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INTEGRABILITY PROPERTIES OF QUAD EQUATIONS CONSISTENT ON THE CUBE

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ABSTRACT

In this Thesis we present some results obtained recently about the integrability properties of the multi-affine partial difference equations Consistent Around the Cube classified by R. Boll. We review some known result and we present for the first time the non-autonomous form on the lattice of some of these equations. Using the so-called Algebraic Entropy test we conjecture that two sub-families of the equations found by Boll, namely the trapezoidal H^4 equations and the H^6 equations are linearizable. By computing the Generalized Symmetries of the trapezoidal H^4 equations and the H^6 equations we propose a non-autonomous generalization of the Q_V equation. Finally we prove the linearizability of these equation by showing that they are Darboux integrable equations and we show how to use this property in order to obtain general solutions.

SOMMARIO

In questa Tesi presentiamo alcuni risultati recentemente ottenuti sulle proprietà di ingregrabilità delle equazioni alle differenze parziali multiaffini Consistenti sul Cubo classificate da R. Boll. Passiamo in rassegna alcuni risultati nuovi e presentiamo la forma nonautonoma sul reticolo di alcune di queste equazioni per la prima volta. Usando il cosiddetto metodo dell'Entropia Algebrica congetturiamo che le due sottofamiglie delle equazioni trovate da Boll, ossia le equazioni H^4 trapezoidali e le equazioni H^6, siano linearizzabili. Calcolando le Simmetrie Generalizzate delle equazioni H^4 trapezoidali e delle equazioni H^6 proponiamo una generalizzazione nonautonoma dell'equazione Q_V. Per ultima cosa dimostriamo come queste equazioni siano linearizzabili mostrando che sono Darboux integrabili e che tramite questa proprietà è possibile scrivere le soluzioni generali.

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INTRODUCTION

The main object of this Thesis are the *multi-affine partial difference equations defined on a quad graph*, i.e. equations for an unknown function $u_{n,m}$ of the two discrete variables $(n, m) \in \mathbb{Z}^2$, which are *multi-affine polynomial* in the unknown function and its shifts in a way that the corresponding points on the $\mathbb{Z}^2$ plane lie on the vertices of a quadrilateral figure. The most known example of this kind of equations is the *discrete wave equation*:

$$u_{n+1,m+1} - u_{n+1,m} - u_{n,m+1} + u_{n,m} = 0,$$

i.e. the lattice analogue of the *wave equation* in the *light-cone coordinates* $u_{xt} = 0$. Indeed the partial difference equations defined on a quad graph are the lattice analogue of the *hyperbolic partial differential equations*.

The property of the multi-affine partial difference equations defined on a quad graph we are interested in is their *integrability*. The most accepted definition of integrability for nonlinear equations (of any kind) is that of the existence of a *Lax pair* [97]. A Lax pair is a linear representation of a nonlinear problem which yield, through the method of the Inverse Scattering Transform the solution of the original nonlinear equation.

Finding a Lax pair for a nonlinear equation is a non-trivial and non-algorithmic task so a preeminent rôle in the theory of Integrable Systems is given to the so-called *integrability criteria* [100, 172]. Integrability criteria are algorithmic procedure that permit to say if a given equation is integrable or not. Integrability criteria are therefore particularly useful in applications.

In the case of the nonlinear multi-affine partial difference equations defined on a quad graph a particularly useful integrability criterion has been discovered: the so-called Consistency Around the Cube. Roughly speaking a multi-affine partial difference equations defined on a quad graph is said to possess the Consistency Around the Cube if it can be extended in an agreeing way on a three dimensional lattice. This means that a single equation is replaced by a sextuple of equations assigned on the faces of a cube, which explains the name, see Chapter 1 for a more formal definition. The usefulness of the Consistency Around the Cube relays in the fact that it provides for an equation possessing it Bäcklund transforms and as a consequence a Lax pair [120, 122].

Beyond its rôle as integrability criterion, i.e. as a method to establish if a *given* equation is or not integrable, the Consistency Around the Cube has been also used for classification purposes in a program aimed to find all the equations possessing it up to some chosen group of transformations. Indeed, since if an equation possess the Consistency Around the Cube we can find its Lax pair, to classify all the equations with this property means to classify a subset of the integrable multi-affine partial difference equations defined on a quad graph. The classification using the Consistency Around the Cube was done in [2]. In [2] were used two technical assumptions: that the given equation possesses the *discrete symmetries of the square* and the *tetrahedron property*. An equation possesses the discrete symmetries of the square if the multi-affine polynomial defining it is symmetric under a particular exchange of its entries, while it possess the tetrahedron property if the equation on the top of the cube does not contains the field $u_{n,m}$. In the following years many authors dealt with the problem of weakening the hypothesis in [2] and extending the classification therein made, e.g. [81, 82]. Most notably the paper [3] opened the way for a complete classification, using as only technical hypothesis the tetrahedron property. A complete classification of the quad graph equations possessing the Consistency Around the Cube and the tetrahedron property was then accomplished by R. Boll in a series of papers culminating in his PhD thesis [20–22]. The result of the classification made by Boll was that there exist three families of equations possessing the Consistency Around the Cube: the Q equations, the H^4 equations, divided in *rhombic* and *trapezoidal*, and the H^6 equations. Let us notice that the Q family was introduced in [2] and their integrability properties are well established. A detailed study of all the lattice equations derived from the *rhombic* H^4 family, including the construction of their three-leg forms, Lax pairs, Bäcklund transformations and infinite hierarchies of generalized symmetries, was presented in [166].

In this Thesis we deal with two families of equations of the Boll's classification possessing the Consistency Around the Cube which were still unstudied, namely the trapezoidal H^4 equations and the H^6 equations. The main result is that these two classes of equations consist of linearizable equations. We will go through the various steps which were made to understand their deep structure in a series of papers. The plan of the Thesis is the following:

CHAPTER 1: We introduce the concept of integrability for finite and infinite dimensional systems with particular attention to the partial difference equations case. In particular we introduce the concept of integrability indicators and we present in a rigorous way the concept of Consistency Around the Cube and its historical development. We present then the explicit construction of the lattice equations from the single-cell equations in the case when the equations on the side of the cube are different. We discuss then how the classification at the single-cell level is preserved when passing to the lattice by introducing explicitly the group

$\text{Möb}^4_{n,m}$. The original part of this Chapter is mainly based on [67].

CHAPTER 2: We introduce the Algebraic Entropy to study the trapezoidal H^4 and the H^6 equation. The fact that the trapezoidal H^4 and the H^6 equations are non-autonomous requires a modification of the usual definition of Algebraic Entropy for lattice equation [155]. We then present the results of the calculation of the Algebraic Entropy for the trapezoidal H^4 and the H^6 equations. To support this evidence we present two examples of direct linearization and an example of the fact that Lax pairs obtained from Consistency Around the Cube for linearizable equations can be fake [27, 77]. The original part of this Chapter is mainly based on [67, 68].

CHAPTER 3: We introduce the Generalized Symmetry method for partial difference equations on the quad graph. We then present the three-point Generalized Symmetries of the trapezoidal H^4 and the H^6 equations and put them in relation with particular cases of the Yamilov discretization of the Kriechever-Novikov equation [168]. Stimulated by the results of these computations we introduce a new partial difference equation which we conjecture to be integrable due to Algebraic Entropy test. We show that this new partial difference equation is a non-autonomous generalization of the so-called Q_V equation [156]. The original part of this Chapter is mainly based on [65, 66].

CHAPTER 4: We present a rigorous proof of the fact that the H^4 and the H^6 equations are linearizable based on the concept of *Darboux integrability* for partial difference equations [5]. This enables us also to construct the general solutions of these equation by solving some linear non-autonomous ordinary difference equations or some non-autonomous discrete Riccati equations. This fact can be taken to be a confirmation of the Algebraic Entropy conjecture [84]. The original part of this Chapter is mainly based on [70, 71].

In the CONCLUSIONS we make some comments on the results obtained in the various Chapters of this Thesis and we discuss some open problems and further developments.

1 INTEGRABILITY AND THE CONSISTENCY AROUND THE CUBE

Here we introduce the main concepts we will deal with. In particular we will introduce the main ingredients in order to construct the lattice representation of the trapezoidal H^4 and H^6 equations, object of this Thesis.

We begin in Section 1.1 by introducing the concept of *integrability*. The in Section 1.2 we introduce the basis of one of the most prolific *integrability indicators* of the last years: the *Consistency Around the Cube*. In Section 1.3 we will give an account of how the Consistency Around the Cube has been used in order to find and classify integrable systems. In particular in Subsection 1.3.2 we will introduce the Boll's classification [20–22] and present and discuss in the following Subsection 1.3.3 the equations yet not studied. In Section 1.4 we will discuss the problem of the embedding in the 2D and 3D lattices. In Section 1.5 we give the proof of how the classification at the level of single cell is preserved once embedded in the full lattice, given originally in [67]. Finally in Section 1.6 we will present the explicit lattice form of the trapezoidal H^4 and H^6 equations as given in [67] which will used everywhere throughout the Thesis.

1.1 THE MEANING OF INTEGRABILITY

In this Thesis we will focus mainly on *two-dimensional partial difference equations* defined on a square lattice for an unknown function $u_{n,m}$ with $(n, m) \in \mathbb{Z}^2$, i.e. relations of the form:

$$Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right) = 0, \quad (n, m) \in \mathbb{Z}^2. \tag{1.1}$$

If the function Q is a multi-affine irreducible polynomial of its arguments we say that (1.1) is a *quad-equation*. This will be the kind of two-dimensional partial difference equation we will consider mostly.

We wish to study the *integrability* of these equations. The notion of integrability comes from Classical Mechanics and roughly means the existence of a "sufficiently" high number of *first integrals*. Indeed given an Hamiltonian system with Hamiltonian $H = H(p, q)$ with N degrees of freedom we say that this system is integrable if there exists N integrals of motions, i.e. N functions H_i $i = 1, \dots, N$, well defined on the phase space, i.e. *analytic* and *single-valued*, which Poisson-commutes with the Hamiltonian:

$$\{H_i, H\} = 0. \tag{1.2}$$

The Hamiltonian, which commutes with itself, is included in the list as $H_1 = H$. These first integrals must be well defined functions on the phase space and *in involution*:

$$\{H_i, H_j\} = 0, \quad i \neq j = 1, \dots, N \tag{1.3}$$

and finally they should be functionally independent:

$$\operatorname{rank} \frac{\partial(H_1, \dots, H_n)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)} = N. \tag{1.4}$$

Indeed the knowledge of these integrals provides the remaining $N - 1$ integrals and this is the content of the famous *Liouville theorem* [112, 161]. When more than N independent integrals exists, we say that the system is *superintegrable* [44, 140]. Note that in this case the additional integrals will not be in involution with the previous ones. Integrability in Classical Mechanics means that the motion is constrained on subspace of the full phase space. With some additional assumptions on the geometric structure of the first integral it is possible to prove that the motion is quasi-periodic on some tori in the phase space [12, 13]. Note that in the classical case is possible to have some regularity on the behaviour of the system even when there exists $M < N$ first integrals. This situation is called *partial integrability* [49, 50, 119].

In the infinite dimensional case, i.e. for partial differential equations, the notion of integrability have been studied and developed in the second half of the XXth Century. In this case one should be able to find *infinitely many conservation laws*. These infinitely many conservations laws can be obtained, for example, from the so-called *Lax pair* [97], that is an overdetermined *linear* problem whose compatibility is ensured if and only if the nonlinear equation is satisfied. A *bona fide* Lax pair can be used to produce the required conservations laws, with the associated generalized symmetries. The form of the Lax pair will be different depending on the kind of equation we are considering. It is worth to note that Lax pairs can be used also in Classical Mechanics [40] and that there exist examples of Lax pairs which are not bona fide [27, 77, 104, 109, 114, 115, 143, 144], the so-called *fake Lax pairs*. These fake Lax pairs cannot provide the infinite sequence of first integrals, so they are useless in proving integrability.

In the case of the partial difference equation (1.1) a Lax Pair is the *overdetermined system of linear equations* for a field vector function $\Phi_{n,m} \in \mathbb{R}^K$:

$$\Phi_{n+1,m} = L_{n,m}(u_{n,m}, u_{n+1,m})\, \Phi_{n,m}, \tag{1.5a}$$

$$\Phi_{n,m+1} = M_{n,m}(u_{n,m}, u_{n,m+1})\, \Phi_{n,m}, \tag{1.5b}$$

where $L_{n,m}$ and $M_{n,m}$ are $K \times K$ matrices, depending possibly on some *spectral parameter* λ[1]. The compatibility of the two equations in (1.5) is:

$$L_{n,m+1}M_{n,m} = M_{n+1,m}L_{n,m}, \tag{1.6}$$

which must be satisfied if and only if we are on the solution of our nonlinear partial difference equation (1.1) for any value of λ. Remark that in (1.6) we have to consider the shifts also in the dependent variable upon which the matrices $L_{n,m}$ and $M_{n,m}$ depend. That is if $L_{n,m} = L_{n,m}\left(u_{n,m}, u_{n+1,m}\right)$ then $L_{n,m+1} = L_{n,m+1}\left(u_{n,m+1}, u_{n+1,m+1}\right)$ and if $M_{n,m} = M_{n,m}\left(u_{n,m}, u_{n,m+1}\right)$ then $M_{n+1,m} = M_{n+1,m}\left(u_{n+1,m}, u_{n+1,m+1}\right)$. This notation will be employed everywhere.

The vector function $\Phi_{n,m}$ is usually called, for historical reasons, the *wave function*. This is because in the case of the Korteweg-deVries equation [92], which is the first example of Lax pair ever produced, the spatial part of the Lax pair is a one-dimensional Schrödinger equation [97].

Example 1.1.1. Let us consider the following couple of equations from [76]:

$$\frac{x_{n+1,m+1}}{x_{n,m+1}} + y_{n,m} = \frac{x_{n+1,m+1}}{x_{n+1,m}}, \tag{1.7a}$$

$$\frac{x_{n+1,m}}{x_{n,m}} + y_{n+1,m} = \frac{x_{n,m+1}}{x_{n,m}}. \tag{1.7b}$$

This is a couple of equations for the unknown functions $x_{n,m}$ and $y_{n,m}$ defined on a $\mathbb{Z}^2$ lattice. This is not a quad equation in the form (1.1), but written as a single equation it is defined on a rectangle. However we can see that the following 2×2 matrices:

$$L_{n,m} = \begin{pmatrix} x_{n+1,m}/x_{n,m} & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{1.8a}$$

$$M_{n,m} = \begin{pmatrix} x_{n,m+1}/x_{n,m} & \lambda \\ \lambda & y_{n,m} \end{pmatrix}, \tag{1.8b}$$

yield the system (1.7) as compatibility condition (1.6). Indeed by computing (1.6) we obtain:

$$\begin{pmatrix} 0 & \lambda\left(\frac{x_{n+1,m+1}}{x_{n,m+1}} + y_{n,m} - \frac{x_{n+1,m+1}}{x_{n+1,m}}\right) \\ -\lambda\left(\frac{x_{n+1,m}}{x_{n,m}} + y_{n+1,m} - \frac{x_{n,m+1}}{x_{n,m}}\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.9}$$

That is the couple of matrices (1.8) is a Lax pair for the system (1.7). □

1 In the case of 2×2 matrices it was proved that any Lax pair where a non-trivial spectral parameter cannot be inserted [104] is fake [75, 76].

If by integrability we mean the existence of a Lax pair (1.5) then the prove of the integrability of an equation is a highly non-trivial task. Indeed there is no general algorithm to produce a Lax pair. For this reason in the years many *Integrability detectors* have been developed. Integrability detectors are algorithmic procedures which are sufficient conditions for integrability, or in some cases *alternative* definition of integrability. In the next Section we will introduce one of the most fruitful integrability detector developed in the past years: the Consistency Around the Cube. Its primary importance comes from the fact that it can produces algorithmically a Lax pair. In Chapter 2 we will introduce another integrability detector the *Algebraic Entropy* which can also discriminate between integrable and linearizable equations. Also *Generalized Symmetries* which we will introduce in Chapter 3 can be used as a definition of integrability. The key importance of Algebraic Entropy and Generalized Symmetries is that they avoid the need of prove (or disprove) the existence of a Lax pair, at difference from the Consistency Around the Cube technique.

As a final important remark we want to stress that integrability is a property affecting the *regularity* of the solutions. In the case of discrete equations given an initial condition we can compute the whole set of solutions with machine precision. The question of regularity can seem futile, but in fact what we want is to understand the behavior of the given equation without having to compute a full sequence of iterates!

It is worth to note that the concept of integrability and the concept of solvability are very different. This can be easily explained on the example of the famous logistic map:

$$u_{n+1} = 4u_n (1 - u_n) . \tag{1.10}$$

Given the initial condition through $u_0 = \sin^2 (c_0/2)$ the solution of this equation is given by [145]:

$$u_n = \sin^2 \left(c_0 2^{n-1}\right) . \tag{1.11}$$

Equation (1.11) is sensitive to the initial condition. Indeed computing the derivative with respect to the initial condition we have:

$$\frac{du_n}{dc_0} = 2^n \sin \left(c_0 2^{n-1}\right) \cos \left(c_0 2^{n-1}\right) . \tag{1.12}$$

Thus we see that the error grows exponentially with n, one of the indicators of "chaos". So the system (1.11) is solvable, but not integrable.

1.2 THE CONSISTENCY AROUND THE CUBE

Here we discuss the basic properties of the so-called Consistency Around the Cube technique, following mainly the exposition in [83].

Let us now to have a quad equation defined on the quad graph depending also on some parameters p and q:

$$Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}, p, q\right) = 0 \tag{1.13}$$

as it is shown in Figure 1.1.

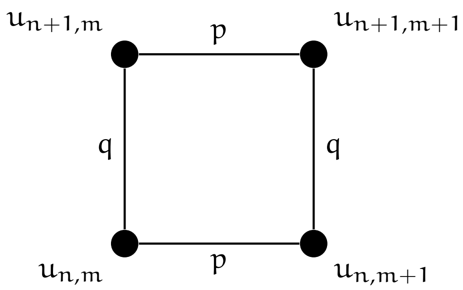

Figure 1.1: A quad-graph.

We say that this equation is Consistent Around the Cube if it can be extended to an equation defined on a three dimensional lattice in a coherent way. To do so first add a third direction $u_{n,m} \to u_{n,m,p}$ and the new three dimensional lattice is depicted in Figure 1.2.

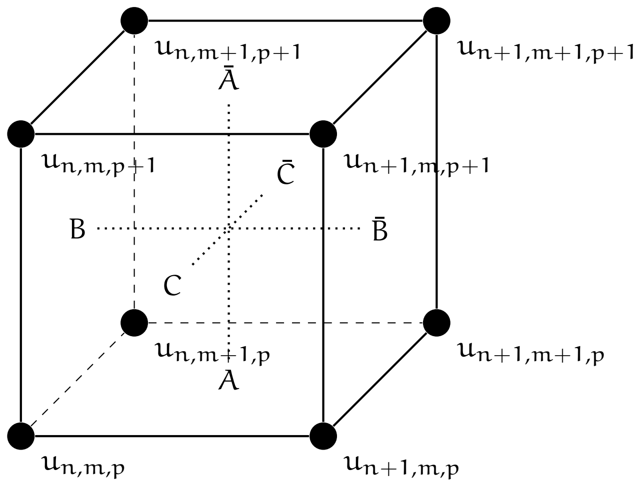

Figure 1.2: The extension of the 2D lattice to a 3D lattice when the equation on the edges are the same.

Then on this new lattice we consider the following equations:

$$A = Q\left(u_{n,m,p}, u_{n+1,m,p}, u_{n,m+1,p}, u_{n+1,m+1,p}, p, q\right) = 0, \tag{1.14a}$$

$$\bar{A} = Q\left(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}, p, q\right) = 0, \tag{1.14b}$$

$$B = Q\left(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}, q, r\right) = 0, \tag{1.14c}$$

$$\bar{B} = Q\left(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}, q, r\right) = 0, \tag{1.14d}$$

$$C = Q\left(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}, r, p\right) = 0, \tag{1.14e}$$

$$\bar{C} = Q\left(u_{n,m+1,p}, u_{n+1,m+1,p}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}, r, p\right) = 0, \tag{1.14f}$$

There is a natural consistency problem: Given the values $u_{n,m,p}$, $u_{n+1,m,p}$, $u_{n,m+1,p}$, $u_{n,m,p+1}$ as in Figure 1.2 we can compute values at $u_{n+1,m+1,p}$, $u_{n+1,m,p+1}$, $u_{n,m+1,p+1}$ uniquely using the equations on the bottom, front and left sides (1.14a,1.14e,1.14c), respectively. Then $u_{n+1,m+1,p+1}$ can be computed from the equations on the top, right and back side equations (1.14b,1.14f,1.14d), and they should all agree. This is the consistency condition.

The Consistency Around the Cube technique represents a rather high level of integrability. Indeed it can be thought as three dimensional version of the multi-dimensional consistency introduced in [35]. In fact it is a kind of Bianchi identity, and was observed to hold for a sequence of Moutard transforms [124]. In its form it was proposed as a property of maps in [122].

The main consequence of the Consistency Around the Cube is that it allows the immediate construction of Lax pair [123] via Bäcklund transformations. We will now give this construction in the way as presented in [120]. The idea is to take the third direction as a spectral direction and generate the field $\Phi_{n,m}$ in (1.5) from the shifts in the third direction. To do so we first solve (1.14e) with respect to $u_{n+1,m,p+1}$ and (1.14c) with respect to $u_{n,m+1,p+1}$ and relabel the third direction parameter $r = \lambda$. We can rewrite $u_{n+1,m,p+1}$, $u_{n,m+1,p+1}$ and $u_{n,m,p+1}$ introducing the inhomogeneous projective variables:

$$u_{n,m,p+1} = \frac{f_{n,m}}{g_{n,m}}, \quad u_{n+1,m,p+1} = \frac{f_{n+1,m}}{g_{n+1,m}}, \qquad u_{n,m+1,p+1} = \frac{f_{n,m+1}}{g_{n,m+1}}. \tag{1.15}$$

Then, due to the multi-linearity assumption, we can write down (1.14e) and (1.14c) as:

$$u_{n+1,m,p+1} = \frac{l_{1,1} u_{n,m,p+1} + l_{1,2}}{l_{2,1} u_{n,m,p+1} + l_{2,2}}, \tag{1.16a}$$

$$u_{n,m+1,p+1} = \frac{m_{1,1} u_{n,m,p+1} + m_{1,2}}{m_{2,1} u_{n,m,p+1} + m_{2,2}}, \tag{1.16b}$$

where $l_{i,j} = l_{i,j}\left(u_{n,m}, u_{n+1,m}\right)$ and $m_{i,j} = m_{i,j}\left(u_{n,m}, u_{n,m+1}\right)$ with $i, j = 1, 2$ are the coefficients with respect to $u_{n,m,p+1}$. Introducing (1.15) into (1.16) we then obtain the following couple of equations:

$$\frac{f_{n+1,m}}{g_{n+1,m}} = \frac{l_{1,1} f_{n,m} + l_{1,2} g_{n,m}}{l_{2,1} f_{n,m} + l_{2,2} g_{n,m}}, \tag{1.17a}$$

$$\frac{f_{n,m+1}}{g_{n,m+1}} = \frac{m_{1,1} f_{n,m} + m_{1,2} g_{n,m}}{m_{2,1} f_{n,m} + m_{2,2} g_{n,m}}, \tag{1.17b}$$

Then if we introduce the vector function:

$$\Phi_{n,m} = \begin{pmatrix} f_{n,m} \\ g_{n,m} \end{pmatrix}, \tag{1.18}$$

we can write interpret (1.17) as matrix relation of the form (1.5) with the matrix L and M given by:

$$L_{n,m} = \gamma_{n,m} \begin{pmatrix} l_{1,1} & l_{1,2} \\ l_{2,1} & l_{2,2} \end{pmatrix}, \quad M_{n,m} = \gamma'_{n,m} \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}, \tag{1.19}$$

where $\gamma_{n,m} = \gamma_{n,m}(u_{n,m}, u_{n+1,m})$ and $\gamma'_{n,m} = \gamma'_{n,m}(u_{n,m}, u_{n,m+1})$ are the so-called *separation constants*. These separation constants arise from the fact that the coefficients in (1.17) are defined in projective sense, i.e. *up a common multiple*. The determination of the separation constants is crucial in the application of the method [23]. To determine the separation constants it is always possible to impose some restrictions on $L_{n,m}$ and $M_{n,m}$, e.g. they can be taken as matrices with unit determinant (members of the special linear group). However this kind of restrictions usually introduces square roots which can make the expressions unmanageable. Therefore usually it is easier to consider a single separation constant $\tau_{n,m}$. From (1.19) writing $L_{n,m} = \gamma_{n,m}\mathcal{L}_{n,m}$ and $M_{n,m} = \gamma'_{n,m}\mathcal{M}_{n,m}$ and taking into account the compatibility condition (1.6), we can write:

$$\tau_{n,m}\mathcal{L}_{n,m+1}\mathcal{M}_{n,m} = \mathcal{M}_{n+1,m}\mathcal{L}_{n,m} \tag{1.20}$$

where we now have a unique separation constant:

$$\tau_{n,m} = \frac{\gamma_{n,m+1}\gamma'_{n,m}}{\gamma_{n,m}\gamma'_{n+1,m}}. \tag{1.21}$$

Note that *a priori* $\tau_{n,m} = \tau_{n,m}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1})$. This kind of reasoning is sufficient if we want to prove that (1.19) gives a Lax pair for the quad equation (1.1) consistent on the cube. On the other hand, if we want to use the obtained Lax pair for constructing soliton solutions or for the usual Inverse Scattering machinery, we need to have the precise form of the separation constants. Separation constants are needed also if we want to prove that a Lax pair is fake, as showed in [68] and will be discussed in Subsection 2.4.1.

At the end we can state the following result *given a quad equation* (1.1) *compatible around the cube, given the matrices* $L_{n,m}$ *and* $M_{n,m}$ *constructed according to* (1.19) *we can find a Lax pair together with the separation constant* $\tau_{n,m}$ *such that the compatibility condition* (1.20) *holds identically on the solutions of* (1.1).

1.3 THE CONSISTENCY AROUND THE CUBE AS A SEARCH METHOD

The Consistency Around the Cube is an algorithmic tool: given a quad equation (1.1) by a few calculations it is possible to tell if it is or it is not consistent around the cube. In the affirmative case it is possible to construct its Lax pair using (1.19). However it is also possible to reverse the reasoning and assume to be given a generic quad equation (1.1) which possesses the Consistency Around the Cube property and the derive the form the equation must have. This means to use the Consistency Around the Cube as a *search method*, and, *in principle*, obtain equations with the desired properties. From the Consistency Around the Cube have been obtained two kinds of results. In a first instance by the papers [2, 81, 82], were produced *autonomous* equations. In a second instance by the papers [3, 20, 22] and in R. Boll Ph.D. Thesis [21], a more general situation was studied and *non-autonomous* equations were produced. We will now list the main findings of these researches, especially we will discuss the generalization introduced in the second phase and, following [67] we will give the explicit formulæ for the construction of the lattice equations in Section 1.4.

1.3.1 *Original attempts*

The first attempt of using the Consistency Around the Cube as a classifying tool for quad equations was made in [2]. The authors used the following hypothesis:

1. The quad equation (1.13) possess the *tetrahedron property*: $u_{n+1,m+1,p+1}$ is independent of $u_{n,m,p}$.
2. The quad equation (1.13) possess the discrete symmetries of the square, i.e. it has the following invariance property:

$$\begin{aligned} &Q\left(u_{n,m},u_{n+1,m},u_{n,m+1},u_{n+1,m+1},p,q\right)\\ &=\mu Q\left(u_{n,m},u_{n,m+1},u_{n+1,m},u_{n+1,m+1},q,p\right)\\ &=\mu' Q\left(u_{n+1,m},u_{n,m},u_{n+1,m+1},u_{n,m+1},p,q\right), \end{aligned} \tag{1.22}$$

where $\mu,\mu'\in\{\pm1\}$.

The outcome of this classification, up to Möbius transformations, are the celebrated ABS equations, namely two families of *autonomous* quad equations. The first one is the H *family* with three members:

$$H_1\colon (u_{n,m}-u_{n+1,m+1})(u_{n,m+1}-u_{n+1,m})-\alpha+\beta=0, \tag{1.23a}$$

$$\begin{aligned} H_2\colon\ &(u_{n,m}-u_{n+1,m+1})(u_{n,m+1}-u_{n+1,m})-\alpha^2+\beta^2\\ &+(\beta-\alpha)(u_{n,m}+u_{n,m+1}+u_{n+1,m}+u_{n+1,m+1})=0, \end{aligned} \tag{1.23b}$$

$$\begin{aligned} H_3\colon\ &\alpha(u_{n,m}u_{n,m+1}+u_{n+1,m}u_{n+1,m+1})\\ &-\beta(u_{n,m}u_{n+1,m}+u_{n,m+1}u_{n+1,m+1})+\delta(\alpha^2-\beta^2)=0, \end{aligned} \tag{1.23c}$$

while the second one is the Q *family* consisting of four members:

$$Q_1: \quad \alpha(u_{n,m}-u_{n+1,m})(u_{n,m+1}-u_{n+1,m+1}) -\beta(u_{n,m}-u_{n,m+1})(u_{n+1,m}-u_{n+1,m+1}) +\delta^2\alpha\beta(\alpha-\beta)=0, \tag{1.24a}$$

$$Q_2: \quad \alpha(u_{n,m}-u_{n+1,m})(u_{n,m+1}-u_{n+1,m+1}) -\beta(u_{n,m}-u_{n,m+1})(u_{n+1,m}-u_{n+1,m+1}) +\alpha\beta(\alpha-\beta)(u_{n,m}+u_{n,m+1}+u_{n+1,m}+u_{n+1,m+1}) -\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0, \tag{1.24b}$$

$$Q_3: \quad (\beta^2-\alpha^2)(u_{n,m}u_{n+1,m+1}+u_{n,m+1}u_{n+1,m}) +\beta(\alpha^2-1)(u_{n,m}u_{n,m+1}+u_{n+1,m}u_{n+1,m+1}) -\alpha(\beta^2-1)(u_{n,m}u_{n+1,m}+u_{n+1,m+1}u_{n+1,m+1}) -\frac{\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)}{4\alpha\beta}=0, \tag{1.24c}$$

$$Q_4: \quad k_0 u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1} - k_1(u_{n,m}u_{n+1,m}u_{n,m+1}+u_{n+1,m}u_{n,m+1}u_{n+1,m+1} + u_{n,m}u_{n,m+1}u_{n+1,m+1}+u_{n,m}u_{n+1,m}u_{n+1,m+1}) + k_2(u_{n,m}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}) - k_3(u_{n,m}u_{n+1,m}+u_{n,m+1}u_{n+1,m+1}) - k_4(u_{n,m}u_{n,m+1}+u_{n+1,m}u_{n+1,m+1}) + k_5(u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1})+k_6=0 \tag{1.24d}$$

with

$$\begin{gathered} k_0=\alpha+\beta,\quad k_1=\alpha\nu+\beta\mu,\quad k_2=\alpha\nu^2+\beta\mu^2, \\ k_3=\frac{\alpha\beta(\alpha+\beta)}{2(\nu-\mu)}-\alpha\nu^2+\beta(2\mu^2-\frac{g_2}{4}), \\ k_4=\frac{\alpha\beta(\alpha+\beta)}{2(\mu-\nu)}-\beta\mu^2+\alpha(2\nu^2-\frac{g_2}{4}), \\ k_5=\frac{g_3}{2}k_0+\frac{g_2}{4}k_1,\quad k_6=\frac{g_2^2}{16}k_0+g_3k_1, \end{gathered} \tag{1.25}$$

where

$$\alpha^2=r(\mu),\quad \beta^2=r(\nu),\quad r(z)=4z^3-g_2z-g_3. \tag{1.26}$$

Here α and β are the parameters associated to the edges of the cell presented in Figure 1.1.

The Q_4 equation given by (1.24d) is in the so-called *Adler's form*, first found in [1]. This equation possesses two other reparametrizations, one in terms of points on a Weierstraß elliptic curve [120] called the *Nijhoff's form*, and one in terms of Jacobi elliptic functions [82], called the *Hietarinta's form*.

In [2] a third family of equations consisting of two members is presented, namely the A *family*. However it was proved in the same paper that such family is included in a sub-family of the Q one through a non-autonomous Möbius transformation. From the discussion in Section 1.4 and in Section 1.5 it will be clear why we can safely omit such family.

After the introduction of the ABS equations J. Hietarinta tried to weaken its hypotheses. First in [81] he made a new classification with no assumption about the symmetry and the tetrahedron property. Therein he found the following new equation:

$$J_1\colon \frac{u_{n,m}+e_2}{u_{n,m}+e_1}\frac{u_{n+1,m+1}+o_2}{u_{n+1,m+1}+o_1}=\frac{u_{n+1,m}+e_2}{u_{n+1,m}+o_1}\frac{u_{n,m+1}+o_2}{u_{n,m+1}+e_1}, \quad (1.27)$$

where e_i and o_i are constants.

Later in [82] he released just the tetrahedron property, maintaining the symmetries of the square and he found the following equations:

$$J_2\colon u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1}=0, \quad (1.28a)$$

$$J_3\colon u_{n,m}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}=0, \quad (1.28b)$$

$$\begin{aligned} J_4\colon & u_{n,m}u_{n+1,m}u_{n,m+1}+u_{n,m}u_{n+1,m}u_{n+1,m+1} \\ & +u_{n,m}u_{n,m+1}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}u_{n+1,m+1} \\ & +u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1}=0. \end{aligned} \quad (1.28c)$$

It is worth to note that all these J equations, namely (1.27) and (1.28), are linear or linearizable [138]. Furthermore it was proved in [69] that the J equations (1.28) are also *Darboux integrable*, shedding light on the origin of integrability. This property will be discussed in Chapter 4, and its discovery was a crucial step which led to a better understanding of the relations between Consistency Around the Cube and linearizability.

1.3.2 *Boll's classification: non-autonomous lattices*

New classes of equations where introduced by a generalization of the Consistency Around the Cube technique given in [3]. Let us astray from the quad equation as equation embedded on a lattice, and just think it as multi-affine relation between some *a priori* independent fields x, x_1, x_2, x_{12} with edge parameters α_1 and α_2:

$$Q\left(x,x_1,x_2,x_{12},\alpha_1,\alpha_2\right)=0. \quad (1.29)$$

The situation is that pictured in Figure 1.3, where the cell is single and yet not embedded in any lattice.

In [3] the authors considered then a more general perspective in the classification problem. They assumed that on faces of the consistency cube, A, B, C and $\bar{A}$, $\bar{B}$ and $\bar{C}$ are *different quad equations* of the form (1.29). Furthermore they made no assumption either of the symmetry

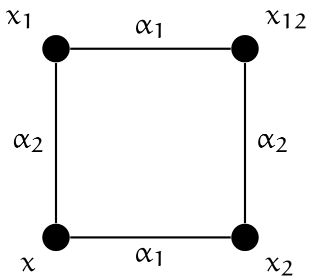

Figure 1.3: The purely geometric quad graph not embedded in any lattice.

of the square (1.22) nor of the tetrahedron property. They considered six-tuples of (a priori different) quad equations assigned to the faces of a 3D cube:

$$A\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) = 0, \tag{1.30a}$$
$$\bar{A}\left(x_3, x_{13}, x_{23}, x_{123}; \alpha_1, \alpha_2\right) = 0, \tag{1.30b}$$
$$B\left(x, x_2, x_3, x_{23}; \alpha_3, \alpha_2\right) = 0, \tag{1.30c}$$
$$\bar{B}\left(x_1, x_{12}, x_{13}, x_{123}; \alpha_3, \alpha_2\right) = 0, \tag{1.30d}$$
$$C\left(x, x_1, x_3, x_{13}; \alpha_1, \alpha_3\right) = 0, \tag{1.30e}$$
$$\bar{C}\left(x_2, x_{12}, x_{23}, x_{123}; \alpha_1, \alpha_3\right) = 0, \tag{1.30f}$$

see Figure 1.4. Such a six-tuple is then defined to be *3D consistent* if, for arbitrary initial data x, x_1, x_2 and x_3, the three values for x_{123} (calculated by using $\bar{A} = 0$, $\bar{B} = 0$ and $\bar{C} = 0$) coincide. As a result in [3] the authors obtained the same Q family equations of [2]. In addition some new quad equations of type H. These new equations turned out to be deformations of those present up above (1.23).

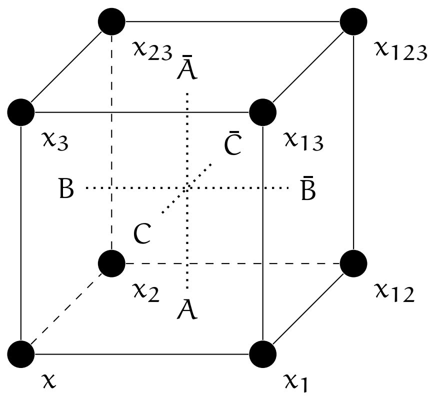

Figure 1.4: Equations on a Cube

The new equations were the *rhombic* H^4 *equations* which possess the symmetries of the *rhombus*:

$$\begin{aligned} Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) &= \mu Q(x, x_2, x_1, x_{12}, \alpha_2, \alpha_1) \\ &= \mu' Q(x_{12}, x_1, x_2, x, \alpha_2, \alpha_1), \end{aligned} \tag{1.31}$$

with $\mu, \mu' \in \{\pm 1\}$ and are given by:

$$ {}_{r}H_1^{\varepsilon} \colon (x - x_{12})(x_1 - x_2) - (\alpha_1 - \alpha_2)(1 + \varepsilon x_1 x_2), \tag{1.32a} $$

$$ \begin{aligned} {}_{r}H_2^{\varepsilon} \colon & (x - x_{12})(x_1 - x_2) + (\alpha_2 - \alpha_1)(x + x_1 + x_2 + x_{12}) \\ & + \varepsilon(\alpha_2 - \alpha_1)(2x_1 + \alpha_1 + \alpha_2)(2x_2 + \alpha_1 + \alpha_2) \\ & + \varepsilon(\alpha_2 - \alpha_1)^3 - \alpha_1^2 + \alpha_2^2 \end{aligned} \tag{1.32b} $$

$$ \begin{aligned} {}_{r}H_3^{\varepsilon} \colon & \alpha_1(x x_1 + x_2 x_{12}) - \alpha_2(x x_2 + x_1 x_{12}) \\ & + (\alpha_1^2 - \alpha_2^2)\left(\delta + \frac{\varepsilon x_1 x_2}{\alpha_1 \alpha_2}\right) \end{aligned} \tag{1.32c} $$

Note that as $\varepsilon \to 0$ these equations reduce to the H family (1.23) discussed before.

As these equations do not possess anymore the symmetry of the square (1.22) they cannot be embedded in a lattice with a fundamental cell of size one, but the size of the elementary cell should be bigger, equal to two. This implies that the corresponding equation on the lattice will be a non-autonomous equation [3]. The explicit form of the non-autonomous equation was displayed in [166]. We will postpone to Section 1.4 the discussion on how this reasoning is carried out since we will present a more general case.

In [20–22], Boll, starting from [3], classified all the consistent equations on the quad graph possessing the tetrahedron property only The results were summarized by Boll in [21] in a set of theorems, from Theorem **3.9** to Theorem **3.14**, listing all the consistent six-tuples configurations (1.30) up to Möb^8, the group of independent Möbius transformations of the eight fields on the vertexes of the consistency three dimensional cube, see Figure 1.4. The essential tool used in the classification where the *bi-quadratics*, i.e. the expression:

$$ h_{ij} = \frac{\partial Q}{\partial x_k}\frac{\partial Q}{\partial x_l} - Q\frac{\partial^2 Q}{\partial x_k\, \partial x_l}, \tag{1.33} $$

where the pair $\{k, l\}$ is the complement of the pair $\{i, j\}$ in $\{0, 1, 2, 12\}$[2]. A bi-quadratic is called *degenerate* if it contains linear factors of the form $x_i - c$, with $c \in \mathbb{C}$ a constant, otherwise a bi-quadratic is called *non-degenerate*. The three families are classified depending on how many bi-quadratics are are degenerate:

- Q family: all the bi-quadratics are non-degenerate,
- H^4 family: four bi-quadratics are degenerate,
- H^6 family: all of the six bi-quadratics are degenerate.

Let us notice that the Q family is the same as presented in [2, 3]. The H^4 equations are divided into two subclasses: the *rhombic* one, which

2 Here $x_0 = x$.

we discussed above, and the *trapezoidal* one The rhombic symmetry is given by (1.31), whereas the trapezoidal one is given by [3]:

$$Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = Q(x_1, x, x_{12}, x_2, \alpha_1, \alpha_2). \quad (1.34)$$

The trapezoidal symmetry is an invariance with respect to the axis parallel to (x_1, x_{12}). There might be a trapezoidal symmetry also with respect to the reflection around an axis parallel to (x_2, x_{12}), but this can be reduced to the previous one by a rotation. So there is no need to treat such symmetry but it is sufficient to consider (1.34).

We remark that a simplest trapezoidal equation appeared, without purpose of classification already in [3].

1.3.3 *The equations on the single cell*

In **Theorems 3.9 – 3.14** [21], Boll classified up to the action of the group Möb^8 every consistent six-tuples of equations with the tetrahedron property. Here we consider all independent quad equations defined on a single cell not of type Q (Q_1^ε, Q_2^ε, Q_3^ε and Q_4) or rhombic H^4 (${}_rH_1^\varepsilon$, ${}_rH_2^\varepsilon$ and ${}_rH_3^\varepsilon$) as these two families have been already studied extensively [2, 3, 102, 139, 166]. By independent we mean that the equations are defined up to the action of the group Möb^4 on the fields, rotations, translations and inversions of the reference system. By reference system we mean those two vectors applied on the point x which define the two oriented directions i and j upon which the elementary square is constructed. The vertex of the square lying on direction i (j) is then indicated by x_i (x_j). The remaining vertex is then called x_{ij}. In Fig. 1.3 one can see an elementary square where $i = 1$ and $j = 2$ or viceversa.

The list we present in the following expands the analogous one given by **Theorems 2.8–2.9** in [21], where the author does not distinguish between different arrangements of the fields x_i, over the four corners of the elementary square. Different choices reflects in different *bi-quadratics patterns* and, for any system presented in **Theorems 2.8–2.9** in [21], it is easy to see that a maximum of three different choices may arise up to rotations, translations and inversions.

The list obtain consists of nine different representatives, three of H^4-type and six of H^6-type. We list them with their quadruples of discriminants, as defined by equation (1.33) and we identify the six-tuple where the equation appears by the theorem number indicated in [21] in the form **3.a.b**, where **b** is the order of the six-tuple into the Theorem **3.a**.

The independent trapezoidal equations of type H^4 are:

${}_tH_1^\varepsilon$, $(\varepsilon^2, \varepsilon^2, 0, 0)$: Eq. B of **3.10.1**.

$$(x - x_2)(x_3 - x_{23}) - \alpha_2(1 + \varepsilon^2 x_3 x_{23}) = 0. \quad (1.35a)$$

${}_tH_2^\varepsilon$, $(1+4\varepsilon x, 1+4\varepsilon x_2, 1, 1)$: Eq. B of **3.10.2**.

$$\begin{aligned}&(x-x_2)(x_3-x_{23})+\alpha_2(x+x_2+x_3+x_{23})\\&+\frac{\varepsilon\alpha_2}{2}(2x_3+2\alpha_3+\alpha_2)(2x_{23}+2\alpha_3+\alpha_2)\\&+(\alpha_2+\alpha_3)^2-\alpha_3^2+\frac{\varepsilon\alpha_2^3}{2}=0.\end{aligned} \tag{1.35b}$$

${}_tH_3^\varepsilon$, $\left(x^2-4\delta^2\varepsilon^2, x_2^2-4\delta^2\varepsilon^2, x_3^2, x_{23}^2\right)$: Eq. B of **3.10.3**.

$$\begin{aligned}&e^{2\alpha_2}(xx_{23}+x_2x_3)-(xx_3+x_2x_{23})\\&-e^{2\alpha_3}\left(e^{4\alpha_2}-1\right)\left(\delta^2+\frac{\varepsilon^2x_3x_{23}}{e^{4\alpha_3+2\alpha_2}}\right)=0.\end{aligned} \tag{1.35c}$$

The independent equations of type H^6 are:

D_1, $(0,0,0,0)$: Eq. A of **3.12.1** and **3.13.1**.

$$x+x_1+x_2+x_{12}=0. \tag{1.36a}$$

This equation is linear and invariant under any exchange of the fields.

${}_1D_2$, $\left(\delta_1^2, (\delta_1\delta_2+\delta_1-1)^2, 1, 0\right)$: Eq. A of **3.12.2**.

$$\delta_2x+x_1+(1-\delta_1)x_2+x_{12}(x+\delta_1x_2)=0. \tag{1.36b}$$

D_3, $(4x,1,1,1)$: Eq. A of **3.12.3**.

$$x+x_1x_2+x_1x_{12}+x_2x_{12}=0. \tag{1.36c}$$

This equation is invariant under the exchange $x_1\leftrightarrow x_2$.

${}_1D_4$, $\left(x^2+4\delta_1\delta_2\delta_3, x_1^2, x_{12}^2, x_2^2\right)$: Eq. A of **3.12.4**.

$$xx_{12}+x_1x_2+\delta_1x_1x_{12}+\delta_2x_2x_{12}+\delta_3=0. \tag{1.36d}$$

This equation is invariant under the simultaneous exchanges $x_1\leftrightarrow x_2$ and $\delta_1\leftrightarrow\delta_2$.

${}_2D_2$, $\left(\delta_1^2, 0, 1\,(\delta_1\delta_2+\delta_1-1)^2\right)$: Eq. C of **3.13.2**.

$$\delta_2x+(1-\delta_1)x_3+x_{13}+x_1(x+\delta_1x_3-\delta_1\lambda)-\delta_1\delta_2\lambda=0. \tag{1.36e}$$

${}_3D_2$, $\left(\delta_1^2, 0, (\delta_1\delta_2+\delta_1-1)^2, 1\right)$: Eq. C of **3.13.3**.

$$\delta_2x+x_3+(1-\delta_1)x_{13}+x_1(x+\delta_1x_{13}-\delta_1\lambda)-\delta_1\delta_2\lambda=0. \tag{1.36f}$$

${}_2D_4$, $(x^2 + 4\delta_1\delta_2\delta_3, x_1^2, x_2^2, x_{12}^2)$: Eq. A of **3.13.5**.

$$xx_1 + \delta_2 x_1 x_2 + \delta_1 x_1 x_{12} + x_2 x_{12} + \delta_3 = 0. \qquad (1.36g)$$

This equation is invariant under the simultaneous exchanges $x_2 \leftrightarrow x_{12}$ and $\delta_1 \leftrightarrow \delta_2$

Let us note that at difference from the rhombic H^4 equations (1.32), which as stated above are ε-deformations of the H equations in the ABS classification [2] the trapezoidal H^4 equations in the limit $\varepsilon \to 0$ keep their discrete symmetry. Such class is then completely new with respect the ABS classification and the "deformed" and the "undeformed" equations share the same discrete symmetry.

Up to now we have written the result of the classification on a single cell and no dynamical system over the entire lattice yet exists. Therefore we pass now to the discussion of the embedding in the 2D and 3D lattices.

1.4 CONSTRUCTION OF THE 2D/3D LATTICE IN THE GENERAL CASE

Let us assume to have a geometric quad equation in the form (1.29). We need to embed it into into a $\mathbb{Z}^2$ lattice with an elementary cell of size greater than one. To do so we have to impose a lattice structure which preserves the properties of the quad equation (1.29). Following [20], one reflects the square with respect to the normal to its right and to the top and then complete a 2×2 cell by reflecting again one of the obtained equation with respect to the other direction. Let us note that, whatsoever side we reflect, the result of the last reflection is the same. Such a procedure is graphically described in Figure 1.5, and at the level of the quad equation this corresponds to constructing the three equations obtained from (1.29) by flipping its arguments:

$$Q = Q(x, x_1, x_2, x_{12}, \alpha_1, \alpha_2) = 0, \qquad (1.37a)$$

$$|Q = Q(x_1, x, x_{12}, x_2, \alpha_1, \alpha_2) = 0, \qquad (1.37b)$$

$$\underline{Q} = Q(x_2, x_{12}, x, x_1, \alpha_1, \alpha_2) = 0, \qquad (1.37c)$$

$$|\underline{Q} = Q(x_{12}, x_2, x_1, x, \alpha_1, \alpha_2) = 0. \qquad (1.37d)$$

By paving the whole $\mathbb{Z}^2$ with such equations we get a partial difference equation, which we can in principle study with the known methods. Since *a priori* $Q \neq |Q \neq \underline{Q} \neq |\underline{Q}$ the obtained lattice will be a four stripe lattice, i.e. an extension of the Black and White lattice considered for example in [3, 85, 166].

Let us notice that if the quad-equation Q possess the symmetries of the square given by (1.22), one has:

$$Q = |Q = \underline{Q} = |\underline{Q} \qquad (1.38)$$

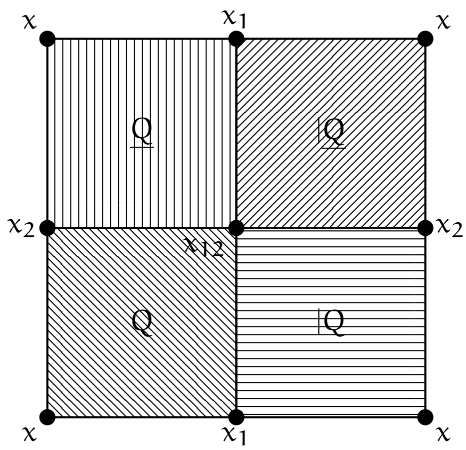

Figure 1.5: The "four colors" lattice

implying that the elementary cell is actually of size one, and one falls into the case of the ABS classification.

On the other hand in the case of the *rhombic symmetry* as given by (1.31), we can see from the explicit form of the rhombic equations themselves (1.32) that holds the relation:

$$Q = |\underline{Q}, \qquad \underline{Q} = |Q. \tag{1.39}$$

This means that in the case of the equations with rhombic symmetries this construction yields the Black and White lattice considered in [3, 85, 166]. The geometric picture of this case is given in Figure 1.6.

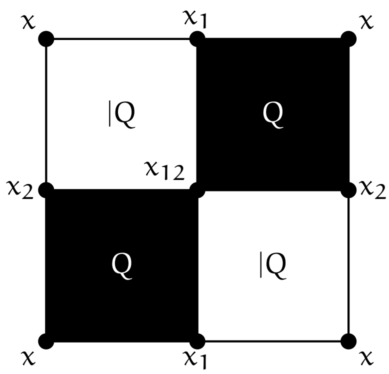

Figure 1.6: Rhombic Black and White lattice

Lastly in the case of quad equations invariant under *trapezoidal symmetry* as given by (1.34) we have the following equality:

$$Q = |Q, \qquad \underline{Q} = |\underline{Q}. \tag{1.40}$$

This means that in the case of the equations with trapezoidal symmetries this construction yields the Black and White lattice considered as stated in [21]. The geometric picture of this case is given in Figure 1.7.

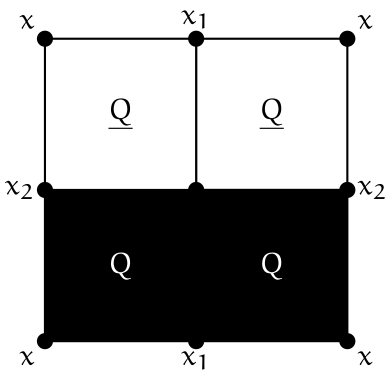

Figure 1.7: Trapezoidal Black and White lattice

Now by paving the whole $\mathbb{Z}^2$ with the elementary cell defined in Figure 1.5 and choosing the origin on the $\mathbb{Z}^2$ lattice in the point x we obtain a lattice equation of the following form:

$$\widetilde{Q}\,[u] = \begin{cases} Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) & n = 2k, m = 2k,\ k \in \mathbb{Z}, \\ |Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) & n = 2k+1, m = 2k,\ k \in \mathbb{Z}, \\ \underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) & n = 2k, m = 2k+1,\ k \in \mathbb{Z}, \\ |\underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) & n = 2k+1, m = 2k+1,\ k \in \mathbb{Z}, \end{cases} \tag{1.41}$$

We could have constructed $\widetilde{Q}\,[u]$ starting from any other point in Figure 1.5 as the origin, but such equations would differ from each other only by a translation, a rotation or a reflection. So in the sense of the discussion made in Subsection 1.3.3 they will be equivalent to (1.41) and we will not discuss them.

Example 1.4.1. In the case of ${}_rH_1^{\varepsilon}$ (1.32a) we have:

$${}_r\widetilde{H}_1^{\varepsilon} = \begin{cases} (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) & \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n+1,m} u_{n,m+1}), & |n| + |m| = 2k, \quad k \in \mathbb{Z}, \\ (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) & \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n,m} u_{n+1,m+1}), & |n| + |m| = 2k+1, \quad k \in \mathbb{Z}, \end{cases} \tag{1.42}$$

where we have used the symmetry properties of the equation ${}_rH_1^{\varepsilon}$. This result coincide with that presented in [166]. □

We shall now consider quad equations which possess the Consistency Around the Cube, since we are concerned about integrability. So let us consider six-tuples of quad equations (1.30) assigned to the faces of a 3D cube as displayed in Figure 1.4.

First let us notice that, without loss of generality, we can assume that, if Q is the consistent quad equation we are interested in, then we may assume that Q is the bottom equation i.e. $Q = A$. Indeed if we are interested in an equation on the side of the cube of Figure 1.4 and these equations are different from A (once made the appropriate substitutions) we may just rotate it and re-label the vertices in an appropriate manner, so that our side equation will become the bottom equation. In this way following again [20] and taking into account the result stated above we may build an embedding in $\mathbb{Z}^3$, whose points we shall label by triples (n, m, p), of the consistency cube. To this end we reflect the consistency cube with respect to the normal of the back and the right side and then complete again with another reflection, just in the same way we did for the square. Using the same notations as in the planar case we see which are the proper equations which must be put on the sides of the "multi-cube". Their form can therefore be described as in (1.41). As a result we end up with Figure

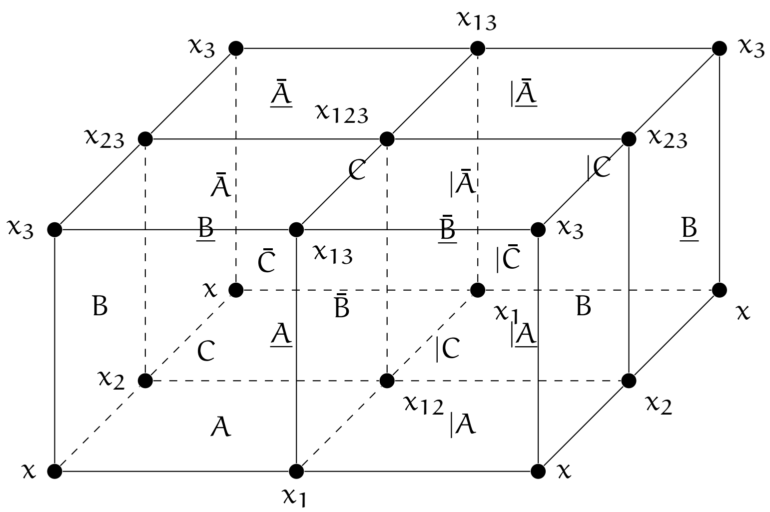

Figure 1.8: The extension of the consistency cube.

1.8, where the functions appearing on the top and on the bottom can be defined as in (1.41)[3] while on the sides we shall have:

$$\underline{B}(x, x_2, x_3, x_{23}) = B(x_2, x, x_{23}, x_3), \tag{1.43a}$$

$$\underline{\bar{B}}(x_1, x_{12}, x_{13}, x_{123}) = \bar{B}(x_{12}, x_1, x_{123}, x_{13}), \tag{1.43b}$$

$$|C(x, x_1, x_3, x_{13}) = C(x_1, x, x_{13}, x_3), \tag{1.43c}$$

$$|\bar{C}(x_2, x_{12}, x_{23}, x_{123}) = \bar{C}(x_{12}, x_2, x_{123}, x_{23}). \tag{1.43d}$$

3 Obviously in the case of $\bar{A}$ one should traslate every point by one unit in the p direction.

From (1.43) we obtain the extension to $\mathbb{Z}^3$ of (1.41). We have a new consistency cube, see Figure 1.8, with equations given by[4]:

$$\widetilde{\bar{A}}\left[u\right] = \begin{cases} \bar{A}(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ |\bar{A}(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ \underline{\bar{A}}(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ |\underline{\bar{A}}(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \end{cases} \tag{1.44a}$$

$$\widetilde{B}\left[u\right] = \begin{cases} B(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}), \\ \bar{B}(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}), \\ \underline{B}(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}), \\ \underline{\bar{B}}(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}), \end{cases} \tag{1.44b}$$

$$\widetilde{\bar{B}}\left[u\right] = \begin{cases} \bar{B}(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}), \\ B(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}), \\ \underline{\bar{B}}(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}), \\ \underline{B}(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}), \end{cases} \tag{1.44c}$$

$$\widetilde{C}\left[u\right] = \begin{cases} C(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}), \\ |C(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}), \\ \bar{C}(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}), \\ |\bar{C}(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}), \end{cases} \tag{1.44d}$$

$$\widetilde{\bar{C}}\left[u\right] = \begin{cases} \bar{C}(u_{n,m+1,p}, u_{n+1,m+1,p}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ |\bar{C}(u_{n,m+1,p}, u_{n+1,m+1,p}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ C(u_{n,m,p}, u_{n+1,m+1,p}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}), \\ |\bar{C}(u_{n,m+1,p}, u_{n+1,m+1,p}, u_{n,m+1,p+1}, u_{n+1,m,+1p+1}), \end{cases} \tag{1.44e}$$

Formula (1.44) means that the "multi-cube" of Figure 1.8 appears as the usual consistency cube of Figure 1.2 with the following identifications:

$$A \rightsquigarrow \widetilde{A}, \quad \bar{A} \rightsquigarrow \widetilde{\bar{A}}, \quad B \rightsquigarrow \widetilde{B}, \quad \bar{B} \rightsquigarrow \widetilde{\bar{B}}, \quad C \rightsquigarrow \widetilde{C}, \quad \bar{C} \rightsquigarrow \widetilde{\bar{C}}. \tag{1.45}$$

4 For the sake of the simplicity of the presentation we have left implied when n and m are even or odd integers. They are recovered by comparing with (1.41).

Example 1.4.2. Let us consider again the equation ${}_{r}H_1^{\varepsilon}$ (1.32a). This equation comes from the six-tuple [3]:

$$A = (x - x_{12})(x_1 - x_2) + (\alpha_1 - \alpha_2)(1 + \varepsilon x_1 x_2), \tag{1.46a}$$
$$\bar{A} = (x_3 - x_{123})(x_{13} - x_{23}) + (\alpha_1 - \alpha_2)(1 + \varepsilon x_3 x_{123}), \tag{1.46b}$$
$$B = (x - x_{23})(x_2 - x_3) + (\alpha_2 - \alpha_3)(1 + \varepsilon x_2 x_3), \tag{1.46c}$$
$$\bar{B} = (x_1 - x_{123})(x_{12} - x_{13}) + (\alpha_2 - \alpha_3)(1 + \varepsilon x_1 x_{123}), \tag{1.46d}$$
$$C = (x - x_{13})(x_1 - x_3) + (\alpha_1 - \alpha_3)(1 + \varepsilon x_1 x_3), \tag{1.46e}$$
$$\bar{C} = (x_2 - x_{123})(x_{12} - x_{23}) + (\alpha_1 - \alpha_2)(1 + \varepsilon x_2 x_{123}), \tag{1.46f}$$

therefore from (1.44, 1.45) we get the following consistent system on the "multi-cube":

$$A = \begin{cases} (u_{n,m,p} - u_{n+1,m+1,p})(u_{n+1,m,p} - u_{n,m+1,p}) \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n+1,m,p} u_{n,m+1,p}), & |n| + |m| = 2k,\ k \in \mathbb{Z}, \\ (u_{n,m,p} - u_{n+1,m+1,p})(u_{n+1,m,p} - u_{n,m+1,p}) \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n,m,p} u_{n+1,m+1,p}), & |n| + |m| = 2k+1,\ k \in \mathbb{Z}, \end{cases} \tag{1.47a}$$

$$\bar{A} = \begin{cases} (u_{n,m,p+1} - u_{n+1,m+1,p+1})(u_{n+1,m,p+1} - u_{n,m+1,p+1}) \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n,m,p+1} u_{n+1,m+1,p+1}), & |n| + |m| = 2k,\ k \in \\ (u_{n,m,p+1} - u_{n+1,m+1,p+1})(u_{n+1,m,p+1} - u_{n,m+1,p+1}) \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n+1,m,p+1} u_{n,m+1,p+1}), & |n| + |m| = 2k+1,\ k \end{cases} \tag{1.47b}$$

$$B = \begin{cases} (u_{n,m,p} - u_{n,m+1,p+1})(u_{n,m+1,p} - u_{n,m,p+1}) \\ \quad -(\alpha_2 - \alpha_3)(1 + \varepsilon^2 u_{n,m+1,p} u_{n,m,p+1}), & |n| + |m| = 2k,\ k \in \mathbb{Z}, \\ (u_{n,m,p} - u_{n,m+1,p+1})(u_{n,m+1,p} - u_{n,m,p+1}) \\ \quad -(\alpha_2 - \alpha_3)(1 + \varepsilon^2 u_{n,m,p} u_{n,m+1,p+1}), & |n| + |m| = 2k+1,\ k \in \mathbb{Z}, \end{cases} \tag{1.47c}$$

$$\bar{B} = \begin{cases} (u_{n+1,m,p} - u_{n,m+1,p+1})(u_{n+1,m+1,p} - u_{n+1,m,p+1}) \\ \quad -(\alpha_2 - \alpha_3)(1 + \varepsilon^2 u_{n+1,m,p} u_{n+1,m+1,p+1}, & |n| + |m| = 2k,\ k \in \\ (u_{n+1,m,p} - u_{n+1,m+1,p+1})(u_{n+1,m+1,p} - u_{n+1,m,p+1}) \\ \quad -(\alpha_2 - \alpha_3)(1 + \varepsilon^2 u_{n+1,m+1,p} u_{n+1,m,p+1}), & |n| + |m| = 2k+1,\ k \end{cases} \tag{1.47d}$$

$$C = \begin{cases} (u_{n,m,p} - u_{n+1,m,p+1})(u_{n+1,m,p} - u_{n,m,p+1}) \\ \quad -(\alpha_1 - \alpha_3)(1 + \varepsilon^2 u_{n+1,m,p} u_{n,m,p+1}), & |n| + |m| = 2k,\ k \in \mathbb{Z}, \\ (u_{n,m,p} - u_{n+1,m,p+1})(u_{n+1,m,p} - u_{n,m,p+1}) \\ \quad -(\alpha_1 - \alpha_3)(1 + \varepsilon^2 u_{n,m,p} u_{n+1,m,p+1}), & |n| + |m| = 2k+1,\ k \in \mathbb{Z}, \end{cases} \tag{1.47e}$$

$$\bar{C} = \begin{cases} (u_{n,m+1,p} - u_{n+1,m+1,p+1})(u_{n+1,m+1,p} - u_{n,m+1,p+1}) \\ \quad -(\alpha_1 - \alpha_3)(1 + \varepsilon^2 u_{n,m+1,p} u_{n+1,m+1,p+1}), & |n| + |m| = 2k,\ k \in \\ (u_{n,m+1,p} - u_{n+1,m+1,p+1})(u_{n+1,m+1,p} - u_{n,m+1,p+1}) \\ \quad -(\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n+1,m+1,p} u_{n,m+1,p+1}), & |n| + |m| = 2k+1,\ k \end{cases} \tag{1.47f}$$

The reader can easily check that the equations in (1.47) possess the Consistency Around the Cube in the form presented in Section 1.2. □

Up to now we showed how, given a quad equation of the form (1.29) which possess the property of the Consistency Around the Cube, it is possible to embed it into a quad equation in $\mathbb{Z}^2$ given by (1.41). Furthermore we showed that this procedure can be extended along the third dimension in such a way that the consistency is preserved. This have been done following [20] and filling the details (which are going to be important).

The quad difference equation (1.41) is not very manageable since we have to change equation according to the point of the lattice we are in. It will be more efficient to have an expression which "knows" by itself in which point we are. This can obtained by going over to non-autonomous equations as was done in the BW lattice case [166].

We shall present here briefly how from (1.41) it is possible to construct an equivalent non-autonomous system, and moreover how to construct the non-autonomous version of CAC (1.44).

We take an equation $\hat{Q}$ constructed by a linear combination of the equations (1.37) with n and m depending coefficients:

$$\begin{aligned}\widehat{Q} = {} & f_{n,m}\, Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) + \\ & + |f_{n,m}\, |Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) + \\ & + \underline{f}_{n,m}\, \underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) + \\ & + |\underline{f}_{n,m}\, |\underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}).\end{aligned} \tag{1.48}$$

We require that it satisfies the following conditions:

1. The coefficients are periodic of period 2 in both directions, since, in the $\mathbb{Z}^2$ embedding, the elementary cell is a 2×2 one.

2. The coefficients are such that they produce the right equation in a given lattice point as specified in (1.41).

Condition 1 implies that any function $\tilde{f}_{n,m}$ in (1.48), i.e. either $f_{n,m}$ or $|f_{n,m}$ or $\underline{f}_{n,m}$ or $|\underline{f}_{n,m}$, solves the two ordinary difference equations:

$$\tilde{f}_{n+2,m} - \tilde{f}_{n,m} = 0, \quad \tilde{f}_{n,m+2} - \tilde{f}_{n,m} = 0, \tag{1.49}$$

whose solution is:

$$\tilde{f}_{n,m} = c_0 + c_1 (-1)^n + c_2 (-1)^m + c_3 (-1)^{n+m} \tag{1.50}$$

with c_i constants to be determined.

The condition 2 depends on the choice of the equation in (1.41) and will give some "boundary conditions" for the function $\tilde{f}$, allowing us to fix the coefficients c_i. For $f_{n,m}$, for example, we have, substituting the appropriate lattice points, the following conditions:

$$f_{2k,2k} = 1, \quad f_{2k+1,2k} = f_{2k,2k+1} = f_{2k+1,2k+1} = 0, \tag{1.51}$$

which yield:

$$f_{n,m} = \frac{1+(-1)^n+(-1)^m+(-1)^{n+m}}{4}. \tag{1.52a}$$

In an analogous manner we obtain the form of the other functions in (1.48):

$$|f_{n,m} = \frac{1-(-1)^n+(-1)^m-(-1)^{n+m}}{4}, \tag{1.52b}$$

$$\underline{f}_{n,m} = \frac{1+(-1)^n-(-1)^m-(-1)^{n+m}}{4}, \tag{1.52c}$$

$$|\underline{f}_{n,m} = \frac{1-(-1)^n-(-1)^m+(-1)^{n+m}}{4}. \tag{1.52d}$$

Then inserting (1.52) in (1.48) we obtain a non-autonomous equation which corresponds to (1.41). Note that finally this quad equation is a quad equation in the form (1.1) with non-autonomous coefficients. We can say that these kind of equations are *weakly non-autonomous*, since as we will see in Chapter 2 and in 4 this kind of non-autonomicity can be easily removed by introducing more components.

If the quad-equation Q possess some discrete symmetries, the expression (1.48) greatly simplify. If an equation Q possess the symmetries of the square, we trivially have, using (1.38), $\widehat{Q} = Q$. This result states that an equation with the symmetry (1.38) is defined on a monochromatic lattice, as expected since we are in the case of the ABS classification [2]. If the equation Q has the symmetries of the *rhombus*, namely (1.39), we get:

$$\widehat{Q} = (f_{n,m} + |\underline{f}_{n,m})Q + (|f_{n,m} + \underline{f}_{n,m})|Q. \tag{1.53}$$

Using (1.52):

$$f_{n,m} + |\underline{f}_{n,m} = F^{(+)}_{n+m}, \quad |f_{n,m} + \underline{f}_{n,m} = F^{(-)}_{n+m}. \tag{1.54}$$

where:

$$F^{(\pm)}_k = \frac{1 \pm (-1)^k}{2}, \quad k \in \mathbb{Z}, \tag{1.55}$$

we obtain:

$$\widehat{Q} = F^{(+)}_{n+m} Q + F^{(-)}_{n+m} |Q. \tag{1.56}$$

This obviosly match with the results in [166].

In the case of *trapezoidal* symmetry (1.40) one obtains:

$$\widehat{Q} = (f_{n,m} + |f_{n,m})Q + (\underline{f}_{n,m} + |\underline{f}_{n,m})\underline{Q}. \tag{1.57}$$

Therefore using that:

$$f_{n,m} + |f_{n,m} = F^{(+)}_m, \quad \underline{f}_{n,m} + |\underline{f}_{n,m} = F^{(-)}_m, \tag{1.58}$$

we obtain:

$$\widehat{Q} = F^{(+)}_m Q + F^{(-)}_m \underline{Q}. \tag{1.59}$$

Example 1.4.3. As an example of such construction let us consider again ${}_rH_1^\varepsilon$ (1.32a). Since we are in the rhombic case [166] we use formula (1.54) and get:

$$
\begin{aligned}
{}_r\widehat{H}_1^\varepsilon &= (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - (\alpha_1 - \alpha_2) \\
&\quad + (\alpha_1 - \alpha_2)\varepsilon^2 \left(F^{(+)}_{n+m}\, u_{n+1,m} u_{n,m+1} + |F^{(-)}_{n+m}\, u_{n,m} u_{n+1,m+1} \right) = 0,
\end{aligned} \tag{1.60}
$$

which corresponds to the case $\sigma = 1$ of [166]. The discussion of the absence of the parameter σ introduced in [166] from our theory is postponed to the end this Section. □

The consistency of a generic system of quad equations is obtained by considering the consistency of the tilded equations as displayed in (1.45). We now construct, starting from (1.45), the non autonomous partial difference equations in the (n, m) variables using the weights $\tilde{f}_{n,m}$, as given in (1.48), applied to the relevant equations. Carrying out such construction, we end with the following six-tuple of equations:

$$
\begin{aligned}
\widehat{A}(u_{n,m,p}, u_{n+1,m,p}, u_{n,m+1,p}, u_{n+1,m+1,p}) &= f_{n,m} A + |f_{n,m}| A \\
&\quad + \underline{f}_{n,m} \underline{A} + |\underline{f}_{n,m}| \underline{A} = 0,
\end{aligned} \tag{1.61a}
$$

$$
\begin{aligned}
\widehat{\bar{A}}(u_{n,m,p+1}, u_{n+1,m,p+1}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}) &= f_{n,m} \bar{A} + |f_{n,m}| \bar{A} + \\
&\quad + \underline{f}_{n,m} \underline{\bar{A}} + |\underline{f}_{n,m}| \underline{\bar{A}} = 0,
\end{aligned} \tag{1.61b}
$$

$$
\begin{aligned}
\widehat{B}(u_{n,m,p}, u_{n,m+1,p}, u_{n,m,p+1}, u_{n,m+1,p+1}) &= f_{n,m} B + |f_{n,m}| \bar{B} + \\
&\quad + \underline{f}_{n,m} \underline{B} + |\underline{f}_{n,m}| \underline{\bar{B}} = 0,
\end{aligned} \tag{1.61c}
$$

$$
\begin{aligned}
\widehat{\bar{B}}(u_{n+1,m,p}, u_{n+1,m+1,p}, u_{n+1,m,p+1}, u_{n+1,m+1,p+1}) &= f_{n,m} \bar{B} + |f_{n,m}| B + \\
&\quad + \underline{f}_{n,m} \underline{\bar{B}} + |\underline{f}_{n,m}| \underline{B} = 0,
\end{aligned} \tag{1.61d}
$$

$$
\begin{aligned}
\widehat{C}(u_{n,m,p}, u_{n+1,m,p}, u_{n,m,p+1}, u_{n+1,m,p+1}) &= f_{n,m} C + |f_{n,m}| C + \\
&\quad + \underline{f}_{n,m} \bar{C} + |\underline{f}_{n,m}| \bar{C} = 0,
\end{aligned} \tag{1.61e}
$$

$$
\begin{aligned}
\widehat{\bar{C}}(u_{n,m+1,p}, u_{n+1,m,p}, u_{n,m+1,p+1}, u_{n+1,m+1,p+1}) &= f_{n,m} \bar{C} + |f_{n,m}| \bar{C} + \\
&\quad + \underline{f}_{n,m} C + |\underline{f}_{n,m}| C = 0,
\end{aligned} \tag{1.61f}
$$

where all the functions on the right hand side of the equality sign are evaluated on the point indicated on the left hand side.

We note that a Lax pair obtained by making use of equations (1.61) will be effectively a pair, since the couples $(\widehat{B}, \widehat{\bar{B}})$ and $(\widehat{C}, \widehat{\bar{C}})$ are related by translation so they are just two different solutions of the same equation. Indeed by using the properties of the functions $\tilde{f}_{n,m}$ we have:

$$\widehat{\bar{B}} = T_n \widehat{B}, \quad \widehat{\bar{C}} = T_m \widehat{C}, \tag{1.62}$$

where T_n is the operator of translation in the n direction, and T_m the operator of translation in the m direction. This allow us to construct Bäcklund transformations and Lax pair in the way explained in Section 1.2 [19, 123].

Example 1.4.4. As a final example we shall derive the non-autonomous side equations for H_1^ε and its Lax pair. We will then confront the result with that obtained in [166]. Considering (1.46, 1.61, 1.62) and using the fact that the equation is rhombic (1.54) we get the following result:

$$\begin{aligned}\widehat{A} = {} & (u_{n,m,p} - u_{n+1,m+1,p})(u_{n+1,m,p} - u_{n,m+1,p}) - (\alpha_1 - \alpha_2)\cdot \\ & \cdot \left[1 + \varepsilon^2 \left(F^{(+)}_{n+m}\, u_{n+1,m,p} u_{n,m+1,p} + F^{(-)}_{n+m}\, u_{n,m,p} u_{n+1,m+1,p}\right)\right] = 0,\end{aligned} \tag{1.63a}$$

$$\begin{aligned}\widehat{\bar{A}} = {} & (u_{n,m,p+1} - u_{n+1,m+1,p+1})(u_{n+1,m,p+1} - u_{n,m+1,p+1}) - (\alpha_1 - \alpha_2)\cdot \\ & \cdot \left[1 + \varepsilon^2 \left(F^{(-)}_{n+m}\, u_{n+1,m,p+1} u_{n,m+1,p+1} + F^{(+)}_{n+m}\, u_{n,m,p+1} u_{n+1,m+1,p+1}\right)\right] = 0\end{aligned} \tag{1.63b}$$

$$\begin{aligned}\widehat{B} = {} & (u_{n,m,p} - u_{n,m+1,p+1})(u_{n,m+1,p} - u_{n,m,p+1}) - (\alpha_2 - \alpha_3)\cdot \\ & \cdot \left[1 + \varepsilon^2 \left(F^{(+)}_{n+m}\, u_{n,m+1,p} u_{n,m,p+1} + F^{(-)}_{n+m}\, u_{n,m,p} u_{n,m+1,p+1}\right)\right] = 0,\end{aligned} \tag{1.63c}$$

$$\begin{aligned}\widehat{C} = {} & (u_{n,m,p} - u_{n+1,m,p+1})(u_{n+1,m,p} - u_{n,m,p+1}) - (\alpha_1 - \alpha_3)\cdot \\ & \cdot \left[1 + \varepsilon^2 \left(F^{(+)}_{n+m}\, u_{n+1,m,p} u_{n,m,p+1} + F^{(-)}_{n+m}\, u_{n,m,p} u_{n+1,m,p+1}\right)\right] = 0,\end{aligned} \tag{1.63d}$$

From the equations $\widehat{B}$ and $\widehat{C}$ we find, the following Lax pair:

$$\begin{aligned}L_{n,m} = {} & \begin{pmatrix} u_{n,m} & \alpha_1 - \alpha_3 - u_{n,m} u_{n+1,m} \\ 1 & -u_{n+1,m} \end{pmatrix} \\ & + (\alpha_1 - \alpha_3)\varepsilon^2 \begin{pmatrix} F^{(+)}_{n+m} u_{n+1,m} & 0 \\ 0 & -F^{(-)}_{n+m} u_{n,m} \end{pmatrix}\end{aligned} \tag{1.64a}$$

$$\begin{aligned}M_{n,m} = {} & \begin{pmatrix} u_{n,m} & \alpha_2 - \alpha_3 - u_{n,m} u_{n,m+1} \\ 1 & -u_{n,m+1} \end{pmatrix} \\ & + (\alpha_2 - \alpha_3)\,\varepsilon^2 \begin{pmatrix} F^{(+)}_{n+m} u_{n,m+1} & 0 \\ 0 & -F^{(-)}_{n+m} u_{n,m} \end{pmatrix},\end{aligned} \tag{1.64b}$$

$$\begin{aligned} L_{n,m+1} &= \begin{pmatrix} u_{n,m+1} & \alpha_1 - \alpha_3 - u_{n,m+1} u_{n+1,m+1} \\ 1 & -u_{n+1,m+1} \end{pmatrix} \\ &+ (\alpha_1 - \alpha_3) \varepsilon^2 \begin{pmatrix} F^{(-)}_{n+m} u_{n+1,m+1} & 0 \\ 0 & -F^{(+)}_{n+m} u_{n,m+1} \end{pmatrix} \end{aligned} \tag{1.64c}$$

$$\begin{aligned} M_{n+1,m} &= \begin{pmatrix} u_{n+1,m} & \alpha_2 - \alpha_3 - u_{n+1,m} u_{n+1,m+1} \\ 1 & -u_{n+1,m+1} \end{pmatrix} \\ &+ (\alpha_2 - \alpha_3) \varepsilon^2 \begin{pmatrix} F^{(-)}_{n+m} u_{n+1,m+1} & 0 \\ 0 & -F^{(+)}_{n+m} u_{n+1,m} \end{pmatrix}, \end{aligned} \tag{1.64d}$$

with the following separation constant (1.21):

$$\tau = \frac{1 + \varepsilon^2 \left(F^{(-)}_{n+m} u_{n,m} + F^{(+)}_{n+m} u_{n+1,m} \right)}{1 + \varepsilon^2 \left(F^{(-)}_{n+m} u_{n,m} + F^{(+)}_{n+m} u_{n,m+1} \right)}. \tag{1.65}$$

This is a Lax pair since $\bar{L} = T_m L$ and $\bar{M} = T_n M$. □

Remark 1.4.1. The Lax pair (1.64) is gauge equivalent to that obtained in [166] with gauge:

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.66}$$

A calculation similar to that of Example 1.4.4 can be done for the other two rhombic equations. Up to gauge transformations, this calculation gives, as expected, the same Lax pairs as in [166]. Indeed the gauge transformations (1.66) is needed for H_1^ε and H_2^ε whereas for H_3^ε we need the gauge:

$$\tilde{G} = \begin{pmatrix} 0 & (-1)^{n+m} \\ -(-1)^{n+m} & 0 \end{pmatrix}. \tag{1.67}$$

Now notice that at the level of the non-autonomous equations the choice of origin of $\mathbb{Z}^2$ in a point different from x in (1.41) would have led to different initial conditions in (1.51), which ultimately lead to the following form for the functions f:

$$f^{\sigma_1,\sigma_2}_{n,m} = \frac{1 + \sigma_1 (-1)^n + \sigma_2 (-1)^m + \sigma_1 \sigma_2 (-1)^{n+m}}{4}, \tag{1.68a}$$

$$|f^{\sigma_1,\sigma_2}_{n,m} = \frac{1 - \sigma_1 (-1)^n + \sigma_2 (-1)^m - \sigma_1 \sigma_2 (-1)^{n+m}}{4}, \tag{1.68b}$$

$$\underline{f}^{\sigma_1,\sigma_2}_{n,m} = \frac{1 + \sigma_1 (-1)^n - \sigma_2 (-1)^m - \sigma_1 \sigma_2 (-1)^{n+m}}{4}, \tag{1.68c}$$

$$|\underline{f}^{\sigma_1,\sigma_2}_{n,m} = \frac{1 - \sigma_1 (-1)^n - \sigma_2 (-1)^m + \sigma_1 \sigma_2 (-1)^{n+m}}{4}, \tag{1.68d}$$

where the two constants $\sigma_i \in \{\pm 1\}$ depends on the point chosen. Indeed if the point is x we have $\sigma_1 = \sigma_2 = 1$, whereas if we choose x_1 we have $\sigma_1 = 1$, $\sigma_2 = -1$, if we choose x_2 then $\sigma_1 = -1$ and $\sigma_2 = 1$ and finally if we choose x_{12} we shall put $\sigma_1 = \sigma_2 = -1$. It is easy to see that in the rhombic and in the trapezoidal case the functions (1.68) collapse to the σ version of the functions $F_k^{(\pm)}$ as given by (1.55):

$$F_k^{(\pm,\sigma)} = \frac{1 \pm \sigma(-1)^k}{2}. \tag{1.69}$$

The final equation will then depend on σ_1 or σ_2, only if rhombic or trapezoidal.

It was proved in [166] that the transformations:

$$v_{n,m} = u_{n+1,m}, \quad w_{n,m} = u_{n,m+1}, \tag{1.70}$$

map a rhombic equation with a certain σ into the same rhombic equation with $-\sigma$. In [166] this fact was used to construct a Lax Pair and Bäcklund transformations. An analogous result can be easily proven for trapezoidal equations: using the transformation $w_{n,m} = u_{n,m+1}$ we can send a trapezoidal equation with a certain σ into the same equation with $-\sigma$. A similar transformation in the n direction would just trivally leave invariant the trapezoidal equation, since there is no explicit dependence on n. However in general, if an equation does not posses discrete symmetries, as it is the case for a H^6 equation, no trasformation like (1.70) would take the equation into itself with different coefficents. We can anyway construct a Lax pair with the procedure explained above, which is then slightly more general than the approach based on the transformations (1.70).

We end this Section by remarking that the choice of the embedding is crucial to determine the integrability properties of the quad equations. Indeed this was proved in [85] where it was shown that is possible to produce many Black and White lattices of consistent yet non-integrable equations. Here we shall give just a very simple example to acquaint the reader about this fact. Let us consider again the (1.32a) equation. Suppose one makes the *trivial embedding* of such equation into a lattice given by the identification:

$$x \to u_{n,m}, \quad x_1 \to u_{n+1,m}, \quad x_2 \to u_{n,m+1}, \quad x_{12} \to u_{n+1,m+1}. \tag{1.71}$$

Then equation (1.32a) with the identification (1.71) becomes the following lattice equation:

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - (\alpha_1 - \alpha_2)(1 + \varepsilon^2 u_{n+1,m} u_{n,m+1}) = 0. \tag{1.72}$$

We apply the algebraic entropy test[5] to (1.72) and we find the following growth in the Nord-East $(-,+)$ direction of the degrees:

$$\{ d_{-,+} \} = \{ 1, 2, 4, 9, 21, 50, 120, 289 \ldots \}. \tag{1.73}$$

This sequence has generating function

$$g_{-,+} = \frac{1 - s - s^2}{s^3 + s^2 - 3s + 1} \tag{1.74}$$

and therefore has a non-zero algebraic entropy given by:

$$\eta_{-,+} = \log\left(1 + \sqrt{2}\right), \tag{1.75}$$

corresponding to the entropy of a non-integrable lattice equation. For more complex examples the reader may refer to [85].

1.5 CLASSIFICATION TOOLS ON THE 2D LATTICE

We have addressed in the previous Section the problem of the construction of the lattices out of single cells equations. In this Section we present a proof of the fact that the classification carried out at the level of single cell is preserved when passing to the lattice.

First of all recall that the classification of quad equations presented in Section 1.3.2, has been carried out up to a Möbius transformations in each vertex:

$$M\colon\ (x, x_1, x_2, x_{12}) \mapsto \left(\frac{a_0 x + b_0}{c_0 x + d_0}, \frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \frac{a_2 x_2 + b_2}{c_2 x_2 + d_2}, \frac{a_{12} x_{12} + b_{12}}{c_{12} x_{12} + d_{12}} \right). \tag{1.76}$$

As in the usual Möbius transformation we have here $(a_i, b_i, c_i, d_i) \in \mathbb{CP}^4 \setminus V(a_i d_i - b_i c_i) \simeq PGL(2, \mathbb{C})$ with $i = 0, 1, 2, 12$, i.e. each set of parameters is defined up to to a multiplication by a number [36]. Obviously as the usual Möbius transformations these transformations will form a group under composition and we shall call such group Möb^4 [14].

On the other hand when dealing with equations defined on the lattice we have to follow the prescription of Section 1.4 and use the representation given by (1.48), i.e. we will have non-autonomous lattice equations. In this Section we prove a result which extends the group Möb^4 to the level of the transformations of the non autonomous lattice equations, and shows that the classification made at the level of single cell equation is preserved up to the action of this group for the equation on the 2D lattice. We will call such group the non-autonomous lifting of Möb^4, and denote it by $\widehat{\text{Möb}}^4$. The group $\widehat{\text{Möb}}^4$

5 For more details on degree of growth, algebraic entropy and related subjects see Chapter 2 and references therein.

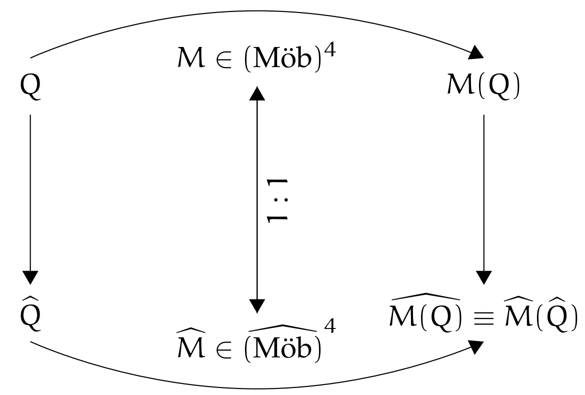

Figure 1.9: The commutative diagram defining $\widehat{\text{Möb}}^4$.

is the non-autonomous counterpart of the group Möb^4 on the four-color lattice and its existence shows that the classification at the level of single cell is preserved passing to the full lattice. Our proof is constructive: *we will first construct a candidate transformation and then we will prove that this is object we are looking for*. What we will do is to fill the entries in the commutative diagram given by Figure (1.9): we prove that the result that we get by acting with a group of transformations $\widehat{\text{Möb}}^4$ is the same of that we obtain if we first transform a single cell equation using $M \in \text{Möb}^4$ and then we construct the non-autonomous quad equation with the prescription of Section 1.4 or viceversa if we first construct the non-autonomous equation and then we transform it using the "non-autonomous lifting" of M, $\widehat{M}$.

Let us first construct, a transformation which will be the candidate of the non-autonomous lifting of $M \in \text{Möb}^4$. Given $M \in \text{Möb}^4$ using the same ideas of Section 1.4 we can construct the following transformation:

$$
\begin{aligned}
M_{n,m} \in \widehat{\text{Möb}}^4 : u_{n,m} \mapsto \quad & f_{n,m}\frac{a_0 u_{n,m}+b_0}{c_0 u_{n,m}+d_0} + |f_{n,m}\frac{a_1 u_{n,m}+b_1}{c_1 u_{n,m}+d_1} \\
& + \underline{f}_{n,m}\frac{a_2 u_{n,m}+b_2}{c_2 u_{n,m}+d_2} + |\underline{f}_{n,m}\frac{a_{12} u_{n,m}+b_{12}}{c_{12} u_{n,m}+d_{12}}.
\end{aligned}
\tag{1.77}
$$

where the functions $\tilde{f}_{n,m}$ are defined by (1.52).

Equation (1.77) gives us a mapping Φ between the group Möb^4 and a set of non-autonomous transformations of the field $u_{n,m}$. Moreover there is a one to one correspondence between an element $M \in \text{Möb}^4$

(1.76) and one $M_{n,m} \in \widehat{\text{Möb}}^4$ (1.77). This correspondence is given by:

$$\Phi\colon \begin{pmatrix} \dfrac{ax+b}{cx+d} \\ \dfrac{a_1x_1+b_1}{c_1x_1+d_1} \\ \dfrac{a_2x_2+b_2}{c_2x_2+d_2} \\ \dfrac{a_{12}x_{12}+b_{12}}{c_{12}x_{12}+d_{12}} \end{pmatrix}^T \longmapsto \begin{array}{l} f_{n,m}\dfrac{au_{n,m}+b}{cu_{n,m}+d} + |f_{n,m}\dfrac{a_1u_{n,m}+b_1}{c_1u_{n,m}+d_1} \\ + \underline{f}_{n,m}\dfrac{a_2u_{n,m}+b_2}{c_2u_{n,m}+d_2} + |\underline{f}_{n,m}\dfrac{a_{12}u_{n,m}+b_{12}}{c_{12}u_{n,m}+d_{12}}, \end{array} \tag{1.78a}$$

and its inverse is given by:

$$\Phi^{-1}\colon \begin{array}{l} f_{n,m}\dfrac{\alpha^{(0)}u_{n,m}+\beta^{(0)}}{\gamma^{(0)}u_{n,m}+\delta^{(0)}} + |f_{n,m}\dfrac{\alpha^{(1)}u_{n,m}+\beta^{(1)}}{\gamma^{(1)}u_{n,m}+\delta^{(1)}} \\ + \underline{f}_{n,m}\dfrac{\alpha^{(2)}u_{n,m}+\beta^{(2)}}{\gamma^{(2)}u_{n,m}+\delta^{(2)}} + |\underline{f}_{n,m}\dfrac{\alpha^{(3)}u_{n,m}+\beta^{(3)}}{\gamma^{(3)}u_{n,m}+\delta^{(3)}} \end{array} \longmapsto \begin{pmatrix} \dfrac{\alpha^{(0)}x+\beta^{(0)}}{\gamma^{(0)}x+\delta^{(0)}} \\ \dfrac{\alpha^{(1)}x_1+\beta^{(1)}}{\gamma^{(1)}x_1+\delta^{(1)}} \\ \dfrac{\alpha^{(2)}x_2+\beta^{(2)}}{\gamma^{(2)}x_2+\delta^{(2)}} \\ \dfrac{\alpha^{(3)}x_{12}+\beta^{(3)}}{\gamma^{(3)}x_{12}+\delta^{(3)}} \end{pmatrix}^T. \tag{1.78b}$$

We have now to prove that $\widehat{\text{Möb}}^4$ is a group and that the mapping (1.78) is actually a group homomorphism.

Using the computational rules given in Table 1.1 we obtain a new representation of the generic element of the group $\widehat{\text{Möb}}^4$:

$$M_{n,m}(u_{n,m}) = \frac{\left\{ \begin{array}{l} \left(a_0 f_{n,m} + a_1|f_{n,m} + a_2\underline{f}_{n,m} + a_{12}|\underline{f}_{n,m}\right) u_{n,m} \\ + b_0 f_{n,m} + b_1|f_{n,m} + b_2\underline{f}_{n,m} + b_{12}|\underline{f}_{n,m} \end{array} \right\}}{\left\{ \begin{array}{l} \left(c_0 f_{n,m} + c_1|f_{n,m} + c_2\underline{f}_{n,m} + c_{12}|\underline{f}_{n,m}\right) u_{n,m} \\ + d_0 f_{n,m} + d_1|f_{n,m} + d_2\underline{f}_{n,m} + d_{12}|\underline{f}_{n,m} \end{array} \right\}}. \tag{1.79}$$

This form of the elements of $\widehat{\text{Möb}}^4$ shows immediately that $\widehat{\text{Möb}}^4$ is a subset of the general non-autonomous Möbius transformation:

$$W_{n,m}\colon u_{n,m} \mapsto \frac{a_{n,m}u_{n,m}+b_{n,m}}{c_{n,m}u_{n,m}+d_{n,m}}. \tag{1.80}$$

From the general rule of composition of two Möbius transformations (1.80) we get:

$$W^1_{n,m}\left(W^0_{n,m}(u_{n,m})\right) = \frac{(a^0_{n,m}a^1_{n,m} + b^1_{n,m}c^0_{n,m})u_{n,m} + a^1_{n,m}b^0_{n,m} + b^1_{n,m}d^0_{n,m}}{(a^0_{n,m}c^1_{n,m} + c^0_{n,m}d^1_{n,m})u_{n,m} + b^0_{n,m}c^1_{n,m} + d^0_{n,m}d^1_{n,m}}. \tag{1.81}$$

Its inverse is given by

$$W^{-1}_{n,m}(u_{n,m}) = \frac{d_{n,m}u_{n,m} - b_{n,m}}{-c_{n,m}u_{n,m} + a_{n,m}}, \tag{1.82}$$

Using the computational rules given in Table 1.1, the formula for the composition (1.81) and the representation (1.79) we find that the composition of two elements $M^{(1)}_{n,m}, M^{(2)}_{n,m} \in \widehat{\text{Möb}}^4$ with parameters $(a^{(i)}_j, b^{(i)}_j, c^{(i)}_j, d^{(i)}_j)$, $j = 0, 1, 2, 12$ and $i = 1, 2$, gives:

$$M^2_{n,m}\left(M^1_{n,m}\left(u_{n,m}\right)\right) = \frac{\left\{\begin{array}{l} \left(a^{(2)}_0 f_{n,m} + a^{(2)}_1 |f_{n,m} + a^{(2)}_2 \underline{f}_{n,m} + a^{(2)}_{12} |\underline{f}_{n,m}\right) u_{n,m} \\ + b^{(2)}_0 f_{n,m} + b^{(2)}_1 |f_{n,m} + b^{(2)}_2 \underline{f}_{n,m} + b^{(2)}_{12} |\underline{f}_{n,m} \end{array}\right\}}{\left\{\begin{array}{l} \left(c^{(2)}_0 f_{n,m} + c^{(2)}_1 |f_{n,m} + c^{(2)}_2 \underline{f}_{n,m} + c^{(2)}_{12} |\underline{f}_{n,m}\right) u_{n,m} \\ + d^{(2)}_0 f_{n,m} + d^{(2)}_1 |f_{n,m} + d^{(2)}_2 \underline{f}_{n,m} + d^{(2)}_{12} |\underline{f}_{n,m} \end{array}\right\}}, \tag{1.83}$$

where the new coefficients $(a^{(2)}_j, b^{(2)}_j, c^{(2)}_j, d^{(2)}_j)$ with $j = 0, 1, 2, 12$ are given by:

$$\begin{aligned}
a^{(2)}_0 &= a^{(0)}_0 a^{(1)}_0 + b^{(1)}_0 c^{(0)}_0, & a^{(2)}_1 &= a^{(0)}_1 a^{(1)}_1 + b^{(1)}_1 c^{(0)}_1, \\
a^{(2)}_2 &= a^{(0)}_2 a^{(1)}_2 + b^{(1)}_2 c^{(0)}_2, & a^{(2)}_{12} &= a^{(0)}_{12} a^{(1)}_{12} + b^{(1)}_{12} c^{(0)}_{12}, \\
b^{(2)}_0 &= a^{(1)}_0 b^{(0)}_0 + b^{(1)}_0 d^{(0)}_0, & b^{(2)}_1 &= a^{(1)}_1 b^{(0)}_1 + b^{(1)}_1 d^{(0)}_1, \\
b^{(2)}_2 &= a^{(1)}_2 b^{(0)}_2 + b^{(1)}_2 d^{(0)}_2, & b^{(2)}_{12} &= a^{(1)}_{12} b^{(0)}_{12} + b^{(1)}_{12} d^{(0)}_{12}, \\
c^{(2)}_0 &= a^{(0)}_0 c^{(1)}_0 + c^{(0)}_0 d^{(1)}_0, & c^{(2)}_1 &= a^{(0)}_1 c^{(1)}_1 + c^{(0)}_1 d^{(1)}_1, \\
c^{(2)}_2 &= a^{(0)}_2 c^{(1)}_2 + c^{(0)}_2 d^{(1)}_2, & c^{(2)}_{12} &= a^{(0)}_{12} c^{(1)}_{12} + c^{(0)}_{12} d^{(1)}_{12}, \\
d^{(0)}_0 &= b^{(0)}_0 c^{(1)}_0 + d^{(0)}_0 d^{(1)}_0, & d^{(0)}_1 &= b^{(0)}_1 c^{(1)}_1 + d^{(0)}_1 d^{(1)}_1, \\
d^{(0)}_2 &= b^{(0)}_2 c^{(1)}_2 + d^{(0)}_2 d^{(1)}_2, & d^{(0)}_{12} &= b^{(0)}_{12} c^{(1)}_{12} + d^{(0)}_{12} d^{(1)}_{12}.
\end{aligned} \tag{1.84}$$

The inverse of (1.79) is given by:

$$M^{-1}_{n,m}(u_{n,m}) = \frac{\left\{\begin{array}{l} (f_{n,m} d + |f_{n,m} d_1 + \underline{f}_{n,m} d_2 + |\underline{f}_{n,m} d_{12}) u_{n,m} \\ - (f_{n,m} b_0 + |f_{n,m} b_1 + \underline{f}_{n,m} b_2 + |\underline{f}_{n,m} b_{12}) \end{array}\right\}}{\left\{\begin{array}{l} - (f_{n,m} c_0 + |f_{n,m} c_1 + \underline{f}_{n,m} c_2 + |\underline{f}_{n,m} c_{12}) u_{n,m} \\ + (f_{n,m} a_0 + |f_{n,m} a_1 + \underline{f}_{n,m} a_2 + |\underline{f}_{n,m} a_{12}) \end{array}\right\}}. \tag{1.85}$$

Thus one has proved that $\widehat{\text{Möb}}^4$ is a group and in fact it is a subgroup of the general autonomous Möbius transformations (1.80).

Let us show that the maps Φ and Φ^{-1} given in (1.78) represent a group homomorphism. They preserve the identity, and from the formula of composition of Möbius transformations in Möb^4 (1.83) we derive the required result.

$\cdot$	$f_{n,m}$	$\vert f_{n,m}$	$\underline{f}_{n,m}$	$\vert\underline{f}_{n,m}$
$f_{n,m}$	$f_{n,m}$	0	0	0
$\vert f_{n,m}$	0	$\vert f_{n,m}$	0	0
$\underline{f}_{n,m}$	0	0	$\underline{f}_{n,m}$	0
$\vert\underline{f}_{n,m}$	0	0	0	$\vert\underline{f}_{n,m}$

Table 1.1: Multiplication rules for the functions $\tilde{f}_{n,m}$ as given by (1.52).

Let us now check if the diagram of Figure 1.9 is satisfied. Let us consider (1.76) and (1.77) and a general *multi-linear* quad equation:

$$\begin{aligned} Q_{\text{gen}}(x, x_1, x_2, x_{12}) = {} & A_{0,1,2,12} x x_1 x_2 x_{12} \\ & + B_{0,1,2} x x_1 x_2 + B_{0,1,12} x x_1 x_{12} \\ & + B_{0,2,12} x x_2 x_{12} + B_{1,2,12} x_1 x_2 x_{12} \\ & + C_{0,1} x x_1 + C_{0,2} x x_2 + C_{0,12} x x_{12} \\ & + C_{1,2} x_1 x_2 + C_{1,12} x_1 x_{12} + C_{2,12} x_2 x_{12} \\ & + D_0 x + D_1 x_1 + D_2 x_2 + D_{12} x_{12} + K \end{aligned} \tag{1.86}$$

where $A_{0,1,2,12}$, $B_{i,j,k}$, $C_{i,j}$, D_i and K with $i, j, k \in \{0, 1, 2, 12\}$ are arbitrary complex constants. The proof that $Q(M(x, x_1, x_2, x_{12})) = \widehat{Q}(M_{n,m}(u_{n,m}))$ where $M \in \text{Möb}^4$ and $M_{n,m} = \Phi(M) \in \widehat{\text{Möb}}^4$ is a computationally very heavy calculation due to the high number of parameters involved (twelve in the transformation[6] and fifteen in the equation (1.86), twenty-seven paramerters in total) and to the fact that rational functions are involved. To simplify the problem it is sufficient to recall that every Möbius

$$m(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \tag{1.87}$$

transformation can be obtained as a superposition of a *translation*:

$$T_a(z) = z + a, \tag{1.88a}$$

dilatation:

$$D_a(z) = az \tag{1.88b}$$

and *inversion*:

$$I(z) = \frac{1}{z}, \tag{1.88c}$$

6 Using that Möbius transformations are projectively defined one can lower the number of parameters from sixteen to twelve, but this implies to impose that some parameters are non-zero and thus all various different possibilities must be taken into account.

i.e.

$$m(z) = \left(T_{a/c} \circ D_{(bc-ad)/c^2} \circ I \circ T_{d/c}\right)(z). \tag{1.89}$$

As the group Möb^4 is obtained by four copies of the Möbius group each acting on a different variable we can decompose each entry $M \in \text{Möb}^4$ as in (1.89). Therefore we need to check $3^4 = 81$ transformations, depending at most on four parameters.We can automatize such proof by making a specific computer program to generate all the possible fundamental transformations in Möb^4 and then check them one by one reducing the computational effort. To this end we used the Computer Algebra System (CAS) `sympy` [150]. This ends the proof of the preservation of the classification when we pass from the cell equation to the lattice equation. The details of this calculation are contained in Appendix A.

1.6 INDEPENDENT EQUATIONS ON THE 2D-LATTICE

In this last Section of this Chapter we present the explict form on the 2D-lattice of the all the independent systems listed in Subsection 1.3.3. The construction of these systems is carried out according to the prescription given in Section 1.4. In light of the preceding Section independence is now understood to be up to the action of the group $\widehat{\text{Möb}}^4$ and rotations, translations and inversions of the reference system. The transformations of the reference system are acting on the discrete indices rather than on the reference frame. For sake of compactness and as the equations are on the lattice we shall omit the hats on the Möbius transformations when clear. Obviously, as in the single lattice case we have nine non-autonomous representatives, three of trapezoidal H^4-type and six of H^6-type. These equations are non-autonomous with two-periodic coefficients which can be given in terms of the functions:

$$F_k^{(\pm)} = \frac{1 \pm (-1)^k}{2}. \tag{1.90}$$

The H^4 type equations are:

$$\begin{aligned} {}_tH_1 : \; & (u_{n,m} - u_{n+1,m}) \cdot (u_{n,m+1} - u_{n+1,m+1}) - \\ & - \alpha_2 \varepsilon^2 \left(F_m^{(+)} u_{n,m+1} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m}\right) - \alpha_2 = 0, \end{aligned} \tag{1.91a}$$

$$
\begin{aligned}
{}_tH_2\colon\ & (u_{n,m}-u_{n+1,m})(u_{n,m+1}-u_{n+1,m+1}) \\
&+\alpha_2(u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1}) \\
&+\frac{\varepsilon\alpha_2}{2}\left(2F_m^{(+)}u_{n,m+1}+2\alpha_3+\alpha_2\right)\left(2F_m^{(+)}u_{n+1,m+1}+2\alpha_3+\alpha_2\right) \\
&+\frac{\varepsilon\alpha_2}{2}\left(2F_m^{(-)}u_{n,m}+2\alpha_3+\alpha_2\right)\left(2F_m^{(-)}u_{n+1,m}+2\alpha_3+\alpha_2\right) \\
&+(\alpha_3+\alpha_2)^2-\alpha_3^2-2\varepsilon\alpha_2\alpha_3(\alpha_3+\alpha_2)=0
\end{aligned}
\tag{1.91b}
$$

$$
\begin{aligned}
{}_tH_3\colon\ & \alpha_2(u_{n,m}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}) \\
&-(u_{n,m}u_{n,m+1}+u_{n+1,m}u_{n+1,m+1})-\alpha_3\left(\alpha_2^2-1\right)\delta^2+ \\
&-\frac{\varepsilon^2(\alpha_2^2-1)}{\alpha_3\alpha_2}\left(F_m^{(+)}u_{n,m+1}u_{n+1,m+1}+F_m^{(-)}u_{n,m}u_{n+1,m}\right)=0,
\end{aligned}
\tag{1.91c}
$$

These equations arise from the B equation of the cases **3.10.1**, **3.10.2** and **3.10.3** in [21] respectively. We remark that we have passed to the *rational* form of the ${}_tH_3^\varepsilon$ (1.91c) by making the identification:

$$
e^{2\alpha_2}\to\alpha_2,\qquad e^{2\alpha_3}\to\alpha_3. \tag{1.92}
$$

The H^6 type equations are:

$$
D_1\colon\ u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1}=0. \tag{1.93a}
$$

$$
\begin{aligned}
{}_1D_2\colon\ & \left(F_{n+m}^{(-)}-\delta_1F_n^{(+)}F_m^{(-)}+\delta_2F_n^{(+)}F_m^{(+)}\right)u_{n,m} \\
&+\left(F_{n+m}^{(+)}-\delta_1F_n^{(-)}F_m^{(-)}+\delta_2F_n^{(-)}F_m^{(+)}\right)u_{n+1,m}+ \\
&+\left(F_{n+m}^{(+)}-\delta_1F_n^{(+)}F_m^{(+)}+\delta_2F_n^{(+)}F_m^{(-)}\right)u_{n,m+1} \\
&+\left(F_{n+m}^{(-)}-\delta_1F_n^{(-)}F_m^{(+)}+\delta_2F_n^{(-)}F_m^{(-)}\right)u_{n+1,m+1}+ \\
&+\delta_1\left(F_m^{(-)}u_{n,m}u_{n+1,m}+F_m^{(+)}u_{n,m+1}u_{n+1,m+1}\right) \\
&+F_{n+m}^{(+)}u_{n,m}u_{n+1,m+1}+F_{n+m}^{(-)}u_{n+1,m}u_{n,m+1}=0,
\end{aligned}
\tag{1.93b}
$$

$$
\begin{aligned}
{}_2D_2\colon\ & \left(F_m^{(-)}-\delta_1F_n^{(+)}F_m^{(-)}+\delta_2F_n^{(+)}F_m^{(+)}-\delta_1\lambda F_n^{(-)}F_m^{(+)}\right)u_{n,m} \\
&+\left(F_m^{(-)}-\delta_1F_n^{(-)}F_m^{(-)}+\delta_2F_n^{(-)}F_m^{(+)}-\delta_1\lambda F_n^{(+)}F_m^{(+)}\right)u_{n+1,m} \\
&+\left(F_m^{(+)}-\delta_1F_n^{(+)}F_m^{(+)}+\delta_2F_n^{(+)}F_m^{(-)}-\delta_1\lambda F_n^{(-)}F_m^{(-)}\right)u_{n,m+1} \\
&+\left(F_m^{(+)}-\delta_1F_n^{(-)}F_m^{(+)}+\delta_2F_n^{(-)}F_m^{(-)}-\delta_1\lambda F_n^{(+)}F_m^{(-)}\right)u_{n+1,m+1} \\
&+\delta_1\left(F_{n+m}^{(-)}u_{n,m}u_{n+1,m+1}+F_{n+m}^{(+)}u_{n+1,m}u_{n,m+1}\right) \\
&+F_m^{(+)}u_{n,m}u_{n+1,m}+F_m^{(-)}u_{n,m+1}u_{n+1,m+1}-\delta_1\delta_2\lambda=0,
\end{aligned}
\tag{1.93c}
$$

$$
\begin{aligned}
{}_3D_2\colon\quad & \left(F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} - \delta_1 \lambda F_n^{(-)} F_m^{(+)}\right) u_{n,m} \\
& + \left(F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} - \delta_1 \lambda F_n^{(+)} F_m^{(+)}\right) u_{n+1,m} \\
& + \left(F_m^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} - \delta_1 \lambda F_n^{(-)} F_m^{(-)}\right) u_{n,m+1} \\
& + \left(F_m^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} - \delta_1 \lambda F_n^{(+)} F_m^{(-)}\right) u_{n+1,m+1} \\
& + \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) \\
& + F_m^{(-)} u_{n,m+1} u_{n+1,m+1} + F_m^{(+)} u_{n,m} u_{n+1,m} - \delta_1 \delta_2 \lambda = 0,
\end{aligned} \tag{1.93d}
$$

$$
\begin{aligned}
D_3\colon\quad & F_n^{(+)} F_m^{(+)} u_{n,m} + F_n^{(-)} F_m^{(+)} u_{n+1,m} + F_n^{(+)} F_m^{(-)} u_{n,m+1} \\
& + F_n^{(-)} F_m^{(-)} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} \\
& + F_n^{(-)} u_{n,m} u_{n,m+1} + F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + \\
& + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \\
& + F_m^{(+)} u_{n,m+1} u_{n+1,m+1} = 0,
\end{aligned} \tag{1.93e}
$$

$$
\begin{aligned}
{}_1D_4\colon\quad & \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) + \\
& + \delta_2 \left(F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1}\right) + \\
& + u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1} + \delta_3 = 0,
\end{aligned} \tag{1.93f}
$$

$$
\begin{aligned}
{}_2D_4\colon\quad & \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) + \\
& + \delta_2 \left(F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1}\right) + \\
& + u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1} + \delta_3 = 0.
\end{aligned} \tag{1.93g}
$$

The equations ${}_1D_2$, ${}_1D_4$, ${}_2D_4$ and D_3 arise from the A equation in the cases **3.12.2**, **3.12.3**, **3.12.4** and **3.13.5** respectively. The equations ${}_2D_2$ and ${}_3D_2$ arise from the C equation in the cases **3.13.2** and **3.13.3** respectively.

To write down the explicit form of the equations in (1.93) we used (1.54, 1.58) and the following identities:

$$
\begin{aligned}
f_{n,m} &= F_n^{(+)} F_m^{(+)}, \quad |f_{n,m} = F_n^{(-)} F_m^{(+)}, \\
\underline{f}_{n,m} &= F_n^{(+)} F_m^{(-)}, \quad |\underline{f}_{n,m} = F_n^{(-)} F_m^{(-)}.
\end{aligned} \tag{1.94}
$$

As mentioned in Section 1.4 if we apply this procedure to an equation of rhombic type we get a result consistent with [166].

We note that the explicit lattice form of these equations was first given in [67], being absent in the works of Boll [20–22].

For the rest of the thesis we will study the integrability properties of the trapezoidal H^4 and H^6 equations in their lattice avatars given by (1.91) and (1.93). The main result, i.e. the proof of the fact that they are *linearizable* equation, will be first stated by considering the heuristic test of the Algebraic Entropy in the Chapter 2, whereas in Chapter 4 we will give a formal proof of the linearizability based on the concept of *Darboux integrability*.

2 ALGEBRAIC ENTROPY AND DIRECT LINEARIZATION

In this Chapter we will focus on the integrability detector called Algebraic Entropy. The first part of this Chapter, consisting of Section 2.1 and of Section 2.2, is mainly a review of known concepts and it is essentially based on the exposition given in [63]. In Section 2.1 we will expand the intuitive idea given in the example of the logistic map (1.10) we presented in 1.1 in order to explain the difference between solvability and integrability. Motivated by the discussion of this example we will introduce the precise definition of Algebraic Entropy for difference equation, differential equations and quad equations. We note that the definition of Algebraic Entropy for quad equations is taken from [67] which is a generalization of that given in [155]. We will address the problem from the theoretical point of view discussing mainly the setting. Then we will derive the main properties of Algebraic Entropy and discuss briefly the geometric meaning of the Algebraic Entropy. In Section 2.2 we will discuss the computational tools that allow us to extract the value of the Algebraic Entropy from finite sequences. The implementation of such algorithm in `python`, introduced in [64], is discussed in Appendix B. The second part of the Chapter, consisting of Section 2.3 and of Section 2.4, is instead an original part based on [67, 68]. In Section 2.3 is presented and discussed the Algebraic Entropy test applied to the trapezoidal H^4 equations (1.91) and to the H^6 equations (1.93). This result shows that the trapezoidal H^4 equations (1.91) and to the H^6 equations (1.93) should be *linearizable equations*. To support the statement made in Section 2.3 in Section 2.4 we present some examples of explicit linearization. In particular we will treat the ${}_tH_1^\varepsilon$ equation (1.91a) and the ${}_1D_2$ equation. In the case of the ${}_tH_1^\varepsilon$ equation (1.91a) we also present a proof of the fact that its Lax pair obtained with procedure presented in Chapter 1 is fake, according to the definition of [77, 78]. Hence the Lax obtained from the Consistency Around the Cube is useless in discussing the integrability of the ${}_tH_1^\varepsilon$ equation (1.91a). This proof was first given in [68].

2.1 DEFINITION AND BASIC PROPERTIES

As we saw in the example of the logistic equation (1.10) the notion of chaos is related with the exponential growth of the solution with respect to the initial condition. Equation (1.10) possess an explicit solution therefore it is easy to understand *a posteriori* its properties. Let us now consider the general logistic map with an arbitrary parameter $r \in \mathbb{R}$:

$$u_{n+1} = r u_n \left(1 - u_n\right). \tag{2.1}$$

We wish to understand how this equation evolves, therefore we fix an initial condition u_0 and we compute the iterates:

$$u_1 = -r u_0^2 + r u_0, \tag{2.2a}$$
$$u_2 = -r^3 u_0^4 + 2r^3 u_0^3 - \left(r^3 + r^2\right) u_0^2 + r^2 u_0, \tag{2.2b}$$
$$u_3 = -r^7 u_0^8 + 4r^7 u_0^7 - \left(6r^7 - 2r^6\right) u_0^6 + \text{l.o.t.}, \tag{2.2c}$$
$$u_4 = -r^{15} u_0^{16} + 8r^{15} u_0^{15} - (28r^{15} + 4r^{14}) u_0^{14} + \text{l.o.t}, \tag{2.2d}$$

where by l.o.t. we mean terms with lower powers in u_0. Then it is clear that there is a regularity in how the degree of the polynomial u_n in u_0, d_n, grows. Indeed one can guess that $d_n = 2^n$ and check this guess for successive iterations. This is a clear indication that our system is chaotic as it was in the solvable case $r = 4$. This should be no surprise since the generic r case is even more general!

This simple example shows how examining the iterates of a recurrence relation can be a good way to extract information about integrability even if we cannot solve the equation explicitly. However in more complicated examples it is usually impossible to calculate explicitly these iterates by hand or even with any state-of-the-art formal calculus software, simply because the expressions one should manipulate are rational fractions of increasing degree of the various initial conditions. The complexity and size of the calculation make it impossible to calculate the iterates.

It was nevertheless observed that "integrable" maps are not as complex as generic ones. This was done primarily experimentally, by an accumulation of examples, and later by the elaboration of the concept of *Algebraic Entropy* for difference equations [18, 33, 39, 141, 153]. In [152, 155] the method was developed in the case of quad equations and then used as a classifying tool [84]. Finally in [32] the same concept was introduced for differential-difference equation and later [157] to the very similar case of differential-delay equations. In our review we will mainly follow [60].

The basic idea, given a rational map, which can be an ordinary difference equation, a differential difference equation or even a partial difference equation, is to examine the growth of the degree of its iterates as we did for (2.1), and extract a canonical quantity, which is an index of complexity of the map. This will be the algebraic entropy (or its avatar the dynamical degree). We will now introduce formally this subject by considering before the case of the *ordinary difference equations* and of the *differential-difference equations*. The case of the ordinary difference equations here is mainly intended to be preparatory for the case of the *partial difference equations*, but in Chapter 3 we will also see the application of the Algebraic Entropy to the differential-difference case.

2.1.1 *Algebraic entropy for ordinary difference equations and differential-difference equations*

In this Section we will consider *ordinary difference equations* i.e. expressions of the form:

$$u_{n+l} = f_n\left(u_{n+l-1}, \ldots, u_{n+l'}\right), \quad l', l, n \in \mathbb{Z},\ l' < l \tag{2.3}$$

where the unknown function u_n is a function of the discrete integer variable $n \in \mathbb{Z}$. The difference equation (2.3) is said to be of *order* $l - l'$ if $\partial f_n / \partial u_{n+l'} \neq 0$ identically. We will also discuss *differential-difference equations*, i.e. equations where the unknown is a function $u_n(t)$ of two variables, one continuous $t \in \mathbb{R}$ and one discrete $n \in \mathbb{Z}$. A differential-difference equations is then given by an expression of the form:

$$u_n^{(p)} = f_n\left(u_{n+l}, \ldots, u_{n+l'}; u_n', \ldots, u_n^{(p-1)}\right), \tag{2.4}$$

where we used the prime notation for derivatives, $u_n^{(p)} = d^p u / dt^p$. Furthermore $l', l, n \in \mathbb{Z}$, with the conditions $l' < l$ and finally $p \in \mathbb{N}$, $p > 1$ otherwise we fall back into the case of ordinary difference equations. A differential-differential equation of the form (2.4) is said to have differential order p and difference order $l - l'$ provided that

$$\frac{\partial f_n}{\partial u_{n+l}} \frac{\partial f_n}{\partial u_{n+l'}} \neq 0. \tag{2.5}$$

Typical example of such equations are the so called *Volterra-like equations*:

$$u_n' = f_n\left(u_{n+1}, u_n, u_{n+1}\right), \quad n \in \mathbb{Z}, \tag{2.6}$$

or *Toda-like equations*:

$$u_n'' = f_n\left(u_n', u_{n+1}, u_n, u_{n+1}\right). \quad n \in \mathbb{Z}, \tag{2.7}$$

Volterra-like equations are first order differential-difference equations in the differential order, while Toda-like equations are second order differential-difference equations in the differential order. Both these classes are second order in the difference order provided that the functions in the right hand side of (2.6) and (2.7) satisfy the condition (2.5).

For theoretical purposes it is usual to consider maps in a projective space rather than in the affine one. One then transforms its recurrence relation into a polynomial map in the homogeneous coordinates of the proper projective space over some closed field[1]:

$$x_i \mapsto \varphi_i(x_k), \tag{2.8}$$

with $x_i, x_k \in \mathcal{IN}$ where $\mathcal{IN}$ is the space of the initial conditions. The recurrence is then obtained by iterating the polynomial map φ_i. It is usual to ask that the map φ_i be *bi-rational*, i.e. it possesses an inverse which is again a rational map. In the affine setting the bi-rationality of the map means that we can solve the relation also for the lower-index variable, e.g. $u_{n+l'}$ in (2.3) and (2.4). This fact is of crucial theoretical importance, as it will be explained at the end of this Subsection.

Example 2.1.1. To clarify the concept of how we can translate a recurrence relation into a projective map we present a very simple example which, however gives the flavor of the method. Let us a consider the most general bilinear first order recurrence relation of the form (2.3):

$$u_{n+1} = \frac{a u_n + b}{c u_n + d}, \tag{2.9}$$

where we are assuming a, b, c, d constants such that $ad - bc \neq 0$. Since (2.9) is a first order difference equations we must convert it into a bi-rational map of $\mathbb{P}^1$ into itself. To this end we can introduce the projective coordinates:

$$u_n = \frac{x}{y}, \quad u_{n+1} = \frac{X}{y}. \tag{2.10}$$

From (2.9) we obtain then:

$$X = y\frac{ax + by}{cx + dy}. \tag{2.11}$$

Since the recurrence relation in the projective plane is given by:

$$\varphi\colon (x, y) \in \mathbb{P}^1 \mapsto (X, y) \in \mathbb{P}^1 \tag{2.12}$$

using (2.11) we obtain:

$$(X, y) = \left(y\frac{ax + by}{cx + dy}, y\right) \simeq (ax + by, cx + dy). \tag{2.13}$$

Therefore we have converted the recurrence relation (2.9) into the following map of $\mathbb{P}^1$ into itself:

$$\varphi\colon (x, y) \mapsto (ax + by, cx + dy). \tag{2.14}$$

This is a *linear* map. The inverse is the clearly given by:

$$\varphi^{-1}\colon (x, y) \mapsto (dx - by, -cx + ay), \tag{2.15}$$

and can also be obtained directly from (2.9) using the substitution (2.10) and then solving with respect to x. □

1 The reader can think this field to be the complex one $\mathbb{C}$, but we will see in Subsection 2.2 that in practice finite fields can be useful.

Now to proceed further we need to specify the space of the initial conditions $\mathcal{IN}$. The space of the initial condition depends on which type of recurrence relation we are considering. Let us enumerate the various cases.

1. If we are dealing with a $l - l'$ order difference equation in the form (2.3) the initial conditions will be just the starting $l - l'$-tuple:

$$\mathcal{IN} = \{u_{l'}, u_{l'+1}, \dots, u_{l-2}, u_{l-1}\}. \tag{2.16}$$

   To obtain the map we just need to pass to homogeneous coordinates in (2.3) and in (2.16). Note that the logistic map we considered above (2.1) is a polynomial map, but it is not bi-rational, since its inverse is algebraic.

2. If we are dealing with a differential-difference equation of the discrete $l - l'$-th order and of the p-th continuous order, the space of initial conditions is infinite dimensional. Indeed, in the case the order of the equation is $l - l'$, we need the initial value of $l - l'$-tuple as a function of the parameter t, but also the value of *all its derivatives*:

$$\mathcal{IN} = \left\{u_{l'}^{(j)}(t), u_{l'+1}^{(j)}(t), \dots, u_{l-2}^{(j)}(t), u_{l-1}^{(j)}(t)\right\}_{j\in\mathbb{N}_0}, \tag{2.17}$$

   where by $u_i^{(j)}(t)$ we mean the j–derivative of $u_i(t)$ with respect to t. We need all the derivatives of $u_i(t)$ and not just the first p because at every iteration the order of the equation is raised by p. To obtain the map one just need to pass to homogeneous coordinates in the equation and in (2.17).

For both kind of equations we are in the position to define the concept of Algebraic Entropy. Indeed if we factors out any common polynomial factors we can say that the degree with respect to the initial conditions is well defined, in a given system of coordinates, although it is not invariant with respect to changes of coordinates. We can therefore form the sequence of degrees of the iterates of the map φ and call it $d_k = \deg \varphi^k$:

$$\underbrace{1, \dots 1}_{l-l'}, d_1, d_2, d_3, d_4, d_5, \dots, d_k, \dots. \tag{2.18}$$

The degree of the bi-rational projective map φ have to be understood as the *maximum of the total polynomial degree in the initial conditions* $\mathcal{IN}$ of the entries of φ. The same definition in the affine case just translates to the *maximum between the degree of the numerator and of the denominator* of the kth iterate in terms of the affine initial conditions. Degrees in the projective and in the affine setting can be different, but the global behavior will be the same due to the properties of homogenization and de-homogenization which is an invertible procedure. Then the entropy of such sequence is defined to be:

$$\eta = \lim_{k\to\infty} \frac{1}{k} \log d_k. \tag{2.19}$$

Such limit exists since from the elementary property of any pair of bi-rational maps φ and ψ:

$$\deg(\psi \circ \varphi) \leqslant \deg\psi \deg\varphi. \tag{2.20}$$

Furthermore the inequality (2.20) proves that there is an upper bound for the Algebraic Entropy:

$$\eta \leqslant \deg\varphi. \tag{2.21}$$

An equation whose Algebraic Entropy is equal to $\deg\varphi$ is said to saturate the bound. Indeed from (2.20) applied to φ^k we obtain $\deg\left(\varphi^k\right) \leqslant k\deg\varphi$ from which (2.21) follows using the definition (2.19). We see then that if $\eta = 0$ we must have

$$d_k \sim k^\nu, \quad \text{with } \nu \in \mathbb{N}_0, \text{ as } k \to \infty. \tag{2.22}$$

We will then have the following classification of equations according to their Algebraic Entropy [84]:

LINEAR GROWTH: The equation is linearizable.

POLYNOMIAL GROWTH: The equation is integrable.

EXPONENTIAL GROWTH: The equation is chaotic.

Furthermore it is easy to see that the Algebraic Entropy is a *bi-rational invariant of such kind of maps*. Indeed if we suppose that we have two *bi-rationally equivalent maps* φ and ψ then there exists a bi-rational map χ such that:

$$\psi = \chi^{-1} \circ \varphi \circ \chi \tag{2.23}$$

implying:

$$\deg\psi^k \leqslant M \deg\varphi^K, \tag{2.24}$$

where $M = \deg\chi \deg\left(\chi^{-1}\right) \in \mathbb{N}$. Since we can obtain an analogous equation as $\varphi = \chi \circ \psi \circ \chi^{-1}$ we conclude that $\eta_\psi = \eta_\varphi$.

To clarify the theory we presented we now give three very simple examples of calculation of the Algebraic Entropy of difference equations[2].

2 We will consider the degrees always computed in the affine setting.

Example 2.1.2 (Hénon map [79]). In this example we consider the so called Hénon map of the plane [79]. It relates the iterates of a two component vector (x_n, y_n) via the recurrence[3]:

$$x_{n+1} = 1 - \alpha x_n^2 + y_n, \tag{2.25a}$$

$$y_{n+1} = \beta x_n. \tag{2.25b}$$

It can be written as a second order difference equation:

$$u_{n+1} = 1 - \alpha u_n^2 + \beta u_{n-1} \tag{2.26}$$

Computing the degrees of the iterates for any non-zero value of the coefficients α and β we get:

$$1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 \ldots \tag{2.27}$$

The Hénon map is of degree 2. The sequence it is not only bounded by 2^k, but it saturates its bound. The sequence is exactly fitted by 2^k and its entropy will be $\log 2$.

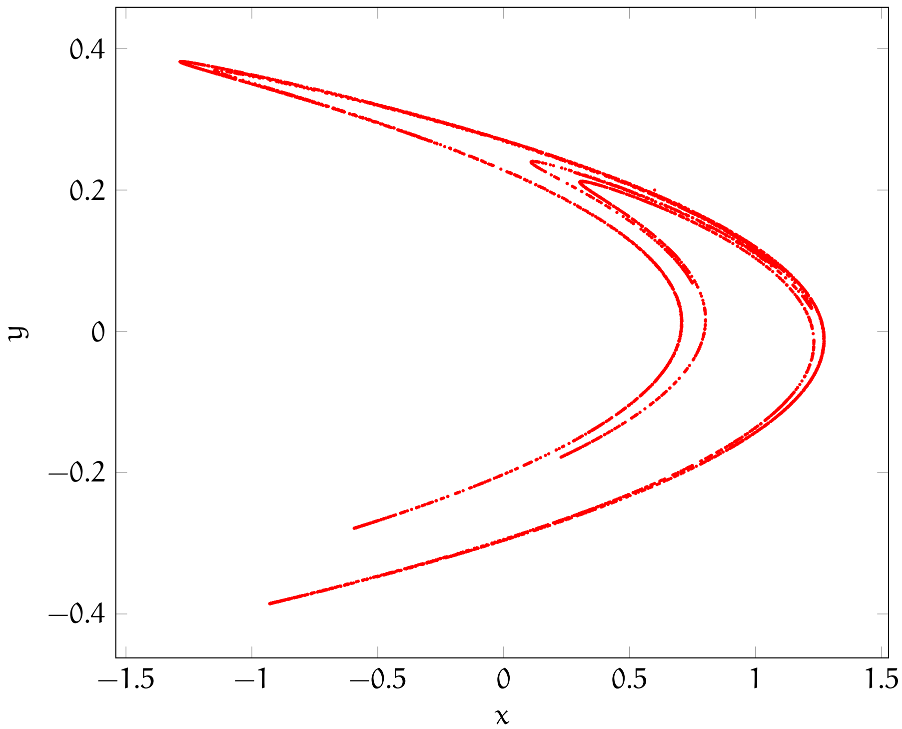

Figure 2.1: The Hénon map in the plane (x, y) with $a = 1.4$ and $b = 0.3$ and the initial conditions $(x_0, y_0) = (0.6, 0.2)$.

The trajectory of the map (2.25) in the plane are displayed in Figure 2.1, which clearly shows the "chaoticity" of the system. □

3 In the original work Hénon used the particular choice of parameters $\alpha = 1.4$ and $\beta = 0.3$

Example 2.1.3 (Non saturating exponential difference equation)*.* Consider the second order nonlinear difference equation:

$$u_{n+1} = \alpha u_n u_{n-1} + \beta u_n + \gamma u_{n-1}, \tag{2.28}$$

where α, β and γ are real constants. Computing the degrees of the iterates for any non-zero choice of the parameters α, β and γ we get:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \ldots \tag{2.29}$$

Also this map is of degree 2, but now its growth is different. We see that from the second iterate there is a drop in the degree, so $d_k < 2^k$ asymptotically. The reader may recognize in this sequence the Fibonacci sequence. The Fibonacci sequence is known to solve the second order linear difference equation:

$$d_{k+1} = d_k + d_{k-1}. \tag{2.30}$$

This means that we have asymptotically:

$$d_k \sim \left(\frac{1+\sqrt{5}}{2}\right)^k \tag{2.31}$$

and the algebraic entropy of the recurrence relation (2.28) will be:

$$\eta = \log\left(\frac{1+\sqrt{5}}{2}\right), \tag{2.32}$$

i.e. the logarithm of the Golden Ratio. □

Example 2.1.4 (Hirota-Kimura-Yahagi equation [86])*.* Consider the nonlinear second order difference equation:

$$u_{n+1} u_{n-1} = u_n^2 + \beta^2. \tag{2.33}$$

This equation possess the first integral [86]:

$$K(u_n, u_{n-1}) = \frac{2 u_n u_{n-1}}{u_n^2 + u_{n-1}^2 + \beta^2}, \tag{2.34}$$

i.e.:

$$K(u_{n+1}, u_n) - K(u_n, u_{n-1}) = 0, \tag{2.35}$$

along the solutions of (2.33). First integrals are a constraint to the motion of a system, so in this case we expect a great drop in the degrees. Indeed we have for every value of the parameter β:

$$1, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28 \ldots \tag{2.36}$$

The degrees appear to grow *linearly*: $d_k = 2k$ and the discrepancy from the saturation start from the third iterate. □

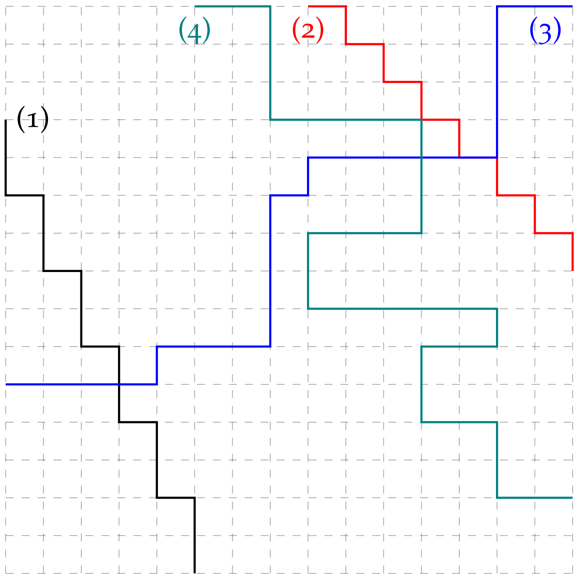

Figure 2.2: Regular and non-regular staircases.

2.1.2 *Algebraic entropy for quad equations*

Now we introduce the concept of Algebraic Entropy for quad equations which was introduced in [152]. We will follow mainly the exposition presented in [155] with some generalizations which were given in [67]. These generalization where introduced explicitly with the scope of calculating the Algebraic Entropy of the trapezoidal H^4 (1.91) and H^6 equations (1.93), due to the peculiar behavior of these equations.

In the case of quad equations the situation is more complicated than in the case of one-dimensional equations. First of all in the one-dimensional case we have to worry only about the evolution in two opposite directions. This was the meaning of the bi-rationality condition as explained in the preceding Subsection. In the two dimensional case we have to worry about four possible directions of evolution corresponding to the four ways we can solve the quad equation. In general, initial conditions can be given along straight lines in the four direction. However usually is preferred to give initial conditions on *staircase* configurations. Examples of staircase-like arrangements of initial values are displayed in Figure 2.2. The evolution from any staircase-like arrangement of initial values is possible in the quadrilateral lattice. This in principle does not rules out configurations in which are present hook-like configurations (see (4) in Figure 2.2). These kind of configurations will require compatibility conditions on the initial data, since they give more than one way to calculate the same value for the dependent variable which is not ensured to be the equal. The staircases need to go from $(n = -\infty, m = -\infty)$ to $(n = \infty, m = \infty)$, or from $(n = -\infty, m = +\infty)$ to $(n = \infty, m = -\infty)$

because the space of initial conditions is infinite. We will restrict ourselves to *regular diagonals* which are staircases with steps of *constant* horizontal length, and *constant* vertical height. Figure 2.2 shows four staircase-like configurations. The ones labeled (1) and (2) are regular diagonals. The one labeled (3) would be acceptable, but we will not consider such configurations. Line (4) is excluded since it may lead to incompatibilities.

Given a line of initial conditions, it is possible to calculate the values $u_{n,m}$ all over the two-dimensional lattice. We have a well defined evolution, since we restrict ourselves to regular diagonals. Moreover, and this is a crucial point, *if we want to evaluate the transformation formula for a finite number of iterations, we only need a regular diagonal of initial conditions with finite extent*.

For any positive integer N, and each pair of relative integers $[\lambda_1, \lambda_2]$, we denote by $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$, a regular diagonal consisting of N steps, each having horizontal size $l_1 = |\lambda_1|$, height $l_2 = |\lambda_2|$, and going in the direction of positive (resp. negative) n_k, if $\lambda_k > 0$ (resp. $\lambda_k < 0$), for $k = 1, 2$. For examples see Figure 2.3.

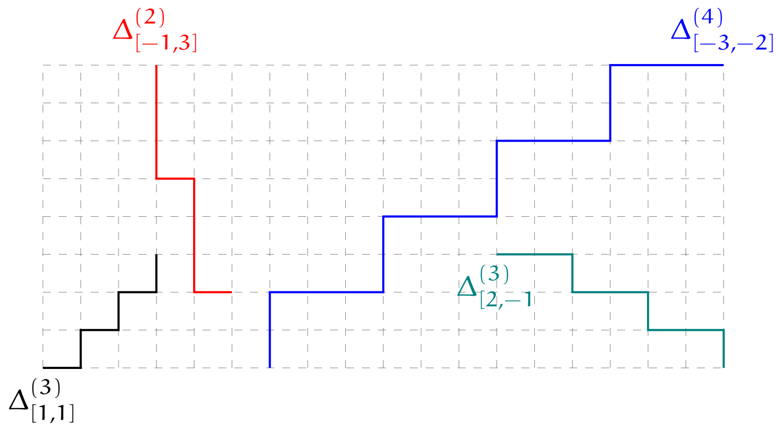

Figure 2.3: Varius kinds of restricted initial conditions.

Suppose we fix the initial conditions on $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. We may calculate u over a rectangle of size $(Nl_1 + 1) \times (Nl_2 + 1)$. The diagonal cuts the rectangle in two halves. One of them uses all initial values, and we will calculate the evolution only on that part. See Figure 2.4.

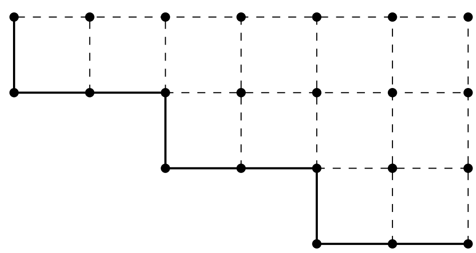

Figure 2.4: The range covered by the initial conditions $\Delta^{(3)}_{[-2,1]}$.

We are now in position to calculate "iterates" of the evolution. Choose some restricted diagonal $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. The total number of initial points is $q = N(l_1 + l_2) + 1$. For such restricted initial data, the natural space where the evolution acts is the projective space $\mathbb{P}^q$ of dimension q. We may calculate the iterates and fill Figure 2.4, considering the q initial values as inhomogeneous coordinates of $\mathbb{P}^q$. Evaluating the degrees of the successive iterates, we will produce double sequences of degrees $d^{(l)}_k$. The simplest possible choice is to apply this construction to the restricted diagonals $\Delta^{(N)}_{[\pm1,\pm1]}$, which we will denote $\Delta^{(N)}_{++}$, $\Delta^{(N)}_{+-}$, $\Delta^{(N)}_{-+}$ and $\Delta^{(N)}_{--}$. We will call them *fundamental diagonals* (the upper index (N) is omitted for infinite lines). The four principal diagonals are shown in Figure 2.5.

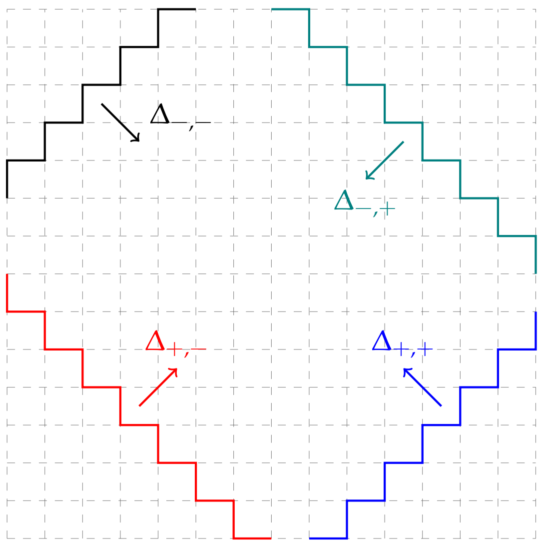

Figure 2.5: The four principal diagonals.

Assuming for example that we are given the $\Delta_{+,-}$ principal diagonal the pattern of degrees is of the form:

$$\begin{array}{cccccc}
1 & d_1^{(N-1)} & d_2^{(N-2)} & \dots & d_{N-1}^{(2)} & d_N^{(1)} \\
1 & 1 & d_1^{(N-2)} & d_2^{(3)} & \dots & d_{N-1}^{(1)} \\
 & 1 & 1 & d_1^{(3)} & d_2^{(2)} & \dots \\
 & & 1 & 1 & d_1^{(2)} & d_2^{(1)} \\
 & & & 1 & 1 & d_1^{(1)} \\
 & & & & 1 & 1
\end{array} \tag{2.37}$$

Therefore to each choice of indices $[\pm1, \pm1]$ we associate a *double* sequence of degrees $d^{(l)}_{k,\pm\pm}$.

We shall call the sequence

$$1, d^{(1)}_{1,\pm\pm}, d^{(1)}_{2,\pm\pm}, d^{(1)}_{3,\pm\pm}, d^{(1)}_{4,\pm\pm}, \dots \tag{2.38}$$

the *principal sequence* of growth. A sequence as

$$1, d^{(i)}_{1,\pm\pm}, d^{(i)}_{2,\pm\pm}, d^{(i)}_{3,\pm\pm}, d^{(i)}_{4,\pm\pm}, \dots \tag{2.39}$$

with $i = 2$ will be a *secondary sequence* of growth, for $i = 3$ a *third*, and so on. The fundamental entropies of the lattice equation are given by:

$$\eta^{(i)}_{\pm\pm} = \lim_{k\to\infty} \frac{1}{k} \log d^{(i)}_{k,\pm,\pm}. \tag{2.40}$$

The existence of this limit can be proved in an analogous ways as for one dimensional systems [18]. We note that *a priori* the degrees along the diagonals of a quad equation do not need to be equal, however in most cases they are. In the cases in which the degrees along the diagonals are not equal it is important to isolate repeating patterns in (2.37), i.e. if there exists a positive integer $\kappa \in \mathbb{N}$ such that:

$$d^{(i+\kappa)}_{k,\pm,\pm} = d^{(i)}_{k,\pm,\pm}, \quad i = 1, \dots, \kappa - 1. \tag{2.41}$$

If we can find such κ we can describe the rate of growth of the quad equation through a *finite* number of sequences and define its Algebraic Entropy of the quad equation as:

$$\eta^{\text{TOT}}_{\pm\pm} = \max_{i=1,\dots,\kappa-1} \eta^{(i)}_{\pm\pm}. \tag{2.42}$$

The definition given in [155] corresponds to (2.42) with $\kappa = 1$, i.e. the degrees are assumed *a priori* equal. For the moment the only known examples of equations with $\kappa > 1$ are the trapezoidal H_4 equations (1.91) and the H_6 equations (1.93). The discussion of the Algebraic Entropy of these equation is postponed to Section 2.3.

2.1.3 *Geometric meaning of the Algebraic Entropy*

One may wonder about the origin of the drop of the degrees which is observed in integrable and linearizable equations. It is actually geometrically very simple, and comes from the singularity structure. We will discuss this problem in the case of difference equations, since the other cases are essentially the same, but they are less intuitive. We recall that to a difference equation (2.3) of order $n = l - l'$ we can associate a map projective map $\varphi \colon \mathbb{P}^n \to \mathbb{P}^n$. A point of homogeneous coordinates $[x] = [x_0, x_1, \dots, x_n]$ is *singular* if all the homogeneous coordinates of the image by φ vanish. The set of these points is thus given by $n + 1$ homogeneous equations This set has co-dimension at least 2: it will be points in $\mathbb{P}^2$, complex curves and points in $\mathbb{P}^3$, and so on. One important point is that, as soon as the map is nonlinear, and this is the case we will be mainly interested in, there are always singular points. Without singular points the drop of degrees just cannot happen! The vanishing of all homogeneous coordinates means that there is no image point in $\mathbb{P}^n$. The mere vanishing of a few, but not all coordinates, means that the image "goes to infinity", but this is harmless, contrary to what happens in affine space. This is what projective space has been invented for: to cope with points at infinity, which are not to be forgotten when one consider algebraic varieties and rational maps. Moreover, using projective space over closed fields simplifies a lot the counting of intersection points by Bezout theorem. The maps we consider are almost invertible. They are diffeomorphisms on a Zariski open set, i.e. they are invertible everywhere except on an algebraic variety. Suppose the map φ and its inverse $\psi = \varphi^{-1}$ are written in terms of homogeneous coordinates. The composed maps $\varphi \circ \psi$ and respectively $\psi \circ \varphi$ are then just multiplication of all coordinates by some polynomial κ_φ and respectively κ_ψ:

$$\varphi \circ \psi\left([x]\right) = \kappa_\varphi\left([x]\right) \mathrm{Id}\left([x]\right), \quad \psi \circ \varphi\left([x]\right) = \kappa_\psi\left([x]\right) \mathrm{Id}\left([x]\right), \tag{2.43}$$

The map φ is clearly not invertible on the image of the variety of equation $\kappa_\varphi\left([x]\right)$. What may happen is that further action of φ on these points leads to images in the singular set of φ. This means that $\kappa_\varphi\left([x]\right)$ (or a piece of it if it is decomposable) has to factorize from all the components of the iterated map. This is the origin of the drop of the degree! This is the link between singularity (in the projective sense) and the degree sequence. A graphical explanation of this procedure is illustrated schematically in Figure 2.6. Figure 2.6 can be explained as follows: the equation of the surface Σ is $\kappa = 0$ and the factor κ appears anew in $\varphi \circ \varphi^{(4)}$. The fifth iterate $\varphi^{(5)}$ is then regular of $\kappa = 0$.

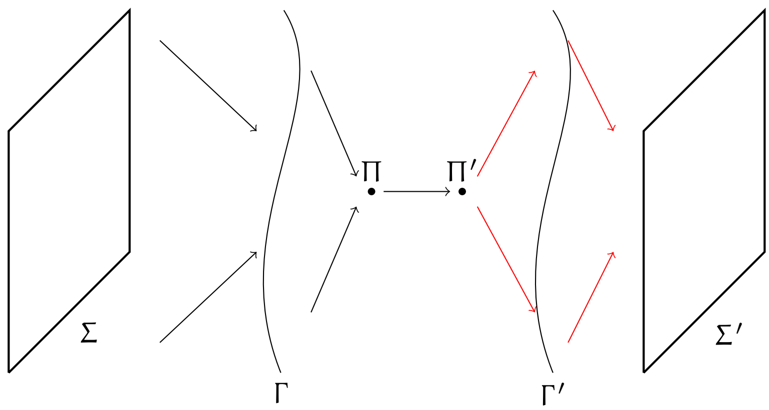

Figure 2.6: A possible blow-down blow-up scheme in $\mathbb{P}^3$.

This is a link to the method of *singularity confinement* [61, 89, 137]. If for all components of the variety $\kappa_\varphi\left([x]\right) = 0$ one encounters singular points of φ in such a way that some finite order iterate of φ define non ambiguously a proper image in $\mathbb{P}^n$, we have "singularity confinement".

The relation of the notion of Algebraic Entropy with the structure of the singularities of maps and lattice equations have been used also as a computational tool. Examples of such approach are given in [34, 142, 151] and more recently in [158].

2.2 COMPUTATIONAL TOOLS

Now that we have introduced the definition of Algebraic Entropy and discussed the origin of the dropping of the degrees, it is time to turn to the computational tools which help us to calculate it.

First of all it is clear from our first example with the logistic map that the complexity of the calculations grow more and more iterates we compute. A good way to reduce the computational complexity of the problem is to consider a particular set of initial condition given by *straight lines* in the appropriate projective space $\mathbb{P}^q$. This correspond to the following choice of inhomogeneous coordinates:

$$u_i = \frac{\alpha_i t + \beta_i}{\alpha_0 t + \beta_0}, \quad u_i \in \mathcal{IN}, \tag{2.44}$$

where α_i and β_i are the constant parameters describing the straight line and t is the "time" parameter i.e. the curve parametrization. In the case differential-difference equation we will assume that the parameter t is the same as the evolution parameter of the problem.

A useful simplification consists in using only integer numbers in the computations. Moreover we want to avoid, *accidental cancellations* i.e. factorizations due numerical substitution. Using generic integers to this end is not recommended, since due to prime number factorization they can introduce such kinds of cancellations. E.g. let us assume that we are given the rational function:

$$R = \frac{\alpha_1 t + \beta_1}{\alpha_0 t + \beta_0} \frac{P(t)}{Q(t)}, \tag{2.45}$$

where P and Q are polynomials in t of degree p and q respectively and with no common factors and α_i, β_i with $i = 0, 1$ are real or complex numbers. Therefore the degree of the numerator of R is $p+1$ and the degree of its denominator is $q+1$. For generic values of the constants α_i and β_i with $i = 0, 1$ obviously there is no factorization. If however to evaluate the polynomial R numerically we choose, maybe randomly, the α_i and the β_i with $i = 0, 1$ as integer it may happen that:

$$\alpha_1 = \lambda\alpha_0, \ \beta_1 = \lambda\beta_0, \ \text{or} \ \alpha_0 = \lambda\alpha_1, \ \beta_0 = \lambda\beta_1, \quad \lambda \in \mathbb{Z}, \tag{2.46}$$

due to integer number factorization. Then in this situation we have in (2.45):

$$R = \lambda \frac{P(t)}{Q(t)}, \ \text{or} \ R = \frac{1}{\lambda} \frac{P(t)}{Q(t)}, \tag{2.47}$$

so in both cases the degrees of numerator and denominator drop by one. This is the mechanism of an accidental cancellation. So since we want to avoid accidental cancellation as in equation (2.47), the best way is to consider only *prime numbers*. If we had chosen the α_i and the β_i with $i = 0, 1$ as prime numbers the situation displayed in equation (2.46) couldn't have happened. This means that we must choose the α_i, β_i in (2.44) and the eventual parameters appearing in the equation as prime numbers. A final simplification which can speed up the calculations is to consider the factorization of the iterates on a finite field $\mathbb{K}_p$, with p prime. This is particularly useful since otherwise we need to perform factorization over the integer domain and the most common algorithms for factorization over integer domain actually at first compute factorization over finite fields [159]. Therefore using the factorization over finite fields we can speed up the evaluation of the iterates.

With these choices we should be able to avoid eventual accidental cancellations and therefore to have a *bona fide* sequence of degrees. Since the result is experimental it is still better to do it more than once, with different initial data, to be sure of the result. In Appendix B a `python` implementation of these ideas is given, including also other useful tools for the post evolution analysis. Let us just mention the fact that the prime number p is chosen adaptively as the first prime after the square plus one of the biggest prime in α_i, β_i and the equation parameters. This program is also presented in [64].

Now let us assume that we have computed our iterations and we are given the finite sequence:

$$d_0, d_1, \ldots, d_N. \tag{2.48}$$

To get the information on the asymptotic behavior of the sequence (2.48) we calculate its generating function [95], i.e. a function $g = g(s)$ such that its Taylor series coincides with the elements of the series. To look for such functions it is important to *use the minimum number of* d_k *possible*. It is reasonable to suppose that such generating function is rational even if it is known that it is not alway the case [15]. Such generating function can therefore be calculated by using Padé approximants [17, 135].

In the *Padé approximant* we represent a function as the ratio of two polynomials, i.e. as a rational function, an idea which dates back to the work of Frobenius [45]. Let us assume that we are given a function $f = f(s)$ analytic in some domain $D \subset \mathbb{C}$ containing $s = 0$[4]. Therefore the function f can be represented as *power series* centered at $s = 0$:

$$f(s) = \sum_{k=0}^{\infty} d_k s^k. \tag{2.49}$$

4 This is always possible to be done up to a translation.

To find the Padé approximant of order $[L : M]$ for the function f means to find a rational function:

$$[L:M](s) = \frac{P^{[L:M]}(s)}{Q^{[L:M]}(s)} = \frac{a_0 + a_1 s + \dots a_L s^L}{1 + b_1 s + \dots b_M s^M}, \tag{2.50}$$

such that its Taylor series centered at $s = 0$ coincides with (2.49) as much as possible. Notice that the choice of $b_0 = 1$ into the denominator (i.e. in the polynomial $Q^{[L:M]}(s)$) is purely conventional, since the ratio of the two polynomial $P^{[L:M]}(s)$ and $Q^{[L:M]}(s)$ is defined up to a common factor. A Padé approximant of order $[L : M]$ has *a priori* $L + 1$ independent coefficients in the numerator and M independent coefficients in denominator, i.e. $L + M + 1$ independent coefficients. This means that given a full Taylor series (2.49) and a Padé approximant of order $[L : M]$ we will have a precision of order s^{L+M+1}:

$$f(s) = [L:M](s) + O\left(s^{L+M+1}\right). \tag{2.51}$$

In the same way as a polynomial is its own Taylor series, a rational function is its own Padé approximant for the correct choice of L and M. This is why the best approach in the search for rational generating functions is the use of Padé approximants. By computing a sufficiently high number of terms if the generating function is rational we will eventually find it.

Without going into the details of such beautiful theory we note that the simplest way to compute the Padé approximant of order $[L : M]$ (2.50) is from Linear Algebra by computing two determinants:

$$Q^{[L:M]}(s) = \begin{vmatrix} d_{L-M+1} & d_{L-M+2} & \dots & d_L & d_{L+1} \\ d_{L-M+2} & d_{L-M+3} & \dots & d_{L+1} & d_{L+2} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{L-1} & d_L & \dots & d_{L+M-2} & d_{L+M-1} \\ d_L & d_{L+1} & \dots & d_{L+M-1} & d_{L+M} \\ s^M & s^{M-1} & \dots & s & 1 \end{vmatrix}, \tag{2.52a}$$

$$P^{[L:M]}(z) = \begin{vmatrix} d_{L-M+1} & d_{L-M+2} & \dots & d_{L+1} \\ d_{L-M+2} & d_{L-M+3} & \dots & d_{L+2} \\ \vdots & \vdots & & \vdots \\ d_{L-1} & d_L & \dots & d_{L+M-1} \\ d_L & d_{L+1} & \dots & d_{L+M} \\ \sum_{k=0}^{L-M} d_k s^{M+k} & \sum_{k=0}^{L-M+1} d_k s^{M+k-1} & \dots & \sum_{k=0}^{L} d_k s^k \end{vmatrix}. \tag{2.52b}$$

The Padé approximant of order $[L : M]$ is then given by [88]:

$$[L : M](s) = \frac{P^{[L:M]}(s)}{Q^{[L:M]}(s)}. \tag{2.53}$$

The interested reader can consult the reference [17] for a complete exposition.

Let us suppose now we have obtained a generating function using a subset of the elements in (2.48). Such generating function is a *predictive tool*, since we can confront the next terms in (2.48) with the successive terms of the Taylor expansion of the generating function. If they agree we are almost surely on the right way. This is the reason why it is important to find the generating function with the minimum number of d_k possible. If we used all the terms in the sequence (2.48) then we need to compute more iterates in order to have a predictive result. Notice that since the sequence (2.48) is a series of degrees, i.e. of positive integers, if the Taylor series of the generating functions give raise to rational numbers it can be immediately discarded as non *bona fide* generated function. A reasonable strategy for finding rational generating functions with Padé approximants is to use Padé approximants of equal order $[j : j]$ which, as stated above, need $2j + 1$ point to be determined. This kinds of computational issues are discussed practically in Appendix B following [64]. Let us remark, that the Taylor coefficients of a rational generating function satisfy a finite order linear recurrence relation [37].

Once we have a generating function we need to calculate the asymptotic behavior of the coefficients of its Taylor series. To do this we will use the *inverse* $\mathcal{Z}$*-transform* [31, 37, 90]. Indeed let $f(\tau)$ be a function expressible as Laurent series of *negative powers* of its argument:

$$f(\tau) = \sum_{k=0}^{\infty} d_k \tau^{-k}. \tag{2.54}$$

The inverse $\mathcal{Z}$-transform of f, which we denote by $\mathcal{Z}^{-1}$, is then defined to be [90]:

$$\mathcal{Z}^{-1}[f(\tau)]_k = \frac{1}{2\pi i}\int_C f(\tau)\tau^{k-1}d\tau, \quad k \in \mathbb{N} \tag{2.55}$$

where C is a simple closed path outside of which $f(\tau)$ is analytic. If f is a rational function, C can be taken as a circle of radius R in the complex τ plane enclosing all the singularities of $f(\tau)$. By the residue theorem [28, 162], this implies, as the rational functions have only a finite numbers of poles $\tau_j \in \mathbb{C}$, $j \in \{1, \ldots, P\}$:

$$\mathcal{Z}^{-1}[f(\tau)] = \sum_{j=1}^{P} \operatorname{Res}_{\tau=\tau_j}\left\{f(\tau)\tau^{k-1}\right\}. \tag{2.56}$$

Given the generating function g which is expressed as the Taylor series

$$g(s) = \sum_{k=0}^{\infty} d_k s^k \tag{2.57}$$

i.e. a series of *positive powers* of s, we have by the definition (2.55):

$$d_k = \mathcal{Z}^{-1}\left[g(\tau^{-1})\right]_k. \tag{2.58}$$

Since we suppose our generating function to be rational, $g\left(\tau^{-1}\right)$ is again a rational function and we can easily compute it using the residue approach (2.56).

Formula (2.56) will be valid asymptotically, and for rational $f(\tau)$ we can estimate for which k it will be valid. Assume that $f(\tau) = \widehat{P}(\tau)/\widehat{Q}(\tau)$ with $\widehat{P}, \widehat{Q} \in \mathbb{C}[\tau]$, i.e. polynomials in τ. Indeed if $\tau = 0$ is a root of $\widehat{Q}$ of order k_0 we must distinguish the cases when $k > k_0 + 1$ and $k \leqslant k_0 + 1$. For $k \leqslant k_0 + 1$ we will have a pole in $\tau = 0$ of order $k_0 + 1 - k$. So the general formula (2.56) will be valid only for $n > k_0 + 1$. In the case of rational generating functions $g(s) = P(s)/Q(s)$, with $P, Q \in \mathbb{C}[s]$ we will introduce a spurious $\tau = 0$ singularity in $g\left(\tau^{-1}\right)$ if we will have $\deg P \geqslant \deg Q$.

From the generating function we can get the Algebraic Entropy, defined by formula (2.19). Recalling the notion of radius of convergence R of a power series [162]:

$$R^{-1} = \lim_{k\to\infty} |d_k|^{\frac{1}{k}} \tag{2.59}$$

and the continuity of the logarithm function we have from (2.19) that the Algebraic Entropy can be always given by the logarithm of the inverse of the smallest root of the denominator of g:

$$\eta = \min_{\{s\in\mathbb{C} \mid Q(s)=0\}} \log|s|^{-1}. \tag{2.60}$$

To conclude this Subsection we examine the growths of Examples 2.1.2, 2.1.3 and 2.1.4 using the generating functions in order to have a more rigorous proof of their growth.

Example 2.2.1 (Growth of the Hénon map). The first and very trivial example is to consider the growth of the Hénon map (2.25) (or equation (2.26)) as given by (2.27). We see that computing the Padé approximant with the first three points, i.e. $[1:1](s)$ we obtain:

$$[1:1](s) = \frac{1-s}{1-2s}. \tag{2.61}$$

This first Padé approximant is already predictive since

$$[1:1](s) = 1 + s + 2s^2 + 4s^3 + 8s^4 + 16s^5 + 32s^5 + O\left(s^6\right). \tag{2.62}$$

So we can conclude $g(s) = [1:1](s)$. This obviously gives the growth as $d_k = 2^{k-1}$ for every $k \geqslant 1$[5]. The only pole of $g(s)$ is then $s_0 = 1/2$ which according to (2.60) yields $\eta = \log 2$. □

Example 2.2.2 (Growth of the map (2.28)). We see that in this case the Padé approximant $[1 : 1](s)$ gives exactly the same result as (2.61), therefore it does not describes the series. The next one $[2 : 2](s)$ instead gives:

$$[2:2](s) = \frac{1}{1-s-s^2}, \tag{2.63}$$

which is predictive. We conclude that in this case $g(s) = [2 : 2](s)$. This gives the growth:

$$d_k = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^k + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^k, \tag{2.64}$$

which, as we said before, has the asymptotic behavior (2.31). The poles of (2.63) are then:

$$s_\pm = \frac{-1 \pm \sqrt{5}}{2} \tag{2.65}$$

and as $|s_+| > |s_-|$ the algebraic entropy is given by (2.32). □

Example 2.2.3 (Growth of the Hirota-Kimura-Yahagi equation). We now consider the very slow growth of equation (2.33) given by (2.36). Again the $[1 : 1](s)$ is the same as in (2.61) and therefore does not describes (2.36) as well as $[2;2](s)$. On the other hand $[3:3](s)$ gives:

$$[3:3](s) = \frac{s^3+s^2-s+1}{(1-s)^2} \tag{2.66}$$

and again this approximant is predictive. So we conclude that $g(s) = [3 : 3](s)$ and that $d_k = 2k-2$. Since the only pole lays on the unit circle we have that the entropy is zero. □

2.3 ALGEBRAIC ENTROPY TEST FOR (1.91, 1.93)

Before considering the Algebraic Entropy of the equations presented in Section 1.6 we discuss briefly the Algebraic Entropy for the rhombic H^4 equations 1.32, which are well known to be integrable [166]. Running the program `ae2d.py` from [64], which is contained in Appendix B, on these equations one finds that they possess only a principal sequence with the following isotropic sequence of degrees:

$$1, 2, 4, 7, 11, 16, 22, 29, 37, \ldots. \tag{2.67}$$

5 We note that in terms of the iterations the first 1 in (2.27) have to be interpreted as d_{-1}, but this is just matter of notation.

Equation	Growth direction			
	$-,+$	$+,+$	$+,-$	$-,-$
${}_tH_1^\varepsilon$	L_1, L_2	L_1, L_2	L_3, L_4	L_3, L_4
${}_tH_2^\varepsilon$	L_5, L_6	L_5, L_6	L_7, L_8	L_7, L_8
${}_tH_3^\varepsilon$	L_5, L_6	L_5, L_6	L_7, L_8	L_7, L_8
D_1	L_0, L_0	L_0, L_0	L_0, L_0	L_0, L_0
${}_1D_2$	L_9, L_{10}	L_9, L_{10}	L_9, L_{10}	L_9, L_{10}
${}_2D_2$	L_{14}, L_{15}	L_{14}, L_{15}	L_{15}, L_{14}	L_{15}, L_{14}
${}_3D_2$	L_{16}, L_{17}	L_{16}, L_{17}	L_{17}, L_{16}	L_{17}, L_{16}
D_3	L_{11}, L_{12}	L_{11}, L_{12}	L_{11}, L_{12}	L_{11}, L_{12}
${}_1D_4$	L_{13}, L_8	L_{13}, L_8	L_{13}, L_8	L_{13}, L_8
${}_2D_4$	L_{18}, L_8	L_{18}, L_8	L_{19}, L_{20}	L_{19}, L_{20}

Table 2.1: Sequences of growth for the trapezoidal H^4 and H^6 equations. The first one is the principal sequence, while the second the secondary. All sequences L_j, $j = 0, \cdots, 20$ are presented in Table 2.3.

To this sequence corresponds the generating function:

$$g = \frac{s^2 - s + 1}{(1-s)^3}, \tag{2.68}$$

which gives, through the application of the $\mathcal{Z}$-transform (2.55), the asymptotic fit of the degrees:

$$d_k = \frac{k(k+1)}{2} + 1. \tag{2.69}$$

Since the growth is quadratic $\eta = 0$ or it can be seen directly from (2.67) using (2.60). This result is a confirmation by the Algebraic Entropy approach of the integrability of the rhombic H^4 equations. Let us note that the growth sequences (2.67) are the same in the Rhombic H^4 equations also when $\varepsilon = 0$, i.e. if we are in the case of the H equations of the ABS classification (1.23).

For the trapezoidal H^4 and H^6 equations the situation is more complicated. Indeed those equations have in every direction two different sequence of growth, the principal and the secondary one, as the coefficients of the evolution matrix (2.37) are two-periodic. The most surprising feature is however that all the sequences of growth are linear. This means that all such equations are not only integrable due to the fact that they possess the Consistency Around the Cube property, but also linearizable.

Instead of presenting here the full evolution matrices (2.37), which would be very lengthily and obscure, we present two tables with the relevant properties. The interested reader will find the full matrices in Appendix C. In Table 2.1 we present a summary of the sequences of growth of both trapezoidal $H^4$and H^6 equations. The explicit sequence of the degrees of growth with generating functions, asymptotic fit of the degrees of growth and entropy is given in Table 2.3. Following the convention in [84] we labeled the different sequences by L_j, with $j \in \{1, 2, \dots, 20\}$.

Observing Table 2.1 and Table 2.3 we may notice the following facts:

- The trapezoidal H^4 equations are not isotropic: the sequences in the $(-,+)$ and $(+,+)$ directions are different from those in the $(+,-)$ and $(-,-)$ directions. These results reflect the symmetry of the equations.

- The H^6 equations, except from ${}_2D_4$ which has the same behaviour as the trapezoidal H^4, are isotropic. Equations ${}_2D_2$ and ${}_3D_2$ exchange the principal and the secondary sequences from the $(-,+)$, $(+,+)$ directions and the $(+,-)$, $(-,-)$ directions.

- All growths, except L_0, L_3, L_4, L_7, L_8, L_{12} and L_{17}, exhibit a highly oscillatory behaviour. They have generating functions of the form:

$$g(s) = \frac{P(s)}{(s-1)^2(s+1)^2}, \tag{2.70}$$

with the polynomial $P(s) \in \mathbb{Z}[s]$. We may write

$$g(s) = P_0(s) + \frac{P_1(s)}{(s-1)^2(s+1)^2}, \tag{2.71}$$

with $P_0(s) \in \mathbb{Z}[s]$ of degree less than P and $P_1(s) = \alpha s^3 + \beta s^2 + \gamma s + \delta$. Expanding the second term in (2.71) in partial fractions we obtain:

$$\begin{aligned} g(s) = P_0(s) + \frac{1}{4}\Bigg[&\frac{\beta+\delta-\alpha-\gamma}{(s+1)^2} + \frac{\alpha+\gamma+\beta+\delta}{(s-1)^2} \\ &+ \frac{2\alpha+\beta-\delta}{s-1} + \frac{2\alpha-\beta+\delta}{s+1}\Bigg]. \end{aligned} \tag{2.72}$$

Expanding the term in square parentheses in Taylor series we find that:

$$g(s) = P_0(s) + \sum_{k=0}^{\infty}\left[A_0 + A_1(-1)^k + A_2 k + A_3(-1)^k k\right] s^k, \tag{2.73}$$

with $A_i = A_i(\alpha, \beta, \gamma, \delta)$ constants. This means the $d_k = A_0 + A_1(-1)^k + A_2 k + A_3(-1)^k k$ for $k > \deg P_0(s)$, and therefore it asymptotically solves a fourth order difference equation. As far as we know, even if some example of behaviour containing terms like $(-1)^k$ are known [84], this is the first time that we observe patterns with oscillations given by $k\,(-1)^k$.

We remark that the usage of the Algebraic Entropy as integrability indicator is actually justified by the existence of of finite order recurrence relations between the degrees d_k. Indeed the existence of such recurrence relations means that from a local property (the sequence of degrees) we may infer a global one (chaoticity/integrability/linearizaribilty) [158].

We finally note that at the moment no other quad equation is known to possess an analogous growth property as the trapezoidal H^4 and H^6 equations. Furthermore is also unknown if more complicated behavior, with higher order periodicities are possible. We note that the existence of two sequence of growth is coherent with the fact that the trapezoidal H^4 and H^6 equations have two-periodic coefficients. So we can conjecture that the existence of multiples growth patterns is linked with the periodicity of the coefficients. In literature equations with higher periodicity have been introduced e.g. in [53], but are still unstudied from the point of view of Algebraic Entropy.

Table 2.2: Sequences of growth, generating functions, analytic expression of the degrees and entropy for the trapezoidal H^4 and H^6 equations

Name	Degrees $\{d_k\}$	Generating function $g(s)$	Degree fit d_k	Entropy η
L_0	$1,1,1,1,1,1,1,1,1,1,1,1,1\dots$	$\dfrac{1}{1-s}$	1	0
L_1	$1,2,2,5,3,8,4,11,5,14,6,17,7\dots$	$\dfrac{s^3+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(-2k+1)+k+\dfrac{3}{4}$	0
L_2	$1,2,4,3,7,4,10,5,13,6,16,7,19\dots$	$\dfrac{-s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(2k-1)+k+\dfrac{5}{4}$	0
L_3	$1,2,2,3,3,4,4,5,5,6,6,7,7\dots$	$\dfrac{-s^2+s+1}{s^3-s^2-s+1}$	$-\dfrac{(-1)^k}{4}+\dfrac{k}{2}+\dfrac{5}{4}$	0
L_4	$1,2,4,5,7,8,10,11,13,14,16,17,19\dots$	$\dfrac{s^2+s+1}{s^3-s^2-s+1}$	$\dfrac{(-1)^k}{4}+\dfrac{3k}{2}+\dfrac{3}{4}$	0
L_5	$1,2,4,6,11,10,19,14,27,18,35,22,43\dots$	$\dfrac{s^6+4s^4+2s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$(-1)^k\left(k-\dfrac{5}{2}\right)+3k-\dfrac{5}{2}$	0
L_6	$1,2,4,7,8,15,12,23,16,31,20,39,24\dots$	$\dfrac{3s^5+s^4+3s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$(-1)^k\left(-k+\dfrac{5}{2}\right)+3k-\dfrac{5}{2}$	0
L_7	$1,2,4,7,11,15,19,23,27,31,35,39,43\dots$	$\dfrac{s^4+s^3+s^2+1}{(s-1)^2}$	$4k-5$	0

Continued on next page

Table 2.2 – *Continued from previous page*

Name	Degrees $\{d_k\}$	Generating function $g(s)$	Degree fit d_k	Entropy η
L_8	$1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 \ldots$	$\dfrac{s^2+1}{(s-1)^2}$	$2k$	0
L_9	$1, 2, 2, 5, 3, 8, 4, 11, 5, 14, 6, 17, 7, \ldots$	$\dfrac{s^3+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(-2k+1)+k+\dfrac{3}{4}$	0
L_{10}	$1, 2, 3, 5, 5, 8, 7, 11, 9, 14, 11, 17, 13, \ldots$	$\dfrac{s^3+s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(-k+1)+\dfrac{5k}{4}+\dfrac{3}{4}$	0
L_{11}	$1, 2, 4, 5, 10, 8, 16, 11, 22, 14, 28, 17, 34, \ldots$	$\dfrac{3s^4+s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(3k-5)+\dfrac{9k}{4}-\dfrac{3}{4}$	0
L_{12}	$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, \ldots$	$\dfrac{s^2+s+1}{(s-1)^2(s+1)}$	$\dfrac{(-1)^k}{4}+\dfrac{3k}{2}+\dfrac{3}{4}$	0
L_{13}	$1, 2, 4, 6, 11, 10, 18, 14, 25, 18, 32, 22, 39, \ldots$	$\dfrac{4s^4+2s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{3(-1)^k}{4}(k-2)+\dfrac{11k}{4}-\dfrac{3}{2}$	0
L_{14}	$1, 2, 3, 3, 6, 4, 9, 5, 12, 6, 15, 7, 18, \ldots$	$\dfrac{s^4-s^3+s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(2k-3)+k+\dfrac{3}{4}$	0
L_{15}	$1, 1, 3, 3, 6, 5, 9, 7, 12, 9, 15, 11, 18, \ldots$	$\dfrac{s^4+s^3+s^2+s+1}{(s-1)^2(s+1)^2}$	$\dfrac{k}{4}\left((-1)^k+5\right)$	0

Continued on next page

Table 2.2 – *Continued from previous page*

Name	Degrees $\{d_k\}$	Generating function $g(s)$	Degree fit d_k	Entropy η
L_{16}	$1, 2, 3, 2, 5, 2, 7, 2, 9, 2, 11, 2, 13, \ldots$	$\dfrac{-2s^3+s^2+2s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{2}(k-1)+\dfrac{k}{2}+\dfrac{3}{2}$	0
L_{17}	$1, 1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, \ldots$	$\dfrac{s^2+1}{(s-1)^2(s+1)}$	$\dfrac{(-1)^k}{2}+k+\dfrac{1}{2}$	0
L_{18}	$1, 2, 4, 5, 11, 9, 19, 13, 27, 17, 35, 21, 43, \ldots$	$\dfrac{s^6+s^5+4s^4+s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$(-1)^k(k-2)+3k-3$	0
L_{19}	$1, 2, 4, 6, 11, 10, 19, 14, 27, 18, 35, 22, 43, \ldots$	$\dfrac{s^6+4s^4+2s^3+2s^2+2s+1}{(s-1)^2(s+1)^2}$	$(-1)^k\left(k-\dfrac{5}{2}\right)+3k-\dfrac{5}{2}$	0
L_{20}	$1, 1, 3, 2, 6, 3, 9, 4, 12, 5, 15, 6, 18, \ldots$	$\dfrac{s^4+s^2+s+1}{(s-1)^2(s+1)^2}$	$\dfrac{(-1)^k}{4}(2k-1)+k+\dfrac{1}{4}$	0

2.4 EXAMPLES OF DIRECT LINEARIZATION

In this Section we considers in detail the ${}_{t}H_{1}^{\varepsilon}$ (1.91a) and ${}_{1}D_{2}$ (1.93b) equations and shows the explicit form of the quadruple of matrices coming from the Consistency Around the Cube, the non-autonomous equations which give the consistency on $\mathbb{Z}^3$ and the effective Lax pair. Finally we confirm the predictions of the algebraic entropy analysis showing how they can be explicitly linearized looking directly at the equation.

2.4.1 *The* ${}_{t}H_{1}^{\varepsilon}$ *equation*

To construct the Lax Pair for (1.91a) we have to deal with **Case 3.10.1** in [21]. The sextuple of equations we consider is:

$$A = \alpha_2 (x - x_1)(x_2 - x_{12}) - \alpha_1 (x - x_2)(x_1 - x_{12}) + \varepsilon^2 \alpha_1 \alpha_2 (\alpha_1 - \alpha_2), \tag{2.74a}$$

$$B = (x - x_2)(x_3 - x_{23}) - \alpha_2 \left(1 + \varepsilon^2 x_3 x_{23}\right) = 0, \tag{2.74b}$$

$$C = (x - x_1)(x_3 - x_{13}) - \alpha_1 \left(1 + \varepsilon^2 x_3 x_{13}\right) = 0, \tag{2.74c}$$

$$\bar{A} = \alpha_2 (x_{13} - x_3)(x_{123} - x_{23}) - \alpha_1 (x_{13} - x_{123})(x_3 - x_{123}), \tag{2.74d}$$

$$\bar{B} = (x_1 - x_{12})(x_{13} - x_{123}) - \alpha_2 \left(1 + \varepsilon^2 x_{13} x_{123}\right) = 0, \tag{2.74e}$$

$$\bar{C} = (x_2 - x_{12})(x_{23} - x_{123}) - \alpha_1 \left(1 + \varepsilon^2 x_{23} x_{123}\right) = 0, \tag{2.74f}$$

In this sextuple (1.91a) originates from the B equation.

We now make the following identifications

$$\begin{array}{lllll} A: & x \to u_{p,n} & x_1 \to u_{p+1,n} & x_2 \to u_{p,n+1} & x_{12} \to u_{p+1,n+1} \\ B: & x \to u_{n,m} & x_2 \to u_{n+1,m} & x_3 \to u_{n,m+1} & x_{23} \to u_{n+1,m+1} \\ C: & x \to u_{p,m} & x_1 \to u_{p+1,m} & x_3 \to u_{p,m+1} & x_{13} \to u_{p+1,m+1} \end{array} \tag{2.75}$$

so that in any equation we can suppress the dependence on the appropriate parametric variables. On $\mathbb{Z}^3$ we get the following triplet of equations:

$$\tilde{A} = \alpha_2 \left(u_{p,n} - u_{p+1,n}\right)\left(u_{p,n+1} - u_{p+1,n+1}\right) - \alpha_1 \left(u_{p,n} - u_{p,n+1}\right)\left(u_{p+1,n} - u_{p+1,n+1}\right) + \varepsilon^2 \alpha_1 \alpha_2 (\alpha_1 - \alpha_2) F_m^{(+)}, \tag{2.76a}$$

$$\tilde{B} = (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) - \alpha_2 - \alpha_2 \varepsilon^2 \left(F_m^{(+)} u_{n,m+1} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m}\right), \tag{2.76b}$$

$$\tilde{C} = \left(u_{p,m} - u_{p+1,m}\right)\left(u_{p,m+1} - u_{p+1,m+1}\right) - \alpha_1 - \alpha_1 \varepsilon^2 \left(F_m^{(+)} u_{p,m+1} u_{p+1,m+1} + F_m^{(-)} u_{p,m} u_{p+1,m}\right). \tag{2.76c}$$

Then with the method in Section 1.4 we find the following Lax Pair[6]:

$$\tilde{L}_{n,m} = \begin{pmatrix} u_{n,m+1} & -u_{n,m}u_{n,m+1}+\alpha_1 \\ 1 & -u_{n,m} \end{pmatrix} - \varepsilon^2\alpha_1 \begin{pmatrix} -F_m^{(-)}u_{n,m} & 0 \\ 0 & F_m^{(+)}u_{n,m+1} \end{pmatrix}, \tag{2.77a}$$

$$\begin{aligned}\tilde{M}_{n,m} &= \alpha_1\left(u_{n,m}-u_{n+1,m}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ &+ \alpha_2\begin{pmatrix} u_{n+1,m} & -u_{n,m}u_{n+1,m} \\ 1 & -u_{n,m}\end{pmatrix} \\ &- \varepsilon^2\alpha_1\alpha_2\left(\alpha_1-\alpha_2\right)F_m^{(+)}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\end{aligned} \tag{2.77b}$$

and the separation constant (1.21) is given by:

$$\tau_{n,m} = \alpha_2\frac{1+\varepsilon^2\left(F_m^{(-)}u_{n,m}^2+F_m^{(+)}u_{n,m+1}^2\right)}{\left(u_{n,m}-u_{n,m+1}\right)^2+\varepsilon^2\alpha_2^2F_m^{(+)}}. \tag{2.78}$$

With a highly non-trivial computation it is however possible to show that the separation constant (2.78) arise from the following *rational* separation constants:

$$\gamma_{n,m} = \frac{1}{1+i\varepsilon\left(F_m^{(-)}u_{n,m}-F_m^{(+)}u_{n,m+1}\right)}, \tag{2.79a}$$

$$\gamma'_{n,m} = \frac{1}{u_{n,m}-u_{n+1,m}+i\alpha_2F_m^{(-)}}. \tag{2.79b}$$

Therefore the Lax pair for the ${}_tH_1^\varepsilon$ equation (1.91a) is completely characterized from the Consistency Around the Cube.

Let us now turn to the linearization procedure. In (1.91a) we must have $\alpha_2 \neq 0$, otherwise the equation becomes

$$\left(u_{n,m}-u_{n+1,m}\right)\left(u_{n,m+1}-u_{n+1,m+1}\right)=0, \tag{2.80}$$

which is factorizable and therefore degenerate. Let us define two new fields for the even and odd values of m in order to write the ${}_tH_1^\varepsilon$ equation (1.91a) as an autonomous system:

$$u_{n,2k}=w_{n,k}, \quad u_{n,2k+1}=z_{n,k}. \tag{2.81}$$

Then the ${}_tH_1^\varepsilon$ equation (1.91a) becomes the following system of coupled autonomous difference equations

$$\left(w_{n,k}-w_{n+1,k}\right)\left(z_{n,k}-z_{n+1,k}\right)=\varepsilon^2\alpha_2 z_{n,k}z_{n+1,k}+\alpha_2, \tag{2.82a}$$

6 In this case for simplicity of calculations the rôle of the L and M matrices are inverted.

$$(w_{n,k+1} - w_{n+1,k+1})(z_{n,k} - z_{n+1,k}) = \varepsilon^2 \alpha_2 z_{n,k} z_{n+1,k} + \alpha_2. \tag{2.82b}$$

Subtracting (2.82b) to (2.82a), we obtain

$$(w_{n,k} - w_{n+1,k} - w_{n,k+1} + w_{n+1,k+1})(z_{n,k} - z_{n+1,k}) = 0. \tag{2.83}$$

At this point the solution of the system bifurcates depending on which factor in (2.83) we choose to annihilate.

CASE 1: $z_{n,k} = f_k$ Let us assume that $z_{n,k} = f_k$ where f_k is an arbitrary function of its argument. Then the second factor in (2.83) is satisfied and from (2.82a) or (2.82b) we have that $\varepsilon \neq 0$ and, solving for f_k, one gets

$$f_k = \pm \frac{i}{\varepsilon}. \tag{2.84}$$

This is essentially a degenerate case, since this solution holds for every value of $w_{n,k}$.

CASE 2: $w_{n,k} = g_n + h_k$ Let us assume that $w_{n,k} = g_n + h_k$ where g_n and h_k are arbitrary functions of their argument and that $z_{n,k} \neq f_k$. Hence (2.82) reduces to the equation:

$$\varepsilon^2 z_{n,k} z_{n+1,k} + \kappa_n (z_{n,k} - z_{n+1,k}) + 1 = 0, \quad , \tag{2.85}$$

where $\kappa_n = (g_{n+1} - g_n)/\alpha_2$. Depending on the value of ε we can distinguish two different cases:

CASE 2.1: $\varepsilon = 0$ If $\varepsilon = 0$, (2.85) implies that $\kappa_n \neq 0$, so that we have:

$$z_{n+1,k} - z_{n,k} = \frac{1}{\kappa_n}. \tag{2.86}$$

Solving this equation we get:

$$z_{n,k} = \begin{cases} j_k + \displaystyle\sum_{l=n_0}^{n-1} \frac{1}{\kappa_l}, & n \geqslant n_0 + 1, \\ j_k - \displaystyle\sum_{l=n}^{n_0-1} \frac{1}{\kappa_l}, & n \leqslant n_0 - 1, \end{cases} \tag{2.87}$$

where $j_k = z_{n_0,k}$ is an arbitrary function of its argument.

CASE 2.2: $\varepsilon \neq 0$ If $\varepsilon \neq 0$, then (2.85) is a discrete Riccati equation which can be linearized through the Möbius transformation

$$z_{n,k} = \frac{i}{\varepsilon} \frac{y_{n,k} - 1}{y_{n,k} + 1} \tag{2.88}$$

to

$$(i\kappa_n - \varepsilon)\, y_{n+1,k} = (i\kappa_n + \varepsilon)\, y_{n,k}. \tag{2.89}$$

Note that (2.89) implies that $\kappa_n \neq \pm i\varepsilon$ because otherwise $y_{n,k} = 0$ and $z_{n,k} = -i/\varepsilon$. Therefore equation (2.89) implies that we have the following solution:

$$y_{n,k} = \begin{cases} j_k \prod_{l=n_0}^{n-1} \dfrac{i\kappa_l + \varepsilon}{i\kappa_l - \varepsilon}, & n \geqslant n_0 + 1, \\ j_k \prod_{l=n}^{n_0-1} \dfrac{i\kappa_l - \varepsilon}{i\kappa_l + \varepsilon}, & n \leqslant n_0 - 1, \end{cases} \tag{2.90}$$

where $j_k = y_{n_0,k}$ is another arbitrary function of its argument.

In conclusion we have always completely integrated the original system.

Remark 2.4.1. Let us note that in the case $\varepsilon = 0$ the (1.91a) can be linearized also without the need to introduce the transformation separating the even and the odd part in m (2.81). Indeed by putting ε in (1.91a) we obtain:

$$(u_{n,m} - u_{n+1,m})\,(u_{n,m+1} - u_{n+1,m+1}) - \alpha_2 = 0, \tag{2.91}$$

and it is easy to see that the Möbius-like transformation depending on the field and on its first order shift

$$u_{n+1,m} - u_{n,m} = \sqrt{\alpha_2}\,\frac{1 - w_{n,m}}{1 + w_{n,m}}, \tag{2.92}$$

brings (1.91a) into the following first order linear equation:

$$w_{n,m+1} + w_{n,m} = 0,. \tag{2.93}$$

Therefore we can write the solution to (2.91) as:

$$u_{n,m} = \begin{cases} k_m + \sqrt{\alpha_2} \sum_{l=n_0}^{l=n-1} \dfrac{1 - (-1)^m w_l}{1 + (-1)^m w_l}, & n \geqslant n_0 + 1, \\ k_m - \sqrt{\alpha_2} \sum_{l=n}^{l=n_0-1} \dfrac{1 - (-1)^m w_l}{1 + (-1)^m w_l}, & n \leqslant n_0 - 1. \end{cases} \tag{2.94}$$

Here $k_m = u_{n_0,m}$ and w_n, are two arbitrary integration functions.

Now we prove that the Lax pair of the ${}_tH_1^\varepsilon$ equation (1.91a) given by equation (2.77) is *fake*. This was presented for the first time in [68]. Indeed we recall that according to the definition in [77, 78] a Lax pair is called $\mathcal{G}$-*fake* if, on solutions of the equation appearing in its compatibility condition, one can remove all dependent variables in the associated nonlinear system from the Lax pair by applying a *gauge transformations*. A gauge transformation for a Lax pair in the form (1.5) is a $K \times K$ invertible matrix $\mathcal{G}_{n,m}$, possibly dependent on the unknown $u_{n,m}$ and its shifts, acting at the level of the wave function $\Phi_{n,m}$ in the following way:

$$\Phi_{n,m} = \mathcal{G}_{n,m}\Psi_{n,m}. \tag{2.95}$$

The transformation (2.95) yields a new Lax pair of the form (1.5) for the new wave function $\Psi_{n,m}$ with new matrices:

$$\mathcal{L}_{n,m} = \mathcal{G}_{n+1,m}^{-1} L_{n,m}\mathcal{G}_{n,m}, \tag{2.96a}$$

$$\mathcal{M}_{n,m} = \mathcal{G}_{n,m+1}^{-1} M_{n,m}\mathcal{G}_{n,m}. \tag{2.96b}$$

We note that a Gauge transformation in the sense of equation (2.95) is a particular kind of symmetry of the Lax pair.

Now that we have stated the required definition, let us return to the Lax pair for the ${}_tH_1^\varepsilon$ equation (1.91a) as given by equation (2.77). First we separate the even and odd part in m in (2.77) defining:

$$\Xi_{n,k} = \Psi_{n,2k}, \quad \Theta_{n,k} = \Psi_{n,2k+1}. \tag{2.97}$$

Substituting m odd into the spectral problem defined from (2.77) we obtain the following constraint:

$$\Theta_{n,k} = \frac{1}{i - \varepsilon z_{n,k}} \begin{pmatrix} z_{n,k} & \alpha_1 - w_{n,k} z_{n,k} \\ 1 & w_{n,k} - \varepsilon^2 \alpha_1 z_{n,k} \end{pmatrix} \Xi_{n,k}. \tag{2.98}$$

Then using (2.98) and its difference consequences we can write down a Lax pair involving only the field $w_{n,k}$ and the wave function $\Xi_{n,k}$:

$$\Xi_{n,k+1} = L_{n,k}^{(1)}\Xi_{n,k}, \quad \Xi_{n+1,k} = M_{n,k}^{(1)}\Xi_{n,k}, \tag{2.99}$$

where:

$$L_{n,k}^{(1)} = -\alpha_1 \begin{pmatrix} 1 & w_{n,k+1} - w_{n,k} \\ 0 & 1 \end{pmatrix}, \tag{2.100a}$$

$$\begin{aligned} M_{n,k}^{(1)} = & \frac{\alpha_1 (w_{n,k} - w_{n+1,k})}{w_{n,k} - w_{n,k+1} + i\varepsilon\alpha_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & + \frac{\alpha_2}{w_{n,k} - w_{n,k+1} + i\varepsilon\alpha_2} \begin{pmatrix} w_{n+1,k} & -w_{n,k} w_{n+1,k} \\ 1 & -w_{n,k} \end{pmatrix} \\ & - \frac{\varepsilon^2 \alpha_1 \alpha_2 (\alpha_1 - \alpha_2)}{w_{n,k} - w_{n,k+1} + i\varepsilon\alpha_2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.100b}$$

The compatibility conditions of (2.99) yields the linear equation:

$$w_{n+1,k+1} - w_{n,k+1} - w_{n+1,k} + w_{n,k} = 0. \tag{2.101}$$

Therefore we have incidentally given another proof of the linearization using the Lax pair.

Finally we see that applying the following gauge transformation:

$$\mathcal{G}_{n,k} = (-\alpha_1)^k \alpha_1^n \begin{pmatrix} 1 & w_{n,k} \\ 0 & 1 \end{pmatrix} \tag{2.102}$$

yields the new matrices obtained from (2.96):

$$\mathcal{L}^{(1)}_{n,k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.103a}$$

$$\begin{aligned} \mathcal{M}^{(1)}_{n,k} = & \frac{w_{n,k} - w_{n+1,k}}{w_{n,k} - w_{n+1,k} + i\varepsilon\alpha_2} \begin{pmatrix} 1 & w_{n,k} - w_{n+1,k} \\ \alpha_2/\alpha_1 & 1 \end{pmatrix} \\ & + \frac{\alpha_2}{w_{n,k} - w_{n+1,k} + i\varepsilon\alpha_2} \begin{pmatrix} 0 & -\varepsilon^2 (\alpha_1 - \alpha_2) \\ 1/\alpha_1 & 0 \end{pmatrix}. \end{aligned} \tag{2.103b}$$

Since (2.103a) is independent on the field $w_{n,k}$ we conclude that the Lax pair (2.100) is a $\mathcal{G}$-fake Lax pair according to the above definition.

We recall that a fake Lax pair do not provide the infinite number of conservation laws needed by the definition of integrability [27, 77, 78]. Since we proved that the ${}_tH_1^\varepsilon$ is linearizable this have to be expected. Indeed linear equations possess a *finite* number of conservation laws [169] and therefore any Lax pair for a linear equation must be fake. Ultimately this example shows that a Lax pair produced from the Consistency Around the Cube property is not *bona fide* in advance, but its properties must be thoroughly checked.

2.4.2 *The* ${}_1D_2$ *equation*

Now we consider the equation ${}_1D_2$ (1.93b). The ${}_1D_2$ equation (1.93b) is given by the sextuple of equations given by **Case 3.12.2** in [21]:

$$A = \delta_2 x + x_1 + (1-\delta_1) x_2 + x_{12} (x + \delta_1 x_2), \tag{2.104a}$$

$$\begin{aligned} B = & (x - x_3)(x_2 - x_{23}) \\ & + \lambda [x + x_3 - \delta_1 (x_2 + x_{23})] + \delta_1 \lambda, \end{aligned} \tag{2.104b}$$

$$\begin{aligned} C = & (x - x_3)(x_1 - x_{13}) + \delta_1 (\delta_1 \delta_2 + \delta_1 - 1) \lambda^2 \\ & - \lambda [(\delta_1 \delta_2 + \delta_1 - 1)(x + x_3) + \delta_1 (x_1 + x_{13})], \end{aligned} \tag{2.104c}$$

$$\bar{A} = \delta_2 x_3 + x_{13} + (1-\delta_1) x_{23} + x_{123} (x + \delta_1 x_{23}), \tag{2.104d}$$

$$\bar{B} \begin{aligned} & = (x_1 - x_{13})(x_{12} - x_{123}) \\ & + \lambda [2\delta_2 (\delta_1 - 1) + (\delta_1 - 1 - \delta_1 \delta_2) - 2\delta_1 x_{12} x_{123}], \end{aligned} \tag{2.104e}$$

$$\bar{C} = (x_1 - x_{23})(x_{12} - x_{123}) - \lambda (2\delta_2 + x_{12} + x_{123}). \tag{2.104f}$$

The triplet of consistent dynamical systems on the 3D-lattice is:

$$\begin{aligned}
\tilde{A} =& \left(F_{p+n}^{(-)} - \delta_1 F_p^{(+)} F_n^{(-)} + \delta_2 F_p^{(+)} F_n^{(+)}\right) u_{p,n} \\
&+ \left(F_{p+n}^{(+)} - \delta_1 F_p^{(-)} F_n^{(-)} + \delta_2 F_p^{(-)} F_n^{(+)}\right) u_{p+1,n} \\
&+ \left(F_{p+n}^{(+)} - \delta_1 F_p^{(+)} F_n^{(+)} + \delta_2 F_p^{(+)} F_n^{(-)}\right) u_{p,n+1} \\
&+ \left(F_{p+n}^{(-)} - \delta_1 F_p^{(-)} F_n^{(+)} + \delta_2 F_p^{(-)} F_n^{(-)}\right) u_{p+1,n+1} \\
&+ \delta_1 \left(F_n^{(-)} u_{p,n} u_{p+1,n} + F_n^{(+)} u_{p,n+1} u_{p+1,n+1}\right) \\
&+ F_{p+n}^{(+)} u_{p,n} u_{p+1,n+1} + F_{p+n}^{(-)} u_{p+1,n} u_{p,n+1},
\end{aligned} \tag{2.105}$$

$$\begin{aligned}
\tilde{B} =& \lambda \left\{ \left[(\delta_1 - 1) F_n^{(+)} - \delta_1 \right] F_p^{(+)} + (\delta_1 - 1 - \delta_1 \delta_2) F_p^{(-)} F_n^{(-)} \right\} (u_{n,m} + u_{n,m+1}) \\
&+ \lambda \left\{ \left[(\delta_1 - 1) F_n^{(-)} - \delta_1 \right] F_p^{(+)} + (\delta_1 - 1 - \delta_1 \delta_2) F_p^{(-)} F_n^{(+)} \right\} (u_{n+1,m} + u_{n+1,m+1}) \\
&- 2\delta_1 \lambda F_p^{(-)} \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) \\
&+ (u_{n,m} - u_{n,m+1})(u_{n+1,m} - u_{n+1,m+1}) \\
&+ \delta_1 \lambda^2 F_p^{(+)} + 2(\delta_1 - 1)\delta_2 \lambda F_p^{(-)},
\end{aligned} \tag{2.106}$$

$$\begin{aligned}
\tilde{C} =& \lambda \left\{ \left[(1 - \delta_1 \delta_2) F_p^{(+)} - \delta_1 \right] F_n^{(+)} + F_p^{(-)} F_n^{(-)} \right\} (u_{p,m} + u_{p,m+1}) \\
&+ \lambda \left\{ \left[(1 - \delta_1 \delta_2) F_p^{(-)} - \delta_1 \right] F_n^{(+)} + F_p^{(+)} F_n^{(-)} \right\} (u_{p+1,m} + u_{p+1,m+1}) \\
&+ (u_{p,m} - u_{p,m+1})(u_{p+1,m} - u_{p+1,m+1}) \\
&+ 2\delta_2 \lambda F_n^{(-)} + \delta_1 (\delta_1 - 1 + \delta_1 \delta_2) \lambda^2 F_n^{(+)}.
\end{aligned} \tag{2.107}$$

We leave out the Lax pair for ${}_1D_2$ as it is too complicate to write down and not worth while the effort for the reader. If necessary one can always write it down using the standard procedure outlined in Section 1.4.

Let us now turn to the linearization procedure. Notice that there is no combination of the parameters δ_1 and δ_2 such that (1.93b) becomes non-autonomous. So we are naturally induced to introduce the following four fields:

$$\begin{aligned}
w_{s,t} &= u_{2s,2t}, & y_{s,t} &= u_{2s+1,2t}, \\
v_{s,t} &= u_{2s,2t+1} & z_{s,t} &= u_{2s+1,2t+1}.
\end{aligned} \tag{2.108}$$

which transform (1.93b) into the following system of four coupled autonomous difference equations:

$$(1-\delta_1) v_{s,t} + \delta_2 w_{s,t} + y_{s,t} + (\delta_1 v_{s,t} + w_{s,t}) z_{s,t} = 0, \tag{2.109a}$$

$$\begin{aligned}
&(1-\delta_1) v_{s+1,t} + \delta_2 w_{s+1,t} + y_{s,t} \\
&\qquad + (\delta_1 v_{s+1,t} + w_{s+1,t}) z_{s,t} = 0,
\end{aligned} \tag{2.109b}$$

$$\begin{aligned}
&(1-\delta_1) v_{s,t} + \delta_2 w_{s,t+1} + y_{s,t+1} \\
&\qquad + (\delta_1 v_{s,t} + w_{s,t+1}) z_{s,t} = 0,
\end{aligned} \tag{2.109c}$$

$$\begin{aligned}(1-\delta_1)\,v_{s+1,t}+\delta_2 w_{s+1,t+1}+y_{s,t+1}&\\ +(\delta_1 v_{s+1,t}+w_{s+1,t+1})\,z_{s,t}&=0.\end{aligned} \tag{2.109d}$$

Let us solve (2.109a) with respect to $y_{s,t}$:

$$y_{s,t}=-(1-\delta_1)\,v_{s,t}-\delta_2 w_{s,t}-(\delta_1 v_{s,t}+w_{s,t})\,z_{s,t} \tag{2.110}$$

and let us insert $y_{s,t}$ into (2.109b) in order to get an equation solvable for $z_{s,t}$. This is possible *iff* $\delta_1 v_{s,t}+w_{s,t}\neq f_t$, with f_t an arbitrary function of t, since in this case the coefficient of $z_{s,t}$ is zero. Then the solution of the system (2.109) bifurcates.

CASE 1: $\delta_1 v_{s,t}+w_{s,t}\neq f_t$ Assume that $\delta_1 v_{s,t}+w_{s,t}\neq f_t$, then we can solve with respect to $z_{s,t}$ the expression obtained inserting (2.110) into (2.109b). We get:

$$z_{s,t}=-\frac{(1-\delta_1)\,(v_{s+1,t}-v_{s,t})+\delta_2\,(w_{s+1,t}-w_{s,t})}{\delta_1\,(v_{s+1,t}-v_{s,t})+w_{s+1,t}-w_{s,t}}. \tag{2.111}$$

Now we can substitute (2.110) and (2.111) together with their difference consequences into (2.109c) and (2.109d) and we get two equations for $w_{s,t}$ and $v_{s,t}$:

$$\begin{aligned}(\delta_1\delta_2+\delta_1-1)\,\big[&w_{s+1,t+1}v_{s,t+1}w_{s,t}+v_{s+1,t+1}w_{s,t+1}w_{s+1,t}\\
&-v_{s+1,t+1}w_{s,t+1}w_{s,t}+w_{s,t+1}v_{s,t}w_{s+1,t+1}\\
&+v_{s,t}w_{s+1,t+1}w_{s+1,t}-w_{s,t+1}v_{s+1,t}w_{s+1,t+1}\\
&-w_{s,t+1}^2v_{s,t}+w_{s,t+1}^2v_{s+1,t}\\
&-w_{s+1,t+1}v_{s,t+1}w_{s+1,t}-v_{s,t}w_{s+1,t+1}w_{s,t}\\
&-v_{s,t}w_{s,t+1}w_{s+1,t}+v_{s,t}w_{s,t+1}w_{s,t}\\
&-\delta_1(v_{s,t}v_{s,t+1}w_{s+1,t}+w_{s,t+1}v_{s,t}v_{s,t+1}\\
&-w_{s+1,t+1}v_{s,t+1}v_{s,t}+w_{s+1,t+1}v_{s,t+1}v_{s+1,t}\\
&+v_{s,t}v_{s+1,t+1}w_{s,t}-v_{s,t}v_{s+1,t+1}w_{s+1,t}\\
&-v_{s,t}v_{s,t+1}w_{s,t}-w_{s,t+1}v_{s+1,t}v_{s,t+1})\big]\end{aligned} \tag{2.112a}$$

$$\begin{aligned}(\delta_1\delta_2+\delta_1-1)\,\big[&w_{s+1,t+1}^2v_{s+1,t}+w_{s+1,t+1}v_{s,t+1}w_{s+1,t}\\
&-w_{s+1,t+1}v_{s,t+1}w_{s,t}-w_{s+1,t+1}^2v_{s,t}\\
&-w_{s,t+1}v_{s+1,t}w_{s+1,t+1}+w_{s,t+1}v_{s,t}w_{s+1,t+1}\\
&-v_{s+1,t+1}w_{s,t+1}w_{s+1,t}+v_{s+1,t}w_{s,t+1}w_{s+1,t}\\
&+v_{s+1,t+1}w_{s,t+1}w_{s,t}-v_{s+1,t}w_{s+1,t+1}w_{s+1,t}\\
&+v_{s+1,t}w_{s+1,t+1}w_{s,t}-v_{s+1,t}w_{s,t+1}w_{s,t}+\\
&\delta_1(-v_{s+1,t}v_{s+1,t+1}w_{s+1,t}+v_{s+1,t}v_{s+1,t+1}w_{s,t}\\
&-w_{s+1,t+1}v_{s,t}v_{s+1,t+1}-v_{s+1,t}v_{s,t+1}w_{s,t}\\
&-v_{s+1,t+1}w_{s,t+1}v_{s+1,t}+w_{s+1,t+1}v_{s+1,t}v_{s+1,t+1}\\
&+v_{s+1,t}v_{s,t+1}w_{s+1,t}+v_{s+1,t+1}w_{s,t+1}v_{s,t})\big]\end{aligned} \tag{2.112b}$$

If

$$\delta_1 (1 + \delta_2) = 1, \tag{2.113}$$

then (2.112) are identically satisfied. If $\delta_1 (1 + \delta_2) \neq 1$, adding (2.112a) and (2.112b), we obtain:

$$(w_{s,t} - w_{s+1,t} - w_{s,t+1} + w_{s+1,t+1}) \cdot (v_{s+1,t} - v_{s,t}) (\delta_1 + \delta_1\delta_2 - 1) = 0. \tag{2.114}$$

Therefore we are in front of a new factorization. Since $\delta_1 + \delta_1\delta_2 \neq 1$ we can divide by $\delta_1 + \delta_1\delta_2 - 1$ and annihilate alternatively the first or the second factor.

If set equal to zero the second factor in (2.114), we get $v_{s,t} = g_t$ with g_t arbitrary function of t alone. Substituting this result into(2.112a) or (2.112b), we obtain $g_t = g_0$, with g_0 constant. Since we do not have any other condition we have that $w_{s,t}$ is an arbitrary function satisfying $w_{s,t} \neq f_t - g_0$ while from (2.111) we obtain $z_{s,t} = -\delta_2$ and finally from (2.115) we have:

$$y_{s,t} = -(1 - \delta_1 - \delta_1\delta_2) g_0. \tag{2.115}$$

Therefore the only non-trivial case is when $\delta_1 + \delta_1\delta_2 \neq 1$, and $v_{s,t} \neq g_t$. In this case we have that $w_{s,t}$ solves the discrete wave equation, i.e. $w_{s,t} = h_s + l_t$. Substituting $w_{s,t}$ into (2.112) we get a single equation for $v_{s,t}$:

$$\begin{aligned}(h_s - h_{s+1}) \big[&(v_{s+1,t+1} - v_{s+1,t}) (h_s + l_{t+1}) \\ &- (v_{s,t+1} - v_{s,t}) (h_{s+1} + l_{t+1}) \\ &+ \delta_1 (v_{s,t} v_{s+1,t+1} - v_{s+1,t} v_{s,t+1}) \big] = 0.\end{aligned} \tag{2.116}$$

which is identically satisfied if $h_s = h_0$, with h_0 a constant. Therefore we have a non-trivial cases only if $h_s \neq h_0$.

CASE 1.1: $\delta_1 = 0$ We have a great simplification if in addition to $h_s \neq h_0$ we have $\delta_1 = 0$. In this case (2.116) is linear:

$$\begin{aligned}&(v_{s+1,t+1} - v_{s+1,t}) (h_s + l_{t+1}) \\ &- (v_{s,t+1} - v_{s,t}) (h_{s+1} + l_{t+1}) = 0.\end{aligned} \tag{2.117}$$

Eq. (2.117) can be easily integrated twice to give:

$$v_{s,t} = \begin{cases} j_s + \sum_{k=t_0}^{t-1} (h_s + l_{k+1}) i_k, & t \geqslant t_0 + 1, \\ j_s - \sum_{k=t}^{t_0-1} (h_s + l_{k+1}) i_k, & t \leqslant t_0 - 1, \end{cases} \tag{2.118}$$

with i_t and $j_s = v_{s,t_0}$ arbitrary integration functions.

CASE 1.2: $\delta_1 \neq 0$, BUT $l_t = l_0$ Now let us suppose $h_s \neq h_0$, $\delta_1 \neq 0$ but let us choose $l_t = l_0$, with l_0 a constant. Performing the translation $\theta_{s,t} = v_{s,t} + (h_s + l)/\delta_1$, from (2.116) we get:

$$\theta_{s,t}\theta_{s+1,t+1} - \theta_{s+1,t}\theta_{s,t+1} = 0. \tag{2.119}$$

Eq. (2.119) is linearizable via a Cole-Hopf transformation $\Theta_{s,t} = \theta_{s+1,t}/\theta_{s,t}$ as $v_{s,t}$ cannot be identically zero. This linearization yields the general solution $\theta_{s,t} = S_s T_t$ with S_s and T_t arbitrary functions of their argument.

CASE 1.3: $\delta_1 \neq 0$, $l_t \neq l_0$ Finally if $h_s \neq h$, $\delta_1 \neq 0$ and $l_t \neq l_0$, we perform the transformation

$$\theta_{s,t} = \frac{1}{\delta_1}\left[(l_t - l_{t+1})\, v_{s,t} - h_s - l_{t+1}\right]. \tag{2.120}$$

Then from (2.116) we get:

$$\theta_{s,t}\left(1 + \theta_{s+1,t+1}\right) - \theta_{s+1,t}\left(1 + \theta_{s,t+1}\right) = 0, \tag{2.121}$$

which, as $v_{s,t}$ cannot be identically zero, is easily linearized via the Cole-Hopf transformation $\Theta_{s,t} = (1 + \theta_{s,t+1})/\theta_{s,t}$ to $\Theta_{s+1,s} - \Theta_{s,t} = 0$ which yields for $\theta_{s,t}$ the linear equation:

$$\theta_{s,t+1} - p_t\theta_{s,t} + 1 = 0, \tag{2.122}$$

where p_t is an arbitrary integration function. Then the general solution is given by:

$$\theta_{s,t} = \begin{cases} \left(u_s - \sum_{l=t_0}^{t-1} \prod_{j=t_0}^{l} p_j^{-1}\right) \prod_{k=t_0}^{t-1} p_k, & t \geqslant t_0 + 1, \\ u_s \prod_{k=t}^{t_0-1} p_k^{-1} + \sum_{l=t}^{t_0-1} \prod_{j=t}^{l} p_j^{-1}, & t \leqslant t_0 - 1, \end{cases} \tag{2.123}$$

where $u_s = \theta_{s,t_0}$ is an arbitrary integration function.

CASE 2: $\delta_1 v_{s,t} + w_{s,t} = f_t$ Let us suppose:

$$w_{s,t} = f_t - \delta_1 v_{s,t}, \tag{2.124}$$

where f_t is an arbitrary function of its argument. Inserting (2.110) and (2.124) and their difference consequences into (2.109), we get:

$$(v_{s+1,t} - v_{s,t})(\delta_1 + \delta_1\delta_2 - 1) = 0, \tag{2.125}$$

and the two relations:

$$\begin{aligned} f_{t+1} z_{s,t+1} &+ \left[\delta_1 (v_{s,t} - v_{s,t+1}) - f_{t+1}\right] z_{s,t} \\ &+ (\delta_1 - 1)(v_{s,t} - v_{s,t+1}) = 0, \end{aligned} \tag{2.126a}$$

$$\begin{aligned} f_{t+1} z_{s,t+1} + [\delta_1 (v_{s+1,t} - v_{s+1,t+1}) - f_{t+1}] z_{s,t} \\ + (\delta_1 - 1)(v_{s+1,t} - v_{s,t+1}) \\ + \delta_1 \delta_2 (v_{s+1,t+1} - v_{s,t+1}) = 0. \end{aligned} \tag{2.126b}$$

Hence in (2.125) we have a bifurcation depending on the factor we choose to annihilate.

CASE 2.1: $\delta_1 (1 + \delta_2) = 1$ If we annihilate the second factor in (2.125) we get $\delta_1 (1 + \delta_2) = 1$, i.e. $\delta_1 \neq 0$. Then adding (2.126a) and (2.126b) we obtain:

$$(v_{s,t} - v_{s+1,t} - v_{s,t+1} + v_{s+1,t+1})(1 - \delta_1 + \delta_1 z_{s,t}) = 0. \tag{2.127}$$

It seems that we are facing a new bifurcation. However annihilating the second factor, i.e. setting $z_{s,t} = 1 - 1/\delta_1$ is trivial, since (2.126) are identically satisfied.

Therefore we may assume that $z_{s,t} \neq 1 - 1/\delta_1$. This implies that $v_{s,t}$ solves a discrete wave equation whose solution is given by:

$$v_{s,t} = h_s + k_t, \tag{2.128}$$

where h_s and k_t are generic integration functions of their argument. Inserting (2.128) in (2.126) we can obtain the following linear equation for $z_{s,t}$:

$$f_{t+1} z_{s,t+1} + (\delta_1 j_t - f_{t+1}) z_{s,t} + (\delta_1 - 1) j_t = 0, \tag{2.129}$$

with $j_t = k_{t+1} - k_t$. This equation can is solved by:

$$\begin{aligned} z_{s,t} = (-1)^t (\delta_1 - 1) \prod_{t'=0}^{t-1} \frac{\delta_1 j_{t'} - f_{t'+1}}{f_{t'+1}} \cdot \sum_{t''=0}^{t-1} \frac{j_{t''} (-1)^{t''}}{f_{t''+1} \prod_{t'=0}^{t''} \frac{\delta_1 j_{t'} - f_{t'+1}}{f_{t'+1}}} \\ + (-1)^t z_{s,0} \prod_{t'=0}^{t-1} \frac{\delta_1 j_{t'} - f_{t'+1}}{f_{t'+1}} \end{aligned} \tag{2.130}$$

CASE 2.2: $v_{s,t} = l_t$ Now we annihilate the first factor in (2.125) i.e. $\delta_1 (1 + \delta_2) \neq 1$ and $v_{s,t} = l_t$, where l_t is an arbitrary function of its argument. From (2.126) we obtain (2.129) with $j_t = l_{t+1} - l_t$.

In conclusion we have always integrated the original system using an explicit linearization through a series of transformations and bifurcations.

As a final remark we observe that every transformation used in the linearization procedure both for the ${}_tH_1^\varepsilon$ (1.91a) equation and for the

${}_1D_2$ (1.93b) equation is bi-rational in the fields and their shifts (like Cole-Hopf-type transformations). This, in fact, has to be expected, since the Algebraic Entropy test is valid only if we allow transformations which preserve the algebrogeometric structure underlying the evolution procedure [158]. Indeed there are examples on one-dimensional lattice of equations chaotic according to the Algebraic Entropy, but linearizable using some transcendental transformations [62]. So exhibiting the explicit linearization and showing that it can be attained by bi-rational transformations is indeed a very strong confirmation of the Algebraic Entropy conjecture [84].

Indeed this does not prevent the fact that in some cases such equations can be linearized through some transcendental transformations. In fact if $\varepsilon = 0$ the ${}_1H_1^{\varepsilon}$ equation (2.91) can be linearized through the transcendental contact transformation:

$$u_{n,m} - u_{n+1,m} = \sqrt{\alpha_2} e^{z_{n,m}}, \tag{2.131}$$

i.e.

$$z_{n,m} = \log \frac{u_{n,m} - u_{n+1,m}}{\sqrt{\alpha_2}}, \tag{2.132}$$

with log standing for the principal value of the complex logarithm (the principal value is intended for the square root too). The transformation (2.131) brings (1.91a) into the following family of first order linear equations:

$$z_{n,m+1} + z_{n,m} = 2i\pi\kappa, \quad \kappa = 0, 1. \tag{2.133}$$

However this kind of transformation does not prove the result of the Algebraic Entropy, as it involves exponential transformations. Therefore the method explained in Subsection 2.4.1 should be considered the correct one.

3 GENERALIZED SYMMETRIES OF THE TRAPEZOIDAL H^4 AND H^6 EQUATIONS AND THE NON-AUTONOMOUS YDKN EQUATION

In this Chapter we introduce the concept of Generalized Symmetries for quad equations. We present the main computational tools for finding a specific class of Generalized Symmetries. Then we will use the developed tools to compute the Generalized Symmetries of the trapezoidal H^4 equations (1.91) and of the H^6 equations (1.93). The main result will then be the fact that all the symmetries we computed are related to a well-known differential-difference equation, the non-autonomous Yamilov discretization of the Krichever-Novikov (YdKN) equation. This identification led us to conjecture and prove, using the Algebraic Entropy test, the existence of a new non-autonomous integrable quad equation which is a generalization of the Q_V equation [156].

The material is structured as follows: In the first part of the Chapter we will introduce the concept needed, in particular in Section 3.1 we will discuss the relationship of the Generalized Symmetries with integrability properties, which is one of the reason for interest in the Generalized Symmetries. To do so we will treat the well known example of the Korteweg-deVries equation. In Section 3.2 we will construct three-point generalized symmetries for quad equations. We will give in particular a method for finding three-point generalized symmetries in the case of equation with two-periodic coefficients. In this second part of the Chapter we present some original results bases on [65, 66, 68]. In particular in Section 3.3 we will present the symmetries of the trapezoidal H^4 equations (1.91) and of the H^6 equation (1.93) constructed with the method explained in 3.2. This Section is mainly based on the exposition in [66, 68]. Then in Section 3.4 we identify the three-point generalized symmetries computed in Section (3.3) with some particular sub-cases of the non-autonomous YdKN equation. Since an analogous result, which for sake of completeness is stated in Appendix E, was known for the rhombic H^4 equation [166], we conclude that the three-point Generalized Symmetries of all the equations belonging to the classification of Boll [3, 20–22], discussed in Chapter 1, are all sub-cases of the YdKN equation. This result extends what was previously known for three-point Generalized Symmetries of the ABS equations [2] which were known to be particular sub-cases of the *autonomous* YdKN equation [102]. In Section 3.5 we discuss the integrability properties of the obtained three-point generalized symmetries seen as differential-difference equations, both with the Master Symmetries approach and with the Algebraic Entropy test. These two last Sections are is essentially based on the results given in [66]. In the last Section 3.6 based on the considerations made in Section 3.4 and on the results of [165] about the Q_V equation [156] and its relation to the autonomous YdKN equation we conjecture the existence of a non-autonomous generalization of this equation. We then find the appropriate candidate for such an extension by providing a non-autonomous extension of the Klein symmetries used in [156] to find the Q_V equation. We then prove the integrability using the Algebraic Entropy test and finally we discuss the three-point generalized symmetries of the obtained equation and their relation with the non-autonomous YdKN equation. This last Section is based on [65].

3.1 GENERALIZED SYMMETRIES AND INTEGRABILITY

At the time of the Franco-Prussian war in 1870 the Norwegian mathematician Sophus Lie considered the question of the invariance of differential equations with respect to continuous infinitesimal transformation, i.e. transformations which can be seen as continuous deformations of the identity. In the successive 30 years Lie developed a theory which includes all the implications of such invariance. The *summa* of the work of Sophus Lie on the subject is contained in [111].

Lie's works on differential equation had a brief moment of success, but soon they were forgotten due to the complexity of the calculations. They were subsequently rediscovered around the middle of the XXth century by Russian mathematicians under the leadership of Ovsiannikov [133, 134]. On the other hand Lie work on continuous groups was a powerful tool in the development of Quantum Mechanics where they were used to show unexpected results about the atomic spectra [160]. However the "physicists" theory of Lie groups turns out to be completely separated from the application of continuous groups to differential equations in such a way that it was possible that somebody knew much of one and completely ignore the other.

The original idea of Sophus Lie was to unify the various methods of solution for differential equations uncovering the underlying geometrical structure in the same spirit as it was done by Evariste Galois for algebraic equations. Lie's theory of infinitesimal point transformation is linear, and one is able to reconstruct the full group of transformation by solving some differential equations.

In his work Lie [110] considered infinitesimal transformations depending also on the first derivatives of the dependent variables, the so-called contact transformations [11, 87]. Later Bäcklund considered transformations depending on finitely many derivatives of the dependent variables adding some closure relations in order to preserve the geometrical structure of the transformations [16]. The first to recognize the possibility of considering also transformations depending on higher order derivatives *without additional conditions* was Emmy Noether in her fundamental paper [125]. The mathematical objects Noether considered was what now we call *Generalized Symmetries*.

In modern times due to their algorithmic nature generalized symmetry have been used both as tools for the study of given systems [11, 41, 48, 87, 91, 93, 94, 130, 132] and as a tool to classify integrable equations. The symmetry approach to integrability has mainly been developed by a group of researchers belonging to the scientific school of A.B. Shabat in Russia. It has been developed in the continuum case [4, 73, 116–118, 146, 147], in the differential-difference case [57, 105, 167, 168] and more recently also in the completely discrete case [106, 107].

Let us start discussing the relation between integrability and generalized symmetries in the case of partial differential equations in two independent variables and one dependent variable. Let us assume we are given a partial differential equation (PDE) of order k in *evolutionary form* for an unknown function $u = u(x,t)$:

$$u_t = f(u, u_1, \ldots, u_k), \tag{3.1}$$

where $u_t = \partial u/\partial t$ and $u_j = \partial^j u/\partial x^j$ for any $j > 0$.

A generalized symmetry of order m for (3.1) is an equation of the form:

$$u_\tau = g(u, u_1, u_2, \ldots u_m) \tag{3.2}$$

compatible with (3.1). Here τ plays the role of the group parameter and of course we are assuming $u = u(x,t,\tau)$. The compatibility condition between (3.1) and (3.2) implies the following PDE for the functions f and g:

$$\frac{\partial^2 u}{\partial t \partial \tau} - \frac{\partial^2 u}{\partial \tau \partial t} = D_t g - D_\tau f = 0\,, \tag{3.3}$$

where D_t, D_τ are the operators of total differentiation corresponding to (3.1, 3.2), defined, together with the operator of total x-derivative D, by

$$D = \frac{\partial}{\partial x} + \sum_{i \geqslant 0} u_{i+1} \frac{\partial}{\partial u_i}, \tag{3.4a}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{i \geqslant 0} D^i f \frac{\partial}{\partial u_i}, \tag{3.4b}$$

$$D_\tau = \frac{\partial}{\partial \tau} + \sum_{i \geqslant 0} D^i g \frac{\partial}{\partial u_i}. \tag{3.4c}$$

When (3.3) is satisfied we say that (3.2) is *a generalized symmetry of order* m for (3.1).

We will now see in a concrete example how from the integrability properties is possible to obtain an infinite sequence of generalized symmetries of the form (3.2). Let us consider the well known Korteweg-de Vries equation (KdV equation) [92]:

$$u_t = 6uu_1 + u_3. \tag{3.5}$$

It is known [97] that the KdV equation (3.5) possess the following linear representation:

$$L_t = [L, M], \tag{3.6}$$

where L and M are differential operators given by:

$$L = -\partial_{xx} + u, \tag{3.7a}$$

$$M = 4\partial_{xxx} - 6u\partial_x - 3\partial_x(u). \tag{3.7b}$$

and by $[,]$ we mean the commutator of two differential operators. We look for a chain of differential operators M_j which correspond to different equations associated with the same L:

$$L_{t_j} = \left[L, M_j\right], \tag{3.8}$$

Since $L_t = u_t$ is a scalar operator one must have $[L, M] = V$ with V scalar operator. If there exists another equation associated to L then there must exists another $\tilde{M}$ such that

$$\left[L, \tilde{M}\right] = \tilde{V} \tag{3.9}$$

where $\tilde{V}$ is another scalar operator.

Now one can relate M and $\tilde{M}$. Noting that the equation $u_t = u_x$ can be written in the form (3.8) with $M_j = \partial_x$ we see that the operators M_j are characterized from the power of the operator ∂_x and that the relation between one and the other is of order two, as the order of the operator L. We can therefore make the general assumption:

$$\tilde{M} = LM + F\partial_x + G, \tag{3.10}$$

where F and G are scalar operators. Using the ansatz (3.10) and imposing to the coefficients of the various differential operators in (3.9) we find that:

$$\tilde{V} = \mathcal{L}V + u_x, \tag{3.11}$$

where:

$$\mathcal{L}V = -\frac{1}{4}V_{xx} + uV - \frac{1}{2}u_x \int_x^\infty V(y)dy. \tag{3.12}$$

This yields an infinite family of equations for any entire function $F(z)$,

$$u_t = F(\mathcal{L})\, u_x. \tag{3.13}$$

The equations (3.13) are associated to the same L operator and, by solving the Spectral Problem associated to it [26], they are shown to be commuting with the original KdV equation (3.5). The operator $\mathcal{L}$ is called *recursion operator* or *Lénard operator* [51, 98, 129]. In particular for $F(\mathcal{L}) = \mathcal{L}^n$ we find a whole hierarchy of generalized symmetries for the KdV equation (3.5). The operator (3.12) is called the *recursion operator* for the KdV equation (3.5).

In general any evolution equation possessing a recursion operator has an infinite hierarchy of equations and if they commute they with the evolution equation are generalized symmetries for the evolution equation, and, then, in an appropriate sense, an evolution equation is integrable. We therefore have a symmetry-based definition of integrability: *An evolution equation is called integrable if it possesses non-constant generalized symmetries of any order* m [131]. Note that this definition *a priori* does not distinguish between C-integrable and S-integrable equations. Indeed both C-integrable and S-integrable equations possess the property of having recursion operators like (3.12), but C-integrable equations are *linearizable* [25]. The most famous C-integrable equation, the Burgers equation, also possesses a recursion operator. The difference between C-integrable and S-integrable equations is in the fact that the S-integrable equations possess infinitely many conservation laws of any order whereas the C-integrable ones only up to the order of the equation itself. Finally we note that since integrable equations in this sense comes in hierarchies and the elements of a hierarchy commute between themselves also the generalized symmetries of an integrable equation are integrable.

3.2 THREE-POINT GENERALIZED SYMMETRIES FOR QUAD EQUATION

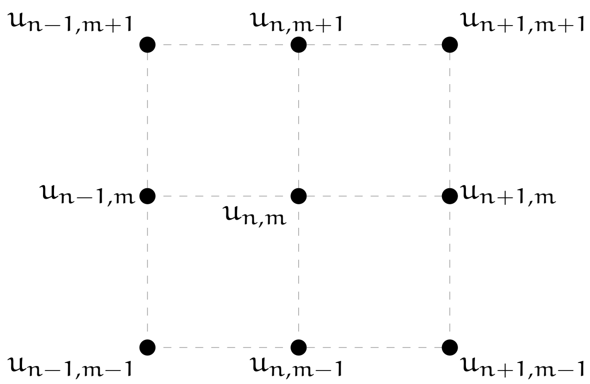

Figure 3.1: The extended square lattice with 9 points.

In this Section we discuss the construction of the most simple generalized symmetries for quad equations (1.1)[1]. Symmetries of this kind were first considered in [139]. The exposition here is mainly based on the original papers [52, 106, 107] and on the review in [63], where a different approach to that used in [139] was developed. The most simple Generalized Symmetries for a quad equation in the form (1.1) are those depending on nine points defined on a square of vertices $u_{n-1,m-1}$, $u_{n-1,m+1}$, $u_{n+1,m+1}$ and $u_{n+1,m-1}$ as depicted in Figure 3.1. *A priori* the symmetry generator can depend on all these points, however by taking into account the difference equation (1.1), we can express the extremal points $u_{n-1,m-1}$, $u_{n-1,m+1}$, $u_{n+1,m+1}$ and $u_{n+1,m-1}$ in terms of the remaining five points $u_{n-1,m}$, $u_{n+1,m}$, $u_{n,m}$, $u_{n,m-1}$ and $u_{n,m+1}$. In general this means that we are taking as independent variables those laying on the coordinate axes, i.e. $u_{n+j,m}$ and $u_{n,m+k}$ with $j, k \in \mathbb{Z}$. This choice of independent variable is acceptable, since these points do not lie on squares. In this way the most general nine points generalized symmetry generator is represented by the infinitesimal symmetry generator

$$\hat{X} = g(u_{n-1,m}, u_{n+1,m}, u_{n,m}, u_{n,m-1}, u_{n,m+1})\partial_{u_{n,m}}. \tag{3.14}$$

As in the case of the differential difference equation the function g is called the *characteristic of the symmetry*. We note that this is not the only possible choice for the independent variables. Another viable choice of independent variables is given by an appropriate restriction of an infinite staircase, e.g. to consider the points $u_{n+1,m-1}$, $u_{n,m-1}$, $u_{n,m}$, $u_{n-1,m}$ and $u_{n-1,m+1}$. This was the choice we made in the case of the Algebraic Entropy, but for the calculation of symmetries the choice of the independent variables on the axes is more convenient.

Now we need to prolong the operator (3.14) in order to apply it on the quad equation (1.1) and construct the determining equations. This prolongation is naturally given by:

$$\begin{aligned}
\mathrm{pr}\hat{X} = {} & g(u_{n-1,m}, u_{n+1,m}, u_{n,m}, u_{n,m-1}, u_{n,m+1})\partial_{u_{n,m}} \\
& + g(u_{n,m}, u_{n+2,m}, u_{n+1,m}, u_{n+1,m-1}, u_{n+1,m+1})\partial_{u_{n+1,m}} \\
& + g(u_{n-1,m+1}, u_{n+1,m+1}, u_{n,m+1}, u_{n,m-1+1}, u_{n,m+2})\partial_{u_{n,m+1}} \\
& + g(u_{n,m+1}, u_{n+2,m+1}, u_{n+1,m+1}, u_{n+1,m}, u_{n+1,m+2})\partial_{u_{n+1,m+1}}.
\end{aligned} \tag{3.15}$$

Applying the prolonged vector field to equation (1.1), we get:

$$\begin{aligned}
& g\frac{\partial Q}{\partial u_{n,m}} + [T_n g]\frac{\partial Q}{\partial u_{n+1,m}} \\
& + [T_m g]\frac{\partial Q}{\partial u_{n,m+1}} + [T_n T_m g]\frac{\partial Q}{\partial u_{n+1,m+1}} = 0,
\end{aligned} \tag{3.16}$$

1 This kind of reasoning in fact can hold for every kind of partial difference equations on the square which are solvable with respect to all its variables.

where $T_n f_{n,m} = f_{n+1,m}$ and $T_m f_{n,m} = f_{n,m+1}$. *A priori* (3.16) contains $u_{n+i,m+j}$ with $i = -1, 0, 1, 2$, $j = -1, 0, 1, 2$.

The invariance condition requires that (3.16) be satisfied on the solutions of (1.1). To be consistent with our choice of independent variables, which now include also $u_{n,m+2}$ and $u_{n+2,m}$, we must use (1.1) and its shifted consequences to express:

- $u_{n+2,m+1} = u_{n+2,m+1}(u_{n+2,m}, u_{n+1,m+1}, u_{n+1,m})$,
- $u_{n+1,m+1} = u_{n+1,m+1}(u_{n+1,m}, u_{n,m+1}, u_{n,m})$,
- $u_{n-1,m+1} = u_{n-1,m+1}(u_{n-1,m}, u_{n,m+1}, u_{n,m})$,

Doing so we reduce the determining equation (3.16) to an equation, written just in terms of independent variables, which thus must be identically satisfied. Differentiating (3.16) with respect to $u_{n,m+2}$ and to $u_{n+2,m}$, we get

$$\frac{\partial^2 T_n T_m g}{\partial u_{n+2,m+1} \partial u_{n+1,m+2}} = T_n T_m \frac{\partial^2 g}{\partial u_{n+1,m} \partial u_{n,m+1}} = 0. \qquad (3.17)$$

Consequently the symmetry coefficient g is the sum of *two simpler functions*,

$$\begin{aligned} g = {} & g_0(u_{n-1,m}, u_{n+1,m}, u_{n,m}, u_{n,m-1}) \\ & + g_1(u_{n-1,m}, u_{n,m}, u_{n,m-1}, u_{n,m+1}). \end{aligned} \qquad (3.18)$$

Introducing this result into the determining equation (3.16) and differentiating it with respect to $u_{n,m+2}$ and to $u_{n-1,m}$, we have that g_1 reduces to

$$g_1 = g_{10}(u_{n-1,m}, u_{n,m}, u_{n,m-1}) + g_{11}(u_{n,m}, u_{n,m-1}, u_{n,m+1}). \qquad (3.19)$$

In a similar way, if we differentiate the resulting determining equation with respect to $u_{n+2,m}$ and to $u_{n,m-1}$, we have that g_0 reduces to

$$g_0 = g_{00}(u_{n-1,m}, u_{n,m}, u_{n,m-1}) + g_{01}(u_{n-1,m}, u_{n+1,m}, u_{n,m}). \qquad (3.20)$$

Combining these results and taking into account the property symmetrical to (3.17), i.e. $\partial^2 g / (\partial u_{n-1,m} \partial u_{n,m-1}) = 0$, we obtain the following form for g:

$$g = g_0(u_{n,m-1}, u_{n,m}, u_{n,m+1}) + g_1(u_{n-1,m}, u_{n,m}, u_{n+1,m}). \qquad (3.21)$$

So, *the infinitesimal symmetry coefficient is the sum of functions that either involve shifts only in n with m fixed or only in m with n fixed* [106, 139].

Let us consider the subcase when the symmetry generator is given by

$$\frac{du_{n,m}}{d\varepsilon} = g_1(u_{n-1,m}, u_{n,m}, u_{n+1,m}). \qquad (3.22)$$

This is a differential-difference equation depending parametrically on m. Setting $u_{n,m} = u_n$ and $u_{n,m+1} = \tilde{u}_n$, the compatible partial difference equation (1.1) turns out to be an ordinary difference equation relating u_n and $\tilde{u}_n$:

$$Q(u_n, u_{n+1}, \tilde{u}_n, \tilde{u}_{n+1}) = 0, \tag{3.23}$$

i.e. a Bäcklund transformation [102] for (3.22). A similar result is obtained in the case of g_0. This result was first presented in [139].

We note that higher order symmetries, i.e. symmetries depending on more lattice points, have the same splitting in simpler symmetries in the two directions [52]. However such higher order symmetries will no longer yield Bäcklund transformations.

To find the specific form of g_1 we have to differentiate the determining equation (3.16) with respect to the independent variables and get from these some *further necessary conditions* on its shape. Let us discuss the case in which the symmetry is given in terms of shifts in the n direction, since the m direction shift case can be treated analogously[2]:

$$X = g\left(u_{n+1,m}, u_{n,m}, u_{n-1,m}\right) \partial_{u_{n,m}}. \tag{3.24}$$

We don't impose restrictions on the explicit dependence of g on the lattice variables, so in principle $g = g_{n,m}$.

To obtain g we have to solve the equation (3.16) which is a *functional equation* since it must be evaluated on the four-tuples $(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1})$ such that the quad equation (1.1) holds. The best way to so is to get some consequences of (3.16) and convert them into a system of *linear partial differential equation*, which we can solve. This will impose restrictions on the form of g and will allow us to solve the functional equation (3.16). Using the assumption that Q is multi-linear we can express $u_{n+1,m+1}$ as:

$$u_{n+1,m+1} = f\left(u_{n+1,m}, u_{n,m}, u_{n,m+1}\right), \tag{3.25}$$

therefore the determining equation takes the form:

$$T_n T_m g = T_n g \frac{\partial f}{\partial u_{n+1,m}} + T_m g \frac{\partial f}{\partial u_{n,m+1}} + g \frac{\partial f}{\partial u_{n,m}} \tag{3.26}$$

Let us start to differentiate (3.26) with g given by (3.24) with respect to $u_{n+2,m}$ which is the higher order shift appearing in (3.26):

$$T_n T_m \frac{\partial g}{\partial u_{n+1,m}} T_n \frac{\partial f}{\partial u_{n+1,m}} = \frac{\partial f}{\partial u_{n+1,m}} T_n \frac{\partial g}{\partial u_{n+1,m}}. \tag{3.27}$$

---

2 In fact the most convenient way for treating the symmetries in the m direction is to consider the transformation $n \leftrightarrow m$ and make the computations in the new n direction. Performing again the same transformation in the obtained symmetry will yield the result.

Define:

$$z = \log \frac{\partial g}{\partial u_{n+1,m}} \tag{3.28}$$

then we can rewrite (3.27) in the form of a conservation law:

$$T_m z = z + \left(T_n^{-1} - \mathrm{Id}\right) \log \frac{\partial f}{\partial u_{n+1,m}} \tag{3.29}$$

We also have another representation for (3.26) which will be useful in deriving another relation similar to (3.29). Let us apply T_n^{-1} to (3.26) then solving the resulting equation for $T_n^{-1} T_m g$ we obtain a equivalent representation for (3.26):

$$\begin{aligned} T_n^{-1} T_m g = & - T_n^{-1} \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right) T_n^{-1} g \\ & + T_n^{-1} \left(1 \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right) T_m g \\ & - T_n^{-1} \left(\frac{\partial f}{\partial u_{n+1,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right) g. \end{aligned} \tag{3.30}$$

Differentiating (3.30) with respect to $u_{n-2,m}$ we obtain:

$$T_n^{-1} T_m \frac{\partial g}{\partial u_{n-1,m}} \frac{\partial u_{n-2,m+1}}{\partial u_{n-2,m}} = -T_n^{-1} \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right) T_n^{-1} \frac{\partial g}{\partial u_{n-1,m}}. \tag{3.31}$$

From the implicit function theorem we get:

$$\frac{\partial u_{n-2,m+1}}{\partial u_{n-2,m}} = T_n^{-1} \frac{\partial u_{n-1,m+1}}{\partial u_{n-1,m}} = -T_n^{-2} \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right). \tag{3.32}$$

So introducing

$$v = \log \frac{\partial g}{\partial u_{n-1,m}} \tag{3.33}$$

we obtain from (3.31) a new equation written in conservation law form:

$$T_m v = v + \left(T_n - \mathrm{Id}\right) \log \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right). \tag{3.34}$$

The equations (3.29) and (3.34) are still functional equations, but we can derive from them a system of first order PDEs. Indeed let s be a function such that $s = s\left(u_{n+1,m}, u_{n,m}, u_{n-1,m}\right)$. Then the function $T_m s = s\left(u_{n+1,m+1}, u_{n,m+1}, u_{n-1,m+1}\right)$ can be annihilated applying the differential operator:

$$\begin{aligned} Y_{-1} = & \frac{\partial}{\partial u_{n,m}} - \frac{\partial f / \partial u_{n,m}}{\partial f / \partial u_{n,m+1}} \frac{\partial}{\partial u_{n+1,m}} \\ & - T_n^{-1} \left(\frac{\partial f / \partial u_{n+1,m}}{\partial f / \partial u_{n,m}} \right) \frac{\partial}{\partial u_{n-1,m}}. \end{aligned} \tag{3.35}$$

On the other hand the function $T_m^{-1} s = s(u_{n+1,m-1}, u_{n,m-1}, u_{n-1,m-1})$ is annihilated by the operator:

$$\begin{aligned} Y_1 = & \frac{\partial}{\partial u_{n,m}} - T_m^{-1} \frac{\partial f}{\partial u_{n,m+1}} \frac{\partial}{\partial u_{n+1,m}} \\ & - T_n^{-1} T_m^{-1} \left(\frac{\partial f}{\partial u_{n,m+1}} \right)^{-1} \frac{\partial}{\partial u_{n-1,m}}. \end{aligned} \tag{3.36}$$

Therefore we can apply the operator Y_{-1} as given by (3.35) to (3.29) and this will give us the linear PDE:

$$Y_{-1} z = -Y_{-1} \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}}. \tag{3.37}$$

Analogously by applying T_m^{-1} to (3.29) we can write

$$T_m^{-1} z = z - T_m^{-1} \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}} \tag{3.38}$$

which applying Y_1 gives us the linear PDE:

$$Y_1 z = Y_1 T_m^{-1} \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}}. \tag{3.39}$$

Since z is independent from $u_{n,m\pm1}$, whereas the coefficients in (3.37,3.39) may depend on it, we may write down a system for the function z:

$$Y_1 z = Y_1 T_m^{-1} \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}}, \tag{3.40a}$$

$$Y_{-1} z = -Y_{-1} \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}}, \tag{3.40b}$$

$$\frac{\partial z}{\partial u_{n,m+1}} = \frac{\partial z}{\partial u_{n,m-1}} = 0. \tag{3.40c}$$

The system (3.40) may not be closed in the general case. If the system (3.40) happens to be not closed it is possible to add an extra equation using Lie brackets:

$$[Y_1, Y_{-1}] z = -\left[Y_1 Y_{-1} T_m^{-1} - Y_{-1} Y_1 \right] \left(T_n^{-1} - \mathrm{Id} \right) \log \frac{\partial f}{\partial u_{n+1,m}}. \tag{3.41}$$

Applying the same line of reasoning we deduce a system of equations also for v:

$$Y_1 v = Y_1 T_m^{-1} (T_n - \mathrm{Id}) \log \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right), \tag{3.42a}$$

$$Y_{-1} v = -Y_{-1} (T_n - \mathrm{Id}) \log \left(\frac{\partial f}{\partial u_{n,m}} \Big/ \frac{\partial f}{\partial u_{n,m+1}} \right), \tag{3.42b}$$

$$\frac{\partial v}{\partial u_{n,m+1}} = \frac{\partial v}{\partial u_{n,m-1}} = 0. \tag{3.42c}$$

As before if the system is not closed we may add the equation obtained using the Lie bracket:

$$[Y_1, Y_{-1}]\, z = -\left[Y_1 Y_{-1} T_m^{-1} - Y_{-1} Y_1\right]\left(T_n^{-1} - \mathrm{Id}\right)\log\left(\frac{\partial f}{\partial u_{n,m}}\Big/\frac{\partial f}{\partial u_{n,m+1}}\right), \tag{3.43}$$

Once we have solved the systems (3.40) and (3.42) (eventually with the aid of the auxiliary equations (3.41) and (3.43)) we insert the obtained values of z and ν into (3.29) and (3.34). In this way we can fix the dependency on the explicit functions of the lattice variables n, m. We can then solve the potential-like equation:

$$\frac{\partial g}{\partial u_{n+1,m}} = e^z, \quad \frac{\partial g}{\partial u_{n-1,m}} = e^\nu. \tag{3.44}$$

Therefore if the compatibility condition

$$\frac{\partial e^z}{\partial u_{n-1,m}} = \frac{\partial e^\nu}{\partial u_{n+1,m}}. \tag{3.45}$$

is satisfied we can write:

$$g = \int e^z du_{n+1,m} + g^{(1)}\left(u_{n,m}\right). \tag{3.46}$$

It only remains to determine the function $g^{(1)}\left(u_{n,m}\right)$. This can be easily done by plugging g as defined by (3.46) into the determining equations (3.26). This determining equation will still be a functional equation, but the only implicit dependence will be in $g^{(1)}\left(f\left(u_{n+1,m}, u_{n,m}, u_{n,m+1}\right)\right)$ and can be annihilated by applying the operator

$$S = \frac{\partial}{\partial u_{n,m}} - \frac{\partial f/\partial u_{n,m}}{\partial f/\partial u_{n+1,m}}\frac{\partial}{\partial u_{n+1,m}}. \tag{3.47}$$

Differentiating in an appropriate way the resulting equation we can determine $g^{(1)}\left(u_{n,m}\right)$ and check its functional form by plugging it back into (3.26).

To conclude this discussion we present two examples of calculations of three-point generalized symmetries using the procedure presented above.

Example 3.2.1 (The H_1 equation (1.23a)). In this example we will consider the H_1 equation as given by (1.23a). We will compute its autonomous three-point symmetries in the n direction using the method we outlined above. These symmetries were first derived with a different technique in [163, 164]. These symmetries appeared also in many other papers, see [139] and references therein for a complete list.

We first have to find the function $z = \log \partial g/\partial u_{n+1,m}$. Using the definition we can write down (3.40a) with f given by solving (1.23a):

$$\begin{aligned}&\left(u_{n,m-1} - u_{n+1,m}\right)^2\frac{\partial z}{\partial u_{n+1,m}} + \left(\alpha_1 - \alpha_2\right)\frac{\partial z}{\partial u_{n,m}}\\&+\left(u_{n-1,m} - u_{n,m-1}\right)^2\frac{\partial z}{\partial u_{n-1,m}} = 2\left(2u_{n,m-1} - u_{n+1,m} - u_{n-1,m}\right).\end{aligned}$$

(3.48)

(3.40c) implies that we can take the coefficients with respect to $u_{n,m-1}$ in (3.48):

$$\begin{aligned} & u_{n+1,m}^2 \frac{\partial z}{\partial u_{n+1,m}} + (\alpha_1 - \alpha_2) \frac{\partial z}{\partial u_{n,m}} \\ & + u_{n-1,m}^2 \frac{\partial z}{\partial u_{n-1,m}} = -2 (u_{n+1,m} + 2u_{n-1,m}), \end{aligned} \tag{3.49a}$$

$$u_{n+1,m} \frac{\partial z}{\partial u_{n+1,m}} + u_{n-1,m} \frac{\partial z}{\partial u_{n-1,m}} = -2, \tag{3.49b}$$

$$\frac{\partial z}{\partial u_{n-1,m}} + \frac{\partial z}{\partial u_{n+1,m}} = 0 \tag{3.49c}$$

This equations can be easily solved to give the form of z:

$$z = \log \left[C^1 (u_{n+1,m} - u_{n-1,m})^{-2} \right], \tag{3.50}$$

being C^1 a constant.

We do the same computations for ν we obtain that ν have to solve exactly the same equations as z, therefore we conclude that:

$$\nu = \log \left[C^2 (u_{n+1,m} - u_{n-1,m})^{-2} \right]. \tag{3.51}$$

being C^2 a new constant of integration. Using the compatibility condition (3.45) we obtain that $C^2 = -C^1$ and integrating (3.44) we have:

$$g = \frac{C^1}{u_{n-1,m} - u_{n+1,m}} + g^{(1)} (u_{n,m}). \tag{3.52}$$

Inserting this form of g into the determining equations (3.26) and applying the operator S (3.47) we obtain the following equation:

$$\begin{aligned} & u_{n,m+1} \frac{dg^{(1)}}{du_{n,m+1}} (u_{n+1,m}) - u_{n,m+1} \frac{dg^{(1)}}{du_{n,m}} (u_{n,m}) \\ & + 2g^{(1)} (u_{n+1,m}) - u_{n+1,m} \frac{dg^{(1)}}{du_{n+1,m}} (u_{n+1,m}) \\ & + u_{n+1,m} \frac{dg^{(1)}}{du_{n,m}} (u_{n,m}) - 2g^{(1)} (u_{n,m+1}) = 0. \end{aligned} \tag{3.53}$$

Differentiating it with respect to $u_{n,m}$ we obtain $d^2 g^{(1)} (u_{n,m}) / du_{n,m}^2 = 0$ which implies $g^{(1)} = C^3 u_{n,m} + C^4$. Substituting this result in (3.53) we obtain the restriction $C^3 = 0$.

In conclusion we have found:

$$g = \frac{C^1}{u_{n-1,m} - u_{n+1,m}} + C^4, \tag{3.54}$$

which satisfy identically the determining equations (3.26). This shows that (3.54) is the most general three-point symmetry in the n direction.

Note that whereas the coefficient of C^1 is a genuine generalized symmetry, the coefficient of C^4 is in fact a *point symmetry*. Since the H_1 equation (1.23a) is invariant under the exchange of variables $n \leftrightarrow m$ we have the symmetry in the m direction:

$$\tilde{g} = \frac{\tilde{C}^1}{u_{n,m-1} - u_{n,m+1}}. \tag{3.55}$$

Therefore equations (3.54, 3.55) represent the most general autonomous five point symmetries of the H_1 equation (1.23a) [139].

It is worth to note that the differential difference equation defined by the equations (3.54) and (3.55) i.e:

$$\frac{du_k}{dt} = \frac{1}{u_{k+1} - u_{k-1}}, \quad k \in \mathbb{Z}, \tag{3.56}$$

is a spatial discretization of the KdV equation [101, 121].

We will return to the symmetry (3.54) in Example 3.2.3 concerning *symmetry reduction*. □

Example 3.2.2 (Autonomous equation with non-autonomous generalized symmetries). Let us consider the quad equation:

$$\begin{aligned} u_{n+1,m+1}u_{n,m}(u_{n+1,m} - 1)(u_{n,m+1} + 1) \\ +(u_{n+1,m} + 1)(u_{n,m+1} - 1) = 0 \end{aligned} \tag{3.57}$$

It was proved in [107] that (3.57) does not admit any autonomous three point generalized symmetry, but it was conjectured there that it might have a non-autonomous symmetry. This conjecture was proved in [52] and we will shall give here such a proof.

We first have to construct the function $z = \log \partial g / \partial u_{n+1,m}$. Using its definition we can write down (3.40) with f obtained from (3.57) as:

$$\begin{aligned} \frac{u_{n-1,m}^2 - 1}{2u_{n,m}} \frac{\partial z}{\partial u_{n-1,m}} + \frac{\partial z}{\partial u_{n,m}} \\ + \frac{2u_{n+1,m}}{u_{n,m}^2 - 1} \frac{\partial z}{\partial u_{n+1,m}} = \frac{u_{n,m}^2 - 2u_{n,m} - 1}{u_{n,m}(u_{n,m}^2 - 1)}, \end{aligned} \tag{3.58a}$$

$$\begin{aligned} \frac{2u_{n-1,m}}{u_{n,m}^2 - 1} \frac{\partial z}{\partial u_{n-1,m}} - \frac{\partial z}{\partial u_{n,m}} + \frac{u_{n+1,m}^2 - 1}{2u_{n,m}} \frac{\partial z}{\partial u_{n+1,m}} \\ = \frac{u_{n,m}^2 u_{n+1,m} + 2u_{n,m}^2 - u_{n+1,m}}{u_{n,m}(1 - u_{n,m}^2)}. \end{aligned} \tag{3.58b}$$

Since in (3.58) there is no dependence on $u_{n,m\pm1}$ we can omit the equations concerning these variables. The system (3.58) is not closed. To close this system one has to add the equation for the Lie bracket (3.41). Solving the obtained system of three equations with respect to the partial derivatives of z we have:

$$\frac{\partial z}{\partial u_{n-1,m}} = 0, \tag{3.59a}$$

$$\frac{\partial z}{\partial u_{n+1,m}} = -\frac{2(u_{n,m}+1)}{u_{n+1,m}(u_{n,m}+1)+1-u_{n,m}}, \tag{3.59b}$$

$$\begin{aligned}\frac{\partial z}{\partial u_{n,m}} &= \frac{1}{u_{n,m}} + \frac{1}{u_{n,m}+1} + \frac{1}{u_{n,m}-1}, \\ &- \frac{2(u_{n+1,m}-1)}{u_{n+1,m}(u_{n,m}+1)+1-u_{n,m}}.\end{aligned} \tag{3.59c}$$

These three equations form an overdetermined system of equations for z which is consistent because is closed. Hence its general solution z is easily found. It contains arbitrary function $C^1_{n,m}$ depending on both discrete variables:

$$z = \log \frac{C^1_{n,m} u_{n,m}(u^2_{n,m}-1)}{(u_{n,m}u_{n+1,m}+u_{n+1,m}-u_{n,m}+1)^2}. \tag{3.60}$$

Substituting (3.60) into the conservation law (3.29) we get:

$$\log \frac{-C^1_{n,m+1}}{C^1_{n,m}} = 0. \tag{3.61}$$

The last equation is solved by $C^1_{n,m} = (-1)^m C^2_n$ where C^2_m is an arbitrary function of one discrete variable.

Therefore by solving equation $z = \log \partial g/\partial u_{n+1,m}$ we find:

$$g = \frac{-(-1)^n C^2_n u_{n,m}(u_{n,m}-1)}{u_{n,m}u_{n+1,m}+u_{n+1,m}-u_{n,m}+1} + g_2(u_{n-1,m}, u_{n,m}). \tag{3.62}$$

with g_2 possibly dependent on the lattice variables n, m. For the further specification consider $v(u_{n-1,m}, u_{n,m}) = \log \partial g/\partial u_{n-1,m} = \log \partial g_2/\partial u_{n-1,m}$ which putted into (3.42) for v gives:

$$\frac{u^2_{n-1,m}-1}{2u_{n,m}} \frac{\partial v}{\partial u_{n-1,m}} + \frac{\partial v}{\partial u_{n,m}} = \frac{2u^2_{n,m}-2u_{n,m}u_{n-1,m}+u_{n-1,m}}{u_{n,m}(u^2_{n,m}-1)}, \tag{3.63a}$$

$$\frac{2u_{n-1,m}}{u^2_{n,m}-1} \frac{\partial v}{\partial u_{n-1,m}} - \frac{\partial v}{\partial u_{n,m}} = -\frac{1-u^2_{n,m}-2u_{n,m}}{u_{n,m}(u^2_{n,m}-1)}. \tag{3.63b}$$

The solution of the system (3.63) is given by:

$$v = \log \frac{C^3_{n,m} u_{n,m}(u^2_{n,m}-1)}{(u_{n,m}u_{n-1,m}-u_{n-1,m}+u_{n,m}+1)^2}. \tag{3.64}$$

Substituting (3.64) in the conservation law (3.34) we obtain the relation

$$\log \frac{-C^3_{n,m+1}}{C^3_{n,m}} = 0, \tag{3.65}$$

whose solution is $C^3_{n,m} = (-1)^m C^4_n$, where C^4_n is an arbitrary function of n.

As a result function g takes the form;

$$g = \frac{-(-1)^m C^2_n u_{n,m}(u_{n,m}-1)}{u_{n,m}u_{n+1,m}+u_{n+1,m}-u_{n,m}+1} + \frac{-(-1)^m C^4_n u_{n,m}(u_{n,m}+1)}{u_{n,m}u_{n-1,m}-u_{n-1,m}+u_{n,m}+1} + g^{(1)}(u_{n,m}). \tag{3.66}$$

Substituting (3.66) into (3.26), applying the operator S defined by (3.47) then dividing by the factor

$$2u_{n,m}\left(u^2_{n+1,m}-1\right)\left(u^2_{n,m+1}-1\right)\left(u_{n+1,m}+1+u_{n,m}u_{n+1,m}-u_{n,m}\right) \tag{3.67}$$

and finally differentiating with respect to $u_{n,m}$ we obtain:

$$\frac{d^2g^{(1)}}{du^2_{n,m}} - \frac{1}{u_{n,m}}\frac{dg^{(1)}}{du_{n,m}} + \frac{g^{(1)}}{u^2_{n,m}} = \frac{(-1)^m\left(u^2_{n+1,m}+1\right)\left(C^2_n - C^4_{n+1}\right)}{(u_{n,m}u_{n+1,m}-u_{n,m}+u_{n+1,m}+1)^2}. \tag{3.68}$$

This equation implies:

$$\frac{d^2g^{(1)}}{du^2_{n,m}} - \frac{1}{u_{n,m}}\frac{dg^{(1)}}{du_{n,m}} + \frac{g^{(1)}}{u^2_{n,m}} = 0, \qquad C^2_n = C^4_{n+1}. \tag{3.69}$$

Therefore $g^{(1)} = C^5_{n,m}u_{n,m} + C^6_{n,m}u_{n,m}\log u_{n,m}$, where the coefficients do not depend on $u_{n,m}$ but might depend on the lattice variables n,m. Substituting this result again into the determining equations, applying the operator S and taking the coefficients with respect to the independent functions $u_{n,m+1}$, $u_{n,m}$, $\log u_{n+1,m}$, $u_{n+1,m}$ we obtain the three equations:

$$\begin{gathered} C^6_{n+1,m}=0, \quad 2C^5_{n+1,m}+\left(C^4_{n+2}-C^4_{n+1}\right)(-1)^m=0, \\ 2C^5_{n+1,m}-\left(3C^4_{n+1}+C^4_{n+2}\right)(-1)^m=0, \end{gathered} \tag{3.70}$$

which, when solved, give us:

$$C^4_{n,m} = C\,(-1)^n, \quad C^5_{n,m} = C\,(-1)^{n+m}, \quad C^6_{n,m}=0. \tag{3.71}$$

By plugging these results into (3.66) we obtain:

$$g = \frac{(-1)^{m+n}Cu_{n,m}(u^2_{n,m}-1)(u_{n+1,m}u_{n-1,m}+1)}{(u_{n,m}u_{n+1,m}+u_{n+1,m}-u_{n,m}+1)(u_{n,m}u_{n-1,m}-u_{n-1,m}+u_{n,m}+1)}. \tag{3.72}$$

From the direct substitution into (3.26) we obtain that (3.72) is a symmetry. □

Generalized symmetries can be used to provide *symmetry reductions*. Suppose we have a three-point generalized symmetry in the form (3.24). We can consider its flux which we recall is the solution of the differential difference equation

$$\frac{du_{n,m}}{d\varepsilon} = g\left(u_{n+1,m}, u_{n,m}, u_{n-1,m}\right) \tag{3.73}$$

and we can consider its *stationary solutions*, i.e. the solutions such that $du_{n,m}/d\varepsilon \equiv 0$:

$$g\left(u_{n+1,m}, u_{n,m}, u_{n-1,m}\right) = 0. \tag{3.74}$$

In this equation m plays the rôle of a parameter and we can consider contemporaneous solutions of the stationary equation (3.74) and of the original quad equation (3.25). This will give raise to families of particular solutions which are known as *symmetry solutions*.

We conclude this Subsection presenting an example of symmetry reduction.

Example 3.2.3 (Reduction of the H_1 equation (1.23a)). Consider the H_1 equation as given by formula (1.23a). In Example 3.2.3 we derived its three point generalized symmetries in both directions. Here we will use the symmetry (3.54) to derive a family of symmetry solutions [24, 136]. First we start by observing that if we assume $C^4 \equiv 0$ we cannot have any stationary solution to the equation

$$\frac{C^1}{u_{n-1,m} - u_{n+1,m}} + C^4 = 0, \tag{3.75}$$

so we must assume $C^4 \neq 0$ and then we can take without loss of generality $C^4 = 1$. This means that the we will have the *linear* stationary equation:

$$u_{n+1,m} - u_{n-1,m} = C^1. \tag{3.76}$$

This equation has solution:

$$u_{n,m} = U_m^{(0)}(-1)^n + U_m^{(1)} + \frac{1}{4}C^1((-1)^n - 1 + 2n), \tag{3.77}$$

which substituted into (1.23a) gives us two equations for the coefficients of $(-1)^n$:

$$U_{m+1}^{(0)} + U_m^{(0)} + \frac{C^1}{2} = 0, \tag{3.78a}$$

$$\begin{gathered}\left(U_{m+1}^{(0)} + U_m^{(0)}\right)^2 - \left(U_{m+1}^{(1)} - U_m^{(1)}\right)^2 \\ + \frac{(C^1)^2}{2} + C^1 U_m^{(0)} + C^1 U_{m+1}^{(0)} = \alpha_1 - \alpha_2.\end{gathered} \tag{3.78b}$$

The solution of (3.78a) is given by:

$$U_m^{(0)} = K_1(-1)^m - \frac{1}{4}\left(1 + (-1)^m\right) C^1, \tag{3.79}$$

which substituted in (3.78b) gives to the equation:

$$U_{m+1}^{(1)} - U_m^{(1)} = \pm\frac{1}{2}\sqrt{-4\alpha_1 + 4\alpha_2 + (C^1)^2} \tag{3.80}$$

whose solution is given by:

$$U_m^{(1)} = K_2 \pm \frac{m}{2}\sqrt{-4\alpha_1 + 4\alpha_2 + (C^1)^2}. \tag{3.81}$$

So inserting (3.79) and (3.81) into (3.77) we obtain:

$$\begin{aligned} u_{n,m} = {}& (-1)^{n+m}\left(K_1 + \frac{C^1}{4}\right) + K_2 \\ & \pm\sqrt{(C^1)^2 - 4\alpha_1 + 4\alpha_2}m - \frac{C^1}{4} + \frac{C^1}{4}n \end{aligned} \tag{3.82}$$

This is our symmetry solution. □

To end this Section we present a convenient way to compute the three-point generalized symmetries in the case of quad equations with two-periodic coefficients:

$$\begin{aligned} & F_n^{(+)}F_m^{(+)}Q^{(+,+)} + F_n^{(+)}F_m^{(-)}Q^{(+,-)} \\ & + F_n^{(-)}F_m^{(+)}Q^{(+,+)} + F_n^{(-)}F_m^{(-)}Q^{(-,-)} = 0, \end{aligned} \tag{3.83}$$

where $Q^{(\pm,\pm)} = Q^{(\pm,\pm)}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1})$ and $F_k^{(\pm)}$ is given by (1.90). Whereas the method outlined above in principle apply to quad equations with two-periodic coefficients (3.83) the computations readily become very cumbersome. So we will give a brief account on how these difficulties can be avoided following the exposition in [63]. The necessity of treating equations like (3.83) as simply as possible comes from the fact that we wish to study the trapezoidal H^4 and H^6 equations which have two-periodic coefficients as shown by their explicit form given in Section 1.6. The three-point generalized symmetries of these equations were first presented in [66] and were computed with the ideas presented above. We recall that the three-point generalized symmetries of the rhombic H^4 equations, which again have two-periodic coefficients, were studied in [166].

Now let us consider a quad equation with two-periodic coefficients of the form (3.83). Since any point (n, m) on the lattice have coordinates which can be even or odd we can derive the equations (3.40,3.42) for the functions z and v and then use the decomposition:

$$\begin{aligned} z = {}& F_n^{(+)}F_m^{(+)}z^{(+,+)} + F_n^{(+)}F_m^{(-)}z^{(+,-)} \\ & + F_n^{(-)}F_m^{(+)}z^{(+,+)} + F_n^{(-)}F_m^{(-)}z^{(-,-)}, \end{aligned} \tag{3.84a}$$

$$\begin{aligned} v = {}& F_n^{(+)}F_m^{(+)}v^{(+,+)} + F_n^{(+)}F_m^{(-)}v^{(+,-)} \\ & + F_n^{(-)}F_m^{(+)}v^{(+,+)} + F_n^{(-)}F_m^{(-)}v^{(-,-)}. \end{aligned} \tag{3.84b}$$

The decomposition (3.84) reduces the problem to the solution of four decoupled systems for the functions $z^{(\pm,\pm)} = z^{(\pm,\pm)}(u_{n+1,m}, u_{n,m}, u_{n-1,m})$ and $v^{(\pm,\pm)} = v^{(\pm,\pm)}(u_{n+1,m}, u_{n,m}, u_{n-1,m})$ by considering the even/odd combinations of discrete variables.

The same decomposition can be used for the function g:

$$\begin{aligned} g &= F_n^{(+)} F_m^{(+)} g^{(+,+)} + F_n^{(+)} F_m^{(-)} g^{(+,-)} \\ &+ F_n^{(-)} F_m^{(+)} g^{(+,+)} + F_n^{(-)} F_m^{(-)} g^{(-,-)}, \end{aligned} \tag{3.85}$$

with $g^{(\pm,\pm)} = g^{(\pm,\pm)}(u_{n+1,m}, u_{n,m}, u_{n-1,m})$, and the relative compatibility conditions (3.45). This will yield the following form for g:

$$\begin{aligned} g &= \Omega_{n,m}(u_{n+1,m}, u_{n,m}, u_{n-1,m}) \\ &+ F_n^{(+)} F_m^{(+)} \varphi^{(+,+)} + F_n^{(+)} F_m^{(-)} \varphi^{(+,-)} \\ &+ F_n^{(-)} F_m^{(+)} \varphi^{(-,+)} + F_n^{(-)} F_m^{(-)} \varphi^{(-,-)}, \end{aligned} \tag{3.86}$$

where $\Omega_{n,m}$ is a known function derived from z and v and $\varphi^{(\pm,\pm)} = \varphi^{(\pm,\pm)}(u_{n,m})$. These functions can be found rewriting the determining equations with the same decomposition.

3.3 THREE-POINT GENERALIZED SYMMETRIES OF THE TRAPEZOIDAL H^4 AND H^6 EQUATIONS

In this Section we apply the method discussed in Section 3.2, especially in its final part, to compute the three-point generalized symmetries of the trapezoidal H^4 equations as given by (1.91) and H^6 equations as given by (1.93). These three-point generalized symmetries were first computed in [66] except for the ${}_tH_1^\varepsilon$ equation whose symmetries were first presented in [68]. Since the computations are very long and tedious we omit them and we present only the final result. The interested reader may refer to Appendix D where we present two examples carried out in the details: the ${}_tH_1^\varepsilon$ in the m direction and the D_3 equation. The three-point generalized symmetries of the other trapezoidal H^4 and H^6 equation can be computed in an analogous way.

3.3.1 *Trapezoidal* H^4 *equations*

We now consider the trapezoidal H^4 equations as given by equation (1.91).

We present first the three-point generalized symmetries of the equations ${}_tH_2^\varepsilon$ (1.91b) and of equations ${}_tH_3^\varepsilon$ (1.91c) are found to be:

$$\widehat{X}_n^{{}_tH_2^\varepsilon} = \left[\frac{(u_{n,m}+\varepsilon\alpha_2^2F_m^{(+)})(u_{n+1,m}+u_{n-1,m})-u_{n+1,m}u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}} - \right. \\ \left. -\frac{u_{n,m}^2-2\varepsilon F_m^{(+)}\alpha_2^2u_{n,m}-\alpha_2^2+4\varepsilon F_m^{(+)}\alpha_2^3+8\varepsilon F_m^{(+)}\alpha_2^2\alpha_3+\varepsilon^2F_m^{(+)}\alpha_2^4}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \tag{3.87a}$$

$$\widehat{X}_m^{{}_tH_2^\varepsilon} = \left[\frac{\left[\frac{1}{2}-\varepsilon(\alpha_2+\alpha_3)F_m^{(+)}\right](u_{n,m+1}+u_{n,m-1})-\varepsilon F_m^{(+)}u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}} - \right. \\ \left. -\frac{\varepsilon F_m^{(-)}u_{n,m}^2-\left[1-2\varepsilon(\alpha_2+\alpha_3)F_m^{(-)}\right]u_{n,m}+\alpha_3+\varepsilon\left(\alpha_2+\alpha_3\right)^2}{u_{n,m+1}-u_{n,m-1}}\right]\partial_{u_{n,m}}, \tag{3.87b}$$

$$\widehat{X}_n^{{}_tH_3^\varepsilon} = \left[\frac{\frac{1}{2}\alpha_2(1+\alpha_2^2)u_{n,m}(u_{n+1,m}+u_{n-1,m})-\alpha_2^2u_{n+1,m}u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}} - \right. \\ \left. -\frac{\alpha_2^2u_{n,m}^2+\varepsilon^2\delta^2(1-\alpha_2^2)^2F_m^{(+)}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \tag{3.87c}$$

$$\widehat{X}_m^{{}_tH_3^\varepsilon} = \left[\frac{\frac{1}{2}\alpha_3u_{n,m}(u_{n,m+1}+u_{n,m-1})-\varepsilon^2F_m^{(+)}u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}} - \right. \\ \left. -\frac{\varepsilon^2F_m^{(-)}u_{n,m}^2+\alpha_3^2\delta^2}{u_{n,m+1}-u_{n,m-1}}\right]\partial_{u_{n,m}}, \tag{3.87d}$$

The symmetries in the n and m directions and the linearization of the ${}_tH_1^\varepsilon$ equation (1.91a) were first presented in [68]. Their peculiarity is that they are written in term of two arbitrary functions of one continuous variable and one discrete index and by arbitrary functions of the lattice variables. This fact is related to the property of the ${}_tH_1^\varepsilon$ of being Darboux integrable with integrals of the first and second order [5] as it was first found in [69] and we will discuss in Chapter 4.

In the direction n we have that the ${}_tH_1^\varepsilon$ possess the following three-point generalized symmetries:

$$\begin{aligned}
\hat{X}_{n,1}^{{}_tH_1^\varepsilon} = \Bigg\{ & \frac{F_m^{(+)}}{(u_{n+1,m}-2u_{n,m}+u_{n-1,m})(u_{n+1,m}-u_{n-1,m})} \\
& \cdot \left[\left((u_{n,m}-u_{n-1,m})^2 + \varepsilon^2 \alpha_2^2 \right) B_n \left(\frac{\alpha_2}{u_{n+1,m}-u_{n,m}} \right) \right. \\
& \left. - \left((u_{n+1,m}-u_{n,m})^2 + \varepsilon^2 \alpha_2^2 \right) B_{n-1} \left(\frac{\alpha_2}{u_{n,m}-u_{n-1,m}} \right) \right] \\
& + F_m^{(-)} \frac{(u_{n+1,m}-u_{n,m})^2 (u_{n,m}-u_{n-1,m})^2}{\begin{pmatrix} (2-2u_{n+1,m}\varepsilon^2 u_{n-1,m}) u_{n,m} \\ +\varepsilon^2 (u_{n-1,m}+u_{n+1,m}) u_{n,m}^2 \\ -u_{n+1,m}-u_{n-1,m} \end{pmatrix} (u_{n-1,m}-u_{n+1,m}) (1+u_{n,m}{}^2\varepsilon^2)} \\
& \cdot \left[B_n \left(\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n+1,m} u_{n,m}} \right) - B_n \left(\frac{u_{n,m}-u_{n-1,m}}{1+\varepsilon^2 u_{n,m} u_{n-1,m}} \right) \right] \Bigg\} \partial_{u_{n,m}},
\end{aligned} \tag{3.88a}$$

$$\begin{aligned}
\hat{X}_{n,2}^{{}_tH_1^\varepsilon} = \Bigg\{ & F_m^{(+)} \left[u_{n,m} - \frac{((u_{n+1,m}-u_{n,m})^2 + \varepsilon^2 \alpha^2)(u_{n,m}-u_{n-1,m})}{(u_{n+1,m}-2u_{n,m}+u_{n-1,m})(u_{n+1,m}-u_{n-1,m})} \right] \\
& -F_m^{(-)} \left[\frac{u_{n,m}u_{n-1,m} + u_{n+1,m}u_{n,m} - u_{n,m}{}^2 - u_{n+1,m}u_{n-1,m}}{(u_{n,m}{}^2\varepsilon^2+1)(-2u_{n+1,m}+2u_{n-1,m})} \right. \\
& \left. + \frac{(u_{n+1,m}-u_{n,m})(u_{n,m}-u_{n-1,m})}{\begin{pmatrix} 2\varepsilon^2 (u_{n-1,m}+u_{n+1,m}) u_{n,m}^2 \\ + (4-4u_{n+1,m}\varepsilon^2 u_{n-1,m}) u_{n,m} - 2u_{n+1,m} - 2u_{n-1,m} \end{pmatrix}} \right] \Bigg\} \partial_{u_{n,m}},
\end{aligned} \tag{3.88b}$$

where $B_n = B_n(\xi)$ is an arbitrary function of its argument. Therefore the most general symmetry in the n direction is given by the combination:

$$\hat{X}_n^{{}_tH_1^\varepsilon} = \hat{X}_{n,1}^{{}_tH_1^\varepsilon} + \alpha \hat{X}_{n,2}^{{}_tH_1^\varepsilon} \tag{3.89}$$

where α is an arbitrary constant.

The general symmetry in the m direction is:

$$\begin{aligned}
\hat{X}_m^{{}_tH_1^\varepsilon} = & \left[F_m^{(+)} B_m \left(\frac{u_{n,m+1}-u_{n,m-1}}{1+\varepsilon^2 u_{n,m+1} u_{n,m-1}} \right) \right. \\
& \left. + F_m^{(-)} \left(1+\varepsilon^2 u_{n,m}^2\right) C_m (u_{n,m+1}-u_{n,m-1}) \right] \partial_{u_{n,m}}.
\end{aligned} \tag{3.90}$$

where $B_m = B_m(\xi)$ and $C_m = C_m(\xi)$ are arbitrary functions of the lattice variable m and of their arguments.

Furthermore the ${}_tH_1^\varepsilon$ equation (1.91a) possesses the following point symmetry:

$$\hat{Y}^{{}_tH_1^\varepsilon} = \left[F_m^{(+)} \kappa_m + F_m^{(-)} \lambda_m \left(1+\varepsilon^2 u_{n,m}^2\right) \right] \partial_{u_{n,m}}, \tag{3.91}$$

where κ_m and λ_m are arbitrary functions of the lattice variable m. On the contrary both the ${}_tH_2^\varepsilon$ equation (1.91b) and the ${}_tH_3^\varepsilon$ equation (1.91c) do not possess point symmetries.

3.3.2 H^6 *equations*

In this Subsection we consider the H^6 equations as given by formula (1.93).

The three forms of the equation D_2 (1.93b,1.93c,1.93d), which we will collectively call ${}_iD_2$ assuming i in $\{1,2,3\}$, possess the following three points generalized symmetries in the n direction and three-points generalized symmetries in the m direction:

$$\hat{X}_n^{{}_1D_2} = \left[\frac{\left(F_n^{(+)}F_m^{(-)} - \delta_1 F_n^{(+)}F_m^{(+)}\right)(u_{n,m+1}+u_{n-1,m})}{u_{n+1,m}-u_{n-1,m}} + \right.$$
$$+\frac{\left(F_n^{(+)}F_m^{(-)} - \delta_1 F_n^{(+)}F_m^{(+)} - \delta_1\delta_2\right)F_n^{(-)}F_m^{(+)}u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}} +$$
$$\left. +\frac{(F_{n+m}^{(+)} - \delta_1 F_m^{(+)} - \delta_1\delta_2 F_n^{(+)}F_m^{(+)})u_{n,m} + \delta_2 F_m^{(-)}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \tag{3.92a}$$

$$\hat{X}_m^{{}_1D_2} = \left[\frac{\delta_1 F_n^{(-)}F_m^{(+)}u_{n,m+1}u_{n,m-1} + F_{n+m}^{(-)}(u_{n,m+1}+u_{n,m-1})}{u_{n,m+1}-u_{n,m-1}} + \right.$$
$$+\frac{\delta_1 F_m^{(+)}u_{n,m+1} + \delta_1\delta_2 F_n^{(-)}F_m^{(+)}u_{n,m-1} + \delta_1 F_n^{(-)}F_m^{(-)}u_{n,m}^2}{u_{n,m+1}-u_{n,m-1}} +$$
$$+\frac{\left[F_{n+m}^{(+)} + \delta_1\left(F_n^{(+)}F_m^{(-)} - F_n^{(-)}F_m^{(-)}\right) + \delta_1\delta_2 F_n^{(-)}F_m^{(-)}\right]u_{n,m}}{u_{n,m+1}-u_{n,m-1}} -$$
$$\left. -\frac{\delta_2(\delta_1-1)F_n^{(-)}}{u_{n,m+1}-u_{n,m-1}}\right]\partial_{unm}, \tag{3.92b}$$

$$\hat{X}_n^{{}_2D_2} = \left[\frac{\left(F_n^{(-)}F_m^{(-)}\delta_1 + F_n^{(-)}F_m^{(-)}\delta_1\delta_2 - F_n^{(-)}F_m^{(-)}\right)u_{n+1,m}}{u_{n+1,m}-u_{n-1,m}} + \right.$$
$$+\frac{\left(F_n^{(+)}F_m^{(+)}\delta_1 - F_n^{(+)}F_m^{(-)}\right)u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}}$$
$$\left. +\frac{\left(\delta_1 F_{n+m}^{(-)} - F_m^{(-)} + \delta_1\delta_2 F_n^{(+)}F_m^{(-)}\right)u_{n,m} - (\delta_1-1)F_m^{(+)}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \tag{3.92c}$$

$$\widehat{X}_m^{2D_2} \left[\frac{F_n^{(-)} F_m^{(-)} \delta_1 u_{n,m+1} u_{n,m-1} + \left(\delta_1 \delta_2 F_n^{(-)} F_m^{(-)} + F_n^{(+)} F_m^{(-)}\right) u_{n,m+1}}{u_{n,m+1} - u_{n,m-1}} + \right.$$
$$+ \frac{+\left(\delta_1 F_n^{(+)} F_m^{(+)} + F_n^{(-)} F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)}\right) u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} +$$
$$+ \frac{\delta_1 F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \left[F_n^{(+)} F_m^{(-)} + (\delta_2 - 1) F_n^{(-)} F_m^{(+)} + F_m^{(+)}\right] u_{n,m}}{u_{n,m+1} - u_{n,m-1}} +$$
$$\left. + \frac{\delta_2 (1 - \delta_2) F_n^{(-)} - \delta_1 \lambda F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \tag{3.92d}$$

$$\widehat{X}_n^{3D_2} = \left[\frac{\left(\delta_1 F_n^{(+)} F_m^{(-)} + \delta_1 \delta_2 F_n^{(+)} F_m^{(-)} - F_n^{(+)} F_m^{(-)}\right) u_{n+1,m}}{u_{n+1,m} - u_{n-1,m}} + \right.$$
$$+ \frac{\left(F_n^{(+)} F_m^{(+)} \delta_1 - F_n^{(-)} F_m^{(-)}\right) u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} +$$
$$\left. + \frac{\left(\delta_1 F_n^{(-)} F_m^{(-)} \delta_2 + F_n^{(-)} \delta_1 - F_m^{(-)}\right) u_{n,m} + (1 - \delta_1) F_m^{(+)}}{u_{n+1,m} - u_{n-1,m}} \right] \partial u_{n,m}, \tag{3.92e}$$

$$\widehat{X}_m^{3D_2} = \left[\frac{(1 - \delta_1 - \delta_1 \delta_2)\, F_n^{(+)} F_m^{(-)} u_{n,m+1}}{u_{n,m+1} - u_{n,m-1}} + \right. \tag{3.92f}$$
$$+ \frac{\left(F_n^{(-)} F_m^{(-)} - F_n^{(+)} F_m^{(+)} \delta_1\right) u_{n,m-1} + \delta_2 F_n^{(-)}}{u_{n,m+1} - u_{n,m-1}} +$$
$$+ \frac{\left(F_m^{(+)} - \delta_1 F_n^{(+)} - \delta_1 \delta_2 F_n^{(+)} F_m^{(+)}\right) u_{n,m}}{u_{n,m+1} - u_{n,m-1}} -$$
$$\left. - \frac{\lambda \delta_1 (1 - \delta_1 - \delta_1 \delta_2) F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{unm}.$$

Furthermore the equations ${}_iD_2$ possess also the following point symmetries:

$$\widehat{Y}_1^{1D_2} = \left(F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(+)}\right) u_{n,m} \partial_{u_{n,m}}, \tag{3.93a}$$
$$\widehat{Y}_2^{1D_2} = \left[\delta_1 F_n^{(+)} F_m^{(+)} + [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(+)} + F_n^{(+)} F_m^{(-)}\right] \partial_{u_{n,m}}, \tag{3.93b}$$
$$\widehat{Y}_1^{2D_2} = \left[\left(F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(+)}\right) u_{n,m} - \right. \tag{3.93c}$$
$$\left. - \lambda F_n^{(+)} F_m^{(-)} + \lambda [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(-)}\right] \partial_{u_{n,m}},$$
$$\widehat{Y}_2^{2D_2} = \left[\delta_1 F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(-)}\right] \partial_{u_{n,m}}, \tag{3.93d}$$

$$\begin{aligned}
\widehat{Y}_1^{3\,D_2} &= \Big[\Big(F_n^{(+)}F_m^{(+)} + F_n^{(+)}F_m^{(-)} + F_n^{(-)}F_m^{(+)}\Big) u_{n,m} - \\
&\quad -\lambda F_n^{(-)}F_m^{(-)} + \lambda[1-\delta_1(1+\delta_2)]F_n^{(-)}F_m^{(-)}\Big]\,\partial_{u_{n,m}}, \qquad (3.93e)\\
\widehat{Y}_2^{3\,D_2} &= \Big[\delta_1 F_n^{(+)}F_m^{(+)} + [1-\delta_1(1+\delta_2)]F_n^{(+)}F_m^{(-)} - F_n^{(-)}F_m^{(-)}\Big]\,\partial_{u_{n,m}}, \qquad (3.93f)
\end{aligned}$$

The D_3 equation (1.93e) admits the following three-point generalized symmetries:

$$\begin{aligned}
\widehat{X}_n^{D_3} &= \left[\frac{F_n^{(+)}F_m^{(+)}u_{n+1,m}u_{n-1,m} + \frac{1}{2}\Big(F_m^{(-)} - F_n^{(-)}F_m^{(+)}\Big) u_{n,m}(u_{n+1,m}+u_{n-1,m})}{u_{n+1,m}-u_{n-1,m}} + \right.\\
&\quad \left. + \frac{F_n^{(-)}F_m^{(+)}u_{n,m}^2 + \Big(F_m^{(-)} - F_n^{(+)}F_m^{(+)}\Big) u_{n,m}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \qquad (3.94a)
\end{aligned}$$

$$\begin{aligned}
\widehat{X}_m^{D_3} &= \left[\frac{F_n^{(+)}F_m^{(+)}u_{n,m+1}u_{n,m-1} + \frac{1}{2}\Big(F_n^{(-)} - F_n^{(+)}F_m^{(-)}\Big) u_{n,m}(u_{n,m+1}+u_{n,m-1})}{u_{n,m+1}-u_{n,m-1}} + \right.\\
&\quad \left. + \frac{F_n^{(+)}F_m^{(-)}u_{n,m}^2 + \Big(F_n^{(-)} - F_n^{(+)}F_m^{(+)}\Big) u_{n,m}}{u_{n,m+1}-u_{n,m-1}}\right]\partial_{u_{n,m}} \qquad (3.94b)
\end{aligned}$$

and the point symmetry:

$$\hat{Y}^{D_3} = \Big[F_n^{(+)}\Big(2F_m^{(+)} + F_m^{(-)}\Big) + F_n^{(-)}\Big] u_{n,m}\partial_{u_{n,m}}. \qquad (3.95)$$

Remark 3.3.1. The equation D_3 (1.93e) is invariant under the exchange $n \leftrightarrow m$ so the symmetry $X_m^{D_3}$ (3.94b) can be obtained from the symmetry $X_n^{D_3}$ (3.94b) performing such exchange.

Finally the two forms of D_4 possess the following three-point generalized symmetries:

$$\begin{aligned}
\widehat{X}_n^{1\,D_4} &= \left[\frac{-\delta_1 F_n^{(+)}u_{n+1,m}u_{n-1,m} - \frac{1}{2}u_{n,m}(u_{n+1,m}+u_{n-1,m})}{u_{n+1,m}-u_{n-1,m}} + \right.\\
&\quad \left. + \frac{-\delta_1 F_n^{(-)}u_{n,m}^2 + \delta_2\delta_3 F_m^{(+)}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}, \qquad (3.96a)
\end{aligned}$$

$$\widehat{X}_m^{1 D_4} = \left[\frac{F_m^{(-)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right.$$
$$\left. + \frac{\delta_2 F_m^{(+)} u_{n,m}^2 - \delta_1 \delta_3 F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \tag{3.96b}$$

$$\widehat{X}_n^{2 D_4} = \left[\frac{-\delta_1 \delta_2 F_n^{(+)} F_m^{(+)} u_{n+1,m} u_{n-1,m} + \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \right.$$
$$\left. + \frac{-\delta_1 \delta_2 F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \delta_3}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \tag{3.96c}$$

$$\widehat{X}_m^{2 D_4} = \left[\frac{\delta_2 F_{n+m}^{(+)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right.$$
$$\left. + \frac{\delta_2 F_{n+m}^{(-)} u_{n,m}^2 - \delta_1 \delta_3 F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \tag{3.96d}$$

and no point symmetries.

3.4 THE NON-AUTONOMOUS YDKN EQUATION

In 1983 Ravil I. Yamilov [167] classified all differential-difference equations of the Volterra class (2.6) using the generalized symmetry method. From the generalized symmetry method one obtains integrability conditions which allow to check whether a given equation is integrable. Moreover in many cases these conditions enable us to classify equations, i.e. to obtain complete lists of integrable equations belonging to a certain class. As integrability conditions are only necessary conditions for the existence of generalized symmetries and/or conservation laws, one then has to prove that the equations of the resulting list really possess generalized symmetries and conservation laws of sufficiently high order. One mainly construct them using Miura-type transformations, finding Master Symmetries (see Section 3.5, or proving the existence of a Lax pair [167–169]. The result of Yamilov classification, up to Miura transformation, is the Toda equation and the so called Yamilov discretization of the Krichever-Novikov equation (YdKN), a differential-difference equation for the unknown function $q_k = q_k(t)$, with $k \in \mathbb{Z}$ and $t \in \mathbb{R}$, depending on six arbitrary coefficients:

$$\frac{dq_k}{dt} = \frac{A(q_k) q_{k+1} q_{k-1} + B(q_k)(q_{k+1} + q_{k-1}) + C(q_k)}{q_{k+1} - q_{k-1}}, \tag{3.97}$$

where:

$$A(q_k) = aq_k^2 + 2bq_k + c, \tag{3.98a}$$

$$B(q_k) = bq_k^2 + dq_k + e, \tag{3.98b}$$

$$C(q_k) = cq_k^2 + 2eq_k + f. \tag{3.98c}$$

The integrability of (3.97) is proven by the existence of point symmetries [108] and of a Master Symmetries [169] from which one is able to explicitly write down an infinite hierarchy of generalized symmetries. The problem of finding the Bäcklund transformation and Lax pair in the general case, where all the six parameters are different from zero, seems to be still open. Some partial results are contained in [6].

In [105] the authors constructed a set of five conditions necessary for the existence of generalized symmetries for a class of differential-difference equations depending only on nearest neighbouring interaction. They used the conditions to propose the integrability of the following non-autonomous generalization of the YdKN:

$$\frac{dq_k}{dt} = \frac{A_k(q_k)q_{k+1}q_{k-1} + B_k(q_k)(q_{k+1} + q_{k-1}) + C_k(q_k)}{q_{k+1} - q_{k-1}}, \tag{3.99}$$

where the now k-dependent coefficients are given by:

$$A_k(q_k) = aq_k^2 + 2b_kq_k + c_k, \tag{3.100a}$$

$$B_k(q_k) = b_{k+1}q_k^2 + dq_k + e_{k+1}, \tag{3.100b}$$

$$C_k(q_k) = c_{k+1}q_k^2 + 2e_kq_k + f, \tag{3.100c}$$

with b_k, c_k and e_k 2-periodic functions. The equation (3.99) was shown to possess non-trivial conservation laws of second and third order and two generalized local symmetries of order i and $i+1$, with $i < 4$.

As it was discussed in Section 3.2 the symmetry generator associated to a three-point symmetry of a quad equation is a differential-difference equation depending parametrically on the other lattice variable, see equation (3.22). Therefore once the generalized symmetries of an equation, or of a class of equations are computed, it is natural to ask if the associated differential-difference equations are of a known type [105, 167–169]. For example let us consider the equations belonging to the ABS classification [2], i.e. the H equations (1.23) and the Q equations (1.24), whose three-point generalized symmetries where computed systematically in [139]. In [102] it was proved that these symmetries are all particular cases of the YdKN equation (3.97). Moreover it was showed in [166] that the three-point generalized symmetries of the rhombic H^4 equations (1.32) are particular instances of the non-autonomous YdKN equation (3.99).

In the previous Section we presented following [66] the three-point generalized symmetries of the trapezoidal H^4 (1.91) and of the H^6 equations, which were previously unknown. It is then natural to ask if there is any relation between these three-point generalized symmetries and the non-autonomous YdKN equation (3.99). The answer is that all the three-point generalized symmetries are somehow related to the non-autonomous YdKN (3.99). For the rest of this Section we will discuss the kind of identification needed for the various equations.

In the case of the ${}_tH_1^\varepsilon$ equation we have symmetries depending on arbitrary functions given by (3.89) and by (3.90). However it can be readily showed that in (3.89) if and only if

$$B_n(\xi) = -\frac{1}{\xi}, \quad \alpha = 0, \tag{3.101}$$

we get a symmetry of YdKN type (3.99):

$$\hat{X}_n^{{}_tH_1^\varepsilon} = \frac{(u_{n+1,m} - u_{n,m})(u_{n,m} - u_{n-1,m}) - F_m^{(+)}\varepsilon^2\alpha_2^2}{u_{n+1,m} - u_{n-1,m}}\partial_{u_{n,m}}. \tag{3.102}$$

In the same way we can notice that the symmetry in the direction m (3.90) if and only if

$$B_m(\xi) = \frac{1}{\xi}, \quad C_m(\xi) = \frac{1}{\xi}, \tag{3.103}$$

becomes a symmetry of the non-autonomous YdKN type (3.99):

$$\hat{X}_m^{{}_tH_1^\varepsilon} = \frac{F_m^{(+)}\left(1 + \varepsilon^2 u_{n,m+1}u_{n,m-1}\right) + F_m^{(-)}\left(1 + \varepsilon^2 u_{n,m}^2\right)}{u_{n,m+1} - u_{n,m-1}}\partial_{u_{n,m}}. \tag{3.104}$$

On the other hand we have that the symmetries of the ${}_tH_2^\varepsilon$ equation (1.91b) and of the ${}_tH_3^\varepsilon$ equation (1.91c) as given by equations (3.87) are naturally identified as particular cases of the non-autonomous YdKN equations (3.99). It is worth to note that, since the only non-autonomous coefficients in the trapezoidal H^4 equations are depending on the lattice variable m, the symmetries in the n direction of these equations actually are sub-cases of the autonomous YdKN equation (3.97).

Turning to the H^6 equations it is easy to check that the symmetries of the ${}_iD_2$ equations (1.93b,1.93c,1.93d) as given by equation (3.92) are not in the form of the YdKN equation (3.99). However a linear combination with point symmetries (3.93):

$$\hat{Z}_j^{{}_iD_2} = \hat{X}_j^{{}_iD_2} + K_{1,j}^{{}_iD_2}\hat{Y}_1^{{}_iD_2} + K_{2,j}^{{}_iD_2}\hat{Y}_2^{{}_iD_2}, \quad j = n, m; \; i = 1,2,3 \tag{3.105}$$

for prescribed values of $K_{1,j}^{{}_iD_2}$ and $K_{2,j}^{{}_iD_2}$ is in the non-autonomous YdKN form (3.99), i.e.

$$\begin{gathered} K_{1,j}^{{}_1D_2} = K_{1,j}^{{}_2D_2} = -K_{1,j}^{{}_3D_2} = \frac{1}{2}, \\ K_{2,j}^{{}_1D_2} = K_{2,j}^{{}_2D_2} = K_{2,j}^{{}_3D_2} = 0. \end{gathered} \tag{3.106}$$

We notice that we are allowed to consider (3.105) as bona fide three-point generalized symmetries since choosing them as new generators just corresponds to choose a new basis for the linear space of the three-point generalized symmetries. On the other hand the symmetries of the D_3 equation (1.93e) as given by equation (3.94) are already sub-cases of the non-autonomous YdKN equation (3.99). Finally we have that also the symmetries of the ${}_iD_4$ equations (1.93f,1.93g) are sub-cases of the non-autonomous YdKN equation (3.99).

Taking into account the result presented in [166], reproduced for sake of completeness in Appendix E, we can then state that *all the three-point generalized symmetries of the equations belonging to the classification made by R. Boll [20–22] are sub-cases of the non-autonomous YdKN equation* (3.99). This result extend what was obtained in [102] for the ABS class of equations. We can then build Table 3.1 where we can show the explicit form of the coefficients of the non-autonomous YdKN equation corresponding to the aforementioned three-point generalized symmetries.

Table 3.1: Identification of the coefficients in the symmetries of the rhombic H^4 equations, trapezoidal H^4 equations and H^6 equations with those of the non autonomous YdKN equation. Here the symmetries of ${}_tH_1^\varepsilon$ in the m direction are the subcase (3.104) of (3.90) while those in the n direction are the subcase (3.102) of (3.89). The symmetries of the ${}_iD_2$ equations (3.92) are combined with the point symmetries (3.93) according to the prescriptions (3.105) and the coefficients given by (3.106).

Eq.	k	a	b_k	c_k	d	e_k	f
${}_rH_1^\varepsilon$	n	0	0	$-\varepsilon F_{n+m}^{(+)}$	0	0	1
	m	0	0	$-\varepsilon F_{n+m}^{(+)}$	0	0	1
${}_rH_2^\varepsilon$	n	0	0	$-4\varepsilon F_{n+m}^{(+)}$	0	$1-4\varepsilon\alpha F_{n+m}^{(-)}$	$2\alpha-4\varepsilon\alpha^2$
	m	0	0	$-4\varepsilon F_{n+m}^{(+)}$	0	$1-4\varepsilon\beta F_{n+m}^{(-)}$	$2\beta-4\varepsilon\beta^2$
${}_rH_3^\varepsilon$	n	0	0	$-\dfrac{\varepsilon F_{n+m}^{(+)}}{\alpha}$	$\dfrac{1}{2}$	0	$\delta\alpha$
	m	0	0	$-\dfrac{\varepsilon F_{n+m}^{(+)}}{\beta}$	$\dfrac{1}{2}$	0	$\delta\beta$
${}_tH_1^\varepsilon$	n	0	0	-1	1	0	$-\varepsilon^2\alpha_2^2F_m^{(+)}$
	m	0	0	$\varepsilon^2F_m^{(+)}$	0	0	2
${}_tH_2^\varepsilon$	n	0	0	-1	1	$\varepsilon\alpha_2^2F_m^{(+)}$	$\alpha_2^2-\varepsilon\alpha_2^2\left(4\alpha_2+8\alpha_3+\varepsilon\alpha_2^2\right)F_m^{(+)}$
	m	0	0	$-\varepsilon F_m^{(+)}$	0	$\dfrac{1}{2}-\varepsilon(\alpha_2+\alpha_3)F_m^{(-)}$	$-\alpha_3-\varepsilon\left(\alpha_2+\alpha_3\right)^2$

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Table 3.1 – *Continued from previous page*

Eq.	k	a	b_k	c_k	d	e_k	f
${}_tH_3^{\varepsilon}$	n	0	0	$-\alpha_2^2$	$\frac{1}{2}\alpha_2(1+\alpha_2^2)$	0	$-\varepsilon^2\delta^2F_m^{(+)}(1-\alpha_2^2)^2$
	m	0	0	$-\varepsilon^2F_m^{(+)}$	$\frac{1}{2}\alpha_3$	0	$-\alpha_3^2\delta^2$
${}_1D_2$	n	0	0	0	0	$\frac{1}{2}[\delta_1(1+\delta_2)-1]F_n^{(+)}F_m^{(+)}$ $+\frac{1}{2}F_n^{(-)}F_m^{(+)}-\frac{1}{2}F_n^{(-)}F_m^{(-)}$	$-\delta_2F_m^{(-)}$
	m	0	0	$-F_n^{(-)}F_m^{(+)}\delta_1$	0	$\frac{1}{2}(\delta_1(1-\delta_2-1)F_n^{(-)}F_m^{(-)}$ $-\frac{1}{2}F_n^{(+)}F_m^{(+)}-\frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)}$	$\delta_2(\delta_1-1)F_n^{(-)}$
${}_2D_2$	n	0	0	0	0	$\frac{1}{2}[1-\delta_1(1+\delta_2)]F_n^{(+)}F_m^{(-)}$ $+\frac{1}{2}F_n^{(-)}F_m^{(-)}-\frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)}$	$(\delta_1-1)F_m^{(+)}$
	m	0	0	$-\delta_1F_n^{(-)}F_m^{(-)}$	0	$\frac{1}{2}[\delta_1(1-\delta_2)-1]F_n^{(-)}F_m^{(+)}$ $-\frac{1}{2}F_n^{(+)}F_m^{(+)}-\frac{1}{2}\delta_1F_n^{(-)}F_m^{(+)}$	$\delta_2\left[\delta_1-1\right]F_n^{(-)}+\lambda\delta_1F_n^{(+)}$

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Table 3.1 – *Continued from previous page*

Eq.	k	a	b_k	c_k	d	e_k	f
${}_3D_2$	n	0	0	0	0	$\frac{1}{2}[\delta_1(1+\delta_2)-1]F_n^{(-)}F_m^{(-)} +\frac{1}{2}F_n^{(+)}F_m^{(-)} + \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$(1-\delta_1)F_m^{(+)}$
	m	0	0	0	0	$\frac{1}{2}[\delta_1(1-\delta_2)-1]F_n^{(+)}F_m^{(+)} -\frac{1}{2}F_n^{(+)}F_m^{(-)} + \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$\delta_1\lambda[-\delta_1(1+\delta_2)]F_n^{(+)} - \delta_2 F_n^{(-)}$
D_3	n	0	0	$F_n^{(+)}F_m^{(+)}$	0	$\frac{1}{2}\left(F_n^{(+)}F_m^{(-)} + F_n^{(-)}F_m^{(-)} - F_n^{(+)}F_m^{(+)}\right)$	0
	m	0	0	$F_n^{(+)}F_m^{(+)}$	0	$\frac{1}{2}\left(F_n^{(-)}F_m^{(+)} + F_n^{(-)}F_m^{(-)} - F_n^{(+)}F_m^{(+)}\right)$	0
${}_1D_4$	n	0	0	$-\delta_1\left(F_n^{(+)}F_m^{(+)} + F_n^{(+)}F_m^{(-)}\right)$	$-\frac{1}{2}$	0	$\delta_2\delta_3F_m^{(+)}$
	m	0	0	$\delta_2\left(F_n^{(+)}F_m^{(+)} + F_n^{(-)}F_m^{(+)}\right)$	$\frac{1}{2}$	0	$-\delta_1\delta_3F_n^{(+)}$
${}_2D_4$	n	0	0	$-F_n^{(+)}F_m^{(+)}\delta_1\delta_2$	$\frac{1}{2}$	0	δ_3
	m	0	0	$\delta_2\left(F_n^{(+)}F_m^{(+)} + F_n^{(-)}F_m^{(-)}\right)$	$\frac{1}{2}$	0	$-\delta_1\delta_3F_n^{(+)}$

3.5 INTEGRABILITY PROPERTIES OF THE NON AUTONOMOUS YDKN EQUATION AND ITS SUB-CASES.

In the previous Section we saw that the fluxes of all the generalized three point symmetries of the H^4 and H^6 equations are eventually related either to the YdKN (3.97) or to the non autonomous YdKN equation (3.99). In this Section we extend those results by considering the ideas in [102] We will use the *Master Symmetries* approach. Master Symmetries are particular kind of symmetries which can generate the whole hierarchy of symmetries of a given equation starting from the given symmetry. The notion of master symmetry has been introduced in [43], see also [42, 46, 47, 126]. In the continuous case master symmetries usually are non-local, i.e. they contain terms involving integrations. In the semi-discrete and discrete case this would correspond to the presence of operators like $(T-\mathrm{Id})^{-1}$, where T is a lattice translation operator, which give raise to infinite summations. Fortunately enough in the semi-discrete and discrete case there are many *local* master symmetries [4, 7, 102, 127, 128, 139, 166, 170, 171, 173], i.e. master symmetries with no such dependence.

Here we will not discuss the general method to find Master Symmetries for differential-difference or quad equation, but we will just present the method we can use to construct the Master Symmetries of the non-autonomous YdKN equation and its sub-cases following [4, 54, 102, 169]. Let us consider differential-difference equations of the form

$$u_{n,\tau} = \varphi_n(\tau, u_{n+k}, u_{n+k-1}, \dots, u_{n+k'+1}, u_{n+k'}), \quad k' < k. \tag{3.107}$$

Now let us suppose we have two such equations (3.107), namely two functions $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$ depending on different "times" τ_1 and τ_2, and depending on u_α for $n+k \geqslant \alpha \geqslant n+k'$ and $n+l \geqslant \alpha \geqslant n+l'$ respectively. We can then require that the two equations commute:

$$u_{n,\tau_1,\tau_2} - u_{n,\tau_2,\tau_1} = D_{\tau_2}\varphi_n^{(1)} - D_{\tau_1}\varphi_n^{(2)}, \tag{3.108}$$

where D_{τ_i} is the *total derivative* with respect to τ_i and it is defined as:

$$D_{\tau_i} = \frac{\partial}{\partial \tau_i} + \sum_{j\in\mathbb{Z}} \varphi_{n+j}^{(i)} \frac{\partial}{\partial u_{n+j}}. \tag{3.109}$$

Let us define a *Lie algebra structure* on the set of functions φ_n of the form (3.107). For any functions $\varphi_n^{(1)}$ and $\varphi_n^{(1)}$, we introduce the equations $u_{n,\tau_1} = \varphi_n^{(1)}$ and $u_{n,\tau_2} = \varphi_n^{(2)}$ and the corresponding total derivatives D_{τ_i} as in (3.109). Then we can define the following operation on the functions $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$:

$$[\varphi_n^{(1)}, \varphi_n^{(2)}] = D_{\tau_2}\varphi_n^{(1)} - D_{\tau_1}\varphi_n^{(2)}. \tag{3.110}$$

The result of this operation is a new function $\varphi_n^{(3)}$ depending on a finite number of shifts of u_n. We prove that the operation $[,]$ is a *Lie bracket*, i.e. that is *bilinear*, *skew-symmetric* and satisfies the *Jacobi identity*:

$$[[\varphi_n^{(1)}, \varphi_n^{(2)}], \varphi_n^{(3)}] = [[\varphi_n^{(1)}, \varphi_n^{(3)}], \varphi_n^{(2)}] + [\varphi_n^{(1)}, [\varphi_n^{(2)}, \varphi_n^{(3)}]]. \quad (3.111)$$

Bilinearity follows from the fact that the total derivative operators (3.109) are linear. It is obviously *skew-symmetric*:

$$[\varphi_n^{(1)}, \varphi_n^{(2)}] = -\left(D_{\tau_1}\varphi_n^{(2)} - D_{\tau_2}\varphi_n^{(1)}\right) = -[\varphi_n^{(2)}, \varphi_n^{(1)}]. \quad (3.112)$$

Finally *Jacobi identity* (3.111) can be checked by direct computation.

A differential-difference equation of the form

$$u_{n,\tau} = g_n(u_{n+k}, u_{n+k-1}, \ldots, u_{n+k'+1}, u_{n+k'}), \quad k' < k. \quad (3.113)$$

is called a *generalized symmetry* for a Volterra-like equation (2.6) if its right hand side *commutes* with the right hand side of the Volterra-like equation, i.e. $[g_n, f_n] = 0$. This generalized symmetry is called *non-trivial* if $k > 1$ and $k' < -1$.

A differential-difference of the first differential-order of the form

$$u_{n,\tau} = \varphi_n(\tau, u_{n+1}, u_n, u_{n-1}), \quad (3.114)$$

where the dependence on τ and on the lattice variable n may be explicit, is called *a Master Symmetry* for a Volterra-like equation (2.6) if the function

$$g_n = [\varphi_n, f_n] \quad (3.115)$$

is the right hand side of a generalized symmetry. This generalized symmetry must be nontrivial, i.e. in (3.113) $k > 1$ and $k' < -1$. The function φ_n satisfies then the following equation:

$$[[\varphi_n, f_n], f_n] = 0. \quad (3.116)$$

Any generalized symmetry (3.113) has a trivial solution: $\varphi_n = g_n$. The Master Symmetry corresponds to a *nontrivial* solution of (3.116).

A practical way of computing Master Symmetries is as follows: Suppose that we are given a Volterra-like equation (2.6). Suppose that there exists a Master Symmetry of the form:

$$u_{n,\tau} = n f_n\left(u_{n+1}, u_n, u_{n-1}\right) \quad (3.117)$$

where f_n is the right hand side of (2.6). Then assume that f_n depends on some constants, say k_i, $i = 1, \ldots, K$, let us replace such constants by functions of the master symmetry "time" τ:

$$k_i \to \kappa_i = \kappa_i\left(\tau\right), \quad i = 1, \ldots, M. \quad (3.118)$$

We can impose the conditions (3.115) and (3.116) so that (3.117) is actually a Master Symmetry. When imposing (3.115) due to the definition of the total derivative D_τ (3.109) we get a set of first order differential equations for the κ_i functions with the initial conditions given by the original constants:

$$\kappa_i'(\tau) = G_i(\kappa_1(\tau), \dots, \kappa_M(\tau)), \quad i = 1, \dots, M, \tag{3.119a}$$
$$\kappa_i(0) = k_i. \tag{3.119b}$$

Then we can derive the symmetries for the original equation (2.6) at any order from the master symmetry (3.117) by putting $\tau = 0$ in the resulting symmetry.

Let us construct the Master Symmetries of the non-autonomous YdKN equation (3.99). Then we can proceed with the method we discussed above. Using the fact that b_k, c_k and e_k are two periodic:

$$b_k = b + (-1)^k \beta, \quad c_k = c + (-1)^k \gamma, \quad e_k = e + (-1)^k \varepsilon. \tag{3.120}$$

and substituting the coefficients a, b, c, γ, d, e, ε and f with function of τ we obtain the following expression for (3.100):

$$\begin{aligned} A_k(q_k,\tau) &= a(\tau)q_k^2 + 2\left[b(\tau) + (-1)^k\beta(\tau)\right] q_k && (3.121a)\\ &\quad + c(\tau) + (-1)^k\gamma(\tau), \\ B_k(q_k,\tau) &= \left[b(\tau) - (-1)^k\beta(\tau)\right] q_k^2 + d(\tau)q_k && (3.121b)\\ &\quad + e(\tau) - (-1)^k\varepsilon(\tau), \\ C_k(q_k,\tau) &= \left[c(\tau) - (-1)^k\gamma(\tau)\right] q_k^2 && (3.121c)\\ &\quad + 2\left[e(\tau) + (-1)^k\varepsilon(\tau)\right] q_k + f(\tau). \end{aligned}$$

From (3.121) we build up the τ-dependent version of (3.99):

$$\frac{dq_k}{dt} = \frac{A_k(q_k,\tau)q_{k+1}q_{k-1} + B_k(q_k,\tau)(q_{k+1}+q_{k-1}) + C_k(q_k,\tau)}{q_{k+1}-q_{k-1}}. \tag{3.122}$$

We make the *ansatz* (3.117) for the Master Symmetry:

$$\frac{dq_k}{d\tau} = n\frac{A_k(q_k,\tau)q_{k+1}q_{k-1} + B_k(q_k,\tau)(q_{k+1}+q_{k-1}) + C_k(q_k,\tau)}{q_{k+1}-q_{k-1}}. \tag{3.123}$$

Commuting the flows of D_t and D_τ we must obtain a five point symmetry, $g_n^{(1)}$ according to (3.115). This obtained function has a purely three point part which is of the same form as (3.122), but with different coefficients. Annihilating this part we obtain some equations for the coefficients a, b, β, c, γ, d, e, ε, f:

$$\frac{da}{d\tau}(\tau) = a(\tau)\,d(\tau) + 2\beta^2(\tau) - 2b^2(\tau), \tag{3.124a}$$

$$\frac{db}{d\tau}(\tau) = -b(\tau)c(\tau) + \beta(\tau)\gamma(\tau) + a(\tau)e(\tau), \tag{3.124b}$$
$$\frac{d\beta}{d\tau}(\tau) = c(\tau)\beta(\tau) - b(\tau)\gamma(\tau) - a(\tau)\varepsilon(\tau), \tag{3.124c}$$
$$\frac{dc}{d\tau}(\tau) = 2b(\tau)e(\tau) - d(\tau)c(\tau) - 2\beta(\tau)\varepsilon(\tau), \tag{3.124d}$$
$$\frac{d\gamma}{d\tau}(\tau) = 2\beta(\tau)e(\tau) - d(\tau)\gamma(\tau) - 2\varepsilon(\tau)b(\tau), \tag{3.124e}$$
$$\frac{dd}{d\tau}(\tau) = -c^2(\tau) + a(\tau)f(\tau) + \gamma^2(\tau), \tag{3.124f}$$
$$\frac{de}{d\tau}(\tau) = -c(\tau)e(\tau) - \gamma(\tau)\varepsilon(\tau) + b(\tau)f(\tau), \tag{3.124g}$$
$$\frac{d\varepsilon}{d\tau}(\tau) = \gamma(\tau)e(\tau) - \beta(\tau)f(\tau) + c(\tau)\varepsilon(\tau), \tag{3.124h}$$
$$\frac{df}{d\tau}(\tau) = f(\tau)d(\tau) - 2e^2(\tau) + 2\varepsilon^2(\tau). \tag{3.124i}$$

If the coefficients satisfy the system (3.124) then it is easy to show that the obtained $g_n^{(1)}$ is a generalized symmetry depending on five-points. We remark that the system (3.124) in its generality it is impossible to solve, but since the right hand is polynomial we are ensured by Cauchy's theorem [162] that such solution always exists in a neighbourhood of $\tau = 0$.

The solutions with the initial conditions given by Table 3.1 will then yield explicit form of the Master Symmetries in *all the relevant sub-cases*. By using the master symmetry constructed above we can construct infinite hierarchies of generalized symmetries of the H^4 and H^6 equations in both directions. Furthermore since for every H^4 and H^6 equation we have $a = b = \beta = 0$ we can in fact use the simpler system where $a(\tau) = b(\tau) = \beta(\tau) = 0$:

$$\frac{dc}{d\tau}(\tau) = -d(\tau)c(\tau), \tag{3.125a}$$
$$\frac{d\gamma}{d\tau}(\tau) = -d(\tau)\gamma(\tau), \tag{3.125b}$$
$$\frac{dd}{d\tau}(\tau) = -c^2(\tau) + \gamma^2(\tau), \tag{3.125c}$$
$$\frac{de}{d\tau}(\tau) = -c(\tau)e(\tau) - \gamma(\tau)\varepsilon(\tau), \tag{3.125d}$$
$$\frac{d\varepsilon}{d\tau}(\tau) = \gamma(\tau)e(\tau) + c(\tau)\varepsilon(\tau), \tag{3.125e}$$
$$\frac{df}{d\tau}(\tau) = f(\tau)d(\tau) - 2e^2(\tau) + 2\varepsilon^2(\tau). \tag{3.125f}$$

As an example in Table 3.2 we list the form of the τ-dependent coefficients in the case of the trapezoidal H^4 equations (1.91). The remaining cases can be computed analogously.

Eq.	Dir.	$c(\tau)$	$\gamma(\tau)$	$d(\tau)$
${}_{t}H_{1}^{\varepsilon}$	n	$\frac{-1}{\tau+1}$	0	$\frac{1}{\tau+1}$
	m	$\frac{\varepsilon^{2}}{2}$	$\frac{\varepsilon^{2}}{2}$	0
${}_{t}H_{2}^{\varepsilon}$	n	$\frac{-1}{\tau+1}$	0	$\frac{1}{\tau+1}$
	m	$-\frac{\varepsilon}{2}$	$-\frac{\varepsilon}{2}$	0
${}_{t}H_{3}^{\varepsilon}$	n	$\frac{\alpha_{2}^{2}\left(\alpha_{2}^{2}-1\right)e^{-\frac{1}{2}\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}}{e^{-\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}-\alpha_{2}^{2}}$	0	$\frac{1}{2}\frac{\alpha_{2}\left(1-\alpha_{2}^{2}\right)\left[e^{-\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}+\alpha_{2}^{2}\right]}{+e^{-\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}-{\alpha_{2}}^{2}}$
	m	$-\frac{\varepsilon^{2}e^{-\frac{1}{2}\alpha_{3}\tau}}{2}$	$-\frac{\varepsilon^{2}e^{-\frac{1}{2}\alpha_{3}\tau}}{2}$	$\frac{\alpha_{3}}{2}$

Eq.	Dir.	$e(\tau)$	$\eta(\tau)$	$f(\tau)$
${}_{t}H_{1}^{\varepsilon}$	n	0	0	$-\varepsilon^{2}\alpha_{2}^{2}F_{m}^{(+)}(\tau+1)$
	m	0	0	2
${}_{t}H_{2}^{\varepsilon}$	n	$\varepsilon{\alpha_{2}}^{2}F_{m}^{(+)}(\tau+1)$	0	$-\alpha_{2}^{2}\left[-1+{\alpha_{2}}^{2}F_{m}^{(+)}(\tau+1)^{2}\varepsilon^{2}+4F_{m}^{(+)}(\alpha_{2}+2\alpha_{3})\varepsilon\right](\tau+1)$
	m	$\frac{1}{2}+\frac{1}{4}(\tau-2\alpha_{2}-2\alpha_{3})\varepsilon$	$-\frac{1}{4}(\tau-2\alpha_{2}-2\alpha_{3})\varepsilon$	$-\frac{1}{4}(\tau-2\alpha_{2}-2\alpha_{3})^{2}\varepsilon-\alpha_{3}-\frac{1}{2}\tau$
${}_{t}H_{3}^{\varepsilon}$	n	0	0	$\frac{\varepsilon^{2}\delta^{2}F_{m}^{(+)}\left(\alpha_{2}^{2}-1\right)\left(e^{-\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}-{\alpha_{2}}^{2}\right)}{e^{-1/2\tau\alpha_{2}\left(\alpha_{2}^{2}-1\right)}}$
	m	0	0	$-\alpha_{3}^{2}\delta^{2}e^{\frac{1}{2}\alpha_{3}\tau}$

Table 3.2: The value of the coefficients of the Master Symmetry (3.123) obtained by solving the system (3.125) in the case of the trapezoidal H^{4} equations (1.91).

Now it is a known fact that Master Symmetries can exists also for linearizable (C-integrable) equations, e.g. the Burgers equation [103], as infinite hierarchies of generalized symmetries exists both for C and S integrable equations. So to end this Section we apply the Algebraic Entropy test, as discussed in Chapter 2, which shows heuristically that the relevant sub-cases of the non-autonomous YdKN equation (3.99) have quadratic growth and therefore are genuine integrable equations.

We look for the sequence of degrees of the iterate map for the non-autonomous YdKN equation (3.99) and in its particular cases found in the previous section. We find for all the cases, except for ${}_{r}H_{1}^{\varepsilon}$, i.e. for the symmetries (3.102, 3.104) of ${}_{t}H_{1}^{\varepsilon}$ and for ${}_{\iota}D_{2}$ equations, the following values:

$$1, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157 \ldots \tag{3.126}$$

This sequence has the following generating function:

$$g(z) = \frac{1 - 2z + 3z^2}{(1 - z)^3}, \tag{3.127}$$

which gives the quadratic fit for the sequence (3.126):

$$d_l = l(l - 1) + 1. \tag{3.128}$$

Therefore the Algebraic Entropy is zero.

For the three-point generalized symmetry in the n direction (E.1a) of the equation ${}_{r}H_{1}^{\varepsilon}$ we have the somehow different situation that the sequence growth is different depending if we consider the even or odd values of the m variable:

$$m = 2k \quad 1, 1, 3, 7, 10, 17, 23, 33, 42, 55, 67, 83, 98, 117 \ldots \tag{3.129a}$$

$$m = 2k + 1 \quad 1, 1, 3, 4, 9, 13, 21, 28, 39, 49, 63, 76, 93, 109 \ldots. \tag{3.129b}$$

These sequences have the following generating functions and asymptotic fits:

$$m = 2k, \quad \begin{aligned} g(z) &= \frac{2z^5 - 3z^4 + 3z^3 + z^2 - z + 1}{(1 - z)^3 (z + 1)}, \\ d_l &= \frac{3}{4} l^2 - l - \frac{5(-1)^l - 21}{8}, \end{aligned} \tag{3.130a}$$

$$m = 2k + 1, \quad \begin{aligned} g(z) &= \frac{(z^2 + z + 1)(2z^2 - 2z + 1)}{(1 - z)^3 (z + 1)}, \\ d_l &= \frac{3}{4} l^2 - \frac{3}{2} l - \frac{5(-1)^l - 19}{8}. \end{aligned} \tag{3.130b}$$

The symmetry in the m direction (E.2b) of the equation ${}_{r}H_{1}^{\varepsilon}$ has the same behavior, obtained by exchanging m with n in formulæ (3.129-3.130).

The three-point generalized symmetry of the equation ${}_tH_1^\varepsilon$ in the n direction has almost the same growth as that obtained for m odd (3.129b, 3.130b) with fit given by:

$$d_l = \frac{3}{4}l^2 - \frac{5}{4}l + (-1)^l \frac{l}{4} + \frac{(-1)^l + 15}{8}. \tag{3.131}$$

Notice the presence of a highly oscillatory term $l(-1)^l$, which, at our knowledge it is observed for the first time. For m even we have the same growth as (3.126). The three-point generalized symmetry of the equation ${}_tH_1^\varepsilon$ in the m direction has the same growth as the even one of ${}_tH_1^\varepsilon$ (3.129a, 3.130a).

For the three-point generalized symmetries (3.92) of the ${}_iD_2$ equation we have different growth according to the even or odd values of the m and n variables. These sequences are slightly lower than in the case of equations H_1^ε, however always corresponding to a quadratic asymptotic fit.

This shows that the whole family of the non autonomous YdKN is integrable (S-integrable) according to the Algebraic Entropy test and they should not be linearizable (C-integrable) even if the equations of which they are symmetries are linearizable.

We conclude this Section by giving an example on how the generalized symmetries criterion of integrability [169] and the Algebraic Entropy give the same result on a non-integrable equation, i.e. an example of how two definitions of integrability coincide. As the symmetries of the ${}_tH_1^\varepsilon$ equation depend on arbitrary functions, not all of them will produce an integrable flux. Let us consider the case of the flux (3.89) when $\varepsilon = 0$, $B_n(\xi) = -1/\xi$ and $\alpha = 1$. This corresponds to the following symmetry:

$$\widehat{X}_n^P = \frac{u_{n+1,m}u_{n-1,m} - u_{n,m}^2}{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}} \partial_{u_{n,m}}. \tag{3.132}$$

Following [169], *necessary condition* for the flux of (3.132) $\frac{du_{n,m}}{dt} = \widehat{X}_n^P u_{n,m}$ to be integrable is that, given

$$p_1 = \log \frac{\partial f_n}{\partial u_{n+1}} = 2 \log \left(\frac{u_{n,m} - u_{n-1,m}}{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}} \right) \tag{3.133}$$

we must have

$$\begin{aligned}
\frac{dp_1}{dt} = & \frac{-2u_{n,m}^2 + 4u_{n,m}u_{n+1,m} - 2u_{n+1,m}^2}{(u_{n+2,m} - 2u_{n+1,m} + u_{n,m})(-u_{n+1,m} + 2u_{n,m} - u_{n-1,m})} \\
& - \frac{2(u_{n-1,m} - u_{n+1,m})(-u_{n+1,m} + u_{n,m})}{(u_{n+1,m} - 2u_{n,m} + u_{n-1,m})^2} \\
& + \frac{2u_{n,m}^2 - 2u_{n,m}u_{n+1,m} - 2u_{n-1,m}u_{n,m} + 2u_{n-1,m}u_{n+1,m}}{(u_{n,m} - 2u_{n-1,m} + u_{n-2,m})(-u_{n+1,m} + 2u_{n,m} - u_{n-1,m})} \\
& = (T-1)g_n
\end{aligned}$$

(3.134)

for any function g_n defined on a finite portion of the lattice, for example such that $g_n = g_n(u_{n+1,m}, u_{n,m}, u_{n-1,m}, u_{n-2,m})$. We search the function g_n using the *partial sum method* [169] and we find an obstruction at the third passage. Then the function g_n does not exists and therefore we conclude that the flux of (3.132) is a non integrable differential-difference equation.

Using the algebraic entropy test on the flux of (3.132) we find the following values for the degrees of the iterates:

$$1, 1, 3, 9, 27, 81, 273, 729 \dots \tag{3.135}$$

which gives us the following generating function:

$$g(s) = \frac{1-2s}{1-3s}, \tag{3.136}$$

and the entropy is clearly non-vanishing $\eta = \log 3$. So the non-integrability result obtained by the Algebraic Entropy test agrees with those obtained by applying the formal generalized symmetry method.

3.6 A NON-AUTONOMOUS GENERALIZATION OF THE Q_V EQUATION

Up to now we constructed the three-point generalized symmetries of the trapezoidal H^4 equations (1.91) and of the H^6 equation (1.93). We then showed that these three-point generalized symmetries are related to some sub-cases of the non-autonomous YdKN equation [105]. Taking into account the results about the rhombic H^4 equations in [166], we were able to produce Table 3.1 where it shown that the three-point generalized symmetries of any quad equation coming from the Boll's classification [20–22] is in the form of the non-autonomous YdKN equation [105]. We then showed how, using known methods, we can construct the Master Symmetries of these equations, generating a hierarchy of equations and that according to the Algebraic Entropy test these are genuine integrable equations since their growth is quadratic.

We finally note that, as was proven in [102] for the YdKN (3.97), no equation belonging to the Boll classification has a generalized symmetry which corresponds to the general non-autonomous YdKN equation (3.99). In all the cases of the Boll classification one has $a = b_k = 0$ as is displayed by Table 3.1.

In [166] it was shown that the Q_V equation, introduced in [156]:

$$
\begin{aligned}
Q_V = & a_1 u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\
& + a_{2,0} \left(u_{n,m} u_{n,m+1} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \right. \\
& \quad \left. + u_{n,m} u_{n+1,m} u_{n+1,m+1} + u_{n,m} u_{n+1,m} u_{n,m+1} \right) \\
& + a_{3,0} \left(u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1} \right) \\
& + a_{4,0} \left(u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1} \right) \\
& + a_{5,0} \left(u_{n+1,m} u_{n+1,m+1} + u_{n,m} u_{n,m+1} \right) \\
& + a_{6,0} \left(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1} \right) \\
& + a_7 = 0
\end{aligned}
\tag{3.137}
$$

admits a symmetry in the direction n

$$
\frac{du_{n,m}}{dt} = \frac{h_n}{u_{n+1,m} - u_{n-1,m}} - \frac{1}{2} \partial_{u_{n+1,m}} h_n. \tag{3.138}
$$

where:

$$
\begin{aligned}
h_n(u_{n,m}, u_{n+1,m}) = & Q_V \partial_{u_{n,m+1}} \partial_{u_{n+1,m+1}} Q_V \\
& - \left(\partial_{u_{n,m+1}} Q_V \right) \left(\partial_{u_{n+1,m+1}} Q_V \right)
\end{aligned}
\tag{3.139}
$$

and a symmetry in the direction m

$$
\frac{du_{n,m}}{dt} = \frac{h_m}{u_{n,m+1} - u_{n,m-1}} - \frac{1}{2} \partial_{u_{n,m+1}} h_m. \tag{3.140}
$$

where:

$$
\begin{aligned}
h_m(u_{n,m}, u_{n,m+1}) = & Q_V \partial_{u_{n+1,m}} \partial_{u_{n+1,m+1}} Q_V \\
& - \left(\partial_{u_{n+1,m}} Q_V \right) \left(\partial_{u_{n+1,m+1}} Q_V \right)
\end{aligned}
\tag{3.141}
$$

which are of the same form as the YdKN (3.97).

The connection formulæ i.e. the relations between the coefficient of Q_V and the coefficients of its YdKN generalized symmetry (3.97) in the n direction YdKN are:

$$a = a_{3,0} a_1 - a_{2,0}^2, \tag{3.142a}$$

$$b = \frac{1}{2} [a_{2,0}(a_{3,0} - a_{5,0} - a_{4,0}) + a_{6,0} a_1], \tag{3.142b}$$

$$c = a_{2,0} a_{6,0} - a_{4,0} a_{5,0}, \tag{3.142c}$$

$$d = \frac{1}{2} [a_{3,0}^2 - a_{4,0}^2 - a_{5,0}^2 + a_1 a_7], \tag{3.142d}$$

$$e = \frac{1}{2} [a_{6,0}(a_{3,0} - a_{4,0} - a_{5,0}) + a_{2,0} a_7], \tag{3.142e}$$

$$f = a_{3,0} a_7 - a_{6,0}^2. \tag{3.142f}$$

The connection formulæ between the coefficient of Q_V and the m direction YdKN (3.97) are:

$$a = a_{5,0} a_1 - a_{2,0}^2, \tag{3.143a}$$

$$b = \frac{1}{2}[a_{2,0}(a_{5,0} - a_{3,0} - a_{4,0}) + a_{6,0}a_1], \tag{3.143b}$$

$$c = a_{2,0}a_{6,0} - a_{4,0}a_{3,0}, \tag{3.143c}$$

$$d = \frac{1}{2}[a_{5,0}^2 - a_{4,0}^2 - a_{3,0}^2 + a_1 a_7], \tag{3.143d}$$

$$e = \frac{1}{2}[a_{6,0}(a_{5,0} - a_{4,0} - a_{3,0}) + a_{2,0}a_7], \tag{3.143e}$$

$$f = a_{5,0}a_7 - a_{6,0}^2. \tag{3.143f}$$

The connection formulæ (3.142,3.143) can be seen as a set of coupled nonlinear algebraic equations between the seven parameters a_1, $a_{2,0}$, $a_{3,0}$, $a_{4,0}$, $a_{5,0}$, $a_{6,0}$ and a_7 of the Q_V (3.137) and the six ones a, b, c, d, e and f of the YdKN equation (3.104). This tells us that the YdKN equation (3.104) with coefficients given by (3.142,3.143) is a three-point generalized symmetry of the Q_V equation (3.137). If a solution of (3.142,3.143) exists, i.e. one is able to express the a_1, $a_{2,0}$, $a_{3,0}$, $a_{4,0}$, $a_{5,0}$, $a_{6,0}$ and a_7 in term of a, b, c, d and f, then the Q_V equation (3.137), maybe after a reparametrization, turns out to be a Bäcklund transformation of the YdKN [6, 99] as explained in Section 3.2. We furthermore remark that in general a and b will be non-zero, so that the three-point generalized symmetries of the Q_V equation (3.137) belong to the class of the general YdKN equation (3.97).

From the results obtained in Section 3.4 and the above remarks we are lead to conjecture the existence a non autonomous generalization of the Q_V equation. To obtain such generalizations we first have to recall how the Q_V equation (3.137) was obtained in [156]. In [156] was showed that the Q_V equation (3.137) is the most general multi-linear equation on a quad graph possessing Klein discrete symmetries, i.e. such that:

$$\begin{aligned} &Q\left(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}\right) = \\ &\quad \tau Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right), \\ &Q\left(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n,m+1}\right) = \\ &\quad \tau' Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right), \end{aligned} \tag{3.144}$$

where $\tau, \tau' = \pm 1$.

We have many possible ways of searching for a generalization of the Q_V equation (3.137). Since we want to generalize the Q_V equation (3.137) to a non-autonomous equation let us consider the most general multi-linear equation in the lattice variables with two-periodic coefficients:

$$\begin{aligned} & p_1 u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ & + p_2 u_{n,m} u_{n,m+1} u_{n+1,m+1} + p_3 u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ & + p_4 u_{n,m} u_{n+1,m} u_{n+1,m+1} + p_5 u_{n,m} u_{n+1,m} u_{n,m+1} \\ & + p_6 u_{n,m} u_{n+1,m} + p_7 u_{n,m+1} u_{n+1,m+1} + p_8 u_{n,m} u_{n+1,m+1} \\ & + p_9 u_{n+1,m} u_{n,m+1} + p_{10} u_{n+1,m} u_{n+1,m+1} + p_{11} u_{n,m} u_{n,m+1} \\ & + p_{12} u_{n,m} + p_{13} u_{n+1,m} + p_{14} u_{n,m+1} + p_{15} u_{n+1,m+1} + p_{16} = 0 \end{aligned} \tag{3.145}$$

i.e. such that the coefficients p_i have the following expression:

$$p_i = p_{i,0} + p_{i,1}(-1)^n + p_{i,2}(-1)^m + p_{i,3}(-1)^{n+m}, \quad i = 1, \dots, 16. \tag{3.146}$$

We will denote the left hand side of (3.145) by $Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}, (-1)^n, (-1)^m\right)$.

A possibility is to require that the function $Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}, (-1)^n, (-1)^m\right)$ to respect a strict discrete Klein symmetry (3.144) and to require that the connection forumulæ provide a non-autonomous YdKN equation (3.99). The requirement of the Klein symmetries gives us:

$$\begin{gathered} p_1 \equiv a_1, \quad p_2 = p_3 = p_4 = p_5 \equiv a_2, \\ p_6 = p_7 \equiv a_3, \quad p_8 = p_9 \equiv a_4, \quad p_{10} = p_{11} \equiv a_5, \\ p_{12} = p_{13} = p_{14} = p_{15} \equiv a_6, \quad p_{16} = a_7 \end{gathered} \tag{3.147}$$

where the a_i are still two-periodic functions of the lattice variables. Choosing the coefficients for example as

$$\begin{gathered} a_1 = 1 + (-1)^n, \quad a_2 = (-1)^n, \\ a_3 = -1 + (-1)^n, \quad a_5 = (-1)^n, \quad a_4 = 1 + 2(-1)^n, \\ a_6 = 1 + (-1)^n, \quad a_7 = 4 + 2(-1)^n, \end{gathered} \tag{3.148}$$

the we have such properties, but this is not the only possible choice. In this case, performing the Algebraic Entropy test the equation turns out to be integrable. Its generalized symmetries, however, are not necessarily in the form of a non-autonomous YdKN equation (3.99). A different non-autonomous choice of the coefficients of (3.145), such that (3.142) is satisfied for the coefficients of the non-autonomous YdKN (3.99), gives, by the algebraic entropy test, a non integrable equation.

A further possibility is to generalize the original Klein symmetry by considering the discrete symmetries possessed by all the equations belonging to the Boll's classification. It is easy to see that all the equations belonging to the Boll's classification possess the following discrete symmetry:

$$\begin{aligned} &Q\left(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}; (-1)^{n}, (-1)^{m}\right) = \\ &\quad \tau Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; -(-1)^{n}, (-1)^{m}\right), \\ &Q\left(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n+1,m}; (-1)^{n}, (-1)^{m}\right) = \\ &\quad \tau' Q\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^{n}, -(-1)^{m}\right). \end{aligned} \tag{3.149}$$

when $\tau = \tau' = 1$. Furthermore it is possible to show that if the quad equation Q is autonomous then the condition (3.149) reduces to the usual Klein discrete symmetry (3.144). Therefore we will say that if a non-autonomous quad equation Q of the form (3.145) satisfies the discrete symmetries (3.149) that Q admits a *non-autonomous discrete Klein symmetry*.

If we impose the non autonomous Klein symmetry condition (3.149) with $\tau = \tau' = 1$ the 64 coefficients of (3.145) turn out to be related among themselves and we can choose among them 16 independent coefficients. In term of the 16 independent coefficients (3.145) reads:

$$\begin{aligned} &a_1 u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ &+ \left[a_{2,0} - (-1)^{n} a_{2,1} - (-1)^{m} a_{2,2} + (-1)^{n+m} a_{2,3}\right] u_{n,m} u_{n,m+1} u_{n+1,m+1} \\ &+ \left[a_{2,0} + (-1)^{n} a_{2,1} - (-1)^{m} a_{2,2} - (-1)^{n+m} a_{2,3}\right] u_{n+1,m} u_{n,m+1} u_{n+1,m+1} \\ &+ \left[a_{2,0} + (-1)^{n} a_{2,1} + (-1)^{m} a_{2,2} + (-1)^{n+m} a_{2,3}\right] u_{n,m} u_{n+1,m} u_{n+1,m+1} \\ &+ \left[a_{2,0} - (-1)^{n} a_{2,1} + (-1)^{m} a_{2,2} - (-1)^{n+m} a_{2,3}\right] u_{n,m} u_{n+1,m} u_{n,m+1} \\ &+ \left[a_{3,0} - (-1)^{m} a_{3,2}\right] u_{n,m} u_{n+1,m} \\ &+ \left[a_{3,0} + (-1)^{m} a_{3,2}\right] u_{n,m+1} u_{n+1,m+1} \\ &+ \left[a_{4,0} - (-1)^{n+m} a_{4,3}\right] u_{n,m} u_{n+1,m+1} \\ &+ \left[a_{4,0} + (-1)^{n+m} a_{4,3}\right] u_{n+1,m} u_{n,m+1} \\ &+ \left[a_{5,0} - (-1)^{n} a_{5,1}\right] u_{n+1,m} u_{n+1,m+1} \\ &+ \left[a_{5,0} + (-1)^{n} a_{5,1}\right] u_{n,m} u_{n,m+1} \\ &+ \left[a_{6,0} + (-1)^{n} a_{6,1} - (-1)^{m} a_{6,2} - (-1)^{n+m} a_{6,3}\right] u_{n,m} \\ &+ \left[a_{6,0} - (-1)^{n} a_{6,1} - (-1)^{m} a_{6,2} + (-1)^{n+m} a_{6,3}\right] u_{n+1,m} \\ &+ \left[a_{6,0} + (-1)^{n} a_{6,1} + (-1)^{m} a_{6,2} + (-1)^{n+m} a_{6,3}\right] u_{n,m+1} \\ &+ \left[a_{6,0} - (-1)^{n} a_{6,1} + (-1)^{m} a_{6,2} - (-1)^{n+m} a_{6,3}\right] u_{n+1,m+1} \\ &+ a_7 = 0 \end{aligned} \tag{3.150}$$

Upon the substitution $a_{2,1} = a_{2,2} = a_{2,3} = a_{3,2} = a_{4,3} = a_{5,1} = a_{6,1} = a_{6,2} = a_{6,3} = 0$ (3.150) reduces to the Q_V equation (3.137). Therefore we will call (3.150) *the non autonomous* Q_V *equation*.

If we impose the non-autonomous Klein symmetry condition (3.149) with the choice $\tau = 1$ and $\tau' = -1$ we get an expression which reduces to (3.150) by multiplying it by $(-1)^{n}$ and redefining the coefficients. In an analogous manner the two remaining cases $\tau = -1$,

$\tau' = 1$ and $\tau = \tau' = -1$ can be identified with the case $\tau = \tau' = 1$ multiplying by $(-1)^n$ and $(-1)^{n+m}$ respectively and redefining the coefficients. Therefore the only equation belonging to the class of the lattice equation possessing the non autonomous Klein symmetries is just the non-autonomous Q_V equation (3.150).

We note that the non-autonomous Q_V equation contains as particular cases the rhombic H^4 equations, the trapezoidal H^4 equations and the H^6 equations. The explicit identification of the coefficients of such equations is given in Table 3.3.

In order to establish if (3.150) is integrable we use the Algebraic Entropy integrability test as explained in Chapter 2, using the program `ae2d.py`, see Appendix B. Applying it to the non autonomous Q_V equation we find in all directions the following sequence of degrees:

$$1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133 \ldots, \tag{3.151}$$

which is the same as for the autonomous Q_V equation [156]. The generating function for the sequence (3.151) is:

$$g(z) = \frac{1+z^2}{(1-z)^3}, \tag{3.152}$$

which implies that we have the following quadratic fit for the growth:

$$d_k = k(k+1) + 1, \tag{3.153}$$

and thus the Algebraic Entropy is zero. This is a strong indication of the integrability of the non autonomous Q_V equation (3.150).

Using (3.138, 3.139) or (3.140, 3.141) with Q_V substituted by its non-autonomous version we get a version of the non-autonomous YdKN (3.99) however the proof that this is effectively a generalized symmetry of the non-autonomous Q_V encounters serious computational difficulties.

We can prove by a direct computation its validity for the following sub-cases:

- When Qv equation is non autonomous with respect to one direction only, either n or m. All the trapezoidal ${}_tH^4$ equations belong to these two sub-classes;
- For all the H^6 equations, which are non autonomous in both directions.

Its validity for the autonomous Q_V and for all the rhombic ${}_rH^4$ equations was already showed respectively in [165] and [166]. However we cannot prove its validity for the general case (3.150).

Here in the following we compute the connection formulæ for the general non autonomous case (3.150). For the n directional symmetry we have:

$$\begin{aligned} a &= a_1 a_{3,0} - a_{2,0}^2 + a_{2,1}^2 - a_{2,2}^2 + a_{2,3}^2 \\ &\quad - (-1)^m (2a_{2,0}a_{2,2} - 2a_{2,1}a_{2,3} + a_1 a_{3,2}), \end{aligned} \tag{3.154a}$$

$$\begin{aligned}
b =& \frac{1}{2}a_{2,0}(a_{3,0}-a_{5,0}-a_{4,0}) \\
&+\frac{1}{2}(a_1 a_{6,0}+a_{2,2}a_{3,2}-a_{2,3}a_{4,3}-a_{2,1}a_{5,1}) \\
&-\frac{1}{2}(-1)^m a_{2,2}(a_{5,0}+a_{3,0}+a_{4,0}) \\
&+\frac{1}{2}(-1)^m(a_{2,3}a_{5,1}+a_1 a_{6,2}+a_{2,0}a_{3,2}+a_{2,1}a_{4,3}),
\end{aligned} \tag{3.154b}$$

$$\begin{aligned}
\beta =& \frac{1}{2}a_{2,1}(a_{3,0}-a_{4,0}+a_{5,0}) \\
&+\frac{1}{2}(a_{2,3}a_{3,2}-a_{2,2}a_{4,3}+a_{2,0}a_{5,1}-a_1 a_{6,1}) \\
&+\frac{1}{2}(-1)^m a_1 a_{6,3}-a_{2,3}(a_{3,0}+a_{4,0}-a_{5,0}) \\
&-\frac{1}{2}(-1)^m(a_{2,1}a_{3,2}+a_{2,0}a_{4,3}-a_{2,2}a_{5,1}),
\end{aligned} \tag{3.154c}$$

$$\begin{aligned}
c =& a_{2,0}a_{6,0}-a_{4,0}a_{5,0}-a_{2,1}a_{6,1}-a_{2,3}a_{6,3}+a_{2,2}a_{6,2} \\
&-(-1)^m[a_{2,2}a_{6,0}-a_{4,3}a_{5,1}-a_{2,3}a_{6,1}+a_{2,0}a_{6,2}-a_{2,1}a_{6,3}],
\end{aligned} \tag{3.154d}$$

$$\begin{aligned}
\gamma =& a_{4,0}a_{5,1}+a_{2,1}a_{6,0}-a_{2,0}a_{6,1}+a_{2,3}a_{6,2}-a_{2,2}a_{6,3} \\
&+(-1)^m[a_{2,2}a_{6,1}-a_{4,3}a_{5,0}-a_{2,3}a_{6,0}-a_{2,1}a_{6,2}+a_{2,0}a_{6,3}],
\end{aligned} \tag{3.154e}$$

$$\begin{aligned}
d =& \frac{1}{2}(a_{3,0}^2-a_{4,0}^2-a_{5,0}^2+a_1 a_7-a_{3,2}^2+a_{4,3}^2+a_{5,1}^2) \\
&-2(-1)^m(a_{2,2}a_{6,0}+a_{2,3}a_{6,1}+a_{2,0}a_{6,2}+a_{2,1}a_{6,3}],
\end{aligned} \tag{3.154f}$$

$$\begin{aligned}
e =& \frac{1}{2}a_{6,0}(a_{3,0}-a_{4,0}-a_{5,0}) \\
&+\frac{1}{2}(a_{2,0}a_7+a_{5,1}a_{6,1}-a_{3,2}a_{6,2}+a_{4,3}a_{6,3}) \\
&+\frac{1}{2}(-1)^m(a_{3,2}a_{6,0}+a_{4,3}a_{6,1}+a_{5,1}a_{6,3}-a_{2,2}a_7) \\
&-\frac{1}{2}(-1)^m a_{6,2}(a_{3,0}+a_{4,0}+a_{5,0}),
\end{aligned} \tag{3.154g}$$

$$\begin{aligned}
\varepsilon =& \frac{1}{2}a_{6,1}(a_{3,0}-a_{4,0}+a_{5,0}) \\
&-\frac{1}{2}(a_{5,1}a_{6,0}-a_{4,3}a_{6,2}+a_{3,2}a_{6,3}+a_{2,1}a_7) \\
&+\frac{1}{2}(-1)^m[a_{4,3}a_{6,0}+a_{3,2}a_{6,1}-a_{5,1}a_{6,2}+a_{2,3}a_7 \\
&+\frac{1}{2}(-1)^m a_{6,3}(a_{5,0}-a_{3,0}-a_{4,0}),
\end{aligned} \tag{3.154h}$$

$$\begin{aligned}
f =& a_{3,0}a_7-a_{6,0}^2-a_{6,2}^2+a_{6,3}^2+a_{6,1}^2 \\
&-(-1)^m(2a_{6,0}a_{6,2}-2a_{6,1}a_{6,3}-a_{3,2}a_7).
\end{aligned} \tag{3.154i}$$

The non-autonomous Q_V is not symmetric in the exchange of n and m so its symmetries in the m direction are different from those in the n direction and so are the connection formulæ. However, for the sake of simplicity, as they are not essentially different from those presented in equation (3.154), we do not write them here but present them in Appendix F.

Table 3.3: Identification of the coefficients of the non autonomous Q_V equation with those of the Boll's equations. Since $a_1 = a_{2,i} = 0$ for all equations considered in this Table these coefficients absent.

Eq.	$a_{3,0}$	$a_{3,2}$	$a_{4,0}$	$a_{4,3}$	$a_{5,0}$	$a_{5,1}$	$a_{6,0}$	$a_{6,1}$	$a_{6,2}$	$a_{6,3}$	a_7
${}_rH_1^\varepsilon$	1	0	$\frac{1}{2}\varepsilon(\alpha-\beta)$	$\frac{1}{2}\varepsilon(\alpha-\beta)$	-1	0	0	0	0	0	$\beta-\alpha$
${}_rH_2^\varepsilon$	1	0	$2\varepsilon(\beta-\alpha)$	$2\varepsilon(\beta-\alpha)$	-1	0	$-(\alpha-\beta)(\varepsilon\alpha+1+\varepsilon\beta)$	0	0	$\varepsilon\left(\beta^2-\alpha^2\right)$	$-(\alpha-\beta)\left(2\varepsilon\alpha^2+\alpha+2\varepsilon\beta^2+\beta\right)$
${}_rH_3^\varepsilon$	α	0	$\frac{1}{2}\frac{\varepsilon\left(\beta^2-\alpha^2\right)}{\alpha\beta}$	$\frac{1}{2}\frac{\varepsilon\left(\beta^2-\alpha^2\right)}{\alpha\beta}$	$-\beta$	0	0	0	0	0	$\delta\left(\alpha^2-\beta^2\right)$
${}_tH_1^\varepsilon$	$-\frac{1}{2}\alpha_2\varepsilon^2$	$-\frac{1}{2}\alpha_2\varepsilon^2$	-1	0	1	0	0	0	0	0	$-\alpha_2$
${}_tH_2^\varepsilon$	$\varepsilon\alpha_2$	$\varepsilon\alpha_2$	-1	0	1	0	$\frac{1}{2}\alpha_2\left[2+\varepsilon(2\alpha_2+\alpha_3)\right]$	0	$\varepsilon\alpha_2\alpha_3+\frac{1}{2}\varepsilon{\alpha_2}^2$	0	$\alpha_2\left[\alpha_2+2\alpha_3+\varepsilon(\alpha_2+\alpha_3)^2\right]$
${}_tH_3^\varepsilon$	$\frac{1}{2}\frac{\varepsilon^2\left(1-{\alpha_2}^2\right)}{\alpha_3\alpha_2}$	$\frac{1}{2}\frac{\varepsilon^2\left(1-{\alpha_2}^2\right)}{\alpha_3\alpha_2}$	α_2	0	-1	0	0	0	0	0	$\delta^2\alpha_3\left(1-\alpha_2^2\right)$
${}_1D_2$	$\frac{1}{2}\delta_1$	$\frac{1}{2}\delta_1$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2)$	$\frac{1}{4}(\delta_2-\delta_1)$	$-\frac{1}{4}(\delta_1+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1+\delta_2)$	0
${}_2D_2$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\delta_1$	$\frac{1}{2}\delta_1$	0	0	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2+\delta_1\lambda)$	$\frac{1}{4}(\delta_1\lambda-\delta_1+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_1\lambda+\delta_2)$	$-\frac{1}{4}(\delta_1+\delta_1\lambda+\delta_2)$	$-\delta_1\delta_2\lambda$
${}_3D_2$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_2+\delta_1\lambda)$	$\frac{1}{4}(\delta_1+\delta_1\lambda+\delta_2)$	$\frac{1}{2}-\frac{1}{4}(\delta_1-\delta_1\lambda+\delta_2)$	$\frac{1}{4}(\delta_1-\delta_1\lambda-\delta_2)$	$-\delta_1\delta_2\lambda$
D_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0
${}_1D_4$	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_2$	1	0	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	0	0	0	0	δ_3
${}_2D_4$	1	0	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_2$	$\frac{1}{2}\delta_1$	$-\frac{1}{2}\delta_1$	0	0	0	0	δ_3

4

DARBOUX INTEGRABILITY AND GENERAL SOLUTIONS

In this Chapter we will discuss the concept of Darboux integrability and its relation with the Consisency Around the Cube. In particular in Section 4.1 we will introduce a definition of Darboux integrability based on the existence of the *first integrals*. We will state the definition using the analogy with the continuous case. Then we will discuss a practical method for finding first integrals in the case of non-autonomous equations with two periodic coefficients proposed in the original work [71] as a generalization of the method proposed in [55, 56, 72]. In Section 4.2 we prove that the three equations found by J. Hietarinta in [82], which are linearizable [138], solving the problem of the Consistency Around the Cube are actually Darboux integrable equations. This was observed in [69] and it was the starting point of the research on the relation between the Consistency Around the Cube and the Darboux integrability. Then in Section 4.3 we show the main result obtained in [71], i.e. that the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93) are Darboux integrable, by reporting their first integrals. We underline that this result can be thought as a formal proof of the heuristic result obtained in Chapter 2 using the Algebraic Entropy, since as it will be discussed in Section 4.1 Darboux integrable equations are naturally linearizable. Finally in Section 4.4 we show how to use the first integrals obtained in Section 4.1 in order to obtain the general solution of the trapezoidal H^4 (1.91) and of the H^6 equations (1.93). This material is based on the discussion made in [70].

4.1 DARBOUX INTEGRABILITY

A nonlinear hyperbolic partial differential equation (PDE) in two variables

$$u_{xt} = f(x, t, u, u_t, u_x) \tag{4.1}$$

is said to be *Darboux integrable* if it possesses two independent first integrals depending only on derivatives with respect to one variable:

$$\begin{aligned} T &= T(x, t, u, u_t, \dots, u_{nt}) \quad \text{s.t} \left.\frac{dT}{dx}\right|_{u_{xt}=f} \equiv 0, \\ X &= X(x, t, u, u_x, \dots, u_{mx}) \quad \text{s.t} \left.\frac{dX}{dt}\right|_{u_{xt}=f} \equiv 0, \end{aligned} \tag{4.2}$$

where $u_{kt} = \partial^k u/\partial t^k$ and $u_{kx} = \partial^k u/\partial x^k$ for every $k \in \mathbb{N}$. The method is based on the linear theory developed by Euler and Laplace [38, 96] extended to the nonlinear case in the 19th and early 20th century [29, 30, 58, 59, 154]. The method was then used at the end of the 20th century mainly by Russian mathematicians as a source of new exactly solvable PDEs in two variable [148, 174–179].

We note that there exists an alternative definition of Darboux integrability. This definition is based on the *stabilization to zero* of the *Laplace chain* of the linearized equation. However it can be proved that the two definitions are equivalent [8–10, 179].

The most famous Darboux integrable equation is the Liouville equation [113]:

$$u_{xt} = e^u \tag{4.3}$$

which possesses the first integrals:

$$X = u_{xx} - \frac{1}{2}u_x^2, \quad T = u_{tt} - \frac{1}{2}u_t^2. \tag{4.4}$$

We remark that the first integrals (4.4) defines two Riccati equations.

In the discrete setting Darboux integrability was introduced in [5] and used to obtain a discrete analogue of the Liouville equation (4.3), namely the Adler-Startsev discretization of the Liouville equation:

$$u_{n,m}u_{n+1,m+1}\left(1+\frac{1}{u_{n+1,m}}\right)\left(1+\frac{1}{u_{n,m+1}}\right) = 1. \tag{4.5}$$

Similarly as in the continuous case we shall say that an equation on the quad graph, possibly non-autonomous, i.e.:

$$Q_{n,m}\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right) = 0, \tag{4.6}$$

is *Darboux integrable* if there exist two independent first integrals, one containing only shifts in the first direction and the other containing only shift in the second direction such that:

$$(T_n - \mathrm{Id})W_2 = 0, \tag{4.7a}$$
$$(T_m - \mathrm{Id})W_1 = 0, \tag{4.7b}$$

where

$$W_1 = W_{1,n,m}(u_{n+l_1,m}, u_{n+l_1+1,m}, \dots, u_{n+k_1,m}), \tag{4.8a}$$
$$W_2 = W_{2,n,m}(u_{n,m+l_2}, u_{n,m+l_2+1}, \dots, u_{n,m+k_2}), \tag{4.8b}$$

holds true on the solution of (4.6). Here l_1, l_2, k_1, k_2 are integers, such that $l_1 < k_1$, $l_2 < k_2$, and T_n, T_m are the shifts operators in the first and second direction, respectively: $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$. Finally Id denotes the identity operator $\mathrm{Id}h_{n,m} = h_{n,m}$.

We notice that the existence of first integrals implies that the following two transformations:

$$u_{n,m} \to \tilde{u}_{n,m} = W_{1,n,m}, \tag{4.9a}$$
$$u_{n,m} \to \hat{u}_{n,m} = W_{2,n,m} \tag{4.9b}$$

bring the quad-equation (4.6) into *trivial linear equations* given by (4.7) [5], namely:

$$\tilde{u}_{n,m+1} - \tilde{u}_{n,m} = 0, \tag{4.10a}$$
$$\hat{u}_{n+1,m} - \hat{u}_{n,m} = 0. \tag{4.10b}$$

Therefore any Darboux integrable equation is linearizable in two different ways. This is the relationship between the Darboux integrability and linearization.

The transformations (4.9) along with the relations (4.10) imply the following relations:

$$W_1 = \xi_n, \quad W_2 = \lambda_m, \tag{4.11}$$

where ξ_n and λ_m are arbitrary functions of the lattice variables n and m respectively. The relations (4.11) can be seen as *ordinary difference equations* which must be satisfied by any solution $u_{n,m}$ of (4.6). The transformations (4.9) and the ordinary difference equations (4.11) may be quite complicated. For this reason we define the *order* of the first integral to be the difference equation obtained from it using (4.11).

Example 4.1.1 (Discrete Liouville equation (4.5)). In the case of the discrete Liouville equation (4.5) the first integrals presented in [5] are:

$$W_1 = \left[1 + \frac{u_{n,m}\left(1 + u_{n-1,m}\right)}{u_{n-1,m}}\right]\left[1 + \frac{u_{n,m}}{u_{n+1,m}\left(1 + u_{n,m}\right)}\right], \tag{4.12a}$$

$$W_1 = \left[1 + \frac{u_{n,m}\left(1 + u_{n,m-1}\right)}{u_{n,m-1}}\right]\left[1 + \frac{u_{n,m}}{u_{n,m+1}\left(1 + u_{n,m}\right)}\right]. \tag{4.12b}$$

These two first integrals are both two-points, second order first integrals.

From its introduction in [5], various papers were devoted to the study of Darboux integrability for quad-equations [55, 56, 72, 74, 149]. In particular in [55, 56, 72] were developed computational methods to compute the first integrals. In [72] was presented a method to compute the first integrals with fixed l_i, k_i of a given autonomous equation. Then in [55] this method was slightly modified and applied to autonomous equations with non-autonomous first integrals. Finally in [56] it was applied to equations with two-periodic coefficients. For the rest of this Section we present a further modification of this method which allows to treat in a simpler way non-autonomous equations with two periodic coefficients. This new modification was first presented in [71].

If we consider the operator

$$Y_{-1} = T_m \frac{\partial}{\partial u_{n,m-1}} T_m^{-1} \tag{4.13}$$

and apply it to the first integral in the n direction (4.7b) we obtain:

$$Y_{-1} W_1 \equiv 0. \tag{4.14}$$

The application of the operator Y_{-1} is to be understood in the following sense: first we must apply T_m^{-1} and then we should express, using equation (4.6), $u_{n+i,m-1}$ in terms of the variables $u_{n+j,m}$ and $u_{n,m-1}$ which for this problem are independent variables. Then we can differentiate with respect to $u_{n,m-1}$ and safely apply T_m [55].

Taking in (4.14) the coefficients at powers of $u_{n,m+1}$, we obtain a system of PDEs for W_1. If this is sufficient to determine W_1 up to arbitrary functions of a single variable, then we are done, otherwise we can add similar equations by considering the "higher-order" operators

$$Y_{-k} = T_m^k \frac{\partial}{\partial u_{n,m-1}} T_m^{-k}, \quad k \in \mathbb{N}, \tag{4.15}$$

which annihilate the difference consequence of (4.7b) given by $T_m^k W_1 = W_1$:

$$Y_{-k} W_1 \equiv 0, \quad k \in \mathbb{N}, \tag{4.16}$$

with the same computational prescriptions as above. We can add equations until we find a non-constant function[1] W_1 which depends on a *single* combination of the variables $u_{n,m+j_1}, \dots, u_{n,m+k_1}$.

If we find a non-constant solution of the equations generated by (4.13) and possibly (4.15) we can insert such solution into (4.7b). In fact given a non-constant W_1 which satisfies (4.16), but does not satisfy (4.7b), we can always construct a first integral out of it. In order to construct a bona fide first integral in [55] it was proposed to look if the first integral and its shifted version can be related through some Möbius transformation:

$$T_m W_1 = \frac{a W_1 + b}{c W_1 + d}. \tag{4.17}$$

This *ansatz* was applied with success on all the examples therein contained.

In the same way the m-integral W_2 can be found by considering the operators

$$Z_{-k} = T_n^k \frac{\partial}{\partial u_{n-1,m}} T_n^{-k}, \quad k \in \mathbb{N} \tag{4.18}$$

1 Obviously constant functions are trivial first integrals.

which are such that:

$$Z_{-k}W_2 \equiv 0, \quad k \in \mathbb{N}. \tag{4.19}$$

In the case of non-autonomous equations with two-periodic coefficients we can assume the decomposition:

$$\begin{aligned} W_i = &F_n^{(+)}F_m^{(+)}W_i^{(+,+)} + F_n^{(-)}F_m^{(+)}W_i^{(-,+)} \\ &+ F_n^{(+)}F_m^{(-)}W_i^{(+,-)} + F_n^{(-)}F_m^{(-)}W_i^{(-,-)} \end{aligned} \tag{4.20}$$

with:

$$F_k^{(\pm)} = \frac{1 \pm (-1)^k}{2} \tag{4.21}$$

and derive from (4.16,4.19) a set of equations for the function $W_i^{(\pm,\pm)}$ by considering the even/odd points on the lattice. The final form of the function W_i will be then fixed explicitly by substituting in (4.7) and separating again.

We note that, when successful, the above procedure gives first integrals depending on arbitrary functions. However this fact has to be understood as a restatement of the trivial property that any function of a first integral is again a first integral. So, in general, one does not need first integrals depending on arbitrary functions. Therefore we can take as the first integral simply a linear function of the arguments of the functions we obtain from a successful application of the above procedure. If there is more than one arbitrary function in the same direction then it sufficient for our purposes to consider linear combination of the arguments of such functions. The reader can find in Appendix G an example carried out in the details.

4.2 DARBOUX INTEGRABILITY AND CONSISTENCY AROUND THE CUBE

Before discussing the Darboux integrability of the trapezoidal H^4 equations (1.91) and of the H^6 equations (1.93) in this Section we would like to underline that previously known linearizable quad equations possessing the Consistency Around the Cube are Darboux integrable by showing their first integrals. In this Section we will consider the three quad equations found by J. Hietarinta in [82] given in formula (1.28). We recall that these equations were found to be linearizable in [138] and were found to be Darboux integrable in [69].

Indeed let us consider the J_2 equation as given by (1.28a) (which is indeed the same as the D_1 equation) is linear and Darboux integrable, being its first integrals given by:

$$W_1 = (-1)^m (u_{n+1,m} + u_{n,m}), \tag{4.22a}$$

$$W_2 = (-1)^n (u_{n,m+1} + u_{n,m}). \tag{4.22b}$$

These two first integrals are first-order, two-point, meaning that the equation itself a first integral.

In the same way the J_3 equation (1.28b) is linearizable and Darboux integrable being its first integrals given by:

$$W_1 = (-1)^m \frac{u_{n+1,m}}{u_{n,m}}, \tag{4.23a}$$

$$W_2 = (-1)^n \frac{u_{n,m+1}}{u_{n,m}}. \tag{4.23b}$$

These two first integrals are first-order, two-point, meaning that the equation itself a first integral.

Finally the J_4 equation (1.28b) is linearizable and Darboux integrable being its first integrals given by:

$$W_1 = (-1)^m \frac{u_{n+1,m} + u_{n,m}}{1 + u_{n,m}u_{n+1,m}}, \tag{4.24a}$$

$$W_2 = (-1)^n \frac{u_{n,m+1} + u_{n,m}}{1 + u_{n,m}u_{n,m+1}}, \tag{4.24b}$$

These two first integrals are first-order, two-point, meaning that the equation itself a first integral.

The idea that these equations should have been Darboux integrable comes from the fact that they admit Generalized Symmetries depending on arbitrary functions [69], like the ${}_tH_1^\varepsilon$ equation (1.91a). Indeed it was shown in [5] that Darboux integrable equations always possess Generalized Symmetries depending on arbitrary functions of order at least equal to the first integrals. Therefore following [69] we tried to reverse the reasoning and prove that these equations were Darboux integrable, knowing their Generalized Symmetries. In this way in [69] it was proved that also the ${}_tH_1^\varepsilon$ equation is Darboux integrable by showing its first integrals.

This observations together with the result of Algebraic Entropy, as reported in Chapter 2, led us to conjecture that *every quad equation possessing the Consistency Around the Cube and linear growth is Darboux integrable*, This encouraged us to check the Darboux integrability of the remaining trapezoidal H^4 and H^6 equations. In Section 4.3 we show how this intuition has been proved by showing all the first integrals of the H^4 equations (1.91) and of the H^6 equations.

We conclude this Section noting that no result about the Darboux integrability of the J_1 equation (1.27) is known. Indeed it is possible to prove using the procedure outlined in Section 4.1 that a candidate two-point first integral in the n direction for the J_1 equation is given by:

$$W_1 = F_m\left(\frac{(u_{n,m} + e_2)(u_{n+1,m} + o_1)}{(u_{n,m} + e_1)(u_{n+1,m} + e_2)}\right). \tag{4.25}$$

However plugging (4.25) into the definition of first integral (4.7b) we obtain that the function F_m becomes a trivial constant. Defining

$$I_n = \frac{(u_{n,m}+e_2)(u_{n+1,m}+o_1)}{(u_{n,m}+e_1)(u_{n+1,m}+e_2)} \tag{4.26}$$

we note that the candidate three-point first integral is given by:

$$W_1 = F_m\left(I_n, I_{n-1}\right). \tag{4.27}$$

However also in this case it is possible to show that plugging (4.27) into (4.7b) the function F_m becomes a trivial constant.

Due to computational issues we were not able to prove or disprove the existence of a first integral in the n direction for the J_1 equation (1.27). In fact we conjecture that the solution of hierarchy of equations (4.16) for the J_1 equation (1.27) is given by:

$$W_1 = F_m\left(\dots, I_{n+1}, I_n, I_{n-1}, \dots\right) \tag{4.28}$$

and that there exists an order N at which this relation defines a first integral for the J_1 equation (1.27), but this kind of computations are too demanding and we were not able to conclude it for technical reasons. The only estimate we know is that $N > 1$. However we note that the dynamical system given constructed for the J_1 equation (1.27) do not respect the prescription of Chapter 1.

4.3 FIRST INTEGRALS FOR THE TRAPEZOIDAL h^4 AND h^6 EQUATIONS

In this Section we show the explicit form of the first integrals for the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93). This integrals are computed with the method presented in Section 4.1. We will not present the details of the calculations since they are algorithmic and they can be implemented in any Computer Algebra System available (we have implemented these conditions in `Maple`). The interested reader can find a worked out example in Appendix G.

We have the following general result:

- The H^4 equations (1.91) possess one first integral in the n direction and two first integrals in the m direction.
- The H^6 equations (1.93) possess two first integrals in the n direction and two first integrals in the m direction.

We believe that these general properties reflect the fact that the H^4 equations (1.91) are two-periodic only in the m direction whereas the H^6 equations (1.93) are two-periodic in both directions.

4.3.1 *Trapezoidal H^4 equations*

We now present the first integrals of the trapezoidal H^4 equations (1.91) in both directions.

4.3.1.1 ${}_tH_1^\varepsilon$ *equation*

Consider the ${}_tH_1^\varepsilon$ equation as given by (1.91a). It has a two-point, first order integral in the n direction:

$$W_1 = F_m^{(+)} \frac{\alpha_2}{u_{n+1,m} - u_{n,m}} + F_m^{(-)} \frac{u_{n+1,m} - u_{n,m}}{1 + \varepsilon^2 u_{n,m} u_{n+1,m}}, \tag{4.29a}$$

and a three-point, second order integral in the m-direction:

$$\begin{aligned} W_2 &= F_m^{(+)} \alpha \frac{1 + \varepsilon^2 u_{n,m+1} u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} \\ &+ F_m^{(-)} \beta \left(u_{n,m+1} - u_{n,m-1}\right), \end{aligned} \tag{4.29b}$$

The integrals (G.8) were first found in [69] from direct computation and later re-derived in [71] using the method explained in Section 4.1. The interested reader will find all the details of its derivation in Appendix G.

4.3.1.2 ${}_tH_2^\varepsilon$ *equation*

Consider the ${}_tH_2^\varepsilon$ equation as given by (1.91b). It has a four-point, third order integral in the n direction:

$$\begin{aligned} W_1 &= F_m^{(+)} \frac{\left(-u_{n+1,m} + u_{n-1,m}\right)\left(u_{n,m} - u_{n+2,m}\right)}{\varepsilon^2 \alpha_2^4 + 4\varepsilon\alpha_2^3 + \left[\left(8\alpha_3 - 2u_{n,m} - 2u_{n+1,m}\right)\varepsilon - 1\right]\alpha_2^2 + \left(u_{n,m} - u_{n+1,m}\right)^2} \\ &- F_m^{(-)} \frac{\left(-u_{n+1,m} + u_{n-1,m}\right)\left(u_{n,m} - u_{n+2,m}\right)}{\left(-u_{n-1,m} + u_{n,m} + \alpha_2\right)\left(u_{n+1,m} + \alpha_2 - u_{n+2,m}\right)} \end{aligned} \tag{4.30a}$$

and a five-point, fourth order integral in the m-direction:

$$\begin{aligned} W_2 &= F_m^{(+)} \alpha \frac{\left(u_{n,m-1} - u_{n,m+1}\right)^2 \left(u_{n,m+2} - u_{n,m}\right)\left(u_{n,m} - u_{n,m-2}\right)}{\left[\left(\alpha_2 + \alpha_3 + u_{n,m-1}\right)^2 \varepsilon - u_{n,m-1} + \alpha_3 - u_{n,m}\right] \cdot \left[\left(\alpha_3 + \alpha_2 + u_{n,m+1}\right)^2 \varepsilon - u_{n,m+1} + \alpha_3 - u_{n,m}\right]} \\ &+ F_m^{(-)} \beta \frac{\begin{bmatrix} \left(u_{n,m-1} - u_{n,m+2} - u_{n,m+1} + u_{n,m-2}\right) u_{n,m} \\ -\varepsilon\left(u_{n,m-2} - u_{n,m+2}\right) u_{n,m}^2 \\ -\left(\alpha_3 + \alpha_2\right)^2 \left(u_{n,m-2} - u_{n,m+2}\right)\varepsilon \\ -2\left(u_{n,m-2} - u_{n,m+2}\right)\left(\alpha_3 + \alpha_2\right)\varepsilon u_{n,m} \\ + \left(-\alpha_3 + u_{n,m+1}\right) u_{n,m-2} + u_{n,m+2}\left(\alpha_3 - u_{n,m-1}\right) \end{bmatrix}}{\left(u_{n,m+2} - u_{n,m}\right)\left(u_{n,m+1} - u_{n,m-1}\right)\left(u_{n,m} - u_{n,m-2}\right)} \end{aligned} \tag{4.30b}$$

Remark 4.3.1. We note that ${}_tH_2^\varepsilon$ equation possesses an autonomous sub-case when $\varepsilon = 0$. We denote this sub-case by ${}_tH_2^{\varepsilon=0}$. In this sub-case the equations defining the first integrals become singular and so the first integrals become simpler:

$$W_1^{\varepsilon=0} = (-1)^m \frac{2\alpha_2 - u_{n-1,m} + 2u_{n,m} - u_{n+1,m}}{u_{n-1,m} - u_{n+1,m}}, \tag{4.31a}$$

$$W_2^{\varepsilon=0} = \frac{(u_{n,m+1} - u_{n,m-1})(u_{n,m+2} - u_{n,m})}{u_{n,m} + u_{n,m+1} - \alpha_3}. \tag{4.31b}$$

The first integral in the n-direction (4.31a) is still non-autonomous, although the equation is autonomous, but now it is a three-point, second order first integral. On the contrary, the first integral in the m-direction (4.31b) is a four-point, third order first integral.

Finally we note that the ${}_tH_2^{\varepsilon=0}$ equation is related to the equation (1) from *List 3* in [55]:

$$(\hat{u}_{n+1,m+1} - \hat{u}_{n+1,m})(\hat{u}_{n,m} - \hat{u}_{n,m+1}) + \hat{u}_{n,m} + \hat{u}_{n+1,m} + \hat{u}_{n,m+1} + \hat{u}_{n+1,m+1} = 0 \tag{4.32}$$

through the transformation:

$$u_{n,m} = -\alpha_2 \hat{u}_{m,n} + \frac{1}{4}\alpha_2 + \frac{1}{2}\alpha_3. \tag{4.33}$$

Note that in this formula (4.33) the two lattice variables are *exchanged*. So it was already known in the literature that the ${}_tH_2^{\varepsilon=0}$ equation was Darboux integrable.

4.3.1.3 ${}_tH_3^\varepsilon$ *equation*

Consider the ${}_tH_2^\varepsilon$ equation as given by (1.91c). It has a four-point, third order integral in the n direction:

$$\begin{aligned} W_1 &= F_m^{(+)} \frac{(u_{n-1,m} - u_{n+1,m})(-u_{n+2,m} + u_{n,m})}{\begin{bmatrix} {\alpha_2}^4 \varepsilon^2 \delta^2 - {\alpha_2}^3 u_{n+1,m} u_{n,m} \\ + \left({u_{n,m}}^2 + u_{n+1,m}^2 - 2\varepsilon^2\delta^2\right){\alpha_2}^2 \\ -\alpha_2 u_{n,m} u_{n+1,m} + \varepsilon^2\delta^2 \end{bmatrix}} \\ &+ F_m^{(-)} \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{\alpha_2 (u_{n+2,m} - \alpha_2 u_{n+1,m})(\alpha_2 u_{n,m} - u_{n-1,m})} \end{aligned} \tag{4.34a}$$

and a five-point, fourth order integral in the m-direction:

$$W_2 = F_m^{(+)} \alpha \frac{(u_{n,m+2}-u_{n,m})(u_{n,m+1}-u_{n,m-1})^2(u_{n,m}-u_{n,m-2})}{(\delta^2\alpha_3^2+u_{n,m-1}^2\varepsilon^2-\alpha_3 u_{n,m-1}u_{n,m})\cdot(\delta^2\alpha_3^2+u_{n,m+1}^2\varepsilon^2-\alpha_3 u_{n,m}u_{n,m+1})} - F_m^{(-)}\beta \frac{\begin{bmatrix} \alpha_3 u_{n,m}^2(u_{n,m+1}-u_{n,m-1}) \\ -\varepsilon^2 u_{n,m}^2(u_{n,m+2}-u_{n,m-2}) \\ +\alpha_3 u_{n,m}(u_{n,m+2}u_{n,m-1}-u_{n,m+1}u_{n,m-2}) \\ -\delta^2{\alpha_3}^2 u_{n,m+2}+\delta^2{\alpha_3}^2 u_{n,m-2}\end{bmatrix}}{(u_{n,m+2}-u_{n,m})(u_{n,m+1}-u_{n,m-1})(u_{n,m}-u_{n,m-2})} \tag{4.34b}$$

Remark 4.3.2. With the same notation as in Remark 4.3.1, we note that also the ${}_tH_3^\varepsilon$ equation has an autonomous sub-case of the equation ${}_tH_3^\varepsilon$ if $\varepsilon = 0$, namely, the ${}_tH_3^{\varepsilon=0}$ equation. In this sub-case the equations defining the first integrals become singular and so the first integrals become simpler:

$$W_1^{\varepsilon=0} = (-1)^m \left[\frac{\alpha_2 u_{n,m}-u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}}+\frac{1}{2}\right], \tag{4.35a}$$

$$W_2^{\varepsilon=0} = \frac{(u_{n,m+2}-u_{n,m})u_{n,m-1}-u_{n,m+1}u_{n,m+2}+\alpha_3\delta^2}{\alpha_3\delta^2-u_{n,m}u_{n,m+1}}. \tag{4.35b}$$

The first integral in the n-direction (4.35a) is still non-autonomous, although the equation is autonomous, but now it is a three-point, second order first integral. On the contrary, the first integral in the m-direction (4.35b) is a four-point, third order first integral.

Finally we note that the ${}_tH_3^{\varepsilon=0}$ equation is related to equation (2) from *List 3* in [55]:

$$\hat{u}_{n+1,m+1}(\hat{u}_{n,m}+b_2\hat{u}_{n,m+1})+\hat{u}_{n+1,m}(b_2\hat{u}_{n,m}+\hat{u}_{n,m+1})+c_4=0 \tag{4.36}$$

through the inversion of two lattice parameters $u_{n,m}=\hat{u}_{m,n}$ and the choice of parameters:

$$b_2 = -\frac{1}{\alpha_2}, \quad c_4 = \frac{\delta^2\alpha_3(1-\alpha_2^2)}{\alpha_2}. \tag{4.37}$$

So it was already known in the literature that the ${}_tH_3^{\varepsilon=0}$ equation was Darboux integrable.

Remark 4.3.3. As a final remark we can say that the ${}_tH_2^\varepsilon$ equation (1.91b) and ${}_tH_3^\varepsilon$ equation (1.91c) possess first integrals of the same order in each direction. Furthermore the first integrals of all the trapezoidal H^4 equations share the important property that in the direction m, which is the direction of the non-autonomous factors $F_m^{(\pm)}$, the W_2 integrals are built up from two different "sub"-integrals.

4.3.2 H^6 *equations*

In this Subsection we present the first integrals of the H^6 equations (1.93) in both direction.

4.3.2.1 ${}_1D_2$ *equation*

In the case of the ${}_1D_2$ equation (1.93b) we have the following second order three-point integrals:

$$\begin{aligned}W_1 &= F_n^{(+)}F_m^{(+)}\alpha\frac{[(1+\delta_2)u_{n,m}+u_{n+1,m}]\,\delta_1-u_{n,m}}{\delta_1\{[(1+\delta_2)u_{n,m}+u_{n-1,m}]\,\delta_1-u_{n,m}\}}\\ &+F_n^{(+)}F_m^{(-)}\alpha\frac{1+(u_{n+1,m}-1)\,\delta_1}{(1+(u_{n-1,m}-1)\,\delta_1)\,\delta_1}\\ &+F_n^{(-)}F_m^{(+)}\beta\,(u_{n+1,m}-u_{n-1,m})\\ &-F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n+1,m}-u_{n-1,m})\,[1-(1-u_{n,m})\,\delta_1]}{\delta_2+u_{n,m}},\end{aligned} \tag{4.38a}$$

$$\begin{aligned}W_2 &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m}+\delta_1 u_{n,m-1}}\\ &+F_n^{(+)}F_m^{(-)}\beta\,(u_{n,m+1}-u_{n,m-1})\\ &-F_n^{(-)}F_m^{(+)}\alpha\frac{u_{n,m+1}-u_{n,m-1}}{1+(\delta_1-1)\,u_{n,m+1}}\\ &-F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{\delta_2+u_{n,m}}.\end{aligned} \tag{4.38b}$$

Remark 4.3.4. We remark that, when $\delta_1 \to 0$, the first integral (4.38a) is singular, since the coefficient at α approaches a constant. In this particular case, it can be shown that the first integrals are given by:

$$\begin{aligned}W_1^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m}-u_{n-1,m}}{u_{n,m}}\\ &-F_n^{(+)}F_m^{(-)}\alpha\,(u_{n+1,m}-u_{n-1,m})\\ &+F_n^{(-)}F_m^{(+)}\beta\,(u_{n+1,m}-u_{n-1,m})\\ &+F_n^{(-)}F_m^{(-)}\beta\frac{u_{n-1,m}-u_{n+1,m}}{\delta_2+u_{n,m}},\end{aligned} \tag{4.39a}$$

$$\begin{aligned}W_2^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m}}\\ &+F_n^{(+)}F_m^{(-)}\beta\,(u_{n,m+1}-u_{n,m-1})\\ &-F_n^{(-)}F_m^{(+)}\alpha\,(u_{n,m+1}-u_{n,m-1})\\ &-F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{\delta_2+u_{n,m}}.\end{aligned} \tag{4.39b}$$

Note that, as the limit in (4.38b) is not singular, then (4.39b) can be obtained directly from (4.38b).

If $\delta_1 \to (1+\delta_2)^{-1}$, then the limit of the first integral (4.38b) is not singular. However, it can be seen from equations defining the first integral in the m-direction that, in this case, there might exist a two-point, first order first integral. Carrying out the computations, we obtain that if $\delta_1 = (1+\delta_2)^{-1}$, the first integrals are given by:

$$\begin{aligned} W_1^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m}}{u_{n-1,m}} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{\delta_2+u_{n+1,m}}{\delta_2+u_{n-1,m}} \\ &+ F_n^{(-)}F_m^{(+)}\beta\left(u_{n+1,m}-u_{n-1,m}\right) \\ &- F_n^{(-)}F_m^{(-)}\beta\frac{\left(u_{n+1,m}-u_{n-1,m}\right)}{\delta_2+1}, \end{aligned} \tag{4.40a}$$

$$\begin{aligned} W_2^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\left[\left(1+\delta_2\right)u_{n,m}+u_{n,m+1}\right] \\ &+ F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m}+\left(1+\delta_2\right)u_{n,m+1}}{\delta_2+1} \\ &- F_n^{(-)}F_m^{(+)}\alpha\frac{\left(\delta_2+1\right)u_{n,m}}{\delta_2+u_{n,m+1}} \\ &- F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}}{\delta_2+u_{n,m}}. \end{aligned} \tag{4.40b}$$

So, unlike the case $\delta_1 \neq (1+\delta_2)^{-1}$, the first integral in the m-direction (4.40b) is a two-point, first order first integral. On the contrary, the first integral in the n-direction (4.40a) is still a three-point, second order first integral, which can be obtained from the complete form (4.38a) by substituting $\delta_1 = (1+\delta_2)^{-1}$.

4.3.2.2 $_2D_2$ *equation*

In the case of the $_2D_2$ equation (1.93c) we have the following second order three-point integrals:

$$\begin{aligned} W_1 &= F_n^{(+)}F_m^{(+)}\alpha\frac{\delta_2+u_{n+1,m}}{\delta_2+u_{n-1,m}} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{\left(1-\left(1+\delta_2\right)\delta_1\right)u_{n,m}+u_{n+1,m}}{\left(1-\left(1+\delta_2\right)\delta_1\right)u_{n,m}+u_{n-1,m}} \\ &+ F_n^{(-)}F_m^{(+)}\beta\frac{\left(u_{n+1,m}-u_{n-1,m}\right)\left(u_{n,m}+\delta_2\right)}{1+\left(-1+u_{n,m}\right)\delta_1} \\ &- F_n^{(-)}F_m^{(-)}\beta\left(u_{n+1,m}-u_{n-1,m}\right) \end{aligned} \tag{4.41a}$$

$$\begin{aligned} W_2 &= F_n^{(+)}F_m^{(+)}\alpha\left(u_{n,m+1}-u_{n,m-1}\right) \\ &- F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{\left(\lambda-u_{n,m}\right)\delta_1-u_{n,m-1}} \\ &- F_n^{(-)}F_m^{(+)}\alpha\frac{u_{n,m+1}-u_{n,m-1}}{1+\left(-1+u_{n,m}\right)\delta_1} \\ &- F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m+1}+\delta_2} \end{aligned} \tag{4.41b}$$

Remark 4.3.5. We remark that if $\delta_1 \to 0$, the first integral (4.41a) is not singular. However it can be seen from equations defining the first integral in the n-direction that in this case there might exist a two-point, first order first integral. Carrying out the computations, we obtain that if $\delta_1 = 0$, the first integrals are given by:

$$
\begin{aligned}
W_1^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\left(\delta_2 + u_{n+1,m}\right)u_{n,m} \\
&- F_n^{(+)}F_m^{(-)}\alpha\left(u_{n+1,m} + u_{n,m}\right) \\
&+ F_n^{(-)}F_m^{(+)}\beta\left(\delta_2 + u_{n,m}\right)u_{n+1,m} \\
&- F_n^{(-)}F_m^{(-)}\beta\left(u_{n+1,m} + u_{n,m}\right),
\end{aligned} \tag{4.42a}
$$

$$
\begin{aligned}
W_2^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\left(u_{n,m+1} - u_{n,m-1}\right) \\
&+ F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m+1}}{u_{n,m-1}} \\
&- F_n^{(-)}F_m^{(+)}\alpha\left(u_{n,m+1} - u_{n,m-1}\right) \\
&- F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + \delta_2}.
\end{aligned} \tag{4.42b}
$$

So, differently from the case $\delta_1 \neq 0$, the first integral in the n-direction (4.42a) is a two-point, first order first integral. On the contrary, the first integral in the m-direction (4.42b) is still a three-point, second order first integral, which can be obtained from the complete form (4.41b) by substituting $\delta_1 = 0$.

Moreover we note that if $\delta_1 \to (1+\delta_2)^{-1}$, the limit of the first integral (4.41b) is not singular. However, it can be seen from equations defining the first integral in the m-direction that in this case there might exist a two-point, first order first integral. Carrying out the computations, we obtain that if $\delta_1 = (1+\delta_2)^{-1}$, the first integrals are given by:

$$
\begin{aligned}
W_1^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{\delta_2 + u_{n+1,m}}{\delta_2 + u_{n-1,m}} \\
&+ F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n+1,m}}{u_{n-1,m}} \\
&- F_n^{(-)}F_m^{(+)}\beta\left(u_{n-1,m} - u_{n+1,m}\right) \\
&- F_n^{(-)}F_m^{(-)}\beta\frac{u_{n+1,m} - u_{n-1,m}}{1+\delta_2},
\end{aligned} \tag{4.43a}
$$

$$
\begin{aligned}
W_2^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\left[\left(1+\delta_2\right)u_{n,m} + u_{n,m+1}\right] \\
&+ F_n^{(+)}F_m^{(-)}\beta\left[u_{n,m} + \left(1+\delta_2\right)u_{n,m+1}\right] \\
&+ F_n^{(-)}F_m^{(+)}\alpha\frac{\delta_2\lambda - \left(1+\delta_2\right)u_{n,m+1} + \lambda u_{n,m}}{u_{n,m} + \delta_2} \\
&+ F_n^{(-)}F_m^{(-)}\beta\frac{\delta_2\lambda - \left(1+\delta_2\right)u_{n,m} + \lambda u_{n,m+1}}{u_{n,m+1} + \delta_2}.
\end{aligned} \tag{4.43b}
$$

So, differently from the case $\delta_1 \neq (1+\delta_2)^{-1}$, the first integral in the m-direction (4.43b) is a two-point, first order first integral. On the contrary, the first integral in the n-direction (4.43a) is still a three-point, second order first integral, which can be obtained from the complete form (4.41a) by substituting $\delta_1 = (1+\delta_2)^{-1}$.

4.3.2.3 ${}_3D_2$ *equation*

In the case of the ${}_3D_2$ equation (1.93d) we have the following second order three-point integrals:

$$\begin{aligned} W_1 &= F_n^{(+)}F_m^{(+)}\alpha\frac{(u_{n-1,m}+\delta_2)\left[1+(u_{n+1,m}-1)\delta_1\right]}{(u_{n+1,m}+\delta_2)\left[1+(u_{n-1,m}-1)\delta_1\right]} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n,m}+(1-\delta_1-\delta_1\delta_2)u_{n-1,m}}{u_{n,m}+(1-\delta_1-\delta_1\delta_2)u_{n+1,m}} \\ &+ F_n^{(-)}F_m^{(+)}\beta\left(u_{n+1,m}-u_{n-1,m}\right)\left(\delta_2+u_{n,m}\right) \\ &- F_n^{(-)}F_m^{(-)}\beta\left(u_{n+1,m}-u_{n-1,m}\right), \end{aligned} \tag{4.44a}$$

$$\begin{aligned} W_2 &= F_n^{(+)}F_m^{(+)}\alpha\left(u_{n,m+1}-u_{n,m-1}\right) \\ &- F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{\lambda(1+\delta_2)\delta_1{}^2-\left[(1+\delta_2)u_{n,m-1}+u_{n,m}+\lambda\right]\delta_1+u_{n,m-1}} \\ &+ F_n^{(-)}F_m^{(+)}\alpha\left(u_{n,m-1}-u_{n,m+1}\right)\left[1+(u_{n,m}-1)\delta_1\right] \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{(\delta_2+u_{n,m+1})\left[1+(1-\delta_1)u_{n,m-1}\right]}. \end{aligned} \tag{4.44b}$$

Remark 4.3.6. We remark that if $\delta_1 \to 0$, the first integral (4.44a) is not singular. However, it can be seen from equations defining the first integral in the n-direction that in this case there might exist a two-point, first order first integral. Carrying out the computations, we obtain that if $\delta_1 = 0$, the first integrals are given by:

$$\begin{aligned} W_1^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha u_{n,m}\left(\delta_2+u_{n+1,m}\right) \\ &- F_n^{(+)}F_m^{(-)}\alpha\left(u_{n+1,m}+u_{n,m}\right) \\ &+ F_n^{(-)}F_m^{(+)}\beta u_{n+1,m}\left(\delta_2+u_{n,m}\right) \\ &- F_n^{(-)}F_m^{(-)}\beta\left(u_{n+1,m}+u_{n,m}\right), \end{aligned} \tag{4.45a}$$

$$\begin{aligned} W_2^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\left(u_{n,m+1}-u_{n,m-1}\right) \\ &+ F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m+1}}{u_{n,m-1}} \\ &- F_n^{(-)}F_m^{(+)}\alpha\left(u_{n,m+1}-u_{n,m-1}\right) \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{\delta_2+u_{n,m-1}}{\delta_2+u_{n,m+1}}. \end{aligned} \tag{4.45b}$$

So, differently from the case $\delta_1 \neq 0$, the first integral in the n-direction (4.45a) is a two-point, first order first integral. On the contrary, the first integral in the m-direction (4.45b) is still a three-point, second order first integral, which can be obtained from the complete form (4.44b) by substituting $\delta_1 = 0$.

Moreover we remark that if $\delta_1 \to (1+\delta_2)^{-1}$, the first integral (4.44a) is singular, since the coefficient at α approaches a constant. In this particular case the first integrals are given by:

$$\begin{aligned} W_1^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+u_{n+1,m})(\delta_2+u_{n-1,m})} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+1)u_{n,m}} \\ &- F_n^{(-)}F_m^{(+)}\beta(u_{n-1,m}-u_{n+1,m})(\delta_2+u_{n,m}) \\ &- F_n^{(-)}F_m^{(-)}\beta(u_{n+1,m}-u_{n-1,m}), \end{aligned} \tag{4.46a}$$

$$\begin{aligned} W_2^{((1+\delta_2)^{-1},\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha(u_{n,m+1}-u_{n,m-1}) \\ &+ F_n^{(+)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m}} \\ &- F_n^{(-)}F_m^{(+)}\alpha\frac{(u_{n,m+1}-u_{n,m-1})(\delta_2+u_{n,m})}{\delta_2+1} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{u_{n,m+1}-u_{n,m-1}}{(\delta_2+u_{n,m+1})(u_{n,m-1}+\delta_2)}. \end{aligned} \tag{4.46b}$$

We point out that the first integral in the m-direction (4.46b) can be obtained from the complete form (4.44b) in the limit $\delta_1 \to (1+\delta_2)^{-1}$.

4.3.2.4 D_3 *equation*

In the case of the D_3 equation as given by (1.93e) we have the following third order four-point integrals:

$$\begin{aligned} W_1 &= F_n^{(+)}F_m^{(+)}\alpha\frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n+1,m}^2-u_{n,m}} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n,m}+u_{n-1,m}} \\ &- F_n^{(-)}F_m^{(+)}\beta\frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n+1,m}-u_{n,m}^2} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n+1,m}+u_{n+2,m}} \end{aligned} \tag{4.47a}$$

$$\begin{aligned} W_2 &= F_n^{(+)}F_m^{(+)}\alpha\frac{(u_{n,m+1}-u_{n,m-1})(u_{n,m+2}-u_{n,m})}{u_{n,m+1}^2-u_{n,m}} \\ &- F_n^{(+)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})(u_{n,m+2}-u_{n,m})}{u_{n,m+1}-u_{n,m}^2} \\ &+ F_n^{(-)}F_m^{(+)}\alpha\frac{(u_{n,m+1}-u_{n,m-1})(u_{n,m+2}-u_{n,m})}{u_{n,m}+u_{n,m-1}} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})(u_{n,m+2}-u_{n,m})}{u_{n,m+1}+u_{n,m+2}} \end{aligned} \tag{4.47b}$$

Remark 4.3.7. The equation D_3 is invariant under the exchange of lattice variables $n \leftrightarrow m$. Therefore its W_2 integral (4.47b) can be obtained from the W_1 one (4.47a) simply by exchanging the index n and m.

4.3.2.5 ${}_1D_4$ *equation*

In the case of the ${}_1D_4$ equation as given by (1.93f) we have the following third order four-point integrals:

$$\begin{aligned} W_1 &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m}^2\delta_1 + u_{n+1,m}u_{n+2,m} + u_{n-1,m}\left(u_{n,m} - u_{n+2,m}\right) - \delta_2\delta_3}{u_{n+1,m}\left(\delta_1 + u_{n,m}\right) - \delta_2\delta_3} \\ &+ F_n^{(+)}F_m^{(-)}\alpha\frac{\left(u_{n,m} - u_{n+2,m} + \delta_1 u_{n+1,m}\right)u_{n-1,m} + u_{n+1,m}u_{n+2,m}}{\left(u_{n,m} + \delta_1 u_{n-1,m}\right)u_{n+1,m}} \\ &+ F_n^{(-)}F_m^{(+)}\beta\frac{\left(u_{n+1,m} - u_{n-1,m}\right)\left(u_{n+2,m} - u_{n,m}\right)}{u_{n,m}^2\delta_1 + u_{n+1,m}u_{n,m} - \delta_2\delta_3} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{\left(u_{n+1,m} - u_{n-1,m}\right)\left(u_{n+2,m} - u_{n,m}\right)}{u_{n,m}\left(u_{n+2,m}\delta_1 + u_{n+1,m}\right)} \end{aligned} \tag{4.48a}$$

$$\begin{aligned} W_2 &= F_n^{(+)}F_m^{(+)}\alpha\frac{\left(u_{n,m+2} - u_{n,m}\right)u_{n,m-1} + \delta_1\delta_3 - \delta_2 u_{n,m+1}^2 - u_{n,m+1}u_{n,m+2}}{\delta_1\delta_3 - u_{n,m}u_{n,m+1} - \delta_2 u_{n,m+1}^2} \\ &- F_n^{(+)}F_m^{(-)}\beta\frac{\left(u_{n,m+1} - u_{n,m-1}\right)\left(u_{n,m+2} - u_{n,m}\right)}{\delta_1\delta_3 - \delta_2 {u_{n,m}}^2 - u_{n,m}u_{n,m+1}} \\ &+ F_n^{(-)}F_m^{(+)}\alpha\frac{\left(u_{n,m} - u_{n,m+2} + \delta_2 u_{n,m+1}\right)u_{n,m-1} + u_{n,m+1}u_{n,m+2}}{\left(u_{n,m} + \delta_2 u_{n,m-1}\right)u_{n,m+1}} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{\left(u_{n,m+1} - u_{n,m-1}\right)\left(u_{n,m+2} - u_{n,m}\right)}{u_{n,m}\left(u_{n,m+2}\delta_2 + u_{n,m+1}\right)} \end{aligned} \tag{4.48b}$$

Remark 4.3.8. The equation ${}_1D_4$ possesses an autonomous sub-case when $\delta_1 = \delta_2 = 0$. In this case the equations defining the first integrals become singular and the first integrals become simpler. If $\delta_3 \neq 0$ we have:

$$W_1^{(0,0,\delta_3)} = (-1)^m \frac{u_{n+1,m} - u_{n-1,m}}{u_{n,m}}, \tag{4.49a}$$

$$W_2^{(0,0,\delta_3)} = (-1)^n \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m}}. \tag{4.49b}$$

They are both non-autonomous, three-point, second order first integrals. Notice that, since this sub-case is such that the equation has the discrete symmetry $n \leftrightarrow m$, the first integral $W_2^{(0,0,\delta_3)}$ can be obtained from $W_1^{(0,0,\delta_3)}$ by using such transformation.

Moreover we notice that the case $\delta_1 = \delta_2 = 0$ $\delta_3 \neq 0$ is linked to the equation (4) with $b_3 = 1$ of *List 3* in [55]:

$$\hat{u}_{n+1,m+1}\hat{u}_{n,m} + \hat{u}_{n+1,m}\hat{u}_{n,m+1} + 1 = 0 \tag{4.50}$$

through the transformation $u_{n,m} = \sqrt{\delta_3}\hat{u}_{n,m}$.

We finally notice that the sub-case where $\delta_1 = \delta_2 = \delta_3 = 0$ possesses two two-point, first order, non-autonomous first integrals:

$$W_1^{(0,0,0)} = (-1)^m \frac{u_{n+1,m}}{u_{n,m}}, \quad W_2^{(0,0,0)} = (-1)^n \frac{u_{n,m+1}}{u_{n,m}}. \tag{4.51}$$

The resulting equation is linked to one of the linearizable and Darboux integrable cases presented in [69, 82].

4.3.2.6 ${}_2D_4$ *equation*

In the case of the ${}_2D_4$ equation as given by (1.93g) we have the following four-point integrals:

$$\begin{aligned} W_1 =& F_n^{(+)}F_m^{(+)}\alpha \frac{\begin{bmatrix} (u_{n,m}-u_{n+2,m}-\delta_1\delta_2 u_{n-1,m})\, u_{n+1,m}^2 \\ +u_{n+1,m}u_{n+2,m}u_{n-1,m}+\delta_3 u_{n-1,m}\end{bmatrix}}{\left(\delta_2 u_{n+1,m}^2\delta_1-\delta_3-u_{n,m}u_{n+1,m}\right)u_{n-1,m}} \\ &- F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n+2,m}u_{n-1,m}+(-u_{n+2,m}+u_{n,m})\,u_{n+1,m}+\delta_3}{u_{n-1,m}u_{n,m}+\delta_3} \\ &- F_n^{(-)}F_m^{(+)}\beta\frac{(u_{n+1,m}-u_{n-1,m})\,(u_{n+2,m}-u_{n,m})\,u_{n,m}}{u_{n+2,m}\,(\delta_2\delta_1 u_{n,m}{}^2-u_{n,m}u_{n+1,m}-\delta_3)} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n+1,m}-u_{n-1,m})\,(u_{n+2,m}-u_{n,m})}{u_{n+1,m}u_{n+2,m}+\delta_3} \end{aligned} \tag{4.52a}$$

$$\begin{aligned} W_2 =& F_n^{(+)}F_m^{(+)}\alpha\frac{(u_{n,m+2}-u_{n,m})\,u_{n,m-1}+\delta_1\delta_3-\delta_2 u_{n,m+1}^2-u_{n,m+1}u_{n,m+2}}{\delta_1\delta_3-\delta_2 u_{n,m+1}^2-u_{n,m}u_{n,m+1}} \\ &- F_n^{(+)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})\,(u_{n,m+2}-u_{n,m})}{\delta_1\delta_3-\delta_2 u_{n,m}^2-u_{n,m}u_{n,m+1}} \\ &+ F_n^{(-)}F_m^{(+)}\alpha\frac{u_{n,m+2}\delta_2 u_{n,m}+u_{n,m-1}u_{n,m}+u_{n,m+1}u_{n,m+2}-u_{n,m}u_{n,m+1}}{u_{n,m+2}\,(\delta_2 u_{n,m}+u_{n,m-1})} \\ &+ F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})\,(u_{n,m+2}-u_{n,m})}{(\delta_2 u_{n,m+1}+u_{n,m+2})\,u_{n,m-1}} \end{aligned} \tag{4.52b}$$

Remark 4.3.9. The equation ${}_2D_4$ possesses an autonomous sub-case when $\delta_1 = \delta_2 = 0$. In this case the equations defining the first integrals become singular and the first integrals become simpler:

$$W_1^{(0,0,\delta_3)} = (-1)^m\left(u_{n,m}u_{n+1,m}+\frac{\delta_3}{2}\right), \quad W_2^{(0,0,\delta_3)} = \left(\frac{u_{n,m+1}}{u_{n,m-1}}\right)^{(-1)^n}. \tag{4.53}$$

Therefore this particular sub-case possesses a two-point, first order integral W_1 and a three-point, second order integral W_2. Both integrals are non-autonomous despite the equation is autonomous. The limit $\delta_3 \to 0$ is in this case regular both in the first integrals and in the equations defining them.

Finally we have that the case $\delta_1 = \delta_2 = 0$ corresponds to the equation (9) of *List 4* in [55]:

$$\hat{u}_{n,m}\hat{u}_{n+1,m} + \hat{u}_{n,m+1}\hat{u}_{n+1,m+1} + c_4 = 0, \tag{4.54}$$

with the identification $u_{n,m} = \hat{u}_{n,m}$ and $c_4 = \delta_3$.

Remark 4.3.10. Besides the general remarks on the structure of the first integrals for the H^6 equations which were given at the beginning of the this Section we would like to note that the first integrals of the H^6 equations are of the same order in both directions at difference of the trapezoidal H^4 equations (except that in the degenerate cases). Furthermore we note that the three forms of the ${}_iD_2$ equations possess first integrals of the second order. In the same way we have that the D_3 and the ${}_iD_4$ equations possess first integrals of the fourth order. This different properties will be important when looking for general solution in Section 4.4.

We conclude this Section noting that together with the results obtained in Section 4.2 we can state that *any quad equation possessing the Consistency Around the Cube with linear growth constructed according to the prescriptions of Chapter 1 is Darboux integrable*. As pointed out in Section 4.2 the J_1 equation (1.27) is not constructed using the prescriptions of Chapter 1, therefore it is different from all the other equations considered. Since no definitive result about the Darboux integrability of the J_1 equation (1.27) is known we cannot say if this result can expand to a wider class of equations including the J_1 equation (1.27).

4.4 GENERAL SOLUTIONS THROUGH THE FIRST INTEGRALS

In Section 4.3 we showed that the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93) are Darboux integrable in the sense of lattice equations, see Section 4.1. In this Section we show that from the knowledge of the first integrals and from the properties of the equations it is possible to construct, maybe after some complicate algebra, the general solutions of the trapezoidal H^4 equations (1.91) and of the H^6 equations (1.93). By general solution we mean a representation of the solution of any of the equations in (1.91) and (1.93) in terms of the right number of arbitrary functions of one lattice variable n or m. Since the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93) are quad equations, i.e. the discrete analogue of second order hyperbolic partial differential equations, the general solution must contain an arbitrary function in the n direction and another one in the m direction.

To obtain the desired solution we will need only the W_1 integrals we presented in Section 4.3 and the relation (4.11), i.e. $W_1 = \xi_n$ with

ξ_n an arbitrary function of n. The equation $W_1 = \xi_n$ can be interpreted as an ordinary difference equation in the n direction depending parametrically on m. Then from every W_1 integral we can derive two different ordinary difference equations, one corresponding to m even and one corresponding to m odd. In both the resulting equation we can get rid of the two-periodic terms by considering the cases n even and n odd and using the general transformation:

$$u_{2k,2l} = v_{k,l}, \quad u_{2k+1,2l} = w_{k,l}, \tag{4.55a}$$

$$u_{2k,2l+1} = y_{k,l}, \quad u_{2k+1,2l+1} = z_{k,l}. \tag{4.55b}$$

This transformation brings both equations to a *system* of *coupled difference equations*. This reduction to a system is the key ingredient in the construction of the general solutions for the trapezoidal H^4 equations (1.91) and for the H^6 equations (1.93).

We note that the transformation (4.55) can be applied to the trapezoidal H^4 equations[2] and H^6 equations themselves. This casts these *non-autonomous equations with two-periodic coefficients* into *autonomous systems of four equations*. We recall that in this way some examples of direct linearization (i.e. without the knowledge of the first integrals) were produced in [67]. Finally we note that if we apply the even/odd splitting of the lattice variables given by equation (4.55) to describe a general solution we will need two arbitrary functions in both directions, i.e. we will need a total of four arbitrary functions.

In practice to construct these general solutions, we need to solve Riccati equations and non-autonomous linear equations which, in general, cannot be solved in closed form. Using the fact that these equations contain arbitrary functions we introduce new arbitrary functions so that we can solve these equations. This is usually done reducing to *total difference*, i.e. to ordinary difference equations which can be *trivially* solved. Let us assume we are given the difference equation:

$$u_{n+1,m} - u_{n,m} = v_n, \tag{4.56}$$

depending parametrically on another discrete index m. Then if I can express the function v_n as a discrete derivative:

$$v_n = w_{n+1} - w_n, \tag{4.57}$$

then the solution of equation (4.56) is simply:

$$u_{n,m} = w_n + \upsilon_m, \tag{4.58}$$

where υ_m is an arbitrary function of the discrete variable m. This is the simplest possible example of reduction to total difference. The

2 In fact, in the case of the trapezoidal H^4 equations (1.91), we use a simpler transformation instead of (4.55), see Section 4.4.3.

general solutions will then be expressed in terms of these new arbitrary functions obtained reducing to total differences and in terms of a finite number of *discrete integrations*, i.e. the solutions of the simple ordinary difference equation:

$$u_{n+1} - u_n = v_n, \tag{4.59}$$

where u_n is the unknown and v_n is an assigned function. We note that the discrete integration (4.59) is the discrete analogue of the differential equation $u'(x) = v(x)$.

To summarize, in this Section we prove the following result:

> *The trapezoidal* H^4 *equations* (1.91) *and* H^6 *equations* (1.93) *are exactly solvable and we can represent the solution in terms of a finite number of discrete integration* (4.59).

The Section is structured in Subsections. In Subsection 4.4.1 we treat the ${}_1D_2$, ${}_2D_2$ and ${}_3D_2$ equations. In this case the construction of the general solution is carried out from the sole knowledge of the first integral. The equation acts only as a compatibility condition for the arbitrary functions obtained in the procedure. The solution obtained is explicit. In Subsection 4.4.2 we treat the D_3, ${}_1D_4$ and ${}_2D_4$ equations. In this case the construction of the general solution is carried out through a series of manipulations in the equation itself and from the knowledge of the first integral. The key point will be that the equations of the first integrals can be reduced to a single linear equation. The solution is obtained up to two discrete integrations, one in every direction. In Subsection 4.4.3 we treat the trapezoidal H^4 equations (1.91). The ${}_tH_1^\varepsilon$ equation (1.91a) is trivial since in the direction n it possess the first integral (4.29a) which is two-point, second order. This is *trivial*, since possessing a first integral of order one means that *the equation itself is a first integral*. For the ${}_tH_2^\varepsilon$ equation (1.91b) and the ${}_tH_3^\varepsilon$ equation (1.91c) the construction of the general solution is carried out reducing the equation to a partial difference equation defined on six points and then using the equations defined by the first integrals. The equations defined by the first integrals are reduced to discrete Riccati equations. The solution will be be given in terms of four discrete integrations.

4.4.1 *The* ${}_iD_2$ *equations* $i = 1, 2, 3$

In this Subsection we construct the general solution of the three forms of the D_2 equation, which we denote collectively as ${}_iD_2$ with $i = 1, 2, 3$ and are given by equations (1.93b), (1.93c) and (1.93d).

4.4.1.1 ${}_1D_2$ *equation*

From Section 4.3 we know that the ${}_1D_2$ equation (1.93b) possesses a three-point, second order first integral W_1 (4.38a). As stated at the beginning of this Section from the relation $W_1 = \xi_n$ this integral defines a three-point, second order ordinary difference equation in the n direction which depends parametrically on m. From the parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

CASE $m = 2l$: If $m = 2l$ we have the following non-autonomous ordinary difference equation:

$$\begin{aligned} &F_n^{(+)} \frac{[(1+\delta_2)\, u_{n,2l} + u_{n+1,2l}]\, \delta_1 - u_{n,2l}}{[(1+\delta_2)\, u_{n,2l} + u_{n-1,2l}]\, \delta_1 - u_{n,2l}} \\ &+ F_n^{(-)} (u_{n+1,2l} - u_{n-1,2l}) = \xi_n, \end{aligned} \tag{4.60}$$

where without loss of generality $\alpha = \delta_1$ and $\beta = 1$. We can easily see, that once solved for $u_{n+1,2l}$ the equation is *linear*:

$$\begin{aligned} u_{n+1,2l} + \frac{F_m^{(+)} (\delta_1 + \delta_1\delta_2 - 1 - \xi_n\delta_1 - \xi_n\delta_1\delta_2 + \xi_n\delta_1)\, u_{n,2l}}{\delta_1} \\ - \left(F_m^{(+)}\xi_n + F_m^{(-)}\right) u_{n-1,2l} - F_m^{(-)}\xi_n = 0. \end{aligned} \tag{4.61}$$

Tackling this equation directly is very difficult, but we can separate again the cases when n is even and odd and convert (4.61) into a system using the standard transformation (4.55a):

$$w_{k,l} - \xi_{2k} w_{k-1,l} = \frac{\delta}{\delta_1} (1 - \xi_{2k})\, v_{k,l}, \tag{4.62a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}, \tag{4.62b}$$

where:

$$\delta = 1 - \delta_1\delta_2 - \delta_1. \tag{4.63}$$

Now we have two *first order* ordinary difference equations. Equation (4.62b) is uncoupled from equation (4.62a). Furthermore, since ξ_{2k} and ξ_{2k+1} are independent functions we can write $\xi_{2k+1} = a_{k+1} - a_k$. So the second equation possesses the trivial solution[3]:

$$v_{k,l} = \alpha_l + a_k. \tag{4.64}$$

Now introduce (4.64) into (4.62a) and solve the equation for $w_{k,l}$:

$$w_{k,l} - \xi_{2k} w_{k-1,l} = \frac{\delta}{\delta_1} (1 - \xi_{2k}) (\alpha_l + a_k). \tag{4.65}$$

3 From now on we use the convention of naming the arbitrary functions depending on k with Latin letters and the functions depending on l by Greek ones.

We define $\xi_{2k} = b_k/b_{k-1}$ and perform the change of dependent variable: $w_{k,l} = b_k W_{k,l}$. Then $W_{k,l}$ solves the equation:

$$W_{k,l} - W_{k-1,l} = \frac{\delta}{\delta_1}\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right)(\alpha_l + a_k). \tag{4.66}$$

The solution of this difference equation is given by:

$$W_{k,l} = \beta_l + \frac{\delta}{\delta_1}\frac{\alpha_l}{b_k} + c_k, \tag{4.67}$$

where c_k is such that:

$$c_k - c_{k-1} = \frac{\delta}{\delta_1}\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) a_k. \tag{4.68}$$

Equation (4.68) is not a total difference, but it can be used to define a_k in terms of the arbitrary functions b_k and c_k:

$$a_k = -\frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}(c_k - c_{k-1}). \tag{4.69}$$

This means that we have the following solution for the system (4.62):

$$v_{k,l} = \alpha_l - \frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}(c_k - c_{k-1}). \tag{4.70a}$$

$$w_{k,l} = b_k(\beta_l + c_k) + \frac{\delta}{\delta_1}\alpha_l, \tag{4.70b}$$

CASE $m = 2l + 1$: If $m = 2l + 1$ we have the following non-autonomous ordinary difference equation:

$$F_m^{(+)} \frac{1 + (u_{n+1,2l+1} - 1)\delta_1}{1 + (u_{n-1,2l+1} - 1)\delta_1} + F_m^{(-)} \frac{(u_{n-1,2l+1} - u_{n+1,2l+1})[1 - \delta_1(1 - u_{n,2l+1})]}{\delta_2 + u_{n,2l+1}} = \xi_n, \tag{4.71}$$

We can easily see that the equation is genuinely nonlinear. However we can separate the cases when n is even and odd and convert (4.71) into a system using the standard transformation (4.55b):

$$z_{k,l} - \xi_{2k} z_{k-1,l} = \left(1 - \frac{1}{\delta_1}\right)(1 - \xi_{2k}), \tag{4.72a}$$

$$y_{k+1,l} - y_{k,l} = \frac{\xi_{2k+1}}{\delta_1}\frac{\delta - 1 + \delta_1(1 - z_{k,l})}{1 - \delta_1(1 - z_{k,l})}, \tag{4.72b}$$

where we used the definition (4.63). This is a system of two *first order* difference equation, and equation (4.72a) is linear and uncoupled from (4.72b). As $\xi_{2k} = b_k/b_{k-1}$ we have that (4.72a) is a total difference:

$$\frac{z_{k,l}}{b_k} - \frac{z_{k-1,l}}{b_{k-1}} = \left(1 - \frac{1}{\delta_1}\right)\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right). \tag{4.73}$$

Hence the solution of (4.73) is given by:

$$z_{k,l} = 1 - \frac{1}{\delta_1} + b_k \gamma_l. \tag{4.74}$$

Inserting (4.74) into (4.72b) and using the definition of ξ_{2k+1} in terms of a_k, i.e. $\xi_{2k+1} = a_{k+1} - a_k$ we obtain:

$$y_{k+1,l} - y_{k,l} = -\left(\frac{1}{\delta_1} + \frac{\delta}{\delta_1^2 b_k \gamma_l}\right)(a_{k+1} - a_k). \tag{4.75}$$

We can then represent the solution of (4.73) as:

$$y_{k,l} = \delta_l + \frac{\delta d_k}{\delta_1^2 \gamma_l} - \frac{a_k}{\delta_1}, \tag{4.76}$$

where d_k satisfies the first order linear difference equation:

$$d_{k+1} - d_k = \frac{a_{k+1} - a_k}{b_k}. \tag{4.77}$$

Inserting the value of a_k given by (4.69) inside (4.77) we obtain that this equation is a total difference. Then d_k is given by:

$$d_k = -\frac{(c_k - c_{k-1}) b_{k-1} \delta_1}{(b_k - b_{k-1})\delta} - \frac{\delta_1 c_k}{\delta}. \tag{4.78}$$

This means that finally we have the following solutions for the fields $z_{k,l}$ and $y_{k,l}$:

$$z_{k,l} = 1 - \frac{1}{\delta_1} + b_k \gamma_l, \tag{4.79a}$$

$$y_{k,l} = \delta_l - \frac{(c_k - c_{k-1}) b_{k-1}}{(b_k - b_{k-1})\delta_1 \gamma_l} - \frac{c_k}{\delta_1 \gamma_l} + \frac{1}{\delta} \frac{b_k b_{k-1} (c_k - c_{k-1})}{b_k - b_{k-1}}. \tag{4.79b}$$

Equations (4.70,4.79) provide the value of the four fields, but we have too many arbitrary functions in the m direction, namely α_l, β_l, γ_l and δ_l. Introducing (4.70,4.79) into (1.93b) and separating the terms even and odd in n and m we obtain two independent equations:

$$(\alpha_l + \delta_1 \delta_l)\gamma_l + \beta_l = 0, \quad (\alpha_{l+1} + \delta_1 \delta_l)\gamma_l + \beta_{l+1} = 0, \tag{4.80}$$

which allow us to reduce by two the number of independent functions in the m direction. Solving equations (4.80) with respect to γ_l and δ_l we obtain:

$$\gamma_l = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \quad \delta_l = \frac{1}{\delta_1} \frac{\beta_l \alpha_{l+1} - \beta_{l+1} \alpha_l}{\beta_{l+1} - \beta_l}. \tag{4.81}$$

Therefore the general solution for the ${}_1D_2$ equation (1.93b) is given by:

$$v_{k,l} = \alpha_l - \frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}\left(c_k - c_{k-1}\right), \tag{4.82a}$$

$$w_{k,l} = b_k\left(\beta_l + c_k\right) + \frac{\delta}{\delta_1}\alpha_l, \tag{4.82b}$$

$$z_{k,l} = 1 - \frac{1}{\delta_1} - b_k\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \tag{4.82c}$$

$$\begin{aligned} y_{k,l} = & \frac{1}{\delta_1}\frac{\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l}{\beta_{l+1} - \beta_l} + \frac{1}{\delta}\frac{b_k b_{k-1}\left(c_k - c_{k-1}\right)}{b_k - b_{k-1}} \\ & + \left[\frac{(c_k - c_{k-1})b_{k-1}}{(b_k - b_{k-1})\delta_1} + \frac{c_k}{\delta_1}\right]\frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l}. \end{aligned} \tag{4.82d}$$

Remark 4.4.1. It is easy to see that the solution (4.82) is ill-defined if $\delta_1 = 0$ or $\delta = 0$. We will treat these two particular cases separately.

CASE $\delta = 0$: If $\delta = 0$ we can solve (4.63) with respect to δ_1:

$$\delta_1 = \frac{1}{1 + \delta_2}. \tag{4.83}$$

The first integral (4.38a) is not singular for δ_1 given by (4.83). The procedure of solution will become different only when we arrive to the systems of ordinary difference equations (4.62) and (4.72). So we will present the solution of the systems in this case.

CASE $m = 2k$: If δ_1 is given by equation (4.83) the system (4.62) becomes:

$$w_{k,l} - \xi_{2k} w_{k-1,l} = 0, \tag{4.84a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}. \tag{4.84b}$$

The system (4.84) is uncoupled and imposing $\xi_{2k} = a_k/a_{k-1}$ and $\xi_{2k+1} = b_{k+1} - b_k$ it is readily solved to give:

$$w_{k,l} = a_k\alpha_l, \tag{4.85a}$$

$$v_{k,l} = b_k + \beta_l. \tag{4.85b}$$

CASE $m = 2k + 1$: If δ_1 is given by equation (4.83) the system (4.72) becomes:

$$\frac{\delta_2 + z_{k,l}}{a_k} = \frac{\delta_2 + z_{k-1,m}}{a_{k-1}}, \tag{4.86a}$$

$$-\frac{y_{k+1,l} - y_{k,l}}{1 + \delta_2} = b_{k+1} - b_k, \tag{4.86b}$$

where we used the fact that $\xi_{2k} = a_k/a_{k-1}$ and $\xi_{2k+1} = b_{k+1} - b_k$. The solution to this system is immediate and it is given by:

$$z_{k,l} = a_k \gamma_l - \delta_2, \tag{4.87a}$$

$$y_{k,l} = (1+\delta_2)(\delta_l - b_k). \tag{4.87b}$$

As in the general case we obtained the expressions of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and δ_l. Substituting the obtained expressions (4.85,4.87) in the equation ${}_1D_2$ (1.93b) with δ_1 given by equation (4.83) separating the even and odd terms we obtain two compatibility conditions:

$$\alpha_l + \gamma_l \beta_l + \gamma_l \delta_l = 0, \quad \alpha_{l+1} + \gamma_l \beta_{l+1} + \gamma_l \delta_l = 0. \tag{4.88}$$

We can solve this equation with respect to γ_l and δ_l and we obtain:

$$\gamma_l = -\frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l}, \quad \delta_l = -\frac{\beta_l \alpha_{l+1} - \beta_{l+1}\alpha_l}{\alpha_{l+1} - \alpha_l}. \tag{4.89}$$

Therefore the general solution for the ${}_1D_2$ equation (1.93b) if δ_1 is given by (4.83) is:

$$w_{k,l} = a_k \alpha_l, \tag{4.90a}$$

$$v_{k,l} = b_k + \beta_l. \tag{4.90b}$$

$$z_{k,l} = -a_k \frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l} - \delta_2, \tag{4.90c}$$

$$y_{k,l} = -(1+\delta_2)\left(\frac{\beta_l \alpha_{l+1} - \beta_{l+1}\alpha_l}{\alpha_{l+1} - \alpha_l} + b_k\right). \tag{4.90d}$$

CASE $\delta_1 = 0$: If $\delta_1 = 0$ the first integral (4.38a) is singular. Then following Remark 4.3.4 the ${}_1D_2$ equation (1.93b) possesses in the direction n three-point, second order integral $W_1^{(0,\delta_2)}$ (4.39a). In order to solve the ${}_1D_2$ in this case equation (1.93b) we use the first integral (4.39a). We start separating the cases even and odd in m.

CASE $m = 2k$: If $m = 2k$ we obtain from the first integral (4.39a):

$$F_n^{(+)} \frac{u_{n+1,2l} - u_{n-1,2l}}{u_{n,2l}} + F_n^{(-)}(u_{n+1,2l} - u_{n-1,2l}) = \xi_n, \tag{4.91}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. This equation is non-linear. Applying the transformation (4.55a) we transform equation (4.91) into the system:

$$w_{k,l} - w_{k-1,l} = \xi_{2k} v_{k,l}, \tag{4.92a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}. \tag{4.92b}$$

The system (4.92) is linear and equation (4.92b) is uncoupled from equation (4.92a). If we put $\xi_{2k+1} = a_{k+1} - a_k$ then equation (4.92b) has the solution:

$$v_{k,l} = a_k + \alpha_l. \tag{4.93}$$

Substituting into (4.92a) we obtain:

$$w_{k,l} - w_{k-1,l} = \xi_{2k}(a_k + \alpha_l). \tag{4.94}$$

Equation (4.94) becomes a total difference if we set:

$$\xi_{2k} = b_k - b_{k-1}, \quad a_k = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} \tag{4.95}$$

and then the solution of the system (4.84) is given by:

$$v_{k,l} = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} + \alpha_l, \tag{4.96a}$$

$$w_{k,l} = c_k + \alpha_l b_k + \beta_l. \tag{4.96b}$$

CASE $m = 2k+1$: If $m = 2k+1$ we obtain from the first integral (4.39a):

$$F_n^{(+)}(u_{n-1,2l+1} - u_{n+1,2l+1}) - F_n^{(-)}\frac{(u_{n+1,2l+1} - u_{n-1,2l+1})}{\delta_2 + u_{n,2l+1}} = \xi_n, \tag{4.97}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. This equation is non-linear. Applying the transformation (4.55b) we transform equation (4.97) into the system:

$$z_{k-1,l} - z_{k,l} = b_k - b_{k-1}, \tag{4.98a}$$

$$y_{k,l} - y_{k+1,l} = (a_{k+1} - a_k)(\delta_2 + z_{k,l}), \tag{4.98b}$$

where we used the values of ξ_{2k} and ξ_{2k+1}. The system is now linear and equation (4.98a) can be already solved to give:

$$z_{k,l} = \gamma_l - b_k. \tag{4.99}$$

Substituting into (4.98b) we obtain:

$$y_{k,l} - y_{k+1,l} = (a_{k+1} - a_k)(\delta_2 + \gamma_l - b_k). \tag{4.100}$$

Then we have that $y_{k,l}$ is given by:

$$y_{k,l} = \delta_l - (\gamma_l + \delta_2)\, a_k + d_k, \tag{4.101}$$

where d_k solves the ordinary difference equation:

$$d_{k+1} - d_k = b_{k+1}\frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1} - b_k\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + c_k.$$

(4.102)

In (4.102) we inserted the value of a_k according to (4.95). Equation (4.102) is a total difference and then d_k is given by:

$$d_k = b_k \frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k. \tag{4.103}$$

Then the solution of the system (4.98) is:

$$z_{k,l} = \gamma_l - b_k, \tag{4.104a}$$

$$y_{k,l} = \delta_l - (\gamma_l + \delta_2)\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + b_k \frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k. \tag{4.104b}$$

As in the general case we obtained the expressions of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and δ_l. Substituting the obtained expressions (4.96,4.104) in the equation ${}_1D_2$ (1.93b) with $\delta_1 = 0$ separating the even and odd terms we obtain two compatibility conditions:

$$(\gamma_l + \delta_2)\alpha_l + \delta_l + \beta_l = 0, \quad (\gamma_l + \delta_2)\alpha_{l+1} + \beta_{l+1} + \delta_l = 0. \tag{4.105}$$

We can solve equation (4.105) with respect to γ_l and δ_l and to obtain:

$$\gamma_l = -\delta_2 - \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \quad \delta_l = \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l}. \tag{4.106}$$

Therefore the general solution for the ${}_1D_2$ equation (1.93b) if $\delta_1 = 0$ is:

$$v_{k,l} = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} + \alpha_l, \tag{4.107a}$$

$$w_{k,l} = c_k + \alpha_l b_k + \beta_l. \tag{4.107b}$$

$$z_{k,l} = -\delta_2 - \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} - b_k, \tag{4.107c}$$

$$y_{k,l} = \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + b_k \frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k. \tag{4.107d}$$

This discussion exhausts the possible cases. For any value of the parameters we have the general solution of the ${}_1D_2$ equation (1.93b).

4.4.1.2 ${}_2D_2$ *equation*

From Section 4.3 we know that the ${}_2D_2$ equation (1.93c) possesses a three-point, second order first integral W_1 (4.41a). As stated at the beginning of this Section from the relation $W_1 = \xi_n$ this integral defines a three-point, second order ordinary difference equation in the n

direction which depends parametrically on m. From this parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

CASE $m = 2l$: If $m = 2l$ we have the following non-autonomous nonlinear ordinary difference equation:

$$\begin{aligned} &F_n^{(+)} \alpha \frac{\delta_2 + u_{n+1,2l}}{\delta_2 + u_{n-1,2l}} \\ &- F_n^{(-)} \beta \frac{(u_{n-1,2l} - u_{n+1,2l})(u_{n,2l} + \delta_2)}{1 + (u_{n,2l} - 1)\delta_1} = \xi_n. \end{aligned} \tag{4.108}$$

Without loss of generality we set $\alpha = 1$ and $\beta = \delta_1$. Then making the transformation

$$u_{n,2l} = U_{n,2l} - \delta_2 \tag{4.109}$$

and putting

$$\delta = \frac{1 - \delta_1 - \delta_1 \delta_2}{\delta_1} \tag{4.110}$$

equation (4.108) is mapped to:

$$F_n^{(+)} \frac{U_{n+1,2l}}{U_{n-1,2l}} - F_n^{(-)} \frac{(U_{n-1,2l} - U_{n+1,2l}) U_{n,2l}}{U_{n,2l} + \delta} = \xi_n. \tag{4.111}$$

From the definition (4.55a) applied to $U_{n,2l}$ instead of $u_{n,2l}$[4] we can separate again the even and the odd part in (4.111). We obtain the following system of two coupled *first-order* ordinary difference equations:

$$W_{k,l} - \xi_{2k} W_{k-1,l} = 0, \tag{4.112a}$$

$$V_{k+1,l} - V_{k,l} = \xi_{2k+1} \left(1 + \frac{\delta}{W_{k,l}} \right). \tag{4.112b}$$

Putting $\xi_{2k} = a_k / a_{k-1}$ the solution to (4.112a) is immediately given by:

$$W_{k,l} = a_k \alpha_l. \tag{4.113}$$

Inserting the value of $W_{k,l}$ from (4.113) into (4.112b) we obtain:

$$V_{k+1,l} - V_{k,l} = \xi_{2k+1} \left(1 + \frac{\delta}{a_k \alpha_l} \right). \tag{4.114}$$

If we define

$$\xi_{2k+1} = b_{k+1} - b_k, \quad a_k = \frac{b_{k+1} - b_k}{c_{k+1} - c_k}, \tag{4.115}$$

4 We will denote the corresponding fields with capital letters.

then (4.114) becomes a total difference. So we obtain the following solutions for the $W_{k,l}$ and the $V_{k,l}$ fields:

$$W_{k,l} = \alpha_l \frac{b_{k+1} - b_k}{c_{k+1} - c_k}, \tag{4.116a}$$

$$V_{k,l} = b_k + \beta_l + \delta \frac{c_k}{\alpha_l}. \tag{4.116b}$$

Inverting the transformation (4.109) we obtain for the fields $w_{k,l}$ and $v_{k,l}$:

$$w_{k,l} = \alpha_l \frac{b_{k+1} - b_k}{c_{k+1} - c_k} + \frac{1}{\delta_1} - 1 - \delta, \tag{4.117a}$$

$$v_{k,l} = b_k + \beta_l + \delta \frac{c_k}{\alpha_l} + \frac{1}{\delta_1} - 1 - \delta. \tag{4.117b}$$

CASE $m = 2l + 1$: If $m = 2l+1$ we have the following non-autonomous ordinary difference equation:

$$F_n^{(+)} \frac{\delta\delta_1 u_{n,2l+1} + u_{n+1,2l+1}}{\delta\delta_1 u_{n,2l+1} + u_{n-1,2l+1}} + F_n^{(-)} \delta_1 \left(u_{n-1,2l+1} - u_{n+1,2l+1}\right) = \xi_n, \tag{4.118}$$

where we already substituted δ as defined in (4.110). Using the standard transformation to get rid of the two-periodic factors (4.55b) we obtain:

$$\frac{\delta\delta_1 y_{k,l} + z_{k,l}}{\delta\delta_1 y_{k,l} + z_{k-1,l}} = \xi_{2k}, \tag{4.119a}$$

$$\delta_1 \left(y_{k,l} - y_{k+1,l}\right) = \xi_{2k+1}. \tag{4.119b}$$

Both equations in (4.119) are linear in $z_{k,l}$ $y_{k,l}$ and their shifts. As $\xi_{2k+1} = b_{k+1} - b_k$ we have that the solution of equation (4.119b) is given by:

$$y_{k,l} = \gamma_l - \frac{b_k}{\delta_1}. \tag{4.120}$$

As $\xi_{2k} = a_k / a_{k-1}$ and $y_{k,l}$ given by (4.120) we obtain:

$$\frac{z_{k,l}}{a_k} - \frac{z_{k-1,l}}{a_{k-1}} = \left(\frac{1}{a_k} - \frac{1}{a_{k-1}}\right) \left(\delta b_k - \delta\delta_1 \gamma_l\right). \tag{4.121}$$

Recalling the definition of a_k in (4.115) we represent $z_{k,l}$ as:

$$z_{k,l} = \delta b_k + \frac{b_{k+1} - b_k}{c_{k+1} - c_k} \left(\delta_l - \delta c_k\right) - \delta\delta_1 \gamma_l. \tag{4.122}$$

Equations (4.117,4.120,4.122) provide the value of the four fields, but we have too many arbitrary functions in the m direction, namely

α_l, β_l, γ_l and δ_l. Inserting (4.117,4.120,4.122) into (1.93c) and separating the terms even and odd in n and m we obtain two independent equations:

$$\delta_1\delta_l + \alpha_l\delta_1^2\gamma_l - {\delta_1}^2\lambda\alpha_l + \beta_l\alpha_l\delta_1 + \delta\alpha_l\delta_1 - \alpha_l + \alpha_l\delta_1 = 0, \tag{4.123a}$$

$$\delta_1\delta_l + \alpha_{l+1}\delta_1^2\gamma_l - \delta_1^2\lambda\alpha_{l+1} + \beta_{l+1}\alpha_{l+1}\delta_1 + \delta\alpha_{l+1}\delta_1 - \alpha_{l+1} + \alpha_{l+1}\delta_1 = 0, \tag{4.123b}$$

which allow us to reduce by two the number of independent functions in the m direction. Solving (4.123) with respect to γ_l and δ_l we find:

$$\gamma_l = -\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1^2} - \frac{\alpha_l(\beta_{l+1} - \beta_l)}{(\alpha_{l+1} - \alpha_l)\delta_1}, \tag{4.124a}$$

$$\delta_l = \alpha_l(\beta_{l+1} - \beta_l) + \frac{\alpha_l^2(\beta_{l+1} - \beta_l)}{\alpha_{l+1} - \alpha_l} \tag{4.124b}$$

Therefore the general solution of the ${}_2D_2$ equation (1.93c) is given by:

$$v_{k,l} = b_k + \beta_l + \delta\frac{c_k}{\alpha_l} + \frac{1}{\delta_1} - 1 - \delta. \tag{4.125a}$$

$$w_{k,l} = \alpha_l\frac{b_{k+1} - b_k}{c_{k+1} - c_k} + \frac{1}{\delta_1} - 1 - \delta. \tag{4.125b}$$

$$\begin{aligned} z_{k,l} &= \delta b_k + \delta\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1} \\ &+ \frac{b_{k+1} - b_k}{c_{k+1} - c_k}\left[\alpha_l(\beta_{l+1} - \beta_l) + \frac{\alpha_l^2(\beta_{l+1} - \beta_l)}{\alpha_{l+1} - \alpha_l} - \delta c_k\right] \\ &+ \frac{\delta\alpha_l(\beta_{l+1} - \beta_l)}{\alpha_{l+1} - \alpha_l}, \end{aligned} \tag{4.125c}$$

$$\begin{aligned} y_{k,l} &= -\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1^2} \\ &- \frac{\alpha_l(\beta_{l+1} - \beta_l)}{(\alpha_{l+1} - \alpha_l)\delta_1} - \frac{b_k}{\delta_1}. \end{aligned} \tag{4.125d}$$

Remark 4.4.2. It is easy to see that the solution of the ${}_2D_2$ equation (1.93c) given by (4.125) is ill-defined if $\delta_1 = 0$. Therefore we have to threat this case separately. Following Remark 4.3.5 we have the ${}_2D_2$ equation (1.93c) with $\delta_1 = 0$ possesses the following two-point, first order first integral in the direction n $W_1^{(0,\delta_2)}$ (4.42a) To solve the ${}_2D_2$ equation (1.93c) with $\delta_1 = 0$ we use the first integral (4.42a). Again we start separating the cases m even and odd in (4.42a).

CASE $m = 2k$: If $m = 2k$ we obtain from the first integral (4.42a):

$$F_n^{(+)}(\delta_2 + u_{n+1,2l})u_{n,2l} + F_n^{(-)}(\delta_2 + u_{n,2l})u_{n+1,2l} = \xi_n,$$

(4.126)

where we have chosen without loss of generality $\alpha = \beta = 1$. Applying the transformation (4.55a) equation (4.126) becomes the system:

$$v_{k,l}(\delta_2 + w_{k,l}) = \xi_{2k}, \tag{4.127a}$$

$$v_{k+1,l}(\delta_2 + w_{k,l}) = \xi_{2k+1}. \tag{4.127b}$$

In this case the system (4.127) do not consist of purely difference equations. Indeed from (4.127a) we can derive immediately the value of the field $w_{k,l}$:

$$w_{k,l} = -\delta_2 + \frac{\xi_{2k}}{v_{k,l}}. \tag{4.128}$$

Inserting (4.128) into (4.127b) we obtain that $v_{k,l}$ solves the equation:

$$v_{k+1,l} - \frac{\xi_{2k+1}}{\xi_{2k}} v_{k,l} = 0. \tag{4.129}$$

Defining:

$$\xi_{2k+1} = \frac{a_{k+1}}{a_k} \xi_{2k}, \tag{4.130}$$

we have that (4.129) becomes a total difference. So we have that the system (4.127) is solved by:

$$v_{k,l} = a_k \alpha_l, \tag{4.131a}$$

$$w_{k,l} = -\delta_2 + \frac{\xi_{2k}}{a_k \alpha_l}. \tag{4.131b}$$

CASE $m = 2k + 1$: If $m = 2k + 1$ we obtain from the first integral (4.42a):

$$F_n^{(+)}(u_{n+1,2l+1} + u_{n,2l+1}) + F_n^{(-)}(u_{n+1,2l+1} + u_{n,2l+1}) = -\xi_n. \tag{4.132}$$

Applying the transformation (4.55b) equation (4.132) becomes the system:

$$y_{k,l} + z_{k,l} = -\xi_{2k}, \tag{4.133a}$$

$$z_{k,l} + y_{k+1,l} = -\frac{a_{k+1}}{a_k} \xi_{2k}, \tag{4.133b}$$

where ξ_{2k+1} is given by (4.130). Equation (4.133a) is not a difference equation and can be solved to give:

$$z_{k,l} = -\xi_{2k} - y_{k,l}, \tag{4.134}$$

which inserted in (4.133b) gives:

$$y_{k+1,l} - y_{k,l} = \left(1 - \frac{a_{k+1}}{a_k}\right) \xi_{2k}. \tag{4.135}$$

Defining:

$$\xi_{2k} = -a_k \frac{b_{k+1} - b_k}{a_{k+1} - a_k} \tag{4.136}$$

equation (4.135) becomes a total difference. Therefore we can write the solution of the system (4.133) as:

$$y_{k,l} = b_k + \beta_l, \tag{4.137a}$$

$$z_{k,l} = a_k \frac{b_{k+1} - b_k}{a_{k+1} - a_k} - b_k - \beta_l. \tag{4.137b}$$

In this case we have the right number of arbitrary functions in both directions. So the solution of the ${}_2D_2$ equation with $\delta_1 = 0$ is given by:

$$v_{k,l} = a_k \alpha_l, \tag{4.138a}$$

$$w_{k,l} = -\delta_2 - \frac{1}{\alpha_l} \frac{b_{k+1} - b_k}{a_{k+1} - a_k}, \tag{4.138b}$$

$$y_{k,l} = b_k + \beta_l, \tag{4.138c}$$

$$z_{k,l} = a_k \frac{b_{k+1} - b_k}{a_{k+1} - a_k} - b_k - \beta_l. \tag{4.138d}$$

Inserting (4.138) into the ${}_2D_2$ equation (1.93c) and separating the even and odd terms we verify that it is a solution.

4.4.1.3 ${}_3D_2$ *equation*

From Section 4.3 we know that the ${}_2D_2$ equation (1.93c) possesses a three-point, second order first integral W_1 (4.41a). As stated at the beginning of this Section from the relation $W_1 = \xi_n$ this integral defines a three-point, second order ordinary difference equation in the n direction which depends parametrically on m. From this parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

CASE $m = 2l$: If $m = 2l$ we have the following non-autonomous nonlinear ordinary difference equation:

$$\begin{aligned} &F_n^{(+)} \frac{(u_{n-1,2l} + \delta_2)\,[1 + (u_{n+1,2l} - 1)\,\delta_1]}{(u_{n+1,2l} + \delta_2)\,[1 + (u_{n-1,2l} - 1)\,\delta_1]} \\ &+ F_n^{(-)} (u_{n+1,2l} - u_{n-1,2l})\,(\delta_2 + u_{n,2l}) = \xi_n \end{aligned} \tag{4.139}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. We can apply the usual transformation (4.55a) in order to separate the even and odd part in (4.139):

$$\frac{1+(w_{k,l}-1)\,\delta_1}{w_{k,l}+\delta_2} = \xi_{2k}\frac{1+(w_{k-1,l}-1)\,\delta_1}{w_{k-1,l}+\delta_2}, \tag{4.140a}$$

$$v_{k+1,l}-v_{k,l} = \frac{\xi_{2k+1}}{(\delta_2+w_{k,l})}. \tag{4.140b}$$

This system of equations is still non-linear, but the equation (4.140a) is uncoupled from (4.140b). Moreover equation (4.140a) is a discrete Riccati equation which can be linearized through the Möbius transformation:

$$w_{k,l} = -\delta_2 + \frac{1}{W_{k,l}}, \tag{4.141}$$

into:

$$W_{k,l} - \xi_{2k}W_{k-1,l} = \frac{\delta_1}{\delta}\,(\xi_{2k}-1)\,, \tag{4.142a}$$

$$v_{k+1,l}-v_{k,l} = \xi_{2k+1}W_{k,l}. \tag{4.142b}$$

where δ is given by equation (4.63).Putting $\xi_{2k} = a_k/a_{k-1}$ we have the following solution for (4.142a):

$$W_{k,l} = a_k\alpha_l - \frac{\delta_1}{\delta}. \tag{4.143}$$

Plugging (4.143) into equation (4.142b) and defining

$$\xi_{2k+1} = b_{k+1}-b_k, \quad a_k = \frac{c_{k+1}-c_k}{b_{k+1}-b_k}, \tag{4.144}$$

we have that equation (4.142b) becomes a total difference. Then the solution of (4.142b) can be written as:

$$v_{k,l} = -\frac{\delta_1}{\delta}b_k + c_k\alpha_l + \beta_l. \tag{4.145}$$

So using (4.141) we obtain the following solution for the original system (4.140):

$$w_{k,l} = \frac{\delta_1 - 1 + \delta}{\delta_1} + \frac{\delta(b_{k+1}-b_k)}{\delta\alpha_m(c_{k+1}-c_k)-(b_{k+1}-b_k)\delta_1}, \tag{4.146a}$$

$$v_{k,l} = -\frac{\delta_1}{\delta}b_k + c_k\alpha_l + \beta_l. \tag{4.146b}$$

case $m = 2l+1$: If $m = 2l+1$ we have the following non-autonomous ordinary difference equation:

$$F_n^{(+)}\frac{u_{n,2l+1}+\delta u_{n-1,2l+1}}{u_{n,2l+1}+\delta u_{n+1,2l+1}} - F_n^{(-)}\left(u_{n+1,2l+1}-u_{n-1,2l+1}\right) = \xi_n.$$

(4.147)

where without loss of generality $\alpha = \beta = 1$ and δ is given by (4.63). Solving with respect to $u_{n+1,2l+1}$ it is immediate to see that the resulting equation is linear. Then separating the even and the odd part using the transformation (4.55b) we obtain the following system of linear, first-order ordinary difference equations:

$$z_{k,l} - \frac{1}{\xi_{2k}} z_{k-1,l} = \frac{1}{\delta}\left(1 - \frac{1}{\xi_{2k}}\right) y_{k,l}, \tag{4.148a}$$

$$y_{k+1} - y_k = -\xi_{2k+1}. \tag{4.148b}$$

As $\xi_{2k+1} = b_{k+1} - b_k$ we obtain immediately the solution of equation (4.148b) as:

$$y_{k,l} = -b_k + \gamma_l. \tag{4.149}$$

Substituting $y_{k,l}$ given by (4.149) into equation (4.148a) being $\xi_{2k} = a_k / a_{k-1}$, we obtain:

$$a_k z_{k,l} - a_{k-1} z_{k-1,l} = \frac{a_k - a_{k-1}}{\delta} (b_k - \gamma_l). \tag{4.150}$$

Then, in the usual way, we can represent the solution as:

$$z_{k,l} = \frac{b_k - \gamma_l}{\delta} + \frac{b_{k+1} - b_k}{c_{k+1} - c_k}\left(\delta_l - \frac{c_k}{\delta}\right), \tag{4.151}$$

where we have used the explicit definition of a_k given in (4.144). So we have the explicit expression for both fields $y_{k,l}$ and $z_{k,l}$.

Equations (4.146,4.149,4.151) provide the value of the four fields, but we have too many arbitrary functions in the m direction, namely α_l, β_l, γ_l and δ_l. Inserting (4.146,4.149,4.151) into (1.93d) and separating the terms even and odd in n and m we obtain we obtain two equations:

$$\delta_l \delta^2 \alpha_l + (\beta_l - \delta_1 \lambda)\,\delta - \delta_1 \gamma_l = 0, \tag{4.152a}$$

$$\delta_l \delta^2 \alpha_{l+1} + (\beta_{l+1} - \delta_1 \lambda)\,\delta - \delta_1 \gamma_l = 0, \tag{4.152b}$$

which allow us to reduce by two the number of independent functions in the m direction. Indeed solving (4.152) with respect to γ_l and δ_l we find:

$$\gamma_l = \frac{\delta}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\right), \tag{4.153a}$$

$$\delta_l = -\frac{1}{\delta}\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}. \tag{4.153b}$$

Therefore the general solution of the $_3D_2$ equation (1.93d) is given by:

$$v_{k,l} = -\frac{\delta_1}{\delta} b_k + c_k \alpha_l, \tag{4.154a}$$

$$w_{k,l} = \frac{\delta_1 - 1 + \delta}{\delta_1} + \frac{\delta(b_{k+1} - b_k)}{\delta\alpha_l(c_{k+1} - c_k) - (b_{k+1} - b_k)\delta_1}, \tag{4.154b}$$

$$z_{k,l} = \frac{b_k}{\delta} - \frac{1}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\right) + \frac{1}{\delta}\frac{b_{k+1} - b_k}{c_{k+1} - c_k}\left(\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} + c_k\right), \tag{4.154c}$$

$$y_{k,l} = -b_k + \frac{\delta}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\right). \tag{4.154d}$$

Remark 4.4.3. It is easy to see that the solution (4.82) is ill-defined if $\delta_1 = 0$ and if $\delta = 0$. We will treat these two particular cases separately.

CASE $\delta = 0$: If $\delta = 0$ we have that δ_1 is given by equation (4.83). In this case the first integral (4.44a) is singular since the coefficient of α goes to a constant. Following Remark 4.3.6 we have that the $_3D_2$ equation with δ_1 given by (4.83) the first integral in the direction n is given by $W_1^{((1+\delta_2)^{-1},\delta_2)}$ (4.46a). This first integral is a three-point, second order first integral. As in the general case we consider separately the m even and odd cases.

CASE $m = 2k$: If $m = 2k$ then the first integral (4.46a) becomes the following non-linear three-point, second order difference equation:

$$F_n^{(+)} \frac{u_{n+1,2l} - u_{n-1,2l}}{(\delta_2 + u_{n+1,2l})(\delta_2 + u_{n-1,2l})} - F_n^{(-)}(u_{n-1,2l} - u_{n+1,2l})(\delta_2 + u_{n,2l}) = \xi_n, \tag{4.155}$$

where without loss of generality $\alpha = \beta = 1$. If we separate the even and the odd part using the general transformation given by (4.55a) we obtain the system:

$$\frac{w_{k,l} - w_{k-1,l}}{(w_{k,l} + \delta_2)(w_{k-1,l} + \delta_2)} = \xi_{2k}, \tag{4.156a}$$

$$(v_{k+1,l} - v_{k,l})(w_{k,l} + \delta_2) = \xi_{2k+1}. \tag{4.156b}$$

This is a system of first order non-linear difference equations. However (4.156a) is uncoupled from (4.140b), and it is a discrete Riccati equation which can be linearized through the Möbius transformation (4.141). This linearize the system (4.156) to:

$$W_{k,l} - W_{k-1,l} = \xi_{2k}, \tag{4.157a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1} W_{k,l}. \tag{4.157b}$$

Defining $\xi_{2k} = a_k - a_{k-1}$ equation (4.157a) is solved by:

$$W_{k,l} = a_k + \beta_l. \tag{4.158}$$

Introducing (4.158) into equation (4.157b) we have:

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}\left(a_k + \alpha_l\right). \tag{4.159}$$

Equation (4.159) becomes a total difference if:

$$\xi_{2k+1} = b_{k+1} - b_k, \quad a_k = \frac{c_{k+1} - c_k}{b_{k+1} - b_k}. \tag{4.160}$$

This yields the following solution of the system (4.156):

$$v_{k,l} = c_k + b_k\alpha_l + \beta_l, \tag{4.161a}$$

$$w_{k,l} = -\delta_2 + \frac{b_{k+1} - b_k}{c_{k+1} - c_k + \alpha_l\left(b_{k+1} - b_k\right)}. \tag{4.161b}$$

CASE $m = 2k+1$: If $m = 2k+1$ the first integral (4.46a) becomes the following nonlinear, three-point, second order difference equation:

$$F_n^{(+)}\frac{u_{n+1,2l+1} - u_{n-1,2l+1}}{\left(\delta_2 + 1\right)u_{n,2l+1}} - F_n^{(-)}\left(u_{n+1,2l+1} - u_{n-1,2l+1}\right) = \xi_n, \tag{4.162}$$

where without loss of generality $\alpha = \beta = 1$. As usual we can separate the even and odd part in n using the transformation (4.55b). This transformation brings equation (4.162) into the following linear system:

$$z_{k,l} - z_{k-1,l} = (\delta_2 + 1)y_{k,l}\left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k}\right), \tag{4.163a}$$

$$y_{k+1,l} - y_{k,l} = -b_{k+1} + b_k, \tag{4.163b}$$

where we used (4.160) and the definition $\xi_{2k+1} = a_{k+1} - a_k$. Equation (4.163b) is readily be solved and gives:

$$y_{k,l} = \gamma_l - b_k. \tag{4.164}$$

Inserting (4.164) into (4.163a) we obtain:

$$z_{k,l} - z_{k-1,l} = \left(\delta_2 + 1\right)\left(\gamma_l - b_k\right)\left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k}\right). \tag{4.165}$$

We can then write for $z_{k,l}$ the following expression:

$$z_{k,l} = -(\delta_2 + 1)\left(\gamma_l\frac{c_{k+1} - c_k}{b_{k+1} - b_k} + d_k\right) + \delta_l, \tag{4.166}$$

where d_k solves the equation:

$$d_k - d_{k-1} = -b_k \left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k} \right). \tag{4.167}$$

Equation (4.167) is a total difference with d_k given by:

$$d_k = b_{k+1} \frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1}. \tag{4.168}$$

Therefore we have the following solution to the system (4.163):

$$y_{k,l} = \gamma_l - b_k, \tag{4.169a}$$

$$z_{k,l} = -(\delta_2 + 1) \left[(\gamma_l + b_{k+1}) \frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1} \right] + \delta_l. \tag{4.169b}$$

Equations (4.161,4.169) provide the value of the four fields, but we have too many arbitrary functions in the m direction, namely α_l, β_l, γ_l and δ_l. Inserting (4.161,4.169) into (1.93d) with δ_1 given by (4.83) and separating the terms even and odd in n and m we obtain we obtain two equations:

$$\gamma_l(\delta_2 + 1)\alpha_l - \lambda + \delta_l + \beta_l(\delta_2 + 1) = 0, \tag{4.170a}$$

$$\gamma_l(\delta_2 + 1)\alpha_{l+1} - \lambda + \delta_l + \beta_{l+1}(\delta_2 + 1) = 0. \tag{4.170b}$$

Solving this compatibility condition with respect to γ_l and δ_l we obtain:

$$\gamma_l = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \tag{4.171a}$$

$$\delta_l = (\delta_2 + 1) \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \lambda. \tag{4.171b}$$

Inserting then (4.171) into (4.161,4.169) we obtain the following expression for the solution of the ${}_3D_2$ when δ_1 is given by (4.83):

$$v_{k,l} = c_k + b_k\alpha_l + \beta_l, \tag{4.172a}$$

$$w_{k,l} = -\delta_2 + \frac{b_{k+1} - b_k}{c_{k+1} - c_k + \alpha_l (b_{k+1} - b_k)}, \tag{4.172b}$$

$$y_{k,l} = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} - b_k, \tag{4.172c}$$

$$z_{k,l} = -(\delta_2 + 1) \left[\left(b_{k+1} - \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} \right) \frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1} \right] + (\delta_2 + 1) \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \lambda. \tag{4.172d}$$

CASE $\delta_1 = 0$: The first integral (4.44a) is not singular when inserting $\delta_1 = 0$. Therefore the procedure of solution will become different only when we arrive to the systems of ordinary difference equations (4.140) and (4.148). So we will present the solution of the systems in this case.

CASE $m = 2k$: If $\delta_1 = 0$ the system (4.140) becomes:

$$w_{k,l} + \delta_2 = \frac{w_{k-1,l} + \delta_2}{\xi_{2k}}, \tag{4.173a}$$

$$v_{k+1,l} - v_{k,l} = \frac{\xi_{2k+1}}{(\delta_2 + w_{k,l})}. \tag{4.173b}$$

The system (4.173) is nonlinear, but equation (4.173a) is uncoupled from equation (4.173a). Defining $\xi_{2k} = a_{k-1}/a_k$ equation (4.173a) is solved by

$$w_{k,l} = -\delta_2 + a_k \alpha_l. \tag{4.174}$$

Substituting $w_{k,l}$ given by (4.174) into equation (4.173b):

$$v_{k+1,l} - v_{k,l} = \frac{\xi_{2k+1}}{a_k \alpha_l}. \tag{4.175}$$

Defining:

$$\xi_{2k+1} = -a_k \left(b_{k+1} - b_k\right), \tag{4.176}$$

we have that equation (4.175) is a total difference. Therefore we have the following solution of the system (4.173):

$$v_{k,l} = \beta_l + \frac{b_k}{\alpha_l}, \tag{4.177a}$$

$$w_{k,l} = -\delta_2 + a_k \alpha_l. \tag{4.177b}$$

CASE $m = 2k+1$: If $\delta_1 = 0$ the system (4.72) becomes:

$$z_{k,l} - \frac{a_k}{a_{k-1}} z_{k-1,l} = \left(\frac{a_k}{a_{k-1}} - 1\right) y_{k,l}, \tag{4.178a}$$

$$y_{k+1,l} - y_{k,l} = -a_k \left(b_{k+1} - b_k\right), \tag{4.178b}$$

where we used (4.176) and $\xi_{2k} = a_{k-1}/a_k$. The system is linear and equation (4.178b) is uncoupled from (4.178a). If we put

$$a_k = -\frac{c_{k+1} - c_k}{b_{k+1} - b_k}, \tag{4.179}$$

then equation (4.178a) becomes a total difference whose solution is:

$$y_{k,l} = c_k + \gamma_l. \tag{4.180}$$

Substituting $y_{k,l}$ given by (4.180) into equation (4.178a) we obtain:

$$\frac{b_{k+1}-b_k}{c_{k+1}-c_k}z_{k,l}-\frac{b_k-b_{k-1}}{c_k-c_{k-1}}z_{k-1,l}=\left(\frac{b_k-b_{k-1}}{c_k-c_{k-1}}-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}\right)(c_k+\gamma_l). \tag{4.181}$$

We can therefore represent the solution as

$$z_{k,l}=\frac{c_{k+1}-c_k}{b_{k+1}-b_k}(d_k+\delta_l)-\gamma_l \tag{4.182}$$

where d_k solves the equation:

$$d_k-d_{k-1}=b_{k+1}-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}c_{k+1}-b_{k-1}+\frac{b_k-b_{k=1}}{c_k-c_{k-1}}c_{k-1}. \tag{4.183}$$

Equation (4.183) is a total difference and d_k is given by:

$$d_k=b_k-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}c_k. \tag{4.184}$$

Therefore we have that the solution of the system (4.178) is given by:

$$y_{k,l}=c_k+\gamma_l, \tag{4.185a}$$

$$z_{k,l}=\frac{c_{k+1}-c_k}{b_{k+1}-b_k}\delta_l-\gamma_l+\frac{b_kc_{k+1}-c_kb_{k+1}}{b_{k+1}-b_k}. \tag{4.185b}$$

Equations (4.177,4.185) we have the value of the four fields, but we have too many arbitrary functions in the m direction, namely α_l, β_l, γ_l and δ_l. Inserting (4.177,4.185) into (1.93d) with $\delta_1=0$ and separating the terms even and odd in n and m we obtain we obtain two equations:

$$\beta_l\alpha_l-\delta_l=0, \quad \beta_{l+1}\alpha_{l+1}-\delta_l=0. \tag{4.186a}$$

We can solve this compatibility conditions with respect to β_l and δ_l we obtain:

$$\beta_l=\frac{\delta_0}{\alpha_l}, \quad \delta_l=\delta_0, \tag{4.187a}$$

where δ_0 is a constant. Inserting then (4.187) into (4.177,4.185) we obtain the following expression for the solution of the $_3D_2$ when $\delta_1=0$:

$$v_{k,l}=\frac{b_k+\delta_0}{\alpha_l}, \tag{4.188a}$$

$$w_{k,l}=-\delta_2-\alpha_l\frac{c_{k+1}-c_k}{b_{k+1}-b_k}, \tag{4.188b}$$

$$y_{k,l} = c_k + \gamma_l, \tag{4.188c}$$
$$z_{k,l} = \frac{c_{k+1} - c_k}{b_{k+1} - b_k}\delta_0 - \gamma_l + \frac{b_k c_{k+1} - c_k b_{k+1}}{b_{k+1} - b_k}. \tag{4.188d}$$

This discussion exhausts the possible cases. So for any value of the parameters we have the general solution of the ${}_3D_2$ equation (1.93d).

4.4.2 *The* D_3 *and the* ${}_iD_4$ *equations,* $i = 1, 2$

In this Subsection construct the general solution of the D_3 equation and of the two forms of the D_4 equation, which we denote collectively as ${}_iD_4$ with $i = 1, 2$. These equations are given by (1.93e), (1.93f) and (1.93g). The procedure we will follow will make use of the first integrals, but in a different way with respect to that used in Subsection 4.4.1 for the ${}_iD_2$ equations since, as explained in the beginning of this Section, we will need to extract some information from the equations.

4.4.2.1 D_3 *equation*

From Section 4.3 we know that the first integrals of the D_3 equation (1.93e) in the n direction is a four-point, third order first integral (4.47a). Therefore as stated at the beginning of Section this integral defines a three-point, third order ordinary difference equation from the relation $W_1 = \xi_n$. However to tackle the problem of finding the general solution in this case we do not start directly from the first integral W_1 (4.47a), but instead we start by looking to the equation itself (1.93e) written as a system. Applying the general transformation (4.55) we obtain the following system:

$$v_{k,l} + w_{k,l}y_{k,l} + w_{k,l}z_{k,l} + y_{k,l}z_{k,l} = 0, \tag{4.189a}$$
$$v_{k,l+1} + y_{k,l}w_{k,l+1} + z_{k,l}w_{k,l+1} + y_{k,l}z_{k,l} = 0, \tag{4.189b}$$
$$v_{k+1,l} + w_{k,l}y_{k+1,l} + w_{k,l}z_{k,l} + z_{k,l}y_{k+1,l} = 0, \tag{4.189c}$$
$$v_{k+1,l+1} + y_{k+1,l}w_{k,l+1} + z_{k,l}w_{k,l+1} + z_{k,l}y_{k+1,l} = 0. \tag{4.189d}$$

From the system (4.189) we have four different way for calculating $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (4.189a) and (4.189c) we obtain the following equation for $v_{k+1,l}$:

$$v_{k+1,l} = \frac{(w_{k,l} + y_{k+1,l})v_{k,l}}{w_{k,l} + y_{k,l}} + \frac{(y_{k,l} - y_{k+1,l})w_{k,l}^2}{w_{k,l} + y_{k,l}}, \tag{4.190}$$

while from (4.189b) and (4.189d) we obtain the following equation for $v_{k+1,l+1}$:

$$v_{k+1,l+1} = \frac{(w_{k,l+1} + y_{k+1,l})v_{k,l+1}}{w_{k,l+1} + y_{k,l}} + \frac{(y_{k,l} - y_{k+1,l})w_{k,l+1}^2}{w_{k,l+1} + y_{k,l}}. \tag{4.191}$$

Equations (4.190) and (4.191) give rise to a compatibility condition between $v_{k+1,l}$ and its shift in the l direction $v_{k+1,l+1}$ which is given by:

$$\begin{pmatrix} y_{k,l}w_{k,l+1} + y_{k+1,l+1}w_{k,l+1} + y_{k+1,l+1}y_{k,l} \\ -y_{k,l+1}w_{k,l+1} - y_{k+1,l}w_{k,l+1} - y_{k+1,l}y_{k,l+1} \end{pmatrix} \cdot (v_{k,l+1} - w_{k,l+1}^2) = 0. \tag{4.192}$$

Discarding the trivial solution $y_{k,l} = w_{k,l}^2$ we obtain the following value for the field $w_{k,l}$:

$$w_{k,l} = -\frac{y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}}{y_{k+1,l} + y_{k,l-1} - y_{k+1,l-1} - y_{k,l}}, \tag{4.193}$$

which makes (4.190) and (4.191) compatible. Then we have to solve the equation with respect to $v_{k+1,l}$:

$$\begin{aligned} v_{k+1,l} = {} & \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}} v_{k,l} \\ & + \frac{(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1})^2}{(y_{k+1,l-1} + y_{k,l} - y_{k+1,l} - y_{k,l-1})(y_{k,l} - y_{k,l-1})}. \end{aligned} \tag{4.194}$$

Making the transformation

$$v_{k,l} = (y_{k,l} - y_{k,l-1})\, V_{k,l} + y_{k,l-1}^2 \tag{4.195}$$

we can reduce (4.194) to the equation:

$$V_{k+1,l} = V_{k,l} + \frac{(y_{k,l-1} - y_{k+1,l-1})^2}{y_{k+1,l-1} + y_{k,l} - y_{k+1,l} - y_{k,l-1}}. \tag{4.196}$$

To go further we need to specify the form of the field $y_{k,l}$. This can be extracted from the first integrals. Consider the equation $W_1 = \xi_n$ with W_1 given as in (4.47a). This relation defines a third order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular if we choose the case when $m = 2l+1$ we have the equation:

$$\begin{aligned} & F_n^{(+)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})\,(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n,2l+1} + u_{n-1,2l+1}} \\ & + F_n^{(-)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})\,(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n+1,2l+1} + u_{n+2,2l+1}} = \xi_n \end{aligned} \tag{4.197}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. Using the transformation (4.55b) equation (4.197) is converted into the system:

$$(y_{k+1,l} - y_{k,l})(z_{k,l} - z_{k-1,l}) = \xi_{2k}(y_{k,l} + z_{k-1,l}), \tag{4.198a}$$
$$(y_{k+1,l} - y_{k,l})(z_{k+1,l} - z_{k,l}) = \xi_{2k+1}(y_{k+1,l} + z_{k+1,l}). \tag{4.198b}$$

This system is nonlinear, but if we solve (4.198b) with respect to $z_{k+1,l}$ and we substitute it together with its shift in the k direction into (4.198a) we obtain a *linear second order, ordinary difference equation* involving *only* the field $y_{k,l}$:

$$\begin{aligned}\xi_{2k-1}y_{k+1,m} - (\xi_{2k}+\xi_{2k-1})\,y_{k,m}&\\ +\xi_{2k}y_{k-1,m}+\xi_{2k}\xi_{2k-1}&=0.\end{aligned} \tag{4.199}$$

First we can lower the order of this equation by one with the potential transformation:

$$Y_{k,l} = y_{k+1,l} - y_{k,l}. \tag{4.200}$$

Indeed we have that $Y_{k,l}$ solves the equation:

$$Y_{k,l} - \frac{\xi_{2k}}{\xi_{2k-1}}Y_{k-1,l} + \xi_{2k} = 0. \tag{4.201}$$

Defining:

$$\xi_{2k} = -a_k\,(b_k - b_{k-1})\,,\quad \xi_{2k-1} = -a_{k-1}\,(b_k - b_{k-1}) \tag{4.202}$$

we obtain that $Y_{k,l}$ can be expressed as:

$$Y_{k,l} = a_k\,(b_k + \alpha_l)\,. \tag{4.203}$$

From (4.200) we have that:

$$y_{k+1,l} - y_{k,l} = a_k\,(b_k + \alpha_l)\,. \tag{4.204}$$

Setting:

$$a_k = c_{k+1} - c_k,\quad b_k = \frac{d_{k+1}-d_k}{c_{k+1}-c_k}, \tag{4.205}$$

we have:

$$y_{k,l} = \alpha_l c_k + d_k + \beta_l. \tag{4.206}$$

Inserting now the obtained value of $y_{k,l}$ from (4.206) into the equation (4.196) we obtain:

$$V_{k+1,l} = V_{k,l} - \frac{(d_k + \alpha_{l-1}c_k - d_{k+1} - \alpha_{l-1}c_{k+1})^2}{(\alpha_l - \alpha_{l-1})(c_{k+1}-c_k)} \tag{4.207}$$

So we get the following solution for $V_{k,l}$

$$V_{k,l} = \gamma_l - \alpha_{l-1}\frac{\alpha_{l-1}c_k + 2d_k}{\alpha_l - \alpha_{l-1}} + \frac{e_k}{\alpha_l - \alpha_{l-1}} \tag{4.208}$$

up to a quadrature for the function e_k:

$$e_{k+1} = e_k - \frac{(d_{k+1}-d_k)^2}{c_{k+1}-c_k}. \tag{4.209}$$

Plugging the obtained value of $v_{k,l}$ we can compute $w_{k,l}$ from (4.193) and finally $z_{k,l}$ from the original system (4.189). In this case we obtain a *single* compatibility condition given by:

$$\begin{aligned}(\alpha_{l+1}-\alpha_l)\gamma_{l+1}-(\alpha_l-\alpha_{l-1})\gamma_l\\ -\alpha_{l-1}\beta_{l-1}+\alpha_l\beta_l+\alpha_l\beta_{l-1}-\alpha_{l-1}\beta_l=0,\end{aligned} \tag{4.210}$$

which can be expressed as

$$\gamma_l=-\frac{\alpha_{l-1}\beta_{l-1}+\delta_l}{\alpha_l-\alpha_{l-1}} \tag{4.211}$$

with δ_l given by the following first order difference equation:

$$\delta_{l+1}=\delta_l+\alpha_l\beta_{l-1}-\alpha_{l-1}\beta_l. \tag{4.212}$$

We underline that this first order difference equation is the discrete analogue of a quadrature $A'(x)=f(x)$.

So, the function $V_{k,l}$ is given by (4.208-4.212), where e_k and δ_l are defined implicitly and can be found by discrete integration. Then the general solution of (1.93e) is constructed explicitly by successive substitution in (4.206), (4.195), (4.193) and (4.189a).

4.4.2.2 ${}_1D_4$ *equation*

From Section 4.3 we know that the first integrals of the ${}_1D_4$ equation (1.93f) in the n direction is a four-point, third order first integral (4.48a). Therefore this integral defines a three-point, third order ordinary difference equation from the relation $W_1=\xi_n$. However to find the general solution in this case again we do not start directly from the first integral W_1 (4.48a), but instead we start by looking to the equation itself (1.93f) written as a system. Applying the general transformation (4.55) to (1.93f) we obtain the following system:

$$v_{k,l}z_{k,l}+w_{k,l}y_{k,l}+\delta_1 w_{k,l}z_{k,l}+\delta_2 y_{k,l}z_{k,l}+\delta_3=0, \tag{4.213a}$$

$$\begin{aligned}y_{k,l}w_{k,l+1}+z_{k,l}v_{k,l+1}\\ +\delta_1 z_{k,l}w_{k,l+1}+\delta_2 y_{k,l}z_{k,l}+\delta_3=0,\end{aligned} \tag{4.213b}$$

$$\begin{aligned}w_{k,l}y_{k+1,l}+v_{k+1,l}z_{k,l}\\ +\delta_1 w_{k,l}z_{k,l}+\delta_2 z_{k,l}y_{k+1,l}+\delta_3=0,\end{aligned} \tag{4.213c}$$

$$\begin{aligned}z_{k,l}v_{k+1,l+1}+y_{k+1,l}w_{k,l+1}\\ +\delta_1 z_{k,l}w_{k,l+1}+\delta_2 z_{k,l}y_{k+1,l}+\delta_3=0.\end{aligned} \tag{4.213d}$$

From the equations (4.213) we have four different way for calculating $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (4.213a) and (4.213c) we obtain the following equation for $v_{k+1,l}$:

$$v_{k+1,l}=\frac{\delta_3+w_{k,l}y_{k+1,l}}{\delta_3+w_{k,l}y_{k,l}}v_{k,l}+\frac{(y_{k+1,l}-y_{k,l})(\delta_1 w_{k,l}^2-\delta_2\delta_3)}{\delta_3+w_{k,l}y_{k,l}} \tag{4.214}$$

while from (4.213b) and (4.213d) we obtain the following equation for $v_{k+1,l+1}$:

$$
\begin{aligned}
v_{k+1,l+1} = {} & \frac{\delta_3 + y_{k+1,l} w_{k,l+1}}{\delta_3 + y_{k,l} w_{k,l+1}} v_{k,l+1} \\
& + \frac{(y_{k+1,l} - y_{k,l})(\delta_1 w_{k,l+1}^2 - \delta_2 \delta_3}{(y_{k,l} w_{k,l+1} + \delta_3}.
\end{aligned}
\tag{4.215}
$$

Equations (4.214) and (4.215) give rise to a compatibility condition between $v_{k+1,l}$ and its shift in the l direction $v_{k+1,l+1}$, given by:

$$
\begin{aligned}
& \left[\begin{array}{c} (y_{k+1,l+1} y_{k,l} - y_{k+1,l} y_{k,l+1}) w_{k,l+1} \\ + \delta_3 (y_{k+1,l+1} + y_{k,l} - y_{k,l+1} - y_{k+1,l}) \end{array} \right] \cdot \\
& (v_{k,l+1} w_{k,l+1} - \delta_2 \delta_3 + \delta_1 w_{k,l+1}^2) = 0.
\end{aligned}
\tag{4.216}
$$

Discarding the trivial solution

$$
v_{k,l} = -\delta_1 w_{k,l} + \frac{\delta_2 \delta_3}{w_{k,l}}
$$

we obtain for $w_{k,l}$:

$$
w_{k,l} = \delta_3 \frac{y_{k+1,l-1} - y_{k+1,l} - y_{k,l-1} + y_{k,l}}{y_{k+1,l} y_{k,l-1} - y_{k+1,l-1} y_{k,l}}
\tag{4.217}
$$

which makes (4.214) and (4.215) compatible. Then we have to solve the following equation for $v_{k,l}$:

$$
\begin{aligned}
v_{k+1,l} = {} & \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}} v_{k,l} \\
& + \frac{\delta_1 \delta_3 (y_{k+1,l-1} - y_{k,l-1} - y_{k+1,l} + y_{k,l})^2}{(y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1})(y_{k,l} - y_{k,l-1})} \\
& - \frac{\delta_2 (y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1})}{y_{k,l} - y_{k,l-1}}
\end{aligned}
\tag{4.218}
$$

Making the transformation:

$$
v_{k,l} = (y_{k,l} - y_{k,l-1}) V_{k,l} + \frac{\delta_1 \delta_3}{y_{k,l-1}} - \delta_2 y_{k,l-1}.
\tag{4.219}
$$

we obtain that $V_{k,l}$ satisfied the difference equation:

$$
V_{k+1,l} = V_{k,l} + \frac{\delta_1 \delta_3 (y_{k,l-1} - y_{k+1,l-1})^2}{y_{k,l-1} y_{k+1,l-1} (y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1})}
\tag{4.220}
$$

At this point without any knowledge of the field $y_{k,l}$ we cannot go further. However as in the D_3 case we can recover the missing information using the first integral W_1 as given in (4.48a). This integral provide us the relation $W_1 = \xi_n$ which is a third order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular if we choose the case when $m = 2l+1$ we have the equation:

$$F_n^{(+)} \frac{(u_{n,2l+1} - u_{n+2,2l+1} + \delta_1 u_{n+1,2l+1})\, u_{n-1,2l+1} + u_{n+1,2l+1} u_{n+2,2l+1}}{(u_{n,2l+1} + \delta_1 u_{n-1,2l+1})\, u_{n+1,2l+1}} + F_n^{(-)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})\,(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n,2l+1}\,(u_{n+2,2l+1}\delta_1 + u_{n+1,2l+1})} = \xi_n. \tag{4.221}$$

where we set without loss of generality $\alpha = \beta = 1$. Using the transformation (4.55b) then (4.221) is converted into the system:

$$\frac{(y_{k,l} - y_{k+1,l} + \delta_1 z_{k,l}) z_{k-1,l} + y_{k+1,l} z_{k,l}}{(y_{k,l} + \delta_1 z_{k-1,l}) z_{k,l}} = \xi_{2k} \tag{4.222a}$$

$$\frac{(y_{k,l} - y_{k+1,l})(z_{k,l} - z_{k+1,l})}{z_{k,l}(z_{k+1,l}\delta_1 + y_{k+1,l})} = \xi_{2k+1} \tag{4.222b}$$

It is quite easy to see that if we solve (4.222b) with respect to $z_{k+1,l}$ and then substitute the result into (4.222a) we obtain a *linear*, second order ordinary difference equation for $y_{k,l}$:

$$\xi_{2k-1} y_{k+1,l} + (1 - \xi_{2k} - \xi_{2k}\xi_{2k-1}) y_{k,l} - (1 - \xi_{2k}) y_{k-1,l} = 0. \tag{4.223}$$

We can solve this equation as in the case of the D_3 equation. First of all let us introduce $Y_{k,l} = a_k y_{k,l} + b_k y_{k-1,l}$ such that $Y_{k+1,l} - Y_{k,l}$ is equal to the left hand side of (4.223). Then we define:

$$\xi_{2k} = -\frac{b_{k+1} - b_k - a_k}{a_{k+1}}, \tag{4.224a}$$

$$\xi_{2k-1} = \frac{-b_k + a_{k+1} + b_{k+1} - a_k}{b_k}. \tag{4.224b}$$

Therefore $y_{k,l}$ must solve the first order equation:

$$a_k y_{k,l} + b_k y_{k-1,l} = \alpha_l. \tag{4.225}$$

The equation (4.225) can be solved if we choose:

$$a_k = \frac{1}{c_k} \frac{1}{d_k - d_{k-1}}, \quad b_k = -\frac{1}{c_{k-1}} \frac{1}{d_k - d_{k-1}}. \tag{4.226}$$

Then the solution of (4.225) is:

$$y_{k,l} = c_k \left(\alpha_l d_k + \beta_l \right). \tag{4.227}$$

Inserting (4.227) into (4.220) we obtain:

$$V_{k+1,l} = V_{k,l} - \frac{\delta_1 \delta_3}{\beta_{l-1}\alpha_l - \beta_l \alpha_{l-1}} \frac{(c_k - c_{k+1})^2}{c_k^2 c_{k+1}^2 (d_{k+1} - d_k)} + \frac{\delta_1 \delta_3 \alpha_{l-1}}{\beta_{l-1}\alpha_l - \beta_l \alpha_{l-1}} \left[\frac{1}{(\alpha_{l-1} d_{k+1} + \beta_{l-1}) c_{k+1}^2} - \frac{1}{(\alpha_{l-1} d_k + \beta_{l-1}) c_k^2} \right]. \tag{4.228}$$

This means that we can represent the solution of $V_{k,l}$ as:

$$V_{k,l} = \gamma_l + \frac{\delta_1\delta_3}{\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1}}\left[\frac{\alpha_{l-1}}{c_k^2(\alpha_{l-1}d_k + \beta_{l-1})} + e_k\right] \tag{4.229}$$

where e_k is defined from the quadrature:

$$e_{k+1} = e_k - \frac{(c_k - c_{k+1})^2}{c_k^2 c_{k+1}^2 (d_{k+1} - d_k)}. \tag{4.230}$$

Using the obtained value of $v_{k,l}$ we can compute $w_{k,l}$ from (4.217) and finally $z_{k,l}$ from the original system (4.213). In this case we obtain a *single* compatibility condition for the arbitrary functions given by:

$$(\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\gamma_l = (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\delta_2. \tag{4.231}$$

This condition can be expressed also as

$$\gamma_l = \frac{\delta_l\delta_2}{\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1}}. \tag{4.232}$$

with δ_l given by solving the following quadrature:

$$\delta_{l+1} = \delta_l + \alpha_l\beta_{l-1} - \alpha_{l-1}\beta_l. \tag{4.233}$$

Let us note that δ_l is the same as in (4.212).

So, the auxiliary function $V_{k,l}$ is given by (4.229-4.233), where e_k and δ_l are defined implicitly and can be found by discrete integration. Then the general solution of (1.93f) is constructed explicitly by successive substitution in (4.227), (4.219), (4.217) and (4.213a).

4.4.2.3 $_2D_4$ *equation*

From Section 4.3 we know that the first integrals of the $_2D_4$ equation (1.93g) in the n direction is a four-point, third order first integral (4.52a). Therefore this integral defines a three-point, third order ordinary difference equation from the relation $W_1 = \xi_n$. However to find the general solution in this case we do not start directly from the first integral W_1 (4.52a), but we start instead by looking to the equation itself (1.93g) written as a system. Applying the general transformation (4.55) to (1.93g) we obtain the following system:

$$v_{k,l}w_{k,l} + \delta_2 w_{k,l}y_{k,l} + \delta_1 w_{k,l}z_{k,l} + y_{k,l}z_{k,l} + \delta_3 = 0, \tag{4.234a}$$

$$v_{k,l+1}w_{k,l+1} + \delta_2 y_{k,l}w_{k,l+1} + \delta_1 z_{k,l}w_{k,l+1} + y_{k,l}z_{k,l} + \delta_3 = 0, \tag{4.234b}$$

$$w_{k,l}v_{k+1,l} + \delta_2 w_{k,l}y_{k+1,l} + \delta_1 w_{k,l}z_{k,l} + z_{k,l}y_{k+1,l} + \delta_3 = 0, \tag{4.234c}$$

$$w_{k,l+1}v_{k+1,l+1} + \delta_2 y_{k+1,l}w_{k,l+1} + \delta_1 z_{k,l}w_{k,l+1} + z_{k,l}y_{k+1,l} + \delta_3 = 0. \tag{4.234d}$$

From the equations (4.234) we have, in principle, four different way for calculating $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (4.234a) and (4.234c) we obtain the following equation for $v_{k+1,l}$:

$$
\begin{aligned}
v_{k+1,l} = & \frac{\delta_1 w_{k,l} + y_{k+1,l}}{\delta_1 w_{k,l} + y_{k,l}} v_{k,l} \\
& - \frac{(y_{k+1,l} - y_{k,l})(\delta_2 w_{k,l}^2 \delta_1 - \delta_3)}{(\delta_1 w_{k,l} + y_{k,l}) w_{k,l}}
\end{aligned}
\tag{4.235}
$$

while from (4.234b) and (4.234d) we obtain the following equation for $v_{k+1,l+1}$:

$$
\begin{aligned}
v_{k+1,l+1} = & \frac{\delta_1 w_{k,l+1} + y_{k+1,l}}{\delta_1 w_{k,l+1} + y_{k,l}} v_{k,l+1} \\
& - \frac{(y_{k+1,l} - y_{k,l})(\delta_2 w_{k,l+1}^2 \delta_1 - \delta_3)}{(\delta_1 w_{k,l+1} + y_{k,l}) w_{k,l+1}}.
\end{aligned}
\tag{4.236}
$$

Equations (4.235) and (4.236) give rise to a compatibility condition between $v_{k+1,l}$ and $v_{k+1,l+1}$ is given by:

$$
\begin{gathered}
\begin{bmatrix}
y_{k+1,l+1} y_{k,l} + \delta_1 \left(y_{k,l} w_{k,l+1} + y_{k+1,l+1} w_{k,l+1} \right) \\
- y_{k+1,l} y_{k,l+1} - \delta_1 \left(y_{k+1,l} w_{k,l+1} - y_{k,l+1} w_{k,l+1} \right)
\end{bmatrix} \cdot \\
\left(v_{k,l+1} w_{k,l+1} + \delta_3 - \delta_1 \delta_2 w_{k,l+1}^2 \right) = 0.
\end{gathered}
\tag{4.237}
$$

Discarding the trivial solution

$$
v_{k,l} = \delta_1 \delta_2 w_{k,l} - \frac{\delta_3}{w_{k,l}}
$$

we obtain:

$$
w_{k,l} = \frac{1}{\delta_1} \frac{y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1}}{y_{k,l-1} + y_{k+1,l} - y_{k,l} - y_{k+1,l-1}}
\tag{4.238}
$$

which makes (4.235) and (4.236) compatible. Then we have to solve the following equation for $v_{k,l}$:

$$
\begin{aligned}
v_{k+1,l} = & \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}} v_{k,l} - \frac{\delta_1 \delta_3 (y_{k,l-1} - y_{k+1,l-1})^2 (y_{k,l} - y_{k,l-1})}{(y_{k+1,l} y_{k,l-1} - y_{k+1,l-1} y_{k,l}) y_{k,l-1}^2} \\
& + \frac{y_{k,l-1}^2 y_{k+1,l} \delta_2 - \delta_3 y_{k+1,l} \delta_1 + \delta_1 \delta_3 y_{k+1,l-1} - y_{k+1,l-1} \delta_2 y_{k,l-1}^2}{(y_{k,l} - y_{k,l-1}) y_{k,l-1}} \\
& - \frac{y_{k+1,l-1} \delta_2 y_{k,l-1}^2 + \delta_1 \delta_3 y_{k+1,l-1} - 2 y_{k,l-1} \delta_3 \delta_1)}{y_{k,l-1}^2}.
\end{aligned}
\tag{4.239}
$$

Making the transformation:

$$
v_{k,l} = \left(y_{k,l} - y_{k,l-1} \right) V_{k,l} + \frac{\delta_1 \delta_3}{y_{k,l}} - \delta_2 y_{k,l}.
\tag{4.240}
$$

we obtain that $V_{k,l}$ satisfied the first order difference equation:

$$V_{k+1,l} = V_{k,l} - \frac{\delta_1\delta_3(y_{k,l}-y_{k+1,l})^2}{y_{k,l}y_{k+1,l}(y_{k+1,l-1}y_{k,l}-y_{k+1,l}y_{k,l-1})}. \quad (4.241)$$

At this point without any knowledge of the field $y_{k,l}$ we cannot go further. However as in the previous two cases we can recover the missing information using the first integral W_1 as given in (4.52a). This integral defines the relation $W_1 = \xi_n$, a third order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular if we choose the case when $m = 2l+1$ we have the equation:

$$\begin{aligned} &-F_n^{(+)}\frac{u_{n+2,2l+1}u_{n-1,2l+1}+(-u_{n+2,2l+1}+u_{n,2l+1})\,u_{n+1,2l+1}+\delta_3}{u_{n-1,2l+1}u_{n,2l+1}+\delta_3} \\ &+F_n^{(-)}\frac{(u_{n+1,2l+1}-u_{n-1,2l+1})\,(u_{n+2,2l+1}-u_{n,2l+1})}{u_{n+1,2l+1}u_{n+2,2l+1}+\delta_3} = \xi_n. \end{aligned} \quad (4.242)$$

where we choose without loss of generality $\alpha = \beta = 1$. Using the transformation (4.55b) the equation (4.242) is converted into the system:

$$\frac{(y_{k+1,l}-y_{k,l})z_{k,l}-y_{k+1,l}z_{k+1,l}-\delta_3}{(z_{k+1,l}y_{k,l}+\delta_3)} = \xi_{2k} \quad (4.243a)$$

$$\frac{(y_{k+1,l}-y_{k,l})(z_{k+1,l}-z_{k,l})}{(y_{k+1,l}z_{k+1,l}+\delta_3)} = \xi_{2k+1} \quad (4.243b)$$

Now if we solve (4.243b) with respect to $z_{k+1,l}$ and then substitute into (4.243a) we obtain a *linear*, second order ordinary difference equation for $y_{k,l}$:

$$\xi_{2k-1}y_{k+1,l} - (1+\xi_{2k}-\xi_{2k}\xi_{2k-1})y_{k,l} + (1+\xi_{2k})y_{k-1,l} = 0. \quad (4.244)$$

We solve this equation using the freedom provided by the functions ξ_{2k} and ξ_{2k+1} similarly as we did in the case of the ${}_1D_4$ equation. Indeed let us introduce the field $Y_{k,l} = a_k y_{k,l} + b_k y_{k-1,l}$ and assume that $Y_{k+1,l} - Y_{k,l}$ equals the left hand side of (4.244). Then we choose:

$$\xi_{2k} = \frac{b_{k+1}-b_k+-a_k}{a_{k+1}}, \quad (4.245a)$$

$$\xi_{2k-1} = -\frac{b_{k+1}-b_k+a_{k+1}-a_k}{b_k}, \quad (4.245b)$$

so that $y_{k,l}$ solves the first order equation

$$a_k y_{k,l} + b_k y_{k-1,l} = \alpha_l. \quad (4.246)$$

If we define

$$a_k = \frac{1}{c_k}\frac{1}{d_k-d_{k-1}}, \quad b_k = -\frac{1}{c_{k-1}}\frac{1}{d_k-d_{k-1}}, \quad (4.247)$$

(4.246) is solved by:

$$y_{k,l} = c_k \left(\alpha_l d_k + \beta_l\right). \tag{4.248}$$

Inserting (4.248) into (4.241) we obtain:

$$\begin{aligned} V_{k+1,l} = V_{k,l} &+ \frac{\delta_3 \delta_1 (c_{k+1} - c_k)^2}{(d_{k+1} - d_k)(\beta_l \alpha_{l-1} - \beta_{l-1}\alpha_l) c_k^2 c_{k+1}^2} \\ &- \frac{\alpha_l \delta_1 \delta_3}{\beta_l \alpha_{l-1} - \beta_{l-1}\alpha_l} \left[\frac{1}{(\alpha_l d_{k+1} + \beta_l) c_{k+1}^2} \right. \\ &\left. - \frac{1}{(\alpha_l d_k + \beta_l) c_k^2} \right], \end{aligned} \tag{4.249}$$

whose solution is:

$$\begin{aligned} V_{k,l} = \gamma_l &- \frac{\alpha_l \delta_1 \delta_3}{(\alpha_l d_k + \beta_l) c_k^2 (\beta_l \alpha_{l-1} - \beta_{l-1}\alpha_l)} \\ &+ \frac{\delta_3 \delta_1 e_k}{\beta_l \alpha_{l-1} - \beta_{l-1}\alpha_l}, \end{aligned} \tag{4.250}$$

where e_k satisfies the quadrature:

$$e_{k+1} = e_k + \frac{(c_{k+1} - c_k)^2}{(d_{k+1} - d_k) c_k^2 c_{k+1}^2}. \tag{4.251}$$

Using the obtained value of $v_{k,l}$ we can compute $w_{k,l}$ from (4.217) and finally $z_{k,l}$ from the original system (4.234). In this case we obtain a *single* compatibility condition for the arbitrary functions given by:

$$(\beta_l \alpha_{l+1} - \beta_{l+1}\alpha_l)\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l \alpha_{l-1})\gamma_l = (\beta_l \alpha_{l+1} - \beta_{l+1}\alpha_l)\delta_2. \tag{4.252}$$

i.e.:

$$\gamma_l = \frac{\delta_l \delta_2}{\beta_{l-1}\alpha_l - \beta_l \alpha_{l-1}}. \tag{4.253}$$

with δ_l satisfying the following quadrature:

$$\delta_{l+1} = \delta_l + \alpha_{l+1}\beta_l - \alpha_l \beta_{l+1}. \tag{4.254}$$

So, the auxiliary function $V_{k,l}$ is given by (4.250-4.254), where e_k and δ_l are defined implicitly by (4.251) and (4.254) and can be found by discrete integration. Then the general solution of (1.93g) is then constructed explicitly by successive substitution in (4.248), (4.240), (4.238) and (4.234a).

4.4.3 *The trapezoidal* H^4 *equations*

In this Subsection we construct the general solution of the trapezoidal H^4 equations (1.91). The procedure we will follow will make use of the first integrals, as in the cases presented in Section 4.4.2. The main difference is that since the H^4 are non-autonomous only in the direction m with the two-periodic non-autonomous factors $F_m^{(\pm)}$ given by (1.90) instead of the general transformation (4.55) we can use the simplified transformation:

$$u_{n,2l} = p_{n,l}, \quad u_{n,2l+1} = q_{n,l}. \tag{4.255}$$

Then to describe the general solution of a H^4 we only need three arbitrary functions: one in the n direction and two in the m direction.

4.4.3.1 *The* ${}_tH_1^\varepsilon$ *equation*

Let us start from the first integral W_1 (4.29a), which is a two-point, first order first integral. This implies that the ${}_tH_1^\varepsilon$ equation (1.91a) can be written as a conservation law. Indeed we can carefully rearrange the terms in (1.91a) and use the properties of the functions $F_m^{(\pm)}$ to rewrite (1.91a) as:

$$(T_m - \mathrm{Id})\left(F_m^{(+)}\frac{\alpha_2}{u_{n+1,m}-u_{n,m}} + F_m^{(-)}\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n,m}u_{n+1,m}}\right) = 0, \tag{4.256}$$

i.e. as a conservation law in the form (4.7b). From (4.256) we can derive the general solution of (1.91a) itself. In fact (4.256) implies:

$$F_m^{(+)}\frac{\alpha_2}{u_{n+1,m}-u_{n,m}} + F_m^{(-)}\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n,m}u_{n+1,m}} = \lambda_n, \tag{4.257}$$

where λ_n is an arbitrary function of n. This is a first order difference equation in the n direction in which m plays the rôle of a parameter. For this reason we can safely separate the two cases m even and m odd.

CASE $m = 2k$ In this case (4.257) reduces to the first order linear equation:

$$u_{n+1,2k} - u_{n,2k} = \frac{\alpha_2}{\lambda_n} \tag{4.258}$$

which has solution:

$$u_{n,2k} = \theta_{2k} + \omega_n \tag{4.259}$$

where θ_{2k} is an arbitrary function and ω_n is the solution of the simple ordinary difference equation

$$\omega_{n+1} - \omega_n = \frac{\alpha_2}{\lambda_n}, \quad \omega_0 = 0. \tag{4.260}$$

CASE $m = 2k+1$ In this case (4.257) reduces to the discrete Riccati equation:

$$\lambda_n \varepsilon^2 u_{n,2k+1} u_{n+1,2k+1} - u_{n+1,2k+1} + u_{n,2k+1} + \lambda_n. \quad (4.261)$$

Using the Möbius transformation:

$$u_{n,2k+1} = \frac{i}{\varepsilon} \frac{1 - v_{n,2k+1}}{1 + v_{n,2k+1}}, \quad (4.262)$$

this equation reduce to the linear equation

$$(i + \varepsilon\lambda_n) v_{n+1,2k+1} - (i - \varepsilon\lambda_n) v_{n,2k+1} = 0. \quad (4.263)$$

If we define:

$$\lambda_n = \frac{i}{\varepsilon} \frac{\kappa_n - \kappa_{n+1}}{\kappa_n + \kappa_{n+1}} \quad (4.264)$$

then we have that the general solution of (4.263) is expressed as:

$$v_{n,2k+1} = \kappa_n \theta_{2k+1}. \quad (4.265)$$

Using (4.262) we then obtain:

$$u_{n,2k+1} = \frac{i}{\varepsilon} \frac{1 - \kappa_n \theta_{2k+1}}{1 + \kappa_n \theta_{2k+1}}. \quad (4.266)$$

So we have the general solution to (1.91a) in the form:

$$u_{n,m} = F_m^{(+)} (\theta_m + \omega_n) + F_m^{(-)} \frac{i}{\varepsilon} \frac{1 - \kappa_n \theta_m}{1 + \kappa_n \theta_m}, \quad (4.267)$$

where ω_n is defined by (4.260) and λ_n is defined by (4.264). This is another proof, different from that given in Subsection 2.4.1, following [67, 69], of the linearization of the equation (1.91a).

It is worth to note that in the case of the ${}_tH_1^\varepsilon$ equation (1.91a) is possible to give simple proof of the linearization also using the first integral in the m direction (4.29b). This kind of linearization was presented in [71] and it is important since it was the first example of linearization obtained using a higher-order fist integral. We leave this discussion to Appendix H.

4.4.3.2 *The ${}_tH_2^\varepsilon$ equation*

From Section 4.3 we know that the first integrals of the ${}_tH_2^\varepsilon$ equation (1.91b) in the n direction is four-point, third order first integral (4.30a). Therefore this integral defines a three-point, third order ordinary difference equation $W_1 = \xi_n$. However this integral is particularly complex, so we start first by inspecting the ${}_tH_2^\varepsilon$ equation (1.91b) itself.

If we apply the transformation (4.255) we can write down the ${}_tH_2^\varepsilon$ equation (1.91b) as the following system of two coupled equations:

$$\begin{aligned}
&(p_{n,l}-p_{n+1,l})(q_{n,l}-q_{n+1,l})\\
&-\alpha_2(p_{n,l}+p_{n+1,l}+q_{n,l}+q_{n+1,l})\\
&+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)\\
&+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2\\
&-\alpha_3^2-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0,
\end{aligned} \tag{4.268a}$$

$$\begin{aligned}
&(q_{n,l}-q_{n+1,l})(p_{n,l+1}-p_{n+1,l+1})\\
&-\alpha_2(q_{n,l}+q_{n+1,l}+p_{n,l+1}+p_{n+1,l+1})\\
&+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)\\
&+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2-\alpha_3^2\\
&-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0.
\end{aligned} \tag{4.268b}$$

We have that equation (4.268a) depends on $p_{n,l}$ and $p_{n+1,l}$ and that equation (4.268b) depends on $p_{n,l+1}$ and $p_{n+1,l+1}$. So we apply the translation operator T_l to (4.268a) to obtain two equations in terms of $p_{n,l+1}$ and $p_{n+1,l+1}$:

$$\begin{aligned}
&(p_{n,l+1}-p_{n+1,l+1})(q_{n,l+1}-q_{n+1,l+1})\\
&-\alpha_2(p_{n,l+1}+p_{n+1,l+1}+q_{n,l+1}+q_{n+1,l+1})\\
&+\frac{\varepsilon\alpha_2}{2}(2q_{n,l+1}+2\alpha_3+\alpha_2)(2q_{n+1,l+1}+2\alpha_3+\alpha_2)\\
&+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2\\
&-\alpha_3^2-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0,
\end{aligned} \tag{4.269a}$$

$$\begin{aligned}
&(q_{n,l}-q_{n+1,l})(p_{n,l+1}-p_{n+1,l+1})\\
&-\alpha_2(q_{n,l}+q_{n+1,l}+p_{n,l+1}+p_{n+1,l+1})\\
&+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)\\
&+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2-\alpha_3^2\\
&-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0.
\end{aligned} \tag{4.269b}$$

The system (4.269) is equivalent to the original system (4.268). We can solve (4.269) with respect to $p_{n,l+1}$ and $p_{n+1,l+1}$:

$$p_{n,l+1}=\frac{\left\{\begin{gathered}
(q_{n,l+1}-\alpha_3)\,q_{n+1,l}-(q_{n,l}-\alpha_3)\,q_{n+1,l+1}\\
-(\alpha_2+\alpha_3)\,(q_{n,l+1}-q_{n,l})-\varepsilon\alpha_3^2\,(q_{n,l+1}-q_{n,l})\\
+\varepsilon\left[{\alpha_3}^2+2\alpha_3\,(q_{n,l}+\alpha_2)-(q_{n,l+1}-q_{n,l})\,q_{n+1,l}\right]q_{n+1,l+1}\\
+\varepsilon\,(\alpha_2+q_{n,l+1})\,(\alpha_2+q_{n,l})\,q_{n+1,l+1}\\
-\varepsilon q_{n+1,l}\left[{\alpha_3}^2-2\,(q_{n,l+1}+\alpha_2)\,\alpha_3+(\alpha_2+q_{n,l+1})\,(\alpha_2+q_{n,l})\right]
\end{gathered}\right\}}{q_{n+1,l+1}-q_{n,l+1}+q_{n,l}-q_{n+1,l}} \tag{4.270a}$$

$$p_{n+1,l+1} = \frac{\left\{\begin{array}{c} (q_{n+1,l}-\alpha_3)\,q_{n,l+1}+(\alpha_3-q_{n+1,l+1})\,q_{n,l} \\ -(\alpha_2+\alpha_3)\,(q_{n+1,l}-q_{n+1,l+1})+\varepsilon\alpha_3^2\,(q_{n+1,l+1}-q_{n+1,l}) \\ +\varepsilon q_{n,l+1}\left[(q_{n+1,l+1}-q_{n+1,l})\,q_{n,l}-\alpha_3^2-2\alpha_3\,(q_{n+1,l}+\alpha_2)\right] \\ -\varepsilon q_{n,l+1}\,(\alpha_2+q_{n+1,l+1})\,(\alpha_2+q_{n+1,l}) \\ +\varepsilon q_{n,l}\left[\alpha_3^2+2\,(\alpha_2+q_{n+1,l+1})\,\alpha_3+(\alpha_2+q_{n+1,l+1})\,(\alpha_2+q_{n+1,l})\right] \end{array}\right\}}{q_{n+1,l+1}-q_{n,l+1}+q_{n,l}-q_{n+1,l}} \tag{4.270b}$$

We see that the right hand sides of (4.270) are functions only of $q_{n,l}$, $q_{n+1,l}$, $q_{n,l+1}$ and $q_{n+1,l+1}$. Moreover (4.270a) and (4.270b) must be compatible. Therefore applying T_l^{-1} to (4.270b) and imposing to the obtained expression to be equal to (4.270a) we find that $q_{n,l}$ must solve the following equation:

$$\begin{gathered} \alpha_2\,(q_{n-1,l+1}-q_{n-1,l}-q_{n+1,l+1}+q_{n+1,l}) \\ +(q_{n,l}-q_{n+1,l})\,q_{n-1,l+1} \\ -(q_{n,l}-q_{n-1,l})\,q_{n+1,l+1} \\ +q_{n,l+1}\,(q_{n+1,l}-q_{n-1,l}) \\ +\varepsilon\alpha_2^2\,(q_{n+1,l+1}-q_{n+1,l}+q_{n-1,l}-q_{n-1,l+1}) \\ +\varepsilon\alpha_2\,(q_{n+1,l+1}-q_{n+1,l}+q_{n-1,l}-q_{n-1,l+1})\,(q_{n,l+1}+2\alpha_3+q_{n,l}) \\ +\varepsilon\,(q_{n+1,l}-q_{n-1,l})\,q_{n+1,l+1}q_{n-1,l+1} \\ +\varepsilon\,(q_{n,l+1}-q_{n,l}+q_{n+1,l})\,q_{n-1,l}q_{n-1,l+1} \\ +\varepsilon\,[2\alpha_3 q_{n+1,l}-(q_{n,l+1}+2\alpha_3)\,q_{n,l}]\,q_{n-1,l+1} \\ +\varepsilon\,(q_{n,l+1}+2\alpha_3)\,q_{n,l}q_{n+1,l+1} \\ -\varepsilon\,(2\alpha_3+q_{n+1,l})\,q_{n-1,l}q_{n+1,l+1} \\ -\varepsilon\,(q_{n,l+1}-q_{n,l})\,q_{n+1,l}q_{n+1,l+1} \\ -\varepsilon\,(2\alpha_3+q_{n,l})\,(q_{n+1,l}-q_{n-1,l})\,q_{n,l+1}=0. \end{gathered} \tag{4.271}$$

This partial difference equation for $q_{n,l}$ is not defined on a quad graph, but it is defined on the six-point lattice shown in Figure 4.1.

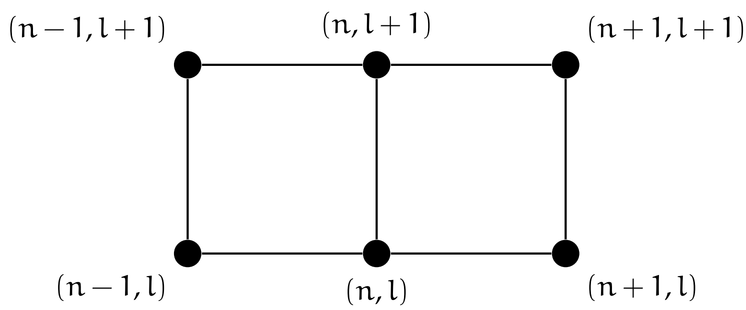

Figure 4.1: The six-point lattice.

Remark 4.4.4. Let us note that if the field $q_{n,l}$ satisfies the following equation, known as the *discrete wave equation*:

$$q_{n+1,l+1} + q_{n,l} - q_{n+1,l} + q_{n,l+1} = 0 \tag{4.272}$$

we cannot define $p_{n,l+1}$ and $p_{n+1,l+1}$ as given in (4.270). In this case we have first to solve equation (4.272) with respect to $q_{n,l}$ and then use the system (4.268) to specify $p_{n,l}$. The discrete wave equation (4.272) is a trivial *linear* Darboux integrable equation, since it possesses the following two-point, first order first integrals:

$$W_1 = q_{n+1,l} - q_{n,l}, \tag{4.273a}$$
$$W_2 = q_{n,l+1} - q_{n,l}. \tag{4.273b}$$

As remarked in the Introduction the existence of a two-point, first order first integral means that the equation is itself a first integral. Therefore the discrete wave equation (4.272) can be alternatively written as $(T_l - \mathrm{Id})\, W_1$ or $(T_n - \mathrm{Id})\, W_2$ with W_1 and W_2 given by (4.273). The solution is readily obtained from (4.273a) which implies $q_{n+1,l} - q_{n,l} = \xi_n$. This equation becomes a total difference setting $\xi_n = a_{n+1} - a_n$, and we get the general solution of equation (4.272):

$$q_{n,l} = a_n + \alpha_l, \tag{4.274}$$

where both a_n and α_l are arbitrary functions of their argument. This is the discrete analogue of d'Alembert method of solution of the continuous wave equation. Substituting (4.274) into (4.269) we obtain the compatibility condition:

$$\alpha_{l+1} - \alpha_l = 0, \tag{4.275}$$

i.e. $\alpha_l = \alpha_0 =$ constant and the system (4.268) is now consistent. Therefore we are left with one equation for $p_{n,l}$, e.g. (4.268a). Therefore inserting (4.274) with $\alpha_l = \alpha_0$ in (4.268a) and solving with respect to $p_{n+1,l}$ we obtain:

$$\begin{aligned} p_{n+1,l} = {} & \frac{a_{n+1} - a_n + \alpha_2}{a_{n+1} - a_n - \alpha_2} p_{n,l} + \frac{\alpha_2 \left(\alpha_2 - a_n + 2\alpha_3 - 2\alpha_0 - a_{n+1}\right)}{\alpha_2 + a_n - a_{n+1}} \\ & + \frac{\alpha_2 \varepsilon \begin{bmatrix} \alpha_2^2 + \left(2\alpha_0 + a_{n+1} + 2\alpha_3 + a_n\right)\alpha_2 \\ \quad + 2\left(a_{n+1} + \alpha_0 + \alpha_3\right)\left(\alpha_3 + a_n + \alpha_0\right) \end{bmatrix}}{\alpha_2 + a_n - a_{n+1}}. \end{aligned} \tag{4.276}$$

Defining through discrete integration a new function b_n such that:

$$\frac{a_{n+1} - a_n + \alpha_2}{a_{n+1} - a_n - \alpha_2} = \frac{b_{n+1}}{b_n}, \tag{4.277}$$

we have that $p_{n,l}$ solves the equation:

$$\frac{p_{n+1,l}}{b_{n+1}} = \frac{p_{n,l}}{b_n} + \frac{(a_n + \alpha_0 - \alpha_3)\, b_{n+1} + b_n\, (\alpha_2 + \alpha_3 - \alpha_0 - a_n)}{b_n b_{n+1}} - \varepsilon \frac{\left[(\alpha_2 + \alpha_0 + a_n + \alpha_3)^2\, b_{n+1} - b_n\, (\alpha_3 + a_n + \alpha_0)^2\right]}{b_n b_{n+1}}. \tag{4.278}$$

Equation (4.278) is solved by:

$$p_{n,l} = b_n\, (\beta_l + c_n)\,, \tag{4.279}$$

where c_n is given by the discrete integration:

$$c_{n+1} = c_n + \frac{(a_n + \alpha_0 - \alpha_3)\, b_{n+1} + b_n\, (\alpha_2 + \alpha_3 - \alpha_0 - a_n)}{b_n b_{n+1}} - \varepsilon \frac{\left[(\alpha_2 + \alpha_0 + a_n + \alpha_3)^2\, b_{n+1} - b_n\, (\alpha_3 + a_n + \alpha_0)^2\right]}{b_n b_{n+1}}. \tag{4.280}$$

This yields the solution of the ${}_tH_2^\varepsilon$ equation (1.91b) when $q_{n,l}$ satisfy the discrete wave equation (4.272).

In the general case we have proved that the ${}_tH_2^\varepsilon$ equation (1.91b) is equivalent to the system (4.268) which in turn is equivalent to the solution of equations (4.270a) and (4.271). However (4.270a) merely defines $p_{n,l+1}$ in terms of $q_{n,l}$ and its shifts. Therefore if we find the general solution of equation (4.271) the value of $p_{n,l}$ will follow. To find such solution we turn to the first integrals. Like in the case of the H^6 equations (1.93) we will find an expression for $q_{n,l}$ using the first integrals, and then we will insert it into (4.271) to reduce the number of arbitrary functions to the right one.

We consider the equation $W_1 = \xi_n$, where W_1 is given by (4.30a), with $k = 2l + 1$:

$$\frac{(u_{n-1,2l+1} - u_{n+1,2l+1})\,(u_{n+2,2l+1} - u_{n,2l+1})}{(u_{n,2l+1} - u_{n-1,2l+1} + \alpha_2)\,(u_{n+1,2l+1} - u_{n+2,2l+1} + \alpha_2)} = \xi_n. \tag{4.281}$$

Using the substitutions (4.255) we have:

$$\frac{(q_{n-1,l} - q_{n+1,l})\,(q_{n+2,l} - q_{n,l})}{(q_{n,l} - q_{n-1,l} + \alpha_2)\,(q_{n+1,l} - q_{n+2,l} + \alpha_2)} = \xi_n. \tag{4.282}$$

This equation contains only $q_{n,l}$ and its shifts. From equation (4.282) it is very simple to obtain a discrete Riccati equation. Indeed the transformation:

$$Q_{n,l} = \frac{q_{n,l} - q_{n-1,l} + \alpha_2}{q_{n+1,l} - q_{n-1,l}} \tag{4.283}$$

brings (4.282) into:

$$Q_{n+1,l} + \frac{1}{\xi_n Q_{n,l}} = 1 \tag{4.284}$$

which is a discrete Riccati equation. Let us assume a_n to be a particular solution of (4.284), then we express ξ_n as:

$$\xi_n = \frac{1}{a_n\,(1 - a_{n+1})}. \tag{4.285}$$

Using the standard linearization of the discrete Riccati equation:

$$Q_{n,l} = a_n + \frac{1}{Z_{n,l}} \tag{4.286}$$

from (4.285) we obtain the following equation for $Z_{n,l}$:

$$Z_{n+1,l} = \frac{a_n Z_{n,l} + 1}{1 - a_{n+1}}. \tag{4.287}$$

Introducing:

$$a_n = \frac{b_{n-1}}{b_n + b_{n-1}} \tag{4.288}$$

we obtain:

$$Z_{n+1,l} - \frac{b_{n-1}b_{n+1} + b_{n-1}b_n}{b_n b_{n+1} + b_{n-1}b_{n+1}} Z_{n,l} = \frac{b_{n-1}b_{n+1} + b_{n-1}b_n + b_n b_{n+1} + b_n^2}{b_n b_{n+1} + b_{n-1}b_{n+1}}. \tag{4.289}$$

If we assume that (4.289) can be written as a total difference, i.e.:

$$(T_n - \mathrm{Id})\,(d_n Z_{n,l} - c_n) = 0, \tag{4.290}$$

we obtain:

$$b_n = c_{n+1} - c_n, \quad d_n = \frac{(c_{n+1} - c_n)(c_n - c_{n-1})}{c_{n+1} - c_{n-1}}. \tag{4.291}$$

So b_n must be a total difference and therefore we can represent $Z_{n,l}$ as:

$$Z_{n,l} = \frac{(c_{n+1} - c_{n-1})(c_n + \alpha_l)}{(c_{n+1} - c_n)(c_n - c_{n-1})}. \tag{4.292}$$

From (4.286) and (4.288) we obtain the form of $Q_{n,l}$:

$$Q_{n,l} = \frac{(c_n - c_{n-1})(c_{n+1} + \alpha_l)}{(c_n + \alpha_l)(c_{n+1} - c_{n-1})}. \tag{4.293}$$

Introducing the value of $Q_{n,l}$ from (4.293) into (4.283) we obtain the following equation for $q_{n,l}$:

$$\frac{q_{n+1,l} - q_{n-1,l}}{q_{n,l} - q_{n-1,l} + \alpha_2} = \frac{(c_n + \alpha_l)(c_{n+1} - c_{n-1})}{(c_n - c_{n-1})(c_{n+1} + \alpha_l)}. \tag{4.294}$$

Performing the transformation:

$$R_{n,l} = (c_n + \alpha_l)\, q_{n,l} \tag{4.295}$$

we obtain the following second order ordinary difference equation for the field $R_{n,l}$:

$$\begin{aligned} R_{n+1,l} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} R_{n,l} \\ + \frac{c_{n+1}-c_n}{c_n-c_{n-1}} R_{n-1,l} - \alpha_2(c_n+\alpha_l)\frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0. \end{aligned} \tag{4.296}$$

Then we can represent the solutions of the equation (4.296) as:

$$R_{n,l} = P_{n,l} + \alpha_l e_n + f_n, \tag{4.297}$$

where e_n and f_n are particular solutions of:

$$\begin{aligned} e_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} e_n \\ + \frac{c_{n+1}-c_n}{c_n-c_{n-1}} e_{n-1} = \alpha_2 \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}, \end{aligned} \tag{4.298a}$$

$$\begin{aligned} f_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} f_n \\ + \frac{c_{n+1}-c_n}{c_n-c_{n-1}} f_{n-1} = \alpha_2 c_n \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}. \end{aligned} \tag{4.298b}$$

$P_{n,l}$ will be then solve the following equation:

$$P_{n+1,l} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} P_{n,l} + \frac{c_{n+1}-c_n}{c_n-c_{n-1}} P_{n-1,l} = 0. \tag{4.299}$$

The equations (4.298a) and (4.298b) are not independent. Indeed defining:

$$A_n = \frac{e_n c_{n-1} - e_{n-1} c_n - f_n + f_{n-1}}{c_n - c_{n-1}}, \tag{4.300}$$

and using (4.298) it is possible to show that the function A_n lies in the kernel of the operator $T_n - \mathrm{Id}$. This implies that $A_n = A_0 = \text{constant}$. We can without loss of generality assume the constant A_0 to be zero, since if we perform the transformation:

$$e_n = \tilde{e}_n - A_0, \tag{4.301}$$

the equation (4.300) is mapped into:

$$\frac{\tilde{e}_n c_{n-1} - \tilde{e}_{n-1} c_n - f_n + f_{n-1}}{c_n - c_{n-1}} = 0. \tag{4.302}$$

Furthermore since (4.298a) is invariant under the transformation (4.301) we can safely drop the tilde in (4.302) and assume that the functions e_n and f_n are solutions of the equations:

$$e_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} e_n + \frac{c_{n+1}-c_n}{c_n-c_{n-1}} e_{n-1} - \alpha_2 \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0, \tag{4.303a}$$

$$f_n - f_{n-1} = e_n c_{n-1} - c_n e_{n-1}. \tag{4.303b}$$

Finally we note that the function e_n can be obtained from (4.303a) by *two* discrete integrations. Indeed defining:

$$E_n = \frac{e_{n+1} - e_n}{c_{n+1} - c_n}, \tag{4.304}$$

and substituting in (4.303a) we obtain that E_n must solve the equation:

$$E_n - E_{n-1} = \alpha_2 \left(\frac{1}{c_{n+1} - c_n} + \frac{1}{c_n - c_{n-1}} \right). \tag{4.305}$$

Note that the right hand side of (4.305) is not a total difference. So the function e_n can be obtained by integrating (4.305) and subsequently integrating (4.304). This provides the value of e_n. The obtained value can be plugged in (4.303b) to give f_n after discrete integration. This reasoning shows that we can obtain the non-arbitrary functions e_n and f_n as result of a finite number of discrete integrations.

Now we turn to the solution of the homogeneous equation (4.299). We can reduce (4.299) to a total difference using the potential substitution $T_{n,l} = P_{n,l} - P_{n-1,l}$:

$$\frac{T_{n+1,l}}{c_{n+1} - c_n} - \frac{T_{n,l}}{c_n - c_{n-1}} = 0. \tag{4.306}$$

This clearly implies:

$$\frac{P_{n,l} - P_{n-1,l}}{c_n - c_{n-1}} = \beta_l, \tag{4.307}$$

where β_l is an arbitrary function. The solution to this equation is given by[5]:

$$P_{n,l} = (c_n + \alpha_l)\, \beta_l + \gamma_l, \tag{4.308}$$

where γ_l is an arbitrary function. Using (4.295,4.297,4.308) we obtain then the following expression for $q_{n,l}$:

$$q_{n,l} = \beta_l + \frac{\gamma_l + \alpha_l e_n + f_n}{c_n + \alpha_l}, \tag{4.309}$$

where e_n and f_n are solutions of (4.303).

Now since the solution we obtained in (4.309) depends on three arbitrary functions in the l direction, namely α_l, β_l and γ_l, there must be a constraint between these functions. This constraint is readily obtained by plugging (4.309) into (4.271). Factorizing the n dependent part away we are left with:

$$\gamma_{l+1} - \gamma_l = -(\alpha_{l+1} - \alpha_l) \left(\alpha_2 + \beta_{l+1} + \beta_l + 2\alpha_3 - \frac{1}{\varepsilon} \right). \tag{4.310}$$

5 The arbitrary functions are taken in a convenient way.

This equation tells us that the function γ_l can be expressed after a discrete integration in terms of the two arbitrary functions α_l and β_l. So the function $q_{n,l}$ is defined by (4.303,4.309,4.310) where the functions e_n, f_n and γ_l are defined implicitly and can be found by discrete integration. The value of $p_{n,l}$ now can be recovered by substituting (4.303,4.309,4.310) into (4.270a) and applying T_l^{-1}. This yields the general solution of the ${}_tH_2^\varepsilon$ equation (1.91b).

Remark 4.4.5. Let us notice that in the case $\varepsilon = 0$ formula (4.310) is singular. However in this case we just express the compatibility condition as $\alpha_{l+1} - \alpha_l = 0$, i.e. $\alpha_l = \alpha_0 =$ constant. It is easy to check that the obtained value of $q_{n,l}$ through formula (4.309) is consistent with the substitution of $\varepsilon = 0$ in (4.268). This means that in the case $\varepsilon = 0$ the value of $q_{n,l}$ is given by

$$q_{n,l} = \beta_l + \frac{\gamma_l + \alpha_0 e_n + f_n}{c_n + \alpha_0}, \tag{4.311}$$

where the functions e_n and f_n are defined implicitly and can be found by discrete integration from (4.303). As in the general case the value of $p_{n,l}$ now can be recovered by substituting (4.303,4.309) into (4.270a) and applying T_l^{-1}. This yields the general solution of the ${}_tH_2^\varepsilon$ equation (1.91b) if $\varepsilon = 0$.

4.4.3.3 *The ${}_tH_3^\varepsilon$ equation*

From Section 4.3 we know that the first integrals of the ${}_tH_3^\varepsilon$ equation (1.91c) in the n direction is a four-point, third order first integral (4.34a). Therefore this integral defines a four-point, third order ordinary difference equation from $W_1 = \xi_n$. This integral is particularly complex, so we start first by inspecting the ${}_tH_3^\varepsilon$ equation (1.91c) itself. If we apply the transformation (4.255) we can write the ${}_tH_3^\varepsilon$ equation (1.91c) as the following system of two coupled equations:

$$\begin{aligned}&\alpha_2(p_{n,l}q_{n+1,l} + p_{n+1,l}q_{n,l})\\ &- p_{n,l}q_{n,l} - p_{n+1,l}q_{n+1,l}\\ &- \alpha_3(\alpha_2^2 - 1)\left(\delta^2 + \varepsilon^2 \frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right) = 0,\end{aligned} \tag{4.312a}$$

$$\begin{aligned}&\alpha_2(q_{n,l}p_{n+1,l+1} + q_{n+1,l}p_{n,l+1})\\ &- q_{n,l}p_{n,l+1} - q_{n+1,l}p_{n+1,l+1}\\ &- \alpha_3(\alpha_2^2 - 1)\left(\delta^2 + \varepsilon^2 \frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right) = 0.\end{aligned} \tag{4.312b}$$

As in the case of the ${}_tH_2^\varepsilon$ equation (1.91b) we have that equation (4.312a) depends on $p_{n,l}$ and $p_{n+1,l}$ and that equation (4.312b) depends on $p_{n,l+1}$ and $p_{n+1,l+1}$. So we can apply the translation operator T_l to (4.312a) to obtain two equations in terms of $p_{n,l+1}$ and $p_{n+1,l+1}$:

$$\begin{aligned}
&\alpha_2(p_{n,l+1}q_{n+1,l+1}+p_{n+1,l+1}q_{n,l+1})\\
&-p_{n,l+1}q_{n,l+1}-p_{n+1,l+1}q_{n+1,l+1}\\
&-\alpha_3(\alpha_2^2-1)\left(\delta^2+\varepsilon^2\frac{q_{n,l+1}q_{n+1,l+1}}{\alpha_3^2\alpha_2}\right)=0,
\end{aligned} \tag{4.313a}$$

$$\begin{aligned}
&\alpha_2(q_{n,l}p_{n+1,l+1}+q_{n+1,l}p_{n,l+1})\\
&-q_{n,l}p_{n,l+1}-q_{n+1,l}p_{n+1,l+1}\\
&-\alpha_3(\alpha_2^2-1)\left(\delta^2+\varepsilon^2\frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right)=0.
\end{aligned} \tag{4.313b}$$

The system (4.313) is equivalent to the original system (4.312). Then since we can assume $\alpha_2, \alpha_3 \neq 0$[6] we can solve (4.313) with respect to $p_{n,l+1}$ and $p_{n+1,l+1}$:

$$p_{n,l+1}=\frac{\left[\begin{array}{c}\alpha_2\left(q_{n+1,l+1}-q_{n+1,l}\right)\left(\varepsilon^2 q_{n,l}q_{n,l+1}+\delta^2{\alpha_3}^2\right)\\ +\delta^2\alpha_2^2\alpha_3^2\left(q_{n,l}-q_{n,l+1}\right)\\ +\varepsilon^2 q_{n+1,l+1}q_{n+1,l}\left(q_{n,l}-q_{n,l+1}\right)\end{array}\right]}{\left(q_{n+1,l+1}q_{n,l}-q_{n+1,l}q_{n,l+1}\right)\alpha_3\alpha_2}, \tag{4.314a}$$

$$p_{n+1,l+1}=\frac{\left[\begin{array}{c}\alpha_2\left(\varepsilon^2 q_{n+1,l+1}q_{n+1,l}+\delta^2{\alpha_3}^2\right)\left(q_{n,l}-q_{n,l+1}\right)\\ +\delta^2\alpha_2^2\alpha_3^2\left(q_{n+1,l+1}-q_{n+1,l}\right)\\ +\varepsilon^2 q_{n,l}q_{n,l+1}\left(q_{n+1,l+1}-q_{n+1,l}\right)\end{array}\right]}{\left(q_{n+1,l+1}q_{n,l}-q_{n+1,l}q_{n,l+1}\right)\alpha_3\alpha_2}. \tag{4.314b}$$

We see that the right hand sides of (4.314) are functions only of $q_{n,l}$, $q_{n+1,l}$, $q_{n,l+1}$ and $q_{n+1,l+1}$. Moreover (4.314a) and (4.314b) must be compatible. Therefore applying T_l^{-1} to (4.314b) and imposing to the obtained expression to be equal to (4.314a) we find that $q_{n,l}$ must solve the following equation:

$$\begin{gathered}
\delta^2\alpha_2^2\alpha_3^2\left[q_{n+1,l+1}q_{n,l}-q_{n,l}q_{n-1,l+1}+q_{n,l+1}\left(q_{n-1,l}-q_{n+1,l}\right)\right]\\
-\alpha_2\left(\varepsilon^2 q_{n,l}q_{n,l+1}+\delta^2\alpha_3^2\right)\left(q_{n+1,l+1}q_{n-1,l}-q_{n-1,l+1}q_{n+1,l}\right)\\
+\varepsilon^2\left[q_{n,l}q_{n+1,l+1}\left(q_{n-1,l}-q_{n+1,l}\right)-q_{n,l+1}q_{n-1,l}q_{n+1,l}\right]q_{n-1,l+1}\\
+\varepsilon^2 q_{n+1,l+1}q_{n+1,l}q_{n,l+1}q_{n-1,l}=0.
\end{gathered} \tag{4.315}$$

As in the case of the ${}_tH_2^\varepsilon$ equation (1.91b) the partial difference equation for $q_{n,l}$ is not defined on a quad graph, but it is defined on the six-point lattice shown in Figure 4.1.

6 If $\alpha_2 = 0$ or $\alpha_3 = 0$ in (4.312) we have that the system becomes trivially equivalent to $q_{n,l} = 0$ and $p_{n,l}$ is left unspecified. Therefore we can discard this trivial case.

Remark 4.4.6. Let us note that if

$$q_{n+1,l+1}q_{n,l} - q_{n+1,l}q_{n,l+1} = 0 \tag{4.316}$$

we cannot define $p_{n,l+1}$ and $p_{n+1,l+1}$ as in (4.314). In this case we first solve equation (4.316) with respect to $q_{n,l}$ and then use the system (4.312) to specify $p_{n,l}$. Indeed equation (4.316) is a trivial Darboux integrable equation, since it possesses the following two-point, first order first integrals:

$$W_1 = \frac{q_{n+1,l}}{q_{n,l}}, \tag{4.317a}$$

$$W_2 = \frac{q_{n,l+1}}{q_{n,l}}. \tag{4.317b}$$

As remarked in the Introduction the existence of a two-point, first order first integral means that the equation is itself a first integral. Therefore the equation (4.316) can be alternatively written as $(T_l - \mathrm{Id})\,W_1$ or $(T_n - \mathrm{Id})\,W_2$ with W_1 and W_2 given by (4.317). From (4.317a) we obtain $q_{n+1,l}/q_{n,l} = \xi_n$. This equation is immediately solved by defining $\xi_n = a_{n+1}/a_n$, and we get the general solution of equation (4.316):

$$q_{n,l} = a_n\alpha_l, \tag{4.318}$$

where both a_n and α_l are arbitrary functions of their argument. Let us note that equation (4.316) is the *logarithmic discrete wave equation*, since it can be mapped into the discrete wave equation (4.272) exponentiating (4.272) and then taking $q_{n,l} \to e^{q_{n,l}}$, and it is a discretization of the hyperbolic partial differential equation:

$$uu_{xt} - u_x u_t = 0, \tag{4.319}$$

which is obtained from the wave equation $v_{xt} = 0$ using the transformation $v = \log u$. Since the transformation connecting (4.272) and (4.316) is not bi-rational, it does not preserves *a priori* its properties [62] (in this case linearization and Darboux integrability). Substituting (4.318) into (4.313) we obtain the compatibility condition:

$$\alpha_{l+1} - \alpha_l = 0, \tag{4.320}$$

i.e. $\alpha_l = \alpha_0 =$ constant and the system (4.312) is now consistent. Therefore we are left with one equation for $p_{n,l}$, e.g. (4.312a). Therefore inserting (4.318) with $\alpha_l = \alpha_0$ in (4.312a) and solving with respect to $p_{n+1,l}$ we obtain:

$$p_{n+1,l} = \frac{\alpha_2 a_{n+1} - a_n}{a_{n+1} - \alpha_2 a_n} p_{n,l} + (\alpha_2^2 - 1)\frac{\delta^2\alpha_3^2\alpha_2 + \varepsilon^2 a_n \alpha_0^2 a_{n+1}}{\alpha_3\alpha_2\alpha_0(\alpha_2 a_n - a_{n+1})}. \tag{4.321}$$

Defining

$$\frac{\alpha_2 a_{n+1} - a_n}{a_{n+1} - \alpha_2 a_n} = \frac{b_{n+1}}{b_n}, \tag{4.322}$$

we have that $p_{n,l}$ solves the equation:

$$\frac{p_{n+1,l}}{b_{n+1}} = \frac{p_{n,l}}{b_n} + \frac{\delta^2\alpha_3^2\alpha_2^2 b_n - b_{n+1}\left(\delta^2\alpha_3^2 + \varepsilon^2 a_n^2\alpha_0^2\right)\alpha_2 + \varepsilon^2 a_n^2\alpha_0^2 b_n}{b_n a_n \alpha_0\alpha_2\alpha_3 b_{n+1}}. \tag{4.323}$$

Note that b_n in (4.322) is given in terms of a_n and a_{n+1} through discrete integration. Equation (4.323) is solved by:

$$p_{n,l} = b_n\left(\beta_l + c_n\right), \tag{4.324}$$

where c_n is given by the discrete integration:

$$c_{n+1} = c_n + \frac{\delta^2\alpha_3^2\alpha_2^2 b_n - b_{n+1}\left(\delta^2\alpha_3^2 + \varepsilon^2 a_n^2\alpha_0^2\right)\alpha_2 + \varepsilon^2 a_n^2\alpha_0^2 b_n}{b_n a_n \alpha_0\alpha_2\alpha_3 b_{n+1}}. \tag{4.325}$$

This yields the solution of the ${}_tH_3^\varepsilon$ equation (1.91c) when $q_{n,l}$ satisfy equation (4.316).

In the general case we have proved that the ${}_tH_3^\varepsilon$ equation (1.91c) is equivalent to the system (4.312) which in turn is equivalent to the solution of equations (4.314a) and (4.315). However equation (4.314a) merely defines $p_{n,l+1}$ in terms of $q_{n,l}$ and its shifts. Therefore if we find the general solution of (4.315) the value of $p_{n,l}$ will follow. To find such solution we turn to the first integrals. Like in the case of the ${}_tH_2^\varepsilon$ equation (1.91b) we will find an expression for $q_{n,l}$ using the first integrals, and then we will insert it into (4.315) to reduce the number of arbitrary functions to the right one.

We consider the equation $W_1 = \xi_n/\alpha_2$[7], where W_1 is given by (4.34a), with $k = 2l+1$:

$$\frac{\left(u_{n+1,2l+1} - u_{n-1,2l+1}\right)\left(u_{n+2,2l+1} - u_{n,2l+1}\right)}{\left(\alpha_2 u_{n,2l+1} - u_{n-1,2l+1}\right)\left(u_{n+2,2l+1} - \alpha_2 u_{n+1,2l+1}\right)} = \xi_n. \tag{4.326}$$

Using the substitutions (4.255) we have:

$$\frac{\left(q_{n+1,l} - q_{n-1,l}\right)\left(q_{n+2,l} - q_{n,l}\right)}{\left(\alpha_2 q_{n,l} - q_{n-1,l}\right)\left(q_{n+2,l} - \alpha_2 q_{n+1,l}\right)} = \xi_n. \tag{4.327}$$

This equation contains only $q_{n,l}$ and its shifts. By the transformation:

$$Q_{n,l} = \frac{\alpha_2 q_{n,l} - q_{n-1,l}}{q_{n+1,l} - q_{n-1,l}} \tag{4.328}$$

equation (4.327) becomes:

$$Q_{n+1,l} + \frac{1}{\xi_n Q_{n,l}} = 1, \tag{4.329}$$

7 The extra α_2 is due to the arbitrariness of ξ_n and is inserted in order to simplify the forumulæ.

which is the same discrete Riccati equation as in (4.284). This means that the solution of (4.329) is given by (4.293) with the appropriate definitions (4.285,4.288,4.291). We can substitute into (4.328) the solution (4.293):

$$\frac{q_{n+1,l} - q_{n-1,l}}{\alpha_2 q_{n,l} - q_{n-1,l}} = \frac{(c_n + \alpha_l)(c_{n+1} - c_{n-1})}{(c_{n+1} + \alpha_l)(c_n - c_{n-1})} \tag{4.330}$$

and we obtain an equation for $q_{n,l}$. Introducing:

$$P_{n,l} = (c_n + \alpha_l)\, q_{n,l} \tag{4.331}$$

we obtain that $P_{n,l}$ solves the equation:

$$P_{n+1,l} - \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}} P_{n,l} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} P_{n-1,l}. = 0. \tag{4.332}$$

Using the transformation:

$$P_{n,l} = \frac{R_{n,l}}{R_{n-1,l}}, \tag{4.333}$$

we can cast equation (4.332) in discrete Riccati equation form:

$$R_{n+1,l} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} \frac{1}{R_{n,l}} = \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}}. \tag{4.334}$$

Let d_n be a particular solution of equation (4.334):

$$d_{n+1} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} \frac{1}{d_n} = \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}}. \tag{4.335}$$

Assuming d_n as the new arbitrary function we can express c_n as the result of *two* discrete integrations. Indeed introducing $z_n = c_n - c_{n-1}$ in (4.335) we have:

$$\frac{z_{n+1}}{z_n} = \frac{(d_{n+1} - \alpha_2) d_n}{\alpha_2 d_n - 1}. \tag{4.336}$$

Equation (4.336) represents the first discrete integration, whereas the second one is given by the definition:

$$c_n - c_{n-1} = z_n. \tag{4.337}$$

Now we can linearize the discrete Riccati equation (4.334) by the transformation:

$$R_{n,l} = d_n + \frac{1}{S_{n,l}}. \tag{4.338}$$

and we get the following *linear* equation for $S_{n,l}$:

$$S_{n+1,l} - \frac{d_n^2 (c_n - c_{n-1})}{c_{n+1} - c_n} S_{n,l} = \frac{d_n (c_n - c_{n-1})}{c_{n+1} - c_n}. \tag{4.339}$$

Defining:

$$d_n = \frac{e_n}{e_{n-1}}, \tag{4.340a}$$

$$f_n - f_{n-1} = \frac{c_n - c_{n-1}}{e_n e_{n-1}}, \tag{4.340b}$$

the solution of (4.339) is:

$$S_{n,l} = \frac{(f_{n-1} + \beta_l) e_{n-1}^2}{c_n - c_{n-1}}. \tag{4.341}$$

Inserting (4.341) and (4.340) into (4.338) we obtain:

$$R_{n,l} = \frac{e_n (f_n + \beta_l)}{e_{n-1}(f_{n-1} + \beta_l)}. \tag{4.342}$$

Inserting the definition of $R_{n,l}$ (4.333) into (4.342) we obtain:

$$\frac{P_{n,l}}{e_n(f_n + \beta_l)} = \frac{P_{n-1,l}}{e_{n-1}(f_{n-1} + \beta_l)}, \tag{4.343}$$

i.e.:

$$P_{n,l} = \gamma_l e_n (f_n + \beta_l). \tag{4.344}$$

Introducing (4.344) into (4.331) we obtain:

$$q_{n,l} = \frac{\gamma_l e_n (f_n + \beta_l)}{c_n + \alpha_l}, \tag{4.345}$$

where f_n is defined by (4.340b), and c_n is given by (4.336) and (4.337), i.e. c_n is the solution of the equation:

$$\frac{c_{n+1} - c_n}{c_n - c_{n-1}} = \frac{e_{n+1} - \alpha_2 e_n}{\alpha_2 e_n - e_{n-1}}, \tag{4.346}$$

and e_n is an arbitrary function.

Now since the solution we obtained in (4.345) depends on three arbitrary functions in the l direction, namely α_l, β_l and γ_l, there must be a constraint between these functions. This constraint is readily obtained by plugging (4.345) into (4.315). Factorizing the n dependent part away we are left with:

$$\alpha_{l+1} - \alpha_l = \frac{\varepsilon^2}{\alpha_2 \delta^2 \alpha_3^2} \gamma_{l+1} \gamma_l (\beta_{l+1} - \beta_l) \tag{4.347}$$

This equation tells us that the function α_l can be expressed after a discrete integration in terms of the two arbitrary functions β_l and γ_l. So the function $q_{n,l}$ is defined by (4.340b,4.345–4.347) where the functions c_n, f_n and α_l are defined implicitly and can be found by discrete integration. The value of $p_{n,l}$ now can be recovered by substituting (4.340b,4.345–4.347) into (4.314a) and applying T_l^{-1}. This yields the general solution of the ${}_tH_3^\varepsilon$ equation (1.91c).

Remark 4.4.7. Let us notice that when $\delta = 0$ equation (4.347) is singular. However in this case the compatibility condition is replaced by $\beta_{l+1} - \beta_l = 0$, i.e. $\beta_l = \beta_0 =$ constant. It is easy to check that the obtained value of $q_{n,l}$ through formula (4.345) is consistent with the substitution of $\delta = 0$ in (4.312). This means that in the case $\delta = 0$ the value of $q_{n,l}$ is given by

$$q_{n,l} = \frac{\gamma_l e_n (f_n + \beta_0)}{c_n + \alpha_l}, \tag{4.348}$$

where the functions c_n and f_n are defined implicitly and can be found by discrete integration from (4.340b) and (4.346) respectively. As in the general case the value of $p_{n,l}$ now can be recovered by substituting (4.340b,4.345,4.346) into (4.314a) and applying T_l^{-1}. This yields the general solution of the ${}_tH_3^\varepsilon$ equation (1.91c) if $\delta = 0$.

CONCLUSIONS

In this Thesis we gave a comprehensive account of recent findings about the equations possessing the Consistency Around the Cube. In particular our main objective was the study of the trapezoidal H^4 equation (1.91) and of the H^6 equations (1.93). Apart from their introduction and classification in [3, 20–22] before the studies of the author and his collaborators little was known about the integrability properties of these equation. These integrability properties were studied from different points of view, since as it was claimed in Section (1.1) does not exists a universal definition of integrability. Therefore in each Chapter of this thesis we gave a self-contained introduction to the methods we used, namely the *Consistency Around the Cube* in Chapter 1, the *Algebraic Entropy* in Chapter 2, *Generalized Symmetries* in Chapter 3 and the *Darboux integrability* in Chapter 4. Therefore we can say that the primary object of this Thesis is to consider the trapezoidal H^4 equation (1.91) and of the H^6 equations (1.93) from the standpoint of different forms of integrability, as each form sheds a different light on the nature of the equations dealt with.

Our main result was stated in Chapter 2 and it is the heuristic proof of the fact that, according to Algebraic Entropy, the trapezoidal H^4 equation (1.91) and of the H^6 equations (1.93) are actually linearizable equations. This shows how different tests of integrability can provide different outcomes, and they must be considered as complementary to each other. To support this heuristic statement we gave some examples of explicit linearization.

In Chapter 3 we showed that the three-point Generalized Symmetries of the trapezoidal H^4 equation (1.91) and of the H^6 equations (1.93) are particular instances of the non-autonomous YdKN equation (3.99). This result completed the identification of the three-point Generalized Symmetries belonging to the ABS and Boll's classification started in [102] and in [166] respectively. Furthermore based on the observation that the most general case of the non-autonomous YdKN (3.99) was not covered by such symmetries we conjectured the existence of a new integrable equations which encloses all the equations coming from the Boll's classification. We then found a candidate equation namely (3.150), which is a non-autonomous generalization of the Q_V equation (3.137) introduced in [156]. This new non-autonomous Q_V equation (3.150) passes the Algebraic Entropy test, but unfortunately we were not able, due to the computational complexity, to prove that its three-point symmetries are given by the general case of the non-autonomous YdKN equation (3.99). The main significance of this new equation is that that all the equations from Boll's classification [20–22] can be seen as particular cases of a single equation. A complete understanding of this equation will result in better comprehension of all its particular cases.

In Chapter 4 we started by showing the Darboux integrability of the equations possessing the Consistency Around the Cube, but not the tetrahedron property found by J. Hietarinta in [82]. These equations were known to be linearizable [138] and this observation, along with the peculiar symmetry structure of the ${}_tH_1^\varepsilon$ equation (1.91a) led us to conjecture that the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93) are Darboux integrable equations. The Darboux integrability of the trapezoidal H^4 equations (1.91) and the H^6 equations (1.93) was then established using a modification of the method proposed in [55]. In the final part of Chapter 4 we showed how from Darboux integrability we can obtain linearization and the general solutions of the trapezoidal H^4 equations (1.91) and of the H^6 equations (1.93). The construction of the general solutions for the trapezoidal H^4 equations and for the H^6 equations is a consequence of the Darboux integrability property and it is something that cannot be inferred from the method used in classify them. These general solutions were obtained in three different ways, but the common feature is that they can be found through some *linear or linearizable (discrete Riccati) equations*. This is the great advantage of the first integral approach with respect to the direct one which was pursued in [67]. The Darboux integrability therefore yields extra information that it is useful to get the final result, i.e. the general solutions. Moreover linearization arises very naturally from first integrals also in the most complicated cases, whereas in the direct approach can be quite tricky, see e.g. the examples in [67]. The linearization of the first integrals is another proof of the intimate linear nature of the H^4 and H^6 equations. This result is even more stronger than Darboux integrability alone, since *a priori* the first integrals do not need to define linearizable equations. We also note that our procedure of construction of the general solution, based on the ideas from [56], is likely to be the discrete version of the procedure of linearization and solutions for continuous Darboux integrable equations presented in [178]. The preeminent rôle of the discrete Riccati equation in the solutions is reminiscent of the importance of the usual Riccati equation in the continuous case. Recall e.g. that the first integrals of the Liouville equation (4.3) are Riccati equations (4.4).

Now, although our knowledge of the trapezoidal H^4 equations (1.91) and of the H^6 equations (1.93) is more deep than before, we would like to end this Thesis mentioning some problems which are still open.

- Prove (or disprove) that the J_1 equation, i.e. the first of the Hietarinta's (1.28) is Darboux integrable. Since it linearizable and possess the Consistency Around the Cube in Chapter 4 we conjectured that it should be Darboux integrable, but it is possible to prove that its first integrals (4.7) must be of order higher than two.

- Prove the validity of the connection formulæ (3.154) and (F.4) and directly compute the three-point Generalized Symmetries of the non-autonomous Q_V equation.

- Find and classify all the non-autonomous equations satisfying the "strict" Klein symmetries (3.144) according to their integrability properties. We believe that in studying such equations both the Algebraic Entropy approach and the Generalized Symmetries approach can be very fruitful. However we remark that, as showed with an example in Chapter 3 the Generalized Symmetries of these equations can depend on more than on three points.

- As mentioned in Chapter 4, Darboux integrable equations admit Generalized Symmetries depending on arbitrary functions [5]. However for the trapezoidal H^4 equations (1.91) and for the H^6 equations (1.93) the explicit form of the symmetries depending on arbitrary functions is known only for the ${}_tH_1^\varepsilon$ equation (1.91a). This poses the challenging problem of finding the explicit form of such generalized symmetries. These symmetries will be highly non-trivial, especially in the case of the ${}_tH_2^\varepsilon$ equation (1.91b) and of the ${}_tH_3^\varepsilon$ equation (1.91c) were the order of the first integrals is particularly high.

As a final remark we note that the majority of the open problems originates directly from the original researches reported in this Thesis.

A

FINAL PART OF THE PROOF OF THE COMMUTATIVE DIAGRAM 1.9

In this Appendix we conclude the proof of fact that the diagram 1.9 is commutative, by checking all the fundamental Möbius transformations. To this end we have written a `python` module `mobgen.py` which contains `python` functions in order to generate all possible Möb^4 and $\widehat{\text{Möb}}^4$ transformations and compare between the results:

`generate_mob4` Generate an element of Möb^4 with prescribed transformations type acting on the four vertices. Types are classified as translations, T, dilation, D, and inversions I.

`generate_mobnm` Generate an element of $\widehat{\text{Möb}}^4$ with prescribed transformations type acting on the four vertices. As abobe types are classified as translations, T, dilation, D, and inversions I.

`tilde_eq_der` Generate the equation on the lattice (1.48) starting from an abstract equation on the geometrical quad graph (1.29).

`equality_check` Checks if the result of the application of an element of Möb^4 and of an element of $\widetilde{\text{Möb}}^4$ is the same. The output is a boolean value.

`generate_type` Generate all possible combinations (with repetition) of four Möbius transformations.

The content of `mobgen.py` is the following:

```
from sympy import *

def generate_type():
    "Generate all possible combinations (with repetition) of four Moebius transformations."
    TT= ['T', 'D', 'I']
    TF = []
    for t1 in TT:
        for t2 in TT:
            for t3 in TT:
                for t4 in TT:
                    TE = [t1,t2,t3,t4]
                    TF = TF + [TE]
    return TF

def generate_mob4(TT,u):
    "Generate an element of Mob4 with prescribed transformations type acting on the four vertices."
```

```
    var('a')
    Mob4 = {}
    Ind = [0,1,2,12]
    for i in range(len(TT)):
        if TT[i] == 'T':
            MobT = u(Ind[i])+a(Ind[i])
        elif TT[i] == 'I':
            MobT = 1/u(Ind[i])
        elif TT[i] == 'D':
            MobT = a(Ind[i])*u(Ind[i])
        else:
            raise ValueError('Not a fundamental type Moebius transformation')
        Mob4[u(Ind[i])] = MobT
    return Mob4

def generate_mobnm(TT,EM,u):
    "Generate an element of Mobnm with prescribed transformations type acting on the four vertices."
    var('a')
    var('n m', integer=True)
    f0 = (1+EM[0]*(-1)**n+EM[1]*(-1)**m+EM[0]*EM[1]*(-1)**(n+m))/4
    f1 = (1-EM[0]*(-1)**n+EM[1]*(-1)**m-EM[0]*EM[1]*(-1)**(n+m))/4
    f2 = (1+EM[0]*(-1)**n-EM[1]*(-1)**m-EM[0]*EM[1]*(-1)**(n+m))/4
    f12 = (1-EM[0]*(-1)**n-EM[1]*(-1)**m+EM[0]*EM[1]*(-1)**(n+m))/4
    F = [f0,f1,f2,f12]
    Mobnm = {}
    Ind = [0,1,2,12]
    Mob0=0
    for i in range(len(TT)):
        if TT[i] == 'T':
            MobT = u(0)+a(Ind[i])
        elif TT[i] == 'I':
            MobT = 1/u(0)
        elif TT[i] == 'D':
            MobT = a(Ind[i])*u(0)
        else:
            raise ValueError('Not a fundamental type Moebius transformation')
        Mob0 += F[i]*MobT
    Mob0=(Mob0.expand()).simplify()
    Mobnm[u(0)]=Mob0
    Mob1 = Mob0.subs({n: n+1,u(0): u(1)},simultaneous=True)
    Mobnm[u(1)]=Mob1
    Mob2 = Mob0.subs({m: m+1,u(0): u(2)},simultaneous=True)
    Mobnm[u(2)]=Mob2
    Mob12 = Mob0.subs({n: n+1,m: m+1, u(0): u(12)},simultaneous=True)
```

```
    Mobnm[u(12)]=Mob12
    return Mobnm

def tilde_eq_der(Q,EM,u):
    bQsub = {u(0): u(1), u(1): u(0), u(2): u(12), u(12): u(2)}
    Qbsub = {u(0): u(2), u(1): u(12), u(2): u(0), u(12): u(1)}
    bQbsub = {u(0): u(12), u(1): u(2), u(2): u(1), u(12): u(0)}

    f0 = (1+EM[0]*(-1)**n+EM[1]*(-1)**m+EM[0]*EM[1]*(-1)**(n+m))
        /4
    f1 = (1-EM[0]*(-1)**n+EM[1]*(-1)**m-EM[0]*EM[1]*(-1)**(n+m))
        /4
    f2 = (1+EM[0]*(-1)**n-EM[1]*(-1)**m-EM[0]*EM[1]*(-1)**(n+m))
        /4
    f12 = (1-EM[0]*(-1)**n-EM[1]*(-1)**m+EM[0]*EM[1]*(-1)**(n+m))
        /4

    Qt = ((f0*Q+f1*Q.subs(bQsub,simultaneous=True)\
    +f2*Q.subs(Qbsub,simultaneous=True)\
    +f12*Q.subs(bQbsub,simultaneous=True)).expand()).simplify()

    return Qt

def equality_check(Q,TT,E,u):
    Mob4sub=generate_mob4(TT,u)
    Mobnmsub=generate_mobnm(TT,E,u)

    Qhat=Q.subs(Mob4sub)
    Qhattilde=tilde_eq_der(Qhat,E,u)

    Qtilde=tilde_eq_der(Q,E,u)
    Qtildehat=Qtilde.subs(Mobnmsub)

    DQ = ((Qhattilde-Qtildehat).expand()).simplify()
    if DQ == 0:
        return True
    else:
        return False
```

All the functions described above are used in following program which generates the most general quad equation as shown in (1.86) and checks if the action of all the possible elements of Möb^4 and $\widehat{\text{Möb}}^4$ yields the same equation.

```
from sympy import *
import time
from mobgen import *

def seconds_to_hours(s):
    if s >= 3600:
        si, sd = divmod(s, 1)
        si = int(si)
        h = si/3600
```

```
        m = (si/60) % 60
        ss = si % 60
        return str(h)+'h'+str(m)+'m'+str(ss+sd)[:5]+'s'
    elif s < 3600 and s >= 60:
        si, sd = divmod(s, 1)
        si = int(si)
        m = (si/60) % 60
        ss = si % 60
        return str(m)+'m'+str(ss+sd)[:5]+'s'
    elif s < 60 and s >10:
        return str(s)[:5]+'s'
    else:
        return str(s)[:4]+'s'

def main():
    #generate all possible Moebius transformations
    MTot = generate_type()
    #generic quad-equation
    var('A B C D K u')
    #constant part
    Qconst = K
    #linear part
    Qlin = D(0)*u(0)+D(1)*u(1)+D(2)*u(2)+D(12)*u(12)
    #quadratic part
    Qquadr = C(0,1)*u(0)*u(1)+C(0,2)*u(0)*u(2)+C(0,12)*u(0)*u(12) \
    +C(1,2)*u(1)*u(2)+C(1,12)*u(1)*u(12)+A(2,12)*u(2)*u(12)
    #cubic part
    Qcub = B(0,1,2)*u(0)*u(1)*u(2)+B(0,1,12)*u(0)*u(1)*u(12)+B(1,2,12)*u(1)*u(2)*u(12)
    #quartic part
    Qquart = A(0,1,2,12)*u(0)*u(1)*u(2)*u(12)
    #the equation
    Q = Qconst + Qlin + Qquadr + Qcub + Qquart
    print Q
    #various possible embeddings
    R = [-1,1]
    EM = []
    for r in R:
        for s in R:
            EP = [r,s]
            EM += [EP]
    for E in EM:
        print E
        #taking trace of the transformation
        N=0
        #taking trace of eventual ''errors''
        Nerr = []
        dt =float(0)
        fo= open('quad_equation_check_'+str(E[0])+'_'+str(E[1])+'.out', 'w')
```

```
        fo.write('We check the equation\n'+latex(Q, mode='equation')+'\n')
        for TT in MTot:
            N += 1
            t0 = time.time()
            test = equality_check(Q,TT,E,u)
            dt0 = time.time()-t0
            print N, test, dt0
            dt += dt0
            fo.write(str(N)+': '+str(TT)[1:-1]+' is '+str(test)+'\n')
            if test == False:
                Nerr += [N]

        fo.write('Operations take '+seconds_to_hours(dt)+', with average time '+seconds_to_hours(dt/81)+'.\n')

        if Nerr != []:
            fo.write('There were errors at'+str(Nerr)[1:-1]+'. Check carefully what happened!')

if __name__ == '__main__':
    main()
```

Running the above program has resulted in a complete verification of the equivalence of the two actions.

B

PYTHON PROGRAMS FOR COMPUTING ALGEBRAIC ENTROPY

In this Appendix we discuss some `python` programs which can compute the iterates of various kind of discrete an semi-discrete equations. In particular in Section B.1 we present the `python` module `ae2d.py` which can be used to compute the degrees of iterates of systems of quad equations, this was presented in [64]. In Section B.2 we present the `python` module `ae_differential_difference.py` which can be used to compute the degrees of the iterates of systems of difference equations and/or differential-difference of arbitrary order. Finally in Section B.3 we present some `python` utilities for the analysis of the obtained sequences. All the proposed modules extensively uses the classes defined by the pure `python` Computer Algebra System `sympy` [150], so to be used they require a basic knowledge of it.

Following the development guidelines of `sympy` [150] all the module are written using the `python3` standards, but they maintain the `python2` compatibility. To this end is used the module `__future__` to redefine the `division` function and the printing system. All the modules were tested on both `python` versions.

B.1 ALGEBRAIC ENTROPY OF SYSTEMS OF QUAD EQUATIONS WITH `ae2d.py`

In this Section we give a brief description of the functions contained in the `python` module `ae2d.py`. The scope of the functions in `ae2d.py` is to calculate the degree of the iterates of a system of quad equations. We underline that `ae2d.py` uses the `multiprocessing` module which permit to take advantage of modern multi-core architectures to evaluate simultaneously the degrees along the diagonals.

B.1.1 *Description of the content of* `ae2d.py`

The `ae2d.py` module contains the following functions[1]:

`up_or_anti_diagonal` Creates a diagonal or anti-diagonal matrix. This is an auxiliary function used in constructing the *evolution matrices* (2.37).

`evolution_matrix` Construct an evolution matrix starting from the degrees of the iterates. This representation relies on the `matrix`

1 The names of the functions follow the naming convention adopted by `sympy` developers.

class defined by `sympy`. This is an auxiliary function automatically used by the `evol` function in case there is the need of displaying the results in the form of an evolution matrix (2.37).

`analyze_evolution_matrix` This function analyze an evolution matrix (2.37) searching for repeating patterns. This is a user-level function and it accepts a single argument `M` which must be a `sympy` matrix. The user has no need to specify which is the direction of the evolution. The function detect the kind of evolution matrix from the fact that an evolution matrix (or it transposed matrix) is a *Hessenberg matrix* [80]. Then the matrix is transformed into a list of lists eliminating the zero entries and ordering the list from the longest to the shortest. Then a pairwise comparison of the lists is carried out by comparing appropriate slice of the two lists. Equal (slice of) lists are then removed. The result of this procedure are the unequal sequences of degrees.

`gen_ics` This function generates the set of the initial conditions for a system of quad equations. The main advantage of `ae2d.py` is that this function, can also handle arbitrary constants and arbitrary functions of the lattice variables including shifts. Following the remarks given in Section 2.2 it generate a staircase of initial values linearly parametrized (2.44). Assuming that we have a system of M quad equations and that we have chosen to perform N iterations we will have the following number of initial conditions needed:

$$N_{\text{ini}} = 4M\left(N+1\right). \tag{B.1}$$

We assign to those values prime numbers.

Next the function has a list F of the arbitrary functions, possibly with their shifts. Then we operate in the following way:

1. Taken an element $f \in F$ check on how many variables it depends. Then we proceed in isolating all the other elements $g \in F$ which has the same symbol, i.e.:

$$f_{l+1} \to f, g_{l+1} \to g \dots \tag{B.2}$$

2. Now consider the case when $f \in F$ depends only upon a single variable. At the end of the preceding step we have formed a list of the form:

$$F_f = \left\{ f_{q+j_1}, \dots, f_{q+j_{k_f}} \right\}. \tag{B.3}$$

Then we define:

$$j_m = \min_{k=1,\dots,k_f} j_k, \quad j_M = \min_{k=1,\dots,k_f} j_k. \tag{B.4}$$

We then form a list F_s of all the triples

$$(f, j_m, j_M). \tag{B.5}$$

So finally we have that to evaluate the functions of a single variable along the evolution we then need to evaluate them on the following number of points:

$$N_s = \sum_{k=1}^{|F_s|} (N + 1 + j_{M_i} - j_{m_i}). \tag{B.6}$$

We choose as values for those functions N_s prime numbers.

3. Now we consider the case in which $g \in F$ depends on two variables. Again from 1 we have a list of the form:

$$F_g = \left\{ g_{l+j_1, m+z_1}, \dots, g_{l+j_{k_g}, m+z_{w_g}} \right\}. \tag{B.7}$$

Now we define:

$$i_m = \min_{k=1,\dots,k_f, w=1,\dots,w_g} \{ j_k, z_w \}, \tag{B.8a}$$

$$i_M = \max_{k=1,\dots,k_f, w=1,\dots,w_g} \{ j_k, z_w \}. \tag{B.8b}$$

We then form a list F_d of all the triples:

$$(g, i_m, i_M). \tag{B.9}$$

So finally we have that to evaluate the functions of a two variables along the evolution we then need to evaluate them on the following number of points:

$$N_d = \sum_{k=1}^{|F_d|} \frac{[N + 1 + 2(i_{M_k} - i_{m_k})][N + 2 + 2(i_{M_k} - i_{m_k})]}{2} + \sum_{k=1}^{|F_d|} [N + 2(i_{M_k} - j_{m_k})] \tag{B.10}$$

We choose as values for those functions N_d prime numbers.

Remark B.1.1. The proposed procedure for the evaluation of the functions of two variables may result in considering more points than needed. However a more accurate procedure of evaluation will be much more complicated, and above all this procedure is quite cheap in term of resource usage.

We needed the following amount of prime numbers

$$N_{TOT} = N_{ini} + N_{ac} + N_s + N_d. \tag{B.11}$$

where N_{ac} is the number of arbitrary constants. The prime numbers are generated with the `sympy` function `sieve` and shuffled before being assigned, so that running the program on the same equation more than once will result in different initial conditions and different values of the arbitrary functions and arbitrary constants.

We underline that the `gen_ics` function is not intended for a user-level usage, but is used automatically by the `evol` function.

`evol` `evol` is the principal user-level function in `ae2d.py`. Its scope is to output a degree sequence or an evolution matrix. Since the whole program is supposed to work at the affine level it returns the degree sequence of each dependent variable. Now we describe carefully its input, its operations and the possible form of its output.

The function `evol` needs two mandatory arguments:

`EQ` The system of quad equations under considerations. It can be entered as a `python` list, or as a `python` dictionary. In the case of a single scalar equation it can be entered plainly.

`Y` The dependent variables. They can be entered as a `python` list, or as a `python` dictionary. In the case of a single dependent variable it can be entered plainly. The entries of the list must be `sympy` `Functions` of two lattice variables, e.g.:

$$\mathsf{Y} = [\mathsf{u}(\mathsf{n},\mathsf{m}), \mathsf{v}(\mathsf{n},\mathsf{m})]. \tag{B.12}$$

Remark B.1.2. The function `evol` can handle only well-determined systems of quad equations so it will check that the number of equations is equal to the number of dependent variables. In negative case it will raise an error. To reduce a system of quad equation to a well-determined one is the up to the user.

Moreover the function `evol` support four optional parameters:

NI Defines the number of iterations. It must be a positive integer, and error is returned otherwise. The default value is `NI=8`.

direction Direction defines the principal diagonal of evolution as shown in Figure 2.5. The possible choices are:

direction="mp" perform the evolution from the $\Delta_{-,+}$ principal diagonal, this is the default argument,

direction="pm" perform the evolution from the $\Delta_{+,-}$ principal diagonal,

direction="mm" perform the evolution from the $\Delta_{-,-}$ principal diagonal,

direction="pp" perform the evolution from the $\Delta_{+,+}$ principal diagonal, direction="all" performs the evolution in all directions. If the evolution in one of the four direction is not possible [64] then the direction is skipped.

If an argument different from the listed ones is used then the program falls back to the default one.

matrix_form This is a boolean parameter. By default it is `false`, and it is automatically activated by the program if it finds different degrees along the diagonals. Setting it to true *ab initio* will cause the result to be outputted in matrix form regardless of the need of such representation.

verbosity_level This parameter sets how much output will be displayed on the `stout` by the program. It has three possible values:

verbosity_level=0 The function displays nothing on the `stout`.

verbosity_level=1 The function displays on the `stout` only the progression of the iterates in the form `i/NI`, where `i` is the `i`-th iterate, this is the default value.

verbosity_level=2 The function displays nothing the `stout`.

If the value of `verbosity_level` is not an integer then it falls back to the default value.

The function then accepts *keyword arguments*. Keyword arguments in this context can be used to set some parameters to a fixed (not arbitrary in the sense of the function `gen_ics`) value. Keyword arguments are immediately substituted into `EQ`.

The function then starts by checking that the given equations are rational quad equations and search for arbitrary constants and functions. Anything not-appearing in the list `Y` is considered here arbitrary. Then the system is solved in the chosen direction, and the function `gen_ics` is recalled in order to generate the initial conditions and the values of the arbitrary functions and constants.

Remark B.1.3. A continuous variable `t` is defined as in (2.44). This implies that it is not possible to call one of the two discrete variable `t`.

Now the actual evolution begins. The function `value_deg` of step evaluation is defined. This function encloses all the relevant step-wise substitutions and degree calculations. The main point is the fact that after the substition a factorization in the finite field $\mathbb{K}_P$ is carried out. The prime number P is chosen to be the first prime after p_{MAX}^2, where p_{MAX} is the biggest prime generated by `gen_ics`. The number of points on the diagonal to be evaluated is then divided by the number of CPU at disposal. The function `evol` then exectute the `value_deg` function in chuncks each one of lenght equal to the number of CPU. This procedure is repeated until the `NI`-th iteration.

At this point the evolution is ended. It is checked if there is the need of a matrix form. If not the result is returned as a list of tuples whose first element is the name of the dependent variable (e.g. `u(l,m)` is returned as `u`) and whose second member is the list of the degrees or an evolution matrix. In the case of scalar equation, since there is no ambiguity, just the list of degrees or an evolution matrix is returned.

B.1.2 *Code*

The content of the file `ae2d.py` is the following:

```python
from __future__ import print_function, division

from sympy.core.symbol import Symbol, Wild
from sympy.core.relational import Equality
from sympy.core.add import Add
from sympy.core.mul import Mul
from sympy.simplify import simplify, numer, denom
from sympy.solvers import solve
from sympy.polys.polytools import degree, factor
from sympy.functions.elementary.miscellaneous import Max
from sympy.ntheory.generate import sieve,nextprime
from sympy.matrices import Matrix
from sympy.functions.elementary.integers import ceiling
from array import array
from random import shuffle
from warnings import warn
from itertools import chain
from multiprocessing import Queue, Process,cpu_count

class WellDefinitenessError(Exception):
    pass

def up_or_anti_diagonal(dim,n,V,Anti=False):
    """Create a matrix which is different from 0 only on an upper
    or lower diagonal."""
    if len(V) !=dim-abs(n):
        raise ValueError("Vector must be of the appropriate
            lenght %s" % str(dim-abs(n)))
    if Anti:
        k1 = 0
        k2 = 1
    else:
        k1 = 1
        k2 = 0
    def f(i,j):
        if i == k1*(j-n) + k2*(dim - 1 - n -j):
            if n>0:
```

```
                return V[i]
            else:
                return V[k1*j+k2*(j+n)]
        else:
            return 0
    return Matrix(dim,dim,f)

def evolution_matrix(VV,DIR):
    """Generate an evolution matrix on
    in all the the possible directions."""
    if type(VV) != list:
        raise TypeError("Argument %s must be a list." % str(VV))
    NI = len(VV)
    for i in range(NI):
        if type(VV[i]) != list:
            raise TypeError("Elements of the argument must be lists: %s is not." % str(VV[i]))
    if DIR[0]*DIR[1] == 1:
        adiag = True
    else:
        adiag = False
    M = up_or_anti_diagonal(NI+1,0,[1]*(NI+1),adiag)+up_or_anti_diagonal(NI+1,-DIR[1],[1]*NI,adiag)
    for i in range(len(VV)):
        M += up_or_anti_diagonal(NI+1,DIR[1]*(i+1),VV[i],adiag)
    return M

def analyze_evolution_matrix(M):
    if not type(M) == MutableDenseMatrix or type(M) == ImmutableMatrix:
        return TypeError("%s expected to be a matrix" % str(M))

    if not M.is_square:
        raise ValueError("Matrix expected to be square.")

    W = []
    K = M.cols
    for i in range(K):
        W.append(list(M[:, i]))
    if M.is_upper_hessenberg:
        for i in range(K):
            W[i].reverse()
            W[i] = W[i][K-1-i:]
    elif M.is_lower_hessenberg:
        for i in range(K):
            W[i] = W[i][i:]
    else:
        M = Matrix([list(M[:, K-1-i]) for i in range(K)]).transpose()
        if M.is_upper_hessenberg:
            for i in range(K):
                W[i].reverse()
```

```
                W[i] = W[i][i:]
        elif M.is_lower_hessenberg:
            for i in range(K):
                W[i] = W[i][K-1-i:]
        else:
            raise ValueError("Matrix is not an evolution matrix.")

    T = []
    for i in range(K):
        T.append((len(W[i]),W[i]))
    S = [sorted(T, key = lambda couple: couple[0], reverse=True)[i][1] for i in range(K-1)]
    patterns = S[:]
    for i in range(len(S)):
        for j in range(i+1,len(S)):
            if S[i][:K-j] == S[j]:
                if S[j] in patterns:
                    patterns.remove(S[j])


    return patterns

def gen_ics(NI,DIR,s,VARS,y,CA=[],CAF=[]):
    """Generate the necessary initial conditions for the calculation of
    the algebraic entropy. If necessary substitute arbitrary constants
    and functions with prime numbers."""
    l = VARS[0]
    m = VARS[1]
    NN = 2*NI
    NEQ = len(y)
    Fl = []
    Fm = []
    Flm = []
    q = Wild("q", exclude=(l,))
    p = Wild("p", exclude=(m,))
    if CAF != []:
        CAFC = CAF[:]
        for C in CAF:
            if not C in CAFC:
                continue
            CL = [K for K in CAF if K.func == C.func]
            CLargs = [K.args for K in CL]
            CAFC = [ K for K in CAFC if K.func != C.func]
            if len(set([len(K.args) for K in CL])) > 1:
                raise ValueError("Arbitray function %s can\'t be function of different number of arguments." % str(C.func))
            if len(CLargs[0]) == 1:
```

```
                CLatom = list(set([list(Ka[0].atoms(Symbol))[0] for Ka in CLargs]))
                if len(CLatom) > 1:
                    raise ValueError("Arbitrary function %s can\'t be function of different arguents." % str(C.func))
                if CLatom == [l]:
                    qv = []
                    for arg in CLargs:
                        qs = arg[0].match(l+q)
                        qv.append(qs[q])
                    Fl.append((C.func,min(qv),max(qv)))
                elif CLatom == [m]:
                    pv = []
                    for arg in CLargs:
                        ps = arg[0].match(m+p)
                        pv.append(ps[p])
                    Fm.append((C.func,min(pv),max(pv)))
                else:
                    raise ValueError("Unknown variable %s in function %s." % (str(CLatom[0]), str(C.func)))
            if len(CLargs[0]) == 2:
                CLatom = list(set([list(Ka[0].atoms(Symbol))[0] for Ka in CLargs]))+\
                    list(set([list(Ka[1].atoms(Symbol))[0] for Ka in CLargs]))
                if CLatom == [l,m]:
                    qpv = []
                    for arg in CLargs:
                        qs = arg[0].match(l+q)
                        qpv.append(qs[q])
                        ps = arg[1].match(m+p)
                        qpv.append(ps[p])
                        qpvm = min(qpv)
                        qpvM = max(qpv)
                        if qpvm == qpvM:
                            if qpvm < 0:
                                Flm.append((C.func,qpvm,0))
                            else:
                                Flm.append((C.func,0,qpvM))
                        else:
                            Flm.append((C.func,qpvm,qpvM))
                else:
                    raise ValueError("Unknown variables %s in function %s." % (str(CLatom)[1:-1], str(C.func)))
            if len(CLargs[0]) > 2:
                raise ValueError("Wrong number of arguments, %s, in arbitrary function %s." \
                    % (str(len(CLargs[0])), str(C.func)))

```

```
        NFl = sum(NI+1+fl[2]-fl[1] for fl in Fl)
        NFm = sum(NI+1+fm[2]-fm[1] for fm in Fm)
        NFlm = sum((NI+1+2*(flm[2]-flm[1]))*(NI+2+2*(flm[2]-flm
            [1]))/2+NI+2*(flm[2]-flm[1]) for flm in Flm)
        NTOT = int(NEQ*(2*NN+4)+len(CA)+NFl+NFm+NFlm)
        if len(sieve._list) < NTOT:
            sieve.extend_to_no(NTOT)
        rp = sieve._list[:NTOT]
        P = nextprime(max(rp)**2)
        print('a big prime',P)
    else:
        NTOT = NEQ*(2*NN+4)+len(CA)
        if len(sieve._list) < NTOT:
            sieve.extend_to_no(NTOT)
        rp=list(sieve._list[:])[:NTOT]
        P = nextprime(max(rp)**2)
        print('a big prime',P)
    shuffle(rp)
    al = []
    be = []
    for i in range(NEQ):
        al.append(rp.pop())
        be.append(rp.pop())
    s1 = DIR[0]
    s2 = DIR[1]
    repl = {}
    for i in range(NEQ):
        for j in range(NN+1):
            ll = int((2*j - (-1)**j + 1)/4)
            mm = int((2*j + (-1)**j - 1)/4)
            ci = (rp.pop()+rp.pop()*s)/(al[i]+be[i]*s)
            repl[y[i](s1*ll,s2*mm)]=ci

    CAs = {}
    for i in range(len(CA)):
        CAs[CA[i]] = rp.pop()

    replfunc = {}

    for fl in Fl:
        for i in range(NI+1+fl[2]-fl[1]):
            replfunc[fl[0](s1*(i+fl[1]))] = rp.pop()
    for fm in Fm:
        for i in range(NI+1+fm[2]-fm[1]):
            replfunc[fm[0](s2*(i+fm[1]))] = rp.pop()
    for flm in Flm:
        for i in range(2*(NI+2*(flm[2]-flm[1]))+1):
            ll = int((2*i - (-1)**i + 1)/4)
            mm = int((2*i + (-1)**i - 1)/4)
            if s1 == 1:
                l_s = s1*(ll+flm[1])
            else:
```

```
                l_s = s1*(ll-flm[2])
            if s2 == 1:
                m_s = s2*(mm-(flm[2]-2*flm[1]))
            else:
                m_s = s2*(mm-(2*flm[2]-flm[1]))
            replfunc[flm[0](l_s,m_s)] = rp.pop()
        for i in range(NI+2*(flm[2]-flm[1])+1):
            for j in range(NI+2*(flm[2]-flm[1])-i):
                if s1 == 1:
                    l_s = s1*(j+flm[1])
                else:
                    l_s = s1*(j-flm[2])
                if s2 == 1:
                    m_s = s2*(j+i+1-(flm[2]-2*flm[1]))
                else:
                    m_s = s2*(j+i+1-(2*flm[2]-flm[1]))
                replfunc[flm[0](l_s,m_s)] = rp.pop()
    return repl,CAs,replfunc,P

def evol(EQ, Y, NI=8, direction="mp", matrix_form = False, verbosity_level=1,modfact=True, **kwargs):
    if not type(NI) == int:
        NI = 8
    if type(EQ) == set or type(EQ) == tuple:
        EQ = list(EQ)
    if type(Y) == set or type(Y) == tuple:
        Y = list(Y)
    if not type(EQ) == list:
        EQ = [EQ]
    if not type(Y) == list:
        Y = [Y]
    if not type(verbosity_level) == int:
        verbosity_level = 1

    NEQ = len(EQ)
    if NEQ != len(Y):
        if NEQ > len(Y):
            raise ValueError("Overdetermined system of equations.")
        else:
            raise ValueError("Underdetermined system of equations.")
    for i in range(NEQ):
        if isinstance(EQ[i],Equality):
            EQ[i] = EQ[i].lhs - EQ[i].rhs
        if kwargs:
            EQ[i] = EQ[i].subs(kwargs)

    y = []

    if not Y[0].is_Function:
        raise TypeError("%s is expected to be function." % Y[0])
```

```
    if len(Y[0].args) != 2:
        raise ValueError("The dependent variable %s must be function of two discrete variables." % str(Y[0].func))
    l = Y[0].args[0]
    m = Y[0].args[1]
    y.append(Y[0].func)

    for i in range(1,NEQ):
        if not Y[i].is_Function:
            raise TypeError("%s is expected to be function." % str(Y[i]))
        if len(Y[i].args) != 2:
            raise ValueError("The dependent variable %s must be function of two discrete variables." % str(Y[i].func))
        if Y[i].args[0] != l or Y[i].args[1] != m:
            raise ValueError("The dependent variable %s must be function of (%s,%s)." % (str(Y[i]), str(l), str(m)))
        y.append(Y[i].func)

    N_core = cpu_count()

    CA = set([])
    CAF = set([])

    Ymp = [yf(l+1,m+1) for yf in y]
    Ypp = [yf(l,m+1) for yf in y]
    Ypm = [yf(l,m) for yf in y]
    Ymm = [yf(l+1,m) for yf in y]
    Ymon = Ymp+Ypp+Ypm+Ymm

    for e in EQ:
        if not e.is_rational_function(str(Ymon)[1:-1]):
            raise TypeError("The equation %s must be a rational function of the dependent variables." % str(e))
        e = numer(e.expand()).expand()
        for g in Add.make_args(e):
            for h in Mul.make_args(g):
                h = h.as_base_exp()
                print(h)
                for w in [ w for w in h if abs(w) != 1 ]:
                    if w.is_Function:
                        if w.func in y:
                            if not w in Ymon:
                                raise ValueError("The function must be defined on the square, got instead %h", str(w))
                        else:
                            CAF = CAF.union(set([w]))
                    else:
```

```
                            CA = CA.union(w.atoms(Symbol))
    if l in CA:
        CA.remove(l)
    if m in CA:
        CA.remove(m)
    CA = list(CA)
    CAF = list(CAF)

    if direction == "mp":
        s1 = -1
        s2 = 1
        print('rank',(Matrix(EQ).jacobian(Ymp)).rank())
        print('n eqs', NEQ)
        if (Matrix(EQ).jacobian(Ymp)).rank() < NEQ:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = solve(EQ,Ymp,dict=True)
        if not len(ySol) == 1:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = ySol[0]
        ySub = [ySol[s] for s in Ymp]
        lmSub = lambda i,j: {l: s1*(j+1), m: s2*(j+i)}
    elif direction == "pp":
        s1 = 1
        s2 = 1
        print('rank',(Matrix(EQ).jacobian(Ypp)).rank())
        if (Matrix(EQ).jacobian(Ypp)).rank() < NEQ:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = solve(EQ,Ypp,dict=True)
        if not len(ySol) == 1:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1),str(s2)))
        ySol = ySol[0]
        ySub = [ySol[s] for s in Ypp]
        lmSub = lambda i,j: {l: s1*j, m: s2*(j+i)}
    elif direction == "pm":
        s1 = 1
        s2 = -1
        if (Matrix(EQ).jacobian(Ypm)).rank() < NEQ:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = solve(EQ,Ypm,dict=True)
        if not len(ySol) == 1:
```

```
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = ySol[0]
        ySub = [ySol[s] for s in Ypm]
        lmSub = lambda i,j: {l : s1*j, m: s2*(j+i+1)}
    elif direction == "mm":
        s1 = -1
        s2 = -1
        if (Matrix(EQ).jacobian(Ymm)).rank() < NEQ:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = solve(EQ,Ymm,dict=True)
        if not len(ySol) == 1:
            raise WellDefinitenessError("The system has not well defined evolution in the %s,%s direction." % (str(s1), str(s2)))
        ySol = ySol[0]
        ySub = [ySol[s] for s in Ymm]
        lmSub = lambda i,j: {l: s1*(j+1), m: s2*(j+i+1)}
    elif direction == "all":
        alldirs = ["mp", "pp", "pm", "mm"]
        ALL = []
        for dire in alldirs:
            try:
                print("Direction "+dire)
                ALL.append((dire,evol(EQ, Y, NI, dire, matrix_form, verbosity_level, **kwargs)))
            except WellDefinitenessError:
                print("Evolution is not well defined in the "+dire+" direction: continuing to the following.")
                continue
        if len(ALL) == 4:
            if matrix_form:
                return tuple([seq[1] for seq in ALL])
            for E in ALL:
                for F in ALL:
                    if E[1] != F[1]:
                        return tuple([seq[1] for seq in ALL])
            return ALL[0][1]
        return tuple(ALL)
    else:
        warn("Direction not detected, switching to default.")
        return evol(EQ, Y,  NI, "mp", matrix_form, **kwargs)

    t = Symbol("t", real=True)
    #bug if a discrete variable is called t!
    CIm = dict()
    CI,CAsub,CAFsub,P = gen_ics(NI,[s1,s2],t,(l,m),y,CA,CAF)
    EQ = [e.subs(CAsub) for e in EQ]
```

```
    ySub = [ys.subs(CAsub) for ys in ySub]
    if verbosity_level > 1:
            print(CI,CAsub,CAFsub)

    DS = [[] for i in range(NEQ)]

    #Internal evaluation function
    def value_deg(i,j,k,CIm,CI,outEQ):
        step = (s1*j,s2*(j+i+1))
        ev = factor((((ySub[k].subs(lmSub(i,j))).subs(CIm,
            simultaneous=True)).subs(CI,simultaneous=True)).subs(
            CAFsub,simultaneous=True),modulus=P)
        d1 = degree(numer(ev),t)
        d2 = degree(denom(ev),t)
        d = Max(d1,d2)
        if verbosity_level > 1:
            print(ev)
            print(d)
        outEQ.put((k,step,ev,d))
        return True


    #Actual evolution algorithm
    for i in range(NI):
        outd = Queue()
        out = Queue()
        if verbosity_level > 0:
            print(str(i+1)+"/"+str(NI))
        processes = [ Process(target=value_deg, args=(i,j,k,CIm,
            CI,out)) for k in range(NEQ) for j in range(NI-i)]
        N_proc = (NI-i)*NEQ
        Cal = ceiling(len(processes)/N_core)
        for h in range(Cal+1):
            for i in range(h*N_core,min((h+1)*N_core,len(
                processes))):
                processes[i].start()
            for i in range(h*N_core,min((h+1)*N_core,len(
                processes))):
                processes[i].join(1)
        OutC = sorted(sorted([out.get() for p in processes], key=
            lambda op: op[1][1]), key=lambda op: op[0])
        CIm = dict(CI)
        CI.clear()
        for k in range(len(OutC)):
            CI[y[OutC[k][0]](OutC[k][1][0],OutC[k][1][1])] = OutC
                [k][2]

        for k in range(NEQ):
            DK = [OutC[q][3] for q in range(len(OutC)) if OutC[q
                ][0] == k]
            print('k,d_k',k,DK)
            if [s1,s2] != [-1,1]:
```

```
                DK.reverse()
            if matrix_form == False:
                for i in range(len(DK)):
                    for j in range(i+1,len(DK)):
                        if DK[i] != DK[j]:
                            matrix_form = True
                            break
                    if matrix_form:
                        break
            DS[k].append(DK)

    if matrix_form:
        if NEQ == 1:
            return evolution_matrix(DS[0],[s1,s2])
        return [(y[k],evolution_matrix(DS[k],[s1,s2])) for k in range(NEQ)]

    if NEQ == 1:
        return [1]+[d[0] for d in DS[0]]

    return [(y[k],[1]+[d[0] for d in DS[k]]) for k in range(NEQ)]
```

B.2 Algebraic entropy for systems of ordinary difference equations and differential difference equations of arbitrary order with `ae_differential_difference.py`

In this Section we give a brief description of the functions contained in the `python` module `ae_differential_difference.py`. This module contains only the function `evol` (not to be confused with the function of the same name in `ae2d.py`) that can be used to compute the sequence of the degrees for systems of ordinary difference equations or systems of differential-difference equations of arbitrary order.

Remark B.2.1. The function `evol` cannot automatically treat systems of mixed ordinary difference equations and differential-difference equations. If this is the case under consideration we suggest to the user to consider the purely difference variables as implicitly depending on "time": $u_n \to u_n(t)$. For this reason from now on we will address to the systems under study as *one-dimensional discrete systems*.

The function `evol` needs two mandatory arguments:

`f`: The system of one-dimensional discrete equations under considerations. It can be entered as a `python` list, or as a `python` dictionary. In the case of a single scalar equation it can be entered plainly.

`y`: The dependent variables. They can be entered as a `python` list, or as a `python` dictionary. In the case of a single dependent variable it can be entered plainly. The entries of the list must be sympy Functions of one lattice variables, e.g.:

$$y = [u(n), v(n)], \tag{B.13}$$

or of one lattice variable and one continuous variable:

$$y = [u(n, t), v(n, t)]. \tag{B.14}$$

The continuous variable is, by convention, the second one.

Remark B.2.2. The function evol can handle only well-determined systems of one-dimensional discrete equations so it will check that the number of equations is equal to the number of dependent variables. In negative case it will raise an error. To reduce a system of one-dimensional discrete equations to a well-determined one is the up to the user.

The function evol also accepts an optional argument NI. NI defines the number of iteration to be performed. Its default value is NI=8.

The function then accepts *keyword arguments*. Keyword arguments in this context can be used to set some parameters to a fixed value. Keyword arguments are immediately substituted into f.

First of all the function evol analyze the equation f in order to determine the order with respect to every variable. Furthermore any symbol (not function) different from those of the dependent variables contained in y is considered an arbitrary constant. Arbitrary functions are not supported by evol.

At this point the system, if consistent, is solved with respect to the highest shifts, otherwise an error is raised. Then the initial conditions in the form (2.44) are generated. If we denote by ∂_e the discrete order, as defined in Section (2.1), of an equation e in f we have that we need, if M is the number of equations:

$$N_{\text{ini}} = 2 \sum_{k=1}^{M} (\partial_{e_k} + 1) \tag{B.15}$$

prime numbers to evaluate them. Then if N_{ac} is the number of arbitrary constants we need:

$$N_{\text{TOT}} = N_{\text{ini}} + N_{\text{ac}} \tag{B.16}$$

prime numbers. The prime numbers are generated with the sympy function sieve and shuffled before being assigned, so that running the program multiple times will yield different initial conditions.

Remark B.2.3. In principle evol cannot perform the evolution in the direction of the *smallest* shifts. However this can be simply obtained by the user performing the change $n \to -n$ into f.

Now the actual evolution begins, and the iterates are computed. The main point is the fact that after the substition a factorization in the finite field $\mathbb{K}_P$ is carried out. The prime number P is chosen to be the next prime after p_{MAX}^2, where p_{MAX} is the biggest prime number generated to evaluate the initial conditions and the arbitrary constants.

The output of the function `evol` is then a dictionary whose keywords are the elements of `y` and whose values are the sequences of degrees.

The content of the file `aedifferential_difference_systems.py` is the following:

```
from __future__ import print_function, division

from sympy import *
from random import shuffle
from collections import defaultdict

def evol(f,y,NI=8, **kwargs):
    if not type(NI) == int:
        NI = 8
    if type(f) == set or type(f) == tuple:
        f = list(f)
    if type(y) == set or type(y) == tuple:
        y = list(y)
    if not type(f) == list:
        f = [f]
    if not type(y) == list:
        y = [y]

    NEQ = len(f)
    if NEQ != len(y):
        if NEQ > len(y):
            raise ValueError("Overdetermined system of equations.")
        else:
            raise ValueError("Underdetermined system of equations.")

    #Convert equalities to single side equations
    #and substitute keywords arguments into the equations
    for i in range(NEQ):
        if isinstance(f[i],Equality):
            f[i] = f[i].lhs - f[i].rhs
        if kwargs:
            f[i] = f[i].subs(kwargs)

    if not y[0].is_Function:
        raise TypeError("%s is expected to be function." % Y[0])
    if len(y[0].args) == 1:
        n = y[0].args[0]
        s = var('s')
```

```
    elif len(y[0].args) == 2:
        n, s = y[0].args
    else:
        raise ValueError("%s must be function of maximum two variables." % y[0])
    Y = [y[0].func]

    for i in range(1,NEQ):
        if not y[i].is_Function:
            raise TypeError("%s is expected to be function." % str(y[i]))
        if y[i].args != y[0].args:
            raise ValueError("The dependent variable %s must be function of\
                (%s,%s)." % (str(y[i]), str(n), str(s)))
        Y.append(y[i].func)

    k = Wild('k', exclude=(n,))

    f = [ numer(simplify(fs,ratio=oo)).expand() for fs in f]
    pprint(f)

    CA = set([])
    kv  = {yf : set([]) for yf in Y}

    for fs in f:
        for g in Add.make_args(fs):
            coeff = S.One
            kspec = None
            for q in Mul.make_args(g):
                q = q.as_base_exp()
                for h in q:
                    if h.is_Derivative:
                        h = h.args[0]
                    if h.is_Function:
                        if h.func in Y:
                            result = h.args[0].match(n + k)
                            if result is not None:
                                kspec = int(result[k])
                                kv[h.func]=kv[h.func] = kv[h.func].union(set([kspec]))
                            else:
                                raise ValueError(
                                    "'%s(%s+k)' expected, got '%s'" % (y, n, h))
                        else:
                            raise ValueError(
                                "'%s' expected, got '%s'" % (Y, h.func))
                    else:
                        coeff *= h
                        CA = CA.union(coeff.atoms(Symbol))
```

```

    if n in CA:
        CA.remove(n)
    if s in CA:
        CA.remove(s)
    if CA:
        CA = list(CA)

    N = {yf : max(kv[yf]) for yf in Y}
    M = {yf : min(kv[yf]) for yf in Y}
    ORD = {yf : N[yf]-M[yf] for yf in Y}

    svars = [y[i].subs(n,n+N[Y[i]]) for i in range(NEQ)]

    if (Matrix(f).jacobian(svars)).rank() < NEQ:
        raise ValueError(
                "'%s' must be of maximal rank with respect
                to '%s'" % (f, svars))
    K = 2*sum(ORD[yf] for yf in Y)+2*NEQ
    NTOT = K + len(CA)
    sieve.extend_to_no(NTOT)
    rp = sieve._list[:NTOT]
    P = nextprime(max(rp)**2)
    shuffle(rp)

    rp0 = rp[0:2*NEQ]
    rp = rp[2*NEQ:]

    CI = {}
    for i in range(NEQ):
        den = rp0.pop()*s+rp0.pop()
        for k in range(M[Y[i]],N[Y[i]]):
            CI[y[i].subs(n,k)] = (rp.pop()*s+rp.pop())/den
    CAs = {}
    for i in range(len(CA)):
        CAs[CA[i]] = rp.pop()

    #Evolution in the right direction.
    if CAs:
        f = [f[al].subs(CAs) for al in range(len(f))]
    ySol = solve(f,svars,dict=True)
    if not len(ySol) == 1:
        raise ValueError(
                "'%s' must be single-valued." % (f))
    ySol = ySol[0]
    yN = [ySol[yks] for yks in svars]


    ds = {y[i] : [1] for i in range(NEQ)}

    for i in range(NI):
        print(str(i)+'/'+str(NI))
```

```
        val = [factor((yN[k].subs(n,i).subs(CI)).doit(),modulus=P
            ) for k in range(NEQ)]
        for k in range(NEQ):
            CI[y[k].subs(n,i+N[Y[k]])] = val[k]
            del CI[y[k].subs(n,i+M[Y[k]])]
            vq = numer(val[k])
            d1 = degree(vq, s)
            vq = denom(val[k])
            d2 = degree(vq,s)
            d = max(d1,d2)
            ds[y[k]] += [d]

    return ds
```

B.3 ANALYSIS OF THE DEGREE SEQUENCES

Once obtained a finite sequence of degrees it must be analyzed using e.g. the methods presented in Section 2.2. To this end we complement the programs for calculating the sequences of degrees with some `python` functions which can compute Padé approximants and the inverse $\mathcal{Z}$-transform. This need comes from the fact that the analytic evaluation of the Padé approximants and of the inverse $\mathcal{Z}$-transform is not yet implemented in `sympy`. We present then three modules: the `pade.py` module which defines the calculation of the Padé approximants, the `pade_rat_gfunc.py` module which computes the generating function of a sequence using Padé approximants and the `z_transform.py` which defines the $\mathcal{Z}$-transform for rational functions and functions to find the coefficients and the algebraic entropy of a given generating function.

B.3.1 *The `pade.py` module*

The `pade.py` contains a single function: `pade_expansion`. This function computes the Padé approximant of order $[L : M]$ using the formula (2.52). It has four arguments, all of them mandatory:

- `vect` This argument must be a list of numbers or a function. If the argument is a function its Taylor expansion of order $L + M + 1$ with respect to `z` is computed.
- `L` This argument must be a positive integer and it is the degree of the numerator of the Padé approximant (2.50).
- `M` This argument must be a positive integer and it is the degree of the denominator of the Padé approximant (2.50).
- `z` This argument is the independent variable.

`vect` can be a list and not a polynomial since it is obvious that to any list of numbers $[a_0, a_1, \ldots, a_N]$ we can associate the polynomial

$$p(z) = a_0 + a_1 z + \ldots a_N z^N \tag{B.17}$$

The content of the file `pade.py` is the following:

```
from __future__ import print_function, division

from sympy.core.expr import Expr
from sympy.polys.polytools import factor
from sympy.matrices import Matrix


def pade_expansion(vect,L,M,z):
    """Calculate the Pade expasion [L/M] of a given function or vector
    in the variable z."""
    #Check if the first argument is a list of values. If not it is assumed
    #to be a symbolic expression.
    if not type(vect) is list:
        taylp=vect.series(z,0,L+M+1).removeO()
        if M==0:
            return taylp
        vect = [taylp.coeff(z,i) for i in range(L+M+1)]
    #Check that we have enough terms in the initial vector
    #to carry out the computations needed for the L/M
    #Pade approximant.
    else:
        if len(vect) != L+M+1:
            raise ValueError("Pade expansion cannon't be computed: length of vector and of degree don't match.")
    if L == 0 and M ==0:
        return vect[0]

    def c(i):
        if i<0 or i >= len(vect):
            return 0
        else:
            return vect[i]

    if M == 0:
        return Poly([c(i) for i in range(0,L+M+1)],z).simplify()
    def cfP(i):
        if M-i>L:
            return 0
        else:
            S = c(0)
            for k in range(1,max(L-M+i+1,1)):
                    S += c(k)*z**k
            return (S*z**(M-i)).simplify()
    vzP = Matrix([[cfP(i) for i in range(M+1)]])
```

```
    vzQ = Matrix([[z**(M-i) for i in range(M+1)]])
    A = Matrix([[c(L-M+i+j) for i in range(M+1)] for j in range
        (1,M+1)])
    QMat = A.col_join(vzQ)
    PMat = A.col_join(vzP)
    QLM = ((QMat).det()).simplify()
    PLM = ((PMat).det()).simplify()
    PA = (PLM/QLM).simplify()

    return factor(PA)
```

B.3.2 *The `pade_rat_gfunc.py` module*

The `pade_rat_gfunc.py` contains a single function: `find_rat_gen_function`. This function has two arguments, both of them mandatory:

`V` A list of numbers.

`s` The independent variable of the generating function.

The algorithm runs as follows: it start with a slice `V'` of the list `V` with three elements, and compute the Padé approximant $g = [1 : 1]$ through the association (B.17). Then computes the Taylor expansion of g up to the length of `V`. If its coefficients agrees with `V` then g is returned. If not we take a slice with two more elements `V'` and compute the Padé approximant $g = [2 : 2]$ through the association (B.17). If its coefficients agrees with `V` then g is returned. This is repeated until one of the two following condition is reached:

1. The coefficients of the Taylor expansion of the Padé approximant g agree with `V`.
2. We cannot form a slice with two elements more than the previous.

In the case 1 g is returned since it is the desired generating function. In the case 2 it is not possible to find a generating function, so the boolean value `False` is return.

The content of the file `pade_rat_gfunc.py` is the following:

```
from __future__ import print_function, division

from sympy.series import series
from sympy.core.symbol import Symbol
from sympy.functions.elementary.integers import floor
from pade import pade_expansion
import warnings

def find_rat_gen_func(V,s):
    """
    Find a rational generating function by using the Pade
        approximant method:
```

```
    taking a slice of the input vector V finds a Pade approximant using
    pade_expansion then check if it reproduces the exact behaviour of the following
    elements of V. If not it takes a bigger slice until the vector V is finished.
    """
    if not type(V) is list:
        raise TypeError("Argument %s expected to be list." % str(V))
    if not s.is_Symbol:
        raise TypeError("Argument %s expected to be symbol." % str(s))
    M = len(V)
    N = floor((M-1)/2)
    j = 1
    while(j<N+1):
        R = pade_expansion(V[:2*j+1],j,j,s)
        ret = True
        Rv = [R.series(s,0,M+1).removeO().coeff(s,i) for i in range(M)]
        if Rv == V:
            return R
        else:
            j = j + 1
    return False
```

B.3.3 *The `z_transform.py` module*

The `z_transform.py` module contains the following functions:

inverse_z_transform This function computes the inverse $\mathcal{Z}$-transform of a rational function. It needs three mandatory arguments:

- `f` A `sympy` rational function. If another kind of function is given the program raise an error.
- `z` The independent variable of the function `z`.
- `n` The discrete target variable.

The $\mathcal{Z}$-transform of `f` is then computed using equation (2.56).

Remark B.3.1. Due to some limitations in `sympy` this program may misbehave in case of high order polynomials in the denominator.

find_coefficients This function compute the asymptotic form of the coefficients of a given ration function. It needs three mandatory arguments:

- `f` A `sympy` rational function. If another kind of function is given the program raise an error.
- `z` The independent variable of the function `z`.

`n` The discrete target variable.

Using formula (2.58) and the function `z_transform` the asymptotic form of the coefficients are reconstructed.

find_entropy This function compute the Algebraic Entropy defined by a ration generating function. It needs two mandatory arguments:

`f` A `sympy` rational function. If another kind of function is given the program raise an error.

`z` The independent variable of the function `z`.

Using formula (2.60) the value of the Algebraic Entropy is computed.

The content of the file `z_transform.py` is the following:

```
from __future__ import print_function, division

from sympy.core.symbol import Symbol
from sympy.simplify import simplify
from sympy.functions.elementary.miscellaneous import Max
from sympy.functions.elementary.complexes import Abs
from sympy.functions.elementary.exponential import log
from sympy.polys.polytools import div, degree
from sympy.solvers import solve
from sympy.series import residue

def inverse_z_transform(f,z,n):
    if not f.is_rational_function(z):
        raise TypeError("Function %s expected to be rational." % str(f))
    if not z.is_Symbol:
        raise TypeError("Input variable %s must be of type symbol." % str(z))
    if not n.is_Symbol:
        raise TypeError("Output variable %s must be of type symbol." % str(n))
    if not n.is_integer:
        n = Symbol(str(n),integer=True)
    f = f.as_numer_denom()
    if degree(f[0],z) >= degree(f[1],z):
        quot,rem = div(f[0],f[1])
        f = (rem,)+f[1:]
    sing = solve(f[1],z)
    transf = 0
    for z0 in sing:
        s=residue((f[0]/f[1]*z**(n-1)).simplify(),z,z0)
        transf = transf + s
    return transf.simplify()

def find_coefficients(f,z,n):
    if not f.is_rational_function(z):
```

```
        raise TypeError("Function %s expected to be rational." % str(f))
    if not z.is_Symbol:
        raise TypeError("Input variable %s must be of type symbol." % str(z))
    if not n.is_Symbol:
        raise TypeError("Output variable %s must be of type symbol." % str(n))
    if not n.is_integer:
        n = Symbol(str(n),integer=True)
    f = f.subs(z,1/z).simplify()
    dn = inverse_z_transform(f,z,n)
    return dn

def find_entropy(f,z):
    if not f.is_rational_function(z):
        raise TypeError("Function %s expected to be rational." % str(f))
    if not z.is_Symbol:
        raise TypeError("Input variable %s must be of type symbol." % str(z))
    f = f.as_numer_denom()
    sing = solve(f[1],z)
    m = 0
    for z0 in sing:
        m = Max(1/Abs(z0),m)
    return log(m).simplify()
```

C

EVOLUTION MATRICES

In this Appendix we give the explicit form of the evolution matrices for the trapezoidal H^4 equations (1.91) and H^6 equations (1.93). Using the program `ae2d.py` contained in Appendix B we performed 32 iterations in each of the four possible direction of growth.

C.1 TRAPEZOIDAL H^4 EQUATIONS (1.91)

C.1.1 ${}_{t}H_1^{\varepsilon}$ *equation* (1.91a)

DIRECTION $-+$:

$$
\begin{matrix}
1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12&13&13&14&14&15&15&16&16&17&17\\
1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41&43&44&46&47\\
0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12&13&13&14&14&15&15&16&16\\
0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41&43&44\\
0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12&13&13&14&14&15&15\\
0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41\\
0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12&13&13&14&14\\
0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38\\
0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12&13&13\\
0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35\\
0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11&12&12\\
0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32\\
0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10&11&11\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9&10&10\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6&7&7\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5&6&6\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4&5&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3&4&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&3&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{matrix}
\tag{C.1}
$$

direction ++:

17	17	16	16	15	15	14	14	13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1
47	46	44	43	41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1
16	16	15	15	14	14	13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0
44	43	41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0
15	15	14	14	13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0
41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0
14	14	13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0
38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0
13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0
35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0
12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0
32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0
11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0
29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	10	9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	9	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	7	6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	6	5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.2)

DIRECTION $+-$:

$$
\begin{matrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 34 & 12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 37 & 13 & 34 & 12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
15 & 40 & 14 & 37 & 13 & 34 & 12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\
16 & 43 & 15 & 40 & 14 & 37 & 13 & 34 & 12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 & 0 & 0 \\
16 & 44 & 15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 \\
17 & 46 & 16 & 43 & 15 & 40 & 14 & 37 & 13 & 34 & 12 & 31 & 11 & 28 & 10 & 25 & 9 & 22 & 8 & 19 & 7 & 16 & 6 & 13 & 5 & 10 & 4 & 7 & 3 & 4 & 2 & 1 & 1 \\
17 & 47 & 16 & 44 & 15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1
\end{matrix}
\tag{C.3}
$$

DIRECTION ——:

$$
\begin{matrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11\\
0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11\\
0&0&0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12\\
0&0&0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12\\
0&0&0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12&34&13\\
0&0&0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12&35&13\\
0&0&0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12&34&13&37&14\\
0&0&0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12&35&13&38&14\\
0&0&0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12&34&13&37&14&40&15\\
0&0&0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12&35&13&38&14&41&15\\
0&0&1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12&34&13&37&14&40&15&43&16\\
0&1&1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12&35&13&38&14&41&15&44&16\\
1&1&2&4&3&7&4&10&5&13&6&16&7&19&8&22&9&25&10&28&11&31&12&34&13&37&14&40&15&43&16&46&17\\
1&2&2&5&3&8&4&11&5&14&6&17&7&20&8&23&9&26&10&29&11&32&12&35&13&38&14&41&15&44&16&47&17
\end{matrix}
\tag{C.4}
$$

C.1.2 ${}_{t}H_2^{\varepsilon}$ *equation* (1.91b)

DIRECTION $-+$:

1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91	95	99	103	107	111	115	119	123
1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62
0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91	95	99	103	107	111	115
0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91	95	99	103	107
0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54
0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91	95	99
0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91
0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83
0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51	55	59
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43	47	51
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35	39	43
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27	31	35
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19	23	27
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11	15	19
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1

(C.5)

DIRECTION ++:

123	119	115	111	107	103	99	95	91	87	83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1
62	60	58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1
115	111	107	103	99	95	91	87	83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0
58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0
107	103	99	95	91	87	83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0
54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0
99	95	91	87	83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0
50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0
91	87	83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0
46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0
83	79	75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0
42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0
75	71	67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
67	63	59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	55	51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	47	43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	39	35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	31	27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	23	19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	15	11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.6)

DIRECTION $+-$:

$$(C.7)$$

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0
83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0
87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0
91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0
95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0
99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0
103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0
107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0
111	56	103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0
115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0
119	60	111	56	103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1
123	62	115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1

DIRECTION ——:

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71\\
0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75\\
0&0&0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79\\
0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83\\
0&0&0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79&44&87\\
0&0&0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83&46&91\\
0&0&0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79&44&87&48&95\\
0&0&0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83&46&91&50&99\\
0&0&0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79&44&87&48&95&52&103\\
0&0&0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83&46&91&50&99&54&107\\
0&0&1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79&44&87&48&95&52&103&56&111\\
0&1&1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83&46&91&50&99&54&107&58&115\\
1&1&2&4&7&8&15&12&23&16&31&20&39&24&47&28&55&32&63&36&71&40&79&44&87&48&95&52&103&56&111&60&119\\
1&2&4&6&11&10&19&14&27&18&35&22&43&26&51&30&59&34&67&38&75&42&83&46&91&50&99&54&107&58&115&62&123
\end{pmatrix}
\quad (C.8)
$$

C.1.3 $_{t}\mathrm{H}_3^{\varepsilon}$ *equation* (1.91c)

DIRECTION $-+$:

$$
\begin{matrix}
1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 & 95 & 99 & 103 & 107 & 111 & 115 & 119 & 123 \\
1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 & 56 & 58 & 60 & 62 \\
0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 & 95 & 99 & 103 & 107 & 111 & 115 \\
0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 & 56 & 58 \\
0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 & 95 & 99 & 103 & 107 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 & 95 & 99 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 & 87 & 91 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 & 79 & 83 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 & 71 & 75 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 & 63 & 67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 & 55 & 59 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 & 47 & 51 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 & 39 & 43 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 & 23 & 27 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 & 15 & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 8 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 7 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{matrix}
\tag{C.9}
$$

DIRECTION ++:

$$
\begin{matrix}
123 & 119 & 115 & 111 & 107 & 103 & 99 & 95 & 91 & 87 & 83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 \\
62 & 60 & 58 & 56 & 54 & 52 & 50 & 48 & 46 & 44 & 42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 \\
115 & 111 & 107 & 103 & 99 & 95 & 91 & 87 & 83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 \\
58 & 56 & 54 & 52 & 50 & 48 & 46 & 44 & 42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 \\
107 & 103 & 99 & 95 & 91 & 87 & 83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 \\
54 & 52 & 50 & 48 & 46 & 44 & 42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
99 & 95 & 91 & 87 & 83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
50 & 48 & 46 & 44 & 42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
91 & 87 & 83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
46 & 44 & 42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
83 & 79 & 75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & 40 & 38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
75 & 71 & 67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
38 & 36 & 34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
67 & 63 & 59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 32 & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
59 & 55 & 51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 28 & 26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
51 & 47 & 43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
26 & 24 & 22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
43 & 39 & 35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 20 & 18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 31 & 27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
27 & 23 & 19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 15 & 11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 8 & 6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 7 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\tag{C.10}
$$

direction +−:

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0	0	0
83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0
87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0	0	0
91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0
95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0	0	0
99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0
103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0	0	0
107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0
111	56	103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1	0	0
115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0
119	60	111	56	103	52	95	48	87	44	79	40	71	36	63	32	55	28	47	24	39	20	31	16	23	12	15	8	7	4	2	1	1
123	62	115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1

(C.11)

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71
0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75
0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79
0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83
0	0	0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79	44	87
0	0	0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83	46	91
0	0	0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79	44	87	48	95
0	0	0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83	46	91	50	99
0	0	0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79	44	87	48	95	52	103
0	0	0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83	46	91	50	99	54	107
0	0	1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79	44	87	48	95	52	103	56	111
0	1	1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83	46	91	50	99	54	107	58	115
1	1	2	4	7	8	15	12	23	16	31	20	39	24	47	28	55	32	63	36	71	40	79	44	87	48	95	52	103	56	111	60	119
1	2	4	6	11	10	19	14	27	18	35	22	43	26	51	30	59	34	67	38	75	42	83	46	91	50	99	54	107	58	115	62	123

(C.12)

C.2 H^6 equations (1.93)

C.2.1 ${}_1D_2$ *equation* (1.93b)

DIRECTION $-+$:

$$
\setcounter{MaxMatrixCols}{40}
\begin{matrix}
1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 & 35 & 13 & 38 & 14 & 41 & 15 & 44 & 16 & 47 & 17 \\
1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 & 23 & 35 & 25 & 38 & 27 & 41 & 29 & 44 & 31 & 47 \\
0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 & 35 & 13 & 38 & 14 & 41 & 15 & 44 & 16 \\
0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 & 23 & 35 & 25 & 38 & 27 & 41 & 29 & 44 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 & 35 & 13 & 38 & 14 & 41 & 15 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 & 23 & 35 & 25 & 38 & 27 & 41 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 & 35 & 13 & 38 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 & 23 & 35 & 25 & 38 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 & 35 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 & 23 & 35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 & 32 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 & 21 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 & 29 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 & 19 & 29 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 & 26 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 & 17 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 & 23 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 & 15 & 23 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 & 20 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 & 13 & 20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 & 17 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 & 11 & 17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 & 14 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 & 9 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 & 11 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 & 7 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 & 8 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 5 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 5 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{matrix}
\tag{C.13}
$$

DIRECTION ++:

17	47	16	44	15	41	14	38	13	35	12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1
47	31	44	29	41	27	38	25	35	23	32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1
16	44	15	41	14	38	13	35	12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0
44	29	41	27	38	25	35	23	32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0
15	41	14	38	13	35	12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0
41	27	38	25	35	23	32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0
14	38	13	35	12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0
38	25	35	23	32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0
13	35	12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0
35	23	32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0
12	32	11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0
32	21	29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0
11	29	10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0
29	19	26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	26	9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
26	17	23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	23	8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	15	20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	20	7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	13	17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	17	6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	11	14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	14	5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	9	11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	11	4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	7	8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	8	3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	5	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.14)

direction $+-$:

$$
\begin{matrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 23 & 32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
38 & 25 & 35 & 23 & 32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
41 & 27 & 38 & 25 & 35 & 23 & 32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\
44 & 29 & 41 & 27 & 38 & 25 & 35 & 23 & 32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 & 0 & 0 \\
16 & 44 & 15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1 & 1 & 0 \\
47 & 31 & 44 & 29 & 41 & 27 & 38 & 25 & 35 & 23 & 32 & 21 & 29 & 19 & 26 & 17 & 23 & 15 & 20 & 13 & 17 & 11 & 14 & 9 & 11 & 7 & 8 & 5 & 5 & 3 & 2 & 1 & 1 \\
17 & 47 & 16 & 44 & 15 & 41 & 14 & 38 & 13 & 35 & 12 & 32 & 11 & 29 & 10 & 26 & 9 & 23 & 8 & 20 & 7 & 17 & 6 & 14 & 5 & 11 & 4 & 8 & 3 & 5 & 2 & 2 & 1
\end{matrix}
\tag{C.15}
$$

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29
0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11
0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32
0	0	0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12
0	0	0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32	23	35
0	0	0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12	35	13
0	0	0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32	23	35	25	38
0	0	0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12	35	13	38	14
0	0	0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32	23	35	25	38	27	41
0	0	0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12	35	13	38	14	41	15
0	0	1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32	23	35	25	38	27	41	29	44
0	1	1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12	35	13	38	14	41	15	44	16
1	1	2	3	5	5	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21	32	23	35	25	38	27	41	29	44	31	47
1	2	2	5	3	8	4	11	5	14	6	17	7	20	8	23	9	26	10	29	11	32	12	35	13	38	14	41	15	44	16	47	17

(C.16)

C.2.2 ${}_2D_2$ *equation* (1.93c)

DIRECTION $-+$:

$$
\begin{matrix}
1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 & 23 & 36 & 25 & 39 & 27 & 42 & 29 & 45 & 31 & 48 \\
1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 & 33 & 13 & 36 & 14 & 39 & 15 & 42 & 16 & 45 & 17 \\
0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 & 23 & 36 & 25 & 39 & 27 & 42 & 29 & 45 \\
0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 & 33 & 13 & 36 & 14 & 39 & 15 & 42 & 16 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 & 23 & 36 & 25 & 39 & 27 & 42 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 & 33 & 13 & 36 & 14 & 39 & 15 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 & 23 & 36 & 25 & 39 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 & 33 & 13 & 36 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 & 23 & 36 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 & 33 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 & 21 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 & 30 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 & 19 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 & 27 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 & 17 & 27 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 & 24 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 & 15 & 24 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 & 21 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 & 13 & 21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 & 18 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 & 11 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 & 15 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 & 9 & 15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 & 12 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 & 7 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 & 9 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 & 5 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 6 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{matrix}
\tag{C.17}
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DIRECTION ++:

48	31	45	29	42	27	39	25	36	23	33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1
17	45	16	42	15	39	14	36	13	33	12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1
45	29	42	27	39	25	36	23	33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0
16	42	15	39	14	36	13	33	12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0
42	27	39	25	36	23	33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0
15	39	14	36	13	33	12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0
39	25	36	23	33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0
14	36	13	33	12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0
36	23	33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0
13	33	12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0
33	21	30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0
12	30	11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0
30	19	27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0
11	27	10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
27	17	24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	24	9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	15	21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	21	8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	13	18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	18	7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	11	15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	15	6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	9	12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	12	5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	7	9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	9	4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	5	6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	6	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.18)

DIRECTION $+-$:

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\begin{matrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 33 & 21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
36 & 13 & 33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
25 & 36 & 23 & 33 & 21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
39 & 14 & 36 & 13 & 33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
27 & 39 & 25 & 36 & 23 & 33 & 21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
42 & 15 & 39 & 14 & 36 & 13 & 33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\
29 & 42 & 27 & 39 & 25 & 36 & 23 & 33 & 21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 & 0 & 0 \\
45 & 16 & 42 & 15 & 39 & 14 & 36 & 13 & 33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1 & 1 & 0 \\
31 & 45 & 29 & 42 & 27 & 39 & 25 & 36 & 23 & 33 & 21 & 30 & 19 & 27 & 17 & 24 & 15 & 21 & 13 & 18 & 11 & 15 & 9 & 12 & 7 & 9 & 5 & 6 & 3 & 3 & 1 & 1 & 1 \\
48 & 17 & 45 & 16 & 42 & 15 & 39 & 14 & 36 & 13 & 33 & 12 & 30 & 11 & 27 & 10 & 24 & 9 & 21 & 8 & 18 & 7 & 15 & 6 & 12 & 5 & 9 & 4 & 6 & 3 & 3 & 2 & 1
\end{matrix}
\tag{C.19}
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DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6	4	9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6	4	9	5	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7	12	9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6	4	9	5	12	6	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7	12	9	15	11
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7	12	9	15	11	18	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6	4	9	5	12	6	15	7	18	8	21
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0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7	12	9	15	11	18	13	21	15	24	17	27	19
0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	3	6	4	9	5	12	6	15	7	18	8	21	9	24	10	27	11	30
0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	6	5	9	7	12	9	15	11	18	13	21	15	24	17	27	19	30	21
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0	0	1	1	1	3	3	6	5	9	7	12	9	15	11	18	13	21	15	24	17	27	19	30	21	33	23	36	25	39	27	42	29
0	1	1	2	3	3	6	4	9	5	12	6	15	7	18	8	21	9	24	10	27	11	30	12	33	13	36	14	39	15	42	16	45
1	1	1	3	3	6	5	9	7	12	9	15	11	18	13	21	15	24	17	27	19	30	21	33	23	36	25	39	27	42	29	45	31
1	2	3	3	6	4	9	5	12	6	15	7	18	8	21	9	24	10	27	11	30	12	33	13	36	14	39	15	42	16	45	17	48

(C.20)

C.2.3 $_3D_2$ *equation* (1.93d)

DIRECTION $-+$:

1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27	29	29	31	31	33
1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2	29	2	31	2
0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27	29	29	31
0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2	29	2
0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27	29
0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2
0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27
0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2
0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25
0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2
0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23
0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2
0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1

(C.21)

DIRECTION ++:

33	31	31	29	29	27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1
2	31	2	29	2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1
31	29	29	27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0
2	29	2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0
29	27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0
2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0
27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0
2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0
25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0
2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0
23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0
2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0
21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.22)

DIRECTION $+-$:

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0
21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0	0	0
23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0	0	0
23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0	0	0
25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0	0	0
25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0	0	0
27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0	0	0
27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0	0	0
29	2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0	0	0
29	29	27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1	0	0
31	2	29	2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1	1	0
31	31	29	29	27	27	25	25	23	23	21	21	19	19	17	17	15	15	13	13	11	11	9	9	7	7	5	5	3	3	1	1	1
33	2	31	2	29	2	27	2	25	2	23	2	21	2	19	2	17	2	15	2	13	2	11	2	9	2	7	2	5	2	3	2	1

(C.23)

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19
0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21
0	0	0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21
0	0	0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23
0	0	0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23
0	0	0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25
0	0	0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25
0	0	0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27
0	0	0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27
0	0	0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2	29
0	0	1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27	29	29
0	1	1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2	29	2	31
1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15	17	17	19	19	21	21	23	23	25	25	27	27	29	29	31	31
1	2	3	2	5	2	7	2	9	2	11	2	13	2	15	2	17	2	19	2	21	2	23	2	25	2	27	2	29	2	31	2	33

(C.24)

c.2.4 D_3 *equation* (1.93e)

direction $-+$:

$$
\begin{matrix}
1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53&34&60&38&67&42&74&46&81&50&88&54&95&58&102&62&109\\
1&1&2&4&6&8&10&12&14&16&18&20&22&24&26&28&30&32&34&36&38&40&42&44&46&48&50&52&54&56&58&60&62\\
0&1&1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53&34&60&38&67&42&74&46&81&50&88&54&95&58&102\\
0&0&1&1&2&4&6&8&10&12&14&16&18&20&22&24&26&28&30&32&34&36&38&40&42&44&46&48&50&52&54&56&58\\
0&0&0&1&1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53&34&60&38&67&42&74&46&81&50&88&54&95\\
0&0&0&0&1&1&2&4&6&8&10&12&14&16&18&20&22&24&26&28&30&32&34&36&38&40&42&44&46&48&50&52&54\\
0&0&0&0&0&1&1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53&34&60&38&67&42&74&46&81&50&88\\
0&0&0&0&0&0&1&1&2&4&6&8&10&12&14&16&18&20&22&24&26&28&30&32&34&36&38&40&42&44&46&48&50\\
0&0&0&0&0&0&0&1&1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53&34&60&38&67&42&74&46&81\\
0&0&0&0&0&0&0&0&1&1&2&4&6&8&10&12&14&16&18&20&22&24&26&28&30&32&34&36&38&40&42&44&46\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&6&11&10&18&14&25&18&32&22&39&26&46&30&53\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{matrix}
\tag{C.25}
$$

DIRECTION ++:

109	62	102	58	95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1
62	60	58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1
102	58	95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0
58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0
95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0
54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0
88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0
50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0
81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0
46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0
74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0
42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0
67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.26)

direction +−:

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0
74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0
46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0
81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0	0	0
50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0
88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0	0	0
54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0
95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0	0	0
58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0
102	58	95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1	1	0
62	60	58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1
109	62	102	58	95	54	88	50	81	46	74	42	67	38	60	34	53	30	46	26	39	22	32	18	25	14	18	10	11	6	4	2	1

(C.27)

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38
0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67
0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
0	0	0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74
0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
0	0	0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74	46	81
0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
0	0	0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74	46	81	50	88
0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54
0	0	0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74	46	81	50	88	54	95
0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
0	1	1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74	46	81	50	88	54	95	58	102
1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62
1	2	4	6	11	10	18	14	25	18	32	22	39	26	46	30	53	34	60	38	67	42	74	46	81	50	88	54	95	58	102	62	109

(C.28)

C.2.5 ${}_1\mathrm{D}_4$ *equation* (1.93f)

DIRECTION $-+$:

$$
\begin{matrix}
1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64&35&70&38&76&41&82&44&88&47&94\\
1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41&43&44&46&47\\
0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64&35&70&38&76&41&82&44&88\\
0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41&43&44\\
0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64&35&70&38&76&41&82\\
0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38&40&41\\
0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64&35&70&38&76\\
0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35&37&38\\
0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64&35&70\\
0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32&34&35\\
0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58&32&64\\
0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29&31&32\\
0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52&29&58\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26&28&29\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46&26&52\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23&25&26\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40&23&46\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20&22&23\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34&20&40\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17&19&20\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28&17&34\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14&16&17\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22&14&28\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11&13&14\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16&11&22\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8&10&11\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10&8&16\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&7&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5&10\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{matrix}
\tag{C.29}
$$

DIRECTION ++:

$$
\begin{matrix}
94 & 47 & 88 & 44 & 82 & 41 & 76 & 38 & 70 & 35 & 64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 \\
47 & 46 & 44 & 43 & 41 & 40 & 38 & 37 & 35 & 34 & 32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 \\
88 & 44 & 82 & 41 & 76 & 38 & 70 & 35 & 64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 \\
44 & 43 & 41 & 40 & 38 & 37 & 35 & 34 & 32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 \\
82 & 41 & 76 & 38 & 70 & 35 & 64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 \\
41 & 40 & 38 & 37 & 35 & 34 & 32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
76 & 38 & 70 & 35 & 64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
38 & 37 & 35 & 34 & 32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
70 & 35 & 64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 34 & 32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & 32 & 58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 31 & 29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
58 & 29 & 52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
29 & 28 & 26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
52 & 26 & 46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
26 & 25 & 23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
46 & 23 & 40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 22 & 20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & 20 & 34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 19 & 17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 17 & 28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 16 & 14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
28 & 14 & 22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 13 & 11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 11 & 16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 10 & 8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 8 & 10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 7 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\tag{C.30}
$$

DIRECTION $+-$:

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0
64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0	0	0
35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0	0	0
70	35	64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0	0	0
38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0	0	0
76	38	70	35	64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0	0	0
41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0	0	0
82	41	76	38	70	35	64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0	0	0
44	43	41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1	0	0
88	44	82	41	76	38	70	35	64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1	1	0
47	46	44	43	41	40	38	37	35	34	32	31	29	28	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1	1
94	47	88	44	82	41	76	38	70	35	64	32	58	29	52	26	46	23	40	20	34	17	28	14	22	11	16	8	10	5	4	2	1

(C.31)

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29
0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58
0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32
0	0	0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64
0	0	0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32	34	35
0	0	0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64	35	70
0	0	0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32	34	35	37	38
0	0	0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64	35	70	38	76
0	0	0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32	34	35	37	38	40	41
0	0	0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64	35	70	38	76	41	82
0	0	1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32	34	35	37	38	40	41	43	44
0	1	1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64	35	70	38	76	41	82	44	88
1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26	28	29	31	32	34	35	37	38	40	41	43	44	46	47
1	2	4	5	10	8	16	11	22	14	28	17	34	20	40	23	46	26	52	29	58	32	64	35	70	38	76	41	82	44	88	47	94

(C.32)

C.2.6 ${}_2D_4$ *equation* (1.93g)

DIRECTION $-+$:

$$
\begin{matrix}
1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 & 46 & 91 & 50 & 99 & 54 & 107 & 58 & 115 & 62 & 123 \\
1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 & 44 & 45 & 48 & 49 & 52 & 53 & 56 & 57 & 60 & 61 \\
0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 & 46 & 91 & 50 & 99 & 54 & 107 & 58 & 115 \\
0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 & 44 & 45 & 48 & 49 & 52 & 53 & 56 & 57 \\
0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 & 46 & 91 & 50 & 99 & 54 & 107 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 & 44 & 45 & 48 & 49 & 52 & 53 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 & 46 & 91 & 50 & 99 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 & 44 & 45 & 48 & 49 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 & 46 & 91 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 & 44 & 45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 & 42 & 83 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 & 40 & 41 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 & 38 & 75 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 & 36 & 37 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 & 34 & 67 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 32 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 & 30 & 59 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 & 28 & 29 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 & 26 & 51 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 & 24 & 25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 & 22 & 43 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 & 20 & 21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 & 18 & 35 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 & 14 & 27 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 & 12 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 & 10 & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 & 8 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 6 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{matrix}
\tag{C.33}
$$

DIRECTION ++:

123	62	115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1
61	60	57	56	53	52	49	48	45	44	41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1
115	58	107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0
57	56	53	52	49	48	45	44	41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0
107	54	99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0
53	52	49	48	45	44	41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0
99	50	91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0
49	48	45	44	41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0
91	46	83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0
45	44	41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0
83	42	75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0
41	40	37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0
75	38	67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
37	36	33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
67	34	59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
33	32	29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	30	51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	28	25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	26	43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	24	21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	22	35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	20	17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	18	27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	16	13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	14	19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	12	9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	10	11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	8	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(C.34)

DIRECTION $+-$:

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0	0	0
42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0	0	0
83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0	0	0
46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0	0	0
91	45	83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0	0	0
50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0	0	0
99	49	91	45	83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0	0	0
54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0	0	0
107	53	99	49	91	45	83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0	0	0
58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1	0	0
115	57	107	53	99	49	91	45	83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1	1	0
62	60	58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	16	14	12	10	8	6	4	2	1	1
123	61	115	57	107	53	99	49	91	45	83	41	75	37	67	33	59	29	51	25	43	21	35	17	27	13	19	9	11	5	4	2	1

(C.35)

DIRECTION ——:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	11
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	5	11	9	19
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
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0	0	0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
0	0	0	0	0	0	0	1	1	2	4	5	11	9	19	13	27	17	35	21	43	25	51	29	59	33	67	37	75	41	83	45	91
0	0	0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
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0	0	0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54
0	0	0	1	1	2	4	5	11	9	19	13	27	17	35	21	43	25	51	29	59	33	67	37	75	41	83	45	91	49	99	53	107
0	0	1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58
0	1	1	2	4	5	11	9	19	13	27	17	35	21	43	25	51	29	59	33	67	37	75	41	83	45	91	49	99	53	107	57	115
1	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62
1	2	4	5	11	9	19	13	27	17	35	21	43	25	51	29	59	33	67	37	75	41	83	45	91	49	99	53	107	57	115	61	123

(C.36)

D EXAMPLES OF CALCULATIONS OF GENERALIZED SYMMETRIES FOR EQUATIONS WITH TWO-PERIODIC COEFFICIENTS

In this Appendix we give two examples of explicit calculations of the three-points generalized symmetries in the case of equation with two periodic coefficients as explained in Section 3.2. The two examples we will give are the D_3 equation (1.93e), which is quite simple, and the ${}_tH_1^\varepsilon$ equation (1.91a) in the direction m, which is instead a more difficult one. These detailed calculations were presented in [63].

D.1 THREE-POINTS GENERALIZED SYMMETRIES OF THE D_3 EQUATION

Consider the D_3 equation as given by formula (1.93e). We wish to compute its three-points generalized symmetries in the direction n. To do so we can apply the method explained in Subsection 3.2. We then first find the system for the function z from (3.40). Considering the case in which $n = 2k$ and $m = 2l$ we find from (3.40) two equations for $z^{(+,+)}$:

$$\begin{aligned}&\frac{u_{n+1,m}+u_{n,m-1}}{u_{n,m}-u_{n,m-1}^2}\frac{\partial z^{(+,+)}}{\partial u_{n+1,m}}+\frac{\partial z^{(+,+)}}{\partial u_{n,m}}\\&+\frac{u_{n-1,m}+u_{n,m-1}}{u_{n,m}-u_{n,m-1}^2}\frac{\partial z^{(+,+)}}{\partial u_{n-1,m}}=\frac{2u_{n,m-1}u_{n-1,m}+u_{n,m-1}^2+u_{n,m}}{(u_{n,m}-u_{n,m-1}^2)(u_{n,m}-u_{n-1,m}^2)},\end{aligned} \tag{D.1a}$$

$$\begin{aligned}&\frac{u_{n+1,m}+u_{n,m+1}}{u_{n,m+1}^2-u_{n,m}}\frac{\partial z^{(+,+)}}{\partial u_{n+1,m}}-\frac{\partial z^{(+,+)}}{\partial u_{n,m}}\\&+\frac{u_{n-1,m}+u_{n,m+1}}{u_{n,m+1}^2-u_{n,m}}\frac{\partial z^{(+,+)}}{\partial u_{n-1,m}}=\frac{u_{n,m+1}^2+2u_{n-1,m}u_{n,m+1}+u_{n,m}}{(u_{n,m}-u_{n-1,m}^2)(u_{n,m}-u_{n,m+1}^2)}.\end{aligned} \tag{D.1b}$$

Taking the coefficients with respect to $u_{n,m\pm1}$ and solving we obtain:

$$z^{(+,+)}=\log\left[C_1^{(+,+)}\left(\frac{u_{n,m}-u_{n-1,m}^2}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right]. \tag{D.2}$$

In an analogous way we obtain the solutions for $z^{(+,-)}$, $z^{(-,+)}$ and $z^{(-,-)}$:

$$z^{(-,+)}=\log\left[C_1^{(-,+)}\left(\frac{u_{n,m}^2-u_{n-1,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right], \tag{D.3a}$$

$$z^{(+,-)} = \log\left[C_1^{(+,-)}\left(\frac{u_{n,m}+u_{n-1,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right], \tag{D.3b}$$

$$z^{(-,-)} = \log\left[C_1^{(-,-)}\left(\frac{u_{n,m}+u_{n-1,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right]. \tag{D.3c}$$

Inserting the resulting value for z (3.84a) obtained from (D.3) into the conservation law (3.29) we derive the following relation between the constants:

$$C_1^{(+,-)} = -C_1^{(+,+)}, \quad C_1^{(-,-)} = C_1^{(-,+)}. \tag{D.4}$$

Proceeding in the same way we can derive the values of the functions $\nu^{(\pm,\pm)}$ from (3.42):

$$\nu^{(+,+)} = \log\left[C_2^{(-,+)}\left(\frac{u_{n,m}-u_{n+1,m}^2}{(u_{n+1,m}-u_{n-1,m})}\right)\right], \tag{D.5a}$$

$$\nu^{(-,+)} = \log\left[C_2^{(-,+)}\left(\frac{u_{n,m}^2-u_{n+1,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right], \tag{D.5b}$$

$$\nu^{(+,-)} = \log\left[C_2^{(+,-)}\left(\frac{u_{n+1,m}+u_{n+1,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right], \tag{D.5c}$$

$$\nu^{(-,-)} = \log\left[C_2^{(-,-)}\left(\frac{u_{n+1,m}+u_{n,m}}{(u_{n+1,m}-u_{n-1,m})^2}\right)\right]. \tag{D.5d}$$

Inserting the resulting value for ν (3.84b) obtained from (D.5) into the conservation law (3.34) we derive the following relation between the constants:

$$C_2^{(+,-)} = -C_2^{(+,+)}, \quad C_2^{(-,-)} = C_2^{(-,+)}. \tag{D.6}$$

We can now insert z and ν into the compatibility conditions in order to find g. From these compatibility conditions we obtain the following relations between the remaining constants:

$$C_2^{(+,+)} = -C_1^{(+,+)}, \quad C_2^{(-,+)} = -C_1^{(-,+)}. \tag{D.7}$$

Integrating the equation for g (3.44) we get:

$$\begin{aligned} g = {} & F_n^{(+)}F_m^{(+)}\left[C_1^{(+,+)}\left(\frac{u_{n-1,m}^2-u_{n,m}}{u_{n+1,m}-u_{n-1,m}}+u_{n-1,m}\right)+\Phi^{(+,+)}\right] \\ & + F_n^{(-)}F_m^{(+)}\left[C_1^{(-,+)}\frac{u_{n-1,m}-u_{n,m}^2}{u_{n+1,m}-u_{n-1,m}}+\Phi^{(-,+)}\right] \\ & + F_n^{(+)}F_m^{(-)}\left[C_1^{(+,+)}\frac{u_{n,m}+u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}}+\Phi^{(+,-)}\right] \\ & + F_n^{(-)}F_m^{(-)}\left[C_1^{(-,+)}\frac{u_{n,m}+u_{n-1,m}}{u_{n-1,m}-u_{n+1,m}}+\Phi^{(-,-)}\right]. \end{aligned} \tag{D.8}$$

Inserting it in the determining equations (3.26) and applying the operator (3.47) we obtain a system of four equations which have to be identically satisfied. The result is $C_1^{(-,+)} = -C_1^{(+,+)}$ and:

$$\Phi^{(+,+)} = K_1 u_{n,m}, \quad \Phi^{(-,+)} = \frac{1}{2}K_1 u_{n,m} - \frac{1}{2}C_1^{(+,+)},$$
$$\Phi^{(+,-)} = \frac{1}{2}K_1 u_{n,m} + \frac{1}{2}C_1^{(+,+)}, \quad \Phi^{(-,-)} = \frac{1}{2}K_1 u_{n,m} + \frac{1}{2}C_1^{(+,+)}, \tag{D.9}$$

where K_1 is another arbitrary constant. Note that the symmetry generated by K_1 is a point symmetry, so the equation (1.93e) in the direction n possess only the genuine three-points symmetry given by:

$$g = \frac{F_n^{(+)}F_m^{(+)}u_{n+1,m}u_{n-1,m} + \frac{1}{2}\left(F_m^{(-)} - F_n^{(-)}F_m^{(+)}\right)u_{n,m}(u_{n+1,m}+u_{n-1,m})}{u_{n+1,m}-u_{n-1,m}}$$
$$+ \frac{F_n^{(-)}F_m^{(+)}u_{n,m}^2 + \left(F_m^{(-)} - F_n^{(+)}F_m^{(+)}\right)u_{n,m}}{u_{n+1,m}-u_{n-1,m}}, \tag{D.10}$$

As a sub-product of these computation we obtain then also the point symmetry given by the coefficients of K_1:

$$g_P = \left[F_n^{(+)}\left(2F_m^{(+)} + F_m^{(-)}\right) + F_n^{(-)}\right]u_{n,m}. \tag{D.11}$$

Note that since (1.93e) is invariant under the exchange $n \leftrightarrow m$ the symmetry in the m direction is given simply by performing such exchange in (D.10), which then gives:

$$\tilde{g} = \frac{F_n^{(+)}F_m^{(+)}u_{n,m+1}u_{n,m-1} + \frac{1}{2}\left(F_n^{(-)} - F_n^{(+)}F_m^{(-)}\right)u_{n,m}(u_{n,m+1}+u_{n,m-1})}{u_{n,m+1}-u_{n,m-1}} +$$
$$+ \frac{F_n^{(+)}F_m^{(-)}u_{n,m}^2 + \left(F_n^{(-)} - F_n^{(+)}F_m^{(+)}\right)u_{n,m}}{u_{n,m+1}-u_{n,m-1}}. \tag{D.12}$$

Clearly these results match with those presented without all the details in Section 3.3.

D.2 THREE-POINTS GENERALIZED SYMMETRIES OF THE ${}_tH_1^\varepsilon$ IN DIRECTION m

Now we consider another interesting example: the ${}_tH_1^\varepsilon$ equation. We will show how to compute the three-points generalized symmetries of the ${}_tH_1^\varepsilon$ equation as given by (1.91a) in the m direction. To safely apply the methods discussed in Section 3.2 we have to perform the exchange of the discrete indices $n \leftrightarrow m$ and then compute the symmetries in the n direction. Applying such exchange to ${}_tH_1^\varepsilon$ equation (1.91a) we obtain:

$$(u_{n,m} - u_{n,m+1})(u_{n+1,m} - u_{n+1,m+1}) - \alpha_2 \\ - \varepsilon^2 \alpha_2 (F_n^{(+)} u_{n+1,m+1} u_{n+1,m} + F_n^{(-)} u_{n,m} u_{n,m+1}) \tag{D.13}$$

Since there are only two two-periodic functions in (D.13) a function w decompose as:

$$w = F_n^{(+)} w^{(+)} + F_n^{(-)} w^{(-)} \tag{D.14}$$

instead of the most general decomposition (3.85). Applying the method explained in Subsection 3.2 we can find the system for the function z decomposed as in (D.14) from (3.40). Considering the case $n = 2k$ we find from (3.40) the system for $z^{(+)}$:

$$-\frac{\alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)}{(u_{n,m}-u_{n,m-1})^2+\varepsilon^2\alpha_2^2}\frac{\partial z^{(+)}}{\partial u_{n+1,m}} + \frac{\partial z^{(+)}}{\partial u_{n,m}} \\ + \frac{\alpha_2\left(1+\varepsilon^2 u_{n-1,m}^2\right)}{(u_{n,m}-u_{n,m-1})^2+\varepsilon^2\alpha_2^2}\frac{\partial z^{(+)}}{\partial u_{n-1,m}} = \frac{2\varepsilon^2 u_{n+1,m}}{(u_{n,m}-u_{n,m-1})^2+\varepsilon^2\alpha_2^2} \tag{D.15a}$$

$$-\frac{\alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)}{(u_{n,m}-u_{n,m+1})^2+\varepsilon^2\alpha_2^2}\frac{\partial z^{(+)}}{\partial u_{n+1,m}} + \frac{\partial z^{(+)}}{\partial u_{n,m}} \\ + \frac{\alpha_2\left(1+\varepsilon^2 u_{n-1,m}^2\right)}{(u_{n,m}-u_{n,m+1})^2+\varepsilon^2\alpha_2^2}\frac{\partial z^{(+)}}{\partial u_{n-1,m}} = \frac{2\varepsilon^2 u_{n+1,m}}{(u_{n,m}-u_{n,m+1})^2+\varepsilon^2\alpha_2^2} \tag{D.15b}$$

Taking the coefficients with respect to $u_{n,m\pm1}$ we can solve this system. This time the system is not overdetermined, but it is underdetermined:

$$z^{(+)} = -\log(1+\varepsilon^2 u_{n+1,m}^2) \\ + \log Z_{n,m}^{(+)}\left(\frac{\varepsilon(u_{n+1,m}-u_{n-1,m})}{1+\varepsilon^2 u_{n+1,m}u_{n-1,m}}.\right) \tag{D.16}$$

The same holds true for $z^{(-)}$:

$$z^{(-)} = \log(1+\varepsilon^2 u_{n,m}^2) \\ + \log Z_{n,m}^{(-)}(-u_{n+1,m}+u_{n-1,m}). \tag{D.17}$$

Using (3.41) we just add equations which are identically satisfied, so all the differential conditions on z are satisfied. Inserting into the conservation law (3.29) the only condition we get is $Z^{(\pm)}_{n,m} = Z^{(\pm)}_{n}$ i.e. the two arbitrary functions can depend only on one lattice variable.

For v we can proceed in the same way and we obtain:

$$v^{(+)} = \log\left[\frac{(1+\varepsilon^2 u_{n+1,m}u_{n-1,m})^2}{1+\varepsilon^2 u_{n+1,m}^2}\right] + \log V^{(+)}_{n,m}\left(\frac{\varepsilon(u_{n+1,m}-u_{n-1,m})}{1+\varepsilon^2 u_{n+1,m}u_{n-1,m}}\right), \tag{D.18a}$$

$$v^{(-)} = \log\left[(1+\varepsilon^2 u_{n,m}^2)V^{(-)}_{n,m}(-u_{n+1,m}+u_{n-1,m})\right]. \tag{D.18b}$$

Using (3.43) we just add equations which are identically satisfied, so we have satisfied all the differential conditions on v. Inserting v into the conservation law (3.34) the only condition we get is $V^{(\pm)}_{n,m} = V^{(\pm)}_{n}$ i.e. the two arbitrary functions depend only on one lattice variable.

Inserting z and v into the compatibility conditions (3.45) we can reduce the number of independent functions from four to two, since we find the following relations:

$$Z^{(+)}_n(\xi) = C^{(1)}_n - \left(1+\xi^2\right) V^{(+)}_n(\xi), \tag{D.19a}$$

$$V^{(-)}_n(\xi) = C^{(2)}_n - Z^{(-)}(\xi). \tag{D.19b}$$

Then defining:

$$G^{(+)}_n(\xi) = \int V^{(+)}_n(\xi)d\xi, \quad G^{(-)}_n(\xi) = \int Z^{(-)}_n(\xi)d\xi, \tag{D.20}$$

we can write the solution for g from (3.44):

$$\begin{aligned} g = {} & \frac{-F^{(+)}_n}{\varepsilon} G_n\left(\frac{\varepsilon(u_{n+1,m}-u_{n-1,m})}{1+\varepsilon^2 u_{n+1,m}u_{n-1,m]}}\right) \\ & + F^{(+)}_n\left[\frac{C^{(1)}_n}{\varepsilon}\arctan(\varepsilon u_{n+1,m}) + \Phi^{(+)}\right] \\ & - F^{(-)}_n(1+\varepsilon^2 u_{n,m}^2)G^{(-)}_n(-u_{n+1,m}+u_{n-1,m}) \\ & + F^{(-)}_n\left[(1+\varepsilon^2 u_{n,m}^2)C^{(2)}_n u_{n-1,m} + \Phi^{(-)}\right], \end{aligned} \tag{D.21}$$

with $\Phi^{(\pm)}_{n,m} = \Phi^{(\pm)}_{n,m}(u_{n,m})$ functions to be determined.

Inserting this form of g into the determining equations we find the following restrictions:

$$C^{(1)}_n = C^{(2)}_n = 0, \quad \Phi^{(+)} = C^{(3)}_n, \quad \Phi^{(-)} = C^{(4)}_n(1+\varepsilon^2 u_{n,m}^2), \tag{D.22}$$

where $C^{(3)}_n$ and $C^{(4)}_n$ are arbitrary functions of the lattice variable n. This means that (D.13) the three-points generalized symmetries are generated by the following function::

$$\begin{aligned} g = {} & F^{(+)}_n\left[\frac{-1}{\varepsilon}G^{(+)}_n\left(\frac{\varepsilon(u_{n+1,m}-u_{n-1,m})}{1+\varepsilon^2 u_{n+1,m}u_{n-1,m}}\right) + C^{(3)}_n\right] \\ & + F^{(-)}_n\left[-(1+\varepsilon^2 u_{n,m}^2)G^{(-)}_n(-u_{n+1,m}+u_{n-1,m}) + (1+\varepsilon^2 u_{n,m}^2)C^{(4)}_n\right] \end{aligned}$$

(D.23)

which depends on *arbitrary functions*. Note that we can separate the true generalized symmetry part from the point part noting that the coefficients of $C_n^{(3)}$ and $C_n^{(4)}$ only depend on $u_{n,m}$ and not from its shifts. Therefore we have the three-points generalized symmetry part:

$$g_{GS} = F_n^{(+)}\left[\frac{-1}{\varepsilon}G_n^{(+)}\left(\frac{\varepsilon(u_{n+1,m}-u_{n-1,m}}{(1+\varepsilon^2 u_{n+1,m}u_{n-1,m]})}\right)\right] \\ + F_n^{(-)}\left[-(1+\varepsilon^2 u_{n,m}^2)G_n^{(-)}(-u_{n+1,m}+u_{n-1,m})\right] \tag{D.24}$$

and in the purely point part:

$$g_P = F_n^{(+)}C_n^{(3)} + F_n^{(-)}(1+\varepsilon^2 u_{n,m}^2)C_n^{(4)}. \tag{D.25}$$

Reversing the transformation $n \leftrightarrow m$ we obtain the result displayed in Section 3.3.

We underline that the property of possessing generalized symmetries depending on arbitrary functions is linked with the fact that (1.91a) is *Darboux integrable* [5] as it was proved in [69] and is discussed in Chapter 4.

E

THREE-POINTS GENERALIZED SYMMETRIES OF THE RHOMBIC H^4 EQUATIONS

E.1 THE RHOMBIC H^4 EQUATIONS

In Chapter 1 we presented the single-cell form of the rhombic H^4 equations in formula (1.32). Applying the prescription of [3, 21, 67] we have (with the identification $\alpha_1 = \alpha$ and $\alpha_2 = \beta$) that the equations (1.32) once written on the $\mathbb{Z}^2_{(n,m)}$ lattice have the form:

$$
\begin{aligned}
{}_rH_1^{\varepsilon}\colon\quad & (u_{n,m}-u_{n+1,m+1})(u_{n+1,m}-u_{n,m+1}) - (\alpha-\beta)\\
&+\varepsilon(\alpha-\beta)\left(F^{(+)}_{n+m}\,u_{n+1,m}u_{n,m+1}+F^{(-)}_{n+m}\,u_{n,m}u_{n+1,m+1}\right)=0,
\end{aligned}
\tag{E.1a}
$$

$$
\begin{aligned}
{}_rH_2^{\varepsilon}\colon\quad & (u_{n,m}-u_{n+1,m+1})(u_{n+1,m}-u_{n,m+1})+\\
&+(\beta-\alpha)(u_{n,m}+u_{n+1,m}+u_{n,m+1}+u_{n+1,m+1})-\alpha^2+\beta^2\\
&-\varepsilon\,(\beta-\alpha)^3-\varepsilon\,(\beta-\alpha)\,\left(2F^{(-)}_{n+m}u_{n,m}+2F^{(+)}_{n+m}u_{n+1,m}+\alpha+\beta\right)\cdot\\
&\cdot\left(2F^{(-)}_{n+m}u_{n+1,m+1}+2F^{(+)}_{n+m}u_{n,m+1}+\alpha+\beta\right)\;=\;0,
\end{aligned}
\tag{E.1b}
$$

$$
\begin{aligned}
{}_rH_3^{\varepsilon}\colon\quad & \alpha(u_{n,m}u_{n+1,m}+u_{n,m+1}u_{n+1,m+1})\\
&-\beta(u_{n,m}u_{n,m+1}+u_{n+1,m}u_{n+1,m+1})+(\alpha^2-\beta^2)\delta\\
&-\frac{\varepsilon(\alpha^2-\beta^2)}{\alpha\beta}\left(F^{(+)}_{n+m}\,u_{n+1,m}u_{n,m+1}+F^{(-)}_{n+m}\,u_{n,m}u_{n+1,m+1}\right)=0,
\end{aligned}
\tag{E.1c}
$$

where $F^{(\pm)}_k$ is still given by formula (1.90). Obviously this formula agrees with that presented in [166].

E.2 THREE-POINTS GENERALIZED SYMMETRIES OF THE RHOMBIC H^4 EQUATIONS

The three-points generalized symmetries of the rhombic H^4 equations as given by (E.1) can be computed with the methods presented in Chapter 3 and were first presented in [166]. Their expression is the following:

$$
\widehat{X}_n^{{}_rH_1^{\varepsilon}} = \frac{1-\varepsilon\left(F^{(+)}_{n+m}u_{n+1,m}u_{n-1,m}+F^{(-)}_{n+m}u^2_{n,m}\right)}{u_{n+1,m}-u_{n-1,m}}\partial_{u_{n,m}},
\tag{E.2a}
$$

$$\widehat{X}_m^{rH_1^\varepsilon} = \frac{1-\varepsilon\left(F^{(+)}_{n+m}u_{n,m+1}u_{n,m-1}+F^{(-)}_{n+m}u_{n,m}^2\right)}{u_{n,m+1}-u_{n,m-1}}\partial_{u_{n,m}}, \tag{E.2b}$$

$$\begin{aligned}\widehat{X}_n^{rH_2^\varepsilon} &= \left[\frac{\left(1-4\varepsilon\alpha F^{(-)}_{n+m}\right)(u_{n+1,m}+u_{n-1,m})-4\varepsilon F^{(+)}_{n+m}u_{n+1,m}u_{n-1,m}}{u_{n+1,m}-u_{n-1,m}}+\right.\\ &\left.+\frac{2\alpha-4\varepsilon\alpha^2-4\varepsilon F^{(-)}_{n+m}u_{n,m}^2+\left(1-4\varepsilon\alpha F^{(-)}_{n+m}\right)u_{n,m}}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}\end{aligned} \tag{E.2c}$$

$$\begin{aligned}\widehat{X}_m^{rH_2^\varepsilon} &= \left[\frac{\left(1-4\varepsilon\beta F^{(-)}_{n+m}\right)(u_{n,m+1}+u_{n,m-1})-4\varepsilon F^{(+)}_{n+m}u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}+\right.\\ &\left.+\frac{2\beta-4\varepsilon\beta^2-4\varepsilon F^{(-)}_{n+m}u_{n,m}^2+\left(1-4\varepsilon\beta F^{(-)}_{n+m}\right)u_{n,m}}{u_{n,m+1}+u_{n,m-1}}\right]\partial_{u_{n,m}}\end{aligned} \tag{E.2d}$$

$$\begin{aligned}\widehat{X}_n^{rH_3^\varepsilon} &= \left[\frac{1}{2}\frac{u_{n,m}(u_{n+1,m}+u_{n-1,m})+2\delta\alpha}{u_{n+1,m}-u_{n-1,m}}-\right.\\ &\left.-\frac{\varepsilon}{\alpha}\frac{\left(F^{(+)}_{n+m}u_{n+1,m}u_{n-1,m}+F^{(-)}_{n+m}u_{n,m}^2\right)}{u_{n+1,m}-u_{n-1,m}}\right]\partial_{u_{n,m}}\end{aligned} \tag{E.2e}$$

$$\begin{aligned}\widehat{X}_m^{rH_3^\varepsilon} &= \left[\frac{1}{2}\frac{u_{n,m}(u_{n,m+1}+u_{n,m-1})+2\delta\beta}{u_{n,m+1}+u_{n,m-1}}-\right.\\ &\left.-\frac{\varepsilon}{\beta}\frac{\left(F^{(+)}_{n+m}u_{n,m+1}u_{n,m-1}+F^{(-)}_{n+m}u_{n,m}^2\right)}{u_{n,m+1}+u_{n,m-1}}\right]\partial_{u_{n,m}}\end{aligned} \tag{E.2f}$$

As stated in [166] the fluxes of the symmetries (E.2) are readily identified with the corresponding cases of the non autonomous YdKN equation (3.99). The explicit coefficients of the corresponding non-autonomous YdKN equation (3.99) are displayed in Table 3.1.

F

CONNECTION FORMULÆ BETWEEN THE NON-AUTONOMOUS Q_V AND THE NON-AUTONOMOUS YDKN IN THE m DIRECTION.

For the sake of completeness let us write down the non autonomous YdKN in the m direction:

$$\frac{du_m}{dt} = \frac{A_m(u_m)u_{m+1}u_{m-1} + B_m(u_m)(u_{m+1} + u_{m-1}) + C_m(u_m)}{u_{m+1} - u_{m-1}}. \tag{F.1}$$

Here the m-dependent coefficients are given by:

$$A_m(u_m) = au_m^2 + 2b_m u_m + c_m, \tag{F.2a}$$

$$B_m(u_m) = b_{m+1}u_m^2 + du_m + e_{m+1}, \tag{F.2b}$$

$$C_m(u_m) = c_{m+1}u_m^2 + 2e_m u_m + f, \tag{F.2c}$$

where b_m, c_m and e_m are 2-periodic functions, i.e.

$$\begin{gathered} b_m = b + \beta(-1)^m, \quad c_m = c + \gamma(-1)^m, \\ e_m = e + \varepsilon(-1)^m. \end{gathered} \tag{F.3}$$

The correlation formulæ read:

$$\begin{aligned} a = {} & a_1 a_{5,0} - a_{2,0}^2 - a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2 \\ & + (-1)^n (2a_{2,0}a_{2,1} - 2a_{2,2}a_{2,3} + a_1 a_{5,1}), \end{aligned} \tag{F.4a}$$

$$\begin{aligned} b = {} & \frac{1}{2} a_{2,0}(a_{5,0} - a_{3,0} - a_{4,0}) \\ & + \frac{1}{2}(a_1 a_{6,0} - a_{2,2}a_{3,2} - a_{2,3}a_{4,3} + a_{2,1}a_{5,1}) \\ & + \frac{1}{2}(-1)^n a_{2,1}(a_{5,0} + a_{3,0} + a_{4,0}) \\ & + \frac{1}{2}(-1)^n (a_{2,3}a_{3,2} + a_1 a_{6,1} + a_{2,0}a_{5,1} + a_{2,2}a_{4,3}), \end{aligned} \tag{F.4b}$$

$$\begin{aligned} \beta = {} & \frac{1}{2} a_{2,2}(a_{4,0} - a_{3,0} - a_{5,0}) \\ & - \frac{1}{2}(a_{2,3}a_{5,1} - a_{2,1}a_{4,3} + a_{2,0}a_{3,2} - a_1 a_{6,2}) \\ & + \frac{1}{2}(-1)^n a_{2,3}(a_{3,0} - a_{4,0} - a_{5,0}) \\ & + \frac{1}{2}(-1)^n (a_1 a_{6,3} + a_{2,1}a_{3,2} - a_{2,0}a_{4,3} - a_{2,2}a_{5,1}), \end{aligned} \tag{F.4c}$$

$$\begin{aligned} c = {} & a_{2,0}a_{6,0} - a_{4,0}a_{3,0} + a_{2,1}a_{6,1} - a_{2,3}a_{6,3} - a_{2,2}a_{6,2} \\ & - (-1)^n [a_{2,2}a_{6,3} + a_{4,3}a_{3,2} + a_{2,3}a_{6,2} - a_{2,0}a_{6,1} - a_{2,1}a_{6,0}], \end{aligned} \tag{F.4d}$$

$$\begin{aligned}\gamma &= a_{2,1}a_{6,3} - a_{4,0}a_{3,2} + a_{2,0}a_{6,2} - a_{2,3}a_{6,1} - a_{2,2}a_{6,0} - \\ &\quad - (-1)^n[a_{2,2}a_{6,1} + a_{4,3}a_{3,0} + a_{2,3}a_{6,0} - a_{2,1}a_{6,2} - a_{2,0}a_{6,3}],\end{aligned} \tag{F.4e}$$

$$\begin{aligned}d &= \frac{1}{2}(a_{5,0}^2 - a_{4,0}^2 - a_{3,0}^2 + a_1 a_7 + a_{3,2}^2 + a_{4,3}^2 - a_{5,1}^2) \\ &\quad + 2(-1)^n(a_{2,1}a_{6,0} + a_{2,0}a_{6,1} + a_{2,3}a_{6,2} + a_{2,2}a_{6,3}),\end{aligned} \tag{F.4f}$$

$$\begin{aligned}e &= \frac{1}{2}a_{6,0}(a_{5,0} - a_{4,0} - a_{3,0}) \\ &\quad + \frac{1}{2}(a_{2,0}a_7 - a_{5,1}a_{6,1} + a_{3,2}a_{6,2} + a_{4,3}a_{6,3}) \\ &\quad - \frac{1}{2}(-1)^n(a_{3,2}a_{6,3} + a_{4,3}a_{6,2} + a_{5,1}a_{6,0} - a_{2,1}a_7) \\ &\quad - \frac{1}{2}(-1)^n a_{6,1}(a_{3,0} + a_{4,0} + a_{5,0}),\end{aligned} \tag{F.4g}$$

$$\begin{aligned}\varepsilon &= \frac{1}{2}\{a_{6,2}(a_{4,0} - a_{5,0} - a_{3,0}) \\ &\quad + \frac{1}{2}(a_{3,2}a_{6,0} - a_{4,3}a_{6,1} + a_{5,1}a_{6,3} + a_{2,2}a_7) \\ &\quad + \frac{1}{2}(-1)^n(a_{4,3}a_{6,0} - a_{3,2}a_{6,1} + a_{5,1}a_{6,2} + a_{2,3}a_7) \\ &\quad + \frac{1}{2}(-1)^n a_{6,3}(a_{3,0} - a_{5,0} - a_{4,0}),\end{aligned} \tag{F.4h}$$

$$\begin{aligned}f &= a_{5,0}a_7 - a_{6,0}^2 + a_{6,2}^2 + a_{6,3}^2 - a_{6,1}^2 \\ &\quad - (-1)^n(2a_{6,2}a_{6,3} - 2a_{6,0}a_{6,1} + a_{5,1}a_7).\end{aligned} \tag{F.4i}$$

G

CALCULATION OF THE FIRST INTEGRALS OF THE ${}_tH_1$ EQUATION

In this Appendix we present a full developed example of calculation of the first integrals of an quad equation using the method discussed in Section 4.1. We will discuss the example of the ${}_tH_1^\varepsilon$ equation (1.91a). We note that the first integrals were first presented in [69] where they were found by direct inspection.

Before proceeding we note that since the H^4 equations in general, and therefore the ${}_tH_1^\varepsilon$ in particular, are non-autonomous only in the direction m though the factors $F_m^{(\pm)}$ we can consider a simplified version of (4.20), in the same spirit of what was done with the Generalized Symmetries, see Appendix E:

$$W_i = F_m^{(+)} W_i^{(+)} + F_m^{(-)} W_i^{(-)}. \tag{G.1}$$

If we assume $W_1 = W_1 \left(u_{n+1,m}, u_{n,m}\right)$ separating the even and odd terms with respect to m in (4.14) we find the following equations:

$$\frac{\partial W_1^{(+)}}{\partial u_{n+1,m}} + \frac{\partial W_1^{(+)}}{\partial u_{n,m}} = 0, \tag{G.2a}$$

$$\left(1+\varepsilon^2 u_{n+1,m}^2\right) \frac{\partial W_1^{(-)}}{\partial u_{n+1,m}} + \left(1+\varepsilon^2 u_{n,m}^2\right) \frac{\partial W_1^{(-)}}{\partial u_{n,m}} = 0. \tag{G.2b}$$

whose solution is:

$$W_1 = F_m^{(+)} F\left(u_{n+1,m} - u_{n,m}\right) + F_m^{(-)} G\left(\frac{u_{n+1,m} - u_{n,m}}{1+\varepsilon^2 u_{n,m} u_{n+1,m}}\right) \tag{G.3}$$

Inserting (G.3) into the functional equation (4.7b) we obtain that F and G must satisfy the following identity:

$$G\left(\xi\right) = F\left(\frac{\alpha_2}{\xi}\right). \tag{G.4}$$

This yields:

$$W_1 = F_m^{(+)} F\left(\frac{\alpha_2}{u_{n+1,m} - u_{n,m}}\right) + F_m^{(-)} F\left(\frac{u_{n+1,m} - u_{n,m}}{1+\varepsilon^2 u_{n,m} u_{n+1,m}}\right). \tag{G.5}$$

For the m direction we may also suppose our first integral to be two-point: $W_2 = W_2 \left(u_{n,m+1}, u_{n,m}\right)$. It easy to see from (4.19) with $k = 1$ that this yields only the trivial solution $W_2 =$ constant. Therefore we consider the case where the integral is three-point: $W_2 =$

$W_2\left(u_{n,m+1}, u_{n,m}, u_{n,m-1}\right)$. From (4.19) with $k=1$, separating the even and odd terms, we obtain:

$$\begin{aligned}&\alpha_2\left(1+\varepsilon^2 u_{n,m+1}^2\right)\frac{\partial W_2^{(+)}}{\partial u_{n,m+1}}\\&-\left[\left(u_{n,m}-u_{n+1,m}\right)^2+\varepsilon^2\alpha_2^2\right]\frac{\partial W_2^{(+)}}{\partial u_{n,m}}\\&+\alpha_2\left(1+\varepsilon^2 u_{n,m-1}^2\right)\frac{\partial W_2^{(+)}}{\partial u_{n,m-1}}=0,\end{aligned}\tag{G.6a}$$

$$\begin{aligned}&\alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)\frac{\partial W_2^{(-)}}{\partial u_{n,m+1}}\\&-\left(u_{n,m}-u_{n+1,m}\right)^2\frac{\partial W_2^{(-)}}{\partial u_{n,m}}\\&+\alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)\frac{\partial W_2^{(-)}}{\partial u_{n,m-1}}=0.\end{aligned}\tag{G.6b}$$

Taking the coefficients with respect to $u_{n+1,m}$ and then solving we have:

$$\begin{aligned}W_2 &= F_m^{(+)}F\left(\frac{1+\varepsilon^2 u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}\right)\\&\quad+F_m^{(-)}G\left(u_{n,m+1}-u_{n,m-1}\right).\end{aligned}\tag{G.7}$$

Inserting (G.7) into (4.7a) we don't have any further restriction on the form of the first integral. So we conclude that we have two independent first integrals in the m direction, as it was observed in [69].

As explained in Section 4.1 since the application of the method was fruitful we obtained first integrals depending on arbitrary functions, which is of course redundant. We can then apply the simplifying assumptions discussed in Section 4.1 and consider the first integrals for the ${}_tH_1^\varepsilon$ as given by linear functions of their arguments:

$$W_1 = F_m^{(+)}\frac{\alpha_2}{u_{n+1,m}-u_{n,m}}+F_m^{(-)}\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n,m}u_{n+1,m}},\tag{G.8a}$$

$$\begin{aligned}W_2 &= F_m^{(+)}\alpha\frac{1+\varepsilon^2 u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}\\&\quad+F_m^{(-)}\beta\left(u_{n,m+1}-u_{n,m-1}\right).\end{aligned}\tag{G.8b}$$

Here α and β are two arbitrary constants coming from the fact that in the m direction we found *two* independent arbitrary functions.

The first integrals for the ${}_tH_1^\varepsilon$ equation (1.91a) were first presented in the form (G.8) in [69]. In Section 4.3 we show these first integrals again in this form.

H

LINEARIZATION OF THE ${}_tH_1^\varepsilon$ EQUATION THROUGH THE FIRST INTEGRAL IN THE DIRECTION m

In this Appendix we consider the linearization of the ${}_tH_1^\varepsilon$ equation (1.91a) through the integral in the direction m, namely W_2 given by formula (4.29b). Note that this case is more interesting since now we are dealing with a three-point, second order integral. Let us assume, without loss of generality that $\alpha = \beta = 1$. Our starting point is the relation $W_2 = \rho_m$ from which we can derive two different equations, one for the even and one for the odd m component of $u_{n,m}$. This will give *a priori* a coupled system. However in this case it easy to see that choosing $m = 2k$ and $m = 2k+1$ we obtain the two equations:

$$1 + \varepsilon^2 u_{n,2k+1} u_{n,2k-1} = \rho_{2k} \left(u_{n,2k+1} - u_{n,2k-1} \right), \tag{H.1a}$$

$$u_{n,2k+2} - u_{n,2k} = \rho_{2k+1}. \tag{H.1b}$$

Therefore the system consists of two *uncoupled* equations.

The first one (H.1a) is a discrete Riccati equation which can be linearized through the non-autonomous Möbius transformation:

$$u_{n,2k-1} = \frac{1}{v_{n,k}} + \alpha_k, \quad \rho_{2k} = \frac{1 + \varepsilon^2 \alpha_{k+1} \alpha_k}{\alpha_{k+1} - \alpha_k} \tag{H.2}$$

from which we obtain:

$$\left(1 + \varepsilon^2 \alpha_{k+1}^2\right) v_{n,k+1} + \varepsilon^2 \alpha_{k+1} = \left(1 + \varepsilon^2 \alpha_k^2\right) v_{n,k} + \varepsilon^2 \alpha_k. \tag{H.3}$$

This equation is a total difference and therefore its solution is given by:

$$v_{n,k} = \frac{\theta_n - \varepsilon^2 \alpha_k}{1 + \varepsilon^2 \alpha_k^2}. \tag{H.4}$$

Putting $\alpha_k = \kappa_{2k-1}$ we obtain the solution for $u_{n,2k-1}$ as:

$$u_{n,2k-1} = \frac{1 + \kappa_{2k-1} \theta_n}{\theta_n - \varepsilon^2 \kappa_{2k-1}}. \tag{H.5}$$

The second equation is just a linear ordinary difference equation which can be written as a total difference performing the substitution $\rho_{2k+1} = \kappa_{2k+2} - \kappa_{2k}$:

$$u_{n,2k} = \omega_n + \kappa_{2k}. \tag{H.6}$$

Therefore we obtain:

$$u_{n,m} = F_m^{(+)} \left(\omega_n + \kappa_m \right) + F_m^{(-)} \frac{1 + \kappa_m \theta_n}{\theta_n - \varepsilon^2 \kappa_m}. \tag{H.7}$$

(H.7) depends on *three* arbitrary function. This is because we started from a second order integral, which is a consequence the discrete equation. This means that there must be a relation between θ_n and ω_n. This relation can be retrieved by inserting (H.7) into (1.91a). So we obtain the following relation:

$$\omega_n - \omega_{n+1} = \alpha_2 \frac{\varepsilon^2 + \theta_n \theta_{n+1}}{\theta_{n+1} - \theta_n}, \tag{H.8}$$

which gives us the final expression for the solution of (1.91a) up to the integration given by (H.8). The general solutions obtained from different first integrals are the same in the sense that one of them can easily be transformed into the other.

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