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# Sign-changing solutions of the Brezis–Nirenberg problem: asymptotics and existence results.

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# Notations

- $\mathbb{N}$ is the set of natural numbers including 0.
- $\mathbb{N}^+$ is the set of positive natural numbers, namely $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}^+$ is the set of positive real numbers, namely $\mathbb{R}^+ = \{x \in \mathbb{R};\ x > 0\}$.
- $\mathbb{R}^N$ is the product $\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{N \text{ times}}$ (it will be always understood that $N \in \mathbb{N}^+$).
- $\mathbb{R}^N_+$ is the product $\underbrace{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{N \text{ times}}$.
- If $\Omega$ is an open set[1] in $\mathbb{R}^N$ and $\alpha$ is a multi-index, namely $\alpha = (\alpha_1, \cdots, \alpha_N)$, with $\alpha_i \in \mathbb{N}$, we denote by $D^\alpha u$ the partial derivative $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$, where $|\alpha| = \sum_{i=1}^N \alpha_i$ is the order of $\alpha$. If $N = 1$ we use the notation $u'$, $u''$ to denote, respectively, the first and second order derivatives.
- $\nabla u$ denotes the gradient of $u$, namely $\nabla u := \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right)$.
- $\Delta$ denotes the Laplace operator, namely $\Delta u := \displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$.
- $C^k(\Omega)$, $k \in \mathbb{N}$, denotes the set of functions $u : \Omega \to \mathbb{R}$ having all derivatives, of order less or equal to $k$, continuous in $\Omega$.
- $C^k(\overline{\Omega})$, $k \in \mathbb{N}$, denotes the set of functions $u \in C^k(\Omega)$ all of whose derivatives of order less or equal to $k$ have continuous extensions to $\overline{\Omega}$.
- $C_0^\infty(\Omega)$ denotes the set of all infinitely-differentiable functions of compact support in $\Omega$.
- We denote by $|\Omega|$ the Lebesgue measure of $\Omega$ and by $\int_\Omega u\ dx$ the integral of a function $u$ in $\Omega$.
- $L^p(\Omega)$, for $1 \le p < +\infty$, denotes the usual Lebesgue space endowed with the norm $|u|_{p,\Omega} := \left(\int_\Omega |u|^p\ dx\right)^{1/p}$. We will write also $|u|_p$ when the set of integration is understood.
- $L^\infty(\Omega)$ is the Lebesgue space of essentially bounded functions endowed with the norm $|u|_{\infty,\Omega} = \operatorname{ess\,sup}_\Omega |u|$. We will write also $|u|_\infty$ when $\Omega$ is understood.

[1]From now on it will be understood that $\Omega$ is an open set in $\mathbb{R}^N$.

- $H_0^1(\Omega)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|_\Omega = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}.$$

  We will write also $\|u\|$ when $\Omega$ is understood.

- $H^1(\Omega)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|_{1,2,\Omega} = \left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx\right)^{1/2}.$$

  We will write also $\|u\|_{1,2}$ when $\Omega$ is understood.

- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathbb{R}^N} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{1/2}.$$

- If $X$, $Y$ are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear maps from $X$ to $Y$.

# Introduction

In this PhD thesis we give contributions to some questions regarding the asymptotic analysis, the existence and nonexistence of sign-changing solutions for the Brezis–Nirenberg problem.

The Brezis–Nirenberg problem is the following semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$, $\lambda$ is a real constant and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$.

Solutions to Problem 0.0.1 corresponds to critical points of the functional

$$J_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2}\int_\Omega u^2 \, dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} \, dx. \tag{0.0.2}$$

Since the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact there are serious difficulties when trying to find critical points of (0.0.2) with the standard variational methods.

We point out that weak solutions of (0.0.1) are classical solution. This is a consequence of a well-known lemma of Brezis and Kato (see for instance [58]).

Problem 0.0.1 is connected to some variational problems in geometry and physics where lack of compactness also occurs. The most known example is the Yamabe's problem but also (0.0.1) is related to the problem of the existence of extremal functions for isoperimetric inequalities (Hardy–Littlewood–Sobolev inequalities, trace inequalities, see [42], [43]), as well as the existence of non-minimal solutions for Yang-Mills functionals (see [61]). For more examples see [17] and the references therein.

For these reasons Problem 0.0.1 has been widely studied over the last decades, and many results for positive solutions have been obtained.

The first existence result for positive solutions of (0.0.1) has been given by Brezis and Nirenberg in their celebrated paper [17], where, in particular the crucial role played by the dimension was enlightened. In fact they proved that:

(i) if $N \geq 4$ positive solutions exist for every $\lambda \in (0, \lambda_1)$, where $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

(ii) if $N = 3$ there exists $\lambda_* = \lambda_*(\Omega) > 0$ such that positive solutions exist for every $\lambda \in (\lambda_*, \lambda_1)$.

When $\Omega = B \subset \mathbb{R}^3$ is a ball they also proved that $\lambda_*(B) = \lambda_1(B)/4$ and a positive solution of (0.0.1) exists if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. Moreover, for any $N \geq 3$, there are no positive solutions of (0.0.1) if $\lambda \geq \lambda_1$, and, if $\Omega$ is strictly star-shaped, Problem (0.0.1) has no solutions for $\lambda < 0$. Hence, from now on we will assume that $\lambda > 0$.

Concerning the case of sign-changing solutions of (0.0.1), several existence results have been obtained if $N \geq 4$. In this case one can get sign-changing solutions for every $\lambda \in (0, \lambda_1(\Omega))$, or even $\lambda > \lambda_1(\Omega)$ (see [20, 23, 28, 29, 6, 24, 25, 54]). More precisely, Capozzi, Fortunato and Palmieri in [20] showed that for $N = 4$, $\lambda > 0$ and $\lambda \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$ in $H_0^1(\Omega)$) Problem 0.0.1 has a nontrivial solution. The same holds if $N \geq 5$ for all $\lambda > 0$ (see also [35]).

The case $N = 3$ presents the same difficulties enlightened before for positive solutions and even more. In fact, it is not yet known, when $\Omega = B$ is a ball in $\mathbb{R}^3$, if there are non radial sign-changing solutions of (0.0.1) when $\lambda$ is smaller than $\lambda_*(B) = \lambda_1(B)/4$. A partial answer to this question posed by H. Brezis has been given in [14].

However, even in the case $N = 4, 5, 6$, some strange phenomenon appears for what concerns radial sign-changing solutions in the ball. Indeed it was first proved by Atkinson, Brezis and Peletier in [6] that for $N = 4, 5, 6$ there exists $\lambda^* = \lambda^*(N)$ such that there are no sign-changing radial solutions of (0.0.1) for $\lambda \in (0, \lambda^*)$. Later this result was proved in [1] in a different way.

From the nonexistence result of [6] (and [1]) some question arise:

**(Q1)** Is it possible to extend, in some way, this result to other bounded domains? In which sense? What are the solutions which play the same role as the radial nodal solutions in the case of the ball?

Some related results which have connections with these questions were later obtained by Ben Ayed, El Mehdi and Pacella who analyzed the asymptotic behavior of low energy sign-changing solutions of (0.0.1) in general bounded domains $\Omega$ in dimension $N = 3$ (see [14]) and $N \geq 4$ (see [13]) as the parameter $\lambda$ tends to the limit value for which nodal solutions exist, which is a $\bar{\lambda} > 0$, if $N = 3$, and $\bar{\lambda} = 0$, if $N \geq 4$.

More precisely, they studied solutions $u_\lambda$ of (0.0.1) whose energy converges to $2S^{N/2}$ ($S$ is the best Sobolev constant in for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$) and proved that:

(i) if $N = 3$ the positive part $u_\lambda^+$ and the negative part $u_\lambda^-$ blow up and concentrate at two different points of $\Omega$, as $\lambda \to \bar{\lambda}$, having each one the asymptotic profile of a standard "bubble" in $\mathbb{R}^3$ (i.e. of a positive solution of the equation $-\Delta U = U^{2^*-1}$ in $\mathbb{R}^3$), where $\bar{\lambda}$ is the infimum of the values of $\lambda$ for which nodal low-energy solutions exist.

(ii) if $N \geq 4$ and the "concentration speeds" of $u_\lambda^+$ and $u_\lambda^-$ are comparable (i.e. their $L^\infty$-norms blow up with the same rate as $\lambda \to 0$) then again $u_\lambda^+$ and $u_\lambda^-$ concentrate at two different points of $\Omega$, as $\lambda \to 0$, having each one the asymptotic profile of a standard "bubble" in $\mathbb{R}^N$.

Since in ii) it was assumed that $u_\lambda^+$ and $u_\lambda^-$ blow up with the same rate, other questions arise:

**(Q2)** If $N \geq 4$ do there exist low-energy nodal solutions such that $u_\lambda^+$ and $u_\lambda^-$ concentrate and blow-up at the same point, as $\lambda \to 0$? In the affirmative case what is their limit profile? Is there a difference between the case $N = 4, 5, 6$ and the case $N \geq 7$?

In this PhD thesis we give answers to **(Q1)** and **(Q2)**.

First, in order to understand what kind of results we could expect, we have analyzed the asymptotic behavior of sign-changing radial solutions in the ball with two nodal regions, as $\lambda$ goes to some limit value obtained by studying the associated ordinary differential equation. In view of the result of Atkinson, Brezis and Peletier the limit value of the parameter $\lambda$ is a strictly positive real number $\bar{\lambda} = \bar{\lambda}(N)$, if $N = 4, 5, 6$ and it is 0, if $N \geq 7$, according with the existence result of Cerami, Solimini and Struwe (see [25]).

The results obtained from the analysis of radial sign-changing solutions in the ball, are the following:

**(R1)** If $N \geq 7$, and $(u_\lambda)$ is a family of radial sign-changing solutions of least energy (i.e. such that $\|u_\lambda\| \to 2S^{N/2}$, as $\lambda \to 0$) then the positive part and the negative part, $u_\lambda^+$ and $u_\lambda^-$, concentrate and blow-up (with different speeds) at the same point, which is the center of the ball, as $\lambda \to 0$, and each limit profile is that of a "standard bubble" [2] in $\mathbb{R}^N$. In other words the whole solution $u_\lambda$ looks like a "tower of two bubbles".

We point out that this result is the first existence result of "bubble towers" solutions for the Brezis–Nirenberg problem, and it is contained in the paper [37].

In order to understand, in the low dimensions $N = 4, 5, 6$, what kind of asymptotic profile we could expect for radial sign-changing solutions, by studying the associated differential equation (as done by [8, 6]) and reasoning as in [37], we have:

**(R2)** If $3 \leq N \leq 6$, and $(u_\lambda)$ is a family of radial sign-changing solutions of (1.1.2) in the unit ball $B_1$ of $\mathbb{R}^N$, having two nodal regions, such that $u_\lambda(0) > 0$, and denoting by $\bar{\lambda} = \bar{\lambda}(N)$ the limit value of the parameter $\lambda$, then:

(i) if $N = 3$, then $\bar{\lambda} = \frac{9}{4}\lambda_1(B_1)$, where $\lambda_1(B_1)$ is the first eigenvalue of $-\Delta$ in $H_0^1(B_1)$, and $u_\lambda^+$ concentrates and blows-up at the center of the ball having the limit profile of standard bubble in $\mathbb{R}^3$, while $u_\lambda^-$ converges to zero uniformly, as $\lambda \to \bar{\lambda}$.

(ii) if $N = 4, 5$, then $\bar{\lambda} = \lambda_1(B_1)$ and $u_\lambda^+$ , $u_\lambda^-$ behave as in the case $N = 3$.

(iii) if $N = 6$, then $\bar{\lambda} \in (0, \lambda_1(B_1))$ and $u_\lambda^+$ behaves as for $N = 3$ while $u_\lambda^-$ converges to the unique positive radial solution of (0.0.1) in $B_1$, as $\lambda \to \bar{\lambda}$.

In particular, for these dimensions, there cannot exist radial sign-changing solutions in the ball having the asymptotic shape of a "tower of two bubbles".

[2]For $\mu > 0$, $x_0 \in \mathbb{R}^N$, we call "standard bubble" the function $\mathcal{U}_{x_0,\mu} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\mathcal{U}_{x_0,\mu}(x) := \frac{[N(N-2)\mu^2]^{(N-2)/4}}{[\mu^2 + |x - x_0|^2]^{(N-2)/2}}$$

These results are collected in the paper [39].

The proof of the results **(R1)**, **(R2)** are quite technically complicated and often rely on the radial character of the problem. We would like to stress that the presence of the lower-order term $\lambda u$ makes our analysis quite different from that performed in [15] for low-energy sign-changing solution of an almost critical problem.

We also point out that, since we consider nodal solutions, our results cannot be obtained by following the proofs for the case of positive solutions ([7], [8],[36], [52]).

In view of **(R1)** is natural to ask whether solutions of (0.0.1) which behave like the radial ones exist in other bounded domains. In [40], we answer positively this question at least in the case of symmetric domains of $\mathbb{R}^N$, for $N \geq 7$. More precisely, applying a variant of of the Lyapunov-Schmidt reduction method, we show that:

**(R3)** If $N \geq 7$ and $\Omega \subset \mathbb{R}^N$ is any bounded smooth domain which is symmetric with respect to $x_1, \dots, x_N$ and such that the center of symmetry $0 \in \Omega$, then, for any $\lambda$ sufficiently small, there exist sign-changing solutions of Problem 0.0.1 of the form

$$u_\lambda(x) = \alpha_N \left[ \left( \frac{d_{1,\lambda}\lambda^{\frac{1}{N-4}}}{d_{1,\lambda}^2 \lambda^{\frac{2}{N-4}} + |x|^2} \right)^{\frac{N-2}{2}} - \left( \frac{d_{2,\lambda}\lambda^{\frac{3N-10}{(N-4)(N-6)}}}{d_{2,\lambda}^2 \lambda^{2\frac{3N-10}{(N-4)(N-6)}} + |x|^2} \right)^{\frac{N-2}{2}} \right] + \Phi_\lambda, \quad (0.0.3)$$

where $d_{i,\lambda} \to \bar{d}_i$, for $i = 1, 2$, as $\lambda \to 0$ and $\Phi_\lambda$ is such that $\Phi_\lambda \to 0$ in $H^1(\Omega)$, as $\lambda \to 0$.

We point out that if one applies directly the finite dimensional reduction method, looking for a sign-changing bubble tower solution of the form 0.0.3 for Problem 0.0.1, then, when solving the associated finite dimensional problem one finds that the reduced functional has not a critical point (see Section 1 of Chapter 3). In order to overcome this difficulty, we have introduced a new idea based on the splitting of the remainder term in two parts. Usually the remainder term is found by solving an infinite dimensional problem, called "the auxiliary equation", here, we look for a remainder term which is the sum of two remainder terms, of different orders. These two functions are found by solving a system of two equation, which is obtained by splitting the the auxiliary equation in an appropriate way. At the end, when solving the finite dimensional problem, we get a reduced functional which has a critical point.

We also observe that the symmetry assumption for $\Omega$ is due only in order to simplify the computations which however, even in the symmetric context, are long and tough. But there is no reason, a priori, for which the previous result should not hold in general domains. Hence, reasoning as in [48], we believe that the symmetry assumption could be removed.

We observe that, in **(R3)**, the assumption $N \geq 7$ on the dimension is not only technically crucial but it also is indeed necessary. In fact, by applying a Pohozaev's identity and fine estimates, we prove the following result:

**(R4)** If $N = 4, 5, 6$, for any smooth bounded domain $\Omega \subset \mathbb{R}^N$, for any $\xi \in \Omega$, then, there cannot exist solutions $u_\lambda$ of Problem 0.0.1 of the form

$$u_\lambda = \mathcal{U}_{\xi,\delta_1} - \mathcal{U}_{\xi,\delta_2} + \Phi_\lambda,$$

where $\delta_i = \delta_i(\lambda)$ for $i = 1, 2$, are such that $\delta_1 \to 0$ and $\delta_2 = o(\delta_1)$, as $\lambda \to 0$, and $\Phi_\lambda$ is such that $\Phi_\lambda \to 0$ in $H^1(\Omega)$, $|\Phi_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$, $|\nabla\Phi_\lambda| = o(\delta_1^{-\frac{N}{2}})$ uniformly in compact subsets of $\Omega$ as $\lambda \to 0$.

This result is hence the counterpart of the nonexistence theorem of Atkinson, Brezis and Peletier if we think "bubble tower" solutions as the solutions which play the role, in general bounded domains, of the radial ones in the ball. This result is contained in [38].

In view of **(R2)** and in order to complete our analysis it would be interesting, for the dimensions $N = 4, 5, 6$, to show that sign-changing solutions of Problem 0.0.1, having an asymptotic profile similar to that of radial ones in the ball, exist in general bounded domains. This is the content of a paper in preparation [41] in which we deal with the cases $N = 4, 5$. By applying a variant of the Lyapunov-Schmidt reduction method we prove the following:

**(R5)** Let $\Omega$ be a smooth bounded domain, which is symmetric with respect to $x_1, \dots, x_N$ and such that the center of symmetry $0 \in \Omega$, then:

(i) if $N = 4$ there exists $\epsilon_0$ such that such that for any $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega) + \epsilon_0)$ there exists a sign-changing solution $u_\lambda$ of (0.0.1) of the form

$$u_\lambda(x) = \left[\alpha_4 \frac{e^{-\frac{1}{\lambda-\lambda_1}} d_{1,\lambda}}{e^{-\frac{2}{\lambda-\lambda_1}} d_{1,\lambda}^2 + |x|^2} - e^{-\frac{1}{\lambda-\lambda_1}} (\lambda - \lambda_1) d_{2,\lambda} e_1\right] + \Phi_\lambda(x) \qquad (0.0.4)$$

where $\alpha_4 = 2\sqrt{2}$, $d_{j,\lambda}$ is a positive function depending on $\lambda$ such that $d_{j,\lambda} \to \bar{d}_j > 0$, for $j = 1, 2$, as $\lambda \to \lambda_1^+$ and $\Phi_\lambda$ is such that $\Phi_\lambda \to 0$ in $H^1(\Omega)$, as $\lambda \to \lambda_1^+$.

(ii) if $N = 5$ there exists $\epsilon_0$ such that such that for any $\lambda \in (\lambda_1(\Omega) - \epsilon_0, \lambda_1(\Omega))$ there exists a sign-changing solution $u_\lambda$ of (0.0.1) of the form

$$u_\lambda(x) = \left[\alpha_5 \left(\frac{(\lambda_1 - \lambda)^{\frac{3}{2}} d_{1,\lambda}}{(\lambda_1 - \lambda)^2 d_{1,\lambda}^2 + |x|^2}\right)^{\frac{3}{2}} - (\lambda_1 - \lambda)^{\frac{3}{4}} d_{2,\lambda} e_1\right] + \Phi_\lambda(x) \qquad (0.0.5)$$

where $\alpha_5 = 15\sqrt{15}$, $d_{j,\lambda}$ is a positive function depending on $\lambda$ such that $d_{j,\lambda} \to \bar{d}_j > 0$, for $j = 1, 2$ as $\lambda \to \lambda_1^-$, and $\Phi_\lambda$ is such that $\Phi_\lambda \to 0$ in $H^1(\Omega)$, as $\lambda \to \lambda_1^-$.

We point out that **(R5)** agrees with the results of Arioli, Gazzola, Grunau, Sassone [4] and Gazzola, Grunau [32]. In fact, analyzing radial sign-changing solutions of (0.0.1) in the unit ball $B_1$, having two nodal regions, they proved that if $N = 4$ then $\lambda_1(B_1)$ is reached from above, while, if $N = 5$ then $\lambda_1(B_1)$ is reached from below. Actually, in the case $N = 4$, Arioli, Gazzola, Grunau, Sassone in [4] proved more: they proved that there are no radial sign-changing solutions in the ball for Problem 0.0.1 when $\lambda \leq \lambda_1(B_1)$. Hence, in view of this result, it would be interesting to prove that, for a generic bounded domain $\Omega$, sign-changing solutions of the form (0.0.4), cannot exist for $\lambda \leq \lambda_1(\Omega)$.

Moreover, we observe that the principal part of the solutions obtained in **(R5)** have a negative part which converges in compact subsets of $\Omega \setminus \{0\}$ (up to multiplication by a vanishing function, as $\lambda \to \lambda_1$) to the first (normalized, positive) eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. Hence, if we could prove that the remainder term $\Phi_\lambda$ is sufficiently small also in the $L^\infty$-norm, as $\lambda \to \lambda_1$, we could exhibit a family of solutions which verifies a conjecture made by Atkinson, Brezis and Peletier in [6]. Indeed, they conjectured that, for $N = 4, 5$, if $(u_\lambda)$ is a family of radial sign-changing solutions of (0.0.1) in the unit ball of $\mathbb{R}^N$, with two nodal regions and such that $u_\lambda(0) > 0$, then, $u_\lambda^-$ converges in compact subsets of $B_1 \setminus \{0\}$, as $\lambda \to \lambda_1$, and up to multiplication to a vanishing function as $\lambda \to \lambda_1$, to the first eigenfunction of $-\Delta$ in the unit ball. We believe that this could be achieved by working with weighted norms, as done in [27, 49].

The remaining case $N = 6$ has been only partially investigated and it is very interesting. In fact, $N = 6$, is a sort of "borderline" dimension, since the corresponding solutions have an asymptotic profile which is the middle between the distinct behaviors seen in **(R1)** and **(R2)**. This peculiarity is reflected also on the technical side since, even in the radial context, there are difficulties when trying to get asymptotic results. By the way, in [39] we have given a characterization of the limit value $\bar{\lambda}$ which seems to be promising.

We conclude observing that these "bubble tower" solutions, found in **(R1)**, **(R3)**, have interest also for the associated parabolic problem, since, as proved in [44], [22], [30], they induce a peculiar blow-up phenomenon for the initial data close to them.

We now briefly describe the content of the thesis. We refer to the first section of each chapter for a detailed description and for the statements of our results.

- **Chapter 1** is devoted to the proof of **(R1)** (see Theorem 1.1.1) and other related asymptotic results concerning radial sign-changing solutions for (0.0.1), when $N \geq 7$, as $\lambda \to 0$. In particular, in Theorem 1.1.2 we determine the blow-up rate of $\|u_\lambda^-\|_\infty$, as $\lambda \to 0$, and get an asymptotic relation between $\|u_\lambda^+\|_\infty$, $\lambda$ and the node $r_\lambda$. Moreover, in Theorem 1.1.3, we show that, up to a positive constant, $u_\lambda$ converges, in $C^1_{loc}(B_1 \setminus \{0\})$, as $\lambda \to 0$, to $G(\cdot, 0)$, where $G = G(x, y)$ is the Green function of $\Delta$ for the unit ball $B_1$ of $\mathbb{R}^N$.

- **Chapter 2** is devoted to the proof of **(R2)** (see Theorem 2.3.2, Theorem 2.4.7, Theorem 2.5.1) and other related asymptotic results concerning radial sign-changing solutions for (0.0.1), with two nodal regions in the ball, when $3 \leq N \leq 6$, as $\lambda \to \bar{\lambda}$, where $\bar{\lambda} = \bar{\lambda}(N) > 0$ is some limit value obtained by analyzing the ordinary differential equation. We also determine (if $N = 3, 4, 5$) and estimate (if $N = 6$) the limit energy of these solutions (see Theorem 2.3.2 and Theorem 2.5.1 and Corollary 2.4.9). In Proposition 2.4.2 and in Theorem 2.5.1 we prove that there cannot exist, in the low dimensions $N = 3, 4, 5, 6$, radial sign-changing solutions of Problem 0.0.1, in the ball, having the asymptotic shape of a "tower of two bubbles" (see also Remark 2.4.3). In Theorem 2.4.8 we give a characterization of the limit value $\bar{\lambda}$ for $N = 6$.

- **Chapter 3** is devoted to the proof of **(R3)** (see Theorem 3.1.1). Moreover, in Theorem 3.1.2, under some additional assumptions, we prove that the solutions obtained in Theorem 3.1.1, have the property that their nodal set does not touch the boundary of $\Omega$, as $\lambda \to 0$.

- **Chapter 4** is devoted to the proof of **(R4)** (see Theorem 4.1.1). In Theorem 4.5.1 we prove that if $N \geq 7$ when considering sign-changing bubble-tower solutions of

Problem 0.0.1, then the concentration speeds found in **(R3)** are the only possible ones.

# Chapter 1

# Asymptotic analysis for radial sign-changing solutions of the Brezis-Nirenberg problem, $N \geq 7$.

## 1.1 Introduction

Here we present and prove the result **(R1)**.

Let $N \geq 3$, $\lambda > 0$ and $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with smooth boundary. We consider the Brezis–Nirenberg problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$. Problem (1.1.1) has been widely studied over the last decades, and many results for positive solutions have been obtained.

The first existence result for positive solutions of (1.1.1) has been given by Brezis and Nirenberg in their classical paper [17], where, in particular, the crucial role played by the dimension was enlightened. They proved that if $N \geq 4$ there exist positive solutions of (1.1.1) for every $\lambda \in (0, \lambda_1(\Omega))$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ on $\Omega$ with zero Dirichlet boundary condition. For the case $N = 3$, which is more delicate, Brezis and Nirenberg [17] proved that there exists $\lambda_*(\Omega) > 0$ such that positive solutions exist for every $\lambda \in (\lambda_*(\Omega), \lambda_1(\Omega))$. When $\Omega = B$ is a ball, they also proved that $\lambda_*(B) = \frac{\lambda_1(B)}{4}$ and a positive solution of (1.1.1) exists if and only if $\lambda \in (\frac{\lambda_1(B)}{4}, \lambda_1(B))$. Moreover, for more general bounded domains, they proved that if $\Omega \subset \mathbb{R}^3$ is strictly star-shaped about the origin, there are no positive solutions for $\lambda$ close to zero. We point out that weak solutions of (1.1.1) are classical solution. This is a consequence of a well-known lemma of Brezis and Kato (see for instance Appendix B of [58]).

The asymptotic behavior for $N \geq 4$, as $\lambda \to 0$, of positive solutions of (1.1.1), minimizing the Sobolev quotient, has been studied by Han [36], Rey [52]. They showed, with different proofs, that such solutions blow up at exactly one point, and they also determined the exact blow-up rate as well as the location of the limit concentration points.

Concerning the case of sign-changing solutions of (1.1.1), several existence results have been obtained if $N \geq 4$. In this case, one can get sign-changing solutions for every $\lambda \in (0, \lambda_1(\Omega))$, or even $\lambda > \lambda_1(\Omega)$, as shown in the papers of Atkinson–Brezis–Peletier [6], Clapp–Weth [24], Capozzi–Fortunato–Palmieri [20]. The case $N = 3$ presents the same difficulties enlightened before for positive solutions and even more. In fact, differently from the case of positive solutions, it is not yet known, when $\Omega = B$ is a ball in $\mathbb{R}^3$, if there are sign-changing solutions of (1.1.1) when $\lambda$ is smaller than $\lambda_*(B) = \lambda_1(B)/4$. A partial answer to this question posed by H. Brezis has been given in [14].

The blow-up analysis of low-energy sign-changing solutions of (1.1.1) has been done by Ben Ayed–El Mehdi–Pacella [14], [13]. In [14] the authors analyze the case $N = 3$. They introduce the number defined by

$$\begin{aligned}\bar\lambda(\Omega) \ := \ \inf\big\{\lambda \in \mathbb{R}^+;\ &\text{Problem (1.1.1) has a sign-changing solution } u_\lambda,\\ &\text{with } \|u_\lambda\|^2_\Omega - \lambda|u_\lambda|^2_{2,\Omega} \leq 2S^{3/2}\big\},\end{aligned}$$

where $\|u_\lambda\|^2_\Omega = \int_\Omega |\nabla u_\lambda|^2\, dx$, $|u_\lambda|^2_{2,\Omega} = \int_\Omega |u_\lambda|^2\, dx$ and $S$ is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. To be precise, they study the behavior of sign-changing solutions of (1.1.1) which converge weakly to zero and whose energy converges to $2S^{3/2}$ as $\lambda \to \bar\lambda(\Omega)$. They prove that these solutions blow up at two different points $\bar a_1$, $\bar a_2$, which are the limit of the concentration points $a_{\lambda,1}$, $a_{\lambda,2}$ of the positive and negative part of the solutions. Moreover, the distance between $a_{\lambda,1}$ and $a_{\lambda,2}$ is bounded from below by a positive constant depending only on $\Omega$ and the concentration speeds of the positive and negative parts are comparable. This result shows that, in dimension 3, there cannot exist, in any bounded smooth domain $\Omega$, sign-changing low-energy solutions whose positive and negative part concentrate at the same point.

In higher dimensions ($N \geq 4$), the same authors, in their paper [13], describe the asymptotic behavior, as $\lambda \to 0$, of sign-changing solutions of (1.1.1) whose energy converges to the value $2S^{N/2}$. Even in this case, they prove that the solutions concentrate and blow up at two separate points, but they need to assume the extra hypothesis that the concentration speeds of the two concentration points are comparable, while in dimension three, this was derived without any extra assumption (see Theorem 4.1 in [14]). They also describe in [13] the asymptotic behavior, as $\lambda \to 0$, of the solutions outside the limit concentration points proving that there exist positive constants $m_1, m_2$ such that

$$\lambda^{-\frac{N-2}{2N-8}} u_\lambda \to m_1 G(x, \bar a_1) - m_2 G(x, \bar a_2) \ \text{ in } C^2_{loc}(\Omega - \{\bar a_1, \bar a_2\}),\ \text{if } N \geq 5,$$

$$\|u_\lambda\|_\infty u_\lambda \to m_1 G(x, \bar a_1) - m_2 G(x, \bar a_2) \ \text{ in } C^2_{loc}(\Omega - \{\bar a_1, \bar a_2\}),\ \text{if } N = 4,$$

where $G(x, y)$ is the Green's function of the Laplace operator in $\Omega$. So for $N \geq 4$ the question of proving the existence of sign-changing low-energy solutions (i.e., such that $\|u_\lambda\|^2_\Omega$ converges to $2S^{N/2}$ as $\lambda \to 0$) whose positive and negative part concentrate and blow up at the same point was left open.

To the aim to contribute to this question as well as to describe the precise asymptotic behavior of radial sign-changing solutions, we consider the Brezis–Nirenberg problem in the unit ball $B_1$, i.e.,

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } B_1,\\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1.2}$$

It is important to recall that Atkinson–Brezis–Peletier [5], Adimurthi–Yadava [1] showed, with different proofs, that for $N = 3, 4, 5, 6$ there exists $\lambda^* = \lambda^*(N) > 0$ such that there is no radial sign-changing solution of (1.1.1) for $\lambda \in (0, \lambda^*)$. Instead, they do exist if $N \geq 7$, as shown by Cerami–Solimini–Struwe in their paper [25]. In Proposition

1.2.1 (see also Remark 1.2.2) we recall this existence result and get the limit energy of such solutions as $\lambda \to 0$.

In view of these results, we analyze the case $N \geq 7$ and $\lambda \to 0$. More precisely, we consider a family $(u_\lambda)$ of least energy sign-changing solutions of (0.0.1). It is easy to see that $u_\lambda$ has exactly two nodal regions. We denote by $r_\lambda \in (0,1)$ the node of $u_\lambda = u_\lambda(r)$ and, without loss of generality, we assume $u_\lambda(0) > 0$, so that $u_\lambda^+$ is different from zero in $B_{r_\lambda}$ and $u_\lambda^-$ is different from zero in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^N;\ r_\lambda < |x| < 1\}$, where $u_\lambda^+ := \max(u_\lambda, 0)$, $u_\lambda^- := \max(0, -u_\lambda)$ are, respectively, the positive and the negative part of $u_\lambda$.

We set $M_{\lambda,+} := \|u_\lambda^+\|_\infty$, $M_{\lambda,-} := \|u_\lambda^-\|_\infty$, $\beta := \frac{2}{N-2}$, $\sigma_\lambda := M_{\lambda,+}^\beta r_\lambda$, $\rho_\lambda := M_{\lambda,-}^\beta r_\lambda$. Moreover, for $\mu > 0$, $x_0 \in \mathbb{R}^N$, let $\mathcal{U}_{x_0,\mu}$ be the function $\mathcal{U}_{x_0,\mu} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\mathcal{U}_{x_0,\mu}(x) := \frac{[N(N-2)\mu^2]^{(N-2)/4}}{[\mu^2 + |x - x_0|^2]^{(N-2)/2}}. \tag{1.1.3}$$

Proposition 1.3.1 states that both $M_{\lambda,+}$ and $M_{\lambda,-}$ diverge, $u_\lambda$ weakly converge to 0 and $\|u_\lambda^\pm\|_{B_1}^2 \to S^{N/2}$, as $\lambda \to 0$. The main results of this chapter are the following:

**Theorem 1.1.1.** *Let $N \geq 7$ and $(u_\lambda)$ be a family of least energy radial sign-changing solutions of (1.1.2) (i.e. $\|u_\lambda^\pm\|_{B_1}^2 \to S^{N/2}$, as $\lambda \to 0$ ) and $u_\lambda(0) > 0$. Consider the rescaled functions $\tilde{u}_\lambda^+(y) := \frac{1}{M_{\lambda,+}} u_\lambda^+ \left( \frac{y}{M_{\lambda,+}^\beta} \right)$ in $B_{\sigma_\lambda}$, and $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^- \left( \frac{y}{M_{\lambda,-}^\beta} \right)$ in $A_{\rho_\lambda}$, where $B_{\sigma_\lambda} := M_{\lambda,+}^\beta B_{r_\lambda}$, $A_{\rho_\lambda} := M_{\lambda,-}^\beta A_{r_\lambda}$. Then:*

**(i)** *$\tilde{u}_\lambda^+ \to \mathcal{U}_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^N)$ as $\lambda \to 0$, where $\mathcal{U}_{0,\mu}$ is the function defined in (1.1.3) for $\mu = \sqrt{N(N-2)}$.*

**(ii)** *$\tilde{u}_\lambda^- \to \mathcal{U}_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^N - \{0\})$ as $\lambda \to 0$, where $\mathcal{U}_{0,\mu}$ is the same as in (i).*

From this theorem, we deduce that the positive and negative parts of $u_\lambda$ concentrate at the origin. Moreover, as a consequence of the preliminary results for the proof of Theorem 1.1.1, we show that $M_{\lambda,+}$ and $M_{\lambda,-}$ are not comparable, i.e., $\frac{M_{\lambda,+}}{M_{\lambda,-}} \to +\infty$ as $\lambda \to 0$, which implies that the speed of concentration and blowup of $u_\lambda^+$ and $u_\lambda^-$ are not the same, and hence, the asymptotic profile of $u_\lambda$ is that of a tower of two "bubbles." Indeed, we are able to determine the exact rate of $M_{\lambda,-}$ and an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and the radius $r_\lambda$ (see also Remark 1.5.5).

**Theorem 1.1.2.** *As $\lambda \to 0$ we have the following:*

**(i)** $M_{\lambda,+}^{2-2\beta} r_\lambda^{N-2} \lambda \to c(N)$,

**(ii)** $M_{\lambda,-}^{2-2\beta} \lambda \to c(N)$,

**(iii)** $\dfrac{M_{\lambda,-}^{2-2\beta}}{M_{\lambda,+}^{2-2\beta} r_\lambda^{N-2}} \to 1$,

*where $c(N) := \frac{c_1^2(N)}{c_2(N)}$, $c_1(N) := \int_0^\infty \mathcal{U}_{0,\mu}^{2^*-1}(s) s^{N-1} ds$, $c_2(N) := 2\int_0^\infty \mathcal{U}_{0,\mu}^2(s) s^{N-1} ds$, $\mu = \sqrt{N(N-2)}$.*

The last result we provide in this chapter is about the asymptotic behavior of the functions $u_\lambda$ in the ball $B_1$, outside the origin. We show that, up to a constant, $\lambda^{-\frac{N-2}{2N-8}} u_\lambda$ converges in $C^1_{loc}(B_1 - \{0\})$ to $G(x,0)$, where $G(x,y)$ is the Green function of the Laplace operator in $B_1$.

**Theorem 1.1.3.** *As $\lambda \to 0$ we have*

$$\lambda^{-\frac{N-2}{2N-8}} u_\lambda \to \tilde{c}(N) G(x,0) \quad \text{in } C^1_{loc}(B_1 - \{0\}),$$

*where $G(x,y)$ is the Green function for the Laplacian in the unit ball, $\tilde{c}(N)$ is the constant defined by $\tilde{c}(N) := \omega_N \frac{c_2(N)^{\frac{N-2}{2N-8}}}{c_1(N)^{\frac{4}{2N-8}}}$, $\omega_N$ is the measure of the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ and $c_1(N)$, $c_2(N)$ are the constants appearing in Theorem 1.1.2.*

We point out that in order to analyze the behavior of the negative part $u_\lambda^-$, which is defined in an annulus, we prove a new uniform estimate (see Propositions 1.4.7), which is of its own interest.

For the sake of completeness, let us mention that our results, as well as those of [15], show a big difference between the asymptotic behavior of radial sign-changing solutions in dimension $N > 2$ and $N = 2$. Indeed, in this last case, the limit problems as well as the limit energies of the positive and negative part of solutions are different (see [34]).

We conclude observing that with similar proofs, it is possible to extend our results to the case of radial sign-changing solutions of (1.1.2) with $k$ nodal regions, $k > 2$, and such that $\|u_\lambda\|^2_{B_1} \to kS^{N/2}$, as $\lambda \to 0$. As expected, the limit profile will be that of a tower of $m$ bubbles with alternating signs. Moreover, with the same methods applied here, we can deduce analogous asymptotic relations as those of Theorem 1.1.2.

The chapter is divided into 6 sections. In Sect. 1.2, we give some preliminary results on radial sign-changing solutions. In Section 1.3, we prove estimates for solutions with two nodal regions and, in particular, prove the new uniform estimate of Proposition 1.4.7.

In Sect. 1.4, we analyze the asymptotic behavior of the rescaled solutions and prove Theorem 1.1.1. Section 1.5 is devoted to the study of the divergence rate of $\|u_\lambda^\pm\|_\infty$, as $\lambda \to 0$ and to the proof of Theorem 1.1.2. Finally, in Sect. 1.6, we prove Theorem 1.1.3.

## 1.2 Preliminary results on radial sign-changing solutions

In this section, we recall or prove some results about the existence and qualitative properties of radial sign-changing solutions of the Brezis–Nirenberg problem (1.1.2).

We start with the following:

**Proposition 1.2.1.** *Let $N \geq 7$, $k \in \mathbb{N}^+$ and $\lambda \in (0, \lambda_1)$, where $\lambda_1 = \lambda_1(B_1)$ is the first eigenvalue of $-\Delta$ in $H^1_0(B_1)$. Then, there exists a radial sign-changing solution $u_{k,\lambda}$ of (1.1.2) with the following properties:*

**(i)** $u_{k,\lambda}(0) > 0$,

**(ii)** *$u_{k,\lambda}$ has exactly $k$ nodal regions in $B_1$,*

**(iii)** $I_\lambda(u_{k,\lambda}) = \frac{1}{2}\left(\int_{B_1} |\nabla u_{k,\lambda}|^2 - \lambda |u_{k,\lambda}|^2 \, dx\right) - \frac{1}{2^*}\int_{B_1} |u_{k,\lambda}|^{2^*} \, dx \to \frac{k}{N} S^{N/2}$ *as $\lambda \to 0$, where $S$ is the best constant for the Sobolev embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$.*

*Proof.* The existence of radial solutions of (1.1.2) satisfying (i) and (ii) is proved in [25]. It remains only to prove (iii). To do this, we need to introduce some notations and recall some facts proved in [25] and [17]. Let $k \in \mathbb{N}^+$ and $0 = r_0 < r_1 < \ldots < r_k = 1$ any partition of the interval $[0,1]$, we define the sets $\Omega_1 := B_{r_1} = \{x \in B_1; |x| < r_1\}$ and, if $k \geq 2$, $\Omega_j := \{x \in B_1; r_{j-1} < |x| < r_j\}$ for $j = 2, \ldots, k$.

Then, we consider the set

$$\mathcal{M}_{k,\lambda} := \Big\{u \in H^1_{0,rad}(B_1);\ \text{there exists a partition } 0 = r_0 < r_1 < \ldots < r_k = 1 \text{ such that: } u(r_j) = 0, \text{for } 1 \le j \le k,\ (-1)^{j-1}u(x) \ge 0, u \not\equiv 0 \text{ in } \Omega_j, \text{ and } \int_{\Omega_j} \left(|\nabla u_j|^2 - u_j^2 - |u_j|^{2^*}\right)\, dx = 0, \text{ for } 1 \le j \le k\Big\},$$

where $H^1_{0,rad}(B_1)$ is the subspace of the radial functions in $H^1_0(B_1)$ and $u_j$ is the function defined by $u_j := u\ \chi_{\Omega_j}$, where $\chi_{\Omega_j}$ denotes the characteristic function of $\Omega_j$. Note that for any $k \in \mathbb{N}^+$ we have $\mathcal{M}_{k,\lambda} \neq \emptyset$, so we define

$$c_k(\lambda) := \inf_{\mathcal{M}_{k,\lambda}} I_\lambda(u).$$

In [25] the authors prove, by induction on $k$, that for every $k \in \mathbb{N}^+$ there exists $u_{k,\lambda} \in \mathcal{M}_{k,\lambda}$ such that $I_\lambda(u_{k,\lambda}) = c_k(\lambda)$ and $u_{k,\lambda}$ solves (1.1.2) in $B_1$. Moreover, they prove that

$$c_{k+1}(\lambda) < c_k(\lambda) + \frac{1}{N} S^{N/2}. \tag{1.2.1}$$

Note that for $k = 1$ $u_{1,\lambda}$ is just the positive solution found in [17], since by the Gidas, Ni and Nirenberg symmetry result [33] every positive solution is radial, and from [2] or [56] we know that positive solutions of (1.1.2) are unique.

To prove (iii) we argue by induction. Since $c_1(0) = \frac{1}{N} S^{N/2}$, by continuity we get that $c_1(\lambda) \to \frac{1}{N} S^{N/2}$, as $\lambda \to 0$, so that (iii) holds for $k = 1$.

Now assume that $c_k(\lambda) \to \frac{k}{N} S^{N/2}$, and let us to prove that $c_{k+1}(\lambda) = I_\lambda(u_{k+1,\lambda}) \to \frac{k+1}{N} S^{N/2}$.

Let us observe that $c_{k+1}(\lambda) \ge (k+1)c_1(\lambda)$. In fact, $w := u_{k+1,\lambda}$ achieves the minimum for $I_\lambda$ over $\mathcal{M}_{k+1,\lambda}$, so that, by definition, it has $k+1$ nodal regions and $w_j := w\chi_{\Omega_j}$ belongs to $H^1_{0,rad}(B_1)$ for all $j = 1, \ldots, k+1$. Since $w \in \mathcal{M}_{k+1,\lambda}$ we have, depending on the parity of $j$, that one between $w_j^+$ and $w_j^-$ is not zero and belongs to $\mathcal{M}_{1,\lambda}$, we denote it by $\tilde{w}_j$. Then, $I_\lambda(\tilde{w}_j) \ge c_1(\lambda)$ for all $j = 1, \ldots, k+1$ and hence

$$c_{k+1}(\lambda) = I_\lambda(w) = \sum_{j=1}^{k+1} I_\lambda(w_j^\pm) \ge (k+1)c_1(\lambda).$$

Combining this with (1.2.1) we get

$$c_k(\lambda) + \frac{1}{N} S^{N/2} > c_{k+1}(\lambda) \ge (k+1)c_1(\lambda).$$

Since by induction hypothesis $c_k(\lambda) \to \frac{k}{N} S^{N/2}$ as $\lambda \to 0$ and we have proved that $c_1(\lambda) \to \frac{1}{N} S^{N/2}$ we get that $c_{k+1}(\lambda) \to \frac{k+1}{N} S^{N/2}$, and the proof is concluded. □

**Remark 1.2.2.** *Let $k \in \mathbb{N}^+$ and $(u_\lambda)$ be a family of solutions of (1.1.2), satisfying (iii) of Proposition 1.2.1, then $\|u_\lambda\|^2_{B_1} = \int_{B_1} |\nabla u_\lambda|^2\, dx \to kS^{N/2}$, as $\lambda \to 0$.*

This comes easily from Proposition 1.2.1, and the fact that $u_\lambda$ belongs to the Nehari manifold $\mathcal{N}_\lambda$ associated with (1.1.2), which is defined by

$$\mathcal{N}_\lambda := \{u \in H^1_0(B_1);\ \|u\|^2_{B_1} - \lambda |u|^2_{2,B_1} = |u|^{2^*}_{2^*,B_1}\}.$$

The first qualitative property we state about any radial sign-changing solution $u_\lambda$ of (1.1.2) is that the global maximum point of $|u_\lambda|$ is located at the origin, which is a well-known fact for positive solutions of (1.1.2), as consequence of [33].

**Proposition 1.2.3.** *Let $u_\lambda$ be a radial solution of (1.1.2), then we have $|u_\lambda(0)| = \|u_\lambda\|_\infty$.*

*Proof.* Since $u_\lambda = u_\lambda(r)$ is a radial solution of (1.1.2), then it solves

$$\begin{cases} u_\lambda'' + \frac{N-1}{r} u_\lambda' + \lambda u_\lambda + |u_\lambda|^{2^*-2} u_\lambda = 0 & \text{in } (0,1) \\ u_\lambda'(0) = 0, \quad u_\lambda(1) = 0. \end{cases} \tag{1.2.2}$$

Multiplying the equation by $u_\lambda'$ we get

$$u_\lambda'' u_\lambda' + \lambda u_\lambda u_\lambda' + |u_\lambda|^{2^*-2} u_\lambda u_\lambda' = -\frac{N-1}{r}(u_\lambda')^2 \leq 0.$$

We rewrite this as

$$\frac{d}{dr}\left[\frac{(u_\lambda')^2}{2} + \lambda\frac{u_\lambda^2}{2} + \frac{|u_\lambda|^{2^*}}{2^*}\right] \leq 0.$$

Which implies that the function

$$E(r) := \frac{(u_\lambda')^2}{2} + \lambda\frac{u_\lambda^2}{2} + \frac{|u_\lambda|^{2^*}}{2^*}$$

is not increasing. So $E(0) \geq E(r)$ for all $r \in (0,1)$, where $E(0) = \lambda\frac{(u_\lambda(0))^2}{2} + \frac{|u_\lambda(0)|^{2^*}}{2^*}$. Assume that $r_0 \in (0,1)$ is the global maximum for $|u_\lambda|$, so we have $u_\lambda'(r_0) = 0$, $|u_\lambda(r_0)| = \|u_\lambda\|_\infty$ and $E(r_0) = \lambda\frac{\|u_\lambda\|_\infty^2}{2} + \frac{\|u_\lambda\|_\infty^{2^*}}{2^*}$.

Now we observe that, for all $\lambda > 0$, the function $g(x) := \frac{\lambda}{2}x^2 + \frac{1}{2^*}x^{2^*}$, defined in $\mathbb{R}^+ \cup \{0\}$, is strictly increasing; thus, we have $E(r_0) \geq E(0)$ and hence, $E(r_0) = E(0)$. Since $g$ is strictly increasing, we get $|u_\lambda(0)| = |u_\lambda(r_0)| = \|u_\lambda\|_\infty$ and we are done. □

A consequence of the previous proposition is the following:

**Corollary 1.2.4.** *Assume $u_\lambda$ is a nontrivial radial solution of (1.1.2). If $0 \leq r_1 \leq r_2 < 1$ are two points in the same nodal region such that $|u_\lambda(r_1)| \leq |u_\lambda(r_2)|$, $u_\lambda'(r_1) = u_\lambda'(r_2) = 0$, then necessarily $r_1 = r_2$.*

*Proof.* Assume by contradiction $r_1 < r_2$. By the assumptions and since the function $g(x) := \frac{\lambda}{2}x^2 + \frac{1}{2^*}x^{2^*}$ is a strictly increasing function (in $\mathbb{R}^+ \cup \{0\}$), we have $E(r_1) = g(|u_\lambda(r_1)|) \leq g(|u_\lambda(r_2)|) = E(r_2)$. But, as proved in Proposition 1.2.3, $E(r)$ is a decreasing function, so necessarily $E(r_1) = g(|u_\lambda(r_1)|) = g(|u_\lambda(r_2)|) = E(r_2)$ from which we get $|u_\lambda(r_1)| = |u_\lambda(r_2)|$. Since $r_1, r_2$ are in the same nodal region from $|u_\lambda(r_1)| = |u_\lambda(r_2)|$ we have $u_\lambda(r_1) = u_\lambda(r_2)$, and thus, there exists $r_* \in (r_1, r_2)$ such that $u_\lambda'(r_*) = 0$, and, since $E(r)$ is a decreasing function, we have $E(r_1) \geq E(r_*) \geq E(r_2)$. From this, we deduce $g(|u_\lambda(r_1)|) \geq g(|u_\lambda(r_*)|) \geq g(|u_\lambda(r_2)|)$, and hence, $u_\lambda(r_1) = u_\lambda(r_*) = u_\lambda(r_2)$. Therefore, $u_\lambda$ must be constant in the interval $[r_1, r_2]$ and, being a solution of (1.1.2), it must be zero in that interval. In fact, since (1.1.2) is invariant under a change of sign, we can assume that $u_\lambda \equiv c > 0$. Then, by the strong maximum principle, $u_\lambda$ must be zero in the nodal region to which $r_1$, $r_2$ belong. This, in turn, implies that $u_\lambda$ is a trivial solution of (1.1.2) which is a contradiction. □

## 1.3 Asymptotic results for solutions with 2 nodal regions

### 1.3.1 General results

Let $(u_\lambda)$ be a family of least energy radial, sign-changing solutions of (1.1.2) and such that $u_\lambda(0) > 0$.

We denote by $r_\lambda \in (0,1)$ the node, so we have $u_\lambda > 0$ in the ball $B_{r_\lambda}$ and $u_\lambda < 0$ in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^N; r_\lambda < |x| < 1\}$. We write $u_\lambda^\pm$ to indicate that the statements hold both for the positive and negative part of $u_\lambda$.

**Proposition 1.3.1.** *We have:*

**(i)** $\|u_\lambda^\pm\|_{B_1}^2 = \int_{B_1} |\nabla u_\lambda^\pm|^2\, dx \to S^{N/2}$, *as* $\lambda \to 0$,

**(ii)** $|u_\lambda^\pm|_{2^*,B_1}^{2^*} = \int_{B_1} |u_\lambda^\pm|^{\frac{2N}{N-2}}\, dx \to S^{N/2}$, *as* $\lambda \to 0$,

**(iii)** $u_\lambda \rightharpoonup 0$, *as* $\lambda \to 0$,

**(iv)** $M_{\lambda,+} := \max_{B_1} u_\lambda^+ \to +\infty$, $M_{\lambda,-} := \max_{B_1} u_\lambda^- \to +\infty$, *as* $\lambda \to 0$.

*Proof.* This proposition is a special case of Lemma 2.1 in [13]. □

Let us recall a classical result, due to Strauss, known as "radial lemma":

**Lemma 1.3.2** (Strauss)**.** *There exists a constant $c > 0$, depending only on $N$, such that for all $u \in H^1_{rad}(\mathbb{R}^N)$*

$$|u(x)| \leq c\, \frac{\|u\|_{1,2}^{1/2}}{|x|^{(N-1)/2}} \quad a.e. \ on \ \mathbb{R}^N, \tag{1.3.1}$$

*where $\|\cdot\|_{1,2}$ is the standard $H^1$-norm.*

*Proof.* For the proof of this result see for instance [63]. □

We denote by $s_\lambda \in (0,1)$ the global minimum point of $u_\lambda = u_\lambda(r)$ (the uniqueness of $s_\lambda$ follows from Corollary 1.2.4), so we have $0 < r_\lambda < s_\lambda$, $u_\lambda^-(s_\lambda) = M_{\lambda,-}$. The following proposition gives an information on the behavior of $r_\lambda$ and $s_\lambda$ as $\lambda \to 0$.

**Proposition 1.3.3.** *We have $s_\lambda \to 0$ (and so $r_\lambda \to 0$ as well), as $\lambda \to 0$.*

*Proof.* Assume by contradiction that $s_{\lambda_m} \geq s_0$ for a sequence $\lambda_m \to 0$ and for some $0 < s_0 < 1$. Then, by Lemma 1.3.2 we get

$$M_{\lambda_m,-} = |u_{\lambda_m}(s_{\lambda_m})| \leq c\frac{\|u_{\lambda_m}\|_{1,2,B_1}^{1/2}}{s_{\lambda_m}^{(N-1)/2}} \leq c\frac{\|u_{\lambda_m}\|_{1,2,B_1}^{1/2}}{s_0^{(N-1)/2}},$$

where $c$ is a positive constant depending only on $N$. Since $|\nabla u_\lambda|_{2,B_1}^2 \to 2S^{N/2}$ as $\lambda \to 0$ it follows that $M_{\lambda_m,-}$ is bounded, which is a contradiction. □

We recall another well-known proposition:

**Proposition 1.3.4.** *Let $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & in\ \mathbb{R}^N \\ u \to 0 & as\ |y| \to +\infty. \end{cases} \tag{1.3.2}$$

*Assume that $u$ has a finite energy $I_0(u) := \frac{1}{2}|\nabla u|_{2,\mathbb{R}^N}^2 - \frac{1}{2^*}|u|_{2^*,\mathbb{R}^N}^{2^*}$ and $u$ satisfies one of these assumptions:*

**(i)** *$u$ is positive (negative) in $\mathbb{R}^N$,*

**(ii)** *$u$ is spherically symmetric about some point.*

*Then, there exist $\mu > 0$, $x_0 \in \mathbb{R}^N$ such that $u$ is one of the functions $\mathcal{U}_{x_0,\mu}$, defined in (1.1.3).*

*Proof.* A sketch of the proof can be found in [25], Proposition 2.2. □

### 1.3.2 An upper bound for $u_\lambda^+$, $u_\lambda^-$

In this section, we recall an estimate for positive solutions of (1.1.2) in a ball and we generalize it to get an upper bound for $u_\lambda^-$ in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^N; r_\lambda < |x| < 1\}$.

**Proposition 1.3.5.** *Let $N \geq 3$ and $u$ be a solution of*

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}} & \text{in } B_R \\ u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{1.3.3}$$

*for some positive $\lambda$. Then, $u(x) \leq w(x, u(0))$ in $B_R$, where*

$$w(x,c) := c\left\{1 + \frac{c^{-1}f(c)}{N(N-2)}|x|^2\right\}^{-(N-2)/2},$$

*and $f : [0,+\infty) \to [0,+\infty)$ is the function defined by $f(y) := \lambda y + y^{\frac{N+2}{N-2}}$.*

*Proof.* The proof is based on the results contained in the papers of Atkinson and Peletier [7], [8]. Since the solutions of (1.3.3) are radial (see [33]) we consider the ordinary differential equation associated with (1.3.3) which, by some change of variable, can be turned into an Emden–Fowler equation. For it is easy to get the desired upper bound. All details are given in the next Proposition 1.3.7. □

**Remark 1.3.6.** *The previous proposition gives an upper bound for $u_\lambda^+$. In fact, taking into account that $u_\lambda^+$ is defined and positive in the ball $B_{r_\lambda}$ and $u_\lambda^+(0) = M_{\lambda,+}$, we have*

$$\begin{aligned} u_\lambda^+(x) &\leq M_{\lambda,+}\left\{1 + \frac{M_{\lambda,+}^{-1}\, f(M_{\lambda,+})}{N(N-2)}|x|^2\right\}^{-(N-2)/2} \\ &= M_{\lambda,+}\left\{1 + \frac{\lambda + M_{\lambda,+}^{\frac{4}{N-2}}}{N(N-2)}|x|^2\right\}^{-(N-2)/2}, \end{aligned} \tag{1.3.4}$$

*for all $x \in B_{r_\lambda}$.*

**Proposition 1.3.7.** *Let $u_\lambda$ be as in Sect. 1.3.1 and $\epsilon \in (0, \frac{N-2}{2})$. There exist $\delta = \delta(\epsilon) \in (0,1)$, $\delta(\epsilon) \to 1$ as $\epsilon \to 0$ and a positive constant $\overline{\lambda} = \overline{\lambda}(\epsilon)$, such that for all $\lambda \in (0, \overline{\lambda})$ we have*

$$u_\lambda^-(x) \leq M_{\lambda,-}\left\{1 + \frac{M_{\lambda,-}^{-1}\, f(M_{\lambda,-})}{N(N-2)}c(\epsilon)|x|^2\right\}^{-(N-2)/2}, \tag{1.3.5}$$

*for all $x \in A_{\delta,\lambda}$, where $A_{\delta,\lambda} := \{x \in \mathbb{R}^N;\ \delta^{-1/N}s_\lambda < |x| < 1\}$, $c(\epsilon) = \frac{2}{N-2}\epsilon$, $s_\lambda$ is the global minimum point of $u_\lambda$, $M_{\lambda,-} = u_\lambda^-(s_\lambda)$ and $f$ is defined as in Proposition 1.3.5.*

*Proof of Proposition 1.3.7.* Let $v_\lambda$ the function defined by $v_\lambda(s) := u_\lambda^-(s+s_\lambda)$, $s \in (0, 1-s_\lambda)$. Since $u_\lambda^-$ is a positive radial solution of (1.1.2) then $v_\lambda$ is a solution of

$$\begin{cases} v_\lambda'' + \frac{N-1}{s+s_\lambda}v_\lambda' + \lambda v_\lambda + v_\lambda^{2^*-1} = 0 & \text{in } (0, 1-s_\lambda) \\ v_\lambda'(0) = 0, \quad v_\lambda(1-s_\lambda) = 0. \end{cases} \tag{1.3.6}$$

To eliminate $\lambda$ from the equation, we make the following change of variable, $\rho := \sqrt{\lambda}\,(s+s_\lambda)$, and we define $w_\lambda(\rho) := \lambda^{-\frac{N-2}{4}} v_\lambda(\frac{\rho}{\sqrt{\lambda}} - s_\lambda) = \lambda^{-\frac{N-2}{4}} u_\lambda^-(\frac{\rho}{\sqrt{\lambda}})$. By elementary computation, we see that $w_\lambda$ solves

$$\begin{cases} w_\lambda'' + \frac{N-1}{\rho} w_\lambda' + w_\lambda + w_\lambda^{2^*-1} = 0 \quad \text{in } (\sqrt{\lambda}\, s_\lambda, \sqrt{\lambda}) \\ w_\lambda'(\sqrt{\lambda}\, s_\lambda) = 0, \quad w_\lambda(\sqrt{\lambda}) = 0. \end{cases} \tag{1.3.7}$$

Making another change of variable, precisely $t := \left(\frac{N-2}{\rho}\right)^{N-2}$, and setting $y_\lambda(t) := w_\lambda\left(\frac{N-2}{t^{\frac{1}{N-2}}}\right)$ we eliminate the first derivative in (1.3.7). Thus, we get

$$\begin{cases} y_\lambda''\, t^k + y_\lambda + y_\lambda^{2^*-1} = 0 & \text{in } \left(\frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}}}, \frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}} s_\lambda^{N-2}}\right), \\ y_\lambda'\left(\frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}} s_\lambda^{N-2}}\right) = 0, \quad y_\lambda\left(\frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}}}\right) = 0. \end{cases} \tag{1.3.8}$$

where $k = 2\dfrac{N-1}{N-2} > 2$. To simplify the notation, we set $t_{1,\lambda} := \frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}}}$, $t_{2,\lambda} := \frac{(N-2)^{N-2}}{\lambda^{\frac{N-2}{2}} s_\lambda^{N-2}}$, $I_\lambda = (t_{1,\lambda}, t_{2,\lambda})$ and $\gamma_\lambda := y_\lambda(t_{2,\lambda}) = \lambda^{-\frac{N-2}{4}} M_{\lambda,-}$. Observe also that $2^* - 1 = 2k-3$.

We write the equation in (1.3.8) as $y_\lambda'' + t^{-k}(y_\lambda + y_\lambda^{2k-3}) = 0$, which is an Emden–Fowler type equation $y'' + t^{-k} h(y) = 0$ with $h(y) := y + y^{2k-3}$. The first step to prove (1.3.5) is the following inequality:

$$(y_\lambda' t^{k-1} y_\lambda^{1-k})' + t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda) \le 0, \quad \text{for all } t \in I_\lambda. \tag{1.3.9}$$

To prove (1.3.9) we differentiate $y_\lambda' t^{k-1} y_\lambda^{1-k}$. Since $y_\lambda'' + t^{-k} h(y_\lambda) = 0$ we get

$$\begin{aligned} & y_\lambda'' t^{k-1} y_\lambda^{1-k} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \;& -t^{-k}(y_\lambda + y_\lambda^{2k-3}) t^{k-1} y_\lambda^{1-k} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \;& -t^{-1} y_\lambda^{2-k} - t^{-1} y_\lambda^{k-2} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \;& -2(k-1) t^{k-2} y_\lambda^{-k}\left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda^2 + \tfrac{1}{2(k-1)} t^{1-k} y_\lambda^{2k-2} - \tfrac12 y_\lambda y_\lambda' + \tfrac12 t (y_\lambda')^2\right) \\ = \;& -2(k-1) t^{k-2} y_\lambda^{-k}\left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \tfrac12 y_\lambda y_\lambda' + \tfrac12 t (y_\lambda')^2\right). \end{aligned}$$

Now, we add and subtract the number $\frac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)$ inside the parenthesis, so we have

$$\begin{aligned} & (y_\lambda' t^{k-1} y_\lambda^{1-k})' \\ = \;& -2(k-1) t^{k-2} y_\lambda^{-k}\left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \tfrac12 y_\lambda y_\lambda' + \tfrac12 t (y_\lambda')^2 - \tfrac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)\right) \\ & - t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda). \end{aligned}$$

Setting $L_\lambda(t) := \frac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \frac12 y_\lambda y_\lambda' + \frac12 t (y_\lambda')^2 - \frac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)$ we get

$$(y'_\lambda t^{k-1} y_\lambda^{1-k})' + t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda) = -2(k-1) t^{k-2} y_\lambda^{-k} L_\lambda(t).$$

If we show that $L_\lambda(t) \geq 0$ for all $t \in I_\lambda$ we get (1.3.9). By definition it is immediate to verify that $L_\lambda(t_{2,\lambda}) = 0$, also by direct calculation, we have $L'_\lambda(t) = \frac{1}{2(k-1)} t^{1-k} y'_\lambda [y_\lambda h'(y_\lambda) - (2k-3)h(y_\lambda)] = \frac{1}{2(k-1)} t^{1-k} y'_\lambda [(4-2k) y_\lambda]$. Since $y_\lambda > 0$, $y'_\lambda \geq 0$ in $I_\lambda$ [1] and $k > 2$ we have $L'_\lambda(t) \leq 0$ in $I_\lambda$, and from $L_\lambda(t_{2,\lambda}) = 0$ it follows $L_\lambda(t) \geq 0$ for all $t \in I_\lambda$.

As second step, we integrate (1.3.9) between $t$ and $t_{2,\lambda}$, for all $t \in I_\lambda$. Then, since $y'_\lambda(t_{2,\lambda}) = 0$ we get

$$-y'_\lambda(t) t^{k-1} y_\lambda^{1-k}(t) + \int_t^{t_{2,\lambda}} s^{k-2} y_\lambda^{-k}(s)\ t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)\ ds \leq 0.$$

We rewrite this last inequality as

$$y'_\lambda(t) t^{k-1} y_\lambda^{1-k}(t) \geq t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda) \int_t^{t_{2,\lambda}} s^{k-2} y_\lambda^{-k}(s)\ ds.$$

Since $u_\lambda^- \leq M_{\lambda,-}$ by definition, it follows $y_\lambda^{-k} \geq \gamma_\lambda^{-k}$, so

$$\begin{aligned} y'_\lambda(t) t^{k-1} y_\lambda^{1-k}(t) &\geq t_{2,\lambda}^{1-k} \gamma_\lambda^{1-k} h(\gamma_\lambda) \int_t^{t_{2,\lambda}} s^{k-2}\ ds \\ &= \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \frac{t_{2,\lambda}^{k-1} - t^{k-1}}{t_{2,\lambda}^{k-1}} \\ &= \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \left[1 - \left(\frac{t}{t_{2,\lambda}}\right)^{k-1}\right]. \end{aligned}$$

Multiplying the first and the last term of the above inequality by $t^{1-k}$ we get

$$\frac{1}{2-k} (y_\lambda^{2-k})'(t) = y'_\lambda(t)\ y_\lambda^{1-k}(t) \geq \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \left(t^{1-k} - \frac{1}{t_{2,\lambda}^{k-1}}\right),$$

for all $t \in I_\lambda$. Integrating this inequality between $t$ and $t_{2,\lambda}$ we have

$$\begin{aligned} \frac{\gamma_\lambda^{2-k}}{2-k} - \frac{y_\lambda^{2-k}(t)}{2-k} &\geq \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \int_t^{t_{2,\lambda}} \left(s^{1-k} - \frac{1}{t_{2,\lambda}^{k-1}}\right) ds \\ &= \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \left(\frac{t_{2,\lambda}^{2-k}}{2-k} - \frac{t^{2-k}}{2-k} - \frac{1}{t_{2,\lambda}^{k-2}} + \frac{t}{t_{2,\lambda}^{k-1}}\right). \end{aligned}$$

We rewrite this last inequality as

$$\begin{aligned} \frac{y_\lambda^{2-k}(t)}{k-2} - \frac{\gamma_\lambda^{2-k}}{k-2} &\geq \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} \left(\frac{t^{2-k}}{k-2} + \frac{t}{t_{2,\lambda}^{k-1}} - \frac{k-1}{k-2} \frac{1}{t_{2,\lambda}^{k-2}}\right) \\ &\geq \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1} t^{2-k} \left[\frac{1}{k-2} + \left(\frac{t}{t_{2,\lambda}}\right)^{k-1} - \frac{k-1}{k-2} \left(\frac{t}{t_{2,\lambda}}\right)^{k-2}\right]. \end{aligned} \quad (1.3.10)$$

[1] $y'_\lambda \geq 0$ because $(u_\lambda^-)'(r) \leq 0$ for $s_\lambda < r < 1$ as we can easily deduce from Corollary 1.2.4.

To the aim of estimating the last term in (1.3.10) we set $s := \left(\frac{t}{t_{2,\lambda}}\right)^{k-1}$ and study the function $g(s) := \frac{1}{k-2} + s - \frac{k-1}{k-2} s^{\frac{k-2}{k-1}}$ in the interval $[0,1]$. Clearly, $g(0) = \frac{1}{k-2} = \frac{N-2}{2} > 0$, $g(1) = 0$ and $g$ is a decreasing function because $g'(s) = 1 - s^{-\frac{1}{k-1}} < 0$ in $(0,1)$. In particular, we have $g(s) > 0$ in $(0,1)$. Let us fix $\epsilon \in (0, \frac{N-2}{2})$, by the monotonicity of $g$ we deduce that there exists only one $\delta = \delta(\epsilon) \in (0,1)$ such that $g(s) > \epsilon$ for all $0 \le s < \delta$, $g(\delta) = \epsilon$ and $\delta \to 1$ as $\epsilon \to 0$. Now remembering that $s = \left(\frac{t}{t_{2,\lambda}}\right)^{k-1}$, we have $\left(\frac{t}{t_{2,\lambda}}\right)^{k-1} < \delta$ if and only if $t < \delta^{\frac{1}{k-1}} t_{2,\lambda}$ and $t_{1,\lambda} < \delta^{\frac{1}{k-1}} t_{2,\lambda}$ if and only if $s_\lambda^{N-2} < \delta^{\frac{1}{k-1}}$ which is true for all $0 < \lambda < \overline{\lambda}$, for some positive number $\overline{\lambda} = \overline{\lambda}(\epsilon)$. Setting $c(\epsilon) := (k-2)\epsilon$, from (1.3.10) and the previous discussion, we have

$$y_\lambda^{2-k}(t) - \gamma_\lambda^{2-k} \ge \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1}\, t^{2-k} c(\epsilon), \tag{1.3.11}$$

for all $t \in (t_{1,\lambda},\ \delta^{\frac{1}{k-1}} t_{2,\lambda})$, $0 < \lambda < \overline{\lambda}$. Now from (1.3.11) we deduce the desired bound for $u_\lambda^-$. In fact, we have

$$y_\lambda^{2-k}(t) \ge \gamma_\lambda^{2-k} + \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1}\, t^{2-k} c(\epsilon),$$

from which, since $k > 2$, we get

$$\begin{aligned} y_\lambda(t) &\le \left(\gamma_\lambda^{2-k} + \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1}\, t^{2-k} c(\epsilon)\right)^{-\frac{1}{k-2}} \\ &= \gamma_\lambda \left(1 + \frac{\gamma_\lambda^{-1} h(\gamma_\lambda)}{k-1}\, t^{2-k} c(\epsilon)\right)^{-\frac{1}{k-2}} \end{aligned} \tag{1.3.12}$$

Now, by definition, we have $y_\lambda(t) = \lambda^{-\frac{N-2}{4}} u_\lambda^-\left(\frac{\rho}{\sqrt{\lambda}}\right) = \lambda^{-\frac{N-2}{4}} u_\lambda^-(s+s_\lambda)$, $\gamma_\lambda = \lambda^{-\frac{N-2}{4}} M_{\lambda,-}$, $k-2 = \frac{2}{N-2}$, $k-1 = \frac{N}{N-2}$, $t = \left(\frac{N-2}{\rho}\right)^{N-2} = \left(\frac{N-2}{\sqrt{\lambda}(s+s_\lambda)}\right)^{N-2}$, in particular $t^{2-k} = t^{-\frac{2}{N-2}} = \left(\frac{\sqrt{\lambda}(s+s_\lambda)}{N-2}\right)^2 = \frac{\lambda(s+s_\lambda)^2}{(N-2)^2}$. Thus, we get

$$\begin{aligned} \frac{\gamma_\lambda^{-1} h(\gamma_\lambda)}{k-1}\, t^{2-k} c(\epsilon) &= \frac{\lambda^{\frac{N-2}{4}} M_{\lambda,-}^{-1}\left(\lambda^{-\frac{N-2}{4}} M_{\lambda,-} + \lambda^{-\frac{N+2}{4}} M_{\lambda,-}^{\frac{N+2}{N-2}}\right)}{\frac{N}{N-2}} c(\epsilon) \frac{\lambda(s+s_\lambda)^2}{(N-2)^2} \\ &= \frac{M_{\lambda,-}^{-1}\left(\lambda M_{\lambda,-} + M_{\lambda,-}^{2^*-1}\right)}{N(N-2)} c(\epsilon)(s+s_\lambda)^2 \\ &= \frac{M_{\lambda,-}^{-1}\ f\left(M_{\lambda,-}\right)}{N(N-2)} c(\epsilon)(s+s_\lambda)^2, \end{aligned}$$

where $f(z) := \lambda z + z^{2^*-1}$. Also, by direct computation, we see that the interval $(t_{1,\lambda},\ \delta^{\frac{1}{k-1}} t_{2,\lambda})$, corresponds to the interval $(\delta^{-\frac{1}{N}} s_\lambda, 1)$ for $s + s_\lambda = \frac{\rho}{\sqrt{\lambda}} = \frac{N-2}{\sqrt{\lambda}\ t^{\frac{1}{N-2}}}$. Thus, from the previous computations and (1.3.12) we have

$$\lambda^{-\frac{N-2}{4}} u_\lambda^-(s+s_\lambda) \le \lambda^{-\frac{N-2}{4}} M_{\lambda,-} \left(1 + \frac{M_{\lambda,-}^{-1}\ f\left(\lambda M_{\lambda,-}\right)}{N(N-2)} c(\epsilon)(s+s_\lambda)^2\right)^{-\frac{N-2}{2}}.$$

Finally, dividing each term by $\lambda^{-\frac{N-2}{4}}$ and setting $r := s + s_\lambda$ we have

$$u_\lambda^-(r) \leq \left(1 + \frac{M_{\lambda,-}^{-1}\ f(\lambda M_{\lambda,-})}{N(N-2)} c(\epsilon) r^2\right)^{-\frac{N-2}{2}},$$

for all $r \in (\delta^{-\frac{1}{N}} s_\lambda, 1)$, which is the desired inequality since $u_\lambda^-$ is a radial function. $\square$

## 1.4 Asymptotic analysis of the rescaled solutions

### 1.4.1 Rescaling the positive part

As in Sect. 1.3, we consider a family $(u_\lambda)$ of least energy radial, sign-changing solutions of (1.1.2) with $u_\lambda(0) > 0$. Let us define $\beta := \frac{2}{N-2}$, $\sigma_\lambda := M_{\lambda,+}^\beta \cdot r_\lambda$; consider the rescaled function $\tilde{u}_\lambda^+(y) = \frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)$ in $B_{\sigma_\lambda}$. The following lemma is elementary but crucial.

**Lemma 1.4.1.** *We have:*

**(i)** $\|u_\lambda^+\|_{B_{r_\lambda}}^2 = \|\tilde{u}_\lambda^+\|_{B_{\sigma_\lambda}}^2$,

**(ii)** $|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*} = |\tilde{u}_\lambda^+|_{2^*,B_{\sigma_\lambda}}^{2^*}$,

**(iii)** $|u_\lambda^+|_{2,B_{r_\lambda}}^2 = \frac{1}{M_{\lambda,+}^{2^*-2}} |\tilde{u}_\lambda^+|_{2,B_{\sigma_\lambda}}^2$

*Proof.* To prove (i) we have only to remember the definition of $\tilde{u}_\lambda$ and make the change of variable $x \to \frac{y}{M_{\lambda,+}^\beta}$. Taking into account that by definition $\nabla_y \tilde{u}_\lambda^+(y) = \frac{1}{M_{\lambda,+}^{1+\beta}} (\nabla_x u_\lambda^+)(\frac{y}{M_{\lambda,+}^\beta})$ and $2 + 2\beta = 2 + \frac{4}{N-2} = N\frac{2}{N-2} = N\beta = 2^*$, we get

$$\begin{aligned} \|u_\lambda^+\|_{B_{r_\lambda}}^2 &= \int_{B_{r_\lambda}} |\nabla_x u_\lambda^+(x)|^2 dx &&= \frac{1}{M_{\lambda,+}^{N\beta}} \int_{B_{\sigma_\lambda}} \left|\nabla_x u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)\right|^2 dy \\ &= \frac{M_{\lambda,+}^{2+2\beta}}{M_{\lambda,+}^{N\beta}} \int_{B_{\sigma_\lambda}} |\nabla_y \tilde{u}_\lambda(y)|^2\, dy &&= \|\tilde{u}_\lambda^+\|_{B_{\sigma_\lambda}}^2. \end{aligned}$$

The proof of (ii) is simpler:

$$\begin{aligned} \int_{B_{r_\lambda}} |u_\lambda^+(x)|^{2^*} dx &= \int_{B_{\sigma_\lambda}} \frac{1}{M_{\lambda,+}^{N\beta}} \left|u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)\right|^{2^*} dy \\ &= \int_{B_{\sigma_\lambda}} |\tilde{u}_\lambda^+(y)|^{2^*} dy. \end{aligned}$$

The proof of (iii) is similar:

$$\begin{aligned} \int_{B_{r_\lambda}} |u_\lambda^+(x)|^2 dx &= \int_{B_{\sigma_\lambda}} \frac{1}{M_{\lambda,+}^{N\beta}} \left|u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)\right|^2 dy \\ &= \int_{B_{\sigma_\lambda}} \frac{1}{M_{\lambda,+}^{N\beta-2}} \left|\frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)\right|^2 dy \\ &= \frac{1}{M_{\lambda,+}^{2^*-2}} \int_{B_{\sigma_\lambda}} |\tilde{u}_\lambda^+(y)|^2 dy. \end{aligned}$$

□

**Remark 1.4.2.** *Obviously, the previous lemma is still true if we consider any radial function* $u \in H^1_{rad}(D)$*, where* $D$ *is a radially symmetric domain in* $\mathbb{R}^N$*, and for any rescaling of the kind* $\tilde{u}(y) := \frac{1}{M} u\left(\frac{y}{M^\beta}\right)$*, where* $M > 0$ *is a constant.*

The first qualitative result concerns the asymptotic behavior, as $\lambda \to 0$, of the radius $\sigma_\lambda = M^\beta_{\lambda,+} \cdot r_\lambda$ of the rescaled ball $B_{\sigma_\lambda}$. From Proposition 1.3.3 we know that $r_\lambda \to 0$ as $\lambda \to 0$, so this result gives also information on the growth of $M_{\lambda,+}$ compared to the decay of $r_\lambda$.

**Proposition 1.4.3.** *Up to a subsequence,* $\sigma_\lambda \to +\infty$ *as* $\lambda \to 0$*.*

*Proof.* Up to a subsequence, as $\lambda \to 0$, we have three alternatives:

**(i)** $\sigma_\lambda \to 0$,

**(ii)** $\sigma_\lambda \to l > 0$, $l \in \mathbb{R}$,

**(iii)** $\sigma_\lambda \to +\infty$.

We will show that (i) and (ii) cannot occur. Assume, by contradiction, that (i) holds then writing $|u^+_\lambda|^{2^*}_{2^*,B_{r_\lambda}}$ in polar coordinates we have

$$\begin{aligned}
|u^+_\lambda|^{2^*}_{2^*,B_{r_\lambda}} &= \omega_N \int_0^{r_\lambda} [u^+_\lambda(r)]^{2^*} r^{N-1} dr \\
&\leq \omega_N \, M^{2^*}_{\lambda,+} \int_0^{r_\lambda} r^{N-1} dr \\
&= \omega_N \, (M^\beta_{\lambda,+})^N \, \frac{r^N_\lambda}{N} \\
&= \frac{\omega_N}{N} \, (M^\beta_{\lambda,+} \, r_\lambda)^N \to 0 \quad \text{as } \lambda \to 0.
\end{aligned}$$

But from Proposition 1.3.1 we know that $|u^+_\lambda|^{2^*}_{2^*,B_{r_\lambda}} \to S^{N/2}$ as $\lambda \to 0$, so we get a contradiction.

Next, assume, by contradiction, that (ii) holds. Since the rescaled functions $\tilde{u}^+_\lambda$ are solutions of

$$\begin{cases} -\Delta u = \frac{\lambda}{M^{2\beta}_\lambda} u + u^{2^*-1} & \text{in } B_{\sigma_\lambda} \\ u > 0 & \text{in } B_{\sigma_\lambda} \\ u = 0 & \text{on } \partial B_{\sigma_\lambda}. \end{cases} \tag{1.4.1}$$

and $(\tilde{u}^+_\lambda)$ is uniformly bounded, then by standard elliptic theory, $\tilde{u}^+_\lambda \to \tilde{u}$ in $C^2_{loc}(B_l)$, where $B_l$ is the limit domain of $B_{\sigma_\lambda}$ and $\tilde{u}$ solves

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in } B_l \\ u > 0 & \text{in } B_l. \end{cases} \tag{1.4.2}$$

Let us show that the boundary condition $\tilde{u} = 0$ on $\partial B_l$ holds. Since $M_{\lambda,+}$ is the global maximum of $u_\lambda$ (see Proposition 1.2.3) then the rescaling $\tilde{u}_\lambda(y) := \frac{1}{M_{\lambda,+}} u_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)$ of the whole function $u_\lambda$ is a bounded solution of

$$\begin{cases} -\Delta u = \frac{\lambda}{M_{\lambda}^{2\beta}} u + |u|^{2^*-2} u & \text{in } B_{M^{\beta}_{\lambda,+}} \\ u = 0 & \text{on } \partial B_{M^{\beta}_{\lambda,+}}. \end{cases}$$

So as before we get that $\tilde{u}_\lambda \to \tilde{u}_0$ in $C^2_{loc}(\mathbb{R}^N)$, where $\tilde{u}_0$ is a solution of $-\Delta u = |u|^{2^*-2}u$ in $\mathbb{R}^N$. Obviously, by definition, we have $\tilde{u}_\lambda(y) = \tilde{u}^+_\lambda(y)$ for all $y \in B_{\sigma_\lambda}$, $\tilde{u}_\lambda(y) = 0$ for all $y \in \partial B_{\sigma_\lambda}$ and $\tilde{u}_\lambda(y) < 0$ for all $y \in B_{M^{\beta}_{\lambda,+}} - \overline{B_{\sigma_\lambda}}$. Passing to the limit as $\lambda \to 0$, since $\overline{B_l}$ is a compact set of $\mathbb{R}^N$ we have $\tilde{u}_\lambda \to \tilde{u}_0$ in $C^2(\overline{B_l})$, now since $\tilde{u} = \tilde{u}_0 > 0$ in $B_l$ and $\tilde{u}_0 = 0$ on $\partial B_l$, it follows $\tilde{u} = 0$ on $\partial B_l$. Since $B_l$ is a ball, by Pohozaev's identity, we know that the only possibility is $\tilde{u} \equiv 0$ which is a contradiction since $\tilde{u}(0) = 1$. So the assertion is proved. □

**Proposition 1.4.4.** *We have:*

$$\tilde{u}^+_\lambda(y) \leq \left\{1 + \frac{1}{N(N-2)}|y|^2\right\}^{-(N-2)/2}, \tag{1.4.3}$$

*for all $y \in \mathbb{R}^N$.*

*Proof.* From (1.3.4) for all $x \in B_{r_\lambda}$ we have

$$u^+_\lambda(x) \leq M_{\lambda,+}\left\{1 + \frac{\lambda + M_{\lambda,+}^{\frac{4}{N-2}}}{N(N-2)}|x|^2\right\}^{-(N-2)/2}.$$

Dividing each side by $M_{\lambda,+}$ and setting $x = \frac{y}{M^{\beta}_{\lambda,+}} = \frac{y}{M_{\lambda,+}^{\frac{2}{N-2}}}$ we get

$$\begin{aligned} \frac{1}{M_{\lambda,+}} u^+_\lambda\left(\frac{y}{M^{\beta}_{\lambda,+}}\right) &\leq \left\{1 + \frac{\lambda + M_{\lambda,+}^{\frac{4}{N-2}}}{M_{\lambda,+}^{\frac{4}{N-2}}\, N(N-2)}|y|^2\right\}^{-(N-2)/2} \\ &= \left\{1 + \frac{\lambda}{M_{\lambda,+}^{\frac{4}{N-2}}\, N(N-2)}|y|^2 + \frac{1}{N(N-2)}|y|^2\right\}^{-(N-2)/2} \\ &\leq \left\{1 + \frac{1}{N(N-2)}|y|^2\right\}^{-(N-2)/2}, \end{aligned}$$

for all $y \in B_{\sigma_\lambda}$. Thus, we have proved (1.4.3) for all $y \in B_{\sigma_\lambda}$. Since $\tilde{u}^+_\lambda$ is zero outside the ball $B_{\sigma_\lambda}$ and the second term in (1.4.3) is independent of $\lambda$, this bound holds in the whole $\mathbb{R}^N$. □

### 1.4.2 An estimate on the first derivative at the node

In this subsection, we prove an inequality concerning $(u^+_\lambda)'(r_\lambda)$ (or $(u^-_\lambda)'(r_\lambda)$) that will be useful in the next sections.

**Lemma 1.4.5.** *There exists a constant $c_1$, depending only on $N$, such that*

$$|(u^+_\lambda)'(r_\lambda) r_\lambda^{N-1}| \leq c_1\, r_\lambda^{\frac{N-2}{2}} \tag{1.4.4}$$

*for all sufficiently small $\lambda > 0$. Since $(u^-_\lambda)'(r_\lambda) = -(u^+_\lambda)'(r_\lambda)$ the same inequality holds for $(u^-_\lambda)'(r_\lambda)$.*

*Proof.* Since $u_\lambda^+ = u_\lambda^+(r)$ is a solution of $-[(u_\lambda^+)' r^{N-1}]' = \lambda u_\lambda^+ r^{N-1} + (u_\lambda^+)^{2^*-1} r^{N-1}$ in $(0, r_\lambda)$ and $(u_\lambda^+)'(0) = 0$ by integration, we get

$$\begin{aligned}(u_\lambda^+)'(r_\lambda) r_\lambda^{N-1} &= -\left[\int_0^{r_\lambda} \lambda u_\lambda^+ r^{N-1} dr + \int_0^{r_\lambda} (u_\lambda^+)^{2^*-1} r^{N-1} dr\right] \\ &= -\left[\frac{\lambda}{\omega_N} \int_{B_{r_\lambda}} u_\lambda^+(x)\, dx + \frac{1}{\omega_N} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^{2^*-1}\, dx\right],\end{aligned}$$

where, as before, $\omega_N$ denotes the measure of the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$. Using Hölder's inequality and observing that $|B_{r_\lambda}| = \frac{\omega_N}{N} r_\lambda^N$ we deduce

$$\left|(u_\lambda^+)'(r_\lambda) r_\lambda^{N-1}\right| \le \frac{\lambda}{(N\,\omega_N)^{\frac{1}{2}}} r_\lambda^{\frac{N}{2}} |u_\lambda^+|_{2,B_{r_\lambda}} + \frac{1}{N^{\frac{N-2}{2N}}\,\omega_N^{\frac{N+2}{2N}}} r_\lambda^{\frac{N-2}{2}} \left[|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*}\right]^{\frac{2^*-1}{2^*}}.$$

From Proposition 1.3.1 we know that both $|u_\lambda^+|_{2,B_{r_\lambda}}$, $|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*}$ are bounded, moreover from Proposition 1.3.3 we have $r_\lambda \to 0$ as $\lambda \to 0$. So there exists a constant $c_1 = c_1(N)$ such that for all sufficiently small $\lambda > 0$ (1.4.4) holds. □

### 1.4.3 Rescaling the negative part

Now, we study the rescaled function $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^-\left(\frac{y}{M_{\lambda,-}^\beta}\right)$ in the annulus $A_{\rho_\lambda} := \{y \in \mathbb{R}^N; M_{\lambda,-}^\beta r_\lambda < |y| < M_{\lambda,-}^\beta\}$, where $\rho_\lambda := M_{\lambda,-}^\beta r_\lambda$. This case is more delicate than the previous one since the radius $s_\lambda$, where the minimum is achieved, depends on $\lambda$. Thus, roughly speaking, we have to understand how $r_\lambda$ and $s_\lambda$ behave with respect to the scaling parameter $M_{\lambda,-}^\beta$. This means that we have to study the asymptotic behavior of $M_{\lambda,-}^\beta r_\lambda$ and $M_{\lambda,-}^\beta s_\lambda$ as $\lambda \to 0$. It will be convenient to consider also the one-dimensional rescaling

$$z_\lambda(s) := \frac{1}{M_{\lambda,-}} u_\lambda^-\left(s_\lambda + \frac{s}{M_{\lambda,-}^\beta}\right),$$

which satisfies

$$\begin{cases} z_\lambda'' + \frac{N-1}{s + M_{\lambda,-}^\beta s_\lambda} z_\lambda' + \frac{\lambda}{M_{\lambda,-}^{2\beta}} z_\lambda + z_\lambda^{2^*-1} = 0 \quad \text{in } (a_\lambda, b_\lambda) \\ z_\lambda'(0) = 0, \quad z_\lambda(0) = 1, \end{cases} \tag{1.4.5}$$

where $a_\lambda := M_{\lambda,-}^\beta \cdot (r_\lambda - s_\lambda) < 0$, $b_\lambda := M_{\lambda,-}^\beta \cdot (1 - s_\lambda) > 0$. We define $\gamma_\lambda := M_{\lambda,-}^\beta s_\lambda$.

Since $s_\lambda \to 0$ as $\lambda \to 0$, we have $b_\lambda \to +\infty$; for the remaining parameters $a_\lambda, \gamma_\lambda$ it will suffice to study the asymptotic behavior of $\gamma_\lambda$ as $\lambda \to 0$.

Up to a subsequence, we have three alternatives:

**(a)** $\gamma_\lambda \to +\infty$,

**(b)** $\gamma_\lambda \to \gamma_0 > 0$,

**(c)** $\gamma_\lambda \to 0$.

**Lemma 1.4.6.** $\gamma_\lambda \to +\infty$ *cannot happen.*

*Proof.* Assume $\gamma_\lambda \to +\infty$; up to a subsequence, we have $a_\lambda \to \bar{a} \leq 0$, as $\lambda \to 0$, where $\bar{a} \in \mathbb{R} \cup \{-\infty\}$.

If $\bar{a} < 0$ or $\bar{a} = -\infty$ then passing to the limit in (1.4.5) as $\gamma_\lambda = M_{\lambda,-}^{\beta} \cdot s_\lambda \to +\infty$ we have that $z_\lambda \to z$ in $C^1_{loc}(\bar{a}, +\infty)$, where $z$ solves the limit problem

$$\begin{cases} z'' + z^{2^*-1} = 0 & \text{in } (\bar{a}, +\infty) \\ z'(0) = 0, \quad z(0) = 1. \end{cases} \tag{1.4.6}$$

Since $z_\lambda \to z$ in $C^1_{loc}(\bar{a}, +\infty)$ and being $z_\lambda > 0$, then by Fatou's lemma we have

$$\liminf_{\lambda \to 0} \int_{a_\lambda}^{b_\lambda} [z_\lambda(s)]^{2^*} ds \geq \int_{\bar{a}}^{+\infty} [z(s)]^{2^*} ds \geq c_1 > 0.$$

In particular, being $a_\lambda < 0$, by the same argument it follows that for all small $\lambda > 0$

$$\int_0^{b_\lambda} [z_\lambda(s)]^{2^*} ds \geq \int_0^{+\infty} [z(s)]^{2^*} ds \geq c_2 > 0.$$

Now, we have the following estimate:

$$\begin{aligned} |u_\lambda^-|_{2^*, A_{r_\lambda}}^{2^*} &= \omega_N \int_{r_\lambda}^1 [u_\lambda^-(r)]^{2^*} r^{N-1} dr &&\geq \omega_N s_\lambda^{N-1} \int_{s_\lambda}^1 [u_\lambda^-(r)]^{2^*} dr \\ &= \omega_N s_\lambda^{N-1} M_{\lambda,-}^{2^*} \int_{s_\lambda}^1 \left[ \frac{1}{M_{\lambda,-}} u_\lambda^-(r) \right]^{2^*} dr &&= \omega_N s_\lambda^{N-1} M_{\lambda,-}^{2^*-\beta} \int_0^{b_\lambda} [z_\lambda(s)]^{2^*} ds \\ &= \omega_N \gamma_\lambda^{N-1} \int_0^{b_\lambda} [z_\lambda(s)]^{2^*} ds &&\geq \omega_N \gamma_\lambda^{N-1} c_2, \end{aligned}$$

having used the change of variable $r = s_\lambda + \frac{s}{M_{\lambda,-}^{\beta}}$. Since $|u_\lambda^-|_{2^*, A_{r_\lambda}}^{2^*} \to S^{N/2}$ while $\gamma_\lambda \to +\infty$, as $\lambda \to 0$, we get a contradiction.

If instead $\bar{a} = 0$ we consider the rescaled function $\tilde{u}_\lambda^-$ which solves

$$\begin{cases} -\Delta \tilde{u}_\lambda = \frac{\lambda}{M_{\lambda,-}^{2\beta}} \tilde{u}_\lambda + \tilde{u}_\lambda^{2^*-1} & \text{in } A_{\rho_\lambda} \\ \tilde{u} = 0 & \text{on } \partial A_{\rho_\lambda}, \end{cases} \tag{1.4.7}$$

and is uniformly bounded. We observe that since $a_\lambda \to 0$ then $\rho_\lambda = a_\lambda + \gamma_\lambda \to +\infty$. By definition, we have $\tilde{u}_\lambda^-(\rho_\lambda) = 0$, $\tilde{u}_\lambda^-(\gamma_\lambda) = 1$, for all $\lambda \in (0, \lambda_1)$. Thus, we have

$$\frac{|\tilde{u}_\lambda^-(\rho_\lambda) - \tilde{u}_\lambda^-(\gamma_\lambda)|}{|\rho_\lambda - \gamma_\lambda|} = \frac{1}{|a_\lambda|} \to +\infty \quad \text{as } \lambda \to 0.$$

From standard elliptic regularity theory, we know that $\tilde{u}_\lambda^-$ is a classical solution, so by the mean value theorem,

$$\frac{|\tilde{u}_\lambda^-(\rho_\lambda) - \tilde{u}_\lambda^-(\gamma_\lambda)|}{|\rho_\lambda - \gamma_\lambda|} = |(\tilde{u}_\lambda^-)'(\xi_\lambda)|,$$

for some $\xi_\lambda \in (\rho_\lambda, \gamma_\lambda)$; thus, $|(\tilde{u}_\lambda^-)'(\xi_\lambda)| \to +\infty$ as $\lambda \to 0$. From Corollary 1.2.4 it follows that $(\tilde{u}_\lambda^-)' > 0$ in $(\rho_\lambda, \gamma_\lambda)$ for all $\lambda > 0$.

By writing (1.4.7) in polar coordinates, we get:

$$(\tilde{u}_\lambda^-)'' + \frac{N-1}{r}(\tilde{u}_\lambda^-)' + \frac{\lambda}{M_{\lambda,-}^{2\beta}}\tilde{u}_\lambda^- + (\tilde{u}_\lambda^-)^{2^*-1} = 0.$$

From this, since $\tilde{u}_\lambda^- > 0$ and $(\tilde{u}_\lambda^-)' > 0$ in $(\rho_\lambda, \gamma_\lambda)$, we get $(\tilde{u}_\lambda^-)'' < 0$ in $(\rho_\lambda, \gamma_\lambda)$. Thus, $(\tilde{u}_\lambda^-)'(\rho_\lambda) > (\tilde{u}_\lambda^-)'(\xi_\lambda) > 0$, for all $\lambda > 0$. In particular, $(\tilde{u}_\lambda^-)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$.

Since, by elementary computation, we have $(\tilde{u}_\lambda^-)'(\rho_\lambda) = \frac{1}{M_{\lambda,-}^{1+\beta}}(u_\lambda^-)'(r_\lambda)$, by Lemma 1.4.5 we get

$$|(\tilde{u}_\lambda^-)'(\rho_\lambda)| \leq c\frac{1}{M_{\lambda,-}^{1+\beta}\ r_\lambda^{N/2}}$$

for a constant c independent from $\lambda$. Remembering that $1+\beta = 1 + \frac{2}{N-2} = \beta \cdot \frac{N}{2}$, and the definition of $\rho_\lambda$ we have the following estimate

$$|(\tilde{u}_\lambda^-)'(\rho_\lambda)| \leq c\frac{1}{\rho_\lambda^{N/2}}.$$

Since $\rho_\lambda \to +\infty$, as $\lambda \to 0$, we deduce that $(\tilde{u}_\lambda^-)'(\rho_\lambda)$ is uniformly bounded, against $(\tilde{u}_\lambda^-)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$. Thus, we get a contradiction. □

Thanks to Lemma 1.4.6 we deduce that $(\gamma_\lambda)$ is a bounded sequence. The following proposition states an uniform upper bound for $\tilde{u}_\lambda^-$.

**Proposition 1.4.7.** *Let us fix $\epsilon \in (0, \frac{N-2}{2})$, and set $\bar{M} := \sup_\lambda \gamma_\lambda$. There exist $h = h(\epsilon)$ and $\bar{\lambda} = \bar{\lambda}(\epsilon) > 0$ such that*

$$\tilde{u}_\lambda^-(y) \leq U_h(y) \tag{1.4.8}$$

*for all $y \in \mathbb{R}^N$, $0 < \lambda < \bar{\lambda}$, where*

$$U_h(y) := \begin{cases} 1 & \text{if } |y| \leq h \\ \left[1 + \frac{1}{N(N-2)}c(\epsilon)|y|^2\right]^{-(N-2)/2} & \text{if } |y| > h, \end{cases} \tag{1.4.9}$$

*with $c(\epsilon) = \frac{2}{N-2}\epsilon$.*

*Proof.* We fix $\epsilon \in (0, \frac{N-2}{2})$, so by Proposition 1.3.7 there exist $\delta = \delta(\epsilon) \in (0,1)$ and $\overline{\lambda}(\epsilon) > 0$ such that

$$u_\lambda^-(x) \leq M_{\lambda,-}\left\{1 + \frac{M_{\lambda,-}^{-1}\ f(M_{\lambda,-})}{N(N-2)}c(\epsilon)|x|^2\right\}^{-(N-2)/2},$$

for all $x \in A_{\delta,\lambda} = \{x \in \mathbb{R}^N;\ \delta^{-1/N}s_\lambda < |x| < 1\}$, for all $\lambda \in (0, \overline{\lambda})$, where $c(\epsilon) = \frac{2}{N-2}\epsilon$. The same proof of Proposition 1.4.4 shows that

$$\tilde{u}_\lambda^-(y) \leq \left\{1 + \frac{1}{N(N-2)}c(\epsilon)|y|^2\right\}^{-(N-2)/2},$$

for all $y \in \tilde{A}_{\delta,\lambda} = \{y \in \mathbb{R}^N;\ M_{\lambda,-}^\beta \delta^{-1/N}s_\lambda < |y| < M_{\lambda,-}^\beta\}$. Now, since by definition $\tilde{u}_\lambda^-$ is uniformly bounded by 1, we get an upper bound defined in the whole annulus $\tilde{A}_{\rho_\lambda} = \{y \in \mathbb{R}^N;\ M_{\lambda,-}^\beta r_\lambda < |y| < M_{\lambda,-}^\beta\}$; to be more precise $\tilde{u}_\lambda^-(y) \leq U_\lambda(y)$, where

$$U_\lambda(y) := \begin{cases} 1 & \text{if } M_{\lambda,-}^\beta r_\lambda < |y| \leq M_{\lambda,-}^\beta \delta^{-1/N}s_\lambda \\ \left[1 + \frac{1}{N(N-2)}c(\epsilon)|y|^2\right]^{-(N-2)/2} & \text{if } M_{\lambda,-}^\beta \delta^{-1/N}s_\lambda < |y| < M_{\lambda,-}^\beta. \end{cases} \tag{1.4.10}$$

Since $\gamma_\lambda = M^\beta_{\lambda,-} s_\lambda \leq \bar{M}$, then setting $h := \delta^{-1/N}\bar{M}$ we get that $\delta^{-1/N} M^\beta_{\lambda,-} s_\lambda \leq h$. Therefore, from (1.4.10), since $\tilde{u}^-_\lambda$ is zero outside $\tilde{A}_{\rho_\lambda}$, we deduce (1.4.8). □

**Lemma 1.4.8.** $\gamma_\lambda \to \gamma_0 > 0$, $\gamma_0 \in \mathbb{R}$, *cannot happen.*

*Proof.* Assume that $\gamma_\lambda \to \gamma_0 > 0$, $\gamma_0 \in \mathbb{R}$. Since $0 < r_\lambda < s_\lambda$ there are only two possibilities for $a_\lambda$. To be precise, up to a subsequence we can have:

**(i)** $a_\lambda \to 0$,

**(ii)** $a_\lambda \to \bar{a} < 0$, $\bar{a} \in \mathbb{R}$.

We will show that both (i) and (ii) lead to a contradiction.

If we assume (i) the same proof of Lemma 1.4.6 gives a contradiction. We point out that now $\rho_\lambda \to \gamma_0$, as $\lambda \to 0$, so as before we get a contradiction since $(\tilde{u}^-_\lambda)'(\rho_\lambda)$ is uniformly bounded, against $(\tilde{u}^-_\lambda)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$.

Assuming (ii) we have $a_\lambda \to \bar{a} < 0$ and $\gamma_\lambda \to \gamma_0 > 0$. We define $m := \bar{a} + \gamma_0$. Clearly, we have $0 \leq m < \gamma_0$ and $\rho_\lambda \to m$ as $\lambda \to 0$. Assume $m > 0$ and consider the rescaling $\tilde{u}^-_\lambda$ in the annulus $A_{\rho_\lambda}$ defined as before. Since $\tilde{u}^-_\lambda$ satisfies (1.4.7) and $(\tilde{u}^-_\lambda)$ is uniformly bounded then passing to the limit as $\lambda \to 0$ we get $\tilde{u}^-_\lambda \to \tilde{u}$ in $C^2_{loc}(\Pi)$, where $\Pi$ is the limit domain $\Pi := \{y \in \mathbb{R}^N; |y| > m\}$ and $\tilde{u}$ is a positive radial solution of

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \Pi \tag{1.4.11}$$

By definition $\tilde{u}^-_\lambda(\gamma_\lambda) = 1$, $(\tilde{u}^-_\lambda)'(\gamma_\lambda) = 0$ for all $\lambda$, so as $\lambda \to 0$ we get $\tilde{u}(\gamma_0) = 1$, $\tilde{u}'(\gamma_0) = 0$ because of the convergence of $\tilde{u}^-_\lambda \to \tilde{u}$ in $C^2(K)$, for all compact subsets $K$ in $\Pi$, and $\gamma_0 > m$. In particular, we deduce that $\tilde{u} \not\equiv 0$. We now show that $\tilde{u}$ can be extended to zero on $\partial\Pi = \{y \in \mathbb{R}^N; |y| = m\}$. Thanks to Lemma 1.4.5 and since we are assuming $m > 0$, which is the limit of $\rho_\lambda$ as $\lambda \to 0$, we get that $(\tilde{u}^-_\lambda)'(\rho_\lambda)$ is uniformly bounded by a constant $M$, and by the monotonicity of $(\tilde{u}^-_\lambda)'$ the same bound holds for $(\tilde{u}^-_\lambda)'(s)$ for all $s \in (\rho_\lambda, \gamma_\lambda)$. It follows that in that interval $\tilde{u}^-_\lambda(s) \leq M(s - \rho_\lambda)$. Passing to the limit as $\lambda \to 0$ we have $\tilde{u}(s) \leq M(s - m)$ for all $s \in (m, \gamma_0)$ which implies $\tilde{u}$ can be extended by continuity to zero on $\partial\Pi$. We use the same notation $\tilde{u}$ to denote this extension.

Observe that $\tilde{u}$ has finite energy, in particular, using Fatou's lemma and thanks to Lemma 1.4.1, Remark 1.4.2, Proposition 1.3.1, we get

$$\int_\Pi |\nabla \tilde{u}|^2 dy \leq \liminf_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\nabla \tilde{u}^-_\lambda|^2 dy = \liminf_{\lambda \to 0} \int_{A_{r_\lambda}} |\nabla u^-_\lambda|^2 dx = S^{N/2}, \tag{1.4.12}$$

$$\int_\Pi |\tilde{u}|^{2*} dy \leq \liminf_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\tilde{u}^-_\lambda|^{2*} dy = \liminf_{\lambda \to 0} \int_{A_{r_\lambda}} |u^-_\lambda|^{2*} dx = S^{N/2}. \tag{1.4.13}$$

Moreover, since $\tilde{u}^-_\lambda \to \tilde{u}$ in $C^2_{loc}(\Pi)$ and thanks to the uniform upper bound given by Proposition 1.4.7, by Lebesgue's theorem, we have

$$\int_\Pi |\tilde{u}|^{2*} dy = \lim_{\lambda \to 0} \int_{A_{r_\lambda}} |u^-_\lambda|^{2*} dx = S^{N/2}. \tag{1.4.14}$$

Since $\tilde{u} \in H^1(\Pi) \cap C^0(\bar{\Pi})$ and is zero on $\partial\Pi$, then $\tilde{u} \in H^1_0(\Pi)$ and thanks to (1.4.12), (1.4.14) it follows that $\tilde{u}$ achieves the best constant in the Sobolev embedding on $\Pi$, which is impossible (see for instance [58], Theorem III.1.2). This ends the proof for the case $m > 0$.

Assume now $m = 0$, then $\tilde{u}_\lambda^-$ converges in $C^2_{loc}(\mathbb{R}^N - \{0\})$ to a radial function $\tilde{u}$ which is a positive bounded solution of

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in} \quad \mathbb{R}^N - \{0\} \tag{1.4.15}$$

Since $\tilde{u}$ is a radial solution of (1.4.15), then integrating $-(\tilde{u}'(r)r^{N-1})' = \tilde{u}^{2^*-1}(r)r^{N-1}$ between $\delta > 0$ sufficiently small and $\gamma_0$ we get

$$\tilde{u}'(\delta)\delta^{N-1} = \int_\delta^{\gamma_0} \tilde{u}^{2^*-1} r^{N-1} dr.$$

Since the right-hand side is a positive and decreasing function of $\delta$, we get $\tilde{u}'(\delta)\delta^{N-1} \to \tilde{l} > 0$ as $\delta \to 0$. Thus, $\tilde{u}'(\delta)$ behaves as $\delta^{1-N}$ near the origin, and this is a contradiction since $\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dy = \omega_N \int_0^{+\infty} |\tilde{u}'(r)|^2 r^{N-1} dr$ is finite, and the proof is complete. □

As a consequence of Lemma 1.4.6 and Lemma 1.4.8 we have proved:

**Proposition 1.4.9.** *Up to a subsequence, we have $\gamma_\lambda \to 0$ as $\lambda \to 0$.*

### 1.4.4 Final estimates and proof of Theorem 1.1.1

From Proposition 1.4.9 we know that, up to a subsequence, $\gamma_\lambda = M^\beta_{-,\lambda} s_\lambda \to 0$ as $\lambda \to 0$. The rescaled function $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^- \left( \frac{y}{M^\beta_{\lambda,-}} \right)$ in the annulus $A_{\rho_\lambda} := \{y \in \mathbb{R}^N; M^\beta_{\lambda,-} r_\lambda < |y| < M^\beta_{\lambda,-}\}$ solves (1.4.7) and the functions $(\tilde{u}_\lambda^-)$ are uniformly bounded. Since $\gamma_\lambda \to 0$ as $\lambda \to 0$, in particular the limit domain of $A_{\rho_\lambda}$ is $\mathbb{R}^N - \{0\}$ and by standard elliptic theory $\tilde{u}_\lambda^- \to \tilde{u}$ in $C^2_{loc}(\mathbb{R}^N - \{0\})$, where $\tilde{u}$ is positive, radial and solves

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \mathbb{R}^N - \{0\} \tag{1.4.16}$$

As in the proof of Lemma 1.4.8 by Fatou's Lemma, it follows that $\tilde{u}$ has finite energy $I_0(\tilde{u}) = \frac{1}{2}|\nabla \tilde{u}|^2_{2,\mathbb{R}^N} - \frac{1}{2^*}|\tilde{u}|^{2^*}_{2^*,\mathbb{R}^N}$. Moreover, thanks to the uniform upper bound (1.4.8), by Lebesgue's theorem, we have

$$\lim_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\tilde{u}_\lambda^-|^{2*} dy = \int_{\mathbb{R}^N} |\tilde{u}|^{2*} dy,$$

so, by Lemma 1.4.1, Remark 1.4.2 and Proposition 1.3.1 we get

$$\int_{\mathbb{R}^N} |\tilde{u}|^{2*} dy = S^{N/2}.$$

The next two lemmas show that the function $\tilde{u} = \tilde{u}(s)$ can be extended to a $C^1([0, +\infty))$ function if we set $\tilde{u}(0) := 1$ and $\tilde{u}'(0) := 0$.

**Lemma 1.4.10.** *We have*

$$\lim_{s \to 0} \tilde{u}(s) = 1.$$

*Proof.* Since $\tilde{u}_\lambda^-$ is a radial solution of (1.4.7) and $\tilde{u}_\lambda^- \leq 1$, then

$$\begin{aligned}
[(\tilde{u}_\lambda^-)' s^{N-1}]' &= -\frac{\lambda}{M^{2\beta}_{\lambda,-}} \tilde{u}_\lambda^-(s) s^{N-1} - [\tilde{u}_\lambda^-(s)]^{2^*-1} s^{N-1} \\
&\geq -\frac{\lambda}{M^{2\beta}_{\lambda,-}} s^{N-1} - s^{N-1} \\
&\geq -2s^{N-1}.
\end{aligned}$$

Integrating between $\gamma_\lambda$ and $s > \gamma_\lambda$ (with $s < M^{\beta}_{\lambda,-}$) we get

$$(\tilde{u}^-_\lambda)'(s)s^{N-1} \geq -2\int_{\gamma_\lambda}^{s} t^{N-1}dt \geq -\frac{2}{N}s^N.$$

Hence, $(\tilde{u}^-_\lambda)'(s) \geq -\frac{2}{N}s$ for all $s \in (\gamma_\lambda, M^{\beta}_{\lambda,-})$. Integrating again between $\gamma_\lambda$ and $s$ we have

$$\tilde{u}^-_\lambda(s) - 1 \geq -\frac{1}{N}(s^2 - \gamma_\lambda^2) \geq -\frac{1}{N}s^2.$$

Hence, $\tilde{u}^-_\lambda(s) \geq 1 - \frac{1}{N}s^N$ for all $s \in (\gamma_\lambda, M^{\beta}_{\lambda,-})$. Since $\gamma_\lambda \to 0$ and $M^{\beta}_{\lambda,-} \to +\infty$, then, passing to the limit as $\lambda \to 0$, we get $\tilde{u}(s) \geq 1 - \frac{1}{N}s^2$, for all $s > 0$. From this inequality and since $\tilde{u} \leq 1$ we deduce $\lim_{s\to 0}\tilde{u}(s) = 1$. □

**Lemma 1.4.11.** *We have*

$$\lim_{s\to 0}\tilde{u}'(s) = 0.$$

*Proof.* As before, from the radial equation satisfied by $\tilde{u}^-_\lambda$, integrating between $\gamma_\lambda$ and $s > \gamma_\lambda$ (with $s < M^{\beta}_{\lambda,-}$) we get

$$-(\tilde{u}^-_\lambda)'(s)s^{N-1} = \frac{\lambda}{M^{2\beta}_{\lambda,-}}\int_{\gamma_\lambda}^{s}\tilde{u}^-_\lambda t^{N-1}dt + \int_{\gamma_\lambda}^{s}(\tilde{u}^-_\lambda)^{2^*-1}t^{N-1}dt.$$

Since $\tilde{u} \leq 1$, and $\gamma_\lambda \to 0$ it follows that for all $\lambda > 0$ sufficiently small

$$|(\tilde{u}^-_\lambda)'(s)s^{N-1}| \leq \frac{\lambda}{M^{2\beta}_{\lambda,-}}\int_{\gamma_\lambda}^{s} t^{N-1}dt + \int_{\gamma_\lambda}^{s} t^{N-1}dt \leq 2\frac{s^N}{N}.$$

Passing to the limit, as $\lambda \to 0$, we get $|\tilde{u}'(s)| \leq 2\dfrac{s}{N}$ for all $s > 0$, hence $\lim_{s\to 0}\tilde{u}'(s) = 0$. □

From Lemma 1.4.10 and Lemma 1.4.11 it follows that the radial function $\tilde{u}(y) = \tilde{u}(|y|)$ can be extended to a $C^1(\mathbb{R}^N)$ function. From now on, we denote by $\tilde{u}$ this extension. Next lemma shows that $\tilde{u}$ is a weak solution of (1.4.16) in the whole $\mathbb{R}^N$.

**Lemma 1.4.12.** *The function $\tilde{u}$ is a weak solution of*

$$-\Delta\tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \mathbb{R}^N \tag{1.4.17}$$

*Proof.* Let us fix a test function $\phi \in C^\infty_0(\mathbb{R}^N)$. If $0 \notin supp(\phi)$ the proof is trivial so from now on we assume $0 \in supp(\phi)$. Let $B(\delta)$ be the ball centered at the origin having radius $\delta > 0$, with $\delta$ sufficiently small such that $supp(\phi) \subset\subset B(1/\delta)$. Applying Green's formula to $\Omega(\delta) := B(1/\delta) - B(\delta)$, since $\tilde{u}$ is a $C^2_{loc}(\mathbb{R}^N - \{0\})$ solution of (1.4.16) and $\phi \equiv 0$ on $\partial B(1/\delta)$, we have

$$\int_{\Omega(\delta)}\nabla\tilde{u}\cdot\nabla\phi\, dy = \int_{\Omega(\delta)}\phi\,\tilde{u}^{2^*-1}\, dy + \int_{\partial B(\delta)}\phi\left(\frac{\partial\tilde{u}}{\partial\nu}\right)\, d\sigma. \tag{1.4.18}$$

We show now that $\int_{\partial B(\delta)}\phi\left(\frac{\partial\tilde{u}}{\partial\nu}\right)\, d\sigma \to 0$ as $\delta \to 0$. In fact since $\tilde{u}$ is a radial function, we have $\frac{\partial\tilde{u}}{\partial\nu}(y) = \tilde{u}'(\delta)$ for all $y \in \partial B(\delta)$, and from this relation, we get

$$\begin{aligned}\left|\int_{\partial B(\delta)}\phi\left(\frac{\partial\tilde{u}}{\partial\nu}\right)\, d\sigma\right| &\leq |\tilde{u}'(\delta)|\int_{\partial B(\delta)}|\phi|\, d\sigma \\ &\leq \omega_N|\tilde{u}'(\delta)|\delta^{N-1}||\phi||_\infty.\end{aligned}$$

Thanks to Lemma 1.4.11 we have $|\tilde{u}'(\delta)|\delta^{N-1} \to 0$ as $\delta \to 0$. To complete the proof, we pass to the limit in (1.4.18) as $\delta \to 0$. We observe that

$$\begin{aligned} |\nabla\tilde{u}\cdot\nabla\phi|\ \chi_{\Omega(\delta)} &\leq |\nabla\tilde{u}|^2\ \chi_{\{|\nabla\tilde{u}|>1\}}|\nabla\phi| + |\nabla\tilde{u}|\ \chi_{\{|\nabla\tilde{u}|\leq 1\}}|\nabla\phi| \\ &\leq |\nabla\tilde{u}|^2\ \chi_{\{|\nabla\tilde{u}|>1\}}|\nabla\phi| +\ \chi_{\{|\nabla\tilde{u}|\leq 1\}}|\nabla\phi|. \end{aligned} \tag{1.4.19}$$

Since $\int_{\mathbb{R}^N} |\nabla\tilde{u}|^2 dy \leq S^{N/2}$ and $\phi$ has compact support, the right-hand side of (1.4.19) belongs to $L^1(\mathbb{R}^N)$. Hence, from Lebesgue's theorem, we have

$$\lim_{\delta\to 0}\int_{\Omega(\delta)} \nabla\tilde{u}\cdot\nabla\phi\ dy = \int_{\mathbb{R}^N} \nabla\tilde{u}\cdot\nabla\phi\ dy. \tag{1.4.20}$$

Since $\phi$ has compact support by Lebesgue's theorem, we have

$$\lim_{\delta\to 0}\int_{\Omega(\delta)} \phi\ \tilde{u}^{2^*-1}\ dy = \int_{\mathbb{R}^N} \phi\ \tilde{u}^{2^*-1}\ dy. \tag{1.4.21}$$

From (1.4.18), (1.4.20), (1.4.21) and since we have proved $\int_{\partial B(\delta)} \phi \left(\frac{\partial\tilde{u}}{\partial\nu}\right)\ d\sigma \to 0$ as $\delta \to 0$ it follows that

$$\int_{\mathbb{R}^N} \nabla\tilde{u}\cdot\nabla\phi\ dy = \int_{\mathbb{R}^N} \phi\ \tilde{u}^{2^*-1}\ dy,$$

which completes the proof. □

Now, we have all the tools to prove Theorem 1.1.1.

*Proof of Theorem 1.1.1.* We start proving (i). By Proposition 1.4.3, arguing as in the previous proofs, we know that $(\tilde{u}_\lambda^+)$ is an equi-bounded family of radial solutions of (1.4.1) and converges in $C^2_{loc}(\mathbb{R}^N)$ to a function $\tilde{u}$ which solves $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$. From (1.4.3) we deduce that $\tilde{u} \to 0$ as $|y| \to +\infty$. To apply Proposition 1.3.4 we have to check that $\tilde{u}$ has finite energy, but this is an immediate consequence of Fatou's lemma and the assumption that $u_\lambda$ has finite energy (for the details see (1.4.12) and (1.4.13)). Thus, $\tilde{u} = \mathcal{U}_{x_0,\mu}$ for some $x_0 \in \mathbb{R}^N$, $\mu > 0$. Since $\tilde{u}$ is a radial function, we have $x_0 = 0$. Moreover, since $\tilde{u}(0) = 1$, by an elementary computation, we see that $\mu = \sqrt{N(N-2)}$.

Now we prove (ii). As we have seen at the beginning of this section, the equi-bounded family $(\tilde{u}_\lambda^-)$ converges in $C^2_{loc}(\mathbb{R}^N - \{0\})$ to a function $\tilde{u}$ which solves (1.4.16). From Lemma 1.4.10 and Lemma 1.4.11 we have that $\tilde{u}$ can be extended to a $C^1(\mathbb{R}^N)$ function such that $\tilde{u}(0) = 1$, $\nabla\tilde{u}(0) = 0$. Moreover, from Lemma 1.4.12 we know that $\tilde{u}$ is a weak solution of (1.4.17) and from Fatou's lemma, as seen in (1.4.12), (1.4.13), we have that $\tilde{u}$ has finite energy. Also, from Proposition 1.4.7 we deduce that $\tilde{u} \to 0$ as $|y| \to +\infty$.

By elliptic regularity (see for instance Appendix B of [58]) since $\tilde{u}$ is a weak solution of (1.4.17) we deduce that $\tilde{u} \in C^2(\mathbb{R}^N)$. Thanks to Proposition 1.3.4, since $\tilde{u}$ is a radial function and $\tilde{u}(0) = 1$, we have $\tilde{u} = \mathcal{U}_{0,\mu}$, where $\mu > 0$ is the same as in (i). □

## 1.5 Asymptotic behavior of $M_{\lambda,+}$, $M_{\lambda,-}$ and proof of Theorem 1.1.2

We know from Proposition 1.3.1 that $M_{\lambda,+}, M_{\lambda,-} \to +\infty$ as $\lambda \to 0$, in addition in the last two sections we have proved that $M^\beta_{\lambda,+} r_\lambda \to +\infty$ while $M^\beta_{\lambda,-} r_\lambda \to 0$, as $\lambda \to 0$. Thus, $\frac{M_{\lambda,+}}{M_{\lambda,-}} \to +\infty$ as $\lambda \to 0$; in other words $M_{\lambda,+}$ goes to infinity faster than $M_{\lambda,-}$. In this section, we determine the order of infinity of $M_{\lambda,-}$ as negative power of $\lambda$ and also an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and the node $r_\lambda$.

**Proposition 1.5.1.** *As $\lambda \to 0$ we have*

**(i)** $M_{\lambda,+}|(u_\lambda^+)'(r_\lambda)|r_\lambda^{N-1} \to c_1(N)$;

**(ii)** $\lambda^{-1}M_{\lambda,+}^{2\beta}r_\lambda^N|(u_\lambda^+)'(r_\lambda)|^2 \to c_2(N)$;

**(iii)** $M_{\lambda,+}^{2-2\beta}r_\lambda^{N-2}\lambda \to c_3(N)$,

*where* $c_1(N) = \int_0^\infty \mathcal{U}_{0,\mu}^{2^*-1}(s)s^{N-1}ds$, $c_2(N) = 2\int_0^\infty \mathcal{U}_{0,\mu}^2(s)s^{N-1}ds$, $c_3(N) = \frac{c_1^2(N)}{c_2(N)}$.

*Proof.* To prove (i) we integrate the equation $-[(u_\lambda^+)'r^{N-1}]' = \lambda u_\lambda^+ r^{N-1} + (u_\lambda^+)^{2^*-1}r^{N-1}$ between 0 and $r_\lambda$ and multiply both sides by $M_{\lambda,+}$. Since $(u_\lambda^+)'(0) = 0$ we have

$$M_{\lambda,+}|(u_\lambda^+)'(r_\lambda)|r_\lambda^{N-1} = \lambda M_{\lambda,+}\int_0^{r_\lambda} u_\lambda^+ r^{N-1}\ dr + M_{\lambda,+}\int_0^{r_\lambda}(u_\lambda^+)^{2^*-1}r^{N-1}\ dr. \tag{1.5.1}$$

We first prove that $\lambda M_{\lambda,+}\int_0^{r_\lambda} u_\lambda^+ r^{N-1}\ dr \to 0$ as $\lambda \to 0$. In fact by the usual change of variable $r = \frac{s}{M_{\lambda,+}^\beta}$ we have

$$\begin{aligned}\lambda M_{\lambda,+}\int_0^{r_\lambda} u_\lambda^+(r)\ r^{N-1}\ dr &= \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\int_0^{M_{\lambda,+}^\beta r_\lambda}\frac{1}{M_{\lambda,+}}u_\lambda^+\left(\frac{s}{M_{\lambda,+}^\beta}\right)s^{N-1}\ ds\\ &= \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\int_0^{M_{\lambda,+}^\beta r_\lambda}\tilde{u}_\lambda^+(s)s^{N-1}\ ds\end{aligned}$$

Thanks to the uniform upper bound (1.4.3) we have

$$\begin{aligned}\lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\int_0^{M_{\lambda,+}^\beta r_\lambda}\tilde{u}_\lambda^+\ s^{N-1}\ ds &\leq \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\int_0^{M_{\lambda,+}^\beta r_\lambda}\left\{1+\frac{1}{N(N-2)}s^2\right\}^{-(N-2)/2}s^{N-1}ds\\ &\leq \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\int_0^1 s^{N-1}ds\\ &+ \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}[N(N-2)]^{(N-2)/2}\int_1^{M_{\lambda,+}^\beta r_\lambda}s^{-(N-2)}s^{N-1}ds\\ &= I_{\lambda,1}+I_{\lambda,2}.\end{aligned}$$

Since $M_{\lambda,+} \to +\infty$ and $\int_0^1 s^{N-1}ds = \frac{1}{N}$ it is obvious that $I_{\lambda,1} \to 0$, as $\lambda \to 0$. Now, we show that the same holds for $I_{\lambda,2}$. In fact, setting $C_1(N) := [N(N-2)]^{(N-2)/2}$ we have

$$\begin{aligned}I_{\lambda,2} &= \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}C_1(N)\int_1^{M_{\lambda,+}^\beta r_\lambda}s\ ds\\ &= \lambda\frac{1}{M_{\lambda,+}^{2^*-2}}C_1(N)\left(\frac{M_{\lambda,+}^{2\beta}r_\lambda^2}{2}-\frac{1}{2}\right)\\ &= \lambda r_\lambda^2\frac{C_1(N)}{2}-\lambda\frac{1}{M_{\lambda,+}^{2^*-2}}\frac{C_1(N)}{2} \to 0,\ \text{as } \lambda \to 0,\end{aligned}$$

since by definition, $2\beta = \frac{4}{N-2} = 2^*-2$. To complete the proof of (i) we show that $M_{\lambda,+}\int_0^{r_\lambda}(u_\lambda^+)^{2^*-1}r^{N-1}\ dr \to \int_0^\infty \mathcal{U}_{0,\mu}^{2^*-1}(s)s^{N-1}ds$ as $\lambda \to 0$. In fact, as before, by the change of variable $r = \frac{s}{M_{\lambda,+}^{\beta}}$ we have

$$\begin{aligned} M_{\lambda,+}\int_0^{r_\lambda} [u_\lambda^+(r)]^{2^*-1}\, r^{N-1}\, dr &= \frac{1}{M_{\lambda,+}^{2^*-1}}\int_0^{M_{\lambda,+}^{\beta} r_\lambda} \left[u_\lambda^+\left(\frac{s}{M_{\lambda,+}^{\beta}}\right)\right]^{2^*-1} s^{N-1}\, ds \\ &= \int_0^{M_{\lambda,+}^{\beta} r_\lambda} [\tilde{u}_\lambda^+(s)]^{2^*-1} s^{N-1}\, ds. \end{aligned}$$

Since $\tilde{u}_\lambda^+ \to \mathcal{U}_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^N)$, in particular we have $[\tilde{u}_\lambda^+(s)]^{2^*-1} \to [\mathcal{U}_{0,\mu}(s)]^{2^*-1}$ as $\lambda \to 0$, for all $s \geq 0$, and thanks to the uniform upper bound (1.4.3), by Lebesgue's dominated convergence theorem, it follows that $\int_0^{M_{\lambda,+}^{\beta} r_\lambda} [\tilde{u}_\lambda^+(s)]^{2^*-1} s^{N-1}\, ds \to \int_0^\infty \mathcal{U}_{0,\mu}^{2^*-1}(s) s^{N-1} ds$ so by (1.5.1) the proof of (i) is complete.

Now, we prove (ii). Applying Pohozaev's identity to $u_\lambda^+$, which solves $-\Delta u = \lambda u + u^{2^*-1}$ in $B_{r_\lambda}$, we have

$$\lambda \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx = \frac{1}{2}\int_{\partial B_{r_\lambda}} (x\cdot \nu)\left(\frac{\partial u_\lambda^+}{\partial \nu}\right)^2 d\sigma,$$

where $\nu$ is the exterior unit normal vector to $\partial B_{r_\lambda}$. Since $u_\lambda^+$ is radial, we have also $\left(\frac{\partial u_\lambda^+}{\partial \nu}\right)^2 = \left[(u_\lambda^+)'(r_\lambda)\right]^2$ so, passing to the unit sphere $\mathbb{S}^{N-1}$, we get

$$\begin{aligned} \lambda \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx &= \frac{1}{2} r_\lambda^{N-1} \int_{\mathbb{S}^{N-1}} r_\lambda \left[(u_\lambda^+)'(r_\lambda)\right]^2 d\omega \\ &= \frac{1}{2}\omega_N r_\lambda^N \left[(u_\lambda^+)'(r_\lambda)\right]^2. \end{aligned}$$

Thus, we have

$$\lambda^{-1} r_\lambda^N \left[(u_\lambda^+)'(r_\lambda)\right]^2 = 2\, \omega_N^{-1} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx. \tag{1.5.2}$$

Now, performing the same change of variable as in (i) we have

$$\begin{aligned} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx &= \frac{1}{M_{\lambda,+}^{2^*-2}} \int_{B_{\sigma_\lambda}} \left[\frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^{\beta}}\right)\right]^2 dy \\ &= \frac{1}{M_{\lambda,+}^{2^*-2}} \int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2 dy, \end{aligned}$$

Thus, we get

$$M_{\lambda,+}^{2\beta} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx = \int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2 dy. \tag{1.5.3}$$

As in (i) since $\tilde{u}_\lambda^+ \to \mathcal{U}_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^N)$ and thanks to the uniform upper bound (1.4.3) we have

$$\int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2 dy \to \int_{\mathbb{R}^N} [\mathcal{U}_{0,\mu}(y)]^2\, dy = \omega_N \int_0^{+\infty} [\mathcal{U}_{0,\mu}(r)]^2 r^{N-1}\, dr.$$

From this, (1.5.2) and (1.5.3) we deduce that $\lambda^{-1} M_{\lambda,+}^{2\beta} r_\lambda^N \left[(u_\lambda^+)'(r_\lambda)\right]^2 \to 2\int_0^{+\infty} [\mathcal{U}_{0,\mu}(r)]^2 r^{N-1}\, dr$, and (ii) is proved.

The proof of (iii) is a trivial consequence of (i) and (ii). □

Now, we state a similar result for $M_{\lambda,-}$.

**Proposition 1.5.2.** *As $\lambda \to 0$ we have the following:*

**(i)** $M_{\lambda,-}|(u_\lambda^-)'(1)| \to c_1(N)$;

**(ii)** $\lambda^{-1}M_{\lambda,-}^{2\beta}\left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^N\right\} \to c_2(N)$;

**(iii)** $\lambda^{-1}M_{\lambda,-}^{2\beta}[(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^N \to 0$;

**(iv)** $M_{\lambda,-}^{2-2\beta}\lambda \to c_3(N)$,

*where $c_1(N)$, $c_2(N)$ and $c_3(N)$ are the constants defined in Proposition 1.5.1.*

*Proof.* The proof of (i) is similar to the proof of (i) of Proposition 1.5.1. Here, we integrate the equation $-[(u_\lambda^-)'r^{N-1}]' = \lambda u_\lambda^- r^{N-1} + (u_\lambda^-)^{2^*-1}r^{N-1}$ between $s_\lambda$ and 1. Since $(u_\lambda^-)'(s_\lambda) = 0$ we have

$$(u_\lambda^-)'(1) = \lambda \int_{s_\lambda}^1 u_\lambda^- \ r^{N-1}\ dr + \int_{s_\lambda}^1 (u_\lambda^-)^{2^*-1} r^{N-1}\ dr.$$

By $M_\lambda^\beta s_\lambda \to 0$ and thanks to the uniform upper bound (1.4.8), arguing like in the proof of (i) of Proposition 1.5.1, we have

$$M_{\lambda,-}\ \lambda \int_{s_\lambda}^1 u_\lambda^- \ r^{N-1}\ dr \to 0$$

and

$$M_{\lambda,-}\int_{s_\lambda}^1 (u_\lambda^-)^{2^*-1} r^{N-1}\ dr = \int_{M_{\lambda,-}^\beta s_\lambda}^{M_{\lambda,-}^\beta} (\tilde{u}_\lambda^-)^{2^*-1} s^{N-1}\ ds \to \int_0^{+\infty} \mathcal{U}_{0,\mu}^{2^*-1} s^{N-1}\ ds,$$

as $\lambda \to 0$. The proof of (i) is complete.

The proof of (ii) is similar to the corresponding one of Proposition 1.5.1. This time we apply Pohozaev's identity to $u_\lambda^-$ in the annulus $A_{r_\lambda} = \{x \in \mathbb{R}^N;\ r_\lambda < |x| < 1\}$ whose boundary has two connected components, namely $\{x \in \mathbb{R}^N;\ |x| = r_\lambda\}$ and the unit sphere $\mathbb{S}^{N-1}$. Thus, we have

$$\begin{aligned}\lambda \int_{A_{r_\lambda}} [u_\lambda^-(x)]^2 dx &= \frac{1}{2}\int_{\partial A_{r_\lambda}} (x\cdot\nu)\left(\frac{\partial u_\lambda^-}{\partial \nu}\right)^2 d\sigma \\ &= \frac{1}{2}\omega_N \left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^N\right\}.\end{aligned}$$

Thus, multiplying each member by $M_{\lambda,-}^{2\beta}$ and rewriting the previous equation, we have

$$\begin{aligned}M_{\lambda,-}^{2\beta}\lambda^{-1}\left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^N\right\} &= 2\omega_N^{-1} M_{\lambda,-}^{2\beta}\int_{A_{r_\lambda}} [u_\lambda^-(x)]^2 dx \\ &= 2\omega_N^{-1} M_{\lambda,-}^{2\beta}\frac{1}{M_{\lambda,-}^{N\beta}}\int_{A_{\sigma_\lambda}}\left[u_\lambda^-\left(\frac{y}{M_{\lambda,-}^\beta}\right)\right]^2 dy \\ &= 2\int_{M_{\lambda,-}^\beta r_\lambda}^{M_{\lambda,-}^\beta} \left[\tilde{u}_\lambda^-(s)\right]^2\ s^{N-1} ds.\end{aligned}$$

Since $2\int_{M^{\beta}_{\lambda,-}r_\lambda}^{M^{\beta}_{\lambda,-}} \left[\tilde{u}^-_\lambda(s)\right]^2 \; s^{N-1}ds \to 2\int_0^\infty \mathcal{U}^2_{0,\mu}(s)s^{N-1}ds$ as $\lambda \to 0$ we are done.

To prove (iii) we write

$$\begin{aligned}\lambda^{-1}M^{2\beta}_{\lambda,-}[(u^-_\lambda)'(r_\lambda)]^2r^N_\lambda &= \frac{\lambda^{-1}M^{2\beta}_{\lambda,-}[(u^-_\lambda)'(r_\lambda)]^2r^N_\lambda}{\lambda^{-1}M^{2\beta}_{\lambda,+}[(u^+_\lambda)'(r_\lambda)]^2r^N_\lambda}\cdot\lambda^{-1}M^{2\beta}_{\lambda,+}[(u^+_\lambda)'(r_\lambda)]^2r^N_\lambda \\ &= \frac{M^{2\beta}_{\lambda,-}}{M^{2\beta}_{\lambda,+}}\cdot\;\lambda^{-1}M^{2\beta}_{\lambda,+}[(u^+_\lambda)'(r_\lambda)]^2r^N_\lambda \to 0\end{aligned}$$

since $\frac{M_{\lambda,-}}{M_{\lambda,+}} \to 0$ and $\lambda^{-1}M^{2\beta}_{\lambda,+}[(u^+_\lambda)'(r_\lambda)]^2r^N_\lambda \to c_2(N)$ as $\lambda \to 0$ (by (ii) of Proposition 1.5.1).

Finally, the proof of (iv) is trivial. In fact from (ii) and (iii) it immediately follows that

$$\lambda^{-1}M^{2\beta}_{\lambda,-}[(u^-_\lambda)'(1)]^2 \to c_2(N).$$

From this and (i), we get (iv). □

**Remark 1.5.3.** *By elementary computation* $2-2\beta = 2-\frac{4}{N-2} = \frac{2N-8}{N-2}$ *so by (iv) of Proposition 1.5.2 we have that* $M_{\lambda,-}$ *is an infinite of the same order as* $\lambda^{-\frac{N-2}{2N-8}}$.

From (iii) of Proposition 1.5.1 and (iv) of Proposition 1.5.2 we deduce the following result which gives an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and $r_\lambda$.

**Proposition 1.5.4.** $\dfrac{M^{2-2\beta}_{\lambda,-}}{M^{2-2\beta}_{\lambda,+}r^{N-2}_\lambda} \to 1,$ *as* $\lambda \to 0.$

*Proof of Theorem 1.1.2.* It suffices to sum up the results contained in Proposition 1.5.1, Proposition 1.5.2 and Proposition 1.5.4. □

**Remark 1.5.5.** *We point out that in order to determine the explicit rate of* $M_{\lambda,+}$ *or, equivalently, that of* $r_\lambda$*, some difficulties arise. The techniques used in the previous proofs of integrating the equation and using the Pohozaev's identity do not seem to be sufficient to this purpose. Nevertheless, as a consequence of the methods applied in Chapter 3 we get, for* $N \geq 7$ *and for all sufficiently small* $\lambda$*, the existence of radial sign-changing solutions of (0.0.1) with the shape of a tower of two bubbles, and the parameters* $\mu_1$*,* $\mu_2$ *of these two bubbles are given. The lowest order bubble diverges as* $\lambda^{-\frac{N-2}{2N-8}}$*, which is the same order of* $M_{\lambda,-}$*, while the other diverges as* $\lambda^{-\frac{(3N-10)(N-2)}{(2N-8)(N-6)}}$*. Moreover, in Chapter 4, we show, under some additional hypotheses, that the previous speeds are the only possible ones, for* $N \geq 7$*. Hence, we conjecture that* $M_{\lambda,+} \simeq \lambda^{-\frac{(3N-10)(N-2)}{(2N-8)(N-6)}}$.

## 1.6 Proof of Theorem 1.1.3

This section is entirely devoted to the proof of Theorem 1.1.3.

*Proof of Theorem 1.1.3.* We want to prove that $\lambda^{-\frac{N-2}{2N-8}}u_\lambda \to \tilde{c}(N)G(x,0)$ in $C^1_{loc}(B_1 - \{0\})$. We begin from the local uniform convergence of $\lambda^{-\frac{N-2}{2N-8}}u_\lambda$. The same argument with some modifications will work for the local uniform convergence of its derivatives. Thanks to the representation formula, since $-\Delta u_\lambda = \lambda u_\lambda + |u_\lambda|^{2^*-2}u_\lambda$ in $B_1$, we have

$$\lambda^{-\frac{N-2}{2N-8}}u_\lambda(x) = -\lambda^{-\frac{N-2}{2N-8}}\lambda\int_{B_1} G(x,y)u_\lambda(y)\ dy - \lambda^{-\frac{N-2}{2N-8}}\int_{B_1} G(x,y)|u_\lambda|^{2^*-2}u_\lambda(y)\ dy. \tag{1.6.1}$$

Since $\lambda^{-\frac{N-2}{2N-8}}\lambda = \lambda^{\frac{N-6}{2N-8}}$, splitting the integrals we have

$$\begin{aligned}\lambda^{-\frac{N-2}{2N-8}}u_\lambda(x) &= -\lambda^{\frac{N-6}{2N-8}}\int_{B_{r_\lambda}} G(x,y)u_\lambda^+(y)\ dy + \lambda^{\frac{N-6}{2N-8}}\int_{A_{r_\lambda}} G(x,y)u_\lambda^-(y)\ dy\\ &\quad -\lambda^{-\frac{N-2}{2N-8}}\int_{B_{r_\lambda}} G(x,y)[u_\lambda^+(y)]^{2^*-1}\ dy + \lambda^{-\frac{N-2}{2N-8}}\int_{A_{r_\lambda}} G(x,y)[u_\lambda^-(y)]^{2^*-1}\ dy\\ &= I_{1,\lambda}+I_{2,\lambda}+I_{3,\lambda}+I_{4,\lambda}.\end{aligned}$$

Let $K$ be a compact subset of $B_1-\{0\}$. We are going to prove that $I_{1,\lambda}, I_{2,\lambda}, I_{3,\lambda} \to 0$ uniformly in $K$, as $\lambda \to 0$. We begin with $I_{1,\lambda}$. For all $x \in K$ we have

$$\begin{aligned}|I_{1,\lambda}| &\leq \left|\lambda^{\frac{N-6}{2N-8}}\int_{B_{r_\lambda}} G(x,y)u_\lambda^+(y)\ dy\right|\\ &= \left|\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{N\beta}}\int_{B_{M_{\lambda,+}^\beta r_\lambda}} G\left(x,\frac{y}{M_{\lambda,+}^\beta}\right)u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)\ dy\right|\\ &\leq \lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\int_{B_{M_{\lambda,+}^\beta r_\lambda}}\left|G\left(x,\frac{y}{M_{\lambda,+}^\beta}\right)\right|\tilde{u}_\lambda^+(y)\ dy.\end{aligned}$$

Since $K$ is a compact subset of $B_1-\{0\}$ and $|\frac{y}{M_{\lambda,+}^\beta}| < r_\lambda$ by an elementary computation, we see that for all $x \in K$, for all $\lambda > 0$ sufficiently small $\left|G\left(x,\frac{y}{M_{\lambda,+}^\beta}\right)\right| \leq c(K)$ for all $y \in B_{M_{\lambda,+}^\beta r_\lambda}$, where $c = c(K)$ is a positive constant depending only on $K$ and $N$. Now, thanks to the uniform upper bound (1.4.3) we have

$$\begin{aligned}&\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\int_{B_{M_{\lambda,+}^\beta r_\lambda}}\left|G\left(x,\frac{y}{M_{\lambda,+}^\beta}\right)\right|\tilde{u}_\lambda^+(y)\ dy\\ \leq\ & c(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\int_{B_{M_{\lambda,+}^\beta r_\lambda}}\left\{1+\frac{1}{N(N-2)}|y|^2\right\}^{-(N-2)/2}\ dy\\ =\ & c(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\omega_N\int_0^{M_{\lambda,+}^\beta r_\lambda}\left\{1+\frac{1}{N(N-2)}s^2\right\}^{-(N-2)/2}s^{N-1}\ ds\\ \leq\ & c_1(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\int_0^{M_{\lambda,+}^\beta r_\lambda}s^{-(N-2)}s^{N-1}\ ds = c_1(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}\int_0^{M_{\lambda,+}^\beta r_\lambda}s\ ds\\ =\ & c_2(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}^{2^*-1}}M_{\lambda,+}^{2\beta}r_\lambda^2 = c_2(K)\lambda^{\frac{N-6}{2N-8}}\frac{1}{M_{\lambda,+}}r_\lambda^2 \to 0,\ \text{as } \lambda \to 0.\end{aligned}$$

Since this inequality is uniform respect to $x \in K$ we have $\|I_{1,\lambda}\|_{\infty,K} \to 0$ as $\lambda \to 0$. The proof that $\|I_{3,\lambda}\|_{\infty,K} \to 0$ is quite similar to the previous one, in fact with small modifications we get the following uniform estimate:

$$\begin{aligned}
|I_{3,\lambda}| &\leq \lambda^{\frac{N-6}{2N-8}} \frac{1}{M_{\lambda,+}} \int_{B_{M^\beta_{\lambda,+} r_\lambda}} \left| G\left(x, \frac{y}{M^\beta_{\lambda,+}}\right)\right| [\tilde{u}^+_\lambda(y)]^{2^*-1}\, dy \\
&\leq c(K)\lambda^{\frac{N-6}{2N-8}} \frac{1}{M_{\lambda,+}} \int_{B_{M^\beta_{\lambda,+} r_\lambda}} \left\{1 + \frac{1}{N(N-2)}|y|^2\right\}^{-(N+2)/2} dy \\
&\leq c(K)\lambda^{\frac{N-6}{2N-8}} \frac{1}{M_{\lambda,+}} \int_{\mathbb{R}^N} \left\{1 + \frac{1}{N(N-2)}|y|^2\right\}^{-(N+2)/2} dy \\
&= c_1(K)\, \lambda^{\frac{N-6}{2N-8}} \frac{1}{M_{\lambda,+}}, \quad \text{as } \lambda \to 0.
\end{aligned}$$

The proof for $I_{2,\lambda}$ is more delicate since for all small $\lambda > 0$ the Green function is not bounded when $x \in K$, $y \in A_{r_\lambda}$. We split the Green function in the singular part and the regular part so that

$$I_{2,\lambda} = \lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} G_{sing}(x,y) u^-_\lambda(y)\, dy + \lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} G_{reg}(x,y) u^-_\lambda(y)\, dy.$$

The singular part of the Green function is given by $\frac{1}{(2-N)\omega_N}\frac{1}{|x-y|^{N-2}}$, we want to show that

$$\lambda^{\frac{N-6}{2N-8}} \frac{1}{(2-N)\omega_N} \int_{A_{r_\lambda}} \frac{1}{|x-y|^{N-2}} u^-_\lambda(y)\, dy \to 0$$

uniformly for $x \in K$. The usual change of variable gives

$$\begin{aligned}
&\lambda^{\frac{N-6}{2N-8}} \frac{1}{(2-N)\omega_N} \int_{A_{r_\lambda}} \frac{1}{|x-y|^{N-2}} u^-_\lambda(y)\, dy \\
&= \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^\beta_{\lambda,-}}|^{N-2}} u^-_\lambda\left(\frac{w}{M^\beta_{\lambda,-}}\right) dw.
\end{aligned}$$

Let $\eta$ be a positive real number such that $\eta < \min\{\frac{d(0,K)}{2}; \frac{d(K,\partial B_1)}{2}\}$, where $d(\cdot,\cdot)$ denotes the Euclidean distance. It is clear that for all $\lambda > 0$ sufficiently small, we have $B(x,\eta) \subset\subset A_{r_\lambda}$, for all $x \in K$. Thus, $B(M^\beta_{\lambda,-}x, M^\beta_{\lambda,-}\eta) \subset\subset \tilde{A}_{r_\lambda}$, for all $x \in K$, and we split the last integral in two parts as indicated below:

$$\begin{aligned}
&\frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^\beta_{\lambda,-}}|^{N-2}} u^-_\lambda\left(\frac{w}{M^\beta_{\lambda,-}}\right) dw \\
&= \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{|M^\beta_{\lambda,-}x - w| < M^\beta_{\lambda,-}\eta} \frac{1}{|x - \frac{w}{M^\beta_{\lambda,-}}|^{N-2}} u^-_\lambda\left(\frac{w}{M^\beta_{\lambda,-}}\right) dw \\
&+ \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\{|M^\beta_{\lambda,-}x - w| \geq M^\beta_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^\beta_{\lambda,-}}|^{N-2}} u^-_\lambda\left(\frac{w}{M^\beta_{\lambda,-}}\right) dw \\
&= \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{|M^\beta_{\lambda,-}x - w| < M^\beta_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|M^\beta_{\lambda,-}x - w|^{N-2}} \tilde{u}^-_\lambda(w)\, dw \\
&+ \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\{|M^\beta_{\lambda,-}x - w| \geq M^\beta_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{M^{(N-2)\beta}_{\lambda,-}}{|M^\beta_{\lambda,-}x - w|^{N-2}} \tilde{u}^-_\lambda(w)\, dw := \tilde{I}_{A,\lambda} + \tilde{I}_{B,\lambda}.
\end{aligned}$$

Let us show that $\tilde{I}_{A,\lambda} \to 0$, uniformly for $x \in K$, as $\lambda \to 0$. First, by making the change of variable $z := w - M^{\beta}_{\lambda,-}x$ we have

$$\tilde{I}_{A,\lambda} = \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \tilde{u}^-_{\lambda}\left(z + M^{\beta}_{\lambda,-}x\right)\, dz.$$

Let us fix $\epsilon \in (0, \frac{N-2}{2})$ and set $C = \frac{2}{N-2}\epsilon$. Thanks to the uniform upper bound (1.4.8), since

$$|M^{\beta}_{\lambda,-}x + z| \geq |M^{\beta}_{\lambda,-}|x| - |z|| = M^{\beta}_{\lambda,-}|x| - |z| \geq M^{\beta}_{\lambda,-}(|x| - \eta) > M^{\beta}_{\lambda,-}\frac{d(0,K)}{2} \geq M^{\beta}_{\lambda,-}\eta, \tag{1.6.2}$$

for all $x \in K$, for all $z$ such that $|z| < \eta M^{\beta}_{\lambda,-}$, then for all sufficiently small $\lambda$ we have

$$\begin{aligned}
|\tilde{I}_{A,\lambda}| &\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(N-2)\omega_N} \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \left[1 + \frac{1}{N(N-2)} C|z + M^{\beta}_{\lambda,-}x|^2\right]^{-(N-2)/2} dz \\
&\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_1 \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \left[M^{2\beta}_{\lambda,-}\eta^2\right]^{-(N-2)/2} dz \\
&= \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_2(K)\omega_N \int_0^{M^{\beta}_{\lambda,-}\eta} r\, dr = \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_2(K)\omega_N \frac{M^{2\beta}_{\lambda,-}\eta^2}{2} \\
&= c_3(K)\frac{\lambda^{\frac{N-6}{2N-8}}}{M_{\lambda,-}} \to 0, \quad \text{as } \lambda \to 0.
\end{aligned}$$

Thus, $\tilde{I}_{A,\lambda} \to 0$, uniformly for $x \in K$, as $\lambda \to 0$. Now, we prove that the same holds for $\tilde{I}_{B,\lambda}$.

$$\begin{aligned}
|\tilde{I}_{B,\lambda}| &\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(N-2)\omega_N} \int_{\{|M^{\beta}_{\lambda,-}x - w| \geq M^{\beta}_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{|\eta|^{N-2}} \tilde{u}^-_{\lambda}(w)\, dw \\
&\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c(K) \int_{\tilde{A}_{r_\lambda}} \tilde{u}^-_{\lambda}(w)\, dw \\
&\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c(K) \int_{|w|\leq h} 1\, dw + \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c(K) \int_{h<|w|<M^{\beta}_{\lambda,-}} \left[1 + \frac{1}{N(N-2)} C|w|^2\right]^{-(N-2)/2} dw \\
&\leq \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_1(K) + \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_2(K) \int_h^{M^{\beta}_{\lambda,-}} r\, dr \\
&= \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_1(K) + \frac{\lambda^{\frac{N-6}{2N-8}}}{M^{2^*-1}_{\lambda,-}} c_2(K) \left(\frac{M^{2\beta}_{\lambda,-}}{2} - \frac{h^2}{2}\right) \to 0, \text{ as } \lambda \to 0,
\end{aligned}$$

having used again (1.4.8). Since this estimate is uniform for $x \in K$ we have proved that $\tilde{I}_{B,\lambda} \to 0$ in $C^0(K)$ and from this and the analogous result for $\tilde{I}_{A,\lambda}$ we have $\lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} G_{sing}(x,y) u^-_{\lambda}(y)\, dy \to 0$ in $C^0(K)$. To complete the proof of $I_{2,\lambda} \to 0$ in $C^0(K)$ it remains to prove that $\lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} G_{reg}(x,y) u^-_{\lambda}(y)\, dy \to 0$ in $C^0(K)$. This is easy because the regular part of the Green function for the ball is uniformly bounded, to be precise let $l(K) := \sup\{d(0,x), x \in K\}$, clearly, being $K$ a compact subset of $B_1 - \{0\}$, we have $l(K) < 1$ and since it is well known that

$$G_{reg}(x,y) = \frac{1}{(2-N)\omega_N} \frac{1}{|(|x||y|)^2 + 1 - 2x \cdot y|^{\frac{N-2}{2}}},$$

we have for all $x \in K$, $y \in A_{r_\lambda}$

$$\begin{aligned} \frac{1}{|(|x||y|)^2 + 1 - 2x \cdot y|^{\frac{N-2}{2}}} &\le \frac{1}{|(1 - |x||y|)^2|^{\frac{N-2}{2}}} \\ &\le \frac{1}{|1 - l(K)|^{N-2}} . \end{aligned} \tag{1.6.3}$$

Thus we have

$$\begin{aligned} \left| \lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} G_{reg}(x,y) u_\lambda^-(y)\ dy \right| &\le c(K) \lambda^{\frac{N-6}{2N-8}} \int_{A_{r_\lambda}} |u_\lambda^-(y)|\ dy \\ &= c(K) \frac{\lambda^{\frac{N-6}{2N-8}}}{M_{\lambda,-}^{2^*}} \int_{\tilde{A}_{r_\lambda}} \left| u_\lambda^- \left( \frac{w}{M_{\lambda,-}^\beta} \right) \right| dw \\ &= c(K) \frac{\lambda^{\frac{N-6}{2N-8}}}{M_{\lambda,-}^{2^*-1}} \int_{\tilde{A}_{r_\lambda}} \left| \tilde{u}_\lambda^-(w) \right|\ dw. \end{aligned}$$

As in the previous case, we see that $c(K) \frac{\lambda^{\frac{N-6}{2N-8}}}{M_{\lambda,-}^{2^*-1}} \int_{\tilde{A}_{r_\lambda}} \left| \tilde{u}_\lambda^-(w) \right|\ dw \to 0$ and the proof of $I_{2,\lambda} \to 0$ in $C^0(K)$ is complete.

Now to end the proof, we need to show that $I_{4,\lambda} \to \tilde{c}(N) G(x,0)$ in $C^0(K)$. We start making the usual change of variable

$$I_{4,\lambda} = \lambda^{-\frac{N-2}{2N-8}} \frac{1}{M_{\lambda,-}} \int_{\tilde{A}_{r_\lambda}} G \left( x, \frac{w}{M_{\lambda,-}^\beta} \right) [\tilde{u}_\lambda^-(w)]^{2^*-1}\ dw.$$

We split the Green function in the singular and the regular part, so that

$$\begin{aligned} I_{4,\lambda} &= \frac{1}{(2-N)\omega_N} \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \int_{\tilde{A}_{r_\lambda}} \frac{1}{\left| x - \frac{w}{M_{\lambda,-}^\beta} \right|^{N-2}} [\tilde{u}_\lambda^-(w)]^{2^*-1}\ dw \\ &+ \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \int_{\tilde{A}_{r_\lambda}} G_{reg} \left( x, \frac{w}{M_{\lambda,-}^\beta} \right) [\tilde{u}_\lambda^-(w)]^{2^*-1}\ dw \end{aligned}$$

We begin with the singular integral which is more delicate. We want to show that

$$\frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\tilde{A}_{r_\lambda}} \frac{1}{\left| x - \frac{w}{M_{\lambda,-}^\beta} \right|^{N-2}} [\tilde{u}_\lambda^-(w)]^{2^*-1}\ dw \to \tilde{c}(N) G_{sing}(x,0) \text{ in } C^0(K). \tag{1.6.4}$$

As in the previous case, we consider the ball $B(M^{\beta}_{\lambda,-}x, M^{\beta}_{\lambda,-}\eta) \subset\subset \tilde{A}_{r_\lambda}$, where $\eta > 0$ is the same as before. Thus, we have

$$\begin{aligned}
& \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\tilde{A}_{r_\lambda}} \frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{N-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1}\, dw \\
= \ & \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{|M^{\beta}_{\lambda,-}x - w| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|M^{\beta}_{\lambda,-}x - w|^{N-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1}\, dw \\
+ \ & \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\{|M^{\beta}_{\lambda,-}x - w| \geq M^{\beta}_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{M^{(N-2)\beta}_{\lambda,-}}{|M^{\beta}_{\lambda,-}x - w|^{N-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1}\, dw \\
:= \ & \tilde{I}_{C,\lambda} + \tilde{I}_{D,\lambda}.
\end{aligned}$$

We show that $\tilde{I}_{C,\lambda} \to 0$ in $C^0(K)$. As before, using the uniform upper bound (1.4.8) and (1.6.2) we get

$$\begin{aligned}
|\tilde{I}_{C,\lambda}| &= \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(N-2)\omega_N} \int_{|z| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \left[\tilde{u}^-_\lambda\left(z + M^{\beta}_{\lambda,-}x\right)\right]^{2^*-1} dz \\
&\leq \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(N-2)\omega_N} \int_{|z| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \left[1 + \frac{1}{N(N-2)} C|z + M^{\beta}_{\lambda,-}x|^2\right]^{-(N+2)/2} dz \\
&\leq \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} c_1 \int_{|z| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{|z|^{N-2}} \left[M^{2\beta}_{\lambda,-}\eta^2\right]^{-(N+2)/2} dz \\
&= \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} c_2(K) \int_0^{M^{\beta}_{\lambda,-}\eta} \frac{M^{(N-2)\beta}_{\lambda,-}}{r^{N-2}} M^{-(N+2)\beta}_{\lambda,-} r^{N-1}\, dr \\
&= \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} c_2(K) \frac{1}{M^{4\beta}_{\lambda,-}} \int_0^{M^{\beta}_{\lambda,-}\eta} r\, dr = \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} c_2(K) \frac{1}{M^{4\beta}_{\lambda,-}} \frac{M^{2\beta}_{\lambda,-}\eta^2}{2} \\
&= c_3(K) \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{M^{2\beta}_{\lambda,-}}.
\end{aligned}$$

Since $\frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}}$ is bounded (see Proposition 1.5.2 (iv) and Remark 1.5.3) then $\tilde{I}_{C,\lambda} \to 0$ uniformly for $x \in K$. Now, we show that $\tilde{I}_{D,\lambda} \to \tilde{c}(N) G_{sing}(x,0)$ in $C^0(K)$. We have

$$\tilde{I}_{D,\lambda} = \frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}} \frac{1}{(2-N)\omega_N} \int_{\left\{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right| \geq \eta\right\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{N-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1}\, dw$$

The first step is to prove that for all $w \in \mathbb{R}^N - \{0\}$

$$\chi(w)_{\left\{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right| \geq \eta\right\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{(2-N)\omega_N} \frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{N-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1} \to G_{sing}(x,0)\,\mathcal{U}^{2^*-1}_{0,\mu}(w), \tag{1.6.5}$$

uniformly for $x \in K$. First, observe that we need only to show that

$$\frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{N-2}} [\tilde{u}^{-}_{\lambda}(w)]^{2^*-1} \to \frac{1}{|x|^{N-2}} \mathcal{U}^{2^*-1}_{0,\mu}(w) \quad \text{in } C^0(K). \tag{1.6.6}$$

In fact, if we fix $w \in \mathbb{R}^N - \{0\}$, and $\lambda > 0$ is sufficiently small so that $w \in \tilde{A}_{r_\lambda}$ and $\frac{w}{M^{\beta}_{\lambda,-}} < \frac{d(0,K)}{2}$ then we have $|x - \frac{w}{M^{\beta}_{\lambda,-}}| \geq \eta$, for all $x \in K$. Hence we get

$$\left| \chi(w)_{\left\{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right| \geq \eta\right\} \cap \tilde{A}_{r_\lambda}} - 1 \right| = \chi(w)_{\left\{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right| < \eta\right\} \cup \tilde{A}^c_{r_\lambda}} = 0,$$

for all $x \in K$, for all $\lambda > 0$ sufficiently small, from which we deduce that

$$\chi(w)_{\left\{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right| \geq \eta\right\} \cap \tilde{A}_{r_\lambda}} \to 1 \quad \text{in } C^0(K).$$

Now, the proof of (1.6.6) is trivial if we show that, for any fixed $w \in \mathbb{R}^N - \{0\}$

$$\left| \frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{N-2}} - \frac{1}{|x|^{N-2}} \right| \leq c(K) \left| \frac{w}{M^{\beta}_{\lambda,-}} \right| \tag{1.6.7}$$

for all $x \in K$ and for all $\lambda > 0$ sufficiently small. This is an elementary computation but for the sake of completeness, we give the proof. We observe that the segment $\sigma\left(x, x - \frac{w}{M^{\beta}_{\lambda,-}}\right)$ joining $x$ and $x - \frac{w}{M^{\beta}_{\lambda,-}}$ is an uniformly bounded set and stays away from the origin. In fact for all $x \in K$, $t \in [0,1]$ and for all $\lambda > 0$ sufficiently small, we have

$$\left| x - t\frac{w}{M^{\beta}_{\lambda,-}} \right| \leq |x| + |t| \left| \frac{w}{M^{\beta}_{\lambda,-}} \right| < 1 + \frac{d(0,K)}{2} \tag{1.6.8}$$

$$\left| x - t\frac{w}{M^{\beta}_{\lambda,-}} \right| \geq \left| |x| - |t| \frac{|w|}{M^{\beta}_{\lambda,-}} \right| \geq d(0,K) - t\frac{d(0,K)}{2} \geq \frac{d(0,K)}{2}. \tag{1.6.9}$$

Thus, setting $g(x) := \frac{1}{|x|^{N-2}}$, by the mean value theorem, we have

$$g\left(x - \frac{w}{M^{\beta}_{\lambda,-}}\right) - g(x) = \nabla g(\xi_{\lambda,x}) \cdot \left(-\frac{w}{M^{\beta}_{\lambda,-}}\right),$$

where $\xi_{\lambda,x}$ lies on $\sigma\left(x, x - \frac{w}{M^{\beta}_{\lambda,-}}\right)$. By (1.6.8) and (1.6.9) we deduce that $|\nabla g(\xi_{\lambda,x})|$ is uniformly bounded[2] and (1.6.7) is proved.

To complete the first part of the proof, we apply Lebesgue's theorem. For all $x \in K$, $w \in \mathbb{R}^N - \{0\}$ we have

[2] by $\|\nabla g\|_{\infty, R(K)}$, where $R(K)$ is the compact annulus $R(K) := \{x \in \mathbb{R}^N;\ \frac{d(0,K)}{2} \leq |x| \leq 1 + \frac{d(0,K)}{2}\}$

$$\begin{aligned}&\left|\chi_{\left\{\left|x-\frac{w}{M^{\beta}_{\lambda,-}}\right|\geq\eta\right\}\cap\tilde{A}_{r_\lambda}}\frac{1}{(2-N)\omega_N}\frac{1}{|x-\frac{w}{M^{\beta}_{\lambda,-}}|^{N-2}}[\tilde{u}^-_\lambda(w)]^{2^*-1}\right|\\ \leq\ &\eta^{-(N-2)}\frac{1}{(N-2)\omega_N}\frac{1}{|x|^{N-2}}[U_h(w)]^{2^*-1}\\ =\ &c_1(K)[U_h(w)]^{2^*-1},\end{aligned}$$

where $U_h$ is the function defined in (1.4.9). Since $(U_h)^{2^*-1}\in L^1(\mathbb{R}^N)$ and thanks to (1.6.5), (iv) of Proposition 1.5.2, by Lebesgue's theorem we deduce (1.6.4), where $G_{sing}(x,0)=\frac{1}{(2-N)\omega_N}\frac{1}{|x|^{N-2}}$, $\tilde{c}(N)=(\lim_{\lambda\to 0}\frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}})\int_{\mathbb{R}^N}\mathcal{U}^{2^*-1}_{0,\mu}(w)\,dw$. It's an elementary computation to see that $\tilde{c}(N)$ equals the expected constant $\omega_N\frac{c_2(N)^{\frac{N-2}{2N-8}}}{c_1(N)^{\frac{4}{2N-8}}}$, where $c_1(N),c_2(N)$ are the constants defined in Proposition 1.5.1. And the proof of (1.6.4) is done.

Finally, we prove that

$$\frac{\lambda^{-\frac{N-2}{2N-8}}}{M_{\lambda,-}}\int_{\tilde{A}_{r_\lambda}}G_{reg}\left(x,\frac{w}{M^{\beta}_{\lambda,-}}\right)[\tilde{u}^-_\lambda(w)]^{2^*-1}\,dw\to\tilde{c}(N)G_{reg}(x,0)\ \text{ in } C^0(K). \tag{1.6.10}$$

Since

$$G_{reg}\left(x,\frac{w}{M^{\beta}_{\lambda,-}}\right)=\frac{1}{(2-N)\omega_N}\frac{1}{\left||x|^2\frac{|w|^2}{M^{2\beta}_{\lambda,-}}+1-2x\cdot\frac{w}{M^{\beta}_{\lambda,-}}\right|^{\frac{N-2}{2}}}$$

by the mean value theorem, repeating a similar argument as in the proof of (1.6.7), we deduce that for any fixed $w\in\mathbb{R}^N-\{0\}$

$$G_{reg}\left(x,\frac{w}{M^{\beta}_{\lambda,-}}\right)\to G_{reg}(x,0)\quad\text{in } C^0(K).$$

Thus, for any $w\in\mathbb{R}^N-\{0\}$ we have

$$G_{reg}\left(x,\frac{w}{M^{\beta}_{\lambda,-}}\right)[\tilde{u}^-_\lambda(w)]^{2^*-1}\to G_{reg}(x,0)\mathcal{U}^{2^*-1}_{0,\mu}(w)\ \text{ in } C^0(K).$$

Thanks to (1.6.3) we know that $G_{reg}\left(x,\frac{w}{M^{\beta}_{\lambda,-}}\right)$ is uniformly bounded, moreover, as we have done in the proof of (1.6.4), thanks to the upper bound (1.4.8), Proposition 1.5.2 we deduce (1.6.10).

To prove the local uniform convergence of $\lambda^{-\frac{N-2}{2N-8}}\nabla u_\lambda$ to $\tilde{c}(N)\nabla G(x,0)$ we simply derive (1.6.1) and repeat the previous proof, taking into account that for $i=1,\dots,n$ we have

$$\partial_{x_i}G_{sing}(x,y)=\frac{1}{\omega_N}\frac{x_i-y_i}{|x-y|^N}.$$

□

# Chapter 2

# Asymptotic analysis for radial sign-changing solutions of the Brezis-Nirenberg problem, $N < 7$.

## 2.1 Introduction

In this chapter we present and prove the result **(R2)**.

We consider the Brezis–Nirenberg problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{2.1.1}$$

where $\lambda > 0$, $2^* = \frac{2N}{N-2}$ and $B_1$ is the unit ball of $\mathbb{R}^N$, $N \geq 3$.

The aim of this chapter is to get asymptotic results for radial sign-changing solutions $u_\lambda$ of (2.1.1) in dimensions $N = 3, 4, 5, 6$. This will give the asymptotic profile of the positive and negative part of $u_\lambda$ as $\lambda$ tends to some limit value.

To motivate our analysis and to explain our results we need to recall a few known results.

The first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983 in the celebrated paper [17]. From their results it came out that the dimension was going to play a crucial role in the study of (2.1.1) in a general bounded domain $\Omega$. Indeed they proved that if $N \geq 4$ there exists a positive solution of (2.1.1) for every $\lambda \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\Delta$ in $\Omega$ with Dirichlet boundary conditions, while if $N = 3$ positive solutions exist only for $\lambda$ away from zero.

Since then several other interesting results were obtained for positive solutions, in particular about the asymptotic behavior of solutions, mainly for $N \geq 5$, because also the case $N = 4$ presents more difficulties compared to the higher dimensional ones.

Concerning the case of sign-changing solutions, existence results hold if $N \geq 4$ both for $\lambda \in (0, \lambda_1(\Omega))$ and $\lambda > \lambda_1(\Omega)$ as shown in [6], [20], [24].

The case $N = 3$ presents even more difficulties than in the study of positive solutions. In particular in the case of the ball is not yet known what is the least value $\bar{\lambda}$ of the parameter $\lambda$ for which sign-changing solutions exist, neither whether $\bar{\lambda}$ is larger or smaller than $\lambda_1(B_1)/4$. This question, posed by H. Brezis, has been given a partial answer in [14].

However it is interesting to observe that in the study of sign-changing solutions even the "low dimensions" $N = 4, 5, 6$ exhibit some peculiarities. Indeed it was first proved by Atkinson, Brezis and Peletier in [5] that if $\Omega$ is the ball $B_1$ there exists $\lambda^* = \lambda^*(N)$ such that there are no radial sign-changing solutions of (2.1.1) for $\lambda \in (0, \lambda^*)$. Later this result was proved in [1] in a different way.

Moreover, for $N \geq 7$ a recent result of Schechter and Zou [54] shows that in any bounded smooth domain there exist infinitely many sign-changing solutions for any $\lambda > 0$. Instead if $N = 4, 5, 6$ only $N + 1$ pairs of solutions, for all $\lambda > 0$, have been proved to exist in [24] but it is not clear that they change sign.

Coming back to radial sign-changing solutions and to the question of existence or nonexistence of them, according to the dimension, as shown by Atkinson, Brezis and Peletier, it is interesting to understand in which way these results can be extended to other bounded domains and to which kind of solutions.

In order to analyze this question let us divide the discussion in two cases: the first one when the dimension $N$ is greater or equal than 7 and the second one when $N < 7$.

In the first case ($N \geq 7$) radial sign-changing solutions $u_\lambda$ exist for all $\lambda > 0$, if the domain is a ball, and analyzing the asymptotic behavior of those of least energy, as $\lambda \to 0$, it is proved in [37] that their limit profile is that of a "tower of two bubbles". This terminology means that the positive part and the negative part of the solutions $u_\lambda$ concentrate at the same point (which is obviously the center of the ball) as $\lambda \to 0$ and each one has the limit profile, after suitable rescaling, of a "standard bubble" in $\mathbb{R}^N$, i.e. of a positive solution of the critical exponent problem in $\mathbb{R}^N$. More precisely the solutions can be written in the following way:

$$u_\lambda = \mathcal{PU}_{\xi,\delta_1} - \mathcal{PU}_{\xi,\delta_2} + w_\lambda, \tag{2.1.2}$$

where $\mathcal{PU}_{\xi,\delta_i}$, $i = 1, 2$ is the projection on $H_0^1(\Omega)$ of the regular positive solution of the critical problem in $\mathbb{R}^N$, centered at $\xi = 0$, with rescaling parameter $\delta_i$ and $w_\lambda$ is a remainder term which converges to zero in $H_0^1(\Omega)$ as $\lambda \to 0$.

Inspired by this result one could then search for solutions of type (2.1.2) in general bounded domains since this kind of solutions can be viewed as the ones which play the same role of the radial solutions in the case of the ball. This has been done recently in [40], where solutions of the type (2.1.2) have been constructed for $\lambda$ close to zero in some symmetric bounded domains (the symmetry makes their construction a bit easier, but the same result should be true in any bounded domain).

On the contrary, coming to the case $N < 7$, in view of the nonexistence result of nodal radial solutions of [6] it is natural to conjecture that, in general bounded domains, there should not be solutions of the form (2.1.2) for $\lambda$ close to zero. Indeed this has been recently proved in [38] if $N = 4, 5, 6$, the case $N = 3$ being obvious.

On the other side, if $N < 7$, radial nodal solutions exist for $\lambda$ bigger than a certain value $\bar{\lambda}_2$ which can be studied by analyzing the associated ordinary differential equation (see [6], [4], [32]).

Therefore, to the aim of getting analogous existence results in other bounded domains, the first step would be to analyze the asymptotic behavior of nodal radial solutions in the ball, for $\lambda \to \bar{\lambda}_2$, in order to understand their limit profile and guess what kind of solutions one can construct in other domains, and for which values of the parameter $\lambda$.

This is the subject of this chapter.

Denoting by $u_\lambda$ a nodal radial solutions of (2.1.1) having two nodal regions and such that $u_\lambda(0) > 0$ we get the following results:

**(i)**: if $N = 6$ then $\bar{\lambda}_2 \in (0, \lambda_1(B_1))$, $\lambda_1(B_1)$ being the first eigenvalue of $-\Delta$ in $H_0^1(B_1)$, and we have that, as $\lambda \to \bar{\lambda}_2$, $u_\lambda^+$ concentrate at the center of the ball, $\|u_\lambda^+\|_\infty \to +\infty$, and a suitable rescaling of $u_\lambda^+$ converges to the standard positive solution of the critical problem in $\mathbb{R}^N$. Instead $u_\lambda^-$ converges to the unique positive solution of (2.1.1) in $B_1$, as $\lambda \to \bar{\lambda}_2$;

**(ii)**: if $N = 4, 5$ then $\bar{\lambda}_2 = \lambda_1(B_1)$ and $u_\lambda^+$ behaves as for the case $N = 6$, while $u_\lambda^-$ converges to zero uniformly in $B_1$;

**(iii)**: if $N = 3$ then $\bar{\lambda}_2 = \frac{9}{4}\lambda_1(B_1)$ and $u_\lambda^+$ behaves as for the case $N = 6$, while $u_\lambda^-$ converges to zero uniformly in $B_1$.

In view of these results we conjecture that, in general bounded domains $\Omega$, for some "limit value" $\bar{\lambda}_2 = \bar{\lambda}_2(N, \Omega)$ there should exist solutions with similar asymptotic profile as $\lambda \to \bar{\lambda}_2$. The number $\bar{\lambda}_2$ should be $\lambda_1(\Omega)$ in dimension $N = 4, 5$.

This chapter is divided in five sections. In Section 2.2 we mainly recall some preliminary results. In Section 2.3 we analyze the asymptotic behavior of the positive part of the solutions, for all dimensions $N = 3, 4, 5, 6$. In Section 2.4 we analyze the negative part in the case $N = 6$ and in Section 2.5 we complete the cases $N = 3, 4, 5$.

## 2.2 Some preliminary results

If $u_\lambda$ is a radial sign-changing solution of (2.1.1) then we can write $u_\lambda = u_\lambda(r)$, where $r = |x|$ and $u_\lambda(r)$ is a solution of the problem

$$\begin{cases} u_\lambda'' + \frac{N-1}{r} u_\lambda' + \lambda u_\lambda + |u_\lambda|^{2^*-2} u_\lambda = 0, & \text{in } (0,1), \\ u_\lambda'(0) = 0, \quad u_\lambda(1) = 0. \end{cases} \tag{2.2.1}$$

We consider the following transformation

$$r \mapsto \left(\frac{N-2}{\sqrt{\lambda}\, r}\right)^{N-2}, \quad u_\lambda \mapsto y(t) := \lambda^{-1/(2^*-2)} u_\lambda \left(\frac{N-2}{\sqrt{\lambda}\, t^{\frac{1}{N-2}}}\right). \tag{2.2.2}$$

It is elementary to see that since $u_\lambda$ is a solution of the differential equation in (2.2.1) then $y = y(t)$ solves

$$y'' + t^{-k}(y + |y|^{2^*-2} y) = 0, \tag{2.2.3}$$

in the interval $\left(\left(\frac{N-2}{\sqrt{\lambda}}\right)^{N-2}, +\infty\right)$, where $k := 2\frac{N-1}{N-2}$. It is clear that the transformation (2.2.2) generates a one-to-one correspondence between solutions of the differential equation in (2.2.1) and solutions of (2.2.3). Equation (2.2.3) is an Emden-Fowler type equation and since $k > 2$ it is well known that, for any $\gamma \in \mathbb{R}$ the problem

$$\begin{cases} y'' + t^{-k} f(y) = 0, & \text{in } (0, +\infty), \\ y(t) \to \gamma, & \text{as } t \to +\infty, \end{cases} \tag{2.2.4}$$

where $f(y) := y + |y|^{2^*-2} y$, has a unique solution defined in the whole $\mathbb{R}^+$ which we denote by $y(t; \gamma)$. Let us recall some results on the functions $y(t; \gamma)$ which are proved in [6].

**Lemma 2.2.1.** *Let $y = y(t, \gamma)$ be a solution of Problem (2.2.4), then:*

*(a) $y$ is oscillatory near $t = 0$;*

*(b) the set $\{|y(\bar{t})|;\ \bar{t}$ extremum point of $y\}$ is an increasing sequence with respect to $t$;*

*(c) the set $\{|y'(t_0)|;\ t_0$ zero of $y\}$ is a decreasing sequence with respect to $t$.*

*Proof.* See Lemma 1 in [6]. □

**Lemma 2.2.2.** *Let $y = y(t, \gamma)$ be a solution of Problem (2.2.4) and let $T > 0$ be one of its zeros, then*

$$|y(t)| < |y'(T)|(T - t),$$

*for all $0 < t < T$.*

*Proof.* See Lemma 2 in [6]. □

We shall denote the sequence of zeros of $y(t; \gamma)$ by $T_n(\gamma)$, ordered backwards, precisely:

$$\cdots < T_3(\gamma) < T_2(\gamma) < T_1(\gamma) < +\infty.$$

We recall some results on the asymptotic behavior of the largest zero $T_1(\gamma)$ and on the slope $y'(T_1(\gamma); \gamma)$ as $\gamma \to +\infty$.

**Lemma 2.2.3.** *Let $y$ be a solution of Problem (2.2.4) and $T_1(\gamma)$ its largest zero, then:*

*(a) if $2 < k < 3$ (which corresponds to $N > 4$), then*

$$T_1(\gamma) = A(k)\gamma^{6-2k}(1 + o(1)) \quad \text{as } \gamma \to +\infty,$$

*where $A(k) := (k-1)^{\frac{k-3}{k-2}} \frac{\Gamma(3-k)/(k-2)\Gamma((k-1)/(k-2))}{\Gamma(2/(k-2))}$, $\Gamma$ is the Gamma function.*

*(b) if $k = 3$ (which corresponds to $N = 4$), then*

$$T_1(\gamma) = 2\log\gamma(1 + o(1)) \quad \text{as } \gamma \to +\infty;$$

*(c) if $k = 4$ (which corresponds to $N = 3$), then there exists $\gamma_0 \in \mathbb{R}^+$ and two positive constants $A, B$ such that*

$$A < T_1(\gamma) < B \quad \text{for all } \gamma \geq \gamma_0.$$

*Proof.* The proof of (a), (b) is contained in [6], Lemma 3 and the proof of (c) is contained in [7], Theorem 3. □

**Lemma 2.2.4.** *For any $k > 2$, let $y$ be a solution of Problem (2.2.4) and $T_1(\gamma)$ its largest zero, then*

$$y'(T_1(\gamma)) = (k-1)^{\frac{1}{k-2}}\gamma^{-1}(1 + o(1)), \quad \text{as } \gamma \to +\infty.$$

*Proof.* See [6], Lemma 4. □

To prove the existence of radial sign-changing solutions of (2.1.1), with exactly two nodal regions, we consider the second zero $T_2(\gamma)$ of $y(t;\gamma)$. If we choose $\lambda = \lambda(\gamma)$ so that $T_2(\gamma) = \left(\frac{N-2}{\sqrt{\lambda}}\right)^{N-2}$, then the inverse transformation of (2.2.2) maps $t = T_2$ in $r = 1$ and $y \mapsto u_\lambda$. Hence, for $\lambda = (N-2)^2 T_2(\gamma)^{-\frac{2}{N-2}}$, we obtain a function $u_\lambda$ which is a radial solution of (2.1.1) having exactly two nodal regions; moreover $u_\lambda(0) = \lambda^{1/(2^*-2)}\gamma$. We observe also that thanks to the invertibility of (2.2.2) every radial sign-changing solution $u_\lambda$ of (2.1.1) with two nodal regions corresponds to a solution $y = y(t;\gamma)$ of (2.2.4) with $\gamma = \lambda^{-1/(2^*-2)} u_\lambda(0)$, $T_2(\gamma) = \left(\frac{N-2}{\sqrt{\lambda}}\right)^{N-2}$.

We are interested in the study of the behavior of the map $\lambda_2 : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $\lambda_2(\gamma) := (N-2)^2 T_2(\gamma)^{-\frac{2}{N-2}}$. Clearly this map is continuous. In [4] (see Proposition 2 and Remark 4), it is proved that for $N = 4$ it holds that $\lim_{\gamma\to 0} \lambda_2(\gamma) = \lambda_2(B_1)$, where $\lambda_2(B_1)$ is the second radial eigenvalue of $-\Delta$ in $H^1_0(B_1)$. Moreover the authors observe that this result holds for all dimensions $N \geq 3$. For the sake of completeness we give a complete proof of this fact. We begin with a preliminary lemma.

**Lemma 2.2.5.** *Let $u_\lambda$ be a radial solution of (2.1.1), then we have $|u_\lambda(0)| = \|u_\lambda\|_\infty$.*

*Proof.* See Proposition 1.2.3 or Lemma 8 in [4]. □

**Proposition 2.2.6.** *Let $N \geq 3$ and $\lambda_2 : \mathbb{R}^+ \to \mathbb{R}^+$ the function defined by $\lambda_2(\gamma) := (N-2)^2 T_2(\gamma)^{-\frac{2}{N-2}}$, where $T_2(\gamma)$ is the second zero of the function $y(t,\gamma)$, $y(t,\gamma)$ is the unique solution of (2.2.4). We have:*

*(a) $\lambda_2(\gamma) < \lambda_2(B_1)$, for all $\gamma \in \mathbb{R}^+$;*

*(b) $\lim_{\gamma\to 0} \lambda_2(\gamma) = \lambda_2(B_1)$,*

*where $\lambda_2(B_1)$ is the second radial eigenvalue of $-\Delta$ in $H^1_0(B_1)$.*

*Proof.* To prove (a) we observe that (a) is equivalent to show that $T_2(\gamma) > \tau_2$ for all $\gamma \in \mathbb{R}^+$, where $\tau_2$ is the second zero of the function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ defined by $\alpha(t) := A_\nu \sqrt{t} J_\nu(2\nu t^{-\frac{1}{2\nu}})$, where $A_\nu := \nu^{-\nu}\Gamma(\nu+1)$, $\nu := \frac{1}{k-2} = \frac{N-2}{2}$, $J_\nu$ is the first kind (regular) Bessel function of order $\nu$, namely $J_\nu(s) := \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+\nu+1)} \left(\frac{s}{2}\right)^{\nu+2j}$. In fact, by a tedious computation, we see that $\alpha$ solves

$$\begin{cases} \alpha'' + t^{-k}\alpha = 0, & \text{in } (0,+\infty), \\ \alpha(t) \to 1, & \text{as } \ t \to +\infty. \end{cases} \tag{2.2.5}$$

Furthermore, let $\tau_2$ be the second zero of $\alpha$, then by elementary computations we see that the function $\varphi_2(x) := \alpha(\tau_2|x|^{-(N-2)})$ solves

$$\begin{cases} -\Delta\varphi_2 = \mu_2\varphi_2 & \text{in } B_1 \\ \varphi_2 = 0 & \text{on } \partial B_1, \end{cases} \tag{2.2.6}$$

with $\mu_2 = (N-2)^2 \tau_2^{-\frac{2}{N-2}}$. Clearly $\mu_2 = \lambda_2(B_1)$. Hence $\lambda_2(\gamma) < \lambda_2(B_1)$ if and only if $T_2(\gamma) > \tau_2$.

To show that $T_2(\gamma) > \tau_2$ for all $\gamma \in \mathbb{R}^+$ first observe that for all $\gamma \in \mathbb{R}^+$ we have $T_1(\gamma) > \tau_1$. In fact, setting $\lambda_1(\gamma) := (N-2)^2 T_1(\gamma)^{-\frac{2}{N-2}}$ as before we have that $\lambda_1(\gamma) < \lambda_1(B_1)$ if and only if $T_1(\gamma) > \tau_1$. Since we know from [17] that equation (2.2.1) has positive solutions only for $\lambda \in (0, \lambda_1(B_1))$ if $N \geq 4$, and only for $\lambda \in (\frac{\lambda_1(B_1)}{4}, \lambda_1(B_1))$ if $N = 3$, we deduce $T_1(\gamma) > \tau_1$ for all $\gamma \in \mathbb{R}^+$. Now we apply the Sturm's comparison theorem to the functions $y(t;\gamma)$, $\alpha(t)$, which are, respectively, solutions of the equations in (2.2.4), (2.2.5). To this end we write $y'' + t^{-k} q_2(t) y = 0$ with $q_2(t) := 1 + |y|^{2^*-2}$ and since $\alpha'' + t^{-k}\alpha = 0$ we set $q_1(t) :\equiv 1$. Clearly $q_2(t) \geq q_1(t)$ for all $t > 0$ (for all $\gamma \in \mathbb{R}^+$), thus $y$ is a Sturm majorant for $\alpha$, and applying the Sturm's comparison theorem in the interval $[\tau_2, \tau_1]$, since $T_1(\gamma) > \tau_1$ we deduce that $T_2(\gamma) \in (\tau_2, \tau_1)$. This concludes the proof of (a).

Let us prove (b). We consider $u_{\lambda_2(\gamma)} = u_{\lambda_2(\gamma)}(r)$ which is a solution of (2.2.1) with exactly one zero in $(0,1)$, and $u_{\lambda_2(\gamma)}(0) = [\lambda_2(\gamma)]^{1/(2^*-2)}\gamma$. Setting $\varphi(x) := u_{\lambda_2}(|x|)$ it is clear that $\varphi$ is the second radial eigenfunction of

$$\begin{cases} -\Delta\varphi = \lambda\varphi + |u_{\lambda_2(\gamma)}|^{2^*-2}\varphi & \text{in } B_1 \\ \varphi = 0 & \text{on } \partial B_1, \end{cases} \tag{2.2.7}$$

with eigenvalue $\lambda = \lambda_2(\gamma)$. Let us denote by $H^1_{0,rad}(B_1)$ the subspace of radially symmetric functions in $H^1_0(B_1)$. Thanks to the variational characterization of eigenvalues and Lemma 2.2.5 we have

$$\begin{aligned} \lambda_2(\gamma) &= \min_{\substack{V\subset H^1_{0,rad}(B_1)\\ \dim V=2}} \max_{\substack{\varphi\in V\\ |\varphi|_2=1}} \left( \int_{B_1} |\nabla\varphi|^2\, dx - \int_{B_1} |u_{\lambda_2(\gamma)}|^{2^*-2}\varphi^2\, dx \right) \\ &> \min_{\substack{V\subset H^1_{0,rad}(B_1)\\ \dim V=2}} \max_{\substack{\varphi\in V\\ |\varphi|_2=1}} \left( \int_{B_1} |\nabla\varphi|^2\, dx - [\lambda_2(\gamma)]^{2/(2^*-2)}\gamma^2 \right) \\ &= \lambda_2(B_1) - [\lambda_2(\gamma)]^{2/(2^*-2)}\gamma^2. \end{aligned} \tag{2.2.8}$$

Since $\lambda_2(\gamma)$ is bounded (because by (a) we have $\lambda_2(\gamma) < \lambda_2(B_1)$ and by definition $\lambda_2(\gamma) > 0$), from (2.2.8), we deduce that $\liminf_{\gamma\to 0} \lambda_2(\gamma) \geq \lambda_2(B_1)$. On the other hand, by the first step we get that $\limsup_{\gamma\to 0} \lambda_2(\gamma) \leq \lambda_2(B_1)$. Hence we deduce that $\lim_{\gamma\to 0} \lambda_2(\gamma) = \lambda_2(B_1)$ and the proof is concluded. ☐

More interesting is the behavior of $\lambda_2(\gamma)$ as $\gamma \to +\infty$. The next result that we recall shows how it strongly depends on the dimension $N$.

**Theorem 2.2.7.** *Let $\lambda_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by $\lambda_2(\gamma) := (N-2)^2 T_2(\gamma)^{-\frac{2}{N-2}}$, where $T_2(\gamma)$ is the second zero of the function $y(t,\gamma)$, being $y(t,\gamma)$ is the unique solution (2.2.4), and let $\lambda_1(B_1)$ be the first eigenvalue of $-\Delta$ in $H^1_0(B_1)$, then:*

*(a) if $N \geq 7$ we have $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = 0$;*

*(b) if $N = 6$ we have $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = \lambda_0$, for some $\lambda_0 \in (0, \lambda_1(B_1))$;*

*(c) if $N = 4$ or $N = 5$ we have $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = \lambda_1(B_1)$;*

*(d) if $N = 3$ we have $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = \frac{9}{4}\lambda_1(B_1) = \frac{9}{4}\pi^2$.*

*Proof.* Statement (a) is a consequence of Theorem B in [25]. Statements (b), (c) are proved in [6], Theorem B. In Section 2.4 we give an alternative proof of (b). Statement (d) is proved in [8]. ☐

Let us define $\lambda_2^\star := \inf\{\lambda_2(\gamma),\ \gamma \in \mathbb{R}^+\}$. Gazzola and Grunau proved in [32] that for $N = 5$ it holds $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = \lambda_1(B_1)^-$, in particular we deduce that for $N = 5$ we have $\lambda_2^\star < \lambda_1(B_1)$ and hence $\lambda_2^\star = \lambda_2(\gamma_0)$ for some $\gamma_0 \in \mathbb{R}^+$. In the same paper it is also proved that for $N = 4$ $\lim_{\gamma\to+\infty} \lambda_2(\gamma) = \lambda_1(B_1)^+$. Recently Arioli, Gazzola, Grunau, Sassone proved in [4] a stronger result: for $N = 4$ we have $\lambda_2(\gamma) > \lambda_1(B_1)$ for all $\gamma \in \mathbb{R}^+$. Thus for $N = 4$, we have $\lambda_2^\star = \lambda_1(B_1)$ and $\lambda_2^\star$ is not achieved.

The asymptotic behavior of $\lambda_2(\gamma)$ as $\gamma \to +\infty$ for $N = 6$ is still unknown. Nevertheless in Section 2.3 we give a characterization of the number $\lambda_0$ appearing in (b) of Theorem 2.2.7.

## 2.3 Energy and asymptotic analysis of the positive part

Let $u_{\lambda_2(\gamma)}$ be the radial solution with exactly two nodal regions of (2.1.1), for $\lambda = \lambda_2(\gamma)$, obtained in the previous section. To simplify the notation we omit the dependence on $\gamma$ and write $u_{\lambda_2}$. We recall that, by definition, for $\gamma \in \mathbb{R}^+$ we have $u_{\lambda_2}(0) > 0$ and we denote by $r_{\lambda_2} \in (0, 1)$ its node.

The aim of this section is to compute the limit energy of the positive part $u^+_{\lambda_2}$, as $\gamma \to +\infty$, as well as, to study the asymptotic behavior of a suitable rescaling of $u^+_{\lambda_2}$. We begin with recalling an elementary but crucial fact:

**Lemma 2.3.1.** *Let $u \in H^1_{0,rad}(B)$, where $B$ is a ball or an annulus centered at the origin of $\mathbb{R}^N$ and consider the rescaling $\tilde{u}(y) := M^{1/\beta} u(My)$, where $M > 0$ is a constant, $\beta := \frac{2}{N-2}$. We have:*

*(i):* $\|u\|^2_B = \|\tilde{u}\|^2_{M^{-1}B}$,

*(ii):* $|u|^{2^*}_{2^*,B} = |\tilde{u}|^{2^*}_{2^*,M^{-1}B}$,

*(iii):* $|u|^2_{2,B} = M^2 |\tilde{u}|^2_{2,M^{-1}B}$.

*Proof.* It suffices to apply the formula of change of variable for the integrals in (i), (ii), (iii). For the details see the proof of Lemma 1.4.1. □

In order to state the main result of this section we introduce some notation. We define the rescaled functions

$$\tilde{u}^+_{\lambda_2}(y) := \frac{1}{M_{\lambda_2,+}} u^+_{\lambda_2}\left(\frac{y}{M^\beta_{\lambda_2,+}}\right), \quad y \in B_{\sigma_{\lambda_2}},$$

where $\beta := \frac{2}{N-2}$, $\sigma_{\lambda_2} = M^\beta_{\lambda_2,+} r_{\lambda_2}$, $M_{\lambda_2,+} := \|u^+_{\lambda_2}\|_{\infty,B_1}$. We observe that thanks to Lemma 2.2.5 and since $u_{\lambda_2}(0) > 0$ we have $M_{\lambda_2,+} = \|u_{\lambda_2}\|_{\infty,B_1} = u_{\lambda_2}(0)$. The following theorem holds for all dimensions $N \geq 3$, here we discuss the case $3 \leq N \leq 6$ (the case $N \geq 7$ has been studied in Chapter 1).

**Theorem 2.3.2.** *Let $N = 3, 4, 5, 6$ and let $u_{\lambda_2}$ be the radial solution with exactly two nodal regions of (2.1.1) with $\lambda = \lambda_2(\gamma)$ obtained in the previous section. Then*

*(i):*

$$J_{\lambda_2}(u^+_{\lambda_2}) \to \frac{1}{N} S^{N/2},$$

*as $\gamma \to +\infty$, where $J_\lambda(u) := \frac{1}{2}\left(\int_{B_1} |\nabla u|^2 - \lambda |u|^2 \, dx\right) - \frac{1}{2^*}\int_{B_1} |u|^{2^*} dx$ is the energy functional related to (2.1.1), $S$ is the best Sobolev constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$.*

*(ii): Up to a subsequence, the rescaled function* $\tilde{u}_\lambda^+$ *converges in* $C^2_{loc}(\mathbb{R}^N)$ *to* $\mathcal{U}_{0,\mu}$*, as* $\gamma \to +\infty$*, where* $\mathcal{U}_{0,\mu}$ *is the solution of the critical exponent problem in* $\mathbb{R}^N$ *centered at* $x_0 = 0$ *and with concentration parameter* $\mu = \sqrt{N(N-2)}$*. We recall that such functions are defined by*

$$\mathcal{U}_{x_0,\mu}(x) := \frac{[N(N-2)\mu^2]^{(N-2)/4}}{[\mu^2 + |x - x_0|^2]^{(N-2)/2}}.$$

*Proof.* We start by proving (i). Let $(u_{\lambda_2})$ be this family of solutions. Since $u^+_{\lambda_2}$ solves $-\Delta u = \lambda_2 u + u^{2^*-1}$ in $B_{r_{\lambda_2}}$ then, considering the rescaling $\hat{u}^+_{\lambda_2}(y) := r^{1/\beta}_{\lambda_2} u^+_{\lambda_2}(r_{\lambda_2} y)$, where $\beta := \frac{2}{N-2}$, we see that $\tilde{u}^+_{\lambda_2}$ solves

$$\begin{cases} -\Delta u = \lambda_2 r^2_{\lambda_2} u + u^{2^*-1} & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{2.3.1}$$

Now we distinguish between two cases: $N = 4, 5, 6$ and $N = 3$.

If $N = 4, 5, 6$, then, from Lemma 2.2.3 we deduce that $r_{\lambda_2} \to 0$ as $\gamma \to +\infty$, in particular this is true for $\lambda_2 r^2_{\lambda_2}$. From [2] we know that $\hat{u}^+_{\lambda_2}$ is unique and it coincides with the solution found in [17], which minimizes the energy $J_{\lambda_2 r^2_{\lambda_2}}$; thus, since $\lambda_2 r^2_{\lambda_2} \to 0$ as $\gamma \to +\infty$ we get that $J_{\lambda_2 r^2_{\lambda_2}}(\hat{u}^+_{\lambda_2}) \to \frac{1}{N} S^{N/2}$. Thanks to Lemma 2.3.1 we get that $J_{\lambda_2}(u^+_{\lambda_2}) = J_{\lambda_2 r^2_{\lambda_2}}(\hat{u}^+_{\lambda_2}) \to \frac{1}{N} S^{N/2}$ as $\gamma \to +\infty$.

Assume now that $N = 3$. As stated in Lemma 2.2.3 we have that $r_{\lambda_2}$ is bounded away from zero. From a well known result of Brezis and Nirenberg (see [17], Theorem 1) we have that (2.3.1) has a positive solution if and only if $\lambda_2 r^2_{\lambda_2} \in (\frac{\pi^2}{4}, \pi^2)$. As $\gamma \to +\infty$ we must have $\lambda_2 r^2_{\lambda_2} \to \frac{\pi^2}{4}$. Hence, the only possibility is that $J_{\lambda_2 r^2_{\lambda_2}}(\hat{u}^+_{\lambda_2}) \to \frac{1}{3} S^{3/2}$ as $\gamma \to +\infty$. As before thanks to Lemma 2.3.1 we have $J_{\lambda_2}(u^+_{\lambda_2}) = J_{\lambda_2 r^{1/\beta}_{\lambda_2}}(\hat{u}^+_{\lambda_2})$ and hence $J_{\lambda_2}(u^+_{\lambda_2}) \to \frac{1}{3} S^{3/2}$ as $\gamma \to +\infty$. The proof of (i) is complete.

We now prove (ii). By definition the rescaled function $\tilde{u}^+_{\lambda_2}$ solves the following problem

$$\begin{cases} -\Delta u = \frac{\lambda_2}{M^{2\beta}_{\lambda_2,+}} u + u^{2^*-1} & \text{in } B_{\sigma_{\lambda_2}}, \\ u > 0 & \text{in } B_{\sigma_{\lambda_2}}, \\ u = 0 & \text{on } \partial B_{\sigma_{\lambda_2}}, \end{cases} \tag{2.3.2}$$

where $\sigma_{\lambda_2} := M^{\beta}_{\lambda_2} r_{\lambda_2}$.

Since the family $(\tilde{u}^+_{\lambda_2})$ is uniformly bounded, then by standard elliptic theory we get that $\tilde{u}^+_{\lambda_2} \to \tilde{u}$ in $C^2_{loc}(B_l)$, where $l$ is the limit of $\sigma_{\lambda_2}$ as $\gamma \to +\infty$. We want to show that

$$\lim_{\gamma \to +\infty} \sigma_{\lambda_2} = +\infty,$$

so that the limit domain is the whole $\mathbb{R}^N$. We can proceed in two different ways: one is to apply directly the estimates contained in Section 2.1, the other one is to apply the methods of Chapter 1. We choose the second approach: arguing as in the proof of Proposition 1.4.3, taking into account that by (i) of Theorem 2.3.2, $J_{\lambda_2}(u^+_{\lambda_2}) \to \frac{1}{N} S^{N/2}$, as $\gamma \to +\infty$, we see that up to a subsequence it cannot happen that $\lim_{\gamma \to +\infty} \sigma_{\lambda_2}$ is finite.

Since $\frac{\lambda_2}{M^{2\beta}_{\lambda_2,+}} \to 0$, as $\gamma \to +\infty$, $\tilde{u}^+_{\lambda_2}$ converges in $C^2_{loc}(\mathbb{R}^N)$ to a positive solution $\tilde{u}$ of

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N \\ u \to 0 & \text{as } |y| \to +\infty. \end{cases}$$

Observe that this holds even in the case $N = 3$, in fact by definition and Remark 2.3.3 we have

$$\frac{\lambda_1(B_{\sigma_{\lambda_2}})}{4} = \frac{\pi^2}{4M^4_{\lambda_2,+}r^2_{\lambda_2}} = \frac{9}{4}\pi^2(1+o(1))\frac{1}{M^4_{\lambda_2,+}} = \frac{\lambda_2}{M^4_{\lambda_2,+}}(1+o(1)) \to 0,$$

as $\gamma \to +\infty$.

Since $\tilde{u}$ is radial and $\tilde{u}(0) = 1$ then $\tilde{u} = \mathcal{U}_{0,\mu}$ where $\mu = \sqrt{N(N-2)}$ (see Proposition 2.2 in [25]). The proof is complete. □

**Remark 2.3.3.** *We observe that for $N = 3$, since $\lambda_2 r^2_{\lambda_2} \to \frac{\pi^2}{4}$ and (d) of Theorem 2.2.7 holds, then, we deduce that $r_{\lambda_2} \to \frac{1}{3}$. On the contrary, if $N = 4, 5, 6$, as seen in the proof of Theorem 2.3.2, we have $r_{\lambda_2} \to 0$ as $\gamma \to +\infty$ (this also holds for $N \geq 7$, as seen in Proposition 1.3.3).*

## 2.4 Asymptotic analysis of the negative part in dimension $N = 6$

In this section we focus on the case $N = 6$ which means to take $k = 5/2$ in (2.2.4). As in [6] we define

$$\begin{aligned} t_0(\gamma) &:= \inf\{t \in (0, +\infty);\ y' > 0 \text{ on } (t, +\infty)\}, \\ y_0(\gamma) &:= y(t_0(\gamma); \gamma). \end{aligned} \tag{2.4.1}$$

We have the following:

**Proposition 2.4.1.** *Assume $k = 5/2$. Then*

*(a) $y_0(\gamma) = -\frac{1}{2}(1 + o(1))$, as $\gamma \to +\infty$;*

*(b) $t_0(\gamma) = (\frac{2}{9}\gamma)^{2/3}(1 + o(1))$, as $\gamma \to +\infty$.*

*Proof.* See [6], Theorem 2. □

Let $u_\lambda$ be any radial solution of (2.1.1) with exactly two nodal regions and without loss of generality assume that $u_\lambda(0) > 0$. We denote by $s_\lambda$ the global minimum point of $u_\lambda$. As in the previous section we set $M_{\lambda,+} := \|u^+_\lambda\|_\infty$, $M_{\lambda,-} := \|u^-_\lambda\|_\infty$, where $u^+_\lambda$, $u^-_\lambda$ are respectively the positive and the negative part of $u_\lambda$. Clearly, by definition, we have $u^-_\lambda(s_\lambda) = M_{\lambda,-}$. In order to estimate the energy of such solutions we need the following preliminary result.

**Proposition 2.4.2.** *Let $N = 6$ and let $(u_\lambda)$ be any family of radial sign-changing solutions of (2.1.1) with exactly two nodal regions and such that $u_\lambda(0) > 0$ for all $\lambda$. Assume that there exists $\lambda_0 \in \mathbb{R}^+$ such that $M_{\lambda,+} \to \infty$ as $\lambda \to \lambda_0$. Then*

$$M_{\lambda,-} \leq \frac{\lambda}{2}(1 + o(1)),$$

*for all $\lambda$ sufficiently close to $\lambda_0$.*

*Proof.* Let $(u_\lambda)$ be such a family of solutions. Since $N = 6$, we have $2^* - 2 = \frac{4}{N-2} = 1$ and thanks to the transformation (2.2.2) we have

$$u_\lambda(r(t)) = \lambda\ y(t;\gamma), \tag{2.4.2}$$

for $t \in \left(\left(\frac{N-2}{\sqrt{\lambda}}\right)^{N-2}, +\infty\right)$, where $\gamma = \lambda^{-1}M_{\lambda,+}$. We observe that the global minimum point $s_\lambda$ corresponds, through the transformation (2.2.2), to the number $t_0(\gamma)$ defined in (2.4.1). In fact by definition we have $u'_\lambda(s_\lambda) = 0$ so it suffices to show that $u'_\lambda(r) < 0$ for all $r \in (0, s_\lambda)$. By Corollary 1.2.4 we know that $u'_\lambda(r) < 0$ for all $r \in (0, r_\lambda)$, and for all $r \in (r_\lambda, s_\lambda)$. Moreover since $u^+_\lambda$ solves (2.1.1) in $B_{r_\lambda}$, then, by Hopf lemma it follows that $u'_\lambda(r_\lambda) < 0$. Now, thanks to the assumptions, as $\lambda \to \lambda_0$ we have $\gamma = \lambda^{-1}M_{\lambda,+} \to +\infty$ and the result follows immediately from (2.4.2) and Proposition 2.4.1. □

**Remark 2.4.3.** *A straight important consequence of Proposition 2.4.2 is that $M_{\lambda,-}$ is uniformly bounded for all $\lambda$ sufficiently close to $\lambda_0$. In particular there cannot exist radial sign-changing solutions of (2.1.1) with the shape of a tower of two bubbles in dimension $N = 6$ (this fact also holds for the dimensions $N = 3, 4, 5$, as we will see later). This is in deep contrast with the case of higher dimensions $N \geq 7$ as seen in Chapter 1.*

**Remark 2.4.4.** *In the case of the solutions obtained in the previous section, thanks to Theorem 2.2.7 we deduce that $M_{\lambda_2(\gamma),-} \leq \frac{\lambda_0}{2}(1 + o(1)) \leq \frac{\lambda_1(B_1)}{2}$ for all sufficiently large $\gamma \in \mathbb{R}^+$.*

In the previous section we have studied the limit energy (see Theorem 2.3.2) of the positive part of the solutions $u_{\lambda_2}$. Here we consider the negative part $u^-_{\lambda_2}$ and prove that its energy $J_{\lambda_2}$ is uniformly bounded as $\gamma \to +\infty$. This is the content of the next proposition.

**Proposition 2.4.5.** *Let $N = 6$. Let $\lambda_2 = \lambda_2(\gamma)$ and $u_{\lambda_2}$ be the radial solution with exactly two nodal regions of (2.1.1) described in Section 2.2. Let $J_\lambda(u) := \frac{1}{2}\left(\int_{B_1} |\nabla u|^2 - \lambda|u|^2\ dx\right) - \frac{1}{2^*}\int_{B_1} |u|^{2^*}\ dx$ be the energy functional related to (2.1.1). Then*

$$J_{\lambda_2}(u^-_{\lambda_2}) \leq \frac{\pi^3}{36}\left(\frac{\lambda_1(B_1)}{2}\right)^3,$$

*for all sufficiently large $\gamma$.*

*Proof.* Since $u^-_{\lambda_2}$ solves $-\Delta u = \lambda_2 u + u^{2^*-1}$ in the annulus $A_{r_{\lambda_2}}$, in particular it belongs to the Nehari manifold $\mathcal{N}_{\lambda_2}$ associated to that equation, which is defined by

$$\mathcal{N}_{\lambda_2} := \{u \in H^1_0(A_{r_{\lambda_2}});\ \|u\|^2_{A_{r_{\lambda_2}}} - \lambda_2|u|^2_{2,A_{r_{\lambda_2}}} = |u|^{2^*}_{2^*,A_{r_{\lambda_2}}}\}. \tag{2.4.3}$$

Hence we deduce that

$$J_{\lambda_2}(u^-_{\lambda_2}) = \frac{1}{6}|u^-_{\lambda_2}|^{2^*}_{2^*,A_{r_{\lambda_2}}}. \tag{2.4.4}$$

Now, thanks to Proposition 2.4.2, (b) of Theorem 2.2.7 and Remark 2.4.4 we have

$$|u^-_{\lambda_2}|^{2^*}_{2^*,A_{r_{\lambda_2}}} = \int_{A_{r_{\lambda_2}}} |u^-_{\lambda_2}|^3\ dx \leq |B_1|\|u^-_{\lambda_2}\|^3_\infty \leq \frac{\pi^3}{6}\left(\frac{\lambda_1(B_1)}{2}\right)^3, \tag{2.4.5}$$

for all sufficiently large $\gamma$. From (2.4.4) and (2.4.5) we deduce the desired relation and the proof is complete. □

**Remark 2.4.6.** *Since $\lambda_2$ is a bounded function, by the same proof of Proposition 2.4.5, but without using (b) of Theorem, 2.2.7 we deduce anyway that $J_{\lambda_2}(u^-_{\lambda_2})$ is uniformly bounded for all sufficiently large $\gamma$.*

We are interested now in studying the asymptotic behavior of the family $(u^-_{\lambda_2})$. More precisely we show that, as $\gamma \to \infty$, the family $(u^-_{\lambda_2})$ converges in $C^2_{loc}(B_1 - \{0\})$ to the unique positive solution $u_0$ of (2.1.1) with $\lambda = \lambda_0$, for some $\lambda_0 \in (0, \lambda_1(B_1))$. We point out that these results will improve the energy estimate of $u^-_{\lambda_2}$ obtained before.

The pointwise convergence of $(u^-_{\lambda_2})$ to $u_0$ is contained in Theorem 3 of [6], but here we use a different approach which is based on the arguments of Chapter 1. Our result is the following:

**Theorem 2.4.7.** *Let $N = 6$, up to a subsequence, we have $\lambda_2(\gamma) \to \lambda_0$, as $\gamma \to +\infty$, for some $\lambda_0 \in (0, \lambda_1(B_1))$, and $(u^-_{\lambda_2})$ converges in $C^2_{loc}(B_1 - \{0\})$ to the unique positive solution $u_0$ of (2.1.1) with $\lambda = \lambda_0$.*

*Proof.* Let us consider the family $(u^-_{\lambda_2})$. These functions solve

$$\begin{cases} -\Delta u = \lambda_2 u + u^2 & \text{in } A_{r_{\lambda_2}}, \\ u > 0 & \text{in } A_{r_{\lambda_2}}, \\ u = 0 & \text{on } \partial A_{r_{\lambda_2}}. \end{cases} \tag{2.4.6}$$

Since $\lambda_2$ is bounded, up to a subsequence we have $\lim_{\gamma \to +\infty} \lambda_2 = \lambda_0$. Thanks to Proposition 2.4.2 we have that $u^-_{\lambda_2}$ is uniformly bounded for all sufficiently large $\gamma$ and by Lemma 2.2.3 and the inverse transformation of (2.2.2) we have $r_{\lambda_2} \to 0$. Hence by standard elliptic theory, up to a subsequence, for any $0 < \delta < 1$, $u^-_{\lambda_2}$ converges in $C^2(\overline{B_1} - B_\delta)$ as $\gamma \to +\infty$ to a solution $u_0$ of

$$\begin{cases} -\Delta u = \lambda_0 u + u^2 & \text{in } B_1 - \{0\}, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

where $B_\delta$ is the ball centered at the origin having radius $\delta$. We now proceed in three steps.

**Step 1:** we have

$$\lim_{r \to 0} u_0(r) = \frac{\lambda_0}{2}. \tag{2.4.7}$$

Since $u^-_{\lambda_2}$ is a radial solution of (2.1.1) and thanks to Proposition 2.4.2, for all sufficiently large $\gamma$, we have

$$u^-_{\lambda_2} \leq \frac{\lambda_0}{2}(1 + o(1)), \tag{2.4.8}$$

and then we deduce that

$$\begin{aligned} [(u^-_{\lambda_2})' r^5]' &= -\lambda_2 u^-_{\lambda_2}(r) r^5 - [u^-_{\lambda}(r)]^2 r^5 \\ &\geq -\lambda_2 \tfrac{\lambda_0}{2}(1 + o(1)) r^5 - \left[\tfrac{\lambda_0}{2}(1 + o(1))\right]^2 r^5 \\ &= -\tfrac{\lambda_0^2}{2}(1 + o(1))^2 r^5 - \tfrac{\lambda_0^2}{4}(1 + o(1))^2 r^5 \\ &\geq -\lambda_0^2\, r^5. \end{aligned}$$

Integrating between $s_{\lambda_2}$ and $r$ (with $s_{\lambda_2} < r < 1$) we get that

$$(u_{\lambda_2}^-)'(r)r^5 \geq -\lambda_0^2 \int_{s_{\lambda_2}}^r t^5 dt \geq -\frac{\lambda_0^2}{6} r^6.$$

Hence $(u_{\lambda_2}^-)'(r) \geq -\frac{\lambda_0^2}{6} r$ for all $r \in (s_{\lambda_2}, 1)$. Integrating again between $s_{\lambda_2}$ and $r$ we have

$$u_{\lambda_2}^-(r) - \frac{\lambda_0}{2}(1 + o(1)) \geq -\frac{\lambda_0^2}{12}(r^2 - s_{\lambda_2}^2) \geq -\frac{\lambda_0^2}{12} r^2.$$

Hence $u_{\lambda_2}^-(r) \geq \frac{\lambda_0}{2}(1 + o(1)) - \frac{\lambda_0^2}{12} r^2$ for all sufficiently large $\gamma$, for all $r \in (s_{\lambda_2}, 1)$. Since $s_{\lambda_2} \to 0$, then, passing to the limit as $\gamma \to \infty$, we get that $u_0(r) \geq \frac{\lambda_0}{2} - \frac{\lambda_0^2}{12} r^2$, for all $0 < r < 1$. From this inequality and (2.4.8) we deduce that $\lim_{r \to 0} u_0(r) = \frac{\lambda_0}{2}$. The proof of Step 1 is complete.

**Step 2:** we have

$$\lim_{r \to 0} u_0'(r) = 0. \tag{2.4.9}$$

As in the previous step, integrating the equation between $s_{\lambda_2}$ and $r$, with $s_{\lambda_2} < r < 1$, we get that

$$-(\tilde{u}_{\lambda_2}^-)'(r)r^5 \quad = \quad \lambda_2 \int_{s_{\lambda_2}}^r u_{\lambda_2}^- t^5 dt + \int_{s_{\lambda_2}}^r (u_{\lambda_2}^-)^2 t^5 dt.$$

Thanks to (2.4.8), for all sufficiently large $\gamma$ we have

$$|(u_{\lambda_2}^-)'(r)r^5| \leq \lambda_2 \frac{\lambda_0}{2}(1 + o(1)) \int_{s_{\lambda_2}}^r t^5 dt + \frac{\lambda_0^2}{4}(1 + o(1))^2 \int_{s_{\lambda_2}}^r t^5 dt \leq \lambda_0^2 \frac{r^6}{6}.$$

Dividing by $r^5$ the previous inequality and passing to the limit, as $\gamma \to +\infty$, we get that

$$|u_0'(r)| \leq \frac{\lambda_0^2}{6} r,$$

for all $0 < r < 1$. Hence $\lim_{r \to 0} u_0'(r) = 0$ and the proof of Step 2 is complete.

From Step 1 and Step 2 it follows that the radial function $u_0(x) = u_0(|x|)$ can be extended to a $C^1(B_1)$ function. We still denote by $u_0$ this extension.
**Step 3:** The function $u_0$ is a weak solution in $B_1$ of

$$-\Delta u = \lambda_0 u + u^2. \tag{2.4.10}$$

Let us fix a test function $\phi \in C_0^\infty(B_1)$. If $0 \notin supp(\phi)$ the proof is trivial so from now on we assume $0 \in supp(\phi)$. Applying Green's formula to $\Omega(\delta) := B_1 - B_\delta$, since $u_0$ is a $C^2(B_1 - \{0\})$-solution of (2.4.10) and $\phi \equiv 0$ on $\partial B_1$, we have

$$\int_{\Omega(\delta)} \nabla u_0 \cdot \nabla \phi \, dx = \lambda_0 \int_{\Omega(\delta)} \phi \, u_0 \, dx + \int_{\Omega(\delta)} \phi \, u_0^2 \, dx + \int_{\partial B_\delta} \phi \left( \frac{\partial u_0}{\partial \nu} \right) \, d\sigma. \tag{2.4.11}$$

We show now that $\int_{\partial B_\delta} \phi \left( \frac{\partial u_0}{\partial \nu} \right) \, d\sigma \to 0$ as $\delta \to 0$. In fact since $u_0$ is a radial function we have $\frac{\partial u_0}{\partial \nu}(x) = u_0'(\delta)$ for all $x \in \partial B_\delta$, and hence we get that

$$\left| \int_{\partial B_\delta} \phi \left( \frac{\partial u_0}{\partial \nu} \right) \, d\sigma \right| \leq |u_0'(\delta)| \int_{\partial B_\delta} |\phi| \, d\sigma \leq \omega_6 |u_0'(\delta)| \delta^5 ||\phi||_\infty.$$

Thanks to (2.4.9) we have $|u_0'(\delta)|\delta^5 \to 0$ as $\delta \to 0$. To complete the proof we pass to the limit in (2.4.11) as $\delta \to 0$. We observe that

$$\begin{aligned} |\nabla u_0 \cdot \nabla \phi| \ \chi_{\Omega(\delta)} &\leq |\nabla u_0|^2 \ \chi_{\{|\nabla u_0|>1\}} |\nabla \phi| + |\nabla u_0| \ \chi_{\{|\nabla u_0| \leq 1\}} |\nabla \phi| \\ &\leq |\nabla u_0|^2 \ \chi_{\{|\nabla u_0|>1\}} |\nabla \phi| + \ \chi_{\{|\nabla u_0| \leq 1\}} |\nabla \phi|. \end{aligned} \tag{2.4.12}$$

We point out that $\int_{B_1} |\nabla u_0|^2 dx$ is finite: this is an easy consequence of the fact that $u_{\lambda_2} \to u_0$ in $C^2_{loc}(B_1 - \{0\})$, the family $(u_{\lambda_2})$ is uniformly bounded, (2.4.3) and Lebesgue's theorem.

Thus, since $\int_{B_1} |\nabla u_0|^2 dx$ is finite and $\phi$ has compact support, the right-hand side of (2.4.12) belongs to $L^1(B_1)$. Hence from Lebesgue's theorem we have

$$\lim_{\delta \to 0} \int_{\Omega(\delta)} \nabla u_0 \cdot \nabla \phi \ dx = \int_{B_1} \nabla u_0 \cdot \nabla \phi \ dx. \tag{2.4.13}$$

Since $\phi$ has compact support by Lebesgue's theorem we have

$$\begin{aligned} \lim_{\delta \to 0} \int_{\Omega(\delta)} \phi \ u_0 \ dx &= \int_{B_1} \phi \ u_0 \ dx, \\ \lim_{\delta \to 0} \int_{\Omega(\delta)} \phi \ u_0^2 \ dx &= \int_{B_1} \phi \ u_0^2 \ dx. \end{aligned} \tag{2.4.14}$$

From (2.4.11), (2.4.13), (2.4.14) and since we have proved $\int_{\partial B(\delta)} \phi \left( \frac{\partial \tilde{u}}{\partial \nu} \right) \ d\sigma \to 0$ as $\delta \to 0$ it follows that

$$\int_{B_1} \nabla u_0 \cdot \nabla \phi \ dx = \lambda_0 \int_{B_1} \phi \ u_0 \ dx + \int_{B_1} \phi \ u_0^2 \ dx,$$

which completes the proof of Step 3.

Thanks to Step 1 - Step 3 we get that $u_0 \in H^1_{0,rad}(B_1)$ is a weak solution of

$$\begin{cases} -\Delta u = \lambda_0 u + u^2 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{2.4.15}$$

In particular, as a consequence of a well known result of Brezis and Kato (for instance see Lemma 1.5 in [17]) it is possible to show that $u_0$ is a classical solution of (2.4.15) (see Appendix B of [58]). Thanks to [2] we know that $u_0$ is the unique positive radial solution of (2.4.15), which is the one found by Brezis and Nirenberg in [17]. Hence we must have $\lambda_0 < \lambda_1(B_1)$ and $J_{\lambda_0}(u_{\lambda_0}) < \frac{1}{6} S^3$. □

Next result gives a characterization of the value $\lambda_0 \in (0, \lambda_1(B_1))$ appearing in Theorem 2.4.7.

**Theorem 2.4.8.** *Let $N = 6$. Let $\lambda_0 := \lim_{\gamma \to +\infty} \lambda_2(\gamma)$. We have that $\lambda_0$ is the unique $\lambda \in (0, \lambda_1(B_1))$ such that $u_\lambda(0) = \frac{\lambda}{2}$, where $u_\lambda$ is the unique positive solution of*

$$\begin{cases} -\Delta u = \lambda u + u^2 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{2.4.16}$$

*Proof.* Thanks to Theorem 2.4.7 and (2.4.7) we have that the set

$$\Gamma := \left\{ \lambda \in (0, \lambda_1(B_1));\ u_\lambda(0) = \frac{\lambda}{2},\ \text{where } u_\lambda \text{ is the unique solution of (2.4.16)} \right\},$$

is not empty since $\lambda_0 \in \Gamma$. We want to prove that $\Gamma = \{\lambda_0\}$. To this end assume that $\bar{\lambda} \in \Gamma$ and $\bar{\lambda} \neq \lambda_0$. In particular the functions $u_{\lambda_0}$ and $u_{\bar{\lambda}}$ are different. Thanks to the definition of $\Gamma$ and applying (2.2.2) (with $2^* - 2 = 1$ because $N = 6$) we get that $u_{\lambda_0}$ and $u_{\bar{\lambda}}$ are respectively transformed to a solution of (2.2.4) with $\gamma = \frac{1}{2}$, but, for a given $\gamma$, the solution of (2.2.4) is unique and this gives a contradiction. □

Now we have all the tools to estimate the energy of the solutions $u_{\lambda_2}$. This is the content of the next result.

**Corollary 2.4.9.** *Let $N = 6$ and let $u_{\lambda_2}$ be the radial solution with exactly two nodal regions of (2.1.1) with $\lambda = \lambda_2(\gamma)$ obtained in Section 2.2. Then*

$$J_{\lambda_2}(u_{\lambda_2}) < \frac{1}{3} S^3,$$

*for all sufficiently large $\gamma \in \mathbb{R}^+$, where $J_\lambda(u) := \frac{1}{2}\left(\int_{B_1} |\nabla u|^2 - \lambda |u|^2\, dx\right) - \frac{1}{2^*}\int_{B_1} |u|^{2^*}\, dx$ is the energy functional related to (2.1.1), $S$ is the best Sobolev constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^6)$ into $L^{2^*}(\mathbb{R}^6)$.*

*Proof.* Let $(u_{\lambda_2})$ be this family of solutions. Observe that $J_{\lambda_2}(u_{\lambda_2}) = J_{\lambda_2}(u^+_{\lambda_2}) + J_{\lambda_2}(u^-_{\lambda_2})$ hence it suffices to estimate separately the energy of the positive and negative part of $u_{\lambda_2}$. The energy of $u^+_{\lambda_2}$ has been determinated in Theorem 2.3.2, and in particular we have $J_{\lambda_2}(u^+_{\lambda_2}) \to \frac{1}{6} S^3$, as $\gamma \to +\infty$.

Now we estimate $J_{\lambda_2}(u^-_{\lambda_2})$. Since $u^-_{\lambda_2}$ solves $-\Delta u = \lambda_2 u + u^{2^*-1}$ in the annulus $A_{r_{\lambda_2}}$, in particular it belongs to the Nehari manifold $\mathcal{N}_{\lambda_2}$ associated to this equation, (see (2.4.3)). Hence we deduce that $J_{\lambda_2}(u^-_{\lambda_2}) = \frac{1}{6} |u^-_{\lambda_2}|^{2^*}_{2^*, A_{r_{\lambda_2}}}$. To complete the proof it will suffice to show that

$$|u^-_{\lambda_2}|^{2^*}_{2^*, A_{r_{\lambda_2}}} \to |u^-_{\lambda_0}|^{2^*}_{2^*, B_1},$$

where $u_0$ is the unique solution of (2.4.15). In fact, thanks to Theorem 2.4.7 we know that, up to a subsequence, $(u^-_{\lambda_2})$ converges in $C^2_{loc}(B_1 - \{0\})$ to the unique solution $u_0$ of (2.4.15). Hence to prove our assertion it suffices to apply Lebesgue's theorem, which clearly holds since $(u^-_{\lambda_2})$ is uniformly bounded as $\gamma \to +\infty$.

Now since $J_{\lambda_2}(u^-_{\lambda_2}) \to J_{\lambda_0}(u_{\lambda_0})$ and $J_{\lambda_0}(u_{\lambda_0}) < \frac{1}{6} S^3$ we deduce the desired relation. □

## 2.5 Asymptotic analysis of the negative part in dimension $N = 3, 4, 5$

Here we prove:

**Theorem 2.5.1.** *Let $N = 3, 4, 5$ and let $(u_\lambda)$ be any family of radial sign-changing solutions of (2.1.1) with exactly two nodal regions and such that $u_\lambda(0) > 0$ for all $\lambda$. Assume that there exists $\bar{\lambda} \in \mathbb{R}^+$ such that $M_{\lambda,+} \to \infty$, as $\lambda \to \bar{\lambda}$. Then:*

*(i):* $M_{\lambda,-} \to 0$, *as* $\lambda \to \bar{\lambda}$;

*(ii):* $(u_\lambda^-)$ *converges to zero uniformly in* $B_1$, *as* $\lambda \to \bar{\lambda}$.

*Proof.* We start by proving (i). Let $(u_\lambda)$ be such a family of solutions. Thanks to the transformation (2.2.2) we have

$$u_\lambda(r(t)) = \lambda^{\frac{1}{2^*-2}}\ y(t;\gamma), \tag{2.5.1}$$

for $t \in \left(\left(\frac{N-2}{\sqrt{\lambda}}\right)^{N-2}, +\infty\right)$, where $\gamma = \lambda^{-\frac{1}{2^*-2}} M_{\lambda,+}$ and $y = y(t;\gamma)$ solves (2.2.4). Clearly, as $\lambda \to \bar{\lambda}$, we have $\gamma \to +\infty$. As in the proof of Proposition 2.4.2 we have that the global minimum point $s_\lambda$ corresponds, through the transformation (2.2.2), to the number $t_0(\gamma)$ defined in (2.4.1).

Hence, thanks to Lemma 2.2.2, it holds

$$|y(t_0(\gamma);\gamma)| < |y'(T_1(\gamma))|\ (T_1(\gamma) - t_0(\gamma)). \tag{2.5.2}$$

For $N = 3$, which corresponds to $k = 4$, by Lemma 2.2.3 we have that $T_1(\gamma)$ is uniformly bounded for all sufficiently large $\gamma$, while, by Lemma 2.2.4 it holds $y'(T_1(\gamma)) = (k-1)^{\frac{1}{k-2}}\gamma^{-1}(1 + o(1))$. Thus, since $0 < t_0(\gamma) < T_1(\gamma)$, from (2.5.2), (2.5.1) we get that $M_{\lambda,-} = \lambda^{\frac{1}{2^*-2}} y(t_0;\gamma) \to 0$ as $\lambda \to \bar{\lambda}$.

For $N = 4$, which corresponds to $k = 3$, by Lemma 2.2.3 we have that $T_1(\gamma) = 2\log(\gamma)(1 + o(1))$ for all sufficiently large $\gamma$, and hence as in the previous case, we get that $M_{\lambda,-} = \lambda^{\frac{1}{2^*-2}} y(t_0;\gamma) \to 0$ as $\lambda \to \bar{\lambda}$. The same happens for $N = 5$ ($k = 8/3$); in fact by Lemma 2.2.3 we have that $T_1(\gamma) = A\gamma^{2/3}(1 + o(1))$ for all sufficiently large $\gamma$, where $A = A(k)$ is a positive constant depending only on $k$ (see Lemma 2.2.3 for its definition). The proof of (i) is complete.

Now we prove (ii). We recall that $u_\lambda^-$ is nonzero in the annulus $A_{r_\lambda}(0) = \{x \in \mathbb{R}^N;\ r_\lambda < |x| < 1\}$ and vanishes outside. Thanks to (i), we have $\|u_\lambda^-\|_{\infty,B_1} = M_{\lambda,-} \to 0$ as $\lambda \to \bar{\lambda}$ and we are done. □

# Chapter 3

# Sign-changing tower of bubbles for the Brezis-Nirenberg problem

## 3.1 Introduction

Here we present and prove the result **(R3)**.

In this chapter we are interested in the construction of solutions to the following problem

$$\begin{cases} -\Delta u = \epsilon u + |u|^{p-1}u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.1.1}$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 7$, $\epsilon$ is supposed to be small and positive while $p+1 = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H^1_0(\Omega)$ into $L^{p+1}(\Omega)$.

The pioneering paper on equation (3.1.1) was written by Brezis and Nirenberg [17] in 1983 where the authors showed that for $N \geq 4$ and $\epsilon \in (0, \lambda_1)$, the problem (3.1.1) has at least one positive solution where $\lambda_1$ denotes the first eigenvalue of $-\Delta$ on $\Omega$.
In the case $N = 3$, a similar result was proved in [17] but only for $\epsilon \in (\lambda^*, \lambda_1)$ with $\lambda^* = \lambda^*(\Omega) > 0$. Moreover by using a version of the Pohozaev Identity the authors showed that $\lambda^*(\Omega) = \frac{1}{4}\lambda_1$ if $\Omega$ is a ball and that no positive solutions exist for $\epsilon \in (0, \frac{1}{4}\lambda_1)$.
Note that, by using again Pohozaev Identity, it is easy to check that problem (3.1.1) has no nontrivial solutions when $\epsilon \leq 0$ and $\Omega$ is star-shaped.
Since then, there has been a considerable number of papers on problem (3.1.1).
We briefly recall some of the main ones.
Han, in [36], proved that the solution found by Brezis and Nirenberg blows-up at a critical point of the Robin's function as $\epsilon$ goes to zero. Conversely, Rey in [52] and in [51] proved that any $C^1$-stable critical point of the Robin's function generates a family of positive solutions which blows-up at this point as $\epsilon$ goes to zero.
After the work of Brezis and Nirenberg, Capozzi, Fortunato and Palmieri [20] showed that for $N = 4$, $\epsilon > 0$ and $\epsilon \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$) problem (3.1.1) has a nontrivial solution. The same holds if $N \geq 5$ for all $\epsilon > 0$ (see also [35]).

The first multiplicity result was obtained by Cerami, Fortunato and Struwe in [23], in which they proved that the number of nontrivial solutions of (3.1.1), for $N \geq 3$, is bounded below by the number of eigenvalues of $(-\Delta, \Omega)$ belonging to $(\epsilon, \epsilon + S|\Omega|^{-2/N})$,

where $S$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$ and $|\Omega|$ is the Lebesgue measure of $\Omega$.
Moreover, if $N \geq 4$, then for any $\epsilon > 0$ and for a suitable class of symmetric domain $\Omega$, problem (3.1.1) has infinitely many solutions of arbitrarily large energy (see Fortunato and Jannelli [31]).
If $N \geq 7$ and $\Omega$ is a ball, then for each $\epsilon > 0$, problem (3.1.1) has infinitely many sign-changing radial solutions (see Solimini [55]).
In the papers [31, 55], the radial symmetry of the domain plays an essential role, therefore their methods do not work for general domains.
Concerning sign-changing solutions, Cerami, Solimini and Struwe showed in [25] that if $N \geq 6$ and $\epsilon \in (0, \lambda_1)$, problem (3.1.1) has a pair of least energy sign-changing solution. In the same paper the authors studied the multiplicity of nodal solutions proving the existence of infinitely many radial solutions when $\Omega$ is a ball centered at the origin.
On the other side, for $3 \leq N \leq 6$ and when $\Omega$ is a ball, it can be proved that there is a $\lambda^* > 0$ such that (3.1.1) has no sign-changing radial solutions for $\epsilon \in (0, \lambda^*)$ (see Atkinson, Brezis and Peletier [5]).
Moreover, Devillanova and Solimini in [28] showed that, if $N \geq 7$ and $\Omega$ is an open regular subset of $\mathbb{R}^N$, problem (3.1.1) has infinitely many solutions for each $\epsilon > 0$.
For low dimensions, namely $N = 4, 5, 6$ and in an open regular subset of $\mathbb{R}^N$, in [29], Devillanova and Solimini proved the existence of at least $N + 1$ pairs of solutions provided $\epsilon$ is small enough. In [24], Clapp and Weth extended this last result to all $\epsilon > 0$.
Neither in [28, 29] nor in [24] there is information on the kind of sign-changing solutions obtained.
Recently, in [54], Schechter and Wenming Zou showed that in any bounded and smooth domain, for $N \geq 7$ and for each fixed $\epsilon > 0$, problem (3.1.1) has infinitely many sign changing solutions.

Concerning the profile of sign-changing solutions some results have been obtained in [14], [13] for low energy solutions, namely solutions $u_\epsilon$ such that $\int_\Omega |\nabla u_\epsilon|^2\, dx \to 2S^{\frac{N}{2}}$, as $\epsilon \to 0$, $S$ being the Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$. More precisely in [14] it is proved that for $N = 3$ these solutions concentrate and blow-up in two different points of $\Omega$, as $\epsilon \to 0$, and have the asymptotic profile of two separate bubbles. A similar result is proved in [13] for $N \geq 4$ but assuming that the blow-up rate of the positive and negative part of $u_\epsilon$ is the same.
Existence of nodal solutions with two nodal regions concentrating in two different points of the domain $\Omega$ as $\epsilon \to 0$ has been obtained in [21], [45] and [12]. So none of these solutions look like tower of bubbles, i.e. superposition of two bubbles with opposite sign concentrating at the same point, as $\epsilon \to 0$. Such a type of solutions is shown to exist for other semilinear problems like the almost critical Lane-Emden problem (see [13], [49], [48]) but not, to our knowledge, for the Brezis-Nirenberg problem with the exception of the case of the ball. If $\Omega$ is a ball, and $N \geq 7$, in a recent paper [37] the asymptotic behaviour as $\epsilon \to 0$ of the least energy nodal radial solution $v_\epsilon$ is analysed and among other things, it is shown that the positive and negative part of $v_\epsilon$ concentrate at the origin. Moreover they have the asymptotic profile of a positive and negative solution of the critical problem in $\mathbb{R}^N$ and the concentration speeds are different.
Hence [37] provides the first example of bubble of towers for the Brezis-Nirenberg problem.
Then the natural question is whether these kind of solutions exist in bounded domains other than the ball.

In this chapter we answer positively this question constructing a sign-changing solution of (3.1.1) in any bounded domain symmetric with respect to $N$ orthogonal hyperplanes.

We next state our result.

**Theorem 3.1.1.** *Let $N \geq 7$ and let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ such that $\Omega$ is symmetric with respect to $x_1, \ldots, x_N$ and $0 \in \Omega$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there exist positive numbers $d_{j\epsilon}$, $j = 1, 2$ and a solution $u_\epsilon$ of problem (3.1.1) of the form*

$$u_\epsilon(x) = \alpha_N \left[ \left( \frac{d_{1\epsilon} \epsilon^{\frac{1}{N-4}}}{d_{1\epsilon}^2 \epsilon^{\frac{2}{N-4}} + |x|^2} \right)^{\frac{N-2}{2}} - \left( \frac{d_{2\epsilon} \epsilon^{\frac{3N-10}{(N-4)(N-6)}}}{d_{2\epsilon}^2 \epsilon^{2\frac{3N-10}{(N-4)(N-6)}} + |x|^2} \right)^{\frac{N-2}{2}} \right] + \Phi_\epsilon, \qquad (3.1.2)$$

*where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, $d_{j\epsilon} \to \bar{d}_j > 0$, as $\epsilon \to 0$, $\Phi_\epsilon \to 0$ in $H^1(\Omega)$, as $\epsilon \to 0$. Moreover $u_\epsilon$ is even with respect to the variables $x_1, \ldots, x_N$.*

We remark that the assumption $N \geq 7$ in our proof is crucial. We believe that it is possible to extend our result to a general domain $\Omega$ with some suitable modifications.

In the case the remainder term converges to zero also in $L^\infty_{loc}(\Omega)$, then, the asymptotic expansion and some energy estimates derived in the course of the proof allow to draw interesting consequences concerning the number and shape of the nodal domains of the solution $u_\epsilon$.

**Theorem 3.1.2.** *Let $N \geq 7$ and assume that the remainder term $\Phi_\epsilon$, appearing in Theorem 3.1.1, is such that $\Phi_\epsilon \to 0$ uniformly in compact subsets of $\Omega$. Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the solution $u_\epsilon$ constructed in Theorem 3.1.1 has precisely two nodal domains $\Omega_\epsilon^1$, $\Omega_\epsilon^2$ such that $\Omega_\epsilon^1$ contains the sphere $\mathcal{S}_\epsilon^1 := \left\{ x \in \mathbb{R}^N ; \ |x| = \epsilon^{\frac{1}{N-4}} \right\}$, $\Omega_\epsilon^2$ contains the sphere $\mathcal{S}_\epsilon^2 := \left\{ x \in \mathbb{R}^N ; \ |x| = \epsilon^{\frac{3N-10}{(N-4)(N-6)}} \right\}$ and $u_\epsilon > 0$ on $\Omega_\epsilon^1$ and $u_\epsilon < 0$ on $\Omega_\epsilon^2$.*
*Consequently, $0 \in \Omega_\epsilon^2$ and $\Omega_\epsilon^1$ is the only nodal domain of $u_\epsilon$ which touches $\partial\Omega$.*

**Remark 3.1.3.** *Under the assumptions of Theorem 3.1.2 it follows that the sign-changing tower of bubble $u_\epsilon$ constructed in Theorem 3.1.1 has two nodal domains and its nodal set does not touch $\partial\Omega$. By this we mean that, denoting by*

$$Z_\epsilon := \{x \in \Omega \ ; \ u_\epsilon(x) = 0\}$$

*the nodal set of $u_\epsilon$ then $\overline{Z}_\epsilon \cap \partial\Omega = \emptyset$.*

The proof of Theorem 3.1.1 is based on the Lyapunov-Schmidt reduction.
To describe the procedure and explain the difficulties which arise when looking for bubble towers of the Brezis-Nirenberg problem, we introduce the functions

$$\mathcal{U}_\delta(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{(\delta^2 + |x|^2)^{\frac{N-2}{2}}}, \qquad \delta > 0 \qquad (3.1.3)$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$. Is is well known (see [9], [19], [59]) that (3.1.3) are the only radial solutions of the equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^N. \qquad (3.1.4)$$

We define $\varphi_\delta$ to be the unique solution to the problem

$$\begin{cases} \Delta\varphi_\delta = 0 & \text{in } \Omega \\ \varphi_\delta = \mathcal{U}_\delta & \text{on } \partial\Omega, \end{cases} \tag{3.1.5}$$

and let

$$\mathcal{P}\mathcal{U}_\delta := \mathcal{U}_\delta - \varphi_\delta \tag{3.1.6}$$

be the projection of $\mathcal{U}_\delta$ onto $H_0^1(\Omega)$, i.e.

$$\begin{cases} -\Delta\mathcal{P}\mathcal{U}_\delta = \mathcal{U}_\delta^p & \text{in } \Omega \\ \mathcal{P}\mathcal{U}_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.7}$$

Finally, let $G(x,y)$ be the Green's function associated to $-\Delta$ with Dirichlet boundary conditions and $H(x,y)$ be its regular part, namely

$$H(x,y) = \frac{1}{|x-y|^{N-2}} - \frac{1}{\gamma_N}G(x,y), \qquad \forall\, x,y \in \Omega, \qquad \text{with } \gamma_N = \frac{1}{N(N-2)\omega_N},$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.
The function $\tau(x) := H(x,x)$, $x \in \Omega$ is called *Robin's function.*

It is well-known that the following expansions holds (see [51])

$$\varphi_\delta(x) = \alpha_N \delta^{\frac{N-2}{2}} H(0,x) + O(\delta^{\frac{N+2}{2}}) \qquad \text{as } \delta \to 0. \tag{3.1.8}$$

Moreover, from elliptic estimates it follows that

$$0 < \varphi_\delta(x) < c\delta^{\frac{N-2}{2}}, \qquad \text{in } \Omega, \tag{3.1.9}$$

$$|\varphi_\delta|_{q,\Omega} \leq C\delta^{\frac{N-2}{2}}, \qquad q \in \left(\frac{p+1}{2}, p+1\right] \tag{3.1.10}$$

and

$$|\nabla\varphi_\delta|_{2,\Omega} \leq C_1\delta^{\frac{N-2}{2}} \tag{3.1.11}$$

see for instance [51], [62] and references therein.

We look for an approximate solution to problem (3.1.1) which is a superposition of two standard bubbles with two different scaling parameters, namely we take $\delta_1 > \delta_2$ and we look for a solution to (3.1.1) of the form

$$u_\epsilon(x) = \mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + \Phi_\epsilon(x) \tag{3.1.12}$$

where the remainder term $\Phi_\epsilon$ is a small function which is even with respect to the variables $x_1, \dots, x_N$.

The Lyapunov-Schmidt reduction allows us to reduce the problem of finding blowing-up solutions to (3.1.1) to the problem of finding critical points of a functional (the reduced energy) which depends only on the concentration parameters.
As announced before in our case some difficulties arise which need some modification of the standard procedure to be overcome.

Indeed, first we remark that the solutions of problem (3.1.1) are the critical points of the functional $J_\epsilon : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx - \frac{\epsilon}{2}\int_\Omega u^2\, dx, \qquad u \in H_0^1(\Omega). \tag{3.1.13}$$

If we apply directly the reduction method looking for a solution of the form (3.1.12) we get that the remainder term is such that

$$\|\Phi_\epsilon\| = O\left(\epsilon^{\frac{N-2}{N-4}+\sigma}\right) \qquad \sigma > 0$$

where $\|\cdot\|$ denotes the $H_0^1(\Omega)$-norm, and that the reduced energy

$$\textit{Reduced Energy} \sim J_\epsilon(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) = C + C_1\tau(0)\delta_1^{N-2} - C_2\epsilon\delta_1^2 + C_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + H.O.T.$$

where $C, C_i$ are some known positive constants.
Since $\delta_1, \delta_2$ are proper power of $\epsilon$ of the form $\delta_j = \epsilon^{\gamma_j} d_j,\ d_j > 0$ , after some easy computations, in order to find a critical point of the reduced energy, we get that

$$\textit{Reduced Energy} \sim C + \epsilon^{\frac{N-2}{N-4}}\left[C_1\tau(0)d_1^{N-2} - C_2 d_1^2 + C_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}\right] + o(\epsilon^{\frac{N-2}{N-4}})$$

with

$$\gamma_1 := \frac{1}{N-4}; \qquad \gamma_2 := \frac{3}{N-4}.$$

However the function

$$\Psi(d_1, d_2) = C_1\tau(0)d_1^{N-2} - C_2 d_1^2 + C_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}$$

has a critical point in $d_1$ but not in $d_2$ and hence in this way we cannot find a solution of our problem.

Hence we use a new idea. We split the remainder term $\Phi_\epsilon$ in two parts:

$$\Phi_\epsilon(x) = \phi_{1,\epsilon}(x) + \phi_{2,\epsilon}(x)$$

such that

$$\|\phi_{2,\epsilon}\| = o(\|\phi_{1,\epsilon}\|), \text{ as } \epsilon \to 0.$$

Usually, the remainder term $\Phi_\epsilon$, solution of the auxiliary equation, is found with a fixed point argument. Here we have to use the Contraction Mapping Theorem twice, since we split the auxiliary equation in a system of two equations. The first one depends only on $\phi_1$ while the second one depends on both $\phi_1, \phi_2$. So we solve the first equation in $\phi_1$ and then the second one finding $\phi_2$. Then we obtain the remainder term $\Phi_\epsilon$ which consists of two terms of different orders. Then we study the finite-dimensional problem, namely the reduced energy that consists of two functions of different orders. The lower term depends only on $d_1$ while the term of higher order depends on $d_1, d_2$. At the end we look for a critical point of this new type of reduced energy. We believe that our strategy can be used also in other contexts.

The outline of the chapter is the following: in Section 3.2 we explain the setting of the problem. In Section 3.3 we look for the remainder term $\Phi_\epsilon$ in a suitable space. In Section 3.4 we study the reduced energy and finally Theorem 3.1.1 and Theorem 3.1.2 are proved in Section 3.5.

## 3.2 Setting of the problem

In what follows we let

$$(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx, \qquad \|u\| := \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$$

as the inner product in $H_0^1(\Omega)$ and its corresponding norm while we denote by $(\cdot, \cdot)_{H^1(\mathbb{R}^N)}$ and by $\| \cdot \|_{1,2,\mathbb{R}^N}$ the scalar product and the standard norm in $H^1(\mathbb{R}^N)$. Moreover we denote by

$$|u|_r := \left( \int_{\Omega} |u|^r \, dx \right)^{\frac{1}{r}}$$

the $L^r(\Omega)$-standard norm for any $r \in [1, +\infty)$. When $A \neq \Omega$ is any Lebesgue measurable subset of $\mathbb{R}^N$, or, when $A = \Omega$ and we need to specify the domain of integration, we will use the alternative notations $\|u\|_A$, $|u|_{r,A}$.

From now on we assume that $\Omega$ is a bounded open set with smooth boundary of $\mathbb{R}^N$, symmetric with respect to $x_1, \ldots, x_N$ and which contains the origin. Moreover we assume that $N \geq 7$.

We define then

$$H_{sim} := \left\{ u \in H_0^1(\Omega) \ ; \ u \text{ is symmetric with respect to each variable } x_k, \ k = 1, \ldots, N \right\},$$

and for $q \in [1, +\infty)$

$$L^q_{sim} := \{ u \in L^q(\Omega) \ ; \ \ u \text{ is symmetric with respect to each variable } x_k, \ k = 1, \ldots, N \}.$$

Let $i^* : L_{sim}^{\frac{2N}{N+2}} \to H_{sim}$ be the adjoint operator of the embedding $i : H_{sim}(\Omega) \to L_{sim}^{\frac{2N}{N-2}}$, namely if $v \in L_{sim}^{\frac{2N}{N+2}}$ then $u = i^*(v)$ in $H_{sim}$ is the unique solution of the equation

$$-\Delta u = v \qquad \text{in } \Omega \qquad u = 0 \qquad \text{on } \partial\Omega.$$

By the continuity of $i$ it follows that

$$\|i^*(v)\| \leq C|v|_{\frac{2N}{N+2}} \qquad \forall v \in L_{sim}^{\frac{2N}{N+2}} \tag{3.2.1}$$

for some positive constant $C$ which depends only on $N$.
Hence we can rewrite problem (3.1.1) in the following way

$$\begin{cases} u = i^* \left[ f(u) + \epsilon u \right] \\ u \in H_{sim} \end{cases} \tag{3.2.2}$$

where $f(s) = |s|^{p-1}s$, $p = \frac{N+2}{N-2}$.
We next describe the shape of the solution we are looking for.
Let $\delta_j = \delta_j(\epsilon)$, for $j = 1, 2$ be positive parameters defined as proper powers of $\epsilon$, multiplied by a suitable positive constant to be determined later, namely

$$\delta_j = \epsilon^{\alpha_j} d_j \qquad \text{with } \ d_j > 0 \tag{3.2.3}$$

and $\alpha_1 := \frac{1}{N-4}$; $\alpha_2 := \frac{3N-10}{(N-4)(N-6)}$.
Fixed a small $\eta > 0$ we impose that the parameters $d_j$ will satisfy

$$\eta < d_j < \frac{1}{\eta} \qquad \text{for } j = 1, 2. \tag{3.2.4}$$

Hence, it is immediate to see that

$$\frac{\delta_2}{\delta_1} = \epsilon^{\frac{2(N-2)}{(N-4)(N-6)}} \frac{d_2}{d_1} \to 0 \qquad \text{as } \epsilon \to 0.$$

We construct solutions to problem (3.1.1), as predicted by Theorem 3.1.1, which are superpositions of copies of the standard bubble defined in (3.1.3) with alternating signs, properly modified (namely we consider the projection of the original bubble into $H_0^1(\Omega)$), centered at the origin which is the center of symmetry of $\Omega$ with parameters of concentrations $\delta_j$. Such an object has the shape of a tower of two bubbles.

Hence the solution to problem (3.1.1) will be of the form

$$u_\epsilon(x) = V_\epsilon(x) + \Phi_\epsilon(x) \tag{3.2.5}$$

where

$$V_\epsilon(x) := \mathcal{P}\mathcal{U}_{\delta_1}(x) - \mathcal{P}\mathcal{U}_{\delta_2}(x). \tag{3.2.6}$$

The term $\Phi_\epsilon$ has to be thought as a remainder term of lower order, which has to be described accurately.
Let $Z_j$ the following functions

$$Z_j(x) := \partial_{\delta_j}\mathcal{U}_{\delta_j}(x) = \alpha_N \frac{N-2}{2} \delta_j^{\frac{N-4}{2}} \frac{|x|^2 - \delta_j^2}{\left(\delta_j^2 + |x|^2\right)^{\frac{N}{2}}}, \quad j = 1, 2. \tag{3.2.7}$$

We remark that the functions $Z_j$ solve the problem (see [16])

$$-\Delta z = p|\mathcal{U}_\delta|^{p-1} z, \qquad \text{in } \mathbb{R}^N. \tag{3.2.8}$$

Let $\mathcal{P}Z_j$ the projection of $Z_j$ onto $H_0^1(\Omega)$. Elliptic estimates give

$$\mathcal{P}Z_j(x) = Z_j(x) - \alpha_N \frac{N-2}{2} \delta_j^{\frac{N-4}{2}} H(0, x) + O(\delta_j^{\frac{N}{2}}), \quad j = 1, 2, \tag{3.2.9}$$

uniformly in $\Omega$.

Let us consider

$$\mathcal{K}_1 := \text{span}\{\mathcal{P}Z_1\} \subset H_{sim}; \qquad \mathcal{K} := \text{span}\{\mathcal{P}Z_j\,;\, j = 1, 2\} \subset H_{sim}$$

and

$$\mathcal{K}_1^\perp := \{\phi \in H_{sim}\,;\, \langle \phi, \mathcal{P}Z_1 \rangle = 0\}\,; \qquad \mathcal{K}^\perp := \{\phi \in H_{sim}\,;\, \langle \phi, \mathcal{P}Z_j \rangle = 0,\ j = 1, 2\}\,.$$

Let $\Pi_1 : H_{sim} \to \mathcal{K}_1$, $\Pi : H_{sim} \to \mathcal{K}$ and $\Pi_1^\perp : H_{sim} \to \mathcal{K}_1^\perp$, $\Pi^\perp : H_{sim} \to \mathcal{K}^\perp$ be the projections onto $\mathcal{K}_1$, $\mathcal{K}$ and $\mathcal{K}_1^\perp$, $\mathcal{K}^\perp$, respectively.
In order to solve problem (3.1.1) we will solve the couple of equations

$$\Pi^\perp \left\{V_\epsilon + \Phi_\epsilon - i^*\left[f(V_\epsilon + \Phi_\epsilon) + \epsilon(V_\epsilon + \Phi_\epsilon)\right]\right\} = 0 \tag{3.2.10}$$

$$\Pi\left\{V_\epsilon + \Phi_\epsilon - i^*\left[f(V_\epsilon + \Phi_\epsilon) + \epsilon(V_\epsilon + \Phi_\epsilon)\right]\right\} = 0. \tag{3.2.11}$$

For any $(d_1, d_2)$ satisfying condition (3.2.4), we solve first the equation (3.2.10) in $\Phi_\epsilon \in \mathcal{K}^\perp$ which is the lower order term in the description of the ansatz.

We start with solving the auxiliary equation (3.2.10). As anticipated in the introduction, we split the remainder term as

$$\Phi_\epsilon = \phi_{1,\epsilon} + \phi_{2,\epsilon}$$

with

$$\|\phi_{2,\epsilon}\| = o(\|\phi_{1,\epsilon}\|), \quad \text{as } \epsilon \to 0.$$

In order to find $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ we solve the following system of equations

$$\begin{cases} \mathcal{R}_1 + \mathcal{L}_1(\phi_1) + \mathcal{N}_1(\phi_1) = 0, \\ \\ \mathcal{R}_2 + \mathcal{L}_2(\phi_2) + \mathcal{N}_2(\phi_1, \phi_2) = 0, \end{cases} \tag{3.2.12}$$

where

$$\mathcal{R}_1 := \Pi_1^\perp\left\{\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon\mathcal{P}\mathcal{U}_{\delta_1}\right]\right\}, \tag{3.2.13}$$

$$\mathcal{R}_2 := \Pi^\perp\left\{-\mathcal{P}\mathcal{U}_{\delta_2} - i^*\left[f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon\mathcal{P}\mathcal{U}_{\delta_2}\right]\right\}, \tag{3.2.14}$$

$$\mathcal{L}_1(\phi_1) := \Pi_1^\perp\left\{\phi_1 - i^*\left[f'(\mathcal{P}\mathcal{U}_1)\phi_1 + \epsilon\phi_1\right]\right\}, \tag{3.2.15}$$

$$\mathcal{L}_2(\phi_2) := \Pi^\perp\left\{\phi_2 - i^*\left[f'(V_\epsilon)\phi_2 + \epsilon\phi_2\right]\right\}, \tag{3.2.16}$$

$$\mathcal{N}_1(\phi_1) := \Pi_1^\perp\{-i^*[f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1]\}, \tag{3.2.17}$$

and

$$\mathcal{N}_2(\phi_1, \phi_2) := \Pi^\perp\{-i^*[f(V_\epsilon + \phi_1 + \phi_2) - f(V_\epsilon) - f'(V_\epsilon)\phi_2 - f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})]\}. \tag{3.2.18}$$

We remark that it is not restrictive to consider $\mathcal{R}_1, \mathcal{L}(\phi_1), \mathcal{N}_1(\phi_1) \in \mathcal{K}_1^\perp$ since only $\delta_1$ appears and it is clear that a solution of (3.2.12) gives a solution of (3.2.10).

Therefore we solve the first equation in (3.2.12) finding a solution $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1)$ and after that we solve the second equation in (3.2.12) (with $\phi_1 = \bar{\phi}_1$) finding also $\bar{\phi}_2 = \bar{\phi}_2(\epsilon, d_1, d_2)$.

Finally let us recall some useful inequality that we will use in the sequel. Since these are known results, we omit the proof.

**Lemma 3.2.1.** *Let $\alpha$ be a positive real number. If $\alpha \leq 1$ there holds*

$$(x + y)^\alpha \leq x^\alpha + y^\alpha,$$

*for all $x, y > 0$. If $\alpha \geq 1$ we have*

$$(x + y)^\alpha \leq 2^{\alpha - 1}(x^\alpha + y^\alpha),$$

*for all $x, y > 0$.*

**Lemma 3.2.2.** *Let $q$ be a positive real number. There exists a positive constant $c$, depending only on $q$, such that for any $a, b \in \mathbb{R}$*

$$||a+b|^q - |a|^q| \leq \begin{cases} c(q)\min\{|b|^q, |a|^{q-1}|b|\} & \text{if } 0 < q < 1, \\ c(q)(|a|^{q-1}|b| + |b|^q) & \text{if } q \geq 1. \end{cases} \tag{3.2.19}$$

*Moreover if $q > 2$ then*

$$\left||a+b|^q - |a|^q - q|a|^{q-2}ab\right| \leq C\left(|a|^{q-2}|b|^2 + |b|^q\right). \tag{3.2.20}$$

**Lemma 3.2.3.** *Let $N \geq 7$. There exists a positive constant $c$, depending only on $p$, such that for any $a, b \in \mathbb{R}$*

$$|f(a+b) - f(a) - f'(a)b| \leq c|b|^p. \tag{3.2.21}$$

**Lemma 3.2.4.** *There exists a positive constant $c$, depending only on $p$, such that for any $a, b \in \mathbb{R}$*

$$|f(a-b) - f(a) + f(b)| \leq c(p)(|a|^{p-1}|b| + |b|^p), \tag{3.2.22}$$

*or*

$$|f(a-b) - f(a) + f(b)| \leq c(p)(|b|^{p-1}|a| + |a|^p). \tag{3.2.23}$$

**Lemma 3.2.5.** *Let $N \geq 7$. There exists a positive constant $c$ depending only on $p$ such that for any $a, b_1, b_2 \in \mathbb{R}$ we get*

$$\left|f(a+b_1) - f(a+b_2) - f'(a)(b_1 - b_2)\right| \leq C\left(|b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2|. \tag{3.2.24}$$

## 3.3 The auxiliary equation: solution of the system (3.2.12)

We first define

$$\theta_1 := \frac{N-2}{N-4}; \qquad \theta_2 := \frac{(N-2)^2}{(N-4)(N-6)}. \tag{3.3.1}$$

We observe that $\theta_2$ is well defined since $N \geq 7$. We also remark that having defined $\delta_j$ as in (3.2.3), $j = 1, 2$, the functions $\mathcal{U}_{\delta_j}$ depend on the parameters $d_j$, $j = 1, 2$.

In this section we solve system (3.2.12). More precisely, the aim is to prove the following result.

**Proposition 3.3.1.** *Let $N \geq 7$. For any $\eta > 0$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (3.2.4), there exists a unique $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1) \in \mathcal{K}_1^\perp$ solution of the first equation of (3.2.12) such that*

$$\|\bar{\phi}_1\| \leq c\epsilon^{\frac{\theta_1}{2}+\sigma}$$

*and there exists a unique solution $\bar{\phi}_2 = \bar{\phi}_2(\epsilon, d_1, d_2) \in \mathcal{K}^\perp$ of the second equation of (3.2.12) (with $\phi_1 = \bar{\phi}_1$) such that*

$$\|\bar{\phi}_2\| \leq c\,\epsilon^{\frac{\theta_2}{2}+\sigma},$$

*for some positive real number $\sigma$ whose choice depends only on $N$. Furthermore, $\bar{\phi}_1$ does not depend on $d_2$ and it is continuously differentiable with respect to $d_1$, $\bar{\phi}_2$ is continuously differentiable with respect to $(d_1, d_2)$.*

In order to prove Proposition 3.3.1 let us first consider the linear operator

$$\mathcal{L}_1 : \mathcal{K}_1^{\perp} \to \mathcal{K}_1^{\perp}$$

defined as in (3.2.15).
The next result provides an a-priori estimate for solutions $\phi \in \mathcal{K}_1^{\perp}$ of $\mathcal{L}_1(\phi) = h$, for some right-hand side $h$ with bounded $\|\cdot\|-$ norm.

**Lemma 3.3.2.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $d_1 \in \mathbb{R}^+$ satisfying* (3.2.4) *for $j = 1$, for all $\phi \in \mathcal{K}_1^{\perp}$ and for all $\epsilon \in (0, \epsilon_0)$ it holds*

$$\|\mathcal{L}_1(\phi)\| \geq c\|\phi\|.$$

*Proof.* For the proof it suffices to repeat with small changes the proof of Lemma 3.1 of [48]. □

Next result states the invertibility of the operator $\mathcal{L}_1$ and provides a uniform estimate on the inverse of the operator $\mathcal{L}_1$.

**Proposition 3.3.3.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that the linear operator $\mathcal{L}_1$ is invertible and $\|\mathcal{L}_1^{-1}\| \leq c$ for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}^+$ satisfying* (3.2.4) *for $j = 1$.*

*Proof.* For the proof it suffices to repeat with small changes the proof of Proposition 3.2 of [48]. □

For the linear operator $\mathcal{L}_2$ we state analogous results.

**Lemma 3.3.4.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $(d_1, d_2) \in \mathbb{R}_+^2$ satisfying* (3.2.4)*, for all $\phi \in \mathcal{K}^{\perp}$ and for all $\epsilon \in (0, \epsilon_0)$ it holds*

$$\|\mathcal{L}_2(\phi)\| \geq c\|\phi\|.$$

*Proof.* For the proof see Lemma 3.1 of [48]. □

**Proposition 3.3.5.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that the linear operator $\mathcal{L}_2$ is invertible and $\|\mathcal{L}_2^{-1}\| \leq c$ for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}_+^2$ satisfying* (3.2.4)*.*

*Proof.* For the proof see Proposition 3.2 of [48]. □

The strategy is to solve the first equation of (3.2.12) by a fixed point argument, finding a unique $\bar{\phi}_1$ and then, substituting $\bar{\phi}_1$ in the second equation of (3.2.12), we obtain an equation depending only on the variable $\phi_2$. Hence, using again a fixed point argument, we solve the second equation of (3.2.12) uniquely.

### 3.3.1 The solution of the first equation of (3.2.12)

The aim is to prove the following proposition.

**Proposition 3.3.6.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}^+$ satisfying condition* (3.2.4) *for $j = 1$, there exists a unique solution $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1)$, $\bar{\phi}_1 \in \mathcal{K}_1^{\perp}$ of the first equation in* (3.2.12) *which is continuously differentiable with respect to $d_1$ and such that*

$$\|\bar{\phi}_1\| \leq c\epsilon^{\frac{\theta_1}{2}+\sigma}, \tag{3.3.2}$$

*where $\theta_1$ is defined in* (3.3.1) *and $\sigma$ is some positive real number whose choice depends only on $N$.*

In order to prove Proposition 3.3.6 we have to estimate the error term $\mathcal{R}_1$ defined in (3.2.13). It holds the following result.

**Proposition 3.3.7.** *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}^+$ satisfying condition* (3.2.4) *for $j = 1$, we have*

$$\|\mathcal{R}_1\| \leq c\,\epsilon^{\frac{\theta_1}{2}+\sigma},$$

*for some positive real number $\sigma$ whose choice depends only on $N$.*

*Proof.* By continuity of $\Pi_1^{\perp}$, by using (3.2.1) and since $\mathcal{P}\mathcal{U}_{\delta_1}$ weakly solves $-\Delta \mathcal{P}\mathcal{U}_{\delta_1} = \mathcal{U}_{\delta_1}^p$ in $\Omega$, it follows that

$$\begin{aligned} \|\mathcal{R}_1\| &= \|\Pi_1^{\perp}\left\{\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon \mathcal{P}\mathcal{U}_{\delta_1}\right]\right\}\| \leq C_1 \|\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon \mathcal{P}\mathcal{U}_{\delta_1}\right]\| \\ &\leq C_2\, |f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon \mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}} \leq C \underbrace{|f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}}_{(I)} + \underbrace{\epsilon\, |\mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}}}_{(II)}. \end{aligned}$$

Let us fix $\eta > 0$. We estimate the terms $(I), (II)$.

*Claim 1:*

$$(I) = O(\epsilon^{\frac{N+2}{2(N-4)}}). \tag{3.3.3}$$

By using (3.1.9), (3.1.10) and by elementary inequalities we get

$$\begin{aligned} \int_\Omega |(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|^{\frac{2N}{N+2}}\, dx &\leq c_1 \int_\Omega |\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}|^{\frac{2N}{N+2}}\, dx + c_2 \int_\Omega |\varphi_{\delta_1}|^{p+1}\, dx \\ &\leq c_3 \delta_1^{\frac{N-2}{2}\frac{2N}{N+2}} \int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2 + |x|^2)^2}\right)^{\frac{2N}{N+2}} dx + c_2 |\varphi_{\delta_1}|_{p+1,\Omega}^{p+1} \\ &= c_3 \delta_1^{\frac{N(N-2)}{N+2}} \int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2 + |x|^2)^2}\right)^{\frac{2N}{N+2}} dx + c_4 \delta_1^N. \end{aligned}$$

Now for $N \geq 7$ we have

$$\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2 + |x|^2)^2}\right)^{\frac{2N}{N+2}} dx = O\left(\delta_1^{\frac{4N}{N+2}}\right).$$

Indeed:

$$\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2 + |x|^2)^2}\right)^{\frac{2N}{N+2}} dx = \delta_1^{\frac{4N}{N+2}} \int_\Omega \frac{1}{(\delta_1^2 + |x|^2)^{\frac{4N}{N+2}}}\, dx \leq \delta_1^{\frac{4N}{N+2}} \int_\Omega \frac{1}{|x|^{\frac{8N}{N+2}}}\, dx,$$

and the last integral is finite since $N > 6$, which implies $\frac{8N}{N+2} < N$. Finally, since $\frac{4N^2(N-2)}{(N+2)^2} > N$, for any $N > 4$, we deduce that

$$\int_\Omega |(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|^{\frac{2N}{N+2}}\, dx = O\left(\delta_1^N\right),$$

and hence

$$|(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2}} = O\left(\delta_1^{\frac{N+2}{2}}\right). \tag{3.3.4}$$

Since $\delta_1 = d_1\epsilon^{\frac{1}{N-4}}$ and $d_1$ satisfies (3.2.4), we get that $|(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2}} = O\left(\epsilon^{\frac{N+2}{2(N-4)}}\right)$ and Claim 1 is proved.

*Claim 2:*

$$(II) = O(\epsilon^{\frac{N-2}{N-4}}). \tag{3.3.5}$$

$$\begin{aligned}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}}\, dx &\leq \int_\Omega \mathcal{U}_{\delta_1}^{\frac{2N}{N+2}}\, dx = \alpha_N^{\frac{2N}{N+2}} \int_\Omega \frac{\delta_1^{-\frac{N(N-2)}{N+2}}}{(1+|\frac{x}{\delta_1}|^2)^{\frac{N(N-2)}{N+2}}}\, dx \\ &= \alpha_N^{\frac{2N}{N+2}} \delta_1^{\frac{4N}{N+2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N(N-2)}{N+2}}}\, dy + o(\delta_1^{\frac{4N}{N+2}}). \end{aligned} \tag{3.3.6}$$

Thus, since $\delta_1 = d_1\epsilon^{\frac{1}{N-4}}$ and $d_1$ satisfies (3.2.4), we get that

$$\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}}\, dx = O\left(\epsilon^{\frac{4N}{(N+2)(N-4)}}\right),$$

and hence

$$\epsilon\left(\int_\Omega P\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}}\, dx\right)^{\frac{N+2}{2N}} = \epsilon O\left(\epsilon^{\frac{2}{N-4}}\right) = O\left(\epsilon^{\frac{N-2}{N-4}}\right).$$

The proof of Claim 2 is complete.

Hence, by (3.3.3) and (3.3.5), we deduce that there exist a constant $c = c(\eta) > 0$ and $\epsilon_0 = \epsilon_0(\eta) > 0$ sufficiently small such that, for all $\epsilon \in (0, \epsilon_0)$ and $d_1 \in \mathbb{R}^+$ satisfying (3.2.4) (with $j = 1$)

$$\|\mathcal{R}_1\| \leq c\,(\epsilon^{\frac{N+2}{2(N-4)}} + \epsilon^{\frac{N-2}{N-4}}) \leq c\epsilon^{\frac{\theta_1}{2}+\sigma},$$

with $\sigma$ such that $0 < \sigma < \frac{2}{N-4}$. □

We are ready to prove Proposition 3.3.6.

*Proof of Proposition 3.3.6.* Let us fix $\eta > 0$ and define $\mathcal{T}_1 : \mathcal{K}_1^\perp \to \mathcal{K}_1^\perp$ as

$$\mathcal{T}_1(\phi_1) := -\mathcal{L}_1^{-1}[\mathcal{N}_1(\phi_1) + \mathcal{R}_1].$$

Clearly solving the first equation of (3.2.12) is equivalent to solving the fixed point equation $\mathcal{T}_1(\phi_1) = \phi_1$.
Let us define the ball

$$B_{1,\epsilon} := \{\phi_1 \in \mathcal{K}_1^\perp;\ \|\phi_1\| \leq r\ \epsilon^{\frac{\theta_1}{2}+\sigma}\} \subset \mathcal{K}_1^\perp$$

with $r > 0$ sufficiently large and $\sigma > 0$.
We want to prove that, for $\epsilon$ small, $\mathcal{T}_1$ is a contraction in the proper ball $B_{1,\epsilon}$, namely we want to prove that, for $\epsilon$ sufficiently small

1. $\mathcal{T}_1(B_{1,\epsilon}) \subset B_{1,\epsilon}$;

2. $\|\mathcal{T}_1\| < 1$.

By Lemma 3.3.2 we get:

$$\|\mathcal{T}_1(\phi_1)\| \le c(\|\mathcal{N}_1(\phi_1)\| + \|\mathcal{R}_1\|) \tag{3.3.7}$$

and

$$\|\mathcal{T}_1(\phi_1) - \mathcal{T}_1(\psi_1)\| \le c(\|\mathcal{N}_1(\phi_1) - \mathcal{N}_1(\psi_1)\|), \tag{3.3.8}$$

for all $\phi_1, \psi_1 \in \mathcal{K}_1^\perp$. Thanks to (3.2.1) and the definition of $\mathcal{N}_1$ we deduce that

$$\|\mathcal{N}_1(\phi_1)\| \le c|f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1|_{\frac{2N}{N+2}}, \tag{3.3.9}$$

and

$$\|\mathcal{N}_1(\phi_1) - \mathcal{N}_1(\psi_1)\| \le c|f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \psi_1) - f'(\mathcal{P}\mathcal{U}_{\delta_1})(\phi_1 - \psi_1)|_{\frac{2N}{N+2}}. \tag{3.3.10}$$

Now we estimate the right-hand term in (3.3.7). Thanks to Lemma 3.2.3 we have the following inequality:

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1| \le c|\phi_1|^p. \tag{3.3.11}$$

Since $p\frac{2N}{N+2} = \frac{2N}{N-2}$ and $|\phi_1^p|_{\frac{2N}{N+2}} = |\phi_1|^p_{\frac{2N}{N-2}}$, from (3.3.11) and the Sobolev inequality we deduce the following:

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1|_{\frac{2N}{N+2}} \le c_1|\phi_1|^p_{\frac{2N}{N-2}} \le c_2\|\phi_1\|^p. \tag{3.3.12}$$

Thanks to (3.3.7), Proposition 3.3.7, (3.3.9), (3.3.12) and since $p > 1$, then, there exist $c = c(\eta) > 0$ and $\epsilon_0 = \epsilon_0(\eta) > 0$ such that

$$\|\phi_1\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma} \Rightarrow \|\mathcal{T}_1(\phi_1)\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma},$$

for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}^+$ satisfying (3.2.4) (with $j = 1$), for some positive real number $\sigma$, whose choice depends only on $N$. In other words $\mathcal{T}_1$ maps the ball $B_{1,\epsilon}$ into itself and (1) is proved.

We want to show that $\mathcal{T}_1$ is a contraction. By using Lemma 3.2.5 we get that for any $\phi_1, \psi_1 \in B_{1,\epsilon}$

$$\left|f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \psi_1) - f'(\mathcal{P}\mathcal{U}_{\delta_1})(\phi_1 - \psi_1)\right| \le C\left(|\phi_1|^{p-1} + |\psi_1|^{p-1}\right)|\phi_1 - \psi_1|.$$

By direct computation $(p-1)\frac{2N}{N+2} = \frac{8N}{(N-2)(N+2)}$, so, since $|\phi_1|^{(p-1)\frac{2N}{N+2}}, |\psi_1|^{(p-1)\frac{2N}{N+2}} \in L^{\frac{N+2}{4}}$, $|\phi_1 - \psi_1|^{\frac{2N}{N+2}} \in L^p$ and $1 = \frac{4}{N+2} + \frac{N-2}{N+2}$ by Hölder inequality we get that

$$\begin{aligned} \left|\left(|\phi_1|^{p-1} + |\psi_1|^{p-1}\right)(\phi_1 - \psi_1)\right|_{\frac{2N}{N+2}} &\le \left[\left(|\phi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} + |\psi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}\right)^{\frac{2N}{N+2}} \left(|\phi_1 - \psi_1|^{\frac{2N}{N-2}}_{\frac{2N}{N-2}}\right)^{\frac{N-2}{N+2}}\right]^{\frac{N+2}{2N}} \\ &= \left(|\phi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} + |\psi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}\right)|\phi_1 - \psi_1|_{\frac{2N}{N-2}}. \end{aligned} \tag{3.3.13}$$

Hence by (3.3.8), (3.3.10), (3.3.13) and Sobolev inequality we get that there exists $L \in (0,1)$ such that

$$\|\phi_1\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma}, \|\psi_1\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma} \Rightarrow \|\mathcal{T}_1(\phi_1) - \mathcal{T}_1(\psi_1)\| \le L\|\phi_1 - \psi_1\|.$$

Hence by the Contraction Mapping Theorem we can uniquely solve $\mathcal{T}_1(\phi_1) = \phi_1$ in $B_{1,\epsilon}$. We denote by $\bar{\phi}_1 \in B_{1,\epsilon}$ this solution. A standard argument shows that $d_1 \to \bar{\phi}_1(d_1)$ is a $C^1$-map (as a map from $\mathbb{R}^+$ to $H_0^1(\Omega)$) (see also [48], [3]). The proof is then concluded. □

### 3.3.2 The proof of Proposition 3.3.1

Before proving Proposition 3.3.1 we need some preliminary results, in particular we need to improve the estimate on the solution $\bar{\phi}_1$ of the first equation of (3.2.12) found in Proposition 3.3.6.

The first preliminary result is an estimate on the error term $\mathcal{R}_2$ defined in (3.2.14).

**Proposition 3.3.8.** *For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying* (3.2.4)*, we have*

$$\|\mathcal{R}_2\| \leq c\, \epsilon^{\frac{\theta_2}{2}+\sigma},$$

*for some positive real number $\sigma$, whose choice depends only on $N$.*

*Proof.* By continuity of $\Pi^\perp$ and by using (3.2.1) we deduce that

$$\begin{aligned} \|\mathcal{R}_2\| &\leq c|f(\mathcal{U}_{\delta_2}) + f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon \mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}} \\ &\leq \underbrace{c|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}}}_{(I)} + \underbrace{c|f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}}}_{(II)} \\ &\quad + \underbrace{c\epsilon|\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}}}_{(III)}. \end{aligned} \tag{3.3.14}$$

Let us fix $\eta > 0$. We begin estimating $(I)$. Let $\rho > 0$ so that $B(0,\rho) \subset \Omega$. We decompose the domain $\Omega$ as $\Omega = A_0 \sqcup A_1 \sqcup A_2$, where $A_0 := \Omega \setminus B(0,\rho)$, $A_1 := B(0,\rho) \setminus B(0, \sqrt{\delta_1\delta_2})$ and $A_2 := B(0, \sqrt{\delta_1\delta_2})$. We evaluate $(I)$ in every set of this decomposition.

Thanks to Lemma 3.2.4 there exists a positive constant $c$ (depending only on $p$) such that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})| \leq c(\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2} + \mathcal{P}\mathcal{U}_{\delta_2}^{p}). \tag{3.3.15}$$

Integrating on $A_0$ and using the usual elementary inequalities (see Lemma 3.2.1) we get that

$$\begin{aligned} &\int_{A_0} |f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|^{\frac{2N}{N+2}}\, dx \\ &\leq C_1 \int_{A_0} (\mathcal{P}\mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}} + \mathcal{P}\mathcal{U}_{\delta_2}^{p+1})\, dx \\ &\leq C_2 \int_{A_0} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2 + |x|^2)^{\frac{4N}{N+2}}} \frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2 + |x|^2)^{\frac{N(N-2)}{N+2}}}\, dx + C_3 \int_{A_0} \frac{\delta_2^N}{(\delta_2^2 + |x|^2)^N}\, dx \\ &\leq C_4 \frac{\delta_1^{\frac{4N}{N+2}}}{\rho^{\frac{8N}{N+2}}} \frac{\delta_2^{\frac{N(N-2)}{N+2}}}{\rho^{2\frac{N(N-2)}{N+2}}} + C_5 \frac{\delta_2^N}{\rho^{2N}} \end{aligned} \tag{3.3.16}$$

and hence we deduce that (recall the choice of $\delta_1, \delta_2$ (see (3.2.3)))

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}, A_0} \leq c\epsilon^{\frac{3N^2-12N-4}{2(N-4)(N-6)}} \leq c\epsilon^{\frac{\theta_2}{2}+\sigma} \tag{3.3.17}$$

where $c$ depends on $\eta$ (and also on $\Omega$, $\rho$, $N$), $\sigma$ is some positive real number (to be precise we can choose $0 < \sigma \le \frac{N^2-4N-4}{(N-4)(N-6)}$).

We evaluate now $(I)$ in $A_1$. By (3.3.15) and the usual elementary inequalities we deduce the following:

$$\int_{A_1} |f(\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2})-f(\mathcal{PU}_{\delta_1})+f(\mathcal{PU}_{\delta_2})|^{\frac{2N}{N+2}}\,dx \le c\int_{A_1} (\mathcal{PU}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{PU}_{\delta_2}^{\frac{2N}{N+2}}+\mathcal{PU}_{\delta_2}^{p+1})\,dx. \tag{3.3.18}$$

Let us estimate every term:

$$\begin{aligned}
&\int_{A_1} \mathcal{PU}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{PU}_{\delta_2}^{\frac{2N}{N+2}}dx \\
\le\ & \int_{A_1} \mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}dx \\
=\ & \alpha_N^{p+1}\int_{A_1} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+|x|^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2+|x|^2)^{\frac{N(N-2)}{N+2}}}\,dx \\
=\ & c_1\int_{\sqrt{\delta_1\delta_2}}^{\rho} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+r^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2+r^2)^{\frac{N(N-2)}{N+2}}}r^{N-1}dr \\
=\ & c_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+\delta_2^2s^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{-\frac{N(N-2)}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}\delta_2^N s^{N-1}ds \\
=\ & c_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{\delta_1^{-\frac{4N}{N+2}}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)^2s^2\right]^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{4N}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}s^{N-1}ds \\
\le\ & c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{(1+s^2)^{\frac{N(N-2)}{N+2}}}s^{N-1}ds \\
\le\ & c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{s^{\frac{N^2-5N+2}{N+2}}}ds = c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\left[\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N^2-6N}{2(N+2)}}-\left(\frac{\delta_2}{\rho}\right)^{\frac{N^2-6N}{(N+2)}}\right] \\
\le\ & c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned} \tag{3.3.19}$$

Moreover

$$\int_{A_1}\mathcal{PU}_{\delta_2}^{p+1}\,dx \le \int_{A_1}\mathcal{U}_{\delta_2}^{p+1}\,dx \le C_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}}\frac{r^{N-1}}{(1+r^2)^N}\,dr \le C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}. \tag{3.3.20}$$

Thanks to the choice of $\delta_1$, $\delta_2$ we have

$$\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} = O(\epsilon^{\frac{N(N-2)}{(N-4)(N-6)}}). \tag{3.3.21}$$

Hence, from (3.3.18), (3.3.19), (3.3.20) and (3.3.21) we deduce that

$$|f(\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2})-f(\mathcal{PU}_{\delta_1})+f(\mathcal{PU}_{\delta_2})|_{\frac{2N}{N+2},A_1} \le c\epsilon^{\frac{(N-2)(N+2)}{2(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{3.3.22}$$

where $c$ depends on $\eta$, $\sigma$ is some positive real number (to be precise we can choose $0 < \sigma \le \frac{2(N-2)}{(N-4)(N-6)}$).

Now we evaluate $(I)$ in $A_2$. To do this we apply (3.2.23) of Lemma 3.2.4, so there exists a constant $c > 0$ such that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})| \leq c(\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\mathcal{P}\mathcal{U}_{\delta_1} + \mathcal{P}\mathcal{U}_{\delta_1}^{p}). \tag{3.3.23}$$

Thanks to (3.3.23) and the usual elementary inequalities we deduce the following:

$$\int_{A_2} |f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|^{\frac{2N}{N+2}}\, dx \leq c\int_{A_2} (\mathcal{P}\mathcal{U}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} + \mathcal{P}\mathcal{U}_{\delta_1}^{p+1})\, dx. \tag{3.3.24}$$

We estimate the first term

$$\begin{aligned}
\int_{A_2} \mathcal{P}\mathcal{U}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx &\leq \int_{A_2} \mathcal{U}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx \\
&= \alpha_N^{p+1} \int_{A_2} \frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2 + |x|^2)^{\frac{4N}{N+2}}} \frac{\delta_1^{\frac{N(N-2)}{N+2}}}{(\delta_1^2 + |x|^2)^{\frac{N(N-2)}{N+2}}}\, dx \\
&= c_1 \int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2 + \delta_1^2 s^2)^{\frac{4N}{N+2}}} \frac{\delta_1^{-\frac{N(N-2)}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}} \delta_1^N s^{N-1} ds \\
&= c_1 \int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{\delta_2^{\frac{4N}{N+2}}}{\left[\left(\frac{\delta_2}{\delta_1}\right)^2 + s^2\right]^{\frac{4N}{N+2}}} \frac{\delta_1^{-\frac{4N}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}} s^{N-1} ds \\
&\leq c_1 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}} \int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{1}{s^{\frac{8N}{N+2}}(1+s^2)^{\frac{N(N-2)}{N+2}}} s^{N-1} ds \\
&\leq c_1 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}} \int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{s^{\frac{N^2-7N-2}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}} ds \\
&\leq c_1 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}} \int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} s^{\frac{N^2-7N-2}{N+2}} ds \\
&= c_2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}} \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N^2-6N}{2(N+2)}} = c_2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned} \tag{3.3.25}$$

By making similar computations as before we get that

$$\int_{A_2} \mathcal{P}\mathcal{U}_{\delta_1}^{p+1}\, dx \leq c_3 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}. \tag{3.3.26}$$

So from (3.3.24) and (3.3.25) we deduce that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}, A_2} \leq c\epsilon^{\frac{(N+2)(N-2)}{2(N-4)(N-6)}} \leq c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{3.3.27}$$

where $c$ depends on $\eta$, $\sigma$ is some positive real number (to be precise we can choose $0 < \sigma \leq \frac{2(N-2)}{(N-4)(N-6)}$). Hence from (3.3.17), (3.3.22) and (3.3.27) we deduce that

$$(I) \leq c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{3.3.28}$$

for some positive constant $c$, for some positive real number $\sigma$ depending only on $N$.
Now by making similar computations as for (I) of Proposition 3.3.7 (see (3.3.4)) we get that

$$(II) = O\left(\delta_2^{\frac{N+2}{2}}\right),$$

and hence we deduce that

$$(II) \leq c\epsilon^{\frac{(3N-10)(N+2)}{2(N-4)(N-6)}} \leq c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

where $c$, $0 < \sigma \leq \frac{N^2-12}{(N-4)(N-6)}$.

It remains to estimate $(III)$.

From (3.3.6), exchanging $\delta_1$ with $\delta_2$ we get:

$$\int_\Omega \mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}\, dx \leq \alpha_N^{\frac{2N}{N+2}} \delta_2^{\frac{4N}{N+2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N(N-2)}{N+2}}}\, dy.$$

Hence we deduce that $(III) \leq c\ \epsilon\delta_2^2$, and thanks to the choice $\delta_2$, by an elementary computation, we get that:

$$(III) \leq c\ \epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}} \leq c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

where $c$, $0 < \sigma \leq \frac{(N-2)^2}{2(N-4)(N-6)}$. Finally, putting together all these estimates we deduce that there exist a positive constant $c = c(\eta) > 0$ and $\epsilon_0 = \epsilon_0(\eta) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}_+^2$ satisfying (3.2.4)

$$\|\mathcal{R}_2\| \leq c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

for some positive real number $\sigma$ (whose choice depends only on $N$). The proof is complete. □

Now we prove a technical result on the behavior of the $L^\infty$-norm of $\bar{\phi}_1$, which will be useful in the sequel.

**Lemma 3.3.9.** *Let $\eta$ be a small positive real number and let $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ be the solution of the first equation in* (3.2.12)*, found in Proposition 3.3.6. Then, as $\epsilon \to 0^+$, we have*

$$|\bar{\phi}_1|_\infty = o(\epsilon^{-\frac{N-2}{2(N-4)}}),$$

*uniformly with respect to $d_1$ satisfying* (3.2.4) *for $j = 1$.*

*Proof.* Let us fix a small $\eta > 0$ and remember that $\delta_1 = \epsilon^{\frac{1}{N-4}} d_1$ (see (3.2.3)), with $d_1$ satisfying (3.2.4) for $j = 1$. We observe that by definition, since $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ solves the first equation of (3.2.12), then, for all $\epsilon > 0$ sufficiently small, there exists a constant $c_\epsilon$ (which depends also on $d_1$) such that $\bar{\phi}_1$ weakly solves

$$-\Delta\bar{\phi}_1 = \epsilon\bar{\phi}_1 + \epsilon\mathcal{P}\mathcal{U}_{\delta_1} + f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar{\phi}_1) - f(\mathcal{U}_{\delta_1}) - c_\epsilon\Delta\mathcal{P}Z_1. \tag{3.3.29}$$

Testing (3.3.29) with $\mathcal{P}Z_1$, taking into account that $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ and the definition of $\mathcal{P}Z_1$, we have that

$$\begin{aligned} c_\epsilon \int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\, dx &= -\epsilon\int_\Omega \bar{\phi}_1\mathcal{P}Z_1\, dx - \epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\mathcal{P}Z_1\, dx \\ &\quad - \int_\Omega [f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_1})]\,\mathcal{P}Z_1\, dx \\ &\quad - \int_\Omega \left[f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar{\phi}_1) - f(\mathcal{P}\mathcal{U}_{\delta_1})\right]\mathcal{P}Z_1\, dx. \end{aligned} \tag{3.3.30}$$

By definition, if we set $\psi := Z_1 - PZ_1$, then $\psi$ is an harmonic function and $\psi = Z_1$ on $\partial\Omega$, therefore, by elementary elliptic estimates, for all sufficiently small $\epsilon > 0$, for any $d_1 \in ]\eta, \frac{1}{\eta}[$ we have that $|\psi|_{\infty,\Omega} \leq C\delta_1^{\frac{N-4}{2}}$, for some positive constant $C = C(N, \Omega)$ depending only on $N$ and $\Omega$, and hence

$$\int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\,dx = \int_\Omega pU_{\delta_1}^{p-1}Z_1^2\,dx - \int_\Omega pU_{\delta_1}^{p-1}\psi Z_1\,dx.$$

Now

$$\begin{aligned}\int_\Omega pU_{\delta_1}^{p-1}Z_1^2\,dx &= c_N\delta_1^{N-4}\delta_1^2\int_\Omega \frac{1}{(\delta_1^2+|x|^2)^2}\frac{(|x|^2-\delta_1^2)^2}{(\delta_1^2+|x|^2)^N}\,dx \\ &= c_N\delta_1^{-2}\int_{\mathbb{R}^N}\frac{(|y|^2-1)^2}{(1+|y|^2)^{N+2}}\,dy + O(\delta_1^{N-2}) \\ &= A_N\delta_1^{-2} + o(1), \quad \text{as } \epsilon \to 0.\end{aligned}$$

By using the property $|\psi|_{\infty,\Omega} \leq C\delta_1^{\frac{N-4}{2}}$, by the same computations, we see that

$$\int_\Omega pU_{\delta_1}^{p-1}\psi Z_1\,dx = O(\delta_1^{N-4}), \quad \text{as } \epsilon \to 0.$$

Therefore, we get that

$$\int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\,dx = A_N\delta_1^{-2} + o(1), \quad \text{as } \epsilon \to 0. \tag{3.3.31}$$

Moreover, reasoning as before, we have

$$\begin{aligned}\int_\Omega Z_1^2\,dx &= c_N\delta_1^{N-4}\int_\Omega \frac{(|x|^2-\delta_1^2)^2}{(\delta_1^2+|x|^2)^N}\,dx \\ &= c_N\int_{\mathbb{R}^N}\frac{(|y|^2-1)^2}{(1+|y|^2)^N}\,dy + O(\delta_1^{N-2}) \\ &= B_N + o(1), \quad \text{as } \epsilon \to 0,\end{aligned}$$

and, by an analogous computation

$$|Z_1|_{\frac{2N}{N-2}} \leq c_N\left[\int_\Omega \delta_1^{\frac{N(N-4)}{N-2}}\frac{||x|^2-\delta_1^2|^{\frac{2N}{N-2}}}{(\delta_1^2+|x|^2)^{\frac{N^2}{N-2}}}\,dx\right]^{\frac{N-2}{2N}} \leq C_N\delta_1^{-1},$$

and hence, since $\mathcal{P}Z_1 = Z_1 - \psi$, by elementary estimates, we get that for all sufficiently small $\epsilon > 0$

$$|\mathcal{P}Z_1|_2^2 \leq 2B_N, \quad |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \leq 2C_N\delta_1^{-1}. \tag{3.3.32}$$

Thanks to (3.3.32), applying Hölder inequality, Poincaré inequality, taking into account of (3.3.4), the asymptotic expansion of $|\mathcal{P}\mathcal{U}_{\delta_1}|_2$ (see Lemma 3.4.6 and its proof), the choice of $\delta_1$ (see (3.2.3)) and since $\bar{\phi}_1 \in B_{1,\epsilon}$, we have the following inequalities

$$\epsilon\int_\Omega |\bar{\phi}_1||\mathcal{P}Z_1|\,dx \leq \epsilon|\bar{\phi}_1|_2|\mathcal{P}Z_1|_2 \leq c_1\epsilon\|\bar{\phi}_1\||\mathcal{P}Z_1|_2 \leq c_2\epsilon^{\frac{\theta_1}{2}+1+\sigma}$$

$$\epsilon \int_\Omega \mathcal{P}\mathcal{U}_{\delta_1} |\mathcal{P}Z_1|\, dx \le \epsilon |\mathcal{P}\mathcal{U}_{\delta_1}|_2 |\mathcal{P}Z_1|_2 \le c\epsilon\delta_1,$$

$$\int_\Omega |f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_1})||\mathcal{P}Z_1|\, dx \le |f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \le c\delta_1^{\frac{N+2}{2}} \delta_1^{-1} = c\delta_1^{\frac{N}{2}}.$$

Moreover, taking into account of Lemma 3.2.3 and Sobolev inequality, we get that

$$\begin{aligned}
&\int_\Omega |f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1})||\mathcal{P}Z_1|\, dx \\
\le\ & |f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \\
\le\ & |f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} + |f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \\
\le\ & c\big||\bar\phi_1|^p\big|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} + |f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \\
\le\ & c_1 \left( |\bar\phi_1|^{\frac{N+2}{N-2}}_{\frac{2N}{N-2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} + |\mathcal{P}\mathcal{U}_{\delta_1}|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} |\bar\phi_1|_{\frac{2N}{N-2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \right) \\
\le\ & c_2 \left( \|\bar\phi_1\|^{\frac{N+2}{N-2}} |\mathcal{P}Z_1|_{\frac{2N}{N-2}} + |\mathcal{P}\mathcal{U}_{\delta_1}|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} \|\bar\phi_1\| |\mathcal{P}Z_1|_{\frac{2N}{N-2}} \right) \\
\le\ & c_3 \epsilon^{\frac{\theta_1}{2}+\sigma} \delta_1^{-1} = c_4 \epsilon^{\frac{N-2}{2(N-4)}+\sigma-\frac{1}{N-4}} \le c_4 \epsilon^{\frac{1}{2}}.
\end{aligned}$$

Thus, from (3.3.30), (3.3.31) and the previous estimates, we get that for all sufficiently small $\epsilon > 0$

$$\begin{aligned}
|c_\epsilon| \ \le\ & \frac{1}{A\delta_1^{-2} + o(1)} \left[ \left| \epsilon \int_\Omega \bar\phi_1 \mathcal{P}Z_1\, dx \right| + \left| \epsilon \int_\Omega \mathcal{P}\mathcal{U}_{\delta_1} \mathcal{P}Z_1\, dx \right| \right. \\
& \left. + \left| \int_\Omega [f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_1})]\, \mathcal{P}Z_1\, dx \right| + \left| \int_\Omega \left[ f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) \right]\, dx \right| \right] \\
& \le c\epsilon^{\frac{2}{N-4}+\frac{1}{2}},
\end{aligned} \tag{3.3.33}$$

uniformly with respect to $d_1$ satisfying $\eta < d_1 < \frac{1}{\eta}$.

We observe that $\bar\phi_1$ is a classical solution of (3.3.29). This comes from the fact that $\bar\phi_1 \in H_0^1(\Omega)$ weakly solves (3.3.29), taking into account the smoothness of $\mathcal{P}\mathcal{U}_{\delta_1}, \mathcal{U}_{\delta_1}, \mathcal{P}Z_1$, from standard elliptic regularity theory and the application of a well-known lemma by Brezis and Kato.

We consider the quantity $\sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar\phi_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right)$, which is defined for all $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ is given by Proposition 3.3.6. We want to prove that

$$\lim_{\epsilon \to 0^+} \sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar\phi_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right) = 0. \tag{3.3.34}$$

It is clear that (3.3.34) implies the thesis. In fact, we recall that, thanks to the definition (3.1.3) and the choice of $\delta_1$ (see (3.2.3)), for any $d_1 \in ]\eta, \frac{1}{\eta}[$ we have

$$\alpha_N \eta^{\frac{N-2}{2}} \epsilon^{-\frac{N-2}{2(N-4)}} < |\mathcal{U}_{\delta_1}|_\infty < \alpha_N \eta^{-\frac{N-2}{2}} \epsilon^{-\frac{N-2}{2(N-4)}}.$$

Hence, by this estimate and (3.3.34), we get that

$$0 \leq \sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \frac{|\bar{\phi}_1|_\infty}{\epsilon^{-\frac{N-2}{2(N-4)}}} = \sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \cdot \frac{|\mathcal{U}_{\delta_1}|_\infty}{\epsilon^{-\frac{N-2}{2(N-4)}}} \right) \leq \sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right) \alpha_N \eta^{-\frac{N-2}{2}} \to 0,$$

as $\epsilon \to 0^+$, and we are done.

In order to prove (3.3.34) we argue by contradiction. Assume that (3.3.34) is false. Then, there exists a positive number $\tau \in \mathbb{R}^+$, a sequence $(\epsilon_k)_k \subset \mathbb{R}^+$, $\epsilon_k \to 0$ as $k \to +\infty$, such that

$$\sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) > \tau, \tag{3.3.35}$$

for any $k \in \mathbb{N}$, where, $\bar{\phi}_{1,k} := \bar{\phi}_1(\epsilon_k, d_1) \in B_{1,\epsilon_k}$ and $\delta_{1,k} := \epsilon_k^{\frac{1}{N-4}} d_1$. We observe that (3.3.35) contemplates the possibility that $\sup_{d_1 \in ]\eta, \frac{1}{\eta}[} \left( \frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) = +\infty$. From (3.3.35), for any $k \in \mathbb{N}$, thanks to the definition of sup, we get that there exists $d_{1,k} \in ]\eta, \frac{1}{\eta}[$ such that

$$\left( \frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) (d_{1,k}) > \frac{\tau}{2}.$$

Hence, if we consider the sequence $\left( \frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \right)_k$, then, up to a subsequence, as $k \to +\infty$, there are only two possibilities:

**(a)** $\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \to +\infty$;

**(b)** $\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \to l$, for some $l \geq \frac{\tau}{2} > 0$.

We will show that (a) and (b) cannot happen.

Assume (a). We point out that, since $\eta > 0$ is fixed, then, $d_{1,k} \in ]\eta, \frac{1}{\eta}[$ for all $k$, in particular this sequence stays definitely away from 0 and from $+\infty$. Hence, in order to simplify the notation of this proof, we omit the dependence from $d_{1,k}$ in $\bar{\phi}_{1,k}(d_{1,k})$ and in $\delta_{1,k}(d_{1,k}) = \epsilon_k^{\frac{1}{N-4}} d_{1,k}$ and thus we simply write $\bar{\phi}_{1,k}$, $\delta_{1,k}$. In particular, we observe that, for any fixed $k$, $\bar{\phi}_{1,k}$ is a function depending only on the space variable $x \in \Omega$.

Then, for any $k \in \mathbb{N}$, let $a_k \in \Omega$ such that $|\bar{\phi}_{1,k}(a_k)| = |\bar{\phi}_{1,k}|_\infty$ and set $M_k := |\bar{\phi}_{1,k}|_\infty$. Thanks to the assumption (a), since $|\mathcal{U}_{\delta_{1,k}}|_\infty = \alpha_N \delta_{1,k}^{-\frac{N-2}{2}} = \alpha_N \epsilon_k^{-\frac{N-2}{2(N-4)}} d_{1,k}^{-\frac{N-2}{2}}$, we get that $M_k \to +\infty$, as $k \to +\infty$. We consider the rescaled function

$$\widetilde{\phi}_{1,k}(y) := \frac{1}{M_k} \bar{\phi}_{1,k} \left( a_k + \frac{y}{M_k^\beta} \right), \qquad \beta = \frac{2}{N-2}$$

defined for $y \in \widetilde{\Omega}_k := M_k^{\frac{2}{N-2}} (\Omega - a_k)$. Moreover let us set

$$\widetilde{\mathcal{P}\mathcal{U}}_{1,k}(y) := \frac{1}{M_k} \mathcal{P}\mathcal{U}_{\delta_{1,k}} \left( a_k + \frac{y}{M_k^\beta} \right); \quad \widetilde{\mathcal{U}}_{1,k}(y) := \frac{1}{M_k} \mathcal{U}_{\delta_{1,k}} \left( a_k + \frac{y}{M_k^\beta} \right);$$

$$\widehat{\mathcal{P}Z}_{1,k}(y) := \frac{1}{M_k^{2\beta+1}} \mathcal{P}Z_{1,k}\left(a_k + \frac{y}{M_k^{\beta}}\right).$$

Since we are assuming (a) it is clear that $|\widetilde{\mathcal{P}\mathcal{U}}_{1,k}|_{\infty,\widetilde{\Omega}_k}, |\widetilde{\mathcal{U}}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. Moreover, thanks to the definition of $Z_1$, and since $\mathcal{P}Z_1 = Z_1 - \psi$, with $|\psi|_{\infty,\Omega} \leq C\delta_1^{\frac{N-4}{2}}$, we have that $|PZ_{1,k}|_\infty \simeq |Z_{1,k}|_\infty \simeq \delta_{1,k}^{-\frac{N}{2}}$, and hence, thanks to (a), we have $\frac{1}{M_k^{2\beta+1}} = o(\delta_{1,k}^{\frac{N+2}{2}})$, which implies that $|\widehat{\mathcal{P}Z}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. In particular, thanks to (3.3.33), the same conclusion holds for $c_{\epsilon_k}(d_{1,k})\widehat{\mathcal{P}Z}_{1,k}$. Taking into account that $2\beta+1 = p$, by elementary computations, we see that $\widetilde{\phi}_{1,k}$ solves

$$\begin{cases} -\Delta\widetilde{\phi}_{1,k} = \frac{\epsilon_k}{M_k^{2\beta}}\widetilde{\phi}_{1,k} + \epsilon_k \frac{\widetilde{\mathcal{P}\mathcal{U}_{\delta_{1,k}}}}{M_k^{2\beta}} + f(\widetilde{\mathcal{P}\mathcal{U}}_{\delta_{1,k}} + \widetilde{\phi}_{1,k}) - f(\widetilde{\mathcal{U}}_{\delta_{1,k}}) + c_{\epsilon_k}(d_{1,k})\widehat{\mathcal{P}Z}_{1,k} & \text{in } \widetilde{\Omega}_k, \\ \widetilde{\phi}_{1,k} = 0 & \text{on } \partial\widetilde{\Omega}_k. \end{cases} \tag{3.3.36}$$

Let us denote by $\Pi$ the limit domain of $\widetilde{\Omega}_k$. Since $M_k \to +\infty$, as $k \to +\infty$, we have that $\Pi$ is the whole $\mathbb{R}^N$ or an half-space. Moreover, since the family $(\widetilde{\phi}_{1,k})_k$ is uniformly bounded and solves (3.3.36), then, by the same proof of Lemma 2.2 of [13], we get that $0 \in \Pi$ (in particular $0 \notin \partial\Pi$), and, by standard elliptic theory, it follows that, up to a subsequence, as $k \to +\infty$, we have that $\widetilde{\phi}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to a function $w$ which satisfies

$$-\Delta w = f(w) \text{ in } \Pi, \quad w(0) = 1 \text{ (or } w(0) = -1), \quad |w| \leq 1 \text{ in } \Pi, \quad w = 0 \text{ on } \partial\Pi. \tag{3.3.37}$$

We observe that, thanks to the definition of the chosen rescaling, by elementary computations (see Lemma 1.4.1), it holds $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_\epsilon} = \|\bar{\phi}_{1,k}\|^2_{\Omega}$. Now, since $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where $c$ depends only on $\eta$ and $\sigma$ is some positive number (see Proposition 3.3.6), we have $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = \|\bar{\phi}_{1,k}\|^2_{\Omega} \to 0$, as $k \to +\infty$. Hence, since $\widetilde{\phi}_{1,k} \to w$ in $C^2_{loc}(\Pi)$, by Fatou's lemma, it follows that

$$\|w\|^2_{\Pi} \leq \liminf_{k\to+\infty} \|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = 0. \tag{3.3.38}$$

Therefore, since $\|w\|^2_{\Pi} = 0$ and $w$ is smooth, it follows that $w$ is constant, and from $w(0) = 1$ (or $w(0) = -1$) we get that $w \equiv 1$ (or $w \equiv -1$) in $\Pi$. But, since $w$ is constant and solves $-\Delta w = f(w)$ in $\Pi$, then necessarily $f(w) \equiv 0$ in $\Pi$, and hence $w$ must be the null function, but this contradicts $w \equiv 1$ (or $w \equiv -1$).

Alternatively, if $\Pi$ is an half-space, by using the boundary condition $w = 0$ on $\partial\Pi$, we contradicts $w \equiv 1$ (or $w \equiv -1$). Hence, the only possibility is $\Pi = \mathbb{R}^N$. In this case, since $w$ solves (3.3.37) and $\|w\|^2_{\Pi} \leq 2S^{N/2}$, it is well known that $w$ cannot be sign-changing and hence, assuming without loss of generality that $w(0) = 1$, $w$ must be a positive function of the form $\mathcal{U}_{\delta_N}$ (see (3.1.3)), for some $\delta_N$ such that $\mathcal{U}_{\delta_N}(0) = 1$, and this contradicts $w \equiv 1$. Hence (a) cannot happen.

Assume (b). Using the same convention on the notation as in previous case, we deduce that there exist two positive uniform constants $c_1$, $c_2$ such that

$$c_1\delta_{1,k}^{-\frac{N-2}{2}} \leq |\bar{\phi}_{1,k}|_\infty \leq c_2\delta_{1,k}^{-\frac{N-2}{2}}, \tag{3.3.39}$$

for all sufficiently large $k$. In particular, it still holds that $M_k \to +\infty$, as $k \to +\infty$. We consider the same rescaled functions $\widetilde{\phi}_{1,k}$ as in (a) and, as before, we denote by $\Pi$ the limit domain of $\widetilde{\Omega}_k$.
Now, up to a subsequence, since $\widetilde{\mathcal{PU}}_{1,k}$ and $\widetilde{\mathcal{U}}_{1,k}$ are uniformly bounded we see that they converge in $C^2_{loc}(\Pi)$ to a bounded function which we denote, respectively, by $\overline{\mathcal{PU}}$ and $\overline{\mathcal{U}}$ (one of them or both could be eventually the null function). In fact $\widetilde{\mathcal{U}}_{1,k}$ is uniformly bounded and solves $-\Delta\widetilde{\mathcal{U}}_{1,k} = \widetilde{\mathcal{U}}^p_{1,k}$ on $\widetilde{\Omega}_k$, and so by standard elliptic theory we get that $\widetilde{\mathcal{U}}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to some non-negative bounded function $\overline{\mathcal{U}}$ which solves $-\Delta\overline{\mathcal{U}} = \overline{\mathcal{U}}^p$ in $\Pi$. Now, taking into account that $\widetilde{\mathcal{U}}_{1,k} \to \overline{\mathcal{U}}$ in $C^2_{loc}(\Pi)$, the same argument applies to $\widetilde{\mathcal{PU}_{1,k}}$, which solves

$$\begin{cases} -\Delta\widetilde{\mathcal{PU}}_{1,k} = \widetilde{\mathcal{U}}^p_{1,k} & \text{in } \widetilde{\Omega}_k, \\ \widetilde{\mathcal{PU}}_{1,k} = 0 & \text{on } \partial\widetilde{\Omega}_k, \end{cases}$$

and hence $\widetilde{\mathcal{PU}}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to some non-negative bounded function $\overline{\mathcal{PU}}$ satisfying $-\Delta\overline{\mathcal{PU}} = \overline{\mathcal{U}}^p$ in $\Pi$, $\overline{\mathcal{PU}} = 0$ on $\partial\Pi$.

We point out that as in (a), but using (3.3.39), we still have $c_{\epsilon_k}(d_{1,k})|\widehat{\mathcal{P}Z}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. Moreover, by the proof of Lemma 2.2 of [13], it also holds that $0 \in \Pi$.

Hence, by standard elliptic theory, we have that $\widetilde{\phi}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to a function $w$ which solves

$$\begin{cases} -\Delta w = f(\overline{\mathcal{PU}} + w) - f(\overline{\mathcal{U}}) & \text{in } \Pi, \\ \\ w = 0 & \text{on } \partial\Pi, \\ \\ w(0) = 1 \text{ (or } w(0) = -1). \end{cases} \tag{3.3.40}$$

As in (3.3.38) we have $\|w\|^2_\Pi = 0$ and hence, since $w$ is smooth, the only possibility is $w \equiv 1$ (or $w \equiv -1$) because of the condition $w(0) = 1$ (or $w(0) = -1$). Moreover, thanks to the definition of the chosen rescaling, it also holds $|\widetilde{\phi}_{1,k}|_{\frac{2N}{N-2},\widetilde{\Omega}_k} = |\bar{\phi}_{1,k}|_{\frac{2N}{N-2},\Omega}$ (for the proof see that of Lemma 1.4.1). Therefore, since $|\bar{\phi}_{1,k}|_{\frac{2N}{N-2},\Omega} \to 0$ (because $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where $c > 0$ depends only on $\eta$) and $\widetilde{\phi}_{1,k} \to w$ in $C^2_{loc}(\Pi)$, as $k \to +\infty$, then, by Fatou's Lemma, it follows that $|w|_{\frac{2N}{N-2},\Pi} = 0$, and thus it cannot happen that $w \equiv 1$ (or $w \equiv -1$). Hence (a) and (b) cannot happen, and the proof is then concluded. □

We are now in position to prove Proposition 3.3.1.

*Proof of Proposition 3.3.1.* Let us fix $\eta > 0$ and let $\bar{\phi}_1 \in \mathcal{K}_1^\perp \cap B_{1,\epsilon}$ be the unique solution of the first equation of (3.2.12) found in Proposition 3.3.6. We define the operator $\mathcal{T}_2 : \mathcal{K}^\perp \to \mathcal{K}^\perp$ as

$$\mathcal{T}_2(\phi_2) := -\mathcal{L}_2^{-1}[\mathcal{N}_2(\bar{\phi}_1, \phi_2) + \mathcal{R}_2].$$

In order to find a solution of the second equation of (3.2.12) we solve the fixed point problem $\mathcal{T}_2(\phi_2) = \phi_2$. Let us define the proper ball

$$B_{2,\epsilon} := \{\phi_2 \in \mathcal{K}^\perp;\ \|\phi_2\| \leq r\ \epsilon^{\frac{\theta_2}{2}+\sigma}\}$$

for $r > 0$ sufficiently large and $\sigma > 0$ to be chosen later.
From Lemma 3.3.4, there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ and $c = c(\eta) > 0$ such that:

$$\|\mathcal{T}_2(\phi_2)\| \le c(\|\mathcal{N}_2(\bar\phi_1, \phi_2)\| + \|\mathcal{R}_2\|), \tag{3.3.41}$$

and

$$\|\mathcal{T}_2(\phi_2) - \mathcal{T}_2(\psi_2)\| \le c(\|\mathcal{N}_2(\bar\phi_1, \phi_2) - \mathcal{N}_2(\bar\phi_1, \psi_2)\|), \tag{3.3.42}$$

for all $\phi_2, \psi_2 \in \mathcal{K}^\perp$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (3.2.4) and for all $\epsilon \in (0, \epsilon_0)$.

We begin with estimating the right hand side of (3.3.41).
Thanks to Proposition 3.3.8 we have that

$$\|\mathcal{R}_2\| \le c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (3.2.4). Thus it remains only to estimate $\|\mathcal{N}_2(\bar\phi_1, \phi_2)\|$. Thanks to (3.2.1) and the definition of $\mathcal{N}_2$ we deduce:

$$\|\mathcal{N}_2(\bar\phi_1, \phi_2)\| \le c|f(V_\epsilon+\bar\phi_1+\phi_2)-f(V_\epsilon)-f'(V_\epsilon)\phi_2-f(\mathcal{PU}_{\delta_1}+\bar\phi_1)+f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2}}. \tag{3.3.43}$$

We estimate the right-hand side of (3.3.43):

$$\begin{aligned}
&|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon) - f'(V_\epsilon)\phi_2 - f(\mathcal{PU}_{\delta_1} + \bar\phi_1) + f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2}} \\
\le\ & |f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|_{\frac{2N}{N+2}} + |(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|_{\frac{2N}{N+2}} \\
&+|f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{PU}_{\delta_1} + \bar\phi_1) + f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2}}
\end{aligned}$$

In order to estimate the last three terms, by Lemma 3.2.2 and Lemma 3.2.3 we deduce that:

$$|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2| \le c|\phi_2|^p \tag{3.3.44}$$

and

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2| \le c|\bar\phi_1|^{p-1}|\phi_2|. \tag{3.3.45}$$

Since $\frac{2N}{N+2} \cdot p = p + 1$ we get that

$$\int_\Omega |f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|^{\frac{2N}{N+2}}\, dx \ \le\ c\int_\Omega |\phi_2|^{p+1}\, dx,$$

and applying Sobolev inequality we deduce that

$$|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|_{\frac{2N}{N+2}} \le c\|\phi_2\|^p. \tag{3.3.46}$$

By (3.3.45) we get that

$$\int_\Omega |(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|^{\frac{2N}{N+2}}\, dx \ \le\ c\int_\Omega |\bar\phi_1|^{(p-1)\frac{2N}{N+2}}|\phi_2|^{\frac{2N}{N+2}}\, dx.$$

We observe that $\phi_1^{(p-1)\frac{2N}{N+2}} \in L^{\frac{N+2}{4}}$, $\phi_2^{\frac{2N}{N+2}} \in L^p$ and $p$, $\frac{N+2}{4}$ are conjugate exponents in Hölder inequality. Moreover $(p-1)\frac{2N}{N+2}\frac{N+2}{4} = p + 1$ so

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|^{\frac{2N}{N+2}}_{\frac{2N}{N+2}} \le c|\bar\phi_1|_{p+1}^{\frac{8N}{(N+2)(N-2)}}|\phi_2|_{p+1}^{\frac{2N}{N+2}},$$

and hence by Sobolev inequality we deduce that

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|_{\frac{2N}{N+2}} \leq c\|\bar\phi_1\|^{\frac{4}{N-2}}\|\phi_2\|. \tag{3.3.47}$$

It remains to estimate the last term. As in the proof of Proposition 3.3.8 we make the decomposition of the domain $\Omega$ as $\Omega = A_0 \sqcup A_1 \sqcup A_2$. Hence we get that:

$$\begin{aligned} & |f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}, A_0} \\ \leq\ & |f(V_\epsilon + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1)|_{\frac{2N}{N+2}, A_0} + |f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}, A_0} \end{aligned}$$

Then, by using the definition of $\delta_1, \delta_2$, the usual elementary inequalities, the computations made in (3.3.16) and Sobolev inequality, we get that

$$\begin{aligned} & |f(V_\epsilon + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1)|_{\frac{2N}{N+2}, A_0} \\ \leq\ & c_1\left(|\mathcal{P}\mathcal{U}_{\delta_2}|^p_{p+1, A_0} + |\mathcal{P}\mathcal{U}^{p-1}_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}, A_0} + \Big||\bar\phi_1|^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}\Big|_{\frac{2N}{N+2}, A_0}\right) \\ \leq\ & c_2\left(\delta_2^{\frac{N+2}{2}} + \delta_1^2\delta_2^{\frac{N-2}{2}} + \|\bar\phi_1\|^{p-1}\delta_2^{\frac{N-2}{2}}\right) \\ \leq\ & c_3\epsilon^{\frac{\theta_2}{2}+\sigma}, \end{aligned}$$

for some $\sigma > 0$.

Moreover, as in the previous estimate, we get that

$$\begin{aligned} |f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}, A_0} &\leq c_1\left(|\mathcal{P}\mathcal{U}^{p-1}_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}, A_0} + |\mathcal{P}\mathcal{U}_{\delta_2}|^p_{\frac{2N}{N+2}, A_0}\right) \\ &\leq c_2\epsilon^{\frac{\theta_2}{2}+\sigma}. \end{aligned}$$

In $A_1$ we argue as in the previous case. The various terms now can be estimated as done in (3.3.19) and (3.3.20) and hence the same conclusion holds.

For $A_2$, by using the usual elementary inequalities, Lemma 3.3.9 and remembering the choice of $\delta_1$, $\delta_2$, we have:

$$\begin{aligned} & |f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}, A_2} \\ \leq\ & |f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f'(V_\epsilon)\bar\phi_1|_{\frac{2N}{N+2}, A_2} + |f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}, A_2} \\ & + |[f'(V_\epsilon) - f'(\mathcal{P}\mathcal{U}_{\delta_1})]\bar\phi_1|_{\frac{2N}{N+2}, A_2} \\ \leq\ & c\Big||\bar\phi_1|^p\Big|_{\frac{2N}{N+2}, A_2} + c|\mathcal{P}\mathcal{U}^{p-1}_{\delta_2}\bar\phi_1|_{\frac{2N}{N+2}, A_2} \\ \leq\ & c|\bar\phi_1|^p_\infty\left(\int_{A_2} 1\,dx\right)^{\frac{N+2}{2N}} + c|\bar\phi_1|_\infty\left(\int_{A_2}\mathcal{U}_{\delta_2}^{\frac{8N}{N^2-4}}\,dx\right)^{\frac{N+2}{2N}} \\ \leq\ & c_1\delta_1^{-\frac{N-2}{2}p}\left(\int_0^{\sqrt{\delta_1\delta_2}} r^{N-1}\,dr\right)^{\frac{N+2}{2N}} + c_2|\bar\phi_1|_\infty\left(\int_{A_2}\frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2 + |x|^2)^{\frac{4N}{N+2}}}\,dx\right)^{\frac{N+2}{2N}} \\ \leq\ & c_3\delta_1^{-\frac{N+2}{2}}(\delta_1\delta_2)^{\frac{N+2}{4}} + c_2|\bar\phi_1|_\infty\delta_2^2\left(\int_{A_2}\frac{1}{|x|^{\frac{8N}{N+2}}}\,dx\right)^{\frac{N+2}{2N}} \end{aligned}$$

$$
\begin{aligned}
&\leq c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}+c_4\delta_1^{-\frac{N-2}{2}}\delta_2^2\left(\int_0^{\sqrt{\delta_1\delta_2}} r^{\frac{N^2-7N-2}{N+2}}\,dr\right)^{\frac{N+2}{2N}}\\
&\leq c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}+c_5\delta_1^{-\frac{N-2}{2}}\delta_2^2\left(\delta_1\delta_2\right)^{\frac{N-6}{4}}=c_6\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\leq c_7\epsilon^{\frac{\theta_2}{2}+\sigma}.
\end{aligned}
$$

Hence, from these estimates, we have

$$
|f(V_\epsilon+\bar\phi_1)-f(V_\epsilon)-f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)+f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}\leq c\epsilon^{\frac{\theta_2}{2}+\sigma}. \tag{3.3.48}
$$

Since $\phi_2\in B_{2,\epsilon}$ and thanks to (3.3.43), (3.3.46), (3.3.47) and (3.3.48) we get that

$$
\|\mathcal{T}_2(\phi_2)\|\leq c\epsilon^{\frac{\theta_2}{2}+\sigma},\qquad \sigma>0
$$

and hence $\mathcal{T}_2$ maps $B_{2,\epsilon}$ into itself .

It remains to prove that $\mathcal{T}_2 : B_{2,\epsilon}\to B_{2,\epsilon}$ is a contraction. Thanks to (3.3.42) it suffices to estimate $\|\mathcal{N}_2(\bar\phi_1,\phi_2)-\mathcal{N}_2(\bar\phi_1,\psi_2)\|$ for any $\psi_2,\phi_2\in B_{2,\epsilon}$. To this end, thanks to (3.2.1), the definition of $\mathcal{N}_2$ and reasoning as in the proof of Proposition 3.3.6 we have:

$$
\|\mathcal{N}_2(\bar\phi_1,\phi_2)-\mathcal{N}_2(\bar\phi_1,\psi_2)\|\leq\epsilon^\alpha\|\phi_2-\psi_2\|,
$$

for some $\alpha>0$.

At the end we get that there exists $L\in(0,1)$ such that

$$
\|\mathcal{T}_2(\phi_2)-\mathcal{T}_2(\psi_2)\|\leq L\|\phi_2-\psi_2\|.
$$

Finally, taking into account that $d_1\to\bar\phi_1(d_1)$ is a $C^1$-map, a standard argument shows that also $(d_1,d_2)\to\bar\phi_2(d_1,d_2)$ is a $C^1$-map. The proof is complete. □

## 3.4 The reduced functional

We are left now to solve (3.2.11). Let $(\bar\phi_1,\bar\phi_2)\in\mathcal{K}_1^\perp\times\mathcal{K}^\perp$ be the solution found in Proposition 3.3.1. Hence $V_\epsilon+\bar\phi_1+\bar\phi_2$ is a solution of our original problem (3.1.1) if we can find $\bar d_\epsilon=(\bar d_{1\epsilon},\bar d_{2\epsilon})$ which satisfies condition (3.2.4) and solves equation (3.2.11).

To this end we consider the reduced functional $\tilde J_\epsilon:\mathbb{R}_+^2\to\mathbb{R}$ defined by:

$$
\tilde J_\epsilon(d_1,d_2):=J_\epsilon(V_\epsilon+\bar\phi_1+\bar\phi_2),
$$

where $J_\epsilon$ is the functional defined in (3.1.13).
Our main goal is to show first that solving equation (3.2.11) is equivalent to finding critical points $(\bar d_{1,\epsilon},\bar d_{2,\epsilon})$ of the reduced functional $\tilde J_\epsilon(d_1,d_2)$ and then that the reduced functional has a critical point. These facts are stated in the following proposition:

**Proposition 3.4.1.** *The following facts hold:*

**(i)** *If $(\bar d_{1,\epsilon},\bar d_{2,\epsilon})$ is a critical point of $\tilde J_\epsilon$, then the function $V_\epsilon+\bar\phi_1+\bar\phi_2$ is a solution of (3.1.1).*

**(ii)** *For any $\eta > 0$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ it holds:*

$$\tilde{J}_\epsilon(d_1, d_2) = \frac{2}{N} S^{N/2} + \epsilon^{\theta_1} \left[ a_1 \tau(0) d_1^{N-2} - a_2 d_1^2 \right] + O(\epsilon^{\theta_1 + \sigma}), \tag{3.4.1}$$

*with*

$$O(\epsilon^{\theta_1+\sigma}) = \epsilon^{\theta_1+\sigma} g(d_1) + \epsilon^{\theta_2} \left[ a_3 \tau(0) \left( \frac{d_2}{d_1} \right)^{\frac{N-2}{2}} - a_2 d_2^2 \right] + o\left(\epsilon^{\theta_2}\right), \tag{3.4.2}$$

*for some function $g$ depending only on $d_1$ (and uniformly bounded with respect to $\epsilon$), where $\theta_1, \theta_2$ are defined in (3.3.1), $\sigma$ is some positive real number (depending only on $N$), $\tau$ is the Robin's function of the domain $\Omega$ at the origin and*

$$a_1 := \frac{1}{2} \alpha_N^{p+1} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+2}{2}}} dy; \quad a_2 := \frac{1}{2} \alpha_N^2 \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{N-2}} dy;$$

$$a_3 := \alpha_N^{p+1} \int_{\mathbb{R}^N} \frac{1}{|y|^{N-2} (1 + |y|^2)^{\frac{N+2}{2}}} dy.$$

*The expansions (3.4.1), (3.4.2) are $C^0$-uniform with respect to $(d_1, d_2)$ satisfying condition (3.2.4).*

**Remark 3.4.2.** *We point out that the term $g$ appearing in (3.4.2) does not depend on $d_2$ and this will be used in the sequel, in particular in (3.5.5).*

The aim of this section is to prove Proposition 3.4.1. First we prove two lemmas about the $C^0$-expansion of the reduced functional $\tilde{J}_\epsilon(d_1, d_2) := J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2)$, where $\bar{\phi}_1 \in \mathcal{K}_1^\perp \cap B_{1,\epsilon}$ and $\bar{\phi}_2 \in \mathcal{K}^\perp \cap B_{2,\epsilon}$ are the functions given by Proposition 3.3.1.

**Lemma 3.4.3.** *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\epsilon(V_\epsilon + \bar{\phi}_1) = J_\epsilon(V_\epsilon) + O(\epsilon^{\theta_1+\sigma}),$$

*with*

$$O(\epsilon^{\theta_1+\sigma}) = \epsilon^{\theta_1+\sigma} g_1(d_1) + O\left(\epsilon^{\theta_2+\sigma}\right), \tag{3.4.3}$$

*for some function $g_1$ depending only on $d_1$ (and uniformly bounded with respect to $\epsilon$), where $\theta_1, \theta_2$ are defined in (3.3.1), $\sigma$ is some positive real number (depending only on $N$). These expansion are $C^0$-uniform with respect to $(d_1, d_2)$ satisfying condition (3.2.4).*

*Proof.* Let us fix $\eta > 0$. By direct computation we immediately see that

$$\begin{aligned} J_\epsilon(V_\epsilon + \bar{\phi}_1) - J_\epsilon(V_\epsilon) = & \tfrac{1}{2} \int_\Omega |\nabla \bar{\phi}_1|^2 \, dx + \int_\Omega \nabla V_\epsilon \cdot \nabla \bar{\phi}_1 \, dx - \tfrac{\epsilon}{2} \int_\Omega |\bar{\phi}_1|^2 \, dx - \epsilon \int_\Omega V_\epsilon \bar{\phi}_1 \, dx \\ & - \tfrac{1}{p+1} \int_\Omega (|V_\epsilon + \bar{\phi}_1|^{p+1} - |V_\epsilon|^{p+1}) \, dx. \end{aligned} \tag{3.4.4}$$

By definition we have

$$\int_\Omega \nabla V_\epsilon \cdot \nabla \bar{\phi}_1 \, dx = \int_\Omega \nabla (\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) \cdot \nabla \bar{\phi}_1 \, dx = \int_\Omega (\mathcal{U}_{\delta_1}^p - \mathcal{U}_{\delta_2}^p) \bar{\phi}_1 \, dx = \int_\Omega [f(\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_2})] \bar{\phi}_1 \, dx,$$

moreover, since $F(s) = \frac{1}{p+1} |s|^{p+1}$ is a primitive of $f$, we can write (3.4.4) as

$$\begin{aligned} J_\epsilon(V_\epsilon+\bar\phi_1)-J_\epsilon(V_\epsilon) &= \tfrac12\|\bar\phi_1\|^2-\tfrac{\epsilon}{2}|\bar\phi_1|_2^2-\epsilon\int_\Omega V_\epsilon\bar\phi_1\,dx+\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_2})]\bar\phi_1\,dx\\ &\quad-\int_\Omega[F(V_\epsilon+\bar\phi_1)-F(V_\epsilon)]\,dx\\ &= \tfrac12\|\bar\phi_1\|^2-\tfrac{\epsilon}{2}|\bar\phi_1|_2^2-\epsilon\int_\Omega V_\epsilon\bar\phi_1\,dx+\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx\\ &\quad-\int_\Omega[F(V_\epsilon+\bar\phi_1)-F(V_\epsilon)-f(V_\epsilon)\bar\phi_1]\,dx\\ &\quad A+B+C+D+E. \end{aligned} \tag{3.4.5}$$

**A,B:** Thanks to Proposition 3.3.1, for all sufficiently small $\epsilon$, we have $\|\bar\phi_1\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma}$, for some $c>0$ and for some $\sigma>0$ depending only on $N$. Hence we deduce that $A=O(\epsilon^{\theta_1+2\sigma})$, $B=O(\epsilon^{\theta_1+2\sigma+1})$. We point out that, since only $\bar\phi_1$ is involved in $A$ and $B$, these terms depend only on $d_1$.

**C:** By definition we have

$$\epsilon\int_\Omega V_\epsilon\bar\phi_1\,dx=\epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\bar\phi_1\,dx-\epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_2}\bar\phi_1\,dx=I_1+I_2.$$

We observe that in the estimate $I_1$ only $\delta_1$ and $\bar\phi_1$ are involved. Hence $I_1$ depends only on $d_1$. Thanks to Hölder inequality, we have the following:

$$|I_1|\le\epsilon|\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}}|\bar\phi_1|_{\frac{2N}{N-2}}$$

Since $N\ge7$ we have $|\mathcal{U}_{\delta_i}|_{\frac{2N}{N+2}}=O(\delta_i^2)$, for $i=1,2$, so from our choice of $\delta_i$ (see (3.2.3)) and since $\|\bar\phi_1\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma}$ we deduce that

$$|I_1|\le c\epsilon(\epsilon^{\frac{2}{N-4}}\epsilon^{\frac{N-2}{2(N-4)}+\sigma})\le c\epsilon^{\theta_1+\sigma}, \tag{3.4.6}$$

for all sufficiently small $\epsilon$. For $I_2$, with similar computations, we get that

$$|I_2|\le\epsilon|\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}}|\bar\phi_1|_{\frac{2N}{N-2}}\le c\epsilon^{1+\frac{2(3N-10)}{(N-4)(N-6)}}\epsilon^{\frac{N-2}{2(N-4)}+\sigma}.$$

Since $N\ge7$ it is elementary to see that $1+\frac{2(3N-10)}{(N-4)(N-6)}+\frac{N-2}{2(N-4)}>\theta_2$. From this we deduce that

$$|I_2|\le c\epsilon^{\theta_2+\sigma},$$

for all sufficiently small $\epsilon$.

**D:** we have

$$\begin{aligned}\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx&=\underbrace{\int_\Omega[f(\mathcal{P}\mathcal{U}_{\delta_1})-f(\mathcal{P}\mathcal{U}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx}_{I_1}+\\ &+\underbrace{\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{P}\mathcal{U}_{\delta_1})]\bar\phi_1\,dx}_{I_2}+\underbrace{\int_\Omega[f(\mathcal{P}\mathcal{U}_{\delta_2})-f(\mathcal{U}_{\delta_2})]\bar\phi_1\,dx}_{I_3}\end{aligned} \tag{3.4.7}$$

We evaluate separately the three terms.
We divide $\Omega$ into the three regions $A_0,A_1,A_2$ (see the proof of Proposition 3.3.8 for their

definition). Then

$$\int_\Omega [f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(V_\epsilon)]\bar{\phi}_1 \, dx = \underbrace{\sum_{j=0}^{1} \int_{A_j} [f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(V_\epsilon)]\bar{\phi}_1 \, dx}_{I_1'}$$

$$\underbrace{-\sum_{j=0}^{1} \int_{A_j} f(\mathcal{P}\mathcal{U}_{\delta_2})\bar{\phi}_1, dx}_{I_1''} + \underbrace{\int_{A_2} [f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(V_\epsilon)]\bar{\phi}_1 \, dx}_{I_1'''}$$

Now, writing $f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(V_\epsilon) = f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(V_\epsilon) + f'(\mathcal{P}\mathcal{U}_{\delta_1})\mathcal{P}\mathcal{U}_{\delta_2} - f'(\mathcal{P}\mathcal{U}_{\delta_1})\mathcal{P}\mathcal{U}_{\delta_2}$, applying the usual elementary inequalities, Hölder inequality and taking into account the computations made in (3.3.16) , (3.3.19), (3.3.20), we get that

$$\begin{aligned}
|I_1'| \leq\ & c|\mathcal{P}\mathcal{U}_{\delta_2}^p|_{\frac{2N}{N+2},A_0}|\bar{\phi}_1|_{\frac{2N}{N-2},A_0} + c|\mathcal{P}\mathcal{U}_{\delta_2}^p|_{\frac{2N}{N+2},A_1}|\bar{\phi}_1|_{\frac{2N}{N-2},A_1} \\
& + c|\mathcal{P}\mathcal{U}_{\delta_1}|^{p-1}_{\frac{2N}{N-2},A_0}|\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2},A_0}|\bar{\phi}_1|_{\frac{2N}{N-2},A_0} + c|\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2},A_1}|\bar{\phi}_1|_{\frac{2N}{N-2},A_1} \\
\leq\ & c_1 \left( \delta_2^{\frac{N+2}{2}} \epsilon^{\frac{\theta_1}{2}+\sigma} + \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} + \delta_1^2 \delta_2^{\frac{N-2}{2}} \epsilon^{\frac{\theta_1}{2}+\sigma} \right. \\
& \left. + \left(\frac{\delta_2}{\delta_1}\right)^2 \left( \int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} r^{\frac{-N^2+5N-2}{N+2}} \, dr \right)^{\frac{N+2}{2N}} \epsilon^{\frac{\theta_1}{2}+\sigma} \right) \\
\leq\ & c_2 \left( \delta_2^{\frac{N+2}{2}} \epsilon^{\frac{\theta_1}{2}+\sigma} + \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} + \delta_1^2 \delta_2^{\frac{N-2}{2}} \epsilon^{\frac{\theta_1}{2}+\sigma} + \left(\frac{\delta_2}{\delta_1}\right)^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} \right) \\
\leq\ & c_3 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} \leq c_4 \epsilon^{\theta_2+\sigma}.
\end{aligned}$$

As before we have

$$|I_1''| \leq \sum_{j=0}^{1} |f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2},A_j}|\bar{\phi}_1|_{\frac{2N}{N-2},A_j} \leq c\epsilon^{\theta_2+\sigma}.$$

Now, by Hölder inequality and reasoning as in (3.3.24), (3.3.25), (3.3.26), we get that

$$\begin{aligned}
|I_1'''| \leq\ & |f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(V_\epsilon)|_{\frac{2N}{N+2},A_2}|\bar{\phi}_1|_{\frac{2N}{N-2},A_2} \\
\leq\ & c_1 \left( |\mathcal{P}\mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2},A_2} + |\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2},A_2} \right) |\bar{\phi}_1|_{\frac{2N}{N-2},A_2} \\
\leq\ & c_2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} \leq c_3 \epsilon^{\theta_2+\sigma}.
\end{aligned}$$

At the end we conclude that

$$|I_1| \leq c\epsilon^{\theta_2+\sigma}.$$

For the remaining two terms of (3.4.7), reasoning as in the proof of Proposition 3.3.7, we get that

$$|f(\mathcal{P}\mathcal{U}_{\delta_i}) - f(\mathcal{U}_{\delta_i})|_{\frac{2N}{N+2}} \leq c\delta_i^{\frac{N+2}{2}}.$$

Hence

$$|I_2| \le |f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} |\bar\phi_1|_{\frac{2N}{N-2}} \le c\epsilon^{\frac{N+2}{2(N-4)}} \epsilon^{\frac{\theta_1}{2}+\sigma} \le c\epsilon^{\theta_1+\sigma},$$

for all sufficiently small $\epsilon$. We remark that $I_2$ depends only on $d_1$ and hence it is sufficient that it is of order $\theta_1 + \sigma$.
At the end

$$|I_3| \le |f(\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}} |\bar\phi_1|_{\frac{2N}{N-2}} \le c\epsilon^{\theta_2+\sigma},$$

for all sufficiently small $\epsilon$.

**E:** We decompose $\Omega$ in the three regions $A_j$, $j = 0, 1, 2$ used before.
For $j = 0, 1$ we have

$$\begin{aligned}
&\int_{A_j} \left[|V_\epsilon + \bar\phi_1|^{p+1} - |V_\epsilon|^{p+1} - (p+1)|V_\epsilon|^{p-1} V_\epsilon \bar\phi_1\right] dx \\
&= \underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + \bar\phi_1|^{p+1} - |\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1|^{p+1} + (p+1)|\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1)\mathcal{P}\mathcal{U}_{\delta_2}\right] dx}_{I_1} \\
&\quad \underbrace{- \int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\right] dx}_{I_2} \\
&\quad \underbrace{+ \int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right] dx}_{I_3} \\
&\quad -(p+1) \underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - \mathcal{P}\mathcal{U}_{\delta_1}^{p}\right] \bar\phi_1 \, dx}_{I_4} \\
&\quad -(p+1) \underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) - \mathcal{P}\mathcal{U}_{\delta_1}^{p}\right] \mathcal{P}\mathcal{U}_{\delta_2} \, dx}_{I_5}.
\end{aligned} \tag{3.4.8}$$

In order to estimate $I_1$, $I_2$, $I_4$ and $I_5$, applying the usual elementary inequalities, we see that

$$|I_1| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2} dx + \int_{A_j} |\bar\phi_1|^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2} dx\right)$$

$$|I_2| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2} dx\right)$$

$$|I_4| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p}|\bar\phi_1| \, dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1| \, dx\right)$$

$$|I_5| \le c\left(\int_{A_j} |\bar\phi_1|^{p}\mathcal{P}\mathcal{U}_{\delta_2} \, dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1| \, dx\right).$$

Now, as seen in the proof of (3.3.16) and thanks to (3.3.20), we have

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} dx \le c \begin{cases} \delta_2^N & \text{if } j = 0 \\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} & \text{if } j = 1, \end{cases}$$

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}^2 \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\,dx \leq c|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_0}^{p-1}|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}^2 \leq c\delta_1^2\delta_2^{N-2}.$$

Moreover, by analogous computations, we get that

$$\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}^2 \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\,dx \leq c_1\left(\frac{\delta_2}{\delta_1}\right)^2 \int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{r^{N-3}}\,dr \leq c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}},$$

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^2|\bar\phi_1|^{p-1}\,dx \leq c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_j}^2\|\bar\phi_1\|^{p-1} \leq c\begin{cases} \delta_2^{N-2}\epsilon^{(p-1)(\frac{\theta_1}{2}+\sigma)} & \text{if } j=0\\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\epsilon^{(p-1)(\frac{\theta_1}{2}+\sigma)} & \text{if } j=1,\end{cases}$$

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^p|\bar\phi_1|\,dx \leq c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_j}^p\|\bar\phi_1\| \leq c\begin{cases} \delta_2^{\frac{N+2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma} & \text{if } j=0\\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma} & \text{if } j=1,\end{cases}$$

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|\bar\phi_1|\,dx \leq c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_0}^{p-1}\|\bar\phi_1\| \leq c\delta_1^2\delta_2^{\frac{N-2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma},$$

and, thanks to (3.3.19), we have

$$\begin{aligned}\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|\bar\phi_1|\,dx &\leq |\mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|_{\frac{2N}{N+2},A_1}|\bar\phi_1|_{\frac{2N}{N-2},A_1}\\ &\leq c_1\left[\int_{A_1}\left(\frac{\delta_2^{\frac{N-2}{2}}\delta_1^2}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}}dx\right]^{\frac{N+2}{2N}}\epsilon^{\frac{\theta_1}{2}+\sigma}\\ &\leq c_2\left(\frac{\delta_2}{\delta_1}\right)^2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma} = c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}\end{aligned}$$

At the end

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|^p\,dx \leq c_1|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}\|\bar\phi_1\|^p \leq c_2\delta_2^{\frac{N-2}{2}}\epsilon^{p(\frac{\theta_1}{2}+\sigma)}$$

and, by using Lemma 3.3.9, we get that

$$\begin{aligned}\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|^p\,dx &\leq c_1|\bar\phi_1|_\infty^{p-1}\left[\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}\,dx\right]^{\frac{N+2}{2N}}|\bar\phi_1|_{p+1,A_1}\\ &\leq c_2\epsilon^{-\frac{2}{N-4}}\delta_2^2\left[\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} r^{\frac{-N^2+5N-2}{N+2}}\,dr\right]^{\frac{N+2}{2N}}\epsilon^{\frac{\theta_1}{2}+\sigma}\\ &= c_3\epsilon^{-\frac{2}{N-4}}\delta_2^2\left[\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}} - \delta_2^{\frac{N-6}{2}}\right]\epsilon^{\frac{\theta_1}{2}+\sigma}\\ &\leq c_4\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}.\end{aligned}$$

In order to estimate $I_3$ we observe that

$$\left|\int_{A_0}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx\right| \leq c_1\left(\|\bar\phi_1\|^2|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1}^{p-1} + \|\bar\phi_1\|^{p+1}\right) \leq c_2\epsilon^{\theta_1+\sigma}, \tag{3.4.9}$$

which is sufficient since this term does not depend on $d_2$.
Moreover

$$\begin{aligned}
&\int_{A_1}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right]dx\\
=&\int_{B(0,\rho)}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right]dx\\
&-\int_{A_2}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right]dx.
\end{aligned}$$

We observe that the first integral in the right-hand side of the previous equation depends only on $d_1$. Hence, as in (3.4.9), we have

$$\left|\int_{B(0,\rho)}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right]dx\right|\le c\epsilon^{\theta_1+\sigma}.$$

Furthermore, by using Lemma 3.3.9, we get that

$$\begin{aligned}
&\left|\int_{A_2}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\bar\phi_1\right]dx\right|\\
\le\;& c_1\left(|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_2}^{p-1}|\bar\phi_1|_{p+1,A_2}^{2}+|\bar\phi_1|_{p+1,A_2}^{p+1}\right)\\
\le\;& c_2\left(\left[\int_{B(0,\sqrt{\frac{\delta_2}{\delta_1}})}\frac{1}{(1+|y|^2)^N}\,dy\right]^{\frac{2}{N}}|\bar\phi_1|_\infty^2\left[\int_{A_2}1\,dx\right]^{\frac{2}{p+1}}+|\bar\phi_1|_\infty^{p+1}\int_{A_2}1\,dx\right)\\
\le\;& c_3\left(\left[\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-1}\,dr\right]^{\frac{2}{N}}\epsilon^{-\frac{N-2}{N-4}}\left[\int_0^{\sqrt{\delta_1\delta_2}}r^{N-1}\,dr\right]^{\frac{2}{p+1}}+\epsilon^{-\frac{N}{N-4}}\int_0^{\sqrt{\delta_1\delta_2}}r^{N-1}\,dr\right)\\
\le\;& c_4\left(\frac{\delta_2}{\delta_1}\epsilon^{-\frac{N-2}{N-4}}(\delta_1\delta_2)^{\frac{N-2}{2}}+\epsilon^{-\frac{N}{N-4}}(\delta_1\delta_2)^{\frac{N}{2}}\right)\\
\le\;& c_5\epsilon^{\theta_2+\sigma}.
\end{aligned}\tag{3.4.10}$$

Now, it remains only to estimate the left-hand side of (3.4.8) for $j=2$. Hence, thanks to the usual elementary inequalities, we get that

$$\begin{aligned}
&\left|\int_{A_2}\left[|V_\epsilon+\bar\phi_1|^{p+1}-|V_\epsilon|^{p+1}-(p+1)|V_\epsilon|^{p-1}V_\epsilon\bar\phi_1\right]dx\right|\\
&\le c\left(\int_{A_2}|V_\epsilon|^{p-1}\bar\phi_1^2\,dx+\int_{A_2}|\bar\phi_1|^{p+1}\,dx\right)\\
&\le c\left(\int_{A_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\bar\phi_1^2\,dx+\int_{A_2}\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\bar\phi_1^2\,dx+\int_{A_2}|\bar\phi_1|^{p+1}\,dx\right)
\end{aligned}$$

For the first and third integrals in the last right-hand side we can reason as in (3.4.10).

For the second integral, using Lemma 3.3.9, we have

$$\begin{aligned}
\int_{A_2} \mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\bar{\phi}_1^2\,dx &\leq c_1|\bar{\phi}_1|_\infty^2 \int_{A_2} \frac{\delta_2^2}{(\delta_2^2+|x|^2)^2}\,dx\\
&\leq c_2\delta_1^{-(N-2)}\delta_2^2\int_{A_2}\frac{1}{|x|^4}\,dx\\
&\leq c_3\delta_1^{-N+2}\delta_2^2\int_0^{\sqrt{\delta_1\delta_2}} r^{N-5}\,dr\\
&\leq c_4\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}
\end{aligned}$$

Finally, summing up all the estimates, we conclude that $|\mathbf{E}| = \epsilon^{\theta_1+\sigma}g(d_1) + O(\epsilon^{\theta_2+\sigma})$.

□

**Lemma 3.4.4.** *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) = J_\epsilon(V_\epsilon + \bar{\phi}_1) + O(\epsilon^{\theta_2+\sigma}),$$

*$C^0$-uniformly with respect to $(d_1, d_2)$ satisfying condition (3.2.4), for some positive real number $\sigma$ depending only on $N$.*

*Proof.* As we have seen in the proof of Lemma 3.4.3, by direct computation we get that

$$\begin{aligned}
&J_\epsilon(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) - J_\epsilon(V_\epsilon+\bar{\phi}_1) = \tfrac{1}{2}\int_\Omega |\nabla\bar{\phi}_2|^2\,dx + \int_\Omega \nabla(V_\epsilon+\bar{\phi}_1)\cdot\nabla\bar{\phi}_2\,dx\\
&\quad -\tfrac{\epsilon}{2}\int_\Omega|\bar{\phi}_2|^2\,dx - \epsilon\int_\Omega(V_\epsilon+\bar{\phi}_1)\bar{\phi}_2\,dx - \tfrac{1}{p+1}\int_\Omega(|V_\epsilon+\bar{\phi}_1+\bar{\phi}_2|^{p+1} - |V_\epsilon+\bar{\phi}_1|^{p+1})\,dx\\
&= -\tfrac{1}{2}\|\bar{\phi}_2\|^2 + \tfrac{\epsilon}{2}|\bar{\phi}_2|_2^2 + \int_\Omega\nabla(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)\cdot\nabla\bar{\phi}_2\,dx\\
&\quad -\epsilon\int_\Omega(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)\bar{\phi}_2\,dx - \int_\Omega f(V_\epsilon+\bar{\phi}_1)\bar{\phi}_2\,dx\\
&\quad -\int_\Omega[F(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) - F(V_\epsilon+\phi_1) - f(V_\epsilon+\bar{\phi}_1)\bar{\phi}_2]\,dx
\end{aligned} \tag{3.4.11}$$

Since $\bar{\phi}_1+\bar{\phi}_2$ is a solution of (3.2.10) we have

$$\Pi^\perp\{V_\epsilon+\bar{\phi}_1+\bar{\phi}_2 - i^*[\epsilon(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) + f(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)]\} = 0,$$

hence, for some $\psi \in \mathcal{K}$, we get that $V_\epsilon+\bar{\phi}_1+\bar{\phi}_2$ weakly solves

$$-\Delta(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) + \Delta\psi - [\epsilon(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) + f(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)] = 0. \tag{3.4.12}$$

Choosing $\bar{\phi}_2$ as test function, since $\bar{\phi}_2 \in \mathcal{K}^\perp$, $\psi \in \mathcal{K}$ we deduce that

$$\int_\Omega \nabla(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)\cdot\nabla\bar{\phi}_2\,dx - \epsilon\int_\Omega(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)\bar{\phi}_2\,dx = \int_\Omega f(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2)\bar{\phi}_2\,dx \tag{3.4.13}$$

Thanks to (3.4.13) we rewrite (3.4.11) as

$$\begin{aligned}
J_\epsilon(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) - J_\epsilon(V_\epsilon+\bar{\phi}_1) &= -\frac{1}{2}\|\bar{\phi}_2\|^2 + \frac{\epsilon}{2}|\bar{\phi}_2|_2^2 + \int_\Omega[f(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) - f(V_\epsilon+\bar{\phi}_1)]\bar{\phi}_2\,dx\\
&\quad -\int_\Omega[F(V_\epsilon+\bar{\phi}_1+\bar{\phi}_2) - F(V_\epsilon+\phi_1) - f(V_\epsilon+\bar{\phi}_1)\bar{\phi}_2]\,dx\\
&= A+B+C+D.
\end{aligned} \tag{3.4.14}$$

**A, B:** Thanks to Proposition 3.3.1, for all sufficiently small $\epsilon$, we have $\|\bar{\phi}_2\| \leq c\epsilon^{\frac{\theta_2}{2}+\sigma}$, for some $c > 0$ and for some $\sigma > 0$ depending only on $N$. Hence we deduce that $A = O(\epsilon^{\theta_2+2\sigma})$, $B = O(\epsilon^{\theta_2+2\sigma+1})$.

**C:** By Lemma 3.2.2 we get

$$\begin{aligned}\left|\int_\Omega [f(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) - f(V_\epsilon + \bar{\phi}_1)]\bar{\phi}_2 \, dx\right| &\leq \int_\Omega |\bar{\phi}_2|^{p+1} \, dx + \int_\Omega |V_\epsilon + \bar{\phi}_1|^{p-1}\bar{\phi}_2^2 \, dx \\ &\leq c\|\bar{\phi}_2\|^{p+1} + c|V_\epsilon|_{p+1}^{p-1}|\bar{\phi}_2|_{p+1}^2 + c|\bar{\phi}_1|_{p+1}^{p-1}|\bar{\phi}_2|_{p+1}^2 \\ &\leq c\epsilon^{\theta_2+\sigma}\end{aligned}$$

for all sufficiently small $\epsilon$.

**D:** Applying Lemma 3.2.2 and Hölder inequality we get that

$$\left|\int_\Omega [F(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) - F(V_\epsilon + \bar{\phi}_1) - f(V_\epsilon + \bar{\phi}_1)\bar{\phi}_2] \, dx\right| \leq c|V_\epsilon|_{p+1}^{p-1}|\bar{\phi}_2|_{p+1}^2 + c|\bar{\phi}_1|_{p+1}^{p-1}|\bar{\phi}_2|_{p+1}^2 + c|\bar{\phi}_2|_{p+1}^{p+1}.$$

Since all the terms from $A$ to $D$ are high order terms with respect to $\epsilon^{\theta_2}$ the proof is complete.

□

In order to prove Proposition 3.4.1 some further preliminary lemmas are needed.

**Lemma 3.4.5.** *Let $\delta_j$ as in (3.2.3) for $j = 1, 2$ and $N \geq 7$. For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, it holds*

$$\frac{1}{2}\int_\Omega |\nabla \mathcal{P}\mathcal{U}_{\delta_j}|^2 \, dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1} \, dx = \frac{1}{N}S^{N/2} + a_1\tau(0)\delta_j^{N-2} + O(\delta_j^{N-1}),$$

*$C^0$-uniformly with respect to $(d_1, d_2)$ satisfying condition (3.2.4), where $a_1 := \frac{1}{2}\alpha_N^{p+1}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \, dy$ and $\tau(0)$ is the Robin's function of the domain $\Omega$ at the origin.*

*Proof.* By using (3.1.6), (3.1.7) and (3.1.8) we have that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla \mathcal{P}\mathcal{U}_{\delta_j}|^2 \, dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1} \, dx = \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p \mathcal{P}\mathcal{U}_{\delta_j} \, dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1} \, dx \\ &= \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p(\mathcal{U}_{\delta_j} - \varphi_{\delta_j}) \, dx - \frac{1}{p+1}\int_\Omega (\mathcal{U}_{\delta_j} - \varphi_{\delta_j})^{p+1} \, dx \\ &= \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^{p+1} \, dx - \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j} \, dx - \frac{1}{p+1}\int_\Omega \mathcal{U}_{\delta_j}^{p+1} \, dx + \int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j} \, dx + O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2 \, dx\right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \mathcal{U}_{\delta_j}^{p+1} \, dx + \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j} \, dx + O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2 \, dx\right)\end{aligned}$$

Now it is easy to see that

$$\int_\Omega \mathcal{U}_{\delta_j}^{p+1} \, dx = \int_{\mathbb{R}^N} \frac{\alpha_N^{p+1}}{(1+|y|^2)^N} \, dy + O(\delta_j^N), \tag{3.4.15}$$

while

$$\begin{aligned}\int_\Omega \mathcal{U}_{\delta_j}^p \varphi_{\delta_j}\, dx &= \int_\Omega \mathcal{U}_{\delta_j}^p \left(\alpha_N \delta_j^{\frac{N-2}{2}} H(0,x) + O(\delta_j^{\frac{N+2}{2}})\right)\, dx \\ &= \alpha_N \delta_j^{\frac{N-2}{2}} \int_\Omega \mathcal{U}_{\delta_j}^p H(0,x)\, dx + O\left(\delta_j^{\frac{N+2}{2}} \int_\Omega \mathcal{U}_{\delta_j}^p\, dx\right) \\ &= \alpha_N^{p+1} \tau(0) \delta_j^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy + O(\delta_j^{N-1}). \qquad (3.4.16)\end{aligned}$$

Moreover

$$O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1} \varphi_{\delta_j}^2\, dx\right) = O(\delta_j^{N-1}). \qquad (3.4.17)$$

Indeed, we get

$$\begin{aligned}\int_\Omega \mathcal{U}_{\delta_j}^{p-1} \varphi_{\delta_j}^2\, dx &= \int_{B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1} \varphi_{\delta_j}^2\, dx + \int_{\Omega \setminus B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1} \varphi_{\delta_j}^2\, dx \\ &\leq c_1 \delta_j^{N-2} \int_{B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1}\, dx + |\varphi_{\delta_j}|_{p+1}^2 \left(\int_{\Omega \setminus B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p+1}\, dx\right)^{\frac{p-1}{p+1}} \\ &\leq c_2 \delta_j^{2N-4} \int_0^{\frac{1}{\sqrt{\delta_j}}} \frac{r^{N-1}}{(1+r^2)^2}\, dr + c_3 \delta_j^{N-2} \left(\int_{\frac{1}{\sqrt{\delta_j}}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N}\, dr\right)^{\frac{p-1}{p+1}} \\ &\leq c_4 \delta_j^{\frac{3N-4}{2}} + c_5 \delta_j^{N-1} \leq c_6 \delta_j^{N-1}.\end{aligned}$$

Hence, from (3.4.15), (3.4.16), (3.4.17) we get the thesis. □

**Lemma 3.4.6.** *Let $\delta_j$ as in* (3.2.3) *for $j = 1, 2$ and $N \geq 7$. For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, it holds*

$$\frac{\epsilon}{2} \int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^2\, dx = a_2 \epsilon \delta_j^2 + O(\epsilon \delta_j^{\frac{N}{2}}),$$

*$C^0$-uniformly with respect to $(d_1, d_2)$ satisfying condition* (3.2.4)*, where $a_2 := \frac{1}{2}\alpha_N^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy$.*

*Proof.* From (3.1.6) we get that

$$\frac{\epsilon}{2} \int_\Omega (\mathcal{P}\mathcal{U}_{\delta_j})^2\, dx = \frac{\epsilon}{2} \int_\Omega (\mathcal{U}_{\delta_j} - \varphi_{\delta_j})^2\, dx = \frac{\epsilon}{2} \int_\Omega \mathcal{U}_{\delta_j}^2\, dx - \epsilon \int_\Omega \mathcal{U}_{\delta_j} \varphi_{\delta_j}\, dx + \frac{\epsilon}{2} \int_\Omega \varphi_{\delta_j}^2\, dx. \qquad (3.4.18)$$

The principal term is the first one, in fact we have:

$$\begin{aligned}\frac{\epsilon}{2} \int_\Omega \mathcal{U}_{\delta_j}^2\, dx &= \frac{\epsilon}{2}\, \alpha_N^2 \int_\Omega \frac{\delta_j^{N-2}}{(\delta_1^2 + |x|^2)^{N-2}}\, dx = \frac{\epsilon}{2}\, \alpha_N^2 \int_\Omega \frac{\delta_j^{-(N-2)}}{(1+|x/\delta_1|^2)^{N-2}}\, dx \\ &= \frac{\epsilon}{2}\, \alpha_N^2 \int_{\Omega/\delta_j} \frac{\delta_j^{-(N-2)}}{(1+|y|^2)^{N-2}} \delta_j^N\, dy = \frac{\epsilon}{2}\, \alpha_N^2 \delta_j^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy + \qquad (3.4.19) \\ &\quad + O\left(\epsilon \delta_j^2 \int_{1/\delta_j}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}}\, dr\right) \\ &= \frac{\epsilon}{2}\, \alpha_N^2 \delta_j^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy + O\left(\epsilon \delta_j^{N-2}\right).\end{aligned}$$

For the remaining terms, by using also (3.1.10), we deduce that

$$\epsilon \int_\Omega \mathcal{U}_{\delta_j} \varphi_{\delta_j}\, dx \le \epsilon |\mathcal{U}_{\delta_j}|_2 |\varphi_{\delta_j}|_2 \le c\epsilon \delta_j \delta_j^{\frac{N-2}{2}} \le c\epsilon \delta_j^{\frac{N}{2}}. \tag{3.4.20}$$

Moreover by using again (3.1.10)

$$\frac{\epsilon}{2}\int_\Omega \varphi_{\delta_j}^2\, dx = \frac{\epsilon}{2}|\varphi_{\delta_j}|_2^2 \le C\epsilon \delta_j^{N-2}$$

and the lemma is proved. □

**Lemma 3.4.7.** *Let $\delta_j$ as in (3.2.3) for $j = 1, 2$ and $N \ge 7$. For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds*

$$\epsilon \int_\Omega \mathcal{P}\mathcal{U}_{\delta_1} \mathcal{P}\mathcal{U}_{\delta_2}\, dx = O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2\right),$$

*$C^0$-uniformly with respect to $(d_1, d_2)$ satisfying condition (3.2.4).*

*Proof.* From (3.1.6) we get that

$$\begin{aligned} \epsilon \int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\ \mathcal{P}\mathcal{U}_{\delta_2}\, dx &= \epsilon \int_\Omega (\mathcal{U}_{\delta_1} - \varphi_{\delta_1})(\mathcal{U}_{\delta_2} - \varphi_{\delta_2})\, dx \\ &= \epsilon \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\, dx - \epsilon \int_\Omega \mathcal{U}_{\delta_1}\varphi_{\delta_2}\, dx - \epsilon \int_\Omega \mathcal{U}_{\delta_2}\varphi_{\delta_1}\, dx + \epsilon \int_\Omega \varphi_{\delta_1}\varphi_{\delta_2}\, dx. \end{aligned} \tag{3.4.21}$$

We analyze every term.

$$\begin{aligned} \epsilon \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\, dx &= \epsilon\ \alpha_N^2 \int_\Omega \frac{\delta_1^{-\frac{N-2}{2}}}{(1 + |x/\delta_1|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2 + |x|^2)^{\frac{N-2}{2}}}\, dx \\ &= \epsilon\ \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1 + |y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2 + \delta_1^2|y|^2)^{\frac{N-2}{2}}}\, dy \\ &= \epsilon\ \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1 + |y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{\left(\left(\frac{\delta_2}{\delta_1}\right)^2 + |y|^2\right)^{\frac{N-2}{2}}}\, dy \\ &\le \epsilon\ \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\Omega/\delta_1} \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}} |y|^{N-2}}\, dy \\ &= \epsilon\ \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}} |y|^{N-2}}\, dy \\ &+ O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{1/\delta_1}^{+\infty} \frac{r^{N-1}}{(1 + r^2)^{\frac{N-2}{2}} r^{N-2}}\, dr\right) \\ &= \epsilon\ \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}} |y|^{N-2}}\, dy + O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^{N-2}\right). \end{aligned} \tag{3.4.22}$$

Hence $\epsilon \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\, dx = O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2\right)$. Thanks to (3.4.20) we deduce that $\epsilon \int_\Omega \mathcal{U}_{\delta_1}\varphi_{\delta_2}\, dx = O\left(\epsilon\ \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$, $\epsilon \int_\Omega \mathcal{U}_{\delta_2}\varphi_{\delta_1}\, dx = O\left(\epsilon\ \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$. Moreover it is clear that $\epsilon \int_\Omega \varphi_{\delta_1}\varphi_{\delta_2}\, dx =$

$O\left(\epsilon\, \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$. Since these last three terms are high order terms compared to $\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2$, we deduce the thesis, and the proof is complete. □

We are ready to prove Proposition 3.4.1.

*Proof of Proposition 3.4.1.* (i): One can reason as Part 1 of Proposition 2.2 of [48].

(ii): Let us fix $\eta > 0$. From Lemma 3.4.3 and Lemma 3.4.4, for all sufficiently small $\epsilon$, we get that

$$J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) = J_\epsilon(V_\epsilon) + \epsilon^{\theta_1+\sigma} g(d_1) + O(\epsilon^{\theta_2+\sigma}),$$

for some $\sigma > 0$. We evaluate $J_\epsilon(V_\epsilon) = J_\epsilon(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})$.

$$\begin{aligned}
J_\epsilon(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}) &= \frac{1}{2}\int_\Omega |\nabla(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})|^2\, dx - \frac{1}{p+1}\int_\Omega |\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}|^{p+1}\, dx \\
&- \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})^2 dx \\
&= \frac{1}{2}\int_\Omega |\nabla \mathcal{PU}_{\delta_1}|^2\, dx + \frac{1}{2}\int_\Omega |\nabla \mathcal{PU}_{\delta_2}|^2\, dx - \int_\Omega \nabla \mathcal{PU}_{\delta_1} \cdot \nabla \mathcal{PU}_{\delta_2}\, dx \\
&- \frac{1}{p+1}\int_\Omega |\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}|^{p+1}\, dx - \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_1})^2\, dx - \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_2})^2\, dx \\
&+ \epsilon \int_\Omega \mathcal{PU}_{\delta_1}\, \mathcal{PU}_{\delta_2}\, dx \\
&= \underbrace{\sum_{j=1}^{2}\left(\frac{1}{2}\int_\Omega |\nabla \mathcal{PU}_{\delta_j}|^2\, dx - \frac{1}{p+1}\int_\Omega \mathcal{PU}_{\delta_j}^{p+1}\, dx\right)}_{(I)} - \underbrace{\sum_{j=1}^{2} \frac{\epsilon}{2}\int_\Omega \mathcal{PU}_{\delta_j}^2\, dx}_{(II)} \\
&\quad \underbrace{+\epsilon\int_\Omega \mathcal{PU}_{\delta_1}\mathcal{PU}_{\delta_2}\, dx}_{(III)} \underbrace{- \int_\Omega \nabla\mathcal{PU}_{\delta_1}\nabla\mathcal{PU}_{\delta_2}\, dx}_{(IV)} \\
&\quad \underbrace{-\frac{1}{p+1}\int_\Omega \left[|\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}|^{p+1} - \mathcal{PU}_{\delta_1}^{p+1} - \mathcal{PU}_{\delta_2}^{p+1}\right] dx}_{(IV)}.
\end{aligned}$$

By Lemma 3.4.5, Lemma 3.4.6 and Lemma 3.4.7 we get

$$\begin{aligned}
(I) &= \frac{2}{N} S^{N/2} + a_1\tau(0)\delta_1^{N-2} + a_1\tau(0)\delta_2^{N-2} + O(\delta_1^{N-1}) + O(\delta_2^{N-1}), \\
(II) &= a_2\epsilon\delta_1^2 + a_2\epsilon\delta_2^2 + O(\epsilon\delta_1^{\frac{N}{2}}) + O(\epsilon\delta_2^{\frac{N}{2}}), \\
(III) &= O\left(\epsilon\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2\right).
\end{aligned}$$

Now since $-\Delta\mathcal{P}\mathcal{U}_{\delta_2} = \mathcal{U}_{\delta_2}^p$ then $\int_\Omega \nabla\mathcal{P}\mathcal{U}_{\delta_1}\nabla\mathcal{P}\mathcal{U}_{\delta_2}\,dx = \int_\Omega \mathcal{U}_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx$ and hence

$$(IV) = \underbrace{-\frac{1}{p+1}\int_\Omega \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\right] dx}_{I_1}$$
$$+ \underbrace{\int_\Omega \left[\mathcal{P}\mathcal{U}_{\delta_2}^p - \mathcal{U}_{\delta_2}^p\right]\mathcal{P}\mathcal{U}_{\delta_1}\,dx}_{I_2}.$$

By (3.1.6) and Lemma 3.2.2 we deduce that

$$|I_2| \le C\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\varphi_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}\,dx + C\int_\Omega \varphi_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx.$$

Now let $\rho > 0$ such that $B(0,\rho) \subset \Omega$.

$$\begin{aligned}
\int_\Omega \varphi_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx &\le \int_\Omega \varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx = \int_{\Omega\setminus B(0,\rho)} \varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx + \int_{B(0,\rho)} \varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx \\
&\le |\varphi_{\delta_2}|_{p+1}^p\left(\int_{\Omega\setminus B(0,\rho)} \mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{1}{p+1}} + C\delta_2^{\frac{N+2}{2}}\int_{B(0,\rho)} \frac{1}{\left(1+\left|\frac{x}{\delta_1}\right|^2\right)^{\frac{N-2}{2}}}\,dx \\
&\le C_1\delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}} + C_2\delta_2^{\frac{N+2}{2}}\delta_1^N\int_0^{\frac{\rho}{\delta_1}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}\,dr \\
&\le C_3\left[\delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}} + \delta_2^{\frac{N+2}{2}}\delta_1^{N-2}\right] \le C_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}}\delta_1^{\frac{N}{2}}.
\end{aligned}$$

Moreover, since $\int_\Omega \mathcal{U}_{\delta_j}^p\,dx = O(\delta_j^{\frac{N-2}{2}})$, we get

$$\begin{aligned}
\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\varphi_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}\,dx &\le \|\varphi_{\delta_2}\|_\infty\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}\,dx \le C\delta_2^{\frac{N-2}{2}}\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}\,dx \\
&\le C\delta_2^{\frac{N-2}{2}}\left(\int_\Omega \mathcal{U}_{\delta_2}^p\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \mathcal{U}_{\delta_1}^p\,dx\right)^{\frac{1}{p}} \\
&\le C_1\delta_2^{\frac{N^2+4N-12}{2(N+2)}}\delta_1^{\frac{(N-2)^2}{2(N+2)}} = C_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{\frac{2(N-2)}{N+2}}\delta_1^{N-2} \\
&\le C_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\delta_1^{N-2}.
\end{aligned}$$

Now let $\rho > 0$ and we decompose the domain $\Omega$ as $\Omega = A_0 \cup A_1 \cup A_2$ where $A_0 = \Omega \setminus B(0,\rho)$, $A_1 = B(0,\rho)\setminus B(0,\sqrt{\delta_1\delta_2})$, $A_2 = B(0,\sqrt{\delta_1\delta_2})$. Then we define

$$L_j := -\frac{1}{p+1}\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\right] dx$$

for $j = 0, 1, 2$.

Now, by using Lemma 3.2.2 and Hölder inequality, we see that

$$\begin{aligned}
|L_0| &\leq \frac{1}{p+1}\left[\int_{A_0}\left(|\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}|^{p+1}-\mathcal{PU}_{\delta_1}^{p+1}\right)dx+\int_{A_0}\mathcal{PU}_{\delta_2}^{p+1}\,dx+\int_{A_0}\mathcal{PU}_{\delta_2}^{p}\mathcal{PU}_{\delta_1}\,dx\right]\\
&\leq C\left(\int_{A_0}\mathcal{PU}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx+\int_{A_0}\mathcal{PU}_{\delta_2}^{p+1}\,dx+\int_{A_0}\mathcal{PU}_{\delta_2}^{p}\mathcal{PU}_{\delta_1}\,dx\right)\\
&\leq C\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx+\int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx+\int_{A_0}\mathcal{U}_{\delta_2}^{p}\mathcal{U}_{\delta_1}\,dx\right)\\
&\leq C\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)^{\frac{1}{p+1}}+C_1\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\\
&\quad +C\left(\int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{1}{p+1}}\\
&\leq C_2\left(\int_{\frac{\rho}{\delta_1}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{p}{p+1}}\left(\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{1}{p+1}}+C_3\delta_2^N\\
&\quad +C_2\left(\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{p}{p+1}}\left(\int_{\frac{\rho}{\delta_1}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{1}{p+1}}\\
&\leq C_4\left(\delta_1^{\frac{N+2}{2}}\delta_2^{\frac{N-2}{2}}+\delta_2^N+\delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}}\right)\leq C_5\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^N.
\end{aligned}$$

Now

$$\begin{aligned}
L_1 &= -\frac{1}{p+1}\int_{A_1}\left[|\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}|^{p+1}-\mathcal{PU}_{\delta_1}^{p+1}+(p+1)\mathcal{PU}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\right]dx\\
&\quad +\int_{A_1}\mathcal{PU}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx-\int_{A_1}\mathcal{PU}_{\delta_2}^{p}\mathcal{PU}_{\delta_1}\,dx-\frac{1}{p+1}\int_{A_1}\mathcal{PU}_{\delta_2}^{p+1}\,dx.
\end{aligned}$$

Applying Lemma 3.2.2 we get

$$\begin{aligned}
&\left|\int_{A_1}\left[|\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}|^{p+1}-\mathcal{PU}_{\delta_1}^{p+1}+(p+1)\mathcal{PU}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\right]dx\right|\\
&\leq C\left(\int_{A_1}\mathcal{PU}_{\delta_1}^{p-1}\mathcal{PU}_{\delta_2}^{2}\,dx+\int_{A_1}\mathcal{PU}_{\delta_2}^{p+1}\,dx\right)\\
&\leq C_1\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}}\frac{1}{r^{N-3}}\,dr+\int_{\sqrt{\frac{\delta_2}{\delta_1}}}^{\frac{\rho}{\delta_2}}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)\leq C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned}$$

Thanks to (3.1.6) and Lemma 3.2.2 we have

$$\begin{aligned}
\int_{A_1}\mathcal{PU}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx &= \int_{A_1}\mathcal{U}_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx+O\left(\int_{A_1}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{PU}_{\delta_2}\,dx\right)+O\left(\int_{A_1}\varphi_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx\right)\\
&= \int_{A_1}\mathcal{U}_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx+O\left(\int_{\Omega}\mathcal{U}_{\delta_1}^{p}\varphi_{\delta_2}\,dx\right)+O\left(\int_{\Omega}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{PU}_{\delta_2}\,dx\right)+\\
&\quad +O\left(\int_{\Omega}\varphi_{\delta_1}^{p}\mathcal{PU}_{\delta_2}\,dx\right)
\end{aligned}$$

By definition we have:

$$
\begin{aligned}
&\int_{A_1} \mathcal{U}_{\delta_1}^p \mathcal{U}_{\delta_2}\, dx \\
= \;& \alpha_N^{p+1} \int_{A_1} \frac{\delta_1^{-\frac{N+2}{2}}}{\left(1+\left|\frac{x}{\delta_1}\right|^2\right)^{\frac{N+2}{2}}} \frac{\delta_2^{-\frac{N-2}{2}}}{\left(1+\left|\frac{x}{\delta_2}\right|^2\right)^{\frac{N-2}{2}}}\, dx \\
= \;& \alpha_N^{p+1} \delta_1^{-\frac{N+2}{2}+N} \delta_2^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{N-2} \int_{\sqrt{\frac{\delta_2}{\delta_1}} \le |x| \le \frac{\rho}{\delta_1}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{|y|^{N-2}}\, dy + o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right) \\
= \;& a_3 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right)
\end{aligned}
$$

Moreover by using (3.1.9) we get

$$
\int_{\Omega} \mathcal{U}_{\delta_1}^p \varphi_{\delta_2}\, dx \le C \delta_2^{\frac{N-2}{2}} \int_{\Omega} \mathcal{U}_{\delta_1}^p\, dx \le C_1 \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}
$$

and by using again (3.1.9) we have

$$
\begin{aligned}
\int_{\Omega} \mathcal{U}_{\delta_1}^{p-1} \varphi_{\delta_1} \mathcal{P}\mathcal{U}_{\delta_2}\, dx \;&\le\; \int_{\Omega} \mathcal{U}_{\delta_1}^{p-1} \varphi_{\delta_1} \mathcal{U}_{\delta_2}\, dx \\
&\le\; C \delta_1^{\frac{N-2}{2}} \left(\int_{\Omega} \mathcal{U}_{\delta_1}^{p+1}\, dx\right)^{\frac{p-1}{p+1}} \left(\int_{\Omega} \mathcal{U}_{\delta_2}^{\frac{p+1}{2}}\, dx\right)^{\frac{2}{p+1}} \le C_1 \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\int_{\Omega} \varphi_{\delta_1}^p \mathcal{P}\mathcal{U}_{\delta_2}\, dx \;&\le\; \int_{\Omega} \varphi_{\delta_1}^p \mathcal{U}_{\delta_2}\, dx = \int_{B(0,\rho)} \varphi_{\delta_1}^p \mathcal{U}_{\delta_2}\, dx + \int_{\Omega \setminus B(0,\rho)} \varphi_{\delta_1}^p \mathcal{U}_{\delta_2}\, dx \\
&\le\; C \delta_1^{\frac{N+2}{2}} \delta_2^{-\frac{N-2}{2}+N} \int_0^{\frac{\rho}{\delta_2}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}\, dr + |\varphi_{\delta_1}|_{p+1}^p \left(\int_{\Omega \setminus B(0,\rho)} \mathcal{U}_{\delta_2}^{p+1}\, dx\right)^{\frac{1}{p+1}} \\
&\le\; C \delta_1^{\frac{N+2}{2}} \delta_2^{\frac{N+2}{2}} + C_1 \delta_1^{\frac{N+2}{2}} \left(\int_{\frac{\rho}{\delta_2}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N}\, dr\right)^{\frac{N-2}{2N}} \\
&\le\; C_2 \delta_1^{\frac{N+2}{2}} \delta_2^{\frac{N-2}{2}}.
\end{aligned}
$$

At the end

$$
\begin{aligned}
\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}^p \mathcal{P}\mathcal{U}_{\delta_1}\, dx \;&\le\; \alpha_N^{p+1} \delta_2^{-\frac{N+2}{2}+N} \delta_1^{-\frac{N-2}{2}} \int_{\sqrt{\frac{\delta_1}{\delta_2}} \le |y| \le \frac{\rho}{\delta_2}} \frac{1}{\left(1+\left|\frac{\delta_2}{\delta_1} y\right|^2\right)^{\frac{N-2}{2}}} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy \\
&\le\; C_1 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\, dr \le C_2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned}
$$

Finally, thanks to Lemma 3.2.2 we get that

$$\begin{aligned}
|L_2| &\leq \frac{1}{p+1}\left\{\left|\int_{A_2}\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}+(p+1)\mathcal{P}\mathcal{U}_{\delta_2}^{p}\mathcal{P}\mathcal{U}_{\delta_1}\right]dx\right|+\int_{A_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}\,dx\right\}\\
&\leq C\left(\int_{A_2}\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\mathcal{P}\mathcal{U}_{\delta_1}^{2}\,dx+\int_{A_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)\leq C\left(\int_{A_2}\mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}^{2}\,dx+\int_{A_2}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)\\
&\leq C_1\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{r^{N-5}}{(1+r^2)^{N-2}}\,dr+\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{r^{N-1}}{(1+r^2)^{N}}\,dr\right)\\
&\leq C_2\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-5}\,dr+\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-1}\,dr\right)\leq C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned}$$

From Lemma 3.4.5 to Lemma 3.4.7 summing up all the terms we get that

$$\begin{aligned}
J_\epsilon(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}) &= \frac{2}{N}S^{N/2}+a_1\tau(0)\delta_1^{N-2}+a_1\tau(0)\delta_2^{N-2}+O(\delta_1^{N-1})+O(\delta_2^{N-1})\\
&+ O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^2\right)+O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right)+a_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\\
&- a_2\epsilon\delta_1^2+O\left(\epsilon\delta_1^{N-2}\right)-a_2\epsilon\delta_2^2+O\left(\epsilon\delta_2^{N-2}\right),
\end{aligned}\tag{3.4.23}$$

where $a_1=\frac{1}{2}\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$, $a_2=\frac{1}{2}\alpha_N^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}\,dy$, $a_3=\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$. Recalling the choice of $\delta_j$, $j=1,2$ we get

$$\begin{aligned}
J_\epsilon(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}) &= \frac{2}{N}S^{N/2}+a_1\tau(0)d_1^{N-2}\epsilon^{\frac{N-2}{N-4}}+a_1\tau(0)d_2^{N-2}\epsilon^{\frac{(3N-10)(N-2)}{(N-4)(N-6)}}\\
&+ O\left(\epsilon^{\frac{N-1}{N-4}}\right)+O\left(\epsilon^{\frac{(3N-10)(N-1)}{(N-4)(N-6)}}\right)+O\left(\epsilon^{\frac{N+2}{N-6}}\right)+O\left(\epsilon^{\frac{(N-2)N}{(N-4)(N-6)}}\right)\\
&+ a_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}-a_2d_1^2\epsilon^{\frac{N-2}{N-4}}+O\left(\epsilon^{\frac{2(N-3)}{N-4}}\right)-a_2d_2^2\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}\\
&+ O\left(\epsilon^{\frac{2(2N^2-13N+22)}{(N-4)(N-6))}}\right)\\
&= \frac{2}{N}S^{N/2}+[a_1\tau(0)d_1^{N-2}-a_2d_1^2]\epsilon^{\frac{N-2}{N-4}}+O\left(\epsilon^{\frac{N-1}{N-4}}\right)\\
&+ \left[a_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}-a_2d_2^2\right]\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}+O\left(\epsilon^{\frac{N+2}{N-6}}\right).
\end{aligned}\tag{3.4.24}$$

We point out that the term $O(\epsilon^{\frac{N-1}{N-4}})$ depends only on $d_1$.

□

## 3.5 proof of Theorems 3.1.1 and 3.1.2

*Proof of Theorem 3.1.1.* Let us set $G_1(d_1) := a_1\tau(0)d_1^{N-2}-a_2d_1^2$, where $a_1$, $a_2$ are the positive constants appearing in Proposition 3.4.1 and $\tau(0)$ is the Robin's function of the domain $\Omega$ at the origin, so by definition it follows that $\tau(0)$ is positive. It's elementary to see that the function $G_1 : \mathbb{R}^+ \to \mathbb{R}$ has a strictly local minimum point at $\bar{d}_1 = \left(\frac{2a_2}{(N-2)a_1\tau(0)}\right)^{\frac{1}{N-4}}$.

Since $\bar{d}_1$ is a strictly local minimum for $G_1$, then, for any sufficiently small $\gamma > 0$ there exists an open interval $I_{1,\sigma_1}$ such that $\overline{I}_{1,\sigma_1} \subset \mathbb{R}^+$, $I_{1,\sigma_1}$ has diameter $\sigma_1$, $\bar{d}_1 \in I_{1,\sigma_1}$ and for all $d_1 \in \partial I_{1,\sigma_1}$

$$G_1(d_1) \geq G_1(\bar{d}_1) + \gamma. \tag{3.5.1}$$

Clearly as $\gamma \to 0$ we can choose $\sigma_1$ so that $\sigma_1 \to 0$.

We set $G_2(d_1, d_2) := a_3\tau(0)\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}} - a_2 d_2^2$, $G_2 : \mathbb{R}^2_+ \to \mathbb{R}$, where $a_3 > 0$ is the same constant appearing in Proposition 3.4.1. If we fix $d_1 = \bar{d}_1$ then $\hat{G}_2(d_2) := G(\bar{d}_1, d_2)$ has a strictly local minimum point at $\bar{d}_2 := \left(\frac{2a_2\bar{d}_1^{\frac{N-2}{2}}}{a_3\tau(0)\frac{N-2}{2}}\right)^{\frac{2}{N-6}}$. As in the previous case there exists an open interval $I_{2,\sigma_2}$ such that $\overline{I}_{2,\sigma_2} \subset \mathbb{R}^+$, $I_{2,\sigma_2}$ has diameter $\sigma_2$, $\bar{d}_2 \in I_{1,\sigma_1}$ and for all $d_2 \in \partial I_{2,\sigma_2}$

$$\hat{G}_2(d_2) \geq \hat{G}_2(\bar{d}_2) + \gamma. \tag{3.5.2}$$

As $\gamma \to 0$ we can choose $\sigma_2$ so that $\sigma_2 \to 0$.

Let us set $K := \overline{I_{1,\sigma_1} \times I_{2,\sigma_2}}$ and let $\eta > 0$ be small enough so that $K \subset ]\eta, \frac{1}{\eta}[\times]\eta, \frac{1}{\eta}[$. Thanks to Proposition 3.3.1, for all sufficiently small $\epsilon$, $\tilde{J}_\epsilon : \mathbb{R}^2_+ \to \mathbb{R}$ is defined and it is of class $C^1$. By Weierstrass theorem we know there exists a global minimum point for $\tilde{J}_\epsilon$ in $K$. Let $(d_{1,\epsilon}, d_{2,\epsilon})$ be that point, we want to show that there exists $\epsilon_1$ such that, for all $\epsilon < \epsilon_1$, $(d_{1,\epsilon}, d_{2,\epsilon})$ lies in the interior of $K$.

Assume by contradiction there exists a sequence $\epsilon_n \to 0$ such that for all $n \in \mathbb{N}$

$$(d_{1,\epsilon_n}, d_{2,\epsilon_n}) \in \partial K.$$

There are only two possibilities:

**(a)** $d_{1,\epsilon_n} \in \partial I_{1,\sigma_1}$, $d_{2,\epsilon_n} \in \overline{I}_{2,\sigma_2}$,

**(b)** $d_{1,\epsilon_n} \in \overline{I}_{1,\sigma_1}$, $d_{2,\epsilon_n} \in \partial I_{2,\sigma_2}$.

Thanks to (ii) of Proposition 3.4.1 we have the uniform expansion

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(\bar{d}_1, d_2) = \epsilon^{\theta_1}\left[G_1(d_1) - G_1(\bar{d}_1)\right] + o\left(\epsilon^{\theta_1}\right). \tag{3.5.3}$$

for all $\epsilon < \epsilon_0$, $(d_1, d_2) \in K$. We point out that we have incorporated the other high order terms in $o\left(\epsilon^{\theta_1}\right)$. Thanks to (3.5.1) and (3.5.3), for all sufficiently small $\epsilon$ we have

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(\bar{d}_1, d_2) > 0, \tag{3.5.4}$$

for all $d_1 \in \partial I_{1,\sigma_1}$, for all $d_2 \in \overline{I}_{2,\sigma_2}$. So for $n$ sufficiently large if (a) holds, since by definition $\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) = \min_K \tilde{J}_{\epsilon_n}$, then

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) \leq \tilde{J}_{\epsilon_n}(\bar{d}_1, d_{2,\epsilon_n}),$$

which contradicts (3.5.4). Assume (b). Thanks to (ii) of Proposition 3.4.1 (see also Remark 3.4.2) we have the uniform expansion

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(d_1, \bar{d}_2) = \epsilon^{\theta_2}\left[G_2(d_1, d_2) - G_2(d_1, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right), \tag{3.5.5}$$

for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in K$.

For $n$ sufficiently large so that $\epsilon_n < \epsilon_0$ we have

$$\begin{aligned}
\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) &= \epsilon^{\theta_2}\left[G_2(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - G_2(d_{1,\epsilon_n}, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right) \\
&= \epsilon^{\theta_2}\left[G_2(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - G_2(\bar{d}_1, d_{2,\epsilon_n}) + G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2)\right. \\
&\qquad \left. G_2(\bar{d}_1, \bar{d}_2) - G_2(d_{1,\epsilon_n}, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right) \\
&= \epsilon^{\theta_2}\left[a_3\tau(0) d_{2,\epsilon_n}^{\frac{N-2}{2}}\left(\frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}} - \frac{1}{\bar{d}_1^{\frac{N-2}{2}}}\right) + G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2)\right. \\
&\qquad \left. + a_3\tau(0)\bar{d}_2^{\frac{N-2}{2}}\left(\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right)\right] + o\left(\epsilon_n^{\theta_2}\right)
\end{aligned} \tag{3.5.6}$$

We observe now that, up to a subsequence, $d_{1,\epsilon_n} \to \bar{d}_1$ as $n \to +\infty$. This is a consequence of the uniform expansion given by (ii) of Proposition 3.4.1, in fact

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(\bar{d}_1, \bar{d}_2) = \epsilon_n^{\theta_1}\left[G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1)\right] + o\left(\epsilon_n^{\theta_1}\right). \tag{3.5.7}$$

Since $(d_{1,\epsilon_n}, d_{2,\epsilon_n})$ is the minimum point we have $\tilde{J}_\epsilon(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_\epsilon(\bar{d}_1, \bar{d}_2) \leq 0$, hence, dividing (3.5.7) by $\epsilon_n^{\theta_1}$, for all sufficiently large $n$ we get that $G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \leq -\frac{o\left(\epsilon_n^{\theta_1}\right)}{\epsilon_n^{\theta_1}}$. On the other side, since $\bar{d}_1$ is the minimum of $G_1$, we get that $G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \geq 0$. So we have proved that

$$0 \leq G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \leq -\frac{o\left(\epsilon_n^{\theta_1}\right)}{\epsilon_n^{\theta_1}},$$

and passing to the limit we deduce that $\lim_{n\to+\infty} G_1(d_{1,\epsilon_n}) = G_1(\bar{d}_1)$. Hence, up to a subsequence, since $\bar{d}_1$ is a strict local minimum, the only possibility is $d_{1,\epsilon_n} \to \bar{d}_1$.

Since we are assuming (b), from (3.5.2) we get that

$$G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2) \geq \gamma.$$

From this last inequality, (3.5.6) and since $(d_{2,\epsilon_n})_n$ is bounded, then, choosing $\bar{n}$ sufficiently large so that $a_3\tau(0) d_{2,\epsilon_n}^{\frac{N-2}{2}}\left|\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right|$ and $a_3\tau(0)\bar{d}_2^{\frac{N-2}{2}}\left|\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right|$ are small enough, we deduce that

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) > 0,$$

for all $n > \bar{n}$. Since $(d_{1,\epsilon_n}, d_{2,\epsilon_n})$ is the minimum point it also holds

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) \leq 0,$$

and we get a contradiction.

To complete the proof we point out that, as observed before, up to a subsequence $d_{1,\epsilon} \to \bar{d}_1$ as $\epsilon \to 0$. With a similar argument we prove that $d_{2,\epsilon} \to \bar{d}_2$. In fact, from the same argument of (3.5.6), since $d_{1,\epsilon} \to \bar{d}_1$ and $(d_{2,\epsilon})_\epsilon$ is bounded, we have

$$\begin{aligned} 0 \geq \frac{\tilde{J}_\epsilon(d_{1,\epsilon}, d_{2,\epsilon}) - \tilde{J}_\epsilon(d_{1,\epsilon}, \bar{d}_2)}{\epsilon^{\theta_2}} &= G_2(d_{1,\epsilon}, d_{2,\epsilon}) - G_2(d_{1,\epsilon}, \bar{d}_2) + \frac{o\left(\epsilon^{\theta_2}\right)}{\epsilon^{\theta_2}} \\ &= a_3\tau(0) d_{2,\epsilon}^{\frac{N-2}{2}} \left( \frac{1}{d_{1,\epsilon}^{\frac{N-2}{2}}} - \frac{1}{\bar{d}_1^{\frac{N-2}{2}}} \right) + G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \\ &\quad + a_3\tau(0) \bar{d}_2^{\frac{N-2}{2}} \left( \frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon}^{\frac{N-2}{2}}} \right) + \frac{o\left(\epsilon^{\theta_2}\right)}{\epsilon^{\theta_2}} \\ &= o(1) + G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2). \end{aligned} \tag{3.5.8}$$

Since $\bar{d}_2$ is a local maximum point for $d_2 \to \hat{G}_2(d_2)$ we have $G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \geq 0$ and so from (3.5.8) we get that

$$0 \leq G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \leq -o(1).$$

Passing to the limit as $\epsilon \to 0$ we deduce that $\hat{G}_2(d_{2,\epsilon}) \to \hat{G}_2(\bar{d}_2)$. Hence, up to a subsequence, since $\bar{d}_2$ is a strict local minimum, the only possibility is $d_{2,\epsilon} \to \bar{d}_2$.
Hence by (i) of Proposition 3.4.1 we have that $V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2$ is a solution of (3.1.1). Moreover, taking into account of (3.1.6), (3.1.10) and (3.1.11), we get that solution obtained is of the form (3.1.2) and the proof is complete. □

We are ready also to prove Theorem 3.1.2. We reason as in [49].

*Proof of Theorem 3.1.2.* Let $u_\epsilon$ be a solution of (3.1.1) as in Theorem 3.1.1 and assume that $\Phi_\epsilon \to 0$ uniformly in compact subsets of $\Omega$. We set

$$\begin{aligned} \tilde{u}_\epsilon(x) &:= \left( \frac{d_{1\epsilon}\epsilon^{\frac{1}{N-4}}}{d_{1\epsilon}^2 \epsilon^{\frac{2}{N-4}} + |x|^2} \right)^{\frac{N-2}{2}} - \left( \frac{d_{1\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}}}{d_{1\epsilon}^2 \epsilon^{2\frac{3N-10}{(N-4)(N-6)}} + |x|^2} \right)^{\frac{N-2}{2}} \\ &= \left( \frac{1}{d_{1\epsilon}\epsilon^{\frac{1}{N-4}} + d_{1\epsilon}^{-1}\epsilon^{-\frac{1}{N-4}}|x|^2} \right)^{\frac{N-2}{2}} - \left( \frac{1}{d_{2\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{3N-10}{(N-4)(N-6)}}|x|^2} \right)^{\frac{N-2}{2}} \end{aligned}$$

Then, by Theorem 3.1.1 and by using the assumption on the remainder term $\Phi_\epsilon$ we get

$$u_\epsilon(x) = \alpha_N \tilde{u}_\epsilon(x)(1 + o(1)), \qquad x \in \Omega, \tag{3.5.9}$$

where $o(1) \to 0$ uniformly on compact subsets of $\Omega$.
We consider the spheres

$$\mathcal{S}_\epsilon^1 := \{x \in \mathbb{R}^N ;\ |x| = \epsilon^{\frac{1}{N-4}}\}$$

and

$$\mathcal{S}_\epsilon^2 := \{x \in \mathbb{R}^N ;\ |x| = \epsilon^{\frac{3N-10}{(N-4)(N-6)}}\}.$$

We may fix a compact subset $K \subset \Omega$ such that $\mathcal{S}_\epsilon^j \subset K$, $j = 1, 2$ and $\epsilon > 0$ sufficiently small.

For $x \in \mathcal{S}_\epsilon^1$ we get

$$\begin{aligned}
\tilde{u}_\epsilon(x) &= \left(\frac{1}{d_{1\epsilon}\epsilon^{\frac{1}{N-4}} + d_{1\epsilon}^{-1}\epsilon^{\frac{1}{N-4}}}\right)^{\frac{N-2}{2}} - \left(\frac{1}{d_{2\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{N+2}{(N-4)(N-6)}}}\right)^{\frac{N-2}{2}} \\
&= \epsilon^{-\frac{N-2}{2(N-4)}}\left[\left(\frac{1}{d_{1\epsilon} + d_{1\epsilon}^{-1}}\right)^{\frac{N-2}{2}} - \left(\frac{1}{d_{2\epsilon}\epsilon^{\frac{2(N-2)}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{8}{(N-4)(N-6)}}}\right)^{\frac{N-2}{2}}\right] \\
&= \epsilon^{-\frac{N-2}{2(N-4)}}\left[\left(\frac{1}{d_{1\epsilon} + d_{1\epsilon}^{-1}}\right)^{\frac{N-2}{2}} + o(1)\right]
\end{aligned}$$

as $\epsilon \to 0$. Hence $\tilde{u}_\epsilon > 0$ on $\mathcal{S}_\epsilon^1$ for $\epsilon$ small.
Analogously if $x \in \mathcal{S}_\epsilon^2$ then

$$\tilde{u}_\epsilon(x) = -\epsilon^{-\frac{(3N-10)(N-2)}{2(N-4)(N-6)}}\left[\left(\frac{1}{d_{2\epsilon} + d_{2\epsilon}^{-1}}\right)^{\frac{N-2}{2}} + o(1)\right]$$

as $\epsilon \to 0$ and hence $\tilde{u}_\epsilon < 0$ on $\mathcal{S}_\epsilon^2$ for $\epsilon$ small.
Since (3.5.9) holds, this implies that $u_\epsilon > 0$ on $\mathcal{S}_\epsilon^1$ and $u_\epsilon < 0$ on $\mathcal{S}_\epsilon^2$ for $\epsilon$ small.
Then $u_\epsilon$ has at least two nodal domains $\Omega_1, \Omega_2$ such that $\Omega_j$ contains the sphere $\mathcal{S}_\epsilon^j$, $j = 1, 2$.
Next we show that $u_\epsilon$ has not more than two nodal domains for $\epsilon$ small.
We remark that by (ii) of Proposition 3.4.1 and by Lemmas 3.4.3, 3.4.4 it follows that

$$J_\epsilon(u_\epsilon) \to \frac{2}{N}S^{\frac{N}{2}}, \qquad \text{as } \epsilon \to 0 \tag{3.5.10}$$

where $J_\epsilon$ is defined in (3.1.13) and $S$ is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, namely

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}.$$

We set $c_\epsilon := \inf_{\mathcal{N}_\epsilon} J_\epsilon$, where $\mathcal{N}_\epsilon$ is the Nehari manifold, which is defined by

$$\mathcal{N}_\epsilon := \left\{u \in H_0^1(\Omega)\ ;\ \int_\Omega |\nabla u|^2\,dx = \int_\Omega |u|^{p+1}\,dx + \epsilon\int_\Omega u^2\,dx\right\}.$$

It is easy to see that $c_\epsilon \to c_0 = \frac{1}{N}S^{\frac{N}{2}}$ as $\epsilon \to 0$ and therefore, by (3.5.10), we get that

$$J_\epsilon(u_\epsilon) < 3c_\epsilon \tag{3.5.11}$$

for $\epsilon$ small enough.
We now suppose by contradiction that $u_\epsilon$ has at least 3 pairwise different nodal domains $\Omega_1, \Omega_2, \Omega_3$.
Let $\chi_i$ be the characteristic function corresponding to the sets $\Omega_i$.
Then $u_\epsilon\chi_i \in H_0^1(\Omega)$ (see [46]). Moreover

$$\begin{aligned}
\int_\Omega |\nabla(u_\epsilon\chi_i)|^2\,dx &= \int_\Omega \nabla u_\epsilon \nabla(u_\epsilon\chi_i) = -\int_\Omega \Delta u_\epsilon(u_\epsilon\chi_i)\,dx \\
&= \int_\Omega |u_\epsilon|^p(u_\epsilon\chi_i)\,dx + \epsilon\int_\Omega u_\epsilon \cdot u_\epsilon\chi_i\,dx \\
&= \int_\Omega |u_\epsilon\chi_i|^{p+1}\,dx + \epsilon\int_\Omega (u_\epsilon\chi_i)^2\,dx
\end{aligned}$$

so that $u_\epsilon \chi_i \in \mathcal{N}_\epsilon$. Since also $u_\epsilon \in \mathcal{N}_\epsilon$ we obtain

$$\begin{aligned} J_\epsilon(u_\epsilon) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |u_\epsilon|^{p+1}\, dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^{3} \int_\Omega |u_\epsilon \chi_i|^{p+1}\, dx \\ &= \sum_{i=1}^{3} J_\epsilon(\chi_i u_\epsilon) \geq 3c_\epsilon \end{aligned}$$

contrary to (3.5.11). The contradiction shows that $u_\epsilon$ has at most two nodal domains for $\epsilon$ small.
This completes the proof. □

# Chapter 4

# A nonexistence result for sign-changing solutions of the Brezis-Nirenberg problem in low dimensions

## 4.1 Introduction

Here we present and prove the result **(R4)**.

In this chapter we study the semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\lambda$ is a positive real parameter and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.

This problem is known as "the Brezis-Nirenberg problem" because the first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983 in the celebrated paper [17]. From their results it came out that the dimension was going to play a crucial role in the study of (4.1.1). Indeed they proved that if $N \geq 4$ there exists a positive solution of (4.1.1) for every $\lambda \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\Delta$ in $\Omega$ with Dirichlet boundary conditions, while if $N = 3$ positive solutions exists only for $\lambda$ away from zero. In particular, in the case of the ball $B$ they showed that there are no positive solutions in the interval $(0, \frac{\lambda_1(B)}{4})$.

Since then several other interesting results were obtained for positive solutions, in particular about the asymptotic behavior of solutions, mainly for $N \geq 5$ because also the case $N = 4$ presents more difficulties compared to the higher dimensional ones.

Concerning the case of sign-changing solutions, existence results hold if $N \geq 4$ both for $\lambda \in (0, \lambda_1(\Omega))$ and $\lambda > \lambda_1(\Omega)$ as shown in [20], [24], [6].

The case $N = 3$ presents even more difficulties than in the study of positive solutions. In particular in the case of the ball is not yet known what is the least value $\bar{\lambda}$ of the parameter $\lambda$ for which sign-changing solutions exist, neither whether $\bar{\lambda}$ is larger or smaller than $\lambda_1(B)/4$. This question, posed by H. Brezis, has been given a partial answer in [14]. However it is interesting to observe that in the study of sign-changing solutions even the "low dimensions" $N = 4, 5, 6$ exhibit some peculiarities. Indeed it was first proved by

Atkinson, Brezis and Peletier in [5] that if $\Omega$ is a ball there exists $\lambda^* = \lambda^*(N)$ such that there are no radial sign-changing solutions of (4.1.1) for $\lambda \in (0, \lambda^*)$. Later this result was reproved in [1] in a different way.

Moreover for $N \geq 7$ a recent result of Schechter and Zou [54] shows that in any bounded smooth domain there exist infinitely many sign-changing solutions for any $\lambda > 0$. Instead if $N = 4, 5, 6$ only $N + 1$ pairs of solutions, for all $\lambda > 0$, have been proved to exist in [24] but it is not clear that they change sign.

Coming back to the nonexistence result of [5] and [1] an interesting question would be to see whether and in which way it could be extended to other bounded smooth domains.

Since the result of [5] and [1] concerns nodal radial solutions in the ball the first issue is to understand what are, in general bounded domains, the sign-changing solutions which play the same role as the radial nodal solutions in the case of the ball. A main property of a radial nodal solution in the ball is that its nodal set does not touch the boundary therefore, a class of solutions to consider, in general bounded domains, could be the one made of functions which have this property.

Moreover, in analyzing the asymptotic behavior of least energy nodal radial solutions $u_\lambda$ in the ball, as $\lambda \to 0$, in dimension $N \geq 7$ (in which case they exist for all $\lambda \in (0, \lambda_1(B))$, see [25]) one can prove (see [37]) that their limit profile is that of a "tower of two bubbles". This terminology means that the positive part and the negative part of the solutions $u_\lambda$ concentrate at the same point (which is obviously the center of the ball) as $\lambda \to 0$ and each one has the limit profile, after suitable rescaling, of a "standard" bubble in $\mathbb{R}^N$, i.e. of a positive solution of the critical exponent problem in $\mathbb{R}^N$. More precisely the solutions $u_\lambda$ can be written in the following way:

$$u_\lambda = \mathcal{P}\mathcal{U}_{\delta_1,\xi} - \mathcal{P}\mathcal{U}_{\delta_2,\xi} + w_\lambda, \tag{4.1.2}$$

where $\mathcal{P}\mathcal{U}_{\delta_i,\xi}$, $i = 1, 2$ is the projection on $H_0^1(\Omega)$ of the regular positive solution of the critical problem in $\mathbb{R}^N$, centered at $\xi = 0$, with rescaling parameter $\delta_i$ and $w_\lambda$ is a remainder term which converges to zero in $H_0^1(\Omega)$.

It is also interesting to observe that, thanks to a recent result of [40], sign-changing bubble-tower solutions exist also in bounded smooth symmetric domains in dimension $N \geq 7$ for $\lambda$ close to zero, and they have the property that their nodal set does not touch the boundary of the domain.

In view of all these remarks we are entitled to assert that in general bounded domains sign-changing solutions which behave as the radial ones in the ball, at least for $\lambda$ close to zero, are the ones which are of the form (4.1.2). Hence a natural extension of the nonexistence result of [5] and [1] would be to show that, in dimension $N = 4, 5, 6$, sign-changing solutions of the form (4.1.2) do not exist in any bounded smooth domain.

This is indeed the main aim of this chapter. Let us also note that in the 3-dimensional case a similar nonexistence result was already proved in [14]. Indeed, in studying the asymptotic behavior of low-energy nodal solutions it was shown in [14] that their positive and negative part cannot concentrate at the same point, as $\lambda$ tends to a limit value $\bar{\lambda} > 0$. In the case $N \geq 4$ this question was left open in [13]. Therefore our results also complete the analysis made in these last two papers.

To state precisely our result let us recall that the functions

$$\mathcal{U}_{\xi,\delta}(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{(\delta^2 + |x - \xi|^2)^{\frac{N-2}{2}}}, \qquad \delta > 0,\ \xi \in \mathbb{R}^N, \tag{4.1.3}$$

$\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, describe all regular positive solutions of the problem

$$\begin{cases} -\Delta U = U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ U(x) \to 0, & \text{as } |x| \to +\infty. \end{cases}$$

Then, denoting by $\mathcal{P}\mathcal{U}_\delta$ their projection on $H_0^1(\Omega)$, and by $\|u\| := \int_\Omega |\nabla u|^2\, dx$ for any $u \in H_0^1(\Omega)$, we have:

**Theorem 4.1.1.** *Let $N = 4, 5, 6$ and $\xi$ a point in the domain $\Omega$. Then, for $\lambda$ close to zero, Problem (4.1.1) does not admit any sign-changing solution $u_\lambda$ of the form (4.1.2) with $\delta_i = \delta_i(\lambda)$, $i = 1, 2$, such that $\delta_2 = o(\delta_1)$, $\|w_\lambda\| \to 0$ and $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$, $|\nabla w_\lambda| = o(\delta_1^{-\frac{N}{2}})$ uniformly in compact subsets of $\Omega$, as $\lambda \to 0$.*

The previous notations mean that $\frac{|w_\lambda|}{\delta_1^{-\frac{N-2}{2}}}, \frac{|\nabla w_\lambda|}{\delta_1^{-\frac{N}{2}}}$ converge to zero as $\lambda \to 0$ uniformly in compact subsets of $\Omega$.

The proof of the above theorem is based on a Pohozaev identity and fine estimates which are derived in a different way in the case $N = 4$ or $N = 5, 6$. We would like to point out that it cannot be deduced by the proof of Theorem 3.1 of [14] which holds only in dimension three.

Concerning the assumption on the $C^1$-norm in compact subsets of $\Omega$ of the remainder term $w_\lambda$, whose gradient is only required not to blow up too fast, in Section 4.4 we show that it is almost necessary.

Note that we do not even require that $w_\lambda \to 0$ uniformly in $\Omega$ neither that it remains bounded as $\lambda \to 0$, but only a control of possible blow-up of $|w_\lambda|$ and $|\nabla w_\lambda|$. We delay to the next sections some further comments and comparisons with the case $N \geq 7$.

Finally in the last section we show that in dimension $N \geq 7$ if $(u_\lambda)$ is a family of solutions of type (4.1.2) with $|w_\lambda|$, $|\nabla w_\lambda|$ as in Theorem 4.1.1 and $\delta_i = d_i \lambda^{\alpha_i}$, for some positive numbers $d_i = d_i(\lambda)$ with $0 < c_1 < d_i < c_2$, for all sufficiently small $\lambda$, and $0 < \alpha_1 < \alpha_2$, then necessarily:

$$\alpha_1 = \frac{1}{N-4}, \quad \alpha_2 = \frac{3N-10}{(N-4)(N-6)}. \tag{4.1.4}$$

In other words we prove that if the concentration speeds are powers of $\lambda$ then necessarily the exponent must be as in (4.1.4). Note that these are exactly the type of speeds assumed in Chapter 3 to construct the tower of bubbles in higher dimensions.

## 4.2 Some preliminary results

**Lemma 4.2.1.** *Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ and let $(\xi, \delta) \in \Omega \times \mathbb{R}^+$. As $\delta \to 0$ it holds:*

$$\mathcal{P}\mathcal{U}_{\xi,\delta}(x) = \mathcal{U}_{\xi,\delta}(x) - \alpha_N \delta^{\frac{N-2}{2}} H(x,\xi) + o(\delta^{\frac{N-2}{2}}), \ x \in \Omega$$

*$C^1$-uniformly on compact subsets of $\Omega$, where $H$ is the regular part of the Green function for the Laplacian. Moreover, setting $\varphi_{\xi,\delta}(x) := \mathcal{U}_{\xi,\delta}(x) - \mathcal{P}\mathcal{U}_{\xi,\delta}(x)$, the following uniform estimates hold:*

**(i)** $0 \leq \varphi_{\xi,\delta} \leq \mathcal{U}_{\xi,\delta}$,

**(ii)** $\|\varphi_{\xi,\delta}\|^2 = O\left((\frac{\delta}{d})^{N-2}\right)$, *where $d = d(\xi, \partial\Omega)$ is the euclidean distance between $\xi$ and the boundary of $\Omega$.*

*Proof.* See [51], Proposition 1 and its proof. □

**Lemma 4.2.2.** *Let $N \geq 4$ and $(u_\lambda)$ be a family of sign-changing solutions of (4.1.1) satisfying*

$$\|u_\lambda\|^2 \to 2S^{N/2}, \quad \text{as } \lambda \to 0.$$

*Then, for all sufficiently small $\lambda > 0$, the set $\Omega \setminus \{x \in \Omega;\ u_\lambda(x) = 0\}$ has exactly two connected components.*

*Proof.* Let us consider the nodal set $Z_\lambda := \{x \in \Omega;\ u_\lambda(x) = 0\}$ and let $\Omega_1$ be a connected component of $\Omega \setminus Z_\lambda$. Multiplying (4.1.1) by $u_\lambda$ and integrating on $\Omega_1$, we get that

$$\int_{\Omega_1} |\nabla u_\lambda|^2\ dx \geq S^{N/2}(1 + o(1)),$$

where we have used the Sobolev embedding and the fact that $\lambda \to 0$ and $\lambda_1(\Omega_1) \int_{\Omega_1} u_\lambda^2\ dx \leq \int_{\Omega_1} |\nabla u_\lambda|^2\ dx$, where $\lambda_1(\Omega_1)$ is the first Dirichlet eigenvalue of $-\Delta$ on $\Omega_1$.

Since $\|u_\lambda\|^2 \to 2S^{N/2}$, as $\lambda \to 0$, then for all sufficiently small $\lambda > 0$ we deduce that $\Omega \setminus Z_\lambda$ can have only two connected components. □

We recall now the Pohozaev identity for solutions of semilinear problems which are not necessarily zero on the boundary. Let $D$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary and consider the equation

$$-\Delta u = f(u) \ \text{ in } D, \tag{4.2.1}$$

where $s \mapsto f(s)$ is a continuos function. Denoting $F(s) := \int_0^s f(t)\ dt$, we have:

**Proposition 4.2.3.** *Let $u$ be a $C^2$-solution of (4.2.1), then*

$$\begin{aligned} &\int_D \left\{ NF(u) - \frac{N-2}{2} u f(u) \right\}\ dx \\ = &\int_{\partial D} \left\{ \sum_{i=1}^N x_i \nu_i \left( F(u) - \frac{1}{2}|\nabla u|^2 \right) + \frac{\partial u}{\partial \nu} \sum_{i=1}^N x_i u_{x_i} + \frac{N-2}{2} u \frac{\partial u}{\partial \nu} \right\}\ d\sigma, \end{aligned} \tag{4.2.2}$$

*where $\nu$ denotes the outer normal to the boundary and $u_{x_i}$ is the partial derivative with respect to $x_i$ of $u$.*

*Proof.* For the proof see [64]. □

The following lemma gives information on the asymptotic behavior of the nodal set $Z_\lambda$ of solutions of (4.1.1) as $\lambda \to 0$.

**Lemma 4.2.4.** *Let $N \geq 4$, $\xi \in \Omega$ and let $(u_\lambda)$ be a family of solutions of (4.1.1), such that $u_\lambda = \mathcal{P}\mathcal{U}_{\xi,\delta_1} - \mathcal{P}\mathcal{U}_{\xi,\delta_2} + w_\lambda$, with $\delta_1 = \delta_1(\lambda)$ and $\delta_2 = \delta_2(\lambda)$ satisfying*

$$\delta_2 = o(\delta_1) \ \text{ and } \ \|w_\lambda\| \to 0, \ \text{as } \lambda \to 0.$$

*Moreover, assume that $w_\lambda$ satisfies $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$ uniformly in compact subsets of $\Omega$. Then, for all small $\epsilon > 0$ there exists $\lambda_\epsilon > 0$ such that the nodal set $Z_\lambda$ is contained in the annular region $A_{r_1,r_2}(\xi) := \{x \in \Omega;\ r_1 < |x - \xi| < r_2\}$, for all $\lambda \in (0, \lambda_\epsilon)$, where $r_1 := \delta_1^{\frac{1}{2}-\epsilon}\delta_2^{\frac{1}{2}+\epsilon}$, $r_2 := \delta_1^{\frac{1}{2}+\epsilon}\delta_2^{\frac{1}{2}-\epsilon}$.*

*Proof.* Without loss of generality we assume that $\xi = 0$. Let us fix a small $\epsilon > 0$ and a compact neighborhood of the origin $K$. Thanks to the assumptions and Lemma 4.2.1, we have the following expansion $u_\lambda(x) = \mathcal{U}_{\delta_1}(x) - \mathcal{U}_{\delta_2}(x) + o(\delta_1^{-\frac{N-2}{2}})$, which is uniform with respect to $x \in K$ and to all small $\lambda > 0$. By definition, for all sufficiently small $\lambda > 0$, we have that $A_{r_1,r_2}(0) \subset K$. For $x$ such that $|x| = r_1$ we have:

$$\begin{aligned}\mathcal{U}_{\delta_1}(x) &= \alpha_N \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2 + \delta_1^{1-2\epsilon}\delta_2^{1+2\epsilon})^{\frac{N-2}{2}}} = \alpha_N \frac{\delta_1^{-\frac{N-2}{2}}}{[1 + (\frac{\delta_2}{\delta_1})^{1+2\epsilon}]^{\frac{N-2}{2}}} \\ &= \alpha_N\, \delta_1^{-\frac{N-2}{2}} - \alpha_N \frac{N-2}{2}\delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{1+2\epsilon} + o\left(\delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{1+2\epsilon}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}_{\delta_2}(x) &= \alpha_N \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2 + \delta_1^{1-2\epsilon}\delta_2^{1+2\epsilon})^{\frac{N-2}{2}}} = \alpha_N \frac{\delta_2^{\frac{N-2}{2}}\delta_1^{-\frac{N-2}{2}+(N-2)\epsilon}\delta_2^{-\frac{N-2}{2}-(N-2)\epsilon}}{[1 + (\frac{\delta_2}{\delta_1})^{1-2\epsilon}]^{\frac{N-2}{2}}} \\ &= \alpha_N \frac{\delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{-(N-2)\epsilon}}{[1 + (\frac{\delta_2}{\delta_1})^{1-2\epsilon}]^{\frac{N-2}{2}}} \\ &= \alpha_N\, \delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{-(N-2)\epsilon} - \alpha_N \frac{N-2}{2}\delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{1-N\epsilon} + o\left(\delta_1^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{1-N\epsilon}\right).\end{aligned}$$

Hence, for $x \in K$, such that $|x| = r_1$, we have

$$u_\lambda(x) = \alpha_N\, \delta_1^{-\frac{N-2}{2}}\left(1 - \left(\frac{\delta_2}{\delta_1}\right)^{-(N-2)\epsilon}\right) + o(\delta_1^{-\frac{N-2}{2}}) < 0$$

for all sufficiently small $\lambda > 0$. On the other hand, by similar computations (just changing the sign of $\epsilon$ in every term of the previous equations), for $x$ such that $|x| = r_2$ we have

$$u_\lambda(x) = \alpha_N\, \delta_1^{-\frac{N-2}{2}}\left(1 - \left(\frac{\delta_2}{\delta_1}\right)^{+(N-2)\epsilon}\right) + o(\delta_1^{-\frac{N-2}{2}}) > 0$$

for all sufficiently small $\lambda > 0$.

From Lemma 4.2.2 and since $u_\lambda$ is a continuos function we deduce that $Z_\lambda \subset A_{r_1,r_2}(0)$ for all sufficiently small $\lambda > 0$. □

## 4.3 Proof of the nonexistence result

We begin considering the case $N = 5, 6$ since the case $N = 4$ requires different estimates.

**Proof of Theorem 4.1.1 for N=5,6.** Arguing by contradiction let us assume that such a family of solutions exists and, without loss of generality set $\xi = 0$. Defining $r := \sqrt{\delta_1\delta_2}$, we apply the Pohozaev formula (4.2.2) to $u_\lambda$ in the ball $B_r = B_r(0)$. Since $u_\lambda$ is a solution of (4.1.1) we set $f(u) := \lambda u + |u|^{p-1}u$, where $p := 2^* - 1$, and hence, using the notation of

Proposition 4.2.3, we have $F(u) = \frac{\lambda}{2}u^2 + \frac{1}{p+1}|u|^{p+1}$. By elementary computations [1] (see the footnote) we get that the left-hand side of (4.2.2) reduces to

$$\lambda \int_{B_r} u_\lambda^2 \, dx.$$

For the right-hand side

$$\int_{\partial B_r} \left\{ \sum_{i=1}^N x_i \nu_i \left( F(u_\lambda) - \frac{1}{2}|\nabla u_\lambda|^2 \right) + \frac{\partial u_\lambda}{\partial \nu} \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} + \frac{N-2}{2} u_\lambda \frac{\partial u_\lambda}{\partial \nu} \right\} \, d\sigma,$$

since $\partial B_r$ is a sphere, we have $\nu_i(x) = \frac{x_i}{|x|}$ for all $x \in \partial B_r$, $i = 1, \dots, N$, and hence $\sum_{i=1}^N x_i \nu_i = |x|$. Furthermore since $\frac{\partial u_\lambda}{\partial \nu} = \nabla u_\lambda \cdot \frac{x}{|x|}$ and $\sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right) |x|$ we get that

$$\frac{\partial u_\lambda}{\partial \nu} \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right) \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right)^2 |x|,$$

$$u_\lambda \frac{\partial u_\lambda}{\partial \nu} = u_\lambda \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right).$$

Thus (4.2.2) rewrites as

$$\begin{aligned} &\lambda \int_{B_r} u_\lambda^2 \, dx \\ = \; &\int_{\partial B_r} \left\{ |x| \left( F(u_\lambda) - \frac{1}{2}|\nabla u_\lambda|^2 \right) + \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right)^2 |x| + \frac{N-2}{2} u_\lambda \left( \nabla u_\lambda \cdot \frac{x}{|x|} \right) \right\} \, d\sigma. \end{aligned} \tag{4.3.1}$$

We estimate the left-hand side of (4.3.1). Let us fix a compact subset $K \subset \Omega$; for $\lambda > 0$ sufficiently small we get that $B_r \subset K$. Thanks to Lemma 4.2.1 we have $\mathcal{P}\mathcal{U}_{\delta_j} = \mathcal{U}_{\delta_j} - \varphi_{\delta_j}$, where $\varphi_{\delta_j} = O\left( \delta_j^{\frac{N-2}{2}} \right)$, for $j = 1, 2$, and this estimate is uniform for $x \in K$, in particular

[1]

$$\begin{aligned} NF(u) - \frac{N-2}{2} u f(u) &= N \left( \frac{\lambda}{2} u^2 + \frac{1}{p+1} |u|^{p+1} \right) - \frac{N-2}{2} (\lambda u^2 + |u|^{p+1}) \\ &= \left( \frac{N}{2} - \frac{N-2}{2} \right) \lambda u^2 + \left( \frac{N}{p+1} - \frac{N-2}{2} \right) |u|^{p+1} \\ &= \lambda u^2. \end{aligned}$$

for $x \in B_r$. Thus, as $\lambda \to 0$, we get that

$$\begin{aligned}
\lambda \int_{B_r} u_\lambda^2 \, dx &= \lambda \int_{B_r} \left( \mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + o(\delta_1^{-\frac{N-2}{2}}) \right)^2 \, dx \\
&= \lambda \int_{B_r} \left( \mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} - \varphi_{\delta_1} + \varphi_{\delta_2} + o(\delta_1^{-\frac{N-2}{2}}) \right)^2 \, dx \\
&= \lambda \int_{B_r} \left( \mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + o(\delta_1^{-\frac{N-2}{2}}) \right)^2 \, dx \\
&= \lambda \int_{B_r} \left( \mathcal{U}_{\delta_1}^2 + \mathcal{U}_{\delta_2}^2 - 2\mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2} + o(\delta_1^{-\frac{N-2}{2}} \mathcal{U}_{\delta_1}) + o(\delta_1^{-\frac{N-2}{2}} \mathcal{U}_{\delta_2}) + o(\delta_1^{-\frac{N-2}{2}}) \right) \, dx \\
&= A + B + C + D + E + F.
\end{aligned} \tag{4.3.2}$$

We estimate every term of the previous decomposition.

$$\begin{aligned}
A &= \lambda \int_{B_r} \alpha_N^2 \frac{\delta_1^{N-2}}{(\delta_1^2 + |x|^2)^{N-2}} \, dx &&= \alpha_N^2 \lambda \int_{B_r} \frac{\delta_1^{-(N-2)}}{(1 + |x/\delta_1|^2)^{N-2}} \, dx \\
&= \alpha_N^2 \lambda \delta_1^2 \int_{B_r/\delta_1} \frac{1}{(1+|y|^2)^{N-2}} \, dy &&\leq \alpha_N^2 \lambda \delta_1^2 |B_r/\delta_1| \\
&= c_N \lambda \delta_1^2 \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N}{2}},
\end{aligned}$$

where we have set $c_N := \alpha_N^2 \frac{\omega_N}{N}$, $\omega_N$ is the measure of the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$.

$$\begin{aligned}
B &= \lambda \int_{B_r} \alpha_N^2 \frac{\delta_2^{N-2}}{(\delta_2^2 + |x|^2)^{N-2}} \, dx = \alpha_N^2 \lambda \int_{B_r} \frac{\delta_2^{-(N-2)}}{(1 + |x/\delta_2|^2)^{N-2}} \, dx \\
&= \alpha_N^2 \lambda \delta_2^2 \int_{B_r/\delta_2} \frac{1}{(1+|y|^2)^{N-2}} \, dy \\
&= \alpha_N^2 \lambda \delta_2^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy + O\left( \lambda \delta_2^2 \int_{\left(\frac{\delta_1}{\delta_2}\right)^{\frac{1}{2}}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}} \, dr \right) \\
&= a_1 \lambda \delta_2^2 + O\left( \lambda \delta_2^2 \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-4}{2}} \right),
\end{aligned}$$

where we have set $a_1 := \alpha_N^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy$. We point out that since $N = 5$ or $N = 6$ the function $\frac{1}{(1+|y|^2)^{N-2}} \in L^1(\mathbb{R}^N)$ while this is not true when $N = 4$.

$$\begin{aligned}
|C| &= \lambda\, \alpha_N^2 \int_{B_r} \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}}\, dx \\
&= \lambda\, \alpha_N^2 \int_{B_r/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+\delta_1^2|y|^2)^{\frac{N-2}{2}}}\, dy \\
&= \lambda\, \alpha_N^2 \int_{B_r/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{\left(\left(\frac{\delta_2}{\delta_1}\right)^2+|y|^2\right)^{\frac{N-2}{2}}}\, dy \\
&\le \lambda\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{B_r/\delta_1} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&= O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_0^{\left(\frac{\delta_2}{\delta_1}\right)^{1/2}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}} r^{N-2}}\, dr\right) \\
&= O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} \delta_1^2\right).
\end{aligned}$$

$$\begin{aligned}
|D| &= o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}}}\, dx\right) \\
&\le o\left(\lambda \int_{B_r} \delta_1^{-(N-2)}\, dx\right) \\
&= o\left(\lambda \delta_1^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).
\end{aligned}$$

$$\begin{aligned}
|E| &= o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}}\, dx\right) \\
&\le o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_2^{\frac{N-2}{2}}}{|x|^{N-2}}\, dx\right) \\
&= o\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).
\end{aligned}$$

$$\begin{aligned}
|F| &= o\left(\lambda \delta_1^{-\frac{N-2}{2}} |B_r|\right) \\
&= o\left(\lambda\ \delta_1\ \delta_2^{\frac{N}{2}}\right).
\end{aligned}$$

Now we estimate the right-hand side of (4.3.1). Remembering that $F(u_\lambda) = \frac{\lambda}{2}u_\lambda^2 + \frac{1}{p+1}|u_\lambda|^{p+1}$ we get that the first term is equal to

$$\int_{\partial B_r} |x| \left(\frac{\lambda}{2}u_\lambda^2 + \frac{1}{p+1}|u_\lambda|^{p+1} - \frac{1}{2}|\nabla u_\lambda|^2\right)\, d\sigma.$$

We observe that by definition of $r$ it is immediate to see that

$$\mathcal{U}_{\delta_1}(x) = \mathcal{U}_{\delta_2}(x),$$

for all $x \in \partial B_r$, and hence we have

$$\begin{aligned}
\int_{\partial B_r} \frac{\lambda}{2} u_\lambda^2 \, |x| \, d\sigma &= \frac{\lambda}{2} \int_{\partial B_r} \left( \mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + o\left( \delta_1^{-\frac{N-2}{2}} \right) \right)^2 \, |x| \, d\sigma \\
&= \frac{\lambda}{2} \int_{\partial B_r} \left[ o\left( \delta_1^{-\frac{N-2}{2}} \right) \right]^2 \, |x| \, d\sigma \\
&= o\left( \lambda \delta_1^{-(N-2)} \int_{\partial B_r} |x| \, d\sigma \right) \\
&= o\left( \lambda \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N}{2}} \delta_1^2 \right).
\end{aligned}$$

As in the previous case we have

$$\begin{aligned}
\frac{1}{p+1} \int_{\partial B_r} |u_\lambda|^{p+1} |x| \, d\sigma &= \frac{1}{p+1} \int_{\partial B_r} |\mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + o(\delta_1^{-\frac{N-2}{2}})|^{p+1} \, |x| \, d\sigma \\
&= \frac{1}{p+1} \int_{\partial B_r} |o(\delta_1^{-\frac{N-2}{2}})|^{p+1} \, |x| \, d\sigma \\
&= o\left( \delta_1^{-N} \int_{\partial B_r} |x| \, d\sigma \right) \\
&= o\left( \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N}{2}} \right).
\end{aligned}$$

To complete the estimate of the first term it remains to analyze

$$-\frac{1}{2} \int_{\partial B_r} |\nabla u_\lambda|^2 |x| \, d\sigma.$$

As before, writing $\mathcal{P}\mathcal{U}_{\delta_j} = \mathcal{U}_{\delta_j} - \varphi_{\delta_j}$ for $j = 1, 2$ we have

$$|\nabla u_\lambda|^2 = |\nabla \mathcal{U}_{\delta_1} - \nabla \mathcal{U}_{\delta_2} - \nabla \varphi_{\delta_1} + \nabla \varphi_{\delta_2} + \nabla w_\lambda|^2 = |\nabla \mathcal{U}_{\delta_1} - \nabla \mathcal{U}_{\delta_2} + \nabla \Phi_\lambda|^2,$$

where we have set $\Phi_\lambda := -\varphi_{\delta_1} + \varphi_{\delta_2} + w_\lambda$. Hence, we get that

$$\begin{aligned}
& -\frac{1}{2} \int_{\partial B_r} |\nabla u_\lambda|^2 |x| \, d\sigma \\
= \; & -\frac{1}{2} \int_{\partial B_r} |\nabla \mathcal{U}_{\delta_1}|^2 \, |x| \, d\sigma - \frac{1}{2} \int_{\partial B_r} |\nabla \mathcal{U}_{\delta_2}|^2 \, |x| \, d\sigma + \int_{\partial B_r} \nabla \mathcal{U}_{\delta_1} \cdot \nabla \mathcal{U}_{\delta_2} \, |x| \, d\sigma \\
& - \int_{\partial B_r} \nabla \mathcal{U}_{\delta_1} \cdot \nabla \Phi_\lambda \, |x| \, d\sigma + \int_{\partial B_r} \nabla \mathcal{U}_{\delta_2} \cdot \nabla \Phi_\lambda \, |x| \, d\sigma - \frac{1}{2} \int_{\partial B_r} |\nabla \Phi_\lambda|^2 \, |x| \, d\sigma \\
= \; & A_1 + B_1 + C_1 + D_1 + E_1 + F_1.
\end{aligned} \tag{4.3.3}$$

By elementary computations, for all $i = 1, \dots, N$, $j = 1, 2$ we have:

$$\frac{\partial \mathcal{U}_{\delta_j}}{\partial x_i}(x) = -\alpha_N (N-2)\delta_j^{\frac{N-2}{2}} \frac{x_i}{(\delta_j^2 + |x|^2)^{\frac{N}{2}}},$$

$$|\nabla \mathcal{U}_{\delta_j}|^2 = \alpha_N^2 (N-2)^2 \delta_j^{N-2} \frac{|x|^2}{(\delta_j^2 + |x|^2)^N}. \tag{4.3.4}$$

Thus, we get that

$$\begin{aligned} A_1 &= -\alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_1^{-(N+2)}}{\left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^N} \int_{\partial B_r} |x|^3 \, d\sigma \\ &= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \frac{\delta_1^{-(N+2)}}{\left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^N} \delta_1^{\frac{N+2}{2}} \delta_2^{\frac{N+2}{2}} \\ &= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+4}{2}}\right). \end{aligned}$$

$$\begin{aligned} B_1 &= -\alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_2^{N-2} \delta_1^{-N} \delta_2^{-N}}{\left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^N} \int_{\partial B_r} |x|^3 \, d\sigma \\ &= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right). \end{aligned}$$

$$\begin{aligned} C_1 &= \alpha_N^2 (N-2)^2 \frac{\delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}} \delta_1^{-N} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{\left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}} \left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}} \int_{\partial B_r} |x|^3 \, d\sigma \\ &= \alpha_N^2 (N-2)^2 \omega_N \frac{\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}}{\left[1 + \left(\frac{\delta_2}{\delta_1}\right)\right]^N} \\ &= \alpha_N^2 (N-2)^2 \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}}\right). \end{aligned}$$

Taking into account the assumptions on the remainder term $w_\lambda$ and thanks to Lemma

4.2.1 we have $|\nabla\Phi_\lambda| = o(\delta_1^{-\frac{N}{2}})$, uniformly on $\partial B_r$. Thus we have the following:

$$\begin{aligned} |D_1| &\leq \int_{\partial B_r} |\nabla\mathcal{U}_{\delta_1}||\nabla\Phi_\lambda||x|\ d\sigma \\ &= o\left(\frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+\delta_1\delta_2)^{\frac{N}{2}}}\delta_1^{-\frac{N}{2}}\int_{\partial B_r}|x|^2\ d\sigma\right) \\ &= o\left(\frac{\delta_1^{\frac{N-2}{2}}\delta_1^{-N}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}}\delta_1^{-\frac{N}{2}}\int_{\partial B_r}|x|^2\ d\sigma\right) \\ &= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+1}{2}}\right). \end{aligned}$$

$$\begin{aligned} |E_1| &\leq \int_{\partial B_r} |\nabla\mathcal{U}_{\delta_2}||\nabla\Phi_\lambda||x|\ d\sigma \\ &= o\left(\frac{\delta_2^{\frac{N-2}{2}}\delta_1^{-\frac{N}{2}}\delta_2^{-\frac{N}{2}}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}}\delta_1^{-\frac{N}{2}}\int_{\partial B_r}|x|^2\ d\sigma\right) \\ &= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-1}{2}}\right). \end{aligned}$$

And finally the last term of (4.3.3) is trivial:

$$|F_1| = o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).$$

Now we analyze the term

$$\int_{\partial B_r}\left(\nabla u_\lambda\cdot\frac{x}{|x|}\right)^2|x|\ d\sigma. \tag{4.3.5}$$

As before we write $u_\lambda = \mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + \Phi_\lambda$ and we have

$$\begin{aligned} \left(\nabla u_\lambda\cdot\frac{x}{|x|}\right)^2|x| &= \left(\nabla\mathcal{U}_{\delta_1}\cdot\frac{x}{|x|}\right)^2|x| + \left(\nabla\mathcal{U}_{\delta_2}\cdot\frac{x}{|x|}\right)^2|x| - 2\left(\nabla\mathcal{U}_{\delta_1}\cdot\frac{x}{|x|}\right)\left(\nabla\mathcal{U}_{\delta_2}\cdot\frac{x}{|x|}\right)|x| \\ &\quad +2\left(\nabla\mathcal{U}_{\delta_1}\cdot\frac{x}{|x|}\right)\left(\nabla\Phi_\lambda\cdot\frac{x}{|x|}\right)|x| - 2\left(\nabla\mathcal{U}_{\delta_2}\cdot\frac{x}{|x|}\right)\left(\nabla\Phi_\lambda\cdot\frac{x}{|x|}\right)|x| \\ &\quad + \left(\nabla\Phi_\lambda\cdot\frac{x}{|x|}\right)^2|x| \end{aligned} \tag{4.3.6}$$

By elementary computations we see that for $j = 1, 2$

$$\begin{aligned} \left(\nabla\mathcal{U}_{\delta_j}\cdot\frac{x}{|x|}\right)^2|x| &= |\nabla\mathcal{U}_{\delta_j}|^2\ |x|, \\ -2\left(\nabla\mathcal{U}_{\delta_1}\cdot\frac{x}{|x|}\right)\left(\nabla\mathcal{U}_{\delta_2}\cdot\frac{x}{|x|}\right)\ |x| &= -2(\nabla\mathcal{U}_{\delta_1}\cdot\nabla\mathcal{U}_{\delta_2})\ |x|, \end{aligned}$$

and for the remaining terms we have

$$\begin{aligned}\left| \pm 2 \left(\nabla \mathcal{U}_{\delta_j} \cdot \frac{x}{|x|}\right)\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)|x|\right| &\leq 2|\nabla \mathcal{U}_{\delta_j}||\nabla \Phi_\lambda||x|,\\ \left|\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)^2 |x|\right| &\leq |\nabla \Phi_\lambda|^2\,|x|.\end{aligned}$$

Thus, in order to estimate (4.3.5) it suffices to apply the estimates of the previous case, and hence we get that

$$\int_{\partial B_r}\left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)^2 |x|\, d\sigma = \alpha_N^2 (N-2)^2 \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right).$$

To complete our analysis of (4.3.1) it remains only to study the term

$$\frac{N-2}{2}\int_{\partial B_r} u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)\, d\sigma.$$

$$\begin{aligned}&\frac{N-2}{2}\int_{\partial B_r} u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)\, d\sigma\\ =\;& \frac{N-2}{2}\int_{\partial B_r} (\mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + \Phi_\lambda)\left[(\nabla \mathcal{U}_{\delta_1} - \nabla \mathcal{U}_{\delta_2} + \nabla \Phi_\lambda)\cdot \frac{x}{|x|}\right] d\sigma\\ =\;& \frac{N-2}{2}\int_{\partial B_r} \Phi_\lambda \left(\nabla \mathcal{U}_{\delta_1} \cdot \frac{x}{|x|}\right) d\sigma - \frac{N-2}{2}\int_{\partial B_r} \Phi_\lambda \left(\nabla \mathcal{U}_{\delta_2} \cdot \frac{x}{|x|}\right) d\sigma \qquad (4.3.7)\\ &+\frac{N-2}{2}\int_{\partial B_r} \Phi_\lambda \left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right) d\sigma\\ =\;& A_2 + B_2 + C_2.\end{aligned}$$

$$\begin{aligned}|A_2| &\leq \alpha_N^2 \frac{(N-2)^2}{2}\frac{\delta_1^{\frac{N-2}{2}}\delta_1^{-N}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}}\int_{\partial B_r} |\Phi_\lambda|\,|x|\, d\sigma\\ &= o\left(\frac{\delta_1^{\frac{N-2}{2}}\delta_1^{-N}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}}\int_{\partial B_r} \delta_1^{-\frac{N-2}{2}}\,|x|\, d\sigma\right)\\ &= o\left(\frac{\delta_1^{-N}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}}\delta_1^{\frac{N}{2}}\delta_2^{\frac{N}{2}}\right)\\ &= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).\end{aligned}$$

$$\begin{aligned} |B_2| &\le \alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}} \int_{\partial B_r} |\Phi_\lambda|\, |x|\, d\sigma \\ &= o\left( \frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)\right]^{\frac{N}{2}}} \int_{\partial B_r} \delta_1^{-\frac{N-2}{2}} |x|\, d\sigma \right) \\ &= o\left( \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \right). \\ |C_2| &\le \frac{(N-2)}{2} \int_{\partial B_r} |\Phi_\lambda||\nabla \Phi_\lambda|\, d\sigma \\ &= o\left( \delta_1^{-\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_1^{\frac{N-1}{2}} \delta_2^{\frac{N-1}{2}} \right) \\ &= o\left( \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-1}{2}} \right). \end{aligned}$$

Summing up all the estimates, from (4.2.2), for all sufficiently small $\lambda > 0$, we deduce the following equation

$$a_1 \lambda \delta_2^2 + o\left(\lambda \delta_2^2\right) = \alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left( \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \right). \tag{4.3.8}$$

From (4.3.8) we deduce that

$$a_1 \lambda \delta_1^{\frac{N-2}{2}} (1+o(1)) = \alpha_N^2 \frac{(N-2)^2}{2} \omega_N \delta_2^{\frac{N-6}{2}} (1+o(1)), \tag{4.3.9}$$

for all sufficiently small $\lambda > 0$. Since $N = 5, 6$ it is clear that (4.3.9) is contradictory, in fact, passing to the limit as $\lambda \to 0$, the left-hand side goes to zero while the right-hand side goes to a constant, when $N = 6$ and diverges to $+\infty$ when $N = 5$. The proof is complete. □

Now we turn to the case $N = 4$

**Proof of Theorem 4.1.1 for N=4.** Again, without loss of generality we assume that $\xi = 0$. We repeat the scheme of the proof for the previous case, but some modification is needed. In fact, since $N = 4$, we have to change the estimate of the term $B$ in (4.3.2):

$$\begin{aligned} B_* &= \lambda \int_{B_r} \alpha_4^2 \frac{\delta_2^2}{(\delta_2^2 + |x|^2)^2}\, dx = \alpha_4^2 \lambda \int_{B_r/\delta_2} \frac{\delta_2^{-2}}{(1+|y|^2)^2} \delta_2^4\, dy \\ &= \alpha_4^2 \lambda \delta_2^2 \int_{B_r/\delta_2} \frac{1}{(1+|y|^2)^2}\, dy = \alpha_4^2 \omega_4 \lambda \delta_2^2 \int_0^{\left(\frac{\delta_1}{\delta_2}\right)} \frac{r^3}{(1+r^2)^2}\, dr \end{aligned}$$

It's elementary to see that

$$\int_0^{\left(\frac{\delta_1}{\delta_2}\right)} \frac{r^3}{(1+r^2)^2}\, dr = O\left( \log\left(\frac{\delta_1}{\delta_2}\right) \right),$$

and hence we have that

$$B_* = O\left(\lambda\delta_2^2 \log\left(\frac{\delta_1}{\delta_2}\right)\right). \tag{4.3.10}$$

Thus, summing up (4.3.10) with the other estimates made in the previous case (in which we take $N = 4$), from (4.2.2), we deduce the following asymptotic relation

$$O\left(\lambda\delta_2^2 \log\left(\frac{\delta_1}{\delta_2}\right)\right) + o\left(\lambda\delta_2^2 \log\left(\frac{\delta_1}{\delta_2}\right)\right) = 2\alpha_4^2\omega_4\left(\frac{\delta_2}{\delta_1}\right) + o\left(\frac{\delta_2}{\delta_1}\right). \tag{4.3.11}$$

It is clear that (4.3.11) gives a contradiction. In fact, dividing each side of (4.3.11) by $\left(\frac{\delta_2}{\delta_1}\right)$ we have

$$O\left(\lambda\delta_1\delta_2 \log\left(\frac{\delta_1}{\delta_2}\right)\right) + o\left(\lambda\delta_1\delta_2 \log\left(\frac{\delta_1}{\delta_2}\right)\right) = 2\alpha_4^2\omega_4 + o(1). \tag{4.3.12}$$

Passing to the limit as $\lambda \to 0$ in (4.3.12), taking into account that $\delta_2 = o(\delta_1)$, we deduce that $0 = 2\alpha_4^2\omega_4$ which is a contradiction. □

**Remark 4.3.1.** *In [13] sign-changing solutions $u_\lambda$ of* (4.1.1) *with low energy were studied, namely solutions such that*

$$\int_\Omega |\nabla u_\lambda|^2 \, dx \to 2S^{N/2}.$$

*For this kind of solutions it is not difficult to show (see [13], Theorem 1.1) that there exist two points $a_1 = a_1(\lambda)$, $a_2 = a_2(\lambda)$ in $\Omega$ (one of them is the global maximum point of $|u_\lambda|$) and two positive real numbers $\delta_1 = \delta_1(\lambda)$, $\delta_2 = \delta_2(\lambda)$, such that for $N \geq 4$, as $\lambda \to 0$, we have*

$$\|u_\lambda - \mathcal{P}\mathcal{U}_{a_1,\delta_1} + \mathcal{P}\mathcal{U}_{a_2,\delta_2}\| \to 0, \quad \delta_i^{-1} d(a_i, \partial\Omega) \to +\infty, \text{ for } i = 1, 2,$$

*where $d(a_i, \partial\Omega)$ is the euclidean distance between $a_i$ and the boundary of $\Omega$. Hence these solutions are of the form* (4.1.2) *but with possibly different concentration points. In [13], assuming that the concentration speeds of $u_\lambda^+$ and $u_\lambda^-$ were comparable, it was proved that the positive and the negative part of $u_\lambda$ had to concentrate in two different points.*

*Since here we assume that the concentration speeds are different, our result also completes the study made in [13].*

## 4.4 About the estimate on the $C^1$-norm of $w_\lambda$

Here we show that the hypotheses of Theorem 4.1.1 on the $C^1$-norm of the remainder term $w_\lambda$ are almost necessary. Indeed we have:

**Theorem 4.4.1.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ with smooth boundary, $N \geq 4$, and let $\xi \in \Omega$. Let $u_\lambda$ a solution of (4.1.1) of the form*

$$u_\lambda = \mathcal{P}\mathcal{U}_{\xi,\delta_1} - \mathcal{P}\mathcal{U}_{\xi,\delta_2} + w_\lambda,$$

*with $\delta_2 = o(\delta_1)$ as $\lambda \to 0$. Assume that the remainder term $w_\lambda$ is uniformly bounded with respect to $\lambda$ in compact subsets of $\Omega$. Then for any open subset $\Omega'' \subset\subset \Omega$ such that $\xi \in \Omega''$ and for all sufficiently small $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, N, \Omega'')$ such that*

$$\|w_\lambda\|_{C^1(\bar{\Omega}'')} \leq C\delta_1^{-\frac{N-2}{2}}\delta_2^{-1+O(\epsilon)},$$

*for all sufficiently small $\lambda > 0$.*

*Proof.* Without loss of generality we assume that $\xi = 0$. By definition $w_\lambda$ satisfies the following:

$$\begin{cases} -\Delta w_\lambda = \lambda w_\lambda + \lambda(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) + \mathcal{U}_{\delta_2}^p - \mathcal{U}_{\delta_1}^p + |u_\lambda|^{2^*-2}u_\lambda & \text{in } \Omega \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4.1}$$

Let us set $f_\lambda := \lambda w_\lambda + \lambda(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) + \mathcal{U}_{\delta_2}^p - \mathcal{U}_{\delta_1}^p + |u_\lambda|^{2^*-2}u_\lambda$. Since $w_\lambda$ and $u_\lambda$ are smooth, applying the Calderón-Zygmund inequality we deduce that for any $p \in (1,\infty)$, for any $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ it holds:

$$\|w_\lambda\|_{2,p,\Omega''} \leq C(|w_\lambda|_{p,\Omega'} + |f_\lambda|_{p,\Omega'}), \tag{4.4.2}$$

where $C$ depends on $\Omega'$, $N$, $p$, $\Omega''$. Thanks to the Sobolev imbedding theorem, for any $\epsilon > 0$, if $p = N + \epsilon$ we have that $W^{2,p}(\Omega)$ is continuously imbedded in $C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \frac{N}{N+\epsilon}$. Let us consider two open subsets $\Omega''$, $\Omega'$ of $\Omega$ such that $0 \in \Omega''$ and $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Thanks to (4.4.1) and (4.4.2), in order to estimate $\|w_\lambda\|_{C^1(\bar{\Omega}'')}$ we have to estimate the following quantities: $|w_\lambda|_{N+\epsilon,\Omega'}$, $|f_\lambda|_{N+\epsilon,\Omega'}$.

Thanks to the assumptions on $w_\lambda$ we deduce immediately that $|w_\lambda|_{N+\epsilon,\Omega'} = O(1)$, uniformly with respect to $\lambda$. For the other term we argue as it follows: we set $g(s) := |s|^{2^*-2}s$, $\Phi_\lambda := w_\lambda + \varphi_2 - \varphi_1$, where $\varphi_j := \mathcal{U}_{\delta_j} - \mathcal{P}\mathcal{U}_{\delta_j}$, for $j = 1,2$, and we write

$$\begin{aligned} & |f_\lambda|_{N+\epsilon,\Omega'} \\ \leq\ & \lambda|w_\lambda|_{N+\epsilon,\Omega'} + \lambda|\mathcal{P}\mathcal{U}_{\delta_1}|_{N+\epsilon,\Omega'} + \lambda|\mathcal{P}\mathcal{U}_{\delta_2}|_{N+\epsilon,\Omega'} + |\mathcal{U}_{\delta_1}^p|_{N+\epsilon,\Omega'} \\ & + |g(\mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + \Phi_\lambda) - g(-\mathcal{U}_{\delta_2})|_{N+\epsilon,\Omega'} \\ \leq\ & \lambda|w_\lambda|_{N+\epsilon,\Omega'} + \lambda|\mathcal{P}\mathcal{U}_{\delta_1}|_{N+\epsilon,\Omega'} + \lambda|\mathcal{P}\mathcal{U}_{\delta_2}|_{N+\epsilon,\Omega'} + |\mathcal{U}_{\delta_1}^p|_{N+\epsilon,\Omega'} \\ & + |g(\mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + \Phi_\lambda) - g(-\mathcal{U}_{\delta_2}) - g'(-\mathcal{U}_{\delta_2})(\mathcal{U}_{\delta_1} + \Phi_\lambda)|_{N+\epsilon,\Omega'} + |g'(-\mathcal{U}_{\delta_2})(\mathcal{U}_{\delta_1} + \Phi_\lambda)|_{N+\epsilon,\Omega'} \\ =\ & A + B + C + D + E + F. \end{aligned}$$

The term $A$ has been estimated before, and hence $\lambda|w_\lambda|_{N+\epsilon,\Omega'} = O(\lambda)$. For $B$ and $C$ we use the following estimates:

$$\begin{aligned} & \int_{\Omega'} \alpha_N^{N+\epsilon} \frac{\delta_j^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_j^2 + |x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx = \alpha_N^{N+\epsilon} \int_{\Omega'/\delta_j} \frac{\delta_j^{-\frac{N-2}{2}(N+\epsilon)+N}}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy \\ =\ & \alpha_N^{N+\epsilon} \delta_j^{\frac{4-N}{2}N - \epsilon\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy \\ & + O\left( \delta_j^{\frac{4-N}{2}N - \epsilon\frac{N-2}{2}} \int_{1/\delta_j}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dr \right). \end{aligned}$$

Thus, for all $\epsilon > 0$ sufficiently small we have

$$\begin{aligned} |\mathcal{P}\mathcal{U}_\delta|_{N+\epsilon,\Omega'} \ \leq\ & \left( \int_{\Omega'} \alpha_N^{N+\epsilon} \frac{\delta_j^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_j^2 + |x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx \right)^{\frac{1}{N+\epsilon}} \\ =\ & \alpha_N \delta_j^{\frac{4-N}{2}+O(\epsilon)} \left( \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy \right)^{\frac{1}{N+\epsilon}} + o\left( \delta_j^{\frac{4-N}{2}+O(\epsilon)} \right). \end{aligned}$$

From this we deduce that $B = O(\lambda\delta_1^{\frac{4-N}{2}+O(\epsilon)})$, $C = O(\lambda\delta_2^{\frac{4-N}{2}+O(\epsilon)})$. Concerning the term $D$, with similar computations we see that

$$\begin{aligned} |\mathcal{P}\mathcal{U}_{\delta_1}^p|_{N+\epsilon,\Omega'} &\leq \left(\int_{\Omega'} \alpha_N^{\frac{N+2}{2}(N+\epsilon)} \frac{\delta_1^{\frac{N+2}{2}(N+\epsilon)}}{(\delta_1^2+|x|^2)^{\frac{N+2}{2}(N+\epsilon)}}\, dx\right)^{\frac{1}{N+\epsilon}} \\ &= \alpha_N^p \delta_1^{-\frac{N}{2}+O(\epsilon)} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}(N+\epsilon)}}\, dy\right)^{\frac{1}{N+\epsilon}} + o\left(\delta_1^{-\frac{N}{2}+O(\epsilon)}\right), \end{aligned}$$

and hence $D = O(\delta_1^{-\frac{N}{2}+O(\epsilon)})$. In order to estimate $E$ we remember that by elementary inequalities we have $|g(u+v) - g(u) - g'(u)v| \leq c|v|^p$, for all $u, v \in \mathbb{R}$, for some constant depending only on $p$, and hence we get that

$$E \leq c||\Phi_\lambda|^p|_{N+\epsilon,\Omega'} = O(1).$$

For the last term we have the following:

$$\begin{aligned} |g'(\mathcal{U}_{\delta_2})\mathcal{U}_{\delta_1}|_{N+\epsilon,\Omega'}^{N+\epsilon} &= p^{N+\epsilon} \int_{\Omega'} \alpha_N^{\frac{N+2}{2}(N+\epsilon)} \frac{\delta_2^{\frac{4}{N-2}\frac{N-2}{2}(N+\epsilon)}}{(\delta_2^2+|x|^2)^{\frac{4}{N-2}\frac{N-2}{2}(N+\epsilon)}} \frac{\delta_1^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx \\ &= p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)} \int_{\Omega'} \frac{\delta_2^{-2(N+\epsilon)}}{(1+|x/\delta_2|^2)^{2(N+\epsilon)}} \frac{\delta_1^{-\frac{N-2}{2}(N+\epsilon)}}{(1+|x/\delta_1|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx \\ &\leq p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)} \delta_1^{-\frac{N-2}{2}(N+\epsilon)} \delta_2^{-2(N+\epsilon)+N} \int_{\Omega'/\delta_2} \frac{1}{(1+|x/\delta_2|^2)^{2(N+\epsilon)}} dy \\ &\leq p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)} \delta_1^{-\frac{N-2}{2}(N+\epsilon)} \delta_2^{-N-2\epsilon} \int_{\Omega'/\delta_2} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}} dy \\ &= p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)} \delta_1^{-\frac{N-2}{2}(N+\epsilon)} \delta_2^{-N-2\epsilon} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}} dy \\ &\quad + O\left(\delta_1^{-\frac{N-2}{2}(N+\epsilon)} \delta_2^{-N-2\epsilon} \int_{1/\delta_2}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{2(N+\epsilon)}}\right). \end{aligned}$$

Hence we get that

$$|g'(\mathcal{U}_{\delta_2})\mathcal{U}_{\delta_1}|_{N+\epsilon,\Omega'} \leq p\alpha_N^{\frac{N+2}{2}} \delta_1^{-\frac{N-2}{2}} \delta_2^{-1+O(\epsilon)} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}} dy\right)^{\frac{1}{N+\epsilon}} + o\left(\delta_1^{-\frac{N-2}{2}} \delta_2^{-1+O(\epsilon)}\right).$$

By the same computations we see that

$$|g'(\mathcal{U}_{\delta_2})\Phi_\lambda|_{N+\epsilon,\Omega'} = O\left(\delta_2^{-1+O(\epsilon)}\right).$$

Thus, we get that

$$|F| \leq c(N,p)\delta_1^{-\frac{N-2}{2}} \delta_2^{-1+O(\epsilon)}.$$

Summing up all these estimates, from (4.4.2) and Sobolev imbedding theorem we deduce that

$$\|w_\lambda\|_{C^1(\bar{\Omega}'')} \leq C\delta_1^{-\frac{N-2}{2}} \delta_2^{-1+O(\epsilon)},$$

where $C$ is a positive constant depending on $\epsilon, N, \Omega'', \Omega'$. □

A straightforward consequence of the previous theorem is the following result:

**Corollary 4.4.2.** *Under the assumptions of Theorem 4.4.1, for all sufficiently small $\epsilon > 0$ we have*

$$\int_{\partial B_r} |\nabla w_\lambda|^2 |x| \, d\sigma \leq C(\epsilon, N) \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-4}{2}} \delta_2^{O(\epsilon)},$$

*for all sufficiently small $\lambda > 0$, where $B_r$ is the ball centered at $\xi$ having radius $r = \sqrt{\delta_1 \delta_2}$.*

## 4.5 Concentration speeds for $N \geq 7$

We consider as in the previous sections sign-changing solutions of Problem 4.1.1 which are of the form $u_\lambda = \mathcal{P}\mathcal{U}_{\delta_1,\xi} - \mathcal{P}\mathcal{U}_{\delta_2,\xi} + w_\lambda$, with $\delta_1 = \delta_1(\lambda)$, $\delta_2 = \delta_2(\lambda)$ satisfying $\delta_2 = o(\delta_1)$ as $\lambda \to 0$. In addition we assume that $\delta_i$, for $i = 1, 2$, is of the form

$$\delta_i = d_i \lambda^{\alpha_i}, \tag{4.5.1}$$

where $d_i = d_i(\lambda)$ is a strictly positive function such that $d_i \to \bar{d}_i > 0$, as $\lambda \to 0$, and the exponents $\alpha_i$ satisfy $0 < \alpha_1 < \alpha_2$. Following the ideas contained in [51] and applying the asymptotic relation (4.3.8), found in the proof of Theorem 4.1.1, we determine precisely the exponents $\alpha_1$, $\alpha_2$ in the case $N \geq 7$. We observe that these speeds are exactly the same used in [40] to construct solutions of (4.1.1) of the form (4.1.2).

**Theorem 4.5.1.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ with smooth boundary, $N \geq 7$, and let $\xi \in \Omega$. Let $u_\lambda$ a solution of (4.1.1) such that $u_\lambda$ is of the form $u_\lambda = \mathcal{P}\mathcal{U}_{\delta_1,\xi} - \mathcal{P}\mathcal{U}_{\delta_2,\xi} + w_\lambda$, where $\delta_i$, for $i = 1, 2$, is of the form (4.5.1) with $\alpha_2 > \alpha_1 > 0$, $w_\lambda \in V_{\lambda,\xi}$, $V_{\lambda,\xi}$ is the subspace of $H_0^1(\Omega)$:*

$$V_{\lambda,\xi} := \left\{ v \in H_0^1(\Omega); \quad (v, \mathcal{P}\mathcal{U}_{\delta_i,\xi})_{H_0^1(\Omega)} = \left( v, \mathcal{P}\frac{\partial \mathcal{U}_{\delta_i,\xi}}{\partial \delta_i} \right)_{H_0^1(\Omega)} = 0, \quad i = 1, 2 \right\}.$$

*Moreover assume that $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$, $|\nabla w_\lambda| = o(\delta_1^{-\frac{N}{2}})$, uniformly in compact subsets of $\Omega$. Then $\alpha_1 = \frac{1}{N-4}$, $\alpha_2 = \frac{3N-10}{(N-4)(N-6)}$.*

In order to prove Theorem 4.5.1 we need some preliminary lemmas. Without loss of generality we assume that $\xi = 0$. The first one is the following:

**Lemma 4.5.2.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ with smooth boundary and assume that $0 \in \Omega$, $N \geq 5$. Then, as $\delta \to 0$, we have*

$$\int_{\partial\Omega} \left( \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial \nu} \right)^2 (x \cdot \nu) \, d\sigma = a_2 \delta^{N-2} + o\left(\delta^{N-2}\right),$$

*for some positive real number $a_2$, depending only on $N$ and $\Omega$.*

*Proof.* We multiply the equation $-\Delta \mathcal{P}\mathcal{U}_\delta = \mathcal{U}_\delta^p$ by $\sum_{i=1}^N x_i \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial x_i}$ and we integrate on $\Omega$. On one hand, integrating by parts we obtain

$$\begin{aligned} &\int_\Omega -\Delta \mathcal{P}\mathcal{U}_\delta \sum_{i=1}^N x_i \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial x_i} \, dx \\ = \; &\left(1 - \frac{N}{2}\right) \int_\Omega |\nabla \mathcal{P}\mathcal{U}_\delta|^2 \, dx - \frac{1}{2} \int_{\partial\Omega} \left( \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial \nu} \right)^2 (x \cdot \nu) \, d\sigma \qquad (4.5.2) \\ = \; &\left(1 - \frac{N}{2}\right) \int_\Omega \mathcal{U}_\delta^p \mathcal{P}\mathcal{U}_\delta \, dx - \frac{1}{2} \int_{\partial\Omega} \left( \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial \nu} \right)^2 (x \cdot \nu) \, d\sigma. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \mathcal{U}_\delta^p \sum_{i=1}^N x_i \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial x_i}\, dx &= -\sum_{i=1}^N \int_\Omega \left( \mathcal{U}_\delta^p + p x_i \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial x_i}\right) \mathcal{P}\mathcal{U}_\delta\, dx \\ &= -N \int_\Omega \mathcal{U}_\delta^p \mathcal{P}\mathcal{U}_\delta\, dx - p\sum_{i=1}^N \int_\Omega x_i \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial x_i} \mathcal{P}\mathcal{U}_\delta\, dx.\end{aligned} \tag{4.5.3}$$

By elementary computations we see that

$$-\sum_{i=1}^N x_i \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial x_i} = \frac{N-2}{2}\mathcal{U}_\delta + \delta \frac{\partial \mathcal{U}_\delta}{\partial \delta},$$

and hence from (4.5.3) we get that

$$\begin{aligned}&\int_\Omega \mathcal{U}_\delta^p \sum_{i=1}^N x_i \frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial x_i}\, dx \\ =\ & -N \int_\Omega \mathcal{U}_\delta^p \mathcal{P}\mathcal{U}_\delta\, dx + p\frac{N-2}{2}\int_\Omega \mathcal{U}_\delta^p \mathcal{P}\mathcal{U}_\delta\, dx + p\delta \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} \mathcal{P}\mathcal{U}_\delta\, dx \\ =\ & \left(1-\frac{N}{2}\right) \int_\Omega \mathcal{U}_\delta^p \mathcal{P}\mathcal{U}_\delta\, dx + p\delta \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} \mathcal{P}\mathcal{U}_\delta\, dx.\end{aligned} \tag{4.5.4}$$

We analyze the last term of (4.5.4). Applying Lemma 4.2.1 and since it is well known that

$$\int_{\mathbb{R}^N} \mathcal{U}_\delta^p \frac{\partial \mathcal{U}_\delta}{\partial \delta}\, dx = 0,$$

we have

$$\begin{aligned}p\delta \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} \mathcal{P}\mathcal{U}_\delta\, dx &= p\delta \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} \mathcal{U}_\delta\, dx - p\alpha_N \delta^{\frac{N}{2}} \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} H(x,0)\, dx \\ &\quad + o\left( \delta^{\frac{N}{2}} \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} H(x,0)\, dx\right) \\ &= -p\delta \int_{\mathbb{R}^N \setminus \Omega} \mathcal{U}_\delta^p \frac{\partial \mathcal{U}_\delta}{\partial \delta}\, dx - p\alpha_N \delta^{\frac{N}{2}} \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} H(x,0)\, dx \\ &\quad + o\left( \delta^{\frac{N}{2}} \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} H(x,0)\, dx\right),\end{aligned} \tag{4.5.5}$$

where $H$ denotes, the regular part of the Green function for the Laplacian. By definition it is easy to see that

$$\begin{aligned}\left| -p\delta \int_{\mathbb{R}^N \setminus \Omega} \mathcal{U}_\delta^p \frac{\partial \mathcal{U}_\delta}{\partial \delta}\, dx \right| &\le \alpha_N^{p+1} \frac{N+2}{2}\, \delta \int_{\mathbb{R}^N \setminus \Omega} \frac{\delta^{\frac{N+2}{2}}}{(\delta^2+|x|^2)^{\frac{N+2}{2}}} \frac{\delta^{\frac{N-2}{2}}\, \left||x|^2-\delta^2\right|}{(\delta^2+|x|^2)^{\frac{N}{2}}}\, dx \\ &\le \alpha_N^{p+1} \frac{N+2}{2} \int_{\mathbb{R}^N \setminus \Omega} \frac{\delta^{N+1}}{|x|^{N+2}} \frac{\left||x|^2-\delta^2\right|}{|x|^N}\, dx \\ &= O\left(\delta^{N+1}\right).\end{aligned} \tag{4.5.6}$$

Moreover, by the usual change of variable and applying the mean value theorem, we have

$$\begin{aligned}
p\alpha_N \delta^{\frac{N}{2}} \int_\Omega \mathcal{U}_\delta^{p-1} \frac{\partial \mathcal{U}_\delta}{\partial \delta} H(x,0)\, dx &= p\alpha_N^{p+1} \delta^{\frac{N-2}{2}} \int_\Omega \frac{\delta^2}{(\delta^2+|x|^2)^2} \frac{\delta^{\frac{N-2}{2}}\left(|x|^2-\delta^2\right)}{(\delta^2+|x|^2)^{\frac{N}{2}}} H(x,0)\, dx \\
&= p\alpha_N^{p+1} \delta^{\frac{N-2}{2}} \int_\Omega \frac{\delta^2}{\delta^4\left(1+|\frac{x}{\delta}|^2\right)^2} \frac{\delta^{\frac{N-2}{2}}\delta^2\left(|\frac{x}{\delta}|^2-1\right)}{\delta^N\left(1+|\frac{x}{\delta}|^2\right)^{\frac{N}{2}}} H(x,0)\, dx \\
&= p\alpha_N^{p+1} \delta^{N-2} \int_{\Omega/\delta} \frac{1}{(1+|y|^2)^2} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N}{2}}} H(\delta y,0)\, dy \\
&= p\alpha_N^{p+1} \delta^{N-2} \int_{\Omega/\delta} \frac{1}{(1+|y|^2)^2} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N}{2}}} H(0,0)\, dy \\
&+ O\left(\delta^{N-1} \int_{\Omega/\delta} \frac{1}{(1+|y|^2)^2} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N}{2}}} (\nabla H(\eta y,0)\cdot y)\, dy\right) \\
&= p\alpha_N^{p+1} \delta^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^2} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N}{2}}} H(0,0)\, dy \\
&+ O\left(\delta^{N-2} \int_{1/\delta}^{+\infty} \frac{r^{N-1}}{(1+r^2)^2} \frac{\left(r^2-1\right)}{(1+r^2)^{\frac{N}{2}}} H(0,0)\, dr\right) \\
&+ O\left(\delta^{N-1} \int_{\Omega/\delta} \frac{1}{(1+|y|^2)^2} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N}{2}}} (\nabla H(\eta y,0)\cdot y)\, dy\right) \\
&= p\alpha_N^{p+1} H(0,0)\delta^{N-2} \int_{\mathbb{R}^N} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N+4}{2}}}\, dy + O(\delta^{N-1}).
\end{aligned} \tag{4.5.7}$$

Finally from (4.5.2)-(4.5.7) we get that

$$\int_{\partial\Omega} \left(\frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial \nu}\right)^2 (x\cdot\nu)\, d\sigma = 2p\alpha_N^{p+1} H(0,0)\delta^{N-2} \int_{\mathbb{R}^N} \frac{\left(|y|^2-1\right)}{(1+|y|^2)^{\frac{N+4}{2}}}\, dy + O(\delta^{N-1}),$$

and the proof is complete. ☐

Another preliminary lemma is the following:

**Lemma 4.5.3.** *Under the assumptions of Theorem 4.5.1, as $\lambda \to 0$, we have*

$$\left|\int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 (x\cdot\nu)\, d\sigma\right| = O(\lambda^2\delta_1^4) + o(\delta_1^{N-2}).$$

*Proof.* The first step is the following:

$$\begin{aligned}
\left|\int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 (x\cdot\nu)\, d\sigma\right| &\leq \int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 |x\cdot\nu|\, d\sigma \\
&\leq \int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 |x|\, d\sigma \\
&\leq c(\Omega) \int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 d\sigma.
\end{aligned}$$

Thus we need to estimate $\displaystyle\int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2 d\sigma$. Let us consider a smooth function $\zeta : \mathbb{R}^N \to \mathbb{R}$ such that $0 \leq \zeta \leq 1$, $\zeta(x) = 0$ for $|x| \leq \frac{1}{2}$ and $\zeta(x) = 1$ for $|x| \geq 1$. We set $\eta(x) := \zeta(\frac{x}{d(0,\partial\Omega)})$. It's elementary to see that $\eta w_\lambda$ is a solution of the following problem

$$\begin{cases} -\Delta(\eta w_\lambda) = \lambda\eta w_\lambda + g_\lambda & \text{in } \Omega \\ \eta w_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5.8}$$

where $g_\lambda = \eta\left(\lambda\mathcal{P}\mathcal{U}_{\delta_1} - \lambda\mathcal{P}\mathcal{U}_{\delta_2} - \mathcal{U}^p_{\delta_1} + \mathcal{U}^p_{\delta_2} + |u_\lambda|^{2^*-2}u_\lambda\right) - 2\nabla\eta\cdot\nabla w_\lambda - w_\lambda\Delta\eta$. Since $\eta w_\lambda$ is a solution of (4.5.8), the following inequality holds (see Appendix C in [51]):

$$\left|\frac{\partial}{\partial\nu}(\eta w_\lambda)\right|^2_{2,\partial\Omega} = \left|\frac{\partial w_\lambda}{\partial\nu}\right|^2_{2,\partial\Omega} \leq C|g_\lambda|^2_{\frac{2N}{N+1},\Omega}, \tag{4.5.9}$$

where $C$ is a positive constant depending only on $\Omega$ and $N$. Hence, in order to complete the proof, it suffices to estimate the $L^{\frac{2N}{N+1}}(\Omega)$-norm of $g_\lambda$. We point out that, thanks to the multiplication by the cut-off function $\eta$, what occurs around the origin does not count anymore and this will make the boundary estimate sharper. By elementary inequalities we get that

$$|g_\lambda| \leq c(p)\eta\left(\lambda\mathcal{U}_{\delta_1} + \lambda\mathcal{U}_{\delta_2} + \mathcal{U}^p_{\delta_1} + \mathcal{U}^p_{\delta_2} + |w_\lambda|^p\right) + 2|\nabla\eta||\nabla w_\lambda| + |\Delta\eta||w_\lambda|.$$

Thus we have to estimate the following quantities:

$\lambda|\eta\mathcal{U}_{\delta_j}|_{\frac{2N}{N+1},\Omega}$, $|\eta\mathcal{U}^p_{\delta_j}|_{\frac{2N}{N+1},\Omega}$, for $j = 1, 2$, and $|\eta|w_\lambda|^p|_{\frac{2N}{N+1},\Omega}$, $|\,|\nabla\eta||\nabla w_\lambda|\,|_{\frac{2N}{N+1},\Omega}$, $|\,|\Delta\eta||w_\lambda|\,|_{\frac{2N}{N+1},\Omega}$.

This is a long computation already made by O. Rey (see Appendix C of [51]), in the case of positive solutions of the form $u_\lambda = \mathcal{P}\mathcal{U}_\delta + w_\lambda$. In that paper it is shown that

$$|\eta\mathcal{U}^p_{\delta_j}|^2_{\frac{2N}{N+1},\Omega} = o\left(\delta_j^{N-2}\right), \quad |\eta\lambda\mathcal{U}_{\delta_j}|^2_{\frac{2N}{N+1},\Omega} = O\left(\lambda^2\delta_j^{N-2}\right),$$

$$\Big||\nabla\eta||\nabla w_\lambda|\Big|^2_{\frac{2N}{N+1},\Omega} = O\left(\|w_\lambda\|^2\right), \quad \Big||\Delta\eta||w_\lambda|\Big|^2_{\frac{2N}{N+1},\Omega} = O\left(\|w_\lambda\|^2\right). \tag{4.5.10}$$

Moreover, by the same computations of Appendix C in [51] we see that

$$\Big|\eta|w_\lambda|^p\Big|^2_{\frac{2N}{N+1},\Omega} = o(\delta_1^{N-2}).$$

In order to complete the proof we need to estimate the quantities in (4.5.10), and hence we have to study the asymptotic behavior of $\|w_\lambda\|$. An estimate for $\|w_\lambda\|$ is contained in [13]; in particular, by the proof of Lemma 3.3 of [13] we see that

$$\|w_\lambda\| \leq c\left[\sum_i\left(\lambda\delta_i^{(N-2)/2} + \delta_i^{N-2}\right) + \epsilon_{12}(\log\epsilon_{12}^{-1})^{(N-2)/N}\right], \tag{4.5.11}$$

where $\epsilon_{12}$ is defined by $\epsilon_{12} := \left(\frac{\delta_1}{\delta_2} + \frac{\delta_2}{\delta_1}\right)^{(2-N)/2}$. Since $\frac{\delta_2}{\delta_1} \to 0$ as $\lambda \to 0$ we see that

$$\epsilon_{12} = \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}.$$

Moreover by the assumptions on the growth of $\nabla w_\lambda$ and $w_\lambda$, and thanks to (4.3.8) we get that $\epsilon_{12}$ is of the same order as $\lambda\delta_2^2$, hence, since $\delta_2 = o(\delta_1)$ as $\lambda \to 0$, we have that

$$\epsilon_{12}(\log \epsilon_{12}^{-1})^{(N-2)/N} = o(\lambda\delta_1^2).$$

Thus, from (4.5.11), and since $N \geq 7$, we deduce that for all sufficiently small $\lambda$ it holds

$$\|w_\lambda\| \leq c(\delta_1^{N-2} + \lambda\delta_1^2). \tag{4.5.12}$$

Summing up all these estimates we deduce the desired relation. □

**Lemma 4.5.4.** *Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ with smooth boundary and assume that $0 \in \Omega$, $N \geq 5$. Then, as $\delta \to 0$, we have*

$$\int_{\partial\Omega} \left(\frac{\partial \mathcal{P}\mathcal{U}_\delta}{\partial \nu}\right)^2 d\sigma = O(\delta^{N-2}).$$

*Proof.* We consider a smooth function $\eta : \mathbb{R}^N \to \mathbb{R}$ having the same properties as the one considered in the previous proof. By elementary computation we see that $\eta\mathcal{P}\mathcal{U}_\delta$ satisfies

$$\begin{cases} -\Delta(\eta\mathcal{P}\mathcal{U}_\delta) = -(\Delta\eta)\mathcal{P}\mathcal{U}_\delta - \nabla\eta \cdot \nabla\mathcal{P}\mathcal{U}_\delta + \eta\mathcal{U}_\delta^p & \text{in } \Omega \\ \eta\mathcal{P}\mathcal{U}_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5.13}$$

Since $\eta\mathcal{P}\mathcal{U}_\delta$ is a solution of (4.5.13), the following inequality holds:

$$\left|\frac{\partial}{\partial\nu}(\eta\mathcal{P}\mathcal{U}_\delta)\right|_{2,\partial\Omega}^2 = \left|\frac{\partial\mathcal{P}\mathcal{U}_\delta}{\partial\nu}\right|_{2,\partial\Omega}^2 \leq C\Big||\Delta\eta|\mathcal{P}\mathcal{U}_\delta + |\nabla\eta\cdot\nabla\mathcal{P}\mathcal{U}_\delta| + \eta\mathcal{U}_\delta^p\Big|_{\frac{2N}{N+1},\Omega}^2, \tag{4.5.14}$$

where $C$ is a positive constant depending only on $\Omega$ and $N$. In order to complete the proof we have to estimate the quantities: $|(\Delta\eta)\mathcal{P}\mathcal{U}_\delta|_{\frac{2N}{N+1}^2,\Omega}$, $|\nabla\eta\cdot\nabla\mathcal{P}\mathcal{U}_\delta|^2_{\frac{2N}{N+1},\Omega}$, $|\eta\mathcal{U}_\delta^p|^2_{\frac{2N}{N+1},\Omega}$. Using the same computations made by O. Rey in [51], and since $\eta \equiv 0$ in a neighborhood of the origin we get that

$$\begin{aligned} &|\eta\mathcal{U}_\delta^p|^2_{\frac{2N}{N+1},\Omega} = o\left(\delta^{N-2}\right), \quad \Big||\nabla\eta||\nabla\mathcal{P}\mathcal{U}_\delta|\Big|^2_{\frac{2N}{N+1},\Omega} = O\left(\|\mathcal{P}\mathcal{U}_\delta\|^2_{\Omega\cap supp(\nabla\eta)}\right), \\ &\Big||\Delta\eta||\mathcal{P}\mathcal{U}_\delta|\Big|^2_{\frac{2N}{N+1},\Omega} = O\left(\|\mathcal{P}\mathcal{U}_\delta\|^2_{\Omega\cap supp(\nabla\eta)}\right). \end{aligned} \tag{4.5.15}$$

Applying Lemma 4.2.1 and taking account of (4.3.4), since $\nabla\eta \equiv 0$ in an open neighborhood of the origin, we have

$$\begin{aligned} \|\mathcal{P}\mathcal{U}_\delta\|^2_{\Omega\cap supp(\nabla\eta)} &= \int_{\Omega\cap supp(\nabla\eta)} |\nabla(\mathcal{U}_\delta - \varphi_\delta)|^2\, dx \\ &\leq \int_{\Omega\cap supp(\nabla\eta)} |\nabla\mathcal{U}_\delta|^2 dx + 2\int_{\Omega\cap supp(\nabla\eta)} |\nabla\mathcal{U}_\delta||\nabla\varphi_\delta| dx \\ &\quad + \int_{\Omega\cap supp(\nabla\eta)} |\nabla\varphi_\delta|^2 dx \\ &= O(\delta^{N-2}). \end{aligned} \tag{4.5.16}$$

From (4.5.14), (4.5.15) and (4.5.16) we deduce that

$$\left|\frac{\partial\mathcal{P}\mathcal{U}_\delta}{\partial\nu}\right|^2_{2,\partial\Omega} = O(\delta^{N-2}),$$

and the proof is complete. □

**Proof of Theorem 4.5.1.** We apply the Pohozaev's identity to $u_\lambda = \mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + w_\lambda$. Since $u_\lambda$ is a solution of Problem 4.1.1 we have

$$\lambda \int_\Omega u_\lambda^2 \, dx = \frac{1}{2} \int_{\partial\Omega} \left( \frac{\partial u_\lambda}{\partial \nu} \right)^2 (x \cdot \nu) \, d\sigma. \tag{4.5.17}$$

For the left-hand side of (4.5.17), as in the previous proofs we set $\Phi_\lambda := w_\lambda - \varphi_{\delta_1} + \varphi_{\delta_2}$, where $\varphi_{\delta_j} = U_{\delta_j} - \mathcal{P}\mathcal{U}_{\delta_j}$ for $j = 1, 2$, and we have

$$\begin{aligned} \lambda \int_\Omega u_\lambda^2 \, dx &= \lambda \int_\Omega (\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + w_\lambda)^2 \, dx \\ &= \lambda \int_\Omega (\mathcal{U}_{\delta_1} - \mathcal{U}_{\delta_2} + \Phi_\lambda)^2 \, dx \\ &= \lambda \int_\Omega \left( \mathcal{U}_{\delta_1}^2 + \mathcal{U}_{\delta_2}^2 - 2\mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2} + 2\mathcal{U}_{\delta_1}\Phi_\lambda - 2\mathcal{U}_{\delta_2}\Phi_\lambda + \Phi_\lambda^2 \right) dx \\ &= A + B + C + D + E + F. \end{aligned} \tag{4.5.18}$$

In order to estimate $A$ and $B$ we use the following

$$\begin{aligned} \lambda \int_\Omega \mathcal{U}_{\delta_j}^2 \, dx &= \lambda \, \alpha_N^2 \int_\Omega \frac{\delta_j^{-(N-2)}}{(1+|x/\delta_j|^2)^{N-2}} \, dx = \lambda \, \alpha_N^2 \int_{\Omega/\delta_j} \frac{\delta_j^{-(N-2)}}{(1+|y|^2)^{N-2}} \delta_j^N \, dy \\ &= \lambda \, \alpha_N^2 \delta_j^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy + O\left( \lambda \delta_j^2 \int_{1/\delta_j}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}} \, dr \right) \\ &= \lambda \, \alpha_N^2 \delta_j^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy + O\left( \lambda \delta_j^{N-2} \right). \end{aligned} \tag{4.5.19}$$

We point out that since we are assuming that $N \geq 5$, the first integral in the last line of (4.5.19) converges. To estimate $C$ we apply the following

$$\begin{aligned} \lambda \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2} \, dx &= \lambda \, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2 + \delta_1^2|y|^2)^{\frac{N-2}{2}}} \, dy \\ &= \lambda \, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{\left( \left(\frac{\delta_2}{\delta_1}\right)^2 + |y|^2 \right)^{\frac{N-2}{2}}} \, dy \\ &\leq \lambda \, \alpha_N^2 \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-2}{2}} \delta_1^2 \int_{\Omega/\delta_1} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}} |y|^{N-2}} \, dy \\ &= \lambda \, \alpha_N^2 \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}} |y|^{N-2}} \, dy \\ &+ O\left( \lambda \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-2}{2}} \delta_1^2 \int_{1/\delta_1}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}} r^{N-2}} \, dr \right) \\ &= \lambda \, \alpha_N^2 \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}} |y|^{N-2}} \, dy + O\left( \lambda \left( \frac{\delta_2}{\delta_1} \right)^{\frac{N-2}{2}} \delta_1^{N-2} \right). \end{aligned} \tag{4.5.20}$$

In order to estimate $D$, $E$, $F$, thanks to (4.5.12), Hölder's inequality and Poincaré's inequality we get that

$$\int_\Omega w_\lambda^2 \leq c_1 \|w_\lambda\|^2 \leq c_2 (\delta_1^{N-2} + \lambda \delta_1^2)^2. \tag{4.5.21}$$

We observe that, by Lemma 4.2.1 and since $N \geq 5$, we have $|\varphi_{\delta_j}|_{2,\Omega} = O\left(\delta_j^{\frac{N-2}{2}}\right) = o(\delta_j)$. Thus, by definition of $\Phi_\lambda$ and (4.5.21) we deduce that

$$\int_\Omega \Phi_\lambda^2 \, dx = \int_\Omega (w_\lambda + \varphi_{\delta_2} - \varphi_{\delta_1})^2 \, dx = o(\delta_1^2), \tag{4.5.22}$$

and hence

$$F = o(\lambda\delta_1^2). \tag{4.5.23}$$

Moreover, by the same computations of (4.5.19) we have $\int_\Omega \mathcal{U}_{\delta_j}^2 = a_1\delta_j^2 + o(\delta_j^2)$, for some positive constant $a_1$. Hence by Hölder's inequality and (4.5.22) we get that

$$|D| = o(\lambda\delta_1^2),$$

and

$$|E| = o(\lambda\delta_1\delta_2) = o(\lambda\delta_1^2).$$

We analyze now the right-hand side of (4.5.17): by definition we have

$$\begin{aligned}
\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_\lambda}{\partial\nu}\right)^2 (x\cdot\nu)\, d\sigma &= \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu} - \frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu} + \frac{\partial w_\lambda}{\partial\nu}\right)^2 (x\cdot\nu)\, d\sigma \\
&= \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\right)^2 (x\cdot\nu)\, d\sigma + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu}\right)^2 (x\cdot\nu)\, d\sigma \\
&\quad - \int_{\partial\Omega}\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu}(x\cdot\nu)\, d\sigma + \int_{\partial\Omega}\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\frac{\partial w_\lambda}{\partial\nu}(x\cdot\nu)\, d\sigma \\
&\quad - \int_{\partial\Omega}\frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu}\frac{\partial w_\lambda}{\partial\nu}(x\cdot\nu)\, d\sigma + \frac{1}{2}\int_{\partial\Omega}\left(\frac{w_\lambda}{\partial\nu}\right)^2 (x\cdot\nu)\, d\sigma \\
&= A_1 + B_1 + C_1 + D_1 + E_1 + F_1.
\end{aligned} \tag{4.5.24}$$

Thanks to Lemma 4.5.2 we have:

$$\begin{aligned}
A_1 &= \frac{a_2}{2}\delta_1^{N-2} + o(\delta_1^{N-2}), \\
B_1 &= \frac{a_2}{2}\delta_2^{N-2} + o(\delta_2^{N-2}).
\end{aligned} \tag{4.5.25}$$

Thanks to Lemma 4.5.4 and applying Hölder inequality we get that

$$\begin{aligned}
|C_1| &\leq \int_{\partial\Omega}\left|\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\right|\left|\frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu}\right| |x\cdot\nu|\, d\sigma \\
&\leq diam(\partial\Omega)\left(\int_{\partial\Omega}\left|\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\right|^2 d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}\left|\frac{\partial \mathcal{PU}_{\delta_2}}{\partial\nu}\right|^2 d\sigma\right)^{\frac{1}{2}} \\
&= O\left(\delta_1^{\frac{N-2}{2}}\delta_2^{\frac{N-2}{2}}\right).
\end{aligned} \tag{4.5.26}$$

Thanks to (4.5.9), Lemma 4.5.3, Lemma 4.5.4 and applying Hölder inequality we get that

$$\begin{aligned}
|D_1| &\leq \int_{\partial\Omega}\left|\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\right|\left|\frac{\partial w_\lambda}{\partial\nu}\right| |x\cdot\nu|\, d\sigma \\
&\leq diam(\partial\Omega)\left(\int_{\partial\Omega}\left|\frac{\partial \mathcal{PU}_{\delta_1}}{\partial\nu}\right|^2 d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}\left|\frac{\partial w_\lambda}{\partial\nu}\right|^2 d\sigma\right)^{\frac{1}{2}} \\
&= o\left(\lambda\delta_1^2\right) + o\left(\delta_1^{N-2}\right).
\end{aligned} \tag{4.5.27}$$

$$\begin{aligned} |E_1| &\leq \int_{\partial\Omega} \left|\frac{\partial \mathcal{P}\mathcal{U}_{\delta_2}}{\partial \nu}\right| \left|\frac{\partial w_\lambda}{\partial \nu}\right| |x \cdot \nu| \, d\sigma \\ &\leq diam(\partial\Omega) \left( \int_{\partial\Omega} \left|\frac{\partial \mathcal{P}\mathcal{U}_{\delta_2}}{\partial \nu}\right|^2 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} \left|\frac{\partial w_\lambda}{\partial \nu}\right|^2 \, d\sigma \right)^{\frac{1}{2}} \\ &= o\left(\lambda \delta_1^2\right) + o\left(\delta_1^{N-2}\right). \end{aligned} \tag{4.5.28}$$

$$|F_1| = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 (x \cdot \nu) \, d\sigma = o\left(\lambda \delta_1^2\right) + o\left(\delta_1^{N-2}\right). \tag{4.5.29}$$

Summing up all the estimates, from (4.5.17) and since $\delta_2 = o(\delta_1)$ as $\lambda \to 0$, we deduce the following equality:

$$a_1 \lambda \delta_1^2 + o(\lambda \delta_1^2) = a_2 \delta_1^{N-2} + o\left(\delta_1^{N-2}\right). \tag{4.5.30}$$

Since $\delta_j$ is of the form (4.5.1), we deduce that $\alpha_1$ must satisfy the equation

$$1 + 2\alpha_1 = (N-2)\alpha_1,$$

and hence we get that $\alpha_1 = \frac{1}{N-4}$. Moreover, from (4.3.8) we deduce that $\alpha_1$, $\alpha_2$ must satisfy the following algebraic equation

$$1 + 2\alpha_2 = \frac{N-2}{2}(\alpha_2 - \alpha_1). \tag{4.5.31}$$

Thus, combining this result with (4.5.31), we get that $\alpha_2 = \frac{3N-10}{(N-4)(N-6)}$ and the proof is complete. □

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