c}_{m-1}(L_m/L_1)\bigr).$$

On the other hand, on $Q_{m-1} \subseteq \mathbb{P}(E)$, the line bundle $L_1 \subseteq f_{m-1}^* E$ is the pullback of the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$, so $\mathrm{c}_1(L_1) = -h_m$. Hence we have

$$\mathrm{c}_m(L_m) = -h_m \mathrm{c}_{m-1}(L_m/L_{m-1})$$

and

$$\begin{aligned}
-f_{m-1*}\bigl(h_m^{2m-1} y_{m-1}(E_{m-1})\bigr) &= -f_{m-1*}\bigl(h_m^{2m-1} g_*\bigl(h_2^2 h_3^4 \dots h_{m-1}^{2m-4} \\
&\qquad \mathrm{c}_{m-1}(L_m/L_1)\bigr)\bigr) \\
&= -f_{1*}\bigl(h_2^2 h_3^4 \dots h_{m-1}^{2m-4} h_m^{2m-1} \mathrm{c}_{m-1}(L_m/L_1)\bigr) \\
&= f_{1*}\bigl(h_2^2 h_3^4 \dots h_{m-1}^{2m-4} h_m^{2m-2} \mathrm{c}_m(L_m)\bigr) \\
&= y_m(E)
\end{aligned}$$

as claimed. ♪

The Edidin–Graham class $y_m \in \mathrm{A}^m_{\mathrm{SO}_n}$ is defined as follows. Take a representation W of SO_n with an open subset U on which SO_n acts freely, and whose complement has codimension larger than m. Call E the vector bundle with an SO_n-structure associated with the SO_n-torsor $U \to U/\mathrm{SO}_n$. Then we set

$$y_m = y_m(E) \in \mathrm{A}^m(U/\mathrm{SO}_n) = \mathrm{A}^m_{\mathrm{SO}_n}\,.$$

It is easy to verify that this is independent of the W and U chosen.

THEOREM 4.4 (R. Field). *If $n = 2m$, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n, y_m]/\big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

REMARK 4.5. Once again, this result can be extended to other quadratic forms (compare with Remark 3.2). Let V' be another n-dimensional vector space over k, with a non-degenerate quadratic form $q' \colon V' \to k$. This induces a non-degenerate quadratic form on the exterior powers $\bigwedge^i V'$. Let us assume that there is an isometry $\bigwedge^n V \simeq \bigwedge^n V'$.

This is equivalent to the following more concrete condition. We will write $\det q' \in k^*/k^{*2}$ for the class in k^*/k^{*2} of the determinant of a matrix representing q' in some basis. Then two n-dimensional quadratic forms have isomorphic top exterior powers if and only if they have the same determinant. Hence the condition above is equivalent to the equality

$$\det q' = (-1)^m \in k^*/k^{*2}.$$

Fix an isometry $\bigwedge^n V \simeq \bigwedge^n V'$. We can construct an $\big(\mathrm{SO}(q'), \mathrm{SO}_n\big)$-bitorsor $I \to \operatorname{Spec} k$, as the scheme representing the functor of isometries $V \simeq V'$ inducing the fixed isometry $\bigwedge^n V \simeq \bigwedge^n V'$. So we deduce the following result: if the condition above is satisfied, there exists a class $y_m \in \mathrm{A}^m_{\mathrm{SO}(q')}$, such that

$$\mathrm{A}^*_{\mathrm{SO}(q')} = \mathbb{Z}[c_2, \dots, c_n, y_m]/\big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

The proof of the theorem will be split into three parts: first we verify that the classes c_i and y_m generate $\mathrm{A}^*_{\mathrm{SO}_n}$, next that the relations holds, and finally that they generate the ideal of relations.

Step 1: The generators. We proceed by induction on m. In the case $m = 1$ the statement says that

$$\mathrm{A}^*_{\mathrm{SO}_1} = \mathbb{Z}[c_2, y_1]/(y_1^2 + c_2) = \mathbb{Z}[y_1]$$

we have seen that $\mathrm{SO}_1 = \mathrm{G_m}$, that y_1 is the first Chern class of the identity character on $\mathrm{G_m}$, and that $c_2 = -y_1^2$.

Suppose $m > 1$. Set $B = \{x \in \mathbb{A}^n \mid q(x) \neq 0\}$ and $C = \{x \in \mathbb{A}^n \setminus \{0\} \mid q(x) = 0\}$. Proceeding precisely as for O_n, one establishes the following results.

(1) Let $e'_1, \dots, e'_n$ be an orthogonal basis of V in which q has the form

$$q(z_1 e'_1 + \cdots + z_n e'_n) = z_1^2 + \cdots + z_m^2 - z_{m+1}^2 - \cdots - z_n^2.$$

Then the stabilizer of $e_1' \in B$ in SO_n is isomorphic to SO_{n-1}, and the composite

$$\mathrm{A}^*_{\mathrm{SO}_n}(B) \longrightarrow \mathrm{A}^*_{\mathrm{SO}_{n-1}}(B) \longrightarrow \mathrm{A}^*_{\mathrm{SO}_{n-1}}(e_1') = \mathrm{A}^*_{\mathrm{SO}_{n-1}}$$

is an isomorphism.

(2) The stabilizer of the pair (e_1, e_{m+1}) is isomorphic to SO_{n-2}. The composite

$$\mathrm{A}^*_{\mathrm{SO}_n}(C) \longrightarrow \mathrm{A}^*_{\mathrm{SO}_{n-2}}(C) \longrightarrow \mathrm{A}^*_{\mathrm{SO}_{n-2}}(e_1) = \mathrm{A}^*_{\mathrm{SO}_{n-2}}$$

is an isomorphism.

Call $i\colon C \subseteq \mathbb{A}^n \setminus \{0\}$ and $j\colon B \subseteq \mathbb{A}^n \setminus \{0\}$ the inclusions. Then we have an exact sequence

$$\mathrm{A}^*_{\mathrm{SO}_n}(C) \xrightarrow{i_*} \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) \xrightarrow{j^*} \mathrm{A}^*_{\mathrm{SO}_n}(B) \longrightarrow 0.$$

By induction hypothesis, we have that $\mathrm{A}^*_{\mathrm{SO}_n}(C) \simeq \mathrm{A}^*_{\mathrm{SO}_{n-2}}$ is generated as a ring by $c_2, \ldots, c_{n-2}$ and y_{m-1}. From this, and from the relation $y_{m-1}^2 - (-1)^{m-1}2^{n-4}c_{n-2}$, we see that $\mathrm{A}^*_{\mathrm{SO}_n}(C)$ is generated as a module over $\mathrm{A}^*_{\mathrm{SO}_n}$ by 1 and y_{m-1}; hence, since i_* is a homomorphism of $\mathrm{A}^*_{\mathrm{SO}_n}$-modules, by the projection formula, we see that the kernel of the pullback $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n\setminus\{0\}) \to \mathrm{A}^*_{\mathrm{SO}_n}(B)$ is generated as an ideal by $i_*1 = [C]$ and i_*y_{m-1}.

As in the case of O_n, we see that the fundamental class $[C] \in \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\})$ is 0, because C is the scheme-theoretic zero-locus of the invariant function q. Furthermore, the images of $c_2, \ldots c_{n-1}$ generate $\mathrm{A}^*_{\mathrm{SO}_n}(U) \simeq \mathrm{A}^*_{\mathrm{SO}_{n-1}}$: and this implies that $c_2, \ldots, c_{n-1}$, together with i_*y_{m-1}, generate $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) = \mathrm{A}^*_{\mathrm{SO}_n}/(c_n)$. Hence $c_2, \ldots, c_n, i_*y_{m-1}$ generate $\mathrm{A}^*_{\mathrm{SO}_n}$. Next, we have a Lemma.

LEMMA 4.6.

$$i_*y_{m-1} = -y_m \in \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}).$$

PROOF. Let W be a representation of SO_n, and U an open set of W on which the action of SO_n is free, and such that the codimension of $W \setminus U$ in W is larger than m. The vector bundle associated with the SO_n-torsor $U \to U/\mathrm{SO}_n$ is $E \overset{\text{def}}{=} (\mathbb{A}^n \times U)/\mathrm{SO}_n$. We set $X \overset{\text{def}}{=} \big((\mathbb{A}^n \setminus \{0\}) \times U\big)/\mathrm{SO}_n$, so that $X \subseteq E$ is the complement of the zero section, while $Y \overset{\text{def}}{=} (C \times U)/\mathrm{SO}_n \subseteq X$ is the closed subscheme consisting of non-zero isotropic vectors, and $Z \overset{\text{def}}{=} X \setminus Y$. By a slight abuse of notation, we will denote $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ the inclusions. Note that there is a tautological section $s\colon X \to E$ defined set-theoretically by $[u, z] \mapsto [u, z, z]$.

Let us first prove that $j^*y_m = 0 \in \mathrm{A}^*_{\mathrm{SO}_n}(B)$. In fact, the tautological section restricted to Z has the property that $q(s(x)) \neq 0$ for all x, and so $j^*y_m(E) = y_m(j^*E) = 0$, due to the following result.

LEMMA 4.7. *Let $(E, q) \to X$ be a rank $n = 2m$ vector bundle with a non-degenerate quadratic form. Suppose that there exists a section $s\colon X \longrightarrow E$ such that $q(s(z)) \neq 0$ for all $z \in X$. Then $y_m(E) = 0$.*

PROOF. Pulling back to the flag variety $Q_1 \to X$, it suffices to show that if $L \subset E$ is a rank m totally isotropic subbundle, then $\mathrm{c}_m(L) = 0$. The quadratic form gives a perfect pairing $L \times E/L \longrightarrow \mathcal{O}_X$, so $L^\vee \simeq E/L$. On the other hand the line subbundle $\langle s \rangle$ generated by s has intersection with L equal to 0 at every point of X; hence the composite $\mathcal{O}_X \xrightarrow{w} E \to E/L$ gives a nowhere vanishing section of E/L, so that

$$\mathrm{c}_m(L) = (-1)^m \mathrm{c}_m(E/L) = 0$$

as claimed. ♨

It follows that $y_m = d \cdot i_* y_{m-1}$ with $d \in \mathbb{Z}$. We will compute d by restricting to a maximal torus; but first observe that since SO_{n-2} is included in SO_n as the stabilizer of the pair (e_1, e_{m+1}), there is an isomorphism

$$\begin{aligned}(\mathbb{A}^n \times U)/\mathrm{SO}_{n-2} &\longrightarrow \mathbb{A}^2 \times \big((\mathbb{A}^{n-2} \times U)/\mathrm{SO}_{n-2}\big)\\ [(z_1, \ldots, z_n), u] &\longmapsto \big((z_1, z_{m+1}), [(z_2, \ldots, z_m, z_{m+2}, \ldots, z_n), u]\big),\end{aligned}$$

and that $y_{m-1} \in \mathrm{A}^*_{\mathrm{SO}_{n-2}}$ is the Edidin-Graham class of the vector bundle $\big(\mathbb{A}^{n-2} \times U\big)/\mathrm{SO}_{n-2} \to U/\mathrm{SO}_{n-2}$.

Now, let $T_m \subset \mathrm{SO}_n$ is, as before, the torus of diagonal matrices with diagonal entries $t_1, \ldots, ,t_m, t_1^{-1}, \ldots , t_m^{-1}$, and x_i is the first Chern class of the i^{th} projection $T_m \to \mathrm{G_m}$.

LEMMA 4.8. *The formulae*

$$\mathrm{c}_n = (-1)^m x_1^2 \ldots x_m^2$$

and

$$y_m = 2^{m-1} x_1 \ldots x_m$$

hold in $\mathrm{A}^*_{T_m} = \mathbb{Z}[x_1, \ldots, x_m]$.

PROOF. Reducing the structure group to T_m, the vector bundle E on U/T_m associated with the standard representation $T_m \hookrightarrow \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n$ splits into a direct sum of line bundles $\Lambda_1 \oplus \cdots \oplus \Lambda_{2m}$, where the i^{th} summand is the subbundle associates with the 1-dimensional subspace $\langle e_i \rangle \subseteq V$. For each $i = 1 , \ldots, m$ we have $\Lambda_{i+n} \simeq \Lambda_i^\vee$. Then E has an admissible maximal totally isotropic subbundle $\Lambda_1 \oplus \cdots \oplus \Lambda_m$, which pulls back to an admissible totally isotropic subbundle on Q_1. The first Chern class of Λ_i in $\mathrm{A}^1(U/T_m) = \mathrm{A}^1_{T_m}$ is x_i, for $i = 1, \ldots, m$, hence

$$\mathrm{c}_m(\Lambda_1 \oplus \cdots \oplus \Lambda_m) = x_1 \ldots x_m \ \in \mathrm{A}^m_{T_m}$$

On the other hand, the top Chern classes of any two admissible totally isotropic subbundles of Q_1 are the same, by [**8**, Theorem 1], so

$$\begin{aligned} y_m &= f_*\big(s \cdot \mathrm{c}_m(\Lambda_1 \oplus \cdots \oplus \Lambda_m)\big)\\ &= (f_* s) x_1 \ldots x_m;\end{aligned}$$

and it is easy to verify that $f_* s = 2^{m-1}$. ♨

It follows that

$$(\mathbb{A}^{n-2} \times U)/T_{m-1} = \Lambda_2 \oplus \cdots \oplus \Lambda_m \oplus \Lambda_2^\vee \oplus \cdots \oplus \Lambda_m^\vee;$$

moreover, since

$$\big(U \times (\mathbb{A}^n \setminus \{0\})\big)/T_m = (\Lambda_1 \oplus \cdots \oplus \Lambda_m \oplus \Lambda_1^\vee \oplus \cdots \oplus \Lambda_m^\vee) \setminus \{0\},$$

we have

$$\begin{aligned} \mathrm{A}^*(X) &= \mathrm{A}^*_{T_m}/(c_n) \\ &= \mathbb{Z}[x_1, \dots, x_m]/(x_1^2 \dots x_m^2) \end{aligned}$$

and our aim is to verify that the equation

$$i_* y_{m-1} = -2^{m-1} x_1 \dots x_m \tag{4.1}$$

holds in $\mathbb{Z}[x_1, \dots, x_m]/(x_i^2 \dots x_m^2)$.

The inclusion of schemes on U/T_m

$$(\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\} \hookrightarrow (\Lambda_1 \oplus \cdots \oplus \Lambda_m \oplus \Lambda_1^\vee \oplus \cdots \oplus \Lambda_m^\vee) \setminus \{0\}$$

induces a surjection of rings

$$\mathbb{Z}[x_1, \dots, x_m]/(x_1^2 \dots x_m^2) \to \mathbb{Z}[x_1, \dots, x_m]/(x_1^2);$$

since $\mathbb{Z}x_1 \dots x_m$ has trivial intersection with the kernel of this map, we can restrict to $(\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\}$ to verify equation 4.1. There is a cartesian diagram

$$\begin{array}{ccc} (\Lambda_1 \setminus \{0\}) \sqcup (\Lambda_1^\vee \setminus \{0\}) & \longrightarrow & (\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\quad i \quad} & X \end{array}$$

We set

$$X' = (\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\},$$

and

$$\begin{aligned} Y' &= Y_1' \sqcup Y_2' \\ &= (\Lambda_1 \setminus \{0\}) \sqcup (\Lambda_1^\vee \setminus \{0\}); \end{aligned}$$

call $i' \colon Y' \hookrightarrow X'$ the inclusion.

Also, form the vector bundle on Y' defined as

$$\begin{aligned} F &\stackrel{\text{def}}{=} \Lambda_2 \oplus \cdots \oplus \Lambda_m \oplus \Lambda_2^\vee \oplus \cdots \oplus \Lambda_m^\vee \\ &= \big\langle s(Y') \big\rangle^\perp / \big\langle s(Y') \big\rangle . \end{aligned}$$

We need to check that

$$i'_* y_{m-1}(F) = -2^{m-1} x_1 \dots x_m \in \mathrm{A}^*(X').$$

For $l = 1$, 2, call $i'_l \colon Y'_l \hookrightarrow X'$ the inclusion, $s'_l \colon Y'_l \to i'^*_l E$ the tautological section, F_l the restriction of F to Y'_l.

Observe that the bundle $\Lambda_2 \oplus \cdots \oplus \Lambda_m$ of F is totally isotropic: however, its inverse image in E is $\Lambda_1 \oplus \dots \Lambda_m$ is $\Lambda_2 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_m$ on Y_1, but is

$\Lambda_2 \oplus \cdots \oplus \Lambda_m \oplus \Lambda_1^\vee$ on Y_2. The first bundle is admissible, the second one is not. Hence we have

$$y_{m-1}(F_1) = 2^{m-2} x_2 \dots x_m \in \mathrm{A}^*(Y_1')$$

and

$$y_{m-1}(F_2) = -2^{m-2} x_2 \dots x_m \in \mathrm{A}^*(Y_1').$$

Since we also have $[Y_1] = -x_1$ and $[Y_2] = x_1$ in $\mathrm{A}^*(X')$, we get

$$\begin{aligned} i_* y_{m-1} &= i_{1*} y_{m-1}(F_1) + i_{2*} y_{m-1}(F_2) \\ &= i_{1*} i_1^* 2^{m-2} x_2 \dots x_m - i_{2*} i_2^* 2^{m-2} x_2 \dots x_m \\ &= x_1 2^{m-2} x_2 \dots x_m + x_1 2^{m-2} x_2 \dots x_m \\ &= 2^{m-1} x_1 \dots x_m \end{aligned}$$

and Lemma 4.6 is proved. ♩

This proves that $c_2, \dots, c_n, y_m$ generate $\mathrm{A}^*_{\mathrm{SO}_n}$.

Step 2: the relations are satisfied. The fact that $2c_i = 0$ when i is odd follows immediately, as for O_n, from the fact that V is self-dual.

To prove that $y_m c_i = 0$, it is sufficient to show that $c_m(L_m) c_i = 0$ in $\mathrm{A}^*(Q_1)$, for any vector bundle E on X, with an SO_n structure, as $y_m c_i = f_{1*}(s \cdot \mathrm{c}_m(L_m) c_i)$. But on Q_1 there is an exact sequence of vector bundles

$$0 \longrightarrow L_m \longrightarrow f^* E \longrightarrow L_m^\vee \longrightarrow 0$$

so the total Chern class $\mathrm{c}({f_1}^* E)$ is $\mathrm{c}(L_m)\mathrm{c}(L_m^\vee)$ and $\mathrm{c}_i(f^* E) = 0$ when i is odd.

Finally, the normal bundle N of C in $\mathbb{A}^n \setminus \{0\}$ is trivial, since the ideal of C is generated by an invariant function on $\mathbb{A}^n - \{0\}$, so

$$\begin{aligned} y_m^2 &= i_* y_{m-1} \cdot i_* y_{m-1} \\ &= i_*(y_{m-1} \cdot i^* i_* y_{m-1}) \\ &= i_*(y_{m-1}^2 \cdot \mathrm{c}_1(N)) \\ &= 0 \end{aligned}$$

in $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) = \mathrm{A}^*_{\mathrm{SO}_n}/(c_n)$, by the projection formula and the self-intersection formula. Hence there is an integer d such that $y_m^2 = d c_n$; we will compute d once again by restricting to a maximal torus. By Lemma 4.8 we have

$$\begin{aligned} y_m^2 &= 2^{2m-2} x_1^2 \dots x_m^2 \\ &= 2^{n-2} (-1)^m c_n \in \mathrm{A}^n_{T_m}; \end{aligned}$$

hence, since c_n is not a torsion element of $\mathrm{A}^*_{T_m}$, we get that $d = 2^{n-2}$, as claimed.

Step 3: the relations suffice. Consider the ideal J in the polynomial ring $\mathbb{Z}[X_2, \ldots, X_n, Y]$ generated by the polynomials $Y^2 - (-1)^m 2^{n-2} X_n$, $2X_{\text{odd}}$, YX_{odd}. Let $P \in \mathbb{Z}[X_2, \ldots, X_n, Y]$ a homogeneous polynomial such that

$$P(c_2, \ldots, c_n, y_m) = 0;$$

we need to show that $P \in J$.

By modifying P by an element of J, we may assume that it is of the form $Q_1 + YQ_2 + R$, where Q_1 and Q_2 are polynomials in the even x_i, while R is a polynomial in the X_i with coefficients that are all 0 or 1, and all of whose non-zero monomial contain some X_i with i odd.

The odd c_i restrict to 0 in $\mathrm{A}^*_{T_m}$, while c_{2j} restricts to the j^{th} symmetric function s_j of $-x_1^2, \ldots, -x_m^2$; also, y_m restricts to $x_1 \ldots x_m$. Hence $P(c_2, \ldots, c_m, y_m) = 0$ restricts to $Q_1(s_2, s_4, \ldots) + x_1 \ldots x_m Q_2(s_2, s_4, \ldots)$; and this is easily seen to imply that $Q_1 = Q_2 = 0$.

Hence P is a polynomial in $X_2, \ldots, X_n$, all of whose coefficients are 0 or 1. Now consider the basis $e'_1, \ldots, e'_n$ of V, and the subgroup $\boldsymbol{\mu}_2^n \subseteq \mathrm{O}_n$ considered in the previous section, consisting of linear transformations that take each e'_i into e'_i or $-e'_i$. The subgroup $\Gamma_n \stackrel{\text{def}}{=} \boldsymbol{\mu}_2^n \cap \mathrm{SO}_n$ consists of the elements $(\epsilon_1, \ldots, \epsilon_n)$ of $\boldsymbol{\mu}_2^n$ such that $\epsilon_1 \ldots \epsilon_n = 1$ in $\boldsymbol{\mu}_2$. The group Γ_n is isomorphic to $\boldsymbol{\mu}_2^{n-1}$; if we call $\eta_i \in \mathrm{A}^1_{\Gamma_n}$ the first Chern class of the restriction to Γ_n of the i^{th} projection $\boldsymbol{\mu}_2^n \to \boldsymbol{\mu}_2 \subseteq \mathrm{G_m}$, then we have

$$\mathrm{A}^*_{\Gamma_n} = \mathbb{Z}[\eta_1, \ldots, \eta_n]/(\eta_1 + \cdots + \eta_n).$$

We have a natural homomorphism $\mathrm{A}^*_{\Gamma_n} \to \mathbb{F}_2[\eta_1, \ldots, \eta_n]/(\eta_1 + \cdots + \eta_n)$, which is an isomorphism in positive degree. If we denote by $r_1, \ldots, r_n$ the elementary symmetric functions of the h_i, we have that c_i restricts to the image of r_i in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$; hence all we need to show is that the images of $r_2, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$. But $r_1, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]$, so $r_2, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[r_1, \ldots, r_n]/(r_1)$; and the homomorphism

$$\mathbb{F}_2[r_1, \ldots, r_n]/(r_1) \longrightarrow \mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$$

is injective, because the extension $\mathbb{F}_2[r_1, \ldots, r_n] \subseteq \mathbb{F}_2[\eta_1, \ldots, \eta_n]$ is faithfully flat. This shows that $P = 0$, and completes the proof of the theorem.

CHAPTER 2

The Chow ring of the classifying space of Spin_8

1. Preliminaries on Clifford algebras and spin groups

We recall some facts about Clifford algebras and representations of spin groups that will be used throughout the paper (for a detailed treatment, see for example [**22**] or [**25**]). Suppose that $n = 2m$ or $n = 2m+1$, and consider a quadratic form $(\mathbb{C}^n, q)$; in a suitable basis $e_1, \dots, e_n$ the form q is given by

$$q(z_1, \dots, z_n) = -z_1^2 - \dots - z_n^2.$$

We will denote with $C_n = C(\mathbb{C}^n, q)$ the Clifford algebra associated to $(\mathbb{C}^n, q)$, and with Spin_n the (complex) spin group included in C_n. There is a double covering $\rho_n : \mathrm{Spin}_n \to \mathrm{SO}_n$, that is the universal covering of SO_n: we will write $\mathbb{C}^n$ also for the representation of Spin_n given by ρ_n.

Set

$$w_i := \frac{\sqrt{-1}e_i + e_{i+m}}{2}, \quad w_i' := \frac{\sqrt{-1}e_i - e_{i+m}}{2}$$

for $i = 1, \dots, m$, and let $W_n \subseteq \mathbb{C}^n$ the subspace generated by $w_1, \dots, w_m$: then W_n is a totally isotropic subspace, and the spinor space is $S_n = \bigwedge^\bullet W_n$. If $n = 2m$, S_n splits as the direct sum of two irreducible representations S_n^+ and S_n^-; we will denote with

$$\sigma_n^\pm : \mathrm{Spin}_n \longrightarrow \mathrm{GL}(S_n^\pm)$$

the two representations. If $n = 2m+1$, S_n is an irreducible representation of Spin_n, denoted with σ_n.

By definition, a *spin representation* of Spin_n is a representation obtained from an irreducible C_n^+-module, where $C_n^+ \subseteq C_n$ is the subalgebra of the elements of even degree. Let $V \subseteq \mathrm{SO}_n$ be the subgroup of diagonal matrices: then V can be regarded as a vector space with a quadratic form. If h the codimension of a maximal isotropic subspace of V, then 2^h, called the Radon-Hurwitz number a_n, and is the dimension of a spin representation of Spin_n ([**16**, Proposition 6.1]).

On $S_n^\pm$ and S_n there are a symmetric form β and an alternating form $\bar\beta$; for $n = 8$, β is Spin_8-invariant and $\sigma_8^\pm$ factorizes through $\mathrm{SO}(S_8^\pm, \beta)$. It follows that Spin_8 has three 8-dimensional representations $\rho_8, \sigma_8^\pm : \mathrm{Spin}_8 \to \mathrm{SO}_8$. One way to express the phaenomenon of triality is that there is an action of the symmetric group $\mathfrak{S}_3$ on Spin_8 acting as the full symmetric group on the set of the isomorphism classes of these three representations: suppose that $\tau \in \mathfrak{S}_3$ is a transposition that exchanges two representations

ρ_1 and ρ_2 of Spin_8. Then the (outer) automorphism $\tau : \mathrm{Spin}_8 \to \mathrm{Spin}_8$ is defined as the lifting of ρ_1 in the following diagram:

$$\begin{array}{ccc} \mathrm{Spin}_8 & \stackrel{\tau}{\dashrightarrow} & \mathrm{Spin}_8 \\ & \searrow^{\rho_1} & \downarrow{\scriptstyle \rho_2} \\ & & \mathrm{SO}_8. \end{array}$$

For more details we refer the reader to [**22**] or [**25**].

2. Pull-backs from $\mathrm{A}^*_{\mathrm{SO}_8}$: Chern and Edidin-Graham classes

For $n = 8$, we will omit the subscript $(\cdot)_8$. The following classes in $\mathrm{A}^*_{\mathrm{Spin}_8}$ are obtained by pulling back along the maps $\rho, \sigma^{\pm} : \mathrm{Spin}_8 \to \mathrm{SO}_8$:

$$y_4 = \rho^* y_4, \quad c_i = \rho^*(c_i),$$
$$y_4^{\pm} = (\sigma^{\pm})^* y_4, \quad c_i^{+} = (\sigma^{\pm})^* c_i,$$

where y_4 is the Edidin-Graham class of SO_8 (defined up to the sign; see Remark 2.1) and c_i is the i-th Chern class of the tautological representation of SO_8 (there should not be confusion between $c_i \in \mathrm{A}^*_{\mathrm{Spin}_8}$ and $c_i \in \mathrm{A}^*_{\mathrm{SO}_8}$).

By triality, the symmetric group $\mathfrak{S}_3$ acts on the isomorphism classes of the representations ρ, σ^+, σ^-. We will use the following notation: let

$$u_1 = \sqrt{-1}e_1 \in \mathbb{C}^8$$
$$u_2 = \frac{1}{\sqrt{2}}(w_2 \wedge w_1 + w_3 \wedge w_4) \in S^+$$
$$u_3 = u_1 u_2 = \frac{1}{\sqrt{2}}(w_2 - w_1 \wedge w_3 \wedge w_4) \in S^-.$$

so that $q(u_1) = \beta(u_2) = \beta(u_3) = 1$. Then we will denote with

1. (12) the transposition that exchanges $\mathbb{C}^8$ and S^+, and acts on S^- as minus the reflexion along $(\mathbb{C}u_3)^{\perp}$;
2. (13) the transposition that exchanges $\mathbb{C}^8$ and S^-, and acts on S^+ as minus the reflexion along $(\mathbb{C}u_2)^{\perp}$;
3. (23) the transposition that exchanges S^+ and S^-, and acts on $\mathbb{C}^8$ as minus the reflexion along $(\mathbb{C}u_1)^{\perp}$.

Let (V, U) a good pair for Spin_8: then the isometry $(12) : \mathbb{C}^8 \simeq S^+$ induces an isometry of vector bundles on U/Spin_8

$$(\mathbb{C}^8 \times U)/\mathrm{Spin}_8 \simeq (S^+ \times U)/\mathrm{Spin}_8.$$

It follows that there is an isomorphism $(12)^* : \mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}(S^+)$, and composing with the isomorphisms $\mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}$ and $\mathrm{A}^*_{\mathrm{Spin}_8}(S^+) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}$ one finds an action on $\mathrm{A}^*_{\mathrm{Spin}_8}$; this action exchanges the Chern and Edidin-Graham classes of $\mathbb{C}^8$ and S^+, namely $c_i^{(12)} = c_i^+$ and $y_4^{(12)} = y_4^+$. In an analogous way, $c_i^{(13)} = c_i^-$ and $y_4^{(13)} = y_4^-$. Moreover, $(23) : \mathbb{C}^8 \simeq \mathbb{C}^8$ reverse the orientation, so $c_i^{(23)} = c_i$ but $y_4^{(23)} = -y_4$.

(Similar formulae obviously hold for the action of $\mathfrak{S}_3$ on S^+ and S^-.)

REMARK 2.1. We define the orientations of $\mathbb{C}^8, S^+, S^-$ according to the following convention.

Suppose that $n = 2m$ (resp. $n = 2m+1$), and choose an appropriate basis $w_1, \dots, w_m, w_1, \dots, w'_m$ (resp. $w_1, \dots, w_m, w_1, \dots, w'_m, e_n$) of $\mathbb{C}^n$ such that with respect to this basis the quadratic form is given by

$$q(x_1, \dots, x_n) = x_1 x_m + \dots + x_m x_n$$
$$(\text{resp. } q(x_1, \dots, x_n) = x_1 x_m + \dots + x_m x_{2m} + x_n^2).$$

Then in the section on the Edidin-Graham construction we choose the class of admissible totally isotropic subbundles the one containing $W_n = \langle w_1, \dots, w_m \rangle$: so we have a preferred orientation for $\mathbb{C}^8$. For $S^\pm$, we determine the orientation of S^+ (resp. S^-) by requiring that the isometry (12) (resp. (13)) be orientation-preserving. Note that since (23) reverses the orientation of $\mathbb{C}^8$, the isometry (23) : $S^+ \simeq S^-$ does not preserve orientation.

Consider the half-spin representation S_4^+ of Spin_6: this corresponds to the tautological representation of SL_4 under the isomorphism $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$. Form the 6-dimensional representation of SL_4 given by $\bigwedge^2 S_4^+$: on $\bigwedge^2 S_4^+$ there is an SL_4-invariant quadratic form given by

$$\bigwedge^2 S_4^+ \times \bigwedge^2 S_4^+ \longrightarrow \mathbb{C}$$
$$(v_1 \wedge v_2, v_3 \wedge v_4) \longmapsto v_1 \wedge v_2 \wedge v_3 \wedge v_4.$$

So we get a map $\mathrm{SL}_4 \to \mathrm{SO}_6$, and the composition

$$\mathrm{Spin}_6 \xrightarrow{\simeq} \mathrm{SL}_4 \longrightarrow \mathrm{SO}_6$$

is the double covering ρ_6: hence $\bigwedge^2 S_4^+ \simeq \mathbb{C}^6$.

Set $v_1 = 1, v_2 = w_1 \wedge w_2, v_3 = w_2 \wedge w_3, v_4 = w_1 \wedge w_3 \in S_6^+$; then restricting to the torus T_{SL_4} we have that v_i is an eigenvector for the character τ_i given by the i-th projection $T_{\mathrm{SL}_4} \subseteq \mathrm{G}_\mathrm{m}^4 \to \mathrm{G}_\mathrm{m}$.

Define $W^+ := W^{(12)}$ and $W^- := W^{(13)}$. Then W^+ is described as follows:

LEMMA 2.2. *We have that*

$$W^+ = \langle w_1 \wedge w_2, w_2 \wedge w_3, w_2 \wedge w_4, w_1 \wedge w_2 \wedge w_3 \wedge w_4 \rangle$$
$$W^- = \langle w_1, w_2, w_3, w_4 \rangle .$$

PROOF. Recall that the isometry $u_3 : \mathbb{C}^8 \to S^+$ is determined by the formula $\beta(v_2, u_3 \cdot v_1) = \beta(v_1 \cdot v_2, u_3)$ for $v_1 \in \mathbb{C}^8, v_2 \in S^+$, where multiplication in the right hand side of the formula is given by the action of $\mathbb{C}^8 \subseteq C_8$

on the spinor space $S^+ \subseteq \bigwedge^\bullet W$. Then from the formulae

$$\beta(w_3 \wedge w_4, u_3 \cdot w_1) = \beta(w_1 \wedge w_3 \wedge w_4, u_3) = 1/\sqrt{2}$$
$$\beta(1, u_3 \cdot w_2) = \beta(w_2, u_3) = -1/\sqrt{2}$$
$$\beta(w_1 \wedge w_4, u_3 \cdot w_3) = \beta(w_3 \wedge w_1 \wedge w_4, u_3) = -1/\sqrt{2}$$
$$\beta(w_1 \wedge w_3, u_3 \cdot w_4) = \beta(w_4 \wedge w_1 \wedge w_3, u_3) = -1/\sqrt{2}$$

we get

$$u_3(\langle w_1 \rangle) = \langle w_1 \wedge w_2 \rangle$$
$$u_3(\langle w_2 \rangle) = \langle w_1 \wedge w_2 \wedge w_3 \wedge w_4 \rangle$$
$$u_3(\langle w_3 \rangle) = \langle w_2 \wedge w_3 \rangle$$
$$u_3(\langle w_4 \rangle) = \langle w_2 \wedge w_4 \rangle .$$

As for W^-, one proceed in a similar way, or apply u_1 to the vectors spanning W^+. ♪

3. Maximal tori of Spin_n and their Chow rings

Fix $n = 2m$ or $n = 2m+1$. Recall that $\mathrm{Spin}_2 = \{a + be_1e_2 : a^2 + b^2 = 1\}$. For $i = 1, \dots, m$ there is a copy of Spin_2 included in Spin_n via

$$\psi_i : a + be_1e_2 \longmapsto a + be_ie_{i+m};$$

moreover, it is easy to see that $\psi_i(\mathrm{Spin}_2) \subseteq T_{\mathrm{Spin}_n}$. Since we have that $\mathrm{Spin}_2 \simeq \mathrm{G_m}$ via $a + be_1e_2 \mapsto a + \sqrt{-1}b$, we obtain maps $\phi_i : \mathrm{G_m} \to T_{\mathrm{Spin}_n}$. Define

$$\phi := \phi_1 \dots \phi_m : \mathrm{G}_\mathrm{m}^m \longrightarrow T_{\mathrm{Spin}_n}.$$

The map ϕ is surjective, and its kernel is generated by elements $(\epsilon_1, \dots, \epsilon_m) \in (\mathbb{Z}/2)^m$ such that $\epsilon_1 \dots \epsilon_m = 1$, that is a $(\mathbb{Z}/2)^{m-1}$.

There is a canonical isomorphism $T_{\mathrm{SO}_n} \simeq \mathrm{G}_\mathrm{m}^m$ given by

$$\mathrm{diag}(a_1, \dots, a_m, a_1^{-1}, \dots, a_m^{-1}) \leftrightarrow (a_1, \dots, a_m),$$

and it is easy to see that the composition

$$\mathrm{G}_\mathrm{m}^m \xrightarrow{\phi} T_{\mathrm{Spin}_n} \xrightarrow{\rho} T_{\mathrm{SO}_n} \simeq \mathrm{G}_\mathrm{m}^m$$

is the map $(a_1, \dots, a_m) \longmapsto (a_1^2, \dots, a_m^2)$.

It follows that there is a chain of inclusion of Chow rings

$$\mathrm{A}^*_{T_{\mathrm{SO}_n}} \xhookrightarrow{\rho^*} \mathrm{A}^*_{T_{\mathrm{Spin}_n}} \xhookrightarrow{\phi^*} \mathrm{A}^*_{\mathrm{G}_\mathrm{m}^m} \simeq \mathbb{Z}[x_1, \dots, x_m]$$

where $x_i := \mathrm{c}_1(\chi_i)$ and χ_i is the character given by the i-th projection $\mathrm{G}_\mathrm{m}^m \to \mathrm{G_m}$. Note that $\mathrm{A}^*_{T_{\mathrm{SO}_n}}$ is identified with the subring $\mathbb{Z}[2x_1, \dots, 2x_m] \subseteq \mathbb{Z}[x_1, \dots, x_m]$.

Let $\chi : T_{\mathrm{Spin}_n} \to \mathrm{G_m}$ a character; if χ acts trivially on the subgroup $\{\pm 1\}$, then it factorizes through T_{SO_n} and so $\mathrm{c}_1(\chi) \in \rho^*(\mathrm{A}^*_{T_{\mathrm{SO}_n}})$. Suppose on the contrary that $\chi|_{\{\pm 1\}}$ is not trivial; then, for any other character χ'

such that $\chi|_{\{\pm 1\}}$ is not trivial, we have that $\chi \otimes \chi'|_{\{\pm 1\}} = 1$, so we have $\mathrm{c}_1(\chi) + \mathrm{c}_1(\chi') \in \rho^*(\mathrm{A}^*_{T_{\mathrm{SO}_n}})$. It follows that if χ is a character that acts non-trivially on $\{\pm 1\}$, then the ring of characters of T_{Spin_n} is isomorphic to

$$\mathbb{C}[\chi_1^2, \dots, \chi_m^2] \otimes \mathbb{C}[\chi]/\big(\chi^2 \otimes f(\chi_1^2, \dots, \chi_m^2)\big)$$

where f is an appropriate function (notice that χ_i^2 factorizes through T_{SO_n}, so is indeed a character of T_{Spin_n}). For example, consider the character $\chi_1 \otimes \cdots \otimes \chi_m : \mathrm{G}_\mathrm{m}^m \to \mathrm{G}_\mathrm{m}$; it is trivial on the kernel of ϕ, so it lift to a character of T_{Spin_n}. But is not trivial on $\{\pm 1\}$, so we can choose it as our χ.

LEMMA 3.1. *Suppose that $n = 2m$: then the Chow ring of the maximal torus of Spin_n is*

$$\mathrm{A}^*_{T_{\mathrm{Spin}_n}} \simeq \mathbb{Z}[u_1, \dots, u_m, u]/\big(u - (u_1 + \cdots + u_m)\big),$$

and $\mathrm{A}^*_{T_{\mathrm{SO}_n}}$ is included in $\mathrm{A}^*_{T_{\mathrm{Spin}_n}}$ as the subring $\mathbb{Z}[u_1, \dots, u_m]$.

PROOF. We define $u := \mathrm{c}_1(\chi_1 \otimes \cdots \otimes \chi_m) = x_1 + \cdots + x_m$: then the lemma follows immediately from the previous discussion. ♩

The preceeding lemma says that $\mathrm{A}^*_{\mathrm{Spin}_n}$ is included in $\mathrm{A}^*_{\mathrm{G}_\mathrm{m}^m} = \mathbb{Z}[x_1, \dots, x_m]$ as the subring generated by $2x_1, \dots, 2x_m, x_1 + \cdots + x_m$. In particular, we have that $\epsilon_1 x_1 + \cdots + \epsilon_m x_m \in \mathrm{A}^*_{\mathrm{Spin}_n}$ for $(\epsilon_1, \dots, \epsilon_m) \in (\mathbb{Z}/2)^m$, and in fact they generate $\mathrm{A}^*_{\mathrm{Spin}_n}$; we are going to give a precise description of $\mathrm{A}^*_{\mathrm{Spin}_n}$ in terms of these elements. Forst, we need an auxiliary result:

LEMMA 3.2. *The vector $w_{i_1} \wedge \cdots \wedge w_{i_k} \in \bigwedge^\bullet W_n$ is an eigenvector for the action of the maximal torus T_{Spin_n}, and T_{Spin_n} acts on the linear subspace $\mathbb{C} \cdot w_{i_1} \wedge \cdots \wedge w_{i_k}$ via the character $\chi_1^{\epsilon_1} \otimes \cdots \otimes \chi_n^{\epsilon_n}$ where*

$$\epsilon_i = \begin{cases} 1 & \text{for } i \in \{i_1, \dots, i_k\} \\ -1 & \text{otherwise.} \end{cases}$$

PROOF. Let

$$(a_1 + b_1 e_1 e_{m+1})(a_2 + b_2 e_2 e_{m+2}) \dots (a_m + b_m e_m e_{2m}) \in T_{\mathrm{Spin}_n}$$

and consider the vector $w_{i_1} \wedge \cdots \wedge w_{i_k}$. Note that $e_i e_{m+i} = -\sqrt{-1}(w_i + w_i')(w_i - w_i')$. Suppose that $i \notin \{i_1, \dots, i_k\}$: then

$$\begin{aligned}(w_i + w_i')(w_i - w_i') w_{i_1} \wedge \cdots \wedge w_{i_k} &= (w_i + w_i') w_i \wedge w_{i_1} \wedge \cdots \wedge w_{i_k} \\ &= w_{i_1} \wedge \cdots \wedge w_{i_k},\end{aligned}$$

so $(a + b e_i e_{i+m}) w_{i_1} \wedge \cdots \wedge w_{i_k} = (a - \sqrt{-1} b) w_{i_1} \wedge \cdots \wedge w_{i_k}$. Suppose now that $i = i_l$: then

$$\begin{aligned}&(w_i + w_i')(w_i - w_i') w_{i_1} \wedge \cdots \wedge w_{i_k} \\ &\qquad = (-1)^l (w_i + w_i') w_{i_1} \wedge \cdots \wedge w_{i_{l-1}} \wedge w_{i_{l+1}} \wedge \cdots \wedge w_{i_k} \\ &\qquad = (-1)^{l+l-1} w_{i_1} \wedge \cdots \wedge w_{i_k} \\ &\qquad = -w_{i_1} \wedge \cdots \wedge w_{i_k}\end{aligned}$$

so $(a + be_i e_{i+m}) w_{i_1} \wedge \cdots \wedge w_{i_k} = (a + \sqrt{-1}b) w_{i_1} \wedge \cdots \wedge w_{i_k}$. Now the lemma follows from the fact that the element $(a_1 + b_1 e_1 e_{m+1}) \dots (a_m + b_m e_m e_{2m})$ corresponds to $(a_1 + \sqrt{-1}b_1) \dots (a_m + \sqrt{-1}b_m) \in \mathrm{G}_\mathrm{m}^m$. ♪

Set

$$X_n := \{\epsilon_1 x_1 + \cdots + \epsilon_m x_m\}_{(\epsilon_1,\dots,\epsilon_m)\in(\mathbb{Z}/2)^m}.$$

Suppose that $n = 2m$; there are two complementary subsets $X_n^+, X_n^- \subseteq X_n$, given by

$$X_n^+ := \{-x_1 - \cdots - x_n + 2(x_{i_1} + \cdots + x_{i_k})\}_{w_{i_1}\wedge\cdots\wedge w_{i_k}\in S_n^+}$$
$$X_n^- := \{-x_1 - \cdots - x_n + 2(x_{i_1} + \cdots + x_{i_k})\}_{w_{i_1}\wedge\cdots\wedge w_{i_k}\in S_n^-}.$$

Proposition 3.3. *(1) Suppose that $n = 2m + 1$. Then*

$$\sum_{i=0}^{2^m} \mathrm{c}_i(S_n) T^i = \prod_{\epsilon_j\in\mathbb{Z}/2} \big(1 + (\epsilon_1 x_1 + \cdots + \epsilon_m x_m)T\big) = \prod_{x\in X_n} (1 + xT)$$

in $\mathrm{A}^*_{T_{\mathrm{Spin}_n}}[T]$.

(2) Suppose that $n = 2m$. Then

$$\sum_{i=0}^{2^{m-1}} \mathrm{c}_i(S_n^+) T^i = \prod_{x\in X_n^+} (1 + xT)$$
$$\sum_{i=0}^{2^{m-1}} \mathrm{c}_i(S_n^-) T^i = \prod_{x\in X_n^-} (1 + xT)$$

in $\mathrm{A}^*_{T_{\mathrm{Spin}_n}}[T]$.

Proof. Immediate from Lemma 3.2. ♪

To simplify the notation, in the following expressions we will write $\sum x_{i_1}^l \dots x_{i_s}^l$ for the "ordered" $\sum_{1\le i_1<\cdots<i_s\le m} x_{i_1}^l \dots x_{i_s}^l$.

COROLLARY 3.4. *We have that*

$$
\begin{aligned}
c_2 &= -4\sum_i x_i^2\\
c_4 &= 16\sum x_i^2x_j^2\\
c_6 &= -64\sum x_i^2x_j^2x_k^2\\
c_8 &= 256x_1^2x_2^2x_3^2x_4^2\\
y_4 &= 128x_1x_2x_3x_4\\
c_2^+ &= -4\sum x_i^2\\
c_4^+ &= 6\sum x_i^4 + 4\sum x_i^2x_j^2 - 48x_1x_2x_3x_4\\
c_6^+ &= -4\sum x_i^6 + 4\sum(x_i^2x_j^4 + x_i^4x_j^2) - 40\sum x_i^2x_j^2x_k^2 + 32(\sum x_i^2)x_1x_2x_3x_4\\
c_8^+ &= (\sum x_i^4 - 2\sum x_i^2x_j^2 + 8x_1x_2x_3x_4)^2\\
y_4^+ &= 8\sum x_i^4 - 16\sum x_i^2x_j^2 + 64x_1x_2x_3x_4
\end{aligned}
$$

in $\mathrm{A}^*_{T_{\mathrm{Spin}_8}}$.

Moreover, suppose that $c_i^+ = f(x_1,x_2,x_3,x_4)$ *and* $y_4^+ = g(x_1,x_2,x_3,x_4)$*: then* $c_i^- = f(x_1,x_2,x_3,-x_4)$ *and* $y_4^- = -g(x_1,x_2,x_3,-x_4)$ *in* $\mathrm{A}^*_{T_{\mathrm{Spin}_8}}$.

PROOF. The formulae for the c_i follow from the fact that the Chern roots of the representation $\mathbb{C}^8$ are $\pm 2x_1, \pm 2x_2, \pm 2x_3, \pm 2x_4$, so c_{2i} is the i-th symmetric polynomial in $-4x_1^2, -4x_2^2, -4x_3^2, -4x_4^2$.

By Proposition 3.3, the Chern roots of the spin representation S_8^+ are

$$
\begin{gathered}
\pm(x_1+x_2+x_3+x_4), \pm(-x_1+x_2+x_3-x_4),\\
\pm(-x_1+x_2-x_3+x_4), \pm(x_1+x_2-x_3-x_4)
\end{gathered}
$$

so c_{2i}^+ is the i-th symmetric polynomial in these indeterminates.

Suppose that $V' \subseteq V$ is an admissible maximal totally isotropic subspace, and that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are Chern roots for V': then by 4.8 of Chapter 1 $y_4(V) = 2^3\alpha_1\alpha_2\alpha_3\alpha_4$. It is easy to check that $2x_1, 2x_2, 2x_3, 2x_4$ (resp. $x_1+x_2+x_3+x_4, -x_1+x_2+x_3-x_4, -x_1+x_2-x_3+x_4, x_1+x_2-x_3-x_4$) are Chern roots for W (resp. W^+, from Lemma 2.2), and an elementary but long computation leads to the desired expressions (we used the program Mathematica to compute them). Note that for c_8^+ we used the fact that by 4.4, Chapter 1, $2^6c_8^+ = (y_4^+)^2$ in $\mathrm{A}^*_{\mathrm{Spin}_8}$.

Finally, it is easy to see that sending x_4 to $-x_4$ exchanges X^+ and X^-, and carries the Chern roots of W^+ in the Chern roots of a non-admissible totally isotropic subspace of S^-, so the last assertion is prooved too. ♪

4. The localization exact sequences

Let

$$C := \{q = 0\} \overset{i}{\hookrightarrow} \mathbb{C}^8 \setminus \{0\}$$

(risp. $C^{\pm} := \{\beta = 0\} \overset{i^{\pm}}{\hookrightarrow} S^{\pm} \setminus \{0\}$) the cone of isotropic vectors in $(\mathbb{A}^8, q)$ (risp. in $(S^{\pm}, \beta)$), and

$$B := \{q \neq 0\} \overset{j}{\hookrightarrow} \mathbb{C}^8 \setminus \{0\}$$

(risp. $B^{\pm} := \{\beta \neq 0\} \overset{j^{\pm}}{\hookrightarrow} S^{\pm} \setminus \{0\}$) the locus in wich q (risp. β) is non-zero.

Note that the transposition (12) (resp. (13)) carries C, B isomorphically onto C^+, B^+ (resp. C^-, B^-).

Then, correspondently, we have 3 localization exact sequences

$$\mathrm{A}^*_{\mathrm{Spin}_8}(C) \xrightarrow{i_*} \mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8 \setminus \{0\}) \xrightarrow{j^*} \mathrm{A}^*_{\mathrm{Spin}_8}(B) \to 0$$

$$\mathrm{A}^*_{\mathrm{Spin}_8}(C^{\pm}) \xrightarrow{i^{\pm}_*} \mathrm{A}^*_{\mathrm{Spin}_8}(S^{\pm} \setminus \{0\}) \xrightarrow{(j^{\pm})^*} \mathrm{A}^*_{\mathrm{Spin}_8}(B^{\pm}) \to 0.$$

We now analyze the terms of the first sequence:

The term $\mathrm{A}^*_{\mathrm{Spin}_8}(B)$. Let $Q := \{q = 1\}$: then we can consider a diagram analogous to the one seen for O_n

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B \\ \downarrow & & \downarrow q \\ \mathrm{G_m} & \xrightarrow{(-)^2} & \mathrm{G_m} \end{array}$$

where $\widetilde{B} \simeq Q \times \mathrm{G_m}$ and $\widetilde{B}/\boldsymbol{\mu}_2 \simeq B$; the action of $\boldsymbol{\mu}_2$ induced on $Q \times \mathrm{G_m}$ is given by $\epsilon(v, t) = (\epsilon v, \epsilon t)$, and it commutes with the one of Spin_8 given by $a(v, t) = (av, t)$. It follows that there is an isomorphism

$$\mathrm{A}^*_{\mathrm{Spin}_8}(B) \simeq \mathrm{A}^*_{\mathrm{Spin}_8 \times \boldsymbol{\mu}_2}(Q \times \mathrm{G_m}).$$

Since the action of $\mathrm{Spin}_8 \times \boldsymbol{\mu}_2$ extends to $Q \times \mathbb{C}^1$,

$$\mathrm{A}^*_{\mathrm{Spin}_8 \times \boldsymbol{\mu}_2}(Q \times \mathrm{G_m}) \simeq \mathrm{A}^*_{\mathrm{Spin}_8 \times \boldsymbol{\mu}_2}(Q)/(\mathrm{c}_1(\chi_{\boldsymbol{\mu}_2})). \tag{4.1}$$

Now, the action of $\mathrm{Spin}_8 \times \boldsymbol{\mu}_2$ on Q is transitive; let Γ the stabilizer of $e_8 \in Q$ in $\mathrm{Spin}_8 \times \boldsymbol{\mu}_2$. Since the stabilizer of e_8 in $\mathrm{SO}_8 \times \boldsymbol{\mu}_2$ is an $\mathrm{SO}_7 \times \boldsymbol{\mu}_2$, included in $\mathrm{SO}_8 \times \boldsymbol{\mu}_2$ via $(M, \epsilon) \mapsto (\epsilon M, \epsilon)$, we obtain a fiber diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \Gamma & \longrightarrow & \mathrm{SO}_7 \times \boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Spin}_8 \times \boldsymbol{\mu}_2 & \xrightarrow{(\rho, \mathrm{id})} & \mathrm{SO}_8 \times \boldsymbol{\mu}_2 & \longrightarrow & 1. \end{array}$$

It follows that $\Gamma \simeq \mathrm{Spin}_7 \times \boldsymbol{\mu}_2$ included in $\mathrm{Spin}_8 \times \boldsymbol{\mu}_2$ via

$$(a, \epsilon) \mapsto (\eta a, \epsilon)$$

where $\eta = e_1 ... e_8$ is the element of order 2 that generates, with -1, the center of Spin_8.

Hence

$$\mathrm{A}^*_{\mathrm{Spin}_8 \times \boldsymbol{\mu}_2}(Q) \simeq \mathrm{A}^*_{\mathrm{Spin}_7 \times \boldsymbol{\mu}_2} \simeq \mathrm{A}^*_{\mathrm{Spin}_7}[\mathrm{c}_1(\chi_{\boldsymbol{\mu}_2})];$$

and combining this with Formula 4.1 we get

$$\mathrm{A}^*_{\mathrm{Spin}_8}(B) \simeq \mathrm{A}^*_{\mathrm{Spin}_7}.$$

It is easy to see that j^* coincides with the restriction from Spin_8 to Spin_7 along the map given by the fiber diagram

$$\begin{array}{ccc} \mathrm{Spin}_7 & \hookrightarrow & \mathrm{Spin}_8 \\ \downarrow & & \downarrow \rho \\ \mathrm{SO}_7 & \hookrightarrow & \mathrm{SO}_8, \end{array}$$

where SO_7 is included in SO_8 as the stabilizer of an element of Q.

With a slight abuse of notation, we will call $j : \mathrm{Spin}_7 \to \mathrm{Spin}_8$ this inclusion.

The term $\mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8 \setminus \{0\})$. We have that

$$\mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8 \setminus \{0\}) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8).$$

The term $\mathrm{A}^*_{\mathrm{Spin}_8}(C)$. The action of Spin_8 on C is transitive (since the action of SO_8 is); let Γ the stabilizer of w_1 in Spin_8. Recall that the stablizer of w_1 in SO_8 is $\mathrm{SO}_6 \ltimes H$, where SO_6 is included in SO_8 as the stabilizer of the pair (w_1, w_1'), and H is a group isomorphic, as a variety, to an affine space, on which SO_6 acts via affine transformations.

In the same way as for SO_n, it is easy to see that $\Gamma \simeq \mathrm{Spin}_6 \ltimes H'$, where Spin_6 is included in Spin_8 as the stabilizer of the pair (w_1, w_1'). Then $H = H'$, and there are isomorphisms

$$\mathrm{A}^*_{\mathrm{Spin}_8}(C) \simeq \mathrm{A}^*_{\mathrm{Spin}_6 \ltimes H} \simeq \mathrm{A}^*_{\mathrm{Spin}_6}.$$

Finally, since $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$,

$$\mathrm{A}^*_{\mathrm{Spin}_8}(C) \simeq \mathbb{Z}[\sigma_2, \sigma_3, \sigma_4]$$

where $\sigma_i = \mathrm{c}_i(S_6^+)$, $S_6^\pm$ are the two half-spin representations of Spin_6.

5. The action of $\mathfrak{S}_3$

There are other 2 inclusions $j^\pm$ of Spin_7 in Spin_8 given by the fiber diagrams

$$\begin{array}{ccc} \mathrm{Spin}_7 & \overset{j^\pm}{\hookrightarrow} & \mathrm{Spin}_8 \\ \downarrow & & \downarrow \sigma^\pm \\ \mathrm{SO}_7 & \hookrightarrow & \mathrm{SO}_8, \end{array}$$

where SO_7 is included in SO_8 as the stabilizer of an element of $Q^\pm$.

Similarly, there are other 2 inclusions

$$i^{\pm} : \mathrm{SL}_4 \longrightarrow \mathrm{Spin}_8.$$

We will denote with $\mathrm{Spin}_7^{\pm}$ and $\mathrm{SL}_4^{\pm}$ the corresponding subgroups of Spin_8. These subgroups are carried one to another by the symmetric group $\mathfrak{S}_3$, with the following geometric interpretation.

5.1. The subgroups $\mathrm{SL}_4, \mathrm{SL}_4^{\pm}$. We have seen that there is an isomorphism of Spin_8-schemes

$$C \simeq \mathrm{Spin}_8/(\mathrm{SL}_4 \times H)$$

where Spin_8 acts on $C \subseteq \mathbb{C}^8$ via ρ and $\mathrm{SL}_4 \simeq \mathrm{Spin}_6$ is included in Spin_8 as the stabilizer of the pair (w_1, w_1'). There is also an ismomorphism of Spin_8-schemes

$$C^+ \simeq \mathrm{Spin}_8/(\mathrm{Spin}_6^+ \times H)$$

where Spin_8 acts on $C^+ = (12)(C) \subseteq S^+ = (12)(\mathbb{C}^8)$ via $\sigma^+ = \rho(12)$ (Remark **??**) and SL_4^+ is included in Spin_8 as the stabilizer of the pair $((12)w_1, (12)w_1')$. Hence $\mathrm{SL}_4^+ = (12)(\mathrm{SL}_4)$ and the map $(12)^* : \mathrm{A}^*_{\mathrm{Spin}_8}(C) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}(C^+)$ corresponds to the pull-back $(12)^* : \mathrm{A}^*_{\mathrm{SL}_4} \simeq \mathrm{A}^*_{\mathrm{SL}_4^+}$. We define the elements $\sigma_i^+ := (12)^*\sigma_i \in \mathrm{A}^*_{\mathrm{SL}_4^+}$.

Exactly in the same way there is an isomorphism $(13)^* : \mathrm{A}^*_{\mathrm{SL}_4} \simeq \mathrm{A}^*_{\mathrm{SL}_4^-}$ and we define $\sigma_i^- := (13)^*\sigma_i \in \mathrm{A}^*_{\mathrm{SL}_4^-}$.

On the other hand, since (23) sends (w_1, w_1') to (w_1', w_1) the subgroup SL_4 is invariant under (23), and so there is an involution

$$(23) : \mathrm{SL}_4 \to \mathrm{SL}_4.$$

We claim that (23) acts on SL_4 as the outer automorphism $\iota : M \mapsto M^\vee$, where we denote with $M^\vee$ the inverse of the transpose of M.

To see this, note that the restriction of (23) to $\mathbb{C}^6 \subseteq \mathbb{C}^8$ reverse also the orientation of $\mathbb{C}^6 \simeq \bigwedge S_6^+$, and so acts on S_6^+ as a linear transformation with determinant -1: so the induced map $\mathrm{SL}_4 \to \mathrm{SL}_4$ is the conjugation by an element of GL_4 with determinant -1, that is not an inner automorphism. Now the claim follows from the fact that $\mathrm{Out}(\mathrm{SL}_4) \simeq \mathbb{Z}/2$ is generated by the involution ι.

In the same fashion, we see that also $(13) : \mathrm{SL}_4^+ \simeq \mathrm{SL}_4^+$ and $(12) : \mathrm{SL}_4^- \simeq \mathrm{SL}_4^-$ correspond to the involution ι.

Lemma 5.1. *(1) Let $\alpha \in \mathrm{A}^k_{\mathrm{SL}_4}$, and set*

$$\alpha^+ := (12)^*\alpha \in \mathrm{A}^k_{\mathrm{SL}_4^+}, \quad \alpha^- := (13)^*\alpha \in \mathrm{A}^k_{\mathrm{SL}_4^-}.$$

Then the action of $\mathfrak{S}_3$ on the pushforwards of these elements is summarized in the following table:

	(12)	(13)	(23)
$i_*\alpha$	$i^+_*\alpha^+$	$i^-_*\alpha^-$	$(-1)^k i_*\alpha$
$i^+_*\alpha^+$	$i_*\alpha$	$(-1)^k i^+_*\alpha^+$	$(-1)^k i^-_*\alpha^-$
$i^-_*\alpha^-$	$(-1)^k i^-_*\alpha^-$	$i_*\alpha$	$(-1)^k i^+_*\alpha^+$

(2) Let $\beta \in \mathrm{A}^k_{\mathrm{Spin}_8}$, and set

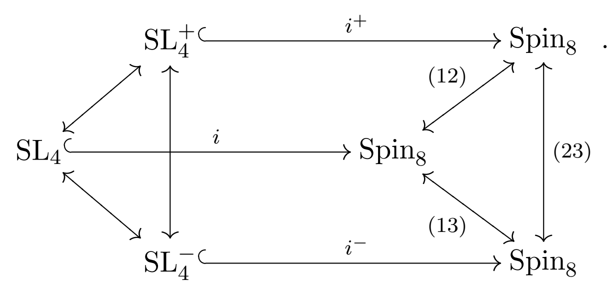
$$\beta^+ := (12)^*\beta, \beta^- := (13)^*\beta \in \mathrm{A}^k_{\mathrm{Spin}_8}.$$

Then the action of $\mathfrak{S}_3$ on the pullbacks of these elements is summarized in the following table:

	(12)	(13)	(23)
$i^*\beta$	$(i^+)^*\beta^+$	$(i^-)^*\beta^-$	$(-1)^k i^*\beta$
$(i^+)^*\beta$	$i^*\beta^+$	$(-1)^k(i^+)^*\beta$	$(-1)^k(i^-)^*\beta^-$
$(i^-)^*\beta$	$(-1)^k(i^-)^*\beta^-$	$i^*\beta$	$(-1)^k(i^+)^*\beta^+$

Proof. First note that $\iota^* : \mathrm{A}^*_{\mathrm{SL}_4} \simeq \mathrm{A}^*_{\mathrm{SL}_4}$ is the automorphism $\sigma_k \mapsto (-1)^k\sigma_k$ (this can be seen for example restricting to a maximal torus), and so it is the multiplication by $(-1)^k$ in degree k: this proves the formulae on the anti-diagonal of the table.

Look at the following diagram in which each square is cartesian:



The other two formulae of the first row of the table follow by definition of α^+, α^-. Finally we prove the remaining formulae in the third column:

$$\begin{aligned}(23)^* i^+_*\alpha^+ &= (12)^*(13)^*(12)^* i^+_*\alpha^+ \\ &= (12)^* i^-_*\alpha^- \\ &= (-1)^k i^-_*\alpha^-,\end{aligned}$$

and similarly for $(23)^* i^-_*\alpha^-$.

The part on the pull-backs is proved similarly. ♪

Remark 5.2. From now on, to simplify the notation, we will omit the $^\pm$ in the elements of $\mathrm{A}^*_{\mathrm{SL}_4^\pm}$: so we write $\sigma_i \in \mathrm{A}^*_{\mathrm{SL}_4^\pm}$ for $\sigma_i^\pm$; this should not create confusion.

LEMMA 5.3. *The restriction of the Chern classes c_i and c_i^+ to SL_4 and SL_4^+ are described in the following table:*

	i^*	$(i^+)^*$
c_{odd}	0	0
c_2	$2\sigma_2$	$2\sigma_2$
c_4	$\sigma_2^2 - 4\sigma_4$	$\sigma_2^2 + 2\sigma_4$
c_6	$-\sigma_3$	$2\sigma_2\sigma_4 - \sigma_3^2$
c_8	0	σ_4^2
c_2^+	$2\sigma_2$	$2\sigma_2$
c_4^+	$\sigma_2^2 + 2\sigma_4$	$\sigma_2^2 - 4\sigma_4$
c_6^+	$2\sigma_2\sigma_4 - \sigma_3^2$	$-\sigma_3$
c_8^+	σ_4^2	0
y_4	0	$8\sigma_4$
y_4^+	$8\sigma_4$	0

Moreover $i^*c_i^+ = i^*c_i^-$ and $i^*y_4^+ = i^*y_4^-$.

PROOF. By Lemma 5.1 $(i^+)^*c_i = (i^*c_i^+)^{(12)}$ and $(i^+)^*c_i^+ = (i^*c_i)^{(12)}$, so it is sufficient to prove only the formulae in the first column of the table. From the cartesian diagram

$$\begin{array}{ccc} \mathrm{Spin}_6 & \xhookrightarrow{\ i\ } & \mathrm{Spin}_8 \\ \downarrow{\scriptstyle \rho_6} & & \downarrow{\scriptstyle \rho} \\ \mathrm{SO}_6 & \xhookrightarrow{\ i'\ } & \mathrm{SO}_8 \end{array}$$

we obtain

$$i^*c_i = i^*\rho^*c_i = \rho_6^*(i')^*c_i,$$

and analogously for y_4. The restriction of $\mathbb{C}^8$ to SO_6 is $\mathbb{C}^6 \oplus \mathbb{C}^2$, where SO_6 acts canonically on the first summand, and trivially on the second one. It follows that $(i')^*c_7 = (i')^*c_8 = 0$ in $\mathrm{A}^*_{\mathrm{SO}_6}$. Moreover, $(i')^*y_4 = 0$ because

$$\big((i')^*y_4\big)^2 = (i')^*(y_4^2) = (i')^*(2^6c_8) = 0.$$

By Remark 2.1, $\mathbb{C}^6 \simeq \bigwedge^2 S_6^+$; since $\mathrm{A}^*_{\mathrm{SL}_4}$ is torsion-free, we can restrict to the maximal torus to verify the formulae. We have that

$$\mathrm{A}^*_{T_{\mathrm{SL}_4}} \simeq \mathbb{Z}[t_1, t_2, t_3, t_4]/(t_1 + t_2 + t_3 + t_4);$$

using the relation $t_4 = -t_1 - t_2 - t_3$, the Chern classes of the characters of the representation $\mathbb{C}^6$ are

$$t_1 + t_2, t_1 + t_3, t_2 + t_3, -(t_1 + t_2), -(t_1 + t_3), -(t_2 + t_3).$$

Since c_{2_i+1} is 2-torsion, we have $\rho_6^*c_{2i-1} = 0$ for $i = 1, 2, 3$. On the other hand, the restriction of $\rho_6^*c_{2i}$ to the torus is the i-th symmetric function in the variables $-(t_1+t_2)^2, -(t_1+t_3)^2, -(t_2+t_3)^2$, for $i = 1, 2, 3$. (Recall that if $\xi_1, ..., \xi_n$ are Chern roots for a fiber bundle E, $\{\xi_{i_1} + \cdots + \xi_{i_m}\}_{1 \le i_1 < \cdots < i_m \le n}$ are Chern roots for $\bigwedge^m E$).

The restrictions of the Chern classes σ_i to the torus are

$$\begin{aligned}\sigma_2 &= -(t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_1t_3 + t_2t_3)\\ \sigma_3 &= -(t_1 + t_2)(t_1 + t_3)(t_2 + t_3)\\ \sigma_4 &= -t_1t_2t_3(t_1 + t_2 + t_3).\end{aligned}$$

At this point we leave to the eager reader to verify the formulae concerning the c_i's.

To prove the remaining formulae, note that the restrictions of S^+ and S^- to Spin_6 are both isomorphic to the direct sum $S_6^+ \oplus S_6^-$, so the restrictions of the Chern and Edidin-Graham classes are the same. Moreover, since $S_6^- \simeq (S_6^+)^\vee$, we have that

$$\begin{aligned}i^*c_i^+ &= \mathrm{c}_i\big(S_6^+ \oplus (S_6^+)^\vee\big)\\ &= \sum_{k+l=i} (-1)^l \mathrm{c}_k(S_6^+)\mathrm{c}_l(S_6^+)\\ &= \sum_{k+l=i} (-1)^l \sigma_k\sigma_l\end{aligned}$$

and these are exactly the expressions for the c_i^+'s of the lemma.

As for y_4^+, since in $\mathrm{A}^*_{\mathrm{Spin}_8}$ the relation $(y_4^+)^2 = 2^6 c_8^+$ holds, restricting to SL_4 we obtain $(i^*y_4^+)^2 = (8\sigma_4)^2$ so we have $i^*y_4^+ = \pm 8\sigma_4$. To determine the sign, we restrict to the maximal torus: since $\mathrm{A}^*_{T_{\mathrm{Spin}_6}} \simeq \mathrm{A}^*_{T_{\mathrm{Spin}_8}}/(x_4)$ by Lemma 3.1, we have that $i^*y_4^+ = y_4 \bmod x_4$, that is

$$\begin{aligned}i^*y_4^+ &= 8(x_1 + x_2 + x_3)(-x_1 + x_2 + x_3)(-x_1 + x_2 - x_3)(x_1 + x_2 - x_3)\\ &= (-2t_1)(2t_3)(-t_4)(2t_2)\\ &= 8\sigma_4.\end{aligned}$$

The formula on the restriction of y_4^- is proved analogously. ♪

Corollary 5.4. *As an $\mathrm{A}^*_{\mathrm{Spin}_8}$-module, $\mathrm{A}^*_{\mathrm{SL}_4}$ is generated by the following 8 elements:*

$$1, \sigma_2, \sigma_3, \sigma_4, \sigma_2\sigma_3, \sigma_2\sigma_4, \sigma_3\sigma_4, \sigma_2\sigma_3\sigma_4.$$

Proof. By Lemma 5.3 we have that

$$\begin{aligned}\sigma_2^2 &= 2(\sigma_2^2 + 2\sigma_4) - (\sigma_2^2 - 4\sigma_4) - 8\sigma_4\\ &= i^*(2c_4^+ - c_4 - y_4^+);\end{aligned}$$

moreover, $\sigma_3^2 = -i^*c_6$ and $\sigma_4^2 = i^*c_8^+$, so $\sigma_2^2, \sigma_3^2, \sigma_4^2 \in i^*\,\mathrm{A}^*_{\mathrm{Spin}_8}$. ♪

5.2. The subgroups $\mathrm{Spin}_7, \mathrm{Spin}_7^\pm$. Before undertaking a similar analysis for the action of $\mathfrak{S}_3$ on $\mathrm{A}^*_{\mathrm{Spin}_7}$ and $\mathrm{A}^*_{\mathrm{Spin}_7^\pm}$ we need some auxiliary results on $\mathrm{A}^*_{\mathrm{Spin}_7}$. Consider the spin representation S_7 of Spin_7, and let $C' \subseteq S_7 \setminus \{0\}$

the cone of isotropic vectors, $B' := (S_7 \setminus \{0\}) \setminus C'$. Then there is an exact localization sequence

$$\mathrm{A}^*_{\mathrm{Spin}_7}(C') \xrightarrow{h} \mathrm{A}^*_{\mathrm{Spin}_7}(S_7 \setminus \{0\}) \xrightarrow{f} \mathrm{A}^*_{\mathrm{Spin}_7}(B') \to 0.$$

Guillot ([**12**, Proposition 6.5]) proved the following facts:

(1) there is an isomorphism
$$\mathrm{A}^*_{\mathrm{Spin}_7}(B') \simeq \mathrm{A}^*_{\mathrm{G}_2}$$
where the exceptional group G_2 is included in Spin_7 as the stabilizer of any point of the quadric $Q' := \{\beta = 1\} \subseteq B'$;

(2) the action of Spin_7 on C' is transitive, and the stabilizer of any point is a semi-direct product $\mathrm{SL}_3 \ltimes H$, where H is a 1-connected solvable group (we conjecture that H is isomorphic, as a variety, to an affine space, and SL_3 acts on H via linear transormations): hence there is an isomorphism
$$\mathrm{A}^*_{\mathrm{Spin}_7}(C') \simeq \mathrm{A}^*_{\mathrm{SL}_3};$$

(3) denote $h : \mathrm{SL}_3 \subseteq \mathrm{Spin}_7$ the inclusion of SL_3 as the stabilizer of a point of C': then the restriction h^* make $\mathrm{A}^*_{\mathrm{SL}_3}$ a finite $\mathrm{A}^*_{\mathrm{Spin}_7}$-module, generated by $\sigma_2, \sigma_3, \sigma_2\sigma_3$, where σ_i is the i-th Chern class of the tautological representation of SL_3.

Define the elements $\xi_3 := h_*\sigma_2, \xi_4 := h_*\sigma_3, \xi_6 := h_*(\sigma_2\sigma_3) \in \mathrm{A}^*_{\mathrm{Spin}_7}$ and set $c_i := \mathrm{c}_i(\mathbb{C}^7), c'_i := \mathrm{c}_i(S_7)$. Then Guillot ([**12**, Proposition 7.3]) proved the following result:

THEOREM 5.5 (P.Guillot). *The Chow ring of* Spin_7 *is generated by the following elements:*
$$\xi_3, \xi_4, \xi_6, c_2, c_4, c_6, c_7, c'_8.$$

As for SL_4 and $\mathrm{SL}_4^{\pm}$, the transposition (12) (resp. (13)) induces an isomorphism $(12)^* : \mathrm{A}^*_{\mathrm{Spin}_7} \simeq \mathrm{A}^*_{\mathrm{Spin}_7^+}$ (resp. $(13)^* : \mathrm{A}^*_{\mathrm{Spin}_7} \simeq \mathrm{A}^*_{\mathrm{Spin}_7^-}$) that corresponds to $(12)^* : \mathrm{A}^*_{\mathrm{Spin}_8}(B) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}(B^+)$ (resp. $(13)^* : \mathrm{A}^*_{\mathrm{Spin}_8}(B) \simeq \mathrm{A}^*_{\mathrm{Spin}_8}(B^+)$). Once fixed the elements ξ_i, c_i, c'_i in $\mathrm{A}^*_{\mathrm{Spin}_7}$, we define $\xi_i^+ := (12)^*\xi_i$ and so on.

The next step is to investigate the relation between SL_3 an SL_4. Composing the inclusions $\mathrm{SL}_3 \subseteq \mathrm{Spin}_7 \subseteq \mathrm{Spin}_8$, we can regard SL_3 as a subgroup of Spin_8. Correspondingly, there are other 2 subgroups $h^+ : \mathrm{SL}_3^+ \subseteq \mathrm{Spin}_7^+$ and $h^- : \mathrm{SL}_3^- \subseteq \mathrm{Spin}_7^-$.

LEMMA 5.6. *The restrictions of the representations* ρ, σ^+, σ^- *of* Spin_8 *to the subgroups* $\mathrm{Spin}_7, \mathrm{Spin}_7^+, \mathrm{Spin}_7^-$ *are summarized in the following table:*

	ρ	σ^+	σ^-
Spin_7	$\rho_7 \oplus 1$	σ_7	σ_7
Spin_7^+	σ_7	$\rho_7 \oplus 1$	σ_7
Spin_7^-	σ_7	σ_7	$\rho_7 \oplus 1$

where $\rho_7 : \mathrm{Spin}_7 \to \mathrm{SO}_7$ *(resp.* $\sigma_7 : \mathrm{Spin}_7 \to \mathrm{SO}_8$*) is the 7-dimensional representation (resp. the spin representation) of* Spin_7*.*

PROOF. First of all, note that thanks to the action of $\mathfrak{S}_3$ it suffices to prove only the formulae on the first row of the table. From the fiber diagram

$$\begin{array}{ccc} \mathrm{Spin}_7 & \xrightarrow{j} & \mathrm{Spin}_8 \\ \downarrow & & \downarrow \rho \\ \mathrm{SO}_7 & \hookrightarrow & \mathrm{SO}_8 \end{array}$$

we obtain $\mathrm{res}^{\mathrm{Spin}_8}_{\mathrm{Spin}_8}\, \rho \simeq \rho_7 \oplus 1$.

We will show that $j : \mathrm{Spin}_7 \subseteq \mathrm{Spin}_8$ is the restriction of the inclusion of Clifford algebras $C_7 \subseteq C_8$ given by the embedding

$$\mathbb{C}^7 = W_7 \oplus W_7' \oplus U \subseteq \mathbb{C}^8 = W_7 \oplus W_7' \oplus U_1 \oplus U_2,$$

where the inclusion $U \subseteq U_1 \oplus U_2$ sends 1 to $1/\sqrt{2}(1,1)$, and the quadratic form on $U_1 \oplus U_2$ is given by the matrix

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

Then it will follow from [**4**, Exercise 20.40] that the restriction of both the two half-spin representations to Spin_7 are isomorphic to the spin representation.

It is easy to see that, with respect to the orthonormal basis $e_1, \dots, e_7$ of $\mathbb{C}^7$ and $e_1, \dots, e_8$ of $\mathbb{C}^8$, the embedding works in the the following way: $e_i \mapsto e_i$, $e_{i+3} \mapsto e_{i+4}$ for $i = 1, 2, 3$, and $e_7 \mapsto \sqrt{-1/2}e_4$. In particular, we have that Spin_7 stabilizes e_8, so the inclusion $\mathrm{Spin}_7 \subseteq \mathrm{Spin}_8$ induced by $C_7 \subseteq C_8$ is j, and we are done. ♩

It follows that $\mathrm{Spin}_7^{\pm}$ acts on $\mathbb{C}^8$ via σ_7: so we can construct a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{A}^*_{\mathrm{Spin}_7^{\pm}}(C) & \longrightarrow & \mathrm{A}^*_{\mathrm{Spin}_7^{\pm}}(\mathbb{C}^8 \setminus \{0\}) & \longrightarrow & \mathrm{A}^*_{\mathrm{Spin}_7^{\pm}}(B) & \longrightarrow & 0 \\ \uparrow (j^{\pm})^* & & \uparrow (j^{\pm})^* & & \uparrow (j^{\pm})^* & & \\ \mathrm{A}^*_{\mathrm{Spin}_8}(C) & \xrightarrow{i_*} & \mathrm{A}^*_{\mathrm{Spin}_8}(\mathbb{C}^8 \setminus \{0\}) & \xrightarrow{j^*} & \mathrm{A}^*_{\mathrm{Spin}_8}(B) & \longrightarrow & 0. \end{array}$$

LEMMA 5.7. *The subgroup* SL_3 *is* $\mathfrak{S}_3$*-invariant, and*

$$\mathrm{SL}_3 = \mathrm{SL}_4 \cap \mathrm{Spin}_7^+ = \mathrm{SL}_4 \cap \mathrm{Spin}_7^-.$$

Moreover, following the notation of Remark 2.1, $h : \mathrm{SL}_3 \subseteq \mathrm{SL}_4$ *includes* SL_3 *in* SL_4 *as the stabilizer of* v_2*.*

PROOF. It is not difficult to see that SL_3^+ is the stabilizer of the pair (w_1, w_1') in Spin_7^+; since SL_4 is the stabilizer of (w_1, w_1') in Spin_8, the statement $\mathrm{SL}_3 = \mathrm{SL}_4 \cap \mathrm{Spin}_7^+$ follows easily.

We have that $\mathrm{res}^{\mathrm{Spin}_8}_{\mathrm{SL}_4} S^+ \simeq S_6^+ \oplus S_6^-$, with $S_6^+ \simeq (S_6^-)^\vee$. On the other side, j^+ includes Spin_7^+ in Spin_8 as the stabilizer of $w_2 \wedge w_1 + w_3 \wedge w_4 \in Q^+ \subseteq \bigwedge^\bullet W$, with $w_2 \wedge w_1 \in S_6^+$, $w_3 \wedge w_4 \in (S_6^+)^\vee$: so $\mathrm{SL}_4 \cap \mathrm{Spin}_7^+$ is the stabilizer in SL_4 of $w_1 \wedge w_2 = v_2 \in S_6^+$. But we have more: since SL_4^+ is the stabilizer of $(w_1 \wedge w_2, w_3 \wedge w_4)$ in Spin_8, we have also that $\mathrm{SL}_3^+ = \mathrm{SL}_4 \cap \mathrm{SL}_4^+$. Applying transposition (12) exchanges SL_4 and SL_4^+, so

$$\mathrm{SL}_3 = (12)\mathrm{SL}_3^+ = (12)(\mathrm{SL}_4 \cap \mathrm{SL}_4^+) = \mathrm{SL}_4^+ \cap \mathrm{SL}_4 = \mathrm{SL}_3^+.$$

Now we apply transposition (23) to the equality $\mathrm{SL}_3 = \mathrm{SL}_4 \cap \mathrm{Spin}_7^+$ and obtain $\mathrm{SL}_3 = \mathrm{SL}_4 \cap \mathrm{Spin}_7^-$. On the other hand, the same argument used for SL_3^+ shows that $\mathrm{SL}_3^- = \mathrm{SL}_4 \cap \mathrm{Spin}_7^-$. ♪

COROLLARY 5.8. *There are commutative diagrams*

$$\begin{array}{ccc} \mathrm{A}^*_{\mathrm{SL}_3} & \xrightarrow{h_*} & \mathrm{A}^*_{\mathrm{Spin}_7}/(c_8') \\ \phi^\pm \uparrow & & \uparrow j^* \\ \mathrm{A}^*_{\mathrm{SL}_4^\pm} & \xrightarrow{i_*^\pm} & \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8^\pm) \end{array}$$

and $\phi^+(\sigma_i) = \sigma_i, \phi^-(\sigma_i) = (-1)^i \sigma_i \in \mathrm{A}^*_{\mathrm{SL}_3}$.

PROOF. By Lemma 5.6 there are commutative diagrams with exacts rows

$$\begin{array}{ccccccc} \mathrm{A}^*_{\mathrm{Spin}_7}(C^\pm) & \longrightarrow & \mathrm{A}^*_{\mathrm{Spin}_7}(S^\pm \setminus \{0\}) & \longrightarrow & \mathrm{A}^*_{\mathrm{Spin}_7}(B^\pm) & \longrightarrow & 0 \\ j^* \uparrow & & (j^\pm)^* \uparrow & & \uparrow & & \\ \mathrm{A}^*_{\mathrm{Spin}_8}(C^\pm) & \xrightarrow{i_*^\pm} & \mathrm{A}^*_{\mathrm{Spin}_8}(S^\pm \setminus \{0\}) & \xrightarrow{j^*} & \mathrm{A}^*_{\mathrm{Spin}_8}(B^\pm) & \longrightarrow & 0. \end{array}$$

We have isomorphisms $\mathrm{A}^*_{\mathrm{Spin}_7}(C^\pm) \simeq \mathrm{A}^*_{\mathrm{SL}_3}$ and $\mathrm{A}^*_{\mathrm{Spin}_8}(C^\pm) \simeq \mathrm{A}^*_{\mathrm{SL}_4^\pm}$, and by Lemma 5.7 the induced pull-back $\mathrm{A}^*_{\mathrm{SL}_4^\pm} \to \mathrm{A}^*_{\mathrm{SL}_3}$ is the one induced by the inclusion of $\mathrm{SL}_3 \subseteq \mathrm{SL}_4^\pm$: it is easy to see that $\mathrm{SL}_3 \subseteq \mathrm{SL}_4^+$ is the usual inclusion, while $\mathrm{SL}_3 \subseteq \mathrm{SL}_4^-$ is the map ι followed by the usual inclusion, whence the assertion on $\phi^\pm$. ♪

6. Informations from mod 2 and Brown-Peterson cohomology

In what follows we will use the same notation for both the complex algebraic group and its real maximal compact subgroup: this convention should not lead to confusion, and it fits well to our aims, since the classifying space of a complex algebraic group is homotopy equivalent to that of its maximal compact subgroup.

6.1. The mod **2 cohomology of the classifying space of spin groups.** We will denote with $w_i \in \mathrm{H}^*(\mathrm{B\,Spin}_n; \mathbb{Z}/2)$ the i-th Stiefel-Whitney class $\mathrm{w}_i(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the representation given by the projection $\mathrm{Spin}_n \to \mathrm{SO}_n$.

Daniel Quillen ([**16**]) described the mod 2 cohomology of the classifying space of Spin_n as follows. The Milnor operations are defined recursively by $\mathrm{Q}_0 = \mathrm{Sq}^1$ and $\mathrm{Q}_i = [\mathrm{Sq}^{2^i}, \mathrm{Q}_{i-1}]$, where Sq^i is the i-th Steenrod operation (see [**14**]).

Theorem 6.1 (D. Quillen). *Let Δ a spin representation of Spin_n, and let 2^h be the Radon-Hurwitz number: then we have that*

$$\mathrm{H}^*(\mathrm{B\,Spin}_n; \mathbb{Z}/2) \simeq \mathbb{Z}/2[w_2, \ldots, w_n, w_h(\Delta)]/(w_2, \mathrm{Q}_0\, w_2, \ldots, \mathrm{Q}_{h-1}\, w_2).$$

Let w'_i (resp. $w_i^{\pm}$) the i-th Stiefel-Whitney class $\mathrm{w}_i(\Delta_7)$ (resp. $\mathrm{w}_i(\Delta_8^{\pm})$), where Δ_7 is the 7-dimensional spin representation of Spin_7 (resp. $\Delta_8^{\pm}$ is the positive/negative 8-dimensional half-spin representation of Spin_8): then we have that

$$\mathrm{H}^*(\mathrm{B\,Spin}_7; \mathbb{Z}/2) \simeq \mathbb{Z}/2[w_4, w_6, w_7, w'_8]$$
$$\mathrm{H}^*(\mathrm{B\,Spin}_8; \mathbb{Z}/2) \simeq \mathbb{Z}/2[w_4, w_6, w_7, w_8, w_8^+].$$

6.2. Complex cobordism and Brown-Peterson cohomology. A *weakly complex manifold* M is a smooth real manifold with a complex vector bundle over M whose underlying real vector bundle is $TM \oplus \mathbb{R}^N$ for some N. In particular, a complex manifold is a weakly complex manifold, but also some odd-dimensional manifolds admit a weakly complex struxture. We identify two weakly complex structures on M if they are homotopic, and we also identify a complex structure on $TM \oplus \mathbb{R}^N$ with the obvious complex structure on $TM \oplus \mathbb{R}^N \oplus \mathbb{R}^2 = TM \oplus \mathbb{R}^{N+2}$.

Let X a topological space: the i-th *complex bordism group* $\mathrm{MU}_i(X)$ is defined as the free abelian group on the set of continuous maps $M \to X$ where M is a closed weakly complex manifold of real dimension i, modulo the relations

$$[M_1 \coprod M_2 \to X] = [M_1 \to X] + [M_2 \to X]$$
$$[\partial N \to X] = 0$$

where N is a weakly complex $i+1$-manifold with boundary with a continuous map $N \to X$.

The groups $\mathrm{MU}_i(X)$ form a generalized homology theory, that is, they satisfy all the formal properties of the ordinary homology, except for the dimension axiom, since $\mathrm{MU}_* = \mathrm{MU}_*(pt) \simeq \mathbb{Z}[x_1, x_2, \ldots]$ with $x_i \in \mathrm{MU}_{2i}$.

As for any generalized homology theory, there is a corresponding cohomology theory, the *complex cobordism* $\mathrm{MU}^*(X)$, which is a ring for any topological space X, and if X is a real compact oriented n-dimensional manifold there is a Poincaré duality $\mathrm{MU}^i(X) \simeq \mathrm{MU}_{2n-i}(X)$ (this isomorphism also holds for noncompact manifolds, with a variant 'á la Borel-Moore' of the (co)bordism ring: see [**20**]).

From now on we will assume that X is a real manifold. There is a natural map $\mathrm{MU}^*(X) \to \mathrm{H}^*(X)$ that sends a cobordism class $[M \to X]$ to the image under this map to the fundamental class of M. This map has an enormous kernel, however Burt Totaro ([**20**]) has shown that we can refine it to a map

$$\mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathbb{Z} \longrightarrow \mathrm{H}^*(X)$$

that has the advantage that its kernel is much smaller than that of the previous one.

Moreover, for any complex algebraic scheme X, Totaro has defined a map

$$\mathrm{A}^*(X) \longrightarrow \mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathbb{Z}$$

that sends the class of an irreducible i-dimensional subvariety $Z \subseteq X$ to the class of the map $\tilde{Z} \to Z \subseteq X$ where $\tilde{Z} \to Z$ is any resolution of singularities of Z (see [**20**, Theorem 3.1]).

The composition of the two maps

$$\mathrm{A}^*(X) \longrightarrow \mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathbb{Z} \longrightarrow \mathrm{H}^*(X)$$

is the usual cycle map.

The Brown-Peterson cohomology is a simplification of complex cobordism; for any prime p, there is a cohomology theory called BP^* (it is conventional not to indicate the number p in the notation), whose coefficient ring is the polynomial ring $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $v_i \in \mathrm{BP}^{-2(p^i-1)}$. The BP^* theory is easier to compute than the MU^* one, but $\mathrm{BP}^*(X)$ carries essentially all the topological information of $\mathrm{MU}^*(X)$; moreover, it can be shown that

$$\mathrm{BP}^*(X) \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(p)} \simeq \mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathbb{Z}_{(p)}$$

so one can define a map

$$\mathrm{A}^*(X)_{(p)} \longrightarrow \mathrm{BP}^*(X) \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(p)} \longrightarrow \mathrm{H}^*(X)_{(p)}$$

whose composition is the cycle map $\mathrm{cl} \otimes \mathbb{Z}_{(p)}$.

Finally, in the same way as for the usual cohomology ring, given a topological group G and a topological G-space X we can define an equivariant Brown-Peterson cohomology ring

$$\mathrm{BP}^*_G(X) := \mathrm{BP}^*\big((X \times \mathrm{E}\,G)/G\big)$$

where $\mathrm{E}\,G \to \mathrm{B}\,G$ is the universal principal G-bundle, and $\mathrm{B}\,G$ is the classifying space for G. We will denote with $\mathrm{BP}^*_G := \mathrm{BP}^*_G(pt)$ the Brown-Peterson cohomology ring of a point: this, as usual, can be identified with the Brown-Peterson cohomology of the classifying space of G.

The main tool to compute the Brown-Peterson cohomology of a space is by means of the Atiyah-Hirzebruch spectral sequence. Suppose that X is a space with the homotopy type of a CW-complex. Then there is a half-plane spectral sequence $E(X)_*^{*,*}$ with

$$E(X)_2^{h,k} \simeq \mathrm{H}^h(X; \mathrm{BP}^k)_{(p)}$$

converging to $\mathrm{BP}^*(X)$. Moreover, some of the differentials of this spectral sequence are given by

$$\mathrm{d}_{2^{i+1}-1}(x) = v_i \otimes \mathrm{Q}_i\, x \mod \big(\text{quotient of } (v_1, \ldots, v_{i-1})E_2 \cap E_r\big).$$

Remark 6.2 (Yagita). To start the Atiyah-Hirzebruch spectral sequence for Brown-Peterson cohomology for a space X, we need to know $\mathrm{H}^*(X; \mathrm{BP}^*)$. If $\mathrm{H}^*(X)$ has only p-torsion (and this will be the case of $\mathrm{G}_2, \mathrm{Spin}_7, \mathrm{Spin}_8$ for $p = 2$), it sufficies to know the mod p cohomology: first note that we can identify $\mathrm{H}^*(X)_{(p)}/p$ as a subgroup of $\mathrm{H}^*(X; \mathbb{Z}/p)$ in the following way (here the Milnor operations Q_i are restricted to $\mathrm{H}^*(X; \mathbb{Z}/p)$):

$$\mathrm{H}^*(X)_{(p)}/p = \ker \mathrm{Q}_0 = (\ker \mathrm{Q}_0 \,/\, \mathrm{im}\, \mathrm{Q}_0) \oplus \mathrm{im}\, \mathrm{Q}_0 \subseteq \mathrm{H}^*(X; \mathbb{Z}/p).$$

Let B the free $\mathbb{Z}_{(p)}$-module generated by the same dimensional generators in $\ker \mathrm{Q}_0 \,/\, \mathrm{im}\, \mathrm{Q}_0$: then, since B is the torsion-free part of $\mathrm{H}^*(X)_{(p)}$ and $\mathrm{im}\, \mathrm{Q}_0$ is the torsion of $\mathrm{H}^*(X)_{(p)}$, we have that

$$\mathrm{H}^*(X)_{(p)} \simeq B \oplus \mathrm{im}\, \mathrm{Q}_0 \,.$$

Now, by the universal coefficient theorem $\mathrm{H}^*(X; \mathrm{BP}^*) = \mathrm{H}^*(X)_{(p)} \otimes_{\mathbb{Z}} \mathrm{BP}^*$ (recall that BP^* is a free $\mathbb{Z}_{(p)}$-module), so we obtain

$$\mathrm{H}^*(X; \mathrm{BP}^*) \simeq (B \oplus \mathrm{im}\, \mathrm{Q}_0) \otimes_{\mathbb{Z}} \mathrm{BP}^* \,.$$

6.3. Brown-Peterson cohomology of $\mathrm{B}\,\mathrm{Spin}_8$**.** The Brown-Peterson cohomology of $\mathrm{B}\,\mathrm{Spin}_n$ has been studied by Akira Kono and Nobuaki Yagita in [**13**]. They compute, among other things, the Brown-Peterson cohomology of some exceptional groups and of Spin_n for $n \leq 10$ using the Atiyah-Hirzebruch spectral sequence.

From now on we will work with $p = 2$. The following result appears in [**13**], but in a different form; here we give a direct proof, also given in an unpublished paper by Yagita (see also [**17**, §7]).

We introduce the following notation: suppose that R is a ring, M is an R-module, and choose elements $x_1, \ldots, x_n \in M$: then we denote with $R\{x_1, \ldots, x_n\}$ the R-submodule of M generated by $x_1, \ldots, x_n$. If $R\{x_1, \ldots, x_n\}$ is a free R-module, we write $R\,\langle x_1, \ldots, x_n\rangle$.

LEMMA 6.3. *There is an isomorphism of $\mathbb{Z}_{(2)}$-modules*

$$\begin{aligned} E(\mathrm{B\,Spin}_8)^{*,*}_\infty = &\mathbb{Z}_{(2)}[c_4, c_6, c_8, c_8^+] \otimes \Big(\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, 2w_8^+, \\ &2w_4w_8^+, v_1w_8, v_1w_8^+, v_1w_8w_8^+, 2w_8w_8^+w_4, 2w_8w_8^+\} \\ &\oplus \big(\mathrm{BP}^*[c_7]\{(c_8 - c_8^+)c_7c_8c_8^+\}\big)/(2, v_1, v_2, v_3, v_4)\Big) \\ &\oplus \mathbb{Z}_{(2)}[c_4, c_6, c_7, c_8] \otimes \big(\mathrm{BP}^*[c_7]\{c_7c_8\}/(2, v_1, v_2, v_3)\big) \\ &\oplus \mathbb{Z}_{(2)}[c_4, c_6, c_7, c_8^+] \otimes \big(\mathrm{BP}^*[c_7]\{c_7c_8^+\}/(2, v_1, v_2, v_3)\big) \\ &\oplus \mathbb{Z}_{(2)}[c_4, c_6] \otimes \big(\mathrm{BP}^*[c_7]\{c_7\}/(2, v_1, v_2, v_3)\big) \\ &\oplus \mathbb{Z}_{(2)}[c_4, c_6, c_8^+] \otimes \big(\mathrm{BP}^*[c_7]\{c_7c_8c_8^+\}/(2, v_1, v_2)\big). \end{aligned}$$

where $2w_i \in \mathrm{BP}^*_{\mathrm{Spin}_8}$ (resp. $2w_i^+ \in \mathrm{BP}^*_{\mathrm{Spin}_8}$) are elements that map to the i-th Stiefel-Whitney class $2\,\mathrm{w}_i(\mathbb{R}^8)$ (resp. $2\,\mathrm{w}_i(\Delta^+)$) under the composition

$$\mathrm{BP}^*_{\mathrm{Spin}_8} \longrightarrow \mathrm{H}^*(\mathrm{B\,Spin}_8) \longrightarrow \mathrm{H}^*(\mathrm{B\,Spin}_8; \mathbb{Z}_2).$$

PROOF. The cohomology of the classifying space of the exceptional Lie group G_2 is computed in [**2**]:

$$\mathrm{H}^*(\mathrm{B\,G}_2; \mathbb{Z}/2) \simeq \mathbb{Z}/2[w_4, w_6, w_7]$$

$$\mathrm{H}^*(\mathrm{B\,G}_2)_{(2)} \simeq \mathbb{Z}_{(2)}[w_4, c_6](\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\}).$$

Let us write $B_{i_1,\dots,i_j} = \mathbb{Z}_{(2)}[c_{i_1}, \dots, c_{i_j}]$, e.g., $B_{4,6} = \mathbb{Z}_{(2)}[c_4, c_6]$ and write $P(n)^* = \mathrm{BP}^*/(2, v_1, \dots, v_{n-1})$ e.g, $P(2)^* = \mathrm{BP}^*/(2, v_1)$. Since $\mathrm{Q}_1(w_4) = w_7$, we have $\mathrm{d}_3(w_4) = v_1 \otimes w_7$. Hence the E_4-term of Atiyah-Hirzebruch spectral sequence is

$$E(\mathrm{B\,G}_2)^{*,*}_4 \cong B_{4,6} \otimes (\mathrm{BP}^*\{1, 2w_4\} \oplus P(2)^*[c_7]\{c_7, w_7\}).$$

Next differential is $\mathrm{Q}_2(w_7) = c_7$ and

$$E(\mathrm{B\,G}_2)^{*,*}_8 \cong B_{4,6} \otimes (\mathrm{BP}^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

which is isomorphic to $E(\mathrm{G}_2)^{*,*}_\infty$.

Next consider the case Spin_7. We have seen that the mod 2 cohomology of Spin_7 is isomorphic to the polynomial ring $\mathbb{Z}/2[w_4, w_6, w_7, w_8']$, so

$$\mathrm{H}^*(\mathrm{B\,Spin}_7; \mathbb{Z}/2) \simeq \mathrm{H}^*(\mathrm{B\,G}_2; \mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8'].$$

Since $\mathrm{Q}_0(w_8) = \mathrm{Q}_1(w_8) = 0$, we see that

$$E(\mathrm{B\,Spin}_7)^{*,*}_4 \simeq E(\mathrm{B\,G}_2)^{*,*}_4[w_8]$$

$$\simeq B_{4,6,8} \otimes (\mathrm{BP}^*\{1, 2w_4\} \oplus P(2)^*[c_7]\{c_7, w_7\})\{1, w_8\}$$

$$\simeq B_{4,6,8} \otimes (\mathrm{BP}^*\{1, 2w_4, w_8, 2w_4w_8\} \oplus P(2)^*[c_7]\{c_7, w_7, c_7w_8, w_7w_8\}).$$

The next differential is $\mathrm{d}_7(w_7) = v_2c_7$ and $\mathrm{d}_7(w_8) = v_2(w_7w_8)$. Hence

$$E(\mathrm{B\,Spin}_7)^{*,*}_8 \simeq$$

$$B_{4,6,8} \otimes (\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8\} \oplus P(3)^*[c_7]\{c_7, w_7w_8\}).$$

Since $Q_3(w_7w_8) = c_7c_8$, we get

$$E(\mathrm{B\,Spin}_7)_{16}^{*,*} \simeq B_{4,6} \otimes P(3)^*[c_7]\{c_7\}\oplus$$

$$B_{4,6,8} \otimes (\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8\} \oplus P(4)^*[c_7]\{c_7c_8\}).$$

This term is also the infity term.

Now we consider the case Spin_8. The mod 2 cohomology is $\mathrm{H}^*(\mathrm{B\,Spin}_8;\mathbb{Z}/2) \simeq \mathrm{H}^*(\mathrm{B\,Spin}_7;\mathbb{Z}/2) \otimes \mathbb{Z}/2[w_8^+]$; since $Q_0(w_8) = Q_1(w_8) = 0$, we see that

$$E(\mathrm{B\,Spin}_8)_4^{*,*} \simeq E(\mathrm{B\,G}_2)_4^{*,*}[w_8, w_8^+]$$

$$\simeq B_{4,6,8,8^+} \otimes (\mathrm{BP}^*\{1, 2w_4\} \oplus P(2)^*[c_7]\{c_7, w_7\})\{1, w_8\} \otimes \{1, w_8^+\})$$

$$\simeq B_{4,6,8,8^+} \otimes (\mathrm{BP}^*\{1, 2w_4, w_8, 2w_4w_8, w_8^+, 2w_4w_8^+, w_8w_8^+, 2w_4w_8w_8^+\}$$

$$\oplus P(2)^*[c_7]\{c_7, w_7, c_7w_8, w_7w_8, c_7w_8^+, w_7w_8^+, c_7w_8w_8^+, w_7w_8w_8^+\}).$$

The next differential is $d_7(w_7) = v_2c_7$ and $d_7(w_8) = v_2(w_7w_8)$, and $d_7(w_8^+)) = v_2c_7w_8'$ and $d_7(w_7w_8w_8^+) = c_7w_8w_8^+$,. Hence

$$E(\mathrm{B\,Spin}_8)_8^{*,*} \simeq$$

$$B_{4,6,8,8^+}\otimes(\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8, 2w_8^+, v_1w_8^+, 2w_4w_8^+, w_8w_8^+, 2w_4w_8w_8^+\}$$

$$\oplus P(3)^*[c_7]\{c_7, w_7w_8, w_7w_8^+, c_7w_8w_8^+\}).$$

Since $Q_3(w_7w_8) = c_7c_8, Q_3(w_7w_8^+) = c_7c_8^+$. $Q_3(w_8w_8^+) = c_8w_7w_8^+ + c_8^+w_7w_8$, we get

$$E(\mathrm{B\,Spin}_8)_{16}^{*,*} \simeq B_{4,6,8,8^+} \otimes (\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8, 2w_8^+, v_1w_8^+,$$

$$2w_4w_8^+, 2w_8w_8^+w_4, 2w_8w_8^+, v_1w_8w_8^+, v_2w_8w_8^+\}$$

$$\oplus P(4)^*[c_7]\{c_7c_8c_8^+, c_8w_7w_8^+ + c_8^+w_7w_8\})$$

$$\oplus B_{4,6,8} \otimes P(4)^*[c_7]\{c_7c_8\} \oplus B_{4,6,8^+} \otimes P(4)^*[c_7]\{c_7c_8^+\} \oplus B_{4,6} \otimes P(3)^*[c_7]\{c_7\}.$$

The next differential is

$$d_{31}(c_8w_7w_8^+ + c_8'w_7w_8) = v_4 \otimes (c_8 + c_8^+)c_8c_7c_8^+.$$

Therefore we get

$$E(\mathrm{B\,Spin}_8)_{32}^{*,*} \simeq B_{4,6,8,8'} \otimes (\mathrm{BP}^*\{1, 2w_4, 2w_4w_8, 2w_8, v_1w_8, 2w_8^+, v_1w_8^+,$$

$$2w_4w_8^+, 2w_8w_8^+w_4, 2w_8w_8^+, v_1w_8w_8^+, v_2w_8w_8^+\}$$

$$\oplus P(5)^*[c_7]\{(c_8 + c_8^+)c_7c_8c_8^+\}) \oplus B_{4,6,7,8} \otimes P(4)^*[c_7]\{c_7c_8\}$$

$$\oplus B_{4,6,7,8^+}\otimes P(4)^*[c_7]\{c_7c_8^+\}\oplus B_{4,6}\otimes P(4)^*[c_7]\{c_7\}\oplus B_{4,6,8^+}\otimes P(3)^*[c_7]\{c_8c_8^+c_7\}.$$

This term is the infinite term and the proof is over. ♪

The first summand of $E(\mathrm{B\,Spin}_8)^{*,*}_\infty$ is isomorphic to

$$\mathrm{BP}^*[c_4, c_6, c_8, c_8^+]\langle y_0, y_4, y_6, y_6^+, y_8, y_8^+, y_{10}, y_{12}, y_{12}^+,$$
$$y_{14}, y_{16}, y_{20}\rangle/(2y_6 - v_1 y_8, 2y_6^+ - v_1 y_8^+, 2y_{10} - v_1 y_{16}, 2y_{14} - v_2 y_8)$$
$$\oplus \mathrm{BP}^*[c_4, c_6, c_8, c_8^+][c_7]\{(c_8 - c_8^+)c_7 c_8 c_8^+\}/(2, v_1, v_2, v_3, v_4)$$

(here the subscripts for the y_i give the cohomological degree). Note that $c_i = w_i^2 \in \mathrm{H}^*(\mathrm{B\,Spin}_8, \mathbb{Z}/2)$ (see for example [**14**, §15]).

So we obtain the following result on the Brown-Peterson cohomology of the classifying space of Spin_8:

THEOREM 6.4 (N.Yagita). *There is an isomorphism of* $\mathbb{Z}_{(2)}[c_4, c_6, c_8, c_8^+]$-*modules*

$$\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)}[c_4, c_6, c_8, c_8^+]\big(\mathbb{Z}_2\left\langle \tilde y_0, \tilde y_4, \tilde y_8, \tilde y_8^+, \tilde y_{12}, \tilde y_{12}^+, \tilde y_{16}, \tilde y_{20}\right\rangle$$
$$\oplus\, \mathbb{Z}/2\left\langle \tilde y_6, \tilde y_6^+, \tilde y_{10}, \tilde y_{14}\right\rangle \oplus \mathbb{Z}/2[c_7]\{c_7\}\big)$$

where we denoted with $\tilde y \in \mathrm{BP}^*_{\mathrm{Spin}_8}$ *a class that is represented by* y *in the page* $E^{*,*}_\infty$.

From this, it is easy to prove the following result:

PROPOSITION 6.5 (N. Yagita). *The cycle map localized at* (2)

$$\mathrm{A}^*_{\mathrm{Spin}_8} \otimes \mathbb{Z}_{(2)} \xrightarrow{\widetilde{\mathrm{cl}}} \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$$

is an isomorphism.

PROOF. Note that since $\mathrm{H}^*(\mathrm{SL}_4; \mathbb{Z})$ has no torsion, $\mathrm{BP}^{\mathrm{odd}}_{\mathrm{SL}_4} = 0$: see [**13**, p. 781]; moreover, by [**13**, p. 797-798] we have also $\mathrm{BP}^{\mathrm{odd}}_{\mathrm{Spin}_7} = \mathrm{BP}^{\mathrm{odd}}_{\mathrm{Spin}_8} = 0$. From the identifications $\mathrm{BP}^*_{\mathrm{Spin}_8}(C) \simeq \mathrm{BP}^*_{\mathrm{SL}_4}$ and $\mathrm{BP}^*_{\mathrm{Spin}_8}(B) \simeq \mathrm{BP}^*_{\mathrm{Spin}_7}$ we obtain the exact localization sequence in the Brown-Peterson theory for the inclusions $C \subseteq (\mathbb{C}^8 \setminus \{0\}) \supseteq B$

$$\mathrm{BP}^*_{\mathrm{Spin}_8}(C) \xrightarrow{i_*} \mathrm{BP}^*_{\mathrm{Spin}_8}(\mathbb{C}^8 \setminus \{0\}) \xrightarrow{j^*} \mathrm{BP}^*_{\mathrm{Spin}_8}(B) \to 0$$

and a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{A}^*_{\mathrm{SL}_4} & \xrightarrow{i_*} & \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8) & \xrightarrow{j^*} & \mathrm{A}^*_{\mathrm{Spin}_7} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \widetilde{\mathrm{cl}}} & & \downarrow{\scriptstyle \widetilde{\mathrm{cl}}} & & \downarrow{\scriptstyle \widetilde{\mathrm{cl}}} & & \\
\mathrm{BP}^*_{\mathrm{SL}_4} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} & \xrightarrow{i_*} & \mathrm{BP}^*_{\mathrm{Spin}_8}/(c_8) \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} & \xrightarrow{j^*} & \mathrm{BP}^*_{\mathrm{Spin}_7} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} & \longrightarrow & 0.
\end{array}$$

Following the notation of [**12**] there is an isomorphism of $\mathbb{Z}_{(2)}$-modules

$$\mathrm{BP}^*_{\mathrm{Spin}_7} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)}[c_4, c_6, c_8, c_8^+]\big(\mathbb{Z}_2\langle \tilde x_0, \tilde x_4, \tilde x_8, \tilde x_{12}\rangle \oplus \mathbb{Z}/2\langle \tilde x_6\rangle \oplus \mathbb{Z}/2[c_7]\{c_7\}\big);$$

since $j^*\tilde y_0 = \tilde x_0, j^*\tilde y_4 = \tilde x_4, j^*\tilde y_6^+ = \tilde x_6, j^*\tilde y_8^+ = \tilde x_8, j^*\tilde y_{12}^+ = \tilde x_{12}$ we have that

$$\tilde y_6, \tilde y_8, \tilde y_{10}, \tilde y_{12}, \tilde y_{14}, \tilde y_{16}, \tilde y_{20} \in \ker j^*$$

and so for dimensional reasons we have the following table of pushforwards in $\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8)$:

α	σ_2	σ_3	σ_4	$\sigma_2\sigma_3$	$\sigma_2\sigma_4$	$\sigma_3\sigma_4$	$\sigma_2\sigma_3\sigma_4$
$i_* \widetilde{\mathrm{cl}}(\alpha)$	$\tilde{y}_6$	$\tilde{y}_8$	$\tilde{y}_{10}$	$\tilde{y}_{12}$	$\tilde{y}_{14}$	$\tilde{y}_{16}$	$\tilde{y}_{20}$

in $\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8)$. Recall that the restriction i^* gives $\mathrm{A}^*_{\mathrm{SL}_4}$ (resp. $\mathrm{BP}^*_{\mathrm{SL}_4}$) the structure of a finite $\mathrm{A}^*_{\mathrm{Spin}_8}$-module (resp. $\mathrm{BP}^*_{\mathrm{Spin}_8}$-module), generated by the elements $1, \sigma_2, \sigma_3, \sigma_4, \sigma_2\sigma_3, \sigma_2\sigma_4, \sigma_3\sigma_4, \sigma_2\sigma_3\sigma_4$ (Corollary 5.4); then the projection formula says exaclty that the pushforward i_* is a morphism of $\mathrm{A}^*_{\mathrm{Spin}_8}$-modules (resp. $\mathrm{BP}^*_{\mathrm{Spin}_8}$-modules). Note also that as a map of $\mathrm{A}^*_{\mathrm{Spin}_8}$-modules, $\ker i_*$ is the submodule generated by 1. Moreover, the map $\widetilde{\mathrm{cl}}$ makes $\mathrm{BP}^* \mathrm{Spin}_8 \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$ an $\mathrm{A}^*_{\mathrm{Spin}_8}$-module: it follows that we have a commutative diagram of $\mathrm{A}^*_{\mathrm{Spin}_8}$-modules with exact rows

$$\begin{array}{ccccccccc}
0 \to & (\mathrm{A}^*_{\mathrm{SL}_4} / \ker i_*) \otimes \mathbb{Z}_{(2)} & \xrightarrow{i_*} & \big(\mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)\big) \otimes \mathbb{Z}_{(2)} & \xrightarrow{j^*} & \mathrm{A}^*_{\mathrm{Spin}_7} \otimes \mathbb{Z}_{(2)} & \to 0 \\
& \downarrow \widetilde{\mathrm{cl}} & & \downarrow \widetilde{\mathrm{cl}} & & \downarrow \widetilde{\mathrm{cl}} & \\
0 \to & \mathrm{BP}^*_{\mathrm{SL}_4} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} / \ker i_* & \xrightarrow{i_*} & \mathrm{BP}^*_{\mathrm{Spin}_8}/(c_8) \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} & \xrightarrow{j^*} & \mathrm{BP}^*_{\mathrm{Spin}_7} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} & \to 0.
\end{array}$$

The right vertical arrow is an isomorphism (because $\mathrm{BP}^*_{\mathrm{SL}_4} = \mathrm{BP}^{\mathrm{even}}_{\mathrm{SL}_4}$), and by [**12**] also the right vertical arrow is an isomorphism: so by the 5-lemma we have that $\big((\mathrm{A}^*_{\mathrm{Spin}_8})_{(2)}\big)/(c_8) \simeq \big(\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}\big)/(c_8)$.

Now let $x \in \mathrm{BP}^i_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$, and choose $\alpha \in (\mathrm{A}^i_{\mathrm{Spin}_8})_{(2)}$ such that $\widetilde{\mathrm{cl}}(\alpha) \bmod c_8 = x \bmod c_8$. Then $\widetilde{\mathrm{cl}}\,\alpha = x + c_8 y$, and by induction on the degree we can suppose that $y = \widetilde{\mathrm{cl}}\,\beta$ with $\beta \in (\mathrm{A}^{i-8}_{\mathrm{Spin}_8})_{(2)}$, and $\widetilde{\mathrm{cl}}(\alpha - c_8\beta) = x$, so $\widetilde{\mathrm{cl}} : (\mathrm{A}^*_{\mathrm{Spin}_8})_{(2)} \to \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$ is surjective. To show injectivity, let $\alpha \in \mathrm{A}^i_{\mathrm{B\,Spin}_8}$ and suppose that $\widetilde{\mathrm{cl}}\,\alpha = 0$: then since $\widetilde{\mathrm{cl}} \bmod c_8$ is an isomorphism we have that $\alpha = c_8\beta$ for some $\beta \in \mathrm{A}^{i-8}_{\mathrm{Spin}_8}$. Since by Theorem 6.4 c_8 is not a zero divisor in $\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$, we obtain that $\widetilde{\mathrm{cl}}\,\beta = 0$; once again, by induction on the degree we deduce that $\beta = 0$ and so $\alpha = 0$.

7. Push-forward classes from $\mathrm{A}^*_{\mathrm{SL}_4}$

Define the following elements in $\mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$:

$$\begin{aligned}
\bar{\zeta}_3 &:= i_*\sigma_2 \\
\bar{\zeta}_4 &:= i_*\sigma_3 \\
\bar{\zeta}_5 &:= i_*\sigma_4 \\
\bar{\zeta}_6 &:= i_*(\sigma_2\sigma_3) \\
\bar{\zeta}_7 &:= i_*(\sigma_2\sigma_4) \\
\bar{\zeta}_8 &:= i_*(\sigma_3\sigma_4)
\end{aligned}$$

$$\bar{\zeta}_{10} := i_*(\sigma_2\sigma_3\sigma_4).$$

For $i = 1, \dots, 6$ there are unique elements $\zeta_i \in \mathrm{A}^*_{\mathrm{Spin}_8}$ such that $\zeta_i \bmod c_8 = \bar{\zeta}_i \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$. To define $\zeta_8, \zeta_{10} \in \mathrm{A}^*_{\mathrm{Spin}_8}$ we need some auxiliary results.

Define the elements

$$\zeta_i^+ := i_*^+\sigma_{i-1}, \quad \zeta_i^- := i_*^-\sigma_{i-1}$$
$$\zeta_6^+ := i_*^+\sigma_2\sigma_3, \quad \zeta_6^- := i_*^-\sigma_2\sigma_3$$

LEMMA 7.1. *The images of the ζ_i under the cycle map $\widetilde{\mathrm{cl}} : \mathrm{A}^*_{\mathrm{Spin}_8} \to \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*}\mathbb{Z}_{(2)}$ are described in the following table:*

α	ζ_3	ζ_3^+	ζ_4	ζ_4^+	ζ_5	ζ_6	ζ_6^+	ζ_7
$\widetilde{\mathrm{cl}}(\alpha)$	$\tilde{y}_6$	$\tilde{y}_6^+$	$\tilde{y}_8$	$\tilde{y}_8^+$	$\tilde{y}_{10}$	$\tilde{y}_{12}$	$\tilde{y}_{12}^+$	$\tilde{y}_{14}$

PROOF. From the exact localization sequences for $\mathrm{A}^*_{\mathrm{Spin}_8}$ and $\mathrm{BP}^*_{\mathrm{Spin}_8}$ it is easy to see that the formulae hold for the $\zeta_i \bmod \mathrm{c}_8$, and so for the ζ_i. The formulae for the ζ_i^+ hold for dimensional reasons. ♪

LEMMA 7.2. *We have that*

$$c_2 = c_2^+ = c_2^-$$
$$c_4^+ - c_4^- = -3\zeta_4$$
$$c_6^+ - c_6^- = -\zeta_6$$

in $\mathrm{A}^*_{\mathrm{Spin}_8}$.

PROOF. By Lemma 5.6 $j^*c_i^+ = j^*c_i^-$, so $c_i^+ - c_i^- \in \operatorname{im} i_*$. By Corollary 5.4 we have that $i_*\,\mathrm{A}^1_{\mathrm{SL}_4} = 0, i_*\,\mathrm{A}^3_{\mathrm{SL}_4} = \mathbb{Z}\zeta_4, i_*\,\mathrm{A}^5_{\mathrm{SL}_4} = \mathbb{Z}\zeta_6$, hence

$$c_2^+ - c_2^- = 0$$
$$c_4^+ - c_4^- = d_4\zeta_4$$
$$c_6^+ - c_6^- = d_6\zeta_6.$$

Moreover, since $c_2 = c_2'$ in $\mathrm{A}^*_{\mathrm{Spin}_7}$, we have that $j^*c_2 = j^*c_2^{\pm}$, and so $c_2 - c_2^{\pm}, c_2^+ - c_2^- \in i_*\,\mathrm{A}^1_{\mathrm{SL}_4} = 0$. Applying the transposition (12) to the last two equations and restricting to Spin_7 we obtain

$$c_4 - c_4' = d_4\xi_4$$
$$c_6 - c_6' = d_6\xi_6$$

in $\mathrm{A}^*_{\mathrm{Spin}_7}$, and by [**12**, Remark p. 19] we obtain that $d_4 = -3$ and $d_6 = -1$. ♪

We know from Lemma 3.1 that $\mathrm{A}^*_{T_{\mathrm{Spin}_6}} \simeq \mathbb{Z}[u_1, u_2, u_3, \bar{u}_3]/(2\bar{u}_3 - (u_1 + u_2 + u_3))$; on the other hand, we have that $\mathrm{A}^*_{T_{\mathrm{SL}_4}} \simeq \mathbb{Z}[t_1, t_2, t_3, t_4]/(t_1 + t_2 + t_3 + t_4)$, where, following notation of Remark 2.1, t_i is the Chern class of the character τ_i. We wish to compare the two expressions: to do this, we need a

LEMMA 7.3. *We have that*

$$W_6 = \langle v_2 \wedge v_4, v_2 \wedge v_3, v_3 \wedge v_4 \rangle ,$$

and the formulae

$$\begin{aligned} u_1 &= 2x_1 = -t_1 - t_3 \\ u_2 &= 2x_2 = t_2 + t_3 \\ u_3 &= 2x_3 = -t_1 - t_2 \end{aligned}$$

hold in $\mathrm{A}^*_{T_{\mathrm{Spin}_6}}$.

PROOF. The restriction of S^+ to T_{Spin_6} is isomorphic to $S_6^+ \oplus S_6^-$, and by Lemma 3.2 the action of T_{Spin_6} on the subspace $\mathbb{C}v_1$ (resp. $\mathbb{C}v_2, \mathbb{C}v_3, \mathbb{C}v_4$) is given by $\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_3^{-1}$ (resp. $\chi_1 \otimes \chi_2 \otimes \chi_3^{-1}, \chi_1^{-1} \otimes \chi_2 \otimes \chi_3, \chi_1 \otimes \chi_2^{-1} \otimes \chi_3$). It follows that T_{Spin_6} acts on $\mathbb{C}v_2 \wedge v_4$ (resp. $\mathbb{C}v_2 \wedge v_3, \mathbb{C}v_3 \wedge v_4$) via χ_1^2 (resp. χ_2^2, χ_3^2). Let $W_3 = \langle w_1, w_2, w_3 \rangle \subseteq \mathbb{C}^6$: then T_{Spin_6} acts on $\mathbb{C}w_i$ via χ_i^2: it follows that (possibly after rescaling the vectors) $w_1 = v_2 \wedge v_4, w_2 = v_2 \wedge v_3$ and $w_3 = v_3 \wedge v_4$.

Since T_{SL_4} acts on $\mathbb{C}v_i \wedge v_j$ via the character $\tau_i \otimes \tau_j$, taking Chern classes of these characters the second assertion is proved too. ♩

LEMMA 7.4. *We have that*

$$\begin{aligned} y_4 - 4\zeta_4 &= y_4^+ - 4\zeta_4^+ = 0 \\ \zeta_4^2 - 4c_8 &= (\zeta_4^+)^2 - 4c_8^+ = 0 \end{aligned}$$

in $\mathrm{A}^*_{\mathrm{Spin}_8}$.

PROOF. Obviously it is sufficient to prove only the formulae for ζ_4, the ones for ζ_4^+ being obtained applying transposition (12).

Consider the cartesian diagram of linear algebraic groups

$$\begin{array}{ccc} \mathrm{Spin}_6 & \xrightarrow{\ i\ } & \mathrm{Spin}_8 \\ \downarrow{\scriptstyle \rho_6} & & \downarrow{\scriptstyle \rho} \\ \mathrm{SO}_6 & \xrightarrow{\ i'\ } & \mathrm{SO}_8. \end{array}$$

Let (U, V) a good pair for Spin_8: then in the diagram

$$\begin{array}{ccc} (U \times C)/\mathrm{Spin}_8 & \xrightarrow{\ i\ } & \big(U \times (\mathbb{A}^8 \setminus \{0\})\big)/\mathrm{Spin}_8 \\ \downarrow & & \downarrow \\ (U \times C)/\mathrm{SO}_6 & \xrightarrow{\ i'\ } & \big(U \times (\mathbb{A}^8 \setminus \{0\})\big)/\mathrm{SO}_8 \end{array}$$

the horizontal arrows are proper and the vertical arrows are flat, so by [**3**, Proposition 1.7] we have that

$$i_* \rho_6^* \alpha = \rho^* i'_* \alpha$$

for $\alpha \in \mathrm{A}^*_{\mathrm{SO}_6}$.

By Lemma 4.8 of Chapter 1, the restriction of $\rho_6^* y_3$ to the torus T_{Spin_6} is $4u_1u_2u_3$, and by Lemma 7.3 this equals $4(t_1+t_2)(t_1+t_3)(t_2+t_3)$ in $\mathrm{A}^*_{T_{\mathrm{SL}_4}}$. On the other hand, it is easily seen that the restriction of σ_3 to T_{SL_4} is $-(t_1+t_2)(t_1+t_3)(t_2+t_3)$, and since $\mathrm{A}^*_{\mathrm{SL}_4}$ injects in $\mathrm{A}^*_{T_{\mathrm{SL}_4}}$ we find

$$\rho_6^* y_3 = -4\sigma_3 \in \mathrm{A}^*_{\mathrm{SL}_4}\,. \tag{7.1}$$

Since by 4.6 of Chapter 1, the relation $i'_* y_3 = -y_4$ holds in $\mathrm{A}^*_{\mathrm{SO}_8}$, we have

$$4 i_* \sigma_3 = i_*(-\rho_6^* y_3) = -\rho^* i'_* y_3 = \rho^* y_4 = y_4.$$

We have that $y_4^2 - 2^6 c_8 = 0 \in \mathrm{A}^*_{\mathrm{SO}_8}$ ([**15**, Section 5.2]), so

$$2^4 \zeta_4^2 = 2^6 c_8 \in \mathrm{A}^*_{\mathrm{Spin}_8};$$

it follows that $\zeta_4^2 - 4c_8 = \alpha$ with α a torsion element. Moreover, since $j^*\zeta_4 = j^* c_8 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_7}$ it must be

$$\alpha \in i_*\,\mathrm{A}^7_{\mathrm{SL}_4} = \mathbb{Z}(2c_4^+ - c_4 - y_4^+)\left\langle \bar\zeta_4 \right\rangle \oplus \mathbb{Z}\left\langle \bar\zeta_8 \right\rangle;$$

since y_4 is not a torsion element, also ζ_4 is not a torsion element; moreover, by Lemmas 5.3 and 7.4 $2\sigma_4 = i^*\zeta_4^+$, so by the projection formula

$$2\bar\zeta_8 = i_*(2\zeta_3\zeta_4) = \zeta_4^+\zeta_4 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

so also $\bar\zeta_8$ is not torsion: it follows that $\alpha = 0$ and the Lemma is proved. ♪

COROLLARY 7.5. *We have that*

$$\zeta_4 = 32 x_1 x_2 x_3 x_4$$
$$\zeta_4^+ = 2\Big(\sum x_i^4 - 2\sum x_i^2 x_j^2 + 8 x_1 x_2 x_3 x_4\Big)$$

in $\mathrm{A}^*_{T_{\mathrm{Spin}_8}}$.

PROOF. Use Lemmas 3.4 and 7.4. ♪

PROPOSITION-DEFINITION 7.6. *There exist unique elements* $\zeta_8, \zeta_{10} \in \mathrm{A}^*_{\mathrm{Spin}_8}$ *such that* $\zeta_i \bmod c_8 = \bar\zeta_i$ *and satisfying the relations*

$$2\zeta_8 = \zeta_4^+\zeta_4 \qquad 2\zeta_{10} = c_2\zeta_8.$$

PROOF. We have seen in the proof of Lemma 7.4 that

$$2\bar\zeta_8 = i_*(2\zeta_3\zeta_4) = \zeta_4^+\zeta_4 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8).$$

Choose $x \in \mathrm{A}^*_{\mathrm{Spin}_8}$ such that $x \bmod c_8 = \bar\zeta_8$: then

$$2x - \zeta_4^+\zeta_4 + dc_8 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}$$

with $d \in \mathbb{Z}$. Since by [**16**, Theorem 6.7] $w_i^+ = w_i^-$ for $i < 8$, we have that

$$\mathrm{cl}\,\zeta_4 = 3\,\mathrm{cl}\,\zeta_4 = \mathrm{cl}(c_4^+ - c_4^-) = (w_4^+)^2 - (w_4^-)^2 = 0;$$

hence

$$0 = \mathrm{cl}(2x - \zeta_4^+\zeta_4 + dc_8) = dw_8^2 \in \mathrm{H}^*(\mathrm{B}\,\mathrm{Spin}_8; \mathbb{Z}/2).$$

It follows from the description of $\mathrm{H}^*(\mathrm{B\,Spin}_8;\mathbb{Z}/2)$ that d must be even; moreover, modifying x by a multiple of c_8 we can assume that $d = 0$, and we can define $\zeta_8 := x$.

Suppose that ζ_8' is another element satisfying the requirements of the Lemma: then $\zeta_8' = \zeta_8 + ac_8$ with $a \in \mathbb{Z}$; moreover, $2\zeta_8' = \zeta_4\zeta_4^+ = 2\zeta_8$ implies that $a = 0$ and $\zeta_8 = \zeta_8'$.

By Lemma 5.3 $2\sigma_2 = i^*c_2$, so by the projection formula

$$2\bar{\zeta}_{10} = i_*(2\zeta_2\zeta_3\zeta_4) = c_2\zeta_8 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8).$$

Choose $x \in \mathrm{A}^*_{\mathrm{Spin}_8}$ such that $x \bmod c_8 = \bar{\zeta}_{10}$: then

$$2x - c_2\zeta_8 + dc_2c_8 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}$$

with $d \in \mathbb{Z}$.

By Lemma 7.4 $4\zeta_8^2 = \zeta_4^2(\zeta_4^+)^2 = 16c_8c_8^+$, hence $\zeta_8^2 = 4c_8c_8^+ + \alpha$ with α a torsion element; it follows that, since $c_2\zeta_8 = 2x + dc_2c_8$,

$$\begin{aligned} 0 &= 2x\zeta_8 - 4c_2c_8c_8^+ - c_2\alpha + dc_8c_2\zeta_8 \\ &= 2x\zeta_8 - 4c_2c_8c_8^+ - c_2\alpha + 2dc_8x + d^2c_2c_8^2 \in \mathrm{A}^*_{\mathrm{Spin}_8}. \end{aligned}$$

Now we map this relation to the Brown-Peterson cohomology of $\mathrm{B\,Spin}_8$; since the cycle map induces an isomorphism $\mathrm{A}^*_{\mathrm{Spin}_8}\otimes\mathbb{Z}_{(2)} \simeq \mathrm{BP}^*_{\mathrm{Spin}_8}\otimes_{\mathrm{BP}^*}\mathbb{Z}_{(2)}$, eventually modifying $\tilde{y}_{16}$ (resp. $\tilde{y}_{20}$) with a multiple of c_8 (resp. c_2c_8), we can suppose that $\widetilde{\mathrm{cl}}\,\zeta_8 = \tilde{y}_{16}$ and $\widetilde{\mathrm{cl}}\,x = \tilde{y}_{20}$. So we obtain

$$0 = 2\tilde{y}_{16}\tilde{y}_{20} - 4\tilde{y}_4c_8c_8^+ - \tilde{y}_4(\widetilde{\mathrm{cl}}\,\alpha) + 2dc_8\tilde{y}_{20} + d^2c_8^2\tilde{y}_4 \in \mathrm{BP}^*_{\mathrm{Spin}_8}\otimes_{\mathrm{BP}^*}\mathbb{Z}_{(2)}.$$

Hence we get the expression for the product $\tilde{y}_{16}\tilde{y}_{20}$ in $\mathrm{BP}^*_{\mathrm{Spin}_8}\otimes_{\mathrm{BP}^*}\mathbb{Z}_{(2)}$:

$$\tilde{y}_{16}\tilde{y}_{20} = 2c_8c_8^+\tilde{y}_4 - dc_8\tilde{y}_{20} - \frac{d^2}{2}c_8^2\tilde{y}_4 + \beta$$

with β a torsion element.

Note that from the additive description of $\mathrm{BP}^*_{\mathrm{Spin}_8}\otimes_{\mathrm{BP}^*}\mathbb{Z}_{(2)}$, the torsion-free submodule is a free $\mathbb{Z}_{(2)}[c_4, c_6, c_8, c_8^+]$-module with basis

$$\tilde{y}_0, \tilde{y}_4, \tilde{y}_8, \tilde{y}_8^+, \tilde{y}_{12}, \tilde{y}_{12}^+, \tilde{y}_{16}, \tilde{y}_{20},$$

so it must be $d^2/2 \in \mathbb{Z}_{(2)}$, that is, d must be even. Modifying x by a multiple of c_2c_8 we can assume that $d = 0$, and we can define $\zeta_{10} := x$.

The uniqueness of ζ_{10} is proved analogously to the one of ζ_8. ♪

Remark 7.7. By [**7**, Theorem 1], for any connected reductive algebraic group G with maximal torus T and Weyl group W there is an isomorphism

$$\mathrm{A}^*_G\otimes\mathbb{Q} \simeq (\mathrm{A}^*_T)^W \otimes \mathbb{Q};$$

it follows that an element $\alpha \in \mathrm{A}^*_G$ is torsion if and only if its restriction to a maximal torus is zero.

Moreover, by [**18**, Theorem 0.1] the torsion index of Spin_8 is 2, so $\mathrm{A}^*_{\mathrm{Spin}_8}$ has only 2-torsion. Hence from now on we will tacitly assume that every torsion element of $\mathrm{A}^*_{\mathrm{Spin}_8}$ is a 2-torsion element.

We will use this facts throughot the paper.

Corollary 7.8. *We have that*

$$\zeta_4 = 32x_1x_2x_3x_4$$
$$\zeta_4^+ = 2(\sum x_i^4 - 2\sum x_i^2x_j^2 + 8x_1x_2x_3x_4)$$
$$\zeta_6 = -64(\sum x_i^2)x_1x_2x_3x_4$$
$$\zeta_6^+ = -4(\sum x_i^2)(\sum x_i^4 - 2\sum x_i^2x_j^2 + 8x_1x_2x_3x_4)$$
$$\zeta_8 = 32(\sum x_i^4 - 2\sum x_i^2x_j^2 + 8x_1x_2x_3x_4)x_1x_2x_3x_4$$
$$\zeta_{10} = -64(\sum x_i^2)(\sum x_i^4 - 2\sum x_i^2x_j^2 + 8x_1x_2x_3x_4)x_1x_2x_3x_4$$

in $\mathrm{A}^*_{T_{\mathrm{Spin}_8}}$.

In particular, the ζ_{even} *are not torsion.*

Proof. The formulas for ζ_4 and ζ_4^+ follow from Lemmas 7.4 and 3.4. Next, note that since $i^*c_2 = 2\sigma_2$, by the projection formula

$$2\zeta_6 = i_*(2\sigma_2\sigma_3) = c_2\zeta_4 \in \mathrm{A}^6_{\mathrm{Spin}_8}/(c_8) = \mathrm{A}^6_{\mathrm{Spin}_8}$$

and applying the permutation (12) we get also $2\zeta_6^+ = c_2^+\zeta_4^+$: using this two relations it is easy to obtain the restrictions for ζ_6 and ζ_6^+.

The restrictions of ζ_8 and ζ_{10} to T_{Spin_8} are easily computed using the relations $2\zeta_8 = \zeta_4\zeta_4^+$ and $2\zeta_{10} = c_2\zeta_8$. ♪

Remark 7.9. By the exact localization sequences for the Chow and Brown-Peterson cohomology rings, we have that $\widetilde{\mathrm{cl}}\,\zeta_8 = \tilde{y}_{16}$ and $\widetilde{\mathrm{cl}}\,\zeta_{10} = \tilde{y}_{20}$ in $\mathrm{BP}^*_{\mathrm{B\,Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8)$. Moreover there is an injection

$$\mathrm{A}^*_{\mathrm{Spin}_8} \xrightarrow{\widetilde{\mathrm{cl}}} \mathrm{BP}^*_{\mathrm{B\,Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$$

which becomes an isomorphism after tensoring with $\mathbb{Z}_{(2)}$. It follows that we can suppose, eventually modifying $\tilde{y}_{16}$ (resp. $\tilde{y}_{20}$) by a multiple of c_8 (resp. $\tilde{y}_4c_8$) that $\widetilde{\mathrm{cl}}\,\zeta_8 = \tilde{y}_{16}$ and $\widetilde{\mathrm{cl}}\,\zeta_{10} = \tilde{y}_{20}$. So we can complete the table of Lemma 7.9:

α	ζ_3	ζ_3^+	ζ_4	ζ_4^+	ζ_5	ζ_6	ζ_6^+	ζ_7	ζ_8	ζ_{10}
$\widetilde{\mathrm{cl}}\,\alpha$	$\tilde{y}_6$	$\tilde{y}_6^+$	$\tilde{y}_8$	$\tilde{y}_8^+$	$\tilde{y}_{10}$	$\tilde{y}_{12}$	$\tilde{y}_{12}^+$	$\tilde{y}_{14}$	$\tilde{y}_{16}$	$\tilde{y}_{20}$

We define elements $\zeta_8^\pm$ and $\zeta_{10}^\pm$ in the same way that for the others. The action of $\mathfrak{S}_3$ on these classes is described in the following Lemma:

Lemma 7.10. *The action of* $\mathfrak{S}_3$ *on the* ζ_i *is described in the following table:*

	(12)	(13)	(23)
ζ_i	ζ_i^+	ζ_i^-	$(-1)^{i+1}\zeta_i$
ζ_i^+	ζ_i	$(-1)^{i+1}\zeta_i^+$	$(-1)^{i+1}\zeta_i$
ζ_i^-	$(-1)^{i+1}\zeta_i^-$	ζ_i	$(-1)^{i+1}\zeta_i^+$

PROOF. Immediate Lemma 5.1. ♪

LEMMA 7.11.

$$\xi_i = j^*\zeta_i^+ = (-1)^{i+1} j^*\zeta_i^-$$

in $\mathrm{A}^*_{\mathrm{Spin}_7}$.

PROOF. Immediate from Lemma 5.8. ♪

8. The $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$-module structure of $\mathrm{A}^*_{\mathrm{Spin}_8}$

The restriction i^* provides $\mathrm{A}^*_{\mathrm{SL}_4}$ of the structure of $\mathrm{A}^*_{\mathrm{Spin}_8}$-module.

PROPOSITION 8.1. *The ring $\mathrm{A}^*_{\mathrm{Spin}_8}$ is generated by the following elements:*

$$c_2, c_4, c_6, c_7, c_8, c_8^+, \zeta_3, \zeta_3^+, \zeta_4, \zeta_4^+, \zeta_5, \zeta_6, \zeta_6^+, \zeta_7, \zeta_8, \zeta_{10}.$$

PROOF. Use the exact localization sequence, noting that by Theorem 5.5 $\mathrm{A}^*_{\mathrm{Spin}_7}$ is generated by $\xi_3, \xi_4, \xi_6, c_2, c_4, c_6, c_7, c_8$, and by Lemma 7.11 $\xi_i = j^*\zeta_i^+$. ♪

LEMMA 8.2. *We have that*

$$c_2\zeta_{odd} = c_2\zeta_{odd}^+ = 2\zeta_{odd} = 2\zeta_{odd}^+ = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}$$

Moreover $\zeta_i\alpha = 0$ in $\mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$ for all $\alpha \in \ker i^*$. In particular, for all the possible choices of i, j the relations

$$\zeta_i\zeta_j = \zeta_i\zeta_{odd}^+ = \zeta_i c_{odd} = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8).$$

PROOF. Since $[C] = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$, we have that $i_*i^*\alpha = \alpha \cdot [C] = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$ for all $\alpha \in \mathrm{A}^*_{\mathrm{Spin}_8}$.

It follows that since by Lemma 5.3 $2\sigma_2, 2\sigma_4, 2\sigma_2\sigma_4 \in i^*\mathrm{A}^*_{\mathrm{Spin}_8}$ we have $2\zeta_3 = 2\zeta_5 = 2\zeta_7 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$, and these relations hold in $\mathrm{A}^*_{\mathrm{Spin}_8}$ because they live in odd degree ≤ 9.

Next, since $i^*c_2 = 2\sigma_2$, using the relation $\sigma_2^2 = i^*(2c_4^+ - c_4 - y_4^+)$ (see the proof of Corollary 5.4) we obtain

$$c_2\zeta_3 = 2i_*i^*(2c_4^+ - c_4 - y_4^+) = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8);$$

the relations $c_2\zeta_5 = c_2\zeta_7 \in (c_8)$ are obtained similarly, and they hold in $\mathrm{A}^*_{\mathrm{Spin}_8}$ because they live in odd degree ≤ 9.

To get the analogous relations for the ζ_{odd}^+ one can apply the transposition (12), noting that $c_2 = c_2^+$ by Lemma 7.2.

Suppose that $\alpha \in \ker i^*$: then by the projection formula we have that

$$(i_*\beta)\alpha = i_*(\beta i^*\alpha) = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

for all $\beta \in \mathrm{A}^*_{\mathrm{SL}_4}$: it follows that

$$\zeta_i\alpha = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

for all i and for $\alpha \in \ker i^*$. Since $\mathrm{c}_1(N) = 0 \in \mathrm{A}^*_{\mathrm{Spin}_6}$ (where N is the normal bundle of C in $\mathbb{A}^8 \setminus \{0\}$), because C is the zero locus of the Spin_8-invariant function q, we have that $i^*i_*\beta = \mathrm{c}_1(N)\beta = 0$, so $\zeta_i \in \ker i^*$. Moreover, since $\mathrm{A}^*_{\mathrm{SL}_4}$ is torsion-free, all torsion elements restrict to zero in $\mathrm{A}^*_{\mathrm{SL}_4}$, that concludes the proof. ♪

LEMMA 8.3. *(1) As a $\mathbb{Z}$-module, $i_* \mathrm{A}^{odd}_{\mathrm{SL}_4}$ is torsion-free.*
(2) As a $\mathbb{Z}$-module, $i_* \mathrm{A}^{even}_{\mathrm{SL}_4}$ is torsion.

PROOF. Let $\alpha \in i_* \mathrm{A}^{\mathrm{odd}}_{\mathrm{SL}_4}$ and suppose that $2\alpha = 0$. By Proposition 8.1 we can write

$$\alpha = \sum_{i=0}^{5} f_{2i}(2c_4^+ - c_4 - y_4^+, c_6, c_8^+)\zeta_{2i} \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

(we set $\zeta_0 = 1, \zeta_2 = 0$). By Lemma 7.11 $j^*(\zeta_4^+ + \zeta_4^-) = 0$ so $\zeta_4^+ + \zeta_4^- = d\zeta_4$ and restricting to T_{Spin_8}, we find $d = 1$: in particular $3\zeta_4^+ + 3\zeta_4^- - 3\zeta_4 = 0$. On the other hand, applying the permutation (13) to the relation $c_4^+ - c_4^- = -3\zeta_4$ (Lemma 7.2) we find $3\zeta_4^- = c_4 - c_4^+$ and substituting this expression in the previous relation we obtain

$$c_4^+ = c_4 + 3\zeta_4^+ - 3\zeta_4. \in \mathrm{A}^*_{\mathrm{Spin}_8} \tag{8.1}$$

It follows that

$$\alpha = \sum_{i=0}^{5} f_{2i}(c_4 + 2\zeta_4^+ - 6\zeta_4, c_6, c_8^+)\zeta_{2i} \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

and so

$$\alpha = \sum_{i=0,3,4,5} f_{2i}(c_4, c_6, c_8^+)\zeta_{2i} \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8, \zeta_4, \zeta_4^+).$$

Mapping in the Brown-Peterson cohomology we obtain

$$\widetilde{\mathrm{cl}}\,\alpha = \sum_{i=0,3,4,5} f_{2i}(c_4, c_6, c_8^+)\tilde{y}_{4i} \in \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8, \tilde{y}_8, \tilde{y}_8^+).$$

On the other hand,

$$\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8, \tilde{y}_8, \tilde{y}_8^+) \simeq \mathbb{Z}_{(2)}[c_4, c_6, c_8^+]\big(\mathbb{Z}_2 \langle \tilde{y}_0, \tilde{y}_4, \tilde{y}_{12}, \tilde{y}_{12}^+, \tilde{y}_{16}, \tilde{y}_{20}\rangle \oplus \mathbb{Z}/2 \langle \tilde{y}_6, \tilde{y}_6^+, \tilde{y}_{10}, \tilde{y}_{14}\rangle \oplus \mathbb{Z}/2[c_7]\{c_7\}\big)$$

so $\widetilde{\mathrm{cl}}\,\alpha \bmod (c_8, \tilde{y}_8, \tilde{y}_8^+)$ is not torsion: hence the element $\widetilde{\mathrm{cl}}\,\alpha \bmod (c_8, \tilde{y}_8, \tilde{y}_8^+)$ must be zero, and this implies that f_0, f_6, f_8, f_{10} are identically zero.

So we have that

$$\alpha = f(c_4 + 2\zeta_4^+ - 6\zeta_4, c_6, c_8^+)\zeta_4.$$

Since by Lemma 7.8 $\mathrm{res}^{\mathrm{Spin}_8}_{T_{\mathrm{Spin}_8}} \zeta_4 \neq 0$, we have that $f(c_4 + 2\zeta_4^+ - 6\zeta_4, c_6, c_8^+)$ is torsion; once again, it is easy to see that

$$f(c_4, c_6, c_8^+) \in \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}/(c_8, \tilde{y}_8, \tilde{y}_8^+)$$

cannot be a torsion element and so $f(c_4, c_6, c_8^+)$ identically zero. So $\alpha = 0$ and the first part is proved.

For the second part, it is sufficient to note that $i_* \mathrm{A}^{\mathrm{even}}_{\mathrm{SL}_4}$ is generated as $\mathrm{A}^*_{\mathrm{Spin}_8}$-module by the ζ_{odd}, and these elements are 2-torsion by Lemma 8.2. ♪

COROLLARY 8.4. *Suppose that $\alpha \in \mathrm{A}^{even}_{\mathrm{Spin}_8}$ is a torsion element that restricts to 0 in $\mathrm{A}^*_{\mathrm{Spin}_7}$: then $\alpha = 0$.*

PROOF. By the exact localization sequence $\ker j^* = \operatorname{im} i_*$; since $\alpha \in \mathrm{A}^{\mathrm{even}}_{\mathrm{Spin}_8}$, this implies that $\alpha \in i_* \mathrm{A}^{\mathrm{odd}}_{\mathrm{SL}_4}$. But by Lemma 8.3 $i_* \mathrm{A}^{\mathrm{odd}}_{\mathrm{SL}_4}$ has not torsion, so $\alpha = 0$. ♪

PROPOSITION 8.5. *The following relations hold in $\mathrm{A}^*_{\mathrm{Spin}_8}$:*

(1) $c_2^2 = 4c_4 + 8\zeta_4^+ - 4\zeta_4$
(2) $c_2\zeta_4 = 2\zeta_6$
(3) $c_2\zeta_4^+ = 2\zeta_6^+$
(4) $c_2\zeta_6 = 2c_4\zeta_4 + 8\zeta_8 - 8c_8$
(5) $c_2\zeta_6^+ = 2c_4\zeta_4^+ + 16c_8^+ - 4\zeta_8$
(6) $c_2\zeta_8 = 2\zeta_{10}$
(7) $c_2\zeta_{10} = 2c_4\zeta_8 + 8c_8^+\zeta_4 - 4c_8\zeta_4^+$
(8) $\zeta_4^2 = 4c_8$
(9) $\zeta_4\zeta_4^+ = 2\zeta_8$
(10) $\zeta_4\zeta_6 = 2c_2c_8$
(11) $\zeta_4\zeta_6^+ = 2\zeta_{10}$
(12) $\zeta_4\zeta_8 = 2c_8\zeta_4^+$
(13) $\zeta_4\zeta_{10} = 2c_8\zeta_6^+$
(14) $(\zeta_4^+)^2 = 4c_8^+$
(15) $\zeta_4^+\zeta_6 = 2\zeta_{10}$
(16) $\zeta_4^+\zeta_6^+ = 2c_2c_8^+$
(17) $\zeta_4^+\zeta_8 = 2c_8^+\zeta_4$
(18) $\zeta_4^+\zeta_{10} = 2c_8^+\zeta_6$
(19) $\zeta_6^2 = 4c_8(c_4 + 2\zeta_4^+ - \zeta_4)$
(20) $\zeta_6\zeta_6^+ = 2c_4\zeta_8 + 8c_8^+\zeta_4 - 4c_8\zeta_4^+$
(21) $\zeta_6\zeta_8 = 2c_8\zeta_6^+$
(22) $\zeta_6\zeta_{10} = c_8(c_4\zeta_4^+ + 12c_8^+ - 2\zeta_8 - 8c_8)$
(23) $(\zeta_6)^2 = 4c_8^+(c_4 + 2\zeta_4^+ - \zeta_4)$
(24) $\zeta_6^+\zeta_8 = 2c_8^+\zeta_6$
(25) $\zeta_6^+\zeta_{10} = c_8^+(2c_4\zeta_4 + 8\zeta_8 - 8c_8)$
(26) $\zeta_8^2 = 4c_8c_8^+$
(27) $\zeta_8\zeta_{10} = 2c_2c_8c_8^+$
(28) $\zeta_{10}^2 = c_8c_8^+(4c_4 + 8\zeta_4^+ - 4\zeta_4)$.

PROOF. First of all, note that by [**12**, Proposition 9.1] all these relations restrict to 0 in $\mathrm{A}^*_{\mathrm{Spin}_7}$, so by Corollary 8.4 it sufficies to prove the formulas in $\mathrm{A}^*_{\mathrm{Spin}_8} \otimes \mathbb{Q}$.

We will use repeatedly Lemmas 5.3 (resp. 3.4 and 7.8) for the restrictions to SL_4 (resp. to T_{Spin_8} during the proof.

By [**12**, Proposition 9.1(6)] the relation $3c_2^2 - 4c_4 - 8c_4' = 0$ holds in $(\mathrm{A}^*_{\mathrm{Spin}_7})_{(2)}$, so $a(3c_2^2 - 4c_4 - 8c_4^+) = b\zeta_4$ in $\mathrm{A}^*_{\mathrm{Spin}_8}$, with $a \in \mathbb{Z} \setminus 2\mathbb{Z}$ and $b \in \mathbb{Z}$. Restricting to T_{Spin_8} we find $b = 12a$, so $a(3c_2^2 - 4c_4 - 8c_4^+ - 12\zeta_4) = 0$; since $\mathrm{A}^*_{\mathrm{Spin}_8}$ has only 2-torsion, we have also that $3c_2^2 - 4c_4 - 8c_4^+ - 12\zeta_4 = 0$.

Substituting Equation 8.1 c_4^+ in the equation $3c_2^2 - 4c_4 - 8c_4^+ - 12\zeta_4 = 0$ we obtain

$$3c_2^2 = 12c_4 + 24\zeta_4^+ - 12\zeta_4 \in \mathrm{A}^*_{\mathrm{Spin}_8}$$

and dividing by 3 we obtain (1).

Formulas (2) and (3) are proved in the proof of Lemma 7.8. Formulas (8) and (14) are implied by Lemma 7.4. Formulas (6) and (9) are proved in the definition of the elements ζ_{10} and ζ_8 respectevly (Lemma 7.6).

Since $\sigma_2 = i^*(c_4 + 2\zeta_4^+) \in \mathrm{A}^*_{\mathrm{SL}_4}$, by the projection formula

$$c_2\zeta_6 = 2i_*(\sigma_2^2\sigma_3) = 2(c_4 + 2\zeta_4^+)\zeta_4 = 2c_4\zeta_4 + 8\zeta_8 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

by formula (9), so we have that $c_2\zeta_6 = 2c_4\zeta_4 + 8\zeta_8 + dc_8$ in $\mathrm{A}^*_{\mathrm{Spin}_8}$; restricting to T_{Spin_8} we find $d = -8$, that proves (4).

To obtain (5), note first that

$$\zeta_8^+ = \zeta_8^{(12)} = \frac{1}{2}(\zeta_4\zeta_4^+)^{(12)} = \frac{1}{2}\zeta_4\zeta_4^+ = \zeta_8; \tag{8.2}$$

then, using Formula 8.1

$$c_2\zeta_6^+ = (c_2\zeta_6)^{(12)} = c_4^+\zeta_4^+ + 4\zeta_8^+ = c_4\zeta_4^+ + 12c_8^+ - 2\zeta_8.$$

We have that

$$\zeta_4^+\zeta_8 = \frac{1}{2}(\zeta_4^+)^2\zeta_4 = 2c_8^+\zeta_4$$

that proves (16); using this equations and the projection formula we obtain

$$c_2\zeta_{10} = i_*(2\sigma_2^2\sigma_3\sigma_4) = 2(c_4 + 2\zeta_4^+)\zeta_8 = 2c_4\zeta_8 + 8c_8^+\zeta_4 \in \mathrm{A}^*_{\mathrm{Spin}_8}/(c_8)$$

so $c_2\zeta_{10} = 2c_4\zeta_8 + 8c_8^+\zeta_4 + c_8(dc_2^2 + ec_4 + f\zeta_4 + g\zeta_4^+)$ with $d, e, f, g \in \mathbb{Z}$: restricting to T_{Spin_8} we have $d = e = f = 0$ and $g = -4$, and we get (7).

Now we can prove most of the relations of the Proposition; we list the proofs in such an order so that each relation is obtained by the previous ones, and leave to the eager reader to complete the computations:

$$\begin{aligned}
\zeta_4\zeta_6 &= \frac{1}{2}c_2\zeta_4^2\\
\zeta_4\zeta_6^+ &= \frac{1}{2}c_2\zeta_4\zeta_4^+ = c_2\zeta_8\\
\zeta_4\zeta_8 &= \frac{1}{2}\zeta_4^2\zeta_4^+\\
\zeta_4\zeta_{10} &= \zeta_4 c_2\zeta_8\\
\zeta_4^+\zeta_6 &= \frac{1}{2}c_2\zeta_4\zeta_4^+\\
\zeta_4^+\zeta_6^+ &= (\zeta_4\zeta_6)^{(12)}\\
\zeta_4^+\zeta_{10} &= \frac{1}{2}\zeta_4^+ c_2\zeta_8\\
\zeta_6^2 &= \frac{1}{2}c_2\zeta_4\zeta_6\\
\zeta_6\zeta_6^+ &= \frac{1}{2}c_2\zeta_4\zeta_6^+\\
\zeta_6\zeta_8 &= \frac{1}{2}\zeta_6\zeta_4\zeta_4^+\\
\zeta_8^2 &= \frac{1}{4}\zeta_4^2(\zeta_4^+)^2\\
\zeta_6\zeta_{10} &= \frac{1}{2}c_2\zeta_6\zeta_8\\
(\zeta_6^+)^2 &= (\zeta_6^2)^{(12)} = 4c_8^+(c_4^+ + 2\zeta_4^+ - \zeta_4)\\
\zeta_6^+\zeta_{10} &= c_2\zeta_6^+\zeta_8\\
\zeta_8\zeta_{10} &= \frac{1}{2}c_2\zeta_8^2\\
\zeta_{10}^2 &= \frac{1}{4}c_2^2\zeta_8^2.
\end{aligned}$$

♪

LEMMA 8.6. *Let A the ideal generated by the elements ζ_{odd}, ζ_3^+. Then*

$$A^2 = c_7 A = 0$$

in $\mathrm{A}^*_{\mathrm{Spin}_8}$.

PROOF. Let $\zeta_i, \zeta_j \in A$: then $\zeta_i\zeta_j, \zeta_i\zeta_3^+ \in \mathrm{A}^{\mathrm{even}}_{\mathrm{Spin}_8}$ and $\zeta_i\zeta_j, \zeta_i\zeta_3^+$ restrict to zero in $\mathrm{A}^*_{\mathrm{Spin}_7}$. By the exact localization sequence, $\zeta_i\zeta_j \in i_* \mathrm{A}^{\mathrm{odd}}_{\mathrm{SL}_4}$; on the other hand, $\zeta_i\zeta_j$ is torsion, so by Corollary 8.4 it must be $0 \in \mathrm{A}^*_{\mathrm{Spin}_8}$. The same argument applies to show that $c_7\zeta_i = c_7^+\zeta_i = 0$, hence applying transposition (12) we have also $c_7\zeta_3^+ = 0$. ♪

COROLLARY 8.7. *As a* $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$*-module,* $\mathrm{A}^*_{\mathrm{Spin}_8}$ *admits the following description:*

$$\mathrm{A}^*_{\mathrm{Spin}_8} \simeq \mathbb{Z}[c_4, c_6, c_8, c_8^+]\big(\left\langle c_2, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_6^+, \zeta_8, \zeta_{10}\right\rangle \oplus \mathbb{Z}/2\left\langle \zeta_3, \zeta_3^+, \zeta_5, \zeta_7\right\rangle \oplus \mathbb{Z}/2[c_7]\{c_7\}\big)$$

PROOF. By Lemma 8.2 we have that $c_2\zeta_{\mathrm{odd}}, \zeta_i\zeta_j, \zeta_i\zeta^+_{\mathrm{odd}}, \zeta_i c_7 \in (c_8)$; applying the same proof to the pushforward i^+_* it is easy to see that

$$c_2\zeta^+_{\mathrm{odd}}, \zeta_i^+\zeta_j^+, \zeta_i^+\zeta_{\mathrm{odd}}, \zeta_i^+ c_7 \in (c_8^+).$$

Moreover,

$$c_2\zeta_7 - \delta_1 c_8^+\zeta_3^+ \in i_*\,\mathrm{A}^{10}_{\mathrm{SL}_4}$$

where $\delta_1 \in \mathbb{Z}/2$ is the indetermined coefficient of [**12**, Proposition 9.1].

By Lemma 8.6 we have that $\zeta_{\mathrm{odd}}\zeta_{\mathrm{odd}} = \zeta_{\mathrm{odd}}\zeta^+_{\mathrm{odd}} = c_7\zeta_{\mathrm{odd}} = c_7\zeta^+_{\mathrm{odd}} = 0$.

Finally, Proposition 8.5 gives the product $c_2\zeta_{\mathrm{even}}$,ζ^+_{even},$\zeta_{\mathrm{even}}\zeta_{\mathrm{even}}$, $\zeta_{\mathrm{even}}\zeta^+_{\mathrm{even}}$: using inductively this results it is easy to see that it is possible to express all the products between the generators $c_2, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_6^+$,$\zeta_8, \zeta_{10}, \zeta_3$,$\zeta_3^+, \zeta_5$,$\zeta_7, c_7$ with expressions in which these generators appears only linearly, with coefficients in $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$, except for the powers of c_7.

Note that by Lemma 7.8 the $\zeta_{\mathrm{even}}, \zeta^+_{\mathrm{even}}$ are not torsion, and by Lemma 3.4 $c_2, c_4, c_6, c_8, c_8^+$ are not torsion, while by Lemma 8.2 the $\zeta_{\mathrm{odd}}, \zeta^+_{\mathrm{odd}}$ are torsion, and $2c_7 = 0 \in \mathrm{A}^*_{\mathrm{SO}_8}$ implies $2c_7 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}$. This proves that the coefficients $\mathbb{Z}$ and $\mathbb{Z}/2$ for the module generators are as stated.

To complete the proof it sufficies to note that there is an injection of $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$-modules

$$\mathrm{A}^*_{\mathrm{Spin}_8} \xrightarrow{\widetilde{\mathrm{cl}}} \mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)} :$$

by the exact localization sequence and Remark 7.9 the module generators of $\mathrm{A}^*_{\mathrm{Spin}_8}$ map to the module generators of $\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$ and the result follows easily comparing the $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$-module structure of $\mathrm{BP}^*_{\mathrm{Spin}_8} \otimes_{\mathrm{BP}^*} \mathbb{Z}_{(2)}$ in Theorem 6.4. ♪

COROLLARY 8.8. *The torsion ideal of* $\mathrm{A}^*_{\mathrm{Spin}_8}$ *is*

$$(\mathrm{A}^*_{\mathrm{Spin}_8})_{tor} = (c_7, \zeta_3^+, \zeta_{odd}).$$

9. The ring structure of $\mathrm{A}^*_{\mathrm{Spin}_8}$

PROPOSITION 9.1. *We have that*

$$\mathrm{A}^*_{\mathrm{Spin}_8}/(\mathrm{A}^*_{\mathrm{Spin}_8})_{tor} \simeq \mathbb{Z}[c_2, c_4, c_6, c_8, c_8^+, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_8, \zeta_{10}]/I$$

where I is the ideal generated by the relations of Proposition 8.5.

PROOF. Let $R := \mathrm{A}^*_{\mathrm{Spin}_8}/(\mathrm{A}^*_{\mathrm{Spin}_8})_{\mathrm{tor}}$: then by Proposition 8.1 and Corollary 8.8 R is generated by the elements

$$c_2, c_4, c_6, c_8, c_8^+, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_8, \zeta_{10}.$$

It follows from Proposition 8.5 that the projection $\mathrm{A}^*_{\mathrm{Spin}_8} \to R$ factorizes through

$$\mathrm{A}^*_{\mathrm{Spin}_8} \longrightarrow \mathbb{Z}[c_2, c_4, c_6, c_8, c_8^+, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_8, \zeta_{10}]/I \xrightarrow{\phi} R;$$

Suppose that $g(c_2, c_4, c_6, c_8, c_8^+, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_8, \zeta_{10}) = 0$ in R: then using the relations of I we can express g in a unique way as

$$g = f_0 + f_2 c_2 + \sum_{i=4,6,8,10} f_i \zeta_i + \sum_{i=4,6} f_i^+ \zeta_i^+$$

with $f_i, f_i^+ \in \mathbb{Z}[c_4, c_6, c_8, c_8^+]$. But then by Corollary 8.7 f_i, f_i^+ must be identically zero, so $g = 0$ and the relations sufficies: hence the map ϕ is an isomorphism. ♪

Corollary 9.2. *The ring $\mathrm{A}^*_{\mathrm{Spin}_8}/(\mathrm{A}^*_{\mathrm{Spin}_8})_{tor}$ can be identified with a subring of $\mathrm{A}^*_{\mathrm{Spin}_8}$: that is, the projection*

$$\mathrm{A}^*_{\mathrm{Spin}_8} \to \mathrm{A}^*_{\mathrm{Spin}_8}/(\mathrm{A}^*_{\mathrm{Spin}_8})_{tor}$$

admits a splitting.

Proof. By Corollary 8.8

$$I \cap (\mathrm{A}^*_{\mathrm{Spin}_8})_{\mathrm{tor}} = I \cap (c_7, \zeta_3^+, \zeta_{\mathrm{odd}}) = (0);$$

it follows that R can be identified with a subring of $\mathrm{A}^*_{\mathrm{Spin}_8}$. ♪

Lemma 9.3. *As an abelian group, $\mathrm{A}^3_{\mathrm{Spin}_8}$ is generated by $\zeta_3, \zeta_3^+, \zeta_3^-$, with the relations*

$$2\zeta_3 = 2\zeta_3^+ = 2\zeta_3^- = 0$$
$$\zeta_3 + \zeta_3^+ + \zeta_3^- = 0.$$

Proof. It is clear that $\zeta_3, \zeta_3^+, \zeta_3^-$ generate $\mathrm{A}^3_{\mathrm{Spin}_8}$, since there is an exact sequence of abelian groups

$$0 \to \mathbb{Z}/2\,\langle\zeta_3\rangle \longrightarrow \mathrm{A}^3_{\mathrm{Spin}_8} \longrightarrow \mathbb{Z}/2\,\langle\xi_3\rangle \to 0.$$

These elements are 2-torsion, since ζ_3 is 2-torsion by Lemma 8.2 and they are exchanged by $\mathfrak{S}_3$. By Corollary 7.11, Lemma 7.10 and Equation 5.1, we have that $j^*\zeta_3^+ = j^*\zeta_3^- = \xi_3$ so

$$j^*(\zeta_3^+ + \zeta_3^-) = 2\xi_3 = 0$$

from which we obtain $\zeta_3^+ + \zeta_3^- \in \operatorname{im} i_*\,\mathrm{A}^2_{\mathrm{SL}_4} \simeq \mathbb{Z}/2 \cdot \zeta_3$, hence $\zeta_3^+ + \zeta_3^- = \delta\zeta_3$ with $\delta \in \mathbb{Z}/2$. Applying transposition (12) we obtain $\zeta_3 + \zeta_3^- = \delta\zeta_3^+$: then by Corollary 5.8

$$\xi_3 = j^*(\zeta_3 + \zeta_3^-) = j^*(\delta\zeta_3^+) = \delta\xi_3 \in \mathrm{A}^*_{\mathrm{Spin}_7}$$

which implies $\delta = 1$. ♪

Lemma 9.4. *The following equations hold in* $\mathrm{A}^*_{\mathrm{Spin}_8}$*:*

$$\begin{aligned} c_3 &= c_3^+ = c_3^- = 0 \\ \zeta_5 &= \zeta_5^+ = \zeta_5^- \\ \zeta_7 &= \zeta_7^+ = \zeta_7^- . \end{aligned}$$

Proof. Lemma 5.6 implies that $j^* c_i^+ = j^* c_i^- = c_i'$ and $j^* c_i = c_i$ in $\mathrm{A}^*_{\mathrm{Spin}_7}$ for $i = 1, \dots, 8$: so

$$c_i^+ - c_i^- \in i_*(\mathrm{A}^{i-1}_{\mathrm{SL}_4}).$$

Since by Proposition 8.1 $i_*(\mathrm{A}^2_{\mathrm{SL}_4}) = \mathbb{Z}\zeta_3$, applying transposition (12) we obtain the equation

$$c_3^+ - c_3 = d\zeta_3^- \in \mathrm{A}^*_{\mathrm{Spin}_8}$$

with $d \in \mathbb{Z}/2$. Since $c_3' = c_3 = 0$ in $\mathrm{A}^*_{\mathrm{Spin}_7}$ ([**12**, proof of Theorem 4.4]), by Corollary 5.8 we obtain $0 = d\xi_3$ and so $d = 0$.

Next, from $\mathrm{A}^5_{\mathrm{Spin}_7} = 0$ we obtain that $\mathrm{A}^5_{\mathrm{Spin}_8}$ is generated by a unique element of 2-torsion $\zeta_5 = \zeta_5^+ = \zeta_5^-$ (this element is not necessarily non-zero).

Finally, by Corollary 5.8 we have that $j^*\zeta_7^+ = h_* \operatorname{res}^{\mathrm{SL}_4}_{\mathrm{SL}_3} \sigma_2\sigma_4 = 0$ so $\zeta_7 \in i_* \mathrm{A}^6_{\mathrm{SL}_4}$, and by Proposition 8.1 we can write

$$\zeta_7^+ = a(2c_4^+ - c_4 - 4\zeta_4^+)\zeta_3 + b\zeta_7 = ac_4\zeta_3 + b\zeta_7$$

with $a, b \in \mathbb{Z}/2$. Then, applying permutation (12), by Lemma 7.10 we obtain $\zeta_7 = ac_4^+\zeta_3^+ + b\zeta_7^+$; it follows that

$$0 = j^*\zeta_7 = ac_4'\xi_3$$

in $\mathrm{A}^*_{\mathrm{Spin}_8}$. Since by [**12**, Propositions 8.3,9.1] $c_4'\xi_3 = c_4\xi_3 \neq 0 \in \mathrm{A}^*_{\mathrm{Spin}_7}$, it must be $a = 0$. It follows that $\zeta_7 = \zeta_7^+$, and by a similar argument $\zeta_7 = \zeta_7^-$. ♪

Lemma 9.5. $(c_8) \cap (c_8^+) = (c_8 c_8^+) \subseteq \mathrm{A}^*_{\mathrm{Spin}_8}$. *In particular, if* $\alpha \in (c_8) \cap (c_8^+)$*, then either* $\alpha = 0$ *or* $\alpha \in \mathrm{A}^{\geq 16}_{\mathrm{Spin}_8}$.

Proof. Suppose that $c_8\alpha = c_8^+\beta$, with $\alpha, \beta \in \mathrm{A}^*_{\mathrm{Spin}_8}$. By Corollary 8.7, we can write

$$\begin{aligned} \alpha &= f_0 + f_2c_2 + \sum_i f_i\zeta_i + \sum_i f_i^+\zeta_i^+ + c_7\sum_i f_i' c_7^i \\ \beta &= g_0 + g_2c_2 + \sum_i g_i\zeta_i + \sum_i g_i^+\zeta_i^+ + c_7\sum_i g_i' c_7^i \end{aligned}$$

with $f_i, f_i^+, f_i', g_i, g_i^+, g_i' \in \mathbb{Z}[c_4, c_6, c_8, c_8^+]$ uniquely determined. It follows that

$$c_8 f_i = c_8^+ g_i \in \mathbb{Z}[c_4, c_6, c_8, c_8^+]$$

that implies $f_i \in (c_8^+)$ and $g_i \in (c_8)$. Similarly, $f_i^+, f_i' \in (c_8^+)$ and $g_i^+, g_i' \in (c_8)$, so $c_8\alpha = c_8c_8^+\gamma$ with $\gamma \in \mathrm{A}^*_{\mathrm{Spin}_8}$. ♪

LEMMA 9.6.

$$(i^+)^*\zeta_4 = 2\sigma_4$$
$$(i^+)^*\zeta_6 = 2\sigma_2\sigma_4$$
$$(i^+)^*\zeta_8 = 0$$
$$(i^+)^*\zeta_{10} = 0$$

in $A^*_{\mathrm{SL}_4^+}$.

PROOF. Since $A^*_{\mathrm{SL}_4^+}$ is torsion-free, we can work with coefficients in $\mathbb{Q}$. Then by Lemmas 7.4, 5.3 and Proposition 8.5

$$(i^+)^*\zeta_4 = \frac{1}{4}(i^+)^*y_4 = 2\sigma_4$$
$$(i^+)^*\zeta_6 = \frac{1}{2}(i^+)^*(c_2\zeta_4) = 2\sigma_2\sigma_4$$
$$(i^+)^*\zeta_8 = \frac{1}{2}(i^+)^*(\zeta_4\zeta_4^+) = 0$$
$$(i^+)^*\zeta_{10} = \frac{1}{2}(i^+)^*(c_2\zeta_8) = 0.$$

♪

PROPOSITION 9.7. *Let* $\alpha \in \left\{c_2, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_8, \zeta_8^+\right\}$, $\beta \in \left\{\zeta_3, \zeta_3^+, \zeta_5, \zeta_7\right\}$*: then*

$$\alpha\beta = 0 \in A^*_{\mathrm{Spin}_8}.$$

PROOF. By Lemma 8.2 we have $c_2\zeta_{\mathrm{odd}} = c_2\zeta^+_{\mathrm{odd}} = 0$ and $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}} \in (c_8)$: then $\zeta_4\zeta_3, \zeta_6\zeta_3 \in A^{\leq 9}_{\mathrm{Spin}_8} \cap (c_8) = \mathbb{Z}c_8$ so $\zeta_4\zeta_3 = \zeta_6\zeta_3 = 0$.

Next, by Lemma 9.4 we have that $\zeta_{\mathrm{even}}\zeta_5 = \zeta_{\mathrm{even}}\zeta_5^+$ and $\zeta_{\mathrm{even}}\zeta_7 = \zeta_{\mathrm{even}}\zeta_7^+$, and using Lemma 9.6 and the projection formula

$$\zeta_4\zeta_3^+ = 2i^+_*(\sigma_2\sigma_4) = 2\zeta_7^+ = 0$$
$$\zeta_6\zeta_3^+ = 2i^+_*(\sigma_2^2\sigma_4) = 2(2c_4 - c_4^+ - y_4)\zeta_5^+ = 0$$
$$\zeta_8\zeta_3^+ = i^+_*(0\cdot\sigma_2) = 0$$
$$\zeta_{10}\zeta_3^+ = i^+_*(0\cdot\sigma_2) = 0$$
$$\zeta_4\zeta_5^+ = 2i^+_*(\sigma_4^2) = 2i^+_*(i^+)^*c_8 = 0$$
$$\zeta_6\zeta_5^+ = 2i^+_*(\sigma_2\sigma_4^2) = 2c_8\zeta_3^+ = 0$$
$$\zeta_8\zeta_5^+ = i^+_*(0\cdot\sigma_4) = 0$$
$$\zeta_{10}\zeta_5^+ = i^+_*(0\cdot\sigma_4) = 0$$
$$\zeta_4\zeta_7^+ = 2i^+_*(\sigma_2\sigma_4^2) = 2c_8\zeta_3^+ = 0$$
$$\zeta_6\zeta_7^+ = 2i^+_*(\sigma_2^2\sigma_4^2) = 2 = i^+_*(i^+)^*\big((2c_4 - c_4^+ - y_4)c_8\big) = 0$$
$$\zeta_8\zeta_7^+ = i^+_*(0\cdot\sigma_2\sigma_4) = 0$$

$$\zeta_{10}\zeta_7^+ = i_*^+(0\cdot\sigma_2\sigma_4) = 0$$

in $\mathrm{A}^*_{\mathrm{Spin}_8}/(c_8^+)$. (We used the fact that $\sigma_2^2 = i^*(2c_4^+ - c_4 - y_4^+)$, and applied transposition (12) to this relation.) It follows that $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}}^+ \in (c_8^+)$; on the other hand, by Lemma 8.2 $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}}^+ \in (c_8)$; by Lemma 9.5 we have that $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}}^+ \in (c_8c_8^+)$; since all this relations live in odd degree ≤ 17, it is easy to see that they must be zero in $\mathrm{A}^*_{\mathrm{Spin}_8}$. It follows that

$$\zeta_{\mathrm{even}}\zeta_5 = \zeta_{\mathrm{even}}\zeta_7 = 0 \in \mathrm{A}^*_{\mathrm{Spin}_8}.$$

Since $\zeta_8\zeta_3^- = (\zeta_8\zeta_3^+)^{(23)} = 0$, by Lemma 9.3 we have

$$\zeta_8\zeta_3 = \zeta_8(\zeta_3^+ + \zeta_3^-) = 0;$$

a similar argument shows that $\zeta_{10}\zeta_3 = 0$.

The relations $\zeta_{\mathrm{even}}^+\zeta_{\mathrm{odd}} = 0$ (resp. $\zeta_{\mathrm{even}}^+\zeta_{\mathrm{odd}}^+ = 0$) are obtained applying transposition (12) to $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}}^+ = 0$ (resp. $\zeta_{\mathrm{even}}\zeta_{\mathrm{odd}} = 0$). ♪

LEMMA 9.8. *The following relations hold in* $\mathrm{A}^*_{\mathrm{Spin}_8}$*:*

$$\begin{aligned}
c_2c_7 &= \delta_1c_6\zeta_3^+ + \delta_3c_6\zeta_3 + \delta_4c_4\zeta_5\\
\zeta_4c_7 &= c_8(\delta_5\zeta_3 + \delta_6\zeta_3^+)\\
\zeta_4^+c_7 &= c_8^+(\delta_2\zeta_3^+ + \delta_7\zeta_3)\\
\zeta_6c_7 &= \delta_8c_8\zeta_5\\
\zeta_6^+c_7 &= \delta_9c_8^+\zeta_5\\
\zeta_8c_7 &= c_8(\delta_{10}c_4\zeta_3 + \delta_{11}c_4\zeta_3^+ + \delta_{12}\zeta_7)\\
\zeta_{10}c_7 &= c_8(\delta_{13}c_4\zeta_5 + \delta_{14}c_6\zeta_3 + \delta_{15}c_6\zeta_3^+)
\end{aligned}$$

with $\delta_i \in \mathbb{Z}/2$ *for* $i = 1,\dots,15$*, and* δ_1, δ_2 *are the same of* [**12**, Proposition 9.1].

Moreover, we have that

$$c_2c_7^2 = \zeta_{even}c_7^2 = \zeta_{even}^+c_7^2 = 0.$$

PROOF. By Lemma 8.2 we have that $\zeta_{\mathrm{even}}c_7 \in (c_8)$, and the corresponding formulae follows from the fact that by Corollary 8.7, as abelian groups there are isomorphism

$$\begin{aligned}
\mathrm{A}^3_{\mathrm{Spin}_8} &\simeq \mathbb{Z}/2\left\langle\zeta_3, \zeta_3^+\right\rangle\\
\mathrm{A}^5_{\mathrm{Spin}_8} &\simeq \mathbb{Z}/2\left\langle\zeta_5\right\rangle\\
\mathrm{A}^7_{\mathrm{Spin}_8} &\simeq \mathbb{Z}/2\left\langle c_4\zeta_3, c_4\zeta_3^+, \zeta_7, c_7\right\rangle\\
\mathrm{A}^9_{\mathrm{Spin}_8} &\simeq \mathbb{Z}/2\left\langle c_4\zeta_5, c_6\zeta_3, c_6\zeta_3^+\right\rangle.
\end{aligned}$$

By [**12**, Proposition 9.1]

$$\begin{aligned} j^*(c_2c_7 + \delta_1 c_6 \zeta_3^+) &= 0 \\ j^*(\zeta_4^+ c_7 + \delta_2 c_8^+ \zeta_3^+) &= 0 \\ j^*(\zeta_6^+ c_7) &= 0 \end{aligned}$$

in $A^*_{\mathrm{Spin}_7}$, and moreover using the same proof of Lemma 8.2 $\zeta_4^+ c_7, \zeta_6^+ c_7 \in (c_8^+)$, so by Lemma 5.3

$$\begin{aligned} &c_2c_7 + \delta_1 c_6 \zeta_3^+ \in i_* \, A^8_{\mathrm{SL}_4} = \mathbb{Z}/2 \langle c_4\zeta_5, c_6\zeta_3 \rangle \\ &\zeta_4^+ c_7 + \delta_2 c_8^+ \zeta_3^+ \in i_* \, A^{10}_{\mathrm{SL}_4} \cap (c_8^+) = \mathbb{Z}/2 \left\langle c_8^+ \zeta_3 \right\rangle \\ &\zeta_6^+ c_7 \in i_* \, A^{12}_{\mathrm{SL}_4} \cap (c_8^+) = \mathbb{Z}/2 \left\langle c_8^+ \zeta_5 \right\rangle \end{aligned}$$

and the remaining formulae follow.

The last assertion follows from the previous ones and the fact that by Lemma 8.6 $c_7 A = 0$. ♪

Recall that by Corollary 8.7 there is an insomorphism of $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$-modules

$$\begin{aligned} A^*_{\mathrm{Spin}_8} \simeq &\mathbb{Z}[c_4, c_6, c_8, c_8^+]\big(\mathbb{Z}\left\langle c_2, \zeta_4, \zeta_4^+, \zeta_6, \zeta_6^+, \zeta_6^+, \zeta_8, \zeta_{10} \right\rangle \\ &\oplus \mathbb{Z}/2 \left\langle \zeta_3, \zeta_3^+, \zeta_5, \zeta_7 \right\rangle \oplus \mathbb{Z}/2[c_7]\{c_7\}\big) \end{aligned}$$

COROLLARY 9.9. *The products between the generators of the $\mathbb{Z}[c_4, c_6, c_8, c_8^+]$-module $A^*_{\mathrm{Spin}_8}$ are determined by Proposition 8.5, Lemma 8.6, Proposition 9.7, and Lemma 9.8.*

From the exact localization sequence we can now determine almost completely the Chow ring of the classifying space of Spin_7 (recall that in [**12**] there is the description of $(A^*_{\mathrm{Spin}_7})_{(2)}$):

PROPOSITION 9.10.

$$A^*_{\mathrm{Spin}_7} \simeq \mathbb{Z}[c_2, c_4, c_6, c_8', \xi_3, \xi_4, \xi_6]/R'$$

where ξ_i are elements of degree i, and R' is the ideal generated from the following elements:

$$\begin{aligned} &2\xi_3 \\ &2c_7 \\ &c_2^2 - 4c_4 - 8\xi_4 \\ &c_2\xi_4 - 2\xi_6 \\ &c_2\xi_6 - 2c_4\xi_4 - 16c_8' \\ &\xi_4^2 - 4c_8' \\ &\xi_4\xi_6 - 2c_2c_8' \\ &\xi_6^2 - 4c_8'(c_4 + 2\xi_4) \end{aligned}$$

$$\begin{aligned}
&\xi_3^2\\
&\xi_3 c_7\\
&c_2\xi_3\\
&\xi_{even}\xi_3\\
&c_2c_7 = \delta_1 c_6\xi_3\\
&\xi_4c_7 + \delta_2 c_8'\xi_3\\
&\xi_6c_7.
\end{aligned}$$

Here $\delta_i \in \mathbb{Z}/2$ are indeterminate coefficients, and δ_1, δ_2 are the same as in [**12**, Proposition 9.1].

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