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Algebraic and Coalgebraic aspects of classical and quantum Hamiltonian systems

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This Thesis is dedicated to my Grandfather.

CONTENTS

1 INTRODUCTION

It is very fascinating how the human being's beliefs dramatically changed over the history, proportionally to the level of his own scientific and technologic knowledge, and how the scientific discovery is going on by making a series of bizarre missteps. Just think at the Geocentric model, believed to be true by the ancient civilizations, supported among the others by the two eminent Greeks intellectuals Aristotle and Ptolemy. In a more modern era, this general thinking started to change when the Polish astronomer Mikolaj Kopernik (Copernicus), in 1543, published his "revolutionary" *De revolutionibus orbium coelestium*. At that time, the sun was believed to be at the center of the solar system, at the center of the Universe, and the six known planets were thought to follow circular orbits around it. More than fifty years later Johannes Kepler, in his *Prodromus dissertationum continens mysterium cosmographicum de admirabili proportione orbium coelestium*, reinterpreted the Copernicus model in a more "geometrical" fashion. His aim was to explain the *God's geometrical plan for the universe*, namely the idea that God created the Universe following geometrical principles based on the five Platonic solids. Between 1609 and 1619, with his three *Kepler's laws* the same scientist achieved one of the most important results of celestial mechanics. Then, circular orbits were replaced by elliptic ones and the sun was posed at the focal point of the elliptical orbit. Some years later, one of the fathers of the scientific revolution, G. Galilei, provided the foundations of his "controversial " *principle of inertia* that, in 1687, found its mathematical formulation thanks to Sir Isaac Newton. His *Philosophiae Naturalis Principia Mathematica*, considered as one of the most important scientific works of all the time, marked the beginning of the modern science. It was the birth of *classical mechanics*. After the formulation of the three Newton's laws, describing the rules that govern the kinematics and the dynamics of bodies, the basic principles underlying such revolutionary theory were established, and a systematic study addressed to achieve a better understanding of the motion was begun. It was the time of the controversy between Newton and Leibniz about the discovery of the infinitesimal calculus. The abstract idea of predicting the motion of a body was then rephrased in terms of a well-defined mathematical problem: given the initial position and velocity of a body, finding out the solution of the three second-order differential equations $\mathbf{F} = m\ddot{\mathbf{x}}$, for the vector $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ describing the position of the body in the space. The theory implies that, given the resulting forces $\mathbf{F} = (F_x, F_y, F_z) \in \mathbb{R}^3$ acting on the body, at fixed initial conditions, there exists a unique vector $\mathbf{x}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ that, $\forall t \in \mathbb{R}$, determines the exact position of the body in the space. It was the era of *determinism*. If the system can be solved exactly, then it is possible to find an analytic expression for the components of the vector $\mathbf{x}$, and the motion is completely solved, i.e. it is possible to predict where the particle will be located at a given time. More than one-hundred years later, the Italian mathematician J. L. Lagrange, with his work *Mecanique analitique*, marked the birth of the modern approach to classical mechanics. The Newton's equations of motion, rephrased in terms of the Euler-Lagrange equations, were recovered starting from a fundamental quantity obtained as a difference between kinetic and potential energy, the Lagrangian. Later, the Irish scientist Sir W. R. Hamilton, who defined the

work of Lagrange *a kind of a scientific Poem*, introduced another equivalent way to describe classical mechanical systems, now known as Hamiltonian's mechanics. The Lagrangians were replaced by the Hamiltonians, the sum of the potential and kinetic energy, and the equivalence between the two approaches was then understood in terms of Legendre tranformations, which connect the two formulations. In this theory, if the Hamiltonian function does not depend explicitly on time, then it is an integral of motion, which means that the Energy of the system is conserved, as it should be for conservative systems. At that time, the central role of the position vector, and consequently of the configuration space, was replaced by a kind of duality between position and momentum of a particle, and the phase space became the fundamental arena of classical systems. The final goal behind such approach to classical mechanics is to find out, at given initial conditions, the trajectories in the phase space of a given model in $\mathbb{R}^N$ (at fixed N). This would be achieved by solving the Hamilton's equations of motion, a set of 2N first-order ordinary differential equations involving the time derivatives of positions and momenta. Anyway, it was already clear at the time that this result is not always reachable, and that there exists a relatively small subset of classical models for which this turns out to be possible. This is the subset of those models that are called *integrable*. The notion of integrability is multifaceted, and has to be explained in each specific field where it appears. The state of the art about integrability can be summarized in the quote of N. Hitchin given in the introduction of [1] where, paraphrasing the answer to a question that was posed to Luiss Armstrong about Jazz, he states that *Integrability is like Jazz, if you have to ask what it is, you will never know*. Surely, one thing that is common to integrable systems is the existence of special properties that the others systems do not possess. For example, in classical mechanics, such systems present a *regular* behaviour: by varying a little bit the initial conditions, it cannot happen something too strange (the motion is not chaotic). The regular motions of integrable systems is a consequence of the existence of conserved quantities in the dynamics (Poisson-commuting with the Hamiltonian), which force the motion to be restricted in a subset of the phase space. As a matter of fact, each conserved quantity drops one degree of freedom of the system and, as a logical consequence, less freedom implies more regularity. Physically, keeping in mind the milestone result of the German mathematician E. Noether on symmetries and conservation laws, this fact reflects the general idea that more symmetries imply a more regular behaviour. The existence of a sufficient number of functionally independent costants of motion (in involution) for an Hamiltonian system is sufficient to ensure the quasi-periodicity of the motion and, as a consequence, the solvability of the Hamilton's equations. These are the integrable systems (in the Liouville sense). At the present days, this concept is understood in terms of the Arnold-Liouville Theorem [2], which encloses in a unique statement the notion of (Liouville) integrability for Hamiltonian systems, i.e. the existence of N involutive constants of motion (in dimension N), and the solvability of the Hamilton's equations. Roughly speaking, the idea behind such result concerns the possibility of rephrasing the original problem in terms of a set of canonical (called action-angle) variables that linearize the Hamilton's equations, and to solve them by means of easy algebraic manipulations and straightforward integrations (in other words by quadratures).

In 1873, another fundamental result was achieved by the French mathematician J. L. F. Bertrand, which proved that among all the central force potentials, there exist only two cases for which all bounded orbits are closed, and they are the *Kepler-Coulomb* and the *harmonic* potentials [3]. This result, known as *Bertrand's theorem*, introduced a new restriction to the

whole set of Hamiltonian systems. The peculiarity of the Kepler-Coulomb and the (isotropic) harmonic oscillator is due to the fact that, they not only define *integrable* systems in the Euclidean space, but they also belong to that subset of Hamiltonian systems that are called, at the present days, *superintegrable* [4]. This theorem has been also generalized to non-Euclidean spaces [5], where the two (multi-parameter) families of Bertrand-Perlick Hamiltonians [6], defining respectively (intrinsic) Kepler-Coulomb and harmonic potentials on their corresponding curved spaces [7], took the place of the original Bertrand systems, the latter recovered in a suitable (flat) limit. The physical reason why superinitegrable systems are so interesting is due to the fact that they possess the maximum possible number of symmetries, and this forces the motion to be regular. The best one can require to a N-dimensional (ND) classical model is to be Maximally Superintegrable (MS). In this case, one deals with an integrable ND Hamiltonian system which is endowed with the maximum possible number of $2N-1$ functionally independent integrals of motion. For such (very special) subset of Hamiltonian systems, their large number of symmetries implies that all finite trajectories are closed and the motion is periodic [4]. Moreover, the notion of solutions by quadratures is replaced by algebraic solvability, meaning that in principle these systems can be solved algebraically, without the need of any differential calculus. The Kepler-Coulomb and the (isotropic) harmonic oscillator potentials are the prototype examples. They are endowed with additional constants of motion, the Laplace-Runge-Lenz vector [8] and the Demkov-Fradkin tensor [9, 10] respectively, which ensure their bounded motion to be periodic. As reasonable to expect, this turned out to be true also for both the curved versions of the Bertrand systems, endowed with a deformed (curved) version of the Laplace-Runge-Lenz and the Demkov-Fradkin tensor respectively [5].

These special properties also found applications in a more modern perspective, where the Hamiltonian mechanics, coming from the old Newton's ideas, made way to the modern view of mechanics. As we know, in the XXth century the concept of determinism was contradicted by the notion of *uncertainty principle* and classical mechanics was replaced by quantum mechanics. Another scientific revolution, based on the ideas of M. Planck, L. De Broglie, W. Heisenberg, N. Bohr, E. Schrödinger just to cite a few, changed once again the entire game. New notions, such as wave function, probability interpretation, observable, average value of operators, Hilbert space, became familiar into the dictionary of Physicists. Solvability in quantum mechanics is related to the solutions of the Schrödinger equation, which means finding spectrum and eigenfunctions of a given spectral problem, whereas the notions of conserved quantities and symmetries are connected to the abstract idea of compatible observables, mathematically rephrased in terms of commuting Hermitian operators. The original definitions given in classical mechanics have been extended also in the quantum theory, where a definition of *quantum integrability* and *quantum superintegrability* has been introduced. The importance of superintegrable quantum systems is related to a conjecture (born out by all known examples [4]) formulated by P. Tempesta, V. Turbiner and P. Winternitz, which states that all MS quantum systems are exactly solvable [11]. It is then possible to calculate spectrum and eigenfunctions in a closed form. This explains the exact solvability of the hydrogen atom and the harmonic oscillator in any dimension and, in turn, the reason why their spectrum is characterized by a (previously called) "accidental degeneracy". Quantum versions of the Laplace-Runge-Lenz vector and the Demkov-Fradkin tensor make them quantum MS systems (the degeneracy is anything but "accidental"). This is a quite general result, deformations of (super)integrable systems (curvature, q-deformations, discrete deformations, ...) should

preserve the fundamental defining properties of the original (undeformed) systems. Clearly, since the superintegrability is a notion linked to exact solvability, it is not difficult to imagine that there exist profound connections between the theory of superintegrable systems and special functions: the Askey scheme [12], organizing hypergeometric orthogonal polynomials, can be derived as a consequence of contractions of two-dimensional second-order superintegrable systems [4]. Moreover, since the exact solvability of quantum systems is often related to the possibility of factorizing the Hamiltonian [13–15], together with an additional condition of shape invariance, it is reasonable that superintegrable systems present also interesting connections with the theory of SUperSYmmetric Quantum Mechanics (SUSYQM) [16, 17].
At the present days, the principal research activity in this area involves the discovery, classification and solution of superintegrable systems in both Euclidean and non-Euclidean space, and characterization of their structure, in particular their underlying symmetry algebra. There is also an increasing interests for discrete superintegrable systems, where finite difference versions of the Schrödinger equation replace the usual standard differential realizations [18]. The physical idea behind such theories is to think about the continuous theory as an approximation of a more fundamental discrete theory, from which the former is recovered in a suitable limit, i.e. when the lattice spacing goes to zero (the continuous limit). This is also compatible with Heisenberg's idea that, at a certain point of the microscopic ladder, a fundamental length would appear in the theory [19]. In the last few years, with the main goal of extending the small set of known discrete models, several superintegrable discrete systems have been constructed, many of them describing superintegrable discrete versions of the harmonic oscillator [20–22]. As for the continuous case, the main difficulties arising in this research grows proportionally to the dimensionality of the problem.

An algebraic method that has been extensively used in the continuous case in order to overcome the difficulties arising from the higher dimensional extensions of superintegrable systems is the so-called *coalgebra symmetry* approach [23–31]. This technique, which works in both classical and quantum mechanics, consists in defining both the Hamiltonians and its constants of motion as functions of the generators of a given algebra endowed with a coproduct map. Once an algebra representation is chosen, then the coproduct can be used to rise the dimension of the representation without losing the superintegrability properties. This is because the map provides, at each application, a set of additional symmetries "*the partial Casimirs*", which help to keep the system superintegrable. A limitation of this technique resides in the fact that it is not possible to obtain the maximal superintegrability, i.e. it is not possible to construct algebraically the entire set of $2N-1$ functionally independent conserved quantities. Anyway, with an $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry one can arrive very close to that result, since $2N-2$ functionally independent first integrals can be constructed. Then, the left quantity can be found by using other techniques.
Another algebraic method introduced to deal with (both classical and quantum) symmetries of superintegrable systems, which has deep roots in the paper [32], is the so-called *factorization method* (or *extended factorization method*) [33, 34]. This approach consists in extending the standard factorization method, which we commonly use in ordinary quantum mechanics, to separable systems depending on several variables [35]. It establishes that, if the Hamiltonian can be separated in a given coordinate system, it is possible to construct for each coordinate two sets of ladder and shift functions (resp. operators in quantum mechanics). Then, if certain

further conditions are satisfied, the additional integrals of motion can be obtained by taking a suitable combination of them.

1.1 OUTLINE OF THE THESIS

This Thesis is the result of a collection of related problems, in the field of superintegrable systems, which have been investigated in different joint collaborations [36–40]. The work is divided in two main parts whose distinction resides both in the purposes and the methodologies. In the first part, dedicated to discrete superintegrable models, the coalgebra symmetry technique represents the main algebraic tool. Whereas, in the second one, specific continuous systems have been investigated in terms of the factorization method and the SUSYQM approach. In both parts, we investigate superintegrable Hamiltonians sharing the same $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry. They are superintegrable deformations, in a suitable sense, of the harmonic oscillator and the Kepler-Coulomb system.

Concerning the internal structure, in each Chapter we added a section *Motivations and definitions* in order to explain the main reasons why we decided to investigate a given problem. This, together with a section *Concluding remarks and open perspectives* which underlying the main results and the future developments, marks the beginning and the end of the individual studies.

Precisely:

- Chapter 2, common for both parts of the Thesis, is devoted to a terse introduction to the basic notions and definitions that will be used throughout the work. After a brief review dedicated to autonomous Hamiltonian systems, the notions of *integrability* and *superintegrability* for classical and quantum Hamiltonians will be introduced, following as a main reference [4]. To be more clear as possible, we will take as prototype examples the harmonic oscillator and the Kepler-Coulomb system. The main aim is to establish a common dictionary needed to understand the main Chapters.

- Chapter 3 is fully dedicated to the coalgebra symmetry and, in particular, to the possibility of applying it in a discrete quantum mechanical framework. After a comprehensive review of the method, we will introduce the building blocks of discrete Quantum Mechanics, a discrete version of the ordinary quantum theory proposed by S. Odake and R. Sasaki [19]. The main goal of the Chapter is to show that the coalgebraic approach to superintegrable systems can be also extended to quantum discrete models. By using the prototype example of the harmonic oscillator, we will introduce a procedure to discretize the one-dimensional Hamiltonian and, once solved the spectral problem on the lattice, we will make use of a discrete representation of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra in order to construct a higher dimensional MS extension of the discrete quantum Hamiltonian [40].

- Chapter 4 is devoted to the application of factorization approaches in both classical and quantum mechanics. After a brief review of the classical factorization method for one-dimensional systems [32], we will use it in order to investigate the classical analog of f-oscillators [41–44], a family of deformed Hamiltonian systems which represents

a generalization of q-oscillators [45, 46]. The main aim is to show how this method allows us to write down the deformed Poisson algebra characterizing the entire family of non-linear oscillators and to contruct its general solution algebraically [39]. The analysis concludes with a discussion about the MS multidimensional generalization of classical f-deformations, once again obtainable by means of the coalgebra symmetry technique.

The final part of the Chapter is instead devoted to a (both classical and quantum) analysis of two prototype examples of N-dimensional MS systems defined on space on nonconstant curvature, the so-called Taub-NUT and Darboux III Hamiltonian systems. These models, which have been extensively studied in the literature for their mathematical as well as physical relevance [47–56], according to the Perlick's classification [5–7], belong to the family of type II, they can be therefore regarded as intrinsic oscillators on their corresponding curved spaces. On the other hand, from an analytic point of view, they represent a one-parameter deformation of the Kepler-Coulomb and harmonic oscillator systems respectively, and their maximal superintegrability is ensured thanks to the existence of a curved version of the Laplace-Runge-Lenz vector as well as of the Demkov-Fradkin tensor. The main aim of the Chapter is to present an algebraic analysis of both models [36–38]. The classical one-dimensional radial dynamics will be investigated by using the classical factorization method, whereas for the quantum case the spectral problem will be solved by means of standard SUSYQM techniques, thus providing new features from the ones already presented in previous works.

- Chapter 5 is devoted to the concluding remarks. A brief review of the main results will be given and the most relevant open problems will be summarized.

2 SUPERINTEGRABILITY IN CLASSICAL AND QUANTUM MECHANICS

Chapter 2 is devoted to a short introduction to the basic notions and definitions that will be used throughout the work. In particular, after a brief review dedicated to autonomous Hamiltonian systems, following mainly the structure of the review paper [4], we will recall the notions of *integrability* and *superintegrability* for classical and quantum Hamiltonian systems.

2.1 SUPERINTEGRABILITY IN CLASSICAL MECHANICS

There exist several ways to define the concept of *Hamiltonian system* and to introduce the notions of *integrability* and *superintegrability* in classical mechanics. Our starting point is the following system of ODE's:

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) \\ \mathbf{y}(t_0) \doteq \mathbf{y_0}\,, \end{cases} \tag{1.1}$$

where we denoted $\dot{\mathbf{y}} \doteq \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}$, with $\mathbf{f}(\mathbf{y}, t) = (f_1(\mathbf{y}, t), \dots, f_d(\mathbf{y}, t)) \in \mathbb{R}^d$, $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$, $\mathbf{y_0} = (y_{01}, \dots, y_{0d}) \in \mathbb{R}^d$. This *dynamical system* defines a trajectory on the phase space manifold $\mathbb{R}^d$ tangent to the vector field $\mathbf{f}(\mathbf{y}, t)$. If the dimension of the phase space is even, i.e. $d = 2N$, and there exists a (smooth) function $H = H(\mathbf{y}, t)$ defined on $\mathbb{R}^{2N} \times \mathbb{R}$, such as the system of ODE's (1.1) can be cast in form:

$$\begin{cases} \dot{\mathbf{y}} = \mathbb{J}\nabla_{\mathbf{y}} H(\mathbf{y}, t) \\ \mathbf{y}(t_0) \doteq \mathbf{y_0}\,, \end{cases} \tag{1.2}$$

where $\mathbb{J} \doteq \begin{pmatrix} 0_N & \mathbb{1}_N \\ -\mathbb{1}_N & 0_N \end{pmatrix}$ is the $2N \times 2N$ skew-symmetric matrix and $\nabla_{\mathbf{y}} \doteq (\partial_{y_1}, \dots, \partial_{y_{2N}})$, then the dynamical system is called *Hamiltonian* [2]. One of the reason why this subclass of dynamical systems has a fundamental importance in physics is related to their connection with the theory of conservative systems in classical mechanics. In fact, in the special case when H is independent of the time variable t, so that $H : \mathbb{R}^{2N} \to \mathbb{R}$, the differential equations (1.2) are autonomous, and the Hamiltonian system is *conservative*. As a matter of fact, the Hamiltonian formalism represents the mathematical structure where the theory of such systems has been developed. This formalism allows to describe the dynamics of a physical system in N dimensions by relating the time derivatives of the position and momentum coordinates to a single function defined on the phase space, the *Hamiltonian* H [8, 57]. In particular, by introducing the pair of coordinates $\mathbf{y} \doteq (\mathbf{x}, \mathbf{p})^T \in \mathbb{R}^{2N}$, which represent the position coordinates of a particle and their (conjugated) momenta, together with $\nabla_{\mathbf{y}} = (\nabla_{\mathbf{x}}, \nabla_{\mathbf{p}}) = (\partial_{x_1}, \dots, \partial_{x_N}, \partial_{p_1}, \dots, \partial_{p_N})$, the dynamical system takes the following form:

$$\begin{cases} \dot{\mathbf{x}} = +\nabla_{\mathbf{p}} H(\mathbf{x}, \mathbf{p}) \\ \dot{\mathbf{p}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \mathbf{p}) \end{cases}, \tag{1.3}$$

together with the initial conditions $(\mathbf{x}(t_0), \mathbf{p}(t_0)) \doteq (\mathbf{x_0}, \mathbf{p_0})$, which are the familiar *Hamilton's equations*. These set of equations, once fixed the $2N$ initial conditions, uniquely[1] determine a collection of points $(x_i(t), p_i(t)) \in \mathcal{M}$ allowed in the motion of the physical system, which is the trajectory in the phase space manifold[2] $\mathcal{M}$, which we can think locally as an open set in R^{2N}. So, the second Newton's law, described by N second-order differential equations in the position coordinates variables $\mathbf{x}(t) \in \mathbb{R}^N$, is now "embedded" into the Hamilton's equations, i.e. a set of $2N$ first-order differential equation in (local) coordinates $(\mathbf{x}(t), \mathbf{p}(t)) \in \mathbb{R}^{2N}$. When the Hamiltonian function is expressed in *natural form*, i.e. $H(\mathbf{x}, \mathbf{p}) = T(\mathbf{x}, \mathbf{p}) + V(\mathbf{x})$, $T : \mathbb{R}^{2N} \to \mathbb{R}$ and $V : \mathbb{R}^N \to \mathbb{R}$ being the kinetic energy and the potential function respectively, then it can be interpreted as the total energy of the mechanical system. In particular, for many physical systems, the Hamiltonian $H : \mathbb{R}^{2N} \to \mathbb{R}$ is defined as:

$$H(\mathbf{x}, \mathbf{p}) \doteq \frac{1}{2m} \sum_{i,j=1}^{N} g^{ij}(\mathbf{x}) p_i p_j + V(\mathbf{x}) \,, \tag{1.4}$$

where $m \in \mathbb{R}^+$ is the mass of the particle and $g^{ij}(\mathbf{x})$ is a contravariant metric tensor of an underlying Riemannian manifold $\mathcal{M}$ for which $g^{-1} \doteq \det(g^{ij}) \neq 0$ and $g^{ij} = g^{ji}$ [4]. The metric of the Riemannian manifold in which the motion takes place is given by the following quadratic form:

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} g_{ij}(\mathbf{x}) \mathrm{d}x^i \mathrm{d}x^j \,, \tag{1.5}$$

$g_{ij}(\mathbf{x})$ being the covariant metric tensor (the inverse of the contravariant metric tensor) such that $\sum_{k=1}^{N} g^{ik} g_{kj} = \delta^i_j$. Let us observe that in the dynamics of a free particle on a Riemannian manifold, for which the Hamiltonian function corresponds to the kinetic energy of the system, i.e. (from now on we shall fix $m = 1$ unless explicitly stated):

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2} \sum_{i,j=1}^{N} g^{ij}(\mathbf{x}) p_i p_j \,, \tag{1.6}$$

the main difference with respect to the Euclidean case, when $g_{ij} = \delta_{ij}$, is related to the fact that the symmetry properties holding in an Euclidean space, such as the translation and rotation invariance coming from the conservation of the linear and angular momenta, are in general not preserved.

A notion that is of crucial importance in the Hamiltonian theory is the one regarding *the conservation of quantities* in the motion of the physical system. In particular, a conserved quantity $f(\mathbf{x}, \mathbf{p}, t) \in \mathbb{R}^{2N} \times \mathbb{R}$ is characterized by the fact that its time derivative is zero:

$$\dot{f} = \frac{\mathrm{d}f(\mathbf{x}, \mathbf{p}, t)}{\mathrm{d}t} = 0 \,, \tag{1.7}$$

which means that there is no variation in time, i.e. $f = c \in \mathbb{R}$ is a constant, or *first integral*. This simple equality can be rephrased in terms of a binary operation playing a central role in the classical Hamiltonian theory. Expanding the l.h.s. of (1.7) we can write:

1 Classical mechanics is a deterministic theory.

2 For a brief introduction to the notion of manifold see Appendix A.

$$\begin{aligned} 0 = \dot{f} &= f_t + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f \\ &= f_t + \nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} H \cdot \nabla_{\mathbf{p}} f \\ &\doteq f_t + \{f, H\}_{(\mathbf{x},\mathbf{p})} \,, \end{aligned} \tag{1.8}$$

where, in the second equality, we made use of the Hamilton's equations and in the third we have defined the *Poisson bracket*, i.e.:

$$\{f, g\}_{(\mathbf{x},\mathbf{p})} \doteq \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{p}} g - \nabla_{\mathbf{p}} f \cdot \nabla_{\mathbf{x}} g \quad \forall f, g \in C^\infty(\mathbb{R}^{2N} \times \mathbb{R}). \tag{1.9}$$

If f, g and h are three smooth functions defined on the phase space manifold $\mathbb{R}^{2N}$, α, β are constants, and we indicate as "$\cdot$" the usual pointwise (symmetric and associative) multiplication of functions on $C^\infty(\mathbb{R}^{2N})$, then the following properties are satisfied:

$$\begin{cases} \{f, g\}_{(\mathbf{x},\mathbf{p})} = -\{g, f\}_{(\mathbf{x},\mathbf{p})} \\ \{\alpha f + \beta g, h\}_{(\mathbf{x},\mathbf{p})} = \alpha\{f, h\}_{(\mathbf{x},\mathbf{p})} + \beta\{g, h\}_{(\mathbf{x},\mathbf{p})} \\ \{h, \{f, g\}_{(\mathbf{x},\mathbf{p})}\}_{(\mathbf{x},\mathbf{p})} + \{g, \{f, h\}_{(\mathbf{x},\mathbf{p})}\}_{(\mathbf{x},\mathbf{p})} + \{h, \{g, f\}_{(\mathbf{x},\mathbf{p})}\}_{(\mathbf{x},\mathbf{p})} = 0 \\ \{\mathcal{F}(f), g\}_{(\mathbf{x},\mathbf{p})} = \mathcal{F}'(f)\{f, g\}_{(\mathbf{x},\mathbf{p})} \\ \{h, f \cdot g\}_{(\mathbf{x},\mathbf{p})} = \{h, f\}_{(\mathbf{x},\mathbf{p})} \cdot g + f \cdot \{h, g\}_{(\mathbf{x},\mathbf{p})} \,. \end{cases} \tag{1.10}$$

The first two properties allows us to call the Poisson bracket a *skew-symmetric bilinear* operation, the third shows that it satisfies the *Jacoby identity*, and the last two are known as the *chain* and *Leibniz* rules respectively. Let us briefly comment that the last rule has to be intended as a *compatibility* condition between the pointwise multiplication and the Poisson brackets, since they are *intertwined* by this property.
As a matter of fact, the set $\mathcal{P}_{\mathbb{R}^N} \doteq (C^\infty(\mathbb{R}^{2N}), \{\cdot,\cdot\}, \cdot)$ define a so-called *Poisson algebra*. In the Euclidean case, physically, we could think it as the algebra of classical observables of a point particle moving on the space $\mathbb{R}^N$, with x_i being the position coordinates, and p_i the corresponding canonical conjugated momenta (for $i = 1, \dots, N$).
The Poisson brackets of the phase space coordinates results in a subalgebra $\mathfrak{h}_c(N) \subset \mathcal{P}_{\mathbb{R}^N}$:

$$\{x_i, x_j\}_{(\mathbf{x},\mathbf{p})} = \{p_i, p_j\}_{(\mathbf{x},\mathbf{p})} = 0 \,, \quad \{x_i, p_j\}_{(\mathbf{x},\mathbf{p})} = \delta_{ij} \,, \tag{1.11}$$

generated by the $2N + 1$ elements $\{x_i, p_i, 1\}_{i=1,\dots,N}$, and the Hamilton's equations can be rephrased as follows:

$$\begin{cases} \dot{x}_i = \{x_i, H\}_{(\mathbf{x},\mathbf{p})} & (i = 1, \dots, N) \\ \dot{p}_i = \{p_i, H\}_{(\mathbf{x},\mathbf{p})} & (i = 1, \dots, N) \,. \end{cases} \tag{1.12}$$

In general, the system of $2N$ equations (1.12) will be solvable if it admits a sufficient number of first integrals. In fact, for each constant of motion we can reduce by one the degrees of freedom of the system. At this level it is straightforward to show that, for autonomous Hamiltonian systems, one (trivial) first integral is:

$$\dot{H} \equiv \partial_t H = 0 \,, \tag{1.13}$$

i.e. the total energy is a conserved quantity: $H = E \in \mathbb{R}$, as expected from Newtonian mechanics. In general, for a given function $f \in C^\infty(\mathcal{M})$ that does not depends on time explicitly, the time-evolution is governed by the equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}_{(\mathbf{x},\mathbf{p})}\,, \tag{1.14}$$

which implies that a classical observable is conserved along the dynamics (is a constant of motion) *if and only if* its Poisson bracket with the Hamiltonian function is zero. This gives an algebraic operational way to establish if a given function of the phase space coordinates is conserved or not. Geometrically, the existence of such integral of the motion forces the trajectory of the mechanical system to lie in an hypersurface of dimension $2N-1$ contained in the $2N$ dimensional phase space.

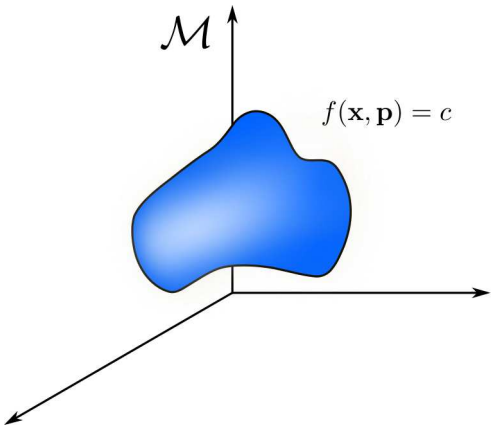


Figure 1.: graphical representation of a level set $f(\mathbf{x},\mathbf{p}) = c \in \mathbb{R}$.

Such a result is a good starting point to introduce the definition of integrability in classical mechanics. In particular, we will say that *an Hamiltonian system is integrable (in the Liouville sense) if it admits N first integrals $f_{i=1,\dots,N}$, with $f_1 = H$, which are in involution, i.e.*:

$$\{f_i, f_j\}_{(\mathbf{x},\mathbf{p})} = 0\,, \quad (1 \leq i\,,\, j \leq N)\,, \tag{1.15}$$

and such as they form a functionally independent set of conserved quantities on some local region of the phase space. Moreover, if the constants of motion are polynomials in the momenta globally defined (except possibly for singularities on a lower dimensional manifold), then the system is called (*polynomially*) *integrable* [4].
The notion of functional independence is related to the rank of the $n \times 2N$ matrix involving the partial derivatives (with respect to the $2N$ coordinates $(\mathbf{x},\mathbf{p}) \in \mathbb{R}^{2N}$) of the set of n smooth functions $f_{k=1,\dots,n}$ defined on some region of the phase space. More precisely, a set of first integrals $\mathcal{O} \doteq \{f_1(\mathbf{x},\mathbf{p}), \dots, f_n(\mathbf{x},\mathbf{p})\}$ is said to be *functionally independent* if the matrix:

$$\mathbb{M} \doteq (\partial_{x_i} f_k, \partial_{p_j} f_k) \qquad (1 \leq i,j \leq N\,;\, 1 \leq k \leq n)\,, \tag{1.16}$$

has rank n throughout the region. In other words, if there exists at least one n-minor of $\mathbb{M}$ (a submatrix of dimension $n \times n$) whose determinant is not zero. In general, if a set is functionally dependent, in some local region of the phase space there exists a (nonzero)

analytic function $\mathcal{F}$ of n variables such as $\mathcal{F}(f_1, \dots, f_n) = 0$ [4], which encodes the functional dependence of the conserved quantities (in this case the rank of the matrix is less than n).
Roughly speaking, we can think at the integrability as the property of a classical system to exhibit regular (quasi-periodic) behaviour, which in turn is a track of its solvability. In particular, a fundamental result holding in the theory of integrable systems is known as *Arnold-Louville theorem*. The basic idea behind such a theorem is to find out a suitable canonical tranformation that linearize the problem. We recall that for a *canonical* change of variables $(\mathbf{x}, \mathbf{p}) \to (\bar{\mathbf{x}}, \bar{\mathbf{p}})$ on the $2N$-dimensional phase space, i.e.:

$$\begin{cases} \bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{x}, \mathbf{p}) \\ \bar{\mathbf{p}} = \bar{\mathbf{p}}(\mathbf{x}, \mathbf{p}) \end{cases} \quad \text{with (local) inverse} \quad \begin{cases} \mathbf{x} = \mathbf{x}(\bar{\mathbf{x}}, \bar{\mathbf{p}}) \\ \mathbf{p} = \mathbf{p}(\bar{\mathbf{x}}, \bar{\mathbf{p}}) \,, \end{cases} \tag{1.17}$$

such as (for $1 \leq i, j \leq N$):

$$\{\bar{x}_i(\mathbf{x}, \mathbf{p}), \bar{x}_j(\mathbf{x}, \mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = \{\bar{x}_i(\mathbf{x}, \mathbf{p}), \bar{x}_j(\mathbf{x}, \mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = 0 \,, \quad \{\bar{x}_i(\mathbf{x}, \mathbf{p}), \bar{p}_j(\mathbf{x}, \mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = \delta_{ij} \,, \tag{1.18}$$

the Hamilton's equations preserve their form, since the Poisson structure remains unchanged. Moreover, it is also known that such transformations can be defined by the means of generating functions [2, 8].
The aforementioned theorem states that [2, 58] for any integrable system defined in a $2N$-dimensional phase space $\mathcal{M}$, which is endowed with $f_{k=1,\dots,N}(\mathbf{x}, \mathbf{p})$ first integrals in involution that are functionally independent on the intersection of the level sets of the N functions $f_k = c_k$, i.e. in the N-dimensional (compact and connected) level surface $\mathcal{M}_f \doteq \{(\mathbf{x}, \mathbf{p}) \in \mathcal{M} : f_k = c_k \in \mathbb{R}\}$, which is diffeomorphic to a torus $\mathbb{T}^N \doteq \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, it is possible to introduce a set of canonical *action-angles variables* (in a neighborhood of $\mathbb{T}^N$): $J_1 \dots J_N$ and $\theta_1 \dots \theta_N$, where $\theta_{k=1,\dots,N} \in [0, 2\pi]$ are cyclic coordinates, in such a way the Hamilton's equations are linearized, i.e.:

$$\begin{cases} \dot{\boldsymbol{\theta}} = +\nabla_{\mathbf{J}} \tilde{H} = \boldsymbol{\omega}(\mathbf{J}) \\ \dot{\mathbf{J}} = -\nabla_{\boldsymbol{\theta}} \tilde{H} = \mathbf{0} \,, \end{cases} \tag{1.19}$$

where the action variables $J_1, \dots, J_N$, and then also the *frequencies* $\boldsymbol{\omega}(\mathbf{J}) = (\omega_1(\mathbf{J}), \dots, \omega_N(\mathbf{J}))$, are first integrals for the new tranformed Hamiltonian $\tilde{H} = H(\mathbf{x}(\mathbf{J}, \boldsymbol{\theta}), \mathbf{p}(\mathbf{J}, \boldsymbol{\theta})) \equiv H(\mathbf{J})$, which is a function of just the action coordinates. In this way the integration is straightforward, the dynamics is characterized by N circular motions with constant angular velocities:

$$\mathbf{J}(t) = \mathbf{J}(0) \qquad \boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \boldsymbol{\omega}(\mathbf{J})t \,. \tag{1.20}$$

This implies that integrable systems are solvable by performing a finite number of integrations and algebraic manipulations of given functions or, in other words, by quadratures. The trajectory (1.20) may be closed or it may cover the torus densely, and this is a consequence of the values that the angular velocities assume. For example, in dimension $N = 2$, the trajectory will be closed if the ratio $\frac{\omega_1}{\omega_2} = \frac{m}{n} \in \mathbb{Q}$ (where $m, n \in \mathbb{N}^*$) and dense otherwise (in this case the motion is quasi-periodic).
An extension of such result has been obtained also in the case of non-commutative algebras of constants of motion where, under suitable hypotesis [59, 60], it has been proved that the trajectory can be also calculated by means of quadratures (see also the recent paper [61]).

Sometimes, Hamiltonian systems can be characterized by a very special property, stronger

than the integrability. This property, called *superintegrability*, requires for such systems to have additional constants of motion besides the usual N that are required in the definition of integrability. In particular, there exists a notion of *minimal* and *maximal* superintegrability, in relation with the total number of first integrals that the Hamiltonian system possesses. Maximal superintegrability (MS) requires the existence of $2N-2$ integrals of motion that, together with the Hamiltonian, form a set of $2N-1$ functionally independent functions on the phase space (the maximum number allowed). More precisely, *a classical Hamiltonian system in N dimensions is (polynomially) superintegrable if it admits $N+m$ with $m=1,\dots,N-1$ functionally independent constants of the motion that are (polynomial in the momenta and) globally defined, except possibly for singularities on a lower dimensional manifold. It is minimally superintegrable if $m=1$ and maximally superintegrable if $m=N-1$* [4].
For example, in dimensions $N=2$, a system will be *superintegrable* if there exist 2 first integrals $f_{i=1,2}$ polynomial in the momenta that, together with the Hamiltonian H, form a functionally independent set $\{H,f_1,f_2\}$ of constants of motion, such that $\{f_i,H\}_{(\mathbf{x},\mathbf{p})}=0$, with $\{f_1,f_2\}_{(\mathbf{x},\mathbf{p})}\neq 0$. Let us observe that the constants of motion are not in mutual involution, but their Poisson brackets will close in general a non-abelian polynomial algebra or, in some exeptional case, finite dimensional Lie Algebras or more complicated structures, such as Kac-Moody algebras [4,62]. The analysis of these algebras represents one of the (sub)domains of research that has its own mathematical interest for the community working on the subject. Superintegrable systems are always presented together with their symmetry algebras. The maximal order (in the momenta) of the first integrals (apart from H) defines the order of the classical superintegrable system (is in some sense its ID). This means that by definition we will have an m-order superintegrable systems if its defining symmetries are of order m in the momenta. Let us remark that several distinct N-subsets of the $2N-1$ functionally independent polynomial constants of the motion for a superintegrable system could be in involution (is the maximum number allowed), and this fact gives rise to the notion of *multi-integrability* [4]. Physically, the importance of classical MS systems is related to the fact that, at least in principle, the trajectories can be calculated without resorting to any differential calculus. In fact, due to their large number of symmetries, they can be solved algebraically: maximal superintegrability restricts trajectories in the phase space to be curves, and implies that all finite trajectories are closed and motion is periodic [4,63]. More precisely, since the existence of a constant of motion forces the trajectory to lie in an hypersurface of dimension $2N-1$, then the existence of $2N-1$ first integrals, let's say $f_1=H,\dots,f_{2N-1}$, restricts the trajectory in the phase space to be the common intersection of such hypersurfaces, let's say $f_1=E, f_j=c_j$, where the real constants E,c_j $(j=2,\dots,2N-1)$ are uniquely determined by imposing the initial conditions [4,64]. In poor words, the existence of an extra set of $m=N-1$ first integrals, besides the N required for the integrability, forces the trajectory to lie in a $2N-(2N-1)=1$ dimensional submanifold of the phase space, i.e. a curve in $\mathcal{M}$.
The two most important models of (maximally) superintegrable systems, which we will take as prototype examples to illustrate the basic definitions we need to introduce, are the *harmonic oscillator* (*HO*) and the *Kepler-Coulomb* (*KC*) system. On one hand, the harmonic oscillator is a universal model in physics: a mass at equilibrium under the influence of any conservative force, in the limit of small perturbations, behaves as a simple harmonic oscillator. On the other hand, the Kepler-Coulomb system is of crucial importance because it describes both the motion of planets around the sun, governed by the three Kepler's laws, and the interaction between

the proton and the electron, governed by the Coulomb attraction force (the hydrogen atom). Their maximal superintegrability is related to the existence of two physical quantities besides the ones related to the rotational symmetry. They are the *Demkov-Fradkin* (DF) tensor [9, 10] and the *Laplace-Runge-Lenz* (LRL) vector [8] respectively. In particular, a milestone result in the theory of superintegrable systems dated back to 1873, due to J. L. F. Bertrand, is the so-called *Bertrand Theorem* [3]. It asserts that *any three-dimensional spherically symmetric natural Hamiltonian system in (a subset of) the Euclidean space, described by the Hamiltonian function*:

$$\begin{cases} H(\mathbf{x},\mathbf{p}) = \frac{1}{2}\sum_{i,j=1}^{3}\delta^{ij}p_ip_j + V(|\mathbf{x}|) = \frac{\mathbf{p}^2}{2} + V(|\mathbf{x}|) \qquad (|\mathbf{x}| \doteq \sqrt{\mathbf{x}^2} = \sqrt{x_1^2+x_2^2+x_3^2}\,) \\ \mathrm{d}s^2 = \sum_{i,j=1}^{3}\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \sum_{i=1}^{3}(\mathrm{d}x^i)^2\,, \end{cases} \tag{1.21}$$

such as for each point of its configuration space $\mathbb{R}^3$ *there exists a stable circular trajectory passing through it, and all whose bounded trajectories are closed, is either a harmonic oscillator (HO) or a Kepler-Coulomb system (KC).*
This means that the two Hamiltonians:

$$\begin{cases} H_{HO}(\mathbf{x},\mathbf{p}) \doteq \frac{\mathbf{p}^2}{2} + V_{HO}(|\mathbf{x}|) = \frac{\mathbf{p}^2}{2} + \frac{1}{2}\omega^2\mathbf{x}^2 \\ H_{KC}(\mathbf{x},\mathbf{p}) \doteq \frac{\mathbf{p}^2}{2} + V_{KC}(|\mathbf{x}|) = \frac{\mathbf{p}^2}{2} - \frac{k}{|\mathbf{x}|}\,, \end{cases} \tag{1.22}$$

where ω, k are two positive constants, which describe the motion in the configuration space $\mathbb{R}^3$ of a point particle (of unit mass) under the influence of the forces:

$$\begin{cases} \mathbf{F}_{HO}(\mathbf{x}) = -\nabla_{\mathbf{x}}V_{HO}(|\mathbf{x}|) = -\omega^2\mathbf{x} \\ \mathbf{F}_{KC}(\mathbf{x}) = -\nabla_{\mathbf{x}}V_{KC}(|\mathbf{x}|) = -\frac{k}{|\mathbf{x}|^3}\mathbf{x}\,, \end{cases} \tag{1.23}$$

are the only two spherically symmetric systems that possess the maximal superintegrability property. As we said, this result is related to the existence of the above mentioned quantities, which provide another set of constants of motion besides the ones generated by the three components of the angular momentum (related to the rotational symmetry), and that can be used to construct the orbit's equation [65]. Explicitly, the LRL vector $\mathcal{R}$, defined as:

$$\mathcal{R} \doteq \mathbf{L}\times\mathbf{p} + \frac{k}{|\mathbf{x}|}\mathbf{x}\,, \tag{1.24}$$

provides the following three additional constants of motion for the *KC* system:

$$\begin{cases} \mathcal{R}_i \doteq \sum_{j=1}^{3}(x_jp_i - x_ip_j)p_j + \frac{k}{|\mathbf{x}|}x_i \qquad (i=1,2,3)\,, \\ \{\mathcal{R}_i, H_{KC}\}_{(\mathbf{x},\mathbf{p})} = 0\,. \end{cases} \tag{1.25}$$

For the sake of clarity we need to mention that the definition (1.24) suffers of a global minus sign from the usual one used in the classical textbook [8]. Clearly, this does not affect the physics of the problem. Concerning the (symmetric) *DF* tensor, it provides the following six conserved quantities for the *HO* system:

$$\begin{cases} \mathrm{I}_{ij} \doteq \frac{p_ip_j}{\omega} + \omega x_ix_j \qquad (i\le j;\, i,j=1,2,3) \\ \{\mathrm{I}_{ij}, H_{HO}\}_{(\mathbf{x},\mathbf{p})} = 0\,. \end{cases} \tag{1.26}$$

Let us observe that both sets of constants of motion are quadratic in the momenta. We are dealing with *second-order superintegrable systems*, a special subclass of superintegrable systems which admits separation of variables both in the classical and the quantum case [4]. Let us recall that the magnitude of the LRL vector $\mathcal{R}$ turns out to be expressible in terms of the Hamiltonian H_{KC} itself and of the modulus squared of the total angular momentum, i.e.:

$$\mathcal{R}^2 = \sum_{i=1}^{3} \mathcal{R}_i^2 = 2\mathbf{L}^2 H_{KC} + k^2 \qquad \text{where} \qquad \mathbf{L}^2 \doteq \sum_{i=1}^{3} \mathrm{L}_i^2 \,, \tag{1.27}$$

where $\mathrm{L}_i \doteq \frac{1}{2}\epsilon_{ijk}\mathcal{L}_{jk}$. Here, we have introduced the usual notation for the rotation generators $\mathcal{L}_{ij} \doteq x_i p_j - x_j p_i$ $(i < j;\, i,j = 1,2,3)$. This relation shows a functional dependence between the Hamiltonian, the angular momentum and the LRL vector. Moreover, it also holds:

$$\mathcal{R} \cdot \mathbf{L} = 0 \,, \tag{1.28}$$

which means that the LRL vector lies in the plane of the orbit (the angular momentum is perpendicular to the orbit).
Thus, at this level, we have a total number of eight conserved quantities $\{H_{KC}, \mathbf{L}^2, \mathbf{L}, \mathcal{R}\}$, together with three functional relations given in (1.27,1.28). This implies that only five of them are independent (as expected).

Concerning the *HO*, a functional relation is given in terms of the trace of the Demkov-Fradkin tensor, which turns out to be proportional to the oscillator Hamiltonian:

$$H_{HO} = \frac{\omega}{2} \sum_{i=1}^{3} \mathrm{I}_{ii} \,. \tag{1.29}$$

Moreover, this tensor is "perpendicular"to the angular momentum in the sense that [65]:

$$\sum_{j=1}^{3} \mathrm{I}_{ij} \mathrm{L}_j = 0 \qquad (i = 1,2,3) \,. \tag{1.30}$$

Thus, we have a total number of eleven conserved quantities $\{H_{HO}, \mathbf{L}^2, \mathbf{L}, \mathrm{I}_{ij}\}$, together with the four functional relations given in (1.29,1.30) and the one relating the components of the angular momentum with its modulus squared. The last functional relation is given by taking the determinant of the DF tensor, which turns out to be zero. This implies that only five of them are independent. Thus, we can choose a functionally independent subset of $2N - 1 = 2 \cdot 3 - 1 = 5$ first integrals, for example:

$$\begin{cases} \mathcal{O}_{KC} \doteq \{H_{KC}, \mathcal{L}_{23}, \mathcal{L}_{13}, \mathcal{L}_{12}, \mathcal{R}_1\} \\ \mathcal{O}_{HO} \doteq \{H_{HO}, \mathcal{L}_{23}, \mathcal{L}_{13}, \mathcal{L}_{12}, \mathrm{I}_{11}\} \,, \end{cases} \tag{1.31}$$

to ensure the maximal superintegrability of the two models. Moreover, the constants $\{H_{KC}, \mathcal{L}_{23}, \mathcal{R}_1\}$, $\{H_{HO}, \mathcal{L}_{23}, \mathrm{I}_{11}\}$ are in mutual involution. As we previously explained, if we are not interested in finding explicitly the functional relations between the constants of motion, it is sufficient to calculate the rank of the 5×6 matrix (1.16), which is equal to 5 for both subsets in (1.31).
Concerning the symmetry algebras of the two MS systems, first of all let us observe that they

are both endowed with an $\mathfrak{so}(3)$ Lie-Poisson symmetry. This is because they are defined on a spherically symmetric space. More precisely, the three functions $\mathcal{L}_{ij}$ satisfy the following $\mathfrak{so}(3)$ Lie-Poisson algebra:

$$\{\mathcal{L}_{12},\mathcal{L}_{13}\}_{(\mathbf{x},\mathbf{p})}=\mathcal{L}_{23}\quad \{\mathcal{L}_{12},\mathcal{L}_{23}\}_{(\mathbf{x},\mathbf{p})}=-\mathcal{L}_{13}\quad \{\mathcal{L}_{13},\mathcal{L}_{23}\}_{(\mathbf{x},\mathbf{p})}=\mathcal{L}_{12}\,. \tag{1.32}$$

Also, we observe that the LRL vector $\mathcal{R}$ closes the Poisson algebra[3]:

$$\{\mathcal{R}_i,\mathcal{R}_j\}_{(\mathbf{x},\mathbf{p})}=-2H_{KC}\mathcal{L}_{ij}\quad (i<j,\ i,j=1,2,3) \tag{1.33}$$

that, together with the Poisson brackets:

$$\{\mathcal{L}_{ij},\mathcal{R}_k\}_{(\mathbf{x},\mathbf{p})}=\delta_{ik}\mathcal{R}_j-\delta_{jk}\mathcal{R}_i\,, \tag{1.34}$$

leads to an $\mathfrak{so}(4)$ *dynamical symmetry algebra*. In fact, by defining the quantities:

$$\tilde{\mathcal{L}}_{0i}\doteq\frac{\mathcal{R}_i}{\sqrt{-2H_{KC}}}\,,\quad \tilde{\mathcal{L}}_{ij}\doteq\mathcal{L}_{ij}\,, \tag{1.35}$$

one finds that the functions $\tilde{\mathcal{L}}_{ij}$ are the generators of an $\mathfrak{so}(4)$ Lie-Poisson algebra, given by:

$$\{\tilde{\mathcal{L}}_{ij},\tilde{\mathcal{L}}_{ik}\}_{(\mathbf{x},\mathbf{p})}=\tilde{\mathcal{L}}_{jk}\quad \{\tilde{\mathcal{L}}_{ij},\tilde{\mathcal{L}}_{jk}\}_{(\mathbf{x},\mathbf{p})}=-\tilde{\mathcal{L}}_{ik}\quad \{\tilde{\mathcal{L}}_{ik},\tilde{\mathcal{L}}_{jk}\}_{(\mathbf{x},\mathbf{p})}=\tilde{\mathcal{L}}_{ij}\quad (i<k<j) \tag{1.36}$$

with $i,j,k=0,1,2,3$. As far as the isotropic harmonic oscillator is concerned, the three angular momentum components L_i, together with components of the DF tensor, lead to the following Poisson brackets:

$$\{\mathrm{L}_i,\mathrm{L}_j\}_{(\mathbf{x},\mathbf{p})}=\epsilon_{ijk}\mathrm{L}_k\,, \tag{1.37}$$

$$\{\mathrm{L}_i,\mathrm{I}_{Jk}\}_{(\mathbf{x},\mathbf{p})}=\epsilon_{ijm}\mathrm{I}_{mk}+\epsilon_{ikm}\mathrm{I}_{jm}\,, \tag{1.38}$$

$$\{\mathrm{I}_{ij},\mathrm{I}_{kl}\}_{(\mathbf{x},\mathbf{p})}=(\delta_{ij}\epsilon_{klm}+\delta_{il}\epsilon_{jkm}+\delta_{jk}\epsilon_{ilm}+\delta_{jl}\epsilon_{ikm})\mathrm{L}_m\,, \tag{1.39}$$

and in particular the five (traceless) components of the DF tensor, together with the three components of $\mathbf{L}$, yields to an $\mathfrak{su}(3)$ *symmetry algebra* [65].

Besides these two prototype examples (which are very special since they are defined on a spherically symmetric space), over the years many other superintegrable systems have been discovered and extensively investigated, also in connection with applications that have been found in condensed matter physics, nuclear physics and celestial mechanics (see [6, 66–69] and references therein). The research involved also a deep interest in quantum systems, where a notion of superintegrability also exists. In particular, an influential conjecture that has been proposed in [11], states that *all superintegrable quantum systems are exactly solvable*, and the solution is given in terms of special functions (basically hypergeometric orthogonal polynomials multiplied by some gauge factor). This conjecture, which works for all known superintegrable systems, has been of crucial importance since established a relation between the concepts of superintegrability and exact solvability in quantum mechanics. Clearly, this fact increased the interest for such quantum systems that, once found (and this step is highly not trivial), can be explicitly solved. In that case, the last step would be find a physical application for the model.

3 Let us point out that this is not a Lie-Poisson algebra, and the reason is related to the presence of the quadratic term involving the Hamiltonian. It can be considered as a Poisson-Lie algebra if we restrict the Hamiltonian on the level surface $H=E\in\mathbb{R}$.

2.2 SUPERINTEGRABILITY IN QUANTUM MECHANICS

We have just seen that in classical mechanics the state of a N-dimensional system is represented in terms of points in phase space, which has the structure of a $2N$-dimensional manifold[4] $\mathcal{M}$, and the observables are C^∞ real-valued functions on $\mathcal{M}$. A state of a quantum mechanical system is instead defined by a vector belonging to an infinite dimensional (separable) complex Hilbert space $\mathcal{H}$, and the observables are described in terms of Hermitian operators on it. One of these observables is the Hamiltonian $\hat{H}$, which we can use to perform the evolution of a quantum observable $\hat{f}$ under the *Heisenberg equation* [71]:

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{f}, \hat{H}]\,. \tag{2.40}$$

This relation has to be thought as the quantum counterpart of (1.14), and suggests a direct correspondence between classical and quantum dynamics [72]. Here, we have introduced the reduced Planck's constant $\hbar \doteq h/2\pi$, and the new skew-symmetric bilinear operation $[\cdot,\cdot]$, *the commutator*. The latter satisfies the following properties: if $\hat{f}_j$ $(j = 1, 2, 3)$ are three operators defined on a given Hilbert space $\mathcal{H}$, and α, β are two complex constants, then it holds:

$$\begin{cases} [\hat{f}_1, \hat{f}_2] = -[\hat{f}_2, \hat{f}_1] \\ [\alpha\hat{f}_1 + \beta\hat{f}_2, \hat{f}_3] = \alpha[\hat{f}_1, \hat{f}_3] + \beta[\hat{f}_2, \hat{f}_3] \\ [\hat{f}_3, [\hat{f}_1, \hat{f}_2]] + [\hat{f}_2, [\hat{f}_3, \hat{f}_1]] + [\hat{f}_1, [\hat{f}_2, \hat{f}_3]] = 0 \\ [\hat{f}_3, \hat{f}_1\hat{f}_2] = [\hat{f}_3, \hat{f}_1]\hat{f}_2 + \hat{f}_1[\hat{f}_3, \hat{f}_2]\,. \end{cases} \tag{2.41}$$

These properties show that the commutator is, like the Poisson bracket, a *skew-symmetric bilinear* operation satisfying the *Jacobi identity* and the *Leibniz rule*. In contrast to classical mechanics, where the product of functions on phase space is both associative and commutative, here the product of operators is still associative but not commutative, and this gives rise to ordering problems. As we know, the commutator replace the Poisson brackets at the quantum level. This means that if a mapping between classical and quantum observables exists, then it has to be defined in such a way to satisfy the "*Poisson brackets* $\to$ *commutators rule*" [73]. For example, in the Euclidean space $\mathbb{R}^N$, the classical position and momentum coordinates $(x_i, p_i) \in \mathcal{M}$ are replaced by self-adjoint operators $(\hat{x}_i, \hat{p}_i)$ defined on a given Hilbert space $\mathcal{H}$:

$$x_i \to \mathcal{Q}_\hbar(x_i) \doteq \hat{x}_i\,, \quad p_i \to \mathcal{Q}_\hbar(p_i) \doteq \hat{p}_i \qquad (i = 1, \dots, N)\,, \tag{2.42}$$

such that:

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0\,, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{\mathbb{1}}\,, \tag{2.43}$$

where $\mathcal{Q}_\hbar : \mathfrak{h}_c(N) \subset \mathcal{P}_{\mathbb{R}^N} \to \mathfrak{h}(N)$ is a quantization map that links the (classical) elements of (1.11) to the (quantum) elements of $\mathfrak{h}(N)$, the abstract $2N+1$ dimensional *Heisenberg algebra* (the basic algebra of quantum observables). Actually, the quantization procedure is a very hard task from the mathematical point of view, and several (still open) problems have been faced. Roughly speaking, the general idea behind such a procedure is to "quantize"

4 Precisely, it is a $2N$-dimensional *symplectic* manifold, i.e. a $2N$-dimensional manifold endowed with a closed (non degenerate) two form $\omega_0 \doteq \sum_{i=1}^N \mathrm{d}x_i \wedge \mathrm{d}p_i$, written in local coordinates x_i, p_i (that always exist because of the Darboux's Theorem [70]).

consistently a Lie subalgebra $\mathcal{U}$ (containing the Heisenberg algebra) of the full *Poisson algebra* $\mathcal{P}_{\mathbb{R}^N} \doteq (C^\infty(\mathbb{R}^{2N}), \{\cdot,\cdot\}, \cdot)$, i.e. to assign at each element of $\mathcal{U}$ a self-adjoint operator on a given Hilbert space $\mathcal{H}$, in such a way for any $f, g \in \mathcal{U} \subset \mathcal{P}_{\mathbb{R}^N}$ some fundamental properties have to be satisfied, among which the well-known Poisson brackets $\rightarrow$ commutators rule, i.e.:

$$\mathcal{Q}_\hbar(\{f,g\}) = -\frac{i}{\hbar}[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] \qquad (\text{together with } \mathcal{Q}_\hbar(1) = \hat{\mathbb{1}})\,, \tag{2.44}$$

which shows that $\mathcal{Q}_\hbar$ is an homomorphism of the corresponding Lie bracket structures. The fact that one is forced to take a subalgebra of $\mathcal{P}_{\mathbb{R}^N}$ is related to the restriction that one has in quantizing the (full) Poisson algebra $\mathcal{P}_{\mathbb{R}^N}$ [74–77]. We will not deal explicitly with such issues because, besides to be very complicated, they are also behind the purposes of this paragraph. As far as we are concerned, a standard representation for the basic observables in ordinary quantum mechanics is given through the so-called *Schrödinger quantization prescription* [72]:

$$\hat{x}_j = x_j\,, \quad \hat{p}_j = -i\hbar\partial_{x_j} \quad (j = 1, \ldots, N)\,, \tag{2.45}$$

for which the $2N+1$ dimensional *Heisenberg algebra* (2.43) is fulfilled. Clearly, the commutation relations (2.43) have to be thought as the quantum version of the classical (canonical) relations (1.11), now expressed in terms of Hermitian operators on a given Hilbert space instead of positions and momentum coordinates on the phase space manifold $\mathcal{M}$.
The notion of Hermiticity is strictly related to the inner product of the Hilbert space where the quantum model is defined. For example, in a standard Euclidean space $\mathbb{R}^N$, the state of a quantum mechanical system with N degrees of freedom, at time t, is characterized by (state) vectors represented by square integrable functions on $\mathcal{H} \doteq L^2(\mathbb{R}^N, d\mu(\mathbf{x}))$, $d\mu(\mathbf{x}) \doteq d\mathbf{x}$ being the measure on $\mathbb{R}^N$. They are complex-valued functions $\psi : \mathbb{R}^N \rightarrow \mathbb{C}$ that are usually normalized in such a way that $||\psi||^2 \doteq \langle\psi,\psi\rangle = 1$, where $\langle,\rangle$ denotes the inner product:

$$\langle\psi,\varphi\rangle \doteq \int_{\mathbb{R}^N} \psi^*(\mathbf{x},t)\varphi(\mathbf{x},t)\,\mathrm{d}\mu(\mathbf{x})\,. \tag{2.46}$$

In this case, if $\hat{f}$ is Hermitian, then:

$$\langle\hat{f}\,\psi,\varphi\rangle = \int_{\mathbb{R}^N} (\hat{f}\,\psi(\mathbf{x},t))^*\varphi(\mathbf{x},t)\,\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbb{R}^N} \psi^*(\mathbf{x},t)(\hat{f}\varphi(\mathbf{x},t))\,\mathrm{d}\mu(\mathbf{x}) = \langle\psi,\hat{f}\,\varphi\rangle\,. \tag{2.47}$$

As we previously said, this subset of operators is of fundamental importance in the quantum theory since they correspond to observable quantities. In particular, unlike what one has in (deterministic) classical mechanics, here one has to deal with probability density and average values of observables [71, 72]. In this sense, the self-adjoint operators $\hat{x}_i, \hat{p}_i$ can be used to calculate the (average) value of positions and momenta $\langle x_i\rangle_t$, $\langle p_i\rangle_t$ in the following way:

$$\begin{aligned}
\langle x_i\rangle_t &\doteq \langle\psi,\hat{x}_i\psi\rangle = \int_{\mathbb{R}^N} x_i|\psi(\mathbf{x},t)|^2\,\mathrm{d}\mu(\mathbf{x})\,,\\
\langle p_i\rangle_t &\doteq \langle\psi,\hat{p}_i\psi\rangle = \int_{\mathbb{R}^N} \psi^*(\mathbf{x},t)(-i\hbar\partial_{x_i}\psi(\mathbf{x},t))\,\mathrm{d}\mu(\mathbf{x})\,.
\end{aligned} \tag{2.48}$$

Here the modulus squared of the wavefunction $|\psi(\mathbf{x},t)|^2$ plays the role of *probability density*. Its physical meaning is clearly explained in [71]: *"when we use a detector that ascertains the presence of the particle within a small volume element $d\mathbf{x}$ around $\mathbf{x}$, the probability of recording a positive result at time t is given by $|\psi(\mathbf{x},t)|^2 d\mathbf{x}$ "*.

The equation that governs the dynamics of quantum mechanical states is the *time-dependent* Schrödinger equation:

$$i\hbar\,\partial_t\psi(\mathbf{x},t) = \hat{H}\psi(\mathbf{x},t)\,, \tag{2.49}$$

where the Hamiltonian $\hat{H}$ is the operator given by:

$$\hat{H} = \hat{H}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2} + V(\hat{\mathbf{x}}) = -\frac{\hbar^2}{2}\Delta_{\mathbf{x}} + V(\mathbf{x})\,, \tag{2.50}$$

where we restricted our consideration to time-independent Hamiltonians. Moreover, the following notation for the Laplacian operator: $\Delta_{\mathbf{x}} \doteq \nabla^2_{\mathbf{x}} = \frac{\partial^2}{\partial^2 x_1} + \cdots + \frac{\partial^2}{\partial^2 x_N}$ has been introduced. Here, the conservation of the energy is related to the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{H}\rangle_t = 0\,, \tag{2.51}$$

which, in analogy with classical mechanics, should imply the existence of a real number E, such that (the expectation value of) the Hamiltonian always takes such a value $\forall t \in \mathbb{R}$. In particular, for the Hamiltonian (2.50), the wavefunction $\psi(\mathbf{x},t)$ oscillates in time according to:

$$\psi(\mathbf{x},t) \sim \exp(-\frac{i}{\hbar}E\,t)\Psi(\mathbf{x})\,. \tag{2.52}$$

This formula is obtained by separation of variables in the time-dependent Schrödinger equation (2.49). In this case, the equation characterizing the so-called "stationary states" is the *time-independent* Schrödinger equation:

$$\hat{H}\,\Psi(\mathbf{x}) = E\,\Psi(\mathbf{x})\,, \tag{2.53}$$

which is an eigenvalue equation for the Hamiltonian operator. For example, in the Euclidean case, equation (2.53) results in:

$$-\frac{\hbar^2}{2}\Delta_{\mathbf{x}}\Psi(\mathbf{x}) + V(\mathbf{x})\Psi(\mathbf{x}) = E\,\Psi(\mathbf{x})\,. \tag{2.54}$$

Similarly to classical mechanics, where we are interested in solving equations of motion to find trajectories in phase space, here the goal is to solve the equation (2.54) in order to find spectrum and eigenfunctions of the quantum system under investigation. Generally, if the system is exactly solvable, the eigenstates will be given in terms of families of polynomials, which will be orthogonal in the domain of definition of the model. Also, the solution of (2.54) will depend on the coordinate system that is chosen to solve it. For example, we can think of the N-dimensional harmonic oscillator expressed in cartesian coordinates, whose solution is factorized in terms of the Hermite polynomials (for each coordinate $x_{j=1,\dots,N}$), multiplied by a factor given by the product of the single gaussians arising as kernels of the lowering operators that factorize the problem. Here, the model is defined in $\mathbb{R}^N$ and the Hilbert space is $\mathcal{H} \doteq L^2(\mathbb{R}^N, d\mathbf{x})$. The same problem, when expressed in (hyper)spherical coordinates, leads to a radial solution given in terms of the generalized Laguerre polynomials, which are orthogonal on the positive semi-line, with respect to a weight function given by a gaussian in the radial coordinate $r > 0$ times a factor r^l, where l is the angular momentum quantum number coming from the angular part of the solution (given in terms of hyperspherical harmonics).

When one deals with systems that are defined on non-Euclidean spaces, i.e. in some Riemannian manifold $\mathcal{M}$, it is possible to define a quantum analog of the classical Hamiltonian (1.4), which takes into account the ordering issues arising in the quantum theory (and in particular the fact that in this case the metric tensor depends on the position operator). The representation of such Hamiltonian operator turns out to be [4]:

$$\hat{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = -\frac{\hbar^2}{2}\Delta_{\mathcal{M}} + V(\mathbf{x})\,, \tag{2.55}$$

where:

$$\Delta_{\mathcal{M}} \doteq \sum_{i,j=1}^{N} \frac{1}{\sqrt{g}}\partial_{x_i}(\sqrt{g}\, g^{ij}\partial_{x_j})\,, \qquad g \doteq \det g_{ij}\,, \tag{2.56}$$

is the so-called *Laplace-Beltrami* operator, which represents a generalization of the Laplace operator, the latter being recovered in the Euclidean case, e.g. when $g_{ij} = \delta_{ij}$. In this case, the time-independent Schrödinger equation reads:

$$-\frac{\hbar^2}{2}\Delta_{\mathcal{M}}\Psi(\mathbf{x}) + V(\mathbf{x})\Psi(\mathbf{x}) = E\,\Psi(\mathbf{x})\,, \tag{2.57}$$

and the inner product will be equipped with the new (hyper)volume measure $\mathrm{d}\mu_g(\mathbf{x}) \doteq \sqrt{g}\,\mathrm{d}\mathbf{x}$.

Now, since our aim is to define the concepts of integrability and superintegrability in quantum mechanics, as we made in the classical case, we need to understand the meaning of quantum conserved quantities in this framework, and how this notion can be useful in the analysis of quantum models. In particular, by taking into account the Heiseinberg equation (2.40), the rate of variation of the expectation value of an observable $\hat{f}$ will be calculated according to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{f}\rangle_t = -\frac{i}{\hbar}\langle[\hat{f}, \hat{H}]\rangle_t\,, \tag{2.58}$$

and this shows that, in full analogy to classical mechanics, there will be no (average) variation on time if and only if $[\hat{f}, \hat{H}] = 0$, i.e. if $\hat{f}$ and $\hat{H}$ are two *compatible* observables. In that case, we can choose a basis of eigenfunctions on the Hilbert space that are eigenstates of both $\hat{f}$ and $\hat{H}$. For this reason, one of the fundamental challenges in quantum mechanics is to find out (observable) quantities commuting with the Hamiltonian to characterize the system [71]. From this point of view, the interest in introducing a notion of integrability also in the quantum framework arose quite naturally, and a rigorous definition of quantum integrability has been formulated. Precisely, it states that *an N-dimensional quantum Hamiltonian system is integrable if there exist N integrals of motion* $\hat{f}_j$, $(j = 1, \ldots, N)$ *satisfying the following conditions:*

1. *they are well-defined self-adjoint operators in the enveloping algebra of the Heisenberg algebra* $\mathfrak{h}(N)$*, or convergent series in the basis vectors* $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$*;*

2. *the integrals* $\hat{f}_j$ $(j = 1, \ldots, N)$ *commute pair-wise;*

3. *they are algebraically independent.*

Moreover, if the quantum integrals are *finite order* partial differential operators, we will speak of *finite order integrability* (which is the quantum analogue of polynomial integrability) [4]. As we previously discussed, in classical mechanics the notion of functional independence

is related to the rank of the matrix (1.16), which is a well-defined (also operative) property. In quantum mechanics, such a notion is not so straightforward and, as the authors claim in [4]: "*there is no upon-agreed operator equivalence for this concept*". In particular, they consider a set of n operators $\hat{f}_{j=1\dots n}$ *algebraically independent*, if there is no nonzero Jordan polynomial that vanishes identically. This means that there is no symmetrized polynomial $\mathcal{P}$ in n non-commuting variables such that $\mathcal{P}(\hat{f}_1, ..., \hat{f}_n) = 0$. We might think at a such algebraic relation as the quantum analog of the classical relations $\mathcal{F}(f_1, \dots, f_n) = 0$, where in that case we recall that $\mathcal{F}$ was a nonzero analytic function defined in some region of the phase space.
In complete analogy to classical mechanics, we can also provide a notion of *quantum superintegrability* (of finite order). We will say that *a quantum system in N dimensions is superintegrable (of finite order) if it admits $N + m$, $m = 1, \dots, N - 1$ algebraically independent finite order partial differential operators $\hat{f}_1 = \hat{H}, \dots, \hat{f}_{N+m}$ in the variables $\mathbf{x}$ globally defined (except for singularities on lower dimensional manifolds), such that $[\hat{f}_j, \hat{H}] = 0$. It is minimally superintegrable if $m = 1$ and maximally superintegrable if $m = N - 1$* [4].
To make clear these points, let us think about the classical oscillator and Kepler-Coulomb problems that we have previously discussed. In these cases, the quantum integrals of motion required for the maximal superintegrability are given by the quantum counterpart of the LRL vector[5] and DF tensor respectively, i.e.:

$$\hat{\mathcal{R}}_i \doteq \frac{1}{2}\sum_{j=1}^{3}(\hat{x}_j\hat{p}_i - \hat{x}_i\hat{p}_j)\hat{p}_j + \frac{1}{2}\sum_{j=1}^{3}\hat{p}_j(\hat{x}_j\hat{p}_i - \hat{x}_i\hat{p}_j) + \frac{k}{|\hat{\mathbf{x}}|}\hat{x}_i \,, \tag{2.59}$$

$$\hat{\mathrm{I}}_{ij} \doteq \frac{\hat{p}_i\hat{p}_j}{\omega} + \omega\hat{x}_i\hat{x}_j \,, \tag{2.60}$$

for $i, j = 1, 2, 3$, where ordering issues have been taken into account. These quantities, together with the angular momentum operators $\hat{\mathcal{L}}_{ij} \doteq \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i$, commute with the corresponding quantum Hamiltonians $\hat{H}_{KC}$ and $\hat{H}_{HO}$:

$$\begin{cases} \hat{H}_{HO}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2} + \frac{1}{2}\omega^2\hat{\mathbf{x}}^2 = -\frac{\hbar^2}{2}\Delta_{\mathbf{x}} + \frac{1}{2}\omega^2\mathbf{x}^2 \\ \hat{H}_{KC}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2} - \frac{k}{|\hat{\mathbf{x}}|} = -\frac{\hbar^2}{2}\Delta_{\mathbf{x}} - \frac{k}{|\mathbf{x}|} \,, \end{cases} \tag{2.61}$$

and we can choose a subset of $2N - 1 = 2 \cdot 3 - 1 = 5$ functionally independent first integrals:

$$\begin{cases} \mathcal{O}_{KC} \doteq \{\hat{H}_{KC}, \hat{\mathcal{L}}_{23}, \hat{\mathcal{L}}_{13}, \hat{\mathcal{L}}_{12}, \hat{\mathcal{R}}_1\} \\ \mathcal{O}_{HO} \doteq \{\hat{H}_{HO}, \hat{\mathcal{L}}_{23}, \hat{\mathcal{L}}_{13}, \hat{\mathcal{L}}_{12}, \hat{\mathrm{I}}_{11}\} \,, \end{cases} \tag{2.62}$$

to determine the maximal superintegrability. Futhermore, the quantum version of the classical Lie-Poisson algebras $\mathfrak{so}(4)$ and $\mathfrak{su}(3)$ can be used to construct the spectrum of the two quantum systems with a group-theoretical approach [79, 80]. Even in this quantum case, by definition, we are dealing with second-order superintegrable systems. In fact, both the (quantum) LRL and DF tensor are represented by second-order partial differential operators.

To sum up:

- in classical mechanics, superintegrability restricts trajectories to a $N - m$ dimensional subspace of the phase space ($0 < m < N$). In the case of maximal superintegrability,

5 To the best of our knowledge, the quantum LRL vector has been proposed for the first time by Pauli in [78].

i.e. when $m = N - 1$, all bounded orbits are closed and the motion is periodic. In principle, the trajectories can be performed algebraically, with no need to calculate any integral. Moreover, as we mentioned, the Bertrand Theorem [3] states that the only two spherically symmetric potentials (in the three dimensional Euclidean space) for which all bounded trajectories are closed, are the harmonic oscillator and the Kepler-Coulomb potentials. This means that no other superintegrable systems on such space are spherically symmetric. Concerning the algebras of the integrals of motion, we saw that they are in general non Abelian. Typically, they are finitely generated (polynomial) algebras [4].

- In quantum mechanics, most of the systems that have been shown to be maximally superintegrable, have also been shown to be exactly solvable. This support the truthfulness of the conjecture proposed in [11]. In particular, the maximal superintegrability leads to an additional "accidental" degeneracy in the spectrum, which is now understood in terms of superintegrability. The polynomial algebras of first integrals can be used to construct the spectra and eigenfunctions of quantum systems (see for example [79–81]).

The search and classification for MS systems has been performed over the years by using many different approaches. A possibility consists in considering a general Hamiltonian function $H(\mathbf{x}, \mathbf{p}) = T(\mathbf{x}, \mathbf{p}) + V(\mathbf{x})$ and imposing the existence of a set of constants of motion $\{f_i\}$ (resp. $\{\hat{f}_i\}$ in the quantum case), under some specific assumptions, such that $\{f_i, H\}_{(\mathbf{x},\mathbf{p})} = 0$ (resp. $[\hat{f}_i, \hat{H}] = 0$). The above constraints turn into a set of determining equations which can be used to completely classify a given class of MS systems. However, the complexity of these determining equations grows in a severe way with the Hamiltonian degrees of freedom, and this partially justifies the abundance of studies of superintegrable systems in two dimensions [82–84]. In order to construct higher dimensional superintegrable systems without tackling the above mentioned issues, Ballesteros et al. [23–25] introduced a novel algebraic approach based on Hopf Algebras, in particular on their coalgebra sector. The main aim of the next chapter is to explain the basic notions of such approach to superintegrability. In particular, we will show how it works in classical and ordinary quantum mechanics and, after an overview of the fundamental ideas behind this algebraic technique, we will discuss the way how to introduced it in a discrete quantum-mechanical framework, which is our original contribution to the field.

3 COALGEBRA SYMMETRY AND SUPERINTEGRABILITY

In order to overcome the technical problems related to the multidimensional extensions of superintegrable systems, a powerful method has been introduced. It is known with the name of *coalgebra symmetry*. This technique consists in defining both the Hamiltonians and its constants of motion as functions of generators of a given algebra equipped with a coproduct. Once an algebra representation is chosen, then the coproduct can be used to rise the dimension of the representation without losing the superintegrability properties. This is because the coproduct provides, at each application, a set of additional symmetries "*the partial Casimirs*", which help to keep the system superintegrable. This method has been successfully applied: several well-known classical integrable and superintegrable systems have been recovered (and reinterpreted from this new "external"point of view), and new families have been discovered, see e.g. [26–30]. This approach to superintegrability is still extensively used in the literature because of its wide range of applicability, both in classical and in quantum mechanics, and there are very recent papers on the subject where the coalgebraic analysis has been applied to give new insights about the already known classifications of superintegrable systems: in [85], for example, it has been shown that a canonical transformation can generate different coalgebraic systems which, once embedded in higher dimensional spaces, generate genuinely new superintegrable systems as deformations, or generalizations, of TTW systems [86, 87] to non-Euclidean spaces. The same philosophy has been reproposed in [88], involving a gauge transformation applied to a two dimensional scalar Hamiltonian. In that case the two-dimensional coalgebraic Hamiltonian, when realized in three dimensions, turned out to be a non-scalar superintegrable Hamiltonian with spin interactions.

The main aim of the chapter is to carry along this path, by showing that the coalgebra symmetry analysis can be succesfully introduced also in a *discrete framework*, where differential operators are replaced by finite difference operators. The chapter is organized as follows, first of all we will review the basics of the coalgebra symmetry approach for superintegrable systems, focusing the analysis on the classical case. Since the approach is purely algebraic, the quantum case is very similar, and conceptually nothing changes. We will introduce the method in its own generality but, to make it as clear as possible, we will explicitly discuss the case of the harmonic oscillator, elucidating its $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry and constructing explicitly its constants of motion for the N-dimensional (both classical and quantum) case. After that, we will introduce our original work, based on the results contained in [40], which show that is possible to introduce the concepts proper of the coalgebra symmetry approach in the realm of discrete quantum superintegrable systems.

3.1 COALGEBRA SYMMETRY AND SUPERINTEGRABLE SYSTEMS

The coalgebra method, introduced by A. Ballesteros and O. Ragnisco in [24], is an algebraic approach that allows to *propagate* the integrability of a (classical or quantum) mechanical

system to any arbitrary dimension starting from a one dimensional problem, that is (trivially) integrable. The key idea is to use Hopf algebras, more precisely their coalgebra sector, in order to construct higher dimensional versions of the system with the help of the *coproduct*. In fact, because of the *homomorphism* and *coassociativity* properties that this map satisfies, it is possible to propagate the integrability by constructing two sets of integrals of motion arising from the so-called *right* and *left* Casimirs of the algebra. In the following, we will review the basic ideas of such approach to superintegrability by taking classical mechanics as "target framework". Anyway, as we will show later, the method equally works in quantum mechanics: in all the construction we have to think the Poisson brackets replaced by the commutators. Before explaining how the method works, let us briefly recall some definitions that will be useful in the following.

3.1.1 Poisson-Hopf algebras and the homomorphism property

First of all we recall that a *Hopf* algebra $\mathfrak{A}$ [89] is a vector space over a field K endowed with the linear applications:

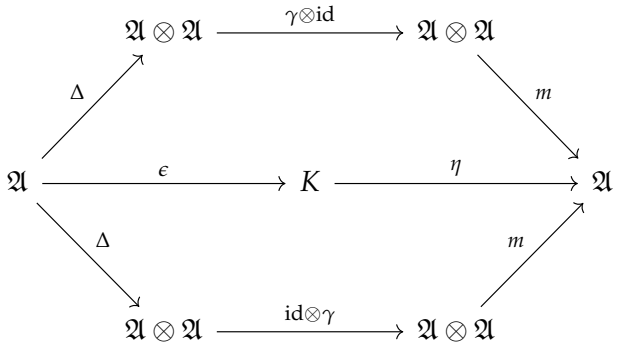
$$\begin{cases} m : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A} & \text{multiplication} \\ \eta : K \to \mathfrak{A} & \text{unit} \\ \Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A} & \text{coproduct} \\ \epsilon : \mathfrak{A} \to K & \text{counit} \\ \gamma : \mathfrak{A} \to \mathfrak{A} & \text{antipode} \end{cases} \tag{1.63}$$

satisfying, $\forall\, u, v, w \in \mathfrak{A}$ and $c \in K$, the following properties:

1. $m(m(u,v),w) = m(u,m(v,w))$
2. $m(\eta(c),u) = m(u,\eta(c)) = cu$
3. $(\mathrm{id} \otimes \Delta)\Delta(u) = (\Delta \otimes \mathrm{id})\Delta(u)$
4. $\begin{cases} (\mathrm{id} \otimes \epsilon)\Delta(u) = u \otimes 1 \\ (\epsilon \otimes \mathrm{id})\Delta(u) = 1 \otimes u \end{cases}$
5. $\Delta(m(u,v)) = m(\Delta(u),\Delta(v))$
6. $\epsilon(m(u,v)) = m(\epsilon(u),\epsilon(v))$
7. $\gamma(m(u,v)) = m(\gamma(v),\gamma(u))$
8. $m((\gamma \otimes \mathrm{id})\Delta(u)) = m((\mathrm{id} \otimes \gamma)\Delta(u)) = \eta(\epsilon(u))\,,$

where id is the identity map on $\mathfrak{A}$, m is the multiplication mapping, i.e. $m(u,v) \doteq u \cdot v$ and the identity η is defined as $\eta(c) \doteq c \cdot 1$. In particular, a vector space endowed with the linear applications (m,η) satisfying the properties 1-2 is called *associative algebra*, while if a vector space is equipped with the homomorphisms (Δ, ϵ) satisfying the properties 3-4 then it is known with the name of *coassociative coalgebra*. Finally, we deal with a *bialgebra* structure when the vector space is endowed with the operations (m, η, Δ, ϵ) satisfying the properties 1-5. Thus, a Hopf algebra is a bialgebra with an antipode, which is an antihomomorphism [24].

Diagrammatically:



As we already mentioned the method we want to introduce deals mainly with coalgebras, i.e. algebras endowed with the coassociative coproduct map Δ. In the classical case, the key role of this game is played by the so-called *Poisson coalgebras*, which are Poisson algebras endowed with a compatible coproduct structure. If $\mathcal{P}$ and $\mathcal{Q}$ are two Poisson algebras, for all $u, v \in \mathcal{P}$ and $w, z \in \mathcal{Q}$, it is possible to define the following Poisson structure on the tensor product space $\mathcal{P} \otimes \mathcal{Q}$ [24]:

$$\{u \otimes w, v \otimes z\}_{(\mathcal{P}\otimes\mathcal{Q})} = \{u, v\}_{(\mathcal{P})} \otimes wz + uv \otimes \{w, z\}_{(\mathcal{Q})} \,. \tag{1.64}$$

In particular, $(\mathcal{P}, \Delta)$ is a Poisson coalgebra if $\mathcal{P}$ is a Poisson algebra and the coproduct map Δ is a Poisson algebra homomorphism between $\mathcal{P}$ and $\mathcal{P} \otimes \mathcal{P}$, which means:

$$\{\Delta(u), \Delta(v)\}_{(\mathcal{P}\otimes\mathcal{P})} = \Delta(\{u, v\}_{(\mathcal{P})}) \quad \forall\, u, v \in \mathcal{P} \,. \tag{1.65}$$

Moreover, the linear maps previously introduced are defined on the generators of $\mathcal{P}$, which we indicate as ξ_α ($\alpha = 1, \ldots, \dim(\mathcal{P})$), as:

$$\begin{cases} \Delta(\xi_\alpha) = \xi_\alpha \otimes 1 + 1 \otimes \xi_\alpha & \Delta(1) = 1 \otimes 1 \\ \epsilon(\xi_\alpha) = 0 & \epsilon(1) = 1 \\ \gamma(\xi_\alpha) = -\xi_\alpha & \gamma(1) = 1 \,. \end{cases} \tag{1.66}$$

Here we can appreciate that we have a full Hopf algebra structure, which means that in general we are dealing with the so-called *Poisson-Hopf algebras*.

Connection with classical mechanics: symplectic representations

Once we have introduced the algebraic structures we are interested in, we want to understand in which way a classical one-dimensional system can be linked to them. The key role of this game is played by the *symplectic representation* of the Poisson algebra. In other words, we want to realize the algebra by means of smooth functions on the phase space $\mathbb{R}^2$. This means that the generators will be represented as:

$$D(\xi_\alpha) \equiv \xi_\alpha(x, p) \qquad \alpha = 1, \ldots, \dim(\mathcal{P}) \,, \tag{1.67}$$

and will close a "1-particle" realization of the Poisson-Lie algebra, which will be given in terms of the usual Poisson brackets, i.e.:

$$\{\xi_\alpha(x, p), \xi_\beta(x, p)\}_{(x,p)} = c^\sigma_{\alpha\beta}\, \xi_\sigma(x, p) \,, \tag{1.68}$$

where $c^{\sigma}_{\alpha\beta}$ are the structure constants of the algebra we are dealing with. Let us remark that fixing the symplectic representation is an important point of the whole construction, since it is a bridge for establishing a connection between a one-dimensional Hamiltonian system, i.e. a smooth function on the phase space manifold expressed in terms of local coordinates $(x, p) \in \mathbb{R}^2$, and the abstract theory of Hopf algebras. If we express the Hamiltonian as a combination of the generators $\xi_\alpha \in \mathcal{P}$ ($\alpha = 1, \ldots, \dim(\mathcal{P}) \doteq d$), the representation that we choose defines the classical system we are dealing with, so the meaning of this choice is of crucial importance from the physical point of view.

$$\text{(abstract theory) } H(\xi_1, \ldots, \xi_d) \quad \xrightarrow{D} \quad H(\xi_1(x,p), \ldots, \xi_d(x,p)) \text{ (classical mechanics)}\,, \tag{1.69}$$

As a matter of fact, once this link has been established, we are able to use all the tools deriving from the abstract theory in such a way to be suitable for dealing with the physics, and in particular for constructing higher dimensional (super)integrable systems. To appreciate the core of this statement, first of all let us explain how to construct a two-dimensional system by using the coproduct map, and how to find out its conserved quantities. The operation is quite simple, if we apply the primitive coproduct to the generators $\xi_\alpha \in \mathcal{P}$ we obtain:

$$\xi^{[2]}_\alpha \doteq \Delta(\xi_\alpha) = \xi_\alpha \otimes 1 + 1 \otimes \xi_\alpha \doteq \xi^{(1)}_\alpha + \xi^{(2)}_\alpha\,, \tag{1.70}$$

where we introduced the notation $\xi_\alpha \otimes 1 \doteq \xi^{(1)}_\alpha$ and $1 \otimes \xi_\alpha \doteq \xi^{(2)}_\alpha$ to indicate the element of the single copy in the tensor product. Thus, the realization of this mapping from $\mathcal{P}$ to $\mathcal{P} \otimes \mathcal{P}$ will generate at a fixed representation two copies of (1.67), given in terms of two pairs of local coordinates $(x_1, p_1) \in \mathbb{R}^2$ and $(x_2, p_2) \in \mathbb{R}^2$, respectively labelled by the superscripts (1) and (2). We can represent this operation as:

$$\begin{aligned} D(\Delta(\xi_\alpha)) \doteq (D \otimes D)\Delta(\xi_\alpha) = D(\xi_\alpha) \otimes 1 + 1 \otimes D(\xi_\alpha) &= D(\xi^{(1)}_\alpha) + D(\xi^{(2)}_\alpha) \\ &\equiv \xi_\alpha(x_1, p_1) + \xi_\alpha(x_2, p_2) \\ &=: \Delta(\xi_\alpha)(x_1, x_2, p_1, p_2)\,. \end{aligned} \tag{1.71}$$

Now, the essential feature of a Poisson coalgebra is that, because of the homomorphism property (1.65), the d functions $\Delta(\xi_\alpha)(x_1, x_2, p_1, p_2)$ define the same Lie-Poisson algebra (1.68), i.e.:

$$\{\Delta(\xi_\alpha), \Delta(\xi_\beta)\}_{(\mathcal{P}\otimes\mathcal{P})} = \Delta(\{\xi_\alpha, \xi_\beta\}_{(\mathcal{P})}) = c^{\sigma}_{\alpha\beta}\Delta(\xi_\sigma)\,, \tag{1.72}$$

where, at a fixed realization, the bracket is:

$$\{f, g\}_{(\mathbf{x},\mathbf{p})} \doteq \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{p}} g - \nabla_{\mathbf{p}} f \cdot \nabla_{\mathbf{x}} g \quad \forall f, g \in C^\infty(\mathbb{R}^4)\,, \tag{1.73}$$

with $\mathbf{x} = (x_1, x_2)$, $\mathbf{p} = (p_1, p_2)$, $\nabla_{\mathbf{x}} = (\partial_{x_1}, \partial_{x_2})$ and $\nabla_{\mathbf{p}} = (\partial_{p_1}, \partial_{p_2})$. Therefore, the new generators will close a "2-particle"realization of the initial algebra. Moreover, we can extend the Hamiltonian to the two-dimensional case by applying the coproduct:

$$H^{[2]} \doteq \Delta(H(\xi_1, \ldots, \xi_d)) = H(\Delta(\xi_1), \ldots, \Delta(\xi_d))\,, \tag{1.74}$$

which means that, when the representation is fixed, we will have:

$$H^{[2]}(\mathbf{x}, \mathbf{p}) \doteq (D \otimes D)\Delta(H(\xi_1, \ldots, \xi_d)) = (D \otimes D)H(\Delta(\xi_1), \ldots, \Delta(\xi_d)) \tag{1.75}$$

$$= H(\Delta(\xi_1)(\mathbf{x}, \mathbf{p}), \ldots, \Delta(\xi_d)(\mathbf{x}, \mathbf{p}))\,, \tag{1.76}$$

which represents the Hamiltonian (smooth) function extended in $\mathbb{R}^4$. Let us now focus on the Casimirs of the algebras, and in particular let us suppose that our initial Lie-Poisson algebra (1.68) is endowed with a Casimir $C(\xi_1, \dots, \xi_d)$ that assumes the value:

$$D(C(\xi_1, \dots, \xi_d)) = C(D(\xi_1), \dots, D(\xi_d)) = c \in \mathbb{R}\,. \tag{1.77}$$

If we apply the coproduct map on the Casimir, we obtain:

$$C^{[2]} \doteq \Delta(C(\xi_1, \dots, \xi_d)) = C(\Delta(\xi_1), \dots, \Delta(\xi_d))\,, \tag{1.78}$$

and this object, once the representation is fixed, can assume nontrivial values: its image under the chosen representation is a function of the phase space coordinates $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^4$, in fact:

$$C^{[2]}(\mathbf{x}, \mathbf{p}) \doteq (D \otimes D)\Delta(C(\xi_1, \dots, \xi_d)) = (D \otimes D)C(\Delta(\xi_1), \dots, \Delta(\xi_d)) \tag{1.79}$$

$$= C(\Delta(\xi_1)(\mathbf{x}, \mathbf{p}), \dots, \Delta(\xi_d)(\mathbf{x}, \mathbf{p}))\,. \tag{1.80}$$

This is a remarkable result because we also have:

$$\{\Delta(C(\xi_1, \dots, \xi_d)), \Delta(\xi_\alpha)\}_{(\mathcal{P}\otimes\mathcal{P})} = \Delta(\{C(\xi_1, \dots, \xi_d), \xi_\alpha\}_{(\mathcal{P})}) = 0 \qquad \forall\, \xi_\alpha \in \mathcal{P}\,, \tag{1.81}$$

because the Casimir, by definition, Poisson-commutes with all the generators of the initial Poisson-Lie algebra. But now, since the Hamiltonian is a combination of the generators of the algebra, it will be also true that:

$$\{\Delta(C(\xi_1, \dots \xi_d)), \Delta(H(\xi_1, \dots \xi_d))\}_{(\mathcal{P}\otimes\mathcal{P})} = \Delta(\{C(\xi_1, \dots \xi_d), H(\xi_1, \dots \xi_d)\}_{(\mathcal{P})}) = 0\,, \tag{1.82}$$

which means that we found an integral of motion for the new two-dimensional Hamiltonian system, represented by the image (under the given symplectic representation) of the coproduct of the Casimir. In fact, by construction, if we calculate the Poisson bracket we obtain:

$$\{C^{[2]}(\mathbf{x}, \mathbf{p}), H^{[2]}(\mathbf{x}, \mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = 0\,. \tag{1.83}$$

Thus, we can conclude that *the realization of the coproduct of any (smooth) function $H(\xi_1, \dots \xi_d)$ of the generators of a coalgebra, with Casimir element $C(\xi_1, \dots \xi_d)$, defines a two-dimensional integrable Hamiltonian.*
This is a good point to introduce an explicit example in order to fix these ideas.

The harmonic oscillator and its coalgebra symmetry: the 2-dimensional case

What we are interested to provide in this section is another characterization of the harmonic oscillator, in terms of an underlying *hidden* symmetry, which is useful for a physical understanding of its superintegrability properties and, more pragmatically, for appreciating the power of the coalgebra symmetry approach. To this aim, let us consider the following oscillator Hamiltonian (smooth) function $H : \mathcal{M} \to \mathbb{R}$, defined on the symplectic manifold $(\mathcal{M} \doteq \mathbb{R}^2, \omega_0 = \mathrm{d}x \wedge \mathrm{d}p)$ (for simplicity, from now on we shall set $m = \omega = 1$):

$$H(x, p) = \frac{p^2 + x^2}{2}\,, \tag{1.84}$$

where x, p are canonical (local) coordinates on $\mathcal{M}$ such as $\{x,p\}=1$. This Hamiltonian can be expressed in terms of the generators ξ_α ($\alpha=\pm,3$) of the three-dimensional $\mathfrak{sl}(2,\mathbb{R})$ Poisson-Lie coalgebra, defined through the Poisson commutation relations:

$$\{\xi_-,\xi_+\}_{(\mathfrak{sl}(2,\mathbb{R}))}=4\xi_3\,,\ \{\xi_3,\xi_\pm\}_{(\mathfrak{sl}(2,\mathbb{R}))}=\pm2\xi_\pm\,, \tag{1.85}$$

and equipped with the (primitive) coproduct map $\Delta:\mathfrak{sl}(2,\mathbb{R})\to\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})$:

$$\Delta(\xi_\alpha)\doteq\xi_\alpha\otimes1+1\otimes\xi_\alpha\ ,\ \Delta(1)\doteq1\otimes1\quad(\text{with }\alpha=\pm,3). \tag{1.86}$$

The Casimir of the algebra is given by $C(\xi_\pm,\xi_3)\doteq\xi_+\xi_--\xi_3^2$. A symplectic realization of (1.85) can be taken in terms of the classical observables:

$$D(\xi_+)\doteq\xi_+(x,p)=p^2\,,\ D(\xi_-)\doteq\xi_-(x,p)=x^2\,,\ D(\xi_3)\doteq\xi_3(x,p)=xp\,, \tag{1.87}$$

it is in fact easy to show that the following Poisson brackets are satisfied:

$$\{\xi_-(x,p),\xi_+(x,p)\}_{(x,p)}=4\xi_3(x,p)\,,\quad\{\xi_3(x,p),\xi_\pm(x,p)\}_{(x,p)}=\pm2\xi_\pm(x,p)\,, \tag{1.88}$$

together with the Casimir $D(C(\xi_\pm,\xi_3))=C(D(\xi_\pm),D(\xi_3))=D(\xi_+)D(\xi_-)-(D(\xi_3))^2=0$, in the given representation. Now, because of this link, it is quite clear that the oscillator Hamiltonian (1.84) can be abstractly rephrased in terms of the coalgebra generators as follows:

$$H(\xi_+,\xi_-)=\frac{\xi_++\xi_-}{2}\,, \tag{1.89}$$

and we have to think it as a function on $\mathfrak{sl}(2,\mathbb{R})$. In particular, since the coproduct map (1.86) defines an homomorphism for the Poisson-Lie algebra (1.85), i.e.:

$$\{\Delta(\xi_-),\Delta(\xi_+)\}_{(\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R}))}=4\Delta(\xi_3)\ ,\quad\{\Delta(\xi_3),\Delta(\xi_\pm)\}_{(\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R}))}=\pm2\Delta(\xi_\pm)\,, \tag{1.90}$$

it is straightforward to extend the classical system to higher dimensions, passing through the symplectic realization that we have chosen. In particular, the generators of the algebra are:

$$\begin{cases}\xi_+^{[2]}(\mathbf{x},\mathbf{p})\doteq(D\otimes D)\Delta(\xi_+)=D(\xi_+)\otimes1+1\otimes D(\xi_+)=p_1^2+p_2^2\\ \xi_-^{[2]}(\mathbf{x},\mathbf{p})\doteq(D\otimes D)\Delta(\xi_-)=D(\xi_-)\otimes1+1\otimes D(\xi_-)=x_1^2+x_2^2\\ \xi_3^{[2]}(\mathbf{x},\mathbf{p})\doteq(D\otimes D)\Delta(\xi_3)=D(\xi_3)\otimes1+1\otimes D(\xi_3)=x_1p_1+x_2p_2\,,\end{cases} \tag{1.91}$$

and the following relations are satisfied (now in $\mathbb{R}^4$):

$$\{\xi_-^{[2]}(\mathbf{x},\mathbf{p}),\xi_+^{[2]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})}=4\xi_3^{[2]}(\mathbf{x},\mathbf{p})\ ,\quad\{\xi_3^{[2]}(\mathbf{x},\mathbf{p}),\xi_\pm^{[2]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})}=\pm2\xi_\pm^{[2]}(\mathbf{x},\mathbf{p})\,. \tag{1.92}$$

Furthermore, it is immediate to show that the two-particle Hamiltonian can be constructed through the coproduct of H, which results in:

$$H^{[2]}\doteq\Delta\left(\frac{\xi_++\xi_-}{2}\right)=\frac{\Delta(\xi_+)+\Delta(\xi_-)}{2}=\frac{\xi_+^{[2]}+\xi_-^{[2]}}{2}\,, \tag{1.93}$$

that implies:

$$H^{[2]}(\mathbf{x},\mathbf{p})\doteq(D\otimes D)\Delta(H(\xi_+,\xi_-))=\frac{\xi_+^{[2]}(\mathbf{x},\mathbf{p})+\xi_-^{[2]}(\mathbf{x},\mathbf{p})}{2}=\frac{p_1^2+p_2^2}{2}+\frac{x_1^2+x_2^2}{2}\,, \tag{1.94}$$

which is the two-dimensional harmonic oscillator Hamiltonian. As far as the Casimir function is concerned, and here is the main point, if we apply the coproduct on it we obtain:

$$\Delta(C(\xi_\pm,\xi_3)) = C(\Delta(\xi_\pm),\Delta(\xi_3)) = \Delta(\xi_+)\Delta(\xi_-) - (\Delta(\xi_3))^2\,, \tag{1.95}$$

or, explicitly:

$$\begin{aligned}\Delta(C) &= (\xi_+\otimes 1 + 1\otimes\xi_+)(\xi_-\otimes 1 + 1\otimes\xi_-) - (\xi_3\otimes 1 + 1\otimes\xi_3)^2\\ &= \xi_+\xi_-\otimes 1 + 1\otimes\xi_+\xi_- + \xi_+\otimes\xi_- + \xi_-\otimes\xi_+ - \xi_3^2\otimes 1 + 1\otimes\xi_3^2 - 2(\xi_3\otimes\xi_3)\\ &= (\xi_+\xi_- - \xi_3^2)\otimes 1 + 1\otimes(\xi_+\xi_- - \xi_3^2) + \xi_+\otimes\xi_- + \xi_-\otimes\xi_+ - 2(\xi_3\otimes\xi_3)\\ &= C\otimes 1 + 1\otimes C + \xi_+\otimes\xi_- + \xi_-\otimes\xi_+ - 2(\xi_3\otimes\xi_3)\,. \end{aligned}\tag{1.96}$$

At this point, fixing the representation, we obtain the "new" Casimir $C^{[2]}(\mathbf{x},\mathbf{p})$ that is:

$$\begin{aligned}C^{[2]}(\mathbf{x},\mathbf{p}) &\doteq (D\otimes D)\Delta(C(\xi_\pm,\xi_3))\\ &= D(C)\otimes 1 + 1\otimes D(C) + D(\xi_+)\otimes D(\xi_-) + D(\xi_-)\otimes D(\xi_+) - 2(D(\xi_3)\otimes D(\xi_3))\\ &= p_1^2x_2^2 + x_1^2p_2^2 - 2x_1p_1x_2p_2 \qquad (1.97)\\ &= (x_1p_2 - x_2p_1)^2\,. \qquad (1.98)\end{aligned}$$

Thus, the image of the Casimir is nothing but the angular momentum squared of the two-dimensional harmonic oscillator. Clearly, we already know that this quantity is conserved because of the rotational symmetry, but the point is that with the *coalgebra technique* it has been obtained algebraically. Also, because of the homomorphism property, we know that this quantity is conserved by construction, since we have:

$$0 = \Delta(\{C,H\}_{(\mathfrak{sl}(2,\mathbb{R}))}) = \{\Delta(C),\Delta(H)\}_{(\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R}))} = \{C^{[2]},H^{[2]}\}_{(\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R}))} \tag{1.99}$$

that is:

$$\{C^{[2]}(\mathbf{x},\mathbf{p}),H^{[2]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = 0\,. \tag{1.100}$$

Therefore, in conclusion, by using this approach we have obtained the conserved quantity that is necessary to ensure the *integrability* of the harmonic oscillator in dimension $N = 2$. Now, the question naturally arises: *it is possible to extend such procedure to higher dimensions*? The answer is affirmative, and it is related to the above mentioned *coassociativity property* of the coproduct.

3.1.2 Higher dimensional superintegrability through coassociativity

As we previously discussed the Poisson coalgebra $(\mathcal{P},\Delta)$ is a unital associative algebra $\mathcal{P}$ endowed with a coproduct map Δ that satisfies the coassociativity property:

$$(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta\,, \tag{1.101}$$

in other words, the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{P} & \xrightarrow{\ \Delta\ } & \mathcal{P}\otimes\mathcal{P}\\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}}\\ \mathcal{P}\otimes\mathcal{P} & \xrightarrow{\ \mathrm{id}\otimes\Delta\ } & \mathcal{P}\otimes\mathcal{P}\otimes\mathcal{P}\end{array}$$

Roughly speaking, this property allows one to define an object on the tensor product space $\mathcal{P}\otimes\mathcal{P}\otimes\mathcal{P}$ in two possible ways, and this is intimately related to the *superintegrability* properties of the system under consideration. Let us explain why. First of all, let us indicate the coproduct $\Delta\equiv\Delta^{[2]}$ to make explicit its connection with the two-particle realization. Now, the point is that the map $\Delta^{[3]}:\mathcal{P}\to\mathcal{P}\otimes\mathcal{P}\otimes\mathcal{P}$ can be defined by using both the *left* and *right* expressions[1]:

$$\Delta^{[3]}\doteq(\mathrm{id}\otimes\Delta^{[2]})\circ\Delta^{[2]}\qquad\Delta_R^{[3]}\doteq(\Delta^{[2]}\otimes\mathrm{id})\circ\Delta^{[2]}\,, \tag{1.102}$$

and this result, since of (1.101), is unique (i.e. $\Delta^{[3]}=\Delta_R^{[3]}$). Once the construction of the "3-particle"system has been performed through coassociativity, this procedure can be generalized to higher dimensional tensor product spaces of $\mathcal{P}$, i.e. we can construct a 4-th order coproduct $\Delta^{[4]}:\mathcal{P}\to\mathcal{P}^{\otimes 4}$ as:

$$\Delta^{[4]}\doteq(\mathrm{id}\otimes\mathrm{id}\otimes\Delta^{[2]})\circ\Delta^{[3]}\qquad\Delta_R^{[4]}\doteq(\Delta^{[2]}\otimes\mathrm{id}\otimes\mathrm{id})\circ\Delta_R^{[3]}\,, \tag{1.103}$$

and the more general *N-th* coproduct map $\Delta^{[N]}:\mathcal{P}\to\mathcal{P}^{\otimes N}$:

$$\Delta^{[N]}\doteq(\overbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}^{N-2}\otimes\Delta^{[2]})\circ\Delta^{[N-1]}\qquad\Delta_R^{[N]}\doteq(\Delta^{[2]}\otimes\overbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}^{N-2})\circ\Delta_R^{[N-1]}\quad(N>2)\,, \tag{1.104}$$

which means that given the $(N-1)$-*th* coproduct, the N-*th* one is obtained by applying $\Delta^{[2]}$ onto the space located at the very right (respectively left) site. Now, due to the coassociativity property, for the N-th order coproduct of any generators in $\mathcal{P}$, the application of these two (left and right) maps will be equivalent, i.e.:

$$\Delta^{[N]}\equiv\Delta_R^{[N]}\,. \tag{1.105}$$

Moreover, $\Delta^{[N]}$ is a Poisson homorphism, now between $\mathcal{P}$ and $\mathcal{P}^{\otimes N}$ [24, 25]:

$$\{\Delta^{[N]}(u),\Delta^{[N]}(v)\}_{(\mathcal{P}^{\otimes N})}=\Delta^{[N]}(\{u,v\}_{(\mathcal{P})})\quad\forall\,u,v\in\mathcal{P}\,. \tag{1.106}$$

Anyway, if one labels from 1 to N the sites of the chain $\mathcal{P}^{\otimes N}$, lower dimensional *m-th* coproducts (with $2\leq m\leq N-1$) will be different in the sense that the *left* coproducts $\Delta^{[m]}$ will contain objects living on the tensor product space $1\otimes\cdots\otimes m$, whereas the *right* coproducts $\Delta_R^{[m]}$ will be defined on the sites $(N-m+1)\otimes(N-m+2)\otimes\cdots\otimes N$. This leads to the possibility of finding out two sets of *left* and *right* Casimir functions that are in involution and Poisson-commute with the N-dimensional Hamiltonian $H^{[N]}$, under a fixed representation of the algebra [24, 25]. So, in short, this is the general idea: *constructing a set of involutive functions in* $\mathcal{P}^{\otimes N}$ *by applying the coproducts* $\Delta^{[m]}$ $(\Delta_R^{[m]})$ *to the Casimirs in* $\mathcal{P}$. Let us explain this in more detail.

Let us suppose that a Poisson-Lie coalgebra $(\mathcal{P},\Delta)$ is characterized by d generators ξ_α ($\alpha=1,\ldots,d$) and r Casimirs $C_i(\xi_1,\ldots,\xi_d)$ ($i=1,\ldots,r$), then using the coproducts it is possible to construct the two following sets composed by the (resp. left and right) functions on the tensor product space of the algebra:

$$\begin{cases}\{\Delta_\Xi^{[1]}(\xi_\alpha),\ldots,\Delta_\Xi^{[m]}(\xi_\alpha),\ldots,\Delta_\Xi^{[N]}(\xi_\alpha)\}\quad(\alpha=1,\ldots,d)\\ \{\Delta_\Xi^{[1]}(C_i),\ldots,\Delta_\Xi^{[m]}(C_i),\ldots,\Delta_\Xi^{[N]}(C_i)\}\quad(i=1,\ldots,r)\,,\end{cases} \tag{1.107}$$

1 Following the notations of [25] we indicate the right application with the subscript R.

where, in what follow, we will indicate with "Ξ" both the right and left applications (1.104), and we are using the convention $\Delta^{[1]} = \mathrm{id}$. Because of the coassociativity property, given a smooth function $H(\xi_1, \dots, \xi_d)$, we can uniquely define the N-dimensional Hamiltonian as the N-*th* coproduct of $H(\xi_1, \dots, \xi_d)$, i.e.:

$$H^{[N]} \doteq \Delta^{[N]}(H(\xi_1, \dots, \xi_d)) = H(\Delta^{[N]}(\xi_1), \dots, \Delta^{[N]}(\xi_d)) = H(\xi_1^{[N]}, \dots, \xi_d^{[N]}), \tag{1.108}$$

where we dropped the subsript Ξ because the two applications are equivalent (see (1.105)). Now, and this is the main point, the two sets of $r \cdot N$ left and right Casimir functions:

$$C_{i,\Xi}^{[m]} \doteq \Delta_{\Xi}^{[m]}(C_i(\xi_1, \dots, \xi_d)) = C_i(\Delta_{\Xi}^{[m]}(\xi_1), \dots, \Delta_{\Xi}^{[m]}(\xi_d)) \quad (m = 1, \dots, N\,;\, i = 1, \dots, r)\,, \tag{1.109}$$

Poisson-commute with the N-*th* coproduct of the generators $\xi_\alpha \in \mathcal{P}$, namely:

$$\{C_{i,\Xi}^{[m]}, \xi_\alpha^{[N]}\}_{(\mathcal{P}^{\otimes N})} = 0 \quad (m = 1, \dots, N\,;\, i = 1, \dots, r\,;\, \alpha = 1, \dots, d)\,, \tag{1.110}$$

where we defined as usual $\xi_\alpha^{[N]} \doteq \Delta^{[N]}(\xi_\alpha)$. This implies that both the left and right Casimirs are in involution within the Hamiltonian $H^{[N]}$ on $\mathcal{P}^{\otimes N}$:

$$\{C_{i,\Xi}^{[m]}, H^{[N]}\}_{(\mathcal{P}^{\otimes N})} = 0 \qquad (m = 1, \dots, N\,;\, i = 1, \dots, r)\,, \tag{1.111}$$

where $C_{i,\Xi}^{[1]} = c_{i,\Xi}$ are constants. Moreover, they themselves are in involution on $\mathcal{P}^{\otimes N}$ [24]:

$$\{C_{i,\Xi}^{[m]}, C_{j,\Xi}^{[l]}\}_{(\mathcal{P}^{\otimes N})} = 0\,, \quad (m, l = 1, \dots, N\,;\, i, j = 1, \dots, r\,;\, \Xi = \text{Left or Right})\,. \tag{1.112}$$

Thus, at the end of the game, *this method will provide us a way to construct higher dimensional version of a given one-dimensional Hamiltonian system, equipped with an underlying coalgebra symmetry, propagating its integrability properties. In particular, it allows us to obtain, by construction, an N-dimensional Hamiltonian $H^{[N]}$ equipped with two sets of $r \cdot N$ integrals of motion represented by the image of the two (left and right) sets of partial Casimirs under a given symplectic representation. Each set is composed by Casimirs that are in mutual involution.*

This construction can be applied to any Poisson coalgebra, and here [24–31] the reader can find a lot of examples, among which the cases of Poisson analogues of quantum algebras and quantum groups. In fact, this algebraic approach has been shown to be extremely useful also in the framework of "q-deformed integrable systems", for which the coproduct map is no longer primitive. Moreover, since it is a purely algebraic approach, it can be applied to noncommutative coalgebras as well, which means that quantum mechanical systems can also be investigated by using this method. For our porpuses we will deal explicitly with the specific case of the $\mathfrak{sl}(2, \mathbb{R})$ coalgebra endowed with the primitive coproduct map, which we already introduced to analyze the classical harmonic oscillator and that will be of interest for dealing with the models studied in this Thesis, which share the same $\mathfrak{sl}(2, \mathbb{R})$ coalgebra symmetry. In this case, if N is the dimension of the ambient space, being the number of Casimirs $r = 1$, such a method will provide to us a total number of $2N - 3$ *left* and *right* Casimir functions (see Table 1) that together with the Hamiltonian itself, at a fixed representation, define a functionally independent set of $2N - 2$ first integrals. This ensures the quasi-maximal superintegrability of the Hamiltonian systems equipped with such a "hidden"coalgebra symmetry. In particular,

we will have that the three N-dimensional generators $\xi^{[N]}_{\pm,3}$ Poisson-commute with an m-fold right and left applications of the coproduct ($2 \leq m \leq N$), i.e.:

$$\begin{cases} \{\xi^{[N]}_{\pm,3}, C^{[m]}\}_{(\mathfrak{sl}(2,\mathbb{R})^{\otimes N})} = \{\xi^{[N]}_{\pm,3}, C_{[m]}\}_{(\mathfrak{sl}(2,\mathbb{R})^{\otimes N})} = 0\,, \\ C^{[m]} \doteq \Delta^{[m]}(C) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{N-m}, \\ C_{[m]} \doteq \underbrace{1 \otimes \cdots \otimes 1}_{N-m} \otimes \Delta^{[m]}(C)\,, \end{cases} \tag{1.113}$$

with $C^{[N]} = C_{[N]}$. This means that we will have a total number of $2N-3$ (nontrivial) Casimirs $C^{[2]} \dots C^{[N]}$, $C_{[2]} \dots C_{[N-1]}$ that, due to the above properties, are in involution with the N-dimensional Hamiltonian $H^{[N]} = \Delta^{[N]}(H(\xi_+, \xi_-, \xi_3)) = H(\xi^{[N]}_+, \xi^{[N]}_-, \xi^{[N]}_3)$. Moreover, each set is composed by functions that are in involution.

Left set of $(N-1)$ Casimirs	**Right set of $(N-1)$ Casimirs**
$C^{[2]} \doteq \Delta^{[2]}(C)$	$C_{[2]} \doteq \Delta^{[2]}_R(C)$
$\vdots$	$\vdots$
$C^{[m]} = \Delta^{[m]}(C) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{N-m}$	$C_{[m]} \doteq \Delta^{[m]}_R(C) = \underbrace{1 \otimes \cdots \otimes 1}_{N-m} \otimes \Delta^{[m]}(C)$
$\vdots$	$\vdots$
$C^{[N]} \doteq \Delta^{[N]}(C)$	$C_{[N]} \doteq \Delta^{[N]}_R(C) = \Delta^{[N]}(C)$

Table 1.: The $2 \cdot (N-1) - 1$ left and right integrals of motion coming from the Casimir C. We remark that one Casimir has to be eliminated in the final calculation because of the equivalence $\Delta^{[N]} \equiv \Delta^{[N]}_R$.

To summarize, in the case of $\mathfrak{sl}(2,\mathbb{R})$, for $N = 2$ the Hamiltonian will be *integrable* with a single constant of motion $C^{[2]} = C_{[2]}$, for $N = 3$ it will be *minimally* superintegrable with three constants of motion $\{C^{[2]}, C^{[3]} = C_{[3]}, C_{[2]}\}$ and for any N it will result quasi-maximally superintegrable with $(2N-3)$ integrals of motion $\{C^{[m]}, C^{[N]} = C_{[N]}, C_{[m]}\}$ for $m = 2, \dots, N-1$. This is the best one can do by using this approach to superintegrable systems endowed with an underlying $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry. If we are interested to ensure the *maximal superintegrability* we need to construct the missing quantity by using alternative methods.
Let us observe that generalizations of the coalgebra method also exist in the literature, and they involve more complicated structures such as comodule algebras or loop algebras. We will not deal explicitly with such algebraic structures, anyhow we refer the interested reader to the papers [90, 91].

The harmonic oscillator and its coalgebra symmetry: the N-dimensional case

Let us come back to the harmonic oscillator in order to see explicitly how the coassociativity property works. We have already shown how to construct the conserved quantity for the 2-dimensional case (the angular momentum squared). Here, we are interested in showing

how to construct an N-dimensional version of the system (1.84) and its $2N-3$ conserved quantities provided by the left and right partial Casimirs.
Let us begin our construction by analyzing the 3-dimensional extension of the system. Since the primitive coproduct Δ of $\mathfrak{sl}(2,\mathbb{R})$ defines a *coassociative* map:

$$\begin{array}{ccc} \mathfrak{sl}(2,\mathbb{R}) & \xrightarrow{\Delta} & \mathfrak{sl}(2,\mathbb{R})^{\otimes 2} \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes \mathrm{id}} \\ \mathfrak{sl}(2,\mathbb{R})^{\otimes 2} & \xrightarrow{\mathrm{id}\otimes\Delta} & \mathfrak{sl}(2,\mathbb{R})^{\otimes 3} \end{array}$$

as we previously explained we are able to define an object on the tensor product space $\mathfrak{sl}(2,\mathbb{R})^{\otimes 3}$ in two possible ways. First of all let us construct the 3-dimensional version of the generators ξ_α ($\alpha = \pm, 3$):

$$\begin{aligned} \xi_\alpha^{[3]} \doteq \Delta^{[3]}(\xi_\alpha) = (\mathrm{id}\otimes\Delta)\Delta(\xi_\alpha) &= (\mathrm{id}\otimes\Delta)(\xi_\alpha\otimes 1 + 1\otimes\xi_\alpha) \\ &= \mathrm{id}(\xi_\alpha)\otimes\Delta(1) + \mathrm{id}(1)\otimes\Delta(\xi_\alpha) \\ &= \xi_\alpha\otimes 1\otimes 1 + 1\otimes(\xi_\alpha\otimes 1 + 1\otimes\xi_\alpha) \\ &= (\xi_\alpha\otimes 1\otimes 1) + (1\otimes\xi_\alpha\otimes 1) + (1\otimes 1\otimes\xi_\alpha)\,, \end{aligned} \tag{1.114}$$

which implies, when we fix the representation, the following realizations:

$$\begin{cases} \xi_+^{[3]}(\mathbf{x},\mathbf{p}) \doteq (D\otimes D\otimes D)\Delta^{[3]}(\xi_+) = p_1^2 + p_2^2 + p_3^2 \\ \xi_-^{[3]}(\mathbf{x},\mathbf{p}) \doteq (D\otimes D\otimes D)\Delta^{[3]}(\xi_-) = x_1^2 + x_2^2 + x_3^2 \\ \xi_3^{[3]}(\mathbf{x},\mathbf{p}) \doteq (D\otimes D\otimes D)\Delta^{[3]}(\xi_3) = x_1p_1 + x_2p_2 + x_3p_3\,, \end{cases} \tag{1.115}$$

where, we now have $\mathbf{x} \doteq (x_1,x_2,x_3) \in \mathbb{R}^3, \mathbf{p} \doteq (p_1,p_2,p_3) \in \mathbb{R}^3$. It is easy to show that the application of the right coproduct leads, as it must be, at the same result. Therefore as expected the Hamiltonian function $H^{[3]}(\mathbf{x},\mathbf{p}) \in C^\infty(\mathbb{R}^6)$ will be given by:

$$\begin{aligned} H^{[3]}(\mathbf{x},\mathbf{p}) \doteq (D\otimes D\otimes D)\Delta^{[3]}(H(\xi_+,\xi_-)) &= \frac{\xi_+^{[3]}(\mathbf{x},\mathbf{p}) + \xi_-^{[3]}(\mathbf{x},\mathbf{p})}{2} \\ &= \frac{p_1^2+p_2^2+p_3^2}{2} + \frac{x_1^2+x_2^2+x_3^2}{2}\,. \end{aligned} \tag{1.116}$$

Now, what we want to find out in the following are the two (left and right) Casimirs $C^{[2]}$ and $C_{[2]}$, together with the Casimir $C^{[3]} = C_{[3]}$. As we know from the theory, they are in involution with the Hamiltonian $H^{[3]}$ on $\mathfrak{sl}(2,\mathbb{R})^{\otimes 3}$. This implies that, when the realization is fixed, we will have three integrals of motion commuting within the Hamiltonian $H^{[3]}(\mathbf{x},\mathbf{p})$ on $\mathcal{M} = \mathbb{R}^6$. Firstly, let us calculate the Casimir $C^{[3]} \doteq \Delta^{[3]}(C) = (\mathrm{id}\otimes\Delta)\Delta(C)$, this will allow

us to understand how this kind of calculations have to be performed. In particular, we have explicitly:

$$\begin{aligned}
C^{[3]} &= (\mathrm{id}\otimes\Delta)(C\otimes 1+1\otimes C+\xi_+\otimes\xi_-+\xi_-\otimes\xi_+-2(\xi_3\otimes\xi_3))\\
&= \mathrm{id}(C)\otimes\Delta(1)+\mathrm{id}(1)\otimes\Delta(C)+\mathrm{id}(\xi_+)\otimes\Delta(\xi_-)+\mathrm{id}(\xi_-)\otimes\Delta(\xi_+)-2(\mathrm{id}(\xi_3)\otimes\Delta(\xi_3))\\
&= (C\otimes 1\otimes 1)+1\otimes(C\otimes 1+1\otimes C+\xi_+\otimes\xi_-+\xi_-\otimes\xi_+-2(\xi_3\otimes\xi_3))\\
&\quad+(\xi_+\otimes(\xi_-\otimes 1+1\otimes\xi_-))+(\xi_-\otimes(\xi_+\otimes 1+1\otimes\xi_+))-2(\xi_3\otimes(\xi_3\otimes 1+1\otimes\xi_3))\\
&= (C\otimes 1\otimes 1)+(1\otimes C\otimes 1)+(1\otimes 1\otimes C)+(1\otimes\xi_+\otimes\xi_-)+(1\otimes\xi_-\otimes\xi_+)\\
&\quad-2(1\otimes\xi_3\otimes\xi_3)+(\xi_+\otimes\xi_-\otimes 1)+(\xi_+\otimes 1\otimes\xi_-)+(\xi_-\otimes\xi_+\otimes 1)+(\xi_-\otimes 1\otimes\xi_+)\\
&\quad-2(\xi_3\otimes\xi_3\otimes 1)-2(\xi_3\otimes 1\otimes\xi_3)\,.
\end{aligned}\tag{1.117}$$

Thus, when we fix the representation $D^{\otimes 3}\Delta^{[3]}(C)$, taking into account that $D(C)=0$, we get the following function on $\mathbb{R}^6$:

$$\begin{aligned}
C^{[3]}(\mathbf{x},\mathbf{p}) &= p_2^2x_3^2+x_2^2p_3^2+x_1^2p_3^2+x_2^2p_1^2+p_1^2x_3^2+x_1^2p_2^2-2x_2p_2x_3p_3-2x_1p_1x_2p_2-2x_1p_1x_3p_3\\
&= (x_1p_2-x_2p_1)^2+(x_2p_3-x_3p_2)^2+(x_1p_3-x_3p_1)^2\,,
\end{aligned}\tag{1.118}$$

where, for example, $D^{\otimes 3}(\xi_-\otimes 1\otimes\xi_+)=D(\xi_-)\otimes D(1)\otimes D(\xi_+)=x_1^2p_3^2$. As we can observe, the Casimir function living on the tensor product space of three $\mathfrak{sl}(2,\mathbb{R})$ algebras is realized in terms of the total angular momentum squared of 3-dimensional Hamiltonian system $H^{[3]}(\mathbf{x},\mathbf{p})$. The other two Casimir functions are realized in terms of the following expressions:

$$\begin{cases}
C^{[2]}\doteq\Delta^{[2]}\otimes 1\overset{D^{\otimes 3}}{\to}C^{[2]}(\mathbf{x},\mathbf{p})=(x_1p_2-x_2p_1)^2 & (\text{site}:1\otimes 2)\\
C_{[2]}\doteq 1\otimes\Delta^{[2]}\overset{D^{\otimes 3}}{\to}C_{[2]}(\mathbf{x},\mathbf{p})=(x_2p_3-x_3p_2)^2 & (\text{site}:2\otimes 3).
\end{cases}\tag{1.119}$$

Therefore, we found the three conserved quantities we were looking for:

$$\begin{cases}
\{C^{[3]}(\mathbf{x},\mathbf{p}),H^{[3]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})}=0\\
\{C^{[2]}(\mathbf{x},\mathbf{p}),H^{[3]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})}=\{C_{[2]}(\mathbf{x},\mathbf{p}),H^{[3]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})}=0\,,
\end{cases}\tag{1.120}$$

where the Poisson brackets are clearly performed on $\mathbb{R}^6$. Thus, since the 3-dimensional Hamiltonian $H^{[3]}(\mathbf{x},\mathbf{p})$ is in involution with the three Casimirs $C^{[2]}(\mathbf{x},\mathbf{p})$, $C_{[2]}(\mathbf{x},\mathbf{p})$ and $C^{[3]}(\mathbf{x},\mathbf{p})=C_{[3]}(\mathbf{x},\mathbf{p})$ we can conclude that the system is *minimally superintegrable* ($3+1=4$ integrals of motion).
The more general N-dimensional case follows directly by considering the N-particle realization of the $\mathfrak{sl}(2,\mathbb{R})$, given in terms of the coproduct $\Delta^{[N]}:\mathfrak{sl}(2,\mathbb{R})\to\mathfrak{sl}(2,\mathbb{R})^{\otimes N}$:

$$\begin{cases}
\xi_+^{[N]}(\mathbf{x},\mathbf{p})\doteq(D^{\otimes N})\Delta^{[N]}(\xi_+)=\sum_{i=1}^N p_i^2=\mathbf{p}^2\\
\xi_-^{[N]}(\mathbf{x},\mathbf{p})\doteq(D^{\otimes N})\Delta^{[N]}(\xi_-)=\sum_{i=1}^N x_i^2=\mathbf{x}^2\\
\xi_3^{[N]}(\mathbf{x},\mathbf{p})\doteq(D^{\otimes N})\Delta^{[N]}(\xi_3)=\sum_{i=1}^N x_ip_i=\mathbf{x}\cdot\mathbf{p}\,,
\end{cases}\tag{1.121}$$

which allows us to construct the N-particle realization of the Hamiltonian in the usual way:

$$H^{[N]}(\mathbf{x},\mathbf{p})\doteq(D^{\otimes N})\Delta^{[N]}(H(\xi_+,\xi_-))=\frac{\xi_+^{[N]}(\mathbf{x},\mathbf{p})+\xi_-^{[N]}(\mathbf{x},\mathbf{p})}{2}=\frac{\mathbf{p}^2+\mathbf{x}^2}{2}\,.\tag{1.122}$$

In particular, performing the calculations, one finds that the $2N-3$ left and right conserved quantities provided by the method are given by:

$$\begin{cases} C^{[m]}(\mathbf{x},\mathbf{p}) = \sum_{1\leq i<j\leq m}(x_ip_j - x_jp_i)^2\,,\; C_{[m]}(\mathbf{x},\mathbf{p}) = \sum_{N-m<i<j\leq N}(x_ip_j - x_jp_i)^2 \\ C^{[N]}(\mathbf{x},\mathbf{p}) = C_{[N]}(\mathbf{x},\mathbf{p}) = \sum_{1\leq i<j\leq N}(x_ip_j - x_jp_i)^2 \text{ (total angular momentum squared)}\,, \end{cases} \tag{1.123}$$

for $m = 2,\dots,N-1$. In fact, it is not difficult to check that on $\mathbb{R}^{2N}$ it holds:

$$\begin{cases} \{C^{[N]}(\mathbf{x},\mathbf{p}), H^{[N]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = 0 \\ \{C^{[m]}(\mathbf{x},\mathbf{p}), H^{[N]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = \{C_{[m]}(\mathbf{x},\mathbf{p}), H^{[N]}(\mathbf{x},\mathbf{p})\}_{(\mathbf{x},\mathbf{p})} = 0 \qquad (m = 2,\dots,N-1)\,. \end{cases} \tag{1.124}$$

Moreover, the set $\mathcal{O} \doteq \{H^{[N]}(\mathbf{x},\mathbf{p}), C^{[m]}(\mathbf{x},\mathbf{p}), C_{[m]}(\mathbf{x},\mathbf{p})\}$ for $(m = 2,\dots,N)$ is functionally independent, providing in this way $2N-2$ conserved quantities required for the *quasi-maximal superintegrability*. Furthermore, each of the two sets $\{H^{[N]}(\mathbf{x},\mathbf{p}); C^{[m]}(\mathbf{x},\mathbf{p})\}$, $\{H^{[N]}(\mathbf{x},\mathbf{p}); C_{[m]}(\mathbf{x},\mathbf{p})\}$ $(m = 2,\dots,N)$ is formed by N functionally independent involutive functions.

Let us observe that the KC system is also endowed with the constants of motion (1.124), and this can be also explained in terms of the coalgebra symmetry. In fact, the KC system admits a coalgebraic N-dimensional version of the form:

$$H^{[N]} \doteq H(\xi_+^{[N]}, \xi_-^{[N]}) = \frac{\xi_+^{[N]}}{2} - \frac{k}{\sqrt{\xi_-^{[N]}}} \qquad (k>0)\,. \tag{1.125}$$

It is easy to see that, under the same symplectic representation (1.121), the coalgebraic Hamiltonian (1.125) reads:

$$H^{[N]}(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{k}{\sqrt{|\mathbf{x}|^2}} \tag{1.126}$$

with $(\mathbf{x},\mathbf{p}) \in \mathbb{R}^{2N}$, which is the Hamiltonian function describing the (cartesian) KC system in N-dimensions. As a matter of fact, the HO and the KC systems represent just two specific examples of a more general N-dimensional family of coalgebraic Hamiltonians, i.e.:

$$H^{[N]} = H(\xi_+^{[N]}, \xi_-^{[N]}, \xi_3^{[N]})\,. \tag{1.127}$$

Once the symplectic representation (1.121) is fixed, then the Hamiltonian can be any smooth function of the form:

$$H^{[N]}(\mathbf{x},\mathbf{p}) = H\Big(\mathbf{p}^2, \mathbf{x}^2, \mathbf{x}\cdot\mathbf{p}\Big) \tag{1.128}$$

and, by construction, it results to be endowed with the integrals of motion (1.123). It is also worth mentioning that a more general family also exists, and it is related to a symplectic realization involving centrifugal terms coming from the representation of the generators ξ_+, i.e. when the representation is such that at the N-dimensional level we have:

$$\begin{cases} \xi_+^{[N]}(\mathbf{x},\mathbf{p}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_+) = \sum_{i=1}^N (p_i^2 + \frac{\beta_i}{x_i^2}) = \mathbf{p}^2 + \sum_{i=1}^N \frac{\beta_i}{x_i^2} \\ \xi_-^{[N]}(\mathbf{x},\mathbf{p}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_-) = \sum_{i=1}^N x_i^2 = \mathbf{x}^2 \\ \xi_3^{[N]}(\mathbf{x},\mathbf{p}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_3) = \sum_{i=1}^N x_ip_i = \mathbf{x}\cdot\mathbf{p}\,, \end{cases} \tag{1.129}$$

with $\beta_i \in \mathbb{R}$. In this case the N-dimensional Hamiltonian function takes the form:

$$H^{[N]}(\mathbf{x},\mathbf{p}) = H\left(\mathbf{p}^2 + \sum_{i=1}^{N} \frac{\beta_i}{x_i^2}, \mathbf{x}^2, \mathbf{x}\cdot\mathbf{p}\right) \tag{1.130}$$

and the following $2N-3$ functions ($m = 2,\dots,N$):

$$\begin{cases} C^{[m]}(\mathbf{x},\mathbf{p}) = \sum_{1\le i<j\le m}\left((x_ip_j - x_jp_i)^2 + \beta_i\frac{x_j^2}{x_i^2} + \beta_j\frac{x_i^2}{x_j^2}\right) + \sum_{i=1}^{N}\beta_i \\ C_{[m]}(\mathbf{x},\mathbf{p}) = \sum_{N-m<i<j\le N}\left((x_ip_j - x_jp_i)^2 + \beta_i\frac{x_j^2}{x_i^2} + \beta_j\frac{x_i^2}{x_j^2}\right) + \sum_{i=N-m+1}^{N}\beta_i \end{cases}, \tag{1.131}$$

Poisson-commute with the family of Hamiltonians $H^{[N]}(\mathbf{x},\mathbf{p})$ defined in (1.130). They are *universal integrals of motion* [92]. Let us notice that the rotational symmetry is recovered when all β_i's are set equal to zero (we have a generalization of the hyperspherical symmetry that characterizes central potentials). This freedom of choice is due to the fact that we can start, at the one-dimensional level, with a symplectic representation of the $\mathfrak{sl}(2,\mathbb{R})$ that includes the term β/x^2 in the generator ξ_+, and the only difference is related to the representation of the Casimir, which in this case results in $D(C) = \beta \neq 0$. Thus, in conclusion, because of the existence of such an underlying coalgebra symmetry the entire family of N-dimensional Hamiltonians previously defined is at least QMS, being the universal integrals of motion nothing but the partial Casimirs of the algebra. Let us mention that a lot of known examples (also MS) belonging to this family have been recovered and also generalized on curved spaces of constant and non-costant curvature [92]. Moreover, new remarkable examples have been constructed [93], also in relation to a (non)-standard quantum deformation of the $\mathfrak{sl}(2,\mathbb{R})$, which has been used to define another family of N-dimensional QMS Hamiltonians on manifold with variable curvature [94].

To summarize, over the years the coalgebra approach became a very well-known method to deal with (both classical and quantum) superintegrable systems, and opened various directions of research, which have been recognized by the scientific community working on the subject of superintegrability (see the introduction of the review paper [4]). Noteworthy results have been obtained also from a geometric point of view, where a notion of coalgebra spaces has been introduced in order to define those manifolds characterized by geodesic flows, i.e. the kinetic energy Hamiltonians corresponding to geodesic motion on such spaces, which are endowed with an $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry [95,96]. In particular, such class of coalgebraic kinetic energy Hamiltonians, defined as [95]:

$$H(\xi_+,\xi_-,\xi_3) \doteq \mathcal{F}(\xi_-)\xi_+ + \mathcal{G}(\xi_-)\xi_3^2 \overset{D^{\otimes N}}{=} \mathcal{F}(\mathbf{x}^2)\mathbf{p}^2 + \mathcal{G}(\mathbf{x}^2)\,(\mathbf{x}\cdot\mathbf{p})^2\,, \tag{1.132}$$

for any arbitrary (smooth) functions $\mathcal{F}$, $\mathcal{G}$, have shown to be very general, as they contain as subcases N-dimensional *spaces of constant curvature* (such as the N-sphere $\mathbb{S}^N$ and the hyperbolic space $\mathbb{H}^N$), *spherically symmetric spaces* and *Darboux spaces* [83,84], the latter implying a non-trivial (actually superintegrable) generalization of the 2-dimensional Darboux surfaces [97]. At this point, the richness and power of the coalgebra symmetry approach should be clear. Furthermore, it should be also clear that the choice of the representation plays a crucial role in the entire game. This fact will represent a crucial point for our construction.

3.2 DISCRETE QUANTUM MECHANICS, COALGEBRA SYMMETRY AND SUPERINTEGRABILITY

3.2.1 Motivations and definitions

As we previously sketched, the coalgebra approach can be used in the very same way in the quantum mechanical framework, clearly when ordering problems have been taken into account. The commutators replace the Poisson brackets and, once a representation of the algebra in terms of (finite order) differential operators have been constructed, the coproduct can be used to rise the dimension of the representation, exactly as we have previously shown for the classical case. What we are interested to investigate in this first part of the work, is the possibility of introducing such (co)algebraic method in a discrete quantum-mechanical framework, where differential operators are replaced by finite difference operators, and where the time-independent Schrödinger equation is no longer represented by a second order differential equation but a second order difference equation involving *shift operators*. In this case, roughly speaking, the quantum theory is defined on a lattice, i.e. on a finite or infinite set of points. The fundamental operators are shifts that allows us to move from a point of the lattice to another. The reason why we decided to investigate such a problem is basically two-fold: on one hand there is an increasing interest on discrete (superintegrable) systems, where the main aim is to enlarge the restricted set of known models, to describe their symmetry algebras and to understand their connection with special functions in terms of families of (discrete) orthogonal polynomials. Some progress has been done during the last decades in this direction, for example in the paper [18] the authors obtained a version of (non)relativistic quantum mechanics in a discrete space-time by applying a method that involves the so-called *umbral calculus* to discretize the Schrödinger equation. They also showed that models which are superintegrable and exactly solvable preserve these properties after discretization. Moreover, several examples of superintegrable discrete models appeared in the last few years [20–22], of which most of them followed the guidelines of previous works based on discrete oscillator models [98–103]. The possibility of introducing a powerful method like the coalgebra symmetry might be very useful to study discrete models from a slightly different point view, offering an alternative way to deal with them, with the purpose of obtaining discrete N-dimensional generalization of superintegrable families defined for continuous systems, or as an elegant way to unify different families of superintegrable systems as different realizations of a common coalgebraic structure.
On the other hand, we are interested in a multidimensional extension of the results concerning *discrete Quantum Mechanics* (dQM), a variant of the ordinary quantum theory proposed by S. Odake and R. Sasaki (see the review paper [19] with references therein), which represents the *target framework* for our coalgebraic approach. As the authors explain in [19], this quantum discrete theory, very well established from a mathematical point of view, can be thought as a deformation of the ordinary Quantum Mechanics (oQM) in the sense that the continuous theory is obtained in a suitable limit, physically speaking when the characteristic lenght scale of the lattice goes to zero. Even if the Hamiltonians are completely different from the ones that we have in the ordinary continuous theory (they are finite difference operators), the probability interpretation, the Hilbert space structure, the continuous real time, and all the physical features characterizing oQM are unchanged. This theory has been widely studied

over the years, mostly by the two aforementioned authors, and remarkable results have been obtained: they proposed a systematic understanding of various orthogonal polynomials satisfying second order difference equations with the pure imaginary shifts (i.e. when the shift on the lattice is in the *pure imaginary* direction), as well as the real shifts (i.e. when the shift is in the *real* direction). The former notion of shift led to the discovery of the so-called *imaginary discrete Quantum Mechanics* (from now on indicated as idQM), whereas the latter led to the definition of *real discrete Quantum Mechanics* (rdQM) and, as we will explicitly discuss in the following, they present very different characteristics. Moreover, they have also recast in this framework new infinite families of exceptional orthogonal polynomials (the exceptional Wilson, Askey-Wilson, Racah and q-Racah) [104, 105], understood as the generic discrete counterpart of the exceptional Laguerre and Jacobi orthogonal polynomials investigated two years before [106–108]. As a matter of fact, one of the main result was the extension of the interwining relations and Crum's theorem [109, 110] in such a discrete framework [111, 112]. This theorem, if rephrased in terms of the factorization method [15], or in the more modern language of supersymmetric Quantum Mechanics (SUSYQM) [16, 17], is well-known: it describes the relationship between the original and the associated Hamiltonian systems (the supersymmetric partners in poor words), which are iso-spectral except for the lowest energy state. Moreover, in total analogy to the ordinary quantum case, they also rephrased the notion of shape invariance for potentials in dQM, thus elucidating their exact solvability from an algebraic point of view.
For all these reasons, it was quite clear to us that this discrete version of the quantum theory was the expected target we would have obtained in terms of coalgebraic discrete representations. To be more precise, it represents a (mathematically) well-established extension of the ordinary Quantum Mechanics, and most of the concepts that we already know for the continuous theory have been reformulated in this framework. Thus, if for example we would look for a discrete representation of a given algebra endowed with a coproduct, in order to introduce coalgebraic structures for discrete models, it would be a good result to obtain a (coalgebraic) Hamiltonian whose representation belongs to such a discrete theory.

Our hope is that, even if the example that we are going to discuss represents just a little step in this direction, the coalgebra symmetry approach could shed some light in the analysis of (superintegrable) discrete models, where remarkable results have been obtained in the continuous classical and quantum cases.

3.2.2 An introduction to the structure of discrete Quantum Mechanics

In this brief section we want to define the very basic notions of *discrete Quantum Mechanics*. In particular, we are interested in introducing the *shift operators*, the explicit form of the *discrete Hamiltonians*, their *factorization* (together with the explicit form of the *ladder operators*) both for idQM and rdQM. Let us point out that we are not going to discuss explicitly the *Crum's Theorem* and the *Shape Invariance* in this discrete case, since they are not crucial in our construction, and we prefer to mantain a straight approach to our results. Anyhow, these subjects are very clearly explained in [19], the reference that we plan to use as a main guide for this introduction, where are also reported most of the results obtained in dQM by the two authors. Therefore, we refer the interested reader to such a reference for detailed explanations.

The shift operators, the domain of definition of the dynamical coordinates x and the inner product of wavefunctions in dQM

The main difference between the oQM and the dQM resides in the fact that the momentum operator $\hat{p} = -i\hbar\partial_x$ (where we keep implicit the given representation D) appears in exponentiated form, so that the fundamental operators of the theory are represented by difference operators called *shifts*, defined as (from now on we shall set $\hbar = 1$):

$$\hat{\mathfrak{T}}^{\pm} \doteq e^{\pm\beta\hat{p}} = e^{\mp i\beta\partial_x}\,, \tag{2.133}$$

where β can be both a *real* or a *pure imaginary* number. These two choices define two types of discrete quantum theories:

$$\begin{cases} \text{idQM} & \text{when} \quad \beta = \gamma \in \mathbb{R}/\{0\}\,, \\ \text{rdQM} & \text{when} \quad \beta = i \in \mathbb{C}/\mathbb{R}\,, \end{cases} \tag{2.134}$$

and the reason of this nomenclature is due to the results of the action of these operators when applied on a wavefunction, which are the following:

$$\begin{cases} \text{idQM}: & e^{\mp i\gamma\partial_x}\Phi(x) = \Phi(x \mp i\gamma)\,, \\ \text{rdQM}: & e^{\pm\partial_x}\Phi(x) = \Phi(x \pm 1)\,. \end{cases} \tag{2.135}$$

In the first case (idQM), the eigenfunctions, the potential functions and all the objects involving the x coordinate are intended as *analytic* functions of x in their domain of definition, which can also include the real axes or some subset of it where the variable x is defined [19]. Conversely, in the second case (rdQM), the wavefunctions take values only on equally spaced lattice points[2] (this is a "true" lattice in the sense that we are commonly used to think about) and the authors, after proper rescaling, define the x variables to take nonnegative integer values. To summarize, the domain of definition of the coordinate x takes continuous or discrete values depending on the version of the theory we are dealing with, i.e.:

$$\begin{cases} \text{idQM}: & x \in \mathbb{R}\,, \quad x \in (x_1, x_2)\,, \\ \text{rdQM}: & x \in \mathbb{Z}_{\geq 0}\,,\ x \in [0, x_{max}]\,, \end{cases} \tag{2.136}$$

where (x_1, x_2) can be finite or $(-\infty, +\infty)$ and $x_{max} = N$ (finite case) or $x_{max} = \infty$. Concerning the inner product of wavefunctions, taking into account the previous definitions, it takes the following form:

$$\begin{cases} \text{idQM}: \langle \Phi_1, \Phi_2 \rangle \doteq \int_{x_1}^{x_2} \mathrm{d}x\, \Phi_1^*(x)\Phi_2(x)\,, \\ \text{rdQM}: \langle \Phi_1, \Phi_2 \rangle \doteq \sum_{x=0}^{x_{max}} \Phi_1^*(x)\Phi_2(x)\,, \end{cases} \tag{2.137}$$

and the wavefuntions are normalized with respect to the usual norm $||\Phi||^2 \doteq \langle \Phi, \Phi \rangle$. Let us mention that the $*$-operation in dQM is defined as follows: if $f(x) = \sum_j a_j x^j$ is a function defined in terms of a polynomial expansion with complex coefficients $a_j \in \mathbb{C}$, then $f^*(x) \doteq \sum_j a_j^* x^j$, where a_j^* is the complex conjugation of a_j. If $f(x) = f^*(x)$, then such a function just takes real values. Moreover, it also (trivially) holds $f(x) = f^{**}(x)$ and $f^*(x) = f^*(x^*)$ [19].

2 Let us mention that orthogonal polynomials involving real shifts are known as *orthogonal polynomial of discrete variable* [113].

Discrete Hamiltonians, factorization and ladder operators

Let us now introduce the definition of Quantum Hamiltonians in dQM. First of all, let us observe that will be considered only Hamiltonian operators having a finite (rdQM with finite N) or semi-infinite number of discrete energy levels (an arbitrary additive constant of the Hamiltonian is chosen in such a way the energy associated to the ground state vanishes). All the Hamiltonians belonging to this theory are semi-positive definite and have a factorized form in terms of two operators, which we will indicate as $\hat{A}$ and $\hat{A}^\dagger$ (the adjoint of $\hat{A}$). In particular, in the case of imaginary discrete Quantum Mechanics they are given by:

$$\begin{cases} \hat{A} \doteq i\big(e^{\frac{\gamma\hat{p}}{2}}\sqrt{\mathcal{V}^*(x)} - e^{-\frac{\gamma\hat{p}}{2}}\sqrt{\mathcal{V}(x)}\big)\,,\ \hat{A}^\dagger = -i\big(\sqrt{\mathcal{V}(x)}e^{\frac{\gamma\hat{p}}{2}} - \sqrt{\mathcal{V}^*(x)}e^{-\frac{\gamma\hat{p}}{2}}\big) \\ \hat{\mathbb{H}} = \hat{A}^\dagger\hat{A} = \sqrt{\mathcal{V}(x)}e^{\gamma\hat{p}}\sqrt{\mathcal{V}^*(x)} + \sqrt{\mathcal{V}^*(x)}e^{-\gamma\hat{p}}\sqrt{\mathcal{V}(x)} - \mathcal{V}(x) - \mathcal{V}^*(x)\,, \end{cases} \tag{2.138}$$

where $\gamma \in \mathbb{R}/\{0\}$, and the (analytic) function $\mathcal{V}(x)$, together with its (always analytic) complex conjugate $\mathcal{V}^*(x)$, are complex-valued functions. As far as the case of real discrete Quantum Mechanics is concerned, the ladder operators and the Hamiltonians are defined as follows:

$$\begin{cases} \hat{A} \doteq \sqrt{\mathcal{B}(x)} - e^{i\hat{p}}\sqrt{\mathcal{D}(x)}\,,\ \hat{A}^\dagger = \sqrt{\mathcal{B}(x)} - \sqrt{\mathcal{D}(x)}e^{-i\hat{p}} \\ \hat{\mathbb{H}} = \hat{A}^\dagger\hat{A} = -\sqrt{\mathcal{B}(x)}e^{i\hat{p}}\sqrt{\mathcal{D}(x)} - \sqrt{\mathcal{D}(x)}e^{-i\hat{p}}\sqrt{\mathcal{B}(x)} + \mathcal{B}(x) + \mathcal{D}(x)\,, \end{cases} \tag{2.139}$$

where the (discrete-valued) functions $\mathcal{B}(x), \mathcal{D}(x)$ are defined in such a way $B(x) > 0$ for $x \geq 0$ (with $\mathcal{B}(N) = 0$ in the finite case) and $\mathcal{D}(x) > 0$ for $x > 0$, together with the condition $\mathcal{D}(0) = 0$, which is necessary in order to drop out the term $\psi(-1)$ when we apply the Hamiltonian to an eigenfunctions $\psi(0)$. Similarly, the condition $B(N) = 0$ is necessary for the finite case. It is interesting to notice that the Hamiltonians in dQM are (formally) self-adjoint, being $\hat{p} = \hat{p}^\dagger$ (a self-adjoint operator). Anyhow, for completeness we have to point out that in the case of idQM the selfadjointness of the Hamiltonian has to be verified by taking into account that, for specific choices of the potential functions, singularities may arise in the calculation of the Cauchy integral (additional integration contours appear) and they have to be cancelled by the zeroes of the ground state wave functions (see appendix A in [114] where three specific examples are discussed). Let us conclude by observing that the expressions for the (finite difference) Hamiltonian operators can be rephrased in a unified way using the following notation:

$$\begin{cases} \hat{\mathbb{H}} = \epsilon\big(\sqrt{\mathcal{V}_+(x)}e^{\beta\hat{p}}\sqrt{\mathcal{V}_-(x)} + \sqrt{\mathcal{V}_-(x)}e^{-\beta\hat{p}}\sqrt{\mathcal{V}_+(x)} - \mathcal{V}_+(x) - \mathcal{V}_-(x)\big)\,, \\ idQM: \quad \epsilon = +1\,, \quad \beta = \gamma\,, \quad \mathcal{V}_+(x) = \mathcal{V}(x)\,, \quad \mathcal{V}_-(x) = \mathcal{V}^*(x)\,, \\ rdQM: \quad \epsilon = -1\,, \quad \beta = i\,, \quad \mathcal{V}_+(x) = \mathcal{B}(x)\,, \quad \mathcal{V}_-(x) = \mathcal{D}(x)\,. \end{cases} \tag{2.140}$$

Thus, in the full theory the time-independent Schrödinger equation results:

$$\hat{\mathbb{H}}\,\Phi(x) = E\,\Phi(x)\,, \tag{2.141}$$

that is:

$$\epsilon\big(\sqrt{\mathcal{V}_+(x)}\sqrt{\mathcal{V}_-(x-i\beta)}\,\Phi(x-i\beta) + \sqrt{\mathcal{V}_-(x)}\sqrt{\mathcal{V}_+(x+i\beta)}\,\Phi(x+i\beta) - (\mathcal{V}_+(x) + \mathcal{V}_-(x))\,\Phi(x)\big) = E\,\Phi(x)\,. \tag{2.142}$$

This second order difference equation, once the values of ϵ, β have been fixed, for specific choices of the potential functions $\mathcal{V}_+(x)$ and $\mathcal{V}_-(x)$, includes a lot of exactly solvable examples of one-dimensional (real and imaginary) discrete quantum models, whose spectra

and eigenfunctions are related to specific families of known orthogonal polynomials, such as for example *Wilson*, *Askey-Wilson*, *Meixner Pollaczek* (idQM) and *Meixner*, *Racah*, *q-Racah* (rdQM) [12], together with new families developed by the two authors, such as the *exceptional Meixner* and *exceptional Meixner-Pollaczek* [19]. Such equation, when a lattice parameter is (in some way) introduced, have to collapse into the continuous Schrödinger equation of the ordinary quantum theory in an appropriate limit. An example of this limit procedure will be clear immediately, when we will present our construction.
Before concluding this brief introduction to the building blocks of dQM, let us comment that the work of the two authors has been remarkable, both for its originality and its complete generality. As a matter of fact, our original contribution can be considered as a specific example belonging to this general framework, which we have selected *ad hoc* in order to convey the notions commonly used in the literature about coalgebra symmetries to this field. In particular, we will explicitly deal with the simplest case of real discrete quantum mechanical system, the one involving the *Charlier orthogonal polynomials*, and in the next section we will explain the reason why we decided to exploit such a particular case.

3.2.3 Coalgebra symmetry and real discrete Quantum Mechanics

As we previously discussed when we were introducing the coalgebra symmetry approach, in this method the fundamental task is to find out a suitable representation of a given algebra endowed with a coproduct, in such a way to use this map to rise the dimension of the (classical or quantum) model we are dealing with. Then, the construction of the integrals of motion is done by taking the images of the partial Casimirs under the given representation. We discussed how in classical mechanics the representation is given in terms of $\mathcal{C}^\infty$ functions on the phase space, whereas in quantum mechanics it is given in terms of (finite order) differential operators defined on some Hilbert space. In this discrete case it is quite clear that we need to look for a quantum discrete representation of the algebra, given in terms of the finite difference operators previously defined. So, the problem we want to deal with is quite clear (even if highly nontrivial): *find out a quantum discrete representation of a given coalgebra*. Now, a question arising almost automatically is: which coalgebra? An important step to answer this question was done when we were analyzing the structure of the *Askey-Scheme* for orthogonal polynomials. In fact, if we take a look at the *Askey Scheme* in Figure 2 (taken from [115]), we can appreciate that the simplest family of (discrete) orthogonal polynomials is represented by the *Charlier polynomials* $C_n(x;a)$ ($x = \{0, 1, 2, \dots \infty\}$), that involve just one free parameter $a > 0$. This parameter governs the contraction from the Charlier polynomials to the Hermite polynomials under the following limit relation [115]:

$$\lim_{a\to\infty} (2a)^{\frac{n}{2}}\, C_n\Big((2a)^{\frac{1}{2}}x + a; a\Big) = (-1)^n\, H_n(x)\,. \tag{2.143}$$

This mathematical relation contains a rich physical meaning, since plays the role of a bridge between two families of hypergeometric orthogonal polynomials, which are respectively discrete (and then described in terms of rdQM) and continuous (then described in oQM). Thus, the idea is basically to treat some quantity proportional to the inverse of the parameter $a > 0$ (that has to be find out) as the lattice spacing, in order to obtain the Hermite polynomials when such a quantity goes to zero (i.e. when $a \to \infty$). Physically speaking, this would represent the continuous limit, when the reticular step goes to zero and the discrete theory collapses

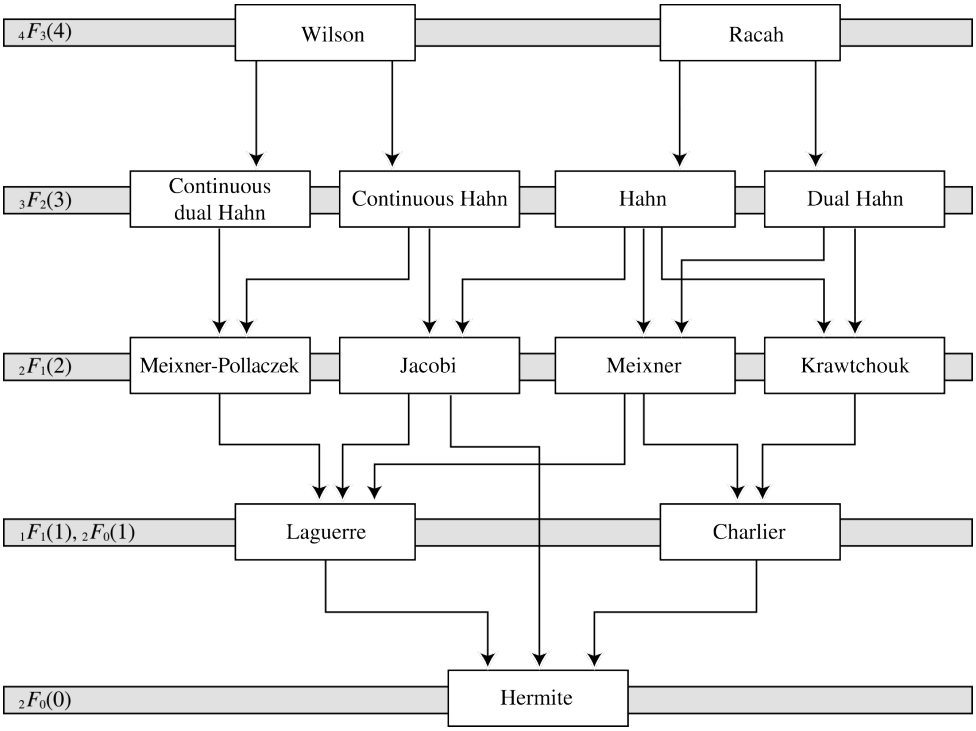


Figure 2.: Askey scheme. The number of free real parameters is zero for Hermite polynomials and increases by one for each row. The scheme terminates at the fifth row, where are involved four free real parameters for the Wilson and Racah polynomials.

to the continuous one. Now, since we know that the Hermite Polynomials (multiplied by a gaussian gauge factor) appear as the solution of the harmonic oscillator expressed in cartesian coordinates, we expect to obtain a (discrete) deformation, in the sense of Sasaki and Odake, of such a system. If we found this model, its solution would be expressed in terms of the Charlier orthogonal polynomials multiplied by some gauge factor. Moreover, and this is the fundamental point, since we know all about the coalgebra symmetry of the harmonic oscillator, in particular we know that it is described by the generators of an $\mathfrak{sl}(2,\mathbb{R})$ (co)algebra, we expect to find a discrete representation of the $\mathfrak{sl}(2,\mathbb{R})$, which would collapse to the differential one in an (appropriate) continuous limit.
An important comment about the Charlier polynomials was done in [19], where the authors claimed: "*The simplest example of rdQM, the Charlier polynomial, has no shiftable parameter. Its shape invariance relation* $\hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} = \hat{\mathbb{1}}$ *gives another realisation of the oscillator algebra*". Thus, strengthened by this further hint, it seemed reasonable to us to choose such a family of (discrete) othogonal polynomials to start connecting the coalgebra symmetry within the framework of discrete Quantum Mechanics. Poorly speaking, we started thinking how to construct such a representation of the oscillator algebra. If we succeed in this, we are able to use the coproduct in order to generalize the discrete system to higher dimensions and to construct its discrete quantum integrals of motion. These integrals, clearly, in the continuous limit must collapse to the ones of the standard quantum harmonic oscillator. As a matter of fact, a discrete representation of such an oscillator algebra can be found in [100], where the Charlier oscillator represent just a particular case of the more general (two-parameters)

Meixner oscillators investigated by the authors. Anyhow, as we will see, we obtained such a representation in a slightly different way, by using a construction which starts from the classical case (we mean classical mechanics). In what follows, we will present this construction. In particular, we will see how it allowed us to obtain a discrete representation of the oscillator algebra. More precisely, we constructed a discrete representation of the Heisenberg algebra $\mathfrak{h}(1)$ and we use it to generate a discrete representation of the $\mathfrak{sl}(2,\mathbb{R})$. Then, since all the results are obtained algebraically, we have been able to introduce the coalgebra symmetry approach. The following results have been obtained in a joint work with D. Riglioni, and are based on the work [40].

3.3 FROM CONTINUOUS TO DISCRETE QUANTUM MECHANICS: THE CHARLIER OSCILLATOR

In order to introduce our construction, let us briefly review the application of the coalgebra symmetry in the quantum case. As we previously discussed the picture is exactly the same, the bracket of the Poisson-Lie algebra are replaced with a new binary operation, the commutator, and the generators are represented by Hermitian operators on the Hilbert Space $\mathcal{H}$, rather than $\mathcal{C}^\infty$ functions on the symplectic manifold $\mathcal{M}$. In particular, by performing the canonical quantization procedure, it is straightforward to map the classical representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra to its quantum counterpart. In fact, by replacing the momentum $p \in \mathbb{R}$ and the coordinate $x \in \mathbb{R}$ with their quantum analogs[3]:

$$p \to \hat{p} = -i\hbar\partial_x \ , \ x \to \hat{x} = x \ , \tag{3.144}$$

we easily find that these operators, together with the central element $\hat{c}$, generate the three dimensional Heisenberg algebra $\mathfrak{h}(1)$:

$$[\hat{p},\hat{c}] = [\hat{x},\hat{c}] = [\hat{c},\hat{c}] = 0 \ , \quad [\hat{x},\hat{p}] = \hat{c} \, , \tag{3.145}$$

where $\hat{c} \doteq i\hbar\,\hat{\mathbb{1}}$. Using these generators, we can construct the quantum representation of the $\mathfrak{sl}(2,\mathbb{R})$, a representation given in terms of differential operators that reads:

$$\begin{cases} \hat{\xi}_+(\hat{x},\hat{p}) \equiv \hat{p}^2 = -\hbar^2\partial_{xx} \, , \\ \hat{\xi}_-(\hat{x},\hat{p}) \equiv \hat{x}^2 = x^2 \, , \\ \hat{\xi}_3(\hat{x},\hat{p}) \equiv \frac{1}{2}[\hat{x},\hat{p}]_+ = -i\hbar(x\partial_x + \frac{1}{2}) \, , \end{cases} \tag{3.146}$$

where we defined $[\hat{x},\hat{y}]_+ \doteq \hat{x}\hat{y} + \hat{y}\hat{x}$, $\forall\ \hat{x},\hat{y} \in \mathcal{H}$. Let us notice that here we took the anticommutator in the definition of the operator $\hat{\xi}_3$, so an ordering has been implicitly chosen. Using this representation, we immediately obtain:

$$[\hat{\xi}_-(\hat{x},\hat{p}),\hat{\xi}_+(\hat{x},\hat{p})] = 4i\hbar\hat{\xi}_3(\hat{x},\hat{p}) \ , \ [\hat{\xi}_3(\hat{x},\hat{p}),\hat{\xi}_\pm(\hat{x},\hat{p})] = \pm 2i\hbar\hat{\xi}_\pm(\hat{x},\hat{p}) \, , \tag{3.147}$$

that is the quantum analog of the classical Poisson-Lie $\mathfrak{sl}(2,\mathbb{R})$ algebra (1.88). In this case, the Casimir of the algebra is represented by the following operator:

$$\hat{C}(\hat{\xi}_\pm,\hat{\xi}_3) \doteq \frac{1}{2}[\hat{\xi}_+,\hat{\xi}_-]_+ - \hat{\xi}_3^2 = (i\hbar)^2\frac{1}{2}(\frac{1}{2}+1)\hat{\mathbb{1}} \, , \tag{3.148}$$

3 To simplify the notation, we will often drop the symbol D to indicate the given representation, except where it is important to emphasize it.

in the given representation. Thus, the quantum harmonic oscillator will be expressed as a function of the quantum generators of the algebra in the following way:

$$\hat{H}(\hat{\xi}_-, \hat{\xi}_+) = \frac{\hat{\xi}_+ + \hat{\xi}_-}{2} \stackrel{D}{=} -\frac{\hbar^2}{2}\partial_{xx} + \frac{x^2}{2} \,. \tag{3.149}$$

The usual factorization procedure will be therefore rephrased in terms of these generators. In particular, we shall define the two (ladder) operators:

$$\begin{cases} \hat{a}(\hat{\xi}_-, \hat{\xi}_3) \doteq \frac{i}{\sqrt{2}}\hat{\xi}_-^{-\frac{1}{2}}(\hat{\xi}_3 + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\xi}_-^{\frac{1}{2}} = \frac{\hbar}{\sqrt{2}}\partial_x + \frac{x}{\sqrt{2}} \\ \hat{a}^\dagger(\hat{\xi}_-, \hat{\xi}_3) \doteq -\frac{i}{\sqrt{2}}\hat{\xi}_-^{-\frac{1}{2}}(\hat{\xi}_3 + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\xi}_-^{\frac{1}{2}} = -\frac{\hbar}{\sqrt{2}}\partial_x + \frac{x}{\sqrt{2}} \,, \end{cases} \tag{3.150}$$

where in the last equality we fixed the representation. Let us notice that it is possible to interpret inverses and roots as Taylor series in the enveloping algebra of the representation for $\mathfrak{sl}(2,\mathbb{R})$ [88]. These two operators, as we know, are such that:

$$[\hat{a}(\hat{\xi}_-, \hat{\xi}_3), \hat{a}^\dagger(\hat{\xi}_-, \hat{\xi}_3)] = \hbar\hat{\mathbb{1}} \,. \tag{3.151}$$

Let us observe that, coalgebraically, the following relation holds $\hat{\xi}_-^{-\frac{1}{2}}(\hat{\xi}_3 + \frac{i\hbar}{2}) \equiv \hat{\xi}_+^{\frac{1}{2}}$. As usual, in terms of the two ladders, the Hamiltonian is factorized as:

$$\hat{H}(\hat{\xi}_-, \hat{\xi}_+) = \frac{\hat{\xi}_+ + \hat{\xi}_-}{2} = \hat{a}^\dagger(\hat{\xi}_-, \hat{\xi}_3)\hat{a}(\hat{\xi}_-, \hat{\xi}_3) + \frac{\hbar}{2} \,. \tag{3.152}$$

The eigenfunctions of the harmonic oscillator will be therefore computed by applying the raising operator $\hat{a}^\dagger(\hat{\xi}_-, \hat{\xi}_3)$ to the kernel of the operator $\hat{a}(\hat{\xi}_-, \hat{\xi}_3)$, which we obtain by solving the first order differential equation $\hat{a}(\hat{\xi}_-, \hat{\xi}_3)\Psi_0(x) = 0$. This leads to the vacuum state:

$$\Psi_0(x) \propto \exp\left(-\frac{x^2}{2\hbar}\right) . \tag{3.153}$$

Then, the (unnormalized) eigenfunctions of the spectral problem will be given by:

$$\Psi_n(x) \propto [\hat{a}^\dagger(\hat{\xi}_-, \hat{\xi}_3)]^n\Psi_0(x) = \left(\frac{\hbar}{2}\right)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{\hbar}}\right)\Psi_0(x) \,, \tag{3.154}$$

where $H_n(x)$ are the *Hermite* orthogonal polynomials, which are orthogonal on the whole real axes with respect to the measure $|\Psi_0(x)|^2 dx = w(x)dx$, $w(x)$ being the *weight function*. In particular, the orthonormality condition reads:

$$O_{nm}^{(Hermite)} \doteq \frac{(\hbar/2)^{-n}}{2^n n!\sqrt{\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x \; \Psi_n^*(x)\Psi_m(x) \doteq \frac{(\hbar/2)^{-n}}{2^n n!\sqrt{\pi\hbar}} \langle \Psi_n, \Psi_m \rangle = \delta_{nm} \,, \tag{3.155}$$

being $\langle , \rangle$ the scalar product on $\mathcal{H} \doteq L^2(\mathbb{R})$. So, we have obtained the known normalized eigenfunctions that solve the spectral problem, i.e.:

$$\Psi_n(x) = \frac{1}{\sqrt{2^n n!}\,(\pi\hbar)^{\frac{1}{4}}} H_n\left(\frac{x}{\sqrt{\hbar}}\right)\Psi_0(x) \,, \tag{3.156}$$

and the corresponding spectrum for the stationary states reads:

$$\hat{H}(\hat{\xi}_-,\hat{\xi}_+)\Psi_n(x) = (\hat{a}^\dagger(\hat{\xi}_-,\hat{\xi}_3)\hat{a}(\hat{\xi}_-,\hat{\xi}_3) + \hbar/2\,)\Psi_n(x) = \hbar(n+1/2)\Psi_n(x)\,. \tag{3.157}$$

As we already said, the advantage of having such a representation is related to the multi-dimensional generalisation of the system, the latter straightforwardly obtainable by means of the coproduct. As an instructive example, we apply this technique to construct the N-dimensional version of the quantum harmonic oscillator (3.149), focusing the analysis on its *superintegrability* properties. To this aim, let us apply the coproduct map Δ to the Hamiltonian:

$$\hat{H}^{[2]} \doteq \Delta(\hat{H}(\hat{\xi}_-,\hat{\xi}_+)) = \Delta\left(\frac{\hat{\xi}_+ + \hat{\xi}_-}{2}\right) = \frac{\Delta(\hat{\xi}_+) + \Delta(\hat{\xi}_-)}{2} \overset{D\otimes D}{=} \sum_{j=1}^{2} \hat{H}_j(\hat{x}_j,\hat{p}_j)\,, \tag{3.158}$$

being $\hat{H}_j(\hat{x}_j,\hat{p}_j) \doteq -\frac{\hbar^2}{2}\partial_{x_j x_j} + \frac{x_j^2}{2}$ in the given representation. The new (abstract) Hamiltonian lives in the tensor product space of two copies of the $\mathfrak{sl}(2,\mathbb{R})$ algebra and, when the representation is fixed, it represents the two-dimensional quantum harmonic oscillator expressed in cartesian coordinates. As already known, this Hamiltonian can be factorized in terms of two copies of the ladder operators:

$$\hat{H}^{[2]} = \sum_{k=1}^{2} \hat{a}_k^\dagger(\hat{\xi}_{(-,k)},\hat{\xi}_{(3,k)})\hat{a}_k(\hat{\xi}_{(-,k)},\hat{\xi}_{(3,k)}) + \hbar\,, \tag{3.159}$$

that are defined as:

$$\begin{cases} \hat{a}_k(\hat{\xi}_{(-,k)},\hat{\xi}_{(3,k)}) \doteq \frac{i}{\sqrt{2}}\hat{\xi}_{(-,k)}^{-\frac{1}{2}}(\hat{\xi}_{(3,k)} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\xi}_{(-,k)}^{\frac{1}{2}} = \frac{\hbar}{\sqrt{2}}\partial_{x_k} + \frac{x_k}{\sqrt{2}} \\ \hat{a}_k^\dagger(\hat{\xi}_{(-,k)},\hat{\xi}_{(3,k)}) \doteq -\frac{i}{\sqrt{2}}\hat{\xi}_{(-,k)}^{-\frac{1}{2}}(\hat{\xi}_{(3,k)} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\xi}_{(-,k)}^{\frac{1}{2}} = -\frac{\hbar}{\sqrt{2}}\partial_{x_k} + \frac{x_k}{\sqrt{2}}\,, \end{cases} \tag{3.160}$$

where in this case we used the following notation for the single copies in the tensor product:

$$\hat{\xi}_{(+,k)} \doteq -\hbar^2\partial_{x_k x_k}\,, \quad \hat{\xi}_{(-,k)} \doteq x_k^2\,, \quad \hat{\xi}_{(3,k)} \doteq \frac{1}{2}[\hat{x}_k,\hat{p}_k]_+ = -i\hbar(x_k\partial_{x_k} + \frac{1}{2}) \quad (k=1,2). \tag{3.161}$$

Clearly, because of the homomorphism property, the coproduct of the generators:

$$\xi_+^{[2]} = \Delta(\hat{\xi}_+) \overset{D\otimes D}{=} \sum_{k=1}^{2} \hat{\xi}_{(+,k)}\,, \quad \xi_-^{[2]} = \Delta(\hat{\xi}_-) \overset{D\otimes D}{=} \sum_{k=1}^{2} \hat{\xi}_{(-,k)}\,, \quad \xi_3^{[2]} = \Delta(\hat{\xi}_3) \overset{D\otimes D}{=} \sum_{k=1}^{2} \hat{\xi}_{(3,k)}\,, \tag{3.162}$$

is such that:

$$[\Delta(\hat{\xi}_-),\Delta(\hat{\xi}_+)] = 4i\hbar\Delta(\hat{\xi}_3)\,, \quad [\Delta(\hat{\xi}_3),\Delta(\hat{\xi}_\pm)] = \pm 2i\hbar\Delta(\hat{\xi}_\pm)\,. \tag{3.163}$$

Moreover, the image of the Casimir operator under such a representation results:

$$\hat{C}^{[2]} \doteq \Delta(\hat{C}) = \Delta(\frac{1}{2}(\hat{\xi}_+\hat{\xi}_- + \hat{\xi}_-\hat{\xi}_+) - \hat{\xi}_3^2) = (\hat{L}_z^2 - \hbar^2)\,, \tag{3.164}$$

where we defined $\hat{L}_z^2 \doteq (-i\hbar x_1\partial_{x_2} + i\hbar x_2\partial_{x_1})^2$. As in the classical case, by construction, the Casimir commutes with the Hamiltonian, i.e. $[\hat{C}^{[2]},\hat{H}^{[2]}] = 0$. This commutation relation is related to the conservation of the *physical* quantity:

$$\hat{L}_z \doteq (\hat{C}^{[2]} + \hbar^2)^{\frac{1}{2}} = (-i\hbar x_1\partial_{x_2} + i\hbar x_2\partial_{x_1}) = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\,, \tag{3.165}$$

that is the angular momentum of the system (where we used the subscript "z" in analogy to the three dimensional case). At this level, our system is *QMS*[4] since it is endowed with $2N-2=2$ conserved quantities, being the dimension of the ambient space $N=2$. The other quantity necessary to ensure the *maximal superintegrability*, which cannot be obtained by using this approach, as we already seen is nothing but the quantum version of the Demkov-Fradkin tensor, which reads ($m=\omega=1$):

$$\hat{\mathrm{I}}_{ij} \doteq \hat{p}_i\hat{p}_j + \hat{x}_i\hat{x}_j\,, \qquad (i,j=1,2)\,. \tag{3.166}$$

This object provides four quantities commuting with the Hamiltonian $\hat{H}^{[2]}$, moreover $[\hat{\mathrm{I}}_{11},\hat{\mathrm{I}}_{22}] = [\hat{\mathrm{I}}_{12},\hat{\mathrm{I}}_{21}] = 0$, being $\hat{\mathrm{I}}_{21} = \hat{\mathrm{I}}_{12}$. This implies that the set $\{\hat{H}^{[2]},\hat{C}^{[2]},\hat{\mathrm{I}}_{ii}\}$ at fixed index i provides 3 algebraically independent observables, the Hamiltonian $\hat{H}^{[2]}$ being $\hat{H}^{[2]} = \frac{1}{2}\sum_{i=1}^{2}\hat{\mathrm{I}}_{ii}$.

Clearly, the spectrum of the two dimensional harmonic oscillator will be performed by applying the raising operators $\hat{a}_k^\dagger(\hat{\xi}_-,\hat{\xi}_3)$ to the kernel of the operators $\hat{a}_k(\hat{\xi}_-,\hat{\xi}_3)$ ($k=1,2$), calculated by solving the first order differential equations $\hat{a}_k(\hat{\xi}_-,\hat{\xi}_3)\Psi_0(x_k)=0$. This leads to the following solutions:

$$\Psi_0(x_k) \propto \exp\left(-\frac{x_k^2}{2\hbar}\right) \quad (k=1,2). \tag{3.167}$$

Thus, the vacuum state of the system can be defined as:

$$\Psi_{(0,0)}(x_1,x_2) \doteq \Psi_0(x_1)\Psi_0(x_2) \propto \exp\left(-\frac{x_1^2+x_2^2}{2\hbar}\right), \tag{3.168}$$

and the corresponding (unnormalized) eigenfunctions of the spectral problem result:

$$\begin{aligned}\Psi_{(n,m)}(x_1,x_2) &\propto [\hat{a}_1^\dagger(\hat{\xi}_{(-,1)},\hat{\xi}_{(3,1)})]^n[\hat{a}_2^\dagger(\hat{\xi}_{(-,2)},\hat{\xi}_{(3,2)})]^m\Psi_{(0,0)}(x_1,x_2)\\ &= \left(\frac{\hbar}{2}\right)^{\frac{n+m}{2}} H_n\left(\frac{x_1}{\sqrt{\hbar}}\right)H_m\left(\frac{x_2}{\sqrt{\hbar}}\right)\Psi_{(0,0)}(x_1,x_2)\,.\end{aligned} \tag{3.169}$$

Finally, the spectrum for the stationary states reads:

$$\hat{H}^{[2]}\Psi_{(n,m)}(x_1,x_2) = \hbar(n+m+1)\Psi_{(n,m)}(x_1,x_2)\,, \tag{3.170}$$

where $n,m=0,1,\dots\infty$ the quantum numbers of the single one-dimensional realization of the harmonic oscillator. Concerning the N-dimensional extension, as we previously discussed in the general theory, it can be straightforwardly constructed by iterating the application of the coproduct, and the coassociativity property can be used to generate the conserved quantities as partial (respectively left and right) Casimirs of the $\mathfrak{sl}(2,\mathbb{R})$. In this case, the N-dimensional quantum Hamiltonian results:

$$\hat{H}^{[N]} \doteq \Delta^{[N]}(\hat{H}(\hat{\xi}_-,\hat{\xi}_+)) = \Delta^{[N]}\left(\frac{\hat{\xi}_+ + \hat{\xi}_-}{2}\right) = \frac{\Delta^{[N]}(\hat{\xi}_+)+\Delta^{[N]}(\hat{\xi}_-)}{2} \overset{D^{\otimes N}}{=} \sum_{j=1}^{N}\hat{H}_j(\hat{x}_j,\hat{p}_j)\,, \tag{3.171}$$

where the generators are given in terms of the following N-dimensional representation:

$$\begin{cases}\hat{\xi}_+^{[N]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_+) = -\hbar^2\sum_{i=1}^{N}\partial_{x_ix_i}\\ \hat{\xi}_-^{[N]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_-) = \sum_{i=1}^{N}x_i^2\\ \hat{\xi}_3^{[N]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) \doteq (D^{\otimes N})\Delta^{[N]}(\xi_3) = -i\hbar\sum_{i=1}^{N}(x_i\partial_{x_i}+\frac{N}{2})\,.\end{cases} \tag{3.172}$$

4 Or integrable, since in dimension $N=2$ the two definitions coincide.

The Hamiltonian (3.171) commutes with the $2N-3$ left and right conserved quantities provided by the coalgebra symmetry. Apart from a (trivial) additive constant $\frac{\hbar^2}{4}m(m-4)$ that appears in the quantum case [52], they are given by:

$$\begin{cases} \hat{C}^{[m]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \sum_{1\le i<j\le m}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2\,,\ \hat{C}_{[m]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \sum_{N-m<i<j\le N}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2 \\ \hat{C}^{[N]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \hat{C}_{[N]}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \sum_{1\le i<j\le N}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2\,, \end{cases} \tag{3.173}$$

and, together with the $^{N(N+1)}/_2$ components of the quantum Demkov-Fradkin tensor, ensure the maximal superintegrability of the quantum oscillator in N dimensions. As we known, in this general case, the solution of the spectral problem is factorized in terms of N single copies of one dimensional harmonic oscillators (in cartesian coordinates): the spectrum is expressed in terms of an additive sum of the quantum numbers of the single realizations, whereas the eigenfunctions are factorized in terms of the product of Hermite polynomials (one for each cartesian variable) multiplied by a gauge gaussian factor involving the N coordinates $x_1 \dots x_N$.

3.3.1 real discrete Quantum Mechanics as a difference realization of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra: the harmonic oscillator on the lattice

What we would point out in this section is an interesting connection between ordinary quantum mechanics (*oQM*) and discrete quantum mechanics (*dQM*). We will show how two oscillator Hamiltonian systems, respectively defined on *oQM* and *rdQM*, can be linked through canonical transformations. Most importantly, we will use such a link in order to introduce, for the first time to the best of our knowledge, the notion of coalgebra symmetry in a quantum-mechanical discrete framework. This link will be used to construct an N-dimensional (maximally) superintegrable quantum discrete version of the harmonic oscillator. In particular, we will show how the maximal superintegrability is ensured thanks to the existence of a discrete version of both the angular momentum and Demkov-Fradkin tensor.
Let us start our construction by recalling the classical oscillator Hamiltonian (smooth) function $H : \mathcal{M} \to \mathbb{R}$, defined on the symplectic manifold $(\mathcal{M} = \mathbb{R}^2,\ \omega_0 = \mathrm{d}x \wedge \mathrm{d}p)$ (where we fix as usual $m = \omega = 1$):

$$H(x,p) = \frac{p^2+x^2}{2}\,, \tag{3.174}$$

where x, p are canonical (local) coordinates on $\mathcal{M}$ such as $\{x,p\} = 1$. Let us introduce the following algebraic transformation, a one-parameter change of variables on the symplectic manifold defined as follows:

$$\begin{cases} \bar{x}(x,p) = x + \lambda\,\dfrac{p^2+x^2}{2} \\ \bar{p}(x,p) = \dfrac{1}{\lambda}\arctan\left(\dfrac{\lambda p}{1+\lambda x}\right) \end{cases} \quad \text{with (local) inverse} \quad \begin{cases} x(\bar{x},\bar{p}) = \dfrac{\sqrt{1+2\lambda\bar{x}}\cos(\lambda\bar{p})-1}{\lambda} \\ p(\bar{x},\bar{p}) = \dfrac{\sqrt{1+2\lambda\bar{x}}\sin(\lambda\bar{p})}{\lambda}\,, \end{cases} \tag{3.175}$$

which maps the classical oscillator Hamiltonian (3.174) to the new Hamiltonian function:

$$H(\bar{x},\bar{p}) = \frac{1+\lambda\bar{x}-\sqrt{1+2\lambda\bar{x}}\cos(\lambda\bar{p})}{\lambda^2}\,, \tag{3.176}$$

where the parameter $\lambda > 0$ and $-\frac{1}{2\lambda} \leq \bar{x} < \infty$, $-\frac{\pi}{2\lambda} < \bar{p} < \frac{\pi}{2\lambda}$. The transformation (3.175) is connected to the identity function being:

$$\begin{cases} \lim_{\lambda \to 0} x(\bar{x}, \bar{p}) = \bar{x} \\ \lim_{\lambda \to 0} p(\bar{x}, \bar{p}) = \bar{p}\,, \end{cases} \tag{3.177}$$

and, by direct computation, we can show that it respects the *canonical* constraints:

$$\{\bar{x}(x,p), \bar{p}(x,p)\}_{(x,p)} = 1\,, \quad \{\bar{x}(x,p), \bar{x}(x,p)\}_{(x,p)} = \{\bar{p}(x,p), \bar{p}(x,p)\}_{(x,p)} = 0\,. \tag{3.178}$$

Moreover:

$$\{x(\bar{x}, \bar{p}), p(\bar{x}, \bar{p})\}_{(\bar{x},\bar{p})} = 1\,. \tag{3.179}$$

This is a crucial observation because, by definition of canonical tranformation, all the algebraic $\mathfrak{sl}(2, \mathbb{R})$ relations characterizing the harmonic oscillator will remain unchanged if expressed in terms of the new coordinates $(\bar{x}, \bar{p})$. In other words, this implies that we can construct a new representation of the $\mathfrak{sl}(2, \mathbb{R})$ Poisson-Lie algebra simply by replacing the old coordinates pair $(x,\, p)$ with the new one $(\bar{x},\, \bar{p})$, giving rise to the following new symplectic representation:

$$\begin{cases} \bar{\xi}_+(\bar{x}, \bar{p}) \doteq \bar{D}(\xi_+) \equiv p^2(\bar{x}, \bar{p}) = \dfrac{(1 + 2\lambda\bar{x})\sin^2(\lambda\bar{p})}{\lambda^2} \\ \bar{\xi}_-(\bar{x}, \bar{p}) \doteq \bar{D}(\xi_-) \equiv x^2(\bar{x}, \bar{p}) = \dfrac{(1 + 2\lambda\bar{x})\cos^2(\lambda\bar{p}) - 2\sqrt{1 + 2\lambda\bar{x}}\cos(\lambda\bar{p}) + 1}{\lambda^2} \\ \bar{\xi}_3(\bar{x}, \bar{p}) \doteq \bar{D}(\xi_3) \equiv x(\bar{x}, \bar{p})p(\bar{x}, \bar{p}) = \dfrac{(1 + 2\lambda\bar{x})\cos(\lambda\bar{p}) - \sqrt{1 + 2\lambda\bar{x}}}{\lambda^2}\sin(\lambda\bar{p})\,, \end{cases} \tag{3.180}$$

such as:

$$\{\bar{\xi}_-(\bar{x}, \bar{p}), \bar{\xi}_+(\bar{x}, \bar{p})\}_{(\bar{x},\bar{p})} = 4\bar{\xi}_3(\bar{x}, \bar{p})\,, \; \{\bar{\xi}_3(\bar{x}, \bar{p}), \bar{\xi}_\pm(\bar{x}, \bar{p})\}_{(\bar{x},\bar{p})} = \pm 2\bar{\xi}_\pm(\bar{x}, \bar{p})\,. \tag{3.181}$$

Clearly, the Hamiltonian (3.176) will be (abstractly) expressed in terms of these classical generators in the usual form (where we use the bar in order to keep in mind that we are using a different symplectic representation):

$$H(\bar{\xi}_-, \bar{\xi}_+) = \frac{\bar{\xi}_+ + \bar{\xi}_-}{2}\,, \tag{3.182}$$

which shows that we just changed representation preserving the coalgebraic structure inaltered. In what follows, our goal is to construct the quantum counterpart of (3.176) in the basis of these new coordinates. For this purpose, we will perform the usual quantisation procedure with the goal of obtaining a discrete representation of the Heisenberg algebra.
Let us replace the classical coordinates pair $(\bar{x},\, \bar{p})$ with their quantum analog:

$$\bar{p} \to \hat{\bar{p}} = -i\hbar\partial_{\bar{x}}\,, \;\; \bar{x} \to \hat{\bar{x}} = \bar{x}\,. \tag{3.183}$$

By requiring a (formal) *Hermiticity* condition in order to construct the quantum analog of (3.175), we obtain:

$$\begin{cases} \hat{x}(\bar{x}, -i\hbar\partial_{\bar{x}}) = \dfrac{\sqrt{1 + 2\lambda(\bar{x} + \lambda\hbar)}\, e^{\lambda\hbar\partial_{\bar{x}}} + \sqrt{1 + 2\lambda\bar{x}}\, e^{-\lambda\hbar\partial_{\bar{x}}} - 2}{2\lambda} \\ \hat{p}(\bar{x}, -i\hbar\partial_{\bar{x}}) = \dfrac{\sqrt{1 + 2\lambda(\bar{x} + \lambda\hbar)}\, e^{\lambda\hbar\partial_{\bar{x}}} - \sqrt{1 + 2\lambda\bar{x}}\, e^{-\lambda\hbar\partial_{\bar{x}}}}{2i\lambda}\,. \end{cases} \tag{3.184}$$

Moreover, it is not difficult to show by direct computation that the following commutation relation, the quantum counterpart of (3.179), holds:

$$[\hat{x}(\bar{x}, -i\hbar\partial_{\bar{x}}), \hat{p}(\bar{x}, -i\hbar\partial_{\bar{x}})] = i\hbar\,\hat{\mathbb{1}}\,, \tag{3.185}$$

which means that we have constructed a discrete quantum representation of the algebra $\mathfrak{h}(1)$. Here is the fundamental point: the transformation that we have performed maps the old oscillator Hamiltonian to a new classical Hamiltonian whose quantum counterpart is defined on the lattice, where the operators $\hat{\mathfrak{T}}^{\pm} \doteq e^{\pm\lambda\hbar\partial_{\bar{x}}}$ act as follows:

$$\hat{\mathfrak{T}}^{\pm} f(\bar{x}) = f(\bar{x} \pm \hbar\lambda). \tag{3.186}$$

The functions $f(\bar{x})$ have now to be thought as functions defined on the set of points characterizing this one-dimensional lattice. This suggests us to interpret the new coordinate $\bar{x}$ as an effective discrete coordinate. As we will immediately show, this construction converges to the framework described in [19]. In particular, our example belongs to the rdQM case, the shift operators being translations on a real lattice, whose origin is fixed at $\bar{x}_0 \doteq -\frac{1}{2\lambda}$ and where the points are equally spaced of the quantity $\hbar\lambda > 0$.

Once we found the new representation of $\hat{x}$ and $\hat{p}$ it is immediate to construct a (difference) realization of the $\mathfrak{sl}(2,\mathbb{R})$ algebra generators, the quantum analog of (3.180), which reads:

$$\begin{cases} \bar{D}(\hat{\bar{\xi}}_+) = \frac{2\left(\lambda^2\hbar+2\lambda\bar{x}+1\right)-\sqrt{2\lambda(\lambda\hbar+\bar{x})+1}\sqrt{2\lambda(2\lambda\hbar+\bar{x})+1}\hat{\mathfrak{T}}^{++}-\sqrt{2\lambda\bar{x}+1}\sqrt{2\lambda(\bar{x}-\lambda\hbar)+1}\hat{\mathfrak{T}}^{--}}{4\lambda^2} \\ \bar{D}(\hat{\bar{\xi}}_-) = \frac{2\left(\lambda^2\hbar+2\lambda\bar{x}+3\right)+\sqrt{2\lambda\bar{x}+1}\left(\sqrt{2\lambda(\bar{x}-\lambda\hbar)+1}\hat{\mathfrak{T}}^{--}-4\hat{\mathfrak{T}}^{-}\right)+\sqrt{2\lambda(\lambda\hbar+\bar{x})+1}\left(\sqrt{2\lambda(2\lambda\hbar+\bar{x})+1}\hat{\mathfrak{T}}^{++}-4\hat{\mathfrak{T}}^{+}\right)}{4\lambda^2} \\ \bar{D}(\hat{\bar{\xi}}_3) = i\frac{\sqrt{2\lambda\bar{x}+1}\left(\sqrt{2\lambda(\bar{x}-\lambda\hbar)+1}\hat{\mathfrak{T}}^{--}-2\hat{\mathfrak{T}}^{-}\right)+\sqrt{2\lambda(\lambda\hbar+\bar{x})+1}\left(2\hat{\mathfrak{T}}^{+}-\sqrt{2\lambda(2\lambda\hbar+\bar{x})+1}\hat{\mathfrak{T}}^{++}\right)}{4\lambda^2}\,, \end{cases} \tag{3.187}$$

where we defined $\hat{\mathfrak{T}}^{++} \doteq e^{2\lambda\hbar\partial_{\bar{x}}}$, $\hat{\mathfrak{T}}^{--} \doteq e^{-2\lambda\hbar\partial_{\bar{x}}}$. These finite difference operators realize a discrete version of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, since the following commutation relations hold:

$$[\hat{\bar{\xi}}_-(\bar{x}, -i\hbar\partial_{\bar{x}}), \hat{\bar{\xi}}_+(\bar{x}, -i\hbar\partial_{\bar{x}})] = 4i\hbar\hat{\bar{\xi}}_3(\bar{x}, -i\hbar\partial_{\bar{x}})\ ,\ [\hat{\bar{\xi}}_3(\bar{x}, -i\hbar\partial_{\bar{x}}), \hat{\bar{\xi}}_\pm(\bar{x}, -i\hbar\partial_{\bar{x}})] = \pm 2i\hbar\hat{\bar{\xi}}_\pm(\bar{x}, -i\hbar\partial_{\bar{x}})\,, \tag{3.188}$$

and the Casimir operator reads $\hat{C}(\hat{\bar{\xi}}_\pm, \hat{\bar{\xi}}_3) \doteq \frac{1}{2}[\hat{\bar{\xi}}_+, \hat{\bar{\xi}}_-]_+ - \hat{\bar{\xi}}_3^2 = (i\hbar)^2\frac{1}{2}(\frac{1}{2}+1)\hat{\mathbb{1}}$. The algebra (3.188) represents the quantum analog of the Lie-Poisson algebra (3.181). This means that, by replacing the classical $\bar{x}$ and $\bar{p}$ coordinates with their quantum analog, and by requiring the Hermiticity of the operators, we have succesfully provided the quantization of the system described by the classical Hamiltonian (3.176), which turns out to be:

$$\hat{H} \doteq \hat{H}(\hat{\bar{\xi}}_-, \hat{\bar{\xi}}_+) = \frac{\hat{\bar{\xi}}_+ + \hat{\bar{\xi}}_-}{2} \overset{\bar{D}}{=} \frac{\left(\lambda^2\hbar + 2\lambda\bar{x} + 2\right) - \sqrt{2\lambda(\lambda\hbar+\bar{x})+1}\,\hat{\mathfrak{T}}^{+} - \sqrt{2\lambda\bar{x}+1}\,\hat{\mathfrak{T}}^{-}}{2\lambda^2}\,. \tag{3.189}$$

The interesting feature to point out is that, since the transformation (3.175) is connected to the identity, the parameter λ acquires a rich physical meaning: *it can be interpreted as a deformation parameter allowing us to recover the original continuous system in the limit* $\lambda \to 0$:

$$\begin{cases} \lim_{\lambda\to 0}\hat{x}(\bar{x}, -i\hbar\partial_{\bar{x}}) = \bar{x} \equiv x \\ \lim_{\lambda\to 0}\hat{p}(\bar{x}, -i\hbar\partial_{\bar{x}}) = -i\hbar\partial_{\bar{x}} \equiv -i\hbar\partial_x \\ \lim_{\lambda\to 0}\hat{H}(\hat{\bar{\xi}}_-, \hat{\bar{\xi}}_+) = -\frac{\hbar^2}{2}\partial_{xx} + \frac{x^2}{2}\,. \end{cases} \tag{3.190}$$

Moreover, the finite difference representation of $\mathfrak{sl}(2,\mathbb{R})$ collapses into the standard differential realization (3.146). Therefore, this approach based on the introduction of a classical canonical tranformation connected to the identity function, allowed us to introduce in a quite natural way a lattice parameter governing the continuous limit. Moreover, the other parameter appearing in the construction, i.e. the reduced Planck's constant $\hbar$, will not be fixed to assume value $\hbar = 1$, since we prefer to mantain the discussion as general as possible. As a matter of fact, it governs the classical limit.

3.3.2 Spectral problem, higher dimensional extension and discrete quantum integrals of motion

So far we have presented a one-dimensional discrete version of the Harmonic oscillator, in the sense that we constructed a finite difference Hamiltonian whose continuous limit collapses to the quantum harmonic oscillator. Thus, it is expected that the solution of the (discrete) spectral problem should converge to the one of the harmonic oscillator in such a limit. This is exactly the case. In particular, since we know how the factorization is expressed in terms of the coalgebra generators, we automatically possess also the factorization of the new (discrete) Hamiltonian. It is sufficient to replace the difference realization with the differential one, in order to introduce the analog of the ladder operators (3.150) on the lattice. Thus, we have all the ingredients to solve the following spectral problem:

$$\hat{\hat{H}}\,\Phi(\bar{x}) = E\,\Phi(\bar{x})\,, \tag{3.191}$$

which is a genuine discrete analog of (3.157). To this aim, let us firstly connect our notations with the ones usually used in the framework of *rdQM*. To this purpose, we need to define the following quantities:

$$\mathcal{B}(\bar{x},\lambda) \doteq \frac{1}{2\lambda^2} > 0\,, \quad \mathcal{D}(\bar{x},\lambda) \doteq \frac{1+2\lambda\bar{x}}{2\lambda^2} \geq 0\,, \tag{3.192}$$

where we immediately see that $\mathcal{D}(\bar{x}_0,\lambda) = 0$. Let us notice that this condition is the same as the one given in [19] that we have previously discussed. In our construction the origin of the lattice is not rescaled to assume the value $\bar{x}_0 = 0$. Clearly, this does not affect the results. By means of these two definitions, it is not difficult to show that the Hamiltonian (3.189) can be cast in the following familiar form:

$$\hat{H}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_+) = -\sqrt{\mathcal{B}(\bar{x},\lambda)}\,\hat{\mathcal{T}}^+\sqrt{\mathcal{D}(\bar{x},\lambda)} - \sqrt{\mathcal{D}(\bar{x},\lambda)}\,\hat{\mathcal{T}}^-\sqrt{\mathcal{B}(\bar{x},\lambda)} + (\mathcal{B}(\bar{x},\lambda) + \mathcal{D}(\bar{x},\lambda) + \hbar/2)\,, \tag{3.193}$$

that is, apart from the constant $\frac{\hbar}{2}$, characteristic of the harmonic oscillator (is the eigenvalue associated to the groundstate), formula (3.193) yields the typical finite difference Hamiltonian in rdQM. In particular, this Hamiltonian operator can be factorized in terms of the following discrete ladder operators:

$$\begin{cases} \hat{a}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3) \doteq \frac{i}{\sqrt{2}}\hat{\bar{\xi}}_-^{-\frac{1}{2}}(\hat{\bar{\xi}}_3 + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\bar{\xi}}_-^{\frac{1}{2}} = \hat{\mathcal{T}}^+\sqrt{\mathcal{D}(\bar{x},\lambda)} - \sqrt{\mathcal{B}(\bar{x},\lambda)}\,, \\ \hat{a}^\dagger(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3) \doteq -\frac{i}{\sqrt{2}}\hat{\bar{\xi}}_-^{-\frac{1}{2}}(\hat{\bar{\xi}}_3 + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\bar{\xi}}_-^{\frac{1}{2}} = \sqrt{\mathcal{D}(\bar{x},\lambda)}\,\hat{\mathcal{T}}^- - \sqrt{\mathcal{B}(\bar{x},\lambda)}\,, \end{cases} \tag{3.194}$$

where we used the $\bar{D}$ representation, as:

$$\hat{H}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_+) = \hat{a}^\dagger(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3)\hat{a}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3) + \frac{\hbar}{2}\,. \tag{3.195}$$

As expected, in the continuous limit, the two discrete ladder operators collapse to the standard first order differential operators (3.150). At this point, to compute the spectral problem, as in the continuous case we need to find the kernel of the lowering operator, now solving the following difference equation:

$$\hat{a}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3)\Phi_0(\bar{x}) = \sqrt{\mathcal{D}(\bar{x}+\lambda\hbar,\lambda)}\Phi_0(\bar{x}+\lambda\hbar) - \sqrt{\mathcal{B}(\bar{x},\lambda)}\Phi_0(\bar{x}) = 0\,. \tag{3.196}$$

The solution of (3.196) is $\Phi_0(\bar{x}) \propto \sqrt{\frac{\left(\frac{1}{2\hbar\lambda^2}\right)^{\left(\frac{\bar{x}}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)}}{\left(\frac{\bar{x}}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)!}}$ and, in turn, it allows us to find the (unnormalized) eigenfunctions of the spectral problem:

$$\Phi_n(\bar{x}) \propto [\hat{a}^\dagger(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3)]^n\Phi_0(\bar{x}) = \left(\frac{-1}{\sqrt{2}\lambda}\right)^n C_n\left(\frac{\bar{x}}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2};\frac{1}{2\hbar\lambda^2}\right)\Phi_0(\bar{x})\,, \tag{3.197}$$

which are given in terms of the *Charlier* orthogonal polynomials $C_n(x;a)$, with $a>0$. As we said, these polynomials belong to the Askey scheme and are orthogonal on the real lattice $x\in[0,1,\dots)$ with respect to the *weight function* $w(x,a) \doteq \frac{a^x}{x!}$ [115].
In our case, the physical coordinate is $\bar{x}$, whose domain of definition is $-\frac{1}{2\lambda} \leq \bar{x} < \infty$ and the parameter a, defined in terms of λ, reads $a \doteq \frac{1}{2\hbar\lambda^2} > 0$. Moreover, for the orthonormality condition it holds:

$$\begin{aligned} O^{(Charlier)}_{nm} &\doteq \left(\frac{-1}{\sqrt{2}\lambda}\right)^{-2n}\frac{\exp\left(-\frac{1}{2\hbar\lambda^2}\right)}{n!\left(\frac{1}{2\hbar\lambda^2}\right)^{-n}}\sum_{j=0}^{\infty}\Phi_n^*\left(-\frac{1}{2\lambda}+j\lambda\hbar\right)\Phi_m\left(-\frac{1}{2\lambda}+j\lambda\hbar\right) \\ &= \left(\frac{-1}{\sqrt{2}\lambda}\right)^{-2n}\frac{\exp\left(-\frac{1}{2\hbar\lambda^2}\right)}{n!\left(\frac{1}{2\hbar\lambda^2}\right)^{-n}}\langle\Phi_n,\Phi_m\rangle \\ &= \delta_{nm}\,, \end{aligned} \tag{3.198}$$

the symbol $\langle,\rangle$ denoting the scalar product on the new Hilbert space $\mathcal{H}$, where the domain $\mathcal{D}_\lambda \doteq [-\frac{1}{2\lambda},\infty)$, and the sum is evaluated on the points of the lattice. They are equally spaced of $\hbar\lambda$. So, the normalized eigenfunctions of the spectral problem (3.191) result:

$$\Phi_n(\bar{x}) = \frac{\exp\left(-\frac{1}{4\hbar\lambda^2}\right)}{\sqrt{n!}\left(\frac{1}{2\hbar\lambda^2}\right)^{-\frac{n}{2}}}C_n\left(\frac{\bar{x}}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2};\frac{1}{2\hbar\lambda^2}\right)\Phi_0(\bar{x})\,. \tag{3.199}$$

Finally, the corresponding spectrum for the stationary states reads:

$$\hat{H}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_+)\Phi_n(\bar{x}) = (\hat{a}^\dagger(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3)\hat{a}(\hat{\bar{\xi}}_-,\hat{\bar{\xi}}_3)+\hbar/2\,)\Phi_n(\bar{x}) = \hbar(n+1/2)\Phi_n(\bar{x})\,. \tag{3.200}$$

In the continuous limit, i.e. for $\lambda\to 0$ at fixed $\hbar$, the discrete spectral problem (3.200) transforms to the standard quantum harmonic oscillator problem (3.157).

At this point, as shown for the continuous case, we can use the coproduct map to perform a multidimensional extension of the system, which is the crucial advantage of having a coalgebraic structure also in the discrete case. Thus, we provide in the following the N-dimensional maximally superintegrable generalization of the discrete Hamiltonian (3.193). For this purpose, let us apply the coproduct map to the Hamiltonian $\hat{H}$:

$$\hat{H}^{[2]} \doteq \Delta(\hat{H}) = \Delta\left(\frac{\hat{\bar{\xi}}_+ + \hat{\bar{\xi}}_-}{2}\right) = \frac{\Delta(\hat{\bar{\xi}}_+)+\Delta(\hat{\bar{\xi}}_-)}{2} \overset{D}{=} \sum_{j=1}^{2}\hat{H}_j(\hat{x}_j,\hat{p}_j)\,, \tag{3.201}$$

where:

$$\hat{H}_j(\hat{\bar{x}}_j, \hat{\bar{p}}_j) \doteq \frac{\left(\lambda^2\hbar + 2\lambda\bar{x}_j + 2\right) - \sqrt{2\lambda(\lambda\hbar + \bar{x}_j) + 1}\,\hat{\mathfrak{T}}_j^+ - \sqrt{2\lambda\bar{x}_j + 1}\,\hat{\mathfrak{T}}_j^-}{2\lambda^2} \quad (j = 1, 2)\,, \tag{3.202}$$

with $\mathfrak{T}_j^{\pm} \doteq e^{\pm\lambda\hbar\partial_{\bar{x}_j}}$. Let us point out that, to keep just a one-parameter limit, we took the same λ in both copies of the tensor product. Anyway, we stress that the construction works also if we take different λ's.

Once again, the new Hamiltonian lives in the tensor product of two copies of the $\mathfrak{sl}(2, \mathbb{R})$ algebra, and represents the two dimensional generalization of the discrete harmonic oscillator expressed in cartesian coordinates. As usual, this system can be rewritten in terms of two copies of the discrete ladder operators:

$$\hat{H}^{[2]} = \sum_{k=1}^{2} \hat{a}_k^\dagger(\hat{\bar{\xi}}_{(-,k)}, \hat{\bar{\xi}}_{(3,k)})\hat{a}_k(\hat{\bar{\xi}}_{(-,k)}, \hat{\bar{\xi}}_{(3,k)}) + \hbar\,, \tag{3.203}$$

which are defined as:

$$\begin{cases} \hat{a}_k(\hat{\bar{\xi}}_{(-,k)}, \hat{\bar{\xi}}_{(3,k)}) \doteq \frac{i}{\sqrt{2}}\hat{\bar{\xi}}_{(-,k)}^{-\frac{1}{2}}(\hat{\bar{\xi}}_{(3,k)} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\bar{\xi}}_{(-,k)}^{\frac{1}{2}} = \hat{\mathfrak{T}}_k^+\sqrt{\mathcal{D}(\bar{x}_k, \lambda)} - \sqrt{\mathcal{B}(\bar{x}_k, \lambda)} \\ \hat{a}_k^\dagger(\hat{\bar{\xi}}_{(-,k)}, \hat{\bar{\xi}}_{(3,k)}) \doteq -\frac{i}{\sqrt{2}}\hat{\bar{\xi}}_{(-,k)}^{-\frac{1}{2}}(\hat{\bar{J}}_{(3,k)} + \frac{i\hbar}{2}) + \frac{1}{\sqrt{2}}\hat{\bar{\xi}}_{(-,k)}^{\frac{1}{2}} = \sqrt{\mathcal{D}(\bar{x}_k, \lambda)}\,\hat{\mathfrak{T}}_k^- - \sqrt{\mathcal{B}(\bar{x}_k, \lambda)}\,, \end{cases} \tag{3.204}$$

where:

$$\begin{cases} \hat{\bar{\xi}}_{(+,k)} \doteq \frac{2\left(\lambda^2\hbar + 2\lambda\bar{x}_k + 1\right) - \sqrt{2\lambda(\lambda\hbar + \bar{x}_k) + 1}\sqrt{2\lambda(2\lambda\hbar + \bar{x}_k) + 1}\hat{\mathfrak{T}}_k^{++} - \sqrt{2\lambda\bar{x}_k + 1}\sqrt{2\lambda(\bar{x}_k - \lambda\hbar) + 1}\hat{\mathfrak{T}}_k^{--}}{4\lambda^2} \\ \hat{\bar{\xi}}_{(-,k)} \doteq \frac{2\left(\lambda^2\hbar + 2\lambda\bar{x}_k + 3\right) + \sqrt{2\lambda\bar{x}_k + 1}\left(\sqrt{2\lambda(\bar{x}_k - \lambda\hbar) + 1}\hat{\mathfrak{T}}_k^{--} - 4\hat{\mathfrak{T}}_k^-\right) + \sqrt{2\lambda(\lambda\hbar + \bar{x}_k) + 1}\left(\sqrt{2\lambda(2\lambda\hbar + \bar{x}_k) + 1}\hat{\mathfrak{T}}_k^{++} - 4\hat{\mathfrak{T}}_k^+\right)}{4\lambda^2} \\ \hat{\bar{\xi}}_{(3,k)} \doteq i\frac{\sqrt{2\lambda\bar{x}_k + 1}\left(\sqrt{2\lambda(\bar{x}_k - \lambda\hbar) + 1}\hat{\mathfrak{T}}_k^{--} - 2\hat{\mathfrak{T}}_k^-\right) + \sqrt{2\lambda(\lambda\hbar + \bar{x}_k) + 1}\left(2\hat{\mathfrak{T}}_k^+ - \sqrt{2\lambda(2\lambda\hbar + \bar{x}_k) + 1}\hat{\mathfrak{T}}_k^{++}\right)}{4\lambda^2}\,, \end{cases} \tag{3.205}$$

being $\hat{\mathfrak{T}}_k^{++} \doteq e^{2\lambda\hbar\partial_{\bar{x}_k}}$, $\hat{\mathfrak{T}}_k^{--} \doteq e^{-2\lambda\hbar\partial_{\bar{x}_k}}$ $(k = 1, 2)$. Also in this discrete case the coproduct of the generators:

$$\Delta(\hat{\bar{\xi}}_+) \overset{\bar{D}\otimes\bar{D}}{=} \sum_{k=1}^{2} \hat{\bar{\xi}}_{(+,k)}\,, \quad \Delta(\hat{\bar{\xi}}_-) \overset{\bar{D}\otimes\bar{D}}{=} \sum_{k=1}^{2} \hat{\bar{\xi}}_{(-,k)}\,, \quad \Delta(\hat{\bar{\xi}}_3) \overset{\bar{D}\otimes\bar{D}}{=} \sum_{k=1}^{2} \hat{\bar{\xi}}_{(3,k)}\,, \tag{3.206}$$

is an homomorphism for the $\mathfrak{sl}(2, \mathbb{R})$ algebra, the commutation relations being:

$$[\Delta(\hat{\bar{\xi}}_-), \Delta(\hat{\bar{\xi}}_+)] = 4i\hbar\Delta(\hat{\bar{\xi}}_3)\,, \quad [\Delta(\hat{\bar{\xi}}_3), \Delta(\hat{\bar{\xi}}_\pm)] = \pm 2i\hbar\Delta(\hat{\bar{\xi}}_\pm)\,. \tag{3.207}$$

Moreover, the Casimir obtained by applying the coproduct results:

$$\hat{\bar{C}}^{[2]} \doteq \Delta(\hat{\bar{C}}) = \Delta\big(\frac{1}{2}(\hat{\bar{\xi}}_+\hat{\bar{\xi}}_- + \hat{\bar{\xi}}_-\hat{\bar{\xi}}_+) - \hat{\bar{\xi}}_3^2\big) = \left(\hat{\bar{L}}_z^2 - \hbar^2\right), \tag{3.208}$$

where, under this difference realization, the angular momentum operator reads:

$$\begin{aligned} \hat{\bar{L}}_z \doteq \frac{i}{2\lambda^2}\Bigg(&\sqrt{2\lambda\bar{x}_1 + 1}\,\hat{\mathfrak{T}}_1^- - \sqrt{2\lambda\bar{x}_2 + 1}\,\hat{\mathfrak{T}}_2^- - \sqrt{2\lambda(\lambda\hbar + \bar{x}_2) + 1}\left(\sqrt{2\lambda\bar{x}_1 + 1}\,\hat{\mathfrak{T}}_{12}^{-+} - \hat{\mathfrak{T}}_2^+\right) \\ &- \sqrt{2\lambda(\bar{x}_1 + \lambda\hbar) + 1}\left(\hat{\mathfrak{T}}_1^+ - \sqrt{1 + 2\bar{x}_2\lambda}\hat{\mathfrak{T}}_{12}^{+-}\right)\Bigg)\,, \end{aligned} \tag{3.209}$$

with $\hat{\mathfrak{J}}_{ij}^{\pm\mp} \doteq \hat{\mathfrak{J}}_i^{\pm}\hat{\mathfrak{J}}_j^{\mp} = e^{\pm\lambda\hbar\partial_{\bar{x}_i}\mp\lambda\hbar\partial_{\bar{x}_j}}$. Thus, we have obtained a discrete representation of the angular momentum operator. Clearly, also in this discrete case, by construction the Casimir is an observable commuting with the Hamiltonian $\hat{H}^{[2]}$, i.e. $[\hat{\hat{C}}^{[2]}, \hat{H}^{[2]}] = 0$. This leads to the conservation of the (discrete) angular momentum, which can be also expressed as:

$$\hat{\hat{L}}_z \doteq \left(\hat{\hat{C}}^{[2]} + \hbar^2\right)^{\frac{1}{2}} = \hat{x}_1(\hat{\bar{x}}_1, \hat{\bar{p}}_1)\hat{p}_2(\hat{\bar{x}}_2, \hat{\bar{p}}_2) - \hat{x}_2(\hat{\bar{x}}_2, \hat{\bar{p}}_2)\hat{p}_1(\hat{\bar{x}}_1, \hat{\bar{p}}_1)\,. \tag{3.210}$$

As expected, in the continuous limit, the discrete angular momentum collapses into the standard differential operator (3.165). So, at this level, we have the *quasi-maximal superintegrability*. In order to ensure the maximal superintegrability, as shown for the standard case, we need to introduce the Demkov-Fradkin tensor, which reads:

$$\hat{\hat{\mathrm{I}}}_{ij} \doteq \hat{p}_i(\hat{\bar{x}}_i, \hat{\bar{p}}_i)\hat{p}_j(\hat{\bar{x}}_j, \hat{\bar{p}}_j) + \hat{x}_i(\hat{\bar{x}}_i, \hat{\bar{p}}_i)\hat{x}_j(\hat{\bar{x}}_j, \hat{\bar{p}}_j)\,, \qquad (i,j=1,2)\,, \tag{3.211}$$

and its discrete representation can be immediately constructed by means of (3.184). In particular, the diagonal elements are given by the one-particle realization of the discrete Hamiltonian (3.202), whereas the off-diagonal elements are given by the following expressions:

$$\begin{aligned}\hat{\hat{\mathrm{I}}}_{12} = \hat{\hat{\mathrm{I}}}_{21} = \frac{1}{2\lambda^2}\Bigg(&2 - \sqrt{2\lambda\bar{x}_2+1}\,\hat{\mathfrak{J}}_2^{-} - \sqrt{2\lambda\bar{x}_1+1}\,\hat{\mathfrak{J}}_1^{-} + \sqrt{2\lambda(\lambda\hbar+\bar{x}_1)+1}\left(\sqrt{2\lambda\bar{x}_2+1}\,\hat{\mathfrak{J}}_{12}^{+-} - \hat{\mathfrak{J}}_1^{+}\right)\\ &+ \sqrt{2\lambda(\lambda\hbar+\bar{x}_2)+1}\left(\sqrt{2\lambda\bar{x}_1+1}\,\hat{\mathfrak{J}}_{12}^{-+} - \hat{\mathfrak{J}}_2^{+}\right)\Bigg)\,.\end{aligned} \tag{3.212}$$

Once again, in the $\lambda \to 0$ limit, the components of the continuous Demkov-Fradkin tensor (3.166) are recovered. To sum up, the set $\{\hat{\hat{H}}^{[2]}, \hat{\hat{C}}^{[2]}, \hat{\hat{\mathrm{I}}}_{ii}\}$ at fixed index i provides 3 observables commuting with the Hamiltonian $\hat{\hat{H}}^{[2]}$, being $\hat{\hat{H}}^{[2]} = \frac{1}{2}\sum_{i=1}^{2}\hat{\hat{\mathrm{I}}}_{ii}$.
In analogy with the continuous case, we can assert that the two-dimensional discrete system we have obtained is *maximally superintegrable*. Also in this case, the eigenfunctions will be constructed by applying the raising operators $\hat{a}_k^{\dagger}(\hat{\bar{\xi}}_-, \hat{\bar{\xi}}_3)$ to the kernel of the operators $\hat{a}_k(\hat{\bar{\xi}}_-, \hat{\bar{\xi}}_3)$ $(k=1,2)$, obtained by solving the difference equations $\hat{a}_k(\hat{\bar{\xi}}_-, \hat{\bar{\xi}}_3)\Phi_0(\bar{x}_k) = 0$, i.e.:

$$\Phi_0(\bar{x}_k) \propto \sqrt{\frac{\left(\frac{1}{2\hbar\lambda^2}\right)^{\left(\frac{\bar{x}_k}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)}}{\left(\frac{\bar{x}_k}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)!}} \quad (k=1,2). \tag{3.213}$$

So, the vacuum state of the discrete system (3.201) results:

$$\Phi_{(0,0)}(\bar{x}_1,\bar{x}_2) \doteq \Phi_0(\bar{x}_1)\Phi_0(\bar{x}_2) \propto \sqrt{\frac{\left(\frac{1}{2\hbar\lambda^2}\right)^{\left(\frac{\bar{x}_1}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)}\left(\frac{1}{2\hbar\lambda^2}\right)^{\left(\frac{\bar{x}_2}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)}}{\left(\frac{\bar{x}_1}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)!\left(\frac{\bar{x}_2}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}\right)!}}\,, \tag{3.214}$$

and the (unnormalized) eigenfunctions of the spectral problem are:

$$\begin{aligned}\Phi_{(n,m)}(\bar{x}_1,\bar{x}_2) &\propto [\hat{a}_1^{\dagger}(\hat{\bar{\xi}}_{(-,1)}, \hat{\bar{\xi}}_{(3,1)})]^n [\hat{a}_2^{\dagger}(\hat{\bar{\xi}}_{(-,2)}, \hat{\bar{\xi}}_{(3,2)})]^m \Phi_{(0,0)}(\bar{x}_1,\bar{x}_2)\\ &= \left(\frac{-1}{\sqrt{2}\lambda}\right)^{n+m} C_n\left(\frac{\bar{x}_1}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}; \frac{1}{2\hbar\lambda^2}\right) C_m\left(\frac{\bar{x}_2}{\lambda\hbar}+\frac{1}{2\hbar\lambda^2}; \frac{1}{2\hbar\lambda^2}\right)\Phi_{(0,0)}(\bar{x}_1,\bar{x}_2)\,.\end{aligned} \tag{3.215}$$

Finally, the corresponding spectrum for the stationary states reads:

$$\hat{\bar{H}}^{[2]}\Phi_{(n,m)}(\bar{x}_1,\bar{x}_2) = \hbar(n+m+1)\Phi_{(n,m)}(\bar{x}_1,\bar{x}_2)\,, \tag{3.216}$$

being $n,m = 0,1,\dots\infty$ the principal quantum numbers of the single one-dimensional realization of the discrete oscillators. At this point of the work it is clear that the superintegrable N-dimensional extension of the system can be immediately obtained. In this discrete case nothing changes conceptually, the N-dimensional Hamiltonian results:

$$\hat{\bar{H}}^{[N]} \doteq \Delta^{[N]}(\hat{\bar{H}}) = \Delta\left(\frac{\hat{\bar{\xi}}_+ + \hat{\bar{\xi}}_-}{2}\right) = \frac{\Delta^{[N]}(\hat{\bar{\xi}}_+) + \Delta^{[N]}(\hat{\bar{\xi}}_-)}{2} \overset{D^{\otimes N}}{=} \sum_{j=1}^{N} \hat{\bar{H}}_j(\hat{\bar{x}}_j,\hat{\bar{p}}_j)\,, \tag{3.217}$$

and the $2N-3$ discrete versions of the conserved quantities (3.173), together with the $\frac{N(N+1)}{2}$ components of the discrete Demkov-Fradkin tensor, obtained by replacing the differential representation with the finite difference one, allow us to obtain a maximally superintegrable N-dimensional extension of the discrete harmonic oscillator. As far as the spectral problem is concerned, in total analogy with the continuous case, it will be factorized in terms of single copies of ladder operators. The eigenfunctions will be expressed in terms of a product of Charlier polynomials (one for each variable) multiplied by the gauge factors arising from the kernels of the lowering operators. Moreover, the spectrum will be expressed as an additive sum of the principal quantum numbers related to the single copy of harmonic oscillators.

Before to conclude this part of the work, it is interesting to point out the connection between the 2-dimensional Hamiltonian (3.201), constructed by means of the coproduct map, and the one discussed in the recent work [22], where the authors also discussed a 2-dimensional superintegrable model based on the Charlier polynomials. To this aim, let us focus on the difference operator:

$$\hat{\mathbb{H}}^{[2]} \doteq \hat{\bar{H}}^{[2]} - \hbar = \sum_{k=1}^{2} \hat{a}_k^{\dagger}(\hat{\bar{\xi}}_{(-,k)},\hat{\bar{\xi}}_{(3,k)})\hat{a}_k(\hat{\bar{\xi}}_{(-,k)},\hat{\bar{\xi}}_{(3,k)})\,, \tag{3.218}$$

and let us perform the following gauge transformation:

$$\hat{\mathbb{H}}_g^{[2]} \doteq \Phi^{-1}_{(0,0)}(\bar{x}_1,\bar{x}_2)\;\hat{\mathbb{H}}^{[2]}\;\Phi_{(0,0)}(\bar{x}_1,\bar{x}_2)\,. \tag{3.219}$$

The resulting discrete Hamiltonian is:

$$\begin{aligned}\hat{\mathbb{H}}_g^{[2]} &= \sum_{k=1}^{2}\left[-\left(\frac{\bar{x}_k}{\lambda}+\frac{1}{2\lambda^2}\right)e^{-\lambda\hbar\partial_{\bar{x}_k}} - \frac{1}{2\lambda^2}e^{\lambda\hbar\partial_{\bar{x}_k}} + \frac{\bar{x}_k}{\lambda}+\frac{1}{\lambda^2}\right] \\ &= \sum_{k=1}^{2}\left[-\left(\frac{\bar{x}_k}{\lambda}+\frac{1}{2\lambda^2}\right)e^{-\lambda\hbar\partial_{\bar{x}_k}} - \frac{1}{2\lambda^2}e^{\lambda\hbar\partial_{\bar{x}_k}} + \left(\frac{\bar{x}_k}{\lambda}+\frac{1}{2\lambda^2}\right)+\frac{1}{2\lambda^2}\right].\end{aligned} \tag{3.220}$$

Now, if we perform the following change of variables:

$$\bar{x}'_k = \frac{\bar{x}_k}{\lambda}+\frac{1}{2\lambda^2}\,,\quad \bar{x}'_k \in [0,\infty)\,, \tag{3.221}$$

then $\partial_{\bar{x}_k} = \dfrac{\partial \bar{x}'_k}{\partial \bar{x}_k}\partial_{\bar{x}'_k} = \lambda^{-1}\partial_{\bar{x}'_k}$ $(k=1,2)$, and the gauged Hamiltonian results:

$$\hat{\mathbb{H}}_g'^{[2]} = \sum_{k=1}^{2}\left[-\bar{x}'_k\, e^{-\hbar\partial_{\bar{x}'_k}} - \frac{1}{2\lambda^2}e^{\hbar\partial_{\bar{x}'_k}} + \bar{x}'_k + \frac{1}{2\lambda^2}\right]. \tag{3.222}$$

At this point, if we set $\hbar = 1$ and define the quantities $\alpha_1^2 = \alpha_2^2 \doteq \frac{1}{2\lambda^2} > 0$, we obtain:

$$\hat{\mathbb{H}}_g'^{[2]} = \sum_{k=1}^{2} \left[-\bar{x}_k' \, e^{-\partial_{\bar{x}_k'}} - \alpha_k^2 e^{\partial_{\bar{x}_k'}} + (\bar{x}_k' + \alpha_k^2) \right], \tag{3.223}$$

where $\bar{x}' = [0, 1, 2, \ldots, \infty)$. This operator is such that:

$$\hat{\mathbb{H}}_g'^{[2]} \, C_n(x_1; \alpha_1^2) C_m(x_2; \alpha_2^2) = (n+m) \, C_n(x_1; \alpha_1^2) C_m(x_2; \alpha_2^2) \quad n, m = 0, 1, 2, \ldots \tag{3.224}$$

where $C_n(x; \alpha_k^2)$ are the Charlier polynomials with parameter $\alpha_k^2 > 0$ for $(k = 1, 2)$. Thus, in this case, we are working in the basis given by the gauged generators:

$$\hat{\bar{\xi}}^{[2]}_{(\sigma,g)} \doteq \Phi^{-1}_{(0,0)} \circ \hat{\bar{\xi}}^{[2]}_{\sigma} \circ \Phi_{(0,0)} \,, \quad (\sigma = \pm, 3). \tag{3.225}$$

Let us notice that the Hamiltonian (3.223) is the one investigated in the paper [22], where the authors have $\alpha_1^2 \neq \alpha_2^2$ (in that paper $\alpha_1^2 = \alpha^2$ and $\alpha_2^2 = \beta^2$). We observe that, since we have chosen the same λ in both copies of the tensor product, we got the following relation: $\alpha_1^2 = \alpha_2^2 = \frac{1}{2\lambda^2}$. This is our particular choice, basically the simplest one, and has been done in order to keep just a one-parameter continuous limit (also in the N-dimensional case). Anyway, we remark once again that the coproduct allows us to introduce different parameters λ, labelled with the number of the site that we are taking in the tensor product. This means that our construction works also if we take $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, i.e. for different values of the Charlier parameters. In this case, the two-dimensional gauged Hamiltonian that we would have obtained is the following (remember $\hbar = 1$):

$$\hat{\mathbb{H}}_g^{[2]} = \sum_{k=1}^{2} \left[-\left(\frac{\bar{x}_k}{\lambda_k} + \frac{1}{2\lambda_k^2} \right) e^{-\lambda_k \partial_{\bar{x}_k}} - \frac{1}{2\lambda_k^2} e^{\lambda_k \partial_{\bar{x}_k}} + \left(\frac{\bar{x}_k}{\lambda_k} + \frac{1}{2\lambda_k^2} \right) + \frac{1}{2\lambda_k^2} \right], \tag{3.226}$$

which, after the following change of variables:

$$\bar{x}_k' = \frac{\bar{x}_k}{\lambda_k} + \frac{1}{2\lambda_k^2} \quad (k = 1, 2) \,, \tag{3.227}$$

would have resulted in:

$$\hat{\mathbb{H}}_g'^{[2]} = \sum_{k=1}^{2} \left[-\bar{x}_k' \, e^{-\partial_{\bar{x}_k'}} - \alpha_k^2 e^{\partial_{\bar{x}_k'}} + (\bar{x}_k' + \alpha_k^2) \right], \tag{3.228}$$

with $\alpha_k^2 = \frac{1}{2\lambda_k^2}$ $(k = 1, 2)$. To conclude, let us mention that in our setting, physically speaking, this choice would have implied a theory defined on a rectangular lattice instead of a squared one (that is the case $\lambda_1 = \lambda_2 = \lambda$). In the N-dimensional case we would have had an N-orthotope instead of an N-cube.

3.4 CONCLUDING REMARKS AND OPEN PERSPECTIVES

Even if this analysis represents just a first step towards a more complete understanding of coalgebraic structures in discrete quantum theories, we hope that the main message of this

research has been delivered: *the coalgebraic symmetry approach can be used also in discrete quantum theories, where finite difference equations replace differential equations: as a main result, in fact, we have succesfully applied the coalgebra symmetry approach in (real) discrete quantum mechanics, providing a new way to investigate discrete quantum models in terms of coalgebraic structures.*

We summarize here the main results of our construction: starting from the classical oscillator Hamiltonian (3.174) and performing the *tranformation* (3.175), we have derived the new classical Hamiltonian (3.176) that, after the application of the canonical quantization procedure (together with the Hermiticity condition), turned out to represent a discrete version of the quantum oscillator system (3.149). We have proved that this Hamiltonian belongs to the realm of real discrete Quantum Mechanics: the Hamiltonian can be cast in form (3.193). The introduction of a new representation for the algebra $\mathfrak{h}(1)$ and, consequently, for the algebra $\mathfrak{sl}(2,\mathbb{R})$ allowed us to keep the formalism of ladder operators in order to compute both the spectrum and the eigenfunctions for the discrete system (3.189). The latters are expressed in the basis of the Charlier orthogonal polynomials. Clearly, the fundamental parameters appearing in our analysis are λ and $\hbar$, governing the continuous and the classical limit respectively, as graphically explained by the following diagram:

$$\begin{array}{ccc} \hat{\bar{H}}(\bar{x}, -i\hbar\partial_{\bar{x}}) & \xrightarrow{\lambda\to 0} & \hat{H}(x, -i\hbar\partial_x) \\ \downarrow{\scriptstyle \hbar\to 0} & & \downarrow{\scriptstyle \hbar\to 0} \\ \bar{H}(\bar{x}, \bar{p}) & \xrightarrow{\lambda\to 0} & H(x,p) \end{array}$$

All our construction has been performed by means of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra, that allowed us to extend the dimension of the system to an arbitrary N, by preserving the superintegrability properties of the Hamiltonian. Let us stress once again that the algebra representation which generates the discrete orthogonal polynomials has been obtained combining two main ingredients: a "classical canonical transformation" together with the standard quantization procedure. This procedure can be therefore regarded as an algorithm that allows to pass from continuous to discrete orthogonal polynomials. In other words, we have discretized the quantum harmonic oscillator by preserving its superintegrability properties. As a byproduct, we proved that *the Charlier orthogonal polynomials share the same coalgebraic structure of the Hermite orthogonal polynomials*. Just to emphasize the usefulness of such construction, as a further result we have immediately obtained a N-dimensional extension of already known results, such as the ones contained in [22]. As a future perspective, we plan to investigate whether these results can be extended also to other families of orthogonal polynomials belonging to the Askey scheme, in order to understand which coalgebraic structures are involved. Moreover, since we showed that the coalgebra symmetry approach can also be used in the realm of discrete quantum mechanics, we will investigate the possibility of constructing (quasi)-maximally and, hopefully, maximally superintegrable generalization of one-dimensional discrete system endowed with an underlying coalgebra symmetry. To be more precise, a further step in that direction could be done, for example, if we would be able to construct a discrete version of the hydrogen atom. A possible strategy could be the following: we know that, at least formally,

the hydrogen atom can be defined in terms of the coalgebraic Hamiltonian (or in terms of coalgebraic related expressions [69]):

$$\hat{H} \doteq \hat{H}(\hat{\xi}_+, \hat{\xi}_-) = \frac{\hat{\xi}_+}{2} - \frac{k}{\sqrt{\hat{\xi}_-}}, \tag{4.229}$$

which, under the quantum (differential) representation (3.146), implies the possibility of extending such a system in N-dimensions preserving its quasi-maximal superitegrability. After that, the components of the quantum version of the Laplace-Runge-Lenz vector provide the way to obtain the maximal superintegrability. In turn, the quantum integrals of motion related to the radial symmetry are the same as the ones of the N-dimensional harmonic oscillator, and are provided by the partial Casimirs of the algebra. So, it is tempting to say that the discrete quantum integrals of motion related to the angular momenta are also commuting observables of a (possible) discrete version of the hydrogen atom, obtainable by replacing the differential representation with the difference one that we have constructed. However, even if this seems to be formally true, there is a technical problem related to the representation of the inverse of the discrete operator $\hat{x}(\bar{x}, -i\hbar\partial_{\bar{x}})$ defined in (3.184), which would be crucial to provide a discrete representation at the hydrogen atom. If we succeed in finding such a representation, the next step would be use it in order to construct a discrete version of the LRL vector. After that, the spectrum and the eigenfunctions of the system could be performed by means of a discrete version of the shape invariance [19]. Along this direction, another interesting point is related to the fact that our discrete representation could also be used to introduce superintegrable discrete versions of coalgebraic oscillators that are defined on curved spaces (see for example [116]). Clearly, all these open problems have to be better understood, and further studies are needed. Work on all these lines is still in progress.

4 FACTORIZATION METHODS, SUPERINTEGRABILITY AND EXACT SOLVABILITY

So far we have discussed a method for dealing with classical and quantum superintegrable systems by using coalgebraic structures. In the general theoretical section we showed how this approach allows us to obtain at most $2N-3$ conserved quantities given by the (left and right) partial Casimirs that, together with the Hamiltonian, form a set of $2N-2$ functionally independent first integrals (the quasi-maximal superintegrability is ensured algebraically). As we said, the other quantity required for the maximal superintegrability has to be found along other routes. Another algebraic method that has been recently introduced to deal with symmetries of superintegrable systems, is the so-called *factorization method*, or *extended factorization method* [33–35]. This approach consists in a revisitation the factorization method in quantum mechanics [13–15]. In particular, it represents an extension of such method to separable systems which depends on several variables. The idea is that, if an Hamiltonian function (resp. operator in the quantum case) admits separation of variables in a certain coordinate system, then for each coordinate it is possible to define two sets of ladder $B_\pm$ and "shift" $A_\pm$ functions (operators) and, if certain further conditions are satisfied, the additional constants of motion can be explicitly constructed by using a combination of them (see [35] for an introduction of such algebraic approach). This method has been shown to be very powerful and, recently, it provided the way to introduce for the first time an exactly solvable anisotropic oscillator defined on curved spaces (in particular on the sphere $\mathbb{S}^2$ and on the hyperbolic plane $\mathbb{H}^2$) [117].

This approach has deep roots in the paper [32], where the subclass of "trivially" maximally (super)integrable classical systems, composed by the one-dimensional ones, has been investigated in terms of a classical version of the factorization method there introduced. As we know, for these conservative systems the Hamiltonians are integrals of motion and the integrability is fulfilled in a trivial way: in dimension $N=1$, the conservation of the energy is sufficient to determine the maximal superintegrability (or, equivalently, integrability) being $2N-1=1$. In quantum mechanics, the solvability of such systems is often related to the existence of a pair of ladder operators, that together with the Hamiltonian, close a spectrum generating algebra [17, 118, 119]. The simplest example is the harmonic oscillator for which, as we discussed explicitly, the ladder operators $a, a^\dagger$ close the Heisenberg algebra, which allows to construct the spectrum and the eigenfunctions for the stationary states algebraically. Let us remark once again that this represents the simplest case since no *shape invariance* is required. More complicated systems, such as for example the hydrogen atom, the Poschl-Teller or the Morse systems just to cite a few, involve more complicated underlying algebraic structures [16]. It is know that by using the theory of SUSYQM, we can solve exactly the quantum problem associated to a physical system if and only if the *superpartner potentials*, related each other by a Riccati-type equation, are not "too different", e.g. if there exists a *shape invariance condition* (SIC) between them.

In [32], with the aim of establishing a bridge to fill the gap between the algebraic structures arising in classical and quantum mechanics, the authors developed a method for solving in

an algebraic way the motion of one-dimensional Hamiltonian systems. The key idea was the introduction of deformed Poisson structures in classical mechanics closed in terms of "ladder operators" (now functions on the phase space) which factorize the Hamiltonian in a suitable way. This structures, which can be viewed as the classical version of the quantum spectrum generating algebras (for this reason the name *classical spectrum generating algebras* (*cSGA*)), can be used to define two time-dependent constants of motion which allow to construct the trajectories on the phase space algebraically. The systems obtained so obtained represent the classical counterpart of the factorizable one-dimensional quantum mechanical systems.

This second part of the Thesis is devoted to a series of works related to the application of factorization methods in classical mechanics and quantum mechanics. The common thread with the first part of the work is the superintegrability, and in particular the fact that the systems that we are going to discuss share the same $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry, being a deformation (in an appropriate sense) of the harmonic oscillator and the Kepler-Coulomb system. Precisely, the aim of the chapter is the following:

- in order to become familiar with such a method, we will start our discussion presenting the results that we have obtained in relation with a family of deformed oscillators. Firstly, we will show how the method works taking as a prototype example the harmonic oscillator. Then, after introducing the motivations and definitions behind our work, we will extend the construction to the classical analog of f-oscillators, a generalization of q-oscillators given in [42]. We will show how this method allows us to write down the deformed Poisson algebra characterizing the entire family of non-linear oscillators and to contruct its general solution algebraically. The original results presented here are based on a joint work with R. Kullock, and are based on the paper [39].

- afterwards, we will present the results regarding the effective one-dimensional radial dynamics of two prototype examples of MS systems defined on space on nonconstant curvature, the so-called Taub-NUT and Darboux III Hamiltonian systems. These models, which have been extensively studied in the literature for their mathematical as well as physical relevance, according to the Perlick's classification [5–7], belong to the family of type II, therefore can be regarded as intrinsic oscillators on such curved spaces. On the other hand, from an analytic point of view, they can be thought as a one-parameter superintegrable deformation of the KC and HO systems respectively, and their maximal superintegrability is ensured thanks to the existence of a deformed version of the LRL vector as well as of the DF tensor in their corresponding curved spaces. The idea of investigating such systems was proposed to me by prof. O. Ragnisco, and the results are based on the papers [36–38].

- finally, we will dedicate the closing section to the investigation of the quantum case. In particular, our aim is to show that the exact solvability of the two aforementioned MS systems is related to the fact that they define two shape invariant potentials (SIP) on their corresponding conformally flat space. We will see that the spectral problem is solved by means of the SUSYQM approach (factorization method and shape invariance) for positive values of the deformation parameters. For negative values of the parameters the problem is still open.

4.1 FACTORIZATION METHOD AND DEFORMED POISSON ALGEBRAS

In order to get the phase space trajectories of one-dimensional classical systems, it is possible to use an approach that allows us to determine the solution algebraically in a direct way. This is the so-called classical factorization method introduced by S. Kuru and J. Negro in [32]. Here, we summarize which is the general idea of the method and introduce the basic notions that we will need in the following. Clearly, for a more detailed discussion, we refer the reader to the original work [32], where a lot of examples are presented and explicitly solved. Let us consider the classical Hamiltonian function $H_0 : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$H_0(x,p) = \frac{p^2}{2m} + V(x)\,, \tag{1.230}$$

where x and p are canonical coordinates such as $\{p,x\} = 1$ and $V(x)$ is the potential term[1]. We deal with Hamiltonian that can be factorised in terms of two complex-conjugate differentiable functions $A_0^{\pm} : \mathcal{M} \to \mathbb{C}$ defined on the phase space $\mathcal{M} = \mathbb{R}^2 \simeq \mathbb{C}$, namely:

$$H_0(x,p) = \frac{p^2}{2m} + V(x) = A_0^+ A_0^- + \gamma(H_0)\,, \tag{1.231}$$

where $A_0^{\pm} = A_0^{\pm}(x,p)$ and $\gamma(H_0)$ is a function depending on just the Hamiltonian H_0. The method requires that the functions $A_0^{\pm}$ and H_0 close the following deformed Poisson algebra:

$$\{A_0^+, A_0^-\} = -i\beta(H_0) \quad , \quad \{H_0, A_0^{\pm}\} = \pm i\alpha(H_0)A_0^{\pm}\,, \tag{1.232}$$

with $\alpha(H_0)$, $\beta(H_0)$, $\gamma(H_0)$ functions to be determined. The crucial point is that, for these systems, it is possible to introduce two *time-dependent* functions $Q_0^{\pm} : \mathcal{M} \times \mathbb{R} \to \mathbb{C}$ defined as:

$$Q_0^{\pm}(x,p,t) = A_0^{\pm}(x,p)\, e^{\mp i\alpha(H_0)t}\,, \tag{1.233}$$

such that:

$$\frac{\mathrm{d}Q_0^{\pm}}{\mathrm{d}t} = \{H_0, Q_0^{\pm}\} + \partial_t Q_0^{\pm} = 0\,, \tag{1.234}$$

which means that $Q_0^{\pm}$ are constants of motion. By using these integrals of motion, whose values given by $q_0^{\pm} = |q_0^{\pm}|e^{\pm i\theta_0}$ are fixed by the initial conditions, it is possible to construct the trajectories $(x(t), p(t)) \in \mathbb{R}^2$ in the phase space algebraically [32]. In particular, because of the factorisation (1.231), the modulus of $q_0^{\pm}$ turns out to be a function of the energy, i.e. $|q_0^{\pm}| = q_0(H_0)|_{H_0=E_0}$. Moreover, for bounded orbits, the frequency of the motion is given by the underlying Poisson algebra. In fact, we can immediately observe from (1.233) that it actually results $\alpha(H_0)|_{H_0=E_0}$. Then, the dynamical information regarding the motion of the system (1.230) is contained into the two algebraic relations:

$$A_0^{\pm}(x,p)\, e^{\mp i\alpha(E_0)t} = q_0(E_0)\, e^{\pm i\theta_0}\,. \tag{1.235}$$

As a matter of fact, by knowing the explicit form of the two functions $A_0^{\pm}$ in terms of x and p, by means of (1.235) we can find $x(t)$ and $p(t)$ as a function of the total (conserved) energy. Thus, in other words, this method allows to solve Hamiltonian differential equations without

1 To be coherent with the conventions of the original work [32], we took the same convention for the Poisson brackets.

the need of performing integrations. This construction will be clear in a moment when the simple example of the harmonic oscillator will be presented.
We should point out that by using this approach we have access to the solution $(x(t), p(t))$ for both bounded and unbounded motion. In fact, as we know there exist systems, such as the Kepler-Coulomb, which are characterized by both bound and unbounded motions, depending on the *energy sector* that we are analyzing. In such cases the complex nature of the factors can change according to the energy sector, and the same happens for the deformed Poisson algebra. We point out that this technique, based on the factorisation of the Hamiltonian in terms of two functions that, together with the Hamiltonian itself, close a Poisson algebra has been considered also to study classical mechanical systems with position-dependent mass [120–122]. Let us discuss in the following how to solve the motion of the harmonic oscillator by using this method.

4.1.1 The standard harmonic oscillator

It is straightforward to obtain the solution $(x(t), p(t))$ of the harmonic oscillator by means of the factorization and its classical spectrum generating algebra. To see this, let us consider the usual Hamiltonian function (in units $m = \omega = 1$):

$$H_0(x, p) = \frac{p^2 + x^2}{2}\,. \tag{1.236}$$

It is trivial to verify that (1.236) is factorised in terms of the two complex functions:

$$A_0^{\pm}(x, p) = \frac{x \mp ip}{\sqrt{2}}\,, \tag{1.237}$$

namely:

$$H_0 = A_0^+ A_0^-\,, \tag{1.238}$$

which implies $\gamma(H_0) = 0$. Moreover, computing the Poisson brackets we obtain the oscillator algebra:

$$\{A_0^+, A_0^-\} = -i \quad , \quad \{H_0, A_0^{\pm}\} = \pm i A_0^{\pm}\,, \tag{1.239}$$

which means $\beta(H_0) = \alpha(H_0) = 1$ (this is because we set $\omega = 1$). Using these relations we are able to define the two "time-dependent" integrals of motion $Q_0^{\pm}$:

$$\begin{cases} Q_0^{\pm}(x, p, t) \doteq A_0^{\pm}(x, p)\, e^{\mp i\alpha(H_0)t} \\ \dot{Q}_0^{\pm}(x, p, t) = 0\,, \end{cases} \tag{1.240}$$

which we can use to construct the solution. In fact, using their polar decomposition we can write:

$$Q_0^{\pm}(x, p, t) \doteq A_0^{\pm}(x, p)\, e^{\mp i\alpha(E_0)t} = q_0(E_0)\, e^{\pm i\theta_0}\,, \tag{1.241}$$

with $q_0^2(E_0) = A_0^+ A_0^- = E_0$, where E_0 is the total (conserved) energy of the system. This means that the two following relations hold:

$$\begin{cases} \frac{x - ip}{\sqrt{2}} = \sqrt{E_0}\, e^{i(\alpha(E_0)t + \theta_0)} \\ \frac{x + ip}{\sqrt{2}} = \sqrt{E_0}\, e^{-i(\alpha(E_0)t + \theta_0)}\,, \end{cases} \tag{1.242}$$

and from these we can immediately obtain the well-known solution of the harmonic oscillator:

$$\begin{cases} x(t) = \sqrt{2E_0}\cos(t+\theta_0) \\ p(t) = -\sqrt{2E_0}\sin(t+\theta_0)\,, \end{cases} \tag{1.243}$$

where E_0 and θ_0 are fixed by the initial conditions (this is the solution of a second-order differential equation). We observe that taking the sum of the square of these two equations we obtain (1.236) restricted to the level surfaces $H_0 = E_0$. Thus, in conclusion, the motion has been straightforwardly solved algebraically. In what follow, we will show how this method can be succesfully used in order to solve the motion of the classical analog of f-oscillators. Thus, let us dedicate the following section to give a concise introduction to such systems.

4.2 f-DEFORMATIONS

4.2.1 Motivations and definitions

The q-deformation is by now a well established subject [123]. As a matter of fact $su_q(2)$ is often the prototype model for the study of quantum groups, deformations of usual groups, and it is related to solutions of the Yang-Baxter equation [124] or, for instance, to (broken) symmetries in solids [125]. Related to this, are the q-bosons [126]. Developed on its own right, they are related to the $su_q(2)$, depending on how one defines it [46]. They can be used as in the undeformed case, for the Jordan-Schwinger construction of the $su(2)$, or they can be seen as the contraction of it [127]. In [42] the authors describe how one of these q-bosons may be written as non-linear expressions from the undeformed algebra. They also generalize this idea to the so-called f-bosons. In the same paper, they show how for the q-bosons oscillator model the classical problem is a non-linear oscillator, with its frequency depending on the initial data. Following the idea, looking into the classical model equivalent to the quantum deformation can be very insightful. In fact, even if in the quantum version we are able to describe its spectrum and stationary states, it is on the classical side that we can get a physical intuition of its real meaning. This idea was further developed in [43], where the classical aspects of these quantum deformations have been considered and applications to quantum optics have been discussed.

For these reasons, we are interested in the classical analog of f-oscillators. In other words, we ask the question: *what physical systems are represented by these spectra?* To give an answer, we will investigate these non-linear systems by implementing the classical factorization method, proposed in this deformed framework. We will show that it is possible to prove that all f-oscillators are in fact a rather simple example of deformations, and we can find out their solutions explicitly. These are nothing more than energy dependent frequencies and amplitudes, and the phase space trajectories are undeformed, except for a change in their radius.

As a matter of fact, the aim of this research is two-fold. On one hand, as we already claimed we are interested to give a physical intuitive interpretation of these kind of systems. On the other hand, we want to provide a first exploration of the connection between f-deformations and classical SGA.

4.2.2 q-deformations and f-deformations

There are a few options for the q-deformation of the Heisenberg algebra (see for example [123] and references within). The one we are currently interested in is[2]:

$$\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger\hat{A} = q^{-\hat{N}}, \tag{2.244}$$

with $q \in \mathbb{R}^+$, $[\hat{N}, \hat{A}^\dagger] = \hat{A}^\dagger$ and $[\hat{N}, \hat{A}] = -\hat{A}$. Moreover, $\hat{A}^\dagger\hat{A} = [\hat{N}]$ and $\hat{A}\hat{A}^\dagger = [\hat{N}+1]$, where $[x]$ is the usual notation for a *q-number*, given by[3]:

$$[x] \doteq \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(\lambda x)}{\sinh\lambda}\,. \tag{2.245}$$

Here we introduced the parameter $\lambda \doteq \log q \in \mathbb{R}$. In particular, this algebra can be represented as non-linear expressions of the usual Heisenberg algebra. Indeed, considering the operators $\hat{a}, \hat{a}^\dagger$:

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1}\,, \qquad \hat{N} = \hat{a}^\dagger\hat{a}\,, \tag{2.246}$$

we can define the new operators:

$$\hat{A} = \hat{a}f(\hat{N})\,, \qquad \hat{A}^\dagger = f(\hat{N})\hat{a}^\dagger, \tag{2.247}$$

where:

$$f(\hat{N}) \doteq \sqrt{\frac{\sinh(\lambda\hat{N})}{\hat{N}\sinh\lambda}} = \sqrt{\frac{[\hat{N}]}{\hat{N}}}\,. \tag{2.248}$$

If we start with an oscillator Hamiltonian for the deformed case, it may be written in terms of the usual coordinates as (in units $\hbar = \omega = 1$):

$$\hat{H} = \hat{A}^\dagger\hat{A} = f^2(\hat{N})\,\hat{a}^\dagger\hat{a} = [\hat{N}]\,, \tag{2.249}$$

leading to the spectrum:

$$E_n = [n] = \frac{\sinh(\lambda n)}{\sinh(\lambda)}\,. \tag{2.250}$$

Notice we choose here a zero vacuum energy for the deformed oscillator. This setting may be generalized by taking an arbitrary "dressing" of the ladder operators and choosing another function $f(\hat{N})$. These are the f-deformations [42, 43]. Clearly, the spectrum of the harmonic oscillator changes accordingly. As an example we could choose $f^2(\hat{N}) = \hat{N}^\lambda$, then:

$$E_n = n^{1+\lambda}\,. \tag{2.251}$$

To understand the physical meaning of these spectra, we will investigate their classical counterpart, by means of the method described in the following section. Before doing this, we should point out that these classical f-deformations have been analyzed in the paper [42]. This is also discussed in [41], where the authors comment about the different coordinates and their physical interpretation for the q-oscillator. Moreover, quite recently it has been shown in [44] that an arbitrary one-dimensional integrable system can be represented as an f-oscillator, and the action-angle variables play a crucial role for this representation.

2 For the Fock space representation, the $q \to q^{-1}$ symmetric expression will be also valid.
3 This expression can be extended to operators in the usual way, by means of formal power series.

4.2.3 The classical q-deformed harmonic oscillator

The q-deformed oscillator is given by the Hamiltonian function:

$$H(x,p) = \frac{\sinh(\lambda \frac{p^2+x^2}{2})}{\sinh\lambda} = \frac{\sinh(\lambda H_0(x,p))}{\sinh\lambda}\,, \tag{2.252}$$

where $\lambda \in \mathbb{R}$ is a deformation parameter such as $\lim_{\lambda\to 0} H(x,p) = H_0(x,p)$. In order to factorize this Hamiltonian we rewrite the latter as follows:

$$H = \frac{\sinh(\lambda H_0)}{H_0 \sinh\lambda} H_0\,. \tag{2.253}$$

It is now straightforward to show that it is possible to factorize (2.253) in terms of the two functions:

$$A^{\pm}(x,p) = \sqrt{\frac{\sinh(\lambda H_0)}{H_0\sinh\lambda}} A_0^{\pm}(x,p) = \sqrt{\frac{H}{H_0}} A_0^{\pm}(x,p)\,, \tag{2.254}$$

so that:

$$H = A^+A^-\,. \tag{2.255}$$

What we have here is essentially a *dressing* of the functions $A_0^{\pm}(x,p)$. In fact, we have introduced the new functions:

$$\begin{cases} A^{\pm} = \mathcal{F}_\lambda(H_0) A_0^{\pm} \\ \mathcal{F}_\lambda(H_0) \doteq \sqrt{\dfrac{\sinh(\lambda H_0)}{H_0\sinh\lambda}}\,, \end{cases} \tag{2.256}$$

such that $\lim_{\lambda\to 0}\mathcal{F}_\lambda(H_0) = 1$, i.e. we recover the functions $A_0^{\pm}$, namely the harmonic oscillator. Notice that we don't take these as actual coordinates of the system: they provide just a convenient factorization of the Hamiltonian. Nevertheless, considering the original q-deformed coordinates of the quantum problem, we could also use them as physical coordinates, with $\sqrt{2}A^{\pm} \doteq X \mp iP$, changing the interpretation of the phase space. This should not be confused with the approach in [42,43], where such transformations are seen as changing the perspective on whether the system is nonlinear or has deformed Poisson brackets.
Now, we have all the ingredients needed to close the deformed Poisson algebra, which results:

$$\{A^+,A^-\} = -i\frac{\lambda}{\sinh\lambda}\cosh(\lambda H_0) \quad , \quad \{H,A^{\pm}\} = \pm i\frac{\lambda}{\sinh\lambda}\cosh(\lambda H_0)A^{\pm}\,. \tag{2.257}$$

We notice immediately that the function $\alpha(H) = \beta(H) = \frac{\lambda}{\sinh\lambda}\cosh(\lambda H_0)$ depends on H_0 and not on H. Clearly, H_0 Poisson commutes with H, which means that it is a constant of motion [42]. Now, it turns out that whenever H_0 can be expressed as a function of the new Hamiltonian H, then we can close the classical SGA and follow the same procedure applied for the standard harmonic oscillator in order to get the trajectories in the phase space, i.e. we can define the time-dependent constants of motion and construct the solution algebraically. In this case, this operation is possible. In fact, by inverting (2.252), we find:

$$H_0 = \frac{1}{\lambda}\operatorname{arcsinh}(H\sinh\lambda)\,, \tag{2.258}$$

and the non-linear Poisson algebra becomes:

$$\{A^+, A^-\} = -i\frac{\lambda}{\sinh\lambda}\sqrt{1+H^2\sinh^2\lambda} \quad , \quad \{H, A^\pm\} = \pm i\frac{\lambda}{\sinh\lambda}\sqrt{1+H^2\sinh^2\lambda}\, A^\pm \,, \tag{2.259}$$

where we used the fact that $\cosh(\operatorname{arcsinh}(x)) = \sqrt{1+x^2}$. This is the cSGA characterizing the q-deformed oscillator. We notice that, in the limit $\lambda \to 0$, the deformed Poisson algebra (2.259) reduces to the "original" oscillator algebra (1.239). At this point, by considering the two functions $Q^\pm(x,p,t) = A^\pm(x,p)\, e^{\mp i\alpha(E)t} = q(E)\, e^{\pm i\theta}$ (in this case $q^2(E) = A^+A^- = E$) and, by taking into account (2.258), we easily arrive to:

$$\begin{cases} \sqrt{\frac{\lambda E}{\operatorname{arcsinh}(E\sinh\lambda)}}\frac{x-ip}{\sqrt{2}} = \sqrt{E}\, e^{i(\alpha(E)t+\theta)} \\ \sqrt{\frac{\lambda E}{\operatorname{arcsinh}(E\sinh\lambda)}}\frac{x+ip}{\sqrt{2}} = \sqrt{E}\, e^{-i(\alpha(E)t+\theta)} \,, \end{cases} \tag{2.260}$$

from which we can immediately write the solution in terms of the total (conserved) energy E:

$$\begin{cases} x(t) = \sqrt{\frac{2}{\lambda}\operatorname{arcsinh}(E\sinh\lambda)}\cos(\alpha(E)t+\theta) \\ p(t) = -\sqrt{\frac{2}{\lambda}\operatorname{arcsinh}(E\sinh\lambda)}\sin(\alpha(E)t+\theta) \,, \end{cases} \tag{2.261}$$

where $\alpha(E) = \frac{\lambda}{\sinh\lambda}\sqrt{1+E^2\sinh^2\lambda}$ is the angular frequency of the motion given by the underlying algebra. It is not difficult to verify that this solution is the same as the one obtained in [42]. We notice also that taking the sum of the square of (2.261) we find equation (2.252) restricted to the level surfaces $H(x,p) = E$. Moreover, in the $\lambda \to 0$ limit, we recover the results obtained for the standard harmonic oscillator.

In comparison with the undeformed case two main differences emerge. The first one, quite obvious, is that the frequency is energy dependent. The second one is related to the amplitude of the oscillations. In fact, it may be written as $\sqrt{2H_0}$, as in the undeformed case. However we have to be careful here with the definition of the energy of the system. For this deformed case, the energy is given by the full Hamiltonian, and so this implies that the amplitude will vary with the energy differently, following the expression above. In fact, in terms of the coordinates (X,P), the solution for the trajectories reads:

$$\begin{cases} X(t) = \sqrt{2E}\cos(\alpha(E)t+\theta) \\ P(t) = -\sqrt{2E}\sin(\alpha(E)t+\theta) \,. \end{cases} \tag{2.262}$$

Here, the amplitude of the trajectories have the same dependence on the energy as in the undeformed case. Therefore, while the energy dependent frequency is always present in the deformed system, the change in the amplitude depends on the physical interpretation.

4.2.4 The classical non-linear f-oscillators

The Hamiltonian function defining the non-linear f-oscillators is given by:

$$H(x,p) = \mathcal{F}_\lambda^2\Big(\frac{x^2+p^2}{2}\Big)\frac{x^2+p^2}{2} = \mathcal{F}_\lambda^2(H_0(x,p))H_0(x,p) \,, \tag{2.263}$$

where $\mathcal{F}_\lambda^2(H_0)$ is a function such that $\lim_{\lambda\to 0}\mathcal{F}_\lambda^2(H_0)=1$. We can factorize this Hamiltonian introducing the following functions on the phase space:

$$A^\pm(x,p) = \mathcal{F}_\lambda(H_0)A_0^\pm(x,p)\,, \tag{2.264}$$

such as:

$$H = \mathcal{F}_\lambda^2(H_0)H_0 = A^+A^-\,. \tag{2.265}$$

In this case, the deformed Poisson algebra reads:

$$\{A^+,A^-\} = -i\frac{\mathrm{d}H}{\mathrm{d}H_0}, \quad \{H,A^\pm\} = \pm i\frac{\mathrm{d}H}{\mathrm{d}H_0}\,A^\pm\,. \tag{2.266}$$

This result is interesting since we know that in order to close the spectrum generating algebra the function $\frac{dH}{dH_0}$ has to be a function of the Hamiltonian H. This implies that we have to require the following condition:

$$\frac{\mathrm{d}H}{\mathrm{d}H_0} = \alpha(H)\,. \tag{2.267}$$

In other words, this means that if we start with the Hamiltonian:

$$H = \mathcal{F}_\lambda^2(H_0)H_0 \doteq \mathcal{H}_\lambda(H_0)\,, \tag{2.268}$$

we need to require the function $\mathcal{H}_\lambda$ to be (differentiable and) invertible, that means there exists the function $\mathcal{H}_\lambda^{-1}:\mathbb{R}\to\mathbb{R}^+$, such that:

$$H_0 = \mathcal{H}_\lambda^{-1}(H)\,. \tag{2.269}$$

Notice that this condition is nothing but the generalization of (2.258) that we used in the particular case of the q-oscillator. If it is satisfied we can perform:

$$\frac{\mathrm{d}H}{\mathrm{d}H_0} = \mathcal{H}'_\lambda(H_0)|_{H_0=\mathcal{H}_\lambda^{-1}(H)} = \mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H))\,, \tag{2.270}$$

to obtain the Poisson algebra:

$$\{A^+,A^-\} = -i\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H)), \quad \{H,A^\pm\} = \pm i\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H))\,A^\pm\,. \tag{2.271}$$

This is the classical SGA underlying the entire family of f-oscillators. In particular, we can immediately observe that the angular frequency of the motion results:

$$\alpha(E) = \mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H))|_{H=E} = \mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(E))\,. \tag{2.272}$$

At this point we can go further to construct the explicit solution $(x(t),p(t))$ for the entire class. Once again we introduce the two integrals of motion $Q^\pm(x,p,t)$ such as:

$$A^\pm(x,p)\,e^{\mp i\alpha(E)t} = q(E)\,e^{\pm i\theta}\,, \tag{2.273}$$

that is:

$$\mathcal{F}_\lambda(\mathcal{H}_\lambda^{-1}(E))\frac{x\mp ip}{\sqrt{2}} = \sqrt{E}\,e^{\pm i(\alpha(E)t+\theta)}\,. \tag{2.274}$$

Now, taking into account (2.268), we can write $\mathcal{F}_\lambda(\mathcal{H}_\lambda^{-1}(E)) = \sqrt{\frac{E}{\mathcal{H}_\lambda^{-1}(E)}}$ to obtain:

$$x \mp ip = \sqrt{2\mathcal{H}_\lambda^{-1}(E)}\, e^{\pm i(\alpha(E)t+\theta)}\,. \tag{2.275}$$

Finally, using the two equations (2.275) we arrive at the solution:

$$\begin{cases} x(t) = \sqrt{2\mathcal{H}_\lambda^{-1}(E)}\cos(\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(E))t+\theta) \\ p(t) = -\sqrt{2\mathcal{H}_\lambda^{-1}(E)}\sin(\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(E))t+\theta)\,. \end{cases} \tag{2.276}$$

This is the classical solution characterizing the entire class of non-linear f-oscillators. We remark that physically speaking, like before, this solution involves for both the frequency and the amplitude a change in the energy dependence. Let us notice that in terms of the coordinates (X, P) we have the amplitude $\sqrt{2E}$. In this general framework, it is straightforward to recover the q-deformed oscillator simply by observing that the function $\mathcal{H}_\lambda$ is given by $\mathcal{H}_\lambda(H_0) = \frac{\sinh(\lambda H_0)}{\sinh\lambda}$. To summarize, we can rewrite the problem and its solution as follows:

$$\textbf{Factorization}: \; H = \mathcal{H}_\lambda(H_0) = A^+A^-\;,\; A^\pm = \sqrt{\frac{\mathcal{H}_\lambda(H_0)}{H_0}}A_0^\pm\;,\; A_0^\pm(x,p) = \frac{x \mp ip}{\sqrt{2}}\,; \tag{2.277}$$

$$\textbf{cSGA}: \; \{A^+, A^-\} = -i\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H))\;,\; \{H, A^\pm\} = \pm i\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H))\,A^\pm\,; \tag{2.278}$$

$$\textbf{Solution}: \; \begin{cases} x(t) = \sqrt{2\mathcal{H}_\lambda^{-1}(E)}\cos(\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(E))t+\theta) \\ p(t) = -\sqrt{2\mathcal{H}_\lambda^{-1}(E)}\sin(\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(E))t+\theta)\,. \end{cases} \tag{2.279}$$

The problem is well-defined if the function $\mathcal{H}_\lambda : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following properties:

1. $\exists\, \lim_{\lambda\to 0}\mathcal{H}_\lambda(H_0) = H_0$, i.e. in this limit $\mathcal{H}_\lambda$ reduces to the identity function;
2. $\mathcal{H}_\lambda$ is at least a class $\mathcal{C}^1(\mathbb{R}^+)$ function, with the first derivative never vanishing in $\mathbb{R}^+$.

These conditions, together with the formulas (2.277)-(2.279), represent the main result of this study. We point out that, depending on the mathematical model of deformation we are considering, the above hypotesis can be restricted to hold in open subdomains of the positive real line.

Before concluding this section we mention that the result we have obtained is intimately connected to the one recently proposed in [44]. In that paper, the author showed that any one-dimensional integrable system can be represented as an f-oscillator, and the (canonical) action-angle variables (J, θ), defined as (the integral is taken along the full period of oscillations):

$$J(H) \doteq \frac{1}{2\pi}\oint p(x,H)\,\mathrm{d}x\;,\quad \theta \doteq \frac{\partial S(x,J)}{\partial J}\,, \tag{2.280}$$

with generating function $S(x,J) \doteq \int^x p(x,H)\,\mathrm{d}x$, are the main ingredients of this representation. The Hamilton's equations in such variables are given by:

$$\dot{J} = -\frac{\partial H(J)}{\partial\theta} = 0\,,\quad \dot{\theta} = \frac{\partial H(J)}{\partial J} \doteq \omega(J)\,. \tag{2.281}$$

In particular, in our case the action variable is $J = H_0 = A_0^+ A_0^-$, and the Hamiltonian function reads:

$$H = \mathcal{H}_\lambda(H_0) = \mathcal{H}_\lambda(J) = A^+ A^- \,, \tag{2.282}$$

where the complex functions $A^\pm$ can be expressed in terms of these variables as [44]:

$$A^\pm = \sqrt{\frac{H(J)}{J}} A_0^\pm \,, \tag{2.283}$$

and the underlying frequency of the motion is $\omega(J) = \frac{\partial H(J)}{\partial J}$, which is equivalent to (2.267) in our notations. What we remark is that in our analysis we are considering the full Hamiltonian H as the function representing the energy of the system. Every quantity has been therefore expressed in terms of this integral of motion. This physical interpretation has a natural algebraic counterpart that automatically arises when we think in terms of deformed Poisson algebras. As a matter of fact, the crucial point is related to the inversion formula (2.269), which connects the old Hamiltonian H_0 with the new deformed one H. This formula, in fact, is the one allowing us to close the deformed Poisson algebra in terms of its three "*natural*" generators $H, A^\pm$.

4.2.5 Another example of deformation

Now, we present a second example of a specific choice of f-deformation parametrized by λ. Let us consider the classical counterpart of (2.251), described by the following Hamiltonian function:

$$H = H_0^{1+\lambda} \,, \tag{2.284}$$

where $\lambda \in (-1, \infty) \subset \mathbb{R}$ is the deformation parameter. The function $\mathcal{H}_\lambda : H_0 \to H = H_0^{1+\lambda}$ is invertible and its inverse results[4]:

$$\mathcal{H}_\lambda^{-1}(H) = H^{\frac{1}{1+\lambda}} \,. \tag{2.285}$$

Then, considering that $\mathcal{H}'_\lambda(H_0) = (1+\lambda) H_0^\lambda$, we find:

$$\mathcal{H}'_\lambda(\mathcal{H}_\lambda^{-1}(H)) = (1+\lambda) H^{\frac{\lambda}{1+\lambda}} \,, \tag{2.286}$$

and the non-linear Poisson algebra results:

$$\{A^+, A^-\} = -i(1+\lambda) H^{\frac{\lambda}{1+\lambda}} \quad , \quad \{H, A^\pm\} = \pm i(1+\lambda) H^{\frac{\lambda}{1+\lambda}} A^\pm \,, \tag{2.287}$$

where $A^\pm = H_0^{\frac{\lambda}{2}} A_0^\pm$. Then, the solution characterizing this non-linear oscillator is:

$$\begin{cases} x(t) = \sqrt{2} E^{\frac{1}{2(1+\lambda)}} \cos\big((1+\lambda) E^{\frac{\lambda}{1+\lambda}} t + \theta\big) \\ p(t) = -\sqrt{2} E^{\frac{1}{2(1+\lambda)}} \sin\big((1+\lambda) E^{\frac{\lambda}{1+\lambda}} t + \theta\big) \,. \end{cases} \tag{2.288}$$

Clearly, in the undeformed (i.e. $\lambda \to 0$) limit, the harmonic oscillator is recovered.

4 We remark that H_0 is a positive definite quantity.

4.2.6 Multidimensional extension and superintegrability of f-deformations

The multidimensional generalization of such systems can be easily obtained by applying the coproduct map to the Hamiltonian $H = \mathcal{H}_\lambda(H_0)$. This is because they can be seen as coalgebraic $\mathfrak{sl}(2,\mathbb{R})$ Hamiltonians. In particular, for "sufficiently good" functions $\mathcal{H}_\lambda$, we can apply the coproduct map $\Delta : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})^{\otimes N}$ in the usual way:

$$\Delta^{[N]}(H) = \mathcal{H}_\lambda(\Delta^{[N]}(H_0))\,, \tag{2.289}$$

where $H_0 = H_0(\xi_+, \xi_-) = \frac{\xi_+ + \xi_-}{2}$ is now a function on the $\mathfrak{sl}(2,\mathbb{R})$ Poisson-Lie coalgebra. This implies that:

$$\Delta^{[N]}(H(\xi_+,\xi_-)) = H(\Delta^{[N]}(\xi_+), \Delta^{[N]}(\xi_-)) = \mathcal{H}_\lambda\left(\frac{\Delta^{[N]}(\xi_-) + \Delta^{[N]}(\xi_+)}{2}\right) \tag{2.290}$$

$$\overset{D^{\otimes N}}{=} \mathcal{H}_\lambda\left(\frac{\mathbf{p}^2 + \mathbf{x}^2}{2}\right), \tag{2.291}$$

where we used the usual representation of the $\mathfrak{sl}(2,\mathbb{R})$. Thus, the superintegrability of such systems is trivially ensured. The same functions obtained by taking the images of the $2N-3$ left and right Casimirs $C^{[m]}(\mathbf{x},\mathbf{p})$,$C_{[m]}(\mathbf{x},\mathbf{p})$ $(m = 2,\dots N)$, together with the components I_{ij} $(i,j = 1,\dots,N)$ of the Demkov-Fradkin tensor, are first integrals of motion also for these f-deformed systems. For the sake of completeness, let us mention that by using the dressed ladder operators $A^\pm = A^\pm(x,p)$, it is possible to write down a deformed version of both the angular momentum and the DF tensor. In particular, within the convention $A_i^\pm \doteq A^\pm(x_i,p_i)$, we can write:

$$\begin{cases} \mathcal{L}_{\lambda ij} = -i\,(A_i^+ A_j^- - A_j^+ A_i^-) = \sqrt{\dfrac{\mathcal{H}_\lambda(H_0(x_i,p_i))\mathcal{H}_\lambda(H_0(x_j,p_j))}{H_0(x_i,p_i)H_0(x_j,p_j)}}\,\mathcal{L}_{ij} \\ \mathrm{I}_{\lambda ij} = A_i^+ A_j^- + A_j^+ A_i^- = \sqrt{\dfrac{\mathcal{H}_\lambda(H_0(x_i,p_i))\mathcal{H}_\lambda(H_0(x_j,p_j))}{H_0(x_i,p_i)H_0(x_j,p_j)}}\,\mathrm{I}_{ij}\,, \end{cases} \tag{2.292}$$

for $i,j = 1,\dots,N$. Clearly, these quantities also Poisson commute with the ND Hamiltonian (2.291) and, in the $\lambda \to 0$ limit, smoothly tranforms to the undeformed angular momentum and DF tensor components.

4.2.7 Concluding remarks and open perspectives

The aim of the present section is to show that the classical factorization method has a natural application in the framework of f-deformations. In fact, by means of the factorization technique we have fully solved the entire class of f-oscillators, writing down an explicit solution for the phase space trajectories, valid for the entire family of deformations. It is interesting to point out that all the information needed to solve this family of Hamiltonian systems, are in fact related to the properties of the function $\mathcal{H}_\lambda$, which maps the old undeformed Hamiltonian onto the new f-deformed ones. In fact, we have shown that in order to close the deformed Poisson algebra in terms of the three (dressed) generators H and $A^\pm$, we automatically obtain a restriction for the class of functions that we can implement to "f-deform" the original system H_0. Regarding the question as it was posed in the introduction it is obvious that,

qualitatively, the entire class of deformations leads to the same physical result. Any such f-deformation on a one-dimesional harmonic oscillator will lead to a change in the frequency and in the amplitude, although the shape of the phase space trajectories will remain the same. We should point out the difference in interpretation when compared to [42, 43]. There, the authors consider H_0 to be the energy of the system, while here we use the whole Hamiltonian. Both quantities are conserved during the evolution. To argue in favor of the interpretation used here, take the quantum expression we started with. There, the spectrum of the total Hamiltonian is the energy of the system. On the other hand, using H_0 would be equivalent to take the energy in the quantum case to be simply the one of the usual harmonic oscillator. Along those lines, we should also note that we take x as the actual coordinate, the one with physical meaning. We could also argue that these are only convenient coordinates, so we could write the deformed system as a nonlinear expression. As we have shown, it is also possible to describe these systems by $\{X, P\}$ coordinates, related in the quantum case to the original f-boson algebra. This, along with an investigation regarding the Poisson algebra, will be further developed in the future.

Actually, many other possible directions could be taken. First of all, since we showed this connection between the Poisson algebra generators and f-deformations, it would be interesting to construct f-deformations for systems other than the harmonic oscillator [32]. In fact, dressing the undeformed generators of the deformed Poisson algebras, we should be able to construct these kind of deformations for other systems.

4.3 FACTORIZATION AND CLASSICAL SGA FOR MS SYSTEMS ON CURVED SPACES

4.3.1 Motivations and definitions

In a paper of 1992 [6] V. Perlick showed that the Bertrand theorem [3] arises naturally also in General Relativity. In particular, the aim of the author was to present the relativistic analogue of Bertrand's theorem, by providing all spherically symmetric and static spacetimes whose bounded trajectories are periodic, giving rise to the so-called *Bertrand spacetimes*. His result can be summarized as follows: let us consider a static $3+1$ dimensional spherically symmetric spacetime $(\mathcal{M}\times\mathbb{R},\eta)$, where $\mathcal{M}$ is a 3-manifold and η is the Lorentzian metric:

$$\eta \doteq \sum_{\mu,\nu=0}^{3}\eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \sum_{i.j=1}^{3} g_{ij}(\mathbf{x})\mathrm{d}x^{i}\mathrm{d}x^{j} - V^{-1}(r)\mathrm{d}t^{2} \tag{3.293}$$

$$= h^{2}(r)\mathrm{d}r^{2} + r^{2}(\mathrm{d}\theta^{2}+\sin^{2}\theta d\varphi^{2}) - V^{-1}(r)\mathrm{d}t^{2} \tag{3.294}$$

$$\doteq g - V^{-1}(r)\mathrm{d}t^{2}\,, \tag{3.295}$$

where g is a Riemannian metric on the 3-manifold in the coordinate system $\boldsymbol{\xi}=(r,\theta,\varphi)$ and $h(r)$, $V(r)$ are two smooth functions. The Lorentzian $(3+1)$-manifold $(\mathcal{M}\times\mathbb{R},\eta)$ is defined to be a *Bertrand spacetime* if, besides to be static and spherically symmetric, for each point of $\mathcal{M}$ there exists a circular ($r=$ *constant*) trajectory passing through it, and such circular trajectories are stable. Under these assumptions, Perlick succeeded in classyfing all such spacetimes. In particular, he obtained two multiparametric families of metrics η_I, η_{II} by deriving the explicit form of the functions $h(r)$, $V(r)$. Precisely, he found that $(\mathcal{M},\eta)$ is a Bertrand spacetime iff:

$$\begin{cases} \eta_I = \dfrac{1}{\beta^2(1+Kr^2)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2) - \left(G+\sqrt{\frac{1}{r^2}+K}\right)^{-1}\mathrm{d}t^2 \\[2ex] \eta_{II}^{\pm} = \dfrac{2(1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4})}{\beta^2((1-Dr^2)^2-Kr^4)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2) + \\ \qquad -\left(G\mp r^2(1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4})^{-1}\right)^{-1}\mathrm{d}t^2\,, \end{cases} \tag{3.296}$$

where D, G and K are real constants, and $\beta=\frac{m}{n}\in\mathbb{Q}$. These two metrics are linked to a pair of Hamiltonian systems whose equations of motion define trajectories that are coincident with the timelike geodesic determined by the metrics η_I and $\eta_{II}^{\pm}$, the so-called *Perlick's systems* of type I and II, namely:

$$\begin{cases} H_I^{(\beta,K,G)} = \beta^2(1+Kr^2)\dfrac{p_r^2}{2} + \dfrac{\mathbf{L}^2}{2r^2} + G + \sqrt{\frac{1}{r^2}+K} \\[2ex] H_{II}^{(\beta,K,D,G,\pm)} = \dfrac{\beta^2((1-Dr^2)^2-Kr^4)}{2(1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}}\dfrac{p_r^2}{2} + \dfrac{\mathbf{L}^2}{2r^2} + G \mp \dfrac{r^2}{1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}}\,, \end{cases} \tag{3.297}$$

where, as usual, $\mathbf{L}^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}$. These Hamiltonians can be therefore regarded as describing particles moving on a non-Euclidean three-dimensional space subjected to an "intrinsic"potential whose physical meaning was apparently not pointed out by Perlick in his original

paper. Neverthless, under the above hypotheses the classification is complete and contains as subcases, for a suitable choice of the parameters, other known families of spherically symmetric superintegrable systems, such as models defined on *flat spaces*, *constant curvature spaces*, *Iway - Katayama* and *Darboux* spaces [128]. These results have been extensively studied in the literature, since they provided the main ingredients for a generalization of the Bertrand theorem [3]. In particular, along this way in a remarkable paper [5], the authors showed that these Hamiltonians are superintegrable, by constructing a generalized version of the Laplace-Runge-Lenz vector, allowing them to propose a theorem that extended the Bertrand result for curved spaces. The theorem asserts that *if H is an Hamiltonian function associated to a Bertrand spacetime, i.e., an autonomous, spherically symmetric natural Hamiltonian system on a Riemannian 3-manifold $(\mathcal{M}, g)$ satisfying the above mentioned conditions, then:*

- *H is of the form (3.297);*
- *The potential $V(r)$ is the intrinsic Kepler or oscillator potential in $(\mathcal{M}, g)$;*
- *H is superintegrable.*

In particular, the second statement of the theorem was firstly investigated in [7], where the authors showed that for any Bertrand's spacetime the potential $V(r)$ is either an intrinsic Kepler–Coulomb or harmonic oscillator potential on its associated Riemannian 3-manifold $(\mathcal{M}, g)$, thus providing a physical interpretation of these Hamiltonians which was missing in the original Perlick's work. Thereafter, the same authors extended the Perlick's results to any arbitrary dimensions N and expressed them in a conformally flat form [54]. In this case timelike geodesics in an $(N+1)$-dimensional spacetime $(\mathcal{M}^N, \eta)$, where $\mathcal{M}^N$ is a N-manifold equipped with metric:

$$\begin{aligned}\eta = g - V^{-1}(r)\mathrm{d}t^2 &= f^2(|\mathbf{x}|)|\mathrm{d}\mathbf{x}|^2 - V^{-1}(|\mathbf{x}|)\mathrm{d}t^2 \\ &= f^2(r)(\mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2_{(N-1)}) - V^{-1}(r)\mathrm{d}t^2\,,\end{aligned} \tag{3.298}$$

where $|\mathbf{x}| = \sqrt{\mathbf{x}^2} = r$, with $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\mathrm{d}\Sigma^2_{(N-1)} \doteq \sum_{j=1}^{N-1} \mathrm{d}\theta_j^2 \prod_{k=1}^{j-1} \sin^2\theta_k$ is the metric of the unit $(N-1)$-sphere $\mathbb{S}^{N-1}$, are related to the trajectories of the N-dimensional classical Hamiltonian in $\mathcal{M}$ given by:

$$\begin{aligned}H(\mathbf{x}, \mathbf{p}) &= \frac{\mathbf{p}^2}{2f^2(|\mathbf{x}|)} + V(|\mathbf{x}|) \\ &= \frac{p_r^2 + \frac{\mathbf{L}^2}{r^2}}{2f^2(r)} + V(r)\,,\end{aligned} \tag{3.299}$$

where $\mathbf{p} = \sqrt{\mathbf{p}^2}$, p_r is the radial momentum canonically conjugated to the radial coordinate r and $\mathbf{L}^2 \doteq \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2\theta_k}$ is the total angular momentum squared. The motion takes place in a Riemannian N-manifold characterized by the metric:

$$g_{ij} = f^2(r)\delta_{ij}, \tag{3.300}$$

with $f(r)$ playing the role of conformal factor of the Euclidean metric $g_0 = |\mathrm{d}\mathbf{x}|^2 \doteq \sum_{j=1}^{N} \mathrm{d}x_j^2$, with scalar (in general non-constant) curvature given by [95]:

$$R(r) = (1-N)\frac{(N-4)f'(r)^2 + f(r)(2f''(r) + 2(N-1)r^{-1}f'(r))}{f(r)^4}\,. \tag{3.301}$$

Here hyperspherical coordinates have been introduced. They are given by a radial coordinate $r \geq 0$ and $N-1$ angular coordinates θ_j, with $\theta_k \in [0,\pi)$ for $1 \leq k \leq N-2$, and $\theta_{N-1} \in [0,2\pi)$, together with their canonical conjugated momenta p_r, p_{θ_j} , $(j = 1,\dots,N-1)$. In the ambient space $\mathbb{R}^N$, they are defined as [52]:

$$\begin{cases} x_j = r\cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k \quad (1 \leq j \leq N-1) \\ x_N = r\prod_{k=1}^{N-1} \sin\theta_k \,, \end{cases} \tag{3.302}$$

where any product $\prod_m^l$ for $m > l$ is assumed to be equal to 1. Together with these set of coordinates, their conjugated momenta are given by:

$$\begin{cases} p_j = p_r \cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k + \dfrac{\cos\theta_j}{r} \sum_{l=1}^{j-1} \dfrac{\prod_{k=l+1}^{j-1} \sin\theta_k}{\prod_{m=1}^{l-1} \sin\theta_m} \cos\theta_l p_{\theta_l} - \dfrac{\sin\theta_j}{r\prod_{k=1}^{j-1} \sin\theta_k} p_{\theta_j} \\ p_N = p_r \prod_{k=1}^{N-1} \sin\theta_k + \dfrac{1}{r} \sum_{l=1}^{N-1} \dfrac{\prod_{k=l+1}^{N-1} \sin\theta_k}{\prod_{m=1}^{l-1} \sin\theta_m} \cos\theta_l p_{\theta_l} \,, \end{cases} \tag{3.303}$$

where the sum $\sum_m^l$ such that $l < m$ is assumed to be zero. Clearly, the notion of intrinsic potential have been consequently extended to the N dimensional case. At the end of the game, the authors showed that the Perlick's Hamiltonians, after a change of variables and various redefinitions of the parameters can be cast in the form [54]:

$$\begin{cases} H_I^{(\beta,k)} = r^2(r^{-\beta} + kr^{\beta})^2 \dfrac{\mathbf{p}^2}{2} + \mu(r^{-\beta} - kr^{\beta}) \\ \\ H_{II}^{(\gamma,\lambda,\delta)} = \dfrac{r^2(r^{-2\gamma} - \lambda^2 r^{2\gamma})^2}{r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta} \dfrac{\mathbf{p}^2}{2} + \dfrac{\nu}{r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta} \,, \end{cases} \tag{3.304}$$

where A,$B \in \mathbb{R}$ are the coupling constants of the potential and β, $\gamma \in \mathbb{Q}$ are rational parameters. The flat harmonic oscillator and Kepler-Coulomb systems are recovered in the particular cases:

$$\begin{cases} H_I^{(1,0)} = \dfrac{\mathbf{p}^2}{2} + \dfrac{\mu}{r} & (\text{KC system}) \\ H_{II}^{(1,0,0)} = \dfrac{\mathbf{p}^2}{2} + \nu r^2 & (HO \text{ system}) \,, \end{cases} \tag{3.305}$$

highlighting the intrinsic nature of the two families of Hamiltonians. In particular, such an intrinsic nature can be understood geometrically from the following definitions. In a N-dimensional spherically symmetric space $\mathcal{M}^N$ with coordinates $(x_1,\dots,x_N) \in \mathbb{R}^N$, endowed with the metric (3.300), the Laplace-Beltrami operator reads:

$$\Delta_{\mathcal{M}^N} = \sum_{i,j=1}^{N} \frac{1}{\sqrt{g}} \partial_{x_i} \sqrt{g}\, g^{ij} \partial_{x_j}, \qquad g \doteq \det g_{ij} \,,$$

The radial symmetric Green function $V(|\mathbf{x}|) = V(r)$ on $\mathcal{M}^N$ (up to multiplicative and additive constants) is defined as the positive nonconstant solution to the following equation [54]:

$$\Delta_{\mathcal{M}^N} V(r) = 0 \quad \text{on} \quad \mathcal{M}^N \backslash \{\mathbf{0}\} \,, \tag{3.306}$$

that is:

$$V(r) = \int^r \frac{\mathrm{d}r'}{r'^2 f(r')}. \tag{3.307}$$

In particular, the *intrinsic KC potential* on the ND space $\mathcal{M}^N$ is defined as:

$$V_{\mathrm{KC}}(r) \doteq a\, V(r) + b, \tag{3.308}$$

whereas the *intrinsic oscillator potential* is defined to be proportional to the inverse square of the KC potential, namely:

$$V_{\mathrm{HO}}(r) \doteq \frac{c}{V^2(r)} + d, \tag{3.309}$$

where a, b, c and d are real constants. By applying these definitions to the two families of Bertrand systems, it follows [7] that type I and type II Hamiltonians (3.304) always define respectively intrinsic KC systems and oscillator potentials [54].

As we mentioned such systems have been shown to be superintegrable in the three-dimensional space thanks to the existence of a generalized version of the LRL vector. In the N-dimensional case $2N-3$ conserved quantities are ensured thanks to an underlying coalgebra symmetry. In fact, the general Hamiltonian (3.299) is defined on $\mathfrak{sl}(2,\mathbb{R})$ coalgebra spaces, being:

$$H(\xi_-^{[N]}, \xi_+^{[N]}) = \frac{\xi_+^{[N]}}{2f^2(\sqrt{\xi_-^{[N]}})} + V(\sqrt{\xi_-^{[N]}}), \tag{3.310}$$

and we know that such systems are QMS by construction thanks to the existence of the left and right Casimirs of the algebra.

Among the systems of the form (3.299), considerable attention has been paid to a couple of special cases. They are two MS N-dimensional Hamiltonian systems associated to a Darboux type-III metric and to a Taub-NUT metric respectively. From now on we will refer to them as the *Darboux III* (*D-III*) and *Taub-NUT* (*TN*) Hamiltonian systems. They have been carefully investigated because of their remarkable properties and a plethora of papers can be found in the literature dedicated to their analysis, both for the classical and the quantum versions [47–56]. On one hand, from the geometrical point of view, both systems pertain to the Perlick's family II, so they can be regarded as intrinsic oscillators on the corresponding spherically symmetric curved space. On the other hand, from an analitycal point of view, they represent a one-parameter superintegrable deformation of the N-dimensional isotropic harmonic oscillator (the one associated to the Darboux space) and of the Kepler-Coulomb system (the one associated to the Taub-NUT space) respectively.

4.3.2 A superintegrable system associated with the Darboux III metric

The classical *D-III* system is defined by the Hamiltonian function:

$$H_\lambda(\mathbf{x},\mathbf{p}) = T_\lambda(\mathbf{x},\mathbf{p}) + V_\lambda(\mathbf{x}) = \frac{\mathbf{p}^2}{2(1+\lambda\mathbf{x}^2)} + \frac{\omega^2\mathbf{x}^2}{2(1+\lambda\mathbf{x}^2)}, \tag{3.311}$$

with $\omega > 0$ and $\lambda \in \mathbb{R}$, where $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$. In this case, the kinetic energy term:

$$T_\lambda(\mathbf{x},\mathbf{p}) = \sum_{i,j=1}^{N} g^{ij}(\mathbf{x}) p_i p_j, \tag{3.312}$$

generates the geodesic motion of a particle with unit mass on a conformally flat space $\mathcal{M}^N = (\mathbb{R}^N; g)$, which is the Riemannian manifold endowed with the conformal metric:

$$\begin{aligned} \mathrm{d}s^2_\lambda = \sum_{i,j=1}^{N} g_{ij}(\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j &= (1+\lambda|\mathbf{x}|^2)|\mathrm{d}\mathbf{x}|^2 \\ &= (1+\lambda r^2)(\mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2_{(N-1)})\,, \end{aligned} \tag{3.313}$$

and non-constant scalar curvature:

$$R_\lambda(r) \doteq R(r)|_{f(r)=\sqrt{1+\lambda r^2}} = -\lambda(N-1)\frac{2N+3\lambda(N-2)r^2}{(1+\lambda r^2)^3}\,. \tag{3.314}$$

Such a curved space represents a N-dimensional spherically symmetric generalization of the Darboux surface of type III [83, 84, 97], which was constructed in [95]. This N-dimensional Hamiltonian has been shown to be *maximally superintegrable* in ref. [47]. In fact, because of the hyperspherical symmetry, it is endowed with an $\mathfrak{so}(N)$ Lie-Poisson symmetry ensuring the existence of the usual $2N-3$ conserved quantities:

$$\begin{cases} C^{[m]}(\mathbf{x},\mathbf{p}) = \sum_{1\leq i<j\leq m}(x_ip_j - x_jp_i)^2 \\ C_{[m]}(\mathbf{x},\mathbf{p}) = \sum_{N-m<i<j\leq N}(x_ip_j - x_jp_i)^2 \quad (m=2,\ldots,N)\,, \end{cases} \tag{3.315}$$

with $C^{[m]} = C_{[m]} = \mathbf{L}^2$. Moreover, other $N(N+1)/2$ conserved quantities are provided by a λ-deformation of the Demkov-Fradkin tensor, whose components:

$$\mathrm{I}_{\lambda ij} = \frac{p_ip_j}{\omega} + (\omega - \frac{2\lambda}{\omega}H_\lambda)x_ix_j \qquad (i,j=1,\ldots N)\,, \tag{3.316}$$

Poisson-commute with the Hamiltonian H_λ. In particular, the elements of the subset $\mathcal{O} = \{H_\lambda, C^{[m]}, C_{[m]}, \mathrm{I}_{\lambda ii}\}$ for $(m=2,\ldots,N)$, for a fixed index i, are functionally independent, providing in this way the $2N-1$ conserved quantities required for the maximal superintegrability. Moreover, each of the three sets $\{H_\lambda; C^{[m]}\}$, $\{H_\lambda; C_{[m]}\}$ $(m=2,\ldots,N)$ and $\{\mathrm{I}_{\lambda ii}\}$ for $(i=1,\ldots,N)$ is formed by N functionally independent functions in involution.
This Hamiltonian system, by using the coordinates (3.302)-(3.303), takes the form:

$$H_\lambda(r,p_r) = T_\lambda(r,p_r) + V_\lambda(r) = \frac{p_r^2 + \frac{\mathbf{L}^2}{r^2}}{2(1+\lambda r^2)} + \frac{\omega^2 r^2}{2(1+\lambda r^2)}\,, \tag{3.317}$$

which, at a fixed value of the total angular momentum $|\mathbf{L}| = l \in \mathbb{R}^+$, describes an effective one-dimensional radial dynamics. Moreover, when the parameter λ tends to zero, it smoothly tranforms to the isotropic harmonic oscillator:

$$\begin{cases} \lim_{\lambda\to 0} H_\lambda(r,p_r) = H_0(r,p_r) = \frac{p_r^2}{2} + \frac{\mathbf{L}^2}{2r^2} + \frac{\omega^2 r^2}{2} \\ \lim_{\lambda\to 0} \mathrm{d}s^2_\lambda = \mathrm{d}s^2_0 = \mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2_{(N-1)} \\ \lim_{\lambda\to 0} R_\lambda(r) = R_0(r) = 0\,, \end{cases} \tag{3.318}$$

which means that it can be regarded as a genuine λ-deformation of the N-dimensional *HO* system. According to the definitions given in (3.309), it easy to show that this Hamiltonian system define an intrinsic oscillator on its underlying Riemannian manifold. In fact, by

considering the conformal factor $f(r) = \sqrt{1+\lambda r^2}$ appearing in the metric, taking into account of the above definitions we can calculate the integral (3.307), thus obtaining:

$$V(r) = -\frac{\sqrt{1+\lambda r^2}}{r}\,, \tag{3.319}$$

which, taking into account of (3.309) under the identification $c = \frac{\omega^2}{2}$ and $d = 0$, gives rise to the potential:

$$V_\lambda(r) = \frac{c}{V^2(r)} + d = \frac{\omega^2 r^2}{2(1+\lambda r^2)}\,, \tag{3.320}$$

which is the one appearing in the Darboux III Hamiltonian H_λ. This intrinsic oscillator for $N = 3$ appears as a particular case of the so-called *multifold Kepler* 3-dimensional Hamiltonians constructed in [129, 130] as generalizations of the *MIC-Kepler* and *Taub-NUT* systems [50]. Geometrically, the Hamiltonian H_λ describes different classes of dynamical systems depending on the sign selected for the deformation parameter λ. In fact, even if the superintegrability properties hold $\forall\lambda \in \mathbb{R}$, the underlying space and the related potential change accordingly to the sign of λ. The Darboux space defines in this way three different underlying manifolds:

$$\begin{cases} \lambda > 0: & \mathcal{M}^N = (\mathbb{R}^N, g) & g_{ij} = (1+\lambda\mathbf{x}^2)\delta_{ij} & \\ \lambda < 0: & \mathcal{M}^N = (\mathcal{B}_{|\mathbf{x}|_c}, g) & g_{ij} = (1-|\lambda|\mathbf{x}^2)\delta_{ij} & \\ & \text{with} \quad \mathcal{B}_{|\mathbf{x}|_c} = [0, |\mathbf{x}|_c) & |\mathbf{x}|_c \doteq 1/\sqrt{|\lambda|} & \text{(interior space)} \\ \lambda < 0: & \mathcal{M}^N = (\mathbb{R}^N/\mathcal{B}_{|\mathbf{x}|_c}, g) & g_{ij} = (|\lambda|\mathbf{x}^2 - 1)\delta_{ij} & \\ & \text{with} \quad \mathbb{R}^N/\mathcal{B}_{r_c} = (r_c, \infty) & & \text{(exterior space)}\,, \end{cases} \tag{3.321}$$

and the scalar curvature changes accordingly [50]. Here $\mathcal{B}_{|\mathbf{x}|_c}$ denotes the open ball centered at the origin of radius $|\mathbf{x}|_c$ (the critical value for the metric and for the Hamiltonian H_λ).

4.3.3 A superintegrable system associated with the Taub-NUT metric

The classical *TN* system is defined by the Hamiltonian function on $\mathbb{R}^{2N}$:

$$H_\eta(\mathbf{x},\mathbf{p}) = T_\eta(\mathbf{x},\mathbf{p}) + V_\eta(\mathbf{x}) = \frac{|\mathbf{x}|}{|\mathbf{x}|+\eta}\frac{\mathbf{p}^2}{2} - \frac{k}{\eta+|\mathbf{x}|}\,, \tag{3.322}$$

where $k, \eta \in \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N$. The kinetic energy term:

$$T_\eta(\mathbf{x},\mathbf{p}) = \sum_{i,j=1}^{N} g^{ij}(\mathbf{x}) p_i p_j\,, \tag{3.323}$$

is the one generating the geodesic motion of a particle with unit mass on a conformally flat space $\mathcal{M}^N = (\mathbb{R}^N/\{0\}; g)$, which is the Riemannian manifold with the conformal metric:

$$\mathrm{d}s^2_\eta = \sum_{i,j=1}^{N} g_{ij}(\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j = \left(1+\frac{\eta}{|\mathbf{x}|}\right)|\mathrm{d}\mathbf{x}|^2 \tag{3.324}$$

$$= \left(1+\frac{\eta}{r}\right)(\mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2_{(N-1)})\,, \tag{3.325}$$

and non-constant scalar curvature:

$$R_\eta(r) \doteq R(r)|_{f(r)=\sqrt{1+\frac{\eta}{r}}} = \eta(N-1)\frac{4(N-1)r+3(N-2)\eta}{4r(r+\eta)^3}\,. \tag{3.326}$$

This system is naturally related to the Taub-NUT system [131–135] since $\mathcal{M}^N$ can be regarded as the (Riemannian) N-dimensional Taub-NUT space [93]. This N-dimensional Hamiltonian has been shown to be *maximally superintegrable* in [49]. In fact, as for the Euclidean KC system, it is endowed with an $\mathfrak{so}(N)$ Lie-Poisson symmetry, due to the fact that it can be constructed on an N-dimensional spherically symmetric space. The maximal superintegrability is ensured thanks to the existence of the usual $2N-3$ right and left conserved quantities given by the Casimirs of the $\mathfrak{sl}(2,\mathbb{R})$ and the N conserved quantities:

$$\mathcal{R}_{\eta i} = \sum_{j=1}^{N}(x_j p_i - x_i p_j)p_j + \frac{(k+\eta H_\eta)}{|\mathbf{x}|}x_i \qquad (i=1,\dots,N), \tag{3.327}$$

which are the components of an N-dimensional η-deformed version of the LRL vector. In particular one can take the subset $\mathcal{O} = \{H_\eta, C^{[m]}, C_{[m]}, \mathcal{R}_{\eta i}\}$ for $(m=2,\dots,N)$ that, for a fixed index i, is functionally independent, providing in this way the $2N-1$ conserved quantities required for the maximal superintegrability. Moreover, each of the two sets $\{H_\eta; C^{[m]}\}$, $\{H_\eta; C_{[m]}\}$ $(m=2,\dots,N)$ is formed by N functionally independent functions in involution. In total analogy with the Euclidean case, the square of the Laplace-Runge-Lenz vector turns out to be radially symmetric, and expressible in terms of the Hamiltonian and of the angular momentum, e.g.:

$$\boldsymbol{\mathcal{R}_\eta}^2 = \sum_{i=1}^{N}\mathcal{R}_{\eta i}^2 = 2\mathbf{L}^2 H_\eta + (k+\eta H_\eta)^2\,. \tag{3.328}$$

Moreover, even in this deformed case, together with the Lie-Poisson algebra $\mathfrak{so}(N)$ commutation relations generated by the $\frac{N(N-1)}{2}$ angular momenta $\mathcal{L}_{ij} = x_i p_j - x_j p_i$ $(i<j;$ $i,j=1,\dots,N)$, namely:

$$\{\mathcal{L}_{ij},\mathcal{L}_{ik}\}_{(\mathbf{x},\mathbf{p})} = \mathcal{L}_{jk} \quad \{\mathcal{L}_{ij},\mathcal{L}_{jk}\}_{(\mathbf{x},\mathbf{p})} = -\mathcal{L}_{ik} \quad \{\mathcal{L}_{ik},\mathcal{L}_{jk}\}_{(\mathbf{x},\mathbf{p})} = \mathcal{L}_{ij} \quad (i<k<j) \tag{3.329}$$

the deformed LRL vector closes the quadratic Poisson algebra:

$$\{\mathcal{R}_{\eta i},\mathcal{R}_{\eta j}\}_{(\mathbf{x},\mathbf{p})} = -2H_\eta\mathcal{L}_{ij} \quad (i<j;\ i,j=1,\dots,N)\,, \tag{3.330}$$

that, together with the Poisson brackets:

$$\{\mathcal{L}_{ij},\mathcal{R}_{\eta k}\}_{(\mathbf{x},\mathbf{p})} = \delta_{ik}\mathcal{R}_{\eta j} - \delta_{jk}\mathcal{R}_{\eta i}\,, \tag{3.331}$$

leads to an $\mathfrak{so}(N+1)$ symmetry algebra, which in this case is obtainable by means of the definitions:

$$\tilde{\mathcal{L}}_{0i} \doteq \frac{\mathcal{R}_{\eta i}}{\sqrt{-2H_\eta}}\,, \quad \tilde{\mathcal{L}}_{ij} \doteq \mathcal{L}_{ij}\,. \tag{3.332}$$

In fact, taking into account (3.332), it holds:

$$\{\tilde{\mathcal{L}}_{ij},\tilde{\mathcal{L}}_{ik}\}_{(\mathbf{x},\mathbf{p})} = \tilde{\mathcal{L}}_{jk} \quad \{\tilde{\mathcal{L}}_{ij},\tilde{\mathcal{L}}_{jk}\}_{(\mathbf{x},\mathbf{p})} = -\tilde{\mathcal{L}}_{ik} \quad \{\tilde{\mathcal{L}}_{ik},\tilde{\mathcal{L}}_{jk}\}_{(\mathbf{x},\mathbf{p})} = \tilde{\mathcal{L}}_{ij} \quad (i<k<j) \tag{3.333}$$

with $i,j,k = 0,\dots,N$. Clearly, in dimensions $N = 3$, if we take the limit $\eta \to 0$ we recover the relations of the original $\mathfrak{so}(4)$ algebra (1.36) of the KC system. Also in this case, by using the coordinates (3.302)-(3.303), we can write the system in radial form:

$$H_\eta(r,p_r) = T_\eta(r,p_r) + V_\eta(r) = \frac{r}{r+\eta}\left(\frac{p_r^2}{2} + \frac{\mathbf{L}^2}{2r^2}\right) - \frac{k}{r+\eta}\,. \tag{3.334}$$

When the parameter η tends to zero, for positive values of the coupling constant k, it smoothly tranforms to the *KC* system:

$$\begin{cases} \lim_{\eta\to 0} H_\eta(r,p_r) = H_0(r,p_r) = \frac{p_r^2}{2} + \frac{\mathbf{L}^2}{2r^2} - \frac{k}{r} \\ \lim_{\eta\to 0} \mathrm{d}s^2_\eta = \mathrm{d}s^2_0 = \mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2_{(N-1)} \\ \lim_{\eta\to 0} R_\eta(r) = R_0(r) = 0\,, \end{cases} \tag{3.335}$$

which implies that it can be regarded as a genuine MS η-deformation of the N-dimensional *KC* system. Once again, according to the definitions given in (3.309), it easy to show that (3.334) defines an intrinsic oscillator on its underlying Riemannian manifold. In fact, by considering the conformal factor $f(r) = \sqrt{1+\frac{\eta}{r}}$ appearing in the metric, taking into account of the above definitions we can calculate the integral (3.307), which gives (up to multiplicative and additive constants):

$$V(r) = -\sqrt{1+\frac{\eta}{r}}\,. \tag{3.336}$$

Taking into account of (3.309) under the identification $c = -d = {}^{k}/_{\eta}$, we obtain the potential:

$$V_\eta(r) = \frac{c}{V^2(r)} + d = -\frac{k}{r+\eta}\,, \tag{3.337}$$

which is the one appearing in the Taub-NUT Hamiltonian H_η. Before concluding, we should also remark that geometrically speaking the Hamiltonian H_η contains different classes of dynamical systems depending on the sign of the parameter η. Analogously to the D-III case, the sign of η determines the domain of definition of the radial coordinate r in $\mathcal{M}^N$:

$$\begin{cases} \eta > 0: & \mathcal{M}^N = (\mathbb{R}^N/\{0\}, g) & g_{ij} = (1+\frac{\eta}{|\mathbf{x}|})\delta_{ij} \\ \eta < 0: & \mathcal{M}^N = (\mathcal{B}_{|\mathbf{x}|_c}, g) & g_{ij} = (1-\frac{|\eta|}{|\mathbf{x}|})\delta_{ij} \\ & \text{with} \quad \mathcal{B}_{|\mathbf{x}|_c} = (|\mathbf{x}|_c, \infty) & |\mathbf{x}|_c \doteq |\eta|\,, \end{cases} \tag{3.338}$$

and the scalar curvature changes accordingly [55]. In this case we have indicated with $\mathcal{B}_{r_c}$ the open ball centered at the origin of radius $|\mathbf{x}|_c = |\eta|$ (the singularity of the metric and of the Hamiltonian H_η).

In what follows, our main goal is to investigate the one-dimensional effective radial dynamics of these two MS Hamiltonian systems by means of the classical factorization method. Following as a guideline the papers [136, 137], where the KC system has been investigated, we will start by presenting the results regarding TN [36]. In particular, we will construct the solution of its radial equation of motion algebraically. Moreover, we will present a standard classical analysis for the closed orbits and we will also discuss a deformed version of the third Kepler's law for the planetary motion. After that, we will describe analogous results for the D-III system based on the work [37], the latter obtained in a joint collaboration with our spanish colleagues.

4.4 THE CLASSICAL TAUB–NUT SYSTEM: FACTORIZATION, CLASSICAL SGA AND SOLUTION OF THE MOTION

As already mentioned we are interested in investigating the one-dimensional effective radial dynamics described by the classical model (3.334). In particular, we want to adapt the results derived in [136, 137] for the KC problem to this η-deformed KC system, therefore our starting Hamiltonian is (in the classical case we will keep $m \neq 0$ in the entire construction):

$$H_\eta = T_\eta(r, p_r) + V_\eta(r) = \frac{r p_r^2}{2m(r+\eta)} + \frac{l^2}{2mr(r+\eta)} - \frac{k}{r+\eta} = \mathcal{K}_\eta(r) H_0 \,, \tag{4.339}$$

where m, k and l are positive constants, $\eta > 0$ is the deformation parameter, p_r is the radial momentum, H_0 is the "undeformed" Kepler-Coulomb Hamiltonian and

$$V_\eta(r) \equiv V_{\text{eff}}(r) = \frac{l^2}{2mr(r+\eta)} - \frac{k}{r+\eta} \,, \qquad \mathcal{K}_\eta(r) \doteq \frac{r}{r+\eta} \,. \tag{4.340}$$

The main idea is to use the classical factorization method in classical mechanics to derive algebraically the classical radial trajectories [32]. To this end, let us multiply the Hamiltonian (4.339) by the factor $r(r+\eta)$:

$$r(r+\eta)H_\eta = r^2\left(\frac{p_r^2}{2m} + \frac{l^2}{2mr^2} - \frac{k}{r}\right) = \frac{1}{2m}(r^2 p_r^2 + l^2 - 2mkr). \tag{4.341}$$

Now, as it has been done in the undeformed case by Kuru and Negro [136, 137], at any $r \neq 0$ we can factorize (4.341) as follows:

$$r^2 p_r^2 - 2mr(k + \eta H_\eta) - 2mr^2 H_\eta = A_\eta^+ A_\eta^- + \gamma(H_\eta) = -l^2 \,, \tag{4.342}$$

where for the time being $A_\eta^\pm = A_\eta^\pm(r, p_r)$ are unknown functions of r, p_r. Paraphrasing the construction of the two aforementioned authors for the undeformed case, we make the following *ansatz* for $A_\eta^\pm$:

$$A_\eta^\pm = \left(\mp i r p_r + a r\sqrt{-H_\eta} + \frac{b(H_\eta)}{\sqrt{-H_\eta}}\right) e^{\pm f_\eta(r, p_r)} \,. \tag{4.343}$$

The "arbitrary function" $f_\eta(r, p_r)$ will be determined by requiring the closure of the Poisson algebra generated by H and $A^\pm$. More precisely, we impose:

$$\{H_\eta, A_\eta^\pm\} = \mp i\alpha(H_\eta) A_\eta^\pm \tag{4.344}$$

$$\{A_\eta^+, A_\eta^-\} = i\beta(H_\eta) \,, \tag{4.345}$$

where the functions α, β wait to be determined. Inserting $A_\eta^\pm$ in (4.342) we get:

$$a = \sqrt{2m} \,, \qquad b(H_\eta) = -\sqrt{\frac{m}{2}}(k + \eta H_\eta) \,, \qquad \gamma(H_\eta) = \frac{m(k+\eta H_\eta)^2}{2H_\eta} \,, \tag{4.346}$$

and requiring that $A_\eta^\pm$ obey the proper Poisson brackets we arrive at:

$$f_\eta(r, p_r) = -i\sqrt{\frac{2}{m}}\frac{r p_r \sqrt{-H_\eta}}{(k - \eta H_\eta)} \,, \quad \alpha(H_\eta) = -\sqrt{\frac{2}{m}}\frac{2H_\eta\sqrt{-H_\eta}}{(k-\eta H_\eta)} \,, \quad \beta(H_\eta) = \sqrt{2m}\,\frac{(k+\eta H_\eta)}{\sqrt{-H_\eta}} \,, \tag{4.347}$$

and finally:

$$A_\eta^\pm = \left(\mp i r p_r + r\sqrt{-2mH_\eta} - \sqrt{\frac{m}{2}}\frac{(k+\eta H_\eta)}{\sqrt{-H_\eta}}\right) e^{\mp i\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{-H_\eta}}{(k-\eta H_\eta)}} \tag{4.348}$$

$$\{H_\eta, A_\eta^\pm\} = \pm i\sqrt{\frac{2}{m}}\frac{2H_\eta\sqrt{-H_\eta}}{(k-\eta H_\eta)}A_\eta^\pm\,, \qquad \{A_\eta^+, A_\eta^-\} = i\sqrt{2m}\,\frac{(k+\eta H_\eta)}{\sqrt{-H_\eta}}\,. \tag{4.349}$$

A mandatory requirement is that in the limit $\eta \to 0$ one gets back the undeformed Poisson algebra that is in fact, for $2m = k = 1$, the result found in [136, 137]. To make the identification even more perspicuous we can introduce the quantity:

$$A_\eta^0 \doteq \sqrt{\frac{m}{2}}\frac{(k+\eta H_\eta)}{\sqrt{-H_\eta}}\,, \tag{4.350}$$

entailing the following $\mathfrak{su}(1,1)$ Poisson-Lie algebra relations:

$$\{A_\eta^0, A_\eta^\pm\} = \mp i A_\eta^\pm\,, \quad \{A_\eta^+, A_\eta^-\} = 2iA_\eta^0\,. \tag{4.351}$$

In analogy to the undeformed case, it is reasonable to think at this algebra as the classical analog of a spectrum generating algebra for the TN system. At this point, we can introduce the two "time-dependent constants of the motion":

$$Q_\eta^\pm = A_\eta^\pm e^{\mp i\alpha(H_\eta)t}\,, \tag{4.352}$$

such that $\frac{\mathrm{d}Q_\eta^\pm}{\mathrm{d}t} = \{Q_\eta^\pm, H\} + \partial_t Q_\eta^\pm = 0$. These dynamical variables take complex values admitting the polar decomposition $Q_\eta^\pm = q_0\, e^{\pm i\theta_0}$ and allowing in fact to determine the motion, which turns out to be bounded for $E = -|E| < 0$. Indeed we have:

$$\left(\mp i r p_r + r\sqrt{2m|E|} - \sqrt{\frac{m}{2}}\frac{(k-\eta|E|)}{\sqrt{|E|}}\right) e^{\mp i\left(\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{|E|}}{(k+\eta|E|)} + \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{(k+\eta|E|)}t\right)} = q_0\, e^{\pm i\theta_0}\,, \tag{4.353}$$

or else:

$$\begin{cases} -ir p_r + r\sqrt{2m|E|} - \sqrt{\frac{m}{2}}\frac{(k-\eta|E|)}{\sqrt{|E|}} = q_0\, e^{i\left(\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{|E|}}{(k+\eta|E|)} + \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{(k+\eta|E|)}t + \theta_0\right)} \\ +ir p - r + r\sqrt{2m|E|} - \sqrt{\frac{m}{2}}\frac{(k-\eta|E|)}{\sqrt{|E|}} = q_0\, e^{-i\left(\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{|E|}}{(k+\eta|E|)} + \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{(k+\eta|E|)}t + \theta_0\right)}\,, \end{cases} \tag{4.354}$$

where $q_0 = q_0(\eta, E) \doteq \sqrt{-l^2 + \frac{m(k-\eta|E|)^2}{2|E|}}$ (from $A_\eta^+ A_\eta^- + \gamma(H_\eta) = q_0^2 + \gamma(H_\eta) = -l^2$). Summing and subtracting (4.354) we obtain:

$$\begin{cases} 2r\sqrt{2m|E|} - \sqrt{2m}\frac{(k-\eta|E|)}{\sqrt{|E|}} = 2q_0\cos\left(\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{|E|}}{(k+\eta|E|)} + \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{(k+\eta|E|)}t + \theta_0\right) \\ r p_r = -q_0\sin\left(\sqrt{\frac{2}{m}}\frac{r p_r\sqrt{|E|}}{(k+\eta|E|)} + \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{(k+\eta|E|)}t + \theta_0\right)\,. \end{cases} \tag{4.355}$$

It is immediate to verify that taking the sum of the square of these two equations we obtain the equation (4.342) restricted to the level surface $H = -|E|$. Finally, thanks to the above relations, we are able to obtain t as a function of r:

$$t(r) = \frac{1}{\Omega_\eta(E)}\left[\arccos\left(-\sqrt{\frac{m}{2}}\frac{\left((k-\eta|E|)-2|E|r\right)}{q_0\sqrt{|E|}}\right)-\sqrt{\frac{2}{m}}\frac{\sqrt{|E|}}{k+\eta|E|}\sqrt{2mr(k-\eta|E|)-2m|E|r^2-l^2}-\theta_0\right], \tag{4.356}$$

where:

$$\Omega_\eta(E) = \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{k+\eta|E|} \equiv \alpha(E) \tag{4.357}$$

is the angular frequency of the motion. Concerning (4.356) it is evident that, due to the presence of the "inverse cosine" function, t is a multivalued function of r defined mod $2\pi/\Omega$. To recover univaluedness, we have to introduce a "uniformization map" which is trivially given by the periodic function $\cos(\Omega t)$. In the limit $\eta \to 0$, the results for the flat Kepler-Coulomb are recovered (see [136, 137]). Thus, at this level, we can say that the radial motion in time of the TN system has been algebraically determined.
A number of plots are reported, showing the behavior of $V_{\text{eff}}(r) \doteq \frac{l^2}{2mr(r+\eta)} - \frac{k}{r+\eta}$ as a function of r, and the orbits on the phase plane (r, p_r) for different values of the deformation parameter (for $2l = m = k = 1$, $E = -1$, in appropriate units).

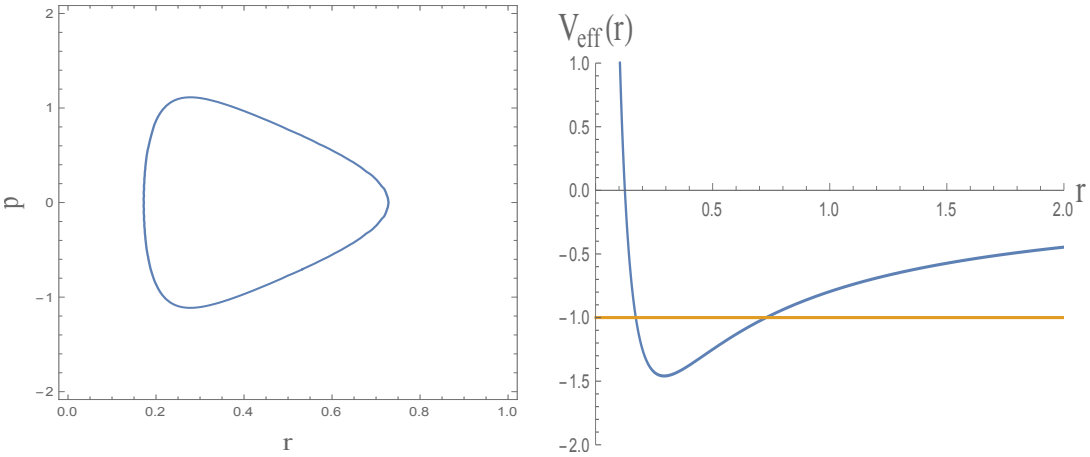


Figure 3.: phase space (r, p_r) and effective potential $V_{\text{eff}}(r)$ for $\eta = 0.1$.

4.4.1 Explicit formula for the orbits and third Kepler's law

As we have shown in the section 4.3.3 dedicated to the TN system, our system is maximally superintegrable and its maximal superintegrability is strictly related to the existence of the Runge-Lenz vector: then, as it happens for the standard Kepler-Coulomb system, we expect that this extra symmetry will play a crucial role in determining the shape of the orbits. As is well known, in the undeformed case the orbits are conic sections, namely ellipses for bounded trajectories. To identify the analytic form of the orbits when $\eta \neq 0$, we will consider the simplest and more *physical* case, corresponding to $\eta > 0$, also in order to present a deformed

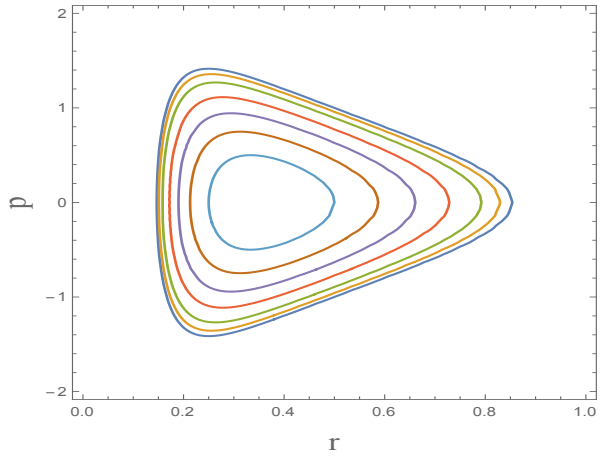


Figure 4.: phase space (r, p_r) for $\eta = 0, 0.01, 0.05, 0.1, 0.15, 0.2, 0.25$.

version of the third Kepler's law. To this end, we will closely follow [8]. In particular, the LRL vector $\mathcal{R}_\eta$, when evaluated *on-shell*, can be written as:

$$\mathcal{R}_\eta = \frac{1}{m}(\mathbf{L} \times \mathbf{p}) + \frac{k - \eta|E|}{|\mathbf{x}|}\mathbf{x}\,. \tag{4.358}$$

Again, we see that its expression is formally identical to the one holding in the flat case and is obtained by letting $k \to k_\eta \doteq k - \eta|E| > 0$. For its square we can write:

$$\mathcal{R}_\eta{}^2 = k_\eta^2 - \frac{2l^2|E|}{m}\,. \tag{4.359}$$

Moreover, it holds:

$$|\mathcal{R}_\eta||\mathbf{x}|\cos\theta = -{}^{l^2}/_{m} + k_\eta|\mathbf{x}|\,, \tag{4.360}$$

where the second equality has been calculated by considering the cartesian components of both $\mathbf{x}$ and $\mathcal{R}_\eta$. At this point, by easy algebraic manipulations, we can write the equation for the orbits in terms of $r \doteq |\mathbf{x}|$ and θ, getting:

$$r(\theta) = \frac{p_\eta}{1 - \epsilon_\eta \cos\theta}\,, \tag{4.361}$$

(p_η being the *parameter* and ϵ_η the *eccentricity* of the ellipses) which is formally the same expression holding in the flat case. But now we have:

$$\begin{cases} p_\eta \equiv p(E,\eta) = {}^{l^2}/_{mk_\eta} \\ \epsilon_\eta \equiv \epsilon(E,\eta) = {}^{|\mathcal{R}_\eta|}/_{k_\eta}\,, \end{cases} \tag{4.362}$$

so that $\epsilon_\eta^2 = 1 - {}^{2|E|l^2}/_{mk_\eta^2}$. Clearly, in the above expression, θ is the angle between the vectors $\mathbf{x}$ and $\mathcal{R}_\eta$. To check whether the third Kepler's law holds in the deformed case as well, we have to compute the ratio ${}^{\tau^2}/_{a^3} = {}^{4\pi^2}/_{a^3\Omega^2}$, where in our deformed case:

$$\Omega_\eta(E) = \frac{4|E|\sqrt{|E|}}{\sqrt{2m}(k + \eta|E|)}\,, \tag{4.363}$$

and a is the larger semi-axis defined as $a = \frac{r_+ + r_-}{2}$. The inversion points $r_\pm$ (where $p_{r_+} = p_{r_-} = 0$) are obtained by taking the roots of the equation:

$$r^2 - \frac{(k - \eta|E|)}{|E|} r + \frac{l^2}{2m|E|} = 0 \;\Rightarrow\; r_{\eta\pm} = \frac{k - \eta|E|}{2|E|} \pm \sqrt{\frac{(k - \eta|E|)^2}{4|E|^2} - \frac{l^2}{2m|E|}}\,, \tag{4.364}$$

entailing:

$$a_\eta = \frac{r_{\eta+} + r_{\eta-}}{2} = \frac{k - \eta|E|}{2|E|}\,. \tag{4.365}$$

In the limit $\eta \to 0$ we recover the larger semi-axis of the flat case, and then:

$$\frac{\tau^2}{a^3} = \frac{4\pi^2 m}{k}\,. \tag{4.366}$$

We remind that the so-called third Kepler's law is obtained by assuming that the ratio $\frac{m}{M}$ between the mass of the planet and the mass of the sun be very small, so that the reduced mass $\mu \doteq {}^{mM}/_{m+M}$ can be identified with the mass of the planet. The substitution $k = GMm$ leads to: $\frac{\tau^2}{a^3} = \frac{4\pi^2 m}{k} = \frac{4\pi^2}{GM}$. In the deformed case the analogous formula reads:

$$\frac{\tau_\eta^2}{a_\eta^3} = 4\pi^2 m \frac{(k + \eta|E|)^2}{(k - \eta|E|)^3}\,. \tag{4.367}$$

The Kepler's third law is then violated as the r.h.s. of (4.367), again assuming $k = GMm$, keeps its dependence upon m and E:

$$\frac{\tau_\eta^2}{a_\eta^3} = 4\pi^2 m \frac{(k + \eta|E|)^2}{(k - \eta|E|)^3} \approx \frac{4\pi^2}{GM}\left(1 + \frac{5\eta|E|}{GMm} + O(\eta^2)\right). \tag{4.368}$$

Incidentally, we observe that the identification of the reduced mass with the mass of the planet might not always be acceptable. In the solar system, for example, the mass of Jupiter is about 0.1% of the mass of the Sun [8].

4.4.2 Explicit evaluation of the trajectory: comparison with algebraic method

For the sake of completeness we present here the explicit derivation of the motion in time by using the standard analytic method [8]. A comparison with the results obtained through the Spectrum Generating Algebra will provide a definite proof of the correctness of the algebraic approach. The starting point is the usual Hamiltonian (4.339):

$$H_\eta = \frac{r}{r + \eta}\left[\frac{p_r^2}{2m} + \frac{l^2}{2mr^2} - \frac{k}{r}\right]. \tag{4.369}$$

The radial momentum p_r is related to the radial component of the velocity through the Hamilton's equation:

$$\dot{r} = \partial_{p_r} H_\eta = \frac{r}{r + \eta}\frac{p_r}{m} \;\Rightarrow\; p_r = \frac{r + \eta}{r} m\dot{r}\,. \tag{4.370}$$

Inserting in the Hamiltonian the expression of p_r in terms of r and $\dot{r}$ we obtain:

$$H_\eta = \frac{r}{r+\eta}\left[\frac{(r+\eta)^2}{r^2}\frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}\right]. \tag{4.371}$$

By solving the above expression with respect to $\dot{r}(t)$ and setting $H_\eta = E$ we get:

$$\dot{r}(t) = \pm\sqrt{\frac{2}{m}}\frac{r}{r+\eta}\sqrt{E + \frac{k+\eta E}{r} - \frac{l^2}{2mr^2}}\,. \tag{4.372}$$

Comparing with the Euclidean case ($\eta = 0$), besides the coupling constant metamorphosis, the essential difference consists in the presence of a nontrivial conformal factor. As a next step, we calculate $t(r)$ by taking the positive branch of the square root:

$$t(r) - t_0 = \sqrt{\frac{m}{2}}\int_{r_0}^{r} \mathrm{d}r \frac{r+\eta}{r\sqrt{E + \frac{k+\eta E}{r} - \frac{l^2}{2mr^2}}} = \sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{E + \frac{k+\eta E}{r} - \frac{l^2}{2mr^2}}} + \sqrt{\frac{m}{2}}\eta\int_{r_0}^{r} \frac{\mathrm{d}r}{r\sqrt{E + \frac{k+\eta E}{r} - \frac{l^2}{2mr^2}}}\,. \tag{4.373}$$

The two integrals involved in the above formula can be conveniently calculated by introducing the so-called *eccentric anomaly* Ψ_η through the relation [8]:

$$r = a_\eta(1 - \epsilon_\eta \cos\Psi_\eta)\,. \tag{4.374}$$

In the previous section we have already shown that $a_\eta = -\frac{k+\eta E}{2E}$ and the eccentricity reads $\epsilon_\eta = \sqrt{1 + \frac{2l^2E}{m(k+\eta E)^2}}$. Let us now pass to the explicit calculation of the two integrals contained in (4.373), setting there $E = -|E| < 0$. It is not too difficult to arrive at the following results:

$$\begin{aligned}\sqrt{\frac{m}{2}}\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{-|E| + \frac{k-\eta|E|}{r} - \frac{l^2}{2mr^2}}} &= \sqrt{\frac{ma_\eta^3}{k-\eta|E|}}\int_0^{\Psi_\eta} \mathrm{d}\Psi'_\eta(1 - \epsilon_\eta\cos\Psi'_\eta) \\ &= \sqrt{\frac{ma_\eta^3}{k-\eta|E|}}(\Psi_\eta - \epsilon_\eta\sin\Psi_\eta)\,,\end{aligned} \tag{4.375}$$

$$\sqrt{\frac{m}{2}}\eta\int_{r_0}^{r} \frac{\mathrm{d}r}{r\sqrt{-|E| + \frac{k-\eta|E|}{r} - \frac{l^2}{2mr^2}}} = \sqrt{\frac{ma_\eta}{k-\eta|E|}}\eta\int_0^{\Psi_\eta}\mathrm{d}\Psi'_\eta = \sqrt{\frac{ma_\eta}{k-\eta|E|}}\eta\Psi_\eta\,. \tag{4.376}$$

Hence, dividing and multiplying the output of the second integral by the same quantity a_η and rearranging the two integrals in a single expression, we get (with the initial data $t_0 = 0$):

$$t(r) = \sqrt{\frac{ma_\eta^3}{k-\eta|E|}}\left(\frac{\eta + a_\eta}{a_\eta}\right)\Psi_\eta - \sqrt{\frac{ma_\eta^3}{k-\eta|E|}}\epsilon_\eta\sin\Psi_\eta\,, \tag{4.377}$$

namely:

$$\begin{aligned}t(r) &= \sqrt{\frac{ma_\eta^3}{k-\eta|E|}}\left(\frac{\eta + a_\eta}{a_\eta}\right)\left[\Psi_\eta - \frac{a_\eta}{\eta + a_\eta}\epsilon_\eta\sin\Psi_\eta\right] \\ &= \frac{1}{\Omega_\eta(E)}\left[\Psi_\eta - \frac{a_\eta}{\eta + a_\eta}\epsilon_\eta\sin\Psi_\eta\right],\end{aligned} \tag{4.378}$$

which is a η-deformed version of the *Kepler's equation* [8]:

$$\Omega_\eta(E)\, t(r) = \Psi_\eta - \frac{a_\eta}{\eta + a_\eta}\epsilon_\eta \sin\Psi_\eta\,. \tag{4.379}$$

The frequency of the motion is given by:

$$\Omega_\eta(E) = \sqrt{\frac{k-\eta|E|}{m a_\eta^3}}\frac{a_\eta}{\eta+a_\eta} = \sqrt{\frac{2}{m}}\frac{2|E|\sqrt{|E|}}{k+\eta|E|}\,, \tag{4.380}$$

which is nothing but the same frequency appearing in the deformed Poisson algebra that we have obtained algebraically. Now we have just to plug in the equation (4.379) the explicit form of Ψ_η and check whether it coincides with the one derived via the algebraic method. By solving for Ψ_η one gets:

$$\Psi_\eta = \arccos\left[\frac{1}{\epsilon_\eta}\left(1-\frac{r}{a_\eta}\right)\right], \tag{4.381}$$

from which it follows:

$$\Omega_\eta(E)t(r) = \arccos\left[\frac{1}{\epsilon_\eta}\left(1-\frac{r}{a_\eta}\right)\right] - \frac{a_\eta}{\eta+a_\eta}\sqrt{\epsilon_\eta^2 - \left(1-\frac{r}{a_\eta}\right)^2}\,. \tag{4.382}$$

Equation (4.382) represents the trajectory calculated through the standard analytic method. On the other hand, the equation (4.356) for the trajectory derived by means of the algebraic method, after easy algebraic manipulations equation acquires the form:

$$\Omega_\eta(E)t(r) = \arccos\left[-\frac{1}{\epsilon_\eta}\left(1-\frac{r}{a_\eta}\right)\right] - \frac{a_\eta}{\eta+a_\eta}\sqrt{\epsilon_\eta^2 - \left(1-\frac{r}{a_\eta}\right)^2}\,. \tag{4.383}$$

In other words, by the algebraic method we get $t(r)$ evaluated for $-\epsilon_\eta$. As we expected, this expression is just the $\eta-$deformation of the results contained in [136, 137].
Here, some comments are in order, which seem to imply a sort of difference between the classical and the quantum case. Namely, according to the results obtained in [55] in the quantum case, for $\eta > 0$, one has a very simple *coupling constant metamorphosis*, amounting just to replace in the Euclidean system k by $k + \eta E$ both in the expressions for the discrete spectrum and for the corresponding eigenfunctions. However, in the classical case, just looking at (4.372), it turns out that we have not just coupling constant metamorphosis. This is because, besides the term $k + \eta E$, there is an extra η dependence in the overall conformal factor. An analogous behaviour, as we will see immediately, is exhibited by the classical D-III.

4.4.3 The case $\eta < 0$: new features

This section is devoted to a terse investigation of the main features arising in the case $k > 0$ for *negative values* of the deformation parameter η. In this case the conformal factor $\frac{r}{r+\eta}$ can be more conveniently written as $\frac{r}{r-|\eta|}$ which *emphasizes* the singularity at $r = |\eta|$. One relevant question is whether the singularity can be overcome or not. In the first case there might be trajectories intersecting the line $r = |\eta|$. In the second case the phase plane $(r, \dot{r})$ will consist

of two non overlapping domains. In particular, for closed orbits one may ask under what conditions the following (mutually excluding) inequalities for the inversion points hold:

$$r_{\eta-} > |\eta|\,,\; r_{\eta+} < |\eta|\,. \tag{4.384}$$

A careful analysis of (4.384) shows that to characterise the corresponding regions one has to look at *both* parameters η and λ, a characteristic lenght scale defined[5] as $\lambda \doteq {}^{l^2}/_{2mk} > 0$, or better at their ratio $\alpha \doteq |\eta|/\lambda$, and at the behaviour of the effective potential:

$$V_{\text{eff}}(r) = \frac{l^2}{2mr(r-|\eta|)} - \frac{k}{r-|\eta|} = -\frac{k}{r}\left(\frac{r-\lambda}{r-\alpha\lambda}\right)\,. \tag{4.385}$$

case $\alpha < 1$

The most interesting situation occurs in the case $\alpha < 1$, where one has indeed two non-overlapping regions separated by the straight-line $r = |\eta|$.

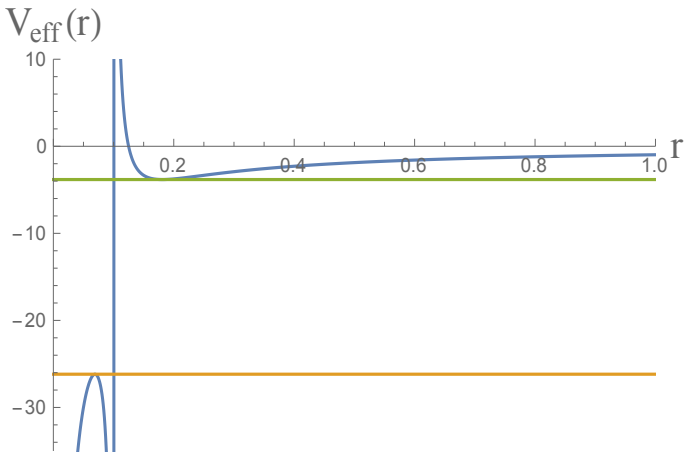


Figure 5.: potential $V_{\text{eff}}(r)$ for $\alpha = \frac{4}{5}$. The straight horizontal lines represent the Energies associated to the critical points.

- In the external domain $r > |\eta|$ the conformal factor is positive. We have a Riemannian manifold with non constant curvature and there will be closed trajectories whenever the energy belongs to the (negative) open interval $\left(0, V_{\text{eff}}(r_+)\right)$, where $V_{\text{eff}}(r_+) = \frac{-k}{\lambda(1+\sqrt{1-\alpha})^2}$ is the value of the effective potential at the critical point $r_+ \doteq \lambda(1+\sqrt{1-\alpha})$ (green line in Figure 5).
- In the internal domain $r < |\eta|$ the conformal factor is *negative* entailing that the kinetic energy is also negative. In order to get a physically significant system we are naturally led to define in this region a new Hamiltonian:

$$\begin{cases} \widetilde{H}_\eta \doteq -H_\eta = \dfrac{r}{|\eta|-r}\dfrac{p_r^2}{2m} + \widetilde{V}_{\text{eff}}(r) \\ \widetilde{V}_{\text{eff}}(r) \doteq \dfrac{l^2}{2mr(|\eta|-r)} - \dfrac{k}{|\eta|-r}\,. \end{cases} \tag{4.386}$$

5 We point out that in the case $l = 0$ the motion takes place on straight lines implying that no two-dimensional closed orbits are allowed.

As shown by Figure 6, after that transformation in the region $0 < r < |\eta|$, *the effective potential acquires a typical "confining" shape*. There will be closed orbits for any positive energy higher than $\tilde{V}_{eff}(r_-)$, where $r_- \doteq \lambda(1 - \sqrt{1-\alpha})$. We point out that the minimum of the potential is a monotonically decreasing function of $|\eta|$, so that it goes to infinity as $|\eta|$ goes to zero: the parameter η governs the shape of the potential.

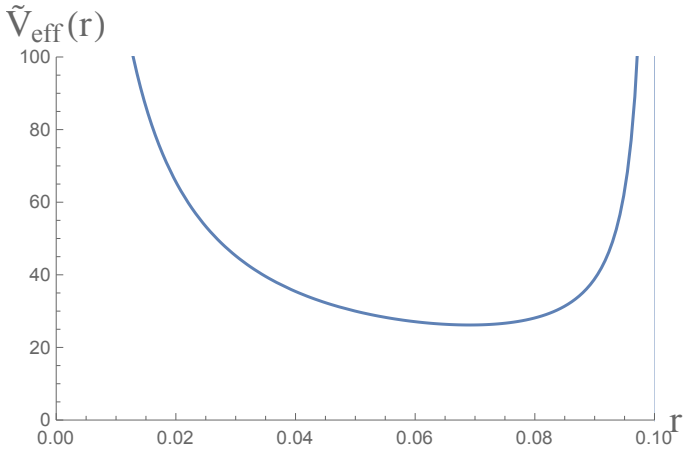


Figure 6.: potential $V_{\text{eff}}(r)$ for the new Hamiltonian system, i.e. $\tilde{V}_{\text{eff}}(r) \doteq -V_{\text{eff}}(r)$ calculated for $\alpha = \frac{4}{5}$. The latter is contained into the region $0 < r < |\eta|$.

In both regions, the we can characterize the solution $t(r)$. In the domain $r > |\eta|$, we have to change the sign of the parameter η in the formula (4.383). Conversely, in the domain $0 < r < |\eta|$, we should factorize the new Hamiltonian $\tilde{H}$ following the same steps as in the case $\eta > 0$.

case $\alpha > 1$

Here the dynamics is certainly less interesting because no closed orbits will come out.

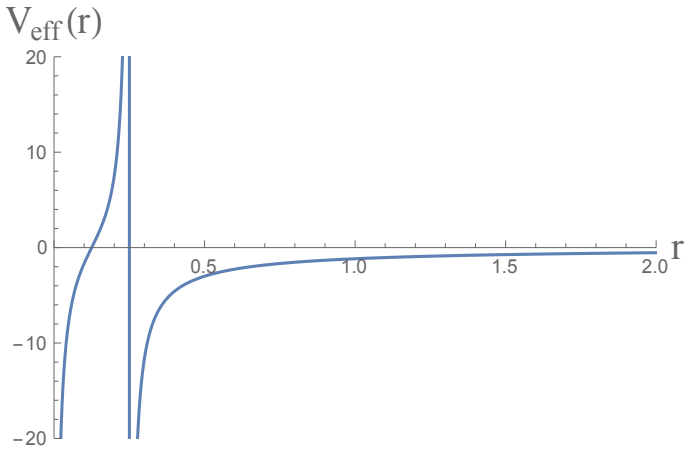


Figure 7.: potential $V_{\text{eff}}(r)$ calculated for $\alpha = 2$.

In the region $r > |\eta|$ the effective potential will be proportional to $-(r-|\eta|)^{-1}$ while in the bounded region $0 < r < |\eta|$ its image is the full real line and furthermore it exhibits an inflection point at the value $\bar{r} = \lambda\left(1-(\alpha-1)^{1/3}+(\alpha-1)^{2/3}\right)$.

case $\alpha = 1$

When the parameter $\alpha = 1$ the situation is totally different from the previous cases and things are definetely less clear. It looks like that the singularity could be overcome, in the sense that it does not affect the potential term. By the way, a plot of the effective potential for this particular value of the parameter α, i.e. $V_{\text{eff}}(r) = -\frac{k}{r}$, shows that the distinction between the two regions ($r > |\eta|$, $r < |\eta|$) seems to vanish, since we have a single continuous line with a monotonically increasing behaviour (see Figure 8).
However, this is not the case because the singularity still appears in the kinetic term, that turns out to be divergent at the boundary $r = |\eta|$. In any case, the dynamics is not so interesting since no closed orbits are allowed.

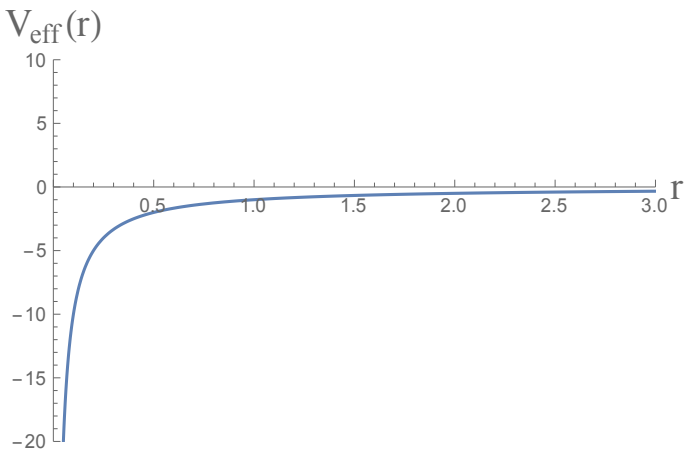


Figure 8.: potential $V_{\text{eff}}(r)$ calculated for $\alpha = 1$. The centrifugal and gravitational contributions add up to give the behaviour $-kr^{-1}$. In this case the singularity in the effective potential disappears.

4.5 THE CLASSICAL D-III OSCILLATOR: FACTORIZATION, CLASSICAL SGA AND SOLUTION OF THE MOTION

4.5.1 The undeformed case: isotropic harmonic oscillator

In this section we want to provide an analogous investigation for the D-III system. In particular, before using the machinery of the classical factorization method to tackle the $\lambda \neq 0$ case, we will firstly review the results concerning the Euclidean case, in order to check our future results in the flat ($\lambda \to 0$) limit. Therefore, let us consider the one-dimensiona radial Hamiltonian:

$$H_0(r,p_r) = T_0(r,p_r) + V_0(r) = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2\,, \tag{5.387}$$

where r is the radial coordinate and p_r is the radial momentum related to the particle of mass m, frequency ω and conserved angular momentum $l > 0$. The effective potential is so given by:

$$V_0(r) \equiv V_{0,\text{eff}}(r) = \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \,. \tag{5.388}$$

Clearly, in order to have closed orbits, the values of the energy have to be confined in the region $V_{0,\text{eff}}(r_{\min}) < E < \infty$, where $V_{0,\text{eff}}(r_{\min}) = l\omega$ is the minimum of the effective potential, i.e. its value calculated at $r_{\min} = \sqrt{\frac{l}{m\omega}}$ (see l.h.s. of Figure 9). Now, we proceed by factorizing the Hamiltonian (5.387) as follows:

$$p_r^2 r^2 + m^2\omega^2 r^4 - 2mr^2 H_0 = A_0^+ A_0^- + \gamma(H_0) = -l^2 \,, \tag{5.389}$$

where $A_0^\pm$ are unknown functions to be determined. In particular, taking into account (5.389), we obtain:

$$A_0^\pm = \mp i r p_r + m\omega r^2 - \frac{H_0}{\omega} \,, \tag{5.390}$$

and then:

$$\gamma(H_0) = -\frac{H_0^2}{\omega^2}. \tag{5.391}$$

The Hamiltonian H_0 and $A_0^\pm$ are functions of the canonical coordinates (r, p_r) and they have to close as usual the following non-linear Poisson algebra:

$$\{H_0, A_0^\pm\} = \mp i\alpha(H_0) A_0^\pm \,, \qquad \{A_0^+, A_0^-\} = i\beta(H_0) \,.$$

In particular, using the expressions (5.390), we obtain that

$$\alpha(H_0) = 2\omega \,, \qquad \beta(H_0) = \frac{4H_0}{\omega} \,.$$

Summarizing, we have the following relations:

$$\{H_0, A_0^\pm\} = \mp i2\omega A_0^\pm \,, \qquad \{A_0^+, A_0^-\} = i\frac{4H_0}{\omega} \,. \tag{5.392}$$

At this point, we can introduce the two *time-dependent constants of motion*:

$$Q_0^\pm = A_0^\pm e^{\mp i\alpha(H_0)t} \,,$$

such that:

$$\frac{\mathrm{d}Q_0^\pm}{\mathrm{d}t} = \{Q_0^\pm, H_0\} + \frac{\partial Q_0^\pm}{\partial t} = 0 \,.$$

As we know, these dynamical variables allow us to determine the motion. In this case it holds:

$$\left(\mp i r p_r + m\omega r^2 - \frac{H_0}{\omega}\right) e^{\mp 2i\,\omega t} = q_0\, e^{\pm i\theta_0} \,. \tag{5.393}$$

By imposing (5.389), taking into accounto (5.391), that is: $A_0^+ A_0^- - \frac{H_0^2}{\omega^2} = q_0^2 - \frac{H_0^2}{\omega^2} = -l^2$, we find:

$$\begin{cases} -ir p_r + m\omega r^2 - \dfrac{H_0}{\omega} = q_0\, e^{i\left(2\,\omega t + \theta_0\right)} \,, \\ ir p_r + m\omega r^2 - \dfrac{H_0}{\omega} = q_0\, e^{-i\left(2\,\omega t + \theta_0\right)} \,, \end{cases} \tag{5.394}$$

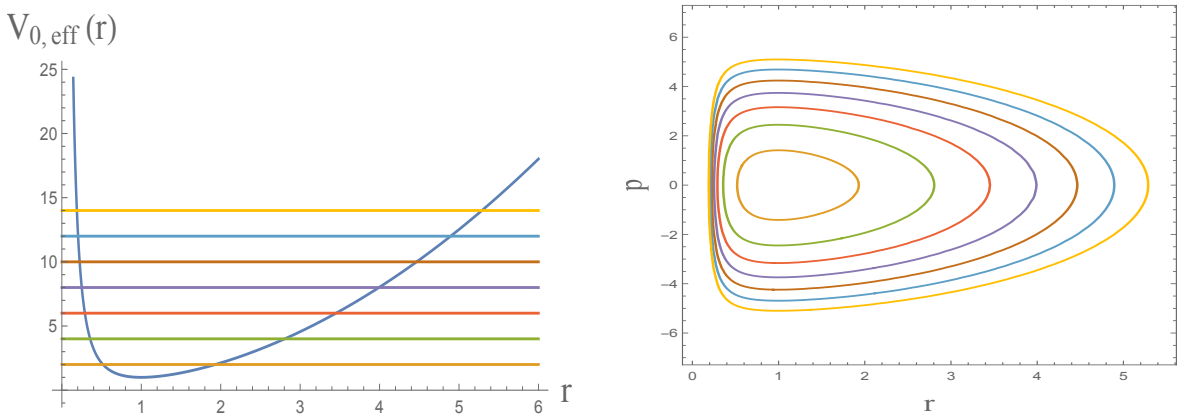


Figure 9.: effective potential $V_{0,\text{eff}}(r)$ (5.388) and phase plane (r, p_r) calculated for $m = \omega = l = 1$ and $E = 2, 4, 6, 8, 10, 12, 14$ (in appropriate units).

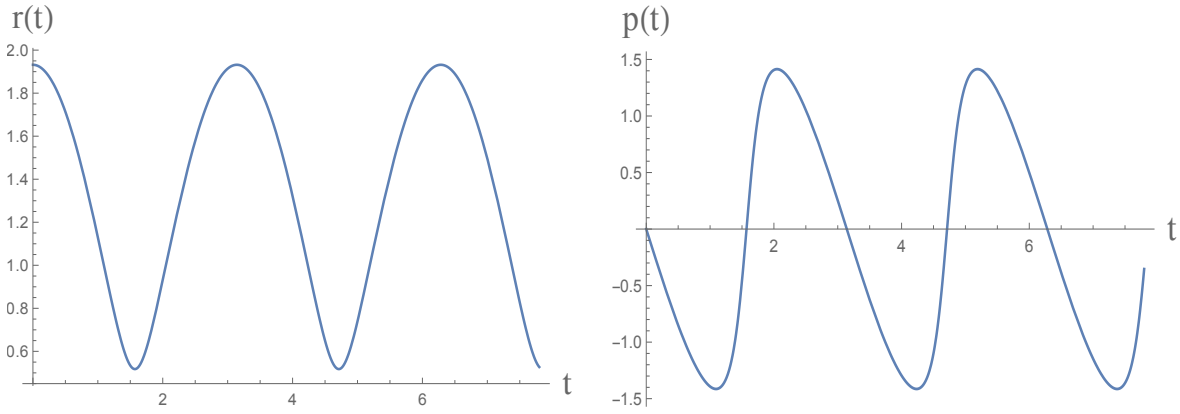


Figure 10.: trajectory $r(t)$ and momentum $p_r(t)$ calculated for $m = \omega = l = 1$ and $E = 2$.

where $q_0 \doteq \sqrt{-l^2 + \frac{H_0^2}{\omega^2}}|_{H_0=E_0}$. Now, since the Hamiltonian does not depend explicitly on time, it is a constant of motion, i.e. the energy of the system $H_0 = E_0$. Therefore, summing and subtracting (5.394) we obtain (on the level surface $H_0 = E_0$):

$$\begin{cases} m\omega r^2 - \dfrac{E_0}{\omega} = q_0 \cos(2\omega t + \theta_0) \ , \\ r p_r = -q_0 \sin(2\omega t + \theta_0) \ . \end{cases} \tag{5.395}$$

Finally, using (5.395) we are able to write r and p as functions of t:

$$\begin{cases} r(t) = \sqrt{\dfrac{E_0}{m\omega^2} + \dfrac{q_0}{m\omega}\cos(2\omega t + \theta_0)} \ , \\ p_r(t) = -\dfrac{q_0 \sin(2\omega t + \theta_0)}{\sqrt{\dfrac{E_0}{m\omega^2} + \dfrac{q_0}{m\omega}\cos(2\omega t + \theta_0)}} \ . \end{cases} \tag{5.396}$$

In this way we have found $r(t)$ and $p_r(t)$ and then the motion has been fully determined by means of the classical factorization method. In the r.h.s. of Figure 9 the phase plane is depicted, while in Figure 10 the curves (5.396) are represented.

In the following, we shall deal with the deformed case using the same procedure. Clearly, the results just obtained in this section have to be recovered in the Euclidean (i.e. $\lambda \to 0$) limit.

4.5.2 The deformed $\lambda > 0$ case: D-III oscillator

In this section we focus our attention on the D-III oscillator Hamiltonian (3.317) written as a one-dimensional radial system, namely:

$$H(r,p_r) = T_\lambda(r,p_r) + V_\lambda(r) = \frac{p_r^2}{2m(1+\lambda r^2)} + \frac{l^2}{2mr^2(1+\lambda r^2)} + \frac{m\omega^2 r^2}{2(1+\lambda r^2)} = \mathcal{F}_\lambda(r) H_0\,, \tag{5.397}$$

where m, ω and l are positive constants, λ is the deformation parameter, H_0 is the 'undeformed' isotropic oscillator Hamiltonian (5.387) and

$$V_\lambda(r) \equiv V_{\text{eff}}(r) = \frac{l^2}{2mr^2(1+\lambda r^2)} + \frac{m\omega^2 r^2}{2(1+\lambda r^2)}\,, \qquad \mathcal{F}_\lambda(r) \doteq \frac{1}{1+\lambda r^2}\,. \tag{5.398}$$

Multiplying both sides of (5.397) by $r^2(1+\lambda r^2) \neq 0$ we get:

$$r^2(1+\lambda r^2)H_\lambda = r^2\left(\frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2\right) = \frac{1}{2m}(r^2 p_r^2 + l^2 + m^2\omega^2 r^4). \tag{5.399}$$

Now, as it has been done in the previous section for the undeformed case, we can factorize (5.399) as:

$$r^2 p_r^2 + m^2 r^4\left(\omega^2 - \frac{2\lambda}{m}H_\lambda\right) - 2mr^2 H_\lambda = A_\lambda^+ A_\lambda^- + \gamma(H_\lambda) = -l^2\,, \tag{5.400}$$

where $A_\lambda^\pm = A_\lambda^\pm(r,p_r)$ are unknown functions of r, p_r. We make the following *ansatz* for $A_\lambda^\pm$:

$$A_\lambda^\pm = \left(\mp i r p_r + mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}H_\lambda} - \frac{H_\lambda}{\sqrt{\omega^2 - \frac{2\lambda}{m}H_\lambda}}\right) e^{\pm f_\lambda(r,p_r)}\,.$$

Once again, we have to require the closure of the Poisson algebra generated by H_λ and $A_\lambda^\pm$:

$$\begin{aligned} \{H_\lambda, A_\lambda^\pm\} &= \mp i\alpha(H_\lambda)A_\lambda^\pm \\ \{A_\lambda^+, A_\lambda^-\} &= i\beta(H_\lambda)\,, \end{aligned} \tag{5.401}$$

where the functions α, β have to be determined. Inserting $A_\lambda^\pm$ in (5.400) we obtain:

$$\gamma(H_\lambda) = -\frac{H_\lambda^2}{\omega^2 - \frac{2\lambda}{m}H_\lambda}\,, \tag{5.402}$$

and requiring that $A_\lambda^\pm$ obey the proper Poisson brackets we arrive at:

$$f_\lambda(r,p_r) = -\frac{i\lambda r p_r\sqrt{\omega^2 - \frac{2\lambda}{m}H_\lambda}}{m(\omega^2 - \frac{\lambda}{m}H_\lambda)}\,, \qquad \alpha(H_\lambda) = \frac{2(\omega^2 - \frac{2\lambda}{m}H_\lambda)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}H_\lambda}\,, \qquad \beta(H_\lambda) = \frac{4H_\lambda}{\sqrt{\omega^2 - \frac{2\lambda}{m}H_\lambda}}\,.$$

Hence we find that:

$$A_\lambda^\pm = \left(\mp i r p_r + m r^2 \sqrt{\omega^2 - \frac{2\lambda}{m} H_\lambda} - \frac{H_\lambda}{\sqrt{\omega^2 - \frac{2\lambda}{m} H_\lambda}}\right) \exp\left\{\mp \frac{i\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m} H_\lambda}}{m(\omega^2 - \frac{\lambda}{m} H_\lambda)}\right\},$$

$$\{H_\lambda, A_\lambda^\pm\} = \mp i \frac{2(\omega^2 - \frac{2\lambda}{m} H_\lambda)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m} H_\lambda} A_\lambda^\pm \,, \qquad \{A_\lambda^+, A_\lambda^-\} = i \frac{4 H_\lambda}{\sqrt{\omega^2 - \frac{2\lambda}{m} H_\lambda}} \,.$$

We notice that in the limit $\lambda \to 0$ one gets back the undeformed Poisson algebra (5.392) as expected. We also point out that the requirement $\omega^2 - \frac{2\lambda}{m} H_\lambda > 0$ implies the upper bound $E < \frac{m\omega^2}{2\lambda}$. Moreover, for bounded motion, the energy has to be greater than the minimum of the effective potential $V_{\text{eff}}(r)$. The latter turns out to be:

$$V_{\text{eff}}(r_{\min}) = \frac{l^2}{m}\left(\sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}} - \lambda\right) \,, \qquad r_{\min}^2 = \frac{l^2}{m^2\omega^2}\left(\lambda + \sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}}\right) \,.$$

As a consequence the energy belongs to the interval (see l.h.s. of Figure 11 and Figure 12):

$$V_{\text{eff}}(r_{\min}) < E < \frac{m\omega^2}{2\lambda} \,. \tag{5.403}$$

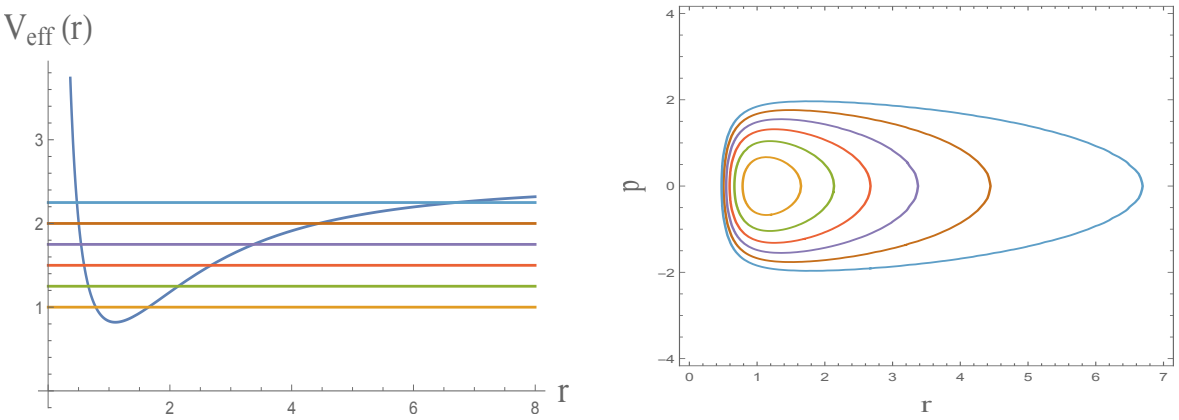


Figure 11.: effective potential $V_{\text{eff}}(r)$ (5.397) and phase plane (r, p_r) calculated for $m = \omega = l = 1$ and $E = 1.00, 1.25, 1.50, 1.75, 2.00, 2.25$. The deformation parameter is $\lambda = 0.20$.

Next, as usual, by defining the two *time-dependent constants of the motion* $Q_\lambda^\pm$ we arrive to the equations (*on-shell*):

$$\left(\mp i r p_r + m r^2 \sqrt{\omega^2 - \frac{2\lambda}{m} E} - \frac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m} E}}\right) \exp\left\{\mp i \left(\frac{\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m} E}}{m(\omega^2 - \frac{\lambda}{m} E)} + \frac{2(\omega^2 - \frac{2\lambda}{m} E)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m} E} t\right)\right\}$$
$$= q\, e^{\pm i\theta} \,,$$

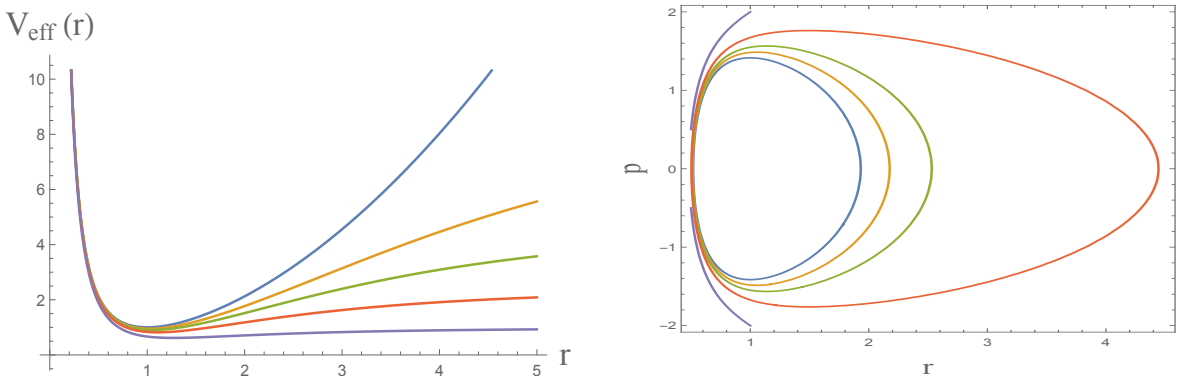


Figure 12.: Effective potential $V_{\text{eff}}(r)$ (5.397) and phase plane (r, p_r) calculated for $m = \omega = l = 1$ and $E = 2$. The deformation parameter take the values $\lambda = 0, 0.05, 0.10, 0.20, 0.50$.

or else:

$$\begin{cases} -irp_r + mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}E} - \dfrac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q\exp\left\{ i\left(\dfrac{\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \dfrac{2\left(\omega^2 - \frac{2\lambda}{m}E\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E}\, t + \theta \right) \right\}, \\ +irp_r + mr^2\sqrt{\omega^2 - \frac{2\lambda}{m}E} - \dfrac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q\exp\left\{ -i\left(\dfrac{\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \dfrac{2\left(\omega^2 - \frac{2\lambda}{m}E\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E}\, t + \theta \right) \right\}, \end{cases} \tag{5.404}$$

where now $q = q(E) = \sqrt{-l^2 + \frac{E^2}{\omega^2 - \frac{2\lambda}{m}E}}$. Once again, after easy manipulations we get:

$$\begin{cases} mr^2\sqrt{\omega^2 - \dfrac{2\lambda}{m}E} - \dfrac{E}{\sqrt{\omega^2 - \frac{2\lambda}{m}E}} = q\cos\left(\dfrac{\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \dfrac{2\left(\omega^2 - \frac{2\lambda}{m}E\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E}\, t + \theta \right), \\ rp_r = -q\sin\left(\dfrac{\lambda r p_r \sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)} + \dfrac{2\left(\omega^2 - \frac{2\lambda}{m}E\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E}\, t + \theta \right). \end{cases}$$

Taking the sum of the square of these two equations we recover (5.400) restricted to the level surface $H_\lambda = E$. Finally, thanks to the above relations, we are able to find t as a function of r:

$$t(r) = \frac{1}{\Omega_\lambda(E)}\left[\arccos\left(\frac{mr^2(\omega^2 - \frac{2\lambda}{m}E) - E}{q\sqrt{\omega^2 - \frac{2\lambda}{m}E}} \right) - \frac{\lambda\sqrt{\omega^2 - \frac{2\lambda}{m}E}}{m(\omega^2 - \frac{\lambda}{m}E)}\sqrt{2mEr^2 - l^2 - m^2r^4\left(\omega^2 - \frac{2\lambda}{m}E\right)} - \theta \right], \tag{5.405}$$

where in this case we have obtained the angular frequency:

$$\Omega_\lambda(E) = 2\frac{\left(\omega^2 - \frac{2\lambda}{m}E\right)^{\frac{3}{2}}}{\omega^2 - \frac{\lambda}{m}E} \equiv \alpha(E). \tag{5.406}$$

In the r.h.s. of Figures 11 and 12 some pictures of the resulting phase space can be found. At this point, in total analogy to the Taub-NUT case, by introducing the quantities:

$$a_\lambda^2 \doteq \frac{r_{\lambda+}^2 + r_{\lambda-}^2}{2} = \frac{E}{m\left(\omega^2 - \frac{2\lambda}{m}E\right)}, \qquad \epsilon_\lambda \doteq \sqrt{1 - \frac{\left(\omega^2 - \frac{2\lambda}{m}E\right)l^2}{E^2}},$$

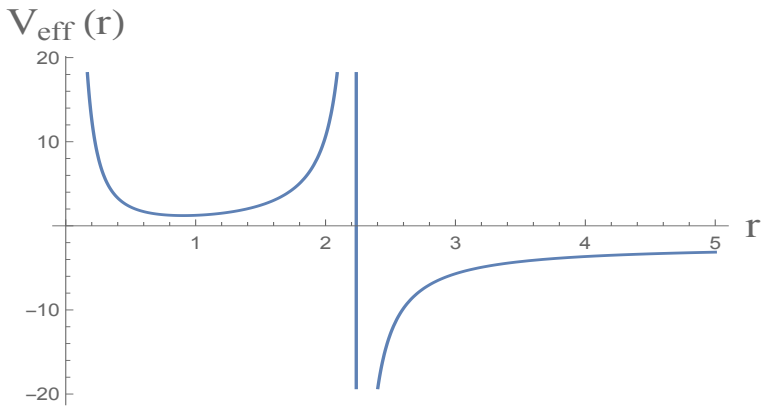


Figure 13.: effective potential $V_{\text{eff}}(r)$ (5.408) calculated for $m = \omega = l = 1$ and $\lambda = -0.20$. The singularity is located at the point $r_c = \sqrt{5} \simeq 2.24$.

where ϵ_λ is a parameter related to the shape of the ellipse and $r^2_{\lambda\pm}$ are the solutions of the quadratic equation obtained by setting $p_r = 0$ in (5.400), we are able to rewrite the equation (5.405) in the following form:

$$\Omega_\lambda(E)t(r) + \theta = \arccos\left[-\frac{1}{\epsilon_\lambda}\left(1 - \left(\frac{r}{a_\lambda}\right)^2\right)\right] - \frac{\lambda a_\lambda^2}{1+\lambda a_\lambda^2}\sqrt{\epsilon_\lambda^2 - \left[1 - \left(\frac{r}{a_\lambda}\right)^2\right]^2}\,, \tag{5.407}$$

whose structure is very similar to the one obtained for the classical Taub-NUT system. In the limit $\lambda \to 0$ the equation (5.407) can be inverted to obtain the trajectory $r(t)$ given in (5.396), showing that it just represents the λ-deformation of the solution of the Euclidean isotropic harmonic oscillator.

4.5.3 The deformed $\lambda < 0$ case

In this case the deformation parameter $\lambda = -|\lambda|$ is negative and the Hamiltonian can be written as follows:

$$H_\lambda(r, p_r) = T_\lambda(r,p) + V_\lambda(r) = \frac{p_r^2}{2m(1-|\lambda|r^2)} + \frac{l^2}{2mr^2(1-|\lambda|r^2)} + \frac{m\omega^2 r^2}{2(1-|\lambda|r^2)}\,. \tag{5.408}$$

where the effective potential is now given by $V_\lambda(r) \equiv V_{\text{eff}}(r)|_{\lambda=-|\lambda|}$. As expected, it exhibits a singularity at the point $r_c \doteq \frac{1}{\sqrt{|\lambda|}}$. As a matter of fact, owing to the above singularity, the domain of definition of the effective potential splits in two subdomains. The first one, corresponding to the punctured open ball $0 < r < r_c$, is characterized by a positive kinetic energy term in the Hamiltonian and the effective potential acquires a typical confining shape. Viceversa, the second one, $r_c < r < \infty$, is characterized by a negative kinetic energy and the effective potential has no critical points (see figure 13 and l.h.s. of figure 14).

The interesting region is the open set $0 < r < r_c$. Indeed, in this case there will be only closed orbits for energy values $V_{\text{eff}}(r_{\text{min}}) < E < \infty$, where:

$$V_{\text{eff}}(r_{\text{min}}) = \frac{l^2}{m}\left(\sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}} + |\lambda|\right), \qquad r^2_{\text{min}} = \frac{l^2}{m^2\omega^2}\left(-|\lambda| + \sqrt{\lambda^2 + \frac{m^2\omega^2}{l^2}}\right).$$

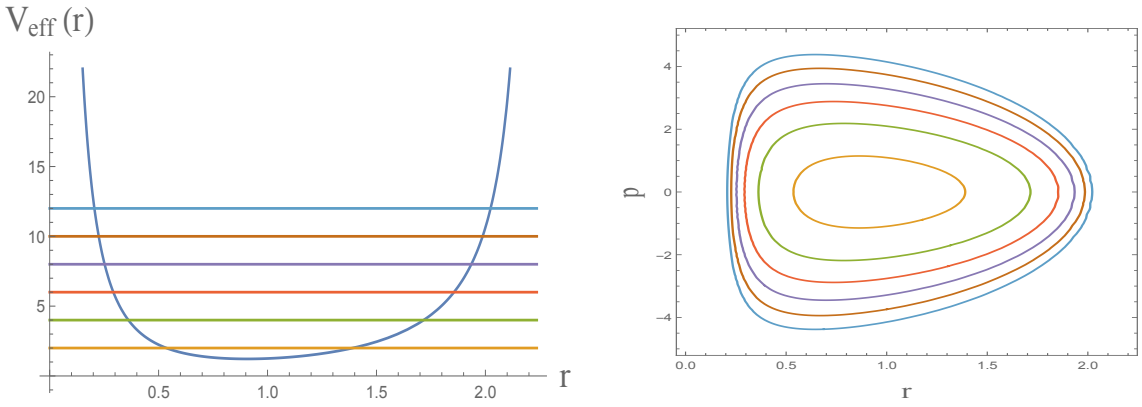


Figure 14.: effective potential $V_{\text{eff}}(r)$ (5.408) and phase plane (r, p_r) calculated for $m = \omega = l = 1$ and $E = 2, 4, 6, 8, 10, 12$ in the region $0 < r < r_c$. The deformation parameter is fixed at the value $\lambda = -0.2$.

In that bounded region the solution can be obtained by considering the negative sign of λ in the entire construction. Clearly, since the term $\omega^2 - \frac{2\lambda}{m}E = \omega^2 + \frac{2|\lambda|}{m}E$ is always positive, no further restrictions on the energy will apply, i.e. the energy has no upper bound, as expected. For this case the phase plane is depicted in the r.h.s. of Figure 14.

Viceversa, in order to obtain the solution in the unbounded region $r > r_c$, as we already noticed in the case of the Taub-NUT system, we should proceed with the factorization of the new Hamiltonian $\tilde{H} = -H$, the latter being defined with an overall change of sign. This allows us to restore a positive kinetic energy and then to obtain a physically meaningful Hamiltonian (see also [50]). However, this case is not so interesting since no closed orbits are allowed. This conclude the analysis of the classical cases.

So far, we have discussed the two MS systems in the classical case by analysing their one-dimensional radial dynamics in terms of the classical factorization method. In what follows, to complete the analysis, we will discuss the quantum case. In particular, we will show that their exact solvability is ensured thanks to the fact that their defining potentials are both shape invariants (in their underlying conformally flat spaces). In particular, our goal is to solve the spectral problem for both systems by means of the quantum factorization method that, if a shape invariance condition for the potentials is satisfied, leads directly to an algebraic construction of both the spectrum and the eigenfunctions, as explained in the literature dedicated to the supersymmetric approach to quantum mechanics [16, 17].

4.6 THE QUANTUM TAUB–NUT AND D–III SYSTEMS: FACTORIZATION AND SHAPE INVARIANCE

As it is well known, for models that are defined on curved spaces, a problem to overcome in order to deal with the quantum case is related to the order ambiguity arising in the quantization of the kinetic terms. This task can be faced by applying three different quantization procedures [51, 52]: the *Schrödinger* quantization, the *Laplace–Beltrami* quantization (which makes use of the Laplace operator on curved spaces) and a *position dependent mass* (PDM)

quantization. We will make use of the first one, the Schrödinger quantization. In particular, it has been shown [48] that by using this prescription to quantize the system H_η, the quantum Taub-NUT keeps the maximal superintegrability property. This means that it is endowed with $(2N-1)$ algebraically independent operators that commute with H_η. This has been also proved for the quantum version of the Hamiltonian H_λ defining the D-III system [52, 53]. The other two approaches, the Laplace-Beltrami and PDM quantizations, also lead to MS quantum Hamiltonians once an additional quantum potential term is added to the initial quantum Hamiltonian, and such potential terms are related through gauge transformations to the Schrödinger quantization [51]. Our aim here, is to look at the quantum problem from the point of view of SUSYQM: we will factorize the two quantum Hamiltonians (in hyperspherical coordinates) and we will solve them by means of the shape invariance: these two systems are exactly solvable because their defining potentials are shape invariant on their corresponding space of nonconstant curvature. As a matter of fact, the coupling constant metamorphosis (CCM) plays a central role in the quantum case, since it allows us to simplify drastically the spectral problem.

4.6.1 The quantum Taub–NUT: superintegrability

Let us consider the quantum Taub-NUT Hamiltonian operator $\hat{H}_\eta$ given by [48]:

$$\hat{H}_\eta = \frac{|\hat{\mathbf{x}}|}{2(\eta+|\hat{\mathbf{x}}|)}\,\hat{\mathbf{p}}^2 - \frac{k}{\eta+|\hat{\mathbf{x}}|} = -\frac{\hbar^2}{2}\frac{|\mathbf{x}|}{|\mathbf{x}|+\eta}\,\Delta_{\mathbf{x}} - \frac{k}{|\mathbf{x}|+\eta}, \tag{6.1}$$

where we are using the standard definitions for the quantum positions $\hat{\mathbf{x}}$ and momenta $\hat{\mathbf{p}}$ operators:

$$\hat{x}_i\,\psi(\mathbf{x}) = x_i\,\psi(\mathbf{x}), \qquad \hat{p}_i\,\psi(\mathbf{x}) = -i\hbar\partial_{x_i}\psi(\mathbf{x}), \qquad [\hat{x}_i,\hat{p}_j] = i\hbar\delta_{ij}\hat{\mathbb{1}}, \qquad i,j=1,\ldots,N,$$

together with the usual conventions:

$$\nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_N}\right), \qquad \Delta_{\mathbf{x}} = \nabla_{\mathbf{x}}^2 = \sum_{i=1}^N \frac{\partial^2}{\partial^2 x_i}, \qquad \mathbf{x}\cdot\nabla_{\mathbf{x}} = \sum_{i=1}^N x_i\frac{\partial}{\partial x_i}.$$

It turns out that for any value of η and k it is verified that:

1. $\hat{H}_\eta$ commutes with the following operators ($m=2,\ldots,N; i=1,\ldots,N$):

$$\hat{C}^{[m]} = \sum_{1\le i<j\le m}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2, \qquad \hat{C}_{[m]} = \sum_{N-m\le i<j\le N}(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2, \tag{6.2}$$

$$\hat{\mathcal{R}}_{\eta i} = \frac{1}{2}\sum_{j=1}^N(\hat{x}_j\hat{p}_i - \hat{x}_i\hat{p}_j)\hat{p}_j + \frac{1}{2}\sum_{j=1}^N\hat{p}_j(\hat{x}_j\hat{p}_i - \hat{x}_i\hat{p}_j) + \frac{\hat{x}_i}{|\hat{\mathbf{x}}|}\left(k+\eta\hat{H}_\eta\right), \tag{6.3}$$

   where $\hat{C}^{[N]} = \hat{C}_{[N]} \equiv \hat{\mathbf{L}}^2$ is the total quantum angular momentum. Furthermore:

$$\hat{\boldsymbol{\mathcal{R}}}_{\boldsymbol{\eta}}{}^2 = \sum_{i=1}^N \hat{\mathcal{R}}_{\eta i}^2 = 2\hat{H}_\eta\left(\hat{\mathbf{L}}^2 + \hbar^2\frac{(N-1)^2}{4}\right) + \left(k+\eta\hat{H}_\eta\right)^2. \tag{6.4}$$

   Each of the two sets $\{\hat{H}_\eta,\hat{C}^{[m]}\}$, $\{\hat{H}_\eta,\hat{C}_{[m]}\}$ $(m=2,\ldots,N)$ is formed by N algebraically independent commuting operators. Moreover, the set $\{\hat{H}_\eta,\hat{C}^{[m]},\hat{C}_{[m]},\hat{\mathcal{R}}_{\eta i}\}$ for $m=2,\ldots,N$, at a fixed index i, is formed by $2N-1$ algebraically independent operators.

2. $\hat{H}_\eta$ is formally self-adjoint on the Hilbert space $L^2(\mathcal{M}^N)$, endowed with the scalar product:

$$\langle \Psi | \Phi \rangle_\eta = \int_{\mathcal{M}^N} \Psi^*(\mathbf{x}) \Phi(\mathbf{x}) \left(1 + \frac{\eta}{|\mathbf{x}|}\right) \mathrm{d}\mathbf{x}. \tag{6.5}$$

We remark that all the above results are well defined for any value of the parameters η and k. Nevertheless, the explicit solution of the quantum Hamiltonian depends on the sign of both of them. In particular, hereafter we shall restrict our considerations for $\eta > 0$, implying $r \in (0, \infty)$ and $k > 0$, which corresponds to the proper curved hydrogen atom potential.

4.6.2 The quantum Taub-NUT: separation in hyperspherical coordinates, factorization and shape invariance

Starting from the Hamiltonian (6.1), and using hyperspherical coordinates, together with:

$$\hat{p}_r = -i\hbar \frac{\partial}{\partial r}, \qquad \hat{p}_{\theta_j} = -i\hbar \frac{\partial}{\partial \theta_j}, \qquad j = 1, \ldots, N-1, \tag{6.6}$$

we obtain the following quantum radial Hamiltonian:

$$\hat{H}_\eta = \frac{r}{2(r+\eta)} \left(\frac{1}{r^{N-1}} \hat{p}_r \, r^{N-1} \, \hat{p}_r + \frac{\hat{\mathbf{L}}^2}{r^2} - \frac{2k}{r} \right), \tag{6.7}$$

where $\hat{\mathbf{L}}^2$ is the square of the total quantum angular momentum operator, now given by:

$$\hat{\mathbf{L}}^2 = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k} \right) \frac{1}{(\sin \theta_j)^{N-1-j}} \, \hat{p}_{\theta_j} (\sin \theta_j)^{N-1-j} \, \hat{p}_{\theta_j}. \tag{6.8}$$

After reordering terms and by inserting the differential operators (6.6) within the Hamiltonian (6.7), we arrive at the following Schrödinger equation:

$$\frac{r}{2(r+\eta)} \left(-\hbar^2 \partial_r^2 - \frac{\hbar^2 (N-1)}{r} \partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} - \frac{2k}{r} \right) \Psi(r, \boldsymbol{\theta}) = E \, \Psi(r, \boldsymbol{\theta}), \tag{6.9}$$

where $\boldsymbol{\theta} \doteq (\theta_1, \ldots, \theta_{N-1})$. By taking into account that the hyperspherical harmonics $Y(\boldsymbol{\theta})$ fulfill the eigenvalue equation:

$$\hat{\mathbf{L}}^2 Y(\boldsymbol{\theta}) = \hat{C}_{[N]} Y(\boldsymbol{\theta}) = \hbar^2 l (l + N - 2) \, Y(\boldsymbol{\theta}), \quad l = 0, 1, 2 \ldots,$$

where l is the total angular momentum quantum number, the equation (6.9) admits a complete set of factorized solutions of the form:

$$\Psi(r, \boldsymbol{\theta}) = \Phi(r) Y(\boldsymbol{\theta}),$$

and, moreover:

$$\hat{C}_{[m]} \Psi = c_m \Psi, \quad m = 2, \ldots, N,$$

where c_m are the eigenvalues of the operators $\hat{C}_{[m]}$ $(m = 2, \ldots, N)$ defined in (6.2), such that c_m are related to the $(N-1)$ quantum numbers of the angular observables in the form:

$$c_k \leftrightarrow l_{k-1}, \quad k = 2, \ldots, N-1, \qquad c_N \leftrightarrow l.$$

Therefore, we can write:

$$Y(\boldsymbol{\theta}) \equiv Y^{c_N}_{c_{N-1},\ldots,c_2}(\theta_1,\theta_2,\ldots,\theta_{N-1}) \equiv Y^{l}_{l_{N-2},\ldots,l_1}(\theta_1,\theta_2,\ldots,\theta_{N-1}),$$

and the radial Schrödinger equation provided by $\hat{H}_\eta$ reads:

$$\frac{r}{r+\eta}\left(-\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+N-2)}{r^2}\right)-\frac{k}{r}\right)\Phi(r) = E\,\Phi(r). \tag{6.10}$$

After performing a "coupling constant metamorphosis", eq. (6.10) can be rewritten as:

$$\left(-\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+N-2)}{r^2}\right)-\frac{k_\eta}{r}\right)\Phi(r) = E\,\Phi(r)\,, \tag{6.11}$$

where we defined $k_\eta \doteq k+\eta E > 0$. This is the equation we want to solve for $E<0$ with $k,\eta>0$ (bound states). To this aim, we will apply standard methods from SUSYQM [16, 17]. In particular, our goal is to factorize the one-dimensional radial Hamiltonian (6.11) in terms of a pair of ladder operators $\hat{A}^{(\alpha)}, \hat{A}^{\dagger(\alpha)}$ (depending on a shape invariance parameter α that has to be found) in order to construct the solution algebraically.
Let us begin our analysis by performing the following "gauge" transformation:

$$\underbrace{w(r)\hat{H}_\eta w^{-1}(r)}_{\hat{H}_{\eta,g}}\underbrace{w(r)\Phi(r)}_{\Phi_g(r)} = E\,\underbrace{w(r)\Phi(r)}_{\Phi_g(r)}, \tag{6.12}$$

with $w(r)\doteq r^{\frac{N-1}{2}}$. This leads to the following gauged Hamiltonian operator:

$$\hat{H}_{\eta,g} = -\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\hbar^2\left(l(l+N-2)+{}^{(N-1)(N-3)}/_{4}\right)}{2r^2}-\frac{k_\eta}{r}\,. \tag{6.13}$$

Thanks to the tranformation (6.12), we got rid of the first derivative term and so we can use the standard SUSYQM techniques in order to factorize the Hamiltonian. As a matter of fact, we are formally dealing with a one-dimensional problem characterized by radial the effective potential (depending on the energy and the deformation parameter):

$$V_{\eta,g}(r) \doteq \frac{\hbar^2\left(l(l+N-2)+{}^{(N-1)(N-3)}/_{4}\right)}{2r^2}-\frac{k_\eta}{r}\,. \tag{6.14}$$

Thus, after subtracting the Hamiltonian of the energy associated to the ground state, let's say $E_{0,\eta}$, we can construct the ladder operators of the gauged Hamiltonian by imposing the Riccati-type equation for the potential $V_{\eta,g}(r)$ in terms of the (super)potential $\mathcal{W}_\eta(r)$, namely:

$$\mathcal{W}^2_\eta(r)-\frac{\hbar}{\sqrt{2}}\mathcal{W}'_\eta(r)+E_{0,\eta} = V_{\eta,g}(r)\,, \tag{6.15}$$

where $\mathcal{W}'_\eta(r)\doteq\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{W}_\eta(r)$. This equation is solved for the following values of the superpotential and eigenvalue $E_{0,\eta}$:

$$\begin{cases}\mathcal{W}_\eta(r) = \dfrac{k_\eta}{\sqrt{2}\hbar(l+\frac{N-1}{2})}-\dfrac{\hbar}{\sqrt{2}r}(l+\frac{N-1}{2})\\[2ex] E_{0,\eta} = -\dfrac{k_\eta^2}{2\hbar^2(l+\frac{N-1}{2})^2}\,.\end{cases} \tag{6.16}$$

It is interesting to notice that $E_{0,\eta} = E_{0,\eta}(E)$ depends on the full energy of the deformed Hamiltonian. Let us also observe that in the flat limit (for $N = 3$) the standard (undeformed) results for the hydrogen atom are recovered [16, 17]. In this η-deformed case the SUSY partner potential is obtained through the superpotential $\mathcal{W}_\eta$ in the usual way:

$$\begin{aligned} U_{\eta,g}(r) &\doteq \mathcal{W}_\eta^2(r) + \frac{\hbar}{\sqrt{2}}\mathcal{W}_\eta'(r) + E_{0,\eta} \\ &= \frac{\hbar^2\left(l(l+N) + {}^{(N-1)(N+1)}/_4\right)}{2r^2} - \frac{k_\eta}{r}\,. \end{aligned} \tag{6.17}$$

This means that the two superpartners are given by:

$$\begin{cases} \tilde{V}_{\eta,g}(r) \doteq V_{\eta,g} - E_{0,\eta} = \dfrac{\hbar^2\left(l(l+N-2) + {}^{(N-1)(N-3)}/_4\right)}{2r^2} - \dfrac{k_\eta}{r} + \dfrac{k_\eta^2}{2\hbar(l+\frac{N-1}{2})^2} \\ \tilde{U}_{\eta,g}(r) \doteq U_{\eta,g} - E_{0,\eta} = \dfrac{\hbar^2\left(l(l+N) + {}^{(N-1)(N+1)}/_4\right)}{2r^2} - \dfrac{k_\eta}{r} + \dfrac{k_\eta^2}{2\hbar(l+\frac{N-1}{2})^2}\,. \end{cases} \tag{6.18}$$

In fact, it is easy to show that by taking the angular momentum quantum number l as the shape invariance parameter the following relation holds[6]:

$$\begin{cases} \tilde{U}_{\eta,g}^{(l)}(r) = \tilde{V}_{\eta,g}^{(l+1)}(r) + R_\eta^{(l)} \\ R_\eta^{(l)} \doteq \dfrac{(l+\frac{N}{2})(k+\eta E)^2}{\hbar^2(l+\frac{N-1}{2})^2(l+\frac{N+1}{2})^2}\,, \end{cases} \tag{6.19}$$

where $R_\eta^{(l)}$ is a reminder (it does not depend on r). Formula (6.19) is known as *shape invariance condition* (SIC) and ensure that the quantum Hamiltonian $\hat{H}_{g,\eta}$ is exactly solvable. The reason why this turns out to be true is due to the fact that the condition (6.19) can be rephrased as a factorizability property of the Hamiltonian in terms of a pair of ladder operators, i.e.:

$$\hat{A}_{g,\eta}^{(l)}\hat{A}_{g,\eta}^{\dagger\,(l)} - \hat{A}_{g,\eta}^{\dagger\,(l+1)}\hat{A}_{g,\eta}^{(l+1)} = R_\eta^{(l)}\hat{\mathbb{1}}\,, \tag{6.20}$$

where the ladder operators are defined through the superpotential as:

$$\hat{A}_{\eta,g}^{(l)} = \frac{\hbar}{\sqrt{2}}\partial_r + \mathcal{W}_\eta^{(l)}(r)\,, \qquad \hat{A}_{\eta,g}^{\dagger\,(l)} = -\frac{\hbar}{\sqrt{2}}\partial_r + \mathcal{W}_\eta^{(l)}(r)\,, \tag{6.21}$$

and factorize the gauged Hamiltonian:

$$\hat{H}_{\eta,g}^{(l)} = \hat{A}_{\eta,g}^{\dagger\,(l)}\hat{A}_{\eta,g}^{(l)} + E_{0,\eta}^{(l)}\,. \tag{6.22}$$

Thus, at this point it is clear what is the basic idea behind the shape invariance: *it represents a generalization of the algebraic method that we commonly use for the harmonic oscillator. Here, the Heisenberg algebra is replaced by the algebraic relation* (6.20). Now, to obtain a factorization of the original Hamiltonian $\hat{H}_\eta$, we only need to perform the inverse gauge transformation:

$$\hat{H}_\eta^{(l)} = w^{-1}(r)\hat{H}_{\eta,g}^{(l)}w(r) = \underbrace{w^{-1}(r)\hat{A}_{\eta,g}^{\dagger\,(l)}w(r)}_{\hat{A}_\eta^{\dagger\,(l)}}\underbrace{w(r)^{-1}\hat{A}_{\eta,g}^{(l)}w(r)}_{\hat{A}_\eta^{(l)}} + E_{0,\eta}^{(l)} \tag{6.23}$$

6 From now on we keep explicit its dependence in the formulae.

where the representation of the ladder operators reads:

$$\hat{A}_{\eta}^{(l)} \doteq \frac{\hbar}{\sqrt{2}}\partial_r + \frac{k_\eta}{\hbar(l+\frac{N-1}{2})} - \frac{\hbar l}{\sqrt{2}r}\,, \qquad \hat{A}_{\eta}^{\dagger\,(l)} \doteq -\frac{\hbar}{\sqrt{2}}\partial_r + \frac{k_\eta}{\hbar(l+\frac{N-1}{2})} - \frac{\hbar(l+N-1)}{\sqrt{2}r} \tag{6.24}$$

As usual, in order to construct the spectrum we need firstly to find the kernel of the lowering operator, which turns out to be:

$$\hat{A}_{\eta}^{(l)}\Phi_{0,\eta}^{(l)}(r) = 0 \quad \Rightarrow \quad \Phi_{0,\eta}^{(l)}(r) \propto r^l \,\exp\left(-\frac{k_\eta r}{\hbar^2(l+\frac{N-1}{2})}\right), \tag{6.25}$$

and we can use the algebraic relation (6.20) (that does not change under gauge transformations) to find the eigenvalues and the eigenfunctions of the Hamiltonian $\hat{H}_\lambda$. In fact, if we apply (6.20) to the ground state calculated at $l+1$ and apply the operator $A_{\eta}^{\dagger\,(l)}$ from the left, we obtain:

$$\hat{A}_{\eta}^{\dagger\,(l)}\hat{A}_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{0,\eta}^{(l+1)}(r) - \hat{A}_{\eta}^{\dagger\,(l)}\hat{A}_{\eta}^{\dagger\,(l+1)}\hat{A}_{\eta}^{(l+1)}\Phi_{0,\eta}^{(l+1)}(r) = R_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{0,\eta}^{(l+1)}(r) \tag{6.26}$$

that is:

$$(\hat{H}_{\eta}^{(l)} - E_{0,\eta}^{(l)})\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{0,\eta}^{(l+1)}(r) = R_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{0,\eta}^{(l+1)}(r)\,. \tag{6.27}$$

This means that, if we define the new eigenfunction $\Phi_{1,\eta}^{(l)}(r) \doteq \hat{A}_{\eta}^{\dagger\,(l)}\Phi_{0,\eta}^{(l+1)}(r)$, we immediately obtain:

$$\begin{cases} \hat{H}_{\eta}^{(l)}\Phi_{1,\eta}^{(l)}(r) = (E_{0,\eta}^{(l)} + R_{\eta}^{(l)})\Phi_{1,\eta}^{(l)}(r) = E_{1,\eta}^{(l)}\Phi_{1,\eta}^{(l)}(r) \\ E_{1,\eta}^{(l)} = -\dfrac{k_\eta^2}{2\hbar^2(1+l+\frac{N-1}{2})^2}\,. \end{cases} \tag{6.28}$$

This procedure can be iterated. For example, to find the second excited state we write:

$$\hat{A}_{\eta}^{\dagger\,(l)}\hat{A}_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{1,\eta}^{(l+1)}(r) - \hat{A}_{\eta}^{\dagger\,(l)}\hat{A}_{\eta}^{\dagger\,(l+1)}\hat{A}_{\eta}^{(l+1)}\Phi_{1,\eta}^{(l+1)}(r) = R_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{1,\eta}^{(l+1)}(r) \tag{6.29}$$

namely:

$$(\hat{H}_{\eta}^{(l)} - E_{0,\eta}^{(l)})\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{1,\eta}^{(l+1)}(r) - \hat{A}_{\eta}^{\dagger\,(l)}(E_{1,\eta}^{(l+1)} - E_{0,\eta}^{(l+1)})\Phi_{1,\eta}^{(l+1)}(r) = R_{\eta}^{(l)}\hat{A}_{\eta}^{\dagger\,(l)}\Phi_{1,\eta}^{(l+1)}(r) \tag{6.30}$$

and, for the new eigenfunction $\Phi_{2,\eta}^{(l)}(r) \doteq \hat{A}_{\eta}^{\dagger\,(l)}\Phi_{1,\eta}^{(l+1)}(r)$, we obtain the eigenvalue:

$$\begin{cases} \hat{H}_{\eta}^{(l)}\Phi_{2,\eta}^{(l)}(r) = (E_{0,\eta}^{(l)} - E_{0,\eta}^{(l+1)} + E_{1,\eta}^{(l+1)} + R_{\eta}^{(l)})\Phi_{2,\eta}^{(l)}(r) = E_{2,\eta}^{(l)}\Phi_{2,\eta}^{(l)}(r) \\ E_{2,\eta}^{(l)} = -\dfrac{k_\eta^2}{2\hbar^2(2+l+\frac{N-1}{2})^2}\,. \end{cases} \tag{6.31}$$

By applying this construction iteratively we obtain the following solution:

$$\begin{cases} \Phi_{n,\eta}^{(l)}(r) \propto r^l \,\exp\left(-\frac{k_\eta r}{\hbar^2(n+l+\frac{N-1}{2})}\right) L_n^{(2l+N-2)}\left(\frac{2k_\eta r}{\hbar^2(n+l+\frac{N-1}{2})}\right) \\ E_{n,\eta}^{(l)} = -\dfrac{k_\eta^2}{2\hbar^2(n+l+\frac{N-1}{2})^2}\,, \end{cases} \tag{6.32}$$

that coincides with the result obtained in [48], here achieved in terms of the shape invariance. In particular, the main point for this system is that the constant k_η depends on the energy. Thus, in order to find the discrete spectrum we need to solve the algebraic equation:

$$E^{(l)}_{n,\eta} = -\frac{(k+\eta E^{(l)}_{n,\eta})^2}{2\hbar^2(n+l+\frac{N-1}{2})^2}\,, \tag{6.33}$$

that leads to:

$$E^{(l)}_{n,\eta} = \frac{-\hbar^2(n+l+\frac{N-1}{2})^2 - \eta k + \sqrt{\hbar^4(n+l+\frac{N-1}{2})^4 + 2\eta k\hbar^2(n+l+\frac{N-1}{2})^2}}{\eta^2}\,. \tag{6.34}$$

In this way, as already noticed, the eigenfunctions $\Phi^{(l)}_n(r)$, can be explicitly obtained by introducing (6.34) into k_η and, next, by substituting the latter in (6.32). Note that in the limit $\eta \to 0$ the entire construction collapses to the standard one, and the well-known expression for the energies of the hydrogen atom is recovered:

$$E^{(l)}_{n,0} = \lim_{\eta\to 0} E^{(l)}_{n,\eta} = -\frac{k^2}{2\hbar^2(n+l+\frac{N-1}{2})^2}\,, \tag{6.35}$$

where $k \doteq e^2/4\pi\epsilon_0$ (ϵ_0 is the electric constant). In conclusion, the general solution of the problem (6.9) is given by:

$$\Psi_{n,l,l_1,\ldots,l_{N-2}}(r,\boldsymbol{\theta}) \propto \Phi^{(l)}_{n,\eta}(r) Y^l_{l_{N-2},\ldots,l_1}(\theta_1\ldots,\theta_{N-1})\,, \tag{6.36}$$

and the eigenfunctions are orthogonal w.r.t. the measure:

$$d\mu(r,\boldsymbol{\theta}) \doteq r^{N-1}\sin^{N-2}(\theta_1)\sin^{N-3}(\theta_2)\ldots\sin(\theta_{N-2})\mathrm{d}r\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_{N-1} \tag{6.37}$$

with $\theta_1,\ldots,\theta_{N-2}\in[0,\pi)$ and $\theta_{N-1}\in[0,2\pi)$ and $r\in(0,\infty)$. It is worth to mention that the effect of the coupling constant metamorphosis, as we can appreciate in (6.37), is to eliminate the conformal factor in the measure, which is now fully-fledged flat (the deformation parameter that is related to the curvature only appears in the new constant k_η).

4.6.3 The quantum Darboux III system: superintegrability

Our aim now is to perform an analogous investigation for the quantum mechanical counterpart of the N-dimensional classical Hamiltonian (3.322). The resulting MS Schrödinger quantization of H_λ is characterized in the following: let $\hat{H}_\lambda$ be the N-dimensional quantum Darboux Hamiltonian given by:

$$\hat{H}_\lambda = \frac{\hat{\mathbf{p}}^2}{2(1+\lambda\hat{\mathbf{x}}^2)} + \frac{\omega^2\hat{\mathbf{x}}^2}{2(1+\lambda\hat{\mathbf{x}}^2)} = -\frac{\hbar^2}{2(1+\lambda\mathbf{x}^2)}\Delta_{\mathbf{x}} + \frac{\omega^2\mathbf{x}^2}{2(1+\lambda\mathbf{x}^2)}. \tag{6.38}$$

For any real value of λ it turns out that:

1. $\hat{H}_\lambda$ commutes with the following observables ($m=,2\ldots,N; i,j=1,\ldots,N$):

$$\hat{C}^{[m]} = \sum_{1\leq i<j\leq m} (\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2, \qquad \hat{C}_{[m]} = \sum_{N-m<i<j\leq N} (\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i)^2, \tag{6.39}$$

$$\hat{\mathrm{I}}_{\lambda ij} = \frac{\hat{p}_i\hat{p}_j}{\omega} + \omega\hat{x}_i\hat{x}_j - \frac{2\lambda}{\omega}\hat{x}_i\hat{x}_j\hat{H}_\lambda(\hat{\mathbf{x}},\hat{\mathbf{p}}), \tag{6.40}$$

where the quantum curved DF tensor is such that $\hat{H}_\lambda = \frac{\omega}{2}\sum_{i=1}^N \hat{\mathrm{I}}_{\lambda ii}$ [51]. Each of the three sets $\{\hat{H}_\lambda, \hat{C}^{[m]}\}$, $\{\hat{H}_\lambda, \hat{C}_{[m]}\}$ $(m = 2,\dots,N)$ and $\{\hat{\mathrm{I}}_{\lambda ii}\}$ $(i = 1,\dots,N)$ is formed by N algebraically independent commuting observables. Moreover, the set $\{\hat{H}_\lambda, \hat{C}^{[m]}, \hat{C}_{[m]}, \hat{\mathrm{I}}_{\lambda ii}\}$ for $m = 2,\dots,N$ with a fixed index i is formed by $2N-1$ algebraically independent observables.

2. $\hat{H}_\lambda$ is formally self-adjoint on the Hilbert space $L^2(\mathcal{M}^N)$, endowed with the scalar product:
$$\langle\Psi|\Phi\rangle_\lambda = \int_{\mathcal{M}^N} \Psi^*(\mathbf{x})\Phi(\mathbf{x})(1+\lambda\mathbf{x}^2)\mathrm{d}\mathbf{x}.$$

Clearly, these results should be adapted to each of the three different underlying manifold described in (3.321). In particular, we consider here the case $\lambda > 0$ and $r \in (0,\infty)$, for bound states (discrete spectrum). We remark that the system has been fully solved in [53].

4.6.4 The quantum Darboux III: separation in hyperspherical coordinates, factorization and shape invariance

Starting from the Hamiltonian (6.38), using hyperspherical coordinates and following the very same steps that we have performed for the TN case, we arrive to the one-dimensional radial equation:

$$\frac{1}{1+\lambda r^2}\left(-\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2}\right) + \frac{1}{2}\omega^2 r^2\right)\Phi(r) = E\,\Phi(r), \tag{6.41}$$

where $\Phi(r)$ is the radial part of the wavefunction $\Psi(r,\boldsymbol{\theta}) = \Phi(r)Y(\boldsymbol{\theta})$, $Y(\boldsymbol{\theta})$ denoting as usual the hyperspherical harmonics. Once again, the mechanism of the CCM allows to simplify the problem, being the new Hamiltonian given by:

$$\left(-\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2}\right) + \frac{1}{2}\omega_\lambda^2 r^2\right)\Phi(r) = E\,\Phi(r), \tag{6.42}$$

where we have introduced the new (energy-dependent) frequency $\omega_\lambda \doteq \sqrt{\omega^2 - 2\lambda E}$, with the additional condition $\omega^2 > 2\lambda E$. Once again, in order to find the ladder operators, let us perform the following gauge transformation:

$$\underbrace{w(r)\hat{H}_\lambda w^{-1}(r)}_{\hat{H}_{\lambda,g}}\underbrace{w(r)\Phi(r)}_{\Phi_g(r)} = E\,\underbrace{w(r)\Phi(r)}_{\Phi_g(r)}, \tag{6.43}$$

for the same gauge function $w(r) = r^{\frac{N-1}{2}}$. This leads to the following gauged Hamiltonian operator:

$$\hat{H}_{\lambda,g} = -\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2\left(l(l+N-2) + {}^{(N-1)(N-3)}/_4\right)}{2r^2} + \frac{1}{2}\omega_\lambda^2 r^2\,, \tag{6.44}$$

which implies that the radial effective potential (depending on the energy and the deformation parameter) is:

$$V_{\lambda,g}(r) \doteq \frac{\hbar^2\left(l(l+N-2)+{}^{(N-1)(N-3)}/_{4}\right)}{2r^2}+\frac{1}{2}\omega_\lambda^2 r^2\,. \tag{6.45}$$

Thus, after the proper shift of the Hamiltonian, we can construct the ladder operators by imposing the usual Riccati-type equation for the potential $V_{\lambda,g}(r)$, namely:

$$\mathcal{W}_\lambda^2(r)-\frac{\hbar}{\sqrt{2}}\mathcal{W}_\lambda'(r)+E_{0,\lambda}=V_{\lambda,g}(r)\,, \tag{6.46}$$

where we introduced the λ-deformed superpotential $\mathcal{W}_\lambda(r)$. This equation is solved for:

$$\begin{cases}\mathcal{W}_\lambda(r)=\dfrac{\omega_\lambda r}{\sqrt{2}}-\dfrac{\hbar}{\sqrt{2}r}(l+{}^{N-1}/_{2})\\ E_{0,\lambda}=\hbar\omega_\lambda(l+{}^{N}/_{2})\,.\end{cases} \tag{6.47}$$

Also in this case, let us notice that $E_{0,\lambda}=E_{0,\lambda}(E)$ depends on the full energy of the λ-deformed Hamiltonian. Let us also observe that in the flat limit (for $N=3$) the standard (undeformed) results for the isotropic oscillator are recovered [16, 17]. In this λ-deformed case the SUSY partner potential obtained through the superpotential $\mathcal{W}_\lambda$ reads:

$$\begin{aligned}U_{\lambda,g}(r) &\doteq \mathcal{W}_\lambda^2(r)+\frac{\hbar}{\sqrt{2}}\mathcal{W}_\lambda'(r)+E_{0,\lambda}\\ &=\frac{\hbar^2\left(l(l+N)+{}^{(N-1)(N+1)}/_{4}\right)}{2r^2}+\frac{1}{2}\omega_\lambda r^2+\hbar\omega_\lambda\,.\end{aligned} \tag{6.48}$$

This means that the two superpartners are given by:

$$\begin{cases}\tilde{V}_{\lambda,g}(r)\doteq V_{\lambda,g}-E_{0,\lambda}=\dfrac{\hbar^2\left(l(l+N-2)+{}^{(N-1)(N-3)}/_{4}\right)}{2r^2}+\frac{1}{2}\omega_\lambda r^2-\hbar\omega_\lambda(l+{}^{N}/_{2})\\ \tilde{U}_{\lambda,g}(r)\doteq U_{\lambda,g}-E_{0,\lambda}=\dfrac{\hbar^2\left(l(l+N)+{}^{(N-1)(N+1)}/_{4}\right)}{2r^2}+\frac{1}{2}\omega_\lambda r^2-\hbar\omega_\lambda(l+{}^{N}/_{2}-1)\,.\end{cases} \tag{6.49}$$

Also in this case, if we take the angular momentum quantum number l as the shape invariance parameter, we obtain the following *shape invariance condition*:

$$\begin{cases}\tilde{U}_{\lambda,g}^{(l)}(r)=\tilde{V}_{\lambda,g}^{(l+1)}(r)+R_\lambda^{(l)}\\ R_\lambda^{(l)}\doteq 2\hbar\omega_\lambda\,,\end{cases} \tag{6.50}$$

where $R_\eta^{(l)}$ is the reminder associated to the SIP. Once again, we can rewrite such a relation in terms of a pair of ladder operators that factorize the Hamiltonian:

$$\hat{A}_{g,\lambda}^{(l)}\hat{A}_{g,\lambda}^{\dagger\,(l)}-\hat{A}_{g,\lambda}^{\dagger\,(l+1)}\hat{A}_{g,\lambda}^{(l+1)}=R_\lambda^{(l)}\hat{\mathbb{1}}\,, \tag{6.51}$$

where the ladder operators are defined through the superpotential as:

$$\hat{A}_{\lambda,g}^{(l)}=\frac{\hbar}{\sqrt{2}}\partial_r+\mathcal{W}_\lambda^{(l)}(r)\,,\qquad \hat{A}_{\lambda,g}^{\dagger\,(l)}=-\frac{\hbar}{\sqrt{2}}\partial_r+\mathcal{W}_\lambda^{(l)}(r)\,, \tag{6.52}$$

and factorize the gauged Hamiltonian as follows:

$$\hat{H}_{\lambda,g}^{(l)}=\hat{A}_{\lambda,g}^{\dagger\,(l)}\hat{A}_{\lambda,g}^{(l)}+E_{0,\lambda}^{(l)}\,. \tag{6.53}$$

Now, if we perform the inverse gauge transformation:

$$\hat{H}^{(l)}_{\lambda} = w^{-1}(r)\hat{H}^{(l)}_{\lambda,g}\,w(r) = \underbrace{w^{-1}(r)\hat{A}^{\dagger\,(l)}_{\lambda,g}w(r)}_{\hat{A}^{\dagger\,(l)}_{\lambda}}\underbrace{w(r)^{-1}\hat{A}^{(l)}_{\lambda,g}w(r)}_{\hat{A}^{(l)}_{\lambda}} + E^{(l)}_{0,\lambda}\,, \tag{6.54}$$

the original Hamiltonian is recovered, and the representation of the ladder operators reads:

$$\hat{A}^{(l)}_{\lambda} \doteq \frac{\hbar}{\sqrt{2}}\partial_r + \frac{\omega_\lambda r}{\sqrt{2}} - \frac{\hbar l}{\sqrt{2}r}\,, \qquad \hat{A}^{\dagger\,(l)}_{\lambda} \doteq -\frac{\hbar}{\sqrt{2}}\partial_r + \frac{\omega_\lambda r}{\sqrt{2}} - \frac{\hbar(l+N-1)}{\sqrt{2}r}\,. \tag{6.55}$$

In this case, the kernel of the lowering operator turns out to be:

$$\hat{A}^{(l)}_{\lambda}\Phi^{(l)}_{0,\lambda}(r) = 0 \quad \Rightarrow \quad \Phi^{(l)}_{0,\lambda}(r) \propto r^l\,\exp\!\left(-\frac{\omega_\lambda r^2}{2\hbar}\right). \tag{6.56}$$

As we know, starting from (6.56) calculated in $l+1$ and by acting with the operator $A^{\dagger\,(l)}_{\lambda}$ from the left, we obtain:

$$\hat{A}^{\dagger\,(l)}_{\lambda}\hat{A}^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r) - \hat{A}^{\dagger\,(l)}_{\lambda}\hat{A}^{\dagger\,(l+1)}_{\lambda}\hat{A}^{(l+1)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r) = R^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r)\,, \tag{6.57}$$

that is:

$$(\hat{H}^{(l)}_{\lambda} - E^{(l)}_{0,\lambda})\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r) = R^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r)\,. \tag{6.58}$$

This means that, for the eigenfunction $\Phi^{(l)}_{1,\lambda}(r) \doteq \hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{0,\lambda}(r)$, we immediately obtain:

$$\begin{cases} \hat{H}^{(l)}_{\lambda}\Phi^{(l)}_{1,\lambda}(r) = (E^{(l)}_{0,\lambda} + R^{(l)}_{\lambda})\Phi^{(l)}_{1,\lambda}(r) = E^{(l)}_{1,\lambda}\Phi^{(l)}_{1,\lambda}(r) \\ E^{(l)}_{1,\lambda} = \hbar\omega_\lambda(2 + l + N/2)\,. \end{cases} \tag{6.59}$$

For the second excited state we have:

$$\hat{A}^{\dagger\,(l)}_{\lambda}\hat{A}^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r) - \hat{A}^{\dagger\,(l)}_{\lambda}\hat{A}^{\dagger\,(l+1)}_{\lambda}\hat{A}^{(l+1)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r) = R^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r) \tag{6.60}$$

namely:

$$(\hat{H}^{(l)}_{\lambda} - E^{(l)}_{0,\lambda})\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r) - \hat{A}^{\dagger\,(l)}_{\lambda}(E^{(l+1)}_{1,\lambda} - E^{(l+1)}_{0,\lambda})\Phi^{(l+1)}_{1,\lambda}(r) = R^{(l)}_{\lambda}\hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r) \tag{6.61}$$

which implies, for the new eigenfunction $\Phi^{(l)}_{2,\lambda}(r) \doteq \hat{A}^{\dagger\,(l)}_{\lambda}\Phi^{(l+1)}_{1,\lambda}(r)$, the following eigenvalue:

$$\begin{cases} \hat{H}^{(l)}_{\lambda}\Phi^{(l)}_{2,\lambda}(r) = (E^{(l)}_{0,\lambda} - E^{(l+1)}_{0,\lambda} + E^{(l+1)}_{1,\lambda} + R^{(l)}_{\lambda})\Phi^{(l)}_{2,\lambda}(r) = E^{(l)}_{2,\lambda}\Phi^{(l)}_{2,\lambda}(r) \\ E^{(l)}_{2,\lambda} = \hbar\omega_\lambda(4 + l + N/2)\,. \end{cases} \tag{6.62}$$

By applying this construction iteratively we obtain the following solution of the D-III oscillator:

$$\begin{cases} \Phi^{(l)}_{n,\lambda}(r) \propto r^l\,\exp\!\left(-\frac{\omega_\lambda r^2}{2\hbar}\right) L_n^{(l+\frac{N-2}{2})}\left(\frac{\omega_\lambda r^2}{\hbar}\right) \\ E^{(l)}_{n,\lambda} = \hbar\omega_\lambda(2n + l + N/2)\,, \end{cases} \tag{6.63}$$

which coincides with the result obtained in [53]. In particular, as it happens in TN case, the new novel feature with respect to the isotropic harmonic oscillator is that the constant ω_λ

depends on the energy and, in order to find the discrete spectrum, we need to solve the (quadratic) algebraic equation:

$$E_{n,\lambda}^{(l)} = \hbar\sqrt{\omega^2 - 2\lambda E_{n,\lambda}^{(l)}}(2n + l + N/2)\,, \tag{6.64}$$

which leads to:

$$E_{n,\lambda}^{(l)} = -\lambda\hbar^2(2n + l + N/2)^2 + \sqrt{\hbar^2\omega^2(2n + l + N/2)^2 + \lambda^2\hbar^4(2n + l + N/2)^4}\,, \tag{6.65}$$

Once again, the eigenfunctions $\Phi_{n,\lambda}^{(l)}(r)$, can be explicitly obtained by introducing (6.65) into k_η and, next, by substituting the latter in (6.63). Note that in the limit $\lambda \to 0$ the entire construction collapses to the one of the isotropic harmonic oscillator, and the well-known expression for the energies is recovered:

$$E_{n,0}^{(l)} = \lim_{\lambda\to 0} E_{n,\lambda}^{(l)} = \hbar\omega(2n + l + N/2)\,. \tag{6.66}$$

In conclusion, the general solution is given by:

$$\Psi_{n,l,l_1,\ldots,l_{N-2}}(r,\boldsymbol{\theta}) \propto \Phi_{n,\lambda}^{(l)}(r)Y^l_{l_{N-2},\ldots,l_1}(\theta_1\ldots,\theta_{N-1})\,, \tag{6.67}$$

and the eigenfunctions are orthogonal w.r.t. to the flat measure $\mathrm{d}\mu(r,\boldsymbol{\theta})$ previously defined.

4.7 CONCLUDING REMARKS AND OPEN PERSPECTIVES

To summarize briefly the results of this section, we have presented a classical and quantum analysis for two specific examples of N-dimensional radially symmetric MS models, the Taub-NUT and the D-III systems. On one hand, in the classical case, we have shown how their one-dimensional effective radial dynamics can be solved by means of the classical factorization method: the classical Poisson algebras close in terms of the deformed Hamiltonians which define the two systems, and the motion in time is solved thanks to the existence of a deformed version of the two time-dependent constants of motion $Q^\pm = Q^\pm(r, p_r, t)$ for both systems. On the other hand, at the quantum level, the separation of the Schrödinger equation in hyperspherical coordinates allowed us to use the quantum factorization method and the shape invariance (for positive values of the deformation parameters) in order to solve the spectral problem for bound states. The main point to emphasize in the latter case is related to the possibility of applying SUSYQM techniques, in the standard way, also on curved spaces. As we mentioned, at least in these two cases, this turns out to be a consequence of the coupling constant metamorphosis phenomenon, which "rephrased" the curved problem into an effective flat problem characterized by a new energy-dependent coupling constant. This is indeed a crucial simplification. In this respect, it is worth mentioning that in [138, 139] similar MS models have been investigated, in that case by means of a deformed version of the shape invariance [140]. We conclude by adding a few remarks, which might be useful to outline some possible future developments. One important question that has to be faced, and hopefully solved, concerns the extension of the approaches used here to the whole classes of systems that have been shown by V. Perlick to be, at a classical level, multiparametric families

of maximally superintegrable deformations of *HO* and *KC*. At the quantum level, this question got an essentially positive answer in the paper [85]. A further issue worthing a deeper analysis has to do with the quantum behaviour of both Taub-NUT and *D-III* Hamiltonians, for negative values of the deformation parameters, such that the potential acquires a typical confining shape: in this case the exact solvability has to be investigated, and the possibility of finding the spectrum in a closed form is still an open problem. Work on this last issue is in progress[7].

7 In the appendix B the interested reader can find an example of an exactly solvable radially symmetric system with confined potential, whose solution is given in terms of the Jacobi orthogonal polynomials. This example has been introduced very recently in order to achieve a better understanding of the behaviour of such ND confined models.

5 CONCLUSIONS

In this Thesis we dealt with different aspects regarding classical and quantum superintegrability, both continuous and discrete. The methodological approach has been of algebraic nature. All the models that have been investigated represent superintegrable deformations of two prototype example of physical systems, the harmonic oscillator and the Kepler-Coulomb. They also share the same coalgebra symmetry. Here we summarize the main results and the open issues which would deserve to be investigated in the future.

1. Chapter 3 contains the main result of the Thesis. We have in fact proved, for the first time to the best of our knowledge, that the coalgebra technique can be succesfully applied in real discrete Quantum Mechanics, thus providing a new intriguing link between superintegrable discrete systems and coalgebra symmetries. The main idea behind such a work was to show the effective usefulness of this method also in the multidimensional generalization of discrete quantum models. In particular our research, focused on the cartesian harmonic oscillator, led us to achieve the multidimensional extension of a discrete quantum system based on the Charlier orthogonal polynomials, the *Charlier oscillator*. As a matter of fact, we have shown that this model share exactly the same coalgebra symmetry of the Hermite oscillator. This is the reason why we have achieved the discrete representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra: *like the standard Hermite oscillator, the Charlier oscillator is described in terms of an* $\mathfrak{sl}(2,\mathbb{R})$ *coalgebraic Hamiltonian*. In particular, after having solved the spectral problem on the lattice using the standard factorization method, simply by replacing the differential representation with its finite-difference counterpart, the coproduct map of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra have been used to extended the discrete system to higher dimensions preserving its superintegrability properties. The discrete quantum integrals of motion, a discrete analog of the angular momentum and the Demkov-Fradkin tensor on the lattice, have been also explicitly constructed.
   A gauge equivalent problem has been investigated in [22], where the authors also described a two-dimensional model based on the Charlier polynomials. As a byproduct of our construction, we have immediately obtained a generalization of their Hamiltonian. To mention some future perspective related to this work, now that we have established such a link between coalgebra symmetries and discrete quantum mechanical systems, it would be interesting to use it in order to provide some new discrete superintegrable model. The analysis should be focused on a generalization of previous results obtained in the continuous case. From this point of view, a still open problem resides in the discrete representation of the inverse of the position operator. This would be crucial to construct, for instance, a coalgebraic discrete version of the hydrogen atom and, consequently, some of its coalgebraic generalizations appeared in the literature.

2. In Chapter 4 we have collected two main results:
   - On one hand, we have proved that the classical factorization method find a natural application in the framework of f-deformations. In particular, we wrote down the

deformed Poisson algebra characterizing the entire class of non-linear oscillators and we found its general solution algebraically. The latter has been constructed thanks to the existence of two time-dependent constants of motion shared by the entire family of deformations. Moreover, we have also shed some light on the physical meaning behind such Hamiltonian systems, they represent in fact a rather simple example of deformations. With respect to the harmonic oscillator, there is nothing more than energy dependent frequencies and amplitudes, and the phase space trajectories are undeformed, except for a change in their radius. In turn, this is the reason why the multidimensional generalization of such systems, which we have also presented, can be easily achieved by means of the primitive coproduct of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra. The superintegrability of such models in ND is trivially ensured, the family of f-deformations share in fact the same first integrals of motion of the standard harmonic oscillator. This is true by construction.
Regarding future perspectives, since we showed this connection between the (dressed) Poisson algebra generators and f-deformations, it would be interesting to construct f-deformed version for systems other than the harmonic oscillator. Also, the quantization of the deformed Poisson algebras obtained in the analysis is another issue that should be investigated.

- On the other hand, by using the same algebraic approach, we have constructed the classical solutions of the motion for two well-known Maximally Superintegrable systems defined on spherically symmetric curved spaces, the Taub-NUT (a deformed Kepler-Coulomb system) and Darboux III (a deformed isotropic oscillator), highlighting new features also on the case of negative values of the deformation parameters. As a matter of fact, for the TN case, in the confined region we have been able to define a new Hamiltonian, characterized by a confined potential, which led us to investigate confined models, mainly for what concern the corresponding quantum system.
  Along these lines, concerning the quantum case, for positive values of the deformation parameters we have shown that the solution of the spectral problem, for both systems, can be achieved by means of the standard SUSYQM approach (quantum factorization method and shape invariance condition). This is because the Schrödinger equation associated to the curved problem, after performing a coupling constant metamorphosis, collapses to a standard Schrödinger equation associated to a flat problem with a new energy dependent coupling constant. This mechanism allowed us to deal with such systems as in the standard undeformed case and, in turn, elucitated the reason behind their exact solvability. Precisely, we have shown that after separation in hyperspherical coordinates the two systems admits a factorization of the Hamiltonian in terms of a pair of ladder operators that, by imposing a shape invariance condition in the angular momentum quantum number, can be used to construct the spectrum and the eigenfunctions algebraically. The main open problem is surely related to the investigation of the quantum case for negative values of the deformation parameters, where the effective potential acquires a confining shape. In that case, in fact, an analytic closed form for the spectrum and the eigenfunctions could not be guaranteed, and numerical solutions could take the place of the exact ones.

Concerning these issues, we should also mention that preliminary results about an exactly solvable ND confined system have been obtained. They are discussed in the appendix B.

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A MANIFOLDS

In this appendix, we would like to give a pedagogical introduction to the notion *manifold*. To this aim, we will strictly follow the reference [141]. We will start the appendix by recalling the notions of maps between sets, open sets, charts and atlas. After that, we will go straight to the definition of manifold. We will conclude the discussion by illustrating some example.

A.1 MAPS BETWEEN SETS

Let us consider two sets $\mathcal{M}$ and $\mathcal{N}$. One defines a map $\varphi : \mathcal{M} \to \mathcal{N}$ as a relationship that assigns at each element of $\mathcal{M}$ exactly one element of $\mathcal{N}$ (it is a generalization of the concept of function). Given two maps $\varphi : \mathcal{M} \to \mathcal{N}$ ed $\phi : \mathcal{N} \to \mathcal{O}$ one defines the composition $(\phi \circ \varphi) : \mathcal{M} \to \mathcal{O}$ as:

$$(\phi \circ \varphi)(x) \doteq \phi(\varphi(x)) \quad \forall\, x \in \mathcal{M}\,. \tag{1.68}$$

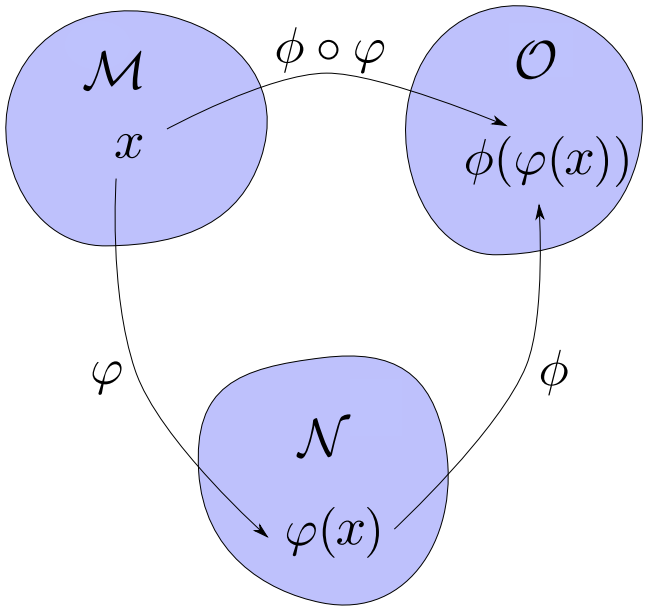

Figure 15.: composition of maps.

1. A map φ is called **one-to-one** (injective) if each element of $\mathcal{N}$ has at most one element of $\mathcal{M}$ mapped into it.

2. A map φ is called **onto** (surjective) if each element of $\mathcal{N}$ has at least one element of $\mathcal{M}$ mapped into it.

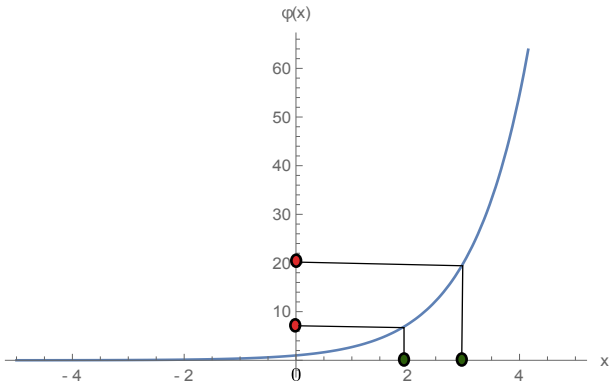

Figure 16.: exponential map.

- The map $\varphi : \mathbb{R} \to \mathbb{R}^+$ $\varphi(x) = \exp(x)$ is both injective and surjective (bijective). If we had considered a different codomain, such as $\varphi(x) : \mathbb{R} \to \mathbb{R}$, we would have lost the surjectivity (this is because the elements belonging to the codomain $\mathbb{R}^-$ does not have elements of $\mathcal{M}$ mapped into them).

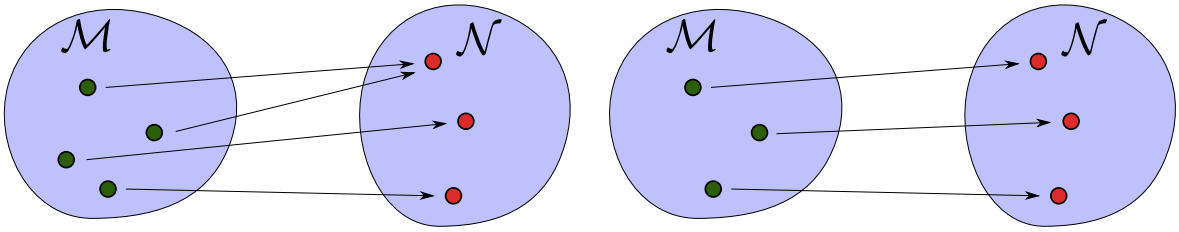

Figure 17.: surjective map (left) and bijective map (right).

- The map $\varphi : \mathbb{R} \to \mathbb{R}$ $\varphi(x) = x(x^2 - 1)$ is surjective but not injective.

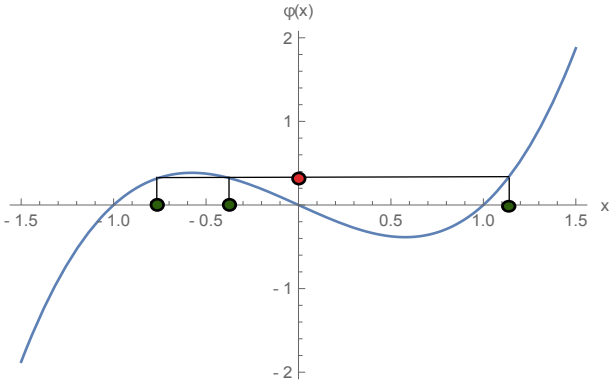

Figure 18.: map $\varphi(x) = x(x^2 - 1)$. Onto, not one-to-one.

- The map $\varphi : \mathbb{R} \to \mathbb{R}$ $\varphi(x) = x^2$ is neither surjective nor injective. If we restrict the codomain to $\mathbb{R}^+$ it becomes surjective.

The set $\mathcal{M}$ is known as the *domain* of the map φ and set of points in $\mathcal{N}$ in which $\mathcal{M}$ is mapped is known as the *image* of φ. For some subset $\mathcal{U} \subset \mathcal{N}$, the set of elements of $\mathcal{M}$ which get mapped to $\mathcal{U}$ is called the *preimage* $\mathcal{U}$ under the map φ, e.g. $\varphi^{-1}(\mathcal{U})$.

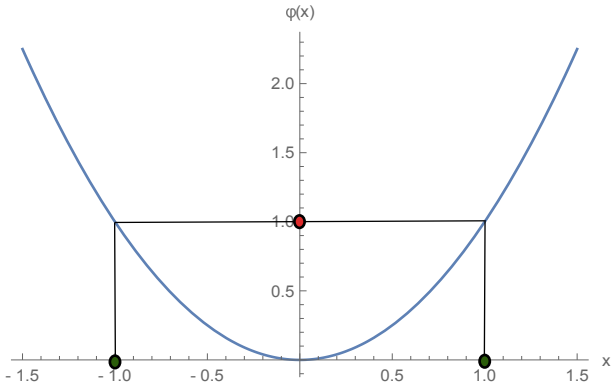

Figure 19.: Map $\varphi(x) = x^2$. Neither onto nor one-to-one.

3. A map that is both one-to-one and onto is also known as **invertible**. In this case it is possible to define the inverse map $\varphi^{-1} : \mathcal{N} \to \mathcal{M}$ so that:

$$(\varphi^{-1} \circ \varphi)(x) \doteq \varphi^{-1}(\varphi(x)) = x \quad \forall\, x \in \mathcal{M}\,. \tag{1.69}$$

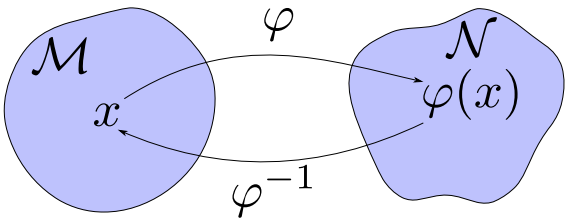

Figure 20.: the inverse of a map.

A simple example of inverse map is given by $\varphi^{-1}(x) = \log(x) : \mathbb{R}^+ \to \mathbb{R}$. This map represents the inverse of the exponential map $\varphi(x) = \exp(x) : \mathbb{R} \to \mathbb{R}^+$. If we had considered as a codomain of the exponential map the entire real axes, being in this case the map not more surjective, we would have lost the invertibility.

4. A map from $\mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces takes m-tuple $(x_1, \ldots, x_m)$ to n-tuple $(x_1, \ldots, x_n)$. It can be thought as a collection of n functions φ_j $(j = 1, \ldots, n)$ of m variables, i.e.:

$$\begin{cases} y_1 = \varphi_1(x_1, \ldots, x_m) \\ \vdots \\ y_n = \varphi_n(x_1, \ldots, x_m)\,. \end{cases} \tag{1.70}$$

Any one of these functions is said to be $\mathcal{C}^p$ if continuous and p-times differentiable. The entire map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is $\mathcal{C}^p$ if each of its component functions is at least $\mathcal{C}^p$. Let us remark that $\mathcal{C}^\infty$ maps are also called *smooth*.

5. Two sets are said to be **diffeomorphic** if there exists a *smooth* map $\varphi : \mathcal{M} \to \mathcal{N}$ whose inverse $\varphi^{-1} : \mathcal{N} \to \mathcal{M}$ is also smooth. Such a map is known as **diffeomorphism**.

A.2 DEFINITION OF MANIFOLD

Before to define what a manifold is, we need to provide the notion of **open set**, where we can introduce a coordinate system, and then sew the open sets together in an appropriate way. An open set is a set formed by an arbitrary union of *open balls*. In other words the set $\mathcal{U} \in \mathbb{R}^n$ is open if $\forall\, x \in \mathcal{U}$, there exists an *open ball* $\mathcal{B}_\epsilon(x)$, centered at x with radius ϵ, completely contained in $\mathcal{U}$. Let us recall that an *open ball* is defined as the set of all points $x \in \mathbb{R}^n$ such that $|x - x_0| < \epsilon$, for some fixed $x_0 \in \mathbb{R}^n$, with $\epsilon \in \mathbb{R}^+$, where it has been defined the distance $|x - x_0| \doteq \sqrt{\sum_{i=1}^n (x_i - x_{0i})^2}$. An open ball is therefore the enterior of an n-sphere centered in x_0. Let us mention that in order to define the concept of open set, we have implicitly introduced the notion of metric (namely a way for measuring distances).

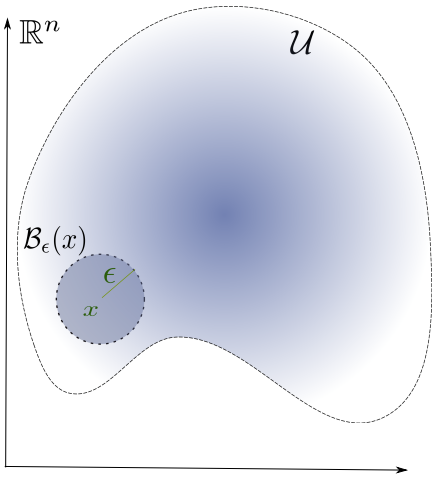

Figure 21.: open set of $\mathbb{R}^n$.

Let us provide now two notions that we need in the following for the definition of manifold, the concept of **chart** and **atlas**.

1. A *chart*, or coordinates system, consist in a subset $\mathcal{U}$ of a set $\mathcal{M}$ together with an injective map $\varphi : \mathcal{U} \to \mathbb{R}^n$ such that the image of $\mathcal{U}$, $\varphi(\mathcal{U})$, is open in $\mathbb{R}^n$ (any map is onto its image so the map φ is invertible). The subset $\mathcal{U} \subset \mathbb{R}^n$ is an open set of $\mathbb{R}^n$.

2. A $\mathcal{C}^\infty$ *atlas* is an indexed collection of charts $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ such that:

   - The union of all the $\mathcal{U}_\alpha$ (i.e. $\bigcup_\alpha \mathcal{U}_\alpha$) is equal to $\mathcal{M}$ (i.e. the $\mathcal{U}_\alpha$ cover the entire set $\mathcal{M}$);
   - The charts are *smoothly* sewn together (*compatibles*). This means that if the two charts overlap, i.e. $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$, then the map $(\varphi_\alpha \circ \varphi_\beta^{-1})$ takes points in $\varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subset \mathbb{R}^n$ onto $\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subset \mathbb{R}^n$, and all of these maps must be smooth where they are defined.

To summarize, a chart is what we normally think of as a coordinate system on some open set, whereas an atlas is a system of charts which are smoothly related on their overlaps.

At this point we have all the ingredients to provide the definition of **manifold**. We will say that *an n-dimensional $\mathcal{C}^\infty$ manifold is a set $\mathcal{M}$ equipped with a maximal atlas, e.g. an atlas which contains every possible compatible chart*.

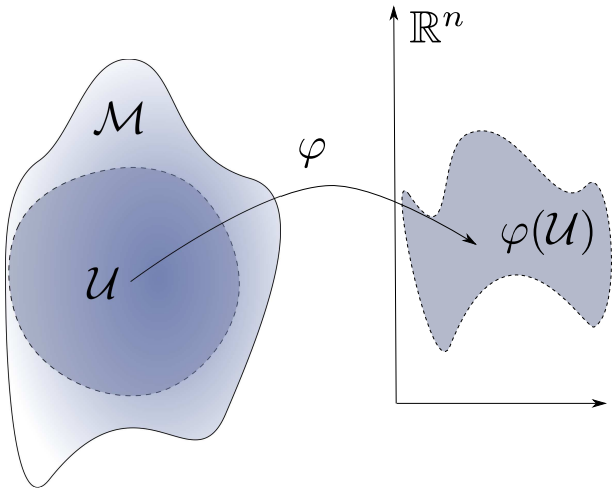

Figure 22.: graphical representation of a chart (coordinate system).

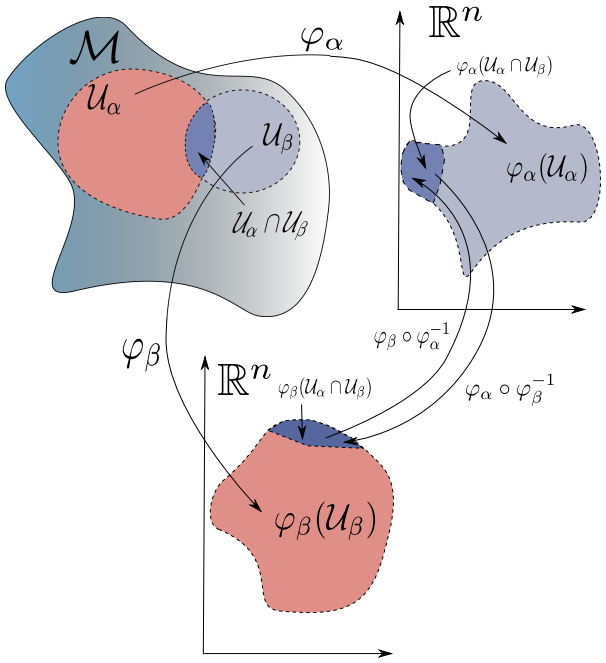

Figure 23.: graphical representation of an atlas.

Intuitively, the notion of a manifold captures the basic idea of a space which may be curved, and characterized by a complicated topology, but in local regions it simply looks like $\mathbb{R}^n$ [141].

A.2.1 Examples of manifolds

Here, some examples of manifolds are listed:

- $\mathbb{R}^n$ itself is a manifold equipped with a single chart (this represents a trivial example since $\mathbb{R}^n$ looks like $\mathbb{R}^n$ globally, not only locally);
- The unit circumference $\mathbb{S}^1$: $\{(x_0, x_1) \in \mathbb{R}^2 \mid x_0^2 + x_1^2 = 1\}$ is a manifold. In order to cover the entire manifold we need at least of two charts:

$$\begin{cases} (\mathcal{U}_1 = (0, 2\pi), \quad \varphi_1 : \mathbb{S}^1 / \{p_n\} \to \mathbb{R}) \\ (\mathcal{U}_2 = (-\pi, \pi), \quad \varphi_2 : \mathbb{S}^1 / \{p_s\} \to \mathbb{R}) \,, \end{cases} \tag{2.71}$$

  where p_n and p_s are the north pole and the south pole respectively.

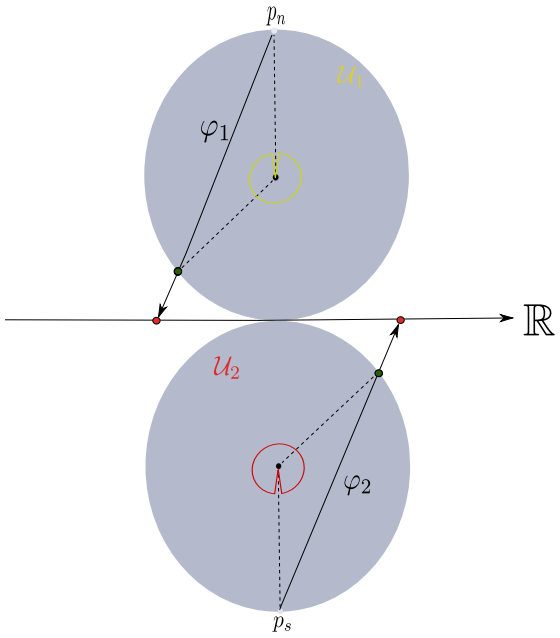

Figure 24.: circumference as a manifold.

  In this way the entire set $\mathcal{M}$ is covered, and the manifold is $(\mathbb{S}^1, \{\mathcal{U}_i, \varphi_i\})$ $(i = 1, 2)$.

- The unit sphere $\mathbb{S}^2$: $\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_3^2 = 1\}$ is a manifold if equipped with two charts φ_i $(i = 1, 2)$ given by the stereographic projections:

$$\begin{cases} (\mathcal{U}_1 = \mathbb{S}^2 / p_n, \quad \varphi_1 : \mathbb{S}^2 / \{p_n\} \to \mathbb{R}^2) \\ (\mathcal{U}_2 = \mathbb{S}^2 / p_s, \quad \varphi_2 : \mathbb{S}^2 / \{p_s\} \to \mathbb{R}^2) \,. \end{cases} \tag{2.72}$$

  The Figure 25 shows the chart $(\mathcal{U}_1, \varphi_1)$, the stereographic projection (with respect the north pole).

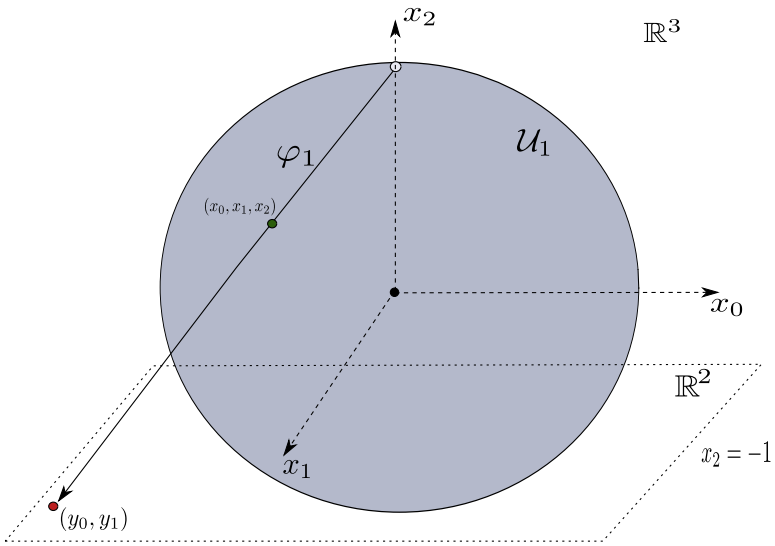

Figure 25.: chart $(\mathcal{U}_1, \varphi_1)$ for the sphere S^2.

A straight line starts from the north pole located at $x_2 = 1$ and reaches the plane situated at $x_2 = -1$, intercepting a point on $\mathbb{S}^2$. When projected to the plane, a pair of cartesian coordinates (y_0, y_1) are assigned to that point. This procedure is defined by the map:

$$\varphi_1(x_0, x_1, x_2) = \left(\frac{2x_0}{1 - x_2}, \frac{2x_1}{1 - x_2}\right) =: (y_0, y_1)\,. \tag{2.73}$$

In total analogy to the circumference S^1, the second chart will be given by the stereographic projection of the sphere, with respect the south pole, to the plane located at $x_2 = 1$. In this case the map $\varphi_2 : \mathbb{S}^2/\{p_s\} \to \mathbb{R}^2$ results:

$$\varphi_2(x_0, x_1, x_2) = \left(\frac{2x_0}{1 + x_2}, \frac{2x_1}{1 + x_2}\right) =: (z_0, z_1)\,. \tag{2.74}$$

When are sewn together, these two charts cover the entire sphere $\mathbb{S}^2$. Moreover, they overlap in the region $-1 < x_2 < 1$. The composition $\varphi_2 \circ \varphi_1^{-1}$ is given by:

$$\varphi_1^{-1}(y_0, y_1) = \left((1 - x_2)\frac{y_0}{2}, (1 - x_2)\frac{y_1}{2}, x_2\right) \Rightarrow \varphi_2(\varphi_1^{-1}(y_0, y_1)) = \left(\frac{4y_0}{y_0^2 + y_1^2}, \frac{4y_1}{y_0^2 + y_1^2}\right), \tag{2.75}$$

whereas the composition $\varphi_1 \circ \varphi_2^{-1}$ results in:

$$\varphi_1(\varphi_2^{-1}(z_0, z_1)) = \left(\frac{4z_0}{z_0^2 + z_1^2}, \frac{4z_1}{z_0^2 + z_1^2}\right), \tag{2.76}$$

and they are $\mathcal{C}^\infty$ in the overlapping domain.

- The n-torus $\mathbb{T}^n$ appearing in the Arnold-Liouville theorem, which results on the cartesian product of n spheres $\mathbb{S}^1$, is a manifold. This is a general result, the cartesian product of manifolds is a also manifold.

B NEGATIVE VALUES OF THE DEFORMATION PARAMETERS: A CASE STUDY

In the section 4.6 we have shown that the exact solvability of the two MS systems on curved space (the Taub-NUT and the D-III) can be solved in terms of the shape invariance. As a matter of fact, the coupling constant metamorphosis allowed us to perform a standard SUSYQM analysis in order to find out the discrete spectrum and the eigenfunctions of the two models. This works for positive values of the deformation parameters. For negative values the situation is not so clear and, indeed, is still an open problem. For example, in [55] an explicit discussion for different values of the sign of the parameters η and k has been presented for the TN, and the problems related to the possibility of obtaining an analytical form of the spectrum and the eigenfunctions, related to the boundary conditions, have been also sketched. Anyway, the model described by the effective potential obtained by an overall change of sign in the original Hamiltonian was not discussed there. The crucial point here is related to the fact that, even if the potential is confined, an analytic closed form for the spectrum and the eigenfunctions is not guaranteed and, at a first glance, it seems reasonable that the solution could be calculated only numerically. A major detailed investigation on these issues, together with our spanish collegues, is in progress. Personally, since I never dealt with such confined models in N-dimensions, I started thinking how to tackle such a problem. In particular, since we are dealing with problems confined on a segment, one reasonable solution would be a set of eigenfunctions associated with (confined) orthogonal polynomials, where I mean polynomials that are orthogonal on a segment. This simple observation led me to investigate, just as a prototype example, the family of Jacobi orthogonal polynomials. The goal of this appendix is to present such analysis.

B.1 A CONFINED EXACTLY SOLVABLE MODEL BASED ON THE JACOBI POLYNOMIALS

In this appendix, in order to make some progress on N-dimensional confined models on curved spaces, we will present an example based on the Jacobi orthogonal polynomials. The basic idea here is to construct an N-dimensional model starting from the one-dimensional secular problem associated to the Jacobi polynomials. The main result of this work can be summarized by the following:

theorem 1. *Let us consider the two-parameter family of N-dimensional (ND) classical Hamiltonian systems given by:*

$$H_\xi = T_\xi(\mathbf{x},\mathbf{p}) + V_\xi(\mathbf{x}) = \left(1-\frac{|\mathbf{x}|}{\xi}\right)\frac{|\mathbf{x}|}{\xi}\,\frac{\mathbf{p}^2}{2} + \frac{k}{\xi-|\mathbf{x}|}, \tag{1.77}$$

where ξ and k are positive *real parameters, $\mathbf{x},\mathbf{p}\in\mathbb{R}^N$ are conjugate coordinates and momenta with $0<|\mathbf{x}|<\xi$. This model, after performing a direct Schrödinger quantization, and a separation of*

variables in (hyper)spherical coordinates defined in the punctured open hyperball $0 < |\mathbf{x}| < \xi$, admits the discrete set of eigenvalues ($n = 0, 1, \ldots; l = 0, 1, \ldots$):

$$E_{n,l} = \frac{\hbar^2}{8\xi^2}\left(1 + 2n + \sqrt{1 + \frac{8k\xi}{\hbar^2}}\right)\left(1 + 2n + 4\left(l + \frac{N-2}{2}\right) + \sqrt{1 + \frac{8k\xi}{\hbar^2}}\right), \tag{1.78}$$

together with the corresponding eigenfunctions:

$$\Psi_{n,l,l_{N-2},\ldots,l_1}(r, \boldsymbol{\theta}) \propto Y^l_{l_{N-2},..,l_1}(\theta_1, \theta_2, ..., \theta_{N-1})\Phi_{n,l}(r), \tag{1.79}$$

where $Y^l_{l_{N-2},..,l_1}(\theta_1, \theta_2, ..., \theta_{N-1})$ are the usual hyperspherical harmonics related to the angular part of the solution and:

$$\Phi_{n,l}(r) = c_{n,l}\ (1 - r/\xi)^{\frac{1}{2}\sqrt{1+\frac{8k\xi}{\hbar^2}}+\frac{1}{2}}(r/\xi)^l\ P_n^{\left(\sqrt{1+\frac{8k\xi}{\hbar^2}}, 2l+N-2\right)}(2r/\xi - 1) \tag{1.80}$$

is the radial solution. Here, $c_{n,l} \doteq \sqrt{\frac{n!\left(2n+2l+N-1+\sqrt{1+\frac{8k\xi}{\hbar^2}}\right)\Gamma\left(n+2l+N-1+\sqrt{1+\frac{8k\xi}{\hbar^2}}\right)}{\xi^{N-2}\Gamma(n+2l+N-1)\,\Gamma\left(n+1+\sqrt{1+\frac{8k\xi}{\hbar^2}}\right)}}$ is a normalization factor and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials with parameters $\alpha \doteq \sqrt{1 + \frac{8k\xi}{\hbar^2}}$ and $\beta \doteq 2l + N - 2$. The eigenfunctions are othonormal w.r.t. the measure:

$$\mathrm{d}\mu(r, \boldsymbol{\theta}) \doteq \frac{r^{N-1}}{r(\xi - r)}\sin^{N-2}(\theta_1)\sin^{N-3}(\theta_2)\ldots\sin(\theta_{N-2})\mathrm{d}r\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_{N-1}, \tag{1.81}$$

with $\theta_1, \ldots, \theta_{N-2} \in [0, \pi)$ and $\theta_{N-1} \in [0, 2\pi)$ and $r \in (0, \xi)$.

Proof. After rephrasing our problem in hyperspherical coordinates (taking into account that $r \in (0, \xi)$) we arrive at the following classical radial Hamiltonian:

$$H_\xi(r, p_r) = T_\xi(r, p_r) + V_\xi(r) = \left(1 - \frac{r}{\xi}\right)\frac{r}{\xi}\left(\frac{p_r^2}{2} + \frac{\mathbf{L}^2}{2r^2}\right) + \frac{k}{\xi - r}, \tag{1.82}$$

where $\mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2\theta_k}$ is the square of the (conserved) total angular momentum. In this case, the effective one-dimensional potential reads:

$$V_{\xi,\mathrm{eff}}(r) = \frac{\xi - r}{r}\frac{\mathbf{L}^2}{2\xi^2} + \frac{k}{\xi - r}. \tag{1.83}$$

Hence:

$$\lim_{r\to 0} V_{\xi,\mathrm{eff}}(r) = +\infty, \qquad \lim_{r\to\xi} V_{\xi,\mathrm{eff}}(r) = +\infty.$$

Observing the Figure 26 we can appreciate that the qualitative behaviour of the effective potential is similar to the one of the confined TN: the deformation parameter governs the shape of the potential, which dissolves in the $\xi \to 0$ limit. Let us notice that in this limit, as expected, the model ceases to exist (the domain of definition of the radial coordinate collapses to a point and the potential dissolves).

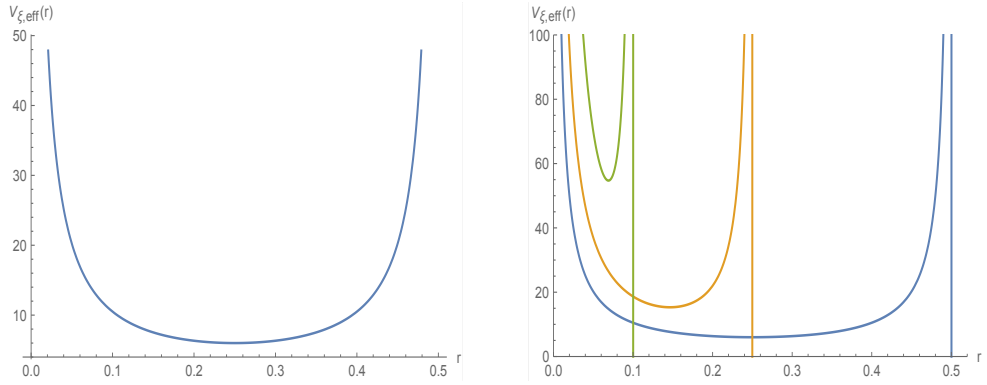

Figure 26.: effective potential $V_{\xi,\text{eff}}(r)$ calculated for $\mathbf{L}^2 = 1$, $k = 1$ and $\xi = 0.5$ (on the left) and for $\mathbf{L}^2 = 1$, $k = 1$, $\xi = 0.5$ (blue), 0.25 (orange), 0.10 (green) (on the right). The vertical lines represent the singularities at $r_c = \xi$.

In order to define the quantum counterpart of the classical Hamiltonian (1.77), we proceed exactly as in the TN and D-III cases by applying the usual Schrödinger quantization prescription. This leads to the quantum Hamiltonian operator:

$$\hat{H}_\xi = \left(1 - \frac{|\hat{\mathbf{x}}|}{\xi}\right)\frac{|\hat{\mathbf{x}}|}{\xi}\,\frac{\hat{\mathbf{p}}^2}{2} + \frac{k}{\xi - |\hat{\mathbf{x}}|} = -\frac{\hbar^2}{2}\left(1 - \frac{|\mathbf{x}|}{\xi}\right)\frac{|\mathbf{x}|}{\xi}\,\Delta_{\mathbf{x}} + \frac{k}{\xi - |\mathbf{x}|}. \tag{1.84}$$

Now, by using the hyperspherical coordinates, together with the usual definition:

$$\hat{p}_r = -\mathrm{i}\hbar\frac{\partial}{\partial r}, \qquad \hat{p}_{\theta_j} = -\mathrm{i}\hbar\frac{\partial}{\partial \theta_j}, \qquad j = 1, \ldots, N-1\,,$$

the quantum radial Hamiltonian corresponding to (1.84) reads:

$$\hat{H}_\xi = \frac{1}{2}\left(1 - \frac{r}{\xi}\right)\frac{r}{\xi}\left(\frac{1}{r^{N-1}}\,\hat{p}_r\, r^{N-1}\,\hat{p}_r + \frac{\hat{\mathbf{L}}^2}{r^2}\right) + \frac{k}{\xi - r}, \tag{1.85}$$

where $\hat{\mathbf{L}}^2$ is the square of the total quantum angular momentum operator, previously defined in (6.8). Once again, after reordering terms and by introducing the differential operators $\hat{p}_{r_j}, \hat{p}_{\theta_j}$ within the Hamiltonian (1.85), we arrive at the following Schrödinger equation:

$$\left(1 - \frac{r}{\xi}\right)\frac{r}{\xi}\left(\frac{-\hbar^2}{2}\partial_r^2 - \frac{\hbar^2(N-1)}{2r}\,\partial_r + \frac{\hat{\mathbf{L}}^2}{2r^2} +\right)\Psi(r,\boldsymbol{\theta}) + \frac{k}{\xi - r}\Psi(r,\boldsymbol{\theta}) = E\,\Psi(r,\boldsymbol{\theta}), \tag{1.86}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{N-1})$. Now, as usual, by taking into account that the hyperspherical harmonics $Y(\boldsymbol{\theta})$ satisfy:

$$\hat{\mathbf{L}}^2 Y(\boldsymbol{\theta}) = \hbar^2 l(l + N - 2)\,Y(\boldsymbol{\theta}), \quad l = 0, 1, 2 \ldots,$$

where l is the angular momentum quantum number, the equation (1.86) admits a complete set of factorized solutions of the form:

$$\Psi(r,\boldsymbol{\theta}) = \Phi(r)Y(\boldsymbol{\theta}),$$

and the radial Schrödinger equation provided by $\hat{H}_\xi$ reads:

$$\left(1-\frac{r}{\xi}\right)\frac{r}{\xi}\left(\frac{-\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+N-2)}{r^2}\right)\right)\Phi(r)+\frac{k}{\xi-r}\Phi(r)=E\,\Phi(r), \quad (1.87)$$

that can be rewritten as:

$$-\frac{\hbar^2}{2\xi^2}(\xi-r)r\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+N-2)}{r^2}\right)\Phi(r)+\frac{k}{\xi-r}\Phi(r)=E\,\Phi(r), \quad (1.88)$$

with $k,\xi>0$ and $0<r<\xi$. By direct computation, it can be proved that the solution of this second-order differential equation is given by (1.80), and that the wavefunctions $\Psi_{n,l}(r,\boldsymbol{\theta})=c_{n,l}Y^l_{l_{N-2},..,l_1}(\theta_1,\theta_2,...,\theta_{N-1})\Phi_{n,l}(r)$ are orthonormal w.r.t. the measure (1.81). □

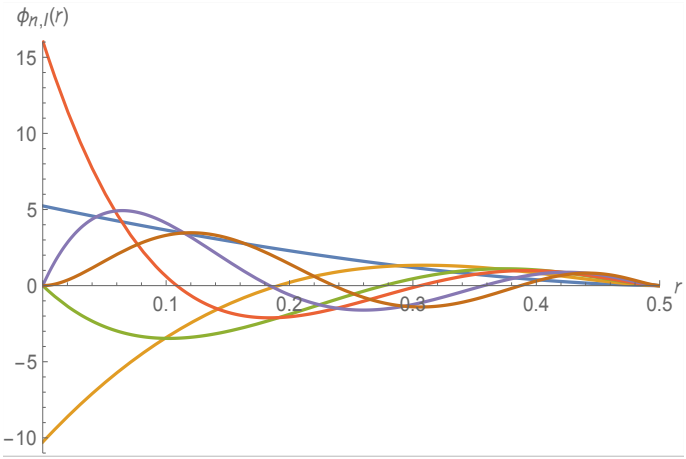

Figure 27.: radial eigenfunctions $\Phi_{0,0}(r)$ (blue), $\Phi_{1,0}(r)$ (orange), $\Phi_{1,1}(r)$ (green), $\Phi_{2,0}(r)$ (red), $\Phi_{2,1}(r)$ (purple) and $\Phi_{2,2}(r)$ (brown), calculated for $k=\hbar=1$, $\xi=0.5$ and $N=3$.

Let us notice that the dimension of the radial wave function is $[\Phi]=L^{\frac{2-N}{2}}$ ($L\equiv$ length) and its square modulus integrated with respect to the radial part of the measure (1.81) is dimensionless as expected. Let us also point out that the spectum cannot be written in terms of a single combination of the quantum numbers n and l. This should be an indicator that the system is not maximally superintegrable. However, the quasi-maximal superintegrability (QMS) is ensured due to the hyperspherical symmetry inside the punctured (open) hyperball $\mathcal{B}^N_\xi \doteq \{x_j>0|\sum_{j=1}^N x_j^2<\xi^2\}$: this system is radially symmetric in that domain of definition. In the following we will conclude our discussion by defining the metric and the scalar curvature characterizing this exactly solvable ND model.

B.2 METRIC AND SCALAR CURVATURE

The classical Hamiltonian:

$$H_\xi=T_\xi(r,p_r)+V_\xi(r)=\frac{1}{f^2_\xi(r)}\left(\frac{p_r^2}{2}+\frac{\mathbf{L}^2}{2r^2}\right)+V_\xi(r), \quad (2.89)$$

describes a particle (with unit mass) on an ND spherically symmetric space:

$$\mathcal{M}^N = (\mathcal{B}^N_\xi, g) \qquad g_{ij} = \frac{\xi^2}{(\xi - |\mathbf{x}|)|\mathbf{x}|}\delta_{ij}\,, \tag{2.90}$$

under the action of the central potential $V_\xi(r) = \frac{k}{\xi - r}$, with $k, \xi > 0$. In hyperspherical coordinates the ND spherically symmetric metric is given by:

$$\begin{cases} \mathrm{d}s^2_\xi = f^2_\xi(r)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{N-1}) \\ \mathrm{d}\Omega^2_{N-1} \doteq \sum_{j=1}^{N-1} \mathrm{d}\theta_j^2 \prod_{k=1}^{j-1} \sin^2\theta_k\,, \end{cases} \tag{2.91}$$

where $\mathrm{d}\Omega^2_{N-1}$ is the metric of the unit $(N-1)$-sphere $\mathbb{S}^{N-1}$. Here $f_\xi(r) = \frac{\xi}{\sqrt{(\xi - r)r}} > 0$, so we have two singularities of the metric in the limit $r \to 0$ and $r \to \xi$. Concerning the scalar curvature, it turns out to be:

$$\begin{aligned} R_\xi(r) &= -(N-1)\frac{(N-4)f'^2_\xi(r) + f_\xi(r)(2f''_\xi(r) + \frac{2(N-1)}{r}f'_\xi(r))}{f^4_\xi(r)}\Big|_{f_\xi(r) = \frac{\xi}{\sqrt{(\xi-r)r}}} \\ &= (N-1)\frac{(N-2)(4r^2 + 3\xi^2) - 4\xi r(2N-3)}{4\xi^2 r(\xi - r)} \quad (0 < r < \xi)\,. \end{aligned} \tag{2.92}$$

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