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Algebraic and Combinatorial Properties of Ranks of Divisors on Finite Graphs

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Abstract

The purpose of this thesis is to study invariants on divisors of finite graphs and algebraic curves. Analysing the construction of the Caporaso's Compactified Jacobian, we present modifications of the algebraic rank with the aim of getting closer to an algebraic interpretation of the combinatorial rank. The new ranks defined here satisfy some important properties and results such as versions of the Riemann-Roch theorem.

Para Alexandrina.

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Contents

Introduction

This work is devoted to the study of the combinatorial properties of the rank of a divisor on a graph and the familiarities with the algebraic rank of divisors on nodal curves.

Motivation and objectives

A divisor on a graph is a formal sum of integers on its vertices. In particular, there is a notion of principal divisor defining equivalence classes of divisors. Baker and Norine in [3] defined the *combinatorial rank* of a divisor motivated by the theory of ranks on algebraic curves. This invariant on a divisor class $[\underline{d}]$ is the maximum integer r such that, for all effective divisors $\underline{e}$ of degree r, the divisor $\underline{d}-\underline{e}$ is linearly equivalent to an effective divisor. The combinatorial rank satisfies a number of interesting properties, more notably it satisfies the Riemann-Roch theorem (see [3]) and the specialization lemma (see [11]). In [9], Caporaso extended the definition of combinatorial rank for weighted graphs and introduced the *algebraic rank* of a divisor on a graph. Given a weighted graph G, the algebraic rank of a divisor class δ on G is defined by

$$r^{\text{alg}}(G,\delta) := \max\left\{ \min\left\{ \max\{r(X,L); \forall L \in \text{Pic}^{\underline{d}}(X)\}; \forall \underline{d} \in \delta \right\}; \forall X \in M^{\text{alg}}(G) \right\},$$

where $M^{\text{alg}}(G)$ is the locus inside the moduli space of stable curves having G as dual graph (see sections 1.2.1 and 1.2.5). This new invariant of divisors on graphs reflects the ranks of line bundles (i.e., its number of independent global sections) with a given multidegree on all the nodal curves dual to the given graph. Applying the divisor theory for graphs to the theory of linear series on singular algebraic curves, several cases were exhibited where the value of the algebraic rank coincides with the value of the combinatorial rank, leading to the conjecture that those two invariants would be equal. Later on, Len, Melo and Caporaso, in [11], were able to prove that the combinatorial rank is an upper bound for the algebraic rank of divisors on finite graphs. However, they also present examples when the inequality is strict, therefore disproving the conjecture (see examples 2.6.1 and 2.6.2). Further examples were considered by Yoav Len in [14] showing that the analogous conjecture in the cases of metrized complexes also fails. By investigating the relation between the rank of divisors on metrized complexes, and the algebraic rank of divisors obtained by forgetting the metrized complex structure, Len showed that, in general, there is no inequality between one and the other.

Given a (connected, reduced, projective) curve X of genus g over an algebraically closed field k, the Jacobian of X, $J(X)$, is the set of isomorphism classes of line bundles of degree 0 on every irreducible component of X. The Jacobian of a singular curve is in general not compact. The problem of presenting a "good" compactification of the Jacobian has been studied by a large number of authors. Here, we are interested in the compactification of the Universal Jacobian constructed by Caporaso in [7], treating the case of a family of curves. This construction is given by admitting not only line bundles of balanced multidegrees on a Deligne-Mumford stable curve X, but also certain line bundles on partial normalizations of X. This leads to a stratification of J_X^d given by the set of nodes that are normalized and the multidegree on this partial normalization.

In Chapter 1 we introduce the theory of divisors on finite graphs in the first section, and dedicate the second section to moduli spaces of curves and universal compactified Jacobians. Chapter 2 focus on the important results related to the combinatorial rank and the algebraic rank.

Contributions

The third chapter of this work is dedicated to the study of different approaches for the algebraic rank. We start but analysing the min-max order in the definition of the algebraic rank, introducing the modifications r^{MAX} and r^{ALG}. For r^{MAX} we consider the maximum of $r^{\max}(X, \underline{d})$ on the curves dual to the graph G, maintaining the same divisor $\underline{d}$. In symbols,

$$r^{\text{MAX}}(G, \underline{d}) := \max\{r^{\max}(X, \underline{d}); \forall X \in M^{\text{alg}}(G)\}.$$

For r^{ALG} we consider the minimum of $r^{\text{MAX}}(G, \delta)$ among the divisors $\underline{d} \in \delta$. We prove that these new ranks satisfy the Riemann-Roch theorem, and that we have the following inequality of ranks: $r^{\text{ALG}}(G, \delta) \leq r_G(\delta)$. Furthermore, we investigate cases where this definition coincides with the original definition of algebraic rank, i.e., when $r^{\text{alg}}(G, \delta) = r^{\text{ALG}}(G, \delta)$, for a given graph G and a divisor class δ on G, e.g., the case when G is a weighted binary graph. We conclude the section proving that r^{ALG} is an upper bound for r^{alg}.

Analysing the counterexamples presented in [11], where strict inequality of ranks occurs, we calculated the r^{MAX} of degenerations of these linear series. We observed that modifications could be made in order to arrive at the exact value of the combinatorial rank of a divisor $\underline{d}$ on a graph G: calculating the rank r^{MAX} of a balanced line bundle on a quasistable curve such that its stabilization is a dual curve of G. In symbols, let δ be a divisor class on G and consider a balanced representative $\underline{d}$ of δ. If $(G, \underline{d})$ is a *weighted exceptional contraction* of $(G', \underline{d}')$, with $\underline{d}'$ balanced, we write $(G', \underline{d}') \geq (G, \underline{d})$. We define the compactified algebraic ranks for a balanced divisor $\underline{d}$ as

$$\overline{r}^b(G, \underline{d}) := \max\{r^{\text{MAX}}(G', \underline{d}') : (G', \underline{d}') \geq (G, \underline{d}) \text{ and } \underline{d}' \text{ is a balanced divisor}\}.$$

We show that $\overline{r}^b$ satisfies the Riemann-Roch theorem and we present a version of the

Specialization lemma. Later we present a generalization for the Dhar decomposition and give a definition of a divisor of *generalized Clifford type* (see Definition 3.3.10). These new definitions allow us to extend the proof of the original inequality between ranks to the case where $\underline{d} \in \mathrm{Div}(G)$ is of generalized Clifford type, S-reduced with respect to all proper subset of vertices in the graph with contain the vertices where $\underline{d}$ possibly has negative degree, obtaining that $r^{\mathrm{MAX}}(G, \underline{d}) \leq r_G(\underline{d})$. This new result helps us to calculate the ranks of a weighted rank for some specific divisors. In Section 3.3.3 we prove that the combinatorial rank equals the balanced algebraic rank of a balanced divisor on a binary graph. In symbols, if G is a binary graph of genus g and $\underline{d}$ is a balanced divisor on G, then $r_G(\underline{d}) = \overline{r}^b(G, \underline{d})$. We also calculate the balanced algebraic rank of a certain balanced divisor on a weighted binary graph, i.e., a graph consisting on two vertices, each of them possibly weighted, joined by k edges.

In the last chapter we point out the future work and expose some conjectures and examples that we can aim to be extended to more general cases, related to the objects that we defined and studied in this work.

Chapter 1

Background theory

The aim of this chapter is to present the theory of divisors on finite graphs and later the moduli spaces of curves and universal compactified Jacobians.

1.1 Graphs and divisors on graphs

Throughout this thesis $G = (V, E, \omega)$ is a finite weighted graph, where $V = V(G)$ denotes the set of vertices of G, $E = E(G)$ its set of edges and $\omega : V \to \mathbb{Z}_{\geq 0}$ its weight function. If $\omega = 0$, G is called *weightless*[1].

If a vertex v is an endpoint of an edge e we say that they are *incident*. When two vertices u and v of G are both incident to an edge $e \in E(G)$, we write $e = uv$ and say that u and v are *adjacent*. When $u = v$, we say that $e = uu$ is a *loop* and we denote by $l(v)$ the number of loops adjacent to v. The number of incident edges of a vertex v is the *valency* of v (or the *degree* of v) and it is denoted by $\text{val}(v)$ (or $\deg(v)$), where loops based at a vertex are counted twice.

The number of edges adjacent to both $v, w \in V(G)$ is denoted by $(v \cdot w)$, or just

[1]In our figures, vertices with weight zero are represented by "$\circ$", while the ones with weight bigger than zero are represented by "$\bullet$".

$v \cdot w$. We set

$$(v \cdot v) = - \sum_{w \in V \setminus \{v\}} (w \cdot v).$$

If $W, Z \subset V(G)$ we write

$$W \cdot Z = \sum_{w \in W, z \in Z} (w \cdot z).$$

Observe that when $v \notin Z$, we have

$$v \cdot Z = \#\{ \text{ edges joing } v \text{ with a vertex of } Z\},$$

by contrast, if $v \in Z$ we have $v \cdot Z \leq 0$.

The *genus* of a connected graph G is

$$\begin{aligned} g(G) &:= \sum_{v \in V(G)} \omega(v) + |E(G)| - |V(G)| + 1 \\ &= \sum_{v \in V(G)} \omega(v) + b_1(G), \end{aligned}$$

where $b_1(G) := |E(G)| - |V(G)| + 1$ is the *first Betti number* of G.

Example 1.1.1 (Weighted binary graph)**.** The *weighted binary graph*, $B_k^{(\omega_1,\omega_2)}$, consists of two vertices v_1, v_2 joined by k edges, i. e., $v_1 \cdot v_2 = k$, such that $\omega_i = \omega(v_i)$, $i = 1, 2$.

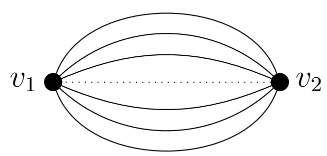


Figure 1.1: The weighted binary graph $B_k^{\omega_1,\omega_2}$.

When $\omega_1 = \omega_2 = 0$, we write $B_k := B_k^{(\omega_1,\omega_2)}$ and say that B_k is a *binary graph*. In this case, the genus of B_k is $g = k - 1$.

A *subgraph* H of G is itself a graph consisting of subsets $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *tree* is a connected undirected graph with no cycles and the edges of a tree are called *branches*. A *spanning tree* T of a connected graph G is a subgraph of G such that T is a tree and contains all vertices of G.

Definition 1.1.2. Let G be a graph, an *exceptional vertex* is a vertex with valency 2 and weight zero. The set of all exceptional vertices of G is denoted by V_G^{exc}.

Definition 1.1.3. Let G be a connected graph of genus g such that $g \geq 2$. We say that G is a *stable* (respectively *semistable*) graph if every vertex of weight zero has valency at least 3 (respectively 2). A semistable graph is *quasistable* is no two exceptional vetices are adjacent.

1.1.1 Divisors on finite graphs

Fixing an ordering $V(G) = \{v_1, \ldots, v_\lambda\}$, we denote by $\text{Div}(G)$ the free $\mathbb{Z}$-module generated by elements of $V(G)$, i.e.,

$$\text{Div}(G) := \left\{ \sum_{i=1}^{\lambda} d_i v_i,\ d_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^{\lambda}.$$

Definition 1.1.4. An element $\underline{d}$ of $\text{Div}(G)$ is called a *divisor* on G. Given a divisor $\underline{d} = \sum_{i=1}^{\lambda} d_i v_i$ its *multidegree* is denoted by $(d_1, \ldots, d_\lambda)$, where $d_i := \underline{d}(v_i)$ is the degree of $\underline{d}$ on the vertex v_i, $i = 1, \ldots, \lambda$.

The *degree* of $\underline{d} = (d_1, \ldots, d_\lambda)$ is the integer $|\underline{d}| := \sum_{i=1}^{\lambda} d_i$. Other notations for the degree of $\underline{d}$ are $\deg(\underline{d})$ and d. A divisor $\underline{d}$ is *effective* if $d_i \geq 0$ for all $v \in V$, we write $\underline{d} \geq 0$. The subset of divisors of degree d is denoted by $\text{Div}^d(G)$ and $\text{Div}_+(G)$ denotes the subset of effective divisors on $\text{Div}(G)$. Furthermore, the subset of effective divisors of degree d is denoted by $\text{Div}_+^d(G)$.

Given $Z \subseteq V(G)$ we write

$$\underline{d}(Z) = \sum_{v \in Z} \underline{d}(v).$$

If $\underline{d}$ is a divisor in a graph G, then given a subgraph G' of G, we denote by $\underline{d}_{G'}$ the divisor in G' such that $\underline{d}_{G'}(v) = \underline{d}(v),\ \forall v \in V(G') \subseteq V(G)$.

The *canonical divisor* in each vertex v of G is defined by

$$\underline{k}_G(v) = \mathrm{val}(v) + 2\omega(v) - 2.$$

Notice that $|\underline{k}_G| = 2g(G) - 2$.

Now, proceeding in analogy with classical geometry, we introduce rational functions on graphs. A rational function f on a graph G is a map $f : V(G) \to \mathbb{Z}$ and the *principal* divisor associated to f is a divisor of the form

$$\mathrm{div}(f) = \sum_{v \in V} \mathrm{ord}_v(f) v,$$

where

$$\mathrm{ord}_v(f) = \sum_{v \neq w} (f(v) - f(w))(v \cdot w).$$

Therefore, if f is constant then its divisor is 0.

Remark 1.1.5. The degree of a principal divisor is zero. Indeed, if f is a rational function on G, we have

$$\begin{aligned} |\mathrm{div}(f)| &= \sum_{v \in V} \left(\sum_{v \neq w} (f(v) - f(w))(v \cdot w) \right) \\ &= \sum_{v \in V} \left(\sum_{v \neq w} f(v)(v \cdot w) \right) - \sum_{v \in V} \left(\sum_{v \neq w} f(w)(v \cdot w) \right). \end{aligned}$$

So, a change of variables on the vertices in the second equality gives that $|\mathrm{div}(f)| = 0$.

By the remark above and noticing that $\mathrm{ord}_v(f) = -\mathrm{ord}_v(-f)$ and $\mathrm{ord}_v(f+g) = \mathrm{ord}(f) + \mathrm{ord}(g)$, we have that the principal divisors form a subgroup of $\mathrm{Div}^0(G)$, denoted by $\mathrm{Prin}(G)$.

We say that two divisors $\underline{d}$ and $\underline{d}'$ are *linearly equivalent* if their difference is a principal divisor, we write $\underline{d} \sim \underline{d}'$. We define

$$\operatorname{Pic}(G) = \operatorname{Div}(G)/\sim .$$

The equivalence class of a divisor $\underline{d}$ is denoted by $[\underline{d}]$. We also use the notation δ for an element of $\operatorname{Pic}(G)$, we write $\underline{d} \in \delta$ if $\underline{d}$ is a representative. Since the principal divisors have degree zero, we have that equivalent divisors have the same degree. We set, for an integer d,

$$\operatorname{Pic}^d(G) = \operatorname{Div}^d(G)/\sim .$$

Given a subset $Z \subset V(G)$, we denote $Z^c := V(G) \setminus Z$ and define the divisor $\underline{t}_Z(v)$ such that

$$\underline{t}_Z(v) := \begin{cases} v \cdot Z & \text{if } v \notin Z \\ -v \cdot Z^c & \text{if } v \in Z. \end{cases}$$

Example 1.1.6. Let G be a graph such that $V(G) = \{v_1, v_2, v_3\}$ with $v_1 \cdot v_2 = 2$ and $v_3 \cdot v_1 = v_3 \cdot v_2 = 1$. Consider the subset $Z = \{v_1, v_2\} \subset V(G)$, we have $\underline{t}_Z = (-1, -1, 2)$. The graph G and the divisors $\underline{t}_{v_1}$, $\underline{t}_{v_2}$ and $\underline{t}_{v_3}$ are pictured as follows:

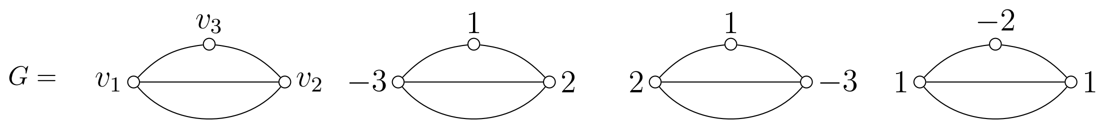


Figure 1.2: The graph G with ordered vertices $\{v_1, v_2, v_3\}$ followed by the divisors $\underline{t}_{v_1}$, $\underline{t}_{v_2}$ and $\underline{t}_{v_3}$ on G.

Remark 1.1.7. The subgroup of principal divisors is generated by divisors of the form $\underline{t}_Z$, for all $Z \subset V(G)$.

Remark 1.1.8. [11, Remark 2.1] Given a $\underline{t} \in \operatorname{Pic}(G)$ there is a decomposition

$$V(G) = Z_0 \sqcup \ldots \sqcup Z_m,$$

with Z_0 and Z_m non-empty, such that

$$\underline{t} = \sum_{i=0}^{m} i\underline{t}_{Z_i}.$$

Indeed, by definition we have $\underline{t} = \underline{t}_{Y_1} + \cdots + \underline{t}_{Y_k}$, where each Y_j is a set of vertices. For each $a \geq 0$, let Y'_a be the set of vertices that are contained in a different such sets (Y'_a may be empty). Then the sets Y'_a are a disjoint cover of V, and $\underline{t} = 0 \cdot \underline{t}_{Y'_0} + \cdots + k \cdot \underline{t}_{Y'_k}$. Let b be the first integer so that Y'_b is non-empty. Since $\sum \underline{t}_{Y'_{i+b}}$, we are done.

This implies that we have $\underline{t}_{|Z_m} \leq (\underline{t}_{|Z_m})_{|Z_m}$. Indeed, pick $v \in Z_m$, we have $\underline{t}_{Z_m}(v) = -Z^c_m \cdot v$; on the other hand

$$\underline{t}(v) = \sum_{i=0}^{m-1} iZ_i \cdot v - mZ^c_m \cdot v \geq (m-1)\sum_{i=0}^{m-1} Z_i \cdot v - mZ^c_m \cdot v = mZ^c_m \cdot v.$$

The equivalence class of a divisor $\underline{d}$ on a weightless graph G can be also described via a solitaire game played on the vertices of G, the *chip-firing* (or the Dollar Game). The goal of this game is to transform, if possible, a given divisor into one that is effective using *chip-firing moves.* There are two types of chip-firing moves: lending moves and borrowing moves.

A *lending move* based at $v \in V(G)$ consists of replacing a divisor $\underline{d}$ by a divisor $\underline{d}'$ in the following way

$$\underline{d}'(v) = \underline{d}(v) - \mathrm{val}(v) \text{ and } \underline{d}'(w) = \underline{d}(w) + v \cdot w,$$

where w is a vertex of G other than v. A *borrowing move* based at $v \in V(G)$ consists of replacing a divisor $\underline{d}$ by a divisor $\underline{d}'$ such that

$$\underline{d}'(v) = \underline{d}(v) + \mathrm{val}(v) \text{ and } \underline{d}'(w) = \underline{d}(w) - v \cdot w,$$

where w is a vertex of G other than v. This way, $\underline{d}$ can be obtained from $\underline{d}'$ by a series of chip-firing moves if and only if $\underline{d} \sim \underline{d}'$. In effect, suppose that $\underline{d} - \underline{d}' = \underline{t}$, with $\underline{t} \in \text{Prin}(G)$. So, by Remark 1.1.8, there is a decomposition $V(G) = Z_0 \sqcup \ldots \sqcup Z_m$ such that $\underline{t} = \sum_{i=0}^{m} i\underline{t}_{Z_i}$. Therefore, firing from the set Z_i i times we obtain the sequence of chip-firing moves. Conversely, $\underline{t}$ is obtained by reconstructing the sets Z_i from chip-firing moves in the obvious manner.

We say that $\underline{d}$ is *winnable* if it is possible to get to an effective configuration starting from $\underline{d}$. Equivalently, $\underline{d}$ is winnable if it is equivalent to an effective divisor.

1.1.2 Reduced divisors and Dhar decomposition

Baker and Norine, in [3], introduced the theory of reduced divisors that we expose in this section.

Definition 1.1.9 (Reduced divisors)**.** Let $\underline{d}$ be a divisor on a weightless and loopless graph G and fix a vertex $u \in V(G)$. We say that $\underline{d}$ is *u-reduced* if

1. $\underline{d}(v) \geq 0$, for all $v \in V(G) \setminus \{u\}$;
2. for every non-empty set $A \subset V(G) \setminus \{u\}$, there exists a vertex $v \in A$ such that $\underline{d}(v) < v \cdot A^c$.

Proposition 1.1.10. *[3, Proposition 3.1] Let G be a weightless loopless graph and fix a vertex $u \in G$. Then for every divisor $\underline{d}$ of G there exists a unique u-reduced divisor $\underline{d}' \in \text{Div}(G)$ on the equivalence class of $\underline{d}$.*

We can define a decomposition on the set of vertices of a graph that gives us a criterion to determine if a divisor is reduced with respect to a given vertex and is very helpful with inductive arguments. This decomposition is called the Dhar decomposition and it does not depend on the loops or the weights of the vertices. Let G be a graph and fix a vertex $u \in V(G)$. Given a divisor $\underline{d} \in \text{Div}(G)$ whose restriction to $V \setminus \{u\}$ is effective, the Dhar decomposition of G associated to $\underline{d}$ with respect to u is denoted by

$$V = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_l \sqcup W.$$

The construction of the decomposition is as follows. Denote $Y_0 := \{u\}$ and $W_0 := V \setminus \{u\}$. If the divisor $\underline{d}+\underline{t}_{W_0}$ is effective then we write $W = W_0$, and the decomposition is $V = Y_0 \sqcup W$. This is the case when $\underline{d}$ is not u-reduced. Otherwise, define

$$Y_1 := \{v \in W_0 : (\underline{d} + \underline{t}_{W_0})(v) < 0\}.$$

Now we iterate the process, defining the sets $Y_0, Y_1, \cdots, Y_j$. Denote $W_j := V \setminus Y_0 \sqcup \ldots \sqcup Y_j$. If the divisor $\underline{d} + \underline{t}_{W_j}$ is effective, then $W = W_j$ and it is done. Otherwise, define

$$Y_{j+1} := \{v \in W_j : (\underline{d} + \underline{t}_{W_j})(v) < 0\}.$$

The graph is finite, so the process will eventually exhaust all the vertices of the graph and stop and then W will be the empty set. This is the case when $\underline{d}$ is u-reduced.

Remark 1.1.11. [11] Given the Dhar decomposition of a graph G associated to a divisor $\underline{d}$ on G with respect to a vertex u of G, vertices in Y_j or W can be characterized as follows

$$v \in Y_j \Leftrightarrow v \notin Y_{j-1} \text{ and } \underline{d}(v) < v \cdot (Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_{j-1}) \text{ for } j = 1, \ldots, l-1.$$

1.1.3 Balanced divisors

Definition 1.1.12. Let G be a semistable graph of genus $g \geq 2$, and let $\underline{d} \in \operatorname{Div}^d G$. Given a subset $Z \subset V(G)$, we define the parameters

$$m_Z(d) := d\,\frac{k_G(Z)}{2g-2} - \frac{(Z \cdot Z^c)}{2} \text{ and } M_Z(d) := d\,\frac{k_G(Z)}{2g-2} + \frac{(Z \cdot Z^c)}{2}.$$

We say that $\underline{d}$ is *semibalanced* if for every $Z \subset V(G)$ the following inequality holds:

$$m_Z(d) \leq \underline{d}(Z) \leq M_Z(d).$$

We say that $\underline{d}$ is *balanced* if it is semibalanced and if for every exceptional vertex v we have $\underline{d}(v) = 1$.

The following definition was adapted by Caporaso and Christ in [10] from the definition at [15].

Definition 1.1.13 (Break divisor)**.** Let G be a graph. A divisor $\underline{d} \in \mathrm{Div}^g(G)$ is called a *break divisor*, if there is a spanning tree T of G such that

$$\underline{d} = \underline{\omega} + \sum_{e_i \in E(G)\setminus E(T),\, v_i \in e_i} (v_i),$$

where $\underline{\omega} := \sum_{v \in V} \omega(v)v$ and by writing $v_i \in e_i$ we mean that for each e_i we choose only one of the end vertices v_i of e_i.

Note that a break divisor is effective by construction. For any spanning tree T we have $|E(G) \setminus E(T)| = b_1(G)$. Thus for a break divisor $\underline{d}$ we have $\deg(\underline{d}) = b_1(G) + \underline{\omega} = g(G)$, so $\underline{d}$ has degree g.

Theorem 1.1.14 ([10])**.** *Let $\delta \in \mathrm{Pic}^g(G)$, then there is a unique $\underline{d} \in \delta$ such that $\underline{d}$ is a break divisor. Therefore the set of break divisors on G is canonically in bijection with* $\mathrm{Pic}^g(G)$.

Proposition 1.1.15 ([10])**.** *Given a divisor $\underline{d} \in \mathrm{Div}^g(G)$, the following are equivalent*

1. *$\underline{d}$ is break divisor.*
2. *$\underline{d}$ is balanced.*

1.2 Moduli spaces of curves and universal compactified Jacobians

1.2.1 Nodal curves

In this work, a curve is a one-dimensional scheme, projective over an algebraically closed field k. But also, unless otherwise specified, the word "curve" stands for a nodal curve, i.e., a reduced (possibly reducible) curve having at most nodes as singularities. For a curve X, we will denote by $g(X)$ the arithmetic genus of X and by ω_X the dualizing sheaf of X.

The *dual graph* G of a curve X is defined such that $V(G)$ is identified with the set of irreducible components of X, $E(G)$ with the set of nodes of X (so that loops correspond to internal nodes of irreducible components) and the value of its weight function on a vertex is the geometric genus of the corresponding component on X (i.e., the genus of its desingularization). We denote by $M^{\text{alg}}(G)$ the set of isomorphism classes of curves having G as dual graph (also referred to as *dual curves* to G). For example, when G has only one vertex, $M^{\text{alg}}(G)$ parametrizes irreducible curves.

We write

$$X = \bigcup_{i=1}^{\lambda} C_i$$

for the irreducible component decomposition of X, and we denote by g_{C_i}, or just g_i, the arithmetic genus of C_i. In some cases, when X is a dual curve to the graph G, it will be more appropriate to write

$$X = \bigcup_{v \in V(G)} C_v,$$

meaning that C_v is the component corresponding to the vertex $v \in V(G)$, and g_v denotes the arithmetic genus of C_v.

Example 1.2.1. Let G be a graph such that $V(G) = \{v_1, v_2\}$ with $\omega(v_1) = 1$, $v_1 \cdot v_2 = 3$ and having one loop at v_2. A curve $X \in M^{\text{alg}}(G)$ has two components C_1 and C_2 such that $C_1 \cap C_2 = \{p_1, p_2, p_3\}$, C_1 has geometric genus 1, and C_2 has one self intersection.

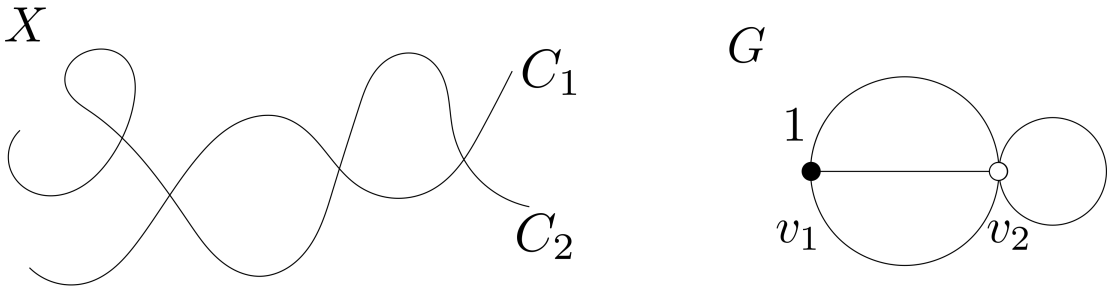


Figure 1.3: The dual curve $X = C_1 \cup C_2$ of the graph G.

Given a proper subcurve Z of X, we denote

$$Z^c := \overline{X \setminus Z}.$$

The Weil divisor $\sum_{n \in Z \cap Z^c} n$ is denoted by $Z \cdot Z^c$, with

$$\delta_Z := \# Z \cap Z^c = \deg Z \cdot Z^c.$$

Recall that the adjunction formula gives

$$w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z.$$

Remark 1.2.2. A curve X has its arithmetic genus equal to the genus of its dual graph.

Let δ be the number of nodes of $X = \bigcup_{v \in V(G)} C_v$, and consider the total normalization of X

$$\nu : X^\nu = \bigsqcup_{v \in V} C_v^\nu \longrightarrow X.$$

The associated map of structure sheaves $\mathcal{O}_X \hookrightarrow \mathcal{O}_{X^\nu}$ yeilds an exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \to \mathcal{N} \tag{1.1}$$

where $\mathcal{N}$ is a skyscraper sheaf supported on the nodes of X. So, (1.1) yields the following exact sequence in cohomology:

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow k^\delta \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X^\nu, \mathcal{O}_{X^\nu}) \longrightarrow 0. \tag{1.2}$$

From (1.2) we obtain

$$g(X) = h^1(X^\nu, \mathcal{O}_{X^\nu}) + \delta - |V| + 1 = \sum_{v \in V} g_v + b_1(G) = g(G).$$

We denote by $\mathrm{Pic}(X)$ the Picard scheme of X parametrizing line bundles on X up to isomorphism. Given a curve X, we denote by K_X the dualizing line bundle on X. For a line bundle $L \in \mathrm{Pic}(X)$, we denote $h^0(X, L) := \dim(H^0(X, L))$ the dimension of the vector space of global sections of L, and

$$r(X, L) := h^0(X, L) - 1.$$

Denote $X = \bigcup_{v \in V(G)} C_v$. The *multidegree* of $L \in \mathrm{Pic}(X)$ is

$$\underline{\deg} L := (\deg_{C_1} L, \ldots, \deg_{C_\lambda} L),$$

and its total degree is

$$\deg L := \sum_{i=1}^{\lambda} \deg_{C_i} L.$$

Notice that $\underline{\deg} L$ can be viewed as a divisor on G setting $\underline{\deg} L(v) = \deg_{C_v} L$. Therefore, there is a surjective group homomorphism

$$\underline{\deg} : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$$

that sends a line bundle to its multidegree. For every curve $X \in M^{\mathrm{alg}}(G)$ we have $\underline{\deg} K_X = \underline{k_G}$.

Given $\underline{d} = (d_1, \dots, d_\lambda) \in \mathbb{Z}^\lambda$, we set

$$\mathrm{Pic}^{\underline{d}}(X) := \{L \in \mathrm{Pic}(X) : \underline{\deg} L = \underline{d}\},$$

the variety of isomorphism classes of line bundles of multidegree $\underline{d}$. If $|\underline{d}| = d_1 + \cdots + d_\lambda$, we have decompositions

$$\mathrm{Pic}(X) = \bigsqcup_{d \in \mathbb{Z}} \mathrm{Pic}^d(X) \quad \text{and} \quad \mathrm{Pic}^d(X) = \bigsqcup_{\substack{|\underline{d}|=d \\ \underline{d} \in \mathrm{Div}(G)}} \mathrm{Pic}^{\underline{d}}(X).$$

Let $\nu : Y \to X$ be some partial normalization, so $Y = \bigsqcup_{i=1}^{\lambda_Y} Y_i$. Let R be the set of nodes normalized by $\nu : Y \to X$ and let $\delta_R = \#R$, and for each node n_i let $\{p_i, q_i\} = \nu^{-1}(n_i)$ be its branches, $i = 1, \dots, \delta_R$. Consider the pull-back map $\nu^* : \mathrm{Pic}\, X \to \mathrm{Pic}\, Y$, we have

$$\mathrm{Pic}\, Y = \prod_{i=1}^{\lambda_Y} \mathrm{Pic}\, Y_i.$$

For every $L' \in \mathrm{Pic}\, Y$ we denote the fiber of ν^* over L' by

$$F_{L'}(X) := \{L \in \mathrm{Pic}\, X : \nu^* L = L'\}.$$

Lemma 1.2.3 (cf. [6])**.** *Using the notation above, we have the isomorphism*

$$F_{L'}(X) \cong (k^*)^{\delta_R - \lambda_Y + 1}.$$

Proof. First, assume that Y is connected. Let $\underline{c} = (c_1, \dots, c_{\delta_R}) \in (k^*)^{\delta_R}$; $\underline{c}$ determines a unique $L \in \mathrm{Pic}\, X$ such that $\nu^* L = L'$. Indeed, for every $j = 1, \dots, \delta_R$ consider the two fibres of L' over p_j and q_j and fix an isomorphism between them. We define $L = L^{\underline{c}}$ on the curve X. $L^{\underline{c}}$ pulls back to L' by gluing L'_{p_j} to L'_{q_j} via the isomorphism $L'_{p_j} \to L'_{q_j}$ given by multiplication by c_j. Conversely, every $L \in F_{L'}(X)$ is of type $L^{\underline{c}}$.

Now, consider the case where Y has λ_Y connect components. The curve X is connected, so we have $\lambda_Y - 1 \leq \delta_R$. There exist some subsets $T \subset \{n_1, \ldots, n_{\delta_R}\}$ such that $\#T = \lambda_Y - 1$ and such that if we remove from the dual graph of X every node that is not in T, the remaining graph is a spanning tree of the dual graph of X.

Fix T and, if necessary, reorder the nodes in $\{n_1, \ldots, n_{\delta_R}\}$ so that

$$\{n_1, \ldots, n_{\delta_R}\} = \{n_1, \ldots, n_{\delta_R - \lambda_Y + 1}\} \cup T.$$

Now, consider the factorization $\nu : Y \xrightarrow{\nu'} Y' \xrightarrow{\nu_T} X$ of ν so that ν' is the partial normalization of X at $\{n_1, \ldots, n_{\delta_R}\} \setminus T$ and ν_T is the normalization at the nodes of Y preimages of the nodes in T. Finally, to construct the finer of $\operatorname{Pic} X \to \operatorname{Pic} Y$ over $M \in \operatorname{Pic} Y$ we proceed as in the previous part.

$\square$

The above lemma gives us that for every $\underline{c} \in (k*)^{\delta_R - \lambda_Y + 1}$ we associate a unique $L^{\underline{c}} \in \operatorname{Pic} Y$. Remark that a section $s \in H^0(Y, M)$ descends to a section $\bar{s} \in H^0(X, L^{\underline{c}})$ if and only if for every $j = 1, \ldots, \delta_R$ we have $s(q_j) = c_j(p_j)$.

1.2.2 Stable curves

Given a nodal curve X, a smooth component E of X is *exceptional* if it is rational and $|E \cap (\overline{X \setminus E})| = 2$. An *exceptional* node is a node that lies on an exceptional component.

Definition 1.2.4. A curve X is called

1. *stable* if every irreducible component C_v of arithmetic genus zero satisfies $|C_v \cap C_u^c| \geq 3$;

2. *semistable* if every irreducible component C_v of arithmetic genus zero satisfies $|C_v \cap C_u^c| \geq 2$;

3. *quasistable* if it is semi-stable and no two excepcional components intersect each other.

Observe that X is stable (resp. semistable, resp. quasistable) if and only if its dual graph G is stable (resp. semistable, resp. quasistable).

Let $\widehat{X}$ be a quasistable curve, there is a unique morphism $\varsigma : \widehat{X} \to st(\widehat{X})$ contracting all the exceptional components of $\widehat{X}$. Observe that the curve $st(\widehat{X})$ is stable. Now, let X be a stable curve and let R be a set of nodes of X, with $\epsilon_R = \#R$. Denote by $\widehat{X}_R$ the blow up of Y at R. The curve $\widehat{X}_R$ is quasitable such that $st(\widehat{X}_R) = X$, where the contracting morphism $\varsigma : \widehat{X}_R \to st(\widehat{X}_R)$ is defined by contracting all the exceptional components of $\widehat{X}_R$ to R. We have the normalization map $\nu : X_R^\nu \to X$, where X_R^ν is the normalization of X at R such that

$$\widehat{X}_R = X_R^\nu \cup (\cup_{r \in R} E_r),$$

with E_r an exceptional component, for $r \in R$.

A *family* of stable (resp. semistable, resp.quasistable) curves is a flat projective morphism $f : X \to B$ whose geometrical fibers are stable (resp. semistable, resp. quasistable) curves. A line bundle of degree d on $f : X \to B$ is a line bundle on X whose restriction to each geometric fiber has degree d.

1.2.3 Linear series on nodal curves

In the following, we present some results concerning global sections on a nodal curve X and how to glue them.

Remark 1.2.5 (cf. [8]). Let R be a set of nodes of X, with $\delta_R = \#R$. Given $L' \in$

$\mathrm{Pic}(X_R^\nu)$, let $L \in F_{L'}(X)$, i.e., $\nu^* L = L'$, then we have the exact sequence

$$0 \longrightarrow L \longrightarrow \nu_* L' \longrightarrow \sum_{n \in R} k_n \longrightarrow 0.$$

Now, consider the associated long exact sequence in cohomology

$$0 \longrightarrow H^0(X, L) \xrightarrow{\alpha} H^0(X_R^\nu, L') \xrightarrow{\beta} k^{\epsilon_R} \longrightarrow H^1(X, L) \longrightarrow H^1(X_R^\nu, L') \longrightarrow 0.$$

From the sequence above we get the bound

$$h^0(X, L) \leq h^0(X_R^\nu, L').$$

Fix $L' \in \mathrm{Pic}\, X_R^\nu$, thus, by Lemma 1.2.3, every $L \in F_{L'}(X)$ is of the form $L^{(\underline{c})}$ for some $\underline{c} \in (k^*)^{\epsilon - \gamma + 1}$. For convenience, we set $c_j = 1$, for $\epsilon - \gamma + 1 \leq j \leq \epsilon$. Let $h^0(X_R^\nu, L') = m$, and consider a basis $s_1, \ldots, s_m$ for $H^0(X_R^\nu, L')$. Let $s \in H^0(X_R^\nu, L')$ be such that $s = \sum_{i=1}^m x_i s_i$ where $x_i \in k$. Now s lies in the image of α if and only if

$$\sum_{i=1}^{m} x_i (s_i(q_2^j) - c_j s_i(q_1^j)) = 0, \ \forall j = 1, \ldots, \epsilon.$$

So we have above a linear system of ϵ homogeneous equations in m unknowns $x_1, \ldots, x_m$. The space of its solutions is identified with $H^0(X, L^{(\underline{c})})$. The space of solutions of the system is a subspace of $H^0(X_R^\nu, L')$ of dimension at least $m - \epsilon$. Hence $h^0(X, L) \geq m - \epsilon$. So we established that

$$h^0(X_R^\nu, L') - \epsilon_R \leq h^0(X, L) \leq h^0(X_R^\nu, L'). \tag{1.3}$$

In the following, we investigate the cases when the equality $h^0(X, L) = h^0(X_R^\nu, L')$ holds.

Fact 1. Given a curve X, a point $p \in X$ and a line bundle $L \in \operatorname{Pic}(X)$, then

$$h^0(X, L) - h^0(X, L(-p)) = 0 \text{ or } 1.$$

If $h^0(X, L) - h^0(X, L(-p)) = 1$, for all points $p \in X$ then L is basepoint-free.

Definition 1.2.6. [8, Definition 1.1] Let Y be a curve, $M \in \operatorname{Pic} Y$ and p and q nonsingular points of Y. We say that p and q are a *neutral pair* of M, and write $p \sim_M q$, if

$$h^0(Y, M(-p)) = h^0(Y, M(-q)) = h^0(Y, M(-p-q)).$$

The relation $p \sim_M q$ is an equivalence relation.

Lemma 1.2.7. *[8, Lemma 1.4] Let Y be a nodal curve, p and q be two nonsingular points of Y and $Y \to X = Y/_{\{p=q\}}$. Let $M \in \operatorname{Pic} Y$ be such that $h^0(Y, M) \neq 0$. There exists $L \in \operatorname{Pic} X$, $\nu^* L = M$, such that*

$$h^0(X, L) = h^0(Y, M)$$

if and only if $p \sim_M q$. If Y is connected, such L is unique (if it exists) if and only if p and q are not base points for M.

Fact 2. [6, Corollary 2.2.5] Let X be the dual curve of G, and assume $\underline{d} = \underline{0}$. Then for every $L \in \operatorname{Pic}^{\underline{0}} X$ we have $h^0(X, L) \leq 1$, and equality holds if and only if $L = \mathcal{O}_X$.

Definition 1.2.8. Given $L \in \operatorname{Pic}(X)$, L is *admissible* if, for every exceptional component E of X, we have $\deg L|_E = 1$.

Lemma 1.2.9 ([17]; [6], Lemma 4.2.5)**.** *Given $R \subset X_{sing}$ and $L^\nu \in \operatorname{Pic} X^\nu_R$. Pick an admissible $\widehat{L} \in \operatorname{Pic} \widehat{X}_R$ such that $\widehat{L}_{|X^\nu_R} = L^\nu$. Then*

$$h^0(\widehat{X}_R, \widehat{L}) = h^0(X^\nu_R, L^\nu).$$

In other words, if $L \in \operatorname{Pic}(\widehat{X}_R)$ is admissible and $i : X^\nu_R \hookrightarrow \widehat{X}_R$ an inclusion map, we have

$$h^0(\widehat{X}_R, L) = h^0(X^\nu_R, L'),$$

for a given $L' = i^*L$.

Proof. Once $\widehat{L}$ is admissible we have $\widehat{L}_{|E} = \mathcal{O}_E(1)$, for every exceptional component E of $\widehat{X}_R$. We know that every section of $H^0(\widehat{X}_R, \widehat{L})$ restricts to a section of L^ν. Conversely, given a pair of points $p_1, p_2 \in \mathbb{P}^1$, consider a trivialization of $\mathcal{O}_{\mathbb{P}^1}(1)$ locally at such points. Now, for any pair $b_1, b_2 \in k$, there exists a unique section $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $s(p_1) = b_i$, for $i = 1, 2$. Therefore, for every section $s_R \in H^0(X^\nu_R, L^\nu)$ extends to a unique section of $H^0(\widehat{X}_R, \widehat{L})$ determined by s_R and by gluing data defining $\widehat{L}$. $\square$

Definition 1.2.10. Let X be a curve of genus g such that X has two smooth rational components intersecting in $g+1$ points. We write $V(G) = \{v_1, v_2\}$ and $X = C_1 \cup C_2$, with $C_i = C_{v_i} \cong \mathbb{P}^1$. Therefore, the dual graph G of X is a binary graph of genus g. We say X is *binary curve*. Recall that $v_1 \cdot v_2 = g + 1$.

Let X be a semistable curve of genus $g \geq 3$ and let $L \in \operatorname{Pic}^{\underline{d}} X$, for a certain multidegree $\underline{d}$. L is called *semibalanced* (respec. *balanced*) if its multidegree is semibalanced (balanced), as defined in Definition 1.1.12. The set of balanced line bundles of degree d on X will be denoted by B^d_X. We say that L (or its multidegree) is *stably balanced* if it is balanced and if for each connected proper subcurve Z of X such that $|\underline{d}_Z| = m_Z(\underline{d})$, Z^c is a union of exceptional components. The set of stably balanced line bundles of degree d on X will be denoted by $\widetilde{B}^d_X$.

Proposition 1.2.11. *[5, Proposition 4.12] Fix $d \in \mathbb{Z}$ and $g \geq 2$.*

1. *Let X be a quasistable curve of genus g and δ multidegree class on X of degree d. Then δ admits a semibalanced representative.*

2. *A balanced multidegree is unique in its equivalence class if and only if it is stably balanced.*

3. *$(d-g+1, 2g-2) = 1$ if and only if for every quasistable curve X of genus g and every $\delta \in \mathrm{Pic}^d(G)$, δ has a unique semibalanced representative.*

Remark 1.2.12. Let X be a quasistable curve and let $L \in \mathrm{Pic}\, X$ be balanced. Then, once L is admissible, by Lemma 1.2.9 we have

$$h^0(X, L) = h^0(X^\nu, L^\nu)$$

for a given $L^\nu = i^*L$, where $i : X^\nu \hookrightarrow X$ is an inclusion map.

The follow result is the Balanced Riemann-Roch, the extension of Riemann's theorem for singular curves in the case where the divisor is balanced.

Theorem 1.2.13. *[8, Theorem 2.3] Let G be a semistable graph of genus $g \geq 2$ and $\underline{d} \in \mathrm{Div}(G)$ a balanced divisor. Given a curve $X \in M^{\mathrm{alg}}(G)$ and a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$, if $d \geq 2g-1$, then*

$$r(X, L) = d - g.$$

Let X be a smooth curve and $L \in \mathrm{Pic}^d(X)$ a line bundle on X with $0 \leq d \leq 2g-2$, then one has the following inequality

$$r(X, L) \leq \frac{d}{2},$$

called *Clifford's inequality.* The equality holds if and only if L is the trivial line bundle, the canonical line bundle or a multiple of a hyper elliptic line bundle.

This is a classical result, but this statement fails for line bundles on nodal curves. As a matter of fact, for any reducible stable curve X and any $0 \leq d \leq 2g-2$ there exist infinitely many $\underline{d}$ of degree d such that for every line bundle L with $\deg(L) = \underline{d}$ one has $r(X, L) > \frac{d}{2}$.

The next result is the Uniform Clifford, it is an extension of the Clifford's theorem for all curves of all degrees, with the additional uniform condition on the degree of each irreducible components of the curve.

Theorem 1.2.14. *[8, Theorem 3.1] Let $X = \cup_{i=1}^{\lambda} C_i$ be a connected curve of genus $g \geq 2$ e let $\underline{d} = (d_1, \dots, d_\lambda) \in \mathrm{Div}^d(X)$ be such that $0 \leq d_i \leq 2g_i$, for every $i = 1, \dots, \lambda$. Then $d \leq 2g$ and for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have*

$$r(X, L) \leq \frac{d}{2}.$$

The following result establishes that, for every balanced multidegree, Clifford's inequality holds for semistable curves with two irreducible components.

Theorem 1.2.15. *[8, Theorem 3.3] Let G be a semistable weighted binary graph of genus $g \geq 2$. Let $0 \leq d \leq 2g$ and $\underline{d} \in \mathrm{Div}^d(G)$ be a balanced divisor. Given $X \in M^{\mathrm{alg}}(G)$ we have that for all $L \in \mathrm{Pic}^{\underline{d}}(X)$,*

$$r(X, L) \leq \frac{d}{2}.$$

1.2.4 Moduli space of curves of genus g

We will denote by M_g the moduli space of smooth curves of genus g. It coarsely represents the functor that associates to a base scheme at families of smooth curves of genus g. The space M_g is a quasiprojective variety of dimension $3g-3$ but it is not proper. It is a classical result of Deligne and Mumford, [12], that M_g can be compactified in a modular way by stable curves. We will denote by $\overline{M_g}$ this compactification, the so called Deligne-Mumford compactification. It parametrizes isomorphism classes of stable curves of genus g, see Section 1.2.2. We denote by $\overline{M}_g^0$ the locus of curves with trivial automorphism group. As proved by Deline and Mumford, $\overline{M}_g$ is a projective variety of dimension $3g-3$ containing M_g as a dense open subvariety.

1.2.5 Compactified Jacobian

Generalized Jacobian of a nodal curve

Let X be a nodal curve, the *generalized Jacobian* of X, denoted by $J(X)$, is a connected algebraic variety identified with $\mathrm{Pic}^{\underline{0}}(X)$, parametrizing line bundles of degree zero on each irreducible component of X. We have

$$J(X) = \mathrm{Pic}^{\underline{0}}(X) = \{L \in \mathrm{Pic}(X) : \underline{\deg} L = (0, \ldots, 0)\}.$$

When every node of X is a separating node, the dual graph of X is a tree. In this case, the pullback map induces an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}\, X_R^\nu$. Such X is said to be a curve of *compact type.*

Remark 1.2.16. $\mathrm{Pic}^{\underline{0}}(X)$ is compact if and only if X is a curve of compact type. In fact, we have the following short exact sequence

$$0 \to (k^*)^{b_1(G)} \to \mathrm{Pic}^{\underline{0}}(X) \xrightarrow{\alpha} \mathrm{Pic}^{\underline{0}}(X_E^\nu) \to 0.$$

The map α is given by taking the pullback of a line bundle along the total normalization $\nu : X_E^\nu \to X$, and $\ker(\alpha)$ consists of the different ways to glue the structural sheaf of X_E^ν over the nodes of X and one checks that up to isomorphism this amount to giving $b_1(G)$ elements of k^*. We have that

$$\mathrm{Pic}^{\underline{0}}(X_E^\nu) = \prod_{i=1}^{\gamma} \mathrm{Pic}^{\underline{0}}(C_i^\nu),$$

so it is compact because it is the product of Picard schemes of smooth curves. On the other hand, the torus $(k^*)^{b_1(G)}$ is not compact if $b_1(G) \neq 0$. Thus $J(X)$ is compact if, and only if, $b_1(G) = 0$, i.e., if G is a tree.

If X is smooth then $\mathrm{Pic}^d(X)$ is isomorphic to an abelian variety. If X is singular, $\mathrm{Pic}^d(X)$ may not be projective.

A compactification of the Universal Jacobian

As mentioned before, the Jacobian of a singular curve is in general not compact. The problem of compactifying the Jacobian of singular curves is natural and therefore there are many different approaches to solve the problem. The question has been considered already in pionnering work of Mayer-Mumford and of Igusa in the 50's, and has been developed later in different generalities by D'Souza, Oda-Seshadri, Altmann-Kleiman, Caporaso, Pandharipande, Simpson, Esteves, etc. We refer the reader to the introduction of Esteves in [13] for an account on the different constructions.

In the present work we will follow the approach of Caporaso in [7]. In loc. cit., the author constructs a modular compactification of the universal Picard variety over the moduli space of stable curves $\overline{M}_g$. The construction is done via a GIT quotient of a suitable Hilbert scheme and therefore consists of a projective variety $\overline{J}_{d,g}$ with a proper map onto $\overline{M}_g$. The fibers of this map over a stable curve X therefore yield compactifications for the Jacobian variety of given degree over X.

The degree d compactified Picard scheme of X, denoted by $\overline{J}^d_X$, is constructed as the GIT-quotient of a certain scheme V_X containing only GIT semistable points, by a certain group G, so that there is a quotient morphism

$$V_X \to \overline{J}^d_X = V_X/G.$$

Equivalence classes of strictly balanced line bundles of degree d on quasistable curves of X correspond to the closed (and GIT-semistable) orbits of G on V_X.

The smooth locus of $\overline{J}_X^d$ consists of the disjoint union of the $\mathrm{Pic}^{\underline{d}}(X)$ with $\underline{d}$ strictly balanced. Points in $\overline{J}_X^d$ consist of line bundles in quasistable modifications of X. More precisely, given a stable curve X, points in $\overline{J}_X^d$ correspond to balanced line bundles on quasistable curves Y having X as stable model. In [7] it is established that for a stable curve X, the compactified Jacobian $\overline{J}_X^d$ is a coarse moduli space for equivalence classes of stably balanced line bundles on quasistable curves having X as stable model.

We recall from [7] that the compactified Jacobians $\overline{J}_X^d$ glue over $\overline{M}_g$ in the sense that there is a proper scheme and a projective morphism

$$\psi_{g,b} : \overline{J}_{d,g} \to \overline{M}_g$$

whose fiber over $[X] \in \overline{M}_g$ is $\overline{J}_X^d / Aut(X)$.

Stratification by partially ordered sets

Definition 1.2.17. Given a set $\mathcal{P}$, we say that the pair $(\mathcal{P}, \leq)$ is *partially ordered set* (*poset* for short) if $\leq$ is a binary relation on $\mathcal{P}$ that is reflexive, antisymmetric and transitive. For $p_1, p_2 \in \mathcal{P}$, we will say p_2 covers p_1 if $p_1 \leq p_2$, $p_1 \neq p_2$ (denoting $p_1 < p_2$) and there is no $p \in \mathcal{P}$ such that $p_1 < p < p_2$.

The *dual poset* $\mathcal{P}^* = (\mathcal{P}, \leq^*)$ is set to be the partial order defined on $\mathcal{P}$ by inverting the order given by $\leq$. A *morphism of posets* is a map $\mu : (\mathcal{P}, \leq) \to (\mathcal{P}', \leq')$ that respects the partial ordering. An *isomorphism of posets* is a bijective morphism of posets whose inverse is also a morphism of posets.

A rank function on a poset $\mathcal{P}$ is a morphism of posets $\rho : \mathcal{P} \to \mathbb{N}$ such that if p_1 covers p_2 in $\mathcal{P}$ then $\rho(p_2)$ covers $\rho(p_1)$ in $\mathbb{N}$. A poset together with a rank function is called a*graded poset.*

Suppose we have a topological space P together with a decomposition $P = P_1 \sqcup \ldots \sqcup P_n$ into disjoint, locally closed subspaces P_i. We will denote by $P^{strat} = \{P_1, \ldots, P_n\}$ the *set of strata* and endow it with a partial order $\leq$ by setting $P_i \leq P_j$ if $P_i \subset \overline{P}_j$. This gives a poset. The decomposition $P = P_1 \sqcup \ldots \sqcup P_n$ of P is called a *stratification* by a partially ordered set $\mathcal{P}$, if the following hold:

1. P_i is locally closed, for all $i = 1, \ldots, n$.

2. If $P_i \cap \overline{P_j} \neq \emptyset$ then $P_i \leq P_j$ in P^{strat}.

3. There is an isomorphism of posets $s : P^{strat} \to \mathcal{P}$.

We say that the M_i are the *strata* of this stratification if P is stratified by $\mathcal{P}$.

A stratification $s : P^{strat} \to \mathcal{P}$ of a topological space P by $\mathcal{P}$ is called a *graded stratification* if the P_i are equidimensional and

$$\begin{array}{rccc} \rho : & \mathcal{P} & \to & \mathbb{N} \\ & p & \mapsto & \dim(s^{-1}(p)) \end{array} ,$$

is a rank function for $\mathcal{P}$. If P and the P_i are algebraic varieties and the P_i are irreducible, viewed as endowed with the Zariski topology, the stratification is called *algebraic*.

The stratification of $\overline{M}_g$ mainly describes the structure of the boundary of a compactification, in this case $\overline{M}_g \setminus M_g$. That is to say, the whole open subset M_g forms a single maximal dimensional stratum.

We denoted by $\mathcal{SG}_g$ the set of stable graphs of genus g.

Definition 1.2.18. We define a partial order on $\mathcal{SG}_g$ by setting $G \leq H$ if for some $S \subset E(G)$, there is a contracting map $G \to H$ such that contracting S in G gives H.

Define $\rho_{\mathcal{SG}_g}(G) = 3g - |E(G)| - 3$, a rank function on $\mathcal{SG}_g$ preserving the covering relation. H has to be obtained from G by contracting a single edge once H covers G.

Recall that

$$M^{\text{alg}}(G) = \{[X] \in \overline{M_g} : G \text{ is the dual graph of X}\},$$

which provides the decomposition

$$\overline{M_g} = \bigsqcup_{G \in \mathcal{SG}_g} M^{\text{alg}}(G).$$

From [6], we have that the above decomposition is an algebraic stratification by the graded poset $\mathcal{SG}_g$ where $\overline{M_g}^{strat} \to \mathcal{SG}_g$ is given by $M^{\text{alg}}(G) \to G$.

The strata of compactified Jacobians

Let $\mathcal{QS}_g$ be the category of quasistable curves. Recall that if X is a quasistable curve and R is a set of nodes of X, then, considering the contracting morphism $\varsigma : \widehat{X}_R \to \varsigma(X)$ contracting all exceptional components of X, the image of ς is a stable curve denoted by $st(\widehat{X}_R)$. Conversely, if X is stable, then $\widehat{X}_R$, the blow up on X at R, is a quasistable curve such that $X = st(\widehat{X}_R)$ and we have $\widehat{X}_R = X_R^\nu \cup (\cup_{r\in R} E_r)$, with E_r an exceptional component, for $r \in R$.

The dual graph of the quasistable curve $\widehat{X}_R$ is denoted by $\widehat{G}_R$, and it is defined by adding one vertex v_r on each edge r of $R \subseteq E(G)$. These new vertices v_r are exceptional vertices. Denote by e_r the edges on $\widehat{G}_R$ adjacent to the exceptional vertex v_r and let $S = \{e_r | r \in R\}$. In this case, there is a contraction

$$\sigma : \widehat{G}_R \to G = \widehat{G}_R / S$$

contracting edges of S, this contraction is not unique and depends only on the choice of the edge e_r for each $r \in R$. Given a divisor $\underline{d} \in \text{Div}(G)$, define the divisor $\widehat{\underline{d}}$ by setting $\widehat{\underline{d}} = \underline{d}(v)$ if $v \in \text{Div}(G)$, and $\widehat{\underline{d}}(v) = 1$ if v is an exceptional vertex on $\widehat{G}_R$.

Denote by $\mathcal{QG}_{g,d}$ the set of elements $(G, \underline{d})$ such that G is a quasistable graph G and $\underline{d} \in \text{Div}(G)$ is a balanced divisor of degree d.

Definition 1.2.19. We define a partial order on $\mathcal{QG}_{g,d}$ by setting $(G', \underline{d}) \geq (G, \underline{d})$ if there exists $R \in E(G')$ such that $(G', \underline{d}')$ is an exceptional weighted contraction of $(G, \underline{d})$ on R, we have

1. The contracting map $\sigma_R : G' \to G$ contracting every edge in R.
2. The morphism $\lambda : \text{Div}(G') \to \text{Div}(G)$ induced by σ_R, where $\lambda_*(\underline{d})(v) = \sum_{u \in \lambda^{-1}(v)} \underline{d}(u)$.

Denote by $J^{\underline{d}_S}$ the subset of stably balanced line bundles on $\widehat{X}_S$ whose restriction to X_S^ν has multidegree $\underline{d}_S$. Equivalently, these are the divisors in G such that $\underline{d}_S$ is strictly balanced on $G \setminus S$. By [7], we have the following decomposition

$$\overline{J}_X^g = \sqcup J^{\underline{d}_S}.$$

Chapter 2

Ranks of divisors on graphs

2.1 The combinatorial rank

This section is dedicated to study the combinatorial invariant in an equivalence class of a divisor defined for weightless and loopless graphs by Baker and Norine, [3], and extended for graphs with weights and loops by Caporaso, [9], the combinatorial rank.

Definition 2.1.1. Let G be a loopless, weightless graph, the *combinatorial rank* is defined by

$$r_G(\underline{d}) = \max\{k : \forall \underline{e} \in \mathrm{Div}_+^k(G) \ \ \exists \ \ \underline{d'} \sim \underline{d} \text{ such that } \underline{d'} - \underline{e} \geq 0\}$$

with $r_G(\underline{d}) = -1$ if the set is empty.

Given a graph G with weights and with loops, consider the graph $G^\bullet$ defined by attaching at each vertex $v \in V(G)$ a number $\omega(v)$ of loops based at v, and then at each loop adding one new vertex in the middle of the edge. The graph $G^\bullet$ is weightless and loopless, and the vertices of G are vertices of $G^\bullet$.

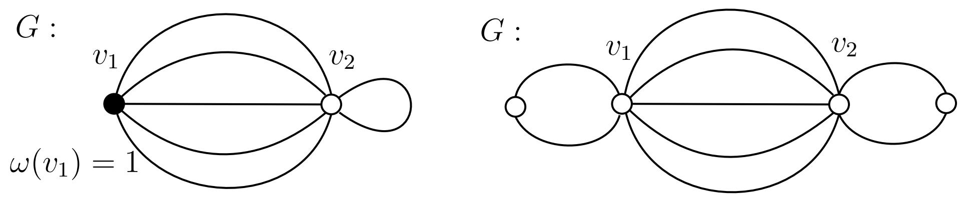


Figure 2.1: The graphs G and $G^\bullet$.

For each $\underline{d} \in \mathrm{Div}(G)$ we can define $\underline{d}^\bullet \in \mathrm{Div}(G^\bullet)$ such that

$$\underline{d}^\bullet(v) = \begin{cases} \underline{d}(v) & \text{if } v \in \mathrm{Div}(G) \\ 0 & \text{otherwise, i.e., if } v \text{ is a new vertex of } G^\bullet. \end{cases}$$

There is a natural injective homomorphism

$$\begin{array}{rccc} \iota: & V(G) & \to & \mathrm{Div}(G^\bullet) \\ & \underline{d} & \mapsto & \underline{d}^\bullet \end{array},$$

which induces an injective homomorphism $\mathrm{Pic}(G) \hookrightarrow \mathrm{Pic}(G^\bullet)$. Therefore, principal divisors on G are principal divisors on $G^\bullet$. For each $v \in V(G)$, set

$$g(v) := \omega(v) + l(v),$$

where $l(v)$ is the number of loops adjacent to v, and denote by $z_v^1, \ldots, z_v^{g(v)}$ the vertices in $V(G^\bullet) \setminus V(G)$ adjacent to v, and by R_v the complete subgraph of $G^\bullet$ whose vertices are $\{v, z_v^1, \ldots, z_v^{g(v)}\}$.

Remark 2.1.2. We can extend Definition 2.1.1 to a weighted graph with loops G, by considering

$$r_G(\underline{d}) := r_{G^\bullet}(\iota(\underline{d})),$$

where $G^\bullet$ is weightless and loopless graph obtained from G by gluing to each vertex $v \in V(G)$ a number of loops equal to $\omega(v)$, then inserting a vertex in every loop, and

ι is the inclusion $\mathrm{Div}(G) \hookrightarrow \mathrm{Div}(G^\bullet)$ defined above.

Remark 2.1.3. The combinatorial rank is constant in an equivalence class.

In the light of the remark above, we can apply the following notation

$$r_G(\delta) = r_G(\underline{d})$$

if $\underline{d}$ is a representative of the divisor class $\delta = [\underline{d}]$.

Remark 2.1.4. For any divisor $\underline{d} \in \mathrm{Div}^d(G)$ we have

$$-1 \leq r_G(\underline{d}) \leq \max\{-1, d\}.$$

Lemma 2.1.5 ([3], Lemma 2.1)**.** *Given $\underline{d}, \underline{d}' \in \mathrm{Div}(G)$ with non-negative combinatorial ranks, we have*

$$r_G(\underline{d}) + r_G(\underline{d}') \leq r_G(\underline{d} + \underline{d}').$$

Proposition 2.1.6. *[11, Lemma 3.8] Let δ be a divisor class on a graph G.*

1. *$r_G(\delta) = -1$ if and only there exists a vertex u whose u-reduced representative $\underline{d} \in \delta$ has $\underline{d}(u) \leq 0$.*

2. *$r_G(\delta) = 0$ if and only if there exists a vertex u whose u-reduced representative $\underline{d} \in \delta$ has $\underline{d}(u) = 0$.*

2.1.1 Riemann-Roch for graphs

The Riemann-Roch for weightless and loopless graphs was proved by Baker and Norine in [[3], Theorem 1.12] and provides a discrete, graph-theoretic version of the classical algebraic Riemann-Roch theorem. In the proof of this result they used the notion of reducedness of a divisor with respect to a vertex of the graph as an important tool. The generalization of this result for all graphs is due to Amini and Caporaso, [1].

Theorem 2.1.7 (Riemann-Roch for graphs)**.** *Given G a loopless, weightless graph, for every divisor $\underline{d} \in \mathrm{Div}(G)$ we have*

$$r_G(\underline{d}) - r_G(\underline{k}_G - \underline{d}) = d - g + 1.$$

Corollary 2.1.8. *[9] Let G be a graph of genus g and let $\underline{d} \in \mathrm{Div}^d(G)$. Then the following facts hold.*

(a) *If $d = 0$, then $r_G(\underline{d}) \leq 0$, and the equality occurs if and only if $\underline{d} \sim \underline{0}$.*

(b) *If $d = 2g - 2$, then $r_G(\underline{d}) \leq g - 1$ and the equality happens if and only if $\underline{d} \sim \underline{k}_G$.*

(c) *If $d < 0$, then $r_G(\underline{d}) = -1$*

(d) *If $d > 2g - 2$, then $r_G(\underline{d}) = d - g$*

The following result is Corollary 3.5 in [3].

Lemma 2.1.9 (Clifford's theorem for graphs)**.** *Suppose that $\underline{d} \in \mathrm{Div}^d(G)$, for any $0 \leq d \leq 2g - 2$, we have*

$$r_G(\underline{d}) \leq \frac{d}{2}.$$

Proof. By Lemma 2.1.5 we have

$$r_G(\underline{d}) + r_G(\underline{k}_G - \underline{d}) \leq r_G(\underline{k}_G) = g - 1.$$

By Theorem 2.1.7 we have

$$2r_G(\underline{d}) \leq d.$$

Proving the lemma for a weightless loopless graph, the extension follows considering the graph $G^\bullet$. □

Definition 2.1.10. Let G be a graph and let $\underline{e}$ be an effective divisor of G. We define the effective divisor $\underline{e}^{\deg}$ on G so that for every $v \in V$ we have

$$\underline{e}^{\deg}(v) = \underline{e}(v) + \min\{\underline{e}(v), g(v)\}.$$

Remark that if G is a weightless and loopless graph then $\underline{e}^{\deg} = \underline{e}$.

The following lemma is a useful tool to find a bound for the combinatorial rank.

Lemma 2.1.11. *[11, Lemma 3.3] Let G be a graph and let $\underline{d} \in \operatorname{Div}(G)$. If for every effective divisor $\underline{e}$ of degree s the divisor $\underline{d} - \underline{e}^{\deg}$ is equivalent to an effective divisor, then $r_G(\underline{d}) \geq s$.*

The following notation was introduced in [11] and helps us to calculate the combinatorial rank of divisors on weighted graphs with loops.

Definition 2.1.12. Let G be a graph and let $\underline{d} \in \operatorname{Div}^+(G)$. We define the effective divisor $\underline{d}_{rk}$ by

$$\underline{d}_{rk}(v) = \underline{d}(v)_{g(v)} := \max\left\{\underline{d}(v) - g(v), \left\lfloor \frac{\underline{d}(v)}{2} \right\rfloor\right\},$$

for every $v \in V(G)$. When G is weightless and loopless, we have $\underline{d}_{rk}(u) = \underline{d}(u)$, for all divisor $\underline{d}$ on G.

For any divisor $\underline{d}$ we define

$$\ell_G(\underline{d}) := \begin{cases} \min\{\underline{d}_{rk}(v), \forall v \in V(G), \} & \text{if } \underline{d} \geq 0 \\ -1, & \text{otherwise.} \end{cases}$$

Proposition 2.1.13. *[11, Proposition 3.10] Let G be a weightless, loopless graph. Let $\underline{d} \in \operatorname{Div}(G)$ be such that for some $u \in V$ with $\underline{d}(u) = \ell_G(\underline{d})$ we have that $\underline{d}$ is u-reduced. Then $r_G(\underline{d}) = \ell_G(\underline{d})$.*

Proof. Suppose that $\ell_G(\underline{d}) = -1$, then the statement is a consequence of [11, Lemma 3.8]. Now, assume $\ell_G(\underline{d}) \geq 0$. By by the definition of $\ell_G(\underline{d})$ we have $r_G(\underline{d}) \geq \ell_G(\underline{d})$, so it suffices to prove

$$r_G(\underline{d}) < \ell_G(\underline{d}) + 1. \tag{2.1}$$

Let u be a vertex as in the statement and consider the effective divisor

$$\underline{e} := (\ell_G(\underline{d}) + 1)u \in \mathrm{Div}(G)$$

of degree $\ell_G(\underline{d}) + 1$. Set $\underline{c} = \underline{d} - \underline{e}$. Now, $\underline{c}$ is u-reduced, hence the previous case yields $r_G(\underline{c}) = -1$. □

The combinatorial rank of a divisor on a (weightless) binary graph was calculated in [11].

Lemma 2.1.14 ([11]). *Let G be a binary graph B_k and $\underline{d} = (d_1, d_2)$ be a divisor on G such that $0 \leq d_1 \leq d_2$. Then, writing $g = k - 1$, we have that*

1. $r_G(\underline{d}) = d_1$, *if* $d_2 \leq g$.

2. $r_G(\underline{d}) = d_1 + d_2 - g$, *if* $d_2 \geq g + 1$.

Proof. If $d_2 \leq g$, we have that $\underline{d}$ is v_1-reduced and $\ell_G(\underline{d}) = \underline{d}(v_1)$, then by Proposition 2.1.13, $r_G(\underline{d}) = d_1$. If $d_2 \geq g + 1$, then $\ell_G(\underline{k}_G - \underline{d}) = -1$. So, by Prop. 2.1.13, we have

$$r_G(\underline{k}_G - (d_1, d_2)) = r_G(g - 1 - d_1, g - 1 - d_2) = -1,$$

Therefore, $r_G(\underline{d}) = d_1 + d_2 - g$ by Riemann-Roch for graphs. □

Proposition 2.1.15. *[11, Proposition 3.17] Let G be any graph. Let $\underline{d} \in \mathrm{Div}(G)$ be such that for some $u \in V$ with $\underline{d}_{rk}(u) = \ell_G(\underline{d})$ we have that $\underline{d}$ is u-reduced. Then $r_G(\underline{d}) = \ell_G(\underline{d})$.*

A direct result of the proposition above is that we can calculate the combinatorial rank of certain divisors on weighted binary graphs.

Corollary 2.1.16. *Let $B_k^{\omega_1,\omega_2}$ be a weighted binary graph, with the order $V(G) = \{v_1, v_2\}$. Let $\underline{d} \in \mathrm{Div}(G)$ be such that $0 \leq d_{rk}(v_1) \leq d_{rk}(v_2)$ and $\underline{d}(v_2) \leq k-1$. Then $r_G(\underline{d}) = \underline{d}_{rk}(v_1)$.*

Proof. From the hypothesis that $\underline{d}_{rk}(v_1) \leq \underline{d}_{rk}(v_2)$ it follows that $\ell_G(\underline{d}) = \underline{d}_{rk}(v_1)$. Secondly, the divisor $\underline{d}$ is v_1-reduced because $\underline{d}_{rk}(v_2) \leq k-1$. So $r_G(\underline{d}) = \underline{d}_{rk}(v_1)$ follows from Proposition 2.1.15. □

The remaining cases occurs when the divisor $\underline{d}$ is such that $\underline{d}(v_2) \geq k$. For a balanced divisor, we expect to have the following:

Conjecture 1. *Let $B_k^{\omega_1,\omega_2}$ be a weighted binary graph, with the order $V(G) = \{v_1, v_2\}$. Let $\underline{d} \in \mathrm{Div}(G)$ be a balanced divisor such that $0 \leq d_{rk}(v_1) \leq d_{rk}(v_2)$ and $\underline{d}(v_2) \geq k$. If $2\omega_i \geq d_i$, for $i = 1, 2$, then*

$$r_G(\underline{d}) = \underline{d}_{rk}(v_1) + \underline{d}_{rk}(v_2) - \frac{k}{2} + 1.$$

Example 2.1.17. If the divisor $\underline{d} \in \mathrm{Div}(B_k^{\omega_1,\omega_2})$ is balanced and $d = g - 1 = k + \omega_1 + \omega_2 - 2$, we have

$$w_i - 1 \leq d_i \leq w_i - 1 + k,$$

for $i = 1, 2$.

2.2 The algebraic rank

Caporaso, in [8], defined an invariant on graphs reflecting the rank of line bundles with a certain degree on all the nodal curves dual to the given graph, the *algebraic rank*. Its construction is given as follows.

We define, for any $\delta \in \mathrm{Pic}^d(G)$ and $\underline{d} \in \delta$,

$$r^{\max}(X, \underline{d}) := \max\{r(X, L); \forall L \in \mathrm{Pic}^{\underline{d}}(X)\},$$

$$r^{\min}(X, \delta) := \min\{r^{\max}(X, \underline{d}); \forall \underline{d} \in \delta\}$$

and we define the algebraic rank by

$$r^{\mathrm{alg}}(G, \delta) := \max\{r^{\min}(X, \delta); \forall X \in M^{\mathrm{alg}}(G)\}.$$

In the above definitions we kept the notations as in [8], except for $r^{\min}(X, \delta)$ for which the author uses the notation $r(X, \delta)$. We say that a pair (X, L) *realizes* $r^{\mathrm{alg}}(G, \delta)$ if $r^{\mathrm{alg}}(G, \delta) = r(X, L)$.

2.2.1 Riemann-Roch for the algebraic rank

The algebraic rank also satisfies the Riemann-Roch theorem.

Remark 2.2.1. Now we introduce a notation used in [11]. Given $L \in \mathrm{Pic}^{\underline{d}}(X)$, denote $L^* = K_X L^{-1}$, so that $L^{**} = L$. Analogously, denote $\underline{d}^* = \underline{\deg} L^* = \underline{k}_G - \underline{d}$ and $d^* = \deg \underline{d}^* = 2g - 2 - d$. Once $\underline{d} \sim \underline{e}$ implies $\underline{d}^* \sim \underline{e}^*$, we have that $\delta^* := [\underline{d}^*]$ is well defined. So $\delta^* \in \mathrm{Pic}^{|\underline{d}^*|}(G)$, $\delta^{**} = \delta$ and $\underline{d}^{**} = \underline{d}$.

Thus we have the bijection $\mathrm{Pic}^{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}^*}(X)$ sending $L \mapsto L^*$; the bijection between the representatives of δ and those of δ^* sending $\underline{d} \mapsto \underline{d}^*$; and the bijection $\mathrm{Pic}^d(X) \to \mathrm{Pic}^{d^*}(X)$ sending $\delta \mapsto \delta^*$.

Theorem 2.2.2. *[11, Proposition 2.6] Let G be a finite graph of genus g. Given a divisor class δ on G of degree d, with $\underline{d} \in \delta$, and a curve $X \in M^{\mathrm{alg}}(G)$. Then*

(a) $r^{\max}(X, \underline{d}) - r^{\max}(X, \underline{k}_G - \underline{d}) = d - g + 1$;

(b) $r^{\min}(X, \delta) - r^{\min}(X, \underline{k}_G - \delta) = d - g + 1$;

(c) $r^{\mathrm{alg}}(G, \delta) - r^{\mathrm{alg}}(G, \underline{k}_G - \delta) = d - g + 1$.

Proof. Given a curve $X \in M^{\mathrm{alg}}(G)$ and a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$. First, we want to prove that

$$r^{\max}(X, \underline{d}) = r(X, L) \Leftrightarrow r^{\max}(X, \underline{d}^*) = r(X, L^*). \tag{2.2}$$

By the algebro-geometric Riemann-Roch applied to L on X it is clear that (2.2) implies (a).

By the bijection $\mathrm{Pic}^{d}(X) \to \mathrm{Pic}^{d^*}(X)$ in Remark 2.2.1, it suffices to prove only one implication of (2.2). So assume $r^{\max}(X, \underline{d}) = r(X, L)$. By contradiction, suppose $r(X, L^*) < r^{\max}(X, \underline{d}^*)$, and let $M^* \in \mathrm{Pic}^{\underline{d}^*}(X)$ be such that $r(X, M^*) = r^{\max}(X, \underline{d}^*)$. Now by Riemann-Roch for X we have

$$r(X, L) = r(X, L^*) + d - g + 1 < r(X, M^*) + d - g + 1 = r(X, M)$$

hence $r^{\max}(X, \underline{d}) = r(X, L) < r(X, M)$, which is impossible as $M \in \mathrm{Pic}^{\underline{d}}(X)$. (2.2) is thus proved, and (a) with it.

Similarly, to prove (b) it suffices that:

$$r^{\max}(X, \underline{d}) = r^{\min}(X, \delta) \Leftrightarrow r^{\max}(X, \underline{d}^*) = r^{\min}(X, \delta^*). \tag{2.3}$$

Assume $r^{\min}(X, \delta) = r^{\max}(X, \underline{d})$ and let $L \in \mathrm{Pic}^{\underline{d}}(X)$ be such that $r(X, L) = r^{\max}(X, \underline{d})$. By (2.2) we have $r(X, L^*) = r^{\max}(X, \underline{d}^*)$, so it suffices to prove that $r(X, L^*) = r^{\min}(X, \delta^*)$. Suppose by contradiction that this is not the case. Then there exists

$\underline{e}^* \in \delta^*$ and $N^* \in \mathrm{Pic}^{\underline{e}^*}(X)$ such that

$$r(X, L^*) > r(X, N^*) = r^{\max}(X, \underline{e}^*).$$

By Riemann-Roch on X we have

$$r(X, L) = r(X, L^*) + d - g + 1 > r(X, N^*) + d - g + 1 = r(X, N).$$

By (2.2) we have

$$r(X, N) = r^{\max}(X, \underline{e}) \leq r^{\max}(X, \underline{d}) = r(X, L),$$

a contradiction with the previous inequality; (4) and (b) are proved.

Now, let $L \in \mathrm{Pic}^{\underline{d}}(X)$ be such that $r(X, L) = r^{\mathrm{alg}}(G, \delta)$. As $r(X, L) = r^{\max}(X, \underline{d}) = r^{\min}(X, \delta)$, by (2.2) and (2.3) we have $r(X, L^*) = r^{\max}(X, \underline{d}^*) = r^{\min}(X, \delta^*)$. By contradiction, suppose there exists a curve $Y \in M^{\mathrm{alg}}(G)$ and a line bundle $P^* \in \mathrm{Pic}^{\underline{e}^*}(Y)$ with $\underline{e}^* \in \delta^*$ such that

$$r(X, L^*) < r^{\mathrm{alg}}(G, \delta^*) = r(Y, P^*) = r^{\max}(Y, \underline{e}^*). \tag{2.4}$$

Arguing as before we get

$$r(X, L) = r(X, L^*) + d - g + 1 < r(Y, P^*) + d - g + 1 = r(Y, P).$$

Now claims (2.2) and (2.3) yield, as $\underline{e} \in \delta$,

$$r(Y, P) = r^{\min}(Y, \delta) \leq r^{\mathrm{alg}}(G, \delta) = r(X, L),$$

contradicting (2.4). The theorem is proved. □

2.3 Inequality of ranks

Caporaso, Len and Melo, in [11], established that the algebraic rank is bounded above by the combinatorial rank of divisors on graphs.

The following lemma is a key step to prove the inequality of ranks.

Lemma 2.3.1 ([11], Lemma 4.1)**.** *Let X be a nodal curve whose dual graph is G. Let L be a line bundle on X, and denote $\underline{d} = \underline{\deg} L$. Suppose that for some $u \in V(G)$, and effective divisor $\underline{e} \in \mathrm{Div}(G)$, the divisor $\underline{d} - \underline{e}^{\deg}$ is u-reduced. Then the space of global sections of L vanishing identically on C_u has dimension at most $|\underline{e}| - \underline{e}(u)$.*

Sketch of the proof. Consider the Dhar decomposition $V = Y_1 \sqcup \ldots \sqcup Y_l$ associated to the u-reduced divisor $\underline{d} - \underline{e}^{\deg}$. Denote, for each $0 \leq j \leq l$, the space of sections of L vanishing on the components of X corresponding to the vertices of $Y_1 \sqcup \ldots \sqcup Y_l$. We want to prove that $\dim \Lambda_0 \leq |\underline{e}| - \underline{e}(u)$. The prove is conducted by induction on $0 \leq j \leq l$, showing that

$$\dim \Lambda_j \leq \sum_{i=j+1}^{l} |\underline{e}_{|Y_i} \cdot|$$

□

Lemma 2.3.2 ([11])**.** *Let $\underline{d} \in \mathrm{Div}(G)$ be such that $r^{\mathrm{alg}}(G, \underline{d}) = k$. Fix a vertex u of G. Given $\underline{e} \in \mathrm{Div}_+^k(G)$ such that $\underline{d} - \underline{e}^{\deg}$ is u-reduced we have that $\underline{d} - \underline{e}^{\deg}$ is effective.*

Sketch of the proof. Since $\underline{d} - \underline{e}^{\deg}$ is u-reduced, we know by definition that this divisor is effective at $V(G) \setminus \{u\}$. And since $r^{\mathrm{alg}}(G, \delta) = k$, there exist $X \in M^{\mathrm{alg}}(G)$ and $L \in \mathrm{Pic}^{\underline{d}}(X)$ such that $r(X, L) \geq s$. Now, consider the following exact sequence

$$0 \to \ker(\pi) \to H^0(X.L) \to H^0(C_u, L_{C_u}) \tag{2.5}$$

where π is the restriction of sections to C_u. Observe that $\ker(\pi)$ is the set of global sections of L vanishing at C_u, hence by Lemma 2.3.1,

$$\dim(\ker(\pi)) \leq s - \underline{e}(u).$$

And from (3.14) we have

$$h^0(C_u, L_{C_u}) \geq h^0(X, L) - \dim(\ker(\pi)) \geq s + 1 - s + \underline{e}(u) = \underline{e}(u) + 1.$$

From Definition 2.1.10, we have that

$$\underline{e}^{\deg}(u) = \underline{e}(u) + \min\{\underline{e}(u), g(u)\}.$$

So, by the Riemann-Roch theorem and Clifford's theorem, $\underline{e}^{\deg}(u)$ is the minimum degree of a line bundle on C_u of rank $\underline{e}(u)$ more precisely

$$\deg_{C_u} L \geq \underline{e}^{\deg}(u),$$

which implies that $\underline{d}(u) \geq \underline{e}^{\deg}(u)$. Proving that $\underline{d}(v) - \underline{e}^{\deg}(v) \geq 0$, for all $v \in V(G)$. □

Theorem 2.3.3 ([11], Theorem 4.2)**.** *Let G be a finite, connected, weighted graph of genus g, and let δ be a divisor class on G, then we have*

$$r^{\text{alg}}(G, \delta) \leq r_G(\delta). \tag{2.6}$$

Sketch of the proof. We can assume $r^{\text{alg}}(G, \delta) = s \geq 0$, once the combinatorial rank is bounded below by -1.

Assuming $s \geq 0$, observe that, if for any effective divisor $\underline{e}$ of degree s there exists a representative $\underline{d}$ of δ such that $\underline{d} - \underline{e}^{\deg}$ is effective, then $r_G(\underline{d}) \geq s$. So, fixing a vertex $u \in V(G)$, let $\underline{e} \in \text{Div}^s_+(G)$. By Proposition 1.1.10 that there exists $\underline{d} \in \delta$ such that $\underline{d} - \underline{e}^{\deg}$ is u-reduced divisor. Therefore, the proof follows from Lemma 2.3.2. □

As a consequence of Theorem 2.3.3 and Clifford inequality for graphs proved in [3], we have the following result establishing that Clifford inequality holds for the algebraic rank.

Proposition 2.3.4 ([11],Proposition 4.6)**.** *Let G be a graph of genus g and $\delta \in \mathrm{Pic}^d(G)$ with $0 \leq d \leq 2g-2$. Then*

$$r^{\mathrm{alg}}(G,\delta) \leq \lfloor d/2 \rfloor.$$

Moreover, $r^{\min}(X,\delta) \leq \lfloor d/2 \rfloor$, for every $X \in M^{\mathrm{alg}}(G)$. In other words, for every X there exists a multidegree $\underline{d} \in \delta$ such that every $L \in \mathrm{Pic}^{\underline{d}}(X)$ satisfies Clifford inequality.

2.4 Specialization lemma

Let $\mathcal{X}$ be a regular 2-dimensional variety, B a regular 1-dimensional variety with a marked point $b_0 \in B$ and let $\phi : \mathcal{X} \to (B, b_0)$ be a regular one-parameter smoothing of a connected curve X, i.e., ϕ is a fibration in curves such that the fiber over b_0 is X and the fibers over $b \in B \setminus \{b_0\}$ are smooth projective curves. The relative Picard scheme of ϕ is written $\mathrm{Pic}_\phi \to B$ so that the fiber of Pic_ϕ over a point $b \in B$ is the Picard schemes of the curve $X_b := \phi^{-1}(b)$. Let L_0 and L_0' be two line bundles on X. If for some D on $\mathcal{X}$ entirely supported on X we have

$$L_0^{-1} \otimes L_0 = \mathcal{O}_{\mathcal{X}}(D)_{|X},$$

then L_0 and L_0' are ϕ-equivalent, and we denote $L_0' \sim_\phi L_0$

The following form of the Specialization Lemma proved by Caporaso, Len and Melo in [11] is a generalization of the one proved by Baker in [2].

Lemma 2.4.1 (Specialization Lemma)**.** *Let* $\phi : \mathcal{X} \to B$ *be a regular one-parameter smoothing of a connected curve* X*. Let* G *be the dual graph of* X*. Then for every* $\mathcal{L} \in \mathrm{Pic}_\phi(B)$ *there exists an open neighbourhood* $U \subset B$ *of* b_0 *such that for every* $b \in U \setminus \{b_0\}$ *we have*

$$r(X_b, \mathcal{L}(b)) \leq r^{\mathrm{alg}}(G, \underline{\deg}\mathcal{L}(b_0)).$$

Proof. For every $\mathcal{L}'(b_0) \in \mathrm{Pic}(X)$ such that $\mathcal{L}'(b_0) \sim_\phi \mathcal{L}(b_0)$, by uppersemicontinuity of h^0, we have, for every b in a neighborhood $U \subset B$ of b_0,

$$r(X_b, \mathcal{L}(b)) \leq r(X, \mathcal{L}'(b_0)).$$

Hence,

$$r(X_b, \mathcal{L}(b)) \leq r^{\max}(X, \underline{\deg}\mathcal{L}'(b_0))$$

Bearing in mind that $\mathcal{L}'(b_0)$ varies in its ϕ-class the values of $\underline{\deg}\mathcal{L}'(b_0)$ cover all the representatives of $[\underline{d}]$, therefore we obtain

$$r(X_b, \mathcal{L}(b)) \leq r(X, \delta).$$

□

2.5 When the ranks are equal

In this section several cases of graphs and divisors where the algebraic rank coincides with the combinatorial rank are exposed.

By Proposition 1.2.11 we have that every divisor class has a balanced representative. The following theorem ensures that, for a divisor of degree $d \geq 2g - 2$, these semibalanced divisors realize the algebraic rank. Furthermore, for such divisors the value of both algebraic and combinatorial rank coincides.

Theorem 2.5.1. *[9, Theorem 2.9] Let G be a semistable graph of genus g. Then, if $d \geq 2g-2$, for every $\delta \in \mathrm{Pic}^d(G)$ the following facts hold.*

(a) *Every semibalanced $\underline{d} \in \delta$ satisfies $r^{\max}(X, \underline{d}) = r_G(\underline{d})$ for every curve $X \in M^{\mathrm{alg}}(G)$.*

(b) *$r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$.*

The proof of the theorem follows from the Balanced Riemmann theorem [8, Theorem 2.3] for singular curves that states that for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have $r(X, L) = d - g$. And the fact that this result and the Clifford's theorem [8, Theorem 4.4] extend to semibalanced multidegrees.

Let G be a weighted graph of genus g and let $\delta \in \mathrm{Pic}^d(G)$. The other cases treated in [9] where the equality $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$ holds are the following:

- $g \leq 1$.
- $d \leq 0$.
- G has only on vertex.
- G is a stable graph of genus 2.

2.5.1 Special curves

Now we describe curves on which line bundles of a fixed multidegree tend to have the highest possible rank, the special curves. The description of such curves that we introduce here is from [11], where it is proved that certain line bundles in such curves realize the algebraic rank and achieve the combinatorial rank value.

In a weightless, loopless graph G we have the following structure maps:

- the *endpoint* map $\epsilon : H(G) \to V(G)$, where $H(G)$ is the set of half edges of G;
- the *gluing* map $\gamma : H(G) \to E(G)$.

The gluing map is surjective and two-to-one and induces an involution on H free from fixed points, denoted by ι. We write $[h, \overline{h}]$ to denote the orbits of ι. These orbits are identified with the edges of G. For any $v \in V(G)$, we denote by $H_v = \epsilon^{-1}(v)$ and $E_v = \gamma(\epsilon^{-1}(v))$ the sets of half-edges and edges adjacent to v. We denote by $H_{v,w} = H_v \cap \gamma^{-1}(E_w)$.

Let $X = \cup_{v\in V(G)} C_v$ be a dual curve to G. We have a set $P_v \subset C_v$ of labeled distinct points of C_v mapping to smooth points of X lying in C_v:

$$P_v := \{p_h; \forall h \in H_v\} = \sqcup_{w\in V(G)} P_{v,w},$$

where

$$P_{v,w} := \{p_h; \forall h \in H_{v,w}\} \subset C_v.$$

We have the following description of X

$$X = \frac{\sqcup_{v\in V(G)} C_v}{p_h = p_{\overline{h}}, \forall h \in H(G)}.$$

Definition 2.5.2 (Special curves)**.** Let G be a weightless, loopless and let $X \in M^{\text{alg}}(G)$. We say that X is *special* if there exists a collection

$$\{\phi_{v,w} : (C_v; P_{v,w}) \to (C_w; P_{w,v}), \forall v, w \in V\},$$

where $\phi_{v,w}$ is an isomorphism of pointed curves such that for every $u, v, w \in V$ and $h \in H_{v,w}$ the following properties hold:

1. $\phi_{v,w}(p_h) = p_{\overline{h}}$;
2. $\phi_{v,w}^{-1} = \phi_{w,v}$;
3. $\phi_{v,u} = \phi_{w,u} \circ \phi_{v,w}$.

If G is not connected, then $X \in M^{\text{alg}}(G)$ is defined to be special if every connected component of X is special.

Remark 2.5.3. [11, Remark 5.4] Let X be a special curve. Then every subcurve of X, and every parcial normalization of X, is special. Moreover, let p be a nonsingular point of X lying in the irreducible component C_u; then for every component C_v of X the curve

$$Y := \frac{X}{p = \phi_{u,v}(p)}$$

is also special. The quotient map $\pi : X \to Y$ describes X as a partial normalization of Y.

Lemma 2.5.4. *[11, Lemma 5.5] For every weightless, loopless graph G, the set $M^{\text{alg}}(G)$ contains a special curve.*

Proposition 2.5.5. *[11, Proposition 5.6] Given a binary graph B_k, we have that $r_G(\delta) = r^{\text{alg}}(G, \delta)$, for every $\delta \in \text{Pic}(G)$.*

2.5.2 Rank-explicit

There are divisors for which its combinatorial rank equals its minimal entry, in the case where the graph is weightless and loop less.

Definition 2.5.6. Let G be a weighted and loopless graph. A divisor $\underline{d}$ in G is called rank-explicit if $\underline{d}$ is u-reduced for some vertex u such that $\underline{d}(u) = \ell_G(\underline{d})$.

In the general case, a divisor $\underline{d}$ is called rank-explicit if $\underline{d}$ is u-reduced for some vertex u such that $\underline{d}_{rk}(u) = \ell_G(\underline{d})$.

Lemma 2.5.7. *[11, Lemma 5.12] Let G be a weightless loopless graph and let $\underline{d} \in \text{Div}(G)$ be a rank-explicit divisor. Then, for every special curve $X \in M^{alg}(G)$, there exists a line bundle $L \in \text{Pic}^{\underline{d}}(G)$ such that $H^0(X, L) \cong H^0(\mathbb{P}^1, (O(\ell_G(\underline{d})))$.*

Theorem 2.5.8. *[11, Theorem 5.13] Let G be a weightless loopless graph. If $\delta \in \mathrm{Pic}(G)$ is rank-explicit, then $r^{\mathrm{alg}}(G, \delta) = r_G(\underline{d})$.*

2.6 Counterexamples

Caporaso, Len and Melo presented in [11] examples where the combinatorial rank is not equal to the algebraic rank. The following is an example of rank-explicit divisor in a weighted binary graph.

Example 2.6.1. [11, Example 5.15] Let G be a weighted graph with $V(G) = \{v_1, v_2\}$, $\omega(v_1) = 1$, $\omega(v_2) = 2$ and such that $v_1 \cdot v_2 > 12$.

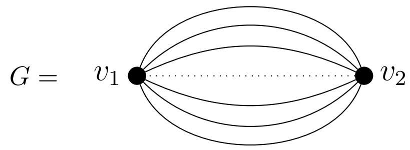


Figure 2.2: The graph G.

Consider the divisor $\underline{d} = (3, 4)$. Observe that $\underline{d}$ is u-reduced, for $u = v_1, v_2$. We have that $r_G(\underline{d}) = 2$.

Claim. Given any curve $X \in M^{\mathrm{alg}}(G)$, we have $r^{\max}(X, \underline{d}) < r_G(\underline{d})$.

Indeed, let $X \in M^{\mathrm{alg}}$, writing $X = C_1 \cup C_2$, where C_i corresponds to the vertex v_i, $i = 1, 2$. Now, consider L a line bundle on $\mathrm{Pic}^{\underline{d}}(X)$ and write $L_i = L_{|C_i}$. Proceeding by contradiction, suppose that $r(X, L) = 2$. By Riemann-Roch, as $\underline{\deg}(L) = (3, 4)$, we have

$$r(C_1, L_1) = r(C_2, L_2) = 2.$$

Observe that the number $|C_1 \cap C_2|$ is large enough so that a section of L_i can be extended to all X. Therefore, the map $\phi_L : X \to \mathbb{P}^2$ defined by L restricts to non-degenerated maps $\phi_1 : C_1 \to \mathbb{P}^2$ and $\phi_2 : C_2 \to \mathbb{P}^2$. The image of ϕ_1 is a irreducible curve of degree 3, so it is a cubic or a line with multiplicity 3. Once ϕ_1 is non-degenerated, the image has to be a cubic.

Likewise, the image of ϕ_2 is a non-degenerated irreducible curve of degree 4. So $\phi_2(C_2)$ is either a quadric or a conic of multiplicity 2. Therefore, $\phi_L(X)$ consists of two distinct irreducible curves of degree 3 and 4, respectively. By Bézout's theorem, they intersect in at most 12 points, which is a contradiction.

Then, in this case, every curve $X \in M^{\mathrm{alg}}(G)$ satisfies $r^{\max}(X, \underline{d}) < r_G(\underline{d}) = 2$ and $r^{\mathrm{alg}}(G, \underline{d}) < r_G(\underline{d})$.

Example 2.6.2. [11, Example 5.16] Let G be a weightless graph such that $V(G) = \{v_1, v_2, v_3\}$ with $v_1 \cdot v_3 = 3$ and $v_2 \cdot v_3 > 6$.

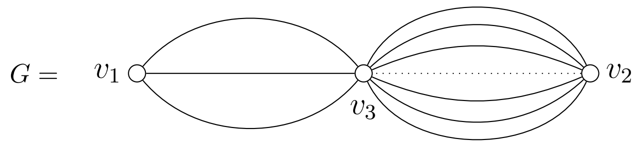


Figure 2.3: The graph G.

Consider $X \in M^{\mathrm{alg}}(G)$ writing $X = C_1 \cup C_2 \cup C_3$, where C_i is the rational curve corresponding to the vertex v_i, $i = 1, 2, 3$. Let $\underline{d} = (1, 2, 3)$ such that the degree of $\underline{d}$ on C_i is i, so we have $r_G(\underline{d}) = 2$. The statement here is that for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have $r(X, L) < 2$. With an analogous analysis as in the precedent example, this statement can be proved by contradiction assuming that $r(X, L) = 2$. The line bundle L defines a non-degenerated map $\phi_L : X \to \mathbb{P}^2$. Now, treating the possible cases for images of ϕ_L, we get a contradiction in each of them. At the end we have that $2 = r_G(\underline{d}) > r^{\mathrm{alg}}(G, \underline{d})$.

The following problem remains unsolved.

Problem 1. What are the cases where in (2.6) we have a strict inequality?

Chapter 3

Different approaches to the algebraic rank

The goal in this chapter is to understand and/or characterize the cases where we have a strict inequality in (2.6). Furthermore, we introduce a number of modifications of the algebraic rank with the aim of getting closer to an algebraic interpretation of the combinatorial rank in the cases where the strict inequality of ranks occurs.

Our modifications will mainly be inspired by the geometry of compactified Jacobians, so we will consider refinements of the curves together with the line bundles.

3.1 Modifications on the min-max order of the algebraic rank

The purpose of this section is to analyse what happens if we change the order of variation of the different objects we consider when computing the algebraic rank, r^{alg}, in particular we will focus on varying the curve first. This will be useful as it will be more appropriate to the modifications we intend to do to the algebraic rank in what follows. Thereunto, we prove that a new version of the Riemann-Roch theorem can be obtained under this new definition.

Given a graph G and a divisor $\underline{d} \in \operatorname{Div} G$, we define

$$r^{\mathrm{MAX}}(G, \underline{d}) := \max\{r^{\max}(X, \underline{d}); \forall X \in M^{\mathrm{alg}}(G)\},$$

and for any $\delta \in \operatorname{Pic}^d(G)$, we define

$$r^{\mathrm{ALG}}(G, \delta) := \min\{r^{\mathrm{MAX}}(G, \underline{d}); \forall \underline{d} \in \delta\}.$$

Theorem 3.1.1 (Riemann-Roch)**.** *Using the notation introduced above, let G be a finite graph of genus g, $\underline{d}$ a divisor of degree d on G, then*

(a) $r^{\mathrm{MAX}}(G, \underline{d}) - r^{\mathrm{MAX}}(G, \underline{k}_G - \underline{d}) = d - g + 1.$

(b) $r^{\mathrm{ALG}}(G, \underline{d}) - r^{\mathrm{ALG}}(G, \underline{k}_G - \underline{d}) = d - g + 1.$

Proof. Here we follow the arguments of the proof of [Proposition 2.6, [11]] and use the notation of Remark 2.2.1. By the same proposition we know that

$$r^{\max}(X, \underline{d}) - r^{\max}(X, \underline{d}^*) = d - g + 1. \tag{3.1}$$

We claim that, given $X \in M^{\mathrm{alg}}(G)$ and $L \in \operatorname{Pic}^{\underline{d}}(X)$, we have

$$r^{\mathrm{MAX}}(G, \underline{d}) = r^{\max}(X, \underline{d}) \iff r^{\mathrm{MAX}}(G, \underline{d}^*) = r^{\max}(X, \underline{d}^*). \tag{3.2}$$

Observe that (3.1) and (3.2) imply that (a) holds. Furthermore, it follows from the bijection described on the discussion after Remark 2.2.1 that it suffices to prove only one implication of (3.2). Assume therefore that $r^{\mathrm{MAX}}(G, \underline{d}) = r^{\max}(X, \underline{d}) = r(X, L)$, for $L \in \operatorname{Pic}^{\underline{d}}(X)$. By (3.1) we have $r^{\max}(X, \underline{d}^*) = r(X, L^*)$. Suppose by contradiction there exists a curve $Y \in M^{\mathrm{alg}}(G)$ and $M \in \mathrm{Pic}^{\underline{d}^*}(Y)$ such that

$$r(Y, M^*) = r^{\max}(Y, \underline{d}^*) = r^{\mathrm{MAX}}(G, \underline{d}^*) > r(X, L^*).$$

In this case, by Riemann-Roch on X, we have

$$r(X, L) = r(X, L^*) + d - g + 1 < r(X, M^*) + d - g + 1 = r(Y, M).$$

By (3.1)

$$r^{\mathrm{MAX}}(G, \underline{d}) = r^{\max}(X, \underline{d}) = r(X, L) \geq r^{\max}(Y, \underline{d}) = r(Y, M).$$

contradicting the previous inequality. Therefore (3.2) is proven.

Now, let $r^{\mathrm{ALG}}(G, \delta) = r(X, L)$. So, by (3.1) and (3.2),

$$r^{\mathrm{MAX}}(G, \underline{d}) = r^{\max}(X, \underline{d}) = r(X, L)$$

gives us

$$r^{\mathrm{MAX}}(G, \underline{d}^*) = r^{\max}(X, \underline{d}^*) = r(X, L^*).$$

By Riemann-Roch on X, to prove (b) it suffices to prove that $r^{\mathrm{ALG}}(G, \delta^*) = r(X, L^*)$. By contradiction, suppose that it exists $Y \in M^{\mathrm{alg}}(G)$, $\underline{e}^* \in \delta^*$ and $N \in \mathrm{Pic}^{\underline{e}^*}(Y)$ such that

$$r(X, L^*) > r^{\mathrm{ALG}}(G, \delta^*) = r^{\mathrm{MAX}}(Y, N^*) = r^{\max}(Y, \underline{e}^*) = r(Y, N^*).$$

By Riemann-Roch on X we have

$$r(X, L) = r(X, L^*) + d - g + 1 > r(X, N^*) + d - g + 1 = r(Y, N).$$

Once $\underline{e} \in \delta$, it follows from (3.1) and (3.2) that

$$r(X, L) = r^{\mathrm{ALG}}(G, \delta) \leq r^{\mathrm{MAX}}(Y, N) = r(Y, N),$$

contradicting the preceding inequality. The theorem is proved. □

Analogous to what happens with the algebraic rank, r^{ALG} is also upper bounded by the combinatorial rank.

Theorem 3.1.2 (Inequality of Ranks)**.** *Given a divisor class δ on a graph G, we have*

$$r^{\text{ALG}}(G,\delta) \leq r_G(\delta).$$

Proof. The inequality above is satisfied once the *min-max* order does not play an important role in the proof of the inequality $r^{\text{alg}}(G,\delta) \leq r_G(\delta)$ on [Theorem 4.2, [11]].

We claim that we can change r^{alg} by r^{ALG} in the original proof. Suppose that $r^{\text{ALG}}(G,\delta) = s$ so we want to prove that $r_G(\delta) \geq s$. Observe that if $r^{\text{ALG}}(G,\delta) = s$, then for all $\underline{d}' \in \delta$, $r^{\text{MAX}}(G,\underline{d}') \geq s$. So if we consider the representative $\underline{d}$ of δ such that $\underline{d} - \underline{e}^{\text{deg}}$ is u-reduced, there exist a curve $X \in M^{\text{alg}}(G)$ and a line bundle $L \in \text{Pic}^{\underline{d}}(X)$ such that $h^0(X,L) \geq s+1$. And this argument is the only one that involves r^{alg} in the proof of Lemma 2.3.2 and consequently the proof of Theorem 2.3.3. □

The following result is analogous to Theorem 2.5.1.

Theorem 3.1.3. *Let G be a semistable graph of genus g. Then, if $d \geq 2g-2$, for every $\delta \in \text{Pic}^d(G)$ the following facts hold:*

(a) *Every semibalanced $\underline{d} \in \delta$ satisfies $r^{\text{MAX}}(G,\underline{d}) = r_G(\underline{d})$.*

(b) $r_G(\delta) = r^{\text{ALG}}(G,\delta)$.

Proof. The existence of semistable representative of δ is given by Proposition 1.2.11. (a) follows directly from Theorem 2.5.1(b) once it states that, if d is at least $2g-2$, every semibalanced $\underline{d} \in \delta$ satisfies $r^{\max}(X,\underline{d}) = r_G(\underline{d})$, for every $X \in M^{\text{alg}}(G)$. For part (b), once r^{ALG} satisfies the Riemann-Roch theorem, if $d \geq 2g-2$ we have

$$r^{\mathrm{ALG}}(G,\delta) \geq d-g. \tag{3.3}$$

So, if $d \geq 2g-1$, by Theorem 2.1.7 we have that $r_G(\underline{d}) = d-g$. Therefore, in this case we have $r_G(\delta) = r^{\mathrm{ALG}}(G,\delta)$. Now, if $d = 2g-2$ then $r_G(\delta) \leq g-1$ with equality if and only if $\delta = [\underline{k}_G]$. If δ is the canonical class then $r_G(\delta) = g-1$ and (b) follows from (3.3) and Theorem 3.1.2. If δ is not the canonical class then $r_G(\delta) = g-2$, on the other hand we have $r^{\mathrm{ALG}}(G,\underline{d}) = g-2$, and we are done. □

Corollary 3.1.4. *Let G be a semistable graph of genus g and let $\underline{d} \in \mathrm{Div}^d(G)$ be a semibalanced divisor. Then the following facts hold.*

(a) *If $d < 0$ then $r^{\mathrm{MAX}}(G,\underline{d}) = -1$.*

(b) *If $d > 2g-2$ then $r^{\mathrm{MAX}}(G,\underline{d}) = d-g$.*

(c) *If $d = 2g-2$ then $r^{\mathrm{MAX}}(G,\underline{d}) \leq g-1$ and the equality happens if and only if $\underline{d} \sim \underline{k}_G$.*

(d) *If $d = 0$ then $r^{\mathrm{MAX}}(G,\underline{d}) \leq 0$ and the equality occurs if and only if $\underline{d} \sim \underline{0}$.*

Proof. Since $\underline{d}$ is semibalanced, By Theorem 3.1.3 we know that

$$r^{\mathrm{MAX}}(G,\underline{d}) = r_G(\underline{d}). \tag{3.4}$$

To prove (a) suppose that $d < 0$, then $\underline{d}^* := \underline{k}_G - \underline{d}$ is a semibalanced divisor of degree greater than $2g-2$. So, by (3.4) and Riemann-Roch Theorem for graphs we have

$$r^{\mathrm{MAX}}(G,\underline{d}^*) = r_G(\underline{d}) = d-g.$$

On the other hand, by Theorem 3.1.1 we have

$$r^{\mathrm{MAX}}(G,\underline{d}^*) = r^{\mathrm{MAX}}(G,\underline{k}_G) - \underline{d}^*) + d - g + 1 = r^{\mathrm{MAX}}(G,\underline{d}) + d - g + 1.$$

Therefore,

$$r^{\mathrm{MAX}}(G,\underline{d}) = d - g - d + g - 1 = -1.$$

Now, suppose that $d > 2g-2$. By Theorem 3.1.3 we have

$$r^{\mathrm{MAX}}(G,\underline{d}) = r_G(\underline{d}) = r_G(\underline{k}_G - \underline{d}) + d - g + 1.$$

But $d > 2g-2$ implies $|\underline{k}_G - \underline{d}| < 0$, so Corollary 2.1.8 implies (b).

If $d = 2g-2$ then by Corollary 2.1.8 we have that $r_G(\underline{d}) = g-1$ if $\underline{d} \sim \underline{k}_G$ and $r_G(\underline{d}) \leq g-2$ otherwise. Therefore (c) follows from (3.4). Now, (d) follows from (c) by Theorem 3.1.1 and (3.4). □

Remark 3.1.5. Observe that examples 2.6.1 and 2.6.2 will work the same for r^{ALG}, meaning that in both situations $r_G(\underline{d}) > r^{\mathrm{ALG}}(G,\underline{d})$. In fact, the min-max order do not play a key role on the proof of the inequalities.

Proposition 3.1.6 (Clifford Inequality)**.** *Let G be a graph of genus g and $\delta \in \mathrm{Pic}^d(G)$ with $0 \leq d \leq 2g-2$, then*

$$r^{\mathrm{ALG}}(G,\delta) \leq \lfloor d/2 \rfloor.$$

Proof. It follows from Theorem 3.1.2 and the Clifford inequality for graphs [Corollary 3.5, [3]]. □

Lemma 3.1.7 (Binary curves)**.** *If G is a binary graph of genus g, given $\delta \in \mathrm{Pic}(G)$, we have $r^{\mathrm{ALG}}(G,\delta) = r_G(\delta)$.*

Proof. The proof of Proposition 2.5.5 gives a stronger statement. Indeed, in the case where $d_2 \leq g$ for every curve $X \in M^{\text{alg}}(G)$, Proposition 2.5.5 states that we have $r^{\min}(X, \delta) = r_G(\delta)$. Consider $(Y, \underline{d})$ the pair that realizes $r^{\text{MAX}}(G, \underline{d})$ and $r^{\min}(Y, \delta)$. Then we have $r^{\max}(Y, \underline{d}) = r_G(\delta)$. Therefore $r^{\text{ALG}}(G, \delta) = r_G(\underline{d}) = r^{\text{alg}}(G, \delta)$.

In the case where $d_2 \geq g + 1$, we have $r(X, L) \geq r_G(\delta)$ for every $X \in M^{\text{alg}}(G)$, every $\underline{d} \in \delta$ and every $L \in \text{Pic}^{\underline{d}}(X)$. So it is clear that we have

$$r_G(\delta) = r^{\text{alg}}(G, \delta) = r^{\text{ALG}}(G, \delta),$$

for all $\delta \in \text{Pic}(G)$. □

A natural question that arises is whether or not r^{alg} equals r^{ALG}. We show in the following that r^{ALG} is an upper bound for r^{alg}. Even though we cold not find a way to prove the equality so far, we believe that the answer to this question is yes, once analysing the problem from the strictly numerical point of view, varying the min-max order could actually change the final result as exemplified below.

Example 3.1.8. Let G be a graph such that $M^{\text{alg}}(G) = \{X_1, X_2\}$ and let $\delta \in \text{Pic}(G)$ such that $\underline{d}_1, \underline{d}_2 \in \delta$. Suppose that the have the following data for $M^{\text{alg}}(G)$ and δ:

δ \ $M^{\text{alg}}(G)$	X_1	X_2
$\underline{d}_1$	$r^{\max}(X_1, \underline{d}_1) = 1$	$r^{\max}(X_2, \underline{d}_1) = 2$
$\underline{d}_2$	$r^{\max}(X_1, \underline{d}_2) = 2$	$r^{\max}(X_2, \underline{d}_2) = 1$

Hence, in principle we could have $r^{\min}(X_1, \delta) = 1 = r^{\min}(X_2, \delta)$ and $r^{\text{MAX}}(G, \underline{d}_1) = 2 = r^{\text{MAX}}(G, \underline{d})$. So, $r^{\text{alg}}(G, \delta) = 1$ but $r^{\text{ALG}}(G, \delta) = 2$. This example is purely numerical so it ignores the fact that there are more dual curves to the graph G and more representatives of δ. And we intuit that it is not possible if we consider the geometry of the problem.

Proposition 3.1.9. *Let G be a graph and let $\delta \in \mathrm{Pic}(G)$, then*

$$r^{\mathrm{alg}}(G,\delta) \leq r^{\mathrm{ALG}}(G,\delta).$$

Proof. Suppose that $r^{\mathrm{alg}}(G,\delta)$ is realized by $(X_1,\underline{d}_1)$, i.e.,

$$r^{\mathrm{alg}}(G,\delta) = r^{\min}(X_1,\delta) = r^{\max}(X_1,\underline{d}_1)$$

and suppose also that $r^{\mathrm{ALG}}(G,\delta)$ is realized by $(X_2,\underline{d}_2)$, i.e.,

$$r^{\mathrm{ALG}}(G,\delta) = r^{\mathrm{MAX}}(G,\underline{d}_2) = r^{\max}(X_2,\underline{d}_2),$$

where $X_1, X_2 \in M^{\mathrm{alg}}(G)$ and $\underline{d}_1, \underline{d}_2 \in [\underline{d}]$. So, since $r^{\mathrm{alg}}(G,\underline{d}) = r^{\max}(X_1,\underline{d}_1)$, we have $r^{\max}(X_1,\underline{d}_1) \leq r^{\max}(X_1,\underline{d}')$, for all $\underline{d}' \in [\underline{d}]$. In particular,

$$r^{\max}(X_1,\underline{d}_1) \leq r^{\max}(X_1,\underline{d}_2). \tag{3.5}$$

Since $r^{\mathrm{ALG}}(G,\underline{d}) = r^{\max}(X_2,\underline{d}_2)$ we have $r^{\max}(X_2,\underline{d}_2) \geq r^{\max}(Y,\underline{d}_2)$, for all $Y \in M^{\mathrm{alg}}(G)$. In particular,

$$r^{\max}(X_2,\underline{d}_2) \geq r^{\max}(X_1,\underline{d}_2). \tag{3.6}$$

By (3.5) and (3.6) we have

$$r^{\max}(X_1,\underline{d}_1) \leq r^{\max}(X_1,\underline{d}_2) \leq r^{\max}(X_2,\underline{d}_2).$$

Therefore,

$$r^{\mathrm{alg}}(G,\underline{d}) \leq r^{\mathrm{ALG}}(G,\underline{d}).$$

□

3.2 Refinement of graphs

The aim of this section is to investigate the relations between the combinatorial and the algebraic rank of a divisor on the dual graph of a curve and the ranks on the dual graph of suitable degenerations of it.

Recall that a quasistable graph is a semistable graph where no two exceptional vertices are adjacent (see Definition 1.1.3).

Definition 3.2.1. Let G be a quasistable graph. A divisor $\underline{d} \in \mathrm{Div}(G)$ is *admissible* if $\underline{d}(v) = 1$, for all $v \in V_G^{\mathrm{exc}}$.

Let G be a graph, we define the graph $\widehat{G}_R$ by adding one vertex on each edge of $R \subseteq E(G)$, these new vertices are exceptional vertices. Define the graph G_R by subtracting from G all the edges of R, such that $V(G_R) = V(G) \subseteq V(\widehat{G}_R)$. When $R = E(G)$, the graph $\widehat{G}_E$ is called the *total resolution* of G. So, if G is a stable graph we have that $\widehat{G}_R$ is a quasistable graph.

Now, we compare the combinatorial and algebraic rank of a divisor and of its admissible desinguarizations. For this we now define some graph contractions that reflect the degenerations in the curves.

Definition 3.2.2. Given graphs G and G' with respective weights ω and ω', we say that (G, ω) is a *weighted contraction* of (G', ω'), if there exist $H \subseteq E(G')$ and a (contracting) map $\sigma : G' \to G$ such that $G = G'/H$, i.e., G is obtained from G' by contracting every edge in H, and for all $v \in V(G)$ we have

$$\omega(v) = g(\sigma^{-1}(v)).$$

The following definition extends weighted contractions at exceptional components to divisors.

Definition 3.2.3. Let G and G' be graphs with respective weights ω and ω' and let $\underline{d}$ be a divisor on G and $\underline{d}'$ be a divisor on G'. We say that $(G, \omega, \underline{d})$ is an *exceptional weighted contraction* of $(G', \omega', \underline{d}')$ if

(a) $V(G') \setminus V(G) \subseteq V_{G'}^{exc}$.

(b) There exists a subset $H \subset E(G')$ obtained by choosing one of the edges between each new vertex and one of its adjacent vertices, and such that (G, ω) is a weighted contraction of (G', ω') by the morphism $\sigma : G' \to G$ contracting every edge of H.

(c) σ induces a morphism

$$\begin{array}{rccc} \eta : & \operatorname{Div} G' & \to & \operatorname{Div} G \\ & \underline{d}' & \mapsto & \eta(\underline{d}') = \underline{d} \end{array} ,$$

with $d(v) = \sum_{v'=\eta^{-1}(v)} \underline{d}'(v')$, for all $v \in V$.

If $(G, \omega, \underline{d})$ is an exceptional weighted contraction of $(G', \omega', \underline{d}')$ we write $(G', \omega', \underline{d}') \geq_E (G, \omega, \underline{d})$, or just $(G', \underline{d}') \geq_E (G, \underline{d})$ if the weights are clear from the context.

Observe that if $F \subset E(G)$ is the empty set, $(G, \omega, \underline{d})$ is an exceptional weighted contraction of itself. When $(G', \omega', \underline{d}') \geq_E (G, \omega, \underline{d})$ and G' is different than G we write $(G', \omega', \underline{d}') >_E (G, \omega, \underline{d})$.

Notice that if $(G', \underline{d}') \geq_E (G, \underline{d})$, G and G' have the same genus and $\underline{d}$ and $\underline{d}'$ have the same degree.

Notation 1. From now on, instead of considering exceptional weighted contractions of all divisors on a graph (as in Definition 3.2.3), we are only considering exceptional contraction of admissible divisors. To emphasize this we write

$$(G', \underline{d}', \omega') \geq (G, \underline{d}, \omega)$$

if $(G', \underline{d}', \omega') \geq_E (G, \underline{d}, \omega)$ and $\underline{d}'$ is admissible.

Example 3.2.4. Let G be the graph as in Example 2.6.2 (Figure 2.3), and let $\widehat{X}_R$ be the blow up of X in one point of $C_1 \cap C_3$. Consider the normalization $\nu : X_R^\nu \to X$. Denoting $R = \{\widehat{v}\}$, the dual graph of $\widehat{X}_R$ is as in Figure 3.1.

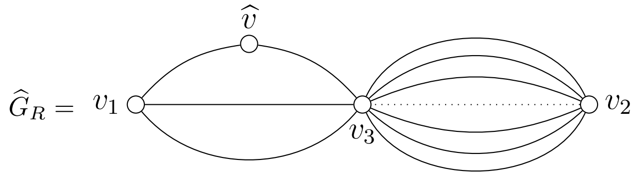


Figure 3.1: The graph G_R dual to the curve $\widehat{X}_R$.

Consider the divisor $\widehat{\underline{d}} = (1, 1, 2, 2)$, with respect to the ordering $V(\widehat{G}) = \{v_1, \widehat{v}, v_3, v_2\}$. One can see that the combinatorial rank of $\widehat{\underline{d}}$ is equal to 2. We claim that the $r^{\max}$ of this divisor is also 2. We have G_R as in Figure 3.2.

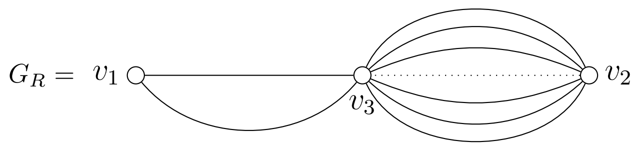


Figure 3.2: The graph G_R, whose dual curves are normalizations of dual curves of $\widehat{G}_R$.

Suppose that $C = C_3 \cup C_2$ is a special curve. The combinatorial rank of the divisor $(2, 2)$ in the dual graph of C is 2, and so is the algebraic rank of $(2, 2)$ once C is a binary curve, by Theorem 2.3.3 (b). Moreover, there is a $L_C \in \operatorname{Pic}^{(2,2)} C$ such that $h^0(C, L_C) = 3$.

Now, let C_{v_1} be the component associated with v_1 in X_R^ν and consider $\mu : Y \to X_R^\nu$ the normalization on all the nodes of $C_{v_1} \cap C$. Let $L \in \operatorname{Pic}^{(1,2,2)}(Y)$ be such that $L|_C = L_C$. Then, given $L' \in \operatorname{Pic}^{(1,2,2)}(X_R^\nu)$ such that $\mu^* L' = L$, by Remark 1.2.5, we have

$$h^0(X_R^\nu, L') \geq h^0(Y, L) - 2 = 2 + 3 - 2 = 3.$$

Given $M \in \operatorname{Pic}^{\widehat{\underline{d}}}(\widehat{X}_R)$, such that $L' = i^* M$, we have by Lemma 1.2.9 that $h^0(\widehat{X}_R, M) =$

$h^0(X^\nu_R, L')$. So $r(\widehat{X}_R, M) \geq 2 = r_{\widehat{G}_R}(\underline{\widehat{d}})$, hence

$$r^{\mathrm{MAX}}(\widehat{G}_R, \underline{\widehat{d}}) = r_{\widehat{G}_R}(\underline{\widehat{d}}) = r_G(\underline{d}).$$

Remark 3.2.5. The divisor $\underline{d} = (1, 2, 3) \in \mathrm{Div}(G)$ in Example 3.2.4 is balanced. Writing $n := v_2 \cdot v_3 > 6$ we have $g(G) = n+1$. The canonical divisor of G is $\underline{k}_G = (1, \mathrm{val}(v_2) - 2, \mathrm{val}(v_3) - 2)$, where $\mathrm{val}(v_2) = n > 6$ and $\mathrm{val}(v_3) = n+3 > 9$. Consider the subsets $Z_i = \{v_i\}$ and $Z_{ij} = \{v_i, v_j\}$, for $i, j = 1, 2, 3$. We have

$$m_{Z_i}(\underline{d}) = 6\frac{\underline{k}_G(v_i)}{2(n+1)-2} - \frac{\mathrm{val}(v_i)}{2} \text{ and } M_{Z_i}(\underline{d}) = 6\frac{\underline{k}_G(v_i)}{2(n+1)-2} + \frac{\mathrm{val}(v_i)}{2},$$

for $i, j = 1, 2, 3$. Thus, since $\underline{d}(v_i) = i$ for $i = 1, 2, 3$, we have

$$m_{Z_1}(\underline{d}) = \frac{3}{2n} - \frac{3}{2} < \underline{d}(v_1) < M_{Z_i}(\underline{d}) = \frac{3}{2n} + \frac{3}{2},$$

$$m_{Z_2}(\underline{d}) = 3\frac{n-2}{2n} - \frac{n}{2} < \underline{d}(v_2) < M_{Z_i}(\underline{d}) = 3\frac{n-2}{2n} + \frac{n}{2},$$

$$m_{Z_3}(\underline{d}) = 3\frac{n+1}{2n} - \frac{n+3}{2} < \underline{d}(v_3) < M_{Z_i}(\underline{d}) = 3\frac{n+1}{2n} + \frac{n+3}{2}.$$

Also, since $n > 6$, $\underline{k}_G(Z_{12}) = n-1$, $\underline{k}_G(Z_{13}) = n+2$ and $\underline{k}_G(Z_{23}) = 2n-1$, we have

$$m_{Z_{12}}(\underline{d}) = 3\frac{n-1}{n} - \frac{n+3}{2} < \underline{d}(Z_{12}) < M_{Z_{12}}(\underline{d}) = 3\frac{n-1}{n} + \frac{n+3}{2},$$

$$m_{Z_{13}}(\underline{d}) = 3\frac{n+2}{n} - \frac{n}{2} < \underline{d}(Z_{13}) < M_{Z_{ij}}(\underline{d}) = 3\frac{n+2}{n} + \frac{n}{2},$$

$$m_{Z_{23}}(\underline{d}) = 3\frac{2n-1}{n} - \frac{3}{2} < \underline{d}(Z_{23}) < M_{Z_{23}}(\underline{d}) = 3\frac{2n-1}{n} + \frac{3}{2},$$

with $\underline{d}(Z_{12}) = 3$, $\underline{d}(Z_{13}) = 4$ and $\underline{d}(Z_{23}) = 5$. Therefore, $\underline{d}$ is balanced.

Now, for $\underline{\widehat{d}} = (1, 1, 2, 2) \in \mathrm{Div}(G')$ with $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$, with analogous calculation we have that the divisor $\underline{d}'$ is also balanced.

Example 3.2.6. Let G be a graph as in Example 2.6.1 (Figure 2.2). Here, we calculate r^{MAX} considering a different desingularization of a curve $X \in M^{\text{alg}}(G)$ considering the geometric genera of its irreducible components. Let (G', ω') be the graph with $V(G') = \{v'_1, v'_2, \widehat{v}\}$ such that $v'_1 \cdot v'_2 > 12$, $v'_2 \cdot \widehat{v} = 2$, which weights are $\omega'(v'_1) = 1, \omega'(v'_1) = 1$ and $\omega'(\widehat{v}) = 0$.

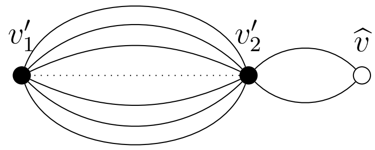
$G' =$ 

Figure 3.3: The weighted graph G' such that $\omega'(v'_1) = 1, \omega'(v'_1) = 1$ and $\omega'(\widehat{v}) = 0$.

Now, let $\underline{d}' = (3, 3, 1)$ be a divisor on G' and observe that $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$. Consider $\widehat{X} \in M^{\text{alg}}(G')$ and $\widehat{L} \in \text{Pic}^{\underline{d}'}(\widehat{X})$, we write

$$\widehat{X} = C_{v'_1} \cup C_{v'_2} \cup C_{\widehat{v}}.$$

By Lemma 1.2.9, we have $h^0(\widehat{X}, \widehat{L}) = h^0(Y, M)$, where $Y = C_{v'_1} \cup C_{v'_2}$ and $Y \sqcup C_{\widehat{v}}$ is the normalization of $\widehat{X}$ at $C_{v'_2} \cap C_{\widehat{v}}$ by μ and $M \in \text{Pic}^{(3,3)}(Y)$ is such that $\mu^* M = \widehat{L}$. Let G'_H be the dual graph of Y. Observe that Y is on the union of two irreducible curves of the same geometric genus, 1, so, there is an isomorphism $C_{v'_1} \cong C_{v'_2}$. Then there exists $M \in \text{Pic}^{(3,3)}(Y)$ such that $r(Y, M) = 2$. By Riemann-Roch on Y, we have

$$r^{\max}(Y, (3, 3)) = 2 = r_G(\underline{d}).$$

Remark 3.2.7. The divisor $\underline{d} \in \text{Div}(G)$ in Example 3.2.6 is balanced. The graph G is a weighted binary graph $B_k^{\omega_1.\omega_2}$ with $k = v_1 \cdot v_2 > 12$ and $\omega_1 = 1$ and $\omega_2 = 2$, its canonical divisor is $\underline{k}_G = (k, k+2)$. So, for $Z_1 = \{v_1\}$ and $Z_2 = \{v_2\}$ we have

$$m_{Z_i}(\underline{d}) = 7\frac{\underline{k}_G(v_i)}{2(k+1)-2} - \frac{k}{2} \text{ and } M_{Z_i}(\underline{d}) = 7\frac{\underline{k}_G(v_i)}{2(k+1)-2} + \frac{k}{2},$$

$i = 1, 2$. So,

$$m_{Z_1}(\underline{d}) = \frac{7-k}{2} < 3 < \frac{7+k}{2} = M_{Z_1}(\underline{d}),$$

$$m_{Z_2}(\underline{d}) = \frac{7}{k} + \frac{7-k}{2} < 4 < \frac{7}{k} + \frac{7+k}{2} = M_{Z_2}(\underline{d}).$$

Therefore, $\underline{d}$ is balanced.

Now, for $\underline{d}' = (3, 3, 1) \in \mathrm{Div}(G')$ with $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$, with analogous calculation we have that the divisor $\underline{d}'$ is also balanced.

From the examples above we can notice that even if the algebraic ranks are strictly smaller than the combinatorial ranks in the respective original curves and divisors, that the ranks of suitable refinements of the curves together with the divisors attain the respective combinatorial ranks.

Example 3.2.8. Let $G = B_k$ be a binary graph of genus $g = k - 1$. For every $\underline{d} = (d_1, d_2)$ divisor in G, with $0 \leq d_1 \leq d_2$, by Proposition 2.5.5 we have that $r_G(\underline{d}) = r^{\mathrm{alg}}(G, \underline{d})$. Consider the binary graph G_R where R has only one edge and consider the divisors $\underline{d}_1 = (d_1 - 1, d_2)$ and $\underline{d}_2 = (d_1, d_2 - 1)$ on G_R. In this case, G_R has genus $g - 1$. So, we have the following cases:

- if $d_2 \leq g$, then by Lemma 2.1.14, $r_G(\underline{d}) = d_1$ and $r_{G_R}(\underline{d}_1) = d_1 - 1$. Therefore, once B_k is binary graph, by Proposition 2.5.5, we have

$$d_1 = r^{\mathrm{alg}}(G, \underline{d}) \neq r^{\mathrm{alg}}(G_R, \underline{d}_1) = d_1 - 1.$$

However, if $d_1 \neq d_2$, for $\underline{d}_2$ we have $r_{G_R}(\underline{d}_2) = d_1 = r^{\mathrm{alg}}(G_R, \underline{d}_2)$. Therefore,

$$r^{\mathrm{alg}}(G, \underline{d}) = r^{\mathrm{alg}}(G_R, \underline{d}_2).$$

- If $d_2 \geq g+1$, then by Lemma 2.1.14, $r_G(\underline{d}) = d_1 + d_2 - g$,

$$r_{G_R}(\underline{d}_1) = d_1 - 1 + d_2 - (g-1) = d_1 + d_2 - g$$

  and

$$r_{G_R}(\underline{d}_2) = d_1 + d_2 - 1 - (g-1) = d_1 + d_2 - g.$$

  We have

$$r^{\mathrm{alg}}(G, \underline{d}) = d_1 + d_2 - g = r^{\mathrm{alg}}(G_R, \underline{d}_1) = r^{\mathrm{alg}}(G_R, \underline{d}_2).$$

In conclusion, $r_G(d)$ is in all cases the maximum of the ranks of these modifications of $\underline{d}$.

We wonder if similar configuration could hold in general. More precisely, we can ask the following question:

Question 1. Let G be a stable graph and $\underline{d} \in [\delta]$ such that

$$r^{\mathrm{ALG}}(G, \delta) = r^{\mathrm{MAX}}(G, \underline{d}),$$

then is it true that there is a triple $(G, \omega, \underline{d})$ such that $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$ and

$$r_G(\underline{d}) = r^{\mathrm{MAX}}(G', \underline{d}')?$$

Notice that if $r^{\mathrm{ALG}}(G, \delta) = r_G(\delta)$ then the answer is yes and it suffices to consider

$$(G', \omega', \underline{d}') = (G, \omega, \underline{d}).$$

Moreover, the answer should be yes as well for the situations illustrated in the examples of this section.

3.3 Balanced compactified rank

Definition 3.3.1 (Compactified algebraic rank, for weighted graphs)**.** Given a graph G of weight ω, we define the *compactified algebraic rank* of a divisor $\underline{d} \in \mathrm{Div}(G)$ as

$$\overline{r}(G, \underline{d}) = \max_{(G',\omega',\underline{d}') \geq (G,\omega,\underline{d})} r^{\mathrm{MAX}}(G', \underline{d}'). \tag{3.7}$$

Remark that $r^{\mathrm{MAX}}(G, \underline{d}) \leq \overline{r}(G, \underline{d})$, since $(G, \underline{d}) \geq (G, \underline{d})$.

Such as in the definition of the algebraic rank, we would like to extend the definition of the compactified algebraic rank for the equivalence class of a divisor. In order to do so we first have to find a divisor in each equivalence class that better adjusts to the value of the combinatorial rank, at least in some special cases of curves. Given a quasistable graph G and $\delta \in \mathrm{Pic}(G)$, by Proposition 1.2.11 we have that there exists a balanced representative divisor $\underline{d}$ of δ. Moreover, for line bundles of semibalanced multidegree we have extensions of Riemann's theorem and partially Clifford's theorem (see [8]). Motivated by this we define the compactified rank for balanced divisors.

Definition 3.3.2 (Balanced compactified rank)**.** Let G be a stable graph (weighted and possibly with loops). Let $\underline{d}$ be a balanced divisor on G. We define the *balanced compactified rank* of $\underline{d}$ as

$$\overline{r}^b(G, \underline{d}) := \max\{r^{\mathrm{MAX}}(G', \underline{d}') : (G', \underline{d}') \geq (G, \underline{d}) \text{ and } \underline{d}' \text{ is a balanced divisor }\}.$$

Remark 3.3.3. Given $\delta \in \mathrm{Pic}(G)$, by Proposition 1.2.11 a balanced representative $\underline{d}$ of δ is unique if and only if $\underline{d}$ is stably balanced. Therefore, in general, Definition 3.3.2 may not be constant in the equivalence class of a divisor so it cannot be extended to the divisor class. However, if $\underline{d} \in \delta$ is stably balanced it is the unique balanced divisor of its equivalence class. An important example of a stably balanced divisor is a break divisor: if G has genus g and $\delta \in \mathrm{Pic}^g(G)$, by Theorem 1.1.14, there is a unique representative $\underline{d}$ of δ that is a break divisor.

Firstly, we compute the balanced compactified rank of the canonical divisor of a stable graph. Observe that the canonical divisor itself is balanced.

Lemma 3.3.4. *Given a stable graph G of genus g, we have*

$$\overline{r}^b(G, \underline{k}_G) = r^{\text{MAX}}(G, \underline{k}_G) = g - 1.$$

Proof. Suppose that the pair $(G', \underline{d}')$ is such that $(G', \underline{d}') \geq (G, \underline{k}_G)$, we know that G' also has genus g and that $\underline{d}'$ also has degree $2g - 2$. Additionally, $\underline{d}'$ is a balanced divisor on G' of degree $2g - 2$, hence, by Corollary 3.1.4 we have

$$r^{\text{MAX}}(G', \underline{d}') \leq g - 1 \text{ and } r^{\text{MAX}}(G, \underline{k}_G) = g - 1.$$

So

$$r^{\text{MAX}}(G', \underline{d}') \leq r^{\text{MAX}}(G, \underline{k}_G).$$

Therefore, since $(G, \underline{k}_G) \geq (G, \underline{k}_G)$, we have that $(G, \underline{k}_G)$ realizes $\overline{r}^b(G, \underline{k}_G)$. □

3.3.1 Riemann-Roch theorem for the balanced compactified rank

The aim of this section is to prove the Riemann-Roch theorem for the balanced compactified rank. Let G be a stable graph and let $\underline{d}$ be a balanced divisor on G. In order to calculate the balanced compactified rank of the divisor $\underline{d}^* := \underline{k}_G - \underline{d}$, $\overline{r}^b(G, \underline{k}_G - \underline{d})$, we calculate $r^{\text{MAX}}(G', \underline{d}'^*)$ for pairs $(G', \underline{d}'^*) \geq (G, \underline{d}^*)$ for which the divisor $\underline{d}'^*$ is balanced. Observe that if the divisor $\underline{d}' \in \text{Div}(G')$ realizes $\overline{r}^b(G, \underline{d})$, the divisor $\underline{k}_{G'} - \underline{d}'$ is not necessarily balanced.

In the following construction we establish that it is sufficient to calculate r^{MAX} on a suitable desingularization of the graph G'. We show that this is true for different configurations of G'.

Construction 1. Suppose that the graph G' is such that $(G', \underline{d}') \geq (G, \underline{d})$. We set $\mathcal{E} = \{\widehat{v}_1, \ldots, \widehat{v}_n\} \subseteq V(G')$ the set of exceptional vertices of G', hence $V(G') = V(G) \cup \mathcal{E}$. Denote by H the set of all edges of G' adjacent to a vertex of $\mathcal{E}$. By the definition of exceptional contraction, there exists a subset F of H and a contraction map $\sigma_F : G' \to G$ contracting all the edges of F. Let $\widehat{v} \in \mathcal{E}$, then there are two possibilities, either there is only one edge in F adjacent to $\widehat{v}$ or there are two edges of F adjacent to $\widehat{v}$. This provides the following decomposition $F = F^1 \sqcup F^2$ where

$$F^1 := \{e \in F : \exists \widehat{v} \in \mathcal{E} \text{ such that } e \text{ is the only edge of } F \text{ adjacent to } \widehat{v}\}$$

and

$$F^2 := \{e \in F : \exists \widehat{v} \in \mathcal{E} \text{ such that } e \text{ is one of the two edges of } F \text{ adjacent to } \widehat{v}\}.$$

In this last situation, $\sigma_F(\widehat{v})$ is a vertex w of G such that $\omega_G(w) > \omega_{G'}(w)$. In order to better visualise this construction, consider the following example of graphs G and G' pictured in Figure 3.4. The graphs G' and G are such that $G' \geq G$, we also have $\omega_G(v_1) = 0$, $\omega_G(v_2) = \omega_2$, and $\omega_{G'}(v_2) = \omega_2 - 1$.

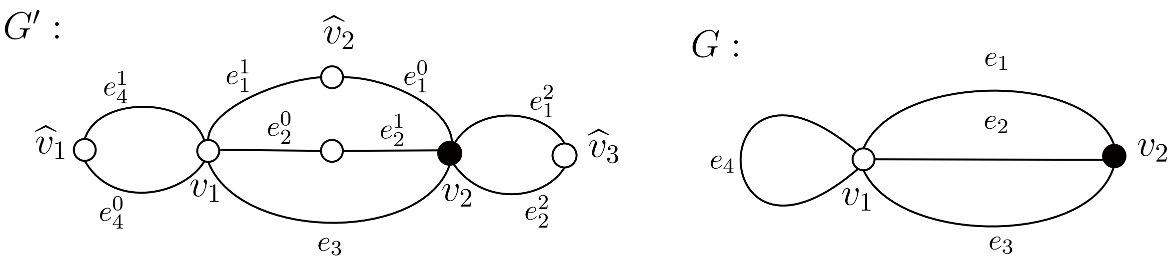


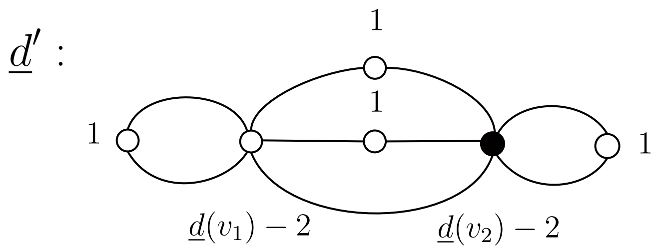
Figure 3.4: The graphs G and G'.

Let $F^0 := H \setminus F$ and notice that $\#F^1 = \#F^0$. In our reference example, we have the sets $H = \{e_1^0, e_1^1, e_2^0, e_2^1, e_1^2, e_2^2, e_4^0, e_4^1\}$, $F = \{e_1^1, e_2^1, e_1^2, e_2^2, e_4^1\}$, $F^1 = \{e_1^1, e_2^1, e_4^1\}$, $F^2 = \{e_1^2, e_2^2\}$ and $F^0 = \{e_1^0, e_2^0, e_4^0\}$.

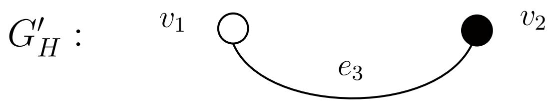
Given a set $A \subseteq E(G')$ and a vertex $v \in V(G')$, we set $\gamma_A(v)$ the number of edges in A adjacent to v. Since $(G',\underline{d}')\geq(G,\underline{d})$ with exceptional contraction σ_F, the divisor $\underline{d}'$ is obtained from $\underline{d}$ as follows

$$\underline{d}'(v)=\begin{cases}1 & \text{if } v\in\mathcal{E},\\ \underline{d}(v)-\gamma_{F^1}(v)-\frac{\gamma_{F^2}(v)}{2} & \text{otherwise.}\end{cases}$$

In our reference example, the divisor $\underline{d}'$ is given as in the figure below.



Now, consider the graph $G'_H=G'-\mathcal{E}$, the graph obtained by subtracting from G' the subset $\mathcal{E}$ and the subset H of all the edges incident to vertices in $\mathcal{E}$. Observe that $g(G'_H)=g(G')-n=g(G)-n$, where $n=|\mathcal{E}|$ is the cardinality of the exceptional vertices in G'.



Consider the divisor $\underline{d}'_H$ in G'_H, we have $\underline{d}'_H(u)=\underline{d}'(u)$ for all $u\in V(G'_H)$. Given $v\in V(G'_H)$ we denote by $\mathrm{val}_{G'_H}(v)$ the valency of v as a vertex of G'_H, if we consider v as a vertex of G' we have that the valency of v as a vertex of G' is $\mathrm{val}_{G'_H}(v)+\gamma_H(v)$. So $\underline{k}_{G'_H}(v)=2\omega_{G'_H}(v)+\mathrm{val}_{G'_H}(v)-2$ and $\underline{d}'_H{}^*:=\underline{k}_{G'_H}-\underline{d}'_H$ is given by

$$\underline{d}'_H{}^*(v)=2\omega_{G'_H}(v)+\mathrm{val}_{G'_H}(v)-2-\underline{d}(v)+\gamma_{F^1}(v)+\frac{\gamma_{F^2}(v)}{2}. \tag{3.8}$$

Set $R=F^0\cup F^2$ and consider G' contracted by a different map $\sigma_R:G'\to G$ contracting all the edges of F^0 and F^2. Recall that $V(G')=V(G)\cup\mathcal{E}$. The divisor $\underline{d}^{*\prime}$ such that

$\sigma_R(\underline{d}^{*\prime}) = \underline{d}^* := \underline{k}_G - \underline{d}$ is given by

$$\underline{d}^{*\prime}(v) = \begin{cases} 1 & \text{if } v \in \mathcal{E}, \\ \underline{k}_{G'}(v) - \underline{d}(v) - \gamma_{F^0}(v) - \frac{\gamma_{F^2}(v)}{2} & \text{otherwise.} \end{cases}$$

Now, consider the graph G'_H. Denote by $\underline{d}^{*\prime}{}_H$ the restriction of the divisor $\underline{d}^{*\prime}$ to the graph G'_H, we have $\underline{d}^{*\prime}{}_H(v) = \underline{k}_{G'}(v) - \underline{d}(v) - \gamma_{F^0}(v) - \frac{\gamma_{F^2}(v)}{2}$, for $v \in V(G'_H)$.Thus,

$$\begin{aligned} \underline{d}^{*\prime}{}_H(v) =& 2\omega_{G'}(v) + \mathrm{val}_{G'}(v) - 2 - \underline{d}(v) - \gamma_{F^0}(v) - \frac{\gamma_{F^2}(v)}{2} \\ =& 2\omega_{G'_H}(v) + (\mathrm{val}_{G'_H}(v) + \gamma_H(v)) - 2 - \underline{d}(v) - \gamma_{F^0}(v) - \frac{\gamma_{F^2}(v)}{2} \\ =& 2\omega_{G'_H}(v) + \mathrm{val}_{G'_H}(v) - 2 - \underline{d}(v) + \gamma_{F^1}(v) + \frac{\gamma_{F^2}(v)}{2}; \qquad (3.9) \end{aligned}$$

since $\gamma_H(v) = \gamma_{F^0}(v) + \gamma_{F^1}(v) + \gamma_{F^2}(v)$. Therefore, (3.9) is equal to (3.8).

Remark 3.3.5. Recall from Section 3.2 that given a quasistable graph G' whose set of edges adjacent to an exceptional vertex is H, the graph G such that $G = G'_H/H$ is stable. So, a dual curve X of G is stable and the blow up of X on H, denoted by $\widehat{X}_H$, is a quasistable curve and has G' as its dual graph. We also consider the curve X^ν_H, the normalization of $\widehat{X}_H$ at H, and whose dual graph is G'_H, defined by subtracting from G all the edges of H.

If $\underline{d} \in \mathrm{Div}(G)$ is balanced, in order to calculate $\overline{r}^b(G, \underline{d})$ we calculate $r^{\mathrm{MAX}}(G', \underline{d}')$ where $\underline{d}'$ is an admissible balanced divisor on G' whose exceptional contraction is $\underline{d}$. Observe that if a line bundle $L' \in \mathrm{Pic}^{\underline{d}'}(G')$ is admissible, hence, by Lemma 1.2.9, we know that

$$h^0(\widehat{X}_H, L') = h^0(X^\nu_H, L^\nu_H) \qquad (3.10)$$

where $L'|_{\widehat{X}_H} = L^\nu_H$. Using the notation in Construction 1, as a result of (3.10), we have

$$r^{\mathrm{MAX}}(G', \underline{d}') = r^{\mathrm{MAX}}(G'_H, \underline{d}'_H),$$

where $\underline{d}'_H \in \mathrm{Div}(G'_H)$ is such that $\underline{d}_H(v) = \underline{d}'(v)$, for all $v \in V(G)$.

Therefore, as a consequence of Construction 1, in order to calculate $\overline{r}^b(G, k_G - \underline{d})$, where $\underline{d}$ is a balanced divisor on G, we can calculate $r^{\mathrm{MAX}}(G'_R, \underline{d}'_H)$, where $\underline{d}'_H$ is the restriction of a divisor $\underline{d}' \in \mathrm{Div}(G')$ to the graph G'_H such that $(G', \underline{d}') \geq (G, \underline{d})$.

Theorem 3.3.6 (Riemann-Roch)**.** *Let G be a stable graph of genus g, $\underline{d} \in B^d_X$, then*

$$\overline{r}^b(G, \underline{d}) - \overline{r}^b(G, \underline{k}_G - \underline{d}) = d - g + 1.$$

Proof. It follows from Remark 3.3.5 that $\overline{r}^b(G, \underline{k}_G - \underline{d})$ is realized by $r^{\mathrm{MAX}}(\widehat{G}_H, \underline{k}_{\widehat{G}_H} - \underline{\widehat{d}}_H)$, with $(\widehat{G}, \underline{\widehat{d}}) \geq (G, d)$ and $G = \widehat{G}/H$. It is sufficient to prove that if $(G', \underline{d}')$ is the pair that realizes $\overline{r}^b(G, \underline{d})$, with $G = G'/R$, then $\overline{r}^b(G, \underline{k}_G - \underline{d})$ is realized by $r^{\mathrm{MAX}}(G', \underline{k}_{G'_R} - \underline{d}'_R)$. Suppose that $\overline{r}^b(G, \underline{k}_G - \underline{d}) = r^{\mathrm{MAX}}(\widehat{G}_H, \underline{k}_{\widehat{G}_H} - \underline{\widehat{d}}_H)$. Then applying Theorem 3.1.1 we have

$$\begin{aligned} r^{\mathrm{MAX}}(G'_R, \underline{d}'_R) - r^{\mathrm{MAX}}(G'_R, \underline{k}_{G'_R} - \underline{d}'_R) &= d - g + 1 \\ &= r^{\mathrm{MAX}}(\widehat{G}_H, \underline{\widehat{d}}_H) - r^{\mathrm{MAX}}(\widehat{G}_H, \underline{k}_{\widehat{G}_H} - \underline{\widehat{d}}_H). \end{aligned}$$

Implying that

$$0 \leq r^{\mathrm{MAX}}(G'_R, \underline{d}'_R) - r^{\mathrm{MAX}}(\widehat{G}_H, \underline{\widehat{d}}_H) = r^{\mathrm{MAX}}(G'_R, \underline{k}_{G'_R} - \underline{d}'_R) - r^{\mathrm{MAX}}(\widehat{G}_H, \underline{k}_{\widehat{G}_H} - \underline{\widehat{d}}_H),$$

Hence

$$r^{\mathrm{MAX}}(G'_R, \underline{k}_{G'_R} - \underline{d}'_R) \geq r^{\mathrm{MAX}}(\widehat{G}_H, \underline{k}_{\widehat{G}_H} - \underline{\widehat{d}}_H).$$

□

Corollary 3.3.7. *Let G be a stable graph of genus g and let $\underline{d} \in \mathrm{Div}^d(G)$ be a balanced divisor, then*

(a) if $d = 2g - 2$, we have $\overline{r}^b(G, \underline{d}) \leq g - 1$, and equality holds if $\underline{d} = \underline{k}_G$.

(b) If $d > 2g-2$, we have $\overline{r}^b(G,\underline{d}) = d-g$.

(c) If $d = 0$, we have $\overline{r}^b(G,\underline{d}) \leq 0$, and equality holds if $\underline{d} = \underline{0}$.

(d) If $d < 0$, we have $\overline{r}^b(G,\underline{d}) = -1$.

Proof. If $d > 2g-2$, by Corollary 3.1.4 we have $r^{\mathrm{MAX}}(G,\underline{d}) = d-g$, and this equality holds for any $(G',\underline{d}') \geq (G,\underline{d})$ because $d = |\underline{d}'|$ and $g = g(G) = g(G')$. Therefore, $\overline{r}^b(G,\underline{d}) = d-g$, and item (b) is proved.

If $\underline{d} < 0$ then $|\underline{d}^*| = |\underline{k}_G(\underline{d}) - \underline{d}| = 2g-2-d > 2g-2$. So, by item (b), it follows that

$$\overline{r}^b(G,\underline{k}_G - \underline{d}) = (2g-2-d) - g = g-d-2.$$

Therefore, by Theorem 3.3.6 we have

$$\overline{r}^b(G,\underline{d}) = \overline{r}^b(G,\underline{k}_G - \underline{d}) + d - g + 1 = (g-d-2) + d - g + 1 = -1.$$

If $d = 2g-2$, we know that $r_G(\underline{d}) \leq g-1$ and the equality holds if and only if $\underline{d} \sim \underline{k}_G$. Since $\underline{d}$ is balanced, we have, by Theorem 3.1.3, that

$$r^{\mathrm{MAX}}(G,\underline{d}) = r_G(\underline{d}) \leq g-1. \tag{3.11}$$

Notice that given $(G',\underline{d}') \leq (G,\underline{d})$ the degree of $\underline{d}'$ equals the degree of $\underline{d}$, hence (3.11) holds for $(G',\underline{d}')$. Therefore, $\overline{r}^b(G,\underline{d}) \leq g-1$. Lemma 3.3.4 asserts that the equality occurs if $\underline{d} = \underline{k}_G$, i.e., $\overline{r}^b(G,\underline{k}_G) = g-1$.

If $d = 0$, then $|\underline{d}^*| = |\underline{k}_G - \underline{d}| = 2g-2$, and it follows from item (a) that

$\overline{r}^b(G, \underline{k}_G - \underline{d}) \leq g - 1$. By Theorem 3.3.6, we have

$$\begin{aligned}\overline{r}^b(G, \underline{d}) &= \overline{r}^b(G, \underline{k}_G - \underline{d}) + d - g + 1 \\ &\leq (g-1) + (2g - 2 - g + 1) \\ &\leq g - 1 - g + 1 = 0.\end{aligned}$$

□

The balanced compactified rank satisfies a version of the Specialization Lemma.

Lemma 3.3.8. *Let $\phi : \mathcal{X} \to B$ be a regular one-parameter smoothing of a connected curve X. Let G be the dual graph of X. Then for every $\mathcal{L} \in \mathrm{Pic}_\phi(B)$ there exists an open neighbourhood $U \subset B$ of b_0 such that for every $b \in U \setminus \{b_0\}$ we have*

$$r(X_b, \mathcal{L}(b)) \leq \overline{r}^b(G, \underline{d}),$$

where $\underline{d}$ is a balanced multidegree equivalent to $\underline{\deg}\mathcal{L}(b_0)$.

Proof. Take $\mathcal{L}'(b_0) \sim_\phi \mathcal{L}(b_0)$ with $\underline{\deg}\mathcal{L}'(b_0) = \underline{d}$ balanced. As explained during the proof of Lemma 2.4.1, we know that, once $\mathcal{L}'(b_0)$ is on the ϕ-class of $\mathcal{L}(b_0)$, by uppersemicontinuity of h^0 and by definition of $r^{\max}$ we have

$$r(X_b, \mathcal{L}(b)) \leq r^{\max}(X, \underline{d})$$

for every b in an open neighbourhood $U \subset B$ of b_0. On the other hand, by definition of r^{MAX}, $r^{\max}(X, \underline{d}) \leq r^{\mathrm{MAX}}(G, \underline{d})$. Once $(G, \underline{d}) \geq (G, \underline{d})$ and $\underline{d}$ is balanced we have $r^{\mathrm{MAX}}(G, \underline{d}) \leq \overline{r}^b(G, \underline{d})$. Therefore,

$$r(X_b, \mathcal{L}(b)) \leq \overline{r}^b(G, \underline{d}).$$

□

3.3.2 Inequality of ranks and generalized Dhar decomposition

Divisors of generalized Clifford type

In [8], Caporaso presents some examples where Clifford's inequality fails for some balanced divisors. In particular, there is an example of a curve with a line bundle of degree $d \geq 3$ that can be seen as a degeneration of a semistable curve.

Example 3.3.9. [8, Example 4.15] Fix an integer $d \geq 3$. Let G be a semistable graph and set the order $V(G) = \{v_1, \ldots, v_d\}$ such $v_i \cdot v_{i+1} = 1 = v_d \cdot v_1$, for $i \geq 1$. Also $\omega(v_i) = 1$, for all $i \geq 1$. Observe that $g = d + 1$.

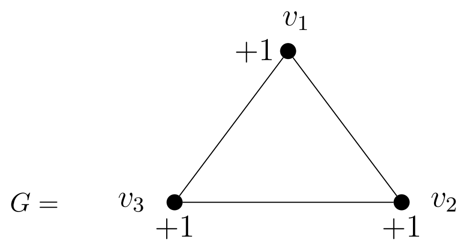


Figure 3.5: The graph G, for $d = 3$.

Consider the divisor $\underline{d} = (1, \ldots, 1)$ on G of degree d. Now, consider the graph G' obtained from G by adding one new exceptional vertex at each edge of G. We write $V(G') = \{v_1, \widehat{v_1}, \cdots, v_d, \widehat{v_d}\}$.

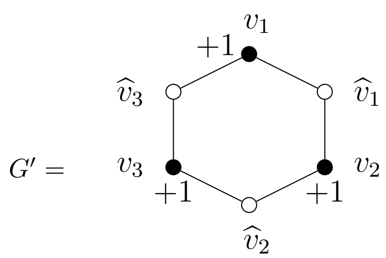


Figure 3.6: The graph G', for $d = 3$.

So, a curve $Y \in M^{\text{alg}}(G')$ is such that $Y = C_1 \cup \cdots \cup C_{2d}$, with $C_{2i} \cong \mathbb{P}^1$ and C_{2i-1} has geometric genus 1, for all i. Notice that Y has $2d$ nodes. Now, set the balanced divisor $\underline{d}' = (0, 1, \ldots, 0, 1)$ on G' and observe that $(G', \underline{d}') \geq (G, \underline{d})$. For any $L \in \operatorname{Pic}^{\underline{d}'}(Y)$, L is such that $L_{C_{2j-1}} \cong \mathcal{O}_{C_{2j-1}}$ and $L_{C_{2j}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Consider the total normalization $Y^\nu = C_1 \sqcup \ldots \sqcup C_{2d}$ of Y. Given a line bundle $L \in \operatorname{Pic}^{\underline{d}'}(Y)$, let $L^\nu \in \operatorname{Pic}(Y^\nu)$ be such that $\nu^* L = L^\nu$. By Remark 1.2.9 we have

$$h^0(Y, L) \geq h^0(Y^\nu, L^\nu) - 2d = d + 2d - 2d = d.$$

Therefore, Clifford's inequality fails for any $L \in \operatorname{Pic}^{\underline{d}'}(X)$ such that the restrictions to the C_i are as above, once $h^0(X, L) > d/2 + 1$.

We know that the combinatorial rank satisfies Clifford's inequality (Theorem 2.1.9), i.e., for any $0 \leq d \leq 2g - 2$, we have

$$r_G(\underline{d}) \leq \frac{d}{2},$$

where $\underline{d} \in \operatorname{Div}^d(G)$. Therefore, in this case, since $d \geq 3$ one has

$$r^{\mathrm{MAX}}(G, \underline{d}) \geq d - 1 > \frac{d}{2} \geq r_G(\underline{d}).$$

Let us now study curves and line bundles for which Clifford's inequality holds not only for the curve but also for each one of its subcurves.

Definition 3.3.10. Let $\underline{d}$ be a divisor on G. We say that $\underline{d}$ is of *generalized Clifford type* if given $X \in M^{\mathrm{alg}}(G)$ for every line bundle $L \in \operatorname{Pic}^{\underline{d}}(X)$ we have that Clifford's Inequality holds for every subcurve Z of X (not necessarily proper). In other words, we have

$$r(Z, L_{|Z}) \geq e_Z \Rightarrow \deg L_{|Z} \geq e_Z^{g(Z)} := e_Z + \min\{e_Z, g(Z)\}.$$

Remark 3.3.11. By Clifford's theorem, if $|V(G)| = 1$, any $\underline{d} \in \operatorname{Div}(G)$, with $0 \leq \deg L \leq 2g$, is of generalized Clifford type. Indeed, if X is a dual curve of G then X is irreducible and any $L \in \operatorname{Pic}^{\underline{d}}(X)$ is such that

$$h^0(X, L) \leq \frac{d}{2} + 1.$$

Example 3.3.12. Let G be the weighted binary graph $B_k^{\omega_1,\omega_2}$. Let $\underline{d} = (d_1, d_2) \in \operatorname{Div}(G)$ be a balanced divisor with $0 \leq d \leq 2g$. By Theorem 1.2.15, we have that for every $X \in M^{\text{alg}}(G)$ and $L \in \operatorname{Pic}^{\underline{d}}(X)$,

$$h^0(X, L) \leq \frac{d}{2} + 1.$$

By Remark 3.3.11, each irreducible component of X satisfies Clifford's theorem. Namely, writing $X = C_1 \cup C_2$, we have

$$h^0(C_i, L_{|C_i}) \leq \frac{d_i}{2} + 1.$$

for $i = 1, 2$. Therefore, $\underline{d}$ is of generalized Clifford type.

Generalized Dhar decomposition

Now, we present a generalization to the Dhar decomposition introduced in Section 1.1.2. Let G be a stable graph and let $\underline{d}$ a divisor in G. Fix a proper subset S of $V(G)$ such that in $V \setminus S$ the divisor $\underline{d}$ is effective. The *generalized Dhar decomposition* of G associated to $\underline{d}$ with respect to S is

$$V = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_l \sqcup W.$$

The construction of the decomposition is the following. Denote by $Y_0 = S$ and $W_0 = V \setminus S$. If the divisor $\underline{d} + \underline{t}_{W_0}$ is effective then we set $W = W_0$ and we have the decomposition $V = Y_0 \sqcup W$. Otherwise, set Y_1 the set of vertices of W_0 where $\underline{d} + \underline{t}_{W_0}$

is negative. Now, we iterate the process, after defining $Y_0, \ldots, Y_{j-1}$, denote

$$W_{j-1} := V \setminus Y_0 \sqcup \ldots \sqcup Y_{j-1}$$

and consider the divisor $\underline{d} + \underline{t}_{W_{j-1}}$. If this divisor is effective then we set $W := W_{j-1}$. Otherwise, define Y_j the set of vertices of W_{j-1} where $\underline{d}+\underline{t}_{W_{j-1}}$ is negative. This process eventually exhausts all the vertices of the graph G, and in this case W will be empty.

Definition 3.3.13. Given a proper subset S of vertices of a graph G, we say that a divisor $\underline{d} \in \mathrm{Div}(G)$ is *S-reduced* if in the Dhar decomposition of G we have that W is empty.

The S-reducedness of a divisor can be characterized analogously to the definition of reducedness with respect to a vertex. Namely, given a proper subset $S \subset V(G)$, a divisor $\underline{d} \in \mathrm{Div}(G)$ is S-reduced if

1. $\underline{d}(v) \geq 0$, for all $v \in, V(G) \setminus S$;
2. For all $A \subset V(G) \setminus S$, there is a $v \in A$ such that $\underline{d}(v) < v \cdot (V(G) \setminus A)$.

Now, we show that the proof given by Baker and Norine in [3] for the existence of a u-reduced divisor can be extend for S-divisors.

Proposition 3.3.14. *Let G be a graph and let $\delta \in \mathrm{Pic}(G)$, then for any proper subset $S \subset V(G)$ the divisor class δ has a S-reduced representative.*

Proof. Given a divisor $\underline{d} \in \mathrm{Div}(G)$ and a proper subset $S \subset V(G)$, the proof starts with the construction of divisor linearly equivalent to $\underline{d}$ that later is proved to be S-reduced. The first step is to construct a divisor in the class of $\underline{d}$ that is effective outside S.

For $v \in V(G)$, let $D(v, S)$ denote the length of the shortest path in G between v and a vertex of S. Let $R_k = \{v \in V(G) : D(v, S) = k\}$ for $0 \leq k \leq d$, so we can thus arrange the vertices of $V(G) \setminus S$ in an order such that every vertex outside S has a neighbour preceding it in this order.

Let $D = \max_{v \in V(G)} D(v, S)$ and define the vectors $\mu_1(\underline{d}) \in \mathbb{Z}^D$ and $\mu_2(\underline{d}) \in \mathbb{Z}^{D+1}$ by

$$\mu_1(\underline{d}) := \left(\sum_{\substack{v \in R_D \\ \underline{d}(v) < 0}} \underline{d}(v), \sum_{\substack{v \in R_{D-1} \\ \underline{d}(v) < 0}} \underline{d}(v), \ldots, \sum_{\substack{v \in R_1 \\ \underline{d}(v) < 0}} \underline{d}(v) \right),$$

$$\mu_2(\underline{d}) := \left(\sum_{v \in R_0} \underline{d}(v), \sum_{v \in R_1} \underline{d}(v), \ldots, \sum_{v \in R_D} \underline{d}(v) \right).$$

Replacing $\underline{d}$ by an equivalent divisor if necessary, we may assume without loss of generality that

$$\mu_1(\underline{d}) = \max_{\underline{d}' \sim \underline{d}} \mu_1(\underline{d}') \quad \text{and} \quad \mu_2(\underline{d}) = \max_{\substack{\underline{d}' \sim \underline{d} \\ \mu_1(\underline{d}) = \mu_1(\underline{d}')}} \mu_2(\underline{d}'),$$

where the maxima are taken in the lexicographic order. The vector $\mu_1(\underline{d})$ tell us how negative $\underline{d}$ is outside S in the order of the subsets S_k's. So, calculating the maximum of the vectors μ_1 in the divisor class of $\underline{d}$ we are making a series of chip-firing moves, starting by the last vertex in the giving order, to orderly make the degree of $\underline{d}$ at no vertex outside S negative. It is easy to see that both maxima are attained.

Claim. The resulting divisor $\underline{d}$ is S-reduced.

Suppose that $\underline{d}(v) < 0$ for some vertex $v \in V(G) \setminus S$. Let v' be adjacent to v such that $D(v', S) < D(v, S)$ and $\underline{d}' = \underline{d} - \underline{t}_{v'}$. Then $\underline{d}'(v) > \underline{d}(v)$, and $\underline{d}'(u) \geq \underline{d}(u)$ for every u such that $D(u, S) \geq D(v, S)$. It follows that $\mu_1(\underline{d}') > \mu_1(\underline{d})$, contradicting the choice of $\underline{d}$. Therefore $\underline{d}(v) \geq 0$ for every $v \in V(G) \setminus S$.

Suppose now that for some non-empty subset $A \subseteq V(G) \setminus S$, we have $\underline{d}(v) \geq (v, A)$ (the out degree of v in A) for every $v \in A$. Let $\underline{d}' = \underline{d} - \underline{t}_A$ and $D_A = \min_{v \in A} D(v, S)$. We have $\underline{d}'(v) \geq \underline{d}(v)$ for all $v \in V(G) \setminus A$ and $\underline{d}'(v) = \underline{d}(v) - (v, A) \geq 0$ for every $v \in A$. Therefore $\mu_1(\underline{d}) = \mu_1(\underline{d}')$, as they are both the zero vector. There must be a vertex

$v' \in V(G)$ such that $D(v,S) < D_A$, and for which v' is adjacent to a vertex of A. It follows that $\underline{d}'(v') > \underline{d}(v')$, and consequently $\mu_2(\underline{d}') > \mu_2(\underline{d})$, once again contradicting the choice of $\underline{d}$. This finishes the proof of claim and of the proposition. □

Notice that we have that the S-reduced representative is unique if $S = \{u\}$, but not in the general case.

Example 3.3.15. Let G be the graph pictured bellow in Figure 3.7. Consider the order $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the divisor $\underline{d} = (0, 2, 1, 1, 2, 1)$. Then the generalized Dhar decoposition of G associated to $\underline{d}$ with respect to $S_1 = \{v_1, v_6\}$ is as follows

$$Y_0 \sqcup Y_1 \sqcup W = \{v_1, v_6\} \sqcup \{v_4\} \sqcup \{v_2, v_3, v_5\}.$$

On the hand, the decomposition with respect to $S_2 = \{v_3, v_5\}$ is

$$Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup Y_3 \sqcup Y_4 = \{v_3, v_5\} \sqcup \{v_1\} \sqcup \{v_4\} \sqcup \{v_2, v_6\}.$$

In this case, since W is empty, the divisor $\underline{d}$ is S_2-reduced.

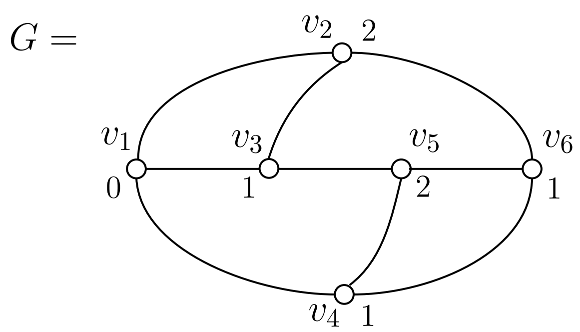


Figure 3.7: The graph G and the divisor $\underline{d} = (0, 2, 1, 1, 2, 1)$.

The inequality of ranks for divisors of generalized Clifford type

Recall that given $X \in M^{\text{alg}}(G)$, we denote $X = \cup_{v \in V(G)} C_v$, where C_v is the component corresponding to the vertex $v \in V(G)$; and given $\underline{d} \in \text{Div}(G)$ and $S \subseteq$

$V(G)$, we write $\underline{d}(S) := \sum_{v\in S}\underline{d}(v)$. If $S \subseteq V(G)$, we denote by C_S the curve $\cup_{v\in S}C_v$. Similarly to Lemma 2.3.1, we have the following generalization.

Lemma 3.3.16. *Let $\underline{d} \in \mathrm{Div}(G)$ and let X be a nodal curve whose dual graph is G and let $L \in \mathrm{Pic}^{\underline{d}}(X)$. Suppose that for some $S \subset V(G)$, and effective divisor $\underline{e} \in \mathrm{Div}(G)$, the divisor $\underline{d} - \underline{e}^{\deg}$ is S-reduced. Then the space of global sections of L vanishing identically on C_S has dimension at most $|\underline{e}| - \underline{e}(S)$.*

Proof. Consider the Dhar decomposition $V = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_l$ associated to the S-reduced divisor $\underline{d} - \underline{e}^{\deg}$. Denote by Λ_j, for each $0 \le j \le l$, the space of sections of L vanishing on the components of X corresponding to the vertices of $Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_l$. We want to prove that $\dim \Lambda_0 \le |\underline{e}| - \underline{e}(S)$. The prove is conducted by induction on $0 \le j \le l$, showing that

$$\dim \Lambda_j \le \sum_{i=j+1}^{l} |\underline{e}_{|Y_i}|. \tag{3.12}$$

For $j = l$, the claim is true, since Λ_l is the space of sections vanishing on the entire curve, and its dimension is 0. Now, assume that (3.12) holds for j and consider Λ_{j-1}.

Let v be any vertex of Y_j, and let D_v be the divisor on C_v consisting exactly of the intersection points of C_v with the components of X corresponding to $Y_0 \sqcup \ldots \sqcup Y_{j-1}$. Notice that Remark 1.1.11 is also valid for the case of S-reduced divisors. So, since $\underline{d} - \underline{e}^{\deg}$ is S-reduced, we have

$$(\underline{d} - \underline{e}^{\deg})(v) - \deg(D_v) < 0.$$

Hence

$$deg_{C_v} L(-D_v) < \underline{e}^{\deg}(v),$$

and therefore, we have

$$h^0(C_v, L(-D_v)) \leq \underline{e}(v).$$

We obtain

$$\dim \left(\bigoplus_{v \in Y_j} H^0(C_v, L(-D_v)) \right) \leq \sum_{v \in Y_j} \underline{e}(v) = |\underline{e}_{|Y_j}|. \tag{3.13}$$

Now, consider the exact sequence

$$0 \longrightarrow \Lambda_j \longrightarrow \Lambda_{j-1} \xrightarrow{\alpha} \bigoplus_{v \in Y_j} H^0(C_v, L(-D_v)),$$

where α is the map restricting a section to each component. Then

$$\dim \Lambda_{j-1} \leq \Lambda_j + \dim \left(\bigoplus_{v \in Y_j} H^0(C_v, L(-D_v)) \right) \leq \sum_{i=j}^{l} |\underline{e}_{|Y_i}|,$$

where the last inequality follows from the induction hypothesis and (3.13). The Lemma is proved. □

Lemma 3.3.17. *Let $\underline{d} \in \mathrm{Div}(G)$ be of generalized Clifford type and such that $r^{\mathrm{MAX}}(G, \underline{d}) = s$. Fix a subset S of $V(G)$. Given $\underline{e} \in \mathrm{Div}^s_+(G)$ such that $\underline{d} - \underline{e}^{\deg}$ is S-reduced we have that $\underline{d}(S) - \underline{e}^{\deg}(S) \geq 0$.*

Proof. The proof follows the proof of Lemma 2.3.2. Since $\underline{d} - \underline{e}^{\deg}$ is S-reduced, we know by definition that this divisor is effective at $V(G) \setminus S$. And since $r^{\mathrm{MAX}}(G, \underline{d}) = s$, there exist $X \in M^{\mathrm{alg}}(G)$ and $L \in \mathrm{Pic}^{\underline{d}}(X)$ such that $r(X, L) \geq s$. Now, consider the following exact sequence

$$0 \to \ker(\pi) \to H^0(X, L) \to H^0(C_S, L_{C_S}) \tag{3.14}$$

where π is the restriction of sections to C_S. Observe that $\ker(\pi)$ is the set of global sections of L vanishing at C_S, hence by Lemma 3.3.16,

$$\dim(\ker(\pi)) \leq s - \underline{e}(S).$$

And from (3.14) we have

$$h^0(C_S, L_{C_S}) \geq h^0(X, L) - \dim(\ker(\pi)) \geq s + 1 - s + \underline{e}(S) = \underline{e}(S) + 1.$$

From Definition 2.1.10, we have that

$$\underline{e}^{\deg}(S) = \underline{e}(S) + \min\{\underline{e}(S), g(S)\}.$$

Since $\underline{d}$ is of generalized Clifford type, $\underline{e}^{\deg}(S)$ is the minimum degree of a line bundle on C_S of rank $\underline{e}(S)$, more precisely

$$\deg_{C_S} L \geq \underline{e}^{\deg}(S),$$

which implies that $\underline{d}(S) \geq \underline{e}^{\deg}(S)$. The proof is complete. □

Theorem 3.3.18. *Let G be a stable graph and let $\underline{d} \in \mathrm{Div}(G)$ be of generalized Clifford type. If $\underline{d}$ is S-reduced with respect to all $S \subsetneq V(G)$ which contain the vertices where $\underline{d}$ possibly has negative degree, then*

$$r^{\mathrm{MAX}}(G, \underline{d}) \leq r_G(\underline{d}).$$

Proof. Let $r^{\mathrm{MAX}}(G, \underline{d}) = s$. If $s = -1$, the result follows from Corollary 3.3.7. So, we can assume $s \geq 0$. We want to prove that $r_G(\underline{d}) \geq s$, by Lemma 2.1.11 it is enough to show that for all $\underline{e} \in \mathrm{Div}^s_+(G)$ there exists a $\underline{d}' \sim \underline{d}$ such that $\underline{d}' - \underline{e}^{\deg} \geq 0$.

Given an effective divisor $\underline{e}$ on G of degree s, set

$$S = \{v \in V(G) : (\underline{d} - \underline{e}^{\deg})(v) < 0\} \subset V(G).$$

Notice that if S is empty then the divisor $\underline{d} - \underline{e}^{\deg}$ is effective and we are done. We want to show that S is proper. Suppose by contradiction that $S = V(G)$, i.e.,

$$(\underline{d} - \underline{e}^{\deg})(v) < 0, \ \forall v \in V(G). \tag{3.15}$$

Recall from Definition 2.1.10 that $\underline{e}^{\deg}(v) = \underline{e}(v) + \min\{\underline{e}(v), g(v)\}$, for all $v \in V(G)$. So, we have

$$|\underline{e}^{\deg}| = \sum_{v \in V(G)} \underline{e}(v) + \sum_{v \in V(G)} \min\{\underline{e}(v), g(v)\} \leq s + \sum_{v \in V(G)} g(v) \leq s + g. \tag{3.16}$$

On the other hand,

$$|\underline{e}^{\deg}| = \sum_{v \in V(G)} \underline{e}(v) + \sum_{v \in V(G)} \min\{\underline{e}(v), g(v)\} \leq 2s. \tag{3.17}$$

Since $r^{\mathrm{MAX}}(G, \underline{d}) = s$, there exist a curve $X \in M^{\mathrm{alg}}(G)$ and a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$ such that $r(X, L) \geq s$. It follows from the fact that $\underline{d}$ is of generalized Clifford type that

$$d = \deg L \geq r(X, L) + \min\{r(X, L), g\} \geq s + \min\{s, g\}. \tag{3.18}$$

Subtracting (3.16) from (3.18) we obtain

$$d - |\underline{e}^{\deg}| \geq \min\{s, g\} - g, \tag{3.19}$$

on the other hand, subtracting (3.17) from (3.18) we obtain

$$d - |\underline{e}^{\deg}| \geq \min\{s, g\} - s. \tag{3.20}$$

So, if $\min\{s, g\} = g$, then, by (3.19), $d - |\underline{e}^{\deg}| \geq 0$. Otherwise, $\min\{s, g\} = s$ and $d - |\underline{e}^{\deg}| \geq 0$ follows from (3.20), which contradicts (3.15).

So we have that S is proper and, by hypothesis, that $\underline{d}$ is S-reduced since S is the set of vertices where $\underline{d} - \underline{e}^{\deg}$ has negative degree. Thus, $\underline{d} - \underline{e}^{\deg}$ is also S-reduced. Then, since $\underline{d}$ is of generalized Clifford type and $\underline{d} - \underline{e}^{\deg}$ is S-reduced, by Lemma 3.3.17 we have $\underline{d}(S) - \underline{e}^{\deg}(S) \geq 0$, implying that S is empty. Therefore, $\underline{d} - \underline{e}^{\deg} \geq 0$ and the proof is complete. □

Corollary 3.3.19. *Let $G = B_k^{\omega_1,\omega_2}$ be a weighted binary graph of genus g and let $0 \leq d \leq 2g$. If $\underline{d} = (d_1, d_2) \in \mathrm{Div}^d(G)$ is a balanced divisor and $d_2 \leq k$, then*

$$r^{\mathrm{MAX}}(G, \underline{d}) \leq r_G(\underline{d}).$$

Proof. We verified in Example 3.3.12 that if $\underline{d}$ is a balanced divisor with $0 \leq d \leq 2g$ on a weighted binary graph $B_k^{\omega_1,\omega_2}$ then $\underline{d} = (d_1, d_2)$ is of generalized Clifford type. In this case, the only non-empty proper subsets of $V(B_k^{\omega_1,\omega_2})$ are $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$. For instance, since $d \geq 0$, thus $d_2 \geq 0$ and since $d_2 \leq k$ we have that $\underline{d}$ is S_1-reduced. Please remark that S_1 is the only proper subset of $V(B_k^{\omega_1,\omega_2})$ containing the vertices where $\underline{d}$ possibly has negative degree. Therefore, it follows from Theorem 3.3.18 that $r^{\mathrm{MAX}}(G, \underline{d}) \leq r_G(\underline{d})$. □

3.3.3 The rank of divisors on binary weighted graphs

The purpose of this section is to establish cases where the combinatorial rank equals the balanced algebraic rank of a divisor on a (weighted) binary graph. We begin by making the following useful remark.

Remark 3.3.20. Let G be a weighted binary graph and $\underline{d} \in \mathrm{Div}(G)$. We know that in order to calculate $\overline{r}^b(G, \underline{d})$ we have to calculate the maximum value of $r^{\mathrm{MAX}}(G', \underline{d}')$ among all $(G', \underline{d}') \geq (G, \underline{d})$, where $\underline{d}'$ is balanced.

An argumentation similar to the one done in Construction 1 shows that $\overline{r}^b(G,\underline{d})$ is calculated considering the r^{MAX} on weighted binary graphs and balanced degrees obtained by partially normalization of the ones weightily contracted to $(G,\underline{d})$. From now on we use the same notation as Construction 1. As discussed in Construction 1 and in Remark 3.3.5, these normalizations correspond to subtracting from G' all the edges of $\pounds$, so the calculations are done in $G'_H = G' - \pounds$, where $\pounds \subset V(G')$ is the set of exceptional vertices of G'. But notice that G'_H is a weighted binary graph (see Figure 3.8).

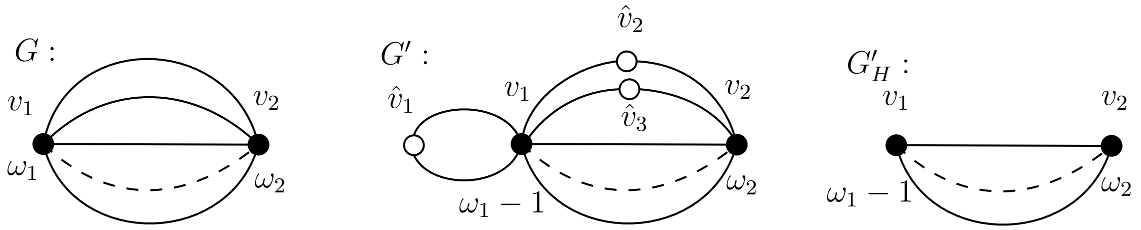


Figure 3.8: The graphs G, G' and G'_H.

In point of fact, we know that the F_2 is the set of edges in the loops in G' being contracted to weights of G. So subtracting from G' all the vertices in $\pounds$ adjacent to edges in F_2 the resulting graph from this contraction has no loops. On the other hand, subtracting from G' all the exceptional vertices adjacent to F_1 results on a weighted graph with the two vertices joined by k' edges, with $k' \leq k$.

Proposition 3.3.21. *Let G be a binary graph and let $\underline{d} \in Div(G)$ be balanced, then*

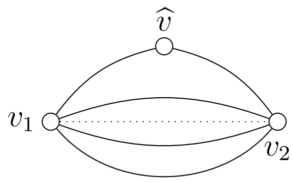
$$\overline{r}^b(G,\underline{d}) = r_G(\underline{d}).$$

Proof. Here we use the same notation of Example 3.2.8, so we consider $\underline{d} = (d_1, d_2)$ with $0 \leq d_1 \leq d_2$. If $d_2 \leq g$, $\underline{d}$ is the unique effective representative on the divisor class, and there exists a special binary curve $X \in M^{\mathrm{alg}}(G)$ such that $r^{\max}(X,\underline{d}) = r_G(\underline{d}) = d_1$.

Moreover $\underline{d}$ is balanced on X once

$$\frac{d-g-1}{2} \leq d_i \leq \frac{d+g+1}{2}, \quad \text{for } i = 1, 2.$$

Let $R = \{e\} \subset E(G)$ and set $\underline{d}'' = (d_1, 1, d_2 - 1) \in \mathrm{Div}^d(\widehat{G}_R)$.

$\widehat{G}_R =$ 

Consider the binary graph G_R and the curve $X_R \in M^{\mathrm{alg}}(G_R)$ the normalization in R of X. The restriction of the divisor $\underline{d}'' = (d_1, 1, d_2 - 1)$ to G_R is $\underline{d}_2 = (d_1, d_2 - 1)$. The divisor $\underline{d}''$ is admissible (and balanced) hence $r^{\max}(\widehat{G}_R, \underline{d}'') = r^{\max}(G_R, \underline{d}_2)$. We have

$$r^{\max}(X, \underline{d}) = r^{\max}(X_R, \underline{d}_2) = r_G(\underline{d}),$$

once G_R is a binary graph and $d_2 - 1 \leq g - 1$. Therefore, by Theorem 3.3.18 and Example 3.3.2 we have $\overline{r}^b(G, \underline{d}) = r_G(\underline{d})$.

When $d_2 \geq g + 1$, the representative $\underline{d}$ of the divisor class is chosen such that is balanced. Set $\underline{d}' = (d_1 - 1, 1, d_2)$ and $\underline{d}'' = (d_1, 1, d_2 - 1)$ balanced divisors on $\widehat{G}_R$. The restrictions of these divisors to G_R are $\underline{d}_1$ and $\underline{d}_2$, respectively. By Example 3.2.8 we know that

$$r_G(\underline{d}) = r_{G_R}(\underline{d}_1) = r_{G_R}(\underline{d}_2) = d_1 + d_2 - g.$$

Pick any curve $X \in M^{\mathrm{alg}}(G)$, then for $X_R \in M^{\mathrm{alg}}(G_R)$, we have

$$r^{\max}(\widehat{X}_R, \underline{d}') = r^{\max}(X_R, \underline{d}_1) \text{ and } r^{\max}(\widehat{X}_R, \underline{d}'') = r^{\max}(X_R, \underline{d}_2).$$

Let $L_R \in \mathrm{Pic}^{\underline{d}_i}(X_R)$, $i = 1, 2$, by [7, Lemma 10(i)], we have

$$r(X_R, L_R) = d_1 + d_2 - 1 - (g - 1) = d_1 + d_2 - g.$$

So,

$$r^{\mathrm{MAX}}(X_R, \underline{d}_1) = r^{\mathrm{MAX}}(X_R, \underline{d}_2) = r_G(\underline{d}).$$

For another $(G', \underline{d}') \geq_{R'} (G, \underline{d})$, we have

$$r^{\mathrm{MAX}}(X_R, \underline{d}_i) \geq r^{\mathrm{MAX}}(X_{R'}, \underline{d}_{G_{R'}}), \quad \text{for } i = 1, 2.$$

Therefore, $\overline{r}^b(G, \underline{d}) = r_G(\underline{d})$. □

Proposition 3.3.22. *Let $G = B_k^{\omega_1,\omega_2}$ be a weighted binary graph, with $\omega_1 = \omega_2$. Let $\underline{d} = (d_1, d_2) \in \mathrm{Div}(G)$ be balanced such that $d_1 \leq d_2 \leq k$. If $0 \leq d \leq 2g$, then*

$$\overline{r}^b(G, \underline{d}) = r_G(\underline{d}). \tag{3.21}$$

Proof. Set the order $V(G) = \{v_1, v_2\}$. Notice that $\underline{d}$ is v_1-reduced, and by balanced hypothesis. If $d_1 < 0$ then it follows from Proposition 2.1.6 that $r_G(\underline{d}) = -1$. Since $d \geq 0$, we have that $d_2 \geq 0$, so by Corollary 3.3.7(d) we have that

$$r^{\mathrm{MAX}}(G, \underline{d}) \leq r_G(\underline{d}) = -1,$$

thus equality.

Hence, we may assume $d_1 \geq 0$, by Corollary 2.1.16 we have

$$r_G(\underline{d}) = \underline{d}_{rk}(v_1) = \max\{\lfloor d/2 \rfloor, d_1 - \omega_1\}.$$

Let C_1 be a curve of genus ω_1 and L_{C_1} be a line bundle on C_1 such that

$$r(C_1, L_{C_1}) = \max\{\lfloor d_1/2 \rfloor, d_1 - \omega_1\}.$$

Since $\omega_1 = \omega_2$, there exists a special curve

$$X = (C_1 \cup C_2)/\{p_i = q_i, i = 1, \dots k\}$$

dual to G such that there is an isomorphism $\phi : C_1 \to C_2$ with $\phi(p_i) = q_i$, $i = 1, \dots k$. Let $L \in \mathrm{Pic}^{\underline{d}}(X)$, we have

$$r(X, L) \geq r(C_1, L_{C_1}) = \max\{\lfloor d_1/2 \rfloor, d_1 - \omega_1\}.$$

So, $r^{\mathrm{MAX}}(G, \underline{d}) \geq \max\{d_1 - \omega_1, \lfloor d/2 \rfloor\}$. By Corollary 3.3.19 we have

$$r^{\mathrm{MAX}}(G, \underline{d}) = r_G(\underline{d}). \tag{3.22}$$

Notice that (3.22) implies that $\overline{r}^b(G, \underline{d}) \geq r_G(\underline{d})$. In order to prove that $\overline{r}^b(G, \underline{d}) = r_G(\underline{d})$ it is enough to prove that for any $(G, \underline{d}', \omega') \geq (G, \underline{d}, \omega)$ we have $r_G(\underline{d}) \geq r^{\mathrm{MAX}}(G', \underline{d}')$.

We know from Remark 3.3.20 that $\overline{r}^b(G, \underline{d})$ is calculated on binary curves and balanced degrees obtained by possibly normalizing dual curves to G'. Using the notation utilized in Construction 1, the weighted contraction depends on the sets F^1 and F^2, they determine if the contraction happens on loops to weights or on edges of F^1 to original edges of G. Either way, it is at the weighted binary graph G'_H where we calculate r^{MAX} of $\underline{d}'_H$, where $\underline{d}'_H(v_i) = \underline{d}'(v_i)$, for $i = 1, 2$ (see Figure 3.8). Thus, for what was discussed here for the weighted binary graph G, the ranks can only decrease once $\underline{d}'(v_2)$ and ω'_i may decrease, and consequently $\min\{\underline{d}'(v_1), \underline{d}'(v_2)\}$ and

$\min\{\underline{d}'(v_1) - \omega'_1, \underline{d}'(v_2) - \omega'_2\}$ may also decrease. Therefore,

$$r^{\text{MAX}}(G', \underline{d}') = r^{\text{MAX}}(G'_H, \underline{d}'_H(v)) \leq r_G(\underline{d}).$$

□

Remark 3.3.23. Let $(G', \omega', \underline{d}')$ and $(G, \omega, \underline{d})$ be triples of (weighted) stable graphs and balanced divisors such that $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$. Let the triple $(G'_H, \omega'_H, \underline{d}'_H)$ be a normalization of the exceptional components of $(G', \omega', \underline{d}')$, then

$$\overline{r}^b(G, \underline{d}) \geq \overline{r}^b(G', \underline{d}') = r^{\text{MAX}}(G'_H, \underline{d}'_H).$$

This is due to the fact that the r^{MAX} of degenerations of $(G', \underline{d}')$ are also calculate when calculating $\overline{r}^b(G, \underline{d})$.

Example 3.3.24. Example 2.6.1 illustrates the case of a balanced divisor $\underline{d}$ in a weighted binary graph whose algebraic rank is strictly smaller than its combinatorial rank. Later, in Example 3.2.6 we show that it is possible to find a triple $(G', \omega', \underline{d}') \geq (G, \omega, \underline{d})$ such that in order to calculate $r^{\text{MAX}}(G', \underline{d}')$ we calculate the normalizations suitable refinement Y (with dual graph G'_H) of the curve X with the divisor $(3,3) \in \text{Div}(G'_H)$ such that

$$r^{\max}(Y, (3,3)) = r_G(\underline{d}) = 2. \tag{3.23}$$

Consider the weighted binary graph $B_k^{2,2}$ with $k > 12$ and let $(4,4) \in \text{Div}(B_k^{2,2})$. Now, let the triple $(G_1, \omega_1, \underline{d}_1)$ be such that $V(G_1) = \{\widehat{v_1}, v_1, v_2\}$ with $v_1 \cdot v_2 = k$ and $v_1 \cdot \widehat{v_1} = 2$, $\omega_1(v_1) = 1$, $\omega_1(v_2) = 2$ and $\omega_1(\widehat{v_1}) = 0$, and $\underline{d}_1 = (1,3,4)$. Observe that $(G_1, \omega_1, \underline{d}_1) \geq (B_k^{2,2}, \omega_{B_k^{2,2}}, (4,4))$ and that the graph G is the normalization of G_1 in its rational components (the two edges whose ends are the vertices v_1 and $\widehat{v_1}$). However, since the graph $B_k^{2,2}$ is a weighted binary graph and

$$0 < |(4,4)| = 8 < 2(k+3) = 2g(B_k^{2,2}),$$

it follows from Proposition 3.3.22 that

$$\overline{r}^b(B_k^{2,2}, (4,4)) = r_{B_k^{2,2}}(4,4) = 2 = r_G(\underline{d}).$$

By Remark 3.3.23, we have that $\overline{r}^b(B_k^{2,2}, (4,4)) \geq \overline{r}^b(G, \underline{d})$. Therefore, it follows from (3.23) and that $\overline{r}^b(G, \underline{d}) = 2 = r_G(\underline{d})$.

In particular, to illustrate another possible degeneration of G consider the complete degeneration on its weights. For instance, let G'' be the weightless graph such that $V(G'') = \{v_1, v_2, \widehat{v_1}, \widehat{v_2}, \widehat{v_3}\}$ such that $v_1 \cdot \widehat{v_1} = v_2 \cdot \widehat{v_2} = v_2 \cdot \widehat{v_3} = 2$ and $v_1 \cdot v_2 = k$, and let $\underline{d}'' = (2,2,1,1,1) \in \text{Div}(G'')$.

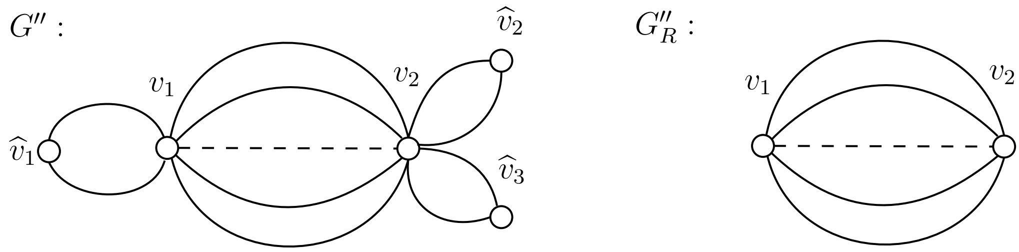


Figure 3.9: The weightless graph G'' and the binary graph G''_R.

We have that $(G'', \omega'', \underline{d}'') \geq (G, \omega, \underline{d})$ and in order to calculate $r^{\text{MAX}}(G'', \underline{d}'')$ we calculate $r^{\text{MAX}}(G''_R, \underline{d}''_R)$, where $R = \{\widehat{v_1}, \widehat{v_2}, \widehat{v_3}\}$. The graph G''_R is a binary graph (see Figure 3.9), therefore Proposition 3.3.21 give us that

$$\overline{r}^b(G''_R, \underline{d}''_R) = r_{G''_R, \underline{d}''_R} = 2.$$

Example 3.3.25. Let $G = B_k^{\omega_1, \omega_2}$ be a weighted binary graph, with the order $V(G) = \{v_1, v_2\}$. Let $\underline{d} \in \text{Div}(G)$ be balanced such that $d_1 = \underline{d}(v_1) \leq 2\omega_1$, $d_2 = \underline{d}(v_2) \leq 2\omega_2$

and $d_2 \geq k$. If d_1, d_2 and k are even integers, we expect to be true that

$$\overline{r}^b(B_k^{\omega_1,\omega_2}, \underline{d}) = \frac{d_1}{2} + \frac{d_2}{2} - \frac{k}{2} + 1. \tag{3.24}$$

In fact, if we consider the graph $\widehat{G}_R$ with $R = V(G)$ and $\underline{d}' = (d_1, 1, \dots, 1, d_2 - k) \in \mathrm{Div}(\widehat{G}_R)$, we have that $(\widehat{G}_R, \underline{d}') \geq (B_k^{\omega_1,\omega_2}, \underline{d})$. Let $X \in M^{\mathrm{alg}}(B_k^{\omega_1,\omega_2})$ be a curve and consider the total normalization $X_E^\nu = C_1 \sqcup C_2 \in M^{\mathrm{alg}}(G')$ of X. Given a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$, let $L^\nu \in \mathrm{Pic}^{\underline{d}'}(X_E^\nu)$ be such that $\nu^* L = L^\nu$. By Uniform Clifford (Theorem 1.2.14), we have

$$h^0(C_1, L^\nu_{C_1}) \leq \frac{d_1}{2} + 1 \quad \text{and} \quad h^0(C_2, L^\nu_{C_2}) \leq \frac{d_2 - k}{2} + 1. \tag{3.25}$$

By Remark 1.2.9, we have

$$r(X, L) \leq \left(\frac{d_1}{2} + 1\right) + \left(\frac{d_2 - k}{2} + 1\right) - 1 = \frac{d_1}{2} + \frac{d_2}{2} - \frac{k}{2} + 1.$$

Therefore, if there is a curve $X \in M^{\mathrm{alg}}(G)$ such that in (3.25) we have equalities, then we would have

$$\overline{r}^b(B_k^{\omega_1,\omega_2}, \underline{d}) \geq \frac{d_1}{2} + \frac{d_2}{2} - \frac{k}{2} + 1.$$

Chapter 4

Conclusion and forthcoming work

In this thesis we addressed the problem of explaining the gap between the combinatorics of rank of divisors on finite graphs and the geometry of ranks of divisors on algebraic curves.

The initial way we approached this problem was defining the rank r^{ALG}. This new rank was proved to satisfy some results analogous to those already existing for r^{alg}, such as the Riemann-Roch theorem and Clifford's theorem. We intuit that these two invariants are equal, we conclude Section 3.1 by proving that r^{alg} is an upper bound for r^{ALG}, the proof was constructed having a strictly numeric point of view. But this perspective ignores the geometry and combinatorics of the objects involved in the problem. So it is natural that we ask the following question.

Question 2. Do we have, for all $\delta \in \mathrm{Pic}(G)$, $r^{\mathrm{alg}}(G,\delta) = r^{\mathrm{ALG}}(G,\delta)$?

Recall that, by Theorem 3.1.3, if G is semistable graph g and $\delta \in \mathrm{Pic}^d(G)$, with $d \geq 2g-2$, then every semibalanced $\underline{d} \in \delta$, in particular balanced, satisfies $r^{\mathrm{MAX}}(G,\underline{d}) = r_G(\underline{d})$. In this case, do we also have $\overline{r}^b(G,\underline{d}) = r_G(\underline{d})$? More generally, we ask the following question:

Question 3. Suppose that $\underline{d} \in \mathrm{Div}(G)$ is balanced, is it true that $r_G(\underline{d}) = r^{\mathrm{MAX}}(G',\underline{d}')$

for some $(G', \underline{d}') \geq (G, \underline{d})$ and $\underline{d}'$ balanced?

The initial problem was to find an algebraic interpretation of the combinatorial rank, but now that we have introduced the balanced compactified rank, we ask if whether or not $\overline{r}^b$ has a combinatorial interpretation.

Regardless of the correlation with the combinatorial rank, we question if the balanced compactified rank carries with it interesting properties related to the geometry of the compactified Jacobian.

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