Facoltà di Scienze Matematiche, Fisiche e Naturali

PhD Thesis in Mathematics

by

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Resonant solutions in the presence of degeneracies for quasi-periodically perturbed systems

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Notations:

- Given a differentiable function $F : \mathbb{R}^n \to \mathbb{R}$, we shall denote by $\partial_j^m = \partial_{x_j}^m$ the m-th derivative of F with respect to the j-th argument i.e. for $m \geq 1$
$$\partial_j^m F := \partial_{x_j}^m F = \frac{\partial^m F}{\partial x_j^m},$$
while $m = 0$ has to be interpreted as $\partial_{x_j}^0 F = F$. If $n = 1$ we shall write also $F'(x) = \partial_1 F(x)$.

- Given a finite set S we denote with $|S|$ its cardinality.

- We set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_*^d := \mathbb{Z}^d \setminus \{\mathbf{0}\}$.

- We denote by $\cdot$ the standard scalar product in $\mathbb{R}^d$, i.e. $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \ldots + x_d y_d$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

- Given a vector $\boldsymbol{x} \in \mathbb{R}^d$ we set $|\boldsymbol{x}| := \|\boldsymbol{x}\|_1 = |x_1| + \ldots + |x_d|$.

- Given a complex number $z \in \mathbb{C}$ we shall denote by z^* its complex conjugate.

- We denote by $\mathbb{1}$ the 2×2 identity matrix.

- The sums and the products over empty sets have to be considered as 0 and 1, respectively.

- Given $x_0 \in \mathbb{R}$ and an interval $(a, b) \subset \mathbb{R}$ such that $x_0 \in (a, b)$, we call half-neighbourhood of x_0 each of the two intervals (a, x_0) and (x_0, b).

- Further notations will be introduced when needed.

Introduction

Melnikov theory studies the fate of homoclinic and periodic orbits of two-dimensional dynamical systems when they are periodically perturbed; see for instance [19, 46] for an introduction to the subject. The problem can be stated as follows. Consider in $\mathbb{R}^2$ a dynamical system of the form

$$\begin{cases} \dot{x} = f_1(x,y) + \varepsilon g_1(x,y,t), \\ \dot{y} = f_2(x,y) + \varepsilon g_2(x,y,t), \end{cases} \tag{1}$$

with f_1, f_2, g_1, g_2 'sufficiently smooth', g_1 and g_2 periodic in t and ε a small parameter, called the *perturbation parameter*. If $f_1 = \partial_y h$ and $f_2 = -\partial_x h$, for a suitable function h, the unperturbed system is Hamiltonian. Assume that for $\varepsilon = 0$ the system (1) admits a homoclinic orbit $u_1(t)$ to a hyperbolic saddle point p, and that the bounded region of the phase space delimited by $\gamma := \{u_1(t) : t \in \mathbb{R}\} \cup \{p\}$, is filled with a continuous family of periodic orbits $u_\delta(t)$, $\delta \in (0,1)$, whose periods tend monotonically to ∞ as $\delta \to 1$; see Figure 1. Because of the assumptions, it is easy to see (as an application of the implicit function theorem) that for $\varepsilon \neq 0$ small enough, the system (1) admits a hyperbolic periodic orbit $\widetilde{u}(t,\varepsilon) = p + O(\varepsilon)$; then one can ask whether the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$ intersect transversely (in turn if this happens it can be used to prove that chaotic motions occur). Another natural question is what happens to the periodic orbits $u_\delta(t)$ when $\varepsilon \neq 0$. In particular one can investigate under which conditions 'periodic orbits persist', that is there are periodic orbits which are close to the unperturbed ones and reduce to them when the perturbation parameter is set equal to zero. If such orbits exist, they are called *subharmonic* or *resonant* orbits.

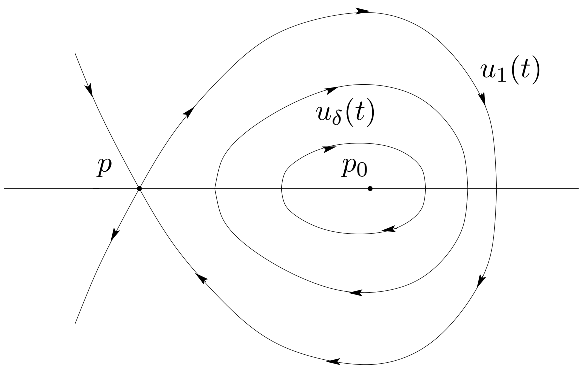

Figure 1: The phase portrait of the unperturbed system: one has $\sup_{t\in\mathbb{R}}\inf_{q\in\gamma}|u_\delta(t)-q|\to 0$ as $\delta\to 1$ and $\sup_{t\in\mathbb{R}}|u_\delta(t)-p_0|\to 0$ as $\delta\to 0$. If T_δ is the period of $u_\delta(t)$ one has $\mathrm{d}T_\delta/\mathrm{d}\delta>0$.

Both the existence of transverse intersections of the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$ and the persistence of periodic orbits are related to the zeroes of suitable functions. More precisely if one define the *Melnikov function* as

$$M(t_0) := \int_{-\infty}^{\infty} \mathrm{d}t \, \big(f_2(u_1(t-t_0)) g_2(u_1(t-t_0),t) - f_1(u_1(t-t_0)) g_1(u_1(t-t_0),t) \big), \tag{2}$$

then if $M(t_0)$ has simple zeroes, then the stable and unstable manifolds of $\widetilde{u}(t,\varepsilon)$ intersect transversely, while if $M(t_0)\neq 0$ for all $t_0\in\mathbb{R}$, no intersection occurs; essentially $M(t_0)$ measures the distance between the two manifolds along the normal to the homoclinic orbit at $u_1(t_0)$.

Concerning the periodic orbits, if the period T_δ of $u_\delta(t)$ is not commensurable with the period T of the functions g_1, g_2, such an orbit will not persist under perturbations. Otherwise, set $T_\delta = mT/n$ and define the *(subharmonic) Melnikov function* as

$$M_{m/n}(t_0) := \int_{0}^{mT} \mathrm{d}t \, \big(f_2(u_\delta(t-t_0)) g_2(u_\delta(t-t_0),t) - f_1(u_\delta(t-t_0)) g_1(u_\delta(t-t_0),t) \big). \tag{3}$$

If $M_{m/n}(t_0)$ has a simple zero then (1) admits a subharmonic orbit $\overline{u}(t,\varepsilon)$ with period mT; in particular, if the functions f_1, f_2, g_1, g_2 are analytic, then $\overline{u}(t,\varepsilon)$ is analytic in both ε and t. If there are no zeroes at all, no periodic solution persists.

The proof of the two claims above is rather standard, and it is essentially based on the application of the implicit function theorem. A possible approach for the case of subharmonic orbits consists in splitting the equations of motion into two separate set of equations, the so-called *range equations* and *bifurcation equations*: one can solve the range equations in terms of the free parameter t_0 and then fix the latter by solving the bifurcation equations, which represent an implicit function problem.

Note that the assumption that the zeroes of the Melnikov functions are simple corresponds to a (generic) non-degeneracy condition on the perturbation. When the zeroes are not simple, the situation is slightly more complicated. In the case of subharmonic orbits, the same result of persistence extends to the more general cases of "topological non-degeneracy", i.e. the existence of an isolated minimum or maximum of the primitive of the Melnikov function [2], which in turn imply the existence of a zero of odd order for the Melnikov function, and interesting new analytical features of the solutions appear [3, 44, 21]; indeed the subharmonic solutions turn out to be analytics in a suitable fractional power of ε rather than ε itself. On the other hand if the zeroes are of even order one cannot predict a priori the persistence of periodic orbits. Finally, if the Melnikov function is identically zero, one has to consider higher order generalisations of it and study the existence and multiplicity of their zeroes to deal with the problem.

If one considers a quasi-periodic perturbation instead of a periodic one, i.e. $g_k(x, y, t) = G_k(x, y, \boldsymbol{\omega} t)$, with $G_k : \mathbb{R}^2 \times \mathbb{T}^d \to \mathbb{R}^2$ and $\boldsymbol{\omega} \in \mathbb{R}^d$, $d \geq 2$, one can still ask whether there exist hyperbolic sets run by quasi-periodic solutions with stable and unstable manifolds which intersect transversely and one can still study the existence of quasi-periodic solutions which are "resonant" with the *frequency vector* $\boldsymbol{\omega}$ of the perturbation; see below – after (4) – for a formal definition of resonant solution for quasi-periodic forcing.

Also in the quasi-periodic case, non-degeneracy assumptions are essential to prove transversality of homoclinic intersections. Existence of a quasi-periodic hyperbolic orbit close to the unperturbed saddle point and of its stable and unstable manifolds follows from general arguments, such as the invariant manifold theorem [29, 47], without even assuming any condition on the frequency vector $\boldsymbol{\omega}$. Palmer generalises Melnikov's method to the case of bounded perturbations [58] using the theory of exponential dichotomies [20]. A suitable generalisation of the Melnikov function for quasi-periodic forcing is also introduced by Wiggins [68]. He shows that if such a function has a simple zero then the stable and unstable manifolds intersect transversely. Then, generalising the Smale-Birkhoff homoclinic theorem to the case of orbits homoclinic to normally hyperbolic tori, he finds that there is an invariant set on which the dynamics of a suitable Poincaré map is conjugate to a subshift of finite type; in turn this yields the existence of chaos. Similar results hold also for more general almost periodic perturbations (which include the quasi-periodic ones as a special case): Meyer and Sell show that also in that case the dynamics near transverse homoclinic orbits behaves as a subshift of finite type [54] and Scheurle, relying on Palmer's results, finds particular solutions which have a random structure [62]; again, to obtain transversality the Melnikov function is assumed to have simple zeroes. Such assumption can be weakened to an assumption of "topological non-degeneracy" (i.e. the existence of an isolated maximum or minimum of the primitive of the Melnikov function) as in the case of subharmonic orbits, and one can deal with the problem by use of a variational approach; see for instance [24, 63, 1, 10].

A natural application for the study of homoclinic intersections, widely studied in the literature, is the quasi-periodically forced Duffing equation [67, 54, 70]. Often, especially in applications, the frequency vector is taken to be two-dimensional, with the two components which are nearly resonant with the proper frequency of the unperturbed system (see for instance [70, 7] and references therein). Then a different approach with respect to [68] is proposed by Yagasaki [70]: first, through a suitable change of coordinates, one arrives at a system with two frequencies, one fast and one slow, and then one uses averaging to reduce the analysis of the original system to that of a perturbation of a periodically forced system for which the standard Melnikov's method applies: the persistence of hyperbolic periodic orbits and their stable and unstable manifolds for the original system is then obtained as a consequence of the invariant manifold theorem. Transversality of homoclic intersections plays also a crucial role in the phenomenon of Arnold diffusion [4, 52]: non-degeneracy assumptions on the perturbation are heavily used in the proofs existing in the literature (see e.g. [27, 36, 26]) in order to find lower bounds on the transversality, which in turn are fundamental to compute the diffusion times along the heteroclinic chains (see e.g. [13, 32, 12, 9, 66]). A physically relevant case, studied within the context of Arnold diffusion, is that with frequency vectors with two fast components [27, 36, 65] or with one component much faster and one component much slower than the proper frequency ('three scale system') [37, 38, 11, 60]. In such cases the homoclinic splitting is exponentially small in the perturbation parameter and this makes the analysis rather delicate, as one has to check that the first order contribution to the splitting (the Melnikov function) really dominates; in particular non-degeneracy conditions on the perturbation are needed once more.

The problem of existence of quasi-periodic orbits close to the center of the unperturbed system is harder and does not follow from the invariant manifold theorem. Second-order approximations for the quasi-periodic solutions close to the centers of a forced oscillator are studied in [7], using the multiple scale technique for asymptotic expansions [57, 48]. But if one wants to really prove the existence of the solution, one must require additional assumption on $\boldsymbol{\omega}$ to deal with the presence of *small divisors*. In [55], Moser considers Duffing's equation with a quasi-periodic driving term and assumes that (i) the system is *reversible*, i.e. it can be written in the form $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$, and there exists an involution $I\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(Ix) = -If(x)$ (so that with $x(t)$ also $Ix(-t)$ is a solution), and (ii) the frequency vector of the driving satisfies some Diophantine condition involving also the proper frequency of the unperturbed system linearised around its center. Then he shows that there exists a quasi-periodic solution, with the same frequency vector as the driving, to a slightly modified equation, in which the coefficient of the linear term is suitably corrected. If one tried to remove the correction then one should deal with an implicit function problem (see [6] for a similar situation), which, without assuming any non-degeneracy condition on the perturbation, would have the same kind of problems as in the present thesis. Quasi-periodically forced Hamiltonian oscillators are also considered in [16], where the persistence of quasi-periodic solutions close to the centers of the unperturbed system is studied, including the case of resonance between the frequency vector of the forcing and the proper frequency. However, again, non-degeneracy conditions are assumed.

On the contrary the problem of persistence of quasi-periodic solutions far from the stationary points, corresponding to the subharmonic solutions of the periodic case, does not seem to have been studied a great deal (we can mention a paper by Xu and Jing [69], who consider Duffing's equation with a two-frequency quasi-periodic perturbation and follow the approach in [70] to reduce the analysis to a one-dimensional backbone system; however the argument used to show the persistence of the two-dimensional tori is incomplete and requires further hypotheses). Again the existence of resonant solutions is related to the zeroes of a suitable function, still called Melnikov function by analogy with the periodic case. If the zeroes are simple, assuming some Diophantine condition on $\boldsymbol{\omega}$, the analysis can be carried out so as to reach conclusions similar to the periodic case, that is the persistence of resonant solutions.

In this thesis we study the problem of the persitence of resonant solutions in the case of zeroes of odd order and additionally investigate what can still be said when the Melnikov function is identically zero. As remarked before, considering non-simple zeroes means removing non-degeneracy – and hence genericity – conditions on the perturbation. This introduces nontrivial technical complications, because one is no longer allowed to separate the small divisor problem plaguing the range equations from the implicit function problem represented by the bifurcation equations.

This thesis continues and extends the analysis started in [22], where more special systems were considered. The method we use is based on the analysis and resummation of the perturbation series through renormalisation group techniques [33, 39, 45, 34, 40, 41]; for other renormalisation group approaches to small divisors problems in dynamical systems see for instance [15, 50, 53, 49, 51]. As in [22], the frequency vector of the perturbation will be assumed to satisfy the *Bryuno condition*; such a condition, originally introduced by Bryuno [17], has been studied recently in several small divisor problems arising in dynamical systems [40, 53, 59, 41, 42, 51] and its relevance is related to the possibility of describing properties of the analyticity domain, such as the radius of convergence, of the solutions in terms of the *Bryuno function* (1.1.3); this has been explicitly showed in some simple cases, such as the Siegel problem [71], the semistandard map [25] and the standard map [8].

With respect to [22], we consider here also non-Hamiltonian systems: what is required on the unperturbed system is a non-degeneracy condition on the frequency map of the periodic solutions (*anisochrony condition*). In the Hamiltonian case, such a condition becomes a convexity condition on the unperturbed Hamiltonian function, analogously to Cheng's paper [18], where the fate of resonant tori is studied without imposing any non-degeneracy condition on the perturbation. In the Hamiltonian case, the main difference with respect to [18] – and what prevents us from simply relying on that result – is that we consider isochronous perturbations (while in [18] the unperturbed Hamiltonian is convex in all action variables) and assume a weaker Diophantine condition on the frequency vector of the perturbation (the Bryuno condition instead of the standard one). Furthermore, as we said, our method covers also the non-Hamiltonian case, where Cheng's approach, based on a sequence of canonical transformations *à la* KAM, does not apply. Finally, in the Hamiltonian "completely degenerate" case (see Hypothesis 3) we are able to prove the existence of a continuum of resonant solutions, which turn out to be analytic in the perturbation parameter. In the Hamiltonian case, we do not require any further assumption on the perturbation (besides analyticity), as in [18]. In the non-Hamiltonian case we shall make some further assumptions. More precisely we shall require that some zeroes of odd order appear at some level of perturbation theory and a suitable positiveness condition holds; see § 1.1 – in particular Hypothesis 4 – for a more formal statement.

Of course, one could also investigate what happens if the non-degeneracy condition on the unperturbed system is completely removed too. However, this would be a somewhat different problem and very likely a non-degeneracy condition could become necessary for the perturbation. Not even in the KAM theory for maximal tori, the fully degenerate case (no assumption on the unperturbed integrable system and no assumption on the perturbation, besides analyticity) has ever been treated in the literature – as far as we know.

The thesis is organised as follows. We consider systems of the form (1) and assume that for $\varepsilon = 0$ there is a family of periodic solutions satisfying the same hypotheses as in the case of periodic forcing. In particular we assume that, in suitable coordinates $(\beta, B) \in \mathbb{T} \times \mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, the unperturbed system reads

$$\begin{cases} \dot{\beta} = \omega_0(B), \\ \dot{B} = 0, \end{cases} \tag{4}$$

with ω_0 analytic and $\partial_B \omega_0(B) \neq 0$ (anisochrony condition) As a particular case we can consider that (B, β) are canonical coordinates (action-angle coordinates), but the formulation we are giving here is more general and applies also to non-Hamiltonian unperturbed systems; see also [5, 44]. Then we add to the vector field a small analytic quasi-periodic forcing term with frequency vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ which satisfy some weak Diophantine condition (Bryuno condition) and concentrate on a periodic solution of the unperturbed system which is resonant with $\boldsymbol{\omega}$, that is a solution with $B = \overline{B}_0$ such that $\omega(\overline{B}_0)\nu_0 = \omega_1\nu_1 + \ldots + \omega_d\nu_d$ for suitable integers $\nu_0, \nu_1, \ldots, \nu_d$. In § 1.1 we state formally our two main results on the persistence of such solution: Theorem 1.1.1 deals with the case in which the Melnikov function vanishes identically and the system is Hamiltonian, while Theorem 1.1.2 takes into account the case in which the perturbation is not Hamiltonian but a zero of odd order appear at some order of perturbation theory. In § 1.2 we shall sketch our strategy for the proof of the two Theorems above and we shall see that a result in the same spirit of [18] (that is the existence of a quasi-periodic solution in the Hamiltonian case, without assumptions on the perturbation besides smallness and analyticity; see Theorem 1.2.1) follows as a corollary of Theorems 1.1.1 and 1.1.2. In § 1.3 we shall introduce some notions from graph theory which will be used in order to prove both Theorems 1.1.1 and 1.1.2.

In Chapter 2 we shall prove Theorem 1.1.1. In particular we shall see how the Hamiltonian structure of the equations of motion is fundamental in order to prove that suitable "cancellations" occur in the perturbative series formally defining the solution. In turn this will imply the convergence of such a series and hence the existence of the solution and its analyticity in the perturbation parameter. The "cancellation mechanism" turns out to be quite similar to the one performed in [23], where Moser's modifying terms theorem [56] (see [6] for a review with our formalism) is proved in Cartesian coordinates instead of action-angle coordinates. It would be interesting to understand the deep reason of such a similarity.

In Chapter 3 we shall prove Theorem 1.1.2. As we shall see, the quasi-periodic solution will be only continuous in the perturbation parameter. In fact, in contrast to the case of periodic perturbations, the quasi-periodic solution is not expected to be analytic in ε nor in some fractional power of ε; already in the non-degenerate (Hamiltonian) case the solution has been proved only to be C^{∞} smooth in ε [33], and analyticity is very unlikely. However under the only Hypotheses of Theorem 1.1.2, no more than continuity in ε can be proved.

A crucial role in both the proofs of Theorems 1.1.1 and 1.1.2 will be played by remarkable identities between classes of diagrams; see Lemmas 2.3.3 and 3.3.8. By exploiting the analogy of the method with the techniques of quantum field theory, one can see the solution as the one-point Schwinger function of a suitable Euclidean field theory – this has been explicitly shown in the case of classical KAM theorem [35]. Then the identities between diagrams, that we prove and use, can be conjectured to reflect suitable Ward identities of the field theory symmetries also in the present case, as it has been pointed out in the case of classical KAM theorem [15]. It would be interesting to confirm the expectation and to determine the Ward identity explicitly.

Contents

1. Main results

In this Chapter we shall state precisely our results and give a sketch of their proofs. Moreover we shall introduce the basic notions from graph theory, which will be used throughout the thesis.

1.1 Statement of the results

Let us consider the ordinary differential equation

$$\begin{cases} \dot{\beta} = \omega_0(B) + \varepsilon F(\boldsymbol{\omega} t, \beta, B), \\ \dot{B} = \varepsilon G(\boldsymbol{\omega} t, \beta, B), \end{cases} \tag{1.1.1}$$

where $(\beta, B) \in \mathbb{T} \times \mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, $F, G : \mathbb{T}^{d+1} \times \mathfrak{B} \to \mathbb{R}$ and $\omega_0 : \mathfrak{B} \to \mathbb{R}$ are real-analytic functions, $\boldsymbol{\omega} \in \mathbb{R}^d$ with $d \geq 2$ and ε is a (small) real parameter called the *perturbation parameter*; hence the *perturbation* (F, G) is quasi-periodic in t with *frequency vector* $\boldsymbol{\omega}$. Without loss of generality we can assume that $\boldsymbol{\omega}$ has rationally independent components. Take the solution for the unperturbed system given by $(\beta(t), B(t)) = (\beta_0 + \omega_0(\overline{B}_0)t, \overline{B}_0)$, with $\overline{B}_0$ such that $\omega_0(\overline{B}_0)$ is *resonant* with $\boldsymbol{\omega}$, i.e. such that there exists $(\overline{\nu}_0, \overline{\boldsymbol{\nu}}) \in \mathbb{Z}^{d+1}$ for which $\omega_0(\overline{B}_0)\,\overline{\nu}_0 + \boldsymbol{\omega} \cdot \overline{\boldsymbol{\nu}} = 0$. We want to study whether for some value of β_0, that is for a suitable choice of the *initial phase*, such a solution can be continued under perturbation.

The resonance condition between $\omega_0(\overline{B}_0)$ and $\boldsymbol{\omega}$ yields a "simple resonance" (or resonance of order 1) for the vector $(\omega_0(\overline{B}_0), \boldsymbol{\omega})$. The main assumptions on (1.1.1) are a Diophantine condition on the frequency vector of the perturbation and a non-degeneracy condition on the unperturbed system. More precisely we shall require that the vector $(\omega_0(\overline{B}_0), \boldsymbol{\omega})$ satisfies the condition

$$\sum_{n \geq 0} \frac{1}{2^n} \log \Big(\inf_{\substack{(\nu_0, \boldsymbol{\nu}) \in \mathbb{Z}^{d+1} \\ (\nu_0, \boldsymbol{\nu}) \nparallel (\overline{\nu}_0, \overline{\boldsymbol{\nu}}), 0 < |(\nu_0, \boldsymbol{\nu})| \leq 2^n}} |\omega_0(\overline{B}_0)\nu_0 + \boldsymbol{\omega} \cdot \boldsymbol{\nu}| \Big)^{-1} < \infty \tag{1.1.2}$$

and that $\omega_0'(\overline{B}_0) \neq 0$.

Up to a linear change of coordinates, we can (and shall) assume $\omega_0(\overline{B}_0) = 0$, so that the vector $\overline{\boldsymbol{\nu}}$, such that $\omega_0(\overline{B}_0)\,\overline{\nu}_0 + \boldsymbol{\omega} \cdot \overline{\boldsymbol{\nu}} = 0$, must be the null vector. Therefore it is not restrictive to formulate the assumptions on $\overline{B}_0$ and $\boldsymbol{\omega}$ as follows.

Hypothesis 1. *$\omega_0(\overline{B}_0) = 0$ and $\boldsymbol{\omega}$ satisfies the Bryuno condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$, where*

$$\mathcal{B}(\boldsymbol{\omega}) = \sum_{n\geq 0} \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})}, \qquad \alpha_n(\boldsymbol{\omega}) = \inf_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^d \\ 0<|\boldsymbol{\nu}|\leq 2^n}} |\boldsymbol{\omega}\cdot\boldsymbol{\nu}|. \tag{1.1.3}$$

Hypothesis 2. *For $\overline{B}_0$ as in Hypothesis 1 one has $\omega_0'(\overline{B}_0) \neq 0$.*

Note that if $\boldsymbol{\omega}$ satisfies the standard Diophantine condition $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \geq \gamma|\boldsymbol{\nu}|^{-\tau}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$, then it also satisfies the Bryuno condition, since $\alpha_m(\boldsymbol{\omega}) \geq \gamma 2^{-m\tau}$ in that case.

Let us write

$$F(\boldsymbol{\psi},\beta,B) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} F_{\boldsymbol{\nu}}(\beta,B), \qquad G(\boldsymbol{\psi},\beta,B) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} G_{\boldsymbol{\nu}}(\beta,B), \tag{1.1.4}$$

and note that, since F and G are real-valued functions, one has

$$F_{-\boldsymbol{\nu}}(\beta,B) = F_{\boldsymbol{\nu}}(\beta,B)^*, \qquad G_{-\boldsymbol{\nu}}(\beta,B) = G_{\boldsymbol{\nu}}(\beta,B)^*. \tag{1.1.5}$$

By analogy with the periodic case, the function $\Gamma^{(1)}_{\mathbf{0}}(\beta) := G_{\mathbf{0}}(\beta,\overline{B}_0)$ will be called the *first order Melnikov function*.

We look for a quasi-periodic solution to (1.1.1) with frequency vector $\boldsymbol{\omega}$, that is a solution of the form $(\beta(t),B(t)) = (\beta_0 + b(t), B_0 + \widetilde{B}(t))$, with

$$b(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} b_{\boldsymbol{\nu}}, \qquad \widetilde{B}(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} B_{\boldsymbol{\nu}}. \tag{1.1.6}$$

Of course the existence of a quasi-periodic solution with frequency $\boldsymbol{\omega}$ in the variables in which $\omega_0(\overline{B}_0) = 0$ implies the existence of a quasi-periodic solution with frequency resonant with $\boldsymbol{\omega}$ in terms of the original variables (that is, before performing the change of variables leading to $\omega_0(\overline{B}_0) = 0$).

If we set $\Phi(t) := \omega_0(B(t)) + \varepsilon F(\boldsymbol{\omega}t,\beta(t),B(t))$ and $\Gamma(t) = \varepsilon G(\boldsymbol{\omega}t,\beta(t),B(t))$, and write

$$\Phi(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} \Phi_{\boldsymbol{\nu}}, \qquad \Gamma(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} \Gamma_{\boldsymbol{\nu}}, \tag{1.1.7}$$

in Fourier space (1.1.1) becomes

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu}) b_{\boldsymbol{\nu}} = \Phi_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu} \neq \mathbf{0}, \tag{1.1.8a}$$

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu}) B_{\boldsymbol{\nu}} = \Gamma_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu} \neq \mathbf{0}, \tag{1.1.8b}$$

$$\Phi_{\mathbf{0}} = 0, \tag{1.1.8c}$$

$$\Gamma_{\mathbf{0}} = 0. \tag{1.1.8d}$$

According to the usual terminology, we shall call (1.1.8a) and (1.1.8b) the *range equations*, while (1.1.8c) and (1.1.8d) will be referred to as the *bifurcation equations*.

We start by looking for a formal solution $(\beta(t), B(t))$, with

$$\begin{aligned} \beta(t) &= \beta(t;\varepsilon,\beta_0) = \beta_0 + \sum_{k\geq 1} \varepsilon^k b^{(k)}(t;\beta_0) = \beta_0 + \sum_{k\geq 1} \varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} b^{(k)}_{\boldsymbol{\nu}}(\beta_0), \\ B(t) &= B(t;\varepsilon,\beta_0) = \overline{B}_0 + \sum_{k\geq 1} \varepsilon^k B^{(k)}(t;\beta_0) = \overline{B}_0 + \sum_{k\geq 1} \varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} B^{(k)}_{\boldsymbol{\nu}}(\beta_0) \end{aligned} \tag{1.1.9}$$

and set $U(t) := \omega_0(B(t)) - \omega_0'(\overline{B}_0)(B(t) - \overline{B}_0)$ and $\phi(t) = U(t) + \varepsilon F(\boldsymbol{\omega}t, \beta(t), B(t))$. Then define recursively for $k \geq 1$

$$\begin{aligned} b^{(k)}_{\boldsymbol{\nu}}(\beta_0) &= \frac{1}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})}\phi^{(k)}_{\boldsymbol{\nu}}(\beta_0) + \frac{\omega_0'(\overline{B}_0)}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Gamma^{(k)}_{\boldsymbol{\nu}}(\beta_0), \qquad \boldsymbol{\nu}\neq\mathbf{0} \\ B^{(k)}_{\boldsymbol{\nu}}(\beta_0) &= \frac{1}{(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})}\Gamma^{(k)}_{\boldsymbol{\nu}}(\beta_0), \qquad \boldsymbol{\nu}\neq\mathbf{0} \\ B^{(k)}_{\mathbf{0}}(\beta_0) &= -\frac{1}{\omega_0'(\overline{B}_0)}\phi^{(k)}_{\mathbf{0}}(\beta_0), \end{aligned} \tag{1.1.10}$$

where we denoted $\Gamma^{(k)}_{\boldsymbol{\nu}}(\beta_0) = [G(\boldsymbol{\omega}t,\beta(t),B(t))]^{(k-1)}_{\boldsymbol{\nu}}$ and $\phi^{(k)}_{\boldsymbol{\nu}}(\beta_0) = [U(t)]^{(k)}_{\boldsymbol{\nu}} + [F(\boldsymbol{\omega}t,\beta(t),B(t))]^{(k-1)}_{\boldsymbol{\nu}}$, with $U^{(1)}_{\boldsymbol{\nu}}(\beta_0) = 0$, so that $\Gamma^{(1)}_{\boldsymbol{\nu}}(\beta_0) = G_{\boldsymbol{\nu}}(\beta_0,\overline{B}_0)$ and $\phi^{(1)}_{\boldsymbol{\nu}}(\beta_0) = F_{\boldsymbol{\nu}}(\beta_0,\overline{B}_0)$, while, for $k \geq 2$,

$$[U(t)]^{(k)}_{\boldsymbol{\nu}} = \sum_{s\geq 2}\frac{1}{s!}\partial^s_B\omega_0(\overline{B}_0) \sum_{\substack{\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu} \\ \boldsymbol{\nu}_i\in\mathbb{Z}^d,\, i=1,\ldots,s}} \sum_{\substack{k_1+\ldots+k_s=k, \\ k_i\geq 1}} \prod_{i=1}^s B^{(k_i)}_{\boldsymbol{\nu}_i}(\beta_0), \tag{1.1.11}$$

and

$$\begin{aligned} [P(\boldsymbol{\omega}t,\beta(t),B(t))]^{(k-1)}_{\boldsymbol{\nu}} &= \sum_{s\geq 1}\sum_{p+q=s} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu} \\ \boldsymbol{\nu}_0,\boldsymbol{\nu}_j\in\mathbb{Z}^d\; j=p+1,\ldots,s \\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*,\, i=1,\ldots,p}} \frac{1}{p!q!}\partial^p_\beta\partial^q_B P_{\boldsymbol{\nu}_0}(\beta_0,\overline{B}_0) \times \\ &\times \sum_{\substack{k_1+\ldots+k_s=k-1, \\ k_i\geq 1}} \prod_{i=1}^p b^{(k_i)}_{\boldsymbol{\nu}_i}(\beta_0) \prod_{i=p+1}^s B^{(k_i)}_{\boldsymbol{\nu}_i}(\beta_0), \qquad P = F, G. \end{aligned} \tag{1.1.12}$$

As we shall prove in Chapter 2 – see Lemma 2.1.3 –, the series (1.1.9), with the coefficients defined as above and arbitrary β_0, turn out to be a formal solution of (1.1.8a)-(1.1.8c): the coefficients $b^{(k)}_{\boldsymbol{\nu}}(\beta_0)$, $B^{(k)}_{\mathbf{0}}(\beta_0)$ and $B^{(k)}_{\boldsymbol{\nu}}(\beta_0)$ are well defined for all $k \geq 1$ and all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$, and solve (1.1.8a)-(1.1.8c) order by order; moreover the functions $b^{(k)}(t;\beta_0)$ and $B^{(k)}(t;\beta_0)$ are analytic and quasi-periodic in t with frequency vector $\boldsymbol{\omega}$.

Note that if there exists $k_0 \geq 1$ such that $\Gamma_{\mathbf{0}}^{(k)}(\beta_0) \equiv 0$ for all $k < k_0$, then the series (1.1.9) with the coefficients $b_{\boldsymbol{\nu}}^{(k)}, B_{\boldsymbol{\nu}}^{(k)}$ defined as in (1.1.10) solve the equations of motion up to order $k_0 - 1$ and moreover $\Gamma_{\mathbf{0}}^{(k_0)}$ is a well-defined function of β_0.

Assume first that the system (1.1.1) is Hamiltonian, i.e. there exists a function

$$H(\boldsymbol{\alpha}, \beta, \boldsymbol{A}, B) := \boldsymbol{\omega} \cdot \boldsymbol{A} + h(B) + \varepsilon f(\boldsymbol{\alpha}, \beta, B), \tag{1.1.13}$$

where $(\boldsymbol{\alpha}, \beta) \in \mathbb{T}^{d+1}$ and $(\boldsymbol{A}, B) \in \mathbb{R}^d \times \mathfrak{B}$, with $\mathfrak{B}$ an open subset of $\mathbb{R}$, are canonically conjugate (action-angle) variables and the functions $f : \mathbb{T}^{d+1} \times \mathfrak{B} \to \mathbb{R}$ and $h : \mathfrak{B} \to \mathbb{R}$ are real-analytic and such that $\omega_0(B) = \partial_1 h(B)$, $\partial_B f(\boldsymbol{\alpha}, \beta, B) = F(\boldsymbol{\alpha}, \beta, B)$ and $-\partial_\beta f(\boldsymbol{\alpha}, \beta, B) = G(\boldsymbol{\alpha}, \beta, B)$, so that the corresponding Hamilton equations for the variables (β, B) are given by

$$\begin{cases} \dot{\beta} = \omega_0(B) + \varepsilon \partial_B f(\boldsymbol{\omega} t, \beta, B), \\ \dot{B} = -\varepsilon \partial_\beta f(\boldsymbol{\omega} t, \beta, B), \end{cases} \tag{1.1.14}$$

which are exactly of the form (1.1.1).

Hypothesis 3. *One has* $\Gamma_{\mathbf{0}}^{(k)}(\beta_0) := [-\partial_\beta f(\beta, B)]_{\mathbf{0}}^{(k-1)} \equiv 0$ *for all* $k \geq 1$.

Then we shall prove the following result.

Theorem 1.1.1. *Consider the system (1.1.14) and assume Hypotheses 1, 2 and 3. Then the series (1.1.9) are convergent for* ε *small enough.*

Next we consider the more general system (1.1.1) and we assume that there exists $k_0 \in \mathbb{N}$ such that all functions $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ are identically zero for $0 \leq k \leq k_0 - 1$, while $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$ is not identically vanishing. Again, by analogy with the periodic case, we shall call the function $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$ the k_0*-th order Melnikov function.*

Hypothesis 4. *There exist* $k_0 \in \mathbb{N}$ *and* $\overline{\beta}_0$ *such that* $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ *vanish identically for* $k < k_0$ *and* $\overline{\beta}_0$ *is a zero of order* $\bar{\mathfrak{n}}$ *for* $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$*, with* $\bar{\mathfrak{n}}$ *odd. Moreover one has* $\varepsilon^{k_0} \omega_0'(\overline{B}_0) \partial_{\beta_0}^{\bar{\mathfrak{n}}} \Gamma_{\mathbf{0}}^{(k_0)}(\overline{\beta}_0) > 0$.

Then we shall prove the following result.

Theorem 1.1.2. *Consider the system (1.1.1) and assume Hypotheses 1, 2 and 4 to be satisfied. Then for* ε *small enough there exists at least one quasi-periodic solution* $(\beta(t), B(t))$ *with frequency vector* $\boldsymbol{\omega}$ *such that* $(\beta(t), B(t)) \to (\overline{\beta}_0, \overline{B}_0)$ *for* $\varepsilon \to 0$.

1.2 Remarks about the results and sketch of their proofs

Quasi-periodic solutions to (1.1.14) with frequency vector $\boldsymbol{\omega}$ describe lower-dimensional tori (d-dimensional tori for a system with $d+1$ degrees of freedom). Such tori are parabolic in the sense that the "normal frequency" vanishes for $\varepsilon = 0$. Theorems 1.1.1 and 1.1.2 imply the following result.

Theorem 1.2.1. *Consider the system (1.1.14) and assume Hypotheses 1 and 2 to be satisfied. Then for ε small enough there exists at least one quasi-periodic solution $(\beta(t), B(t))$ with frequency vector $\boldsymbol{\omega}$.*

Proof. If all the coefficients $\Gamma_{\mathbf{0}}^{(k)} = -[\partial_\beta f]_{\mathbf{0}}^{(k-1)}$ vanish identically for all $k \geq 1$ we simply apply Theorem 1.1.1. Otherwise there exists $k_0 \geq 1$ such that all the coefficients $\Gamma_{\mathbf{0}}^{(k)}(\beta_0)$ vanish identically for all $k < k_0$ while $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0)$ is not identically zero and hence we can solve the equations of motion up to order k_0 without fixing the parameter β_0. Moreover one has $\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0) = \partial_{\beta_0} g^{(k_0)}(\beta_0)$ with

$$g^{(k_0)}(\beta_0) := [\overline{B}\,\dot{\overline{b}}]_{\mathbf{0}}^{(k_0)} - [h(\overline{B}_0 + \overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)} - [f(\boldsymbol{\omega}t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})]_{\mathbf{0}}^{(k_0-1)},$$

because, if we denote

$$\overline{b} = \sum_{k=1}^{k_0-1} b^{(k)}, \qquad \overline{B} = \sum_{k=1}^{k_0-1} B^{(k)}.$$

one has

$$\begin{aligned}
\partial_{\beta_0}[f(\boldsymbol{\omega}t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})]_{\mathbf{0}}^{(k_0-1)} &= [\partial_\beta f(\boldsymbol{\omega}t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})(1 + \partial_{\beta_0}\overline{b})]_{\mathbf{0}}^{(k_0-1)} \\
&\quad + [\partial_B f(\boldsymbol{\omega}t, \beta_0 + \overline{b}, \overline{B}_0 + \overline{B})\partial_{\beta_0}\overline{B}]_{\mathbf{0}}^{(k_0-1)} \\
&= -\Gamma_{\mathbf{0}}^{(k_0)} - [\dot{\overline{B}}\partial_{\beta_0}\overline{b}]_{\mathbf{0}}^{(k_0)} + [\dot{\overline{b}}\partial_{\beta_0}\overline{B}]_{\mathbf{0}}^{(k_0)} \\
&\quad - [\omega_0(\overline{B}_0 + \overline{B} + B^{(k_0)})\partial_{\beta_0}(\overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)} \\
&= -\Gamma_{\mathbf{0}}^{(k_0)} + \partial_{\beta_0}[\overline{B}\,\dot{\overline{b}}]_{\mathbf{0}}^{(k_0)} - \partial_{\beta_0}[h(\overline{B}_0 + \overline{B} + B^{(k_0)})]_{\mathbf{0}}^{(k_0)}.
\end{aligned}$$

Since $g^{(k_0)}$ is analytic and periodic, and then it has at least a maximum β_0' and a minimum β_0''. Then Hypothesis 4 holds. Indeed, if $\varepsilon^{k_0}\omega_0'(\overline{B}_0) > 0$ one can choose $\overline{\beta}_0 = \beta_0''$, while if $\varepsilon^{k_0}\omega_0'(\overline{B}_0) < 0$ one can choose $\overline{\beta}_0 = \beta_0'$ and hence in both cases Hypothesis 4 is satisfied. Therefore the existence of a quasi-periodic solution with frequency vector $\boldsymbol{\omega}$ follows from Theorem 1.1.2. ∎

Theorem 1.2.1 can be seen as the counterpart of Cheng's result [18] in the case in which all "proper frequencies" are fixed (isochronous case) and the perturbation does not depend on the actions conjugated to the "fast angles" (otherwise one should add a correction like in [56]); moreover, with respect to [18], a weaker Diophantine condition is assumed on the proper frequencies.

The proofs of Theorems 1.1.1 and 1.1.2 are organised as follows.

We first introduce a convenient graphical representation for the coefficents $b_{\boldsymbol{\nu}}^{(k)}(\beta_0), B_{\boldsymbol{\nu}}^{(k)}(\beta_0)$ in (1.1.10) and we shall use it in order to prove that they are well defined.

Then we shall see that, under Hypothesis 3 and if the system is Hamiltonian, there are some suitable "cancellations" which will yield the convergence of the series (1.1.9), so that Theorem 1.1.1 will follow.

On the other hand we are not able to prove the same "cancellations" for the system (1.1.1) without Hypothesis 3 and the assumption that the system is Hamiltonian. Hence, in order to prove Theorem 1.1.2, besides the system (1.1.8) we shall consider first the system described by the range equations

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})b_{\boldsymbol{\nu}} = \Phi_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu}\neq\mathbf{0}, \tag{1.2.1a}$$

$$(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})B_{\boldsymbol{\nu}} = \Gamma_{\boldsymbol{\nu}}, \quad \boldsymbol{\nu}\neq\mathbf{0}, \tag{1.2.1b}$$

i.e. with no condition for $\boldsymbol{\nu}=\mathbf{0}$, and we shall prove that, if some further conditions (to be specified later on) are found to be satisfied, it is possible to find, for ε small enough and arbitrary β_0, B_0, a solution

$$(\beta_0 + b(t), B_0 + \widetilde{B}(t)), \tag{1.2.2}$$

to the system (1.2.1), with $b(t)$ and $\widetilde{B}(t)$ as in (1.1.6) depending on the free parameters $\varepsilon, \beta_0, B_0$; such a solution is obtained via a 'resummation procedure', starting from the formal solution of the range equations (1.2.1). The conditions mentioned above can be illustrated as follows. The resummation procedure turns out to be well-defined if the small divisors of the resummed series can be bounded proportionally to the square of the small divisors of the formal series. However, it is not obvious at all that this is possible, since the latter are of the form $(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})^{-1}$ with $\boldsymbol{\nu}\in\mathbb{Z}^d_*$, while the small divisors of the resummed series are of the form $(\det((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\beta_0,B_0)))^{-1}$, for suitable 2×2 matrices $\mathcal{M}^{[n]}$ (see § 3.2). The bound on the small divisors of the resummed series is difficult to check without assuming any non-degeneracy condition on the perturbation. Therefore we replace $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)$ with $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)\xi_n(\det(\mathcal{M}^{[n]}(0;\varepsilon,\beta_0,B_0)))$, for suitable 'cut-off functions' ξ_n, in such a way that the bound automatically holds. The introduction of the cut-offs changes the series in such a way that if on the one hand the modified series are well-defined, on the other hand in principle they no longer solve the range equations: this turns out to be the case only if one can prove that the cut-offs can be removed. So, the last part of the proof consists in showing that, by suitably choosing the parameters β_0, B_0 as continuous functions of ε, this occurs and moreover, for the same choice of β_0, B_0, the bifurcation equations (1.1.8c) and (1.1.8d) hold; hence for such β_0, B_0, the function (1.2.2) is a solution of the whole system (1.1.1).

1.3 Graph and trees: a short introduction

For the proof of both Theorems 1.1.1 and 1.1.2 we shall use a graphical representation for the coefficients of the solutions. The use of a graphical representation for the coefficients of perturbative series is quite common in quantum field theory. In the context of KAM theory it was originally introduced by Gallavotti in [31], inspired by a pioneering paper by Eliasson [28] and thereafter has been used in many other related papers; see [43] for a review. We now introduce some basic facts from the graph theory; see for instance [14]. Then we shall see how to use them for our purpose.

A graph G is an ordered pair $G = (V, L)$, where $V = V(G)$ is a non-empty set whose elements are called *vertices* and $L = L(G)$ a family of unordered couples of elements of $V(G)$, whose elements are called *lines* (or *edges*). Given two vertices $v, w \in V(G)$, the couple $\ell = (v, w)$ can appear more then once in this family. We shall say that a graph G is *simple* if any couple appears only once in $L(G)$. A graph G is *finite* if $|V(G)|, |L(G)| < \infty$. We can represent a (finite) graph G as a set of points (the vertices) and lines connecting them; see Figure 1.1.

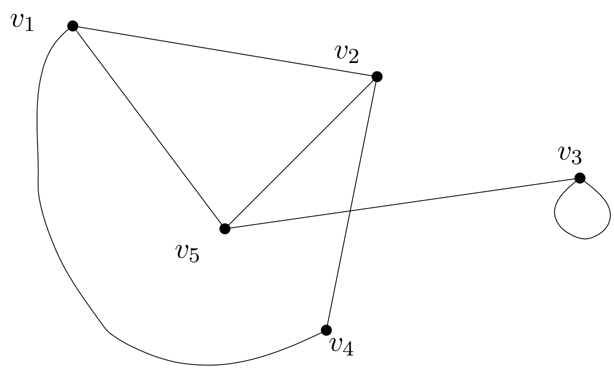

Figure 1.1: A representation for the simple graph $G = (V, L)$ with $V = V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $L = L(G) = \{(v_1, v_2), (v_1, v_4), (v_1, v_5), (v_2, v_4), (v_2, v_5), (v_3, v_3), (v_3, v_5)\}$.

A *planar* graph is a (finite) graph G which can be drawn on a plane without lines crossing.

Remark 1.3.1. Note that the graph G represented in Figure 1.1 is planar: indeed also the drawning in Figure 1.2 is a representation of the same G.

Given a graph G and two vertices $v, w \in V(G)$ we shall say that v, w are *connected* if either $(v, w) \in L(G)$, or there exist $v_0 = v, v_1, \ldots, v_{n-1}, v_n = w \in V(G)$ such that $\mathcal{P} := \{(v_0, v_1), \ldots, (v_{n-1}, v_n)\} \subseteq L(G)$; we shall say that $\mathcal{P}$ is a *path* connecting v to w. A graph G is *connected* if for any $v, w \in V(G)$ either $(v, w) \in L(G)$ or there exists a path connecting them, i.e. if all couples of vertices are connected.

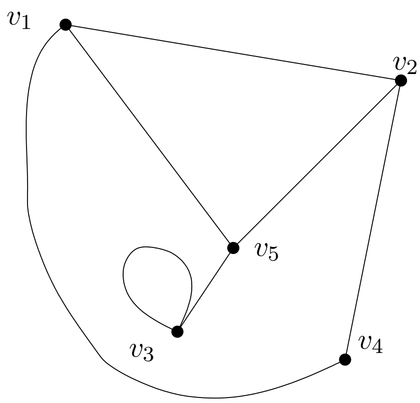

Figure 1.2: Another representation for the graph G in Figure 1.1.

A graph G has a *loop* if there exists $v \in V(G)$ such that either $(v, v) \in L(G)$ or there exists a path $\mathcal{P}$ connecting v to itself.

A *subgraph* S of a graph G is an ordered pair $S = (V, L)$ such that $V = V(S) \subseteq V(G)$, $L = L(S) \subseteq L(G)$ and S is itself a graph.

An *oriented* graph G is an ordered pair (V, L), where $V = V(G)$ is a non-empty set (of vertices) and $L = L(G)$ a family of ordered couples (*oriented lines*) of $V(G)$; clearly, all the definitions above can be suitably adapted also for oriented graphs: in particular a path should be consistent with the orientation.

If G is an oriented graph, we shall say that the line $\ell = (v, w) \in L(G)$ *exits* the vertex v and *enters* w. We can represent an oriented line ℓ as a line with an arrow superimposed; see Figure 1.3. Given an oriented graph G, any subgraph S inherit the orientation of G. We shall say that a line $\ell = (v, w) \in L(G)$ *enters* a subgraph S of G if $v \in V(G) \setminus V(S)$ while $w \in V(S)$; analogously we shall say that $\ell' = (v', w')$ *exits* S if $v' \in V(S)$ while $w' \in V(G) \setminus V(S)$. Note that if ℓ enters or exits S then $\ell \notin L(S)$.

A *tree-graph* $\mathfrak{T}$ is a finite planar connected graph with no loops. A *rooted tree-graph* is a tree-graph with a (unique) special vertex called *root* which induces a natural orientation on the lines, towards or away from the root.

From now on we shall consider only rooted tree-graphs $\mathfrak{T}$ in which the lines are oriented toward the root and the root has only one entering line (the *root line* $\ell_{\mathfrak{T}}$): we shall call them "tree-graphs" for simplicity. Given a tree-graph $\mathfrak{T}$, we shall call *nodes* all the vertices of $\mathfrak{T}$ except the root r and denote $N(\mathfrak{T}) = V(\mathfrak{T}) \setminus \{r\}$; see Figure 1.4.

Remark 1.3.2. Given a tree-graph $\mathfrak{T}$, a line ℓ is uniquely determined by the vertex v which it exits, so we may write $\ell = \ell_v$.

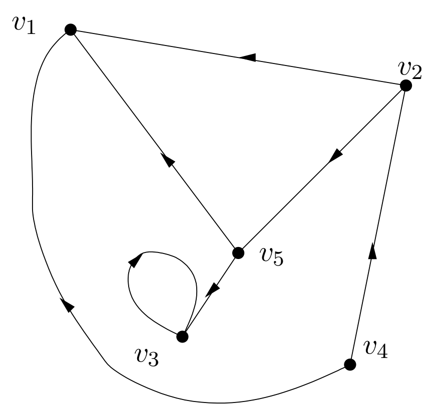

Figure 1.3: A representation of an oriented graph G. Note that, because of the orientation, the line $\ell = (v_3, v_3)$ is the only loop.

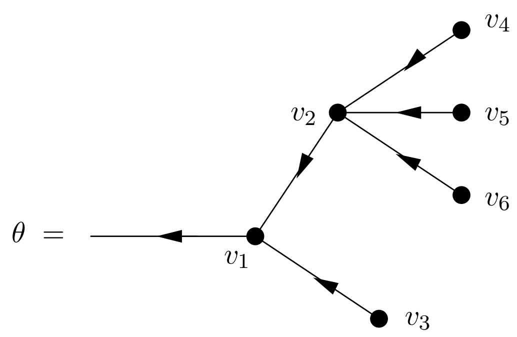

Figure 1.4: A representation θ of a tree-graph $\mathfrak{T}$; we do not drawn the root r of $\mathfrak{T}$ (i.e. the end-point of the line exiting v_1) to stress that $r \notin N(\mathfrak{T})$.

The orientation on a tree-graph $\mathfrak{T}$ provides a partial ordering relation on the vertices: given $v, w \in V(\mathfrak{T})$, we shall write $v \preceq w$ if there is a path connecting v to w; for instance in Figure 1.4 one has $v_4 \prec v_2 \prec v_1$, $v_5 \prec v_2 \prec v_1$, $v_6 \prec v_2 \prec v_1$, and $v_3 \prec v_1$. If $\ell = \ell_v$ and $\ell' = \ell_{v'}$ we may write $w \prec \ell$ if $w \preceq v$, $\ell \prec w$ if $v \prec w$ and $\ell \prec \ell'$ if $v \prec v'$. Give a tree-graph $\mathfrak{T}$ and two distinct lines $\ell \prec \ell' \in L(T)$ we shall denote by $\mathcal{P}(\ell', \ell)$ the unique path connecting ℓ to ℓ'.

Given a tree-graph $\mathfrak{T}$, for all $v \in N(\mathfrak{T})$ denote by s_v the number of lines entering the node v. Note that any subgraph S of any tree-graph $\mathfrak{T}$ has at most one exiting line: we may denote such a line by ℓ_S.

Remark 1.3.3. One has $\displaystyle\sum_{v \in N(\mathfrak{T})} s_v = |N(\mathfrak{T})| - 1$.

A *labelled tree-graph* is a tree-graph together with a label function defined on $N(\mathfrak{T})$ and $L(\mathfrak{T})$. In the following we shall call tree-graphs tout court the tree-graphs with labels, and we shall use the term *unlabelled tree-graphs* for the tree-graphs without labels.

We shall say that two representations θ_1, θ_2 of a tree-graph $\mathfrak{T}$ are *equivalent* if they can be obtained from each other by continuously deforming the lines without lines crossing. We shall call *tree* any equivalence class of representations of a tree-graph. Of course all the definitions and notations above can be suitably adapted also for trees; in particular, with some abuse of notation, we shall call *subgraph* of a tree θ (representing a tree-graph $\mathfrak{T}$) the portion of θ representing a subgraph of $\mathfrak{T}$. Our aim is to represent the solutions as "sum over trees" (in a sense that will be clear later on) hence their labels shall depend on the model under study. In what follows we shall see how to use trees in order to prove both Theorems 1.1.1 and 1.1.2.

2. Proof of Theorem 1.1.1

In this Chapter we shall use the trees introduced in § 1.3 to represent the formal solutions (1.1.9). As we shall see, this will allow us to prove the convergence of the formal power series under the assumption that the system is Hamiltonian and that Hypothesis 3 holds, and hence Theorem 1.1.1 will follow.

2.1 Labels and tree values

We want to associate labels with the nodes and the lines of a tree in such a way that each tree represents a contribution to the coefficients $b_{\boldsymbol{\nu}}^{(k)}, B_{\boldsymbol{\nu}}^{(k)}$ appearing in (1.1.9).

Given a tree θ, we associate with each node v a *mode label* $\boldsymbol{\nu}_v \in \mathbb{Z}^d$, a *component label* $h_v \in \{\beta, B\}$ and an *order label* $k_v \in \{0,1\}$ with the constraint that $k_v = 1$ if $\boldsymbol{\nu}_v \neq \mathbf{0}$. With each line $\ell = \ell_v$, $\ell \neq \ell_\theta$, we associate a *component label* $h_\ell \in \{\beta, B\}$ with the constraint that $h_{\ell_v} = h_v$, and a *momentum label* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$ with the constraint that $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ if $h_\ell = \beta$. We associate with the root-line ℓ_θ a component label $h_{\ell_\theta} \in \{\beta, B, \Gamma, \Phi\}$ and a momentum label $\boldsymbol{\nu}_{\ell_\theta} \in \mathbb{Z}^d$ with the following constraints. Call v_0 the node which ℓ_θ exists: then (i) $h_{\ell_\theta} = B, \Gamma$ if $h_{v_0} = B$ while $h_{\ell_\theta} = \beta, \Phi$ if $h_{v_0} = \beta$, and (ii) $\boldsymbol{\nu}_{\ell_\theta} \neq \mathbf{0}$ for $h_{\ell_\theta} = \beta$ while $\boldsymbol{\nu}_{\ell_\theta} = \mathbf{0}$ for $h_{\ell_\theta} = \Gamma, \Phi$. Moreover we require $k_{v_0} = 1$ if $\ell = \ell_{v_0}$ is such that either $\ell = \ell_\theta$ and $h_\ell = \Gamma$, or $h_\ell = B$ and $\boldsymbol{\nu}_\ell \neq \mathbf{0}$.

We shall call *total component* and *total momentum* of θ the component and the momentum of ℓ_θ respectively. For any node $v \in N(\theta)$ we denote by p_v and q_v the numbers of lines with component β and B, respectively, entering the node v, of course $s_v = p_v + q_v$.

If $k_v = 0$ for some $v \in N(\theta)$ we force also $p_v = 0$ and $q_v \geq 1$ if $h_v = \beta$, while we force $p_v = 0$ and $q_v \geq 2$ if $h_v = B$.

We impose the following *conservation law*

$$\boldsymbol{\nu}_\ell = \sum_{v \prec \ell} \boldsymbol{\nu}_v \tag{2.1.1}$$

and we call *order* of θ the number

$$k(\theta) = \sum_{v \in N(\theta)} k_v. \tag{2.1.2}$$

More generally given any subgraph T of a tree θ we call *order* of T the number

$$k(T) = \sum_{v \in N(T)} k_v. \tag{2.1.3}$$

Lemma 2.1.1. *Let T be a subgraph of any tree θ. Then one has $|N(T)| \leq 4k(T) - 2$.*

Proof. We shall prove the result by induction on $k = k(T)$. For $k = 1$ the bound is trivially satisfied as a direct check shows. Assume then the bound to hold for all $k' < k$. Call v the node which ℓ_T (possibly ℓ_θ) exits, $\ell_1, \ldots, \ell_{s_v}$ the lines entering v and $T_1, \ldots, T_{s_v}$ the subgraphs of T with exiting lines $\ell_1, \ldots, \ell_{s_v}$. If $k_v = 1$ then by the inductive hypothesis one has

$$|N(T)| = 1 + \sum_{i=1}^{s_v} |N(T_i)| \leq 1 + 4(k-1) - 2s_v \leq 4k - 3.$$

If $k_v = 0$ and $h_v = B$ then one has $s_v = q_v \geq 2$ and hence

$$|N(T)| = 1 + \sum_{i=1}^{q_v} |N(T_i)| \leq 1 + 4k - 2q_v \leq 4k - 3.$$

If $k_v = 0$ and $h_v = \beta$, then if $q_v \geq 2$ one can reason as in the previous case. Otherwise, since the line $\ell = \ell_w$ entering v is such that $h_\ell = B$, either $k_w = 1$ or $k_w = 0$ and $q_w \geq 2$. call $\ell'_1, \ldots, \ell'_{s_w}$ the lines entering w and $T'_1, \ldots, T'_{s_v}$ the subgraphs of T with exiting lines $\ell'_1, \ldots, \ell'_{s_w}$. In the first case one has

$$|N(T)| = 2 + \sum_{i=1}^{s_w} |N(T'_i)| \leq 2 + 4(k-1) - 2s_w \leq 4k - 3,$$

while in the second case one has

$$|N(T)| = 2 + \sum_{i=1}^{q_w} |N(T'_i)| \leq 2 + 4k - 2q_w \leq 4k - 2.$$

Therefore the bound follows. ∎

Remark 2.1.2. If T is a subgraph of any tree θ, one has $\sum_{v \in N(T)} s_v \leq 4k(T)$. In particular one has also $\sum_{v \in N(\theta)} s_v \leq 4k(\theta)$.

Given a tree θ we associate with each node v a *node factor*

$$\mathcal{F}_v = \begin{cases} \dfrac{1}{q_v!}\partial_1^{q_v}\omega_0(\overline{B}_0), & k_v = 0, \\ \dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v+1}f_{\boldsymbol{\nu}_v}(\beta_0,\overline{B}_0), & k_v = 1, \quad h_v = \beta, \\ -\dfrac{1}{p_v!q_v!}\partial_\beta^{p_v+1}\partial_B^{q_v}f_{\boldsymbol{\nu}_v}(\beta_0,\overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} \neq \mathbf{0}, \\ \dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v+1}f_{\boldsymbol{\nu}_v}(\beta_0,\overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} = \mathbf{0}, \quad h_{\ell_v} = B, \\ -\dfrac{1}{p_v!q_v!}\partial_\beta^{p_v+1}\partial_B^{q_v}f_{\boldsymbol{\nu}_v}(\beta_0,\overline{B}_0), & k_v = 1, \quad h_v = B, \quad \boldsymbol{\nu}_{\ell_v} = \mathbf{0}, \quad h_{\ell_v} = \Gamma, \end{cases} \tag{2.1.4}$$

and with each line ℓ a *propagator*

$$\overline{\mathcal{G}}_\ell := \begin{cases} \dfrac{1}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell}, & \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ -\dfrac{1}{\omega_0'(\overline{B}_0)}, & \boldsymbol{\nu}_\ell = \mathbf{0}, \quad h_\ell = B, \\ 1, & \boldsymbol{\nu}_\ell = \mathbf{0}, \quad h_\ell = \Gamma, \Phi \end{cases} \tag{2.1.5}$$

and define the *value* of any subgraph S of any tree θ as

$$\mathscr{V}(S) = \left(\prod_{v\in N(S)} \mathcal{F}_v\right)\left(\prod_{\ell\in L(S)} \overline{\mathcal{G}}_\ell\right). \tag{2.1.6}$$

All the labels (and the constraints) above and the definitions of both the node factors and the propagators reflect the form of the coefficients (1.1.10), taking into account the expressions (1.1.11) and (1.1.12). Indeed if we denote by $\mathscr{T}_{k,\boldsymbol{\nu},h}$ the set of trees with order k, total momentum $\boldsymbol{\nu}$ and total component h, one has (at least formally)

$$\begin{aligned} b_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta\in\mathscr{T}_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta), \qquad \boldsymbol{\nu}\in\mathbb{Z}^d_*, \\ B_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta\in\mathscr{T}_{k,\boldsymbol{\nu},B}} \mathscr{V}(\theta), \qquad \boldsymbol{\nu}\in\mathbb{Z}^d, \\ \Phi_{\mathbf{0}}^{(k)} &= [\omega_0(B(t))]_{\mathbf{0}}^{(k)} + [\partial_B f(\boldsymbol{\omega}t,\beta(t),B(t))]_{\mathbf{0}}^{(k-1)} = \sum_{\theta\in\mathscr{T}_{k,\mathbf{0},\Phi}} \mathscr{V}(\theta), \\ \Gamma_{\mathbf{0}}^{(k)} &= [-\partial_\beta f(\boldsymbol{\omega}t,\beta(t),B(t))]_{\mathbf{0}}^{(k-1)} = \sum_{\theta\in\mathscr{T}_{k,\mathbf{0},\Gamma}} \mathscr{V}(\theta), \end{aligned} \tag{2.1.7}$$

as a direct check shows, where the notation (1.1.12) have been used. Here and henceforth in this Chapter we shall not write explicitely the dependence on the parameter β_0; note however that $\mathscr{V}(\theta)$ depends on β_0 only through the node factors.

Lemma 2.1.3. *The coefficients $b_{\boldsymbol{\nu}}^{(k)}, B_{\boldsymbol{\nu}}^{(k)}, \Phi_{\mathbf{0}}^{(k)}$ and $\Gamma_{\mathbf{0}}^{(k)}$ are well defined for all $k \geq 1$.*

Proof. Set

$$\varepsilon_n = \varepsilon_n(\boldsymbol{\omega}) := \frac{1}{2^n} \log \frac{1}{\alpha_n(\boldsymbol{\omega})}, \tag{2.1.8}$$

and note that by Hypothesis 2, $\varepsilon_n \to 0$ as $n \to \infty$. Moreover the analyticity of f and h implies that there exist positive constants F_1, F_2, ξ such that for all $v \in N(\theta)$ one has

$$|\mathcal{F}_v| \leq F_1 F_2^{s_v+1} \mathrm{e}^{-\xi|\boldsymbol{\nu}_v|}.$$

Hence, using Lemma 2.1.1 and Remark 2.1.2, for all $\theta \in \mathscr{T}_{k,\boldsymbol{\nu},h}$ one has

$$\begin{aligned}
|\mathscr{V}(\theta)| &\leq C_0^k \left(\prod_{v \in N(\theta)} \mathrm{e}^{-\xi|\boldsymbol{\nu}_v|} \right) \left(\prod_{\ell \in L(\theta)} \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|} \right) = C_0^k \mathrm{e}^{-\xi \sum_{v \in N(\theta)} |\boldsymbol{\nu}_v|} \left(\prod_{\ell \in L(\theta)} \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|} \right) \\
&= C_0^k \mathrm{e}^{-\xi \sum_{v \in N(\theta)} |\boldsymbol{\nu}_v|/2} \left(\mathrm{e}^{-\xi \sum_{v \in N(\theta)} |\boldsymbol{\nu}_v|/2|L(\theta)|} \right)^{|L(\theta)|} \left(\prod_{\ell \in L(\theta)} \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|} \right) \\
&\leq C_0^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} \prod_{\ell \in L(\theta)} \mathrm{e}^{-\xi|\boldsymbol{\nu}_\ell|/8k} \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|} \\
&\leq C_0^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} \prod_{\ell \in L(\theta)} \mathrm{e}^{-\xi 2^{\overline{n}_\ell}/16k} \frac{1}{\alpha_{\overline{n}_\ell}(\boldsymbol{\omega})} = C_0^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} \prod_{\ell \in L(\theta)} \mathrm{e}^{(-\xi/16k + \varepsilon_{\overline{n}_\ell}) 2^{\overline{n}_\ell}},
\end{aligned} \tag{2.1.9}$$

where C_0 is a suitable positive constant and we have set $\overline{n}_\ell = n(\boldsymbol{\nu}_\ell) := \inf\{n \geq 0 \,:\, |\boldsymbol{\nu}_\ell| \leq 2^n\}$. The sum over all the shapes and all the labels except the mode labels is bounded by a constant to the power k, and hence one has

$$\sum_{\theta \in \Theta_{k,\boldsymbol{\nu}}} |\mathscr{V}(\theta)| \leq \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} C^k \left(\sum_{n \geq 0} \mathrm{e}^{(-\xi/16k + \varepsilon_n) 2^n} \right)^{4k} \leq \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2} C^k D(k)^k \tag{2.1.10}$$

where $C > 0$ is a suitable constant and $D(k)$ is a constant depending on k. Therefore the assertion follows. ■

Remark 2.1.4. The constant $D(k)$ grows with k (for instance if $\boldsymbol{\omega}$ is Diophantine one has $D(k) \approx k$) and hence the bound (2.1.10) is not enough to obtain the convergence of the power series.

2.2 Clusters, self-energy clusters and dimensional bounds

From the proof of Lemma 2.1.3 emerges that it may be (and in fact it is) convenient to associate with each line ℓ a further label in order to control the "size" of the small divisor $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell$. Roughly we would like to associate with a line ℓ a "scale" label n if $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell \approx \alpha_n(\boldsymbol{\omega})$; to be more precise, since the sequence $\{\alpha_n(\boldsymbol{\omega})\}_{n\geq 0}$ is only non-increasing, for the scales to be uniquely defined one should take a decreasing subsequence $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$ and say that ℓ has scale n if $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$ is of order $\alpha_{m_n}(\boldsymbol{\omega})$. It would be tempting to use a sharp partition through step functions with supports $[\alpha_{m_n}(\boldsymbol{\omega}),\alpha_{m_{n-1}}(\boldsymbol{\omega}))$, in order to associate with a line ℓ a scale n if $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|\in[\alpha_{m_n}(\boldsymbol{\omega}),\alpha_{m_{n-1}}(\boldsymbol{\omega}))$. However, it turns out to be more convenient using a smooth partition through compact support functions Ψ_n (because we have to take derivatives of quantities involving such functions). Therefore we shall proceed as follows.

With each line $\ell\in L(\theta)$ we associate a *scale label* n_ℓ such that $n_\ell=-1$ if $\boldsymbol{\nu}_\ell=\mathbf{0}$, while $n_\ell\in\mathbb{Z}_+$ if $\boldsymbol{\nu}_\ell\neq\mathbf{0}$. So far there is no relation between non-zero momenta and scale labels: a constraint will appear later on.

We denote by $\Theta_{k,\boldsymbol{\nu},h}$ the set of trees with order k, total momentum $\boldsymbol{\nu}$ and total component h: we used a different notation for the set of trees to stress that if $\theta\in\Theta_{k,\boldsymbol{\nu},h}$, each $\ell\in L(\theta)$ carries the further label n_ℓ.

To take into account the scale labels we slightly change the definition of the value of a tree. More precisely, for any $\theta\in\Theta_{k,\boldsymbol{\nu},h}$ we define $\mathscr{V}(\theta)$ as in (2.1.6) but with new propagators which depend on the scale labels as follows.

Let us introduce the sequences $\{m_n,p_n\}_{n\geq 0}$, with $m_0=0$ and, for all $n\geq 0$, $m_{n+1}=m_n+p_n+1$, where $p_n:=\max\{q\in\mathbb{Z}_+ : \alpha_{m_n}(\boldsymbol{\omega})<2\alpha_{m_n+q}(\boldsymbol{\omega})\}$. Then the subsequence $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$ of $\{\alpha_m(\boldsymbol{\omega})\}_{m\geq 0}$ is decreasing. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a C^∞ function, non-increasing for $x\geq 0$ and non-decreasing for $x<0$, such that

$$\chi(x)=\begin{cases}1, & |x|\leq 1/2,\\ 0, & |x|\geq 1.\end{cases}\tag{2.2.1}$$

Set $\chi_{-1}(x)=1$ and $\chi_n(x)=\chi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$ for $n\geq 0$. Set also $\psi(x)=1-\chi(x)$, $\psi_n(x)=\psi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$, and $\Psi_n(x)=\chi_{n-1}(x)\psi_n(x)$, for $n\geq 0$; see Figure 2.1.

Lemma 2.2.1. *For all $x\neq 0$ and for all $p\geq 0$ one has*

$$\psi_p(x)+\sum_{n\geq p+1}\Psi_n(x)=1.$$

Proof. For fixed $x\neq 0$ let $N=N(x):=\min\{n : \chi_n(x)=0\}$ and note that $\max\{n : \psi_n(x)=0\}\leq N-1$. Then if $p\leq N-1$

$$\psi_p(x)+\sum_{n\geq p+1}\Psi_n(x)=\psi_{N-1}(x)+\chi_{N-1}(x)=1,$$

while if $p\geq N$ one has

$$\psi_p(x)+\sum_{n\geq p+1}\Psi_n(x)=\psi_p(x)=1.$$

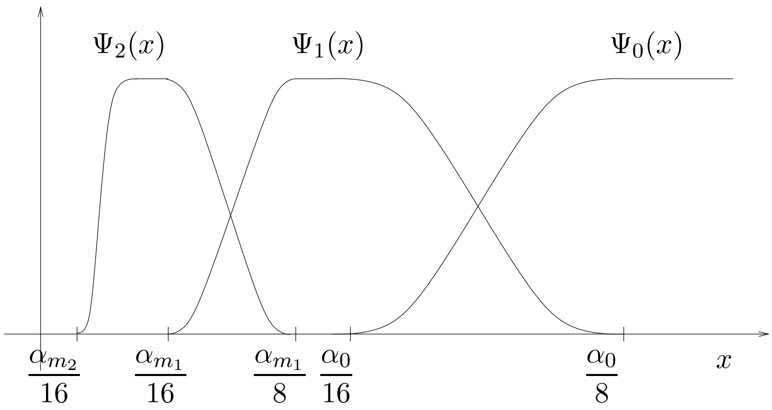

Figure 2.1: Graphs of some of the C^∞ functions $\Psi_n(x)$ partitioning the unity in $\mathbb{R}\setminus\{0\}$; here $\alpha_m = \alpha_m(\boldsymbol{\omega})$. The function $\chi_0(x) = \chi(8x/\alpha_0)$ is given by the sum of all functions $\Psi_n(x)$ for $n \geq 1$.

In both cases the assertion follows. ■

We associate with each line a propagator

$$\mathcal{G}_\ell := \begin{cases} \dfrac{\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell}, & n_\ell \geq 0, \\ -\dfrac{1}{\omega_0'(\overline{B}_0)}, & n_\ell = -1, \quad h_\ell = B, \\ 1, & n_\ell = -1, \quad h_\ell = \Gamma, \Phi. \end{cases} \tag{2.2.2}$$

Note that, although we have changed the propagators, the identities (2.1.7) still hold because of Lemma 2.2.1.

Remark 2.2.2. Given a tree θ such that $\mathscr{V}(\theta) \neq 0$, for any line $\ell \in L(\theta)$ with $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ one has $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \neq 0$, and hence

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{16} < |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| < \frac{\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega})}{8}, \tag{2.2.3}$$

where $\alpha_{m_{-1}}(\boldsymbol{\omega})$ has to be interpreted as $+\infty$. Note also that $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \neq 0$ implies

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| < \frac{1}{8}\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega}) < \frac{1}{4}\alpha_{m_{n_\ell-1}+p_{n_\ell-1}}(\boldsymbol{\omega}) = \frac{1}{4}\alpha_{m_{n_\ell}-1}(\boldsymbol{\omega}) < \alpha_{m_{n_\ell}-1}(\boldsymbol{\omega}),$$

and hence, by definition of $\alpha_m(\boldsymbol{\omega})$, one has $|\boldsymbol{\nu}_\ell| > 2^{m_{n_\ell}-1}$. Moreover, by the definition of $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n\geq 0}$, the number of scales which can be associated with a line ℓ in such a way that the propagator does not vanishes is at most 2. The same considerations apply to any subgraph of θ.

A *cluster* T on scale n is a maximal subgraph of a tree θ such that all the lines have scales $n' \leq n$ and there is at least one line with scale n. The lines entering the cluster T and the line coming out from it (unique if existing at all) are called the *external* lines of T. A *self-energy cluster* is a cluster T such that (i) T has only one entering line ℓ'_T and one exiting line ℓ_T, (ii) either $n=-1$ and $\mathcal{P}(\ell_T,\ell'_T)=\emptyset$ or $n\geq 0$ and one has $n_\ell\geq 0$ and $\boldsymbol{\nu}_\ell\neq\boldsymbol{\nu}_{\ell'_T}$ for all $\ell\in\mathcal{P}(\ell_T,\ell'_T)$, and (iii) one has $\boldsymbol{\nu}_{\ell_T}=\boldsymbol{\nu}_{\ell'_T}$ and hence

$$\sum_{v\in N(T)}\boldsymbol{\nu}_v=\mathbf{0}. \tag{2.2.4}$$

For any self-energy cluster T, set $\mathcal{P}_T=\mathcal{P}(\ell_T,\ell'_T)$. More generally, if T is a subgraph of θ with only one entering line ℓ' and one exiting line ℓ, we set $\mathcal{P}_T=\mathcal{P}(\ell,\ell')$. We shall say that a subgraph T constituted by only one node v with $\boldsymbol{\nu}_v=\mathbf{0}$ such that v has only one entering line, is also a self-energy cluster on scale -1. If a self-energy cluster is on a scale $n\geq 0$ then $|N(T)|\geq 2$ and $k(T)\geq 2$, as it is easy to check.

Remark 2.2.3. Given a self-energy cluster T, the momenta of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\nu}_{\ell'_T}$ because of the conservation law (2.1.1). More precisely, for all $\ell\in\mathcal{P}_T$ one has $\boldsymbol{\nu}_\ell=\boldsymbol{\nu}^0_\ell+\boldsymbol{\nu}_{\ell'_T}$ with

$$\boldsymbol{\nu}^0_\ell:=\sum_{\substack{w\in N(T)\\ w\prec\ell}}\boldsymbol{\nu}_w,$$

while all the other momenta in T do not depend on $\boldsymbol{\nu}_{\ell'_T}$.

We shall say that two self-energy clusters T_1,T_2 have the same *structure* if setting $\boldsymbol{\nu}_{\ell'_{T_1}}=\boldsymbol{\nu}_{\ell'_{T_2}}=\mathbf{0}$ one has $T_1=T_2$. This provides an equivalence relation on the set of all self-energy clusters. From now on we shall call self-energy clusters tout court such equivalence classes. We denote by $\mathfrak{S}^k_{n,u,e}$ the set of self-energy clusters with order k, scale n and such that $h_{\ell'_T}=e$ and $h_{\ell_T}=u$, with $e,u\in\{\beta,B\}$.

We shall say that a line ℓ is *resonant* if there exist two self-energy clusters T, T', such that $\ell_{T'}=\ell=\ell'_T$, otherwise ℓ is *non-resonant*. Given any subgraph S of any tree θ, we denote by $\mathfrak{N}^*_n(S)$ the number of non-resonant lines on scale $\geq n$ in S. Define also, for any line $\ell\in\theta$, the *minimum scale* of ℓ as

$$\zeta_\ell:=\min\{n\in\mathbb{Z}_+:\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq 0\}$$

and denote by $\mathfrak{N}^\bullet_n(S)$ as the number of non-resonant lines $\ell\in L(S)$ such that $\zeta_\ell\geq n$. By definition, if $\mathscr{V}(S)\neq 0$, for each line $\ell\in L(S)$ either $n_\ell=\zeta_\ell$ or $n_\ell=\zeta_\ell+1$. For any subgraph S of any tree θ, define also

$$K(S):=\sum_{v\in N(S)}|\boldsymbol{\nu}_v|. \tag{2.2.5}$$

Then one can prove the following results, which are based on the idea of Siegel [64].

Lemma 2.2.4. *For all $h\in\{\beta,B,\Phi,\Gamma\}$, $\boldsymbol{\nu}\in\mathbb{Z}^d$, $k\geq 1$ and for any $\theta\in\Theta_{k,\boldsymbol{\nu},h}$ with $\mathscr{V}(\theta)\neq 0$,one has $\mathfrak{N}^\bullet_n(\theta)\leq 2^{-(m_n-3)}K(\theta)$ for all $n\geq 0$.*

Proof. We want to prove by induction that

$$\mathfrak{N}_n^{\bullet}(\theta) \leq \max\{2^{-(m_n-3)}K(\theta) - 2, 0\}. \tag{2.2.6}$$

First of all note that $\mathfrak{N}_n^{\bullet}(\theta) \geq 1$ implies $\mathfrak{N}_n^{*}(\theta) \geq 1$ i.e. there is a line ℓ with $n_\ell \geq n$ and hence $K(\theta) \geq |\boldsymbol{\nu}_\ell| \geq 2^{m_n-1}$.

Set $\zeta_0 := \zeta_{\ell_\theta}$, $n_0 := n_{\ell_\theta}$ and $\boldsymbol{\nu} := \boldsymbol{\nu}_{\ell_\theta}$, and note that either $n_0 = \zeta_0$ or $n_0 = \zeta_0 + 1$. If $\zeta_0 < n$ the bound (2.2.6) follows from the inductive hypothesis. If $\zeta_0 \geq n$, call $\ell_1, \ldots, \ell_r$ the lines with minimum scale $\geq n$ closest to ℓ_θ and $\theta_1, \ldots, \theta_r$ the subtrees with root lines $\ell_1, \ldots, \ell_r$, respectively. If $r = 0$ the bound trivially holds. If $r \geq 2$, by the inductive hypothesis one has $\mathfrak{N}_n^{\bullet}(\theta) = 1 + \mathfrak{N}_n^{\bullet}(\theta_1) + \ldots + \mathfrak{N}_n^{\bullet}(\theta_r) \leq 1 + 2^{-(m_n-3)}K(\theta) - 2r \leq 2^{-(m_n-3)}K(\theta) - 3$, so that the bound follows once more.

If $r = 1$ call T the subgraph with exiting line ℓ_θ and entering line ℓ_1. Then either T is a self-energy cluster or $K(T) \geq 2^{m_n-1}$. This can be proved as follows. Set $\boldsymbol{\nu}_1 := \boldsymbol{\nu}_{\ell_1}$. If T is not a cluster, then it must contain at least one line ℓ' on scale $n_{\ell'} = n$, so that if $\ell' \notin \mathcal{P}_T$ and θ' is the subtree with root line ℓ' one has $K(T) \geq K(\theta') \geq 2^{m_n-1}$, while if $\ell \in \mathcal{P}_T$ then $\boldsymbol{\nu}_{\ell'} \neq \boldsymbol{\nu}_1$ (because $\zeta_{\ell'} = n - 1$ and $\zeta_{\ell_1} \geq n$), so that

$$|\boldsymbol{\omega} \cdot (\boldsymbol{\nu}_{\ell'} - \boldsymbol{\nu}_1)| \leq |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'}| + |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_1| \leq \frac{1}{4}\alpha_{m_{n-1}}(\boldsymbol{\omega}) < \alpha_{m_n-1}(\boldsymbol{\omega})$$

since both ℓ', ℓ_1 are on scale $\geq n$, and this implies $K(T) \geq |\boldsymbol{\nu}_{\ell'} - \boldsymbol{\nu}_1| \geq 2^{m_n-1}$. If T is a cluster then either (i) $\boldsymbol{\nu}_1 \neq \boldsymbol{\nu}$ so that $K(T) \geq |\boldsymbol{\nu} - \boldsymbol{\nu}_1| \geq 2^{m_n-1}$ or (ii) $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ and there is a line $\ell \in \mathcal{P}_T$ with $n_\ell = -1$ so that $K(T) \geq |\boldsymbol{\nu}_\ell^0| = |\boldsymbol{\nu}| \geq 2^{m_n-1}$, or (iii) $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ and T is a self-energy cluster, otherwise there would be a line $\ell' \in \mathcal{P}_T$ with $\boldsymbol{\nu}_{\ell'} = \boldsymbol{\nu}_1$, which is incompatible with $\zeta_{\ell'} \leq n - 1$ and $\zeta_{\ell_1} \geq n$.

Therefore, if $K(T) \geq 2^{m_n-1}$, the inductive hypothesis yields the bound (2.2.6). If $K(T) < 2^{m_n-1}$ then T is a self-energy cluster (and hence $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$). In such a case call θ_1 the tree with root line ℓ_1; by construction $\mathfrak{N}_n^{\bullet}(\theta) = 1 + \mathfrak{N}^{\bullet}(\theta_1)$. We can repeat the argument above: call $\ell'_1, \ldots, \ell'_{r'}$ the lines with minimum scale $\geq n$ closest to ℓ_1 and $\theta'_1, \ldots, \theta'_{r'}$ the subtrees with root lines $\ell'_1, \ldots, \ell'_{r'}$, respectively. Again the case $r' = 0$ is trivial. If $r' \geq 2$ then $\mathfrak{N}_n^{\bullet}(\theta) = 2 + \mathfrak{N}_n^{\bullet}(\theta'_1) + \ldots + \mathfrak{N}_n^{\bullet}(\theta'_{r'}) \leq 2 + 2^{-(m_n-3)}K(\theta) - 2r \leq 2^{-(m_n-3)}K(\theta) - 2$, so yielding the bound. Therefore the only case which does not imply immediately the bound (2.2.6) through the inductive hypothesis is when ℓ_1 exits a subgraph T' with only one entering line ℓ'_1 on minimum scale $\geq n$. Set $\boldsymbol{\nu}'_1 := \boldsymbol{\nu}_{\ell'_1}$ and call θ'_1 the tree with root line ℓ'_1. As before we have that either $K(T') \geq 2^{m_n-1}$ or T' is a self-energy cluster. If $K(T') \geq 2^{m_n-1}$ then $\mathfrak{N}_n^{\bullet}(\theta) = 2 + \mathfrak{N}_n^{\bullet}(\theta'_1)$ and one can reason as before to obtain the bound by relying on the inductive hypothesis. If T' is a self-energy cluster then ℓ_1 is a resonant line and $\mathfrak{N}_n^{\bullet}(\theta) = 1 + \mathfrak{N}_n^{\bullet}(\theta'_1)$.

One can iterate again the argument until either one reaches a case which can be dealt with through the inductive hypothesis or one obtains $\mathfrak{N}_n^{\bullet}(\theta) = 1 + \mathfrak{N}_n^{\bullet}(\theta'')$, for some tree θ'' which has no line ℓ with $\zeta_\ell \geq n$. Thus $\mathfrak{N}_n^{\bullet}(\theta'') = 0$ and the bound (2.2.6) follows. ∎

Lemma 2.2.5. *For all* $e, u \in \{\beta, B\}$, $n \geq 0$, $k \geq 1$ *and for any* $T \in \mathfrak{S}^k_{n,u,e}$ *with* $\mathscr{V}(T) \neq 0$, *one has* $K(T) > 2^{m_n - 1}$ *and* $\mathfrak{N}^\bullet_p(T) \leq 2^{-(m_p-3)} K(T)$ *for all* $0 \leq p \leq n$.

Proof. Consider a self-energy cluster $T \in \mathfrak{S}^k_{n,u,e}$. First of all we prove that $K(T) \geq 2^{m_n-1}$. Indeed T contains at least one line ℓ on scale n. If $\ell \in L(T) \setminus \mathcal{P}_T$ then $K(T) \geq |\boldsymbol{\nu}_\ell| \geq 2^{m_n-1}$, while if $\ell \in \mathcal{P}_T$ then $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell'_T}$ (otherwise T would not be a self-energy cluster). But then $K(T) \geq |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell'_T}| \geq 2^{m_n-1}$ as both ℓ, ℓ'_T are on scale $\geq n$. Define $\mathcal{C}(n,p)$ as the set of subgraphs T of θ with only one entering line ℓ'_T and one exiting line ℓ_T both on minimum scale $\geq p$, such that $L(T) \neq \emptyset$ and $n_\ell \leq n$ for any line $\ell \in L(T)$. We prove by induction on the order the bound

$$\mathfrak{N}^\bullet_p(T) \leq 2^{-(m_p-3)} K(T) \tag{2.2.7}$$

for all $T \in \mathcal{C}(n,p)$ and all $0 \leq p \leq n$. Consider $T \in \mathcal{C}(n,p)$, $p \leq n$: call $\ell_1, \ldots, \ell_r$ the lines with minimum scale $\geq p$ closest to ℓ_T. The case $r = 0$ is trivial. If $r \geq 1$ and none of such lines is along the path $\mathcal{P}_T$ then the bound follows from (2.2.6). If one of such lines, say ℓ_1, is along the path $\mathcal{P}_T$, then denote by $\theta_2, \ldots, \theta_r$ the subtrees with root lines $\ell_2, \ldots, \ell_r$, respectively, and by T_1 the subgraph with exiting line ℓ_1 and entering line ℓ'_T. One has $\mathfrak{N}^\bullet_p(T) \leq 1 + \mathfrak{N}^\bullet_p(T_1) + \mathfrak{N}^\bullet_p(\theta_2) + \ldots + \mathfrak{N}^\bullet_p(\theta_r)$. By construction $T_1 \in \mathcal{C}(n,p)$, so that the bound (2.2.7) follows by the inductive hypothesis for $r \geq 2$.

If $r = 1$ then call T_0 the subgraph with exiting line ℓ_T and entering line ℓ_1. By reasoning as in the proof of Lemma 2.2.4 we find that either $K(T_0) \geq 2^{m_p-1}$ or T_0 is a self-energy cluster. Since $\mathfrak{N}^\bullet_p(T) \leq 1 + \mathfrak{N}^\bullet_p(T_1)$, if $K(T_0) \geq 2^{m_p-1}$ the bound follows once more. If on the contrary T_0 is a self-energy cluster we can iterate the construction: call $\ell'_1, \ldots, \ell'_{r'}$ the lines with minimum scale $\geq p$ closest to ℓ_1. If either $r' = 0$ or no line among $\ell'_1, \ldots, \ell'_{r'}$ is along the path $\mathcal{P}_{T_1}$, the bound follows easily. Otherwise if a line, say ℓ'_1 is along the path $\mathcal{P}_{T_1}$ and $r' \geq 2$ one has $\mathfrak{N}^\bullet_p(T) \leq 2 + \mathfrak{N}^\bullet_p(T'_1) + \mathfrak{N}^\bullet_p(\theta'_2) + \ldots + \mathfrak{N}^\bullet_p(\theta'_{r'})$, where T'_1 is the subgraph with exiting line ℓ'_1 and entering line ℓ'_T, and hence $\mathfrak{N}^\bullet_p(T) \leq 2 + 2^{-(m_p-3)} K(T) - 2$, by the inductive hypothesis, so that (2.2.7) follows.

If $r' = 1$ let T'_0 be the subgraph with exiting line ℓ_1 and entering line ℓ'_1. If $K(T'_0) \geq 2^{m_p-1}$ then the inductive hypothesis implies once more the bound (2.2.7), while if $K(T'_0) < 2^{m_p-1}$ then, by the same argument as above, T'_0 must be a self-energy cluster, so that ℓ_1 does not contribute to $\mathfrak{N}^\bullet_p(T)$, i.e. $\mathfrak{N}^\bullet_p(T) \leq 1 + \mathfrak{N}_p(T''_1)$ where T''_1 is the subgraph with exiting line ℓ'_1 and entering line ℓ'_T. Again we can iterate the argument until either one finds a subgraph T'' with $K(T'') \geq 2^{m_p-1}$, so that the inductive hypothesis compels the bound (2.2.7) for T, or one obtains $\mathfrak{N}^\bullet_p(T) \leq 1 + \mathfrak{N}_p(T'')$ for some subgraph T'' which has no line on minimum scale $\geq p$, so that $\mathfrak{N}^\bullet_p(T) \leq 1$. ■

Remark 2.2.6. Inequality (2.2.3) has been repeatedly used in the proof of both Lemmas 2.2.4 and

2.2.5. Actually the proof works – as one can easily check – under the weaker condition that

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{32} < |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| < \frac{\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega})}{4} \tag{2.2.8}$$

as long as $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\neq 0$. This observation will be used later on.

The key point in the proof of the two lemmas above is that in order to have a non-resonant line $\ell\in L(\theta)$ with large scale (hence with large propagator) the subtree θ_ℓ 'preceding' ℓ (i.e. the subtree whose nodes and lines are the nodes and lines of θ preceding ℓ) must be such that $K(\theta_\ell)$ is large: this suggest us to control the product of the propagators of the non-resonant lines with the product of the node factors of the nodes preceding such lines. More precisely we have the following result.

Lemma 2.2.7. *For any tree $\theta\in\Theta_{k,\boldsymbol{\nu},h}$ and any self-energy cluster $T\in\mathfrak{S}^k_{n,u,e}$ denote by $L_{NR}(\theta)$ and $L_{NR}(T)$ the sets of non-resonant lines in θ and T respectively, and set*

$$\mathscr{V}_{NR}(\theta) := \Bigg(\prod_{v\in N(\theta)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in L_{NR}(\theta)}\mathcal{G}_\ell\Bigg),$$
$$\mathscr{V}_{NR}(T) := \Bigg(\prod_{v\in N(T)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in L_{NR}(T)}\mathcal{G}_\ell\Bigg),$$

Then

$$|\mathscr{V}_{NR}(\theta)| \le c_1^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}, \tag{2.2.9a}$$
$$|\mathscr{V}_{NR}(T)| \le c_2^k \mathrm{e}^{-\xi K(T)/2}, \tag{2.2.9b}$$

for some positive constants c_1, c_2.

Proof. We first prove (2.2.9a). One has

$$\begin{aligned}\prod_{\ell\in L_{NR}(\theta)}|\mathcal{G}_\ell| &\le \prod_{n\ge 0}\left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}\right)^{\mathfrak{N}^\bullet_n(\theta)} \le \left(\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})}\right)^{4k-2}\prod_{n\ge n_0+1}\left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}\right)^{\mathfrak{N}^\bullet_n(\theta)}\\ &\le \left(\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})}\right)^{4k-2}\prod_{n\ge n_0+1}\left(\frac{16}{\alpha_{m_q}(\boldsymbol{\omega})}\right)^{2^{-(m_n-3)}K(\theta)}\\ &\le D(n_0)^{4k-2}\exp(\xi(n_0)K(\theta)),\end{aligned}$$

with

$$D(n_0)=\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})},\qquad \xi(n_0)=8\sum_{n\ge n_0+1}\frac{1}{2^{m_n}}\log\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}.$$

Then, by Hypothesis 2, one can choose n_0 such that $\xi(n_0) \le \xi/2$, so that, since

$$\prod_{v \in N(\theta)} |\mathcal{F}_v| \le C_0^k \mathrm{e}^{-\xi K(\theta)},$$

(see the proof of Lemma 2.1.3) the bound (2.2.9a) follows. To obtain (2.2.9b) one can reason in the same way, simply with T playing the role of θ. ■

What emerges from Lemma 2.2.7 is that, if we could ignore the resonant lines, the convergence of the series (1.1.9) would immediately follows. On the contrary, the presence of resonant lines may be a real obstruction for the convergence: indeed if ℓ is a resonant line, both the self-energy cluster T which ℓ enters and the self-energy cluster T' which ℓ exits may be on scale $n \ll n_\ell$ and hence the factor $\approx \mathrm{e}^{-2^n}$ coming from the product of the node factors of the nodes in T, T' is not enough to control the propagator $\mathcal{G}_\ell$ for which we only have the bound $16/\alpha_{m_{n_\ell}}(\boldsymbol{\omega})$. Moreover in principle a tree can contain a "chain" of self-energy clusters (see Section 2.3 for a formal definition) and hence of resonant lines which implies accumulation of small divisors. Therefore one needs a 'gain factor' proportional to $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell$ for each resonant line ℓ, in order to prove the convergence of the power series (1.1.9).

2.3 Cancellations and convergence

Here we shall see that, for the system (1.1.14) and under Hypothesis 3, there are suitable "cancellations" which allow us to prove the convergence of the series (1.1.9) and hence Theorem 1.1.1.

First of all we note that if T is a self-energy cluster, we can (and shall) write $\mathscr{V}(T) = \mathscr{V}_T(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T})$ to stress the dependence on $\boldsymbol{\nu}_{\ell'_T}$ – see Remark 2.2.3.

Remark 2.3.1. Write $\mathscr{V}_{T,NR}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T}) := \mathscr{V}_{NR}(T)$. By using Remark 2.2.6 one can show that also $\partial_x^j \mathscr{V}_{T,NR}(\tau x)$ admits the same bound as $\mathscr{V}_{T,NR}(x)$ in (2.2.9b) for $j = 0, 1, 2$ and $\tau \in [0, 1]$, possibly with a different constant c_2. This will be used later on (in Appendix A).

For all $k \ge 0$, define

$$\begin{aligned} &M_{u,e}^{(k)}(x,n) := \sum_{T \in \mathfrak{S}_{n,u,e}^k} \mathscr{V}_T(x), \qquad \mathcal{M}_{u,e}^{(k)}(x,n) := \sum_{p=-1}^{n} M_{u,e}^{(k)}(x,p), \\ &\mathcal{M}_{u,e}^{(k)}(x) := \lim_{n \to \infty} \mathcal{M}_{u,e}^{(k)}(x,n). \end{aligned} \tag{2.3.1}$$

Remark 2.3.2. One has

$$\begin{aligned} &\mathcal{M}_{\beta,\beta}^{(0)}(x,n) = \mathcal{M}_{B,B}^{(0)}(x,n) = \mathcal{M}_{B,\beta}^{(0)}(x,n) = 0, \\ &\mathcal{M}_{\beta,B}^{(0)}(x,n) = \omega_0'(\overline{B}_0), \qquad \mathcal{M}_{\beta,\beta}^{(1)}(x,n) = \partial_\beta \partial_B f_{\mathbf{0}} = -\mathcal{M}_{B,B}^{(1)}(x,n), \end{aligned} \tag{2.3.2}$$

for all $n \geq -1$ and all $x \in \mathbb{R}$.

Lemma 2.3.3. *For all $k \geq 1$ one has*

$$\mathcal{M}^{(k)}_{B,\beta}(0) + \sum_{k_1+k_2=k} \mathcal{M}^{(k_1)}_{B,B}(0)\partial_{\beta_0} B^{(k_2)}_{\mathbf{0}} = 0, \tag{2.3.3a}$$

$$\mathcal{M}^{(k)}_{\beta,\beta}(0) + \sum_{k_1+k_2=k} \mathcal{M}^{(k_1)}_{\beta,B}(0)\partial_{\beta_0} B^{(k_2)}_{\mathbf{0}} = 0, \tag{2.3.3b}$$

$$\mathcal{M}^{(k)}_{\beta,\beta}(0) = -\mathcal{M}^{(k)}_{B,B}(0). \tag{2.3.3c}$$

Proof. Both (2.3.3a) and (2.3.3b) follow from the fact that

$$\partial_{\beta_0}\Gamma^{(k)}_{\mathbf{0}} = \mathcal{M}^{(k)}_{B,\beta}(0) + \sum_{k_1+k_2=k} \mathcal{M}^{(k_1)}_{B,B}(0)\partial_{\beta_0} B^{(k_2)}_{\mathbf{0}}, \tag{2.3.4a}$$

$$\partial_{\beta_0}\Phi^{(k)}_{\mathbf{0}} = \mathcal{M}^{(k)}_{\beta,\beta}(0) + \sum_{k_1+k_2=k} \mathcal{M}^{(k_1)}_{\beta,B}(0)\partial_{\beta_0} B^{(k_2)}_{\mathbf{0}}, \tag{2.3.4b}$$

which can be obtained as follows. First of all let us write

$$\partial_{\beta_0}\left(\sum_{\theta\in\Theta_{k,\mathbf{0},\Gamma}} \mathscr{V}(\theta)\right) = \sum_{\theta\in\Theta_{k,\mathbf{0},\Gamma}} \sum_{\substack{v\in N(\theta)\\ k_v=1}} \partial_{\beta_0}\mathcal{F}_v \left(\prod_{\substack{v'\in N(\theta)\\ v'\neq v}} \mathcal{F}_{v'}\right)\left(\prod_{\ell\in L(\theta)} \mathcal{G}_\ell\right), \tag{2.3.5}$$

where we have used the fact that $\mathscr{V}(\theta)$ depends on β_0 only through the node factors of the nodes with $k_v = 1$. Each summand in the r.h.s. of (2.3.5) differs from $\mathscr{V}(\theta)$ because we applied a further derivative (with respect to β_0) to the node factor of a node $v \in N(\theta)$. This can be graphically represented as the same tree θ, but with a further line ℓ' (carrying a $\mathbf{0}$-momentum) entering the node v, and hence can be seen as a subgraph S of some tree. If there is no line with $\mathbf{0}$-momentum on the path $\mathcal{P}(\ell_\theta, \ell')$, then $\mathscr{V}(S)$ is a contribution to $\mathcal{M}^{(k)}_{B,\beta}(0)$. Otherwise let ℓ be the line on $\mathcal{P}(\ell_\theta, \ell')$ with $\boldsymbol{\nu}_\ell = \mathbf{0}$ which is closest to ℓ_θ i.e. such that $\boldsymbol{\nu}_{\ell''} \neq \mathbf{0}$ for all $\ell'' \in \mathcal{P}(\ell_\theta, \ell)$. Call T the subgraph between ℓ_θ and ℓ and note that, by the constraints on the labels, $h_\ell = B$. Call also S' the subgraph between ℓ and ℓ'. Then $\mathscr{V}(T)$ is a contribution to $\mathcal{M}^{(k(T))}_{B,B}(0)$ while $\mathscr{V}(S')$ is a contribution to $\partial_{\beta_0} B^{k(S')}_{\mathbf{0}}$. On the other hand it is easy to realise that each contribution to

$$\mathcal{M}^{(k)}_{B,\beta}(0) + \sum_{k_1+k_2=k} \mathcal{M}^{(k_1)}_{B,B}(0)\partial_{\beta_0} B^{(k_2)}_{\mathbf{0}}$$

is of the form described above and hence (2.3.4a) is proved. To obtain (2.3.4b) we can reason analogously. Therefore (2.3.3a) and (2.3.3b) follow from the fact that $\Phi^{(k)}_{\mathbf{0}} \equiv 0$ by construction while $\Gamma^{(k)}_{\mathbf{0}} \equiv 0$ by Hypothesis 3.

To obtain (2.3.3c) one can reason as follows. For any self-energy cluster T, denote by $\overline{N}(T)$ the set of nodes $v \in N(T)$ such that $\ell_v \in \mathcal{P}_T \cup \{\ell_T\}$ and the line $\ell'_v \in \mathcal{P}_T \cup \{\ell'_T\}$ entering v has component $h_{\ell'_v} = h_{\ell_v}$. Let $T \in \mathfrak{S}^k_{n,\beta,\beta}$ and consider the self-energy cluster $T' \in \mathfrak{S}^k_{n,B,B}$ obtained from T by changing all the component labels of the lines in $\mathcal{P}_T \cup \{\ell_T\} \cup \{\ell'_T\}$ and reversing their orientation; in particular the entering line ℓ'_T of T becomes the exiting line $\ell_{T'}$ of T' and, vice versa, the exiting line ℓ_T of T becomes the entering line $\ell'_{T'}$ of T'. Since the external lines $\ell'_T, \ell_T, \ell'_{T'}$ and $\ell_{T'}$ carry the same momentum $\boldsymbol{\nu}$, any line $\ell' \in \mathcal{P}_{T'}$ has momentum $\boldsymbol{\nu}'' = \boldsymbol{\nu} - \boldsymbol{\nu}'$ if $\boldsymbol{\nu}'$ is the momentum of the corresponding line in $\mathcal{P}_T$ so that, when computing at $\boldsymbol{\nu} = \mathbf{0}$, the corresponding propagator changes sign (see (2.2.2) and recall that $n_\ell \geq 0$ for all $\ell \in \mathcal{P}_T$). Moreover, for any $v \in \overline{N}(T)$ the node factor $\mathcal{F}_v$ changes sign when regarded as a node in $\overline{N}(T')$, see (2.1.4). All the other factors remains the same i.e. we can write $\mathscr{V}_T(0) = \mathfrak{A}(T)\,\mathscr{V}(\mathcal{P}_T)$ and $\mathscr{V}_{T'}(0) = \mathfrak{A}(T')\,\mathscr{V}(\mathcal{P}_{T'})$, where

$$\mathscr{V}(\mathcal{P}_T) := \Bigg(\prod_{v\in\overline{N}(T)} \mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in\mathcal{P}_T} \mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\Bigg)$$

and analogous for $\mathscr{V}(\mathcal{P}_{T'})$, while $\mathfrak{A}(T) = \mathfrak{A}(T')$. Now, one has

$$\prod_{v\in\overline{N}(T)} \mathcal{F}_v = \sigma \prod_{v\in\overline{N}(T')} \mathcal{F}_v$$

with $\sigma = \pm 1$. If $\sigma = 1$, then $|\overline{N}(T)| = |\overline{N}(T')|$ is even and hence there is an odd number of lines in $\mathcal{P}_T$. If on the contrary $\sigma = -1$, then there is an even number of lines in $\mathcal{P}_T$. In both cases the assertion follows. ∎

Lemma 2.3.4. *For all $k \geq 1$ one has*

$$\partial_x \mathcal{M}^{(k)}_{B,\beta}(0) = \partial_x \mathcal{M}^{(k)}_{\beta,B}(0) = 0, \tag{2.3.6a}$$

$$\partial_x \mathcal{M}^{(k)}_{\beta,\beta}(0) = \partial_x \mathcal{M}^{(k)}_{B,B}(0). \tag{2.3.6b}$$

Proof. One reason along the same lines as the proof of (2.3.3c) in Lemma 2.3.3. Let $T \in \mathfrak{S}^k_{n,\beta,B}$ and consider the self-energy cluster $T' \in \mathfrak{S}^k_{n,\beta,B}$ obtained from T by changing all the component labels of the lines in $\mathcal{P}_T \cup \{\ell_T\} \cup \{\ell'_T\}$ and reversing their orientation. The derivative ∂_x acts on the propagator on some line $\ell \in \mathcal{P}_T$. After differentiation, when computing the propagators at $x = 0$, any line $\ell' \in \mathcal{P}_{T'}$ turns out to have momentum $\boldsymbol{\nu}' = -\boldsymbol{\nu}$, if $\boldsymbol{\nu}$ is the momentum of the corresponding line in $\mathcal{P}_T$ and the corresponding propagator changes sign, except the differentiated propagator $\partial_x \mathcal{G}_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + x)\big|_{x=0}$, which is even in its argument. Moreover, for any $v \in \overline{N}(T)$ (we use the same notations as in the proof of Lemma 2.3.3) the node factor $\mathcal{F}_v$ changes sign when regarded as a node in $\overline{N}(T')$, while all the other factors remains the same. If $|\overline{N}(T)| = |\overline{N}(T')|$ is even (resp. odd) then there is an even (resp. odd) number of lines in $\mathcal{P}_T$, but, as we said, the differentiated propagator does not change sign. Therefore the two contributions have the same modulus but different signs, so that, once summed together, they gives zero. Therefore (2.3.6a) is proved.

To prove (2.3.6b) reason as in proving (2.3.3c): the only difference is that, as in the previous case, the differentiated propagator does not change sign, so that the two contributions are equal to (and not the opposite of) each other. ∎

Remark 2.3.5. Note that the Hamiltonian structure is fundamental in order to prove both the identity (2.3.3c) and the identities (2.3.6).

Given $p \geq 2$ self-energy clusters $T_1, \dots, T_p$ of any tree θ, with $\ell'_{T_i} = \ell_{T_{i+1}}$ for $i = 1, \dots, p-1$ and ℓ_{T_1}, ℓ'_{T_p} being non-resonant, we say that $C = \{T_1, \dots, T_p\}$ is a *chain*. Define $\ell_0(C) := \ell_{T_1}$ and $\ell_i(C) := \ell'_{T_i}$ for $i = 1, \dots, p$ and set $n_i(C) = n_{\ell_i(C)}$ for $i = 0, \dots, p$; we also call $k(C) := k(T_1) + \ldots + k(T_p)$ the *total order* of the chain C and $p(C) = p$ the *length* of C. Given a chain $C = \{T_1, \dots, T_p\}$ we define the *value* of C as

$$\mathscr{V}_C(x) = \prod_{i=1}^{p} \mathscr{V}_{T_i}(x). \tag{2.3.7}$$

We denote by $\mathfrak{C}(k; h, h'; n_0, \dots, n_p)$ the set of all chains $C = \{T_1, \dots, T_p\}$ with total order k and with fixed labels $h_{\ell_0(C)} = h$, $h_{\ell_p(C)} = h'$ and $n_i(C) = n_i$ for $i = 0, \dots, p$.

Remark 2.3.6. Let ℓ be a resonant line. Then there exists a chain C such that $\ell = \ell_i(C)$ for some $i = 1, \dots, p(C) - 1$. If there exists a minimal self-energy cluster T containing ℓ, then T contains the whole chain C and all lines $\ell_0(C), \dots, \ell_p(C)$ (this follows from the fact that, by definition of self-energy cluster, $\boldsymbol{\nu}^0_{\ell'} \neq \mathbf{0}$ for all $\ell' \in \mathcal{P}_T$). In particular $L(T)$ contains the two non-resonant lines $\ell_0(C)$ and $\ell_p(C)$, with $\zeta_{\ell_0(C)} = \zeta_{\ell_p(C)} = \zeta_\ell$.

Lemma 2.3.7. *For all $p \geq 2$, all $k \geq 1$, all $h, h' \in \{\beta, B\}$ and all $\overline{n}_0, \dots, \overline{n}_p \in \mathbb{Z}_+$ such that $\Psi_{\overline{n}_i}(x) \neq 0$, $i = 0, \dots, p$, one has*

$$\left| \sum_{C \in \mathfrak{C}(k; h, h'; \overline{n}_0, \dots, \overline{n}_p)} \mathscr{V}_C(x) \right| \leq B^k |x|^{p-1}, \tag{2.3.8}$$

for some constant $B > 0$.

The proof is rather long and technical, so that we prefer to perform it in Appendix A.

The bound (2.3.8) provides exactly the gain factor which is needed in order to prove the convergence of the power series. Indeed given a tree θ, sum together the values of the trees obtained from θ by replacing each maximal chain C (i.e. each chain which is not contained inside any other chain) with any other chain which has the same total order, the same length, the same scale labels associated with the lines $\ell_0(C), \dots, \ell_p(C)$ and the same component labels associated with the lines $\ell_0(C)$ and $\ell_p(C)$; in other words, if $C \in \mathfrak{C}(k; h, h'; \overline{n}_0, \dots, \overline{n}_p)$ for some values of the labels, sum over all possible chain belonging to the set $\mathfrak{C}(k; h, h'; \overline{n}_0, \dots, \overline{n}_p)$. Then we can bound the product of the propagators of the non-resonant lines outside the maximal chains thanks to Lemma 2.2.5, while the product of the propagators of the lines $\ell_1(C), \dots, \ell_{p-1}(C)$ of any chain C times the sum of the corresponding chain values is bounded through Lemma 2.3.7. Therefore Theorem 1.1.1 follows.

Remark 2.3.8. We obtained the convergence of the power series (1.1.9) for any β_0 and any ε small enough. Thus the solution turns out to be analytic in both ε, β_0. Moreover, since the solution is parameterised by $\beta_0 \in \mathbb{T}$, in that case the full resonant torus survives. Of course, such a situation is highly non-generic and hence very unlikely.

3. Proof of Theorem 1.1.2

In this Chapter we shall prove Theorem 1.1.2. The main problem is that if the system is not Hamiltonian, then there is no reason for the symmetries (2.3.3c) and (2.3.6) to hold. Moreover in order to obtain (2.3.3a) one needs Hypothesis 3, while in Theorem 1.1.2 we assume Hypothesis 4 so that Hypothesis 3 is obvioulsy not satisfied. Unfortunately the symmetries (2.3.3c) and (2.3.6) and the identity (2.3.3a) are fundamental in order to obtain the 'gain factor' proportional to the propagator of the resonant lines: indeed we are not able to provide any gain factor for the case considered here. Therefore we shall use a different approach.

3.1 Preliminary (heuristics) considerations

Let us come back to the range equations (1.2.1) and start by looking for a quasi-periodic solution which can be formally written as

$$\begin{aligned}\beta(t;\varepsilon,\beta_0,B_0) &= \beta_0 + b(t;\varepsilon,\beta_0,B_0) = \beta_0 + \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}t} b^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0),\\ B(t;\varepsilon,\beta_0,B_0) &= B_0 + \widetilde{B}(t;\varepsilon,\beta_0,B_0) = B_0 + \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}t} B^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0),\end{aligned} \tag{3.1.1}$$

where a different notation for the Taylor coefficients has been used with respect to (1.1.9) to stress that now we are considering $B_0 = B_{\mathbf{0}}$ as a parameter. If we define recurively for $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_*$

$$\begin{aligned} b^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) &:= \Phi^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0),\\ B^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0) &:= \Gamma^{\{k\}}_{\boldsymbol{\nu}}(\beta_0,B_0),\end{aligned} \tag{3.1.2}$$

where we have set

$$\Phi_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) := \sum_{s\geq 1}\frac{1}{s!}\partial_B\omega_0(B_0)\sum_{\substack{\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*}}\sum_{\substack{k_1+\ldots+k_s=k\\ k_i\geq 1}}\prod_{i=1}^{s}B_{\boldsymbol{\nu}_i}^{\{k_i\}}(\beta_0,B_0)$$
$$+\sum_{s\geq 1}\sum_{p+q=s}\sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d,\boldsymbol{\nu}_i\in\mathbb{Z}^d_*}}\frac{1}{p!q!}\partial_\beta^p\partial_B^q F_{\boldsymbol{\nu}_0}(\beta_0,B_0)$$
$$\times\sum_{\substack{k_1+\ldots+k_s=k-1\\ k_i\geq 1}}\prod_{i=1}^{p}b_{\boldsymbol{\nu}_i}^{\{k_i\}}(\beta_0,B_0)\prod_{i=p+1}^{s}B_{\boldsymbol{\nu}_i}^{\{k_i\}}(\beta_0,B_0), \tag{3.1.3}$$
$$\Gamma_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) := \sum_{s\geq 1}\sum_{p+q=s}\sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d,\boldsymbol{\nu}_i\in\mathbb{Z}^d_*}}\frac{1}{p!q!}\partial_\beta^p\partial_B^q G_{\boldsymbol{\nu}_0}(\beta_0,B_0)$$
$$\times\sum_{\substack{k_1+\ldots+k_s=k-1\\ k_i\geq 1}}\prod_{i=1}^{p}b_{\boldsymbol{\nu}_i}^{\{k_i\}}(\beta_0,B_0)\prod_{i=p+1}^{s}B_{\boldsymbol{\nu}_i}^{\{k_i\}}(\beta_0,B_0),$$

for all $k\geq 1$ and all $\boldsymbol{\nu}\in\mathbb{Z}^d$, then (3.1.1) turns out to be a formal solution to the range equations (1.2.1) (this can be proved essentially as in Lemma 2.1.3 once one has written such coefficients as a "sum over trees" – see below). Note that we can see the formal expansion (1.1.9) as obtained from (3.1.1) by solving the bifurcation equation (1.1.8c) and further expanding $B_0 = B_{\mathbf{0}}(\varepsilon,\beta_0)$.

We can represent the coefficients $b_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0), B_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0)$ as a "sum over trees" similar to the one in Chapter 2; of course there are some differences between the two representations: for instance there should be no line ℓ with $\mathbf{0}$-momentum since the subtree with root-line ℓ represent a contribution to $B_{\mathbf{0}}$. More precisely, given a tree θ we associate with each node v a mode label $\boldsymbol{\nu}_v\in\mathbb{Z}^d$, a component label $h_v=\beta,B$ and an order label $k_v=0,1$ with the constraint that $k_v=1$ if $h_v=B$. With each line $\ell=\ell_v$ we associate a component label $h_\ell=h_v$, a momentum label $\boldsymbol{\nu}_\ell\in\mathbb{Z}^d_*$ with the conservation law (2.1.1) and a scale label $n_\ell\in\mathbb{Z}_+$. We still call order of any subgraph T the number (2.1.2) and still denote by $\Theta_{k,\boldsymbol{\nu},h}$ the set of trees with order k, total momentum $\boldsymbol{\nu}$ and total component h although they have different constraints on the labels.

If we associate with each node v a node factor

$$\mathcal{F}_v=\mathcal{F}_v(\beta_0,B_0) := \begin{cases}\dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v}F_{\boldsymbol{\nu}_v}(\beta_0,B_0), & h_v=\beta, k_v=1,\\ \dfrac{1}{q_v!}\partial^{q_v}\omega_0(B_0), & h_v=\beta, k_v=0,\\ \dfrac{1}{p_v!q_v!}\partial_\beta^{p_v}\partial_B^{q_v}G_{\boldsymbol{\nu}_v}(\beta_0,B_0), & h_v=B, k_v=1.\end{cases} \tag{3.1.4}$$

and a propagator

$$\mathcal{G}_\ell = \begin{cases} \dfrac{\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)}{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell}, & \ell\neq\ell_\theta, \\ 1, & \ell=\ell_\theta, \end{cases} \tag{3.1.5}$$

with the functions $\Psi_n(x)$ defined as in § 2.2 and define the value of θ as in (2.1.6), then it strightforward to see that

$$\begin{aligned} (\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})b_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) &= \sum_{\theta\in\Theta_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta), \\ (\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})B_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) &= \sum_{\theta\in\Theta_{k,\boldsymbol{\nu},B}} \mathscr{V}(\theta). \end{aligned} \tag{3.1.6}$$

We define clusters as in § 2.2 while we slight change the definition of self-energy clusters to take into account that now there is no line with **0**-momentum. Namely a self-energy cluster is a cluster T with only one entering line ℓ_T' and one exiting line ℓ_T such that $\boldsymbol{\nu}_{\ell_T'}=\boldsymbol{\nu}_{\ell_T}$. Again we denote by $\mathfrak{S}^k_{n,u,e}$ the set of self-energy clusters T with order k, scale n and such that $h_{\ell_T}=u$, $h_{\ell_T'}=e$ (with some abuse of notation). Call "resonant line" any line exiting a self-energy cluster; then one can adapt all the results in § 2.1 and 2.2 to the present case. Unfortunately, as we said, we are not able to provide any gain factor for the resonant lines (which in turn would imply the convergence of the series). Therefore we try a resummation procedure which can be roughly described as follows.

If we take the "tree expansion" in the r.h.s of (3.1.6), we can distinguish between contribution in which the root-line is resonant, that we can write as

$$\begin{pmatrix} \sum_{\substack{T\in\mathfrak{S}^{k'}_{n,\beta,\beta}\\ k'<k}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) & \sum_{\substack{T\in\mathfrak{S}^{k'}_{n,\beta,B}\\ k'<k}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \\ \sum_{\substack{T\in\mathfrak{S}^{k'}_{n,B,\beta}\\ k'<k}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) & \sum_{\substack{T\in\mathfrak{S}^{k'}_{n,B,B}\\ k'<k}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \end{pmatrix} \begin{pmatrix} b_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) \\ B_{\boldsymbol{\nu}}^{\{k\}}(\beta_0,B_0) \end{pmatrix}, \tag{3.1.7}$$

from the others. If we shift (3.1.7) to the l.h.s. of (3.1.6), sum over k and set

$$M(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon) := \sum_{k\geq1}\varepsilon^k \begin{pmatrix} \sum_{T\in\mathfrak{S}^{k}_{n,\beta,\beta}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) & \sum_{T\in\mathfrak{S}^{k}_{n,\beta,B}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \\ \sum_{T\in\mathfrak{S}^{k}_{n,B,\beta}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) & \sum_{T\in\mathfrak{S}^{k}_{n,B,B}} \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \end{pmatrix}, \tag{3.1.8}$$

we obtain formally

$$((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu} - M(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon))) \begin{pmatrix} b_{\boldsymbol{\nu}}(\beta_0,B_0) \\ B_{\boldsymbol{\nu}}(\beta_0,B_0) \end{pmatrix} = \sum_{k\geq1}\varepsilon^k \begin{pmatrix} \sum_{\theta\in\Theta^*_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta) \\ \sum_{\theta\in\Theta^*_{k,\boldsymbol{\nu},B}} \mathscr{V}(\theta) \end{pmatrix}, \tag{3.1.9}$$

where we denoted by $\Theta^*_{k,\boldsymbol{\nu},h}$ the set of trees with order k, total momentum $\boldsymbol{\nu}$ and total component h, whose root-line is non-resonant.

This suggests us that if we replace the "propagators" $\mathcal{G}_\ell \mathbb{1}$ with the matrix $((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-M(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon))^{-1}$ we obtain a "tree expansion" in which the self-energy clusters (and hence the resonant lines) are not allowed. The problem in proceeding this way is that $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon)$ is itself a sum over self-energy clusters which in principle contain other self-energy clusters, and hence we cannot find 'good bounds' for $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu},\varepsilon)$. To deal with such a difficulty we should start with the momenta $\boldsymbol{\nu}$ such that $\Psi_0(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\neq 0$, then pass to those such that $\Psi_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\neq 0$ and so on. To simplify the exposition let us write $u_0+u=(\beta_0,B_0)+(b(t;\varepsilon,\beta_0),\widetilde{B}(t;\varepsilon,\beta_0,B_0))$ and say that a vector $\boldsymbol{\nu}\in\mathbb{Z}^d_*$ is on scale n if $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|\in[\alpha_{m_n}(\boldsymbol{\omega}),\alpha_{m_{n-1}}(\boldsymbol{\omega}))$ with $\alpha_{m_n}(\boldsymbol{\omega})$ defined as in § 2.2. Let us write

$$u=\sum_{n\geq 0}u^{[n]},\qquad u^{[n]}=\sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^d_*\\ \boldsymbol{\nu}\text{ on scale }n}}\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}u^{[n]}_{\boldsymbol{\nu}} \tag{3.1.10}$$

and assume that $u^{[\geq n+1]}:=\sum_{m\geq n+1}u^{[m]}$ is small with respect to $u^{[\leq n]}:=u-u^{[\geq n+1]}$. If we denote $P(u)=(\varepsilon F(u_0+u),\varepsilon G(u_0+u))$ we can write for $\boldsymbol{\nu}$ on scale 0

$$\begin{aligned}(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})u^{[0]}_{\boldsymbol{\nu}}&=P_{\boldsymbol{\nu}}(u_0)+\sum_{\boldsymbol{\nu}'\in\mathbb{Z}^d_*}P'_{\boldsymbol{\nu}-\boldsymbol{\nu}'}(u_0)u^{[\geq 0]}_{\boldsymbol{\nu}'}+O(\|u\|^2)\\ &=P_{\boldsymbol{\nu}}(u_0)+P'_{\mathbf{0}}(u_0)u^{[0]}_{\boldsymbol{\nu}}+\sum_{\boldsymbol{\nu}'\in\mathbb{Z}^d_*\setminus\{\boldsymbol{\nu}\}}P'_{\boldsymbol{\nu}-\boldsymbol{\nu}'}(u_0)u^{[\geq 0]}_{\boldsymbol{\nu}'}+O(\|u\|^2),\end{aligned} \tag{3.1.11}$$

for some norm $\|\cdot\|$. Note that $\mathcal{M}_{-1,\mathbf{0}}(u_0):=P'_{\mathbf{0}}(u_0)$ is the 2×2 matrix whose components are the self-energy clusters on scale -1. Since $P'_{\boldsymbol{\nu}-\boldsymbol{\nu}'}\sim\mathrm{e}^{-|\boldsymbol{\nu}-\boldsymbol{\nu}'|}$ we can shift $\mathcal{M}_{-1,\mathbf{0}}(u_0)u^{[0]}_{\boldsymbol{\nu}}$ in the l.h.s. of (3.1.11) and write

$$((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}_{-1,\mathbf{0}}(u_0))u^{[0]}_{\boldsymbol{\nu}}=P_{\boldsymbol{\nu}}(u_0)+Q_0(u^{[\geq 1]})+\text{ corrections}, \tag{3.1.12}$$

for suitable Q_0. If the operator with kernel $(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}_{-1,\mathbf{0}}(u_0)$ were invertible with 'good bounds' on its inverse we could apply some implicit function theorem in order to obtain $u^{[0]}=u^{[0]}(u^{[\geq 1]})$: since $\det(\mathcal{M}_{-1,\mathbf{0}})\to 0$ as $\varepsilon\to 0$ we can assume that ε is so small that this can be done. Then we pass to the equations for $u^{[1]}$ which can be written in the form

$$\begin{aligned}(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})u^{[1]}_{\boldsymbol{\nu}}&=P_{\boldsymbol{\nu}}(u_0+u^{[0]}(0))+\sum_{\boldsymbol{\nu}'\in\mathbb{Z}^d_*}\mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}'}(u_0+u^{[0]}(0))u^{[\geq 1]}_{\boldsymbol{\nu}'}+O(\|u^{[\geq 1]}\|^2)\\ &=P_{\boldsymbol{\nu}}(u_0+u^{[0]}(0))+\mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[0]}(0))u^{[1]}_{\boldsymbol{\nu}}+\\ &\qquad+\sum_{\boldsymbol{\nu}'\in\mathbb{Z}^d_*\setminus\{\boldsymbol{\nu}\}}\mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}'}(u_0+u^{[0]}(0))u^{[\geq 1]}_{\boldsymbol{\nu}'}+O(\|u^{[\geq 1]}\|^2),\end{aligned} \tag{3.1.13}$$

where $\mathcal{M}_0(u_0+u^{[0]}(0)):=P'(u_0+u^{[0]}(0))(\mathbb{1}+\partial u^{[0]}(0))$; if instead of the "propagators" $1/(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}$ we use $((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}_{-1,\mathbf{0}}(u_0))^{-1}$, we can regard $\mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[0]}(0))$ as the 2×2 matrix whose components are the sums of the values of all self-energy clusters on scale 0 which do not contain any self-energy cluster. Under the assumption that $u^{[0]}$ is well defined, Lemma 2.2.7 suggests that $\mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}'} \sim \mathrm{e}^{-|\boldsymbol{\nu}-\boldsymbol{\nu}'|}$ so that we write

$$((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[0]}(0)))u^{[1]}_{\boldsymbol{\nu}} = P_{\boldsymbol{\nu}}(u_0+u^{[0]}(0)) + Q_1(u^{[\geq 2]}) + \text{ corrections}, \tag{3.1.14}$$

for suitable Q_1. Again if the operator with kernel $(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}_{0,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[0]}(0))$ were invertible with 'good bounds' on its inverse, we could reason as done for $u^{[0]}$ and obtain $u^{[1]} = u^{[1]}(u^{[\geq 2]})$; moreover we would like to iterate the procedure.

The problem is that, as the scale increases $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|$ gets smaller and smaller while, defining $\mathcal{M}_n(u_0+u^{[\leq n]}(0)) := P'(u_0+u^{[\leq n]}(0))(\mathbb{1}+\partial u^{[\leq n]}(0))$, we only know that $\det(\mathcal{M}_{n,\boldsymbol{\nu},\boldsymbol{\nu}}) \to 0$ as $\varepsilon \to 0$, so that for certain n it can happen that $|\det((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}_{n,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[\leq n]}(0)))|$ is "too small". However if we assume that the iteration can be performed, and this turns out to be the case if we assume a bound like

$$|\det((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} - \mathcal{M}_{n,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[\leq n]}(0)))| \geq \frac{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}{2} \tag{3.1.15}$$

for all $n \geq 0$ and all $\boldsymbol{\nu}$ on scale n, then one can prove

$$\mathcal{M}_{n,\boldsymbol{\nu},\boldsymbol{\nu}}(u_0+u^{[\leq n]}(0))) = \mathcal{M}_{n,\mathbf{0},\mathbf{0}}(u_0+u^{[\leq n]}(0))) + D_n(u_0+u^{[\leq n]}(0)))(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) + O(\varepsilon|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|^2) \tag{3.1.16}$$

where the matrix $D_n(u_0+u^{[\leq n]}(0)))$ has purely imaginary entries, so that assuming (3.1.15) is tantamount to require

$$|(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - \det(\mathcal{M}_{n,\mathbf{0},\mathbf{0}}(u_0+u^{[\leq n]}(0)))| \geq \frac{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}{2}. \tag{3.1.17}$$

for all $n \geq 0$ and all $\boldsymbol{\nu}$ on scale n; moreover we can prove the remarkable identity

$$\mathcal{M}_{n,\mathbf{0},\mathbf{0}}(u_0+u^{[\leq n]}(0)) = [\partial P(u_0+u)(\mathbb{1}+\partial_{u_0}u)]_{\mathbf{0}} + O(e^{-2^n}). \tag{3.1.18}$$

On the other hand, the matrix $[\partial P(u_0+u)(\mathbb{1}+\partial_{u_0}u)]_{\mathbf{0}}$ has the form (assume $k_0 = 1$ in Hypothesis 4 for simplicity)

$$\begin{pmatrix} O(\varepsilon) & \omega_0'(B_0)+O(\varepsilon) \\ \varepsilon\partial_{\beta_0}\Gamma^{(1)}(\beta_0)+O(\varepsilon^2) & O(\varepsilon) \end{pmatrix} \tag{3.1.19}$$

so that Hypothesis 4 ensures $\det([\partial P(u_0+u)(\mathbb{1}+\partial_{u_0}u)]_{\mathbf{0}}) \leq 0$ for u_0 close enough to $(\overline{\beta}_0, \overline{B}_0)$. In other words (3.1.18), (3.1.19) and Hypothesis 4 would imply the bound (3.1.17) and hence (3.1.15). Of course we cannot proceed in this way, so that the idea is to add a cut-off

$$\xi_n = \begin{cases} 0, & \text{if } (\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} \text{ is "too close" to } \mathcal{M}_{n,\mathbf{0},\mathbf{0}}, \\ 1, & \text{if } (\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1} \text{ is "far enough" from } \mathcal{M}_{n,\mathbf{0},\mathbf{0}}, \end{cases} \tag{3.1.20}$$

in order to recursively define for all $n \geq 0$

$$\widetilde{u}_{\boldsymbol{\nu}}^{[n]} = ((\mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu})\mathbb{1} - \widetilde{\mathcal{M}}_{n-1,\boldsymbol{\nu},\boldsymbol{\nu}}\xi_{n-1})^{-1} \left[P_{\boldsymbol{\nu}}(u_0 + u^{[\leq n-1]}(0)) + \widetilde{Q}_n(u^{[\geq n+1]}) \right], \tag{3.1.21}$$

for suitable $\widetilde{Q}_n$, where $\widetilde{\mathcal{M}}_n := P'(u_0 + \widetilde{u}^{[\leq n]}(0))(\mathbb{1} + \partial\widetilde{u}^{[\leq n]}(0))$. Of course the well defined function

$$\widetilde{u} = \sum_{n \geq 0} \widetilde{u}^{[n]}, \qquad \widetilde{u}^{[n]} = \sum_{\substack{\boldsymbol{\nu} \in \mathbb{Z}^d_* \\ \boldsymbol{\nu} \text{ on scale } n}} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} \widetilde{u}_{\boldsymbol{\nu}}^{[n]}, \tag{3.1.22}$$

may no longer solve the range equation: it happens only if one can prove $\xi_n = 1$ for all $n \geq 0$. As we shall see, this can be made possible by suitably choosing the parameter u_0. In what follows we shall make this procedure more precise.

3.2 Resummed series

To take into account the considerations in § 3.1, we perform a "tree expansion" different from both the one in Chapter 2 and the one at the beginning of § 3.1.

More precisely, given a tree θ we associate the labels with the nodes and the lines of θ as follows.

With each node $v \in N(\theta)$ we associate a *mode label*, a *component label* and an *order label* as in § 2.1, while with each line $\ell = \ell_v$ we associate a pair of *component labels* $(e_\ell, u_\ell) \in \{\beta, B\}^2$, with the constraint that $u_\ell = h_v$, and a *momentum* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d_*$, except for the root line which can have either zero momentum or not, i.e. $\boldsymbol{\nu}_{\ell_\theta} \in \mathbb{Z}^d$. For any line ℓ, we call e_ℓ and u_ℓ the *e-component* and the *u-component* of ℓ, respectively.

We denote by p_v and q_v the numbers of lines with e-component β and B, respectively, entering the node v, and set $s_v = p_v + q_v$. If $k_v = 0$ for some $v \in N(\theta)$ we force also $h_v = \beta$, $p_v = 0$ and $q_v \geq 1$.

We still impose the conservation law

$$\boldsymbol{\nu}_\ell = \sum_{v \prec \ell} \boldsymbol{\nu}_v \tag{3.2.1}$$

and still call order of any subgraph T of a tree θ the number

$$k(T) = \sum_{v \in N(T)} k_v. \tag{3.2.2}$$

Finally, we associate with each line ℓ also a scale label n_ℓ as in § 2.2; note that now one can have $n_\ell = -1$ only if ℓ is the root line of θ.

We do not change the definition of cluster given in § 2.2, while, from now on, a *self-energy cluster* is a cluster T such that (i) T has only one entering line ℓ'_T and one exiting line ℓ_T, (ii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$

and hence

$$\sum_{v \in N(T)} \boldsymbol{\nu}_v = \mathbf{0}. \tag{3.2.3}$$

We shall say that a subgraph T constituted by only one node v with $\boldsymbol{\nu}_v = \mathbf{0}$ such that v has only one entering line is a self-energy cluster on scale -1. Note that if a self-energy cluster is on a scale $n \geq 0$ then $|N(T)| \geq 2$ and $k(T) \geq 1$, as is easy to check.

A *left-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least one line on scale n, (ii) ℓ'_T is on scale $n+1$ and ℓ_T is on scale n, and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Analogously a *right-fake cluster* T on scale n is a connected subgraph of a tree θ with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all the lines in T have scale $\leq n$ and there is in T at least one line on scale n, (ii) ℓ'_T is on scale n and ℓ_T is on scale $n+1$, and (iii) one has $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$. Roughly speaking, a left-fake (respectively right-fake) cluster T fails to be a self-energy cluster (or even a cluster) only because the exiting (respectively the entering) line is on scale equal to the scale of T.

A *renormalised tree* is a tree in which no self-energy clusters appear; analogously a *renormalised subgraph* is a subgraph of a tree θ which does not contains any self-energy cluster. Note that if T is a renormalised self-energy cluster and $N(T) \geq 2$ then $k(T) \geq 2$; moreover the bound $\sum_{v \in N(T)} s_v \leq 4k(T) - 2 \leq 4k(T)$ provided by Lemma 2.1.1 holds also for any renormalised subgraph T of any tree θ with the new constraints on the labels.

Given a tree θ we call *total momentum* of θ the momentum associated with ℓ_θ and *total component* of θ the e-component of ℓ_θ. We denote by $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ the set of renormalised trees with order k, total momentum $\boldsymbol{\nu}$ and total component h; the sets of renormalised self-energy clusters, renormalised left-fake clusters and renormalised right-fake clusters T on scale n such that $u_{\ell_T} = u$ and $e_{\ell'_T} = e$ will be denoted by $\mathfrak{R}_{n,u,e}$, $\mathfrak{LF}_{n,u,e}$ and $\mathfrak{RF}_{n,u,e}$, respectively.

For any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ we associate with each node $v \in N(\theta)$ a *node factor*

$$\mathcal{F}_v = \mathcal{F}_v(\beta_0, B_0) := \begin{cases} \dfrac{1}{p_v! q_v!} \partial_\beta^{p_v} \partial_B^{q_v} F_{\boldsymbol{\nu}_v}(\beta_0, B_0), & h_v = \beta, k_v = 1, \\ \dfrac{1}{q_v!} \partial^{q_v} \omega_0(B_0), & h_v = \beta, k_v = 0, \\ \dfrac{1}{p_v! q_v!} \partial_\beta^{p_v} \partial_B^{q_v} G_{\boldsymbol{\nu}_v}(\beta_0, B_0), & h_v = B, k_v = 1. \end{cases} \tag{3.2.4}$$

With each line $\ell = \ell_v$ we associate a *propagator* $\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)$ formally defined recursively as

$$\begin{aligned}\mathcal{G}^{[n]}(x;\varepsilon,\beta_0,B_0) &= \begin{pmatrix}\mathcal{G}^{[n]}_{\beta,\beta}(x;\varepsilon,\beta_0,B_0) & \mathcal{G}^{[n]}_{\beta,B}(x;\varepsilon,\beta_0,B_0)\\ \\ \mathcal{G}^{[n]}_{B,\beta}(x;\varepsilon,\beta_0,B_0) & \mathcal{G}^{[n]}_{B,B}(x;\varepsilon,\beta_0,B_0)\end{pmatrix}\\ &:= \Psi_n(x)\Big((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0)\Big)^{-1},\end{aligned} \tag{3.2.5}$$

where

$$\mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0) := \sum_{q=-1}^{n-1}\chi_q(x)M^{[q]}(x;\varepsilon,\beta_0,B_0), \tag{3.2.6}$$

where, for $n\geq -1$, $M^{[n]}(x;\varepsilon,\beta_0,B_0)$ is the 2×2 matrix

$$M^{[n]}(x;\varepsilon,\beta_0,B_0) := \begin{pmatrix}M^{[n]}_{\beta,\beta}(x;\varepsilon,\beta_0,B_0) & M^{[n]}_{\beta,B}(x;\varepsilon,\beta_0,B_0)\\ \\ M^{[n]}_{B,\beta}(x;\varepsilon,\beta_0,B_0) & M^{[n]}_{BB}(x;\varepsilon,\beta_0,B_0)\end{pmatrix}, \tag{3.2.7}$$

with (formally)

$$M^{[n]}_{u,e}(x;\varepsilon,\beta_0,B_0) := \sum_{T\in\mathfrak{R}_{n,u,e}}\varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0,B_0), \tag{3.2.8}$$

the functions Ψ_n,χ_n defined as in § 2.2, and $\mathscr{V}_T(x;\varepsilon,\beta_0,B_0)$ is the *renormalised value* of T, defined as

$$\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) := \Bigg(\prod_{v\in N(T)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in L(T)}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)\Bigg). \tag{3.2.9}$$

Note that, differently from Chapter 2, here $\mathscr{V}_T$ depends on ε – because the propagators do –; on the other hand it depends on $x=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T}$ only through the propagators associated with the lines $\ell\in\mathcal{P}_T$ (see Remark 2.2.3).

Set $\mathcal{M} := \{\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)\}_{n\geq -1}$. We call *self-energies* the matrices $\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0)$.

Remark 3.2.1. One has

$$\partial_c\mathcal{G}^{[n]}_{e,u}(x;\varepsilon,\beta_0,B_0) = \left(\mathcal{G}^{[n]}(x;\varepsilon,\beta_0,B_0)\partial_c\mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0)\Big((\mathrm{i}x)\mathbb{1}-\mathcal{M}^{[n-1]}(x;\varepsilon,\beta_0,B_0)\Big)^{-1}\right)_{e,u}$$

for both $c=\beta_0,B_0$.

Setting also $\mathcal{G}^{[-1]}(0;\varepsilon,\beta_0,B_0)=\mathbb{1}$, for any subgraph S of any $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ define the *renormalised value* of S as

$$\mathscr{V}(S;\varepsilon,\beta_0,B_0) := \Bigg(\prod_{v\in N(S)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in L(S)}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)\Bigg). \tag{3.2.10}$$

Note that, differently from Chapter 2, here the value of a tree depends on both B_0, β_0; moreover it depends on such parameters also through the propagators.

We define

$$\begin{aligned} b_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0,B_0) &:= \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \\ B_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0,B_0) &:= \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},B}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \end{aligned} \tag{3.2.11}$$

for any $\boldsymbol{\nu}\neq\mathbf{0}$, and

$$\begin{aligned} \Phi_{\mathbf{0}}^{[k]}(\varepsilon,\beta_0,B_0) &:= \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},\beta}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \\ \Gamma_{\mathbf{0}}^{[k]}(\varepsilon,\beta_0,B_0) &:= \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},B}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0). \end{aligned} \tag{3.2.12}$$

Set (again formally)

$$\begin{aligned} b^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0) &:= \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} b_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0,B_0), \\ \widetilde{B}^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0) &:= \sum_{k\geq 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} B_{\boldsymbol{\nu}}^{[k]}(\varepsilon,\beta_0,B_0), \end{aligned} \tag{3.2.13}$$

and

$$\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon;\beta_0,B_0) := \sum_{k\geq 0}\varepsilon^k \Phi_{\mathbf{0}}^{[k]}(\varepsilon,\beta_0,B_0), \qquad \Gamma^{\mathcal{R}}_{\mathbf{0}}(\varepsilon;\beta_0,B_0) := \sum_{k\geq 0}\varepsilon^k \Gamma_{\mathbf{0}}^{[k]}(\varepsilon,\beta_0,B_0), \tag{3.2.14}$$

and define $\beta^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)=\beta_0+b^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)$ and $B^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)=B_0+\widetilde{B}^{\mathcal{R}}(t;\varepsilon,\beta_0,B_0)$. Set also $\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},h}=\{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h} : n_\ell\leq n \text{ for all } \ell\in L(\theta)\}$, and define

$$\begin{aligned} \Phi_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon;\beta_0,B_0) &:= \sum_{k\geq 0}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R},n}_{k,\mathbf{0},\beta}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \\ \Gamma_{\mathbf{0}}^{\mathcal{R},n}(\varepsilon;\beta_0,B_0) &:= \sum_{k\geq 0}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R},n}_{k,\mathbf{0},B}} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0). \end{aligned} \tag{3.2.15}$$

The series (3.2.13) and (3.2.14) will be called the *resummed series*.

Remark 3.2.2. One has

$$\mathcal{M}^{[-1]}(x;\varepsilon,\beta_0,B_0) = M^{[-1]}(x;\varepsilon,\beta_0,B_0) = \begin{pmatrix} \varepsilon\partial_{\beta_0}F_{\mathbf{0}}(\beta_0,B_0) & \omega_0'(B_0)+\varepsilon\partial_{B_0}F_{\mathbf{0}}(\beta_0,B_0) \\ \\ \varepsilon\partial_{\beta_0}G_{\mathbf{0}}(\beta_0,B_0) & \varepsilon\partial_{B_0}G_{\mathbf{0}}(\beta_0,B_0) \end{pmatrix},$$

where $\omega_0'(B_0)\neq 0$ for B_0 close enough to $\overline{B}_0$ by Hypothesis 2. In particular $\mathcal{M}^{[-1]}(x;\varepsilon,\beta_0,B_0)$ does not depend on x and is a real-valued matrix.

Remark 3.2.3. If T is a renormalised left-fake (respectively right-fake) cluster, we can (and shall) write $\mathscr{V}(T;\varepsilon,\beta_0,B_0) = \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T};\varepsilon,\beta_0,B_0)$ to stress that the propagators of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T}$. In particular one has

$$\sum_{T\in\mathfrak{LF}_{n,u,e}} \varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = \sum_{T\in\mathfrak{RF}_{n,u,e}} \varepsilon^{k(T)}\,\mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = M^{[n]}_{u,e}(x;\varepsilon,\beta_0,B_0).$$

For any renormalised subgraph S of any tree θ we denote by $\mathfrak{N}_n(S)$ the number of lines on scale $\geq n$ in S, and set

$$K(S) = \sum_{v\in N(S)} |\boldsymbol{\nu}_v|.$$

Then we have the following results.

Lemma 3.2.4. *For any $h\in\{\beta,B\}$, $\boldsymbol{\nu}\in\mathbb{Z}^d$, $k\geq 1$ and for any $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ such that $\mathscr{V}(\theta;\varepsilon,\beta_0,B_0)\neq 0$, one has $\mathfrak{N}_n(\theta)\leq 2^{-(m_n-2)}K(\theta)$ for all $n\geq 0$.*

Proof. First of all we note that if $\mathfrak{N}_n(\theta)\geq 1$, then there is at least one line ℓ with $n_\ell = n$ and hence $K(\theta)\geq|\boldsymbol{\nu}_\ell|\geq 2^{m_n-1}$ (see Remark 2.2.2). Now we prove the bound $\mathfrak{N}_n(\theta)\leq\max\{2^{-(m_n-2)}K(\theta)-1,0\}$ by induction on the order.

If the root line of θ has scale $n_{\ell_\theta}<n$ then the bound follows by the inductive hypothesis. If $n_{\ell_\theta}\geq n$, call $\ell_1,\ldots,\ell_r$ the lines with scale $\geq n$ closest to ℓ_θ (that is such that $n_{\ell'}<n$ for all lines $\ell'\in\mathcal{P}(\ell_\theta,\ell_i)$, $i=1,\ldots,r$). If $r=0$ then $\mathfrak{N}_n(\theta)=1$ and $|\boldsymbol{\nu}|\geq 2^{m_n-1}$, so that the bound follows. If $r\geq 2$ the bound follows once more by the inductive hypothesis. If $r=1$, then ℓ_1 is the only entering line of a cluster T which is not a renormalised self-energy cluster as $\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$, and hence $\boldsymbol{\nu}_{\ell_1}\neq\boldsymbol{\nu}$. But then

$$|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1})|\leq|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|+|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}|\leq\frac{1}{4}\alpha_{m_{n-1}}(\boldsymbol{\omega})<\alpha_{m_{n-1}+p_{n-1}}(\boldsymbol{\omega})=\alpha_{m_n-1}(\boldsymbol{\omega}),$$

as both ℓ_θ and ℓ_1 are on scale $\geq n$, so that one has $K(T)\geq|\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell_1}|\geq 2^{m_n-1}$. Now, call θ_1 the subtree of θ with root line ℓ_1. Then one has $\mathfrak{N}_n(\theta)=1+\mathfrak{N}_n(\theta_1)\leq 1+\max\{2^{-(m_n-2)}K(\theta_1)-1,0\}$, so that $\mathfrak{N}_n(\theta)\leq 2^{-(m_n-2)}(K(\theta)-K(T))\leq 2^{-(m_n-2)}K(\theta)-1$, again by induction. ∎

Lemma 3.2.5. *For any $e,u\in\{\beta,B\}$, $n\geq 0$ and for any $T\in\mathfrak{R}_{n,u,e}$ such that $\mathscr{V}_T(x;\varepsilon,\beta_0,B_0)\neq 0$, one has $K(T)>2^{m_n-1}$ and $\mathfrak{N}_p(T)\leq 2^{-(m_p-2)}K(T)$ for $0\leq p\leq n$.*

Proof. We first prove that for all $n\geq 0$ and all $T\in\mathfrak{R}_{n,u,e}$, one has $K(T)\geq 2^{m_n-1}$. In fact if $T\in\mathfrak{R}_{n,u,e}$ then T contains at least a line on scale n. If there is $\ell\in L(T)\backslash\mathcal{P}_T$ with $n_\ell=n$, then $K(T)\geq|\boldsymbol{\nu}_\ell|>2^{m_n-1}$ (see Remark 2.2.2). Otherwise, let $\ell\in\mathcal{P}_T$ be the line on scale n which is closest to ℓ'_T. Call $\widetilde{T}$ the subgraph (actually the cluster) consisting of all lines and nodes of T preceding ℓ. Then $\boldsymbol{\nu}_\ell\neq\boldsymbol{\nu}_{\ell'_T}$, otherwise $\widetilde{T}$ would be a renormalised self-energy cluster. Therefore $K(T) > |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell'_T}| > 2^{m_n-1}$ as both ℓ, ℓ'_T are on scale $\geq n$.

Given a tree θ, call $\mathcal{C}(n,p)$ the set of renormalised subgraphs T of θ with only one entering line ℓ'_T and one exiting line ℓ_T both on scale $\geq p$, such that $L(T) \neq \emptyset$ and $n_\ell \leq n$ for any $\ell \in L(T)$. Note that $\mathfrak{R}_{n,u,e} \subset \mathcal{C}(n,p)$ for all $n, p \geq 0$ and $u, e \in \{\beta_0, B_0\}$. We prove that $\mathfrak{N}_p(T) \leq \max\{K(T)2^{-(m_p-2)} - 1, 0\}$ for $0 \leq p \leq n$ and all $T \in \mathcal{C}(n,p)$. The proof is by induction on the order. Call $N(\mathcal{P}_T)$ the set of nodes in T connected by lines in $\mathcal{P}_T$. If all lines in $\mathcal{P}_T$ are on scale $< p$, then $\mathfrak{N}_p(T) = \mathfrak{N}_p(\theta_1) + \ldots + \mathfrak{N}_p(\theta_r)$ if $\theta_1, \ldots, \theta_r$ are the subtrees with root line entering a node in $N(\mathcal{P}_T)$, and hence the bound follows from (the proof of) Lemma 3.2.4. If there exists a line $\ell \in \mathcal{P}_T$ on scale $\geq p$, call T_1 and T_2 the subgraphs of T such that $L(T) = \{\ell\} \cup L(T_1) \cup L(T_2)$. Note that if $L(T_1), L(T_2) \neq \emptyset$, then $T_1, T_2 \in \mathcal{C}(n,p)$. Hence, by the inductive hypothesis one has

$$\mathfrak{N}_p(T) = 1 + \mathfrak{N}_p(T_1) + \mathfrak{N}_p(T_2) \leq 1 + \max\{2^{-(m_p-2)}K(T_1) - 1, 0\} + \max\{2^{-(m_p-2)}K(T_2) - 1, 0\}.$$

If both $\mathfrak{N}_p(T_1), \mathfrak{N}_p(T_2)$ are zero the bound follows as $K(T) \geq 2^{m_p-1}$, while if both are non-zero one has $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}(K(T_1) + K(T_2)) - 1 = 2^{-(m_p-2)}K(T) - 1$. Finally if only one is zero, say $\mathfrak{N}_p(T_1) \neq 0$ and $\mathfrak{N}_p(T_2) = 0$, then $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)}K(T_1) = 2^{-(m_p-2)}K(T) - 2^{-(m_p-2)}K(T_2)$. On the other hand, in such a case T_2 is a cluster and hence $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell'_T}$, which implies $K(T_2) \geq 2^{m_p-1}$. The same argument can be used in the case $\mathfrak{N}_p(T_1) = 0$ and $\mathfrak{N}_p(T_2) \neq 0$. ■

Remark 3.2.6. Lemmas 3.2.4 and 3.2.5 are the counterpart of Lemmas 2.2.4 and 2.2.5 adapted to the present case and their proofs contain essentially the same ingredients. In particular, as in the proofs of Lemmas 2.2.4 and 2.2.5, inequality (2.2.3) is repeatedly used in the proof of both Lemmas above; however, also in this case one can use instead (2.2.8).

3.3 A suitable Ansatz

Here we shall prove that, under the assumption that the propagators $\mathcal{G}^{[n]}_{e,u}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}; \varepsilon, \beta_0, B_0)$ are bounded proportionally to $1/(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2$, the series (3.2.13) converge and solve the range equations (1.2.1): the key point is that now self-energy clusters (and hence resonant lines) are not allowed and hence a result of that kind is expected. Then, in the next section, we shall see that the assumption is justified at least along a curve $(\beta_0(\varepsilon), B_0(\varepsilon))$ satisfying also the bifurcation equations (1.1.8c) and (1.1.8d). We shall not write the dependence on $\varepsilon, \beta_0, B_0$ unless needed.

Definition 3.3.1. *We shall say that $\mathcal{M}$ satisfies property 1 if one has*

$$\Psi_{n+1}(x) \left|\det\left((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n]}(x)\right)\right| \geq \Psi_{n+1}(x)x^2/2,$$

for all $n \geq -1$.

Definition 3.3.2. *We shall say that* $\mathcal{M}$ *satisfies property 1-p if one has*

$$\Psi_{n+1}(x)\left|\det\Big((\mathrm{i}x)\mathbb{1} - \mathcal{M}^{[n]}(x)\Big)\right| \geq \Psi_{n+1}(x)x^2/2.$$

for $-1 \leq n < p$.

Lemma 3.3.3. *Assume* $\mathcal{M}$ *to satisfy property 1-p. Then, for* $0 \leq n \leq p$ *and* ε *small enough, the self-energies are well defined and one has*

$$|M^{[n]}_{u,e}(x)| \leq |\varepsilon|\, K_1 \mathrm{e}^{-K_2 2^{m_n}}, \tag{3.3.1a}$$

$$|\partial^j_x M^{[n]}_{u,e}(x)| \leq |\varepsilon|\, C_j \mathrm{e}^{-\overline{C}_j 2^{m_n}}, \qquad j = 1,2, \tag{3.3.1b}$$

for some constants $K_1, K_2, C_1, C_2, \overline{C}_1$ *and* $\overline{C}_2$.

Proof. We shall prove first (3.3.1a) by induction on n. Let $n \leq p$ and $T \in \mathfrak{R}_{n,u,e}$. The analyticity of F, G and ω_0 implies that there exist positive constants F_1, F_2, ξ such that for all $v \in N(T)$ one has

$$|\mathcal{F}_v| \leq F_1 F_2^{s_v} \mathrm{e}^{-\xi|\boldsymbol{\nu}_v|}.$$

Note that

$$\prod_{v\in N(T)} \mathrm{e}^{-\frac{1}{4}\xi|\boldsymbol{\nu}_v|} = \exp\Big(-\frac{1}{4}\xi\, K(T)\Big) < \exp\Big(-\frac{1}{8}\xi\, 2^{m_n}\Big),$$

by Lemma 3.2.5. Moreover by property 1-p and the inductive hypothesis, one has (for instance)

$$\begin{aligned}|\mathcal{G}^{[n']}_{\beta,\beta}(x)| &\leq \frac{2}{x^2}\Big(|\mathrm{i}x| + \big|\mathcal{M}^{[n'-1]}_{B,B}(x)\big|\Big)\\ &\leq \frac{2}{x^2}\Big(|x| + P_1 + |\varepsilon|^2 K_1 \sum_{q=0}^{n'-1} \mathrm{e}^{-K_2 2^{m_q}}\Big) \leq \gamma_0\, \alpha_{m_{n'}}(\boldsymbol{\omega})^{-2}\end{aligned}$$

for all $0 \leq n' \leq n$ and for a suitable constant γ_0, where we used that any renormalised self-energy cluster T on scale ≥ 0 has at least two nodes and hence $k(T) \geq 2$, and that there exists $P_1 \geq 0$ such that $|\mathcal{M}^{[-1]}_{u,e}| \leq P_1$ (see Remark 3.2.2). Of course one can reason analogously for $\mathcal{G}^{[n']}_{\beta,B}(x)$, $\mathcal{G}^{[n']}_{B,\beta}(x)$ and

$\mathcal{G}^{[n']}_{\beta,B}(x)$. Hence by Lemmas 3.2.5 and 2.1.1 (see the comments after (3.2.3)) one can bound

$$
\begin{aligned}
\prod_{\ell\in L(T)} |\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| &\le \prod_{q\ge 0}\left(\frac{\gamma_0}{\alpha_{m_q}(\boldsymbol{\omega})^2}\right)^{\mathfrak{N}_q(T)} \le \left(\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}\right)^{4k(T)-2}\prod_{q\ge n_0+1}\left(\frac{\gamma_0}{\alpha_{m_q}(\boldsymbol{\omega})^2}\right)^{\mathfrak{N}_q(T)}\\
&\le \left(\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}\right)^{4k(T)-2}\prod_{q\ge n_0+1}\left(\frac{\gamma_0^{1/2}}{\alpha_{m_q}(\boldsymbol{\omega})}\right)^{2^{-(m_q-3)}K(T)}\\
&\le \left(\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2}\right)^{4k(T)-2}\exp\left(8K(T)\sum_{q\ge n_0+1}\frac{1}{2^{m_q}}\log\frac{\gamma_0^{1/2}}{\alpha_{m_q}(\boldsymbol{\omega})}\right)\\
&\le D(n_0)^{4k(T)-2}\exp(\xi(n_0)K(T)),
\end{aligned}
$$

with

$$
D(n_0)=\frac{\gamma_0}{\alpha_{m_{n_0}}(\boldsymbol{\omega})^2},\qquad \xi(n_0)=8\sum_{q\ge n_0+1}\frac{1}{2^{m_q}}\log\frac{\gamma_0^{1/2}}{\alpha_{m_q}(\boldsymbol{\omega})}.
$$

Then, by Hypothesis 1, one can choose n_0 such that $\xi(n_0)\le\xi/2$. Furthermore, Lemma 2.1.1 ensures also that the sum over the other labels is bounded by a constant to the power $k(T)$, and hence one can bound, for some positive constants C and K_0,

$$
|M^{[n]}_{u,e}(x)| \le \sum_{T\in\mathfrak{R}_{n,u,e}} |\varepsilon|^{k(T)}|\mathscr{V}_T(x)| \le \sum_{T\in\mathfrak{R}_{n,u,e}} |\varepsilon|^{k(T)}C^{k(T)}\mathrm{e}^{-K_0K(T)} \le \sum_{k\ge 2}|\varepsilon|^kC^k\mathrm{e}^{-K_2 2^{m_n}}, \tag{3.3.2}
$$

with $K_2=K_0/2$, then (3.3.1a) is proved for ε small enough. Now we prove (3.3.1b), again by induction on n. For $n=0$ the bound is obvious. Assume then (3.3.1b) to hold for all $n'<n$. For any $T\in\mathfrak{R}_{n,u,e}$ such that $\mathscr{V}_T(x)\ne 0$ one has

$$
\partial_x\mathscr{V}_T(x)=\sum_{\ell\in\mathcal{P}_T}\left(\prod_{v\in N(T)}\mathcal{F}_v\right)\left(\partial_x\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell)\prod_{\ell'\in L(T)\setminus\{\ell\}}\mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'})\right), \tag{3.3.3}
$$

where $x_\ell=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell=x+\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell$ and

$$
\begin{aligned}
\partial_x\mathcal{G}^{[n_\ell]}(x_\ell)=\frac{d}{dx}\mathcal{G}^{[n_\ell]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell+x)&=\partial_x\Psi_{n_\ell}(x_\ell)\Big((\mathrm{i}x_\ell)\mathbb{1}-\mathcal{M}^{[n_\ell-1]}(x_\ell)\Big)^{-1}\\
&-\Psi_{n_\ell}(x_\ell)\Big((\mathrm{i}x)\mathbb{1}-\mathcal{M}^{[n_\ell-1]}(x_\ell)\Big)^{-2}\Big(\mathrm{i}\mathbb{1}-\partial_x\mathcal{M}^{[n_\ell-1]}(x_\ell)\Big).
\end{aligned}
$$

One has

$$
|\partial_x\Psi_{n_\ell}(x_\ell)|\le|\partial_x\chi_{n_\ell-1}(x_\ell)|+|\partial_x\psi_{n_\ell}(x_\ell)|\le\frac{B_1}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})},
$$

for some positive constant B_1 and, by (3.3.1a), the inductive hypothesis and Hypothesis 1,

$$\begin{aligned}|\partial_x \mathcal{M}^{[n_\ell-1]}_{u,e}(x_\ell)| &\leq \sum_{q=0}^{n_\ell-1} |(\partial_x \chi_q(x_\ell)) M^{[q]}_{u,e}(x_\ell)| + \sum_{q=0}^{n_\ell-1} |\partial_x M^{[q]}_{u,e}(x_\ell)| \\ &\leq |\varepsilon| B_1 K_1 \sum_{q\geq 0} \frac{1}{\alpha_{m_q}(\boldsymbol{\omega})} \mathrm{e}^{-K_2 2^{m_q}} + |\varepsilon| C_1 \sum_{q\geq 0} \mathrm{e}^{-\overline{C}_1 2^{m_q}} \leq |\varepsilon| B_2,\end{aligned}$$

for some positive constant B_2. Hence the differentiated propagator $\partial_x \mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell)$ can be bounded by $\gamma_1 \alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-4}$ for some constant γ_1. Possibly redefining the constant γ_1, also the propagators of the lines $\ell' \neq \ell$ in (3.3.3) can be bounded by $\gamma_1\, \alpha_{m_{n_{\ell'}}}(\boldsymbol{\omega})^{-4}$, and hence, at the cost of replacing the previous bound $\gamma_0 \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for the propagators $\mathcal{G}^{[n]}(x)$ with $\gamma_1 \alpha_{m_n}(\boldsymbol{\omega})^{-4}$, one can reason as in the proof of (3.3.1a) to obtain (3.3.1b) for $j = 1$. For $j = 2$ one can reason analogously. ■

Remark 3.3.4. From the proof of Lemma 3.3.3 it follows that if $\mathcal{M}$ satisfies property 1-p the matrices $\mathcal{M}^{[n]}(x)$ and $\mathcal{G}^{[n]}(x)$ are well defined for all $-1 \leq n \leq p$. In particular there exists $\gamma_0 > 0$ such that $|\mathcal{G}^{[n]}_{e,u}(x)| \leq \gamma_0\, \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $0 \leq n \leq p$. Moreover if $\mathcal{M}$ satisfies property 1, the same considerations apply for all $n \geq 0$.

Lemma 3.3.5. *Assume $\mathcal{M}$ to satisfy property 1-p. Then one has*

$$\mathcal{M}^{[n]}(-x) = \mathcal{M}^{[n]}(x)^* \tag{3.3.4}$$

for all $-1 \leq n \leq p$.

Proof. We shall prove the result by induction on n. For $n = -1$ the result is obvious; see Remark 3.2.2. Assume (3.3.4) to hold up to scale $n-1$. Then, by definition, one has also $\mathcal{G}^{[q]}(-x) = \mathcal{G}^{[q]}(x)^*$ for all $0 \leq q \leq n$. For any renormalised self-energy cluster T contributing to $M^{[n]}(x)$, consider the renormalised self-energy cluster T' obtained from T by replacing the mode labels $\boldsymbol{\nu}_v$ with $-\boldsymbol{\nu}_v$ and changing the sign of the momentum of the entering line. Then the node factors are changed into their complex conjugated, and this holds also for the propagators because of the conservation law (3.2.1). Then $\mathscr{V}_{T'}(-x) = \mathscr{V}_T(x)^*$. This is enough to prove the assertion. ■

Lemma 3.3.6. *Assume $\mathcal{M}$ to satisfy property 1-p. Then, for $0 \leq n \leq p$ and ε small enough, one has*

$$\left| M^{[n]}_{u,e}(x) - M^{[n]}_{u,e}(0) - x\, \partial_x M^{[n]}_{u,e}(0) \right| \leq |\varepsilon|\, K_3 \mathrm{e}^{-\overline{K}_4 2^{m_n}} x^2 \tag{3.3.5}$$

for some constants K_3 and K_4.

Proof. For $x^2 > |\varepsilon|$ the bound follows trivially from Lemma 3.3.3: thus, we may assume in the following $x^2 \le |\varepsilon|$. Consider a self-energy cluster T whose value $\mathscr{V}_T(x)$ contributes to $M^{[n]}_{u,e}(x)$ through (3.2.8) and set $\mathcal{A}_T(x) = \mathscr{V}_T(x) - \mathscr{V}_T(0) - x\,\partial_x \mathscr{V}_T(0)$. Define also

$$\overline{n} = \min\{n \in \mathbb{Z}_+ : K(T) \le 2^{m_n}\}.$$

Let us distinguish between the two cases: (a) $2^{m_{\overline{n}}-1} < K(T) \le 2^{m_{\overline{n}}}$ and (b) $2^{m_{\overline{n}-1}} < K(T) \le 2^{m_{\overline{n}}-1}$.

In case (a), if $\alpha_{m_{\overline{n}}}(\boldsymbol{\omega}) \le 4|x|$ then one can bound $|\mathcal{A}_T(x)| \le |\mathscr{V}_T(x)| + |\mathscr{V}_T(0)| + |x\,\partial_x \mathscr{V}_T(0)|$. As soon as $\Psi_{n_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) \ne 0$ for all $\ell \in L(T)$, by (the proof of) Lemma 3.3.3 – see in particular (3.3.2) – each contribution can be bounded as

$$\begin{aligned} |\varepsilon|^{k(T)} C^k \mathrm{e}^{-K_0 K(T)} &\le |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/2)K(T)} \mathrm{e}^{-(K_0/2)2^{m_{\overline{n}}-1}} \\ &\le |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/2)K(T)} \alpha_{m_{\overline{n}}}(\boldsymbol{\omega})^2 \le 16\,x^2 |\varepsilon|^{k(T)} C^k \mathrm{e}^{-(K_0/4)2^{m_n}}, \end{aligned}$$

possibly redefining the constants C, K_0. If on the contrary $\alpha_{m_{\overline{n}}}(\boldsymbol{\omega}) > 4|x|$, one can reason as follows. For any line $\ell \in L(T)$ one has $|\boldsymbol{\nu}^0_\ell| \le K(T) \le 2^{m_{\overline{n}}}$ and hence $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge \alpha_{m_{\overline{n}}}(\boldsymbol{\omega})$. Then for all $\tau \in [0,1]$

$$\frac{5}{4}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| + |x| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + \tau x\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right| - |x| \ge \frac{3}{4}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell\right|.$$

In particular $(5/4)|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| \ge (3/4)|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell|$ and therefore

$$2\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell\right| \ge \left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + \tau x\right| \ge \frac{1}{2}\left|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell\right|. \tag{3.3.6}$$

This implies that the sizes of the propagators in $\mathscr{V}_T(\tau x)$ ‘do not change too much’ with respect to $\mathscr{V}_T(x)$: in particular (2.2.3) yields the bound (2.2.8) and hence, by Remark 3.2.6, Lemmas 3.2.4 and 3.2.5 still hold, so as to obtain $|\partial_1^2 \mathscr{V}_T(\tau x)| \le C'(C'')^{k(T)} \mathrm{e}^{-K_2 2^{m_n}}$, where ∂_1 denotes the derivative with respect to the (only) argument, for some constants C' and C''. Then

$$|\mathcal{A}_T(x)| \le \left| x^2 \int_0^1 \mathrm{d}\tau\,(1-\tau)\,\partial_1^2 \mathscr{V}_T(\tau x) \right| \le x^2 C'(C'')^k \mathrm{e}^{-K_2 2^{m_n}}, \tag{3.3.7}$$

By summing over all possible self-energy values contributing to $M^{[n]}_{u,e}(x)$ the bound (3.3.5) follows.

In case (b), if $\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}) \le 8|x|$ then one can bound $|\mathcal{A}_T(x)| \le |\mathscr{V}_T(x)| + |\mathscr{V}_T(0)| + |x\,\partial_x \mathscr{V}_T(0)|$ and use that $K(T) > 2^{m_{\overline{n}-1}}$ to obtain

$$\mathrm{e}^{-(K_0/2)K(T)} \le \mathrm{e}^{-(K_0/2)2^{m_{\overline{n}-1}}} \le \alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega})^2 \le 64 x^2.$$

If $\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}) > 8|x|$, for any line $\ell \in L(T)$ one has $|\boldsymbol{\nu}^0_\ell| \le K(T) \le 2^{m_{\overline{n}}-1}$ and hence

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \ge \alpha_{m_{\overline{n}}-1}(\boldsymbol{\omega}) > \frac{1}{2}\alpha_{m_{\overline{n}-1}}(\boldsymbol{\omega}).$$

Then one can reason as done in case (a) and obtain (3.3.6) for all $\tau \in [0,1]$: in turn this yields the bound (3.3.7) and hence the bound (3.3.5) follows once more. ∎

Remark 3.3.7. From (3.3.1) and Lemmas 3.3.5 and 3.3.6 it follows that if property 1-p (respectively property 1) is satisfied then for all $n \leq p$ (respectively for all $n \geq -1$) one has $\mathcal{M}^{[n]}(x) = \mathcal{M}^{[n]}(0) + \partial_x \mathcal{M}^{[n]}(0)x + O(\varepsilon x^2)$, where $\mathcal{M}^{[n]}(0)$ is a real-valued matrix, while $\partial_x \mathcal{M}^{[n]}(0)$ is a purely imaginary one. In particular this implies that if $\Psi_{n+1}(x)\,|x^2 - \det(\mathcal{M}^{[n]}(0))| \geq \Psi_{n+1}(x)x^2/2$ for all $-1 \leq n < p$ (respectively for all $n \geq -1$) then property 1-p (respectively property 1) holds.

The following result will be crucial to check, in the forthcoming § 3.4, that property 1 is satisfied by $\mathcal{M}$. The proof follows the lines of that for Lemma 4.8 in [22] and it is deferred to Appendix B.

Lemma 3.3.8. *Assume $\mathcal{M}$ to satisfy property 1-p. Then*

$$\mathcal{M}^{[p]}(0) = \begin{pmatrix} \partial_{\beta_0}\Phi_{\mathbf{0}}^{\mathcal{R},p} + e_{p,\beta,\beta} & \partial_{B_0}\Phi_{\mathbf{0}}^{\mathcal{R},p} + e_{p,\beta,B} \\ \\ \partial_{\beta_0}\Gamma_{\mathbf{0}}^{\mathcal{R},p} + e_{p,B,\beta} & \partial_{B_0}\Gamma_{\mathbf{0}}^{\mathcal{R},p} + e_{p,B,B} \end{pmatrix}, \tag{3.3.8}$$

with $|e_{p,u,e}| \leq |\varepsilon| A_1 \mathrm{e}^{-A_2 2^{m_{p+1}}}$, $u, e = \beta, B$ for suitable positive constantes A_1 and A_2.

Lemma 3.3.9. *Assume $\mathcal{M}$ to satisfy property 1. Then the series (3.2.13) and (3.2.14) with the coefficients given by (3.2.11) and (3.2.12) respectively, converge for ε small enough.*

Proof. Let $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$. By Remark 3.3.4 one can bound $|\mathcal{G}^{[n]}_{e,u}(x)| \leq \gamma_0\, \alpha_{m_n}(\boldsymbol{\omega})^{-2}$ for all $n \geq 0$, and hence by Lemma 3.2.4 one can reason as in the proof of the bound (3.3.1a) so as to obtain

$$\sum_{\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}} |\mathscr{V}(\theta)| \leq C_0 \overline{C}_0^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2},$$

for some constants C_0 and $\overline{C}_0$, which is enough to prove the assertion. ∎

Lemma 3.3.10. *Assume $\mathcal{M}$ to satisfy property 1. Then for ε small enough the function (3.2.13), with the coefficients given by (3.2.11), solve the equations (1.2.1).*

Proof. We shall prove that, the functions $b^{\mathcal{R}}, B^{\mathcal{R}}$ satisfy the range equation (1.2.1), i.e. we shall check that $f^{\mathcal{R}} := (b^{\mathcal{R}}, B^{\mathcal{R}}) = g\,\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}})$, where g is the pseudo-differential operator with kernel $g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = 1/(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}$, and $\Xi(\boldsymbol{\omega}t, f^{\mathcal{R}}) := \big(\omega(B^{\mathcal{R}}) + \varepsilon F(\boldsymbol{\omega}t, \beta^{\mathcal{R}}, B^{\mathcal{R}}), \varepsilon G(\boldsymbol{\omega}t, \beta^{\mathcal{R}}, B^{\mathcal{R}})\big)$. We can write the Fourier coefficients of $b^{\mathcal{R}}$ and $B^{\mathcal{R}}$ as

$$b^{\mathcal{R}}_{\boldsymbol{\nu}} = \sum_{n\geq 0} b^{[n]}_{\boldsymbol{\nu}}, \qquad b^{[n]}_{\boldsymbol{\nu}} = \sum_{k\geq 1}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}(n)} \mathscr{V}(\theta),$$
$$B^{\mathcal{R}}_{\boldsymbol{\nu}} = \sum_{n\geq 0} B^{[n]}_{\boldsymbol{\nu}}, \qquad B^{[n]}_{\boldsymbol{\nu}} = \sum_{k\geq 1}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},B}(n)} \mathscr{V}(\theta),$$

where $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}(n)$ is the subset of $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ such that $n_{\ell_\theta} = n$. Set also $\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n) := \overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}(n) \times \overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu},B}(n)$ and, for $(\theta,\theta') \in \overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$, define $\mathscr{V}(\theta,\theta') := (\mathscr{V}(\theta),\mathscr{V}(\theta'))$.

Using Lemmas 2.2.1 and 3.3.9, in Fourier space one can write

$$\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t,f^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\geq 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t,f^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\geq 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(\mathcal{G}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}))^{-1}\mathcal{G}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t,f^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big)\mathcal{G}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t,f^{\mathcal{R}})]_{\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big)\sum_{k\geq 1}\varepsilon^k\sum_{(\theta,\theta')\in\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)}\mathscr{V}(\theta,\theta'),
\end{aligned}$$

where $\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ differs from $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ as it also includes couples where the root line of one or both of is the exiting line of a renormalised self-energy cluster. If we separate such couples from the others, we obtain

$$\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})[\Xi(\boldsymbol{\omega}t,f^{\mathcal{R}})]_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big)f^{[n]}_{\boldsymbol{\nu}} \\
&\quad+\sum_{n\geq 0}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p\geq n}\sum_{q=-1}^{n-1}M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})f^{[p]}_{\boldsymbol{\nu}}+\sum_{n\geq 1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p=0}^{n-1}\sum_{q=-1}^{p-1}M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})f^{[p]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big)f^{[n]}_{\boldsymbol{\nu}}+\sum_{p\geq 0}\Bigg(\sum_{q=-1}^{p-1}M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n\geq q+1}\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg)f^{[p]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}\Big)f^{[n]}_{\boldsymbol{\nu}}+\sum_{n\geq 0}\Bigg(\sum_{q=-1}^{n-1}M^{[q]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\chi_q(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg)f^{[n]}_{\boldsymbol{\nu}}\Bigg] \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Bigg[\sum_{n\geq 0}\Big((\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})\mathbb{1}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big)f^{[n]}_{\boldsymbol{\nu}}+\sum_{n\geq 0}\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})f^{[n]}_{\boldsymbol{\nu}}\Bigg] \\
&= \sum_{n\geq 0}f^{[n]}_{\boldsymbol{\nu}} = f^{\mathcal{R}}_{\boldsymbol{\nu}},
\end{aligned}$$

so that the proof is complete. ■

Remark 3.3.11. If $\mathcal{M}$ satisfies property 1, one can define

$$\mathcal{M}^{[\infty]}(x) := \lim_{n\to\infty}\mathcal{M}^{[n]}(x),$$

and, form Lemma 3.3.8, one has

$$\mathcal{M}^{[\infty]}(0) = \begin{pmatrix} \partial_{\beta_0}\Phi^{\mathcal{R}}_{\mathbf{0}} & \partial_{B_0}\Phi^{\mathcal{R}}_{\mathbf{0}} \\ \\ \partial_{\beta_0}\Gamma^{\mathcal{R}}_{\mathbf{0}} & \partial_{B_0}\Gamma^{\mathcal{R}}_{\mathbf{0}} \end{pmatrix}. \tag{3.3.9}$$

Note that (3.3.9) is pretty much the same equality provided by Lemma 4.8 in [22], adapted to the present case.

Remark 3.3.12. If we take the formal expansion of the functions $\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon, \beta_0, B_0)$, $\Gamma^{\mathcal{R}}_{\mathbf{0}}(\varepsilon, \beta_0, B_0)$ and $\mathcal{M}^{[\infty]}_{u,e}(0; \varepsilon, \beta_0, B_0)$, $u, e \in \{\beta, B\}$, we obtain tree expansions where the self-energy clusters are allowed, as in Chapter 2. Then it is easy to prove the identity (3.3.9) to any perturbation order: the proof would follow the lines of the proof of the identities (2.3.4). In particular, if one expands

$$\det\left(\sum_{k=0}^{k_0-1} \varepsilon^k \left[\mathcal{M}^{[\infty]}\Big(0; \varepsilon, \beta_0, \overline{B}_0 + \sum_{h=1}^{k_0-1} \varepsilon^h B_0^{(h)} + O(\varepsilon^{k_0})\Big)\right]^{(k)}\right) = \sum_{k=0}^{k_0-1} \varepsilon^k \delta^{(k)} + O(\varepsilon^{k_0}),$$

one has $\delta^{(k)} = \delta^{(k)}(\beta_0) \equiv 0$ for all $k = 0, \dots k_0 - 1$, if the coefficients $B_0^{(h)} = B_{\mathbf{0}}^{(h)}(\beta_0)$ are defined as in (1.1.10). Moreover, for such an expansion, if one writes

$$\det\left(\sum_{k=0}^{k_0-1} \varepsilon^k \left[\mathcal{M}^{[n]}(0, \beta_0, \overline{B}_0 + \sum_{h=1}^{k_0-1} \varepsilon^h B_0^{(h)} + O(\varepsilon^{k_0})\Big)\right]^{(k)}\right) = \sum_{k=0}^{k_0-1} \varepsilon^k \delta_n^{(k)} + O(\varepsilon^{k_0})$$

one has

$$\left|\sum_{k=0}^{k_0-1} \varepsilon^k \delta_n^{(k)}\right| \le A_1 \mathrm{e}^{-A_2 2^{mn}}$$

for some positive constants A_1, A_2. However, under Hypotheses 1, 2 and 4, we are not able to prove the convergence of the series and we need to introduce some resummation procedure to give a meaning to the series.

Lemma 3.3.13. *Assume $\mathcal{M}$ to satisfy property 1. Then there exists $B_0 = B_0(\varepsilon, \beta_0)$, C^∞ in both ε, β_0, such that $B_0(\varepsilon, \beta_0) \to \overline{B}_0$ for $\varepsilon \to 0$, and $\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon, \beta_0, B_0(\varepsilon, \beta_0)) \equiv 0$ for any β_0 and ε small enough.*

Proof. One has $\Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon; \beta_0, B_0) = \omega_0(B_0) + O(\varepsilon)$ and it is C^∞ in its arguments because of the assumption that $\mathcal{M}$ satisfies property 1. Then, by Hypothesis 2 one can apply the implicit function theorem to obtain the result. In particular one has

$$B_0(\varepsilon, \beta_0) = \overline{B}_0 + \sum_{h=1}^{k_0} \varepsilon^h B_{\mathbf{0}}^{(h)}(\beta_0) + O(\varepsilon^{k_0+1}),$$

where the coefficients $B_{\mathbf{0}}^{(h)}(\beta_0)$ coincide with those defined in (1.1.10). ∎

Lemma 3.3.14. *Assume $\mathcal{M}$ to satisfy property* 1 *and set* $g(\varepsilon,\beta_0) := \Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon;\beta_0,B_0(\varepsilon,\beta_0))$, *where* $B_0(\varepsilon,\beta_0)$ *is the* C^∞ *function referred to in Lemma 3.3.13. Then there exists a continuous curve* $\beta_0 = \beta_0(\varepsilon)$ *such that* $g(\varepsilon,\beta_0(\varepsilon)) = 0$ *and moreover, at least in a suitable half-neighbourhood of* $\varepsilon = 0$, *one has* $\det\left(\mathcal{M}^{[\infty]}(0;\varepsilon,\beta_0(\varepsilon),B_0(\varepsilon,\beta_0(\varepsilon)))\right) \le 0$.

Proof. Using the same argument in the proof of Lemma 4.11 in [22], as property 1 and Hypothesis 4 imply $g(\varepsilon,\beta_0) = \varepsilon^{k_0}\overline{g}(\varepsilon,\beta_0) = \varepsilon^{k_0}\left(\Gamma_{\mathbf{0}}^{(k_0)}(\beta_0) + O(\varepsilon)\right)$ and $\omega_0'(B_0(\varepsilon,\beta_0))$ has the same sign of $\omega_0'(\overline{B}_0)$ for ε small enough, one can find a continuous curve $\beta_0 = \beta_0(\varepsilon)$ defined at least in a suitable half-neighbourhood of $\varepsilon = 0$ such that $g(\varepsilon,\beta_0(\varepsilon)) \equiv 0$ and $\partial_{\beta_0} g(\varepsilon,\beta_0(\varepsilon))\omega_0'(B_0(\varepsilon,\beta_0(\varepsilon))) \ge 0$. Indeed Hypothesis 4 implies that there exist two half-neighbourhood V_+, V_- of $\overline{\beta}_0$ such that $\overline{g}(0,\beta_0) > 0$ for $\beta_0 \in V_+$ while $\overline{g}(0,\beta_0) < 0$ for $\beta_0 \in V_-$. Hence, by continuity, for all $\beta_0 \in V_+$ there exists a neighbourhood $U_+(\beta_0)$ of $\varepsilon = 0$ such that $\overline{g}(\varepsilon,\beta_0) > 0$ for $\varepsilon \in U_+(\beta_0)$ and, for the same reason for all $\beta_0 \in V_-$ there exists a neighbourhood $U_-(\beta_0)$ of $\varepsilon = 0$ such that $\overline{g}(\varepsilon,\beta_0) < 0$ for all $\varepsilon \in U_-(\beta_0)$. Therefore, again by continuity, there is a continuous curve $\beta_0 = \beta_0(\varepsilon)$, defined in a suitable neighbourhood $U = (\overline{\varepsilon},\overline{\varepsilon})$ such that $\beta_0(0) = \overline{\beta}_0$ and $\overline{g}(\varepsilon,\beta_0(\varepsilon)) \equiv 0$. Moreover, again by continuity and Hypothesis 4, we have $\partial_{\beta_0} g(\varepsilon,\beta_0(\varepsilon))\omega_0'(B_0(\varepsilon,\beta_0(\varepsilon))) \ge 0$, at least in a half-neighbourhood of $\varepsilon = 0$. On the other hand one has

$$\partial_{\beta_0} g(\varepsilon,\beta_0) = \partial_2\Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon;\beta_0,B_0(\varepsilon,\beta_0)) + \partial_3\Gamma_{\mathbf{0}}^{\mathcal{R}}(\varepsilon;\beta_0,B_0(\varepsilon,\beta_0))\partial_{\beta_0}B_0(\varepsilon,\beta_0),$$

so that

$$\det\left(\mathcal{M}^{[\infty]}(0;\varepsilon,\beta_0(\varepsilon),B_0(\varepsilon,\beta_0(\varepsilon)))\right) = -\partial_{\beta_0} g(\varepsilon,\beta_0(\varepsilon))\left(\omega_0'(B_0(\varepsilon,\beta_0(\varepsilon))) + O(\varepsilon)\right)$$

and then the assertion follows. In particular if k_0 is even, the curve $\beta_0(\varepsilon)$ above can be defined in a whole neighbourhood of $\varepsilon = 0$. ∎

By the results above it follows that, if property 1 is satisfied, choosing $\beta_0 = \beta_0(\varepsilon)$ and $B_0 = B_0(\varepsilon,\beta_0(\varepsilon))$ as above, the series (3.2.13) solve the equation of motion (1.1.1).

However we still have to prove that property 1 is satisfied; in the forthcoming § 3.4 we shall see that it is possible to fix $\beta_0 = \beta_0(\varepsilon)$ and $B_0 = B_0(\varepsilon,\beta_0(\varepsilon))$ in such a way that both property 1 is satisfied and the bifurcation equations are solved.

3.4 Fixing the initial phase

In this section, we shall complete the proof of Theorem 1.1.2 by showing that, under Hypotheses 1, 2 and 4 and by suitably choosing β_0, B_0, $\mathcal{M}$ turns out to satisfy property 1.

Define the C^∞ non-increasing "cut-off" function ξ such that

$$\xi(x) = \begin{cases} 1, & x \le 1/2, \\ 0, & x \ge 1, \end{cases} \tag{3.4.1}$$

and set $\xi_{-1}(x) = 1$ and $\xi_n(x) = \xi(2^8 x/\alpha^2_{m_{n+1}}(\boldsymbol{\omega}))$ for all $n \ge 0$. Set also

$$B_0(\varepsilon, \beta_0, B_0') := \overline{B}_0 + \sum_{h=1}^{k_0-1} \varepsilon^h B_{\mathbf{0}}^{(h)}(\beta_0) + \varepsilon^{k_0} B_0' \tag{3.4.2}$$

where the coefficients $B_{\mathbf{0}}^{(h)}(\beta_0)$ are defined as in (1.1.10) and k_0 is as in Hypothesis 4. For all $n \ge 0$ we define recursively the *regularised propagators* as

$$\overline{\mathcal{G}}^{[n]} = \overline{\mathcal{G}}^{[n]}(x; \varepsilon, \beta_0, B_0') := \Psi_n(x) \Big((\mathrm{i}x)\mathbb{1} - \overline{\mathcal{M}}^{[n-1]}(x; \varepsilon, \beta_0, B_0') \xi_{n-1}(\Delta_{n-1}) \Big)^{-1}, \tag{3.4.3}$$

where

$$\overline{\mathcal{M}}^{[n-1]}(x; \varepsilon, \beta_0, B_0') := \sum_{q=-1}^{n-1} \chi_q(x) \overline{M}^{[q]}(x; \varepsilon, \beta_0, B_0'), \tag{3.4.4}$$

with the 2×2 matrix $\overline{M}^{[q]}(x; \varepsilon, \beta_0, B_0')$ defined so as

$$\overline{M}^{[q]}_{u,e}(x; \varepsilon, \beta_0, B_0') := \sum_{T \in \mathfrak{R}_{q,u,e}} \varepsilon^{k(T)} \overline{\mathscr{V}}_T(x; \varepsilon, \beta_0, B_0'), \tag{3.4.5}$$

where

$$\overline{\mathscr{V}}_T(x; \varepsilon, \beta_0, B_0') := \Bigg(\prod_{v \in N(T)} \widetilde{\mathcal{F}}_v \Bigg) \Bigg(\prod_{\ell \in L(T)} \overline{\mathcal{G}}^{[n_\ell]}_{e_\ell, u_\ell} \Bigg), \tag{3.4.6}$$

with $\widetilde{\mathcal{F}}_v = \mathcal{F}_v(\beta_0, B_0(\varepsilon, \beta_0, B_0'))$ and

$$\Delta_{n-1} = \Delta_{n-1}(\varepsilon, \beta_0, B_0') := D_{n-1}(\varepsilon, \beta_0, B_0') - \sum_{k=0}^{k_0-1} \varepsilon^k \left[D_{n-1}(\varepsilon, \beta_0, B_0') \right]^{(k)},$$

with

$$D_{n-1}(\varepsilon, \beta_0, B_0') := \det \Big(\overline{\mathcal{M}}^{[n-1]}(0; \varepsilon, \beta_0, B_0') \Big).$$

For any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$, define also, for all $k \ge 0$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $h = \beta, B$,

$$\overline{\mathscr{V}}(\theta; \varepsilon, \beta_0, B_0') := \Bigg(\prod_{v \in N(T)} \widetilde{\mathcal{F}}_v \Bigg) \Bigg(\prod_{\ell \in L(T)} \overline{\mathcal{G}}^{[n_\ell]}_{e_\ell, u_\ell} \Bigg).$$

Finally, set $\overline{\mathcal{M}} := \{\overline{\mathcal{M}}^{[n]}(x; \varepsilon, \beta_0, B_0')\}_{n \ge -1}$, and $\overline{\mathcal{M}}^{\xi} := \{\overline{\mathcal{M}}^{[n]}(x; \varepsilon, \beta_0, B_0') \xi_n(\Delta_n)\}_{n \ge -1}$.

Lemma 3.4.1. *For ε small enough, $\overline{\mathcal{M}}^{\xi}$ satisfies property 1.*

Proof. We shall prove that $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-p for all $p \geq 0$, by induction on p. For $p = 0$ it is obvious if ε is small enough. Assume then that $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-p. Then we can repeat almost word by word the proofs of Lemmas 3.3.3 and 3.3.5, so as to obtain

$$\overline{\mathcal{M}}^{[p]}(x;\varepsilon,\beta_0,B_0') = \overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0') + x\,\partial_x\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0') + x^2\int_0^1 \mathrm{d}\tau\,(1-\tau)\,\partial_x^2\overline{\mathcal{M}}^{[n]}(\tau x;\varepsilon,\beta_0,B_0'),$$

with $\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0')$ a real-valued matrix, $\partial_x\overline{\mathcal{M}}^{[p]}(0;\varepsilon,\beta_0,B_0')$ a purely imaginary one and

$$\left| x^2\int_0^1 \mathrm{d}\tau\,(1-\tau)\,\partial_x^2\overline{\mathcal{M}}^{[n]}(\tau x;\varepsilon,\beta_0,B_0')\right| \leq C\,|\varepsilon| x^2$$

for some constant C, by Lemma 3.3.6. Then we only have to prove that – see Remark 3.3.7 –

$$\Psi_{p+1}(x)\left|x^2 - D_p(\varepsilon,\beta_0,B_0')\xi_p(\Delta_p)^2\right| \geq \Psi_{p+1}(x)\frac{x^2}{2}.$$

Note that, by the definition of Δ_p, one has

$$\sum_{k=0}^{k_0-1}\varepsilon^k\left[D_p(\varepsilon,\beta_0,B_0')\right]^{(k)} = \sum_{k=0}^{k_0-1}\varepsilon^k\delta_p^{(k)}$$

with the coefficients $\delta_p^{(k)}$ as in Remark 3.3.12, and hence $\overline{\mathcal{M}}^{\xi}$ satisfies property 1-($p+1$) by the definition of the function ξ_p. ∎

Set

$$\overline{\mathcal{M}}^{[\infty]}(x;\varepsilon,\beta_0,B_0') := \lim_{n\to\infty}\overline{\mathcal{M}}^{[n]}(x;\varepsilon,\beta_0,B_0'), \tag{3.4.7}$$

and define

$$\overline{\Phi}(\varepsilon,\beta_0,B_0') := \sum_{k\geq 0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},\beta}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'), \qquad \overline{\Gamma}(\varepsilon,\beta_0,B_0') := \sum_{k\geq 0}\varepsilon^k\sum_{\theta\in\Theta^{\mathcal{R}}_{k,\mathbf{0},B}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'). \tag{3.4.8}$$

Lemma 3.4.2. *One has*

$$[\overline{P}(\varepsilon,\beta_0,B_0')]^{(k)} = [P^{\mathcal{R}}_{\mathbf{0}}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))]^{(k)}, \qquad P = \Phi,\Gamma,$$

for all $k = 0,\dots,k_0$.

Proof. Set $\Theta^{\mathcal{R}(n)}_{k,\boldsymbol{\nu},h} := \{\theta\in\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},h} : \exists\ell\in L(\theta)$ such that $n_\ell = n\}$ and write

$$\overline{P}(\varepsilon,\beta_0,B_0') = \sum_{k\geq 0}\varepsilon^k\sum_{n\geq 0}\sum_{\theta\in\Theta^{\mathcal{R}(n)}_{k,\mathbf{0},h}}\overline{\mathscr{V}}(\theta,\varepsilon,\beta_0,B_0'),$$

with $h = \beta, B$ for $P = \Phi, \Gamma$ respectively, and note that if $\theta \in \Theta^{\mathcal{R}(n)}_{k,\boldsymbol{\nu},h}$ one has

$$\prod_{v\in N(\theta)} |\widetilde{\mathcal{F}}_v| \le E_1^{|N(\theta)|} \mathrm{e}^{-E_2 2^{m_n}},$$

for some constants E_1, E_2. Moreover one can write formally

$$\overline{\mathcal{G}}_{n_\ell} = \Psi_{n_\ell}(x) g_{n_\ell-1}(x)\Big(\mathbb{1} + \sum_{m\ge1}\Big(g_{n_\ell-1}(x)\widetilde{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0')\xi_{n_\ell-1}(\Delta_{n_\ell-1})\Big)^m\Big),$$

with

$$g_{n_\ell-1}(x) = \frac{1}{(\mathrm{i}x)^2}\begin{pmatrix} \mathrm{i}x & \omega_0'(\overline{B}_0)\xi_{n_\ell-1}(\Delta_{n_\ell-1}) \\ 0 & \mathrm{i}x \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0') := \overline{\mathcal{M}}^{[n_\ell-1]}(x;\varepsilon,\beta_0,B_0') - \begin{pmatrix} 0 & \omega_0'(\overline{B}_0) \\ 0 & 0 \end{pmatrix} = O(\varepsilon),$$

and we can write $\xi_{n_\ell-1}(\Delta_{n_\ell-1}) = 1 + \xi'_{n_\ell-1}(\Delta^*)\Delta_{n_\ell-1}$ for some Δ^*, where $\Delta_{n_\ell-1} = O(\varepsilon^{k_0})$ and

$$|\xi'_{n_\ell-1}(\Delta^*)| \le \frac{E_3}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^2} \le \frac{E_3}{\alpha_{m_n}(\boldsymbol{\omega})^2},$$

for some positive constant E_3 independent of n. Hence the assertion follows. ■

Introduce the C^∞ functions $\hat{\Phi}(\varepsilon,\beta_0,B_0)$, $\widetilde{\Phi}(\varepsilon,\beta_0,B_0)$, $\hat{\Gamma}(\varepsilon,\beta_0,B_0)$ and $\widetilde{\Gamma}(\varepsilon,\beta_0,B_0)$ such that
(1) the first k_0 coefficients of the Taylor expansion in ε of both the functions $\hat{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ and $\widetilde{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ coincide with those of $\overline{\Phi}(\varepsilon,\beta_0,B_0')$,
(2) the first k_0 coefficients of the Taylor expansion in ε of both the functions $\hat{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ and $\widetilde{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))$ coincide with those of $\overline{\Gamma}(\varepsilon,\beta_0,B_0')$,
(3) one has

$$\overline{\mathcal{M}}^{[\infty]}(0;\varepsilon,\beta_0,B_0') = \begin{pmatrix} \partial_2\hat{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Phi}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \\ \partial_2\hat{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Gamma}(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \end{pmatrix}. \tag{3.4.9}$$

Define also, for all $n \ge -1$ the C^∞ functions $\hat{\Phi}_n(\varepsilon,\beta_0,B_0)$, $\widetilde{\Phi}_n(\varepsilon,\beta_0,B_0)$, $\hat{\Gamma}_n(\varepsilon,\beta_0,B_0)$, $\widetilde{\Gamma}_n(\varepsilon,\beta_0,B_0)$ such that

$$\overline{\mathcal{M}}^{[n]}(0;\varepsilon,\beta_0,B_0') = \begin{pmatrix} \partial_2\hat{\Phi}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Phi}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \\ \partial_2\hat{\Gamma}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) & \partial_3\widetilde{\Gamma}_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0')) \end{pmatrix}, \tag{3.4.10}$$

and

$$|P_n(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))-P(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,B_0'))|\leq A_P|\varepsilon|\,\mathrm{e}^{-B_P 2^{m_n}},\qquad P=\hat\Phi,\widetilde\Phi,\hat\Gamma,\widetilde\Gamma, \tag{3.4.11}$$

for some constants A_P, B_P. Then, by reasoning as in the proofs of Lemmas 3.3.13 and 3.3.14, we can find $\widetilde B_0=\widetilde B_0(\varepsilon,\beta_0)$ and $\widetilde\beta_0=\widetilde\beta_0(\varepsilon)$ such that
(i) $\hat\Phi(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0)))\equiv 0$ for all β_0 and ε small enough,
(ii) $\hat\Gamma(\varepsilon,\widetilde\beta_0(\varepsilon),B_0(\varepsilon,\widetilde\beta_0(\varepsilon),\widetilde B_0(\varepsilon,\widetilde\beta_0(\varepsilon))))\equiv 0$ for all ε small enough, and
(iii)

$$\partial_3\widetilde\Phi(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0)))\partial_{\beta_0}\hat\Gamma(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0)))\Big|_{\beta_0=\widetilde\beta_0(\varepsilon)}\geq 0,$$

at least in a suitable half-neighbourhood of $\varepsilon=0$.

Lemma 3.4.3. *Set $\widetilde C(\varepsilon)=(\widetilde\beta_0(\varepsilon),\widetilde B_0(\varepsilon,\widetilde\beta_0(\varepsilon)))$ with $\widetilde B_0(\varepsilon,\beta_0)$ and $\widetilde\beta_0(\varepsilon)$ as above. Then, along $\widetilde C(\varepsilon)$ one has $\xi_n(\Delta_n)\equiv 1$ for all $n\geq -1$.*

Proof. We shall prove the result by induction on n. For $n=-1$ it is obvious. Assume then $\xi_p(\Delta_p)\equiv 1$ for all $p=-1,\dots,n-1$ along $\widetilde C(\varepsilon)$ and set $C(\varepsilon)=(\widetilde\beta_0(\varepsilon),B_0(\varepsilon,\widetilde C(\varepsilon)))$. Hence $\overline{\mathcal G}^{[p]}(x;\varepsilon,\widetilde C(\varepsilon))\equiv\mathcal G^{[p]}(x;\varepsilon,C(\varepsilon))$ for all $p=0,\dots,n$ and thence $\overline{\mathcal M}^{[n]}(x;\varepsilon,\widetilde C(\varepsilon))\equiv\mathcal M^{[n]}(x;\varepsilon,C(\varepsilon))$. In particular $\mathcal M$ satisfies property 1-n so that, using Lemma 3.3.8 one has

$$\overline{\mathcal M}^{[n]}(0;\varepsilon,\widetilde C(\varepsilon))=\begin{pmatrix}\partial_2\Phi^{\mathcal R,n}_{\mathbf 0}(\varepsilon,C(\varepsilon))+e_{n,\beta,\beta} & \partial_3\Phi^{\mathcal R,n}_{\mathbf 0}(\varepsilon,C(\varepsilon))+e_{n,\beta,B}\\ \\ \partial_2\Gamma^{\mathcal R,n}_{\mathbf 0}(\varepsilon,C(\varepsilon))+e_{n,B,\beta} & \partial_3\Gamma^{\mathcal R,n}_{\mathbf 0}(\varepsilon,C(\varepsilon))+e_{n,B,B}\end{pmatrix},$$

with $|e_{n,u,e}|\leq|\varepsilon|A_1e^{-A_2 2^{m_{n+1}}}$, $u,e=\beta,B$. On the other hand one has

$$\partial_2\hat\Phi(\varepsilon,C(\varepsilon))=-\partial_3\hat\Phi(\varepsilon,C(\varepsilon))\;\partial_{\beta_0}B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0))\Big|_{\beta_0=\widetilde\beta_0(\varepsilon)},$$

$$\partial_2\hat\Gamma(\varepsilon,C(\varepsilon))=\partial_{\beta_0}\hat\Gamma(\varepsilon,\beta_0,B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0)))\Big|_{\beta_0=\widetilde\beta_0(\varepsilon)}-\partial_3\hat\Gamma(\varepsilon,C(\varepsilon))\;\partial_{\beta_0}B_0(\varepsilon,\beta_0,\widetilde B_0(\varepsilon,\beta_0))\Big|_{\beta_0=\widetilde\beta_0(\varepsilon)},$$

so that, without writing explicitly the dependence on $(\varepsilon,C(\varepsilon))$, one has

$$\overline{\mathcal M}^{[n]}(0;\varepsilon,\widetilde C(\varepsilon))=\begin{pmatrix}-\partial_3\Phi^{\mathcal R,n}_{\mathbf 0}\partial_{\beta_0}B_0+\gamma_n & \partial_3\Phi^{\mathcal R,n}_{\mathbf 0}+e_{n,\beta,B}\\ \\ \partial_{\beta_0}\Gamma^{\mathcal R,n}_{\mathbf 0}-\partial_3\Gamma^{\mathcal R,n}_{\mathbf 0}\partial_{\beta_0}B_0+\gamma_n' & \partial_3\Gamma^{\mathcal R,n}_{\mathbf 0}+e_{n,B,B}\end{pmatrix},$$

with $|\gamma_n|,|\gamma_n'|\leq|\varepsilon|C_1e^{-C_2 2^{m_{n+1}}}$ for some C_1,C_2. Hence

$$\Delta_n=-\partial_{\beta_0}\Gamma^{\mathcal R,n}_{\mathbf 0}\partial_3\Phi^{\mathcal R,n}_{\mathbf 0}+c_n=-\partial_{\beta_0}\hat\Gamma_n\partial_3\widetilde\Phi_n+c_n'=-\partial_{\beta_0}\Gamma\partial_3\widetilde\Phi+c_n''\leq c_n'',$$

with $|c_n|,|c_n'|,|c_n''|\leq|\varepsilon|D_1e^{-D_2 2^{m_{n+1}}}$ for some constants D_1 and D_2, so that the assertion follows by the definition of ξ_n. ∎

Lemma 3.4.4. *Let $\widetilde{C}(\varepsilon)$ be as in Lemma 3.4.3 and set $C(\varepsilon) = (\widetilde{\beta}_0(\varepsilon), B_0(\varepsilon, \widetilde{C}(\varepsilon)))$. One can choose the functions $\hat{\Phi}, \widetilde{\Phi}, \hat{\Gamma}, \widetilde{\Gamma}$ such that $\hat{\Phi}(\varepsilon, C(\varepsilon)) = \widetilde{\Phi}(\varepsilon, C(\varepsilon)) = \Phi^{\mathcal{R}}_{\mathbf{0}}(\varepsilon, C(\varepsilon)) \equiv 0$ and $\hat{\Gamma}(\varepsilon, C(\varepsilon)) = \widetilde{\Gamma}(\varepsilon, C(\varepsilon)) = \Gamma^{\mathcal{R}}_{\mathbf{0}}(\varepsilon, C(\varepsilon)) \equiv 0$. In particular $(\beta(t,\varepsilon), B(t,\varepsilon)) = C(\varepsilon) + (b^{\mathcal{R}}(t;\varepsilon, C(\varepsilon)), B^{\mathcal{R}}(t;\varepsilon, C(\varepsilon)))$ defined in (3.2.13) solves the equation of motion (1.1.1)*

Proof. For any $\hat{\Phi}$, $\hat{\Gamma}$ there is a curve $C(\varepsilon)$ along which $\mathcal{M} = \overline{\mathcal{M}} = \overline{\mathcal{M}}^{\xi}$ (hence $\mathcal{M}$ satisfies property 1) and $\hat{\Phi}(\varepsilon, C(\varepsilon)) = \hat{\Gamma}(\varepsilon, C(\varepsilon)) \equiv 0$. By Remark 3.3.11 also $\Phi^{\mathcal{R}}_{\mathbf{0}}$ and $\Gamma^{\mathcal{R}}_{\mathbf{0}}$ are among the primitives of $\mathcal{M}^{[\infty]}_{\beta,\beta}$ and $\mathcal{M}^{[\infty]}_{B,\beta}$ respectively, and then the assertion follows. ■

Lemma 3.4.4 completes the proof of Theorem 1.1.2: indeed the function $(\beta(t,\varepsilon), B(t,\varepsilon))$ is a quasi-periodic solution to (1.1.1) with frequency vector $\boldsymbol{\omega}$ and, by construction, it reduces to $(\overline{\beta}_0, \overline{B}_0)$ as ε tends to 0.

A. Proof of Lemma 2.3.7

We say that a self-energy cluster T is *isolated* if both its external lines are non-resonant and that is *relevant* if it is not isolated. As will emerge from the proof, it is convenient to introduce a further label $\mathfrak{d}_T \in \{0,1\}$ to be associated with each relevant self-energy cluster T. We shall see later how to fix such a label: for the time being we consider it as an abstract label and we define the *subchains* as follows. Given $p \geq 2$ relevant self-energy clusters $T_1, \dots, T_p$ of a tree θ, with $\ell'_{T_i} = \ell_{T_{i+1}}$ for all $i = 1, \dots, p$, we say that $C = \{T_1, \dots, T_p\}$ is a subchain if $\mathfrak{d}_{T_i} = 1$ for $i = 1, \dots, p$, the line ℓ_{T_1} either is non-resonant or enters a relevant self-energy cluster T_0 with $\mathfrak{d}_{T_0} = 0$ and the line ℓ'_{T_p} either is non-resonant or exits a relevant self-energy cluster T_{p+1} with $\mathfrak{d}_{T_{p+1}} = 0$. We say that a relevant self-energy cluster T is a *link* if $\mathfrak{d}_T = 1$.

Given a subchain $C = \{T_1, \dots, T_p\}$, the relevant self-energy clusters T_i are called the *links* of C. Define $\ell_0(C) := \ell_{T_1}$ and $\ell_i(C) := \ell'_{T_i}$ for $i = 1, \dots, p$ and set $n_i(C) = n_{\ell_i(C)}$ for $i = 0, \dots, p$. The lines $\ell_0(C), \dots, \ell_p(C)$ are the *chain-lines* of C: we call $\ell_1(C), \dots, \ell_{p-1}(C)$ the *internal chain-lines* of C and $\ell_0(C), \ell_p(C)$ the *external chain-lines* of C. For future convenience we also set $\ell_C = \ell_0(C)$ and $\ell'_C = \ell_p(C)$. We also call $k(C) := k(T_1) + \ldots + k(T_p)$ the *total order* of the subchain C and $p(C) = p$ the *length* of C. Note that for all $i = 1, \dots, p-1$ one has $\zeta_{\ell_i(C)} = \zeta_{\ell_C} = \zeta_{\ell'_C}$ if $\mathscr{V}(\theta) \neq 0$.

We denote by $\mathfrak{C}_1(k; h, h'; n_0, \dots, n_p)$ the set of all subchains $C = \{T_1, \dots, T_p\}$ with total order k and with fixed labels $h_{\ell_0(C)} = h$, $h_{\ell_p(C)} = h'$ and $n_i(C) = n_i$ for $i = 0, \dots, p$.

If all relevant self-energy clusters T of θ carried a label $\mathfrak{d}_T = 1$ the definition of subchain would reduce to that of chain in § 2.3. We want to prove the bound (2.3.8). The sum is over all chains $C = \{T_1, \dots, T_p\}$ in $\mathfrak{C}(k; h, h'; n_0, \dots, n_p)$; then we set $\mathfrak{d}_{T_i} = 1$ for $i = 1, \dots, p$, so that we can replace $\mathfrak{C}(k; h, h'; n_0, \dots, n_p)$ with $\mathfrak{C}_1(k; h, h'; n_0, \dots, n_p)$. Thus in (2.3.8) we can write

$$\sum_{C \in \mathfrak{C}(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \mathscr{V}_C(x) = \sum_{C \in \mathfrak{C}_1(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \mathscr{V}_C(x) = \sum_{\substack{h_1,\dots,h_{p-1} \in \{\beta, B\} \\ k_1+\ldots+k_p=k}} \prod_{i=1}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x, n_i), \tag{A.1}$$

where $h_0 = h$, $h_p = h'$ and $n_i = \min\{\overline{n}_{i-1}, \overline{n}_i\} - 1$ for $i = 1, \dots, p$; of course $|n_i - n_j| \leq 1$ for all $i, j = 1, \dots, p$ and $k_i \geq 0$ for $i = 1, \dots, p$; see Figure A.1.

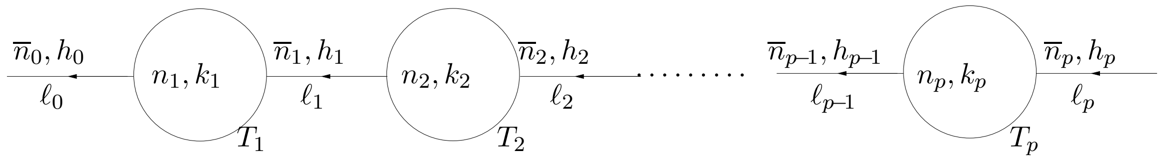

Figure A.1: A subchain C of length p with links $T_1, \ldots, T_p$ and chain-lines $\ell_0, \ldots, \ell_p$; summing over all possible C with $h_0 = h$, $h_p = h'$, $k_1 + \ldots + k_p = k$ and $\overline{n}_1, \ldots, \overline{n}_p$ fixed, one obtains a graphical representation of (A.1).

For all $k \geq 1$, all $n \geq -1$ and all $h, h' \in \{\beta, B\}$ let us write

$$\mathcal{M}^{(k)}_{h,h'}(x, n) = \sum_{\delta \in \Delta} \mathcal{M}^{(k)}_{h,h'}(x, n, \delta) \tag{A.2}$$

where $\Delta := \{\mathcal{L}, \partial, \partial^2, \mathcal{R}\}$ is a set of labels and

$$\begin{aligned} &\mathcal{M}^{(k)}_{h,h'}(x, n, \mathcal{L}) := \mathcal{M}^{(k)}_{h,h'}(0) \qquad \mathcal{M}^{(k)}_{h,h'}(x, n, \partial) := x\partial\mathcal{M}^{(k)}_{h,h'}(0), \\ &\mathcal{M}^{(k)}_{h,h'}(x, n, \partial^2) := x^2 \int_0^1 \mathrm{d}\tau (1-\tau)\partial^2 \mathcal{M}^{(k)}_{h,h'}(\tau x), \\ &\mathcal{M}^{(k)}_{h,h'}(x, n, \mathcal{R}) := \mathcal{M}^{(k)}_{h,h'}(x, n) - \mathcal{M}^{(k)}_{h,h'}(x), \end{aligned} \tag{A.3}$$

so that we can decompose the sum in (A.1) as

$$\sum_{\delta_1, \ldots, \delta_p \in \Delta} \sum_{\substack{h_1, \ldots, h_{p-1} \in \{\beta, B\} \\ k_1 + \ldots + k_p = k}} \prod_{i=1}^{p} \mathcal{M}^{(k_i)}_{h_{i-1}, h_i}(x, n_i, \delta_i). \tag{A.4}$$

There are several contributions to (A.4) which sum up to zero. This holds for all contributions with $\delta_j = \delta_{j+1} = \mathcal{L}$ for some $j = 1, \ldots, p-1$. Indeed one can write such contributions as

$$\begin{aligned} &\sum_{\delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_p \in \Delta} \sum_{\substack{h_1, \ldots, h_{j-1}, h_{j+1}, \ldots h_{p-1} \in \{\beta, B\} \\ k_1 + \ldots + k_{j-1} + \overline{k} + k_{j+2} + \ldots + k_p = k}} \prod_{\substack{i=1 \\ i \neq j}}^{p} \mathcal{M}^{(k_i)}_{h_{i-1}, h_i}(x, n_i, \delta_i) \; \times \\ &\times \left(\sum_{k_j + k_{j+1} = \overline{k}} \mathcal{M}^{(k_j)}_{h_{j-1}, \beta}(0) \mathcal{M}^{(k_{j+1})}_{\beta, h_{j+1}}(0) + \sum_{k_j + k_{j+1} = \overline{k}} \mathcal{M}^{(k_j)}_{h_{j-1}, B}(0) \mathcal{M}^{(k_{j+1})}_{B, h_{j+1}}(0) \right), \end{aligned} \tag{A.5}$$

and by Lemma 2.3.3 one has (for instance)

$$\begin{aligned}
&\sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,B}(0)\mathcal{M}^{(k_{j+1})}_{B,\beta}(0) \\
&\qquad = \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,B}(0)\left(-\sum_{k'+k''=k_{j+1}} \mathcal{M}^{(k')}_{BB}(0)\partial_{\beta_0}B_0^{(k'')}\right) \\
&\qquad = \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{\beta,\beta}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) + \sum_{k_j+k_{j+1}=\overline{k}} \mathcal{M}^{(k_j)}_{BB}(0)\mathcal{M}^{(k_{j+1})}_{\beta,\beta}(0) = 0;
\end{aligned}$$

one can reason in the same way also for the cases $(h_{j-1},h_{j+1}) \neq (\beta,\beta)$. By (2.3.6a) of Lemma 2.3.4, also the contributions with $\delta_j = \partial$ and $h_{j-1} \neq h_j$ for some $j = 1,\dots,p$ sum up zero. Finally we obtain zero also when we sum together all contributions with $\delta_j = \ldots = \delta_{j+q} = \partial$, $h_{j-1} = \ldots = h_{j+q} = \overline{h}$ for some $\overline{h} \in \{\beta, B\}$ and $\delta_{j-1} = \delta_{j+q+1} = \mathcal{L}$ for some $j = 2,\dots,p-1$ and $q = 0,\dots,p-1-j$. Indeed we can write the sum of such contributions as

$$\begin{aligned}
&\sum_{\substack{h_1,\dots,h_{j-2},\overline{h},h_{j+q+1},\dots,h_{p-1}=\beta,B \\ k_1+\ldots+k_p=k}} \left(\prod_{i=1}^{j-2} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right) \mathcal{M}^{(k_{j-1})}_{h_{j-2},\overline{h}}(x,n_{j-1},\mathcal{L}) \times \\
&\qquad \times \left(\prod_{i=j}^{j+q} \mathcal{M}^{(k_i)}_{\overline{h},\overline{h}}(x,n_i,\partial)\right) \mathcal{M}^{(k_{j+q+1})}_{\overline{h},h_{j+q+1}}(x,n_{j+q+1},\mathcal{L}) \left(\prod_{i=j+q+2}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right) \\
&= \sum_{k''\le k} \sum_{k_j+\ldots+k_{j+q}=k''} \left(\prod_{i=j}^{j+q} \mathcal{M}^{(k_i)}_{\beta,\beta}(x,n_i,\partial)\right) \times \\
&\qquad \times \sum_{\substack{k_1+\ldots+k_{j-1}+k_{k+q+1}+\ldots+k_p=k-k' \\ h_1,\dots,h_{j-2},h_{j+q+1},\dots,h_{p-1}=\beta,B}} \left(\prod_{i=1}^{j-2} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right) \times \\
&\qquad \times \sum_{\overline{h}=\beta,B} \mathcal{M}^{(k_{j-1})}_{h_{j-2},\overline{h}}(x,n_{j-1},\mathcal{L}) \mathcal{M}^{(k_{j+q+1})}_{\overline{h},h_{j+q+2}}(x,n_{j+q+1},\mathcal{L}) \left(\prod_{i=j+q+2}^{p} \mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i)\right),
\end{aligned}$$

and the last sum is zero by the same argument used for (A.5); note that we used (2.3.6b) to extract a common factor $\mathcal{M}^{(k_i)}_{\beta,\beta}(x,n_i,\partial)$ in the third line.

We say that a cluster T is a *fake cluster* on scale n if it is a connected subgraph of a tree with only one entering line ℓ'_T and one exiting line ℓ_T such that (i) all lines in T have scale $\le n$ and there is at least one line on T with scale n and (ii) the lines ℓ_T and ℓ'_T carry the same momentum; note that a fake cluster can fail to be a self-energy cluster only because there the scales of the external lines have no relation with n (and hence it can even fail to be a cluster). Denote by $\mathfrak{S}^{*k}_{m,u,e}$ the set of fake cluster with order k, scale m and such that $h_{\ell'_T} = e$ and $h_{\ell_T} = u$.

In (A.4) we can expand

$$\mathcal{M}^{(k_i)}_{h_{i-1},h_i}(x,n_i,\delta_i) = \sum_{T_i\in\mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)} \mathscr{V}_{T_i}(x,\delta_i), \qquad i=1,\dots,p,$$

where we have set

$$\mathfrak{S}^{*k}_{u,e}(n,\delta) := \begin{cases} \bigcup\limits_{m\ge -1} \mathfrak{S}^{*k}_{m,u,e} & \delta=\mathcal{L},\partial,\partial^2, \\ \bigcup\limits_{m>n} \mathfrak{S}^{k*}_{m,u,e}, & \delta=\mathcal{R}, \end{cases} \tag{A.6}$$

for all $k\ge 0$, $n\ge 0$ and $u,e\in\{\beta,B\}$, and defined

$$\mathscr{V}_T(x,\delta) := \begin{cases} \mathscr{V}_T(0), & \delta=\mathcal{L}, \\ x\,\partial\,\mathscr{V}_T(0), & \delta=\partial, \\ -\,\mathscr{V}_T(x), & \delta=\mathcal{R}, \\ x^2\displaystyle\int_0^1 \mathrm{d}\tau(1-\tau)\partial_x^2\,\mathscr{V}_T(\tau x), & \delta=\partial^2, \end{cases} \tag{A.7}$$

with $\mathscr{V}_T(x)$ defined as for self-energy clusters in § 2.3. Denote by $\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)$ the set of fake clusters $\{T_1,\dots,T_p\}$ with $T_i\in\mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)$ for any choice of the labels $\{k_i,n_i,\delta_i\}_{i=1}^p$ and $\{h_i\}_{i=0}^p$ with the following constraints (see Figure A.2):

(i) $k_1+\ldots+k_p=k$,

(ii) $n_i<\min\{\overline{n}_{i-1},\overline{n}_i\}$ for $i=1,\dots,p$,

(iii) $h_0=h$, $h_p=h'$,

(iv) if $\delta_i=\mathcal{L}$ for $i=2,\dots,p-1$, then $\delta_{i-1},\delta_{i+1}\neq\mathcal{L}$,

(v) if $\delta_i=\partial$ for $i=1,\dots,p$, then $h_{i-1}=h_i$,

(vi) if $\delta_j=\delta_{j+1}=\ldots=\delta_{j+q}=\partial$ for some $j\in\{2,\dots,p-1\}$ and some $q\in\{0,\dots,p-1-j\}$ and $\delta_{j-1}=\mathcal{L}$, then $\delta_{j+q+1}\neq\mathcal{L}$.

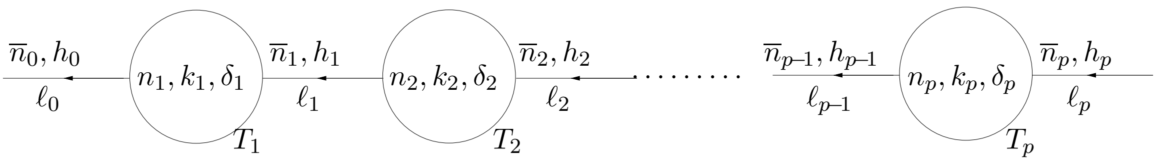

Figure A.2: A $*$-chain $C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)$: the labels satisfy the constraints listed in the text.

We call $*$-*chain* any set $C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)$ and $*$-*links* the sets $T_1,\dots,T_p$. By the discussion between (A.4) and (A.6), we can write (A.4) – and hence the sum in (2.3.8) – as

$$\sum_{C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\dots,\overline{n}_p)} \mathscr{V}_C(x), \tag{A.8}$$

where

$$\mathscr{V}_C(x) := \prod_{i=1}^{p} \mathscr{V}_{T_i}(x,\delta_i). \tag{A.9}$$

With each $*$-chain C summed over in (A.8) we associate a *depth* label $D(C)=0$; if $C=\{T_1,\dots,T_p\}$ we associate with each T_i the same depth label as C, i.e. $D(T_i)=D(C)=0$ for $i=1,\dots,p$; the introduction of such a label is due to the fact that we are performing an iterative construction and we want to keep track of the iteration step by means of the depth label.

Given a $*$-chain $C=\{T_1,\dots,T_p\}$, for all $i=1,\dots,p$ and all $\ell\in L(T_i)$ there exist $q\geq 1$ relevant self-energy clusters $T_i=T_i^{(0)}\supset T_i^{(1)}\supset\ldots\supset T_i^{(q-1)}$, with $T_i^{(j)}$ a maximal relevant self-energy cluster inside $T_i^{(j-1)}$ for all $j=0,\dots,q-1$ and $T_i^{(q-1)}$ is the minimal relevant self-energy cluster containing ℓ. Note that both q and the relevant self-energy clusters $T^{(1)},\dots,T^{(q)}$ depend on ℓ, even though we are not making explicit such a dependence. We call $\{T_i^{(j)}\}_{j=0}^{q}$ the *cloud* of ℓ and $\{T_i^{(j)}\}_{j=1}^{q}$ the *internal cloud* of ℓ. Of course if $q=0$ the internal cloud of ℓ is the empty set.

In (A.9) consider first a factor $\mathscr{V}_{T_i}(x,\delta_i)$ with $\delta_i=\mathcal{L},\mathcal{R}$. Assign a label $\mathfrak{d}_T=1$ with each maximal relevant self-energy cluster T contained inside T_i. Denote by $\mathfrak{C}_0(T_i)$ the set of maximal subchains C' contained inside T_i. For each $C_j\in\mathfrak{C}_0(T_i)$ there are labels $k_j^{(i)},h_j^{(i)},h_j^{(i)\prime},\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)}$ such that $C_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})$. Call $\mathring{T}_i$ the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets $T_i'\in\mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i,\delta_i)$ obtained from T_i by replacing, for all $j=1,\dots,|\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C_j'\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})$. Note that $\mathring{T}_i'=\mathring{T}_i$ for all $T_i'\in\mathfrak{F}(T_i)$. If we sum together all contributions in $\mathfrak{F}(T_i)$ we obtain

$$\sum_{T_i'\in\mathfrak{F}(T_i)} \mathscr{V}_{T'}(x,\delta_i) = a(\delta_i) \sum_{\substack{C_j'\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\dots,\overline{n}_{j,p_j}^{(i)})\\ 1\leq j\leq|\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x,\delta_i) \prod_{j=1}^{|\mathfrak{C}_0(T_i)|} \mathscr{V}_{C_j'}(x_{\ell_{C'},\delta_i}), \tag{A.10}$$

where

$$\mathscr{V}_{\mathring{T}_i}(x,\delta_i) := \left(\prod_{v\in N(\mathring{T}_i)} \mathcal{F}_v\right)\left(\prod_{\ell\in L(\mathring{T}_i)} \mathcal{G}_{n_\ell}(x_{\ell,\delta_i})\right) \tag{A.11}$$

and

$$a(\delta) = \begin{cases} 1, & \delta=\mathcal{L},\\ -1, & \delta=\mathcal{R}, \end{cases} \qquad x_{\ell,\delta} = \begin{cases} \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell^0, & \delta=\mathcal{L},\\ \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell, & \delta=\mathcal{R}, \end{cases} \tag{A.12}$$

for all $\ell \in L(T')$ with $T' \in \mathfrak{F}(T_i)$.

Now consider a set T_i in (A.9) with $\delta_i = \partial, \partial^2$. Write

$$\mathscr{V}_{T_i}(x,\partial) = x \sum_{\ell \in L(T_i)} \partial_x \mathcal{G}_\ell(x_\ell^0) \Bigg(\prod_{v \in N(T_i)} \mathcal{F}_v \Bigg) \Bigg(\prod_{\substack{\ell' \in L(T_i) \\ \ell' \neq \ell}} \mathcal{G}_{\ell'}(x_{\ell'}^0) \Bigg), \tag{A.13}$$

with $x_\ell^0 := \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell^0$, and $\mathscr{V}_{T_i}(x,\partial^2)$ as in the last line of (A.7), with

$$\begin{aligned} \partial_x^2 \mathscr{V}_{T_i}(\tau x) = & \sum_{\ell_1 \neq \ell_2 \in L(T)} \Big(\partial_x \mathcal{G}_{n_{\ell_1}}(x_{\ell_1}(\tau)) \Big) \Big(\partial_x \mathcal{G}_{n_{\ell_2}}(x_{\ell_2}(\tau)) \Big) \Bigg(\prod_{\ell \in L(T) \setminus \{\ell_1,\ell_2\}} \mathcal{G}_{n_\ell}(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T)} \mathcal{F}_v \Bigg) \\ & + \sum_{\ell_1 \in L(T)} \Big(\partial_x^2 \mathcal{G}_{n_{\ell_1}}(x_{\ell_1}(\tau)) \Big) \Bigg(\prod_{\ell \in L(T) \setminus \{\ell_1\}} \mathcal{G}_{n_\ell}(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T)} \mathcal{F}_v \Bigg), \end{aligned} \tag{A.14}$$

where $x_\ell(\tau) = x_\ell^0 + \tau x$ if $\ell \in \mathcal{P}_T$ and $x_\ell = x_\ell^0$ otherwise. To simplify the notations we associate with each line ℓ a label $d_\ell = 0, 1, 2$, which denotes the number of derivatives acting on the corresponding propagator, and set

$$\overline{\mathcal{G}}_\ell(x) = \partial_x^{d_\ell} \mathcal{G}_{n_\ell}(x); \tag{A.15}$$

then we rewrite

$$\mathscr{V}_{T_i}(x,\partial) = x \sum_{\ell_1 \in L(T_i)} \Bigg(\prod_{\ell \in L(T_i)} \overline{\mathcal{G}}_\ell(x_\ell^0) \Bigg) \Bigg(\prod_{v \in N(T_i)} \mathcal{F}_v \Bigg), \tag{A.16a}$$

$$\partial_x^2 \mathscr{V}_{T_i}(\tau x) = \sum_{\ell_1,\ell_2 \in L(T_i)} \Bigg(\prod_{\ell \in L(T_i)} \overline{\mathcal{G}}_\ell(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T_i)} \mathcal{F}_v \Bigg), \tag{A.16b}$$

with the constraint $\sum_{\ell \in L(T)} d_\ell = 1, 2$ for $\delta_i = \partial, \partial^2$ respectively.

Each summand in (A.16a) can be regarded as the value of a fake cluster T_i in which the propagator of a line ℓ_1 has been differentiated. Given a relevant self-energy cluster T contained inside T_i, we set $\mathfrak{d}_T = 0$ if either (1) T belongs to the cloud of ℓ_1 or (2) T is a relevant self-energy cluster with either (2a) ℓ_T non-resonant and $\ell'_T = \ell_{T'}$ for some T' belonging to the cloud of ℓ_1 or (2b) ℓ'_T is non-resonant and $\ell_T = \ell'_{T''}$ for some T'' belonging to the cloud of ℓ_1. Moreover we set $\mathfrak{d}_T = 1$ if T is any other maximal relevant self-energy cluster in T_i or in T' with $\mathfrak{d}_{T'} = 0$. If $\mathfrak{C}_0(T_i) = \{C_1, C_2, \ldots\}$, each C_j belongs to $\mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)})$ for suitable values of the labels. Call $\mathring{T}_i$ the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets $T'_i \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i, \delta_i)$ obtained from T_i by replacing, for all $j = 1, \ldots, |\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C'_j \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)})$. Note that $\mathring{T}'_i = \mathring{T}_i$ for all $T'_i \in \mathfrak{F}(T_i)$. Note also that both $\mathfrak{F}(T_i)$ and $\mathring{T}_i$ depend on the choice of the line ℓ_1, although we are not writing explicitly such a dependence. By summing together all contributions obtained by choosing first the line ℓ_1 and hence the fake clusters belonging to the corresponding family $\mathfrak{F}(T_i)$, we obtain for $\delta_i = \partial$

$$\begin{aligned} & x \sum_{\ell_1 \in L(T_i)} \sum_{T_i' \in \mathfrak{F}(T_i)} \Bigg(\prod_{\ell \in L(T_i')} \overline{\mathcal{G}}_\ell(x_\ell^0) \Bigg) \Bigg(\prod_{v \in N(T_i')} \mathcal{F}_v \Bigg) \\ & \quad = \sum_{\ell_1 \in L(T_i)} x \sum_{\substack{C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)}) \\ 1 \le j \le |\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x, \partial) \prod_{j=1}^{|\mathfrak{C}_0(T_i)|} \mathscr{V}_{C'_j}(x^0_{\ell_{C_j}}), \end{aligned} \tag{A.17}$$

with

$$\mathscr{V}_{\mathring{T}_i}(x, \partial) = \Bigg(\prod_{\ell \in L(\mathring{T}_i)} \overline{\mathcal{G}}_\ell(x_\ell^0) \Bigg) \Bigg(\prod_{v \in N(\mathring{T}_i)} \mathcal{F}_v \Bigg). \tag{A.18}$$

We deal in a similar way with (A.16b). Indeed, each summand can be regarded as the value of a fake cluster T_i in which the propagators of two lines ℓ_1, ℓ_2 (possibly coinciding) have been differentiated. As in the previous case we associate a label $\mathfrak{d}_T = 0$ with (1) the relevant self-energy clusters T of the clouds of both ℓ_1, ℓ_2 and (2) the relevant self-energy clusters T such that either (2a) ℓ_T is non-resonant and $\ell'_T = \ell_{T'}$ for some T' belonging to the cloud of ℓ_1 or ℓ_2 or (2b) ℓ'_T is non-resonant and $\ell_T = \ell'_{T'}$ for some T' belonging to the cloud of ℓ_1 or ℓ_2 or (2c) both ℓ_T, ℓ'_T are resonant and $\ell_T = \ell'_{T'}$, $\ell'_T = \ell_{T''}$ for some T', T'' belonging to the clouds of ℓ_1, ℓ_2. Moreover we associate a label $\mathfrak{d}_T = 1$ with the maximal relevant self-energy clusters T in T_i or in any T' with $\mathfrak{d}_{T'} = 0$. Then we reason as in the previous case, by defining $\mathring{T}_i$ as the set of nodes and lines obtained from T_i by removing all nodes and lines of the subchains in $\mathfrak{C}_0(T_i)$ and $\mathfrak{F}(T_i)$ the family of all possible sets $T_i' \in \mathfrak{S}^{*k_i}_{h_{i-1},h_i}(n_i, \delta_i)$ obtained from T_i by replacing, for all $j = 1, \ldots, |\mathfrak{C}_0(T_i)|$, each subchain C_j with any subchain $C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)})$. Then for $\delta_i = \partial^2$ we obtain

$$\begin{aligned} & x^2 \int_0^1 \mathrm{d}\tau (1-\tau) \sum_{\ell_1 \ell_2 \in L(T_i)} \sum_{T_i' \in \mathfrak{F}(T_i)} \Bigg(\prod_{\ell \in L(T_i')} \overline{\mathcal{G}}_\ell(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(T_i')} \mathcal{F}_v \Bigg) \\ & \quad = \sum_{\ell_1 \ell_2 \in L(T_i)} x^2 \int_0^1 \mathrm{d}\tau (1-\tau) \sum_{\substack{C_j' \in \mathfrak{C}_1(k_j^{(i)}; h_j^{(i)}, h_j^{(i)\prime}; \overline{n}_{j,0}^{(i)}, \ldots, \overline{n}_{j,p_j}^{(i)}) \\ 1 \le j \le |\mathfrak{C}_0(T_i)|}} \mathscr{V}_{\mathring{T}_i}(x, \partial^2) \prod_{j=1}^{|\mathfrak{C}_0(T_i)|} \mathscr{V}_{C'_j}(x_{\ell_{C_j}}(\tau)), \end{aligned} \tag{A.19}$$

with

$$\mathscr{V}_{\mathring{T}_i}(x, \partial^2) = \Bigg(\prod_{\ell \in L(\mathring{T}_i)} \overline{\mathcal{G}}_\ell(x_\ell(\tau)) \Bigg) \Bigg(\prod_{v \in N(\mathring{T}_i)} \mathcal{F}_v \Bigg), \tag{A.20}$$

where $x_\ell(\tau) = x_\ell^0 + \tau x$ if $\ell \in \mathcal{P}_{T_i}$ and $x_\ell(\tau) = x_\ell^0$ otherwise.

By summarising we obtained

$$\begin{aligned}
\sum_{C\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \mathscr{V}_C(x) &= \sum_{\{T_1,\ldots,T_p\}\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \prod_{i=1}^{p} \mathscr{V}_{T_i}(x,\delta_i) \\
&= \sum_{\{T_1,\ldots,T_p\}\in\mathfrak{C}^*(k;h,h';\overline{n}_0,\ldots,\overline{n}_p)} \Bigg(\prod_{\substack{i=1\\ \delta_i=\partial}}^{p} \sum_{\ell_{i,1}\in L(T_i)}\Bigg)\Bigg(\prod_{\substack{i=1\\ \delta_i=\partial^2}}^{p} \sum_{\ell_{i,1},\ell_{i,2}\in L(T_i)}\Bigg)\times \\
&\quad\times\Bigg(\prod_{i=1}^{p} a(x_{\ell_{T_i}},\delta_i)\Bigg)\Bigg(\prod_{\substack{i=1\\ \delta_i=\partial^2}}^{p}\int_0^1 \mathrm{d}\tau_i\,(1-\tau_i)\Bigg)\times \\
&\quad\times\prod_{i=1}^{p}\frac{1}{|\mathfrak{F}(T_i)|}\mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}},\delta_i)\sum_{C_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\ldots,\overline{n}_{j,p_j}^{(i)})}\mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))),
\end{aligned} \tag{A.21}$$

with $x_{\ell_{T_i}} = x$ by construction and

$$\mathscr{V}_{\mathring{T}_i}(x,\delta_i) = \Bigg(\prod_{\ell\in L(\mathring{T}_i)} \overline{\mathcal{G}}_\ell(x_\ell(\tau(\delta_i)))\Bigg)\Bigg(\prod_{v\in N(\mathring{T}_i)}\mathcal{F}_v\Bigg), \tag{A.22}$$

where we have defined $x_\ell(\tau) := x_\ell^0 + \tau x$ and

$$\tau_i(\delta) := \begin{cases} 0, & \delta=\mathcal{L},\\ 1, & \delta=\mathcal{R},\\ 0, & \delta=\partial,\\ \tau_i, & \delta=\partial^2,\end{cases} \qquad a(x,\delta) := \begin{cases} 1, & \delta=\mathcal{L},\\ -1, & \delta=\mathcal{R},\\ x, & \delta=\partial,\\ x^2, & \delta=\partial^2.\end{cases} \tag{A.23}$$

The factors $1/|\mathfrak{F}(T_i)|$ have been introduced in (A.21) to avoid overcountings. The last sums in (A.21) have the same form as the sum (A.1), so that we can iterate the procedure, by writing

$$\sum_{C_j\in\mathfrak{C}_1(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\ldots,\overline{n}_{j,p_j}^{(i)})}\mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))) = \sum_{C_j\in\mathfrak{C}^*(k_j^{(i)};h_j^{(i)},h_j^{(i)\prime};\overline{n}_{j,0}^{(i)},\ldots,\overline{n}_{j,p_j}^{(i)})}\mathscr{V}_{C_j}(x_{\ell_{C_j}}(\tau_i(\delta_i))) \tag{A.24}$$

for $i = 1,\ldots,p$ and $j = 1,\ldots,|\mathfrak{C}_0(T_i)|$, with $\mathscr{V}_T(x)$ defined as in (A.9). Now we associate with each $*$-chain C_j a depth label $D(C_j) = 1$, and if $C_j = \{T_1^{(j)},\ldots,T_{p_j}^{(j)}\}$ we associate with each $T_i^{(j)}$ the same depth label as C_j, i.e. $D(T_i^{(j)}) = D(C_j)$. More generally, by pursuing the construction, in order to keep track of the iteration step, we associate a depth label $D(C) = d$ with each $*$-chain C which appears at the d-th step and the same depth label $D(T) = d$ with each $*$-link T of C. Since at each step the order of the chains is decreased, sooner or later the procedure stops.

To make the notation more uniform, for any $*$-link T such that $\mathfrak{C}_0(T)=\emptyset$ we write $\mathring{T}=T$. Given any $*$-link T with $\mathring{T}\neq T$, if two lines $\ell,\ell'\in L(\mathring{T})$ are such that there exists a maximal link T' in T with $\ell_{T'}=\ell$ and $\ell'_{T'}=\ell'$, we say that the two lines are *consecutive* and we write $\ell'\prec\ell$.

At the end of the procedure described above we obtain a sum of terms of the form

$$\left(\prod_{\substack{i\in I\\ \delta_{T_i}=\partial^2}}\int_0^1 \mathrm{d}\tau_i(1-\tau_i)\right)\prod_{i\in I} a(x_{\ell_{T_i}}(\underline{\tau}),\delta_i)\,\mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}}(\underline{\tau}),\delta_i), \tag{A.25}$$

where the following notations have been used:

(i) $I=\{1,2,\dots,N\}$ for some $N\in\mathbb{N}$;

(ii) $\{T_i\}_{i\in I}$ are $*$-links such that (1) for all T_i with $D(T_i)=d$, $d\geq 1$, there exists a $*$-link T_j, $j\in I$, with $D(T_j)=d-1$ and two consecutive lines $\ell'\prec\ell\in L(\mathring{T}_j)$ with the same labels as ℓ'_{T_i},ℓ_{T_i}, respectively, and conversely (2) for all T_j with $D(T_j)=d$, $d\geq 0$, and all pairs of consecutive lines $\ell'\prec\ell\in L(\mathring{T}_j)$, there is a $*$-link T_i, $i\in I$, with $D(T_i)=d+1$, such that ℓ'_{T_i},ℓ_{T_i} have the same labels as ℓ',ℓ, respectively (roughly we can imagine to 'fill the holes' of all $\mathring{T}$ such that $D(T)=0$ with all $\mathring{T}'$ with $D(T')=1$, then 'fill the remaining holes' with all $\mathring{T}''$ with $D(T'')=2$ and so on up to the $*$-links of maximal depth which have no 'holes');

(iii) $\underline{\tau}=(\tau_1(\delta_1),\dots,\tau_N(\delta_N))$, with $\tau_i(\delta_i)$ defined as in (A.23), and for all $\ell\in L(\mathring{T}_1)\cup\ldots\cup L(\mathring{T}_N)$ we have set

$$x_\ell(\underline{\tau}):=x^0_\ell+\tau_{i_d}(\delta_{T_{i_d}})\left(x^0_{\ell_d}+\tau_{i_{d-1}}(\delta_{T_{i_{d-1}}})\left(x^0_{\ell_{d-1}}+\tau_{i_{d-2}}(\delta_{T_{i_{d-2}}})(\ldots+\tau_{i_0}(\delta_{T_{i_0}})x)\right)\right), \tag{A.26}$$

where T_{i_d} is the minimal $*$-link (with depth d) containing ℓ and ℓ_j is the line in the $*$-link $T_{i_{j-1}}$ with depth $j-1$ corresponding to the entering line of T_{i_j}, for $j=1,\dots,d$.

Recall that each propagator is differentiated at most twice and note that, for T such that $\delta_T=\partial$, there is a line $\ell\in L(\mathring{T})$ with $d_\ell=1$. Then, when bounding the product of propagators, instead of

$$\prod_{\ell\in L(T)}\frac{\gamma_0}{2}\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-1} \tag{A.27}$$

for some positive constant γ_0, we have (A.27) times an extra factor

$$c_1\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-1}|x_{\ell_T}|, \tag{A.28}$$

for suitable constant c_1. Analogously, for T such that $\delta_T=\partial^2$, either there are two lines ℓ_1,ℓ_2 with $d_{\ell_1}=d_{\ell_2}=1$ or one line ℓ_1 with $d_{\ell_1}=2$; in both cases we obtain (A.27) times an extra factor

$$c_1\alpha_{m_{n_{\ell_1}}}(\boldsymbol{\omega})^{-1}\alpha_{m_{n_{\ell_2}}}(\boldsymbol{\omega})^{-1}|x_{\ell_T}|^2. \tag{A.29}$$

On the other hand we have no gain factor coming from the $*$-links with label $\delta = \mathcal{L}, \mathcal{R}$, or from the relevant self-energy clusters T' with $\mathfrak{d}_{T'} = 0$. In order to deal with such lines we need some preliminary results.

Given a $*$-link T, define $\mathring{T}$ as before and denote by $L_R(\mathring{T})$ the set of resonant lines in $\mathring{T}$. Set
(i) $L_{NR}(\mathring{T}) := L(\mathring{T}) \setminus L_R(\mathring{T})$,
(ii) $L_D(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, d_\ell > 0\}$,
(iii) $L_0^1(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, \ell = \ell_{T'}$ for some relevant self-energy cluster $T' \subset T$ with $\mathfrak{d}_{T'} = 0\}$,
(iv) $L_0^2(\mathring{T}) := \{\ell \in L_R(\mathring{T}) \,:\, \ell = \ell'_{T'}$ for some relevant self-energy cluster $T' \subset T$ with $\delta_{T'} = 0\}$;
(v) $L_0(\mathring{T}) := L_0^1(\mathring{T}) \cup L_0^2(\mathring{T})$;
(vi) $L_R^*(\mathring{T}) = L_D(\mathring{T}) \cup L_0(\mathring{T})$.
Of course $L_D(\mathring{T}) = L_0(\mathring{T}) = \emptyset$ if $\delta_T = \mathcal{L}, \mathcal{R}$. Given a $*$-link T we say that $\ell \in L(\mathring{T})$ is *maximal in* T' if T' is the minimal relevant self-energy cluster contained in $\mathring{T}$ (with $\mathfrak{d}_{T'} = 0$) such that $\ell \in L(T')$. If there is no such relevant self-energy cluster we say that ℓ is *maximal in* T. Given a $*$-link T with $\delta_T = \partial, \partial^2$, we denote by $L_M(T)$ the set of lines which are maximal in T; for any relevant self-energy cluster T' with $\mathfrak{d}_{T'} = 0$ we denote by $L_M(T')$ the set of lines which are maximal in T'.

Lemma A.1. *Let T be any $*$-link with $\delta_T = \partial, \partial^2$. For all relevant self-energy clusters T' contained in $\mathring{T}$ one has $q_0(T') := |L_M(T') \cap L_0(\mathring{T})| \leq 4$. Moreover*

$$q_1(T') := \sum_{\ell \in L_M(T')} d_\ell \leq \min\{4 - q_0(T'), 2\}.$$

The same hold if we replace T' with the $*$-link T.

Proof. If $\delta_T = \partial$ there is at most one line $\ell \in L(\mathring{T})$ such that $d_\ell = 1$. Let $\{T_j\}_{j=0}^m$ be the cloud of ℓ, where we denoted $T_0 = T$, so that $L_0(\mathring{T}) \subseteq \{\ell_{T_1}, \ell'_{T_1}, \ldots, \ell_{T_m}, \ell'_{T_m}\}$. Then T_m is the minimal relevant self-energy cluster containing ℓ and $L_M(T_j) \cap L_0(\mathring{T}) \subseteq \{\ell_{T_{j+1}}, \ell'_{T_{j+1}}\}$, $j = 0, \ldots, m-1$. Hence $q_0(T_j) \leq 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, m-1$, while $q_0(T_m) = 0$ and $q_1(T_m) = 1$. If $\delta_T = \partial^2$ and there is one line $\ell \in L(\mathring{T})$ with $d_\ell = 2$ one can reason as in the previous case, denoting by $\{T_j\}_{j=0}^m$ the cloud of ℓ and hence obtaining $q_0(T_j) \leq 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, m-1$, while $q_0(T_m) = 0$ and $q_1(T_m) = 2$. If there are two lines $\ell'_1, \ell'_2 \in L(\mathring{T})$ with $d_{\ell'_1} = d_{\ell'_2} = 1$ one proceeds as follows. Call $\{T_j^{(i)}\}_{j=0}^{m_i}$ the cloud of ℓ'_i, $i = 1, 2$, with $T_0^{(1)} = T_0^{(2)} = T_0 = T$, and set

$$r := \max\{j \geq 0 \,:\, T_j^{(1)} = T_j^{(2)} =: T_j\}.$$

If $r = m_1 = m_2$ then again $q_0(T_j) \leq 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, while $q_0(T_r) = 0$ and $q_1(T_r) = 2$. If $r = m_1 < m_2$ then $q_0(T_j) \leq 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, $q_0(T_r) \leq 2$ and

$q_1(T_r) = 1$, $q_0(T_j^{(2)}) \leq 2$ and $q_1(T_j^{(2)}) = 0$ for $j = r+1, \ldots, m_2$, and $q_0(T_{m_2}) = 0$ and $q_1(T_{m_2}) = 1$. Finally if $r < \min\{m_1, m_2\}$ then $L_M(T_r) \cap L_0(\mathring{T}) \subseteq \{\ell_{T_{r+1}^{(1)}}, \ell'_{T_{r+1}^{(1)}}, \ell_{T_{r+1}^{(2)}}, \ell'_{T_{r+1}^{(2)}}\}$, so that $q_0(T_j) \leq 2$ and $q_1(T_j) = 0$ for $j = 0, \ldots, r-1$, $q_0(T_r) \leq 4$ and $q_1(T_r) = 0$, while $q_0(T_j^{(i)}) \leq 2$ and $q_1(T_j^{(i)}) = 0$ for $j = r+1, \ldots, m_i$, and $q_0(T_{m_i}) = 0$ and $q_1(T_{m_i}) = 1$, $i = 1, 2$. ■

Define the multiplicity (function) of a non-injective map as the cardinality of its pre-image sets [30, 61].

Lemma A.2. *Let T be a $*$-link with $\delta_T = \partial, \partial^2$. There exists an application $\Lambda : L_R^*(\mathring{T}) \to L_{NR}(\mathring{T})$ with multiplicity at most 2 such that $\zeta_\ell = \zeta_{\Lambda(\ell)}$.*

Proof. By Lemma A.1 there are at most four lines $\ell_1, \ell_2, \ell_3, \ell_4 \in L_R^*(\mathring{T})$ such that, if T_i' denote the minimal relevant self-energy cluster containing ℓ_i, then $T_1' = \ldots = T_4'$. Moreover by Remark 2.3.6 if ℓ is a resonant line, then the minimal relevant self-energy cluster containing ℓ contains also two non-resonant lines ℓ_1', ℓ_2' with the same minimum scale as ℓ. Therefore the assertion follows. ■

Now consider a $*$-link T contributing to (A.25) with largest depth, say D. Since T does not contain any resonant line, by (2.2.9b) and Remark 2.3.1 we have

$$|\mathscr{V}_{T_i}(x_{\ell_{T_i}}(\underline{\tau}), \mathcal{R})| \leq c_2^{k(T_i)} \mathrm{e}^{-\xi K(T_i)/2} \leq c_3 c_2^{k(T_i)} |x_{\ell_{T_i}}(\underline{\tau})|^2, \tag{A.30}$$

for some positive constants c_2 and c_3; we have also used that $K(T_i) \geq 2^{m_{n_{\ell_{T_i}}}-1}$ for $\delta_{T_i} = \mathcal{R}$ and $|x_{\ell_{T_i}}(\underline{\tau})| \geq \alpha_{m_{n_{\ell_{T_i}}}}(\boldsymbol{\omega})$ if $\Psi_{n_{\ell_{T_i}}}(x_{\ell_{T_i}}(\underline{\tau})) \neq 0$. Therefore we can bound

$$|a(x_{\ell_T}(\underline{\tau}), \delta_T)||\mathscr{V}_T(x_{\ell_T}(\underline{\tau}), \delta_T)| \leq \begin{cases} c_4^{k(T)}, & \delta = \mathcal{L}, \\ c_4^{k(T)} |x_{\ell_T}(\underline{\tau})|, & \delta_T = \partial, \\ c_4^{k(T)} |x_{\ell_T}(\underline{\tau})|^2, & \delta = \partial^2, \mathcal{R}, \end{cases} \tag{A.31}$$

for some constant c_4. Now consider a $*$-link T contributing to (A.25) with depth $D-1$. For each resonant line $\ell \in L_R(\mathring{T})$ denote by T' the minimal relevant self-energy cluster containing ℓ (set $T' = T$ if there is no minimal relevant self-energy cluster containing ℓ). For all resonant lines $\ell' \in L_M(T') \setminus L_0(\mathring{T})$, there is a subchain $C \in \mathfrak{C}_0(T)$ for which ℓ' is an internal chain-line, such that C is uniquely associated (see the comments after (A.25)) with a $*$-chain $C^* = \{T_{i_1}, \ldots, T_{i_{p(C)}}\}$ with depth D contributing to (A.25), whose value can be bounded by

$$\left| \prod_{j=1}^{p(C)} a(x_{\ell_{T_{j_i}}}(\underline{\tau}), \delta_i)\, \mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_{j_i}}}(\underline{\tau}), \delta_{j_i}) \right| \leq |x_{\ell_{T_{j_i}}}(\underline{\tau})|^{p(C)-1} c_5^{k(C)}, \tag{A.32}$$

for some $c_5 \geq 0$, and this can be obtained as follows. If $\delta_{T_{i_j}} \neq \mathcal{L}$ for all $j = 1, \ldots, p(C)$ or there is only one $j = 1, \ldots, p(C)$ such that $\delta_{T_{i_j}} = \mathcal{L}$, then (A.32) trivially follows from (A.31). Otherwise if $1 \leq j < j' \leq p(C)$ are such that $\delta_{T_{i_j}} = \delta_{T_{i_{j'}}} = \mathcal{L}$ then there is at least one $j'' = j+1, \ldots, j'-1$ such that $\delta_{T_{i_{j''}}} = \partial^2, \mathcal{R}$; recall the constraints (i)–(vi) after (A.7). Then (A.32) follows.

Moreover by Lemma A.1, if there are $R_n^{\bullet}(T')$ resonant lines in $L_M(T')$ with minimum scale n, we have an overall gain $\sim \alpha_{m_n}(\boldsymbol{\omega})^{R_n^{\bullet}(T')-q_0(T')}$ and on the other hand the product of the propagators of such resonant lines can be bounded proportionally to $\alpha_{m_n}(\boldsymbol{\omega})^{-(R_n^{\bullet}(T')+q_1(T'))}$, with $q_0(T')+q_1(T') \leq 4$. Therefore, by Lemma A.2, if we replace the bound for the propagators of each non-resonant line $\ell \in L_M(T')$, $\zeta_\ell = n$, with $c_6 \alpha_{m_n}(\boldsymbol{\omega})^{-3}$ for some positive constant c_6, we have exactly a gain factor which is enough to compensate each propagator of the resonant lines with minimum scale n in $L_M(T')$. But since we can reason in the same way for all n and all resonant lines in T, if we replace the bound for the propagators of each $\ell \in L_{NR}(\mathring{T})$ with $c_6 \alpha_{m_{n_\ell}}(\boldsymbol{\omega})^{-3}$ we obtain a gain proportional to $\alpha_{m_{n_{\ell'}}}(\boldsymbol{\omega})^{1+d_{\ell'}}$ for any $\ell' \in L_R(\mathring{T})$, and hence we can use again Lemma 2.2.5 in order to obtain the bound (A.31) also for the $*$-links with depth $D-1$.

Then we pass to the $*$-links with depth $D-2$ and reason in the same way as above and so on. When we arrive to the $*$-links with depth 0 we only have to recall that they are associated with the maximal relevant self-energy clusters T of θ, which all have label $\mathfrak{d}_T = 1$. Hence we can bound (A.25) as

$$\left| \prod_{i=1}^{N} a(x_{\ell_{T_i}}(\underline{\tau}), \delta_i)\, \mathscr{V}_{\mathring{T}_i}(x_{\ell_{T_i}}(\underline{\tau}), \delta_i) \right| \leq c_7^{k(T_1)+\ldots+k(T_N)} |x|^{p-1} = c_7^{k} |x|^{p-1},$$

for some positive constant c_7. We have still to sum over all the possible contributions of the form (A.25). To take into account the scale labels n_ℓ, $\ell \in L(\mathring{T}_1) \cup \ldots \cup L(\mathring{T}_N)$ simply recall that for each momentum $\boldsymbol{\nu}_\ell$ only 2 scale labels are allowed; see Remark 2.2.2. To sum over the mode labels $\boldsymbol{\nu}_v$, $v \in N(\mathring{T}_1) \cup \ldots \cup N(\mathring{T}_N)$ we can neglect the constraints and use a factor $\mathrm{e}^{-(\xi/4)|\boldsymbol{\nu}_v|}$ for each v. Moreover the component labels h_ℓ are 2. Finally the sum over all possible unlabelled chains with order k is bounded by a constant to the power k. Therefore the assertion follows.

B. Proof of Lemma 3.3.8

Throughout this appendix, for the sake of simplicity, we shall omit the adjective "renormalised" referred to trees, self-energy clusters, left-fake clusters and right-fake clusters.

We shall prove explicitly only the bound

$$|\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0)-\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)|\le|\varepsilon|\,A_1e^{-A_2 2^{m_{p+1}}},\tag{B.1}$$

as the others relations in (3.3.8) can be proved exactly in the same way.

We want to compute $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$, with $\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$ given by the first line of (3.2.15). We start by considering trees $\theta\in\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ such that

$$\max_{\ell\in\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}}\{n\in\mathbb{Z}_+:\Psi_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\ne0\}\le p,\tag{B.2}$$

and shall see later how to deal with trees in $\Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ for which the condition (B.2) is not satisfied (see case **7** at the end).

First of all, for any tree θ set

$$\partial_v\mathscr{V}(\theta;\varepsilon,\beta_0,B_0):=\partial_{\beta_0}\mathcal{F}_v\Bigg(\prod_{w\in N(\theta)\setminus\{v\}}\mathcal{F}_w\Bigg)\Bigg(\prod_{\ell\in L(\theta)}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\beta_0,B_0)\Bigg),\tag{B.3}$$

and

$$\begin{aligned}\partial_\ell\mathscr{V}(\theta;\varepsilon,\beta_0,B_0)&:=\partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)\Bigg(\prod_{v\in N(\theta)}\mathcal{F}_v\Bigg)\Bigg(\prod_{\lambda\in L(\theta)\setminus\{\ell\}}\mathcal{G}^{[n_\lambda]}_{e_\lambda,u_\lambda}(x_\lambda;\varepsilon,\beta_0,B_0)\Bigg)\\&=\mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0,B_0)\,\partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{,e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)\,\mathcal{B}_\ell(\theta;\varepsilon,\beta_0,B_0),\end{aligned}\tag{B.4}$$

where $x_\ell:=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$, $\partial_{\beta_0}\mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(x_\ell;\varepsilon,\beta_0,B_0)$ is written according to Remark 3.2.1 and

$$\mathcal{A}_\ell(\theta,x_\ell;\varepsilon,\beta_0,B_0):=\Bigg(\prod_{\substack{v\in N(\theta)\\ v\not\prec\ell}}\mathcal{F}_v\Bigg)\Bigg(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\not\preceq\ell}}\mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(x_{\ell'};\varepsilon,\beta_0,B_0)\Bigg),\tag{B.5a}$$

$$\mathcal{B}_\ell(\theta;\varepsilon,\beta_0,B_0):=\Bigg(\prod_{\substack{v\in N(\theta)\\ v\prec\ell}}\mathcal{F}_v\Bigg)\Bigg(\prod_{\substack{\ell'\in L(\theta)\\ \ell'\prec\ell}}\mathcal{G}^{[n_{\ell'}]}_{e_{\ell'},u_{\ell'}}(x_{\ell'};\varepsilon,\beta_0,B_0)\Bigg).\tag{B.5b}$$

Let us define in the analogous way $\partial_v \mathscr{V}_T(x;\varepsilon,\beta_0,B_0)$ and $\partial_\ell \mathscr{V}_T(x;\varepsilon,\beta_0 B_0)$ for any self-energy cluster T, and let us write

$$\partial_{\beta_0} \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) = \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) + \partial_L \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \tag{B.6}$$

where

$$\partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) := \sum_{v\in N(\theta)} \partial_v \mathscr{V}(\theta;\varepsilon,\beta_0,B_0), \tag{B.7}$$

and

$$\partial_L \mathscr{V}(\theta;\varepsilon,\beta_0,B_0) := \sum_{\ell\in L(\theta)} \partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0,B_0). \tag{B.8}$$

Let us also write

$$\partial_{\beta_0} \mathscr{V}_T(x;\varepsilon,\beta_0,B_0) = \partial_N \mathscr{V}_T(x;\varepsilon,\beta_0,B_0) + \partial_L \mathscr{V}_T(x;\varepsilon,\beta_0,B_0), \tag{B.9}$$

for any $T \in \mathfrak{R}_{n,u,e}$, $n \geq 0$ and $u,e \in \{\beta, B\}$, where the derivatives ∂_N and ∂_L are defined analogously with the previous cases (B.7) and (B.8), with $N(T)$ and $L(T)$ replacing $N(\theta)$ and $L(\theta)$, respectively, so that we can split

$$\begin{aligned}
\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0) &= \partial_N\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0) + \partial_L\Phi^{\mathcal{R},p}_{\mathbf{0}}(x;\varepsilon,\beta_0,B_0),\\
\partial_{\beta_0}M^{[n]}(x;\varepsilon,\beta_0,B_0) &= \partial_N M^{[n]}(x;\varepsilon,\beta_0,B_0) + \partial_L M^{[n]}(x;\varepsilon,\beta_0,B_0),\\
\partial_{\beta_0}\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0) &= \partial_N\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0) + \partial_L\mathcal{M}^{[n]}(x;\varepsilon,\beta_0,B_0),
\end{aligned} \tag{B.10}$$

again with obvious meaning of the symbols.

Remark B.1. We can interpret the derivative ∂_v as all the possible ways to attach an extra line ℓ (with $\boldsymbol{\nu}_\ell = \mathbf{0}$ and $u_\ell = \beta$) to the node v, so that

$$\sum_{k\geq 0} \varepsilon^{k+1} \sum_{\theta\in\Theta^{\mathcal{R},p}_{k+1,\mathbf{0}}} \partial_N \mathscr{V}(\theta;\varepsilon,\beta_0,B_0),$$

produces contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0)$.

In order to compute $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon,\beta_0,B_0)$, we have to study the derivative (B.6) for any $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$. The terms (B.7) produce immediately contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0;\varepsilon,\beta_0,B_0)$ by Remark B.1. Thus, we have to study the derivatives $\partial_\ell \mathscr{V}(\theta;\varepsilon,\beta_0,B_0)$ appearing in the sum (B.8). From now on, we shall not write any longer explicitly the dependence on ε,β_0 and B_0, in order not to overwhelm the notation.

For any $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ satisfying the condition (B.2) and for any line $\ell \in L(\theta)$, either there is only one scale n such that $\Psi_n(x_\ell) \neq 0$ (and in that case $\Psi_n(x_\ell) = 1$ and $\Psi_{n'}(x_\ell) = 0$ for all $n' \neq n$) or there exists only one $0 \le n \le p-1$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$. To help following the argument below, we divide the discussion into several steps (cases **1** to **7**), marking the end of each step with a white box (□).

1. If $\Psi_n(x_\ell) = 1$ one has

$$\begin{aligned} \partial_\ell \mathscr{V}(\theta) &= \mathcal{A}_\ell(\theta, x_\ell) \left(\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell) \left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell) \right)^{-1} \right)_{e_\ell, u_\ell} \mathcal{B}_\ell(\theta) \\ &= \mathcal{A}_\ell(\theta, x_\ell) \left(\mathcal{G}^{[n]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\mathcal{G}^{[n]}(x_\ell) \right)_{e_\ell, u_\ell} \mathcal{B}_\ell(\theta), \end{aligned} \tag{B.11}$$

with $\mathcal{A}_\ell(\theta, x_\ell)$ and $\mathcal{B}_\ell(\theta)$ defined in (B.5). □

Remark B.2. Note that if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (B.11), the term with $\partial_N \mathcal{M}^{[n-1]}(x_\ell)$ is a contribution to $\mathcal{M}^{[p-1]}_{\beta,\beta}(0)$ and hence to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$.

If there is only one $0 \le n \le p-1$ such that $\Psi_n(x_\ell)\Psi_{n+1}(x_\ell) \neq 0$, then $\Psi_n(x_\ell) + \Psi_{n+1}(x_\ell) = 1$ and $\chi_q(x_\ell) = 1$ for all $q = -1, \ldots, n-1$, so that $\psi_{n+1}(x_\ell) = 1$ and hence $\Psi_{n+1}(x_\ell) = \chi_n(x_\ell)$. Moreover it can happen only (see Remark 2.2.2) $n_\ell = n$ or $n_\ell = n+1$.

2. Consider first the case $n_\ell = n+1$. One has

$$\partial_\ell \mathscr{V}(\theta) = \mathcal{A}_\ell(\theta, x_\ell) \left(\mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n]}(x_\ell)((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell))^{-1} \right)_{e_\ell, u_\ell} \mathcal{B}_\ell(\theta), \tag{B.12}$$

with

$$\begin{aligned} &\mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n]}(x_\ell)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\ &\quad = \mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_\ell)\big(\Psi_n(x_\ell) + \Psi_{n+1}(x_\ell)\big)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\ &\qquad + \mathcal{G}^{[n+1]}(x_\ell)\partial_{\beta_0}M^{[n]}(x_\ell)\,\chi_n(x_\ell)\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1} \\ &\quad = \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n} \partial_{\beta_0}M^{[q]}(x_\ell)\right)\mathcal{G}^{[n+1]}(x_\ell) + \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n-1} \partial_{\beta_0}M^{[q]}(x_\ell)\right)\mathcal{G}^{[n]}(x_\ell) \\ &\qquad + \mathcal{G}^{[n+1]}(x_\ell)\left(\sum_{q=-1}^{n-1} \partial_{\beta_0}M^{[q]}(x_\ell)\right)\mathcal{G}^{[n]}(x_\ell)M^{[n]}(x_\ell)\mathcal{G}^{[n+1]}(x_\ell), \end{aligned} \tag{B.13}$$

where we have used that $\chi_n(x_\ell) = \Psi_{n+1}(x_\ell)$ and

$$\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1}\big(\mathbb{1} + M^{[n]}(x_\ell)\Psi_{n+1}(x_\ell)\big((ix_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1}\big)^{-1} = \big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell)\big)^{-1}.$$

We represent graphically the three contributions in (B.13) as in Figure B.1: we represent the derivative ∂_{β_0} as an arrow pointing toward the graphical representation of the differentiated quantity. □

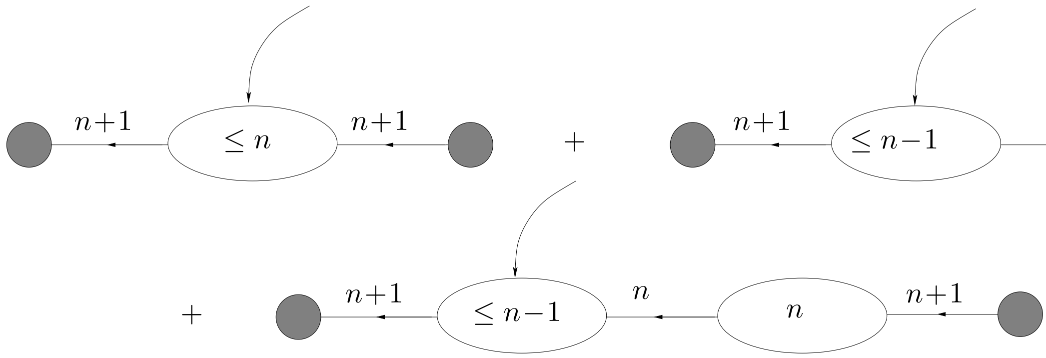

Figure B.1: Graphical representation of the derivative $\partial_\ell \mathscr{V}(\theta)$ according to (B.13).

Remark B.3. Note that the $M^{[n]}(x_\ell)$ appearing in the latter line of (B.13) has to be interpreted (see Remark 3.2.3) as the matrix with components

$$\sum_{T \in \mathfrak{L}\mathfrak{F}_{n,u,e}} \varepsilon^{k(T)} \mathscr{V}_T(x_\ell).$$

Note also that, again, if we split $\partial_{\beta_0} = \partial_N + \partial_L$ in (B.13), all the terms with $\partial_N M^{[q]}(x_\ell)$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$.

Now consider the case $n_\ell = n$. We distinguish among several cases (see Remark B.4 below for the meaning of "removal" and "insertion" of left-fake clusters):

(a) ℓ *does not* exit any left-fake cluster and one *can* insert a left-fake cluster, together its entering line, between ℓ and the node ℓ exists without creating any self-energy cluster (case **3** below);

(b) ℓ *does not* exit any left-fake cluster and one *cannot* insert any left-fake cluster between ℓ and the node ℓ exists because this way a self-energy cluster would appear (case **4** below);

(c) ℓ *does* exit a left-fake cluster and one *can* remove the left-fake cluster, together its entering line, without creating a self-energy cluster (case **3** below);

(d) ℓ *does* exit a left-fake cluster and one *cannot* remove the left-fake cluster because a self-energy cluster would be produced (case **5** below).

Remark B.4. Here and henceforth, if S is a subgraph with only one entering line $\ell'_S = \ell_v$ and one exiting line ℓ_S, by saying that we "remove" S together with ℓ'_S, we mean that we change u_{ℓ_S} into h_v and we also reattach the line ℓ_S to the node v (so that ℓ_S becomes the line exiting v). Analogously, whenever we "insert" a subgraph S with only one entering line ℓ' between a line ℓ and the node v which ℓ exits, we mean that we set $u_{\ell'} = h_v$ and change u_ℓ into h_w if $w \in N(S)$ is the node to which we reattach ℓ (and ℓ becomes the line ℓ_w exiting S).

3. If ℓ is not the exiting line of a left-fake cluster, set $\bar{\theta} = \theta$; otherwise, if ℓ is the exiting line of a left-fake cluster T, define – if possible – $\bar{\theta}$ as the tree obtained from θ by removing T and ℓ'_T In both cases, define – if possible – $\tau_1(\bar{\theta}, \ell)$ as the set constituted by all the renormalised trees θ' obtained from $\bar{\theta}$ by inserting a left-fake cluster, together with its entering line, between ℓ and the node v which ℓ exits; see Figure B.2.

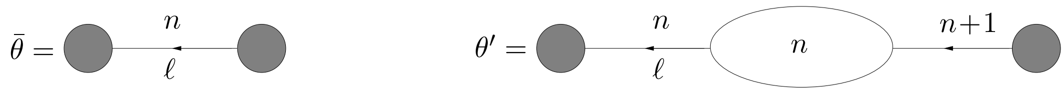

Figure B.2: The renormalised tree $\bar{\theta}$ and the renormalised trees θ' of the set $\tau_1(\bar{\theta}, \ell)$ associated with $\bar{\theta}$.

Remark B.5. The construction of the set $\tau_1(\bar{\theta}, \ell)$ could be impossible if the removal or the insertion of a left-fake cluster T, together with its entering line ℓ'_T, would produce a self-energy cluster. We shall see later (see cases **4** and **5** below) how to deal with these cases.

Then one has

$$\partial_\ell \mathscr{V}(\bar{\theta}) + \partial_\ell \sum_{\theta' \in \tau_1(\bar{\theta}, \ell)} \mathscr{V}(\theta') = \mathcal{A}_\ell(\bar{\theta}, x_\ell) \left(\partial_{\beta_0} \mathcal{G}^{[n]}(x_\ell) \left(\mathbb{1} + M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \right) \right)_{e_\ell, u_\ell} \mathcal{B}_\ell(\bar{\theta}), \tag{B.14}$$

with

$$\begin{aligned}
&\partial_{\beta_0} \mathcal{G}^{[n]}(x_\ell) \left(\mathbb{1} + M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \right) \\
&\quad = \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \Psi_{n+1}(x_\ell) \left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell) \right)^{-1} \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \Psi_{n+1}(x_\ell) \left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell) \right)^{-1} M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\quad = \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\qquad - \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \chi_n(x_\ell) \left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell) \right)^{-1} M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \Psi_{n+1}(x_\ell) \left((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell) \right)^{-1} M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\quad = \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \\
&\qquad + \mathcal{G}^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{M}^{[n-1]}(x_\ell) \mathcal{G}^{[n]}(x_\ell) M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell),
\end{aligned} \tag{B.15}$$

where we have used that

$$\big(\mathbb{1} - \Psi_{n+1}(x_\ell)\big((ix_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell)\big)^{-1}M^{[n]}(x_\ell)\big)^{-1}\big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n-1]}(x_\ell)\big)^{-1} = \big((\mathrm{i}x_\ell)\mathbb{1} - \mathcal{M}^{[n]}(x_\ell)\big)^{-1}.$$

Also in this case, if we split $\partial_{\beta_0} = \partial_N + \partial_L$, all the terms with $\partial_N\mathcal{M}^{[n-1]}$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$ – see Remark B.2. Again, we can represent graphically the three contributions obtained inserting (B.15) in (B.14); see Figure B.3. □

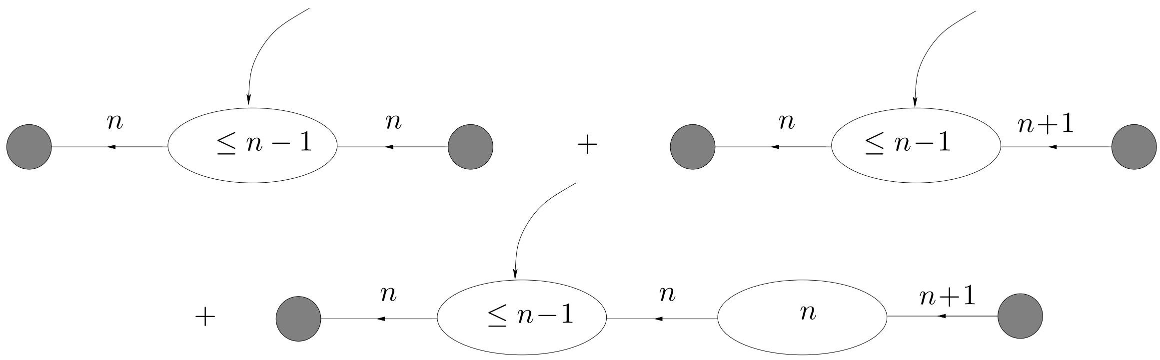

Figure B.3: Graphical representation of the three contributions in the last two lines of (B.15).

4. Assume now that ℓ is not the exiting line of a left-fake cluster and the insertion of a left-fake cluster, together with its entering line, produces a self-energy cluster. Note that this can happen only if ℓ is the entering line of a right-fake cluster T. Let $\overline{\ell}$ be the exiting line (on scale $n+1$) of the right-fake cluster T, call $\overline{\theta}$ the tree obtained from θ by removing T and ℓ and call $\tau_2(\overline{\theta},\overline{\ell})$ the set of trees θ' obtained from $\overline{\theta}$ by inserting a right-fake cluster, together with its entering line, before $\overline{\ell}$; see Figure B.4.

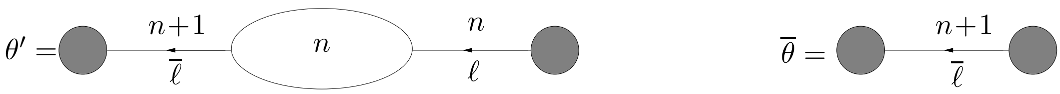

Figure B.4: The trees θ' of the set $\tau_2(\overline{\theta},\overline{\ell})$ obtained from $\overline{\theta}$ when $\ell \in L(\theta)$ enters a right-fake cluster.

By construction one has

$$\begin{aligned}\mathscr{V}(\overline{\theta}) &= \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell)\,\mathcal{G}^{[n+1]}_{e_{\overline{\ell}},u_{\overline{\ell}}}(x_{\overline{\ell}})\,\mathcal{B}_{\overline{\ell}}(\overline{\theta})\\ \sum_{\theta'\in\tau_2(\overline{\theta},\overline{\ell})}\mathscr{V}(\theta') &= \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell)\,\Big(\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\,M^{[n]}(x_{\overline{\ell}})\,\mathcal{G}^{[n]}(x_{\overline{\ell}})\Big)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta}),\end{aligned}$$

where $u_{\overline{\ell}}$ denotes the u-component of $\overline{\ell}$ as line in $\overline{\theta}$, and we have used that $x_\ell = x_{\overline{\ell}}$.

Consider the contribution to $\partial_{\overline{\ell}}\mathscr{V}(\overline{\theta})$ given by

$$\mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_{\overline{\ell}})\Big(\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\partial_L M^{[n]}(x_{\overline{\ell}})\mathcal{G}^{[n+1]}(x_{\overline{\ell}})\Big)_{e_{\overline{\ell}},u_{\overline{\ell}}}\mathcal{B}_{\overline{\ell}}(\overline{\theta}), \tag{B.16}$$

arising from (B.13). For $u, e, e' \in \{\beta, B\}$ and $T \in \mathfrak{RF}_{n,u,e'}$ call $\mathfrak{R}_{n,u,e}(T)$ the subset of $\mathfrak{R}_{n,u,e}$ such that if $T' \in \mathfrak{R}_{n,u,e}(T)$ the exiting line $\ell_{T'}$ exits also the renormalised right-fake cluster T; note that the entering line ℓ of T must be also the exiting line of some renormalised left-fake cluster T'' contained in T'; see Figure B.5.

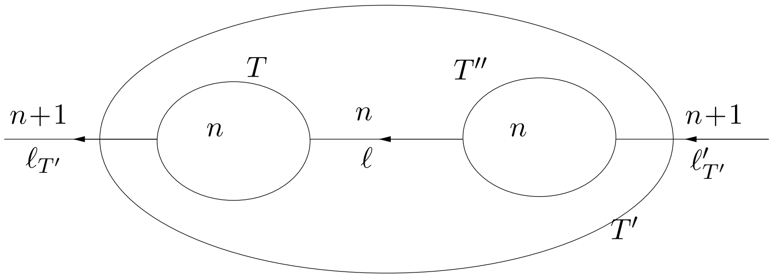

Figure B.5: A self-energy cluster $T' \in \mathfrak{R}_n(T)$.

Define $\mathfrak{M}^{[n]}(x_{\overline{\ell}})$ as the 2×2 matrix with components

$$\mathfrak{M}^{[n]}_{u,e}(x_{\overline{\ell}}) = \sum_{e'=\beta,B} \sum_{T \in \mathfrak{RF}_{n,u,e'}} \sum_{T' \in \mathfrak{R}_{n,u,e}(T)} \varepsilon^{k(T')} \mathscr{V}_{T'}(x_{\overline{\ell}}) \tag{B.17}$$

and consider the contribution $\mathfrak{M}^{[n]}(x_{\overline{\ell}})$ to $M^{[n]}(x_{\overline{\ell}})$ in (B.16). Let us pick up the term with the derivative acting on the line ℓ: one has

$$\begin{aligned} &\partial_\ell \sum_{\theta' \in \tau_2(\overline{\theta}, \overline{\ell})} \mathscr{V}(\theta') + \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell) \left(\mathcal{G}^{[n+1]}(x_\ell)\, \partial_\ell \mathfrak{M}^{[n]}(x_{\overline{\ell}}) \mathcal{G}^{[n+1]}(x_\ell) \right)_{e_{\overline{\ell}}, u_{\overline{\ell}}} \mathcal{B}_{\overline{\ell}}(\overline{\theta}) \\ &= \mathcal{A}_{\overline{\ell}}(\overline{\theta}, x_\ell) \left(\mathcal{G}^{[n+1]}(x_\ell) M^{[n]}(x_\ell) \partial_{\beta_0} \mathcal{G}^{[n]}(x_\ell) \left(\mathbb{1} + M^{[n]}(x_\ell) \mathcal{G}^{[n+1]}(x_\ell) \right) \right)_{e_{\overline{\ell}}, u_{\overline{\ell}}} \mathcal{B}_{\overline{\ell}}(\overline{\theta}), \end{aligned} \tag{B.18}$$

where we have used again that $x_\ell = x_{\overline{\ell}}$. Thus, one can reason as in (B.15), so as to obtain the sum of three contributions, as represented in Figure B.6. □

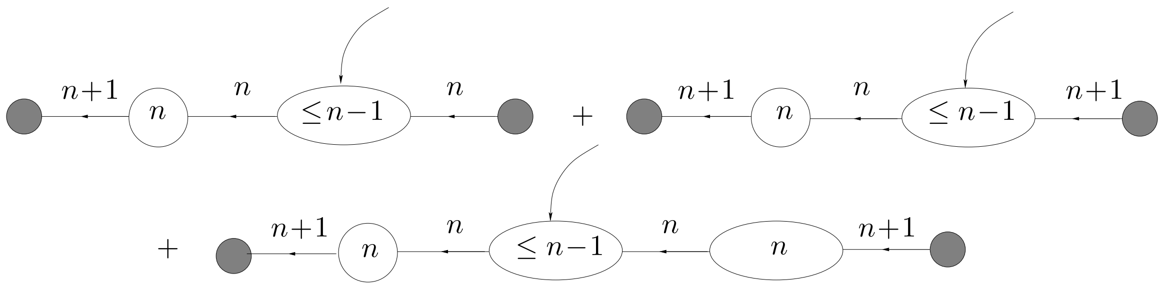

Figure B.6: Graphical representation of the three contributions arising from (B.18).

5. Finally, consider the case in which ℓ is the exiting line of a left-fake cluster, T_0 and the removal of T_0 and ℓ'_{T_0} (see Remark B.4) creates a self-energy cluster.

Set (for a reason that will become clear later) $\theta_0 = \theta$ and $\ell_0 = \ell$. Then there is a maximal $m \geq 1$ such that there are $2m$ lines $\ell_1, \dots, \ell_m$ and $\ell'_1, \dots \ell'_m$, with the following properties:

(i) $\ell_i \in \mathcal{P}(\ell_{\theta_0}, \ell_{i-1})$, for $i = 1, \dots, m$,

(ii) $n_{\ell_i} = n + i < \max\{p : \Psi_p(x_{\ell_i}) \neq 0\} = n + i + 1$, for $i = 0, \dots, m-1$, while $n_m := n_{\ell_m} = n + m + \sigma$, with $\sigma \in \{0, 1\}$,

(iii) $\boldsymbol{\nu}_{\ell_i} \neq \boldsymbol{\nu}_{\ell_{i-1}}$ and the lines preceding ℓ_i but not ℓ_{i-1} are on scale $\leq n + i - 1$, for $i = 1, \dots, m$,

(iv) $\boldsymbol{\nu}_{\ell'_i} = \boldsymbol{\nu}_{\ell_i}$, for $i = 1, \dots, m$,

(v) if $m \geq 2$, ℓ'_i is the exiting line of a left-fake cluster T_i, for $i = 1, \dots, m-1$,

(vi) $\ell'_i \prec \ell'_{T_{i-1}}$ and all the lines preceding $\ell'_{T_{i-1}}$ but not ℓ'_i are on scale $\leq n + i - 1$, for $i = 1, \dots, m$,

(vii) $n'_m := n_{\ell'_m} = n + m + \sigma'$ with $\sigma' \in \{0, 1\}$.

Note that one cannot have $\sigma = \sigma' = 1$, otherwise the subgraph between ℓ_m and ℓ'_m would be a self-energy cluster. Note also that (ii), (iv) and (v) imply $n_{\ell'_i} = n + i$ for $i = 1, \dots, m-1$ if $m \geq 2$. Call S_i the subgraph between ℓ_{i+1} and ℓ_i and S'_i the cluster between ℓ'_{T_i} and ℓ'_{i+1}, for all $i = 0, \dots, m-1$. For $i = 1, \dots, m$, call θ_i the tree obtained from θ_0 by removing everything between ℓ_i and the part of θ_0 preceding ℓ'_i, and note that, if $m \geq 2$, properties (i)–(vii) hold for θ_i but with $m - i$ instead of m, for all $i = 1, \dots, m-1$.

For $i = 1, \dots, m$, call R_i the self-energy cluster obtained from the subgraph of θ_{i-1} between ℓ_i and ℓ'_i, by removing the left-fake cluster T_{i-1} together with ℓ'_{T_i}. Note that $L(R_i) = L(S_{i-1}) \cup \{\ell_{i-1}\} \cup L(S'_{i-1})$ and $N(R_i) = N(S_{i-1}) \cup N(S'_{i-1})$; see Figure B.7.

For $i = 0, \dots, m-1$, given $\ell', \ell \in L(\theta_i)$, with $\ell' \prec \ell$, call $\mathcal{P}^{(i)}(\ell, \ell')$ the path of lines in θ_i connecting ℓ' to ℓ (hence $\mathcal{P}^{(i)}(\ell, \ell') = \mathcal{P}(\ell, \ell') \cap L(\theta_i)$). For any $i = 0, \dots, m-1$ and any $\ell \in \mathcal{P}^{(i)}(\ell_i, \ell'_m)$, let $\tau_3(\theta_i, \ell)$ be the set of all renormalised trees which can be obtained from θ_i by replacing each left-fake cluster preceding ℓ but not ℓ'_m with all possible left-fake clusters. Set also $\tau_3(\theta_{m-1}, \ell'_m) = \theta_{m-1}$.

Note that, by construction,

$$\begin{aligned} &\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m}) \mathcal{G}^{[n_m]}_{e_{\ell_m}, u_{\ell_m}}(x_{\ell_m}) \mathscr{V}(S_{m-1}) = \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}}), \\ &\mathscr{V}(S'_{m-1}) \mathcal{G}^{[n'_m]}_{e_{\ell'_m}, u_{\ell'_m}}(x_{\ell_m}) \mathcal{B}_{\ell_m}(\theta_m) = \mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}). \end{aligned} \tag{B.19}$$

One among cases **1**–**4** holds for $\ell_m \in L(\theta_m)$, so that we can consider the contribution to $\partial_{\ell_m} \mathscr{V}(\theta_m)$ (together with other contributions as in **3** and **4**, if necessary) given by – see (B.11), (B.13) and (B.15) –

$$\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m}) \Big(\mathcal{G}^{[n_m]}(x_{\ell_m}) \partial_{\ell_{m-1}} \mathscr{V}_{R_m}(x_{\ell_m}) \mathcal{G}^{[n'_m]}(x_{\ell_m}) \Big)_{e_{\ell_m}, u_{\ell'_m}} \mathcal{B}_{\ell_m}(\theta_m).$$

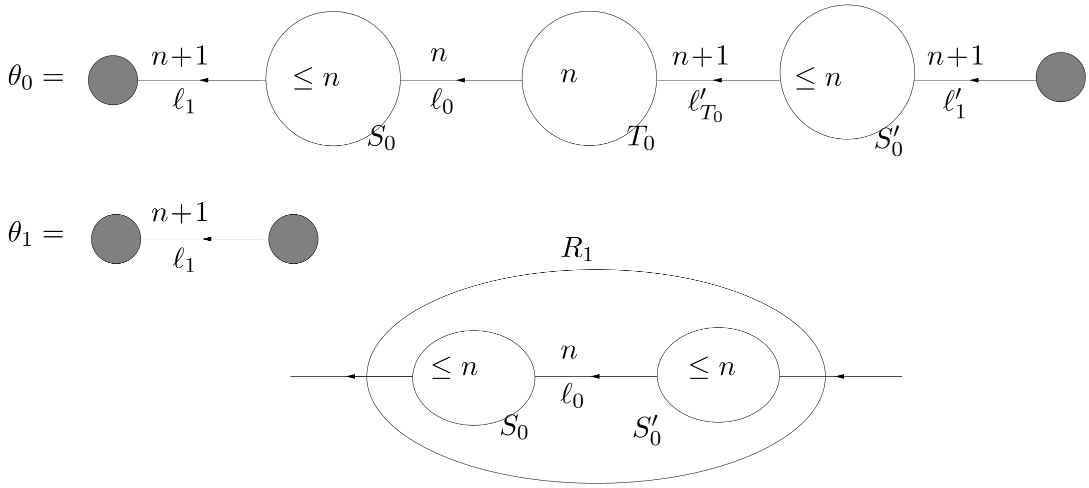

Figure B.7: The renormalised trees θ_0 and θ_1 and the self-energy cluster R_1 in case **5** with $m = 1$ and $\sigma = \sigma' = 0$. Note that the set S_0' is a cluster, but not a self-energy cluster.

Then one has

$$\begin{aligned}
&\mathcal{A}_{\ell_m}(\theta_m, x_{\ell_m})\Big(\mathcal{G}^{[n_m]}(x_{\ell_m})\partial_{\ell_{m-1}}\mathscr{V}_{R_m}(x_{\ell_m})\mathcal{G}^{[n'_m]}(x_{\ell_m})\Big)_{e_{\ell_m},u_{\ell'_m}}\mathcal{B}_{\ell_m}(\theta_m) + \partial_{\ell_{m-1}}\sum_{\theta'\in\tau_3(\theta_{m-1},\ell_{m-1})}\mathscr{V}(\theta';\varepsilon,\beta_0) \\
&= \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\partial_{\beta_0}\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}}) \qquad\qquad \text{(B.20)}\\
&\qquad\times\Big(\mathbb{1} + M^{[n+m-1]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}),
\end{aligned}$$

where we have shortened $e, u' = e_{\ell_{m-1}}, u_{\ell'_{T_{m-1}}}$ to simplify notation. By reasoning as in (B.15), this gives

$$\begin{aligned}
&\mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}) \\
&\quad+ \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}) \\
&\quad+ \mathcal{A}_{\ell_{m-1}}(\theta_{m-1}, x_{\ell_{m-1}})\Big(\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}})\partial_{\beta_0}\mathcal{M}^{[n+m-2]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m-1]}(x_{\ell_{m-1}}) \qquad \text{(B.21)}\\
&\qquad\qquad\times M^{[n+m-1]}(x_{\ell_{m-1}})\mathcal{G}^{[n+m]}(x_{\ell_{m-1}})\Big)_{e,u'}\mathcal{B}_{\ell'_{T_{m-1}}}(\theta_{m-1}).
\end{aligned}$$

where again $e, u' = e_{\ell_{m-1}}, u_{\ell'_{T_{m-1}}}$.

Then, for $i = m-1, \dots, 1$ we recursively reason as follows. Set

$$\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i, \ell'_{i+1})) := \sum_{\theta'\in\tau_3(\theta_i,\ell'_{i+1})}\mathcal{B}_{\ell'_{T_i}}(\theta')$$

and note that

$$\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\mathcal{G}^{[n+i]}_{e_{\ell_i},u_{\ell_i}}(x_{\ell_i})\mathscr{V}(S_{i-1}) = \mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}}),\\
&\mathscr{V}(S'_{i-1})\Big(\mathcal{G}^{[n+i]}(x_{\ell_i})M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell'_{i-1}},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) = \mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).
\end{aligned} \tag{B.22}$$

Consider the contribution

$$\begin{aligned}
\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\Big(&\mathcal{G}^{[n+i]}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}^{[n+i]}(x_{\ell_i})\\
&\times M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell_i},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})),
\end{aligned} \tag{B.23}$$

obtained at the $(i+1)$-th step of the recursion. By (B.22) one has (see Figure B.8)

$$\begin{aligned}
&\mathcal{A}_{\ell_i}(\theta_i, x_{\ell_i})\Big(\mathcal{G}^{[n+i]}(x_{\ell_i})\partial_{\ell_{i-1}}\mathscr{V}_{R_i}(x_{\ell_i})\mathcal{G}^{[n+i]}(x_{\ell_i})\\
&\qquad\times M^{[n+i]}(x_{\ell_i})\mathcal{G}^{[n+i+1]}(x_{\ell_i})\Big)_{e_{\ell_i},u_{\ell'_{T_i}}}\mathcal{B}_{\ell'_{T_i}}(\tau_3(\theta_i,\ell'_{i+1})) + \partial_{\ell_{i-1}}\sum_{\theta'\in\tau_3(\theta_{i-1},\ell_{i-1})}\mathscr{V}(\theta')\\
&= \mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}})\,\Big(\partial_{\beta_0}\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\\
&\qquad\times\Big(\mathbb{1} + M^{[n+i-1]}(x_{\ell_{i-1}})\mathcal{G}^{[n+i]}(x_{\ell_{i-1}})\Big)\Big)_{e_{\ell_{i-1}},u_{\ell'_{T_{i-1}}}}\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)),
\end{aligned} \tag{B.24}$$

which produces, as in (B.21), the contribution

$$\begin{aligned}
\mathcal{A}_{\ell_{i-1}}(\theta_{i-1}, x_{\ell_{i-1}})\Big(&\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\partial_{\ell_{i-2}}\mathscr{V}_{R_{i-1}}(x_{\ell_{i-1}})\mathcal{G}^{[n+i-1]}(x_{\ell_{i-1}})\\
&\times M^{[n+i-1]}(x_{\ell_{i-1}})\mathcal{G}^{[n+i]}(x_{\ell_{i-1}})\Big)_{e_{\ell_{i-1}},u_{\ell'_{T_{i-1}}}}\mathcal{B}_{\ell'_{T_{i-1}}}(\tau_3(\theta_{i-1},\ell'_i)).
\end{aligned} \tag{B.25}$$

Hence we can proceed recursively from θ_m up to θ_0, until we obtain

$$\begin{aligned}
&\mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell_0})\mathcal{G}^{[n]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1))\\
&\quad+\mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell-0})\mathcal{G}^{[n+1]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1))\\
&\quad+\mathcal{A}_{\ell_0}(\theta_0, x_{\ell_0})\Big(\mathcal{G}^{[n]}(x_{\ell_0})\partial_{\beta_0}\mathcal{M}^{[n-1]}(x_{\ell_0})\mathcal{G}^{[n]}(x_{\ell_0})M^{[n]}(x_{\ell_0})\mathcal{G}^{[n+1]}(x_{\ell_0})\Big)_{e_{\ell_0},u_{\ell'_{T_0}}}\mathcal{B}_{\ell'_{T_0}}(\tau_3(\theta_0,\ell'_1)).
\end{aligned} \tag{B.26}$$

Once again, if we split $\partial_{\beta_0} = \partial_N + \partial_L$, all the terms with $\partial_N\mathcal{M}^{[n-1]}$ are contributions to $\mathcal{M}^{[p]}_{\beta,\beta}(0)$. □

6. We are left with the derivatives $\partial_L M^{[q]}(x)$, $q \le n$, when the differentiated propagator is not one of those used along the cases **4** or **5**; see for instance (B.18), (B.20) and (B.24). One can reason as in the

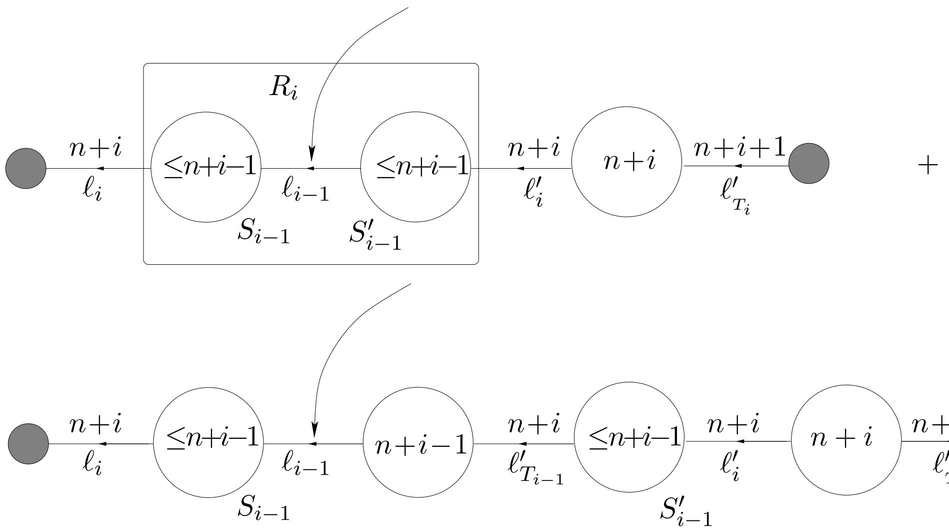

Figure B.8: Graphical representation of the left hand side of (B.24).

case $\partial_L \mathscr{V}(\theta)$, by studying the derivatives $\partial_\ell \mathscr{V}_T(x_\ell)$ and proceed iteratively along the lines of cases **1** to **5** above, until only lines on scales 0 are left. In that case the derivatives $\partial_{\beta_0}\mathcal{G}^{[0]}(x_\ell)$ produce derivatives

$$\partial_{\beta_0} M^{[-1]}(x) = \begin{pmatrix} \varepsilon\partial^2_{\beta_0} F_{\mathbf{0}}(\beta_0, B_0) & \varepsilon\partial^2_{\beta_0,B_0} F_{\mathbf{0}}(\beta_0, B_0) \\ \\ \varepsilon\partial^2_{\beta_0} G_{\mathbf{0}}(\beta_0, B_0) & \varepsilon\partial^2_{\beta_0,B_0} G_{\mathbf{0}}(\beta_0, B_0) \end{pmatrix}$$

(see Remarks 3.2.1 and 3.2.2). Therefore, for $n = -1$, in the splitting (B.10), there are no terms with the derivatives ∂_ℓ, and the derivatives ∂_v can be interpreted as said in Remark B.1. $\square$

7. By construction, each contribution to $\mathcal{M}^{[p-1]}_{\beta,\beta}(0)$ appears as one term among those considered in the discussion above, that is among the contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}(\varepsilon, \beta_0, B_0)$ arising from the trees $\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ satisfying the condition (B.2). Of course, when computing $\partial_{\beta_0}\mathscr{V}(\theta)$ for such trees, also some contributions to $M^{[p]}_{\beta,\beta}(0)$ have been produced. Call $W^{[p]}$ the contributions to $M^{[p]}_{\beta,\beta}(0)$ which are not obtained in the previous steps. Define also $R^{[p]}$ as the sum of the contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}$ such that

$$\partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]} = \mathcal{M}^{[p-1]}_{\beta,\beta}(0) + \left(M^{[p]}_{\beta,\beta}(0) - W^{[p]}\right), \tag{B.27}$$

where we have used that $\mathcal{M}^{[p]}_{\beta,\beta}(0) = \mathcal{M}^{[p-1]}_{\beta,\beta}(0) + M^{[p]}_{\beta,\beta}(0)$ – see definition (3.2.6) and use that $\chi_q(0) = 1$ for all $q \geq -1$. Hence $\partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]}$ represents the sum of all contributions to $\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}}$ used in **1**–**6**. One can write

$$\partial_{\beta_0}\Phi^{\mathcal{R},p}_{\mathbf{0}} = \partial_{\beta_0}\Phi^{\mathcal{R},p-1}_{\mathbf{0}} + R^{[p]} + S^{[p]}, \tag{B.28}$$

for a suitable $S^{[p]}$: by construction $S^{[p]}$ takes into account all contributions arising from the trees

$\theta \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}$ which do not satisfy the condition (B.2), i.e. such that

$$\max_{\ell \in \Theta^{\mathcal{R},p}_{k,\mathbf{0},\beta}} \{n \in \mathbb{Z}_+ : \Psi_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0\} = p + 1. \tag{B.29}$$

Such trees have been excluded in the discussion above, because on the one hand they would produce the remaining contributions to $M^{[p]}_{\beta,\beta}(0)$, on the other hand they would equally produce contributions to $M^{[p+1]}_{\beta,\beta}(0)$. Therefore, by combining (B.27) and (B.28), we obtain $\partial_{\beta_0} \Phi^{\mathcal{R},p}_{\mathbf{0}} = \mathcal{M}^{[p]}_{\beta,\beta}(0) + \big(S^{[p]} - W^{[p]}\big)$, where both $W^{[p]}$ and $S^{[p]}$ arise from trees containing at least one line ℓ on scale p and such that $\Psi_{p+1}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0$: for such a line ℓ one has $|\boldsymbol{\nu}_\ell| > 2^{m_{p+1}-1}$ by Remark 2.2.2. Therefore, one has $\max\big\{|S^{[p]}|, |W^{[p]}|\big\} \le |\varepsilon| \, D_1 e^{-D_2 2^{m_{p+1}}}$, for some constants D_1, D_2, and this is enough to prove the bound (B.1). □

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