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Friedrich-Alexander-Universität
Erlangen-Nürnberg

Naturwissenschaftliche
Fakultät

BI-NATIONALLY-SUPERVISED DOCTORAL THESIS
IN MATHEMATICS
(XXIV-Ciclo)

Kazhdan-Lusztig combinatorics in the moment graph setting

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2011/2012

Kazhdan-Lusztig Kombinatorik und Impulsgraphen

Zusammenfassung

Impulsgraphen, sowie Kazhdan-Lusztig-Polynome, liegen an der Schnittstelle von algebraischer Kombinatorik, Darstellungstheorie und geometrischer Darstellungstheorie. Während die Kombinatorik der Kazhdan-Lusztig Theorie schon seit dem grundlegenden Artikel von Kazhdan und Lusztig (1979), das diese Polynome definiert, untersucht wurde, wurden Impulsgraphen noch nicht als kombinatorische Objekte betrachtet.

Das Ziel dieser Arbeit ist, zuerst eine axiomatische Theorie über Impulsgraphen und Garben auf ihnen zu entwickeln und anschließend diese auf die Untersuchung einer fundamentalen Klasse von Impulsgraphen, nämlich die -regulären und parabolischen- Bruhat–Impulsgraphen, anzuwenden. Sie sind mit jeder symmetrisierbaren Kac–Moody Algebra verbunden und die zugehörigen Braden–MacPherson Garben beschreiben die projektiven unzerlegbaren Objekten der - regulären oder singulären- Kategorie $\mathcal{O}$. Dies ist für uns der wichtigste Grund, zusammen mit ihrem inneren kombinatorischen Interesse, Bruhat–Impulsgraphen zu untersuchen.

Im ersten Kapitel definieren und beschreiben wir die Kategorie der k-Impulsgraphen auf einem Gitter. Zentraler Punkt dieses Teils ist die Definition des Begriffs von Morphismus.

Das zweite Kapitel ist über die Kategorie der Garben auf einem k-Impulsgraph. Wir definieren die Pullback und Push-Forward Funktoren und wir beweisen, dass sie gute Eigenschaften haben. Wie in der klassischen Garbentheorie ist der Pullback links-adjungiert zum Push-Forward. Außerdem zeigen wir, dass der Pullback eines Isomorphismus die wichtigste Klasse von Garben auf einem k-Impulsgraph, genauer die unzerlegbaren Braden-MacPherson Garben, erhält. Dieses Ergebnis wird ein grundlegendes Instrument im Kapitel 5 werden.

Im folgenden Kapitel betrachten wir die Familie von Bruhat k-Impulsgraphen, die mit einer symmetrisierbaren Kac-Moody-Algebra verbunden sind. Das interessanteste Ergebnis des ersten Teils dieses Kapitels ist die Realisierung von parabolischen Bruhat–Impulsgraphen als Quotienten - im Sinne von Kapitel 1 - des regulären Bruhat–Impulsgraph. Im zweiten Teil des Kapitels untersuchen wir den affinen Fall. Insbesondere bekommen wir eine explizite Beschreibung von gewissen endlichen Intervallen des Bruhat–Impulsgraphs die zur affine Grassmannschen assoziiert sind.

In Kapitel 4 verallgemeinern wir eine Kategorifizierung von Fiebig der Hecke-Algebra auf den parabolischen Fall. Der grundlegende Schritt ist die Definition eines involutiven Automorphismus der Strukturalgebra des parabolischen Bruhat-Impulsgraphen.

In den letzten zwei Kapiteln kategorifizieren wir gewisse Eigenschaften der Kazhdan–Lusztig-Polynome durch Garben auf Impulsgraphen. Insbesondere werden die Ergebnisse

aus Kapitel 5 über die Kombinatorik und Eigenschaften der unzerlegbaren Braden-MacPherson Garbe sowie die Aussagen aus Kapitel 1 über den Pullback Funktor angewendet. Der Beweis des Hauptergebnisses des Kapitels 6 ist ziemlich aufwendig und benutzt neben Ergebnissen von Fiebig die bis dahin entwickelten Techniken aus den vorherigen Kapiteln.

Kazhdan-Lusztig combinatorics in the moment graph setting

Abstract

Moment graphs, as well as Kazhdan-Lusztig polynomials, straddle the intersection of algebraic combinatorics, representation theory and geometric representation theory. While the combinatorial core of the Kazhdan-Lusztig theory has been investigated for thirty years, after the seminal paper of Kazhdan-Lusztig (1979) where these polynomials were defined, moment graphs have not yet been studied as combinatorial objects.

The aim of this thesis is first to develop an axiomatic theory of moment graphs and sheaves on them and then to apply it to the study of a fundamental class of moment graphs: the -regular and parabolic- Bruhat (moment) graphs. They are attached to any symmetrisable Kac-Moody algebra and the associated indecomposable Braden-MacPherson sheaves describe the indecomposable projective objects in the corresponding deformed -regular or singular- category $\mathcal{O}$. This is for us the most important reason to consider Bruhat graphs, together with their intrinsic combinatorial interest.

In the first chapter, we define and describe the category of k-moment graphs on a lattice. The fundamental point of this part is the definition of the notion of morphism.

The second chapter is about the category of sheaves on a k-moment graphs. We give the definition of the pullback and push-forward functors and we prove that they have nice properties. In particular, as in classical sheaf theory, the pullback ist left adjoint to the push-forward. Moreover, we show that the pullback of an isomorphism preserves the most important class of sheaves on a k-moment graph: the indecomposable Braden-MacPherson sheaves. This result will be a fundamental tool in Chapter 5.

In the following chapter we study the family of Bruhat (k-moment) graphs associated to a simmetrisable Kac-Moody algebra. The most interesting result of the first part of this chapter is the realisation of parabolic Bruhat graphs as quotients - in the sense of Chapter 1- of the regular one. The second part of the chapter is devoted to the study of the affine case. In particular, we describe certain finite intervals of the Bruhat graph corresponding to the Affine Grassmannian in a very precise way.

In Chapter 4 we generalise to the parabolic setting a categorification, due to Fiebig, of the Hecke algebra. The fundamental step is the definition of an involutive automorphism of the structure algebra of parabolic moment Bruhat graphs.

In the last two chapters we categorify some properties of Kazhdan-Lusztig polynomials via sheaves on moment graphs. In particular, the techniques we use in Chapter 5 play only with combinatorics, intrinsic properties of the indecomposable Braden-MacPherson sheaf and the pullback property we proved in Chapter 1. The proof of the main result of Chapter 6 is quite complicated. We need to use almost all the machinery we developped in the

previous chapters together with results due to Fiebig.

Contents

Introduction

Moment graphs appeared for the first time in 1998, in the remarkable paper of Goresky, Kottwitz and MacPherson (cf.[20]). Their aim was to describe the equivariant cohomology of a "nice" projective algebraic variety X, where "nice" means that an algebraic torus T acts equivariantly formally (cf.[[20], §1.2]) on X with finitely many 1-dimensional orbits and finitely many fixed points (all isolated). In these hypotheses, they proved that $H_T(X)$ can be described using data coming from the 1-skeleton of the T-action. In particular, such data were all contained in a purely combinatorial object: the associated *moment graph*. After Goresky, Kottwitz and MacPherson's paper, several mathematicians, as Lam, Ram, Shilling, Shimozono, Tymozcko, used moment graphs in Schubert calculus (cf.[33], [34], [42], [43], [44]).

In 2001, Braden and MacPherson gave a combinatorial algorithm to compute the T-equivariant intersection cohomology of the variety X, having a T-invariant Whitney stratification (cf. [7]). In order to do that, they associated to any moment graph an object that they called *canonical sheaf*; we will refer hereafter to it also as *Braden-MacPherson sheaf*. Even if their algorithm was defined for coefficients in characteristic zero, it works in positive characteristic too. In this case, Fiebig and Williamson proved that, under certain assumptions, it computes the stalks of indecomposable parity sheaves (cf.[19]), that are a special class of constructible (with respect to the stratification of X) complexes in $D^b_T(X;k)$, the T-equivariant bounded derived category of X over the local ring k. Parity sheaves have been recently introduced by Juteau, Mautner and Williamson (cf.[25]), in order to find a class of objects being the positive characteristic counterpart of intersection cohomology complexes. Indeed, intersection cohomology complexes play a very important role in geometric representation theory thanks to the decomposition theorem, that in general fails in characteristic p, while for parity sheaves holds.

The introduction of moment graph techniques in representation theory is due to the fundamental work of Fiebig. In particular, he associated a moment graph to any Coxeter system and defined the corresponding category of *special modules*, that turned out to be equivalent to a combinatorial category defined by Soergel in [41] (cf.[13]). The advantage of Fiebig's approach is that, as we have already pointed out, the objects he uses may be defined in any characteristic and so they may be applied in modular representation theory. In particular, they provided a totally new approach to Lusztig's conjecture (cf.[37]) on the characters of irreducible modules of semisimple, simply connected, reductive algebraic groups over fields of characteristic bigger than the corresponding Coxeter number (cf.[18]). This conjecture was proved to be true if the characteristic of the base field is *big enough* (by combining [31], [27] and [1]), in the sense that it is true in the limit, while Fiebig's work provided an explicit -but still huge!- bound (cf.[16]). The characteristic zero analog of Lusztig's conjecture, stated by Kazhdan and Lusztig a year before in [29], and proved a couple of years later, independently, by Brylinski-Kashiwara (cf.[8]) and Beilinson -Bernstein (cf.[4]), admits a new proof in this moment graph setting (cf.[14]). In an ongoing project Fiebig and Arakawa are working on the Feigin-Frenkel conjecture on the restricted category $\mathcal{O}$ for affine Kac-Moody algebras at the critical level via sheaves on moment graphs (cf.[2], [3]). A very recent paper of Shan, Varagnolo and Vasserot uses moment graphs to prove the parabolic/singular Koszul duality for the category $\mathcal{O}$ of affine Kac-Moody algebras (cf.[39]), showing that the role played by these objects in representation theory is getting more and more important.

The aim of this thesis is first to develop an axiomatic theory of moment graphs and sheaves on them and then to apply it to the study of a fundamental class of moment graphs: the -regular and parabolic- Bruhat (moment) graphs. They are attached to any symmetrisable Kac-Moody algebra and the associated indecomposable Braden-MacPherson sheaves give the indecomposable projective objects in the corresponding deformed -regular or singular- category $\mathcal{O}$ (cf.[[14],§6]). This is for us the most important reason to consider Bruhat graphs, together with their intrinsic combinatorial interest.

Thesis organisation

Here we describe the structure of our dissertation and present briefly the main results.

From now on, Y will denote a lattice of finite rank, k a local ring such that $2 \in k^{\times}$ and $Y_k := Y \otimes_{\mathbb{Z}} k$.

In the first chapter, we develop a theory of moment graphs. In order to do that, we first had to choose if we were going to work with moment graphs on a vector space (as Goresky-Kottwitz and MacPherson do in [20]) or on a lattice. The first possibility would enable us to associate a moment graph to any Coxeter system (cf.[13]), while the second one has the advantage that a modular theory could be developed (cf. [18]). We decided to work with moment graphs on a lattice, because our results of Chapter 5 and Chapter 6 in characteristic zero categorify properties of Kazhdan-Lusztig polynomials, while in positive characteristic they give also information about the stalks of indecomposable parity sheaves ([19]). Thus, from now on we will speak of k-moment graphs, where k is any local ring with $2 \in k^{\times}$. However, our proofs can be adapted to moment graphs on a vector space, by slightly modifying some definitions.

After recalling the definition of *k-moment graph* on a lattice Y, following [16], we introduce the new concept of *homomorphism* between two k-moment graphs on Y. This is given by nothing but an order-preserving map of oriented graphs together with a collection of automorphisms of the k-module Y_k, satisfying some technical requirements (see §1.2). In this way, once proved that the composition of two homomorphisms of k-moment graphs is again a homomorphism of k-moment graphs (see Lemma 1.2.1), we get the category $\mathbf{MG}(Y_k)$ of k-moment graphs on the lattice Y and in the rest of the chapter we describe some properties of it.

The following chapter is about the category $\mathbf{Sh}_k(\mathcal{G})$, of *sheaves* on the k-moment graph $\mathcal{G}$. We start with recalling some concepts and results from [7], [14], [15], [19]; in particular, the definition of *canonical* or *Braden-MacPherson* sheaves. Even if these objects are not sheaves in the algebro-geometric sense but only combinatorial and commutative algebraic objects, we define *pull-back* and *push-forward* functors (see §2.2). Let $f : \mathcal{G} \to \mathcal{G}'$ be a homomorphism of k-moment graphs on Y, then we are able to prove that, as in algebraic geometry, the adjunction formula holds.

Proposition 0.0.1. *Let $f \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$, then f^* is left adjoint to f_*, that is for all pairs of sheaves $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$ and $\mathcal{H} \in \boldsymbol{Sh}_k(\mathcal{G}')$ the following equality holds*

$$Hom_{\boldsymbol{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) = Hom_{\boldsymbol{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F}) \tag{1}$$

We end the chapter with proving a fundamental property of canonical sheaves, namely we show that, if $f : \mathcal{G} \to \mathcal{G}'$ is an isomorphism, then the pullback functor f^* preserves indecomposable Braden-MacPherson sheaves (see Lemma 2.2.2). This result will provide us with an important technique to compare indecomposable canonical sheaves on different k-moment graphs, that we will use in Chapter 5.

Let $\mathfrak{g}$ be a Kac-Moody algebra, then there is a standard way to associate to $\mathfrak{g}$ certain k-moment graphs on its coroot lattice (cf. [15]), the corresponding *regular* and *parabolic* (k-moment) *Bruhat graphs*. Denote by $\mathcal{W}$ the Weyl group of $\mathfrak{g}$, that it is in particular a Coxeter group. Let $\mathcal{S}$ be its set of simple reflections, then, for any subset $J \subset \mathcal{S}$ there is exactly one parabolic Bruhat graph, that we denote $\mathcal{G}^J$. These are the main objects of Chapter 3. After giving some examples, we prove that all parabolic k-moment Bruhat graphs associated to $\mathfrak{g}$ are nothing but quotients of its regular Bruhat graph (see Corollary 3.1.2). We then focus our attention on the case of $\mathfrak{g}$ affine Kac-Moody algebra. The most interesting parabolic Bruhat graph attached to $\mathfrak{g}$ is the one corresponding to the Affine Grassmannian, that we denote $\mathcal{G}^{\mathrm{par}} = \mathcal{G}^{\mathrm{par}}(\mathfrak{g})$, and we consider it in §3.2.2. Once showed that the set of vertices of $\mathcal{G}^{\mathrm{par}}$ may be identified with the set of alcoves in the fundamental Weyl chamber $\mathcal{C}^+$, we study finite intervals of $\mathcal{G}^{\mathrm{par}}$ *far enough* in $\mathcal{C}^+$. We are able to describe these intervals in a very precise way (see Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.1, Lemma 3.2.4). In particular, we notice that the set of edges is naturally bipartite and this gives rise to the definition of a new k-moment graph attached to $\mathfrak{g}$: the *stable* moment graph (see §3.2.3), that is a subgraph of $\mathcal{G}^{\mathrm{par}}$.

In Chapter 4, we generalise a construction of Fiebig. Let $\mathfrak{g}$ be a Kac-Moody algebra, then we may consider the attached Bruhat graphs. In the case of the regular Bruhat graph

$\mathcal{G} = \mathcal{G}^{\emptyset}(\mathfrak{g})$, Fiebig defined translation functors on the category of $\mathbb{Z}$-graded $\mathcal{Z}$-modules, where $\mathcal{Z}$ is the structure algebra (see §2.1.1) of $\mathcal{G}$. Moreover, in [18] he considered a subcategory $\mathcal{H}$ of the category of $\mathbb{Z}$-graded $\mathcal{Z}$-modules and proved that it gives a categorification of the Hecke algebra $\mathbf{H}$ of $\mathcal{W}$. In a similar way, for any parabolic moment graph $\mathcal{G}^J$ attached to $\mathfrak{g}$, we are able to define translation functors $\{{}^s\theta\}_{s\in\mathcal{S}}$ and the category $\mathcal{H}^J$. Actually, if $\mathbf{H}^J$ is the parabolic Hecke algebra defined by Deodhar in [9], this admits an action of the regular Hecke algebra $\mathbf{H}$. Recall that Kazhdan and Lusztig in [29] defined the *canonical basis* of $\mathbf{H}$, that we denote, following Soergel's notation, by $\{\underline{H}_x\}_{x\in\mathcal{W}}$. Then, if $\langle\mathcal{H}^J\rangle$ is the Grothendieck group of $\mathcal{H}^J$, we may define a *character map* $h : \langle\mathcal{H}^J\rangle \to \mathbf{H}^J$ (see §4.2.2) and, for any simple reflection $s \in \mathcal{S}$, we get the following commutative square (see Proposition 4.2.1).

$$\begin{tikzcd}
\langle\mathcal{H}^J\rangle \arrow[r, "h"] \arrow[d, "{}^s\theta\circ\{1\}"'] & \mathbf{M}^J \arrow[d, "\underline{H}_s\cdot"] \\
\langle\mathcal{H}^J\rangle \arrow[r, "h"] & \mathbf{M}^J
\end{tikzcd},$$

where $\{1\}$ denotes the degree shift functor on the $\mathbb{Z}$-graded category $\mathcal{H}^J$.

In Chapter 5 we report and expand results that have been already presented in our paper [35]. We were motivated by the *multiplicity conjecture* (cf. [16]), a conjectural formula relating the stalks of the indecomposable Braden-MacPherson sheaves on a Bruhat graph $\mathcal{G}^J$ to the corresponding Deodhar's parabolic Kazhdan-Lusztig polynomials for the parameter $u = -1$ (cf. [9]), that we denote $\{m^J_{x,y}\}$ as Soergel does in [40]. The aim of this chapter is then to lift properties of the $m^J_{x,y}$'s to the level of canonical sheaves, that is to categorify some well-know equalities concerning Kazhdan-Lusztig polynomials. We mainly use three strategies to get our claims:

- *Technique of the pullback.* We look for isomorphisms of k-moment graphs and then, via the pullback functor (see Lemma 2.2.2), we get the desired equality between stalks of Braden-MacPherson sheaves (see §5.2).

- *Technique of the set of invariants.* For any $s \in \mathcal{S}$ we define an involution σ_s of the set of local sections of a canonical sheaf on an s-invariant interval of $\mathcal{G}$. In this case, the study of the space of the invariants gives us the property we wanted to show (see §5.3).

- *Flabbiness of the structure sheaf.* It is known (cf. [17]) that the so-called *structure sheaf* (see Example 2.1.1) is isomorphic to an indecomposable Braden-MacPherson sheaf if and only if it is flabby and this is the case if and only if the corresponding Kazhdan-Lusztig polynomials evaluated in 1 are all 1. We prove in a very explicit way that the structure sheaf is flabby to categorify the fact that the associated polynomials evaluated in 1are 1 (see§5.1 and §5.4).

The aim of the last chapter is to describe indecomposable canonical sheaves on finite intervals of $\mathcal{G}^{\mathrm{par}}$ far enough in $\mathcal{C}^+$. Our motivation comes from the multiplicity conjecture together with a result, due to Lusztig (cf. [37]), telling us that for any pair of alcoves $A, B \subset \mathcal{C}^+$ there exists a polynomial $q_{A,B}$, called the *generic polynomial* of the pair A, B, such that

$$\varinjlim_{\mu \in \mathcal{C}^+} m^{\mathrm{par}}_{A+\mu, B+\mu} = q_{A,B}. \tag{2}$$

Actually, this result relates the Hecke algebra of the affine Weyl group $\mathcal{W}^a$ to its periodic module $\mathbf{M}$. Our interest in $\mathbf{M}$ is motivated now by the fact that $\mathbf{M}$ governs the representation theory of the affine Kac-Moody algebra, whose Weyl group is $\mathcal{W}^a$, at the critical level.

Suppose that $A, B, A+\mu, B+\mu$ are alcoves far enough in the fundamental chamber. Then results of §3.2.2 show that the two moment graphs $\mathcal{G}^{\mathrm{par}}_{|[A,B]}$ and $\mathcal{G}^{\mathrm{par}}_{|[A+\mu,B+\mu]}$ are in general not isomorphic, while there is always an isomorphism of moment graphs between $\mathcal{G}^{\mathrm{stab}}_{|[A,B]}$ and $\mathcal{G}^{\mathrm{stab}}_{|[A+\mu,B+\mu]}$. Since the stable moment graph is a subgraph of $\mathcal{G}^{\mathrm{par}}$, there is a monomorphism $\mathcal{G}^{\mathrm{stab}} \hookrightarrow \mathcal{G}^{\mathrm{par}}$. The following diagram summarises this situation:

$$\begin{tikzcd}
\mathcal{G}^{\mathrm{par}}_{|[A,B]} & \mathcal{G}^{\mathrm{par}}_{|[A+\mu,B+\mu]} \\
\mathcal{G}^{\mathrm{stab}}_{|[A,B]} \arrow[u, hook, "i"] \arrow[r] & \mathcal{G}^{\mathrm{stab}}_{|[A+\mu,B+\mu]} \arrow[u, hook, "i_\mu"']
\end{tikzcd}$$

We then get a functor $\cdot^{\mathrm{stab}} := i^* : \mathbf{Sh}_{\mathcal{G}^{\mathrm{par}}_{|[A,B]}} \to \mathbf{Sh}_{\mathcal{G}^{\mathrm{stab}}_{|[A,B]}}$. The main theorem of this chapter is the following one.

Theorem 0.0.1. *The functor* $\cdot^{stab} : \boldsymbol{Sh}_k(\mathcal{G}^{par}_{|[A,B]}) \to \boldsymbol{Sh}_k(\mathcal{G}^{stab}_{|[A,B]})$ *preserves indecomposable Braden-MacPherson sheaves.*

The stabilisation property, that is the categorification of Equality (2), follows by applying the technique of the pullback to the previous result.

In the case of $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, we are able to prove the claim via the third technique we quoted above, that is, for any finite interval of $\mathcal{G}^{\mathrm{stab}}$, we show that in characteristic zero its structure sheaf is flabby, so it is invariant by weight translation for all integral weights $\mu \in \mathcal{C}^+$. On the other hand, we know already that the structure sheaf for the affine Grassmannian is flabby (see §5.4) and this concludes the $\mathfrak{sl}_2$-case.

For the general case, we apply a localisation technique due to Fiebig (that we recall in Chapter 4), which enables us to use the $\mathfrak{sl}_2$-case, together with results of [18].

Perspectives

Since the theory of sheaves on moment graphs is related to geometry, representation theory and algebraic combinatorics, we briefly present three possible applications or developments of this theory on which we are interested in, one for each of these fields.

Equivariant cohomology of affine Bott-Samelson varieties.

In a joint project with Stéphane Gaussent and Michael Ehrig (cf.[12]), we try to generalise to the affine setting the paper [21] of Härterich, where the author describes the T-equivariant cohomology of Bott-Samelson varieties in terms of Braden-MacPherson sheaves on the corresponding Schubert varieties.

Periodic patterns and the Feigin-Frenkel conjecture.

The Feigin-Frenkel conjecture provides a character formula involving Lusztig's periodic polynomials (cf.[37]). In [28], Kato related these polynomials to the generic polynomials. In particular, he showed that generic polynomials are sum of periodic polynomials with certain multiplicities. We believe that a natural development of the results we got in Chapter 5 is to prove an analog of this periodicity property for canonical sheaves. It should correspond to a filtration of the space of global sections of the indecomposable Braden-MacPherson sheaf. In an ongoing project with Peter Fiebig we try to understand this phenomenon and to apply it to get a further step in the proof of the Feigin-Frenkel conjecture. The representation theory of affine Kac-Moody algebras at the critical level is very complicated and, thanks to the fundamental work of Frenkel and Gaitsgory, it is related to the geometric Langlands correspondence.

Moment graphs and Littelmann path model.

In 2008, during the Semester " Combinatorial Representation Theory" at the MSRI of Berkeley, Ram conjectured a connection between the Littelmann path model and affine Kazhdan-Lusztig polynomials (the so–called "Théorève"). Since in characteristic zero the multiplicity conjecture is proved, our hope is that we may get a better understanding of this connection via the study of indecomposable Braden-MacPherson sheaves, by applying results we obtained in this thesis.

Acknowledgements

I am grateful to my advisor P. Peter Fiebig, for his guidance and support during these two years.

I would also like to thank my coadvisor Lucia Caporaso, the coordinator of the Doctoral School of Mathematics of the University of Roma Tre Luigi Chierchia, the former coordinator Renato Spigler, the coordinator of the Doctoral School of Natural Sciences of the Friedrich-Alexander-Universität of Erlangen-Nürnberg Wolfgang Achtziger and the former coordinator Eberhard Bänsch; they all made this binationally supervised PhD thesis possible.

I would like to acknowledge Corrado De Concini, the advisor of my master thesis, for his encouragement and for helpful discussions also during my PhD studies.

This dissertation would not have been written without Andrea Maffei, who drew my attention to the work of Peter Fiebig, and Francesco Esposito, without whose help and support I would never even have started my PhD.

I also owe thanks to Rocco Chirivi, Michael Ehrig and Geordie Williamson for useful conversations. I am grateful to Vladimir Shchigolev for reading carefully a preliminary version of this thesis. His comments and suggestions certainly improved this dissertation.

Many thanks go to Rollo Jenkins for help with the language.

I would like to thank the INdAM and the Frauenbeauftragte der Erlangen-Nürnberg Universität for funding part of my stay at the Friedrich-Alexander-Universität of Erlangen-Nürnberg.

Finally, I am grateful to the Hausdorff Research Institute for Mathematics in Bonn for its hospitality during the programme "On the Interaction of Representation Theory with Geometry and Combinatorics", where I spent a very fruitful period. Many thanks go to the organisers of this programme, Steffen König, Peter Littelmann, Jan Schröer and Catharina Stroppel, who gave me the opportunity to take part in it.

Chapter 1

The category of k-moment graphs on a lattice

Moment graphs were introduced by Goresky, Kottwitz and MacPherson in 1998, in order to give a combinatorial description of the T-equivariant cohomology of a complex algebraic variety X equipped with an action of a complex torus T, satisfying some technical assumptions (cf.[20]). A couple of years later, Braden and MacPherson, in [7], used moment graphs to compute the T-equivariant intersection cohomology of X. Since 2006, thanks to the seminal work of Fiebig (cf.[13],[14],[18],[16],[17]), moment graphs have become a powerful tool in representation theory as well. Even if in the last years moment graphs appeared in several papers, a proper "moment graph theory" has not been developped yet. The aim of this section is to define the category of moment graphs on a lattice and to discuss some examples and properties of it.

1.1 Moment graphs

In [20] and [7], moment graphs were constructed from a geometrical datum, but it is actually possible to give an axiomatic definition.

Definition 1.1.1 ([16])**.** *Let Y be a lattice of finite rank. A* moment graph on the lattice *Y is given by $(\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, where:*

(MG1) $(\mathcal{V}, \mathcal{E})$ *is a directed graph without directed cycles nor multiple edges,*

(MG2) $\trianglelefteq$ *is a partial order on* $\mathcal{V}$ *such that if* $x, y \in \mathcal{V}$ *and* $E : x \to y \in \mathcal{E}$*, then* $x \trianglelefteq y$*,*

(MG3) $l : \mathcal{E} \to Y \setminus \{0\}$ *is a map called the* label function*.*

Following Fiebig's notation ([16]), we will write $x \text{ --- } y$ if we are forgetting about the orientation of the edge.

Studying complex algebraic varieties, Braden, Goresky, Kottwitz and MacPherson considered moment graphs only in characteristic zero, while they turned out to be very important in prime characteristic (see [18], [19]).

From now on, k will be a local ring such that 2 is an invertible element. Moreover, for any lattice Y of finite rank, we will denote by $Y_k := Y \otimes_{\mathbb{Z}} k$.

Definition 1.1.2. *Let $\mathcal{G}$ be a moment graph on the lattice Y. We say that $\mathcal{G}$ is a k-*moment graph *on Y if all labels are non-zero in Y_k*

Definition 1.1.3. *[19] The pair $(\mathcal{G}, k)$ is called a GKM-*pair *if all pairs E_1, E_2 of distinct edges containing a common vertex are such that $k \cdot l(E_1) \cap k \cdot l(E_2) = \{0\}$.*

1.1.1 Examples

Example 1.1.1. *The* empty *k-moment graph is given by the graph having empty set of vertices. All the other data are clearly uniquely determined. We will denote it by $\emptyset$.*

Example 1.1.2 (cf.[16])**.** *A* generic *k-moment graph is a moment graph having a unique vertex. As in the previous example, all the other data are uniquely determined.*

Example 1.1.3 (cf.[16])**.** *A* subgeneric *k-moment graph on Y is a moment graph having two vertices and an (oriented) edge, labelled by a non-zero element $\chi \in Y$, such that $\chi \otimes 1$ is non-zero in Y_k too.*

Example 1.1.4. *We recall here the construction, due to Braden an MacPherson, appeared in [7]. Let G be an irreducible complex projective algebraic variety, with an algebraic action of a complex torus $T \cong (\mathbb{C}^*)^d$. Denote moreover by $X^*(T)$ the character lattice of the torus. If G has a T-invariant Whitney stratification by affine spaces and the action of T is nice enough (see [7], §1.1), then the associated moment graph is defined as follows. Thanks to the technical assumptions made by Braden and MacPherson, any 1-dimensional orbit turns out to be a copy of $\mathbb{C}^*$, whose closure contains exactly two fixed points. Thus, it makes sense to declare that the set of vertices, resp. of edges, of the associated moment graph is given by the set of fixed points, resp. of 1-dimensional orbits, with respect this T-action. Moreover, the assumptions on the variety imply that any stratum contains exactly one fixed point. Then, taken any two (distinct) fixed points, x, y, that is two vertices of the graph we are building, we set $x \leq y$ if and only if the closure of the stratum corresponding to y contains the stratum corresponding to x.*

Now, we want to label all edges of the graph, in order to record more informations about the torus action. Let E be an edge. Any point z of the one-dimensional orbit E has clearly the same stabilizer $Stab_T(z)$ in T, that is the kernel of a character $\chi \in X^*(T)$. We then set $l(E) := \chi$. We obtain in this way a moment graph on $X^*(T)$.

In Chapters 3, 4, 5 and 6 we will focus our attention on a class of moment graphs associated to a symmetrisable Kac-Moody algebra: the Bruhat graphs. These graphs are nothing but an example of the Braden-MacPherson construction for the associated flag variety that we described above (cf.[19], §7).

1.2 Morphisms of k-moment graphs

In this section, we give the definition of morphism between two k-moment graphs. Since a k-moment graph is an ordered graph, whose edges are labeled by (non-zero) elements of Y having non-zero image in Y_k, a morphism will be given by a morphism of oriented graphs together with a family of automorphisms of Y_k.

Definition 1.2.1. *A morphism between two k-moment graphs*

$$f:(\mathcal{V},\mathcal{E},\trianglelefteq,l)\to(\mathcal{V}',\mathcal{E}',\trianglelefteq',l')$$

is given by $(f_{\mathcal{V}},\{f_{l,x}\}_{x\in\mathcal{V}})$, *where*

(MORPH1) $f_{\mathcal{V}}:\mathcal{V}\to\mathcal{V}'$ *is any map of posets such that, if* $x —— y\in\mathcal{E}$, *then either* $f_{\mathcal{V}}(x) —— f_{\mathcal{V}}(y)\in\mathcal{E}'$, *or* $f_{\mathcal{V}}(x)=f_{\mathcal{V}}(y)$. *For a vertex* $E:x —— y\in\mathcal{E}$ *such that* $f_{\mathcal{V}}(x)\neq f_{\mathcal{V}}(y)$, *we will denote* $f_{\mathcal{E}}(E):=f_{\mathcal{V}}(x) —— f_{\mathcal{V}}(y)$.

(MORPH2) For all $x\in\mathcal{V}$, $f_{l,x}:Y_k\to Y_k\in Aut_k(Y_k)$ *is such that, if* $E:x —— y\in\mathcal{E}$ *and* $f_{\mathcal{V}}(x)\neq f_{\mathcal{V}}(y)$, *the following two conditions are verified:*

(MORPH2a) $f_{l,x}(l(E))=h\cdot l'(f_{\mathcal{E}}(E))$, *for some* $h\in k^{\times}$

(MORPH2b) $\pi\circ f_{l,x}=\pi\circ f_{l,y}$, *where* π *is the canonical quotient map* $\pi:Y_k\to Y_k/l'(f_{\mathcal{E}}(E))Y_k$.

If $f:\mathcal{G}=(\mathcal{V},\mathcal{E},\trianglelefteq,l)\to\mathcal{G}'=(\mathcal{V}',\mathcal{E}',\trianglelefteq',l')$ and $g:\mathcal{G}'\to\mathcal{G}''=(\mathcal{V}'',\mathcal{E}'',\trianglelefteq'',l'')$ are two morphisms of k-moment graphs, then there is a natural way to define the composition. Namely, $g\circ f:=(g_{\mathcal{V}'}\circ f_{\mathcal{V}},\{g_{l',f_{\mathcal{V}}(x)}\circ f_{l,x}\}_{x\in\mathcal{V}})$.

Lemma 1.2.1. *The composition of two morphisms between k-moment graphs is again a morphism, and it is associative.*

Proof. The only conditions to check are *(MORPH2a)* and *(MORPH2b)*. Suppose that $E:x —— y\in\mathcal{E}$ and $g_{\mathcal{V}'}\circ f_{\mathcal{V}}(x)\neq g_{\mathcal{V}'}\circ f_{\mathcal{V}}(y)$, that is $f_{\mathcal{V}}(x)\neq f_{\mathcal{V}}(v)$ and $g_{\mathcal{V}'}(f_{\mathcal{V}}(x))\neq g_{\mathcal{V}'}(f_{\mathcal{V}}(v))$. If $f_{l,x}(l(E))=h'\cdot l'(f_{\mathcal{E}}(E))$ and $g_{l',f_{\mathcal{V}}(x)}(l'(f_{\mathcal{E}}(E)))=h''\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E))$, with $h',h''\in k^{\times}$, then

$$(g_{l',f_{\mathcal{V}}(x)}\circ f_{l,x})(l(E))=g_{l',f_{\mathcal{V}}(x)}(h'\cdot l'(f_{\mathcal{E}}(E)))=h'\cdot h''\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E))=\tilde{h}\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E)),$$

and clearly $\tilde{h}=h'\cdot h''\in k^{\times}$.

Moreover,

$$\begin{aligned}&(g_{l',f_{\mathcal{V}}(x)}\circ f_{l,x})(\lambda)=\\&=g_{l',f_{\mathcal{V}}(x)}(f_{l,y}(\lambda)+n'l'(f_{\mathcal{E}}(E))=\\&=g_{l',f_{\mathcal{V}}(x)}(f_{l,y}(\lambda))+n'\cdot h''\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E))=\\&=g_{l',f_{\mathcal{V}}(y)}(f_{l,y}(\lambda))+n'''\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E))+n'\cdot h''\cdot l''(g_{\mathcal{E}'}\circ f_{\mathcal{E}}(E))\end{aligned}$$

where $n, n'' \in k$.

Finally, the associativity follows from the definition. $\square$

For any k-moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, we set $\mathrm{id}_{\mathcal{G}} = (\mathrm{id}_{\mathcal{V}}, \{\mathrm{id}_{Y_k}\}_{x \in \mathcal{V}})$. Thus we may give the following definition

Definition 1.2.2. *We denote by $\boldsymbol{MG}(Y_k)$ the category whose objects are the k-moment graphs on Y and whose morphisms are as in Def.1.2.1.*

1.2.1 Mono-, epi- and isomorphisms

Here we characterise some particular morphisms of k-moment graphs: monomorphisms, epimorphisms and isomorphisms in $\mathbf{MG}(Y_k)$.

Lemma 1.2.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l), \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l') \in \boldsymbol{MG}(Y_k)$ and $f \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$.*

(i) f is a monomorphism if and only if $f_{\mathcal{V}}$ is an injective map of sets (satisfying condition (MORPH1))

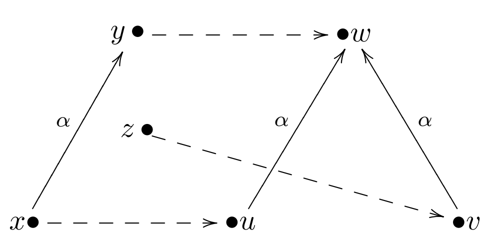
(ii) f is an epimorphism if and only if $f_{\mathcal{V}}$ is a surjective map of sets (satisfying condition (MORPH1))

Proof.

(i) f is a monomorphism if and only if, for any pair of parallel morphisms $g_1, g_2 : \mathcal{H} \to \mathcal{G}$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. Then, f is a monomorphism if and only if $f_{\mathcal{V}}$ is a monomorphisms in the category of sets and, for any $x \in \mathcal{V}$, $f_{l,x}$ is a monomorphism in the category of the k-modules, but by definition it is an automorphism of Y_k, so this condition is empty.

(ii) As in *(i)*, we conclude easily that f is an epimorphism if and only if $f_{\mathcal{V}}$ s a surjective map of sets. $\square$

Example 1.2.1. *Consider the following map between graphs*



If we set $f_{l,x} = f_{l,y} = f_{l,w} = id_{Y_k}$, we get an homomorphism of k-moment graphs that is, by Lemma 1.2.2, a monomorphism and an epimorphism.

A map between sets, that is both injective and bijective, is an isomorphism. Here, we show that such a property does not hold for a homomorphism of k-moment graphs, even if it is given by a map between the sets of vertices and an automorphism of Y_k. This is actually not surprising, since k-moment graphs will play in our theory (see next chapter) the role that topological spaces play in sheaf theory and not all bijective continuous maps between topological spaces are homeomorphisms.

Lemma 1.2.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l), \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l') \in \boldsymbol{MG}(Y_k)$ and $f = (f_{\mathcal{V}}, \{f_{l,x}\}_{x\in\mathcal{V}}) \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$. f is an isomorphism if and only if the following two conditions hold:*

(ISO1) $f_{\mathcal{V}}$ is an isomorphism of posets

(ISO2) for all $u \to w \in \mathcal{E}'$, there exists an edge $x \to y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = w$.

Proof. At first, we show that a homomorphism satisfying *(ISO1)* and *(ISO2)* is invertible. Denote by $f^{-1} := (f'_{\mathcal{V}'}, \{f'_{l',u}\}_{u\in\mathcal{V}'})$, where we set $f'_{\mathcal{V}'} := f_{\mathcal{V}}^{-1}$ and $f'_{l',u} := f^{-1}_{l,f_{\mathcal{V}}^{-1}(u)}$. We have to verify that f^{-1} is well-defined, that is we have to check conditions *(MORPH2a)* and *(MORPH2b)*. Suppose there exists an edge $F : u \to w \in \mathcal{E}'$, then, by *(iii)*, there is an edge $E : x \to y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = w$. Since f satisfies *(MORPH2a)*, $f_{l,x}(l(E)) = h \cdot l'(F)$ for $h \in k^{\times}$ and we get

$$f'_{l',u}(l'(F)) = f^{-1}_{l,f_{\mathcal{V}}^{-1}(u)}(l'(F)) = f^{-1}_{l,x}(l'(F)) = h^{-1} \cdot l(E)$$

Now, let $\mu \in Y_K$ and take $\lambda := f^{-1}_{l,y}(\mu)$. By *(MORPH2a)*, $\mu = f_{l,y}(\lambda) = f_{l,x}(\lambda) + r \cdot l'(F)$ for some $r \in k$. It follows

$$\begin{aligned} f'_{l',u}(\mu) = f^{-1}_{l,x}(\mu) = f^{-1}_{l,x}(f_{l,x}(\lambda) + rl'(F)) = \\ = \lambda + r \cdot f^{-1}_{l,x}(l'(F)) = f^{-1}_{l,y}(\mu) + r \cdot h^{-1} \cdot l(E) = \\ = f'_{l',w}(\mu) + r' \cdot l(E) \end{aligned}$$

Suppose f is an isomorphism. If *(ISO1)* is not satisfied, then $f_{\mathcal{V}}$, and hence f, is not invertible. Moreover, *(ISO1)* implies that for all $u \to v \in \mathcal{E}'$, there exists at most one $x \to y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = v$ (otherwise $f_{\mathcal{V}}$ would not be injective). Now, let f be the following homomorphism, (we do not care about the $f_{l,x}$'s)

$$\begin{tikzcd} & y \bullet \arrow[rr, dashed] & & \bullet w \\ x \bullet \arrow[rr, dashed] & & \bullet u \arrow[ur, "\alpha"] & \end{tikzcd}$$

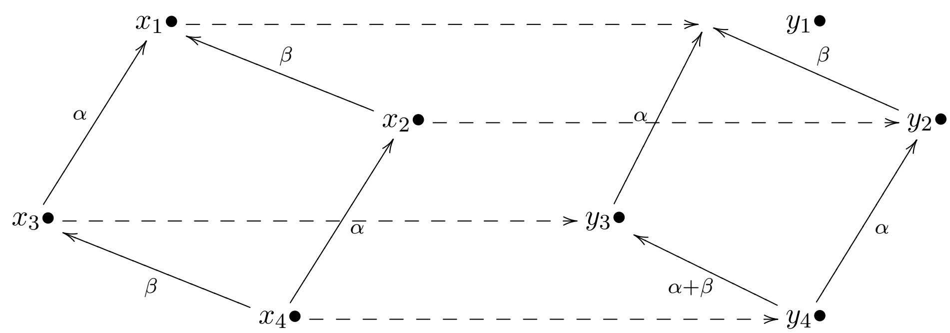
Condition *(ISO1)* holds, but f is not invertible, since $f_{\mathcal{V}}^{-1}(u) \neq f_{\mathcal{V}}^{-1}(w)$ but $f_{\mathcal{V}}^{-1}(u) \text{———} f_{\mathcal{V}}^{-1}(w) \notin \mathcal{E}$ (this contradicts *(MORPH1)*). □

Example 1.2.2. *All the generic k-moment graphs are in the same isomorphism class in $\boldsymbol{MG}(Y_k)$. Then, we will say in the sequel* the *generic k-moment graph and we will denote it by $\{pt\}$.*

Example 1.2.3. *If k is a field, then all the subgeneric k-moment graphs are isomorphic.*

Example 1.2.4. *The homomorphism in Ex. 1.2.1 is surjective and injective but is not an isomorphism.*

Example 1.2.5. *Let α, β be a basis of Y_k. Consider the following morphism of graphs $(f_{\mathcal{V}}, f_{\mathcal{E}})$:*



Define

$$f_{l,x_1} := \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \beta \end{cases} \qquad f_{l,x_2} := \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \beta \end{cases} \qquad f_{l,x_3} := \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \alpha + \beta \end{cases} \qquad f_{l,x_4} := \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \alpha + \beta \end{cases}$$

We have to show that these data define a morphism of k-moment graphs. Condition (MORPH2a) is trivially satisfied. Moreover, for any pair $a, b \in k$,

$$\begin{aligned} &f_{l,x_1}(a\alpha + b\beta) - f_{l,x_2}(a\alpha + b\beta) = 0 \\ &f_{l,x_3}(a\alpha + b\beta) - f_{l,x_4}(a\alpha + b\beta) = 0 \\ &f_{l,x_1}(a\alpha + b\beta) - f_{l,x_3}(a\alpha + b\beta) = -b\alpha = -b \cdot l(x_3 \to x_1) \\ &f_{l,x_2}(a\alpha + b\beta) - f_{l,x_4}(a\alpha + b\beta) = -b\alpha = -b \cdot l(x_4 \to x_2) \end{aligned}$$

Then, condition (MORPH2b) holds too. Since the $f_{l,x}$ are all automorphisms of Y_k, f is an isomorphism.

Lemma 1.2.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l), \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l') \in \boldsymbol{MG}(Y_k)$. Then, any isomorphism $f = (f_{\mathcal{V}}, \{f_{l,x}\}) \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$ can be written, in a unique way, as composition of two isomorphisms $f = g \circ t$ with $g = (f_{\mathcal{V}}, \{id_{Y_k}\})$ and $t = (id_{\mathcal{V}}, \{f_{l,x}\})$.*

Proof. We have to show that there exists a k-moment graph $\mathcal{H}$ such that $t \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{H})$, $g \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{H}, \mathcal{G}')$ and the following diagram commutes

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "f"] \arrow[dr, "t"'] & & \mathcal{G}' \\ & \mathcal{H} \arrow[ur, "g"'] & \end{tikzcd} \tag{1.1}$$

Define $\mathcal{H}$ as the k-moment graph, whose set of vertices, set of edges and partial order are the same as $\mathcal{G}$ and, for any edge $x \to y \in \mathcal{E}$, the label function is defined as follows

$$l_{\mathcal{H}}(x \to y) := l'(f_{\mathcal{V}}(x) \to f_{\mathcal{V}}(y))$$

Now, it is easy to check that $t \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{H})$ and $g \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{H}, \mathcal{G}')$. Clearly, Diagram (1.1) commutes. Observe that $\mathcal{H}$ is not the only k-moment graph having the desired properties, but this does not affect the uniqueness of the decomposition of f. □

1.2.2 Automorphisms

For any $\mathcal{G} \in \mathbf{MG}(Y_k)$, denote by $\mathrm{Aut}(\mathcal{G})$ the automorphisms group of $\mathcal{G}$. Moreover, we set

$$T := \{f \in \mathrm{Aut}(\mathcal{G}) \mid f = (\mathrm{id}_{\mathcal{V}}, \{f_{l,x}\})\} \tag{1.2}$$

$$G := \{f \in \mathrm{Aut}(\mathcal{G}) \mid f = (f_{\mathcal{V}}, \{\mathrm{id}_{Y_k}\})\} \tag{1.3}$$

Lemma 1.2.5. *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$, then T and G is are normal subgroups of $(Aut(\mathcal{G}), \circ)$. Moreover, $Aut(\mathcal{G}) = T \times G$.*

Proof. For any $f \in \mathrm{Aut}(\mathcal{G})$ and $t \in T$,

$$f^{-1}tf = (f_{\mathcal{V}}^{-1} \circ \mathrm{id}_{\mathcal{V}} \circ f_{\mathcal{V}}, \{f^{-1}_{l,f_{\mathcal{V}}(x)} \circ t_{l,f_{\mathcal{V}}(x)} \circ f_{l,x}\}) = (\mathrm{id}_{\mathcal{V}}, \{f^{-1}_{l,f_{\mathcal{V}}(x)} \circ t_{l,f_{\mathcal{V}}(x)} \circ f_{l,x}) \in T$$

For any $f \in \mathrm{Aut}(\mathcal{G})$ and $g \in G$,

$$f^{-1}gf = (f_{\mathcal{V}}^{-1} \circ g_{\mathcal{V}} \circ f_{\mathcal{V}}, \{f^{-1}_{l,f_{\mathcal{V}}(x)} \circ \mathrm{id}_{Y_k} \circ f_{l,x}\}) = (f_{\mathcal{V}}^{-1} \circ g_{\mathcal{V}} \circ f_{\mathcal{V}}, \{\mathrm{id}_{Y_k}\}) \in G$$

Now, $T \cap G = \{\mathrm{id}_{\mathcal{G}}\}$ and the second statement follows by Lemma 1.2.1. □

1.3 Basic constructions in $\mathbf{MG}(Y_k)$

1.3.1 Subgraphs and subobjects

Definition 1.3.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l), \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l') \in \boldsymbol{MG}(Y_k)$. We say that $\mathcal{G}'$ is a k-moment subgraph of $\mathcal{G}$ if*

(SUB1) $\mathcal{V}' \subseteq \mathcal{V}$

(SUB2) $\mathcal{E}' \subseteq \mathcal{E}$

(SUB3) $\trianglelefteq' = \trianglelefteq_{|_{\mathcal{V}'}}$

(SUB4) $l' = l_{|_{\mathcal{E}'}}$

Lemma 1.3.1. *Any k-moment subgraph of $\mathcal{G}$ is a representative of a subobject of $\mathcal{G}$.*

Proof. We have to show that, for any $\mathcal{G}'$, k-moment subgraph of $\mathcal{G}$, there exists a monomorphism $i : \mathcal{G}' \to \mathcal{G}$. Define i as $i_{\mathcal{V}'}(x) := x$ and $i_{l',x} = \mathrm{id}_{Y_k}$ for any $x \in \mathcal{V}'$. From Lemma1.2.2 *(i)*, it follows that i is a monomorphism. □

1.3.2 Quotient graphs

Definition 1.3.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$ and $\sim$ an equivalence relation on $\mathcal{V}$. We say that $\sim$ is* $\mathcal{G}$-compatible *if the following conditions are satisfied:*

(EQV1) $x_1 \sim x_2$ implies $x_1 \sim x$ for all $x_1 \trianglelefteq x \trianglelefteq x_2$

(EQV2) for all $x_1, y_1 \in \mathcal{V}$, if $x_1 \not\sim y_1$ and $x_1 \to y_1 \in \mathcal{E}$, then for any $x_2 \sim x_1$ there exists a unique $y_2 \in \mathcal{V}$ such that $y_2 \sim y_1$, $x_2 \to y_2$ and $l(x_1 \to y_1) = l(x_2 \to y_2)$.

Definition 1.3.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$ and let $\sim$ be a* $\mathcal{G}$-compatible *equivalence relation. We define the oriented labeled graph* quotient of $\mathcal{G}$ by $\sim$, *and we denote it by $\mathcal{G}/\!\sim = (\mathcal{V}_\sim, \mathcal{E}_\sim, \trianglelefteq_\sim, l_\sim)$, in the following way*

(QUOT1) $\mathcal{V}_\sim$ is a set of representatives of the equivalence classes

(QUOT2) $\mathcal{E}_\sim = \{([x] \to [y]) \,|\, x \not\sim y,\ \exists x_1 \sim x,\ y_1 \sim y$ with $x_1 \to y_1\}$

(QUOT3) $\trianglelefteq_\sim$ is the transitive closure of the relation $[x] \trianglelefteq_\sim [y]$ if $[x] \to [y] \in \mathcal{E}_\sim$

(QUOT4) If $[x] \to [y]$ and $x_1 \sim x$, $y_1 \sim y$ are such that $x_1 \to y_1$, we set $l_\sim([x] \to [y]) = l(x_1 \to y_1)$.

Lemma 1.3.2. *The graph $\mathcal{G}/\!\sim$ is a k-moment graph on Y.*

Proof. The only condition to be checked is that $\mathcal{G}/\!\sim$ has no oriented cycles, but it follows easily from *(EQV1)* and *(EQV2)*. Indeed, suppose there were an oriented cycle

$$[x_1] \to [x_2] \to \ldots \to [x_n] \to [x_1]$$

By *(QUOT2)* and *(EQV2)*, this means that there exists the following path on the graph $\mathcal{G}$:

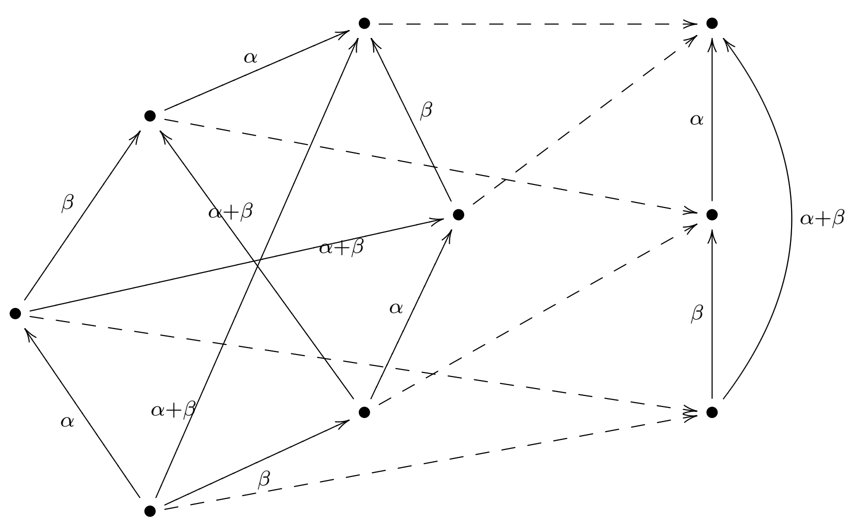
$$x_1 \to x_2' \to \ldots \to x_n' \to x_1',$$

for certain $x_i' \sim x_i$. But now we would get a sequence $x_1 \trianglelefteq x_2' \trianglelefteq \ldots \trianglelefteq x_n' \trianglelefteq x_1'$, with $x_1 \sim x_1'$ and , by *(EQV1)*, it would follow $[x_1] = [x_i]$ for all i. □

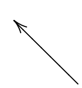
Lemma 1.3.3. *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$ and let $\sim$ be a $\mathcal{G}$-compatible equivalence relation. Then the quotient of $\mathcal{G}$ by $\sim$ is a representative of a quotient of $\mathcal{G}$.*

Proof. Suppose $\mathcal{G}' = \mathcal{G}/\sim$ and define $p = (p_{\mathcal{V}}, p_{\mathcal{E}}, \{p_{l,x}\}) \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$ as $p_{\mathcal{V}}(x) := [x]$, where $[x]$ is the representative of the equivalence class of x and $p_{l,x} = \mathrm{id}_{Y_k}$ for any $x \in \mathcal{V}$. By Lemma1.2.2 *(ii)*, this is an epimorphism. $\square$

Example 1.3.1. *Consider the following map of graphs*



Set $f_{l,x} = id_{Y_k}$ for any vertex x. This is an epimorphism of k-moment graphs and it is clear that the graph on the right is a quotient of the left one by the (compatible) relation $x \sim y$ if and only if x and y are connected by an edge having the following direction



1.3.3 Initial and terminal objects

Remark 1.3.1. *For any $\mathcal{G} \in \boldsymbol{MG}(Y_k)$, $|Hom_{\boldsymbol{MG}(Y_k)}(\emptyset, \mathcal{G})| = 1$, then $\emptyset$ is an initial object.*

Lemma 1.3.4. *If $|Aut_k(Y_k)| > 1$, there are no terminal objects in $\boldsymbol{MG}(Y_k)$.*

Proof. Since in the category of sets the terminal objects are the singletons, all k-moment graphs with more than one vertex cannot be terminal. Let $\mathcal{G} \in \mathbf{MG}(Y_k)$ be a k-moment graph with at least one vertex and let $f \in Hom_{\mathbf{MG}(Y_k)}(\mathcal{G}, \{\mathrm{pt}\})$. Then, $f_{\mathcal{V}}$ is uniquely determined, but, for any vertex x of $\mathcal{G}$, $f_{l,x}$ can be any automorphism of Y_k. Indeed, since $\{\mathrm{pt}\}$ does not have edges, conditions *(MORF3a)* and *(MORF3b)* are empty. $\square$

It follows

Corollary 1.3.1. *$\mathbf{MG}(Y_k)$ is not an additive category.*

Proof. This is because there are no zero objects in $\mathbf{MG}(Y_k)$. Observe, that this is true also if $Y \cong \mathbb{Z}^0$. Indeed, in this case the generic graph is the (unique) terminal object but it is not initial. □

Products

Lemma 1.3.5. *If $|Aut_k(Y_k)| > 1$, $\mathbf{MG}(Y_k)$ has no products.*

Proof. Suppose $\mathbf{MG}(Y_k)$ had products. Then, for any $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(Y_k)$ it would exist the product $(\mathcal{G} \times \mathcal{G}, \{p_1, p_2\})$. In particular, there would exist a $g \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{G} \times \mathcal{G})$ such that the following diagram commutes

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "\mathrm{id}_{\mathcal{G}}"'] \arrow[d, "g"] \arrow[dr, "\mathrm{id}_{\mathcal{G}}"] & \\ \mathcal{G} & \mathcal{G} \times \mathcal{G} \arrow[l, "p_1"] \arrow[r, "p_2"'] & \mathcal{G} \end{tikzcd} \tag{1.4}$$

Let $\mathcal{G}$ be the generic graph and let x be unique vertex. Then, from (1.4), we would get the following commutative diagram

$$\begin{tikzcd} & Y_k \arrow[dl, "\mathrm{id}_{Y_k}"'] \arrow[d, "g_{l,x}"] \arrow[dr, "\mathrm{id}_{Y_k}"] & \\ Y_k & Y_k \arrow[l, "p^1_{l',g_{\mathcal{V}}(x)}"] \arrow[r, "p^2_{l',g_{\mathcal{V}}(x)}"'] & Y_k \end{tikzcd} \tag{1.5}$$

(where we denoted $p_i = (p^i_{\mathcal{V}'}, p^i_{\mathcal{E}'}, \{p^i_{l',y}\})$). The commutativity of the triangles in (1.5) implies $g_{l,x} = (p^1_{l',g_{\mathcal{V}}(x)})^{-1} = (p^2_{l',g_{\mathcal{V}}(x)})^{-1}$, that is $p^1_{l',g_{\mathcal{V}}(x)} = p^2_{l',g_{\mathcal{V}}(x)} =: p_{l',g_{\mathcal{V}}(x)}$.

Now, choose $f \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{G})$ such that $f_{l,x} \neq \mathrm{id}_{Y_k}$ (such an $f_{l,x}$ exists, since we have by hypothesis $|\mathrm{Aut}_k(Y_k)| > 1$). There would exists an $h \in \mathrm{Hom}_{\mathbf{MG}(Y_k)}(\{\mathrm{pt}\}, \{\mathrm{pt}\} \times \{\mathrm{pt}\})$ such that the following diagram commutes

$$\begin{tikzcd} & \{\mathrm{pt}\} \arrow[dl, "\mathrm{id}_{\{\mathrm{pt}\}}"'] \arrow[d, "h"] \arrow[dr, "f"] & \\ \{\mathrm{pt}\} & \{\mathrm{pt}\} \times \{\mathrm{pt}\} \arrow[l, "p_1"] \arrow[r, "p_2"'] & \{\mathrm{pt}\} \end{tikzcd}$$

But this is impossible; indeed, the diagram above would give us the following commutative diagram

$$\begin{tikzcd} & Y_k \arrow[dl, "\mathrm{id}_{Y_k}"'] \arrow[d, dashed, "h_{l,x}"] \arrow[dr, "f_{l,x}"] & \\ Y_k & Y_k \arrow[l, "p_{l',g_{\mathcal{V}}(x)}"] \arrow[r, "p_{l',g_{\mathcal{V}}(x)}"'] & Y_k \end{tikzcd}$$

□

Coproducts

Definition 1.3.4. *Let* $\{\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j, \trianglelefteq_j, l_j)\}_{j\in J}$ *be a family of objects in* ***MG***(Y_k). *Then* $\coprod_{j\in J} \mathcal{G}_j = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l))$ *is defined as follows:*

(PROD1) $\mathcal{V}$ *is given by the disjoint union* $\coprod_{j\in J} \mathcal{V}_j = \bigcup_{j\in J}\{(v,j) \mid v \in \mathcal{V}_j\}$

(PROD2) (x,j) —— (y,i) *if only if* $i = j$ *and* x —— $y \in \mathcal{E}_i$

(PROD3) $(x,j) \trianglelefteq (y,i)$ *if and only if* $i = j$ *and* $x \trianglelefteq_j y$

(PROD4) $l\,((x,j)$ —— $(y,j)) := l_j(x$ —— $y)$

We get:

Lemma 1.3.6. ***MG***(Y_k) *has finite coproducts*

Proof. Denote by $i_j : \mathcal{G}_j \to \coprod_{j\in J} \mathcal{G}_j$ the morphism given by $i_{j\mathcal{V}}(v) = (v,j)$ and $f_{l,x} = \mathrm{id}_{Y_k}$ for any $x \in \mathcal{V}_j$. Then, for any $\mathcal{H} \in \mathbf{MG}(Y_k)$ with a family of morphisms $f_j : \mathcal{G}_j \to \mathcal{H}$ there exists a unique morphism $f : \coprod_{j\in J} \mathcal{G}_j \to \mathcal{H}$ such that $f_j = f \circ i_j$. In particular, f is given by $f_{\coprod_{j\in J} \mathcal{V}_i}((x,j)) = f_j(x)$ and $f_{l,(x,j)} = (f_j)_{l,x}$. □

Chapter 2

The category of sheaves on a k-moment graph

The notion of sheaf on a moment graph is due to Braden and MacPherson (cf.[7]) and it has been used by Fiebig in several papers (cf. [13],[14],[18],[16],[17]). In the first part of this chapter, we recall the definition of category of sheaves on a k-moment graph and we present two important examples, namely, the structure sheaf and the canonical sheaf (cf.[7]). In the second part, for any homomorphism of k-moment graphs f, we define the *pullback* functor f^* and the *push-forward* functor f_*. These two functors turn out to be adjoint (see Proposition 2.2.1). We prove that, if f is a k-isomorphism, then the canonical sheaf turns out to be preserved by f^*. This result will be an important tool in the categorification of some equalities coming from Kazhdan-Lusztig theory (see Chapter 5).

2.1 Sheaves on a k-moment graph

For any finite rank lattice Y and any local ring k (with $2 \in k^*$), we denote by $S = \mathrm{Sym}(Y)$ its symmetric algebra and by $S_k := S \otimes_{\mathbb{Z}} k$ its extension. S_k is a polynomial ring and we provide it with the grading induced by the setting $(S_k)_{\{2\}} = Y_k$. From now on, all the S_k-modules will be finitely generated and $\mathbb{Z}$-graded. Moreover, we will consider only degree zero morphisms between them. Finally, for $j \in \mathbb{Z}$ and M a graded S_k-module we denote by $M\{j\}$ the graded S_k-module obtained from M by shifting the grading by j, that is $M\{j\}_{\{i\}} = M_{\{j+i\}}$.

Definition 2.1.1 ([7])**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then a* sheaf $\mathcal{F}$ on $\mathcal{G}$ *is given by the following data* $(\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$

(SH1) for all $x \in \mathcal{V}$, $\mathcal{F}^x$ is an S_k-module;

(SH2) for all $E \in \mathcal{E}$, $\mathcal{F}^E$ is an S_k-module such that $l(E) \cdot \mathcal{F}^E = \{0\}$;

(SH3) for $x \in \mathcal{V}$, $E \in \mathcal{E}$, $\rho_{x,E} : \mathcal{F}^x \to \mathcal{F}^E$ is a homomorphism of S_k-modules defined if x is in the border of the edge E.

Remark 2.1.1. *We may consider the following topology on $\mathcal{G}$ (cf. [7],§1.3 or [24], §2.4). We say that a subgraph $\mathcal{H}$ of $\mathcal{G}$ is open, if whenever a vertex x is $\mathcal{H}$, then also all the edges adjacent to x are in $\mathcal{H}$. With this topology, the object we defined above is actually a proper sheaf of S_k-modules on $\mathcal{G}$. Anyway, we will not work with this topology in what follows.*

Example 2.1.1 (cf. [7], §1)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then its structure sheaf $\mathscr{Z}$ is given by*

- *for all $x \in \mathcal{V}$, $\mathscr{Z}^x = S_k$*
- *for all $E \in \mathcal{E}$, $\mathscr{Z}^E = S_k/l(E) \cdot S_k$*
- *for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$, such that x is in the border of the edge E, $\rho_{x,E} : S_k \to S_k/l(E) \cdot S_k$ is the canonical quotient map*

Definition 2.1.2. *[15] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$ and let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, $\mathcal{F}' = (\{\mathcal{F}'^x\}, \{\mathcal{F}'^E\}, \{\rho'_{x,E}\})$ be two sheaves on it. A* morphism *$\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ is given by the following data*

(i) for all $x \in \mathcal{V}$, $\varphi^x : \mathcal{F}^x \to \mathcal{F}'^x$ is a homomorphism of S_k-modules

(ii) for all $E \in \mathcal{E}$, $\varphi^E : \mathcal{F}^E \to \mathcal{F}'^E$ is a homomorphism of S_k-modules such that, for any $x \in \mathcal{V}$ on the border of $E \in \mathcal{E}$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}^x \arrow[r, "\rho_{x,E}"] \arrow[d, "\varphi^x"] & \mathcal{F}^E \arrow[d, "\varphi^E"] \\
\mathcal{F}'^x \arrow[r, "\rho'_{x,E}"] & \mathcal{F}'^E
\end{tikzcd}$$

Definition 2.1.3. *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$. We denote by $\boldsymbol{Sh}_k(\mathcal{G})$ the category, whose objects are the sheaves on $\mathcal{G}$ and whose morphisms are as in Def.2.1.2.*

Remark 2.1.2. *If $\mathcal{G} = \{pt\}$, then $\boldsymbol{Sh}_k(\mathcal{G})$ is equivalent to the category of finitely generated $\mathbb{Z}$-graded S_k-modules.*

2.1.1 Sections of a sheaf on a moment graph

Even if $\mathbf{Sh}_k(\mathcal{G})$ is not a category of sheaves in the topological meaning, we may define, following [14], the notion of sections.

Definition 2.1.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\}) \in \boldsymbol{Sh}_k(\mathcal{G})$ and $\mathcal{I} \subseteq \mathcal{V}$. Then the* set of sections of $\mathcal{F}$ over $\mathcal{I}$ *is denoted $\Gamma(\mathcal{I}, \mathcal{F})$ and defined as*

$$\Gamma(\mathcal{I}, \mathcal{F}) := \left\{ (m_x)_{x \in \mathcal{I}} \in \bigoplus_{x \in \mathcal{I}} \mathcal{F}^x \mid \forall\, x \text{ --- } y \in \mathcal{E} \;\; \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \right\}$$

We will denote $\Gamma(\mathcal{F}) := \Gamma(\mathcal{V}, \mathcal{F})$, that is the set of global sections of $\mathcal{F}$.

Example 2.1.2. *A very important example is given by the set of global sections of the structure sheaf $\mathscr{Z}$ (cf. Ex. 2.1.1). In this case, we get the* structure algebra*:*

$$\mathcal{Z} := \Gamma(\mathscr{Z}) = \left\{ (z_x)_{x\in\mathcal{V}} \in \bigoplus_{x\in\mathcal{V}} S_k \,|\, \forall\, E : x \text{ --- } y \in \mathcal{E} \;\; z_x - z_y \in l(E)\cdot S_k \right\} \tag{2.1}$$

Goresky, Kottwitz and MacPherson proved in [20] that, if $\mathcal{G}$ is as in Ex. 1.1.4, i.e. it describes the algebraic action of the complex torus T on the irreducible complex variety X, then $\mathcal{Z}$ is isomorphic, as graded S_k-module, to the T-equivariant cohomology of X. It is easy to check that, for any $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$, the k-structure algebra $\mathcal{Z}$ acts on $\Gamma(\mathcal{F})$ via componentwise multiplication. We will focus our attention on a subcategory of the category of $\mathbb{Z}$-graded $\mathcal{Z}$-modules from Chapter 4.

2.1.2 Flabby sheaves on a k-moment graph

After Braden and MacPherson ([7]), we define a topology on the set of vertices of a k-moment graph $\mathcal{G}$. We state a result about a very important class of flabby (with respect to this topology) sheaves: the *BMP*-sheaves. This notion, due to Fiebig and Williamson (cf. [19]), generalizes the original construction of Braden and MacPherson.

Definition 2.1.5. *([7]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \boldsymbol{MG}(Y_k)$, then the* Alexandrov topology *on $\mathcal{V}$ is the topology, whose basis of open sets is given by the collection $\{\trianglerighteq x\} := \{y \in \mathcal{V} \,|\, y \trianglerighteq x\}$, for all $x \in \mathcal{V}$.*

A classical question in sheaf theory is to ask if a sheaf is flabby, that is whether any local section over an open set extends to a global one or not. In order to characterise the objects in $\mathbf{Sh}_k(\mathcal{G})$ having this property, we need some notation.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(Y_k)$. For any $x \in \mathcal{V}$, we denote (cf. [14], §4.2)

$$\mathcal{E}_{\delta x} := \big\{ E \in \mathcal{E} \mid E : x \to y \big\}$$

$$\mathcal{V}_{\delta x} := \big\{ y \in \mathcal{V} \mid \exists E \in \mathcal{E}_{\delta x} \text{ such that } E : x \to y \big\}$$

Consider $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$ and define $\mathcal{F}^{\delta x}$ to be the image of $\Gamma(\{\triangleright x\}, \mathcal{F})$ under the composition u_x of the following maps

$$\begin{tikzcd} \Gamma(\{\triangleright x\}, \mathcal{F}) \arrow[r, hook] \arrow[rrr, bend right, "u_x"'] & \bigoplus_{y\triangleright x} \mathcal{F}^y \arrow[r] & \bigoplus_{y\in\mathcal{V}_{\delta x}} \mathcal{F}^y \arrow[r, "\oplus\rho_{y,E}"] & \bigoplus_{E\in\mathcal{E}_{\delta x}} \mathcal{F}^E \end{tikzcd} \tag{2.2}$$

Moreover, denote

$$d_x := (\rho_{x,E})^T_{E\in\mathcal{E}_{\delta x}} : \mathcal{F}^x \longrightarrow \bigoplus_{E\in\mathcal{E}_{\delta x}}\mathcal{F}^E$$

Observe that $m \in \Gamma(\{\triangleright x\}, \mathcal{F})$ can be extended, via m_x, to a section $\tilde{m} = (m, m_x) \in \Gamma(\{\trianglerighteq x\}, \mathcal{F})$ if and only if $d_x(m_x) = u_x(m)$. This fact motivates the following result, due to Fiebig, that gives a characterization of the flabby objects in $\mathbf{Sh}_k(\mathcal{G})$.

Proposition 2.1.1 ([14], Prop. 4.2). *Let $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$. Then the following are equivalent:*

(i) $\mathcal{F}$ is flabby with respect to the Alexandrov topology, that is for any open $\mathcal{I} \subseteq \mathcal{V}$ the restriction map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{I}, \mathcal{F})$ is surjective.

(ii) For any vertex $x \in \mathcal{V}$ the restriction map $\Gamma(\{\trianglerighteq x\}, \mathcal{F}) \to \Gamma(\{\triangleright x\}, \mathcal{F})$ is surjective.

(iii) For any vertex $x \in \mathcal{V}$ the map $\oplus_{E\in\mathcal{E}_{\delta x}}\rho_{x,E} : \mathcal{F}^x \to \bigoplus_{E\in\mathcal{E}_{\delta x}} \mathcal{F}^E$ contains $\mathcal{F}^{\delta x}$ in its image.

2.1.3 Braden-MacPherson sheaves

We introduce here the most important class of sheaves on a k-moment graph. We recall the definition given by Fiebig and Williamson in [19].

Definition 2.1.6 ([19], Def. 6). *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$ and let $\mathscr{B} \in \boldsymbol{Sh}_k(\mathcal{G})$. We say that $\mathscr{B}$ is a* Braden-MacPherson sheaf *if it satisfies the following properties:*

(BMP1) for any $x \in \mathcal{V}$, $\mathscr{B}^x$ is a graded free S_k-module

(BMP2) for any $E : x \to y \in \mathcal{E}$, $\rho_{y,E} : \mathscr{B}^y \to \mathscr{B}^E$ is surjective with kernel $l(E) \cdot \mathscr{B}^y$

(BMP3) for any open set $\mathcal{I} \subseteq \mathcal{V}$, the map $\Gamma(\mathscr{B}) \to \Gamma(\mathcal{I}, \mathscr{B})$ is surjective

(BMP4) for any $x \in \mathcal{V}$, the map $\Gamma(\mathscr{B}) \to \mathscr{B}^x$ is surjective

Hereafter, Braden-MacPherson sheaves will be referred to also as *BMP*-sheaves or *canonical* sheaves. An important theorem, characterising Braden-MacPherson sheaves, is the following one.

Theorem 2.1.1 ([19], Theor. 6.3). *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$*

(i) For any $w \in \mathcal{V}$, there is up to isomorphism unique Braden-MacPherson sheaf $\mathscr{B}(w) \in \boldsymbol{Sh}_k(\mathcal{G})$ with the following properties:

> *(BMP0) $\mathscr{B}(w)$ is indecomposable in $\boldsymbol{Sh}_k(\mathcal{G})$*
> *(BMP1a) $\mathscr{B}(w)^w \cong S_k$ and $\mathscr{B}(w)^x = 0$, unless $x \leq w$*

(ii) Let $\mathscr{B}$ be a Braden-MacPherson sheaf. Then, there are $w_1, \ldots, w_r \in \mathcal{V}$ and $l_1 \ldots l_r \in \mathbb{Z}$ such that

$$\mathscr{B} \cong \mathscr{B}(w_1)[l_r] \oplus \ldots \oplus \mathscr{B}(w_r)[l_r]$$

If $\mathscr{B}$ is an indecomposable *BMP*-sheaf, that is $\mathscr{B} = \mathscr{B}(w)$ for some $w \in \mathcal{V}$, then conditions *(BMP3)* and *(BMP4)* may be replaced by the following condition (cf. [7], Theor.1.4)

(BMP3') for all $x \in \mathcal{V}$, with $x \lhd w$, $d_x : \mathscr{B}(w)^x \to \mathscr{B}(w)^{\delta x}$ is a projective cover in the category of graded S_k-modules

Remark 2.1.3. *If X is a complex irreducible algebraic variety with an algebraic action of a torus T, as in Ex. 1.1.4, the associated k-moment graph turns out to have a unique maximal vertex, that we denote by w. For $k = \mathbb{C}$, Braden and MacPherson proved in [7] that the space of global sections of the sheaf $\mathscr{B}(w)$ can be identified with the T-equivariant intersection cohomology of X. In positive characteristic, Fiebig and Williamson related $\mathscr{B}(w)$ to a (very special) indecomposable object in the T-equivariant constructible bounded derived category of sheaves on X with coefficients in k: a parity sheaf. Parity sheaves have been recently defined by Juteau, Mautner and Williamson (cf. [25]) and they have applications in many situations arising in representation theory.*

Remark 2.1.4. *Canonical sheaves are strictly related to important conjectures in representation theory. We will (briefly) discuss this connection in Chapter 5.*

We end this section with a result, that connects structure sheaves and canonical sheaves.

Proposition 2.1.2 ([17], Prop)**.** *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)^+$ and let w be its highest vertex. Then $\mathscr{B}(w) \cong \mathscr{Z}$ if and only if $\mathscr{Z}$ is flabby.*

Remark 2.1.5. *The structure sheaf of a k-moment graph $\mathcal{G}$ is not in general flabby. Actually, if $\mathcal{G}$ is as in Ex.1.1.4, the flabbiness of its structure sheaf is equivalent to the k-smoothness of the variety X (cf. [19]). Indeed, if X is rationally smooth, its intersection cohomology coincides with its ordinary cohomology.*

2.2 Direct and inverse images

Let $f = (f_{\mathcal{V}}, \{f_{l,x}\}) : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \unlhd, l) \to \mathcal{G}' = (\mathcal{V}, \mathcal{E}, \unlhd, l)$ be a homomorphism of k-moment graphs. We want to define, in analogy with classical sheaf theory, two functors

$$\begin{tikzcd} \mathbf{Sh}_k(\mathcal{G}) \arrow[rr, bend right, "f_*"'] & & \mathbf{Sh}_k(\mathcal{G}') \arrow[ll, bend right, "f^*"'] \end{tikzcd}$$

From now on, for any $\varphi \in \mathrm{Aut}_k(Y_k)$, we will denote by φ also the automorphism of S_k that it induces.

We need a lemma, in order to make consistent the definitions we are going to give.

Lemma 2.2.1. *Let $s \in S_k$, $f \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$, $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$ and $\mathcal{H} \in \boldsymbol{Sh}_k(\mathcal{G}')$. Let $E : x \text{---} y \in \mathcal{E}$ and $F : f_\mathcal{V}(x) \text{---} f_\mathcal{V}(y) \in \mathcal{E}'$, then*

(i) the twisted actions of S_k on $\mathcal{F}^E$ defined via $s \centerdot m_E := f_{l,x}^{-1}(s) \cdot m_E$ and $s \centerdot m_E := f_{l,y}^{-1}(s) \cdot m_E$ coincide on $\mathcal{F}^E / l'(F) \centerdot \mathcal{F}^E$ ($\cdot$ denotes the action of S_k on $\mathcal{F}^E$ before the twist). Moreover, $l'(F) \centerdot \mathcal{F}^E = \{0\}$ in both cases.

(ii) the twisted actions of S_k on $\mathcal{H}^F$ defined via $s \centerdot n_F := f_{l,x}(s) \cdot n_F$ and $s \centerdot n_F := f_{l,y}(s) \cdot n_F$ coincide on $\mathcal{H}^F / l(E)\mathcal{H}^F$ ($\cdot$ denotes the action of S_k on $\mathcal{F}^E$ before the twist). Moreover, $l(E) \centerdot \mathcal{H}^F = \{0\}$ in both cases.

Proof. It is enough to prove the claim for $s \in (S_k)_{\{2\}} = Y_k$, since S_k is a k-algebra generated by Y_k.

(i) The statement follows from *(MORPH2a)*, *(MORPH2b)* and the computations we made in the proof of Lemma 1.2.3.

(ii) It is an immediate consequence of conditions *(MORPH2a)*, *(MORPH2b)*. $\square$

If φ is an automorphism of S_k, for any S_k-module M, we will denote $\mathrm{Tw}_\varphi : M \to M$ the map sending M to M and twisting the action of S_k on M by φ.

2.2.1 Definitions

Definition 2.2.1. *Let $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$, then $f_*\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G}')$ is defined as follows*

(PUSH1) for any $u \in \mathcal{V}'$,

$$(f_*\mathcal{F})^u := \Gamma(f_\mathcal{V}^{-1}(u), \mathcal{F})$$

and the structure of S_k-module is given by $s \centerdot (m_x)_{x \in f_\mathcal{V}^{-1}(u)} := (s \cdot m_x)_{x \in f_\mathcal{V}^{-1}(u)}$

(PUSH2) for any $u \in \mathcal{V}'$,

$$(f_*\mathcal{F})^F := \bigoplus_{E : f_\mathcal{E}(E) = F} \mathcal{F}^E$$

and the action of S_k is twisted in the following way: $s \centerdot (m_E)_{E : f_\mathcal{E}(E) = F} := (f_{l,x}^{-1}(s) \cdot m_E)_{E : f_\mathcal{E}(E) = F}$, where x is on the border of E

(PUSH3) for all $u \in \mathcal{V}'$ and $F \in \mathcal{E}'$, such that u is in the border of the edge F,$(f_*\rho)_{u,F}$ is defined as the composition of the following maps:

$$\Gamma(f_\mathcal{V}^{-1}(u), \mathcal{F}) \hookrightarrow \bigoplus_{x : f_\mathcal{V}(x) = u} \mathcal{F}^x \xrightarrow{\oplus \rho_{x,E}} \bigoplus_{E : f_\mathcal{V}(E) = F} \mathcal{F}^E \xrightarrow{Tw} \bigoplus_{E : f_\mathcal{V}(E) = F} \mathcal{F}^E,$$

where $Tw = \oplus Tw_{f_{l,x}^{-1}}$. We call f_* direct image *or* push-forward *functor.*

Definition 2.2.2. *Let $\mathcal{H} \in \boldsymbol{Sh}_k(\mathcal{G}')$, then $f^*\mathcal{H} \in \boldsymbol{Sh}_k(\mathcal{G})$ is defined as follows*

(PULL1) for all $x \in \mathcal{V}$, $(f^*\mathcal{H})^x := \mathcal{H}^{f_\mathcal{V}(x)}$ and $s \in S_k$ acts on it via $f_{l,x}(s)$

(PULL2) for all $E : x \text{——} y \in \mathcal{E}$

$$(f^*\mathcal{H})^E = \begin{cases} \mathcal{H}^{f_{\mathcal{V}}(x)}/l(E)\mathcal{H}^{f_{\mathcal{V}}(x)} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y) \\ \mathcal{H}^{f_{\mathcal{E}}(E)} & \text{otherwise} \end{cases}$$

and of $s \in S_k$ *acts on* $(f^*\mathcal{H})^E$ *via* $f_{l,x}(s)$.

(PULL3) for all $x \in \mathcal{V}$ *and* $E \in \mathcal{E}$, *such that* x *is in the border of the edge* E,

$$(f^*\rho)_{x,E} = \begin{cases} \text{canonical quotient map} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y) \\ Tw_{f_{l,x}^{-1}} \circ \rho_{f_{\mathcal{V}}(x), f_{\mathcal{E}}(E)} \circ Tw_{f_{l,x}^{-1}} & \text{otherwise} \end{cases}$$

We call f^* inverse image *or* pullback *functor.*

Example 2.2.1. *Let* $\mathcal{G} \in \boldsymbol{MG}(Y_k)$ *and let* $p : \mathcal{G} \to \{pt\}$ *be the homomorphism of* k*-moment graphs having* $p_{l,x} = id_{Y_k}$ *for all* x, *vertex of* $\mathcal{G}$. *Then, for any* $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$ $p_*(\mathcal{F}) = \Gamma(\mathcal{F})$. *Moreover* $p^*(S_k) = \mathscr{Z}$, *the structure sheaf of* $\mathcal{G}$.

2.2.2 Adjunction formula

Proposition 2.2.1. *Let* $f \in Hom_{\boldsymbol{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$, *then* f^* *is left adjoint to* f_*, *that is for all pair of sheaves* $\mathcal{F} \in \boldsymbol{Sh}_k(\mathcal{G})$ *and* $\mathcal{H} \in \boldsymbol{Sh}_k(\mathcal{G}')$ *the following equality holds*

$$Hom_{\boldsymbol{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) = Hom_{\boldsymbol{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F}) \tag{2.3}$$

Proof. Take $\varphi \in \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F})$, that is $\varphi = (\{\varphi^x\}_{x \in \mathcal{V}}, \{\varphi^E\}_{E \in \mathcal{E}})$ such that, for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$ such that x is on the border of E, the following diagram commutes

$$\begin{tikzcd} (f^*\mathcal{H})^x \arrow[r, "\varphi^x"] \arrow[d, "(f^*\rho)_{x,E}"'] & \mathcal{F}^x \arrow[d, "\rho_{x,E}"] \\ (f^*\mathcal{H})^E \arrow[r, "\varphi^E"'] & \mathcal{F}^E \end{tikzcd} \tag{2.4}$$

We want to show that there is a bijective map $\gamma : \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) \to \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$ and it is given by $\varphi = (\{\varphi^x\}_{x \in \mathcal{V}}, \{\varphi^E\}_{E \in \mathcal{E}}) \mapsto \psi = (\{\psi^u\}_{u \in \mathcal{V}'}, \{\psi^F\}_{F \in \mathcal{E}'})$, where

$$\psi^u := (\oplus_{x \in f_{\mathcal{V}}^{-1}(u)} \varphi^x)^T, \qquad \psi^F := \oplus_{E \in f_{\mathcal{E}}^{-1}(F)} \varphi^E$$

We start with verifying that this map is well-defined. We have to show that for any $h \in \mathcal{H}^u$, $\psi^u(h) \in (f_*\mathcal{F})^u = \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F})$, that is, for any $x, y \in f_{\mathcal{V}}^{-1}(u)$ such that $E : x \text{——} y \in \mathcal{E}$, $\rho_{x,E}(\varphi^y(h)) = \rho_{y,E}(\varphi^y(h))$.

From Diagram (2.4), we get the following commutative diagram

$$\begin{tikzcd}
(f^*\mathcal{H})^x = \mathcal{H}^{f_\mathcal{V}(x)} = \mathcal{H}^u \arrow[r, "\varphi^x"] \arrow[d, "(f^*\rho')_{x,E}"] & \mathcal{F}^x \arrow[d, "\rho_{x,E}"] \\
(f^*\mathcal{H})^E = \mathcal{H}^u/l(E)\mathcal{H}^u \arrow[r, "\varphi^E"] & \mathcal{F}^E \\
(f^*\mathcal{H})^y = \mathcal{H}^{f_\mathcal{V}(y)} = \mathcal{H}^u \arrow[r, "\varphi^y"] \arrow[u, "(f^*\rho')_{y,E}"] & \mathcal{F}^y \arrow[u, "\rho_{x,E}"]
\end{tikzcd} \tag{2.5}$$

But $(f^*\rho')_{y,E} = (f^*\rho')_{x,E}$ by definition (they are both the canonical projection) and we obtain

$$\rho_{x,E} \circ \varphi^x = \varphi^E \circ (f^*\rho')_{x,E} = \varphi^E \circ (f^*\rho')_{y,E} = \rho_{y,E} \circ \varphi^y$$

It is clear that the map $\gamma : \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) \to \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$ we defined is injective. To conclude our proof, we have to show the surjectivity of γ.

Suppose $\psi = (\{\psi^u\}_{u\in\mathcal{V}'}, \{\psi^F\}_{F\in\mathcal{E}'}) \in \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$, where, for all $u \in \mathcal{V}'$ and $F \in \mathcal{E}'$ such that u is on the border of F, the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}^u \arrow[r, "\psi^x"] \arrow[d, "\rho'_{u,F}"] & \Gamma(f_\mathcal{V}^{-1}(u), \mathcal{F}) \arrow[d, "\oplus(\mathrm{Tw}_{f_{l,x}} \circ \rho_{x,E})"] \\
(f^*\mathcal{H})^F \arrow[r, "\psi^E"'] & \bigoplus_{E\in f_\mathcal{E}^{-1}(F)} \mathcal{F}^E
\end{tikzcd} \tag{2.6}$$

We claim that there exist $\varphi = (\{\varphi^x\}) \in \mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F})$ such that $\gamma(\varphi) = \psi$.

For any $x \in \mathcal{V}$, let us consider $u := f_\mathcal{V}(x)$ and define φ^x as the composition of the following maps

$$\begin{tikzcd}
\mathcal{H}^u \arrow[r, "\psi^y"] \arrow[rrr, bend right, "\varphi^x"'] & \Gamma(f_\mathcal{V}^{-1}(u), \mathcal{F}) \arrow[r, hook] & \bigoplus_{y\in f_\mathcal{V}^{-1}(u)} \mathcal{F}^y \arrow[r, two heads] & \mathcal{F}^x
\end{tikzcd}$$

For any $E : x \text{ --- } y \in \mathcal{E}$ such that $f_\mathcal{V}(x) \neq f_\mathcal{V}(y)$, that is there exists an edge $F \in \mathcal{E}'$ such that $f_\mathcal{E}(E) = F$, we define φ^E as the composition of the following maps

$$\begin{tikzcd}
\mathcal{H}^F \arrow[r, "\psi^F"] \arrow[rrr, bend right, "\varphi^E"'] & \bigoplus_{L\in f_\mathcal{E}^{-1}(F)} \mathcal{F}^L \arrow[r, "\mathrm{Tw}_{f_{l,y}}"] & \bigoplus_{L\in f_\mathcal{E}^{-1}(F)} \mathcal{F}^L \arrow[r, two heads] & \mathcal{F}^E
\end{tikzcd}$$

Now, it is clear that $\gamma(\varphi) = \psi$. Indeed, if $u \notin f_\mathcal{V}(\mathcal{V})$, then $\psi^u = 0$ and the claim is trivial. Otherwise, $u \in f_\mathcal{V}(\mathcal{V})$ and we get the following diagram, with Cartesian squares

$$\begin{tikzcd}
\mathcal{H}^u \arrow[r, "\psi^y"] \arrow[d, dashed, "\rho'_{u,F}"'] \arrow[rrr, bend left, "\varphi^x"] & \Gamma(f_{\mathcal{V}}^{-1}(u),\mathcal{F}) \arrow[r, hook] \arrow[d, dashed, "(f_*\rho)_{y,F}"] & \bigoplus_{y\in f_{\mathcal{V}}^{-1}(u)} \mathcal{F}^y \arrow[r, two heads] \arrow[d, dashed, "\oplus\rho_{z,L}"] & \mathcal{F}^x \arrow[d, dashed, "\rho_{x,E}"] \\
\mathcal{H}^F \arrow[r, "\psi^F"] \arrow[rrr, bend right, "\varphi^E"'] & \bigoplus_{L\in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L \arrow[r, "\mathrm{Tw}_{f_{l,y}}"] & \bigoplus_{L\in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L \arrow[r, two heads] & \mathcal{F}^E
\end{tikzcd}$$

□

As application of the previous proposition, we get the following corollary.

Corollary 2.2.1. *Let $\mathcal{G} \in \boldsymbol{MG}(Y_k)$ and let $\mathscr{Z}$, resp. $\mathcal{Z}$, be its structure sheaf, resp. its structure algebra. Then the functors $\Gamma(-), Hom_{\boldsymbol{Sh}_k(\mathcal{G})}(\mathscr{Z},-) : \boldsymbol{Sh}_k(\mathcal{G}) \to \mathcal{Z} -$ modules are naturally equivalent. In particular, we get the following isomorphism of S_k-modules*

$$\mathcal{Z} \cong End_{\boldsymbol{Sh}_k(\mathcal{G})}(\mathscr{Z}).$$

Proof. Consider the homomorphism $p : \mathcal{G} \to \{\mathrm{pt}\}$, where we set $p_{l,x} = \mathrm{id}_{Y_k}$ for all x, vertex of $\mathcal{G}$. The structure sheaf of $\{\mathrm{pt}\}$ is just a copy of S_k and, for all $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$, by Prop. 2.2.1, we get

$$\mathrm{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(p^*S_k, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Sh}_k(\{\mathrm{pt}\})}(S_k, p_*\mathcal{F})$$

Bu we have already noticed in Example 2.2.1 that $p^*S_k \cong \mathscr{Z}$ and $p_*\mathcal{F} = \Gamma(\mathcal{F})$. Moreover, that $\mathrm{Hom}_{S_k}(S_k, \mathcal{Z}) \cong \mathcal{Z}$ and we get the claim.

□

2.2.3 Inverse image of Braden-MacPherson sheaves.

The following lemma tells us that the pullback functor f^* preserves canonical sheaves if f is an isomorphism.

Lemma 2.2.2. *Let $\mathcal{G}, \mathcal{G}' \in \boldsymbol{MG}(Y_k)^+$. Let w, resp. w', be the (unique) maximal vertex of $\mathcal{G}$, resp. $\mathcal{G}'$, and let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be an isomorphism. If $\mathscr{B}(w)$ and $\mathscr{B}'(w')$ are the corresponding indecomposable* BMP*-sheaves, then $\mathscr{B}(w) \cong f^*\mathscr{B}'(w')$ in $\boldsymbol{Sh}_k(\mathcal{G})$.*

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$ and $f = (f_{\mathcal{V}}, \{f_{l,x}\})$.

Notice that $\mathcal{I} \subseteq \mathcal{V}$ is an open subset if and only if $\mathcal{I}' := f_{\mathcal{V}}(\mathcal{I}) \subseteq \mathcal{V}'$ is an open subset. We prove that $\mathscr{B}(w)_{|_{\mathcal{I}}} \cong f^*\mathscr{B}'(w')_{|_{\mathcal{I}'}}$ by induction on $|\mathcal{I}| = |\mathcal{I}'|$, for $\mathcal{I}$ open.

If $|\mathcal{I}| = |\mathcal{I}'| = 1$, we have $\mathcal{I} = \{w\}$ and $\mathcal{I}' = \{w'\}$. In this case $\mathscr{B}(w)^w = S_k$, $\mathscr{B}'(w')^{w'} = S_k$ and the isomorphism $\varphi^w : \mathscr{B}(w)^w \to \mathscr{B}'(w')^{w'}$ is just given by the twisting of the S_k-action, coming from the automorphism of S_k, induced by the automorphism $f_{l,w}$ of Y_k.

Now let $|\mathcal{I}| = |\mathcal{I}'| = n > 1$ and $y \in \mathcal{I}$ be a minimal element. Obviously, $y' := f_{\mathcal{V}}(y)$ is also a minimal element for $\mathcal{I}'$. Moreover, for any $E \in \mathcal{E}$ we set $E' := f_{\mathcal{E}}(E)$.

First of all, observe that $z \in \mathcal{V}_{\delta y}$ if and only if $z' := f_{\mathcal{V}}(z) \in \mathcal{V}'_{\delta y'}$. By the inductive hypothesis, for all $x \rhd y$ there exists an isomorphism $\varphi^x : \mathscr{B}(w)^x \xrightarrow{\sim} \mathscr{B}'(w')^{x'}$ such that $\varphi^x(s \cdot m) = f_{l,x}(s) \cdot \varphi^x(m)$, for $s \in S_k$ and $m \in \mathscr{B}(w)^x$. Moreover, if $E \notin \mathcal{E}_{\delta y}$ and x is on the border of E with $x \rhd y$, by the inductive hypothesis we have an isomorphism $\varphi^E : \mathscr{B}(w)^E \xrightarrow{\sim} \mathscr{B}'(w')^{E'}$ such that $\varphi^E(s \cdot n) = f_{l,x}(s) \cdot \varphi^E(n)$, for $s \in S_k$ and $n \in \mathscr{B}(w)^E$ and such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{B}(w)^x \arrow[r, "\rho_{x,E}"] \arrow[d, "\varphi^x"] & \mathscr{B}(w)^E \arrow[d, "\varphi^E"] \\
\mathscr{B}'(w')^{x'} \arrow[r, "\rho'_{x',E'}"] & \mathscr{B}'(w')^{E'}
\end{tikzcd}$$

Now, if $E : y \longrightarrow x$ and $E' : y' \longrightarrow x'$, then

$$\mathscr{B}(w)^E \cong \mathscr{B}(w)^x / l(E)\mathscr{B}(w)^x \quad \text{and} \quad \mathscr{B}(w')'^{E'} \cong \mathscr{B}'(w')^{x'} / l'(E')\mathscr{B}'(w')^{x'}.$$

By assumption, $f_{l,x}(l(E)) = h{\cdot}l'(E')$ for some invertible element $h \in k^{\times}$ and $\varphi^x(l(E)\mathscr{B}(w)^x) = f_{l,x}(l(E)) \cdot \mathscr{B}'(w')^{x'} = l'(E')\mathscr{B}'(w')^{x'}$. Thus the quotients are also isomorphic and so there exists $\varphi^E : \mathscr{B}(w)^E \xrightarrow{\sim} \mathscr{B}'(w')^{E'}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{B}(w)^x \arrow[r, "\rho_{y,E}"] \arrow[d, "\varphi^x"] & \mathscr{B}(w)^E \arrow[d, "\varphi^E"] \\
\mathscr{B}'(w')^{x'} \arrow[r, "\rho'_{y',E'}"] & \mathscr{B}'(w')^{E'}
\end{tikzcd}$$

Now we have to construct $\mathscr{B}(w)^{\delta y}$ and $\mathscr{B}'(w')^{\delta y'}$. Observe that $(\varphi^x)_{x \rhd y}$ induces an isomorphism of S_k-modules between the sets of sections $\Gamma(\{\rhd y\}, \mathscr{B}(w)) \cong \Gamma(\{\rhd' y'\}, \mathscr{B}'(w'))$ and, from what we have observed above, the following diagram commutes:

$$\begin{tikzcd}
\Gamma(\{\rhd y\}, \mathscr{B}(w)) \arrow[r] \arrow[d, "\oplus_{x \rhd y}\varphi^x"] \arrow[rrr, bend left, "u_y"] & \bigoplus_{x \rhd y} \mathscr{B}(w)^x \arrow[r] \arrow[d, "\oplus_{x \rhd y}\varphi^x"] & \bigoplus_{x \in \mathcal{V}_{\delta y}} \mathscr{B}(w)^x \arrow[r, "\oplus \rho_{x,E}"] \arrow[d, "\oplus_{x \in \mathcal{V}_{\delta y}} \varphi_x"] & \bigoplus_{E \in \mathcal{E}_{\delta y}} \mathscr{B}(w)^E \arrow[d, "\oplus_{E \in \mathcal{E}_{\delta y}} \varphi^E"] \\
\Gamma(\{\rhd' y'\}, \mathscr{B}'(w')) \arrow[r] \arrow[rrr, bend right, "u'_{y'}"'] & \bigoplus_{x' \rhd' y'} \mathscr{B}'(w')^{x'} \arrow[r] & \bigoplus_{x' \in \mathcal{V}_{\delta y'}} \mathscr{B}'(w')^{x'} \arrow[r, "\oplus \rho'_{x',E'}"'] & \bigoplus_{E \in \mathcal{E}_{\delta y'}} \mathscr{B}'(w')^{E'}
\end{tikzcd}$$

It follows that there exists an isomorphism of S_k-modules $\mathscr{B}(w)^{\delta y} \cong \mathscr{B}'(w')^{\delta y'}$ and by the unicity of the projective cover we obtain $\mathscr{B}(w)^{y} \cong \mathscr{B}'(w')^{y'}$. This proves the statement. $\square$

The lemma above will be a very useful tool in Chapter 5.

Chapter 3

Moment graphs associated to a symmetrisable Kac-Moody algebra

The aim of this chapter is to recall standard notions related to the theory of Weyl groups and to study some classes of moment graphs coming from this theory. At first, we will define regular and parabolic Bruhat graphs associated to a symmetrisable Kac-Moody algebra. In particular, we will see that parabolic Bruhat graphs are quotients of the regular ones in the sense of §1.3.2. The second part of this section is devoted to the affine and affine Grassmannian cases. The main result of this chapter is a characterisation of finite intervals of the moment graph associated to the affine Grassmannian (see §3.2.2 and §3.2.3) that motivates the definition of the *stable moment graph*.

3.1 Bruhat graphs

Here, we define a very important class of moment graphs: the Bruhat graphs. As unlabelled oriented graphs, moment graphs were introduced by Dyer in 1991 (cf.[10]) in order to study some properties of the Bruhat order on a Coxeter group; already in 1993, he considered them as edge–labelled oriented graphs. Actually, he was labelling the edges by reflections of the Coxeter group (cf.[11]), instead of the corresponding positive coroots (see Def.3.1.1). Even if his definition seems equivalent to ours, the extra structure coming from the whole root lattice turns out to be fundamental when we are considering morphisms between two Bruhat (k-moment) graphs (see §1.2). An important (and still open) conjecture, the so-called *combinatorial invariance conjecture* (due to Lusztig and Dyer, independently), states that the Kazhdan-Lusztig polynomial $h_{x,y}$ (see §4.2.1) only depends on the interval $[x, y]$ in the Bruhat graph. As moment graphs, Bruhat graphs constitute a very important example and in fact they have been introduced already in [7].

We start by recalling some notation from [26]. Let $\mathfrak{g}$ be a symmetrisable Kac-Moody algebra, that is the Lie algebra $\mathfrak{g}(A)$ associated to a symmetrisable generalised Cartan matrix

A, and $\mathfrak{h}$ its Cartan subalgebra. Let $\Pi = \{\alpha_i\}_{i=1,\dots,n} \subset \mathfrak{h}^*$, resp. $\Pi^\vee = \{{\alpha_i}^\vee\}_{i=1,\dots,n} \subset \mathfrak{h}$, be the set of simple roots, resp. coroots; let Δ, resp. Δ_+, resp. Δ_+^{re} be the root system, resp. the set of positive roots, resp. the set of positive real roots; and let $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$, resp. $Q^\vee = \sum_{i=1}^n \mathbb{Z}{\alpha_i}^\vee$, be the root lattice, resp. the coroot lattice. For any $\alpha \in \Delta$, we denote by $s_\alpha \in GL(\mathfrak{h}^*)$ the reflection, whose action on $v \in \mathfrak{h}^*$ is given by

$$s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha \tag{3.1}$$

Let $\mathcal{W} = \mathcal{W}(A)$ be the Weyl group associated to A, that is the subgroup of $GL(\mathfrak{h}^*)$ generated by the set of simple reflections $\mathcal{S} = \{s_\alpha | \alpha \in \Pi\}$. Recall that $(\mathcal{W}, \mathcal{S})$ is a Coxeter system (cf. [26], §3.10).

However, $\mathcal{W}$ can be seen also as subgroup of $GL(\mathfrak{h})$, by the setting, for any $\lambda \in \mathfrak{h}$

$$s_\alpha(\lambda) := \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \tag{3.2}$$

We will denote by $\mathcal{T} \subset \mathcal{W}$ the set of reflections, that is

$$\mathcal{T} = \left\{ s_\alpha \,|\, \alpha \in \Delta_+^{\mathrm{re}} \right\} = \left\{ wsw^{-1} \,|\, w \in \mathcal{W},\ s \in \mathcal{S} \right\} \tag{3.3}$$

Hereafter we will write α_t to denote the positive real root corresponding to the reflection $t \in \mathcal{T}$. Finally, denote by $\ell : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ the length function and by $\leq$ the Bruhat order on $\mathcal{W}$.

3.1.1 Regular Bruhat graphs

Definition 3.1.1. *Let $(\mathcal{W}, \mathcal{S})$ be as above. Then the* regular Bruhat (moment) graph $\mathcal{G} = \mathcal{G}(\mathfrak{g}) = (\mathcal{V}, \mathcal{E}, \leq, l)$ *associated to $\mathfrak{g}$ is a moment graph on $Q^\vee$ and it is given by*

(i) $\mathcal{V} = \mathcal{W}$, *that is the Weyl group of* $\mathfrak{g}$

(ii)
$$\begin{aligned} \mathcal{E} &= \left\{ x \to y \,|\, x < y\,,\, \exists\, \alpha \in \Delta_+^{re} \text{ such that } y = s_\alpha x \right\} \\ &= \{ x \to y \,|\, x < y\,,\, \exists\, t \in \mathcal{T} \text{ such that } y = tx \} \end{aligned}$$

(iii) $l(x \to s_\alpha x) := \alpha^\vee$

Remark 3.1.1. *Such a moment graph has an important geometric meaning. If G is the Kac-Moody group, whose Lie algebra is $\mathfrak{g}$, and $B \subset G$ is a standard Borel subgroup, then there is an algebraic action of a maximal torus $T \subset B$ on the flag variety $\mathcal{B} = G/B$ (cf. [32]). Moreover, the stratification coming from the Bruhat decomposition is T-invariant and satisfies all the assumptions of [[7],§1]. It turns out that this is a particular case of Example 1.1.4. In fact, the vertices are the 0-dimensional orbits with respect to the T-action, while the edges represent the 1-dimensional orbits (cf.§2.1 of [19]). The partial order on the set of vertices is induced by the Bruhat decomposition $\mathcal{B} = \bigsqcup_{w \in \mathcal{W}} X_w$, where, indeed, $\overline{X_w} = \bigsqcup_{y \leq w} X_y$.*

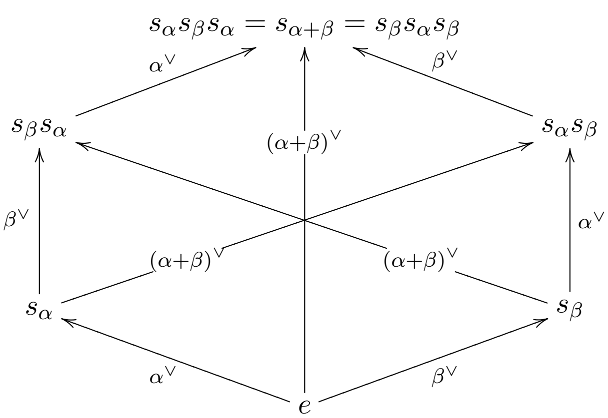
Example 3.1.1. *If $\mathfrak{g} = \mathfrak{sl}_2$, then the corresponding root system is $A_1 = \{\pm\alpha\}$ and $\mathcal{W} = S_2$.*

The associated Bruhat moment graph is the following subgeneric graph (see Example 1.1.3).

$$e \bullet \xrightarrow{\alpha^\vee} \bullet s_\alpha$$

For any local ring k, this graph is clearly a k-moment graph and $(\mathcal{G}(\mathfrak{sl}_2), k)$ is trivially a GKM-pair.

Example 3.1.2. *If $\mathfrak{g} = \mathfrak{sl}_3$, then the corresponding root system is $A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, $\mathcal{W} = S_3$. In this case, we get the following Bruhat graph.*



3.1.2 Parabolic Bruhat graphs

We introduce a class of Bruhat graphs, that generalises the one we described in §3.1.1. In order to do this, we need some combinatorial results.

Let $\mathcal{W}$ be a Weyl group and let $\mathcal{S}$ be its set of simple reflections. For any subset $J \subseteq \mathcal{S}$, we denote $\mathcal{W}_J := \langle J \rangle$ and $\mathcal{W}^J = \{w \in \mathcal{W} \,|\, ws > w \;\forall s \in J\}$. The following results hold.

Proposition 3.1.1 ([5], Prop. 2.4.4)**.**

(i) Every $w \in \mathcal{W}$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in \mathcal{W}^J$ and $w_J \in \mathcal{W}_J$.

(ii) For this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$.

Corollary 3.1.1 ([5], Cor. 2.4.5)**.** *Each left coset $w\mathcal{W}_J$ has a unique representative of minimal length.*

It follows that $\mathcal{W}^J$ is a set of representatives for the equivalence classes in $\mathcal{W}/\mathcal{W}_J$.

In order to make consistent Definition 3.1.2, we prove the following lemma.

Lemma 3.1.1. *Let $\mathcal{W}, \mathcal{S}, J$ be as before. Let $x, y, z \in \mathcal{W}$ and let $y^J = z^J \neq x^J$. If there exist α, $\beta \in \Delta_+^{re}$ such that $x = s_\alpha y = s_\beta z$, then $\alpha = \beta$ and so $y = z$.*

Proof. Take $v \in \mathfrak{h}^*$ such that $\mathcal{W}_J = \mathrm{Stab}_{\mathcal{W}}(v)$ (such a v exists thanks to [[26], Prop. 3.2(a)). By hypothesis, $z^J = y^J$ and then there exists a $w \in \mathcal{W}_J$ such that $z = yw$. It follows

$$s_\alpha y(v) = x(v) = s_\beta yw(v) = s_\beta y(v)$$

That is

$$y(v) - \langle y(v), \alpha^\vee \rangle \alpha = y(v) - \langle y(v), \beta^\vee \rangle \beta$$

This equality holds if and only if $\langle y(v), \alpha^\vee \rangle \alpha = \langle y(v), \beta^\vee \rangle \beta$. But this is the case if and only if $\langle y(v), \alpha^\vee \rangle = \langle y(v), \beta^\vee \rangle = 0$ or α is a multiple of β.

If it were $\langle y(v), \alpha^\vee \rangle = 0$, then $\langle v, y^{-1}(\alpha)^\vee \rangle = 0$ too. But this would imply that $s_{y^{-1}(\alpha)} = y^{-1} s_\alpha y \in \mathrm{Stab}_{\mathcal{W}}(v) = \mathcal{W}_J$, that is there would exist a $u \in \mathcal{W}_J$ such that $s_\alpha = yuy^{-1}$. But then we would get $x = s_\alpha y = (yuy^{-1})y = yu$, that is $x^J = y^J$. This contradicts the hypotheses.

If α is a multiple of β, then $\alpha = \pm\beta$ and, since $\alpha, \beta \in \Delta_+^{\mathrm{re}}$, we get $\alpha = \beta$. □

Definition 3.1.2. *Let $\mathcal{W}$, $\mathcal{S}$ and J be as above. Then the* parabolic Bruhat (moment) graph $\mathcal{G}^J = \mathcal{G}(\mathcal{W}^J) = (\mathcal{V}, \mathcal{E}, \leq, l)$ *associated to $\mathcal{W}^J$ is a moment graph on $Q^\vee$ and it is given by*

(i) $\mathcal{V} = \mathcal{W}^J$

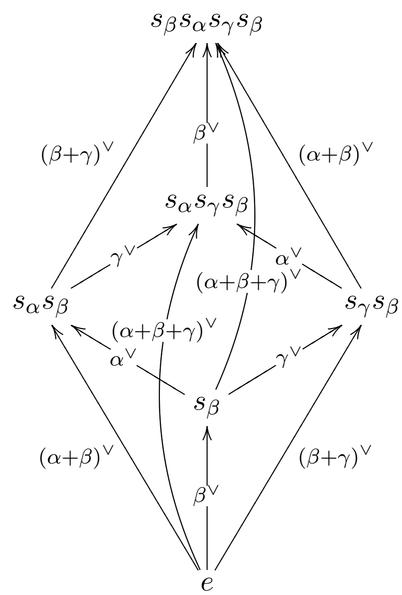
(ii) $\mathcal{E} = \{x \to y \mid x < y\,,\, \exists\, \alpha \in \Delta_+^{re},\, \exists w \in \mathcal{W}_J \text{ such that } ywx^{-1} = s_\alpha\}$

(iii) $l(x \to s_\alpha x w^{-1}) := \alpha^\vee$, *well–defined by Lemma 3.1.1.*

Remark 3.1.2. *Clearly,* $\mathcal{G}(\mathcal{W}^\emptyset) = \mathcal{G}(\mathfrak{g})$.

Remark 3.1.3. *The moment graph we defined describes a geometric situation similar to the one of Remark 3.1.1, once replaced the flag variety with the corresponding partial flag variety (cf. [32]).*

Example 3.1.3. *Let $\mathfrak{g} = \mathfrak{sl}_4$. In this case, $\Delta = A_3$, $\Pi = \{\alpha, \beta, \gamma\}$, $\mathcal{W} = S_4$ and $\mathcal{S} = \{s_\alpha, s_\beta . s_\gamma\}$, where $s_\alpha s_\gamma = s_\gamma s_\alpha$. If we chose $J = \{s_\alpha, s_\gamma\}$, the associated parabolic Bruhat* *graph $\mathcal{G}^J$ is the following octahedron.*



3.1.3 Parabolic graphs as quotients of regular graphs

Here we show that, if $\mathcal{W}, \mathcal{S}$ and J are as in the previous section, then $\mathcal{G}^J$ is a quotient of $\mathcal{G}$ by a $\mathcal{G}$-compatible relation (cf. §1.3.2). To give this characterisation of parabolic Bruhat graphs, we recall two well-known results.

The first one is the so-called *lifting Lemma* and it is a classical tool in combinatorics of Coxeter groups.

Lemma 3.1.2 ([22], Lemma 7.4)**.** *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. Let $s \in \mathcal{S}$ and $v, u \in \mathcal{W}$ be such that $vs < v$ and $u < v$.*

(i) If $us < u$, then $us < vs$.

(ii) If $us > u$, then $us \leq v$ and $u \leq vs$. Thus, in both cases, $us \leq v$.

We will use this lemma several times in what follows.

The following proposition tells that the poset structure of $\mathcal{W}$ is preserved in $\mathcal{W}^J$.

Proposition 3.1.2 ([5], Prop.2.5.1)**.** *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, $J \subseteq \mathcal{S}$ and $x, y \in \mathcal{W}$. If $x \leq y$, then $x^J \leq y^J$.*

Using the previous results, we get

Lemma 3.1.3. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system and $J \subseteq \mathcal{S}$. If $y^J \in \mathcal{W}^J$, $y_J \in \mathcal{W}_J$, $t \in \mathcal{T}$ are such that $(y^J)^{-1} t y^J \notin \mathcal{W}_J$. Then, $ty^J < y^J$ if and only if $ty^J y_J < y^J y_J$.*

Proof. We prove the lemma by induction on $\ell(y_J)$. If $l(y_J) = 0$, there is nothing to prove.

Suppose $ty^J \leq y^J$ and let $\ell(y_J) > 0$. Then there exists a simple reflection $s \in J$ such that $y_J s < y_J$, that is $y^J y_J s < y^J y_J$. Now, by the inductive hypothesis $t(y^J y_J s) < y^J y_J s$ and, from Lemma 3.1.2, it follows $ty^J y_J = (ty^J y_J s)s \leq y^J y_J$.

Viceversa, suppose $ty^J y_J \leq y^J y_J$ and $\ell(y_J) > 0$. Then there exists a simple reflection $s \in J$ such that $y_J s < y_J$, that is $y^J y_J s < y^J y_J$. By hypothesis, $ty^J y_J < y^J y_J$. If $ty^J y_J s < ty^J y_J$, by Lemma 3.1.2 (i), we get $ty^J y_J s < y^J y_J s$ and the claim follows from the inductive hypothesis. Otherwise, $ty^J y_J s > ty^J y_J$ and, by Lemma 3.1.2 (ii), $ty^J y_J s \leq y^J y_J$ and $ty^J y_J \leq y^J y_J s$. If it were $ty^J y_J s \not< y^J y_J s$, then $ty^J y_J s > y^J y_J s$ (because they are comparable) and so $y^J y_J s < ty^Y y_J s \leq y^J y_J$, that would imply $ty^J y_J s = y^J y_J$. But this is a contradiction, since they are not even in the same equivalence class. Thus we get $ty^J y_J s < y^J y_J s$ and hence, from the inductive hypothesis, the statement.

□

Lemma 3.1.4. *Let $\mathfrak{g}$ be a symmetrisable Kac-Moody algebra, $\mathcal{W}$ its Weyl group with $\mathcal{S}$, the set of simple reflections, and let $J \subseteq \mathcal{S}$. Let $\mathcal{G}$ be the Bruhat graph associated to $\mathfrak{g}$, then the equivalence relation on its set of vertices $\mathcal{V}$, given by $x \sim y$ if and only if $x^J = y^J$, is $\mathcal{G}$-compatible.*

Proof. We have to check conditions *(EQV1)* and *(EQV2)*.

(EQV1) From Proposition 3.1.2, if $x \leq y$ and $x^J = y^J$, then for all $z \in [x, y]$, $x^J \leq z^J \leq y^J = x^J$, that is $z^J = x^J$.

(EQV2) Let $x_1, y_1 \in \mathcal{W}$ and $t \in \mathcal{T}$ be such that $x_1 \not\sim y_1 \mathcal{W}_J$ and $x_1 \to y_1 = tx_1 \in \mathcal{E}$. If $x_2 \sim x_1$, that is $x_2 = x_1 w$ for some $w \in \mathcal{W}_J$, then we set $y_2 := y_1 w$, clearly $x_2 \text{ --- } y_2 = tx_2 \in \mathcal{E}$ and $l(x_2 \text{ --- } y_2) = l(x_1 \to y_1) = \alpha_t$. By Lemma 3.1.1, y_2 is the only element equivalent to y_1 and connected to x_2. Finally, from Lemma 3.1.3, it follows that $x_2 < y_2$. □

Corollary 3.1.2. *Let $\mathfrak{g}$ be a symmetrisable Kac-Moody algebra, $\mathcal{W}$ its Weyl group with $\mathcal{S}$, the set of simple reflections, and let $J \subseteq \mathcal{S}$. Let $\mathcal{G}$ be the Bruhat graph associated to $\mathfrak{g}$ and $\mathcal{G}^J$ the one associated to $\mathcal{W}^J$. Then $\mathcal{G}^J$ is the quotient of $\mathcal{G}$ by the $\mathcal{G}$-compatible equivalent relation defined in the previous lemma (in the sense of §1.3.2).*

We will denote by $p_J : \mathcal{G} \to \mathcal{G}^J$ the epimorphism given by $(p_J)_{\mathcal{V}}(x) := x^J$ and $(p_J)_{l,x} = \mathrm{id}$ for all $x \in \mathcal{W}$.

Example 3.1.4. *Let $\mathfrak{g} = \mathfrak{sl}_3$ and $J = \{s_\alpha\} \subset \mathcal{S} = \{s_\alpha, s_\beta\}$. Then Example 1.3.1 describes the parabolic Bruhat graph $\mathcal{G}^J$ as quotient of the regular one (see Example 3.1.2).*

3.2 The affine setting

We want now to focus our attention on the affine case.

Let A be a generalised Cartan matrix of affine type of order $l+1$ and rank l. Let us enumerate its rows and columns from 0 to l (as Kac in [[26], §6.1] does), and denote by $\dot{A}$ the matrix obtained from A by deleting the 0-th row and the 0-th column. Then the Weyl group $\mathcal{W}^a$ of $\mathfrak{g} = \mathfrak{g}(A)$ is the affinization of the (finite) Weyl group $\mathcal{W}^f$ of $\dot{\mathfrak{g}} = \mathfrak{g}(\dot{A})$ (cf. [26], Chapter 1). Take $\dot{\Delta}$ to be the root system of $\dot{\mathfrak{g}}$, and $\dot{\Pi}$ and $\dot{\Delta}_+$ the corresponding set of simple and of positive roots, respectively. It turns out that the set of real roots of $\mathfrak{g}$ has a nice description in terms of the root system of $\dot{\mathfrak{g}}$. Let $\delta \in \mathfrak{h}^*$ be such that $A\delta = 0$ and $\delta = \sum_{i=0}^{r} a_i\alpha_i$, where $\Pi = \{\alpha_i\}_{i=0,\dots,r}$ and the $a_i \in \mathbb{Z}_{>0}$ are relatively prime (such an element exists and it is unique by point b) of Theorem 5.6 in [26]). Then (cf. [26], Proposition 6.3)

$$\Delta^{\mathrm{re}} = \left\{ \alpha + n\delta \,|\, \alpha \in \dot{\Delta},\, n \in \mathbb{Z} \right\} \tag{3.4}$$

and

$$\Delta^{\mathrm{re}}_+ = \left\{ \alpha + n\delta \,|\, \alpha \in \dot{\Delta},\, n \in \mathbb{Z}_{>0} \right\} \cup \dot{\Delta}_+ \tag{3.5}$$

It follows that $\mathcal{W}^a$ is generated by the set of affine reflections

$$\mathcal{T}^a = \{ s_\beta \,\big|\, \beta \in \Delta^{\mathrm{re}}_+ \} = \{ s_{\alpha,n} \,|\, \alpha \in \dot{\Delta},\, n \in \mathbb{Z}_{>0} \} \cup \{ s_{\alpha,0} \,\big|\, \alpha \in \dot{\Delta}_+ \}.$$

Explicitly, the action of $\mathcal{W}^a$ on $\dot{\mathfrak{h}}^* \oplus \delta\mathbb{C}$ is given by

$$s_{\alpha,n}\big((\lambda, r)\big) = (s_\alpha(\lambda), -n\langle\lambda, \alpha^\vee\rangle + r) \tag{3.6}$$

For a given real root $\alpha + n\delta$, we want now to describe the corresponding coroot $(\alpha+\delta)^\vee$. We have a decomposition of the Cartan subalgebra as $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$, while $\mathfrak{h}^* = \dot{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ (cf. [[26], §6.2]), where $\langle \delta, \dot{\mathfrak{h}} \oplus \mathbb{C}c\rangle = 0$. Because $\mathfrak{g}$ is symmetrizable, by [[26], Lemma 2.1], there is a bilinear form $(,)$ that induces an isomorphism $\nu : \dot{\mathfrak{h}} \to \dot{\mathfrak{h}}^*$ such that we may identify $\alpha^\vee$ and $\frac{2\alpha}{(\alpha,\alpha)}$. Then,

$$(\alpha + n\delta)^\vee = \alpha^\vee + \frac{2n}{(\alpha,\alpha)}\, c = \frac{2}{(\alpha,\alpha)}\,(\alpha + n\, c)\,. \tag{3.7}$$

3.2.1 The affine Weyl group and the set of alcoves

We recall briefly a description of $\mathcal{W}^a$ as a group of affine transformations of $\dot{\mathfrak{h}}^*_{\mathbb{R}}$, the $\mathbb{R}$-span of $\alpha_1, \dots, \alpha_l$. This is obtained by identifying $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ with the affine space $\dot{\mathfrak{h}}^*_{-1}$ mod $\mathbb{R}\delta$, where

$$\dot{\mathfrak{h}}^*_{-1} := \left\{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \,\middle|\, \langle\lambda, c\rangle = -1 \right\}$$

Namely, it is possible to define an action of the affine Weyl group on $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ as follows

$$s_{\alpha,n}(\lambda) = \lambda - \left(\langle\lambda, \alpha^\vee\rangle - \frac{2n}{(\alpha,\alpha)} \right) \alpha = s_{\alpha,0}(\lambda) + n\alpha^\vee \tag{3.8}$$

Denote by $\dot{Q}^\vee$ the coroot lattice of $\dot{\mathfrak{g}}$ and by T_μ the translation by $\mu \in \dot{Q}^\vee$, that is the linear transformation defined as $T_\mu(\lambda) = \lambda + \mu$ for any $\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}}$. This is an element of the affine Weyl group, since $T_{n\alpha^\vee} = s_{\alpha,n}s_\alpha$. It is easy to check that for any $w \in \mathcal{W}^a$ and for any $\mu \in \dot{Q}^\vee$ we have $wT_\mu w^{-1} = T_{w(\mu)}$, so the group of translations by an element of the coroot lattice turns out to be a normal subgroup. A well known fact is that $\mathcal{W}^a = \mathcal{W}^f \ltimes \dot{Q}^\vee$ (cf.[[22], Proposition 4.2]).

If θ is the (unique) highest root of $\dot{\Delta}$, then a minimal set of generators for $\mathcal{W}^a$ is given by $\mathcal{S}^a = \{s_{\alpha_i,0}\}_{i=1,\dots,l} \cup \{s_{\theta,1}\}$, where $\mathcal{S}^f := \{s_{\alpha_i,0}\}_{i=1,\dots,l}$ is the set of simple reflections of $\mathcal{W}^f$. Let us set $s_0 := s_{\theta,1}$ and call it the *affine simple reflection*.

Denote by

$$H_{\alpha,n} := \left\{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle = 2\frac{n}{(\alpha,\alpha)}\right\} = \left\{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid (\lambda,\alpha) = n\right\}$$

and observe that the affine reflection $s_{\alpha,n}$ fixes pointwise such a hyperplane. We call *alcoves* the connected components of

$$\dot{\mathfrak{h}}^*_{\mathbb{R}} \setminus \bigcup_{\substack{\alpha\in\dot{\Delta}_+\\ n\in\mathbb{Z}}} H_{\alpha,n}$$

and denote by $\mathcal{A}$ the set of all alcoves.

The *dominant* -or *fundamental*- *(Weyl) chamber* is

$$\mathcal{C}^+ := \{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee\rangle > 0 \ \forall \alpha \in \dot{\Delta}_+\}$$

and an element $\lambda \in \mathcal{C}^+$ is called *dominant weight*. We denote by $\mathcal{A}^+$ the set of all alcoves contained in $\mathcal{C}^+$ and by

$$\Pi_0 = \left\{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee \rangle < \frac{2}{(\alpha,\alpha)} \ \ \forall \alpha \in \Pi\right\} = \left\{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid 0 < (\lambda,\alpha) < 1 \ \ \forall \alpha \in \Pi\right\}$$

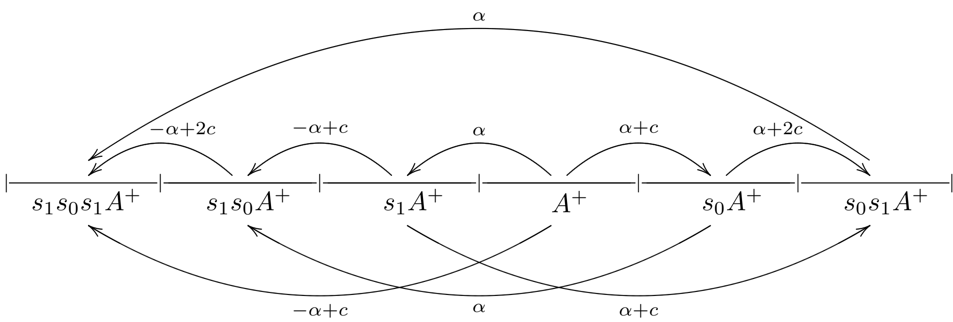
the *fundamental box*.

We state now a 1-1 correspondence between $\mathcal{W}^a$ and $\mathcal{A}$ (cf. [[22], Theorem 4.8]). In order to do that, we fix an alcove A^+, that is the unique alcove in $\mathcal{A}^+$ which contains the null vector in its closure. A^+ is usually called *fundamental alcove* and it has the property that every element $\lambda \in A^+$ is such that $0 < (\lambda,\alpha) \leq 1$ for all $\alpha \in \dot{\Delta}_+$ (cf. [[22], §4.3]).

The affine Weyl group $\mathcal{W}^a$ acts on the left (by (3.8)) simply transitively on $\mathcal{A}$ (cf. [22],§4.5) and so we obtain

$$\begin{array}{ccc} \mathcal{W}^a & \xrightarrow{1-1} & \mathcal{A} \\ w & \mapsto & wA^+. \end{array} \tag{3.9}$$

Example 3.2.1. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. By (3.5), we know that $\Delta^{re}_+ = \{\pm\alpha + n\delta \mid n \in \mathbb{Z}_{>0}\} \cup \{\alpha\}$, where α is the (unique) positive root of $\mathfrak{sl}_2$ and $(\alpha,\alpha) = 2$. The corresponding Bruhat graph is an infinite graph, whose vertices are given by the words in two letters ($s_1 := s_\alpha$ and s_0) without repetitions. Two elements are connected if and only if the difference between their lengths is odd and in this case the edge is oriented from the shorter to the longer one. Thanks to the correspondence (3.9),* *we may identify the set of vertices with the set of alcoves of $\mathfrak{g}$. If we restrict the Bruhat graph to the interval $[A^+, s_1s_0s_1]$, we get the following*



We observe here that each wall of A^+ is fixed by exactly one reflection $s \in \mathcal{S}^a$. We say that such a wall is the s-wall of A^+. In general every $A \in \mathcal{A}$ has one and only one wall in the $\mathcal{W}^a$-orbit of the s-wall of A^+. This is called *s-wall* of A.

The affine Weyl group acts on itself by right multiplication, so it makes sense to define a right action of $\mathcal{W}^a$ on $\mathcal{A}$. It is of course enough define such an action for the generators of the group. Thus for each alcove A let As be the unique alcove having in common with A the s-wall.

Two partial orders on the set of alcoves

Here we want to provide the set of alcoves with two partial orders (cf. [37]).

First of all, the Bruhat order on $\mathcal{W}^a$ induces a partial order on $\mathcal{A}$. Indeed, for all alcoves $A, B \in \mathcal{A}$ with $A = xA^+$, $B = yA^+$, $x, y \in \mathcal{W}^a$ we may set

$$A \leq B \iff x \leq y.$$

We still call it *Bruhat order*.

We observe that in general if we look at two fixed alcoves it is not obvious at all if they are comparable with respect to the Bruhat order without knowing the corresponding elements in $\mathcal{W}^a$.

Now, we recall Lusztig's definition of a nicer partial order $\preccurlyeq$ on $\mathcal{A}$, in the sense that for all pair of alcoves we will be able to say if they are comparable and, in case, to establish which one is the bigger one.

Each $H \in \bigcup_{\substack{\alpha \in \dot{\Delta}_+ \\ n \in \mathbb{Z}}} H_{\alpha,n}$ divides $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ in two half spaces: $\dot{\mathfrak{h}}^*_{\mathbb{R}} = H^+ \cup H \cup H^-$, where H^+ is the half space that intersects every translate of $\mathcal{C}^+$. Let $A \in \mathcal{A}$, if H is the reflecting hyperplane between A and As, $s \in \mathcal{S}^a$, we consider the partial order generated by

$$A \preccurlyeq As \quad \text{if } A \in H^-.$$

We notice that it is not clear in general how $\leq$ and $\preccurlyeq$ are related. Actually, denoting by X the *lattice of (finite) integral coweights*, that is

$$X^\vee := \{\lambda \in \dot{\mathfrak{h}}^*_{\mathbb{R}} \mid (\lambda, \alpha) \in \mathbb{Z}\ \forall \alpha \in \dot{\Delta}\}, \tag{3.10}$$

we have

Proposition 3.2.1 ([40],claim 4.4)**.** *Far enough inside $\mathcal{A}^+$, $\leq$ and $\preccurlyeq$ coincide, that is for all $\lambda \in X^\vee \cap \mathcal{C}^+$, $A, B \in \mathcal{A}$ the following are equivalent:*

1. *$A \preccurlyeq B$;*
2. *$n\lambda + A \leq n\lambda + B$ for $n >> 0$.*

Because of this result we call $\preccurlyeq$ *generic Bruhat order.* Remark that $\preccurlyeq$ is invariant under translation by coweights.

The periodic moment graph

In section 3.1, we associated to any simmetrisable Kac-Moody algebra $\mathfrak{g}$ with Weyl group $\mathcal{W}$ its regular Bruhat graph $\mathcal{G}(\mathfrak{g})$. If $\mathfrak{g}$ is moreover affine, that is its Weyl group is an affine Weyl group $\mathcal{W}^a$, we may give the following definition.

Definition 3.2.1. *The* periodic moment graph $\mathcal{G}^{per} = \mathcal{G}^{per}(\mathfrak{g}) = (\mathcal{V}, \mathcal{E}, \preccurlyeq, l)$ *associated to $\mathcal{W}^a$ is a moment graph on $Q^\vee$ and it is given by*

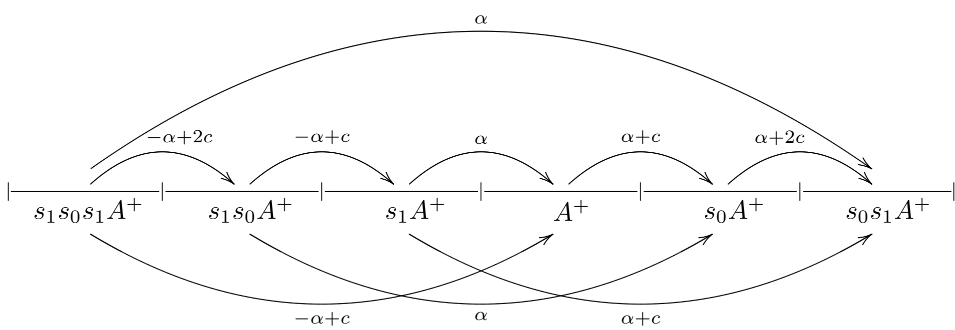
(i) $\mathcal{V} = \mathcal{A}$, *the set of alcoves of* $\mathcal{W}^a$

(ii) $\mathcal{E} = \left\{ xA^+ \to yA^+ \mid xA^+ \preccurlyeq yA^+ \,,\, \exists\, \alpha \in \Delta_+^{re} \text{ such that } y = s_\alpha x \right\}$

(iii) $l(xA^+ \to s_\alpha x A^+) := \alpha^\vee$

Remark 3.2.1. *We identified $\mathcal{W}^a$ and $\mathcal{A}$ by (3.9) and so $\mathcal{G}(\mathcal{W}^a)$ and $\mathcal{G}^{per}(\mathcal{W}^a)$ coincide as labeled* unoriented *graphs.*

Example 3.2.2. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. If we restrict the corresponding periodic moment graph to the interval $[s_1 s_0 s_1, s_0 s_1 A^+]$, we get the following moment graph.*



3.2.2 Parabolic moment graphs associated to the affine Grassmannian

We consider in this section a very important class of parabolic moment graphs: the ones associated to the Affine Grassmannians, that is $\mathcal{G}^J$, where $\mathcal{W} = \mathcal{W}^a$ is an affine Weyl group and J is the corresponding set of *finite* simple reflections, that is we are modding out by the finite Weyl group.

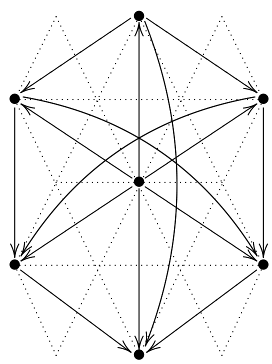
There are actually two descriptions of this graph: one identifies the set of vertices with the coroot lattice $\dot{Q}^\vee$, while the other identifies the set of vertices with $\mathcal{A}^+$, the set of alcoves in the fundamental chamber. Hereafter, we will denote this graph by $\mathcal{G}^{\mathrm{par}}$.

Since $\mathcal{W}_J = \mathrm{Stab}_{\mathcal{W}}(0)$ and $\mathcal{W}.0 = \dot{Q}^\vee$, $\mathcal{W}_J$ is in bijection with the coroot lattice via the mapping $w \mapsto w(0)$ and clearly there exist an element $w \in \mathcal{W}_J$ and a reflection $t \in \mathcal{T}$ such that $x_J = ty_Jw$ if and only if $x_J(0) = ty_J(0)$. And we get in this way the first description.

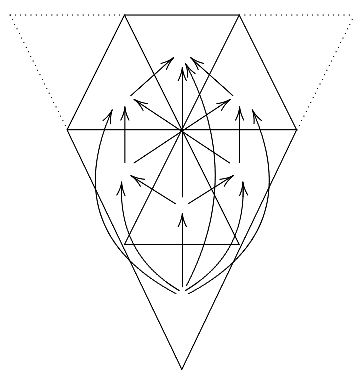
On the other hand, $\mathcal{W}^J$ is evidently in bijection with $\mathcal{W}_J \setminus \mathcal{W}$ via the mapping $w^J \mapsto (w^J)^{-1}$ modulo $\mathcal{W}_J$. The set of minimal representatives for the equivalence classes, under the correspondence (3.9), is given by the set $\mathcal{A}^+$ of the alcoves in the fundamental chamber. Moreover, we will connect $xA^+, yA^+ \in \mathcal{A}^+$ if and only if there exist an element of the finite Weyl group $w \in \mathcal{W}_J$ and an affine reflection $t \in \mathcal{T}$ such that $x = wyt$, that is $x^{-1} = ty^{-1}w^{-1}$.

Example 3.2.3. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$. Let us consider the interval $[e, s_\beta s_\alpha s_\beta s_0] \subset \mathcal{W}^J$ then the two descriptions of $\mathcal{G}^{par}$ are as follows (we omit the labels).*

(i) Description via the coroot lattice



(ii) Description via the set of alcoves in $\mathcal{C}^+$



As we can see in the previous example, in the description of $\mathcal{G}^{\mathrm{par}}$ via the alcoves in the fundamental chamber, the set of edges seems to have a very complex structure, while in the other one the order on the set of vertices is hard to understand. Since we are interested in the study of intervals, the description via the coroot lattice turns out to be not that useful for our purposes, unless $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. We will show later that finite intervals of $\mathcal{G}^{\mathrm{par}}$ "far enough" in $\mathcal{C}^+$ have surprisingly a very regular structure.

The $\widehat{\mathfrak{sl}_2}$ case

If $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, it is actually possible to give a very explicit description of $\mathcal{G}^{\mathrm{par}}$. In this case we may identify the finite root with the finite coroot lattice and then the set of vertices is $\mathcal{V} = \mathbb{Z}\alpha$. For any pair $n, m \in \mathbb{Z}$, it is easy to check that

$$s_{\alpha,n+m}(n\alpha) = m\alpha, \tag{3.11}$$

then $\mathcal{G}^{\mathrm{par}}$ is a fully connected graph. Notice that, even if (3.11) holds for any pairs of integers n and m, we do not allow loops, so $n \neq m$ always. Moreover, by (3.7) and (3.11), it follows

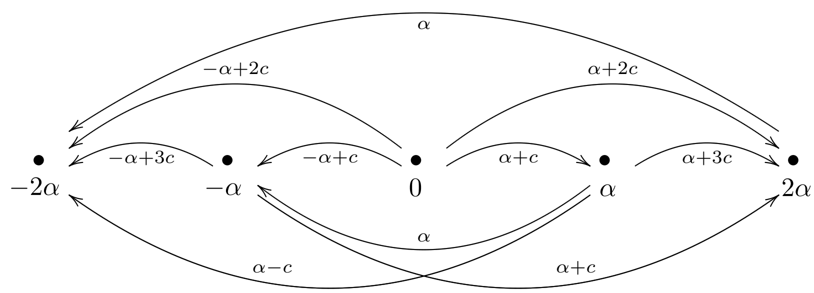
$$l(n\alpha \text{ --- } m\alpha) = \begin{cases} \alpha + (n+m)c & \text{if } n+m \geq 0 \\ -\alpha - (n+m)c & \text{if } n+m < 0 \end{cases} \tag{3.12}$$

Finally, observe that $\alpha = s_0(0)$ and $-\alpha = s_\alpha s_0(0)$. In particular,

$$n\alpha = \begin{cases} (s_\alpha s_0)^n(0) & \text{if } n \leq 0 \\ s_0(s_\alpha s_0)^{n-1}(0) & \text{if } n > 0 \end{cases}$$

We conclude that, for any pair of $n \neq m \in \mathbb{Z}$, $n\alpha < m\alpha$ if and only if $|n| < |m|$ or $n = -m > 0$.

Example 3.2.4. *The interval* $[0, -2\alpha]$ *of* $\mathcal{G}^{par}$ *looks like in the following picture*



3.2.3 Parabolic intervals far enough in the fundamental chamber

In this paragraph, we will consider only the description of $\mathcal{G}^{\mathrm{par}}$ in which the set of vertices coincides with $\mathcal{A}^+$. Our goal is to study the structure of finite intervals of $\mathcal{G}^{\mathrm{par}}$ far enough in the fundamental chamber. In this section, k is any field of characteristic zero.

Definition 3.2.2. *Let* $\lambda, \mu \in \mathcal{C}^+$. *We say that*

(i) λ *is* strongly linked *to* μ *if* $\lambda = \mu + x\alpha$, *for some* $x \in \mathbb{R}$ *and* $\alpha \in \dot{\Delta}_+$

(ii) λ *is* linked *to* μ *if* $\lambda = w(\mu + n\alpha)$, *for some* $n \in \mathbb{R}$, $\alpha \in \dot{\Delta}_+$ *and* $w \in \mathcal{W}^f$

Remark that the fundamental chamber $\mathcal{C}^+$ is a fundamental domain with respect to the left action of the finite Weyl group (cf. [22], §1.12), so the element in point *(ii)* is unique.

Proposition 3.2.2. *There exists a* $K > 0$, *depending only on the root system* $\dot{\Delta}$, *such that if* $\lambda \in \mathcal{C}^+$ *and* d_λ *is the minimum of distances from* λ *to the borders of* $\mathcal{C}^+$, *then all* $\mu \in \mathcal{C}^+$ *linked to* λ *and such that* $|\lambda - \mu| < K \cdot d_\lambda$ *are strongly linked to* λ.

Proof. For any $\lambda \in \mathcal{C}^+$ and any positive finite root $\alpha \in \dot{\Delta}_+$ we denote by $r_{\lambda,\alpha}$ the line $\{\lambda + \alpha x \mid x \in \mathbb{R}\} \subseteq \dot{\mathfrak{h}}^*_{\mathbb{R}}$. It is clear that the set of dominant weights strongly linked to λ corresponds to $(\bigcup_{\alpha \in \dot{\Delta}_+} r_{\lambda,\alpha}) \bigcap \mathcal{C}^+$. On the other hand, we may describe the set of $\mu \in \mathcal{C}^+$ linked to λ as follows. Fix $\alpha \in \dot{\Delta}_+$ and consider the line $r_{\lambda,\alpha}$. Each time that such a line hits a wall of $\mathcal{C}^+$ reflects it off the wall and goes on this way. Denote by $\widetilde{r}_{\lambda,\alpha}$ the piecewise linear path inside of $\mathcal{C}^+$ so obtained. Now $\bigcup_{\alpha \in \dot{\Delta}_+} \widetilde{r}_{\lambda,\alpha}$ is the set of dominant weights linked to λ.

Thus it is enough to show that there exists a $K > 0$ such that if $\mu \in \widetilde{r}_{\lambda,\alpha}$ and $|\lambda - \mu| < K \cdot d_\lambda$, then $\mu \in r_{\lambda,\alpha}$. Notice that the finite Weyl group acts on $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ as a group of orthogonal transformations, hence we may reduce to show that for all $w \in \mathcal{W}^f \setminus \{e, s_\alpha\}$, the distance of the weight $w(\lambda)$ from the line $r_{\lambda,\alpha}$ is not less then than $K \cdot d_\lambda$. Moreover, one may think of this reduction as an "unfolding" back $\widetilde{r}_{\lambda,\alpha}$ to $r_{\lambda,\alpha}$ and considering the conjugates of λ instead of λ.

Since the distance of $w(\lambda)$ from the line $r_{\lambda,\alpha}$ is the minimum of the distances of $w(\lambda)$ from $\lambda + x\alpha$ for $x \in \mathbb{R}$, we have to show that $|\lambda - x\alpha - w(\lambda)|^2 \geq K^2 d_\lambda^2$ for all $x \in \mathbb{R}$. Computing the square norm, and denoting $\lambda^w := \lambda - w(\lambda)$, we have:

$$|\alpha|^2 x^2 + 2(\lambda^w, \alpha)x + |\lambda^w|^2 + K^2 d_\lambda^2 \geq 0 \quad \forall x \in \mathbb{R}$$

Hence this is equivalent to showing that the discriminant $D^w = (\lambda^w, \alpha)^2 - |\alpha|^2|\lambda^w|^2 + |\alpha|^2 K^2 d_\lambda^2 \leq 0$.

First notice that $D^{s_\alpha w} = D^w$, since $\lambda^{s_\alpha w} = \lambda - w(\lambda) + \langle w(\lambda), \alpha^\vee \rangle \alpha = \lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha$, hence:

$$\begin{aligned}
D^{s_\alpha w} &= (\lambda^{s_\alpha w}, \alpha)^2 - |\alpha|^2 |\lambda^{s_\alpha w}|^2 + |\alpha|^2 K^2 d_\lambda^2 = \\
&= (\lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha, \alpha)^2 - |\alpha|^2 (\lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha, \lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha) + |\alpha|^2 K^2 d_\lambda^2 = \\
&= (\lambda^w, \alpha)^2 + 2\langle w(\lambda), \alpha^\vee \rangle |\alpha|^2 (\lambda^w, \alpha) + \langle w(\lambda), \alpha^\vee \rangle^2 |\alpha|^4 + \\
&- |\alpha|^2 |\lambda^w|^2 - 2|\alpha|^2 (\langle w(\lambda), \alpha^\vee \rangle \alpha, \lambda^w) - \langle w(\lambda), \alpha^\vee \rangle^2 |\alpha|^4 + |\alpha|^2 K^2 d_\lambda^2 = \\
&= (\lambda^w, \alpha)^2 - |\alpha|^2 |\lambda^w|^2 + |\alpha|^2 K^2 d_\lambda^2 = D^w
\end{aligned}$$

Now if $w^{-1}(\alpha)$ is a finite negative root, then clearly $(s_\alpha w)^{-1}(\alpha) \in \dot{\Delta}_+$, hence, using the invariance property just proved, in what follows we may assume that $w \in \mathcal{W}^f \setminus \{e, s_\alpha\}$ is such that $w^{-1}(\alpha) \in \dot{\Delta}_+$.

Denote now by $\dot{\Delta}^w_+$ the set of positive roots sent to negative roots by w^{-1}, let C^w be the (closed convex rational) cone $\langle \dot{\Delta}^w_+ \rangle_{\mathbb{R}^+}$ generated by the elements of $\dot{\Delta}^w_+$ and notice that α is not in $\pm C^w$. Indeed, α is not in C^w since all elements of this cone are sent to non-negative linear combination of negative roots by w^{-1} and, on the other hand, α is a positive root while all elements in $-C^w$ are non-negative linear combinations of negative roots.

Let L^w be the set of weights λ^w, where λ runs in $\mathcal{C}^+$ and fix a reduced expression $s_{i_1} \ldots s_{i_r}$, with $s_{i_j} := s_{\alpha_{i_j}}$, for $\alpha_j \in \dot{\Pi}$. Then we have $w(\lambda) = \lambda - (a_1 \beta_{i_1} \ldots a_r \beta_{i_r})$, where $\beta_j = s_{i_1} \ldots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, \ldots, r$. Notice that $a_{i_j} \geq 0$ for all j since $\lambda \in \mathcal{C}^+$ and, moreover, $\{\beta_{i_1}, \ldots, \beta_{i_r}\} = \dot{\Delta}^w_+$. This shows that $L^w \subseteq C^w$.

Let $\pi : \dot{\mathfrak{h}}^*_{\mathbb{R}} \setminus \{0\} \to \mathbb{P}(\dot{\mathfrak{h}}^*_{\mathbb{R}})$ the quotient map to the projective space of $\dot{\mathfrak{h}}^*_{\mathbb{R}}$. Given two non-zero vectors $u, v \in \dot{\mathfrak{h}}^*_{\mathbb{R}}$ we denote by $[u, v]$ the angle between them; clearly this symbol depends only on the lines generated by u and v up to sign to change and up to supplementary angles. In particular, the map $\mathbb{P}(\dot{\mathfrak{h}}^*_{\mathbb{R}})^2 \to \mathbb{R}$ defined by $(\pi[u], \pi[v]) \mapsto \cos^2[u, v]$ is well–defined.

Since C^w is a closed convex rational cone we have that $\pi(C^w \setminus \{0\})$ is closed in $\mathbb{P}(\dot{\mathfrak{h}}^*_{\mathbb{R}})$. Hence the map $\pi(C^w \setminus \{0\}) \to \mathbb{R}$ sending $\pi(\mu) \mapsto \cos^2[\mu, \alpha]$ achieves a maximal value $M^{\alpha,w}$ and this maximal value is less than 1 since $\pi(\alpha) \notin \pi(C^w \setminus \{0\})$. In particular we have $\cos^2[\lambda^w, \alpha] \leq M^{\alpha,w} < 1$ for all $\lambda \in \mathcal{C}^+ \setminus \{0\}$ since $L^w \subseteq C^w$.

Finally, since there are only a finite number of pairs (α, w), we have $M := \max M^{\alpha,w} < 1$. Now notice that $w(\lambda) \notin \mathcal{C}^+$, because $w \neq e$, so $|\lambda^w| \geq d_\lambda$, as the segment from λ to $w(\lambda)$ must cross a wall of $\mathcal{C}^+$.

We are now in a position to conclude the proof. We have to show $D^w \leq 0$. Since

$$(\lambda^w, \alpha) = |\lambda^w||\alpha| \cos[\lambda^w, \alpha],$$

our inequality becomes $\cos^2[\lambda^w, \alpha] \leq 1 - K^2 d_\lambda^2 / |\lambda^w|^2$. But we have $\cos^2[\lambda^w, \alpha] \leq M < 1$ and $1 - K^2 d_\lambda^2/|\lambda^w|^2 > 1 - K^2$. Hence it is enough to choose K such that $M \leq 1 - K^2$. This finishes the proof.

$\square$

Let ρ be half the sum of the finite positive coroots, that is $\rho = \frac{1}{2}\sum_{\alpha \in \dot{\Delta}_+} \alpha^\vee$. Moreover, for any alcove $A \in \mathcal{A}$, let us denote by c_A its centroid.

By using Proposition 3.2.2, together with the identification $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for all $\alpha \in \dot{\Delta}$, we get the following characterisation of finite intervals of $\mathcal{G}^{\text{par}}$ that are far enough from the walls of the dominant chamber.

Lemma 3.2.1. *Let $A, B \in \mathcal{A}^+$, then there exists an integer $n_0 = n_0(A, B)$ such that for any $\lambda \in X \cap n\rho + \mathcal{C}^+$, with $n \geq n_0$, for any pair $C, D \in [A + \lambda, B + \lambda]$ there is an edge C —— D if and only if*

(i) either $D = Ct$ for some $t \in \mathcal{T}$

(ii) or $D = C + a\alpha$ for some $a \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \dot{\Delta}_+$.

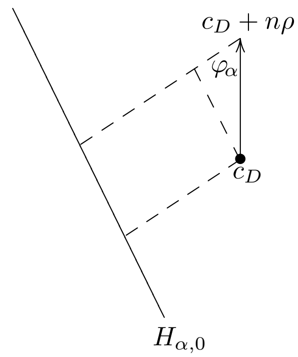
Proof. Observe first that the statement is true for $\mathfrak{g} = \mathfrak{sl}_2$. Indeed, from §3.2.2, it follows that $n_0 = 0$ satisfies already the requirements.

We may then suppose $\mathfrak{g} \neq \mathfrak{sl}_2$. The claim will follow once we prove that there exists an $n_0 \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$, $n \geq n_0$ and for all pair $E, F \in [A + n\rho, B + n\rho]$, we have

$$|c_E - c_F| < K \cdot d_{c_E}$$

where $K > 0$ was defined in Proposition 3.2.2. Indeed, the statement is equivalent to show that for all $n > n_0$, for any $\alpha \in \dot{\Delta}_+$ we have $(A + n\rho + \mathbb{R}\alpha^\vee) \cap w([A + n\rho, B + n\rho]) = \emptyset$ unless $w = e, s_\alpha$, but this is the case if and only if for all $n > n_0$, for any $\alpha \in \dot{\Delta}_+$ we have $\big(A + n\rho + \mathbb{R}w(\alpha^\vee)\big) \cap [A + n\rho, B + n\rho] = \emptyset$ unless $w = e, s_\alpha$.

For any finite simple root $\alpha \in \dot{\Pi}$, let us denote by $d_{c_D,\alpha}$ the distance between c_D and the hyperplane $H_{\alpha,0}$. Let φ_α be the angle between ρ and α, then we get the following picture



As we can see in the picture above, we have

$$d_{c_{D+n\rho},\alpha} = d_{c_D,\alpha} + n|\rho| \cdot \cos(\varphi_\alpha)$$

Moreover for all $D \in [A,B]$

$$d_{c_{D+n\rho}} = \min_\alpha \{d_{c_{D+n\rho},\alpha}\}$$

Let us denote $r := \max_{D,E\in[A,B]} |c_D - c_E|$ and let $H \in [A,B]$ and $\beta \in \dot{\Pi}$ such that $\min_{D\in[A,B]} d_{c_{D+\rho}} = d_{c_H,\beta} + |\rho| \cos\varphi_\beta$ (that is $\min_{D\in[A,B]} d_{c_{D+n\rho}} = d_{c_H,\beta} + n|\rho|\cos\varphi_\beta$ for all $n > 0$). Since $\mathfrak{g} \neq \mathfrak{sl}_2$, for any $\gamma \in \dot{\Pi}$ it holds $\cos\varphi_\gamma \neq 0$ and we may set

$$m := \frac{r}{K|\rho| \cdot \cos\varphi_\beta}$$

Define $n_0 = \lceil m \rceil$. Now, for any pair of alcoves $E, F \in [A,B]$ and for any $n \in \mathbb{Z}$, $n > n_0$

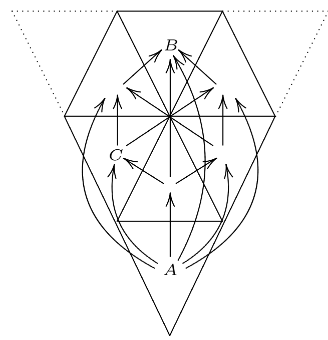
$$\begin{array}{rcl}
|c_{E+n\rho} - c_{F+n\rho}| & = & |c_E - c_F| \\
& \leq & r \\
& = & m \cdot K|\rho| \cdot \cos\varphi_\beta \\
& \leq & n_0 \cdot K|\rho| \cdot \cos\varphi_\beta \\
& \leq & n_0 \cdot K|\rho| \cdot \cos\varphi_\beta + K \cdot d_{c_H,\beta} \\
& = & K \cdot \min_{D\in[A,B]} d_{c_{D+n_0\rho}} \\
& \leq & K \cdot d_{c_{E+n_0\rho}} \\
& < & K \cdot d_{c_{E+n\rho}}
\end{array}$$

□

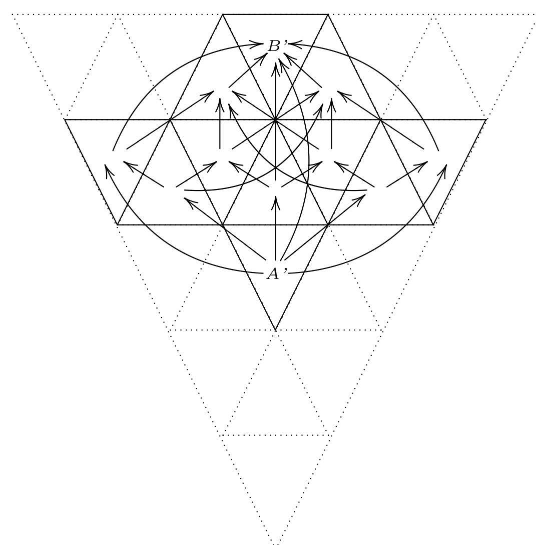
We say that the edges of type (i), that is given by reflections, are *stable*, while the ones of type (ii), that is given by translations, are *non-stable*. We denote the corresponding sets $\mathcal{E}_S$, resp. $\mathcal{E}_{NS}$.

Example 3.2.5. *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $A = A^+$, $B = s_0 s_1 s_2 s_1 A^+$. Then in the interval $[A,B]$ of $\mathcal{G}^{par}$*

there are edges that are neither stable nor non-stable, as the one between A *and* $C = s_0 s_1 A^+$.



It is enough to translate the interval of $\alpha + \beta$ *to get the structure described in Lemma 3.2.1.*



Lemma 3.2.2. *For any pair* $A, B \in \mathcal{A}^+$, $B \leq A$ *and for any pair* $\lambda_1 = n_1\rho$, $\lambda_2 = n_2\rho \in X \cap n\rho + \mathcal{C}^+$ $(n_1, n_2 \geq n_0(A, B))$ *then* $\mathcal{G}^{par}_{|[A+\lambda_1, B+\lambda_1]}$ *and* $\mathcal{G}^{par}_{|A+\lambda_2, B+\lambda_2]}$ *are isomorphic as oriented graphs.*

Proof. Set $\mu := \lambda_2 - \lambda_1$. The isomorphism we are looking for is given by $C \mapsto C + \mu$. Observe that, by Proposition 3.2.1, the Bruhat order coincides in the fundamental chamber with the generic one and so it is invariant by weight translation; then the map we have just defined is an isomorphism of posets. Moreover C is connected to D in $\mathcal{G}^{\text{par}}_{|A+\lambda_1, B+\lambda_1]}$ if and only if $C + \mu$ is connected to $D + \mu$ in $\mathcal{G}^{\text{par}}_{|A+\lambda_2, B+\lambda_2]}$, indeed:

(i) $D = Ct$ for some $t \in \mathcal{T}$ if and only if $D = rC$ for some $r \in \mathcal{T}$, that is if and only if there exist $\alpha \in \dot{\Delta}_+$ and $n \in \mathbb{Z}$ such that $D = s_{\alpha,n}(C)$. It is now easy to check that this is the case if and only if $D + \mu = s_{\alpha,n+(\mu,\alpha)}(C + \mu)$, that is there exists a reflection $t' \in \mathcal{T}$ such that $D + \mu = (C + \mu)t'$.

(ii) $D = C + a\alpha$ if and only if $D + \mu = C + a\alpha + \mu = (C + \mu) + a\alpha$. $\square$

Remark 3.2.2. *We want to stress the fact that in Lemma 3.2.2 we are* not *proving the existence of an isomorphism of moment graphs, but only between the underlying oriented graphs, that is we are not considering labels. Our first hope was that we could find a collection of* $\{f_{l,C}\}_{C\in[A+\lambda_1,B+\lambda_1]}$ *satisfying condition (MORPh2a) and (MORPH2b). In the next two paragraphs, we will see that it is not the case. In particular, it turns out that the labels of stable edges are invariant by coroot translation (cf. Lemma 3.2.1), while the ones of non-stable edges are not (cf. Lemma 3.2.4).*

From now one we will denote by $w \in \mathcal{W}^a$ the corresponding alcove $wA^+ \in \mathcal{A}$, thanks to the identification (3.9) of the affine Weyl group with the set of alcoves. In particular, if wA^+ is contained in the fundamental chamber, we will write $w \in \mathcal{A}^+$.

Stable edges

Let $|\mathcal{S}^a| = n$ and fix a numbering of the simple reflections. We define the permutation $\sigma_{A,\mu} \in S_n$, for $A \in \mathcal{A}$ and $\mu \in \dot{X}^\vee$, in the following way: $\sigma_{A,\mu}(i) = j$ if the image under the translation by μ of the s_i–th wall of A is the s_j–th wall of $A+\mu$ (cf.§3.2.1). Let $\widetilde{\mathcal{W}^a}$ the extended affine Weyl group, that is $\widetilde{\mathcal{W}^a} = \mathcal{W}^a \ltimes \Omega$, where $\Omega := \dot{X}^\vee / \dot{Q}^\vee$ (cf. [38]).

Lemma 3.2.3. *For any* $\mu \in \dot{X}^\vee$ *the permutation defined above is independent on* $A \in \mathcal{A}$*, i.e. there exists* $\sigma_\mu \in S_n$ *such that* $\sigma_{A,\mu} = \sigma_\mu$ *for any alcove A.*

Proof. We know that $T_{k\alpha^\vee} = s_{\alpha,k}s_{\alpha,0}$ for any $\alpha \in \dot{\Delta}$. Since we are reflecting twice in the same direction (orthogonal to α), the walls of $A + k\alpha^\vee$ have the same numbering as the ones of A.

Thus for any $\mu \in \dot{X}^\vee$ there exists an element $\omega \in \Omega$ and roots $\alpha_1, \dots \alpha_r \in R$ such that $T_\mu = \omega s_{\alpha_1,k_1}s_{\alpha_1}\dots s_{\alpha_r,k_r}s_{\alpha_r}$ and the numbering of the walls of $A+\mu$ only depends on ω. □

We get the following lemma.

Corollary 3.2.1. *Let* $x \in \mathcal{W}^f$, $t \in \mathcal{T}$, $\mu \in X^\vee$ *be such that* $x, xt, T_\mu x, T_\mu xt \in \mathcal{A}^+$*. Then,* $l(T_\mu x -\!\!-\, T_\mu xt) = \sigma_\mu(l(x -\!\!-\, xt))$.

Non-stable edges

Now we describe how labels of non-stable edges change. In order to do that we need the following result

Proposition 3.2.3 ([22],Proposition 4.1)**.** *Let* $z = T_{z(0)}v$*, where* $z(0) \in \dot{Q}^\vee$ *and* $v \in \mathcal{W}^f$*. Then, for any* $\alpha \in \dot{\Delta}_+$ *and for any* $n \in \mathbb{Z}$,

$$zs_{\alpha,n}z^{-1} = s_{\pm v(\alpha),r} \quad \textit{with } r = \pm\big(\langle z(0), v(\alpha)^\vee\rangle + n\big), \tag{3.13}$$

where the signs are such that $r > 0$ *or* $r = 0$ *and* $\pm v(\alpha) \in \dot{\Delta}_+$.

We may now prove

Lemma 3.2.4. *Let* $x \in \mathcal{A}^+$ *and* $x = T_{x(0)}w$*, where* $w \in \mathcal{W}^f$.

(i) If $\alpha \in \dot{\Delta}_+$ *and* $T_{a\alpha^\vee} x \in \mathcal{A}^+$*, then*

$$l(x \text{ --- } T_{a\alpha^\vee} x) = \pm w^{-1}(\alpha^\vee) \mp \frac{2}{(\alpha,\alpha)} \left(\langle x(0), \alpha^\vee\rangle + a\right) c, \tag{3.14}$$

where $\mp\big(\langle x(0), \alpha^\vee\rangle + a\big) > 0$ *or* $\big(\langle x(0), \alpha^\vee\rangle + a\big) = 0$ *and* $\pm w^{-1}(\alpha) \in \dot{\Delta}_+$.

(ii) Let $y = T_{a\alpha^\vee} x$*, for some* $a \in \mathbb{Z}$ *and* $\alpha \in \dot{\Delta}_+$*. Let moreover* $\mu \in X^\vee$*,* $\omega \in \Omega$ *and* $\gamma \in \dot{Q}^\vee$ *be such that* $T_\mu = \omega T_\gamma$*. Then, if* $y, T_\mu x, T_\mu y \in \mathcal{C}^+$*,*

$$l(T_\mu x \text{ --- } T_\mu y) = \sigma_\mu(l(x \text{ --- } y)) \mp \frac{2\langle \gamma, \alpha^\vee\rangle}{(\alpha,\alpha)} c, \tag{3.15}$$

where $\mp\big(\langle \gamma + x(0), \alpha^\vee\rangle + a\big) > 0$ *or* $\big(\langle \gamma + x(0), \alpha^\vee\rangle + a\big) = 0$ *and* $\pm\sigma_\mu(w^{-1}(\alpha)) \in \dot{\Delta}_+$.

Proof.

(i) Since $T_{a\alpha^\vee} x = s_{\alpha,a} s_{\alpha,0} x$, we have to determine the positive root corresponding to the reflection $x^{-1} s_{\alpha,0} s_{\alpha,a} s_{\alpha,0} x$.

Since $s_{\alpha,0} s_{\alpha,a} s_{\alpha,0} = s_{\alpha,-a}$, by Proposition 3.2.3 with $z = x^{-1}$, $v = w^{-1}$, $z(0) = -w^{-1}(x(0))$ and $n = -a$, we get

$$\begin{aligned} x^{-1} s_{\alpha,0} s_{\alpha,a} s_{\alpha,0} x &= s_{\pm w^{-1}(\alpha), \pm(\langle -w^{-1}(x(0)), w^{-1}(\alpha)^\vee\rangle - a)} \\ &= s_{\pm w^{-1}(\alpha), \mp(\langle x(0), \alpha^\vee\rangle + a)}. \end{aligned}$$

The result follows from (3.7) and the fact that $w(\alpha)^\vee = w(\alpha^\vee)$ for all $\alpha \in \dot{\Delta}$ and $w \in \mathcal{W}^f$.

(ii) Observe that $T_\mu x \text{ --- } T_\mu y = T_{a\alpha^\vee}(T_\mu x)$. If $x = T_{x(0)} w$, then $T_\mu x = T_{\mu + x(0)} w = \omega T_{\gamma + x(0)} w$ and we may apply point (i) of this Lemma with $T_\mu x$ instead of x. So, if $\pm w^{-1}(\alpha) \in \dot{\Delta}_+$, $n = -a$, we get

$$\begin{aligned} l(T_{k\gamma^\vee} x \text{ --- } T_{k\gamma^\vee} y) &= \pm\sigma_\mu(w^{-1}(\alpha^\vee)) \mp \tfrac{2}{(\alpha,\alpha)}(\langle \gamma + x(0), \alpha^\vee\rangle + a) c \\ &= \sigma_\mu(l(x \text{ --- } y)) \mp \tfrac{k\langle \gamma, \alpha^\vee\rangle}{(\alpha,\alpha)} c. \end{aligned}$$

□

Stable moment graphs

Let $\mathcal{G}^{\mathrm{par}}$ be the same moment graph as before. We define here the *stable moment graph* $\mathcal{G}^{\mathrm{stab}}$ as follows. This is the moment graph having as set of vertices the alcoves in the fundamental chamber (that we identify with the corresponding elements of the Weyl group), equipped with the Bruhat order (that here coincides with the generic one); we connect two vertices if and only if there exists a reflection $t \in \mathcal{T}^a$ such that $y = xt$, and in this case we set $l(x \text{ --- } xt) := \alpha_t{}^\vee$.

Then we have:

Lemma 3.2.5. *For any interval* $[y, w]$ *and for any* $\mu \in X^\vee$ *there exists an isomorphism of* k*-moment graphs* $\mathcal{G}^{stab}_{|[y,w]} \longrightarrow \mathcal{G}^{stab}_{|[y,w]+\mu}$ *for all* k.

Proof. Since the order on the set of vertices of $\mathcal{G}^{\text{stab}}$ is invariant by weight translation, we have an isomorphism of posets induced by the mapping $z \mapsto z + \mu$. This map induces also a bijection between set of edges, as we have already seen in the proof of Lemma 3.2.2.

The permutation of Lemma 3.2.3 gives an automorphism of the root system and then an induced automorphism of the coweight lattice. Since it depends only on the (finite intergral) coweight μ, we can set $f_{l,x} = \sigma_\mu$ for any x and this gives us an isomorphism of k-moment graphs for any k. $\square$

Chapter 4

Modules over the parabolic structure algebra

Let $\mathcal{Z}$ be the structure algebra (see §2.1.1) of a regular Bruhat graph $\mathcal{G}$. In [13], Fiebig defined translation functors on the category $\mathcal{Z}$-mod, that is the category of $\mathbb{Z}$-graded $\mathcal{Z}$-modules that are torsion free and finitely generated over S_k. Using it, he defined inductively a full subcategory $\mathcal{H}$ of $\mathcal{Z}$-mod and he proved that $\mathcal{H}$, in characteristic zero, is equivalent to a category introduced by Soergel in [41]. In [18], Fiebig showed that $\mathcal{H}$ categorifies the Hecke algebra $\mathbf{H}$ (and the periodic module $\mathbf{M}$), using translation functors. The aim of this chapter is to define translation functors in the parabolic setting and to extend some results of [18].

4.1 Translation functors

Let $\mathcal{W}$ be a Weyl group, let $\mathcal{S}$ be its set of simple reflections and let $J \subseteq \mathcal{S}$. Hereafter we will keep the notation we used in §3.1.2.

For all $s \in \mathcal{S}$, Fiebig defined in [13] an involutive automorphism σ_s of the structure algebra of a regular Bruhat graph. In a similar way, we will define an involution ${}_s\sigma$ for a fixed simple reflection $s \in \mathcal{S}$ on the structure algebra $\mathcal{Z}^J$ of the parabolic Bruhat (k-moment) graph $\mathcal{G}^J$. In this chapter, we suppose that $(\mathcal{G}^J, k)$ is a GKM-pair (see Definition 1.1.3).

Let $x, y \in \mathcal{W}^J$. Notice that $l(x \text{ --- } y) = {\alpha_t}^\vee$ if and only if $l(sx \text{ --- } sy) = s({\alpha_t}^\vee)$, because $sxw(sy)^{-1} = sxwy^{-1}s = sts$, where $w \in \mathcal{W}_J$. From now on, if $x \in \mathcal{W}$, we will write $\overline{x}$ instead of x^J.

Denote by τ_s the automorphism of the symmetric algebra S_k induced by the mapping $\lambda \mapsto s(\lambda)$ for all $\lambda \in Q^\vee$. For any $(z_x)_{x\in\mathcal{W}^J} \in \mathcal{Z}^J$, we set ${}_s\sigma\big((z_x)_{x\in\mathcal{W}^J}\big) = (z'_x)_{x\in\mathcal{W}^J}$, where $z'_x := \tau_s(z_{\overline{sx}})$. This is again an element of the structure algebra from what we have observed above.

Let us denote by ${}^s\mathcal{Z}^J$ the space of invariants with respect to the automorphism ${}_s\sigma$ and by ${}^{-s}\mathcal{Z}^J$ the space of anti-invariants. We denote moreover by $\overline{{\alpha_s}^\vee}$ the element of $\mathcal{Z}^J$ whose components are all equal to ${\alpha_s}^\vee$. We obtain the following decomposition of $\mathcal{Z}^J$ as ${}^s\mathcal{Z}^J$-module.

Lemma 4.1.1. $\mathcal{Z}^J = {}^s\mathcal{Z}^J \oplus \overline{{\alpha_s}^\vee} \cdot {}^s\mathcal{Z}^J$.

Proof. (We follow [13], Lemma 5.1). Because ${}_s\sigma$ is an involution and $char(k) \neq 2$, we get $\mathcal{Z}^J = {}^s\mathcal{Z}^J \oplus {}^{-s}\mathcal{Z}^J$. Since $\overline{\alpha_s} \in \mathcal{Z}^J$ and $s(\alpha_s{}^\vee) = -\alpha_s{}^\vee$, it follows ${}_s\sigma(\overline{\alpha_s{}^\vee}) = -\overline{\alpha_s{}^\vee}$ and so $\overline{\alpha_s{}^\vee} \cdot {}^s\mathcal{Z}^J \subseteq {}^{-s}\mathcal{Z}^J$ and we now have to prove the other inclusion, that is every element $z \in {}^{-s}\mathcal{Z}^J$ is divisible by $\overline{\alpha_s{}^\vee}$ in ${}^{-s}\mathcal{Z}^J$.

If $z = (z_x) \in {}^{-s}\mathcal{Z}^J$, then, for all $x \in \mathcal{W}^J$, $z_x = -\tau_s(z_{\overline{sx}}) \equiv -z_{\overline{sx}} \ (\bmod\ \alpha_s{}^\vee)$ and $z_x \equiv z_{\overline{sx}}\ (\bmod\ \alpha_s{}^\vee)$. It follows that $2z_x \equiv 0\ (\bmod\ \alpha_s{}^\vee)$, that is $\alpha_s{}^\vee$ divides z_x in S_k, as $char(k) \neq 2$. We have now to verify that $z' := \overline{(\alpha_s{}^\vee)^{-1}} \cdot z \in \mathcal{Z}$, that is $z'_x - z'_{\overline{tx}} \equiv 0 (\bmod\ \alpha_t{}^\vee)$ for any $x \in \mathcal{W}^J$ and $t \in \mathcal{T}$. If $\overline{tx} = \overline{sx}$, there is nothing to prove; on the other hand, if $\overline{tx} \neq \overline{sx}$, we get the following.

$$\alpha_s{}^\vee \cdot (z'_x - z'_{\overline{tx}}) = z_x - z_{\overline{tx}} \equiv 0\ (\bmod\ \alpha_t{}^\vee)$$

Since $(\mathcal{G}^J, k)$ is a GKM-pair, $\alpha_s{}^\vee \not\equiv 0\ (\bmod\ \alpha_t{}^\vee)$ and we obtain $z'_x - z'_{\overline{tx}} \equiv 0\ (\bmod\ \alpha_t{}^\vee)$. □

4.1.1 Left translation functors

In order to define translation functors, we need an action of S_k on ${}^s\mathcal{Z}^J$ and $\mathcal{Z}^J$.

Lemma 4.1.2. *For any $\lambda \in Q^\vee$ and any $x \in \mathcal{W}^J$, let us set*

$$c(\lambda)^J_x := \sum_{x_J \in \mathcal{W}_J} x x_J(\lambda). \tag{4.1}$$

Then $c(\lambda)^J := (c(\lambda)^J_x)_{x \in \mathcal{W}^J} \in {}^s\mathcal{Z}^J$.

Proof. It is clear that, if $c(\lambda)^J \in \mathcal{Z}^J$, then it is invariant. So we only have to prove that $c(\lambda)^J \in \mathcal{Z}^J$, that is $c(\lambda)^J_x - c(\lambda)^J_{\overline{tx}} \equiv 0\ (\bmod\ \alpha_t{}^\vee)$. Since for any x_J there exists an element y_J such that $x x_J = t\,\overline{tx}\,y_J$ (cf. Lemma 3.1.4), then

$$\begin{aligned} \textstyle\sum_{x_J \in \mathcal{W}_J} x x_J(\lambda) - \sum_{x_J \in \mathcal{W}_J} \overline{tx}\, x_J(\lambda) &= \textstyle\sum_{y_J \in \mathcal{W}_J} t\,\overline{tx}\, y_J(\lambda) - \sum_{y_J \in \mathcal{W}_J} \overline{tx}\, y_J(\lambda) \\ &= t\left(\textstyle\sum_{y_J \in \mathcal{W}_J} \overline{tx}\, y_J(\lambda)\right) - \sum_{y_J \in \mathcal{W}_J} \overline{tx}\, y_J(\lambda) \\ &= \left(\textstyle\sum_{y_J \in \mathcal{W}_J} \langle \alpha_t, \overline{tx}\, y_J(\lambda)\rangle\right) \alpha_t{}^\vee \\ &\equiv 0\ (\bmod\ \alpha_t{}^\vee), \end{aligned}$$

since $\alpha^\vee$ is a multiple of α. □

For any $x \in \mathcal{W}^J$, denote by η_x the automorphism of the symmetric algebra S_k induced by the mapping $\lambda \mapsto c(\lambda)^J_x$ for all $\lambda \in Q^\vee$. Now, by Lemma 4.1.2, the action of S_k on $\mathcal{Z}^J$ given by

$$p.(z_x)_{x \in \mathcal{W}^J} = (\eta_x(p) z_x) \quad p \in S_k\,,\ z \in \mathcal{Z}^J, \tag{4.2}$$

preserves ${}^s\mathcal{Z}^J$. Thus any $\mathcal{Z}^J$-module and any ${}^s\mathcal{Z}^J$-module has an S_k-module structure as well. Let $\mathcal{Z}^J$-mod, resp. ${}^s\mathcal{Z}^J$-mod, be the category of $\mathbb{Z}$-graded $\mathcal{Z}^J$-modules, resp. ${}^s\mathcal{Z}^J$-modules, that are torsion free and finitely generated over S_k.

The *translation on the wall* is the functor ${}^{s,on}\theta : \mathcal{Z}^J$ -mod $\to {}^s\mathcal{Z}^J$-mod defined by the mapping $M \mapsto \mathrm{Res}^{{}^s\mathcal{Z}^J}_{\mathcal{Z}^J}$.

The *translation out of the wall* is the functor ${}^{s,out}\theta : {}^s\mathcal{Z}^J$ -mod $\to \mathcal{Z}^J$-mod defined by the mapping $N \mapsto \mathrm{Ind}^{{}^s\mathcal{Z}^J}_{\mathcal{Z}^J}$. Observe that this functor is well-defined thanks to Lemma 4.1.1.

By composition, we get a functor ${}^s\theta := {}^{s,out}\theta \circ {}^{s,on}\theta : \mathcal{Z}^J$-mod$\to \mathcal{Z}^J$-mod that we call *(left) translation functor*.

Remark 4.1.1. *We want to stress the fact that, if $J = \emptyset$, the translation functor we defined does not coincide with the one defined by Fiebig in [13]. Indeed, we are twisting the action of S_k, while in [13] S_k acts in the usual way, that is $p.(z_x) = (p \cdot z_x)$.*

The following proposition describes the first properties of ${}^s\theta$.

Proposition 4.1.1. (1) *The functors from ${}^s\mathcal{Z}^J$-mod to ${}^s\mathcal{Z}$-mod mapping $M \mapsto \mathcal{Z}^J\{2\} \otimes_{{}^s\mathcal{Z}^J} M$ and $M \mapsto Hom_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, M)$ are naturally equivalent.*

(2) *The functor ${}^s\theta = \mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} - : \mathcal{Z}^J - mod \to \mathcal{Z}^J - mod$ is selfadjoint up to a shift.*

Proof. (cf. [41], Proposition 5.10, and [13], Proposition. 5.2) By Lemma 4.1.1, $\{\overline{1}, \overline{\alpha_s}\}$ is a ${}^s\mathcal{Z}^J$-basis for $\mathcal{Z}^J$. Let $\overline{1}^*, \overline{\alpha_s}^* \in \mathrm{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J)$ a ${}^s\mathcal{Z}^J$-basis dual to $\overline{1}$ and $\overline{\alpha_s}$. We have an isomorphism of ${}^s\mathcal{Z}^J$-modules $\mathcal{Z}^J\{2\} \cong \mathrm{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J)$ defined by the mapping $1 \mapsto \overline{\alpha_s}^*$ and $\overline{\alpha_s} \mapsto 1^*$, since $\deg(1) - 2 = -2 = \deg(\overline{\alpha_s}^*)$ and $\deg(\overline{\alpha_s}) - 2 = 0 = \deg \overline{1}^*$. Now statement (1) follows from the fact that $\mathcal{Z}^J$ is of finite rank over ${}^s\mathcal{Z}^J$ and so $\mathrm{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, -) = \mathrm{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J) \otimes_{{}^s\mathcal{Z}^J} -$.

Now the second claim follows easily, since $\mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} -$ and $\mathrm{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, -)$ are, resp., left and right adjoint to the restriction functor.

□

Using the selfadjointness of ${}^s\theta$, we get the following corollary.

Corollary 4.1.1. ${}^s\theta : \mathcal{Z}^J - \mathrm{mod} \to \mathcal{Z}^J - \mathrm{mod}$ *is exact.*

4.1.2 Parabolic special modules

As in [13], we define, inductively, a full subcategory of $\mathcal{Z}^J$-mod.

Let $B_e \in \mathcal{Z}^J$-mod be the free S_k-module of rank one on which $z = (z_x)_{x \in \mathcal{W}^J}$ acts via multiplication by z_e.

Definition 4.1.1.

(i) The category of special $\mathcal{Z}^J$-modules *is the full subcategory $\mathcal{H}^J$ of $\mathcal{Z}^J$-mod whose objects are isomorphic to a direct summand of a direct sum of modules of the form ${}^{s_{i_1}}\theta \circ \ldots \circ {}^{s_{i_r}}\theta(B_e)\{n\}$, where $s_{i_1}, \ldots, s_{i_r} \in \mathcal{S}$ and $n \in \mathbb{Z}$.*

(ii) The category of special ${}^s\mathcal{Z}^J$-modules *is the full subcategory ${}^s\mathcal{H}^J$ of ${}^s\mathcal{Z}^J$-mod whose objects are isomorphic to a direct summand of ${}^{s,on}\theta(M)$ for some $M \in \mathcal{H}^J$.*

Let Ω be a finite subset of $\mathcal{W}^J$. Then, we set

$$\mathcal{Z}^J(\Omega) := \left\{ (z_x) \in \prod_{x \in \Omega} S_k \;\middle|\; \begin{array}{c} z_x \equiv z_y \;\; (\bmod\ \alpha_t{}^\vee) \\ \text{if } \exists\, w \in \mathcal{W}_J \text{ s.t. } y\,w\,x^{-1} = t \in \mathcal{T} \end{array} \right\}$$

If $\Omega \subseteq \mathcal{W}^J$ is s-invariant with respect to the left multiplication by s, that is $s\Omega = \Omega$, we may restrict ${}_s\sigma$ to it. We denote by ${}^s\mathcal{Z}^J(\Omega) \subseteq \mathcal{Z}^J(\Omega)$ the space of invariants and, using Lemma 4.1.1, we get a decomposition $\mathcal{Z}^J(\Omega) = {}^s\mathcal{Z}^J(\Omega) \oplus \overline{\alpha_s{}^\vee} \cdot {}^s\mathcal{Z}^J(\Omega)$.

In the following lemma we prove, the *finiteness* of the special $\mathcal{Z}^J$-modules, as Fiebig does in [18] for special $\mathcal{Z}$-modules.

Lemma 4.1.3.

(i) Let $M \in \mathcal{H}^J$. Then there exists a finite subset $\Omega \subset \mathcal{W}^J$ and an action of $\mathcal{Z}^J(\Omega)$ such that $\mathcal{Z}^J$ acts on M via the canonical map $\mathcal{Z}^J \to \mathcal{Z}^J(\Omega)$.

(ii) Let $s \in \mathcal{S}$ and let N be an object in ${}^s\mathcal{H}^J$. Then there exists a finite s-invariant subset $\Omega \subset \mathcal{W}^J$ and an action of ${}^s\mathcal{Z}^J(\Omega)$ on N such that ${}^s\mathcal{Z}^J$ acts on N via the canonical map ${}^s\mathcal{Z}^J \to {}^s\mathcal{Z}^J(\Omega)$.

Proof. (we follow [18]) We prove *(i)* by induction. It holds clearly for B_e, since $\mathcal{Z}^J$ acts on it via the map $\mathcal{Z}^J \to \mathcal{Z}^J(\{e\})$. Now we have to show that if the claim is true for $M \in \mathcal{H}^J$, then it holds also for ${}^s\theta(M)$. Suppose $\mathcal{Z}^J$ acts via the map $\mathcal{Z}^J \to \mathcal{Z}^J(\Omega)$ over M. Observe that we may assume Ω s-invariant, as we can eventually consider $\Omega \cup s\Omega$, that is still finite. In this way the ${}^s\mathcal{Z}^J$-action on ${}^s\theta M$ via ${}^s\mathcal{Z}^J \to {}^s\mathcal{Z}^J(\Omega)$ and so we obtain ${}^s\theta M := \mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} M = \mathcal{Z}^J(\Omega) \otimes_{{}^s\mathcal{Z}^J(\Omega)} M$.

Claim *(ii)* follows directly from claim *(i)*. □

4.1.3 Decomposition and subquotients of modules on $\mathcal{Z}^{\mathbf{J}}$

We recall some notation from [14]. Let $S_k^{\emptyset} := S_k[\alpha^{-1} \mid \alpha \in \Delta]$ and, for any $M \in \mathcal{Z}^J - \text{mod}$, $M^{\emptyset} := M \otimes_{S_k} S_k^{\emptyset}$. By [[18], Lemma 3.1], there is a decomposition $M^{\emptyset} := M \cap \bigoplus_{x \in \mathcal{W}^J} M^{\emptyset,x}$ and so a canonical inclusion $M \subseteq \bigoplus_{x \in \mathcal{W}^J} M^{\emptyset,x}$. For all subset $\Omega \subseteq \mathcal{W}^J$, we may define:

$$M_{\Omega} := M \cap \bigoplus_{x \in \Omega} M^{\emptyset,x},$$

$$M^{\Omega} := M/M_{\mathcal{W}^J \setminus \Omega} = \operatorname{im}\left(M \to M^{\emptyset} = \bigoplus_{x \in \Omega} M^{\emptyset,x}\right).$$

For any $x \in \mathcal{W}^J$, we define

$$M_{[x]} := \ker\left(M^{\{\geq x\}} \to M^{\{>x\}}\right)$$

If $x \neq \overline{sx}$ and $x < xs$, we set moreover

$$M_{[x,sx]} := \ker\left(M^{\{\geq x\}} \to M^{\{\geq x\} \setminus \{sx\}}\right)$$

Lemma 4.1.5 describes the action of ${}^s\theta$ on the subquotients $M_{[x]}$'s. This is important in order to show that $\mathcal{H}^J$ categorifies the parabolic Hecke algebra. Actually, to prove Lemma 4.1.5, we need a combinatorial result.

Lemma 4.1.4. *Let $x \in \mathcal{W}^J$ and $t \in \mathcal{S}$. If $tx \notin \mathcal{W}^J$, then $\overline{tx} = x$.*

Proof. If $tx \notin \mathcal{W}^J$, then there exists a simple reflection $r \in J$ such that $txr < tx$ and, since $x \in \mathcal{W}^J$, $xr > x$. Using (the left version of) Lemma 3.1.2 *(i)* with $s = t$, $v = xr$ and $u = tx$, we get $txr < x$. Applying Lemma 3.1.2 *(i)* with $s = r$, $v = x$ and $u = txr$ it follows $tx > x$. Finally, from Lemma 3.1.2 *(ii)* we obtain $txr \leq x$, that, together with $x < xr$, gives $txr = x$. □

Lemma 4.1.5. *Let $s \in \mathcal{S}$ and $x \in \mathcal{W}^J$, then*

$$({}^s\theta M)_{[x]} \cong \begin{cases} M_{[x]}\{-2\} \oplus M_{[sx]}\{-2\} & \textit{if} \quad sx \in \mathcal{W}^J, sx > x \\ M_{[x]} \oplus M_{[sx]} & \textit{if} \quad sx \in \mathcal{W}^J, sx < x \\ M_{[x]}\{-2\} \oplus M_{[x]} & \textit{if} \quad sx \notin \mathcal{W}^J \end{cases}$$

Proof. (cf. [18]) By Lemma 4.1.4, if $sx \notin \mathcal{W}^J$, then $\overline{sx} = x$ and $M_{[x]} \in {}^s\mathcal{Z}^{\mathrm{par}}$-mod, so by Lemma 4.1.1 we get $\mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} M_{[x]} = M_{[x]}\{-2\} \oplus M_{[x]}$.

If $x \neq sx$, we have a short exact sequence $0 \to M_{[x]} \to M_{[x,sx]} \to M_{[sx]} \to 0$ and, since ${}^s\theta$ is exact (see Corollary 4.1.1), ${}^s\theta M_{x,sx} = ({}^s\theta M)_{[x,sx]} = {}^s\theta M_{[x]} \oplus {}^s\theta M_{[sx]}$. Moreover ${}^s\theta M_{[x,sx]} = \mathcal{Z}^J(\{x, sx\}) \otimes_{{}^s\mathcal{Z}^J(\{x,sx\})} M_{[x,sx]}$ and the two isomorphisms follow by taking in mind that $\mathcal{Z}^J(\{x, sx\})_{[x]} \cong S\{-2\}$ if $x \lhd sx$, while $\mathcal{Z}^J(\{x, sx\})_{[x]} \cong S_k$ if $x \rhd sx$. □

Using induction, we get the following corollary

Corollary 4.1.2. *Let $M \in \mathcal{H}^J$. Then for any $x \in \mathcal{W}^J$, $M_{[x]}$ is isomorphic to a finite direct sum of shifted copies of S_k.*

4.2 Special modules and Hecke algebras

In the first part of this section we recall the definition, due to Deodhar, of the parabolic Hecke algebra $\mathbf{H}^J$ and of its canonical basis. To the Bruhat order on $\mathcal{W}^J$ we associate, as in [[18], §4.5] a character map and in this way we get a map from the Grothendieck group of $\mathcal{H}^J$ to $\mathbf{H}^J$. Finally, we extend Proposition 4.3 of [18] to the parabolic setting, describing the action of the translation functors on the character (up to a shift) via the multiplication by elements of the canonical basis.

4.2.1 Hecke algebras

We start by giving the definition of the Hecke algebra associated to a Coxeter system $(\mathcal{W}, \mathcal{S})$, that is a quantisation of the group algebra of $\mathcal{W}$. We adopt the notation (and the renormalisation) of Soergel [40].

Denote by $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$ the ring of Laurent polynomials in the variable v over $\mathbb{Z}$.

Definition 4.2.1. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, then its Hecke algebra $\mathbf{H} = \mathbf{H}(\mathcal{W})$ is the free $\mathcal{L}$-module having basis $\{H_x \mid x \in \mathcal{W}\}$, subject to the following relations:*

$$H_s H_w = \begin{cases} H_{sx} & \text{if} \quad sx > s \\ (v^{-1} - v)H_x + H_{sx} & \text{if} \quad sx < x \end{cases} \tag{4.3}$$

It is well known that there exists exactly one such an associative $\mathcal{L}$-algebra (cf.[6] or [22]).

It is easy to verify that H_x is invertible for any $x \in \mathcal{W}$ and this allows us to define an involution on $\mathbf{H}$. This is the unique ring homomorphism $- : \mathbf{H} \to \mathbf{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$.

In [29] Kazhdan and Lusztig showed the existence of a nicer basis for $\mathbf{H}$, the so-called *canonical basis*, that they used to define complex representations of the Hecke algebra. The entries of the change of basis matrix were given by a family of polynomials in $\mathbb{Z}[v]$: the *Kazhdan-Lusztig polynomials*. In [9] Deodhar generalised this construction to the parabolic setting. Kazhdan-Lusztig polynomials and their parabolic analog will be the object of the next chapter.

Parabolic Hecke algebra and Kazhdan-Lusztig polynomials

Let us take $J \subseteq \mathcal{S}$. We recall Deodhar's construction, following [[40], §3]. Let $\mathbf{H} = \mathbf{H}(\mathcal{W})$ be the Hecke algebra of $\mathcal{W}$, then for any simple reflection $s \in \mathcal{S}$, by (4.3), we have $(H_s)^2 = (v^{-1} - v)H_s + H_e$, that is $(H_s + v)(H_s - v^{-1}) = 0$. If $u \in \{v^{-1}, -v\}$ and $\mathbf{H}_J := \mathbf{H}(\mathcal{W}_J)$ is the Hecke algebra of $\mathcal{W}_J$, then we may define a map of $\mathcal{L}$-modules $\varphi_u : \mathbf{H}_J \to \mathcal{L}$ by $H_s \mapsto u$. This provides a structure of $\mathbf{H}_J$ -bimodule to $\mathcal{L}$, that we denote by $\mathcal{L}(u)$.

Consider now $\mathbf{M}^J := \mathcal{L}(v^{-1}) \otimes_{\mathbf{H}_J} \mathbf{H}$ and $\mathbf{N}^J := \mathcal{L}(-v) \otimes_{\mathbf{H}_J} \mathbf{H}$. It is easy to verify that the map $- : \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H} \to \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H}$ sending $a \otimes H \mapsto \overline{a \otimes H} := \overline{a} \otimes \overline{H}$ is a ring homomorphism.

For $u \in \{v^{-1}, -v\}$ denote by $H_w^{J,u} := 1 \otimes H_w \in \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H}$. We are now able to state Deodhar's result.

Theorem 4.2.1 ([9])**.** **1.** *For all $w \in \mathcal{W}^J$ there exists a unique element $\underline{H}_w^{J,v^{-1}} \in \mathbf{M}^J$ such that:*

(i) $\overline{\underline{H}_w^{J,v^{-1}}} = \underline{H}_w^{J,v^{-1}}$

(ii) $\underline{H}_w^{J,v^{-1}} = \sum_{y \in \mathcal{W}^J} m_{y,w}^J H_y^{J,v^{-1}}$,

where the $m_{y,w}^J$ are such that $m_{w,w}^J = 1$ and $m_{y,w}^J \in v\mathbb{Z}[v]$ if $y \neq w$.

2. *For all $w \in \mathcal{W}^J$ there exists a unique element $\underline{H}_w^{J,-v} \in \mathbf{N}^J$ such that:*

(i) $\overline{\underline{H}_w^{J,-v}} = \underline{H}_w^{J,-v}$

(ii) $\underline{H}_w^{J,-v} = \sum_{y \in \mathcal{W}^J} n_{y,w}^J H_y^{J,-v}$,

where the $n_{y,w}^J$ are such that $n_{w,w}^J = 1$ and $n_{y,w}^J \in v\mathbb{Z}[v]$ if $y \neq w$.

The polynomials $m_{y,w}^J$ and $n_{y,w}^J$ are called *parabolic Kazhdan-Lusztig polynomials* with respect to the parameter v^{-1}, resp. $-v$, while $\{\underline{H}_w^{J,v^{-1}}\}_{w \in \mathcal{W}^J}$ is the *canonical basis*.

If $J = \emptyset$, then $\mathbf{M}^J = \mathbf{N}^J = \mathbf{H}$ and, for any pair of elements $y, w \in \mathcal{W}$, we will denote $h_{y,w} = m_{y,w}^{\emptyset} = n_{y,w}^{\emptyset}$ the corresponding regular Kazhdan-Lusztig polynomial.

We end this paragraph by recalling that the left multiplication by $\underline{H}_s$ for $s \in \mathcal{S}$, on $\mathbf{H}^J$ is given by (cf. [[40],§3])

$$\underline{H}_s \cdot H_x^{J,v^{-1}} = \begin{cases} H_{sx}^{J,v^{-1}} + vH_x^{J,v^{-1}} & \text{if} \quad sx \in \mathcal{W}^J, sx > x \\ H_{sx}^{J,v^{-1}} + v^{-1}H_x^{J,v^{-1}} & \text{if} \quad sx \in \mathcal{W}^J, sx < x \\ (v + v^{-1})H_x^{J,v^{-1}} & \text{if} \quad sx \notin \mathcal{W}^J \end{cases} \tag{4.4}$$

4.2.2 Character maps

Let M be a $\mathbb{Z}$-graded, free and finitely generated S_k-module; then $M \cong \bigoplus_{i=1}^n S_k\{j_i\}$, for some $j_i \in \mathbb{Z}$. We can associate to M its *graded rank*, that is the following Laurent polynomial.

$$\underline{\mathrm{rk}} M := \sum_{i=1}^{n} v^{-j_i} \in \mathbb{Z}[v, v^{-1}].$$

This is well-defined, because the j_i's are uniquely determined, up to the order.

Let $\langle \mathcal{H}^J \rangle$ be the Grothendieck group of $\mathcal{H}^J$

and let $M \in \mathcal{H}^J$, then by Corollary 4.1.2, we may define a map $h : \langle \mathcal{H}^J \rangle \to \mathbf{M}^J$ as follows.

$$h(M) := \sum_{x \in \mathcal{W}^J} v^{\ell(x)} \underline{rk}\, M_{[x]} H_x^{J,v^{-1}} \in \mathbf{M}^J$$

Proposition 4.2.1. *For each $M \in \mathcal{H}^J$ and for any $s \in \mathcal{S}$ we have $h({}^s\theta M\{1\}) = \underline{H}_s \cdot h(M)$, that is the following diagram is commutative*

$$\begin{tikzcd}
\langle \mathcal{H}^J \rangle \arrow[r, "h"] \arrow[d, "{}^s\theta\circ\{1\}"'] & \mathbf{M}^J \arrow[d, "\underline{H}_s\cdot"] \\
\langle \mathcal{H}^J \rangle \arrow[r, "h"] & \mathbf{M}^J
\end{tikzcd}$$

Proof. (cf. [18], Proposition 4.3) By Lemma 4.1.5, for any $x \in \mathcal{W}^J$ we have

$$\underline{\mathrm{rk}}({}^s\theta M)_{[x]} = \begin{cases} v^2\left(\underline{\mathrm{rk}}M_{[x]} + \underline{\mathrm{rk}}M_{[sx]}\right) & \text{if} \quad sx \in \mathcal{W}^J, sx > x \\ \underline{\mathrm{rk}}M_{[x]} + \underline{\mathrm{rk}}M_{[sx]} & \text{if} \quad sx \in \mathcal{W}^J, sx < x \\ (v^2+1)\underline{\mathrm{rk}}M_{[x]} & \text{if} \quad sx \notin \mathcal{W}^J \end{cases}$$

Then,

$$\begin{array}{rcl} h({}^s\theta M\{1\}) & = & \sum_{x\in\mathcal{W}^J} v^{\ell(x)-1}\underline{\mathrm{rk}}({}^s\theta M)_{[x]} H_x^{J,v^{-1}} \\ & = & \sum_{\substack{x\in\mathcal{W}^J, sx\in\mathcal{W}^J \\ sx>x}} v^{\ell(x)+1}\left(\underline{\mathrm{rk}}M_{[x]} + \underline{\mathrm{rk}}M_{[sx]}\right) H_x^{J,v^{-1}} \\ & + & \sum_{\substack{x\in\mathcal{W}^J, sx\in\mathcal{W}^J \\ sx<x}} v^{\ell(x)-1}\left(\underline{\mathrm{rk}}M_{[x]} + \underline{\mathrm{rk}}M_{[sx]}\right) H_x^{J,v^{-1}} \\ & + & \sum_{x\in\mathcal{W}^J, sx\notin\mathcal{W}^J} (v^{\ell(x)+1} + v^{\ell(x)-1})\underline{\mathrm{rk}}M_{[x]} H_x^{J,v^{-1}} \end{array}$$

Finally,

$$\begin{array}{rcl} \underline{H}_s \cdot h(M) & = & \sum_{x\in\mathcal{W}^J} v^{\ell(x)}(\underline{\mathrm{rk}}M_{[x]})\underline{H}_s \cdot H_x^{J,v^{-1}} \\ & = & \sum_{\substack{x\in\mathcal{W}^J sx\in\mathcal{W}^J \\ sx>x}} v^{\ell(x)}(\underline{\mathrm{rk}}M_{[x]})(H_{sx}^{J,v^{-1}} + vH_x^{J,v^{-1}}) \\ & + & \sum_{\substack{x\in\mathcal{W}^J, sx\in\mathcal{W}^J \\ sx<x}} v^{\ell(x)}(\underline{\mathrm{rk}}M_{[x]})(H_{sx}^{J,v^{-1}} + v^{-1}H_x^{J,v^{-1}}) \\ & + & \sum_{x\in\mathcal{W}^J, sx\notin\mathcal{W}^J} v^{\ell(x)}\underline{\mathrm{rk}}M_{[x]}(v+v^{-1})H_x^{J,v^{-1}} \\ & = & \sum_{\substack{x\in\mathcal{W}^J, sx\in\mathcal{W}^J \\ sx>x}} \left[(v^{\ell(x)}v\,\underline{\mathrm{rk}}M_{[x]}) + (v^{\ell(sx)}\underline{\mathrm{rk}}M_{[sx]})\right] H_x^{J,v^{-1}} \\ & + & \sum_{\substack{x\in\mathcal{W}^J, sx\in\mathcal{W}^J \\ sx<x}} \left[(v^{\ell(x)}v^{-1}\,\underline{\mathrm{rk}}M_{[x]}) + (v^{\ell(sx)}\underline{\mathrm{rk}}M_{[sx]})\right] H_x^{J,v^{-1}} \\ & + & \sum_{x\in\mathcal{W}^J, sx\notin\mathcal{W}^J} (v^{\ell(x)+1} + v^{\ell(x)-1})\underline{\mathrm{rk}}M_{[x]} H_x^{J,v^{-1}} \\ & = & h({}^s\theta M\{1\}) \end{array}$$

□

4.3 Localisaton of special $\mathcal{Z}^{\mathrm{par}}$-modules

In this section, we focus our attention on the affine Grassmannian case. In particular, we consider finite intervals of $\mathcal{G}^{\mathrm{par}}$ *far enough* in the fundamental chamber, whose description has been given in §3.2.3. Hereafter, we denote by $\mathcal{W}^{\mathrm{par}}$ the set of minimal representatives for the equivalence classes of $\mathcal{W}^a/\mathcal{W}^f$ and by $\mathcal{Z}^{\mathrm{par}}$ the structure algebra corresponding to this parabolic setting.

Let $\beta \in \dot{\Delta}_+$, we consider the following localisation of the symmetric algebra S_k:

$$S_k^\beta := S_k[(\alpha + n\delta)^{-1} \mid \alpha \in \dot{\Delta}_+ \setminus \{\beta\},\, n \in \mathbb{Z}] \tag{4.5}$$

Fiebig used this localisation in [18], in order to relate the category of regular special modules to a category introduced by Andersen, Jantzen and Soergel in [1].

Let us denote by $\mathcal{W}_\beta$ the subgroup of $\mathcal{W}^a$ generated by the affine reflections $s_{\beta,n}$, for $n \in \mathbb{Z}$, and by $\mathcal{W}^\beta$ the set of orbits for the left action of $\mathcal{W}_\beta$ on $\mathcal{W}^{\mathrm{par}}$. Remark that the group $\mathcal{W}_\beta$ is isomorphic to $\widehat{\mathfrak{sl}_2}$, the Weyl group of $\widetilde{A_1}$. For any subset $\Omega \subseteq \mathcal{W}^{\mathrm{par}}$, let us write moreover $\mathcal{Z}^{\mathrm{par},\beta}(\Omega) := \mathcal{Z}^{\mathrm{par}}(\Omega) \otimes_{S_k} S_{k,\beta}$. We get then an analog of the decomposition we used in §4.1.3.

Lemma 4.3.1 (cf. [18], Lemma 3.1)**.** *Let $\Omega \subset \mathcal{W}^{par}$ be finite, then*

$$\mathcal{Z}^{par,\beta}(\Omega) = \left\{ (z_x) \in \bigoplus_{x\in\Omega} S_k^\beta \;\middle|\; \begin{array}{c} z_x \equiv z_y \;\;(\bmod\ (\beta+n\delta)^\vee) \\ \text{if } \exists\, w \in \mathcal{W}^f, n \in \mathbb{Z} \text{ s.t. } y\, w\, x^{-1} = s_{\beta,n} \end{array} \right\} = \bigoplus_{\Theta\in\mathcal{W}_\beta} \mathcal{Z}^{par,\beta}(\Omega\cap\Theta)$$

Proof. Omitted, since Fiebig's proof of [[18], Lemma 3.1] works exactly the same in this parabolic setting too. □

For $M \in \mathcal{H}^{\mathrm{par}}$, we set $M^\beta := M \oplus_{S_k} S_{k,\beta}$. Because any special module is a module on $\mathcal{Z}(\Omega)$ for some $\Omega \subset \mathcal{W}^{\mathrm{par}}$ finite (see Lemma 4.1.3), the decomposition of the previous Lemma gives us the following decomposition.

$$M^\beta = \bigoplus_{\Theta\in\mathcal{W}^\beta} M^{\beta,\Theta} \tag{4.6}$$

In the following Lemma we show that this localisation procedure preserves special modules. In particular, we prove that, under the localisation, a special module having support on a finite interval far enough in the fundamental chamber splits in a direct sum of special modules for the parabolic structure algebra of the Bruhat graph of $\widetilde{A_1}$.

Lemma 4.3.2. *Let $M \in \mathcal{H}^{par}$ such that $\mathcal{Z}^{par}$ acts on it via $\mathcal{Z}^{par}(\mathcal{I})$, for $\mathcal{I}$ a finite interval far enough in $\mathcal{C}^+$ and $M^\beta = \bigoplus_{\Theta\in\mathcal{W}^\beta} M^{\beta,\Theta}$. Then, for any $\Theta \in \mathcal{W}^\beta$, $M^{\beta,\Theta}$ is isomorphic to a $\mathcal{Z}^{par}(\mathfrak{sl}_2)$-special module.*

Proof. We prove by induction that any $M^{\beta,\Theta}$ is a special module for the structure algebra of $\mathcal{G}^{\mathrm{par}}_{|\Theta}$. If $M = B_e$, there is nothing to prove. Suppose the lemma holds for $M \in \mathcal{H}^{\mathrm{par}}$; we have to show that it is true also for ${}^s\theta(M) = \bigoplus_{\Theta\in\mathcal{W}^\beta} {}^s\theta(M)^{\beta,\Theta}$.

Thus it is enough to show it for an $M^{\beta,\Theta}$. In order to do this, we follow the proof of [[18], Lemma 3.5]. If $\Theta = \Theta s$, then ${}^s\theta(M)^{\beta,\Theta} = M^{\beta,\Theta} \otimes_{{}^s\mathcal{Z}^{\mathrm{par}\beta}(\Theta)} \mathcal{Z}^{\mathrm{par},\beta}(\Theta)$, since, by Lemma 4.1.3, the inclusion ${}^s\mathcal{Z}^{par,\beta}(\Omega) \subset \mathcal{Z}^{par,\beta}(\Omega)$ contains ${}^s\mathcal{Z}^{\mathrm{par},\beta}(\Theta) \subset \mathcal{Z}^{\mathrm{par},\beta}(\Theta)$ as a direct summand. Otherwise, $\Theta \neq \Theta s$ and the inclusion ${}^s\mathcal{Z}^{\mathrm{par},\beta}(\Theta \cup \Theta s) \subset \mathcal{Z}^{\mathrm{par},\beta}(\Theta) \oplus \mathcal{Z}^{\mathrm{par},\beta}(\Theta s)$ is an isomorphism on each direct summand. It follows, ${}^s\theta(M)^{\mathrm{par},\beta} = M^{\beta,\Theta} \oplus M^{\beta,\Theta s}$. In both cases, we get the claim by induction because $\mathcal{Z}^{\mathrm{par},\beta}$ acts on $M^{\beta,\Theta}$ via $\mathcal{Z}^{\mathrm{par},\beta}(\mathcal{I}\cap\Theta)$ and clearly $\mathcal{Z}^{\mathrm{par},\beta}(\mathcal{I}\cap\Theta) = \mathcal{Z}^{\mathrm{par}}(\mathcal{I}\cap\Theta)$.

Now the statement follows since by Lemma 3.2.1, for any finite interval $\mathcal{I}$ far enough in the fundamental chamber and any $\Theta \in \mathcal{W}^\beta$, $\mathcal{I}\cap\Theta$ is isomorphic (as moment graph) to a finite interval of the parabolic Bruhat graph of $\widetilde{A_1}$. □

Chapter 5

Categorification of Kazhdan-Lusztig equalities

In 1979 Kazhdan and Lusztig ([29]) introduced a family of polynomials $\{h_{x,y}\}$ indexed by pairs of elements in a Coxeter group $\mathcal{W}$ with $\mathcal{S}$, the set of simple reflections. Some years later, Deodhar generalised this notion to the parabolic setting, defining two families of polynomials $\{m^J_{x,y}\}$ and $\{n^J_{x,y}\}$, where x and y are now varying in $\mathcal{W}^J$, for $J \subseteq \mathcal{S}$ (see §4.2.1). If $\mathcal{W}$ was a Weyl group, these polynomials were related to the intersection cohomology of the corresponding (partial) Schubert variety (cf. Appendix A of [29] and [30]) and to the representation theory of the complex Lie algebras (cf.[29]), resp. of the semisimple, simply connected, reductive algebraic groups over a field of positive characteristic (cf.[36]), whose Weyl group is $\mathcal{W}$.

The following conjecture motivates this chapter.

Conjecture 5.0.1 ([16], Conjecture 4.4)**.** *Let $y, w \in \mathcal{W}^J$ and let k be such that $(\mathcal{G}^J_{|_{[y,w]}}, k)$ is a GKM-pair. Then $\underline{\mathrm{rk}}\,(\mathscr{B}(w)^J)^y = v^{\ell(y)-\ell(w)} \cdot m^J_{y,w}$.*

This conjecture is proved in characteristic zero and in this case it is equivalent to Kazhdan-Lusztig's conjecture (cf.[14]). In characteristic p it is proved for p bigger than a huge (but explicit) lower bound and it implies Lusztig's conjecture (cf.[18],[16]). Anyway, this conjecture motivates this chapter: we try to interpret combinatorial properties of Kazhdan-Lusztig polynomials in term of Braden-MacPherson sheaves. We have already presented the results of Sections 5.2 and 5.3 in the preprint [35].

5.1 Short-length intervals

We try here to illustrate the philosophy of this chapter by computing the stalks of the canonical sheaves on Bruhat intervals having length ≤ 2.

For any pair of elements $y, w \in \mathcal{W}$ such that $y \leq w$ and $\ell(w) - \ell(y) \leq 2$, it is know that $h_{y,w} = v^{\ell(w)-\ell(y)}$. If conjecture 5.0.1 is true, then $\underline{\mathrm{rk}}\,\mathscr{B}(w)^y = 1$, that is $\mathscr{B}(w)^y \cong S_k$ if $(\mathcal{G}, k)$ is a GKM-pair. Clearly, there is nothing to prove if $y = w$. If $\ell(y) = \ell(w) - 1$, then $y = tw$ for some $t \in \mathcal{T}$ and the associated moment graph is a subgeneric graph with the edge labeled by $\alpha_t{}^\vee$. In this case, it is clear that $\mathscr{B}(w)^{\delta y} = S_k/\alpha_t S_k$, whose projective cover is obviously S_k.

Suppose now $\ell(w) - \ell(y) = 2$. Then the Bruhat graph restricted to the interval $\mathcal{I} = [y, w]$ has to be of the following shape (cf. [[5], Lemma 2.7.3]).

$$\begin{tikzcd}
& w & \\
s_\alpha y \arrow[ur, "\gamma^\vee"] & & s_\beta y \arrow[ul, "\delta^\vee"'] \\
& y \arrow[ul, "\alpha^\vee"] \arrow[ur, "\beta^\vee"'] &
\end{tikzcd}$$

For some $\alpha, \beta, \gamma, \delta \in \Delta^{\mathrm{re}}_+$.

By Proposition 2.1.2, showing that $\mathscr{B}(w)^x \cong S_k$ for all $x \in [y, w]$ is equivalent to showing that the corresponding structure sheaf is flabby. We know already that $\mathscr{B}(w)^x \cong S_k$ for $x \in (y, w]$, so we have only to prove $\mathscr{B}(w)^y \cong S_k$. In particular, the claim will follow once we prove that all sections $z = (z_w, z_{s_\alpha y}, z_{s_\alpha y}) \in \Gamma(\mathcal{I} \setminus \{y\}, \mathcal{A})$ are extensible. By definition, there exist $p, q_1, q_2 \in S_k$ such that

$$z_w = p, \quad z_{s_\alpha y} = p + \gamma^\vee \cdot q_1, \quad z_{s_\beta y} = p + \delta^\vee \cdot q_2$$

Clearly, there exists an element $z_y \in S_k$ extending z if and only if there exist $q_3, q_4 \in S_k$ such that

$$z_{s_\alpha y} + \alpha^\vee \cdot q_3 = z_{s_\beta y} + \beta^\vee \cdot q_4$$

Now, by hypothesis, $s_\gamma s_\alpha = s_\delta s_\beta$, that is $s_\beta = s_\delta s_\gamma s_\alpha$, so, for all $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$,

$$\lambda - \langle \lambda, \alpha^\vee \rangle \alpha = \lambda - \langle \lambda, \beta^\vee \rangle \beta - \langle s_\beta(\lambda), \delta^\vee \rangle \delta - \langle s_\delta s_\beta(\lambda), \gamma^\vee \rangle \gamma$$

Because of the GKM-property, $\beta \neq \pm\alpha, \pm\delta$ and so it is always possible to find a $\mu \in \mathfrak{h}^*_{\mathbb{R}}$ such that

$$\langle \mu, \beta^\vee \rangle = 0, \quad \langle \mu, \alpha^\vee \rangle \neq 0, \quad \langle s_\beta(\mu), \delta^\vee \rangle = \langle \mu, s_\beta(\delta^\vee) \rangle \neq 0$$

Then, we might write $\alpha = a_1 \delta + a_2 \gamma$ with $a_1, a_2 \in \mathbb{R}$ and $a_1 \neq 0$. Analogously, we get $\beta = b_1 \delta + b_2 \gamma$ with $b_1, b_2 \in \mathbb{R}$ and $b_2 \neq 0$. Thus, if $a_2 = 0$, it is easy to check that

$$q_3 = a_1^{-1}(q_2 - b_1 b_2^{-1} q_1) \qquad q_4 = b_1^{-1} q_1$$

satisfy the requirements. While, for $a_2 \neq 0$, we set

$$q_3 = a_2^{-1} \left((1 - b_2 a_1 a_2^{-1}(b_1 + b_2)^{-1}) q_1 - b_2 (b_1 + b_2)^{-1} q_2 \right) \qquad q_4 = a_2^{-1}(b_1 + b_2)^{-1}(a_1 q_1 + a_2 q_2)$$

Thus we get the following lemma.

Lemma 5.1.1. *Let $y, w \in \mathcal{W}$ be such that $y \leq w$ and $\ell(w) - \ell(y) \leq 2$. If $(\mathcal{G}_{|_{[y,w]}}, k)$ is a GKM-pair, then $\mathscr{B}(w)^y \cong S_k$.*

5.2 Technique of the pullback

Let $\mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{t}$ be a symmetrisable KacMoody algebra, a Borel subalgebra and a Cartan subalgebra. Let Π, resp. $\Pi^\vee$, be the corresponding set of simple roots, resp. of simple coroots. From now on, we denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, l, \leq)$ the regular Bruhat graph we defined in §3.1.1.

In this section, we apply Lemma 2.2.2 in order to lift some equalities concerning KL-polynomials to the moment graph setting. In particular, we will define isomorphisms of k-moment graphs to get isomorphisms between stalks of the corresponding Braden-MacPherson sheaves.

5.2.1 Inverses

Kazhdan and Lusztig gave an inductive formula to calculate the KL-polynomials ((2.2.c) of [29]). From such a formula it follows easily (cf. Exercise 12, Chap.5 of [5]) that, for any pair $y, w \in \mathcal{W}$, one has

$$h_{y,w} = h_{y^{-1},w^{-1}}. \tag{5.1}$$

We translate this equality to an isomorphism of stalks of indecomposable canonical sheaves.

Lemma 5.2.1. *Let $\mathcal{W}$ be a Weyl group. The anti-involution on $\mathcal{W}$ defined by the mapping $x \mapsto x^{-1}$ induces an automorphism of the k-moment Bruhat graph $\mathcal{G}$ for any k.*

Proof. $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ defined by the mapping $x \mapsto x^{-1}$ is obviously a bijection. Moreover, for each pair of elements $x, y \in \mathcal{W}$, $x \leq y$ if and only if $x^{-1} \leq y^{-1}$. So $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is a bijection of posets.

Observe that there exists a reflection $t \in \mathcal{T}$ such that $y = tx$ if and only if $y^{-1} = rx^{-1}$, where $r = x^{-1}tx \in \mathcal{T}$. So $x \text{ --- } y \in \mathcal{E}$ if and only if $x^{-1} \text{ --- } y^{-1} \in \mathcal{E}$.

Thus, for every $x \in \mathcal{W}$ and any $\lambda \in Q^\vee$, we set $f_{l,x}(\lambda) := x^{-1}(\lambda)$. Let $E : x \text{ --- } y = tx$ and recall that, for any $w \in \mathcal{W}$ and $\alpha \in \Delta^{\mathrm{re}}$, $w(\alpha)^\vee = w(\alpha^\vee)$ (cf. [26], §5.1). Then we get the following.

(a) $f_{l,x}(l(x \text{ --- } tx)) = x^{-1}(\alpha_t)^\vee = x^{-1}(\alpha_t{}^\vee) = \pm l(x^{-1} \text{ --- } y^{-1})$, where $\pm x^{-1}(\alpha_t) \in \Delta^{\mathrm{re}}_+$, because $x^{-1}(\alpha_t) = \pm\alpha_{x^{-1}tx}$ (cf. [26], §5.1).

(b)

$$\begin{aligned} f_{l,y}(\lambda) &= y^{-1}(\lambda) \\ &= x^{-1}(t\lambda) \\ &= x^{-1}(\lambda) - \langle \alpha_t, \lambda\rangle x^{-1}(\alpha_t)^\vee \\ &\equiv x^{-1}(\lambda) \pmod{x^{-1}(\alpha_t)^\vee} \\ &= f_{l,x}(\lambda) \pmod{x^{-1}(\alpha_t)^\vee} \end{aligned}$$

This proves that we have an automorphism of the k-moment graph $\mathcal{G}$ for any k. ◻

From the lemma above we get the following corollary.

Corollary 5.2.1. *Let $w \in \mathcal{W}$. Denote by $\mathcal{G}$ the corresponding Bruhat graph and let f be as in Lemma 5.2.1. Then $\mathscr{B}(w) \cong f^*\mathscr{B}(w^{-1})$ as k-sheaves on $\mathcal{G}$ for any k.*

Proof. First observe that $y \not\leq w$ if and only if $y^{-1} \not\leq w^{-1}$. So if $y \not\leq w$, $\mathscr{B}(w)^y = 0 = \mathscr{B}(w^{-1})^{y^{-1}}$.

By Lemma 5.2.1, $f_{\mathcal{V}} : x \mapsto x^{-1}$ induces a k-isomorphism between the two complete subgraphs $\mathcal{G}_w$ and $\mathcal{G}_{w^{-1}}$, so we may apply Lemma 2.2.2; the statement follows. ◻

5.2.2 Multiplying by a simple reflection. Part I

Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $y \leq w$, $ws < w$ and $y \not\leq ws$. In these hypotheses Kazhdan and Lusztig observed (proof of Theor. 4.2 of [29]) that

$$h_{y,w} = h_{ys,ws}. \tag{5.2}$$

In order to interpret (5.2) in our moment graph setting we will use the lifting Lemma, to define an isomorphism of k-moment graphs.

Lemma 5.2.2. *Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $y \leq w$, $ws < w$ and $y \not\leq ws$, then for any k there is an isomorphism of k-moment graphs $\mathcal{G}_{|_{[y,w]}} \xrightarrow{\sim} \mathcal{G}_{|_{[ys,ws]}}$.*

Proof. We show that $f_{\mathcal{V}} : [y,w] \to [ys, ws]$, $x \mapsto xs$ is a bijection of posets inducing the identity map on the labels.

We verify that if $x \in [y,w]$ then $xs \in [ys, ws]$. We see that $xs < x$; indeed, if it were not the case, by Lemma 3.1.2 (ii) $x \leq ws$, but this implies that $y \leq ws$. In particular, this holds for y, that is $ys < y$. Now, by Lemma 3.1.2 (i);

$$xs < x \ , \ ws < w \ \Rightarrow \ xs \leq ws$$

$$ys < y \ , \ xs < x \ \Rightarrow \ ys \leq xs.$$

We now show that if $z \in [ys, ws]$ then $zs \in [y,w]$. Observe that $zs > z$; indeed, $ys < z$, $y = (ys)s > ys$ and if $zs < z$, then by Lemma 3.1.2 (ii), with $u = ys$ and $v = z$, we would get $y = (ys)s \leq z \leq ws$.

Moreover, $z \leq ws < w$ and, by Lemma 3.1.2 (ii),

$$zs > z \ , \ ws < w \ \Rightarrow \ zs \leq w.$$

$$y = (ys)s > ys \ , \ z = (zs)s < zs \ \Rightarrow \ y \leq zs.$$

This completes the proof that $f_{\mathcal{V}}$ maps $[y,w]$ to $[ys, ws]$.

Let $x, z \in [y,w]$, then $x \leq z$ if and only if $xs \leq zs$. Indeed, we have already proved that $xs < x$ and $zs < z$ so, by Lemma 3.1.2 (i), with $u = x$ and $v = z$, we have $xs \leq zs$. On the other hand, $x = (xs)s > xs$ and it follows from Lemma 3.1.2 (ii) with $u = xs$ and $v = z$ that $x = (xs)s \leq z$.

Finally from what we proved above, for each $t \in \mathcal{T}$ we have that $x, tx \in [y,w]$ if and only if $xs, txs \in [ys, ws]$.This means that we have a bijection between sets of edges such that $f_{\mathcal{E}}(x \xrightarrow{\alpha} tx) = xs \xrightarrow{\alpha} txs$.

Therefore $f = (f_{\mathcal{V}}, \{Id_{Y_k}\}_{x \in \mathcal{V}})$ is an isomorphism of k-moment graphs for any k.

□

So we have:

Corollary 5.2.2. *Consider $y, w \in \mathcal{W}$ such that $ws < w$, $y \not\leq ws$ for some $s \in \mathcal{S}$. Let f be as in Lemma 5.2.2, then $\mathscr{B}(w) \cong f^*\mathscr{B}(ws)$ as k-sheaves on $\mathcal{G}_{|_{[y,w]}}$ for any k.*

Proof. The statement follows by combining Lemma 5.2.2 and Lemma 2.2.2 . □

We recollect the results of this section:

Theorem 5.2.1. *Let $y, w \in \mathcal{W}$, then*

(i) $\mathscr{B}(w)^y \cong \mathscr{B}(w^{-1})^{y^{-1}}$.

Let $s \in \mathcal{S}$ be such that $ws < w$ and $y \not\leq ws$, then

(ii) $\mathscr{B}(w)^y \cong \mathscr{B}(ws)^{ys}$

All isomorphisms are isomorphisms of (finitely generated, $\mathbb{Z}$-graded) S_k-modules, for any k.

Proof.

(i) This follows from Corollary 5.2.1, since two k-sheaves are isomorphic only if their stalks are pairwise isomorphic.

(ii) As before, the isomorphism descends from the isomorphism of k-sheaves we obtained in Corollary 5.2.2.

□

5.3 Invariants

Clearly not all equalities concerning Kazhdan-Lusztig polynomials come from k-isomorphisms of the underlying Bruhat graphs. In this section we develop another technique and, as in the previous section, we apply it in order to categorify two well-known properties of these polynomials.

5.3.1 Multiplying by a simple reflection. Part II

Another property that Kazhdan and Lusztig in [29] (2.3.g) proved is that if $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ are such that $y \leq w$ and $ws < w$, then

$$h_{y,w} = v^c h_{ys,w}, \tag{5.3}$$

where $c = 1$ if $sy > y$ and $c = -1$ otherwise.

It is clear that in this case there is no hope of finding any k-isomorphism of moment graphs, since the two Bruhat intervals $[y, w]$ and $[ys, w]$ obviously have different cardinality.

The goal of this section is to prove the following theorem.

Theorem 5.3.1. *For any pair $y, w \in \mathcal{W}$ and for any $s \in \mathcal{S}$ such that $ws < w$ and $ys, y \leq w$, there exist*

- *an isomorphism of S_k-modules $\varphi^y : \mathscr{B}(w)^y \to \mathscr{B}(w)^{ys}$*
- *a family of isomorphisms of S_k-modules $\varphi^E : \mathscr{B}(w)^E \to \mathscr{B}(w)^{Es}$, where $E : y \text{ --- } x \in \mathcal{E}$ and $Es : ys \text{ --- } xs \in \mathcal{E}$*

such that the following diagram commutes

$$\begin{tikzcd} \mathscr{B}(w)^y \arrow[r, "\varphi^y"] \arrow[d, "\rho_{y,E}"'] & \mathscr{B}(w)^{ys} \arrow[d, "\rho_{ys,Es}"] \\ \mathscr{B}(w)^E \arrow[r, "\varphi^E"] & \mathscr{B}(w)^{Es} \end{tikzcd} \tag{5.4}$$

and such that $\varphi^{ys} = (\varphi^y)^{-1}$.

5.3.2 Two preliminary lemmata

In order to prove our claim, we need two combinatorial lemmata.

Recall that

$$\mathcal{T} = \{s_\alpha \mid \alpha \in R^+\} = \{wsw^{-1} \mid w \in \mathcal{W},\, s \in \mathcal{S}\}$$

and, for all $x, y \in \mathcal{W}$, denote

$$G_L(x, y) := \big\{t \in \mathcal{T} \mid tx \in (x, y]\big\}$$

Lemma 5.3.1. *Let $w, y \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ws < w$ and $ys < y$, then*

$$G_L(ys, w) = G_L(y, w) \cup \big\{ysy^{-1}\big\}.$$

Proof. We show that for all $t \in G_L(y,w)$ we have $ys < tys \leq w$ as well, i.e. $t \in G_L(ys,w)$. Indeed, if $tys > ty$, then $ys < y < ty < tys$ and, by Lemma 3.1.2 (ii) with $u = ty$ and $v = w$, $tys \leq w$. Otherwise, $tys < ty \leq w$, $y < ty$, $ys < y$ and, by Lemma 3.1.2 (i) with $u = y$ and $v = ty$, we obtain $ys < tys$.

Clearly, $ysy^{-1} \in G_L(ys,w)$ and this completes the proof that the set on the right hand side is a subset of the one on the left.

Now we verify that if $t \in \mathcal{T}$, $tys \in [ys,w]$ and $ty \notin [y,w]$, then $t = ysy^{-1}$. Indeed, by Lemma 3.1.2 with $u = tys$ and $v = w$, $tys \leq w$ and, if $ty \notin [y,w]$, then $ty < y$. Moreover, $ys < y$ and so, by Lemma 3.1.2 (ii) with $u = ty$ and $v = y$, $tys \leq y$. So $ys < tys \leq y$ and we know that $[ys,y] = \{ys,y\}$. Thus $tys = y$, that is, $t = ysy^{-1}$. □

Lemma 5.3.2. *Let $w,y \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ys < y$ and $ws < w$, then the set $[ys,w] \setminus \{ys,y\}$ is stabilised by the mapping $x \mapsto xs$.*

Proof. Notice that $ys < y \leq w$, so it makes sense to write $[ys,w]$. Let $\mathcal{I} := [ys,w] \setminus \{ys,y\}$ and let $x \in \mathcal{I}$. If $xs > x$, then obviously $ys < xs$ and, by Lemma 3.1.2 (ii) with $u = x$ and $v = w$, $xs \leq w$. On the other hand, if $xs < x$, then $xs < w$ and, by applying Lemma 3.1.2 (ii) with $u = ys$ and $v = x$, $ys \leq xs$. Then, in both cases $xs \in [ys,w]$ and, since $xs \neq y$ and $xs \neq ys$, we get $x \in \mathcal{I}$.

Finally, if $x \in \mathcal{I}$, then $xs \neq y$. Indeed $xs = y$ if and only if $x = ys \notin \mathcal{I}$. □

5.3.3 Proof of the main theorem

We will prove Theorem 5.3.1 by induction on $n = \ell(w) - \ell(y)$.

If $n = 0$, then $y = w$ and there is nothing to prove. If $n > 0$ and $ys > y$, then $\ell(w) - \ell(ys) = n-1$ and by induction we get the desired isomorphisms.

Now, we may suppose $n > 0$ and $ys < y$. Let $\mathcal{I} = [ys,w] \setminus \{y,ys\}$. From the inductive hypothesis, for any $x \in \mathcal{I}$ we get

- an isomorphism of S_k-modules $\varphi^x : \mathscr{B}(w)^x \to \mathscr{B}(w)^{xs}$
- a family of isomorphisms of S_k-modules $\varphi^F : \mathscr{B}(w)^F \to \mathscr{B}(w)^{Fs}$, where $F : x \text{ —— } z \in \mathcal{E}^{\delta y}$ and $Fs : xs \to zs \in \mathcal{E}^{\delta ys}$

such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{B}(w)^x & \xrightarrow{\varphi^x} & \mathscr{B}(w)^{xs} \\ \downarrow{\scriptstyle \rho_{x,F}} & & \downarrow{\scriptstyle \rho_{xs,Fs}} \\ \mathscr{B}(w)^F & \xrightarrow{\varphi^F} & \mathscr{B}(w)^{Fs} \end{array} \tag{5.5}$$

and such that $\varphi^{xs} = (\varphi^x)^{-1}$.

Observe that our claim will follow, once we prove that there is an isomorphism of S_k-modules $\varphi^y : \mathscr{B}(w)^y \to \mathscr{B}(w)^{ys}$ compatible with the restriction maps. Indeed, for $E : y \to x \in \mathcal{E}_{\delta y}$ there exists exactly one $Es : ys \to xs \in \mathcal{E}_{\delta ys}$, and φ^E would already have been given. If $E : ys \to y$, then we could set $\varphi^E = Id$. Finally, for $x \neq ys$, there exists an edge $E : x \to y \in \mathcal{E}$ if and only if there is $Es : xs \to ys \in \mathcal{E}$ (cf. Lemma 5.3.1) and in this case $\mathscr{B}(w)^E \cong \mathscr{B}(w)^y / l(E) \cdot \mathscr{B}(w)^y \cong \mathscr{B}(w)^{ys} / l(Es) \cdot \mathscr{B}(w)^{ys}$, since $E = Es$.

We will get φ^y by defining a surjective map from $\mathscr{B}(w)^y$ to $\mathscr{B}(w)^{\delta ys}$. Since $\mathscr{B}(w)^{ys}$ is the projective cover of the S_k-module $\mathscr{B}(w)^{\delta ys}$, and, since $\mathrm{rk}_{S_k}\mathscr{B}(w)^y \leq \mathrm{rk}_{S_k}\mathscr{B}(w)^{ys}$ (cf. Lemma 3.12. of [15]), Theorem 5.3.1 will follow from the unicity of the projective cover.

Invariants

By Lemma 5.3.2, $\mathcal{I}$ is invariant with respect to the right multiplication by s and we may define an automorphism σ_s of the set of local sections of the Braden-MacPherson sheaf on $\mathcal{I}$ as follows. Let $m = (m_x) \in \Gamma(\mathcal{I}, \mathscr{B}(w))$, then we set $\sigma_s(m) = (m'_x)$, where $m'_x := \varphi^{xs}(m_{xs})$. Since the φ^x's are, by definition, compatible with the restriction maps (see Diagram (5.5)), $\sigma_s(m) \in \Gamma(\mathcal{I}, \mathscr{B}(w))$. Moreover, for any $x \in \mathcal{I}$, $\varphi^{xs} = (\varphi^x)^{-1}$ and so σ_s is an involution.

Let us denote by Γ^s the submodule of σ_s-invariant elements of $\Gamma(\mathcal{I}, \mathscr{B}(w))$, and by Γ^{-s} the elements $m \in \Gamma(\mathcal{I}, \mathscr{B}(w))$ such that $\sigma_s(m) = -m$.

Let us consider $c_s := (c_{s,x}) \in \bigoplus_{x\in\mathcal{W}} S_k$, where $c_{s,x} := x(\alpha_s{}^\vee)$; then $c_s \in \mathcal{Z}$ and so it acts on $\Gamma(\mathcal{I}, \mathscr{B}(w))$ via componentwise multiplication.

Lemma 5.3.3. *Let $(\mathcal{G}_{|\mathcal{I}}, k)$ be a GKM-pair, then we have $\Gamma(\mathcal{I}, \mathscr{B}(w)) = \Gamma^s \oplus c_s \cdot \Gamma^s$.*

Proof. (We follow [18], Lemma 2.4).

By definition, σ_s is an involution and 2 is an invertible element in k, then we get $\Gamma(\mathcal{I}, \mathscr{B}(w)) = \Gamma^s \oplus \Gamma^{-s}$.

Let $m \in \Gamma^s$, then $\sigma_s(c_s \cdot m) = -(c_s \cdot m)$, i.e. $c_s \cdot \Gamma^s \subseteq \Gamma^{-s}$. Indeed, $s(\alpha_s{}^\vee) = -\alpha_s{}^\vee$ and so for any $x \in \mathcal{I}$ we have

$$(c_{s,x} \cdot m_x)' = xs(\alpha_s{}^\vee) \cdot m_x = x(-\alpha_s{}^\vee) \cdot m_x = -c_{s,x} \cdot m_x.$$

We have to prove the other inclusion, that is, every element $m \in \Gamma^{-s}$ can be divided by $(x(\alpha_s{}^\vee))_{x\in\mathcal{I}}$ in $\Gamma(\mathcal{I}, \mathscr{B}(w))$.

If $m = (m_x) \in \Gamma^{-s}$ then $m_x = -\varphi^{xs}(m_{xs})$ and so $\rho_{xs,xs\to x}(m_{xs}) = -\rho_{x,xs\to x}(m_x)$, since the following diagram commutes:

$$\begin{tikzcd}
\mathscr{B}(w)^{xs} \arrow[r, "\varphi^{xs}"] \arrow[d, "\rho_{xs,xs\to x}"'] & \mathscr{B}(w)^{x} \arrow[d, "\rho_{x,xs\to x}"] \\
\mathscr{B}(w)^{xs\to x} \arrow[r, "\varphi^{xs\to x}"'] & \mathscr{B}(w)^{xs\to x}
\end{tikzcd}$$

But m is a section so $\rho_{xs,xs\to x}(m_{xs}) = \rho_{x,xs\to x}(m_x)$. It follows that $2\rho_{x,xs\to x}(m_x) = 0$; moreover, by definition of the canonical sheaf, $\ker \rho_{x,xs\to x} = \alpha^\vee_{xsx^{-1}}\mathscr{B}(w)^x$, that is, $\alpha^\vee_{xsx^{-1}}$ divides m_x in $\mathscr{B}(w)^x$.

Notice that $\alpha_{xsx^{-1}}{}^\vee = \pm x(\alpha_s{}^\vee) = \pm c_{s,x}$, i.e. $c_s^{-1} \cdot m \in \bigoplus_{x\in\mathcal{I}} \mathscr{B}(w)^x$. We have to verify that $\rho_{x,x—tx}(c_{s,x}^{-1} m_x) = \rho_{tx,x—tx}(c_{s,tx}^{-1} m_{tx})$ for all $t \in \mathcal{T}$:

$$(c_{s,tx}c_{s,x})(\rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}) - \rho_{x,x—tx}(c_{s,x}^{-1}m_x)) \tag{5.6}$$
$$= c_{s,x}(\rho_{tx,x—tx}(m_{tx})) - c_{s,tx}(\rho_{x,x—tx}(m_x)) \tag{5.7}$$
$$= (c_{s,x} - c_{s,tx})\rho_{tx,x—tx}(m_{tx}) + c_{s,tx}(\rho_{tx,x—tx}(m_{tx}) - \rho_{x,x—tx}(m_x)). \tag{5.8}$$

The term on line (5.8) is divisible by $\alpha_t{}^\vee$; indeed, $c_{s,x} - c_{s,tx} = x(\alpha_s{}^\vee) - x(\alpha_s{}^\vee) + \langle\alpha_t, x(\alpha_s{}^\vee)\rangle\alpha_t{}^\vee \equiv 0 \pmod{\alpha_t{}^\vee}$ and $\rho_{tx,x—tx}(m_{tx}) - \rho_{x,x—tx}(m_x) = 0$.

Using the GKM-property $c_{s,tx}c_{s,x} = tx(\alpha_s{}^\vee)\cdot x(\alpha_s{}^\vee)$ is a multiple of $\alpha_t{}^\vee$ if and only if $xsx^{-1} = t$, that is $xs = tx$. So, $m_x = -\varphi^{xs}(m_{tx})$, $c_{s,tx} = -c_{s,x}$ and, considering that diagram (5.3.3) commutes, we obtain

$$\begin{aligned} \rho_{x,x—tx}(c_{s,t}^{-1}m_x) &= -c_{s,tx}^{-1}\,\rho_{x,x—tx}(m_x) \\ &= -c_{s,tx}^{-1}\,(-\rho_{tx,x—tx}(m_{tx})) \\ &= \rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}) \end{aligned}$$

Otherwise, $xsx^{-1} \neq t$ and $\alpha_t{}^\vee$ divides $\rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}) - \rho_{x,x—tx}(c_{s,x}^{-1}m_x)$ and so

$$\rho_{x,x—tx}(c_{s,x}^{-1}m_x) = \rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}).$$

□

Building $\mathscr{B}(w)^{\delta ys}$

Let us denote

$$\begin{tikzcd} \Gamma(\mathcal{I},\mathscr{B}(w)) \arrow[r, hook] \arrow[rrr, bend right, "\pi_1"'] & \bigoplus_{x\in\mathcal{I}}\mathscr{B}(w)^x \arrow[r] & \bigoplus_{x\in\mathcal{V}_{\delta y}}\mathscr{B}(w)^x \arrow[r, "\oplus\rho_{x,E}"] & \bigoplus_{E\in\mathcal{E}_{\delta y}}\mathscr{B}(w)^E \end{tikzcd}$$

Recall that $\mathscr{B}(w)^{\delta y} = u_y(\Gamma(\{>y\},\mathscr{B}(w)))$, where u_y was defined as the composition of the following maps

$$\begin{tikzcd} \Gamma(\{>y\},\mathscr{B}(w)) \arrow[r, hook] \arrow[rrr, bend right, "u_y"'] & \bigoplus_{x>y}\mathscr{B}(w)^x \arrow[r] & \bigoplus_{x\in\mathcal{V}_{\delta y}}\mathscr{B}(w)^x \arrow[r, "\oplus\rho_{x,E}"] & \bigoplus_{E\in\mathcal{E}_{\delta y}}\mathscr{B}(w)^E \end{tikzcd}$$

Remark 5.3.1. *Since $\mathscr{B}(w)$ is flabby and $\mathcal{I}$ and $\{>y\}$ are both open sets, we get*

$$\pi_1(\Gamma(\mathcal{I},\mathscr{B}(w))) = u_y(\Gamma(\{>y\},\mathscr{B}(w))) = \mathscr{B}(w)^{\delta y} \tag{5.9}$$

Now, let us denote

$$\begin{tikzcd} \Gamma(\mathcal{I},\mathscr{B}(w)) \arrow[r, hook] \arrow[rrr, bend right, "\pi_2"'] & \bigoplus_{x\in\mathcal{I}}\mathscr{B}(w)^x \arrow[r] & \bigoplus_{x\in\mathcal{V}^{\delta y}}\mathscr{B}(w)^{xs} \arrow[r, "\oplus\rho_{xs,Es}"] & \bigoplus_{E\in\mathcal{E}_{\delta y}}\mathscr{B}(w)^{Es} \end{tikzcd}$$

and define $\widetilde{\mathscr{B}(w)^{\delta ys}} := \pi_2(\Gamma(\mathcal{I},\mathscr{B}(w)))$.

Lemma 5.3.4.

(i) $\mathscr{B}(w)^{\delta y} = \pi_1(\Gamma(\mathcal{I},\mathscr{B}(w))) = \pi_1(\Gamma^s)$

(ii) $\widetilde{\mathscr{B}(w)^{\delta ys}} = \pi_2(\Gamma(\mathcal{I},\mathscr{B}(w))) = \pi_2(\Gamma^s)$

Proof.

(i) Let $m \in \Gamma(\mathcal{I}, \mathscr{B}(w))$. Then, by Lemma 5.3.3, $m = m' + c_s \cdot m''$, with $m', m'' \in \Gamma^s$ and, if $m' = (m'_x)$, $m'' = (m''_x)$,

$$\pi_1(m) = \big(\rho_{x,E}(m'_x)\big)_{x \in \mathcal{V}: y \to x \in \mathcal{E}} + \big(\rho_{x,E}(x({\alpha_s}^\vee) \cdot m''_x)\big)_{x \in \mathcal{V}: y \to x \in \mathcal{E}}$$

If $E : y \to x \in \mathcal{E}_{\delta y}$, then there exists a reflection $t \in \mathcal{T}$ such that $x = ty$ and we have

$$x({\alpha_s}^\vee) = ty({\alpha_s}^\vee) = y({\alpha_s}^\vee) + \langle \alpha_t, y({\alpha_s}^\vee)\rangle {\alpha_t}^\vee \tag{5.10}$$

But, by definition, $\rho_{x,E}$ is a surjective map whose kernel is $l(E) \cdot \mathscr{B}(w)^x = {\alpha_t}^\vee \cdot \mathscr{B}(w)^x$ and

$$\rho_{x,E}(x({\alpha_s}^\vee) \cdot m''_x) = \rho_{x,E}(y({\alpha_s}^\vee) \cdot m''_x) + \langle \alpha_t, y({\alpha_s}^\vee)\rangle \rho_{x,\mathcal{E}}({\alpha_t}^\vee \cdot m''_x) = \rho_{x,E}(y({\alpha_s}^\vee) \cdot m''_x)$$

We conclude that $\pi_1(m) = \pi_1(m' + \overline{y({\alpha_s}^\vee)} \cdot m'')$, where $\overline{y({\alpha_s}^\vee)}$ is the element of the structure algebra, whose components are all equal to $y({\alpha_s}^\vee)$. Clearly, $m' + \overline{y({\alpha_s}^\vee)} \cdot m'' \in \Gamma^s$ and we get the claim.

(ii) As in *(i)*. $\square$

Lemma 5.3.5. *There is an isomorphism of S_k-modules $\tau : \mathscr{B}(w)^{\delta y} \to \widetilde{\mathscr{B}(w)}^{\delta ys}$ given by $(m_E)_{E \in \mathcal{E}_{\delta y}} \mapsto (\varphi^E(m_E))_{E \in \mathcal{E}_{\delta y}}$, that is for all $m \in \Gamma^s$, $\tau \circ \pi_1(m) = \pi_2(m)$.*

Proof. $(m_E)_{E \in \mathcal{E}_{\delta y}} \in \mathscr{B}(w)^{\delta y}$ if and only if there exists an element $m \in \Gamma(\{> y\}, \mathscr{B}(w))$ such that $u_y(m) = (m_E)_{E \in \mathcal{E}_{\delta y}}$. We have already noticed that this is the case if and only if there is an element $m' \in \Gamma(\mathcal{I}, \mathscr{B}(w))$ such that $\pi_1(m') = (m_E)_{E \in \mathcal{E}_{\delta y}}$. From the previous lemma, we know that this is equivalent to the existence of an $\widetilde{m} \in \Gamma^s$ such that $\pi_1(\widetilde{m}) = (m_E)_{E \in \mathcal{E}_{\delta y}}$. But, since the squares in the following diagram are all commutative,

$$\begin{tikzcd}
\Gamma^s \arrow[r] \arrow[d, "Id"'] \arrow[rr, bend left, "\pi_1|_{\Gamma^s}"] & \bigoplus_{x \in \mathcal{V}_{\delta y}} \mathscr{B}(w)^x \arrow[r, "\oplus \rho_{x,E}"] \arrow[d, "\oplus \varphi^x"] & \bigoplus_{E \in \mathcal{E}_{\delta y}} \mathscr{B}(w)E \arrow[d, "\oplus \varphi^x"] \\
\Gamma^s \arrow[r] \arrow[rr, bend right, "\pi_2|_{\Gamma^s}"'] & \bigoplus_{x \in \mathcal{V}_{\delta y}} \mathscr{B}(w)^{xs} \arrow[r, "\oplus \rho_{xs,Es}"'] & \bigoplus_{E \in \mathcal{E}_{\delta y}} \mathscr{B}(w)^{Es}
\end{tikzcd}$$

we get $(\varphi^E(m_E))_{E \in \mathcal{E}_{\delta y}} = \varphi^E \circ \pi_1(\widetilde{m}) = \pi_2(\widetilde{m}) \in \widetilde{\mathscr{B}(w)}^{\delta ys}$.

Analogously, $(m_{Es})_{E \in \mathcal{E}_{\delta y}} \in \widetilde{\mathscr{B}(w)}^{\delta ys}$ if and only if $((\varphi^E)^{-1}(m_{Es}))_{E \in \mathcal{E}_{\delta y}} \in \mathscr{B}(w)^{\delta y}$.

$\square$

Let us denote by $\rho : \mathscr{B}(w)^y \to \mathscr{B}(w)^y / {\alpha_s}^\vee \cdot \mathscr{B}(w)^y$ the canonical quotient map.

Lemma 5.3.6. *We have*

$$\mathscr{B}(w)^{\delta ys} = \Big\{ \big(\tau \circ \pi_1(m_y), \rho(m_y)\big) \in \widetilde{\mathscr{B}(w)}^{\delta ys} \oplus (\mathscr{B}(w)^y / {\alpha_s}^\vee \cdot \mathscr{B}(w)^y) \Big\} \tag{5.11}$$

Proof.

$$\begin{aligned}\mathscr{B}(w)^{\delta ys} &= u_{ys}\big(\Gamma(\{> ys\}, \mathscr{B}(w))\big)\\ &= u_{ys}\left(\left\{(m, m_y) \in \Gamma(\mathcal{I}, \mathscr{B}(w)) \oplus \mathscr{B}(w)^y \mid u_y(m_{|_{\{>y\}}}) = d_y(m_y)\right\}\right)\end{aligned}$$

by Remark 5.3.1

$$\begin{aligned}&= u_{ys}\big(\{(m, m_y) \in \Gamma(\mathcal{I}, \mathscr{B}(w)) \oplus \mathscr{B}(w)^y \mid \pi_1(m) = d_y(m_y)\}\big)\\ &= \big\{\big(\pi_2(m), \rho(m_y)\big) \mid m \in \Gamma(\mathcal{I}, \mathscr{B}(w)),\ m_y \in \mathscr{B}^y_w,\ \pi_1(m) = d_y(m_y)\big\}\end{aligned}$$

by Lemma 5.3.4

$$= \big\{\big(\pi_2(m), \rho(m_y)\big) \mid m \in \Gamma^s,\ m_y \in \mathscr{B}(w)^y,\ \pi_1(m) = d_y(m_y)\big\}$$

by Lemma 5.3.5

$$\begin{aligned}&= \big\{\big(\tau \circ \pi_1(m), \rho(m_y)\big) \mid m \in \Gamma^s,\ m_y \in \mathscr{B}(w)^y,\ \pi_1(m) = d_y(m_y)\big\}\\ &= \big\{\big(\tau \circ d_y(m_y), \rho(m_y)\big) \mid m_y \in \mathscr{B}(w)^y\big\}\end{aligned}$$

□

From the lemma above, it follows immediately, that there is a surjective map of S_k-modules $\mathscr{B}(w)^y \to \mathscr{B}(w)^{\delta ys}$ given by $m_y \mapsto (\tau \circ d_y(m_y), \rho(m_y))$ and this concludes the proof of Theorem 5.3.1.

5.3.4 Rational smoothness and p-smoothness of the flag variety.

We have an easy corollary of Theorem 5.3.1. Recall that if $\mathcal{W}$ is finite, then there exists a unique element of maximal length (cf. [[22], §1.8]) and we denote it by w_0.

Corollary 5.3.1. *Let $\mathcal{W}$ be a finite Weyl group and w_0 its longest element. Let k be such that $(\mathcal{G}(\mathcal{W}), k)$ is a GKM-pair. Then $\mathscr{B}(w_0)^y \cong S_k$ for any $y \in \mathcal{W}$ and any k.*

Proof. We proceed by induction on $n = \ell(w_0) - \ell(y)$. If $n = 0$, by definition, $\mathscr{B}(w_0)^{w_0} \cong S_k$. If $n \geq 1$ then there exists a simple reflection $s \in \mathcal{S}$ such that $ys > y$ (so, $\ell(w_0) - \ell(ys) = n - 1$). Actually, $w_0 s < w_0$ for any $s \in \mathcal{S}$ and, by Theorem 5.3.1 and inductive hypothesis, we have $\mathscr{B}(w_0)^y \cong \mathscr{B}(w_0)^{ys} \cong S_k$. □

Remark 5.3.2. *If $k = \mathbb{Q}$ the result above corresponds to the (rational) smoothness of flag varieties, while if k is a field of characteristic p it gives their p-smoothness (cf. [19]). Our proof is based only on the definition of canonical sheaf; we do not use Fiebig's multiplicity one results (see [17]), nor the geometry of the corresponding flag varieties.*

5.3.5 Parabolic setting

Let $J \subseteq \mathcal{S}$ be such that $\mathcal{W}_J = \langle J \rangle$ is finite with longest element w_J. Let $\mathcal{W}^J$ be the set of minimal representatives of the equivalence classes $\mathcal{W}/\mathcal{W}_J$. For $w \in \mathcal{W}^J$, denote by $\mathscr{B}(ww_J)$, resp. $\mathscr{B}^J(w)$, the corresponding indecomposable canonical sheaf on $\mathcal{G}$, resp. on $\mathcal{G}^J$. It is now easy to see that:

Lemma 5.3.7. *Let $\mathcal{W}_J$ and w_J be as above and consider $x, w \in \mathcal{W}^J$ such that $y \leq w$, then $\mathscr{B}(ww_J)^x \cong \mathscr{B}(ww_J)^{xu}$ for any $u \in \mathcal{W}_J$.*

Proof. We proceed by induction on $\ell(u)$. Clearly there is nothing to prove if $\ell(u) = 0$. If $\ell(x) > 0$ then there exists an $s \in \mathcal{S}$ such that $us < u$ and so by the inductive hypothesis, we get $\mathscr{B}(ww_J)^x = \mathscr{B}(ww_J)^{xus}$. Now for any $s \in J$, $ww_Js < ww_J$ and by Theorem 5.3.1 we obtain the claim. □

Theorem 5.3.2. *Let $(\mathcal{G}_{ww_J}, k)$ be a GKM-pair and let $\mathcal{W}^J$ and w_J be as above. If $y, w \in \mathcal{W}^J$ and $y \leq w$, then there is an isomorphism of S_k-modules*

$$\mathscr{B}(ww_J)^{yw_J} \cong \mathscr{B}^J(w)^y.$$

Proof. We proceed by induction on $n = \ell(w) - \ell(y)$. If $n = 0$ the statement is trivial. Suppose we have a collection of isomorphisms of S_k-modules $\eta_x : \mathscr{B}^J(w)^x \to \mathscr{B}(ww_J)^{xw_J}$ for any x such that $\ell(w) - \ell(x) < n$.

There is a natural injective homomorphism,

$$j : \Gamma(\{> y\}, \mathscr{B}(w)^J) \to \Gamma(\{> yw_J\}, \mathscr{B}(ww_J)),$$

defined by setting $(m_x)_{x\in(y,w]\subset\mathcal{W}^J} \mapsto (\widetilde{m_z})_{z\in(yw_J,ww_J]\subset\mathcal{W}}$, where $\widetilde{m_z} := \psi^z(\eta_x(m_x))$ if $z \in x\mathcal{W}_J$ and $\psi^z : \mathscr{B}(ww_J)^x \to \mathscr{B}(ww_J)^z$ denotes the isomorphism in Lemma 5.3.7.

We will show that such a homomorphism induces an isomorphism $\mathscr{B}(ww_J)^{\delta\, yw_J} \cong \mathscr{B}^J(w)^{\delta y}$. Then, by the unicity of the projective cover, the statement will follow.

Let $z \in (yw_J, ww_J]$, $z = xu$, for some $x > y \in \mathcal{W}^J$, $u \in \mathcal{W}_J$ and $u = s_1 \dots s_r$ a reduced expression with $s_i \in J$ for every i. Moreover, let $(n_v) \in \Gamma(\{> yw_J\}, \mathscr{B}(ww_J))$. We prove by induction on $\ell(u) = r$ that there exists a section $(p_v) \in \Gamma(\{> yw_J\}, \mathscr{B}(ww_J))$ such that $p_{xs_1\dots s_i} = \psi^{xs_1\dots s_i}(\eta_x(m_x))$ for some $m_x \in \mathscr{B}^J(w)^x$ for any $i = 0, \dots, r$ and such that $u_{yw_J}((p_v)) = u_{yw_J}((n_v))$.

For the base step we have $r = 0$ and there is nothing to prove.

If $z = (xs_1s_2 \dots s_{r-1})s_r$ then, by the inductive hypothesis, there exists a section $(q_v) \in \Gamma(\{> yw_J\}, \mathscr{B}(ww_J))$ and an element $m_x \in \mathscr{B}^J(w)^x$ such that $q_{xs_1\dots s_i} = \psi^{xs_1\dots s_i}(\eta_x(m_x))$ and $u_y((q_v)) = u_y((n_v))$ for $i = 0, \dots, r-1$. Thus, by Lemma 5.3.4, the element $(p_v) \in \bigoplus_{v>yw_J} \mathscr{B}(w)^y$ such that

$$p_{ys_1\dots s_{r-1}s_r} = \varphi^{ys_1\dots s_{r-1}}(p_{ys_1\dots s_{r-1}})$$

and

$$p_{xs_1\dots s_i} = q_{xs_1\dots s_i} = \psi^{xs_1\dots s_i}(\eta_x(m_x)) \quad \forall i < r$$

is a section on $\{> yw_J\}$ and verifies $u_{yw_J}((\widetilde{n_v})) = u_{yw_J}((n_v))$.

Finally, from the proof of Lemma 5.3.7 it follows that

$$\varphi^{ys_1\dots s_{r-1}}(p_{ys_1\dots s_{r-1}}) = \varphi^{ys_1\dots s_{r-1}}(\psi^{ys_1\dots s_{r-1}}(\eta_x(m_x)) = \psi^{xs_1\dots s_r}(\eta_x(m_x)).$$

□

Corollary 5.3.2. *Let $(\mathcal{G}_{|_{\leq ww_J}}, k)$ be a GKM-pair and let $p_J : \mathcal{G} \to \mathcal{G}^J$ the quotient map we defined in §3.1.3. Then, $p_J^*\mathscr{B}(w)^J \cong \mathscr{B}(ww_J)$.*

The theorem above is just the categorification of the following theorem, due to Deodhar:

Theorem 5.3.3 ([9])**.** *Let $\mathcal{W}$ be a Weyl group with $\mathcal{S}$, the set of simple reflections, and $J \subseteq \mathcal{S}$ such that $\mathcal{W}_J$ is finite. Let w_J be the longest element of $\mathcal{W}_J$ and $y, w \in \mathcal{W}^J$, then $m^J_{y,w} = h_{yw_J,ww_J}$.*

5.4 Affine Grassmannian for A_1

Using the inductive formula (2.2.c) of [29], it is easy to show that, if $\mathcal{W}$ is the infinite dihedral group, then $h_{y,w} = v^{l(y)-l(w)}$ for all $y, w \in \mathcal{W}$. Let us consider $J = \{s_\alpha\}$, then, from Theorem 5.3.3, it follows $m^J_{y,w} = v^{l(y)-l(w)}$ for any pair $y, w \in \mathcal{W}^J$. In this section we categorify this property. In particular, we prove that the structure sheaves on all finite intervals of the moment graph associated to the affine Grassmannian of $\widehat{\mathfrak{sl}_2}$ (cf. §3.2.2) are flabby. As in §3.2.2, we will denote by $\mathcal{G}^{\mathrm{par}}$ the corresponding moment graph, while, for a vertex $w \in \mathcal{W}^J$, $\mathscr{B}^{\mathrm{par}}(w)$ is the indecomposable canonical sheaf.

Recall that the set of vertices is in this case totally ordered, so we may enumerate the vertices as follows, once identified the finite root α with the corresponding coroot $\alpha^\vee$: $v_0 = 0$, $v_1 = \alpha$, $v_2 = -\alpha$, ... , $v_h = (-1)^{h+1}[\frac{h+1}{2}]\alpha$, ...

From now on we denote the edges as $E_{h,k} : (v_h \longrule v_k)$ and the labels as $l_{h,k} := l(E_{h,k})$; we write moreover $l_{h,k} = \alpha + n_{h,k}c$. Actually, the label of an edge $E_{h,k}$ is by definition $\pm l_{h,k}$; however, there exists an isomorphic k-moment graph with same sets of vertices and edges, but this other label function and, by Lemma 2.2.2, the corresponding indecomposable canonical sheaves are isomorphic.

We will prove in several steps that, if $v_j \leq v_i$ and $(\mathcal{G}^{\mathrm{par}}_{|_{[v_j,v_i]}}, k)$ is a GKM-pair, then $(\mathscr{B}^{\mathrm{par}}(v_i))^{v_j} \cong S_k$ by induction on $i - j$.

Fix once and for all $\mathcal{I} = \{v_i, v_{i-1}, \dots, v_{j+1}\}$.

Lemma 5.4.1. *Let $r \in \mathbb{N}$ be such that $r < i-j$. If $(\mathcal{G}^{par}, k)$ is a GKM-pair, and $z \in \Gamma(\mathcal{I}, \mathscr{B}^{par}(v_i))_{\{r\}}$, then z is uniquely determined by its first $r+1$ components, that is the restriction map $\Gamma(\mathcal{I}, \mathscr{B}^{par}(v_i))_{\{r\}} \longrightarrow \Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{par}(v_i))_{\{r\}}$ is injective.*

Proof. Let $z \in \Gamma(\mathcal{I}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}}$ such that $z_{v_i} = z_{v_{i-1}} = \ldots = z_{v_{i-r}} = 0$. Observe that for any $j+1 \leq h < i-r \leq k \leq i$ one has $z_{v_h} \equiv z_{v_k} = 0 \,(\bmod\ \alpha + n_{h,k}c)$.

By the GKM-property it follows that all the polynomials $MCD(\alpha+n_{h,k}c, \alpha+n_{h,l}c) = 1$ for any $i-r \leq k \neq l \leq i$. Since S_k is an UFD, z_{v_h} has to be divisible by $(\alpha+n_{h,i-r}c)(\alpha+n_{h,i-r+1}c)\dots(\alpha+n_{h,i}c)$. This is a polynomial of degree $r+1$ while z_{v_h} was a polynomial of degree r, so $z_{v_h} = 0$. □

Lemma 5.4.2. *Let $r \in \mathbb{N}$ be such that $r < i - j$. We have $dim_k\Gamma(\mathcal{I}, \mathscr{B}^{par}(v_i))_{\{r\}} = \binom{r+2}{2}$.*

Proof. By Lemma 5.4.1, $\dim_k\Gamma(\mathcal{I}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}} = \dim_k\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}}$.

Clearly, $\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}} \subseteq \bigoplus_0^r (S_k)_{\{r\}}$ and $\dim_k \bigoplus_0^r (S_k)_{\{r\}} = (r+1)^2$.

By definition an element $m \in \bigoplus_0^r (S_k)_{\{r\}}$ is in $\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}}$ if it satisfies some (linear) conditions given by the labels of the edges. If we prove that such conditions are linearly independent, then we know that

$$\dim_k\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}} = \dim_k \bigoplus_0^r (S_k)_{\{r\}} - \sharp \text{ edges}.$$

We noticed in §3.2.2 that in the $\widehat{\mathfrak{sl}_2}$ case all the vertices are connected, so the number of vertices is equal to the number of pairs of different elements in a set with $r+1$ elements, that is $\binom{r+1}{2}$. Then,

$$\dim_k\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}} = (r+1)^2 - \binom{r+1}{2} = \binom{r+2}{2}.$$

Hence now we show that the conditions are linearly independent.

Let $i-r \leq h < k \leq 1$ and define the element $(m^{(h,k)}) \in \bigoplus_0^r (S_k)_{\{r\}}$ in the following way:

$$m_{v_l}^{(h,k)} := \begin{cases} c\prod_{m\in\{i,i-1,\dots,i-r\}\setminus\{h,k\}}(\alpha+n_{h,m}c) & \text{if l =h} \\ 0 & \text{otherwise} \end{cases}$$

Now $m_{v_l}^{(h,k)} = m_{v_m}^{(h,k)}$ for any $l, m \neq h$ and $c\prod(\alpha+n_{h,m}c) \equiv 0(\text{mod } \alpha+n_{h,m}c)$. By the GKM-property, $l_{h,k}$ does not divides $m_h^{(h,k)}$, while $m_k^{(h,k)} = 0$.

So for any condition coming from the edge $E_{l,m}$ we built a $r+1$-tuple which verifies all conditions except the $E_{l,m}$-th. It follows that all conditions are linearly independent.

□

Denote by $m_\alpha, m_c \in \Gamma(\mathcal{I}, \mathscr{B}^{\text{par}}(v_i))_{\{1\}}$ the constant sections $m_{\alpha,v} = \alpha$, $m_{c,v} = c$ for all $v \in \mathcal{I}$. Denote moreover by $u_{v_j} := \oplus\rho_{v_h,E_{h,j}}$, where $\rho_{v_h,E_{h,j}} : S_k \to S_k/(E_{h,j}\cdot S_k)$ are just the canonical quotient maps.

Lemma 5.4.3. *Let $r \in \mathbb{N}$ and let $(\mathcal{G}^{par}, k)$ be a GKM-pair. The vector subspace of $(\mathscr{B}_{v_i}^{par})^{cv_j}$ generated by*

$$u_{v_j}(m_\alpha^r), u_{v_j}(m_\alpha^{r-1}m_c)\dots u_{v_j}(m_\alpha m_c^{r-1}), u_{v_j}(m_c^r)$$

has dimension equal to $r+1$ if $r < i-j$ or dimension equal to $i-j$ otherwise.

Proof. As first notice that $(\mathscr{B}^{\text{par}}(v_i))^{E_{j,k}} = S_k/(l_{j,k}\cdot S_k) \cong k[c]$ by the mapping $\alpha \mapsto -n_{j,k}c$. Then

$$u_{v_j}(m_\alpha^k m_c^{t-k}) = ((-n_{j,i})^k, (-n_{j,i-1})^k, \dots, (-n_{j,j+1}^k))c.$$

We obtain the following matrix

$$N = \begin{pmatrix} 1 & 1 & \dots & 1 \\ -n_{j,i} & -n_{j,i-1} & \dots & -n_{j,j+1} \\ (-n_{j,i})^2 & (-n_{j,i-1})^2 & \dots & (-n_{j,j+1})^2 \\ \vdots & \vdots & & \vdots \\ (-n_{j,i})^t & (-n_{j,i-1})^t & \dots & (-n_{j,j+1})^t \end{pmatrix}$$

By the GKM-property it follows that $n_{j,k} \neq n_{j,h}$ for all pair $j+1 \leq k \neq h \leq i$ and N is a Vandermonde matrix. In particular, such a matrix is not singular and so it has maximal rank, i.e. $rk(N) = t+1$ if $t < i-j$ and $rk(N) = i-j$ otherwise. □

Lemma 5.4.4. *There exists a section $m_0 \in \Gamma(\mathcal{I}, \mathscr{B}^{par}(v_i))_{\{1\}}$ such that $u_{v_j}(m_0) = 0$ and $m_{0,v} \neq 0$ for all $v \in \mathcal{I}$.*

Proof. Let $v_j = r\alpha$. Define $m_{0,v_h} := (r-s)l_{j,h} = (r-s)(\alpha+(r+s)c)$ if $v_h = s\alpha$.

Notice that $(m_0) \in \Gamma(\mathcal{I}, \mathscr{B}^{\text{par}}(v_i))$; indeed for any pair of vertices $v_h = s\alpha$, $v_k = t\alpha$, one has $l_{h,k} = \alpha + (s+t)c$ and

$$\begin{aligned} m_{0,v_h} - m_{0,v_k} &= (r-s)(\alpha+(r+s)c) - (r-t)(\alpha+(r+t)c) = \\ &= -s\alpha - s^2c + t\alpha + t^2c = \alpha(t-s) + c(t^2-s^2) = \\ &= (t-s)(\alpha+(s+t)c) \equiv 0 \quad (\text{ mod } \alpha+(s+t)c). \end{aligned}$$

Moreover, by definition $m_{0,v_h} \neq 0$ for any $v_h \in \mathcal{I}$ and $u_{v_j}((m_0)) = 0$.

□

Lemma 5.4.5. *Let $r \in \mathbb{N}$ be such that $r < i-j$. The collection of monomials $\{m_\alpha^l m_c^h m_0^k \mid l,h,k \geq 0,\ l+h+k=r\}$ is a basis of $\Gamma(\mathcal{I}, \mathscr{B}^{par}(v_i))_{\{r\}}$.*

Proof. Since the number of monomials in three variables of degree r is $\binom{r+2}{2}$ and by Lemma 5.4.2 $\dim_k \Gamma(\mathcal{I}, \mathscr{B}^{\mathrm{par}}(v_i)) = \binom{r+2}{2}$ as well, it is enough to prove that all monomial in m_α, m_c, m_0 are linearly independent. We prove the claim by induction on r.

Let $r=1$. If $xm_\alpha + ym_c + zm_0 = 0$, then clearly $0 = u_{v_j}(xm_\alpha + ym_c + zm_0) = xu_{v_j}(m_\alpha) + yu_{v_j}(m_c) + zu_{v_j}(m_0)$. By Lemma 5.4.4 $u_{v_j}(m_0) = 0$, so $xu_{v_j}(m_\alpha) + yu_{v_j}(m_c) = 0$. But by Lemma 5.4.3 $u_{v_j}(m_\alpha)$ and $u_{v_j}(m_c)$ generate a vector space of dimension 2, then $x=y=0$. Finally, from $zm_0 = 0$ and Lemma 5.4.4 it follows $z=0$.

Now let $r>1$. Let $z = \sum_{l+m+n=r} x_{l,m,n} m_\alpha^l m_c^m m_0^n = 0$. We can write $z = z_1 + z_0 m_0$, z_1 is such that m_0 does not appear. Then by Lemma 5.4.3 $u_{v_j}(z) = u_{v_j}(z_1) + u_{v_j}(z_0)u_{v_j}(m_0) = u_{v_j}(z_1) = 0$. From Lemma 5.4.3 we know that all $u_{v_j}(m_\alpha^l m_c^{r-l})$ are linearly independent and so

$$0 = u_{v_j}(z_1) = u_{v_j}(\sum_{l+m=r} x_{l,m,0} m_\alpha^l m_c^m) = \sum_{l+m=r} x_{l,m,0} u_{v_j}(m_\alpha^l m_c^m)$$

implies $x_{l,m,0} = 0$ for all pair l,m, i.e. $c_1 = 0$. Thus we obtain $c_0 m_0 = 0$ and we conclude by Lemma 5.4.4 that $c_0 = 0$. Finally, $0 = c_0 = \sum_{l+m+n=r-1} x_{l,m,n+1} m_\alpha^l m_c^m m_0^n$ is a linear combination of monomials in m_α, m_c, m_0 of degree $r-1$ and so by the inductive hypothesis we have $x_{l,m,n+1} = 0$ for all l,m,n.

□

Theorem 5.4.1. *If $v_j \leq v_i$ and $(\mathcal{G}^{par}_{|_{[v_j,v_i]}}, k)$ is a GKM-pair, then $(\mathscr{B}^{par}(v_i))^{v_j} \cong S_k$.*

Proof. We prove that $(\mathscr{B}^{\mathrm{par}}(v_i))^{\delta v_i}$ coincides with the u_{v_i} image of the ring generated by m_α and m_c. If $r < i-j$, by 5.4.5, $\Gamma(\mathcal{I}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}}$ is generated by $\{m_\alpha^l m_c^h m_0^k \mid l,h,k \geq 0,\ l+h+k=r\}$. From 5.4.4 it follows $(\mathscr{B}^{\mathrm{par}}(v_i))^{\delta v_i} = u_{v_i}(\Gamma(\mathcal{I}, \mathscr{B}^{\mathrm{par}}(v_i))_{\{r\}})$ is contained in the ring generated by $u_{v_i}((m_\alpha))$ and $u_{v_i}((m_c))$.

Otherwise, $r \geq i-j$ and $\bigoplus_{E_{i,k} \in \mathcal{E}_{\delta v_i}} (\mathscr{B}^{\mathrm{par}}(v_i))^{E_{i,k}} \cong k[c]^{i-j}$, having dimension $i-j$. Then by Lemma 5.4.3 $u_{v_i}(m_\alpha)$ and $u_{v_i}(m_c)$ generate $(\mathscr{B}^{\mathrm{par}}(v_i))^{\delta v_i}$

Thus we have a surjective map $S_k \to \bigoplus_{E_{i,k} \in \mathcal{E}_{\delta v_i}} (\mathscr{B}^{\mathrm{par}}(v_i))^{E_{i,k}}$ by the mapping $\alpha \mapsto m_\alpha$ and $c \mapsto m_c$. Then $(\mathscr{B}^{\mathrm{par}}(v_i))^{v_j} \cong S_k$. □

Remark 5.4.1. *If $k = \mathbb{Q}$, this result corresponds to the rational smoothness of the corresponding (partial) Richardson variety.*

Chapter 6

The stabilisation phenomenon

In [37], Lusztig proved that the affine parabolic Kazhdan-Lusztig polynomials stabilise. Quoting Soergel's reformulation (cf.[[40],Theorem 6.1]), the parabolic Kazhdan-Lusztig polynomials $m^{\mathcal{S}^f}_{A,B}$ indexed by pairs of alcoves far enough in the fundamental chamber stabilise, in the sense that, for any pair of alcoves A, B, there exists a polynomial $q_{A,B}$ with integer coefficients such that

$$\lim_{\mu\in\mathcal{C}^+} m^{\mathcal{S}^f}_{A+\mu,B+\mu} = q_{A,B}$$

The $q_{A,B}$'s are called *generic polynomials* and turn out to have a realisation very similar to the one of the regular Kazhdan-Lusztig polynomials. Indeed, Lusztig in [37] associated to every affine Weyl group $\mathcal{W}^a$ its periodic module $\mathbf{M}$, that is the free $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$-module with set of generators -or standard basis- indexed by the set of all alcoves $\mathcal{A}$. It is possible to define an involution and to prove that there exists a self-dual basis of $\mathbf{M}$: the *canonical basis*. In this setting, the generic polynomials are the coefficients of the change basis matrix. Our interest in the periodic module is motivated by the fact that $\mathbf{M}$ governs the representation theory of the affine Kac-Moody algebra, whose Weyl group is $\mathcal{W}^a$, at the critical level (cf. [3]).

The aim of this chapter is to study the behaviour of indecomposable Braden-MacPherson sheaves on finite intervals of the parabolic Bruhat graph far enough in $\mathcal{C}^+$ (cf. §3.2.3).

6.1 Statement of the main theorem

Let $\mathcal{G}^{\mathrm{par}}$ denote the parabolic moment graph associated to the affine Grassmannian, whose set of vertices we identify with the set of alcoves in the fundamental chamber (cf. §3.2.2), and let $\mathcal{I} = [B, A]$ be an interval far enough in the fundamental chamber. Inspired by [[37], Proposition 11.15], we claim that, for all $\mu \in X^\vee \cap \mathcal{C}^+$,

$$\mathscr{B}(A)^B \cong \mathscr{B}(A+\mu)^{B+\mu}. \tag{6.1}$$

We showed in §3.2.3 that $\mathcal{G}^{\mathrm{par}}_{|_{[B,A]}}$ is in general not isomorphic to $\mathcal{G}^{\mathrm{par}}_{|_{[B+\mu,A+\mu]}}$ as moment graph, so we cannot use the pullback technique we developed in §5.2 to get the isomorphism of S_k-modules above. On the other hand, we proved in Lemma 3.2.5 that, for all $\mu \in X^\vee$, there is an isomorphism of k-moment graphs

$$\tau_\mu : \mathcal{G}^{\mathrm{stab}}_{|_{[B,A]}} \to \mathcal{G}^{\mathrm{stab}}_{|_{[B+\mu,A+\mu]}}$$

Thus, by Lemma 2.2.2, we get an isomorphism between the indecomposable canonical sheaf $\mathscr{B}(A)$ on $\mathcal{G}^{\mathrm{stab}}_{|_{[B,A]}}$ and $\tau_\mu^*\mathscr{B}(A+\mu)$, the pullback of the indecomposable Braden-MacPherson sheaf $\mathscr{B}(A+\mu)$ on $\mathcal{G}^{\mathrm{stab}}_{|_{[B+\mu,A+\mu]}}$.

For any finite interval $\mathcal{J}$ far enough in the fundamental chamber, consider the monomorphism $i_{\mathcal{J}} : \mathcal{G}^{\mathrm{stab}}_{|_{\mathcal{J}}} \hookrightarrow \mathcal{G}^{\mathrm{par}}_{|_{\mathcal{J}}}$, given by $(i_{\mathcal{J}})_{\mathcal{V}} = \mathrm{id}_{\mathcal{V}}$ and $i_{\mathcal{J},l,x} = \mathrm{id}$ for all $x \in \mathcal{J}$. We get the functor $\cdot^{\mathrm{stab}}$: $\mathbf{Sh}_{\mathcal{G}^{\mathrm{par}}_{|_{\mathcal{J}}}} \to \mathbf{Sh}_{\mathcal{G}^{\mathrm{stab}}_{|_{\mathcal{J}}}}$, defined by the setting $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{stab}} := i_{\mathcal{J}}^*(\mathcal{F})$. The goal of this chapter is to prove the following result.

Theorem 6.1.1. *For all finite intervals far enough in the fundamental chamber, the functor* $\cdot^{stab}$: $\boldsymbol{Sh}_{\mathcal{G}^{par}_{|_{\mathcal{J}}}} \to \boldsymbol{Sh}\mathcal{G}^{stab}_{|_{\mathcal{J}}}$ *preserves indecomposable Braden-MacPherson sheaves.*

We will prove this theorem via explicit calculations in the $\widehat{\mathfrak{sl}_2}$ case, while for the general case we will need deep results and methods developed by Fiebig in [18].

Once proved Theorem 6.1.1, we get Equality 6.1 by applying Lemma 3.2.5.

6.2 The subgeneric case

In this section, $\mathcal{G}^{\mathrm{stab}}$, resp. $\mathcal{G}^{\mathrm{par}}$, denote the parabolic moment graph, resp. the stable moment graph, for the $\widetilde{A_1}$ root system. Moreover, we suppose that k has characteristic zero and we write S instead of S_k.

We have already proved that for any two vertices v, w with $v \leq w$ the stalk of the Braden-MacPherson sheaf on $\mathcal{G}^{\mathrm{par}}_{\leq w}$ is $\mathscr{B}(w)^v \cong S$, that is equivalent to the flabbiness of the structure sheaf on $\mathcal{G}^{\mathrm{par}}_{\leq w}$. In order to show that the functor $^{\mathrm{stab}}$ preserves indecomposable canonical sheaves, it is in this case enough to verify that, for any vertex w, the structure sheaf $\mathcal{A}$ on $\mathcal{G}^{\mathrm{stab}}_{\leq w}$ is still flabby.

Recall that the set of vertices of $\mathcal{G}^{\mathrm{par}}$ (and so of $\mathcal{G}^{\mathrm{stab}}$) can be identified with the finite (co)root lattice, that is $\mathbb{Z}\alpha$, where $\alpha = \alpha^\vee$ is the positive (co)root of A_1. Moreover, $\mathcal{G}^{\mathrm{par}}$ is a complete graph and the label function is given, up to a sign, by $l(h\alpha \text{ --- } k\alpha) = \alpha + (h+k)c$. By definition, we get $\mathcal{G}^{\mathrm{stab}}$ from $\mathcal{G}^{\mathrm{par}}$ by deleting the non-stable edges, then $h\alpha \text{ --- } k\alpha \in \mathcal{E}^{\mathrm{stab}}$ if and only if $\mathrm{sgn}(h) = -\mathrm{sgn}(k)$ (where, by convention, we set $\mathrm{sgn}(0) = -$).

Lemma 6.2.1. *Let $r \in \mathbb{Z}_{>0}$. If $n \in \mathbb{Z}$, set, for any $h \in \mathbb{Z}$, with $h\alpha \leq n\alpha$,*

$$
{}^{\mathrm{e}}z^r_{n\alpha,h\alpha} := \begin{cases} 0 & \text{if} \quad |h| \in [|n|-r+1, |n|] \\ \prod_{i=0}^{r-1}\left[\big(-\alpha + (|n|-h-i)c\big)(|n|-h-i)\right] & \text{if} \quad h \in (0, |n|-r] \\ \prod_{i=0}^{r-1}\left[\big(\alpha + (|n|+h-i)c\big)(|n|+h-i)\right] & \text{if} \quad h \in [r-|n|, 0] \end{cases}
$$

Then ${}^{\mathrm{e}}z^r_{n\alpha} = ({}^{\mathrm{e}}z^r_{n\alpha,h\alpha}) \in \Gamma(\{\leq n\alpha\}, \mathcal{A})_{\{r\}}$.

Proof. We verify that, for any $h, k \in \mathbb{Z}$ such that $h\alpha, k\alpha \leq n\alpha$, if $h\alpha \text{ --- } k\alpha$ is an edge, then

$$
{}^{\mathrm{e}}z^r_{n\alpha,h\alpha} - {}^{\mathrm{e}}z^r_{n\alpha,k\alpha} \equiv 0 \pmod{\alpha + (h+k)c} \tag{6.2}
$$

We may clearly suppose $h > 0$ and $k \leq 0$.

Let at first consider $h \in [|n|-r+1, n]$. If $-k \in [|n|-r+1, n]$, then ${}^{\mathrm{e}}z^r_{n\alpha,h\alpha} = {}^{\mathrm{e}}z^r_{n\alpha,k\alpha} = 0$ and there is nothing to prove. Otherwise, $k \in [r-|n|, 0]$ and

$$ {}^{\mathrm{e}}z^r_{n\alpha,h\alpha} - {}^{\mathrm{e}}z^r_{n\alpha,k\alpha} = 0 - \prod_{i=0}^{r-1} \big[\big(\alpha + (|n|+k-i)c\big)(|n|+k-i)\big]. \tag{6.3} $$

Now, $\alpha + (h+k)c$ divides $\prod_{i=0}^{r-1}\big[\big(\alpha+(|n|+k-i)c\big)(|n|+k-i)\big]$ if and only if there exists an $i \in [0, r-1]$ such that $|n| - i = h$, i.e. $h - |n| = -i$. But we supposed $h \in [|n|-r+1, n]$ that is, precisely, $h - |n| \in [-r+1, 0]$.

Let consider the case $h \in (0, |n|-r]$. If $-k \in [|n|-r+1, n]$, then

$$ {}^{\mathrm{e}}z^r_{n\alpha,h\alpha} - {}^{\mathrm{e}}z^r_{n\alpha,k\alpha} = \prod_{i=0}^{r-1} \big[\big(-\alpha + (|n|-h-i)c\big)(|n|-h-i)\big] - 0. \tag{6.4} $$

Now, $\alpha + (h+k)c$ divides $\prod_{i=0}^{r-1}\big[\big(-\alpha+(|n|-h-i)c\big)(|n|-h-i)\big]$ if and only if there exists an $i \in [0, r-1]$ such that $|n| - i = -k$, i.e. $-k - |n| = -i$. But we supposed $-k \in [|n|-r+1, |n|]$ that is, precisely, $-k - |n| \in [-r+1, 0]$.

Otherwise, $k \in [r-|n|, 0]$ and

$$
\begin{aligned}
{}^{\mathrm{e}}z^r_{n\alpha,h\alpha} - {}^{\mathrm{e}}z^r_{n\alpha,k\alpha} &= \textstyle\prod_{i=0}^{r-1}\big[\big(-\alpha+(|n|-h-i)c\big)(|n|-h-i)\big] \\
&\quad - \textstyle\prod_{i=0}^{r-1}\big[\big(\alpha+(k+|n|-i)c\big)(k+|n|-i)\big] \\
&\equiv \textstyle\prod_{i=0}^{r-1}[(k+h+|n|-h-i)(|n|-h-i)c] \\
&\quad - \textstyle\prod_{i=0}^{r-1}[(-k-h+k+|n|-i)(|n|+k-i)c] = \qquad \pmod{\alpha+(h+k)c} \\
&= c^r \textstyle\prod_{i=0}^{r-1}[(k+|n|-i)(|n|-h-i) - (-h+|n|-i)(|n|+k-i)] \\
&= 0
\end{aligned}
$$

□

Lemma 6.2.2. *Let $r \in \mathbb{Z}_{>0}$. If $n \in \mathbb{Z}$, for any $h \in \mathbb{Z}$, such that $h\alpha \le n\alpha$, we set*

$$ {}^{o}z^r_{n\alpha,h\alpha} := \begin{cases} 0 & \text{if} \quad |h| \in [|n|-r+2, |n|] \\ \prod_{i=0}^{r-1}\big[\big(-\alpha+(|n|-h-i)c\big)(|n|-h-i+1)\big] & \text{if} \quad h \in (0, |n|-r+1] \\ \prod_{i=0}^{r-1}\big[\big(\alpha+(|n|+h-i+1)c\big)(|n|+h-i)\big] & \text{if} \quad h \in [r+n-1, 0] \end{cases} $$

Then ${}^{o}z^r_{n\alpha} = ({}^{o}z^r_{n\alpha,h\alpha}) \in \Gamma(\{\le n\alpha\}, \mathcal{A})_{\{r\}}$.

Proof. The proof is very similar to the one of the previous lemma and therefore we omit it. □

Define ${}^{\mathrm{e}}z^0_{n\alpha} := (1)_{h\alpha \le n\alpha}$.

Lemma 6.2.3. *Let $r \in \mathbb{Z}_{\ge 0}$, $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ be such that $m\alpha \le n\alpha$. For all $z \in \Gamma([m\alpha, n\alpha], \mathcal{A})_{\{r\}}$, there exist ${}^{o}s^i_k, {}^{e}s^j_k \in S_{\{i\}}$, with $i \in [0, r]$, $j \in (0, r]$ and k such that $k\alpha \in [m\alpha, n\alpha]$, such that*

$$ z = \sum_{j=1}^{r} {}^{e}s^j_k ({}^{e}z^{r-j}_{k\alpha})_{k\alpha \in [m\alpha, n\alpha]} + \sum_{i=0}^{r} {}^{o}s^i_k ({}^{o}z^r_{k\alpha})_{k\alpha \in [m\alpha, n\alpha]}. \tag{6.5} $$

Proof. Let $h\alpha$ be the maximal vertex in $[m\alpha, n\alpha]$ such that $z_{h\alpha} \neq 0$. We prove the statement by induction on $l = \sharp[m\alpha, h\alpha]$.

If such a vertex does not exists, that is $l = 0$, then $z = (0)$ and there is nothing to prove.

We should consider four cases: $n > 0$ and $l > 0$; $n > 0$ and $l \leq 0$; $n \leq 0$ and $l > 0$; $n \leq 0$ and $l \leq 0$. Actually, we will verify only the first case, since the others can be proven in a very similar way.

Let $n > 0$ and $h > 0$. If $h = n$, then we set $z' = z - z^{\mathrm{e}}_{n\alpha} z^{0}_{n\alpha}$ and the result follows from the inductive hypothesis. Otherwise, $h < n$ and then $\prod_{i=0}^{n-h-1}(-\alpha + (n-h-i)c)$ divides $z_{h\alpha}$ in S and we may set

$$ {}^{\mathrm{e}}s_h^{r-n+h} := \prod_{i=0}^{n-h-1} [(-\alpha + (n-h+i)c)(n-h-i)]^{-1} \cdot z_{h\alpha} \in S_{\{r-n+h\}}. \tag{6.6}$$

Now $z' := z - {}^{\mathrm{e}}s_h^{r-n+h+1} \cdot ({}^{\mathrm{e}}z_{k\alpha}^{n-h})_{k\alpha\in[m\alpha,n\alpha]} \in \Gamma([m\alpha, n\alpha], \mathcal{A})_{\{r\}}$ has the property that $z'_{k\alpha} = 0$ for all $k \in [h\alpha, n\alpha]$ and we get the statement from the inductive hypothesis.

□

Corollary 6.2.1. *For any $n \in \mathbb{Z}$, the structure sheaf $\mathcal{A}$ on $\mathcal{G}^{stab}_{\leq n\alpha}$ is flabby.*

Proof. We have to show that every local section $z \in \Gamma(\mathcal{I}, \mathcal{A})$, with $\mathcal{I}$ open can be extended to a global section $\widetilde{z} \in \Gamma(\mathcal{G}^{\mathrm{stab}}_{\leq n\alpha}, \mathcal{A})$. Since the set of vertices of $\mathcal{G}^{\mathrm{stab}}$ is totally ordered, then any open set of $\mathcal{G}^{\mathrm{stab}}_{\leq n\alpha}$ is actually an interval, that is there exists an $m \in \mathbb{Z}$ such that $\mathcal{I} = [m\alpha, n\alpha]$.

Suppose $z \in \Gamma(\mathcal{I}, \mathcal{A})_{\{r\}}$, then by Lemma 6.2.3, we can write

$$ z = \sum_{j=1}^{r} {}^{\mathrm{e}}s_k^{j}({}^{\mathrm{e}}z_{k\alpha}^{r-j})_{k\alpha\in[m\alpha,n\alpha]} + \sum_{i=0}^{r} {}^{\mathrm{o}}s_k^{i}({}^{\mathrm{o}}z_{k\alpha}^{r})_{k\alpha\in[m\alpha,n\alpha]}. \tag{6.7}$$

By Lemma 6.2.1 and Lemma 6.2.2, z is a sum of extensible sections, and so it is extensible as well.

□

Finally, we get the following theorem.

Theorem 6.2.1. *Let $\mathfrak{g} = \mathfrak{sl}_2$. In this case, for all finite intervals $\mathcal{I}$, the functor $\cdot^{stab}$ preserves indecomposable canonical sheaves.*

6.3 General case

In order to prove our claim, we have to show that, for any interval $\mathcal{I}$ far enough in the fundamental chamber, if $\mathscr{B}$ is an indecomposable Braden-MacPherson sheaf on $\mathcal{G}^{\mathrm{par}}_{|\mathcal{I}}$, then $\mathscr{B}^{\mathrm{stab}}$ is indecomposable and satisfies properties *(BMP1)*,*(BMP2)*,*(BMP3)*,*(BMP4)*. Observe that properties *(BMP1)*, *(BMP2)* are trivial and *(BMP4)* comes from the fact that $\Gamma(\mathcal{I}, \mathcal{F}) \hookrightarrow \Gamma(\mathcal{I}, \mathcal{F}^{\mathrm{stab}})$ for any $\mathcal{F} \in \mathbf{Sh}(\mathcal{G}^{\mathrm{par}}_{|\mathcal{I}})$, so we only have to show that $\mathscr{B}^{\mathrm{stab}}$ is a flabby indecomposable sheaf on $\mathcal{G}^{\mathrm{stab}}_{|\mathcal{I}}$.

6.3.1 Flabbiness

It is possible to define a functor $\cdot^{\text{per}} : \mathbf{Sh}(\mathcal{G}) \to \mathbf{Sh}(\mathcal{G}^{\text{per}})$ in a very easy way. Let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}\{\rho_{x,E}\})$, then we set $(\mathcal{F}^{\text{per}})^x = \mathcal{F}^x$ for any $x \in \mathcal{V}$, $(\mathcal{F}^{\text{per}})^E = \mathcal{F}^E$for any $E \in \mathcal{E}$ and $\rho^{\text{per}}_{x,E} = \rho_{x,E}$. A fundamental step in the proof of the flabbiness of $\mathscr{B}^{\text{stab}}$ consists in showing that $\cdot^{\text{per}}$ maps canonical shaves to flabby sheaves. In order to get this, we will combine several results of Fiebig that we are going to recall.

Hereafter we will consider translation functors on the category of $\mathcal{Z}$-modules, where $\mathcal{Z}$ is the structure algebra of $\mathcal{G}$. From now on θ_s will denote the translation functor defined by Fiebig in [13]. The definition is analogous to the one we have given in Chapter 4. We will moreover denote by $\mathcal{H}$ the corresponding category of special modules. Thus the following theorem holds.

Theorem 6.3.1 ([13]). *Let $M \in \mathcal{Z}-mod$. Then $M \in \mathcal{H}$ if and only if it is isomorphic to the space of global sections of a BradenMacPherson sheaf on $\mathcal{G}$.*

In [14], Fiebig defined the localisation functor $\mathscr{L} : \mathcal{Z}(\mathcal{K}) - \text{mod} \to \mathbf{Sh}(\mathcal{K})$, for all k-moment graphs $\mathcal{K}$, that is left adjoint to the functor of global section $\Gamma : \mathbf{Sh}_{\mathcal{K}} \to \mathcal{Z}(\mathcal{K}) - \text{mod}$ (cf. [[14], Theorem 3.5]).

Using Fiebig's terminology, we may now say that an object $M \in \mathcal{Z} - \text{mod}$ is flabby if the corresponding sheaf $\mathscr{L}(M)$ is flabby. So our claim is equivalent to the fact that $\mathscr{L}(\Gamma(\mathcal{F}^{\text{per}})) = \mathscr{L}(\Gamma(\mathcal{F}))^{\text{per}}$ is flabby if $\mathcal{F}$ is a Braden MacPherson sheaf. We will prove it using translation functors.

When we defined translation functors, we did not use the partial order on the set of vertices, since the structure algebra does not depend on it. Thus it makes sense to speak of the translation functor $\theta_s^{\text{per}} : \mathcal{Z}(\mathcal{G}^{\text{per}})\text{-mod} \to \mathcal{Z}(\mathcal{G}^{\text{per}})\text{-mod}$ and this clearly coincides with $\theta_s : \mathcal{Z}(\mathcal{G})\text{-mod} \to \mathcal{Z}(\mathcal{G})\text{-mod}$. Then also the corresponding categories of special modules (see §4) coincide, but, because of this different order, we get a different topology on the set of vertices and so $M \in \mathcal{H} = \mathcal{H}^{\text{per}}$ could be such that $\mathscr{L}(M)$ is flabby in $\mathbf{Sh}(\mathcal{G})$, while $\mathscr{L}(M)^{\text{per}}$ is not in $\mathbf{Sh}(\mathcal{G}^{\text{per}})$. In [13] Fiebig proved the following fact (used actually in the proof of Theorem 6.3.1).

Theorem 6.3.2 ([13]). *$\theta_s : \mathcal{Z}(\mathcal{G}) - mod \to \mathcal{Z}(\mathcal{G}) - mod$ preserves flabby objects.*

The proof of the theorem above is rather long, so we omit it. However we want to point out the fact that in order to get the previous result Fiebig used only three properties of the Bruhat order, namely

(1) The elements w and tw are comparable for all $w \in \mathcal{W}^a$ and $t \in \mathcal{T}^a$. The relations between all such pairs w, tw generate the partial order.

(2) We have $[w, ws] = \{w, ws\}$ for all $w \in \mathcal{W}^a$ and $s \in \mathcal{S}^a$ such that $w < ws$.

(3) For $x, y \in \mathcal{W}^a$ such that $x < xs$ and $y \leq xs$ we have $ys \leq xs$. For $x, y \in \mathcal{W}^a$ such that $xs < x$ and $xs \leq y$ we have $xs \leq ys$.

Since Lusztig in [37] proved that the generic order has also these properties, we get

Theorem 6.3.3. *$\theta_s : \mathcal{Z}(\mathcal{G}^{per}) - mod \to \mathcal{Z}(\mathcal{G}^{per}) - mod$ preserves flabby objects.*

We are now ready to conclude.

Proposition 6.3.1. *Let $\mathcal{F}$ be a Braden-MacPherson sheaf on $\mathcal{G}$ then $\mathcal{F}^{per}$ is a flabby sheaf on $\mathcal{G}^{par}$.*

Proof. We want to show that $F = \Gamma(\mathcal{F})$ is flabby. By Theorem 6.3.1, we know that $F \in \mathcal{H}$, so we may prove our result by induction. If $F = B_e$, there is nothing to prove. We have to show now that, if the claim is true for $M \in \mathcal{H}$, then it holds also for $\theta_s(M)$, that, again by Theorem 6.3.1, is still isomorphic to the global sections of a Braden-MacPherson sheaf on $\mathcal{G}$. But now by the inductive hypothesis we get that M is a flabby object in $\mathcal{Z}(\mathcal{G}^{\mathrm{per}})$-mod and so, by applying Theorem 6.3.3, $\theta_s(M) = \theta_s^{\mathrm{per}}(M)$ is also a flabby object in $\mathcal{Z}(\mathcal{G}^{\mathrm{per}})$-mod.

□

Decomposition of the functor $\cdot^{\mathrm{stab}}$

The functor $\cdot^{\mathrm{stab}}$ may be obtained as composition of the five following functors.

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{G}^{\mathrm{par}}_{|\mathcal{I}}) \arrow[r, "i_*"] \arrow[rrrrr, bend right, "\cdot^{\mathrm{stab}}"'] & \mathbf{Sh}(\mathcal{G}^{\mathrm{par}}) \arrow[r, "p^*_{\mathrm{par}}"] & \mathbf{Sh}(\mathcal{G}) \arrow[r, "\cdot^{\mathrm{per}}"] & \mathbf{Sh}(\mathcal{G}^{\mathrm{per}}) \arrow[r, "j^*"] & \mathbf{Sh}(\mathcal{G}^{\mathrm{per}}_{|\mathcal{I}}) \arrow[r, "\cdot^{\mathrm{opp}}"] & \mathbf{Sh}(\mathcal{G}^{\mathrm{stab}}_{|\mathcal{I}}) \end{tikzcd}$$

Where

- $i : \mathcal{G}^{\mathrm{par}}_{|\mathcal{I}} \hookrightarrow \mathcal{G}^{\mathrm{par}}$ and $j : \mathcal{G}^{\mathrm{stab}}_{|\mathcal{I}} \hookrightarrow \mathcal{G}^{\mathrm{stab}}$ are the inclusions of subobjects
- $p_{\mathrm{par}} : \mathcal{G} \to \mathcal{G}^{\mathrm{par}}$ is the quotient homomorphism we defined in §3.1.3
- $\cdot^{\mathrm{opp}}$ is the pullback of the isomorphism of moment graphs $f : \mathcal{G}^{\mathrm{stab}}_{|\mathcal{I}} \to \mathcal{G}^{\mathrm{per}}_{|\mathcal{I}}$ defined as $f_{\mathcal{V}} = \mathrm{id}$ and $f_{l,x}(\lambda) = x^{-1}(\lambda)$ for all $x \in \mathcal{I}$ and $\lambda \in \dot{Q}^{\vee}$ (this is proved to be an isomorphism in Lemma 5.2.1).

Now, it is clear that i_* and j^* map flabby sheaves to flabby sheaves. Moreover, p^*_{par}, resp. $\cdot^{\mathrm{opp}}$, by Corollary 5.3.2, resp. Lemma 2.2.2, preserves Braden-MacPherson sheaves, and so, in particular, the flabbiness. Finally, Proposition 6.3.1 tells us that also the functor $\cdot^{\mathrm{per}}$ preserves the flabbiness. It follows that if we apply $\cdot^{\mathrm{stab}}$ to a Braden-MacPherson sheaf we get a flabby sheaf on $\mathcal{G}^{\mathrm{stab}}_{|\mathcal{I}}$, as we wished. Thus we obtain the following result.

Theorem 6.3.4. *Let $\mathcal{F} \in \boldsymbol{Sh}(\mathcal{G}^{par}_{|\mathcal{I}})$ be a Braden-MacPherson sheaf, then $\mathcal{F}^{per} \in \boldsymbol{Sh}(\mathcal{G}^{stab}_{|\mathcal{I})}$ is a flabby sheaf.*

6.3.2 Indecomposability

Here we prove the only step missing in the proof of Theorem 6.1.1.

Proposition 6.3.2. *Let $\mathcal{I}$ be a finite interval of $\mathcal{G}^{par}$ far enough in $\mathcal{C}^+$ and let $\mathscr{B} \in \boldsymbol{Sh}(\mathcal{G}^{par}_{|\mathcal{I}})$ be an indecomposable Braden-MacPherson sheaf. Then $\mathscr{B}^{stab}$ is also indecomposable as sheaf on $\mathcal{G}^{stab}_{|\mathcal{I}}$.*

Proof. Since $\mathscr{B}$ is indecomposable, by Theorem 2.1.1, $\mathscr{B} = \mathscr{B}(w)$ for some $w \in \mathcal{I}$, that implies $\mathscr{B}(w)^x = 0 = \mathscr{B}^{\mathrm{stab},x}$ for all $x > w$ ($x \in \mathcal{I}$) and $\mathscr{B}(w)^w \cong S \cong \mathscr{B}^{\mathrm{stab},w}$. Suppose that $\mathscr{B}^{\mathrm{stab}} = \mathcal{C} \oplus \mathcal{D}$, then for what we have just observed, we may take $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C}^x = \mathcal{D}^x = 0$ for all $x > w$, $\mathcal{C}^w \cong S$ and $\mathcal{D}^w = 0$. Let $y \in \mathcal{I}$ be a maximal vertex such that $\mathcal{D}^y \neq 0$. For any $E : y \to z \in \mathcal{E}_{\delta y}$, by definition of Braden-MacPherson sheaf, $\rho_{z,E} : \mathscr{B}(w)^z = \mathscr{B}^{\mathrm{stab},z} = \mathcal{C}^z \to \mathscr{B}^E = \mathscr{B}^{\mathrm{stab},E}$ is surjective with kernel $l(E) \cdot \mathscr{B}^z = l(E)\mathcal{C}^z$ and this implies $\mathcal{D}^E = 0$.

We now localise $\Gamma(\mathscr{B})$ at a finite simple root β, as we have done in §4.3. Remark that, since we are representing the parabolic Bruhat graph using alcoves, we are taking the quotient of $\mathcal{G}^{\mathrm{opp}}$ instead of $\mathcal{G}$. It means that we have to twist the action of S on any vertex x by x^{-1}. However, once the action of the symmetric algebra is twisted, all the results in §4.3 still work in the same way. By combining Theorem 6.3.1 and Lemma 4.3.2 we know that $\mathscr{L}(\Gamma(\mathscr{B})^{\beta})$ is a direct sum of Braden-MacPherson sheaves on certain moment graphs, each one of them isomorphic to a finite interval of the parabolic Bruhat graph for $\widetilde{A_1}$. From the definition of $\mathscr{L}$, it follows that $\mathscr{L}(\Gamma(\mathscr{B}^{\mathrm{stab}})^{\beta}) = (\mathscr{L}(\Gamma(\mathscr{B})^{\beta}))^{\mathrm{stab}}$.

We have already proved that $\rho_{y,E}(\mathcal{D}^y) = 0$ for any $E \in \mathcal{E}_{\delta y} \cap \mathcal{E}_S$ and we want to show that $\rho_{y,E}(\mathcal{D}^y) =$ for any $E \in \mathcal{E}_{\delta y}$. If it were not the case, there would be a non–stable edge $F \in \mathcal{E}_{\delta y} \cap \mathcal{E}_{NS}$ such that $\rho_{y,E}(D^y) \neq 0$. Let $\beta \in \dot{\Delta}_+$ be such that $l(F) = \beta + n\delta$ for some $n \in \mathbb{Z}$. Localising at β, we would get $\rho^{\beta}_{y,F}(\mathcal{D}^{y,\beta}) \neq 0$ and from the $\widetilde{A_1}$ case, it follows that $\rho^{\beta}_{y,E}(\mathcal{D}^{y,\beta}) \neq 0$ for all $E \in \mathcal{E}_{\delta y}$ in β-direction, but we proved that this is not the case.

We are now ready to conclude. From what we showed, it follows that $u_y(\mathcal{C}^y) = \mathscr{B}^{\delta y}$ and this implies $\mathcal{D}^y = 0$, since $(\mathscr{B}^y, u_y)$ is a projective cover of $\mathscr{B}^{\delta y}$.

$\square$

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