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Renormalization group flow of scalar models in gravity

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"Sic unumquicquid paulatim protrahit aetas in medium ratioque in luminis erigit oras"
De rerum natura, Lucretius

Contents

Abstract

In this Ph.D. thesis we will study the issue of renormalizability of gravitation in the context of the renormalization group (RG), employing both perturbative and non-perturbative techniques. In particular, we will focus on different gravitational models and approximations in which a central role is played by a scalar degree of freedom, since their RG flow is easier to analyze. More specifically, we will consider two types of situations: on the one hand, we will study scalar field theories obtained as conformally reduced toy models of gravity, that is, by neglecting the contribution of the spin-2 degrees of freedom of the metric in the quantization procedure. On the other hand, we will study scalar-tensor theories as dynamically equivalent theories of higher derivative models.

We restrict our interest in particular to two quantum gravity approaches that have gained a lot of attention recently, namely the asymptotic safety scenario for gravity and the Hořava-Lifshitz quantum gravity. In the so-called asymptotic safety conjecture the high energy regime of gravity is controlled by a non-Gaussian fixed point which ensures non-perturbative renormalizability and finiteness of the correlation functions. We will then investigate the existence of such a non trivial fixed point using the functional renormalization group, a continuum version of the non-perturbative Wilson's renormalization group. In particular we will quantize the sole conformal degree of freedom, which is an approximation that has been shown to lead to a qualitatively correct picture. The use of such a scalar toy model, moreover, will help us to investigate in a non-perturbative way the role that non-local operators have in the emergence of a symmetry-broken phase in the infrared.

The question of the existence of a non-Gaussian fixed point in an infinite-dimensional parameter space, that is for a generic $f(R)$ theory, cannot however be studied using such a conformally reduced model. Hence we will study it by quantizing a dynamically equivalent scalar-tensor theory, i.e. a generic Brans-Dicke theory with $\omega = 0$ in the local potential approximation. We will then debate the breaking of the equivalence at a quantum level.

Finally, we will investigate, using a perturbative RG scheme, the asymptotic freedom of the Hořava-Lifshitz gravity, that is an approach based on the emergence of an anisotropy between space and time which lifts the Newton's constant to a marginal coupling and explicitly preserves unitarity. In particular we will evaluate the one-loop correction in 2+1 dimensions quantizing only the conformal degree of freedom. For this dimensionality it is in fact the only physical one. We obtain in this way the first results on the RG flow of the model in the ultraviolet.

Sommario

In questa tesi di dottorato studieremo la rinormalizzabilità di teorie gravitazionali nel contesto del gruppo di rinormalizzazione (RG), impiegando sia tecniche perturbative che non perturbative. In particolare, ci concentreremo su diversi modelli gravitazionali ed approssimazioni in cui un ruolo centrale viene svolto da una grado di libertà scalare; il flusso di rinormalizzazione di questi modelli risulta infatti più semplice da analizzare. Più nello specifico prenderemo in considerazione due situazioni: in una studieremo teorie di campo scalari ottenute come riduzioni conformi di teorie gravitazionali, ovvero modelli semplificati in cui non vengono quantizzati i gradi di libertà tensoriali della metrica, nell'altra studieremo teorie scalare-tensore come teorie dinamicamente equivalenti di modelli contenenti operatori con derivate di ordine superiore.

In particolare, ci limiteremo allo studio di due approcci di gravità quantistica che recentemente hanno ricevuto molta attenzione, ovvero lo scenario di *asymptotic safety* per la gravità e la gravità quantistica di Hořava-Lifshitz. Secondo la congettura nota come *asymptotic safety* l'interazione gravitazionale è controllata ad alte energie da un punto fisso non Gaussiano il quale garantisce la rinormalizzabilità non perturbativa della teoria e la finitezza delle funzioni di correlazione. Studieremo quindi l'esistenza di tale punto fisso utilizzando il gruppo di rinormalizzazione funzionale, ovvero una versione funzionale del gruppo di rinormalizzazione di Wilson, quantizzando in particolare soltanto il grado di libertà conforme. E' stato infatti dimostrato come tale semplificazione conduca ad un diagramma di fase della teoria qualitativamente corretto. L'uso di questo modello approssimato, inoltre, ci permette di studiare in maniera non perturbativa il ruolo che operatori non locali svolgono nell'emergenza di una fase rotta nel limite infrarosso. La questione dell'esistenza di un punto fisso non Gaussiano in uno spazio dei parametri infinito dimensionale, ovvero per una generica teoria $f(R)$, non può essere, tuttavia, indagata utilizzando un modello conformalmente ridotto. Per questo motivo quantizzeremo una teoria dinamicamente equivalente, ovvero una teoria di Brans-Dicke con $\omega = 0$ nell'approssimazione di potenziale locale, e discuteremo la non equivalenza delle due teorie a livello quantistico.

Per concludere, utilizzando tecniche perturbative studieremo la libertà asintotica della gravità di Hořava-Lifshitz, un approccio basato sull'emergenza di una anisotropia tra spazio e tempo, a causa della quale promuove la costante di Newton viene promossa a parametro marginale, preservando simultaneamente l'unitarietà. In particolare, calcoleremo in 2+1 dimensioni l'azione effettiva ad un loop, quantizzando il solo grado di libertà conforme (l'unico fisico per questa dimensionalità) ottenendo cosí i primi risultati sul flusso di rinormalizzazione del modello ad alte energie.

Zusammenfassung

In dieser Doktorarbeit werden wir das Renormierungsproblem von Gravitationstheorien im Kontext der Renormierungsgruppe (RG) unter Anwendung von perturbativen und nicht-perturbativen Methoden untersuchen. Insbesondere werden wir uns auf verschiedene Gravitationsmodelle und Näherungen konzentrieren, in welchen die zentrale Rolle von einem skalaren Freiheitsgrad eingenommen wird, da sein RG Fluss einfacher zu analysieren ist. Wir werden zwei Fälle genauer betrachten: einerseits werden wir skalare Feldtheorien aus konform reduzierten vereinfachten Gravitationsmodellen untersuchen, indem der Beitrag der Gravitonen bei der Quantisierung vernachlässigt wird. Andererseits untersuchen wir Skalar-Tensortheorien als dynamische Äquivalentstheorie zu höheren Ableitungstheorien.

Wir konzentrieren uns besonders auf zwei Ansätze für Quantengravitation, die in letzter Zeit viel Aufmerksamkeit erhalten haben, nämlich den asymptotisch sicheren Fall der Gravitation und die Hořava-Lifshitz Quantengravitation. Das Prinzip der Asymptotischen Sicherheit beruht auf der Annahme, dass das hochenergetische Gravitationsregime von einem nicht-Gaußschen Fixpunkt bestimmt wird, der nicht-perturbative Renormierung und Endlichkeit der Korrelationsfunktionen sicherstellt. Wir werden die Existenz eines solchen nicht-trivialen Fixpunktes mit Hilfe der funktionalen Renormierungsgruppe, einer kontinuierlichen Version der nicht-perturbativen Wilson Renormierungsgruppe, untersuchen. Insbesondere werden wir den einzigen konformen Freiheitsgrad quantisieren. In diesem Fall konnte gezeigt werden, dass dies eine qualitativ gute Näherung darstellt. Diese skalare Feldtheorie hilft uns auf nicht-perturbative Weise die Rolle der nicht-lokalen Operatoren zu verstehen, die diese bei der Entstehung einer Symmetrie-gebrochenen Phase im Infraroten haben.

Die Frage nach der Existenz eines nicht-Gaußschen Fixpunktes in einem unendlich-dimensionalen Parameterraum, das heißt für eine generische $f(R)$-Theorie, kann jedoch nicht mit einem solchen konform reduzierten Model analysiert werden. Deshalb werden wir es untersuchen, indem wir eine skalare dynamische Äquivalentstheorie, das heißt eine generische Brans-Dicke Theorie in der lokal-Potential Näherung mit $\omega = 0$, quantisieren. Wir werden dann die Äquivalenzbrechung auf dem Quantenlevel debattieren.

Schließlich werden wir mittels einer perturbativen RG Methode die asymptotische Freiheit der Hořava-Lifshitz Gravitationstheorie analysieren. Diese Gravitationstheorie beruht auf der Entstehung einer Anisotropie zwischen Raum und Zeit, die Newtons Konstante zu einer marginalen Koppelung werden lässt und explizit die Unitarität bewahrt. Insbesondere werden wir die Einschleifenkorrektur in 2+1 Dimensionen berechnen, indem wir nur den konformen Freiheitsgrad, den einzigen physikalischen für diese Dimensionalität, quantisieren. Wir erhalten somit die ersten Resultate über den RG Fluss dieses Models im ultravioletten Regime.

Introduction

The search for a theory of quantum gravity is one of the most intriguing unsolved puzzles in theoretical physics. Over the years, quantum field theory has shown a remarkable success in describing the physics of electromagnetic, strong and weak interactions. Its application to the description of gravity at the quantum level, however, has encountered many obstacles of both technical and fundamental nature. A naive quantization of general relativity leads in fact to a perturbatively non-renormalizable theory. Although the issue of renormalizability can be cured by adding higher-derivative operators, such models suffer from a lack of unitarity. For this reason, the scientific community tends nowadays to consider other frameworks, e.g. string theory, as more eligible candidates for a theory of quantum gravity.

Quantum field theory, however, has still not been ruled out. Two approaches in particular, namely the asymptotic safety scenario for gravity and the Hořava-Lifshitz quantum gravity, have recently received a lot of attention for their attempt to reconcile quantum field theory and gravity.

In this Ph.D. thesis we will study the issue of the renormalizability in those two approaches in the context of the renormalization group (RG), employing both perturbative and non-perturbative techniques. More specifically, we will focus on the study of gravitational models and approximations in which a central role is played by a scalar degree of freedom. Their RG flow is in fact easier to analyze, and this will allow us to investigate important issues otherwise difficult to examine.

We will consider two types of situations: on the one hand, we will study scalar field theories as toy models obtained by conformal reductions of gravitational theories. That is, models of gravity in which we neglect the contribution of the spin-2 degrees of freedom of the metric in the quantization procedure. On the other hand, we will study scalar-tensor theories as dynamically equivalent theories of higher-derivative models.

The asymptotic safety scenario for gravity is a conjecture proposed in the 70s by Weinberg [1, 2, 3, 4, 5]. In this approach the gravitational interaction is assumed to flow in the ultraviolet towards a strongly coupled regime, where gravity is controlled by a non-Gaussian fixed point which ensures non-perturbative renormalizability and finiteness of the correlation functions. The theory assumes then the form of an effective field theory in which just a finite number of relevant parameters needs to be fine tuned. The existence of this non-trivial fixed point has been confirmed in four dimension for various truncations of the effective action.

We will concentrate on the study of a scalar toy model of gravity obtained by a conformal reduction of the Hilbert-Einstein action. In this toy model we will quantize the sole conformal degree of freedom of the metric, thus neglecting the quantum contributions coming from spin-2 modes. Previous results have shown that the use of this approximation leads to a qualitatively correct picture [6, 7], making it an interesting model with which to test ideas and techniques.

We will hence inquire about the existence of the non-Gaussian fixed point by using the functional renormalization group (fRG) [8, 9, 10], a continuum version of the non-perturbative Wilson's renormalization group, that has been widely employed in statistical mechanics and quantum field theory to study strongly interacting theories. Specifically, we will examine the phase diagram of the toy model using a fRG proper time scheme, that has shown in literature a high precision in the evaluation of universal quantities. The use of such a scalar toy model, moreover, will permit us to investigate in a non-perturbative way the role that non-local operators (e.g. powers of the spacetime volume) have in the emergence of a symmetry-broken phase in the infrared. Hence, we will examine more in detail the ultraviolet fixed-point structure of conformally reduced asymptotically safe quantum gravity for a non-polynomial truncation.

The question of the existence of a non-Gaussian fixed point for a generic $f(R)$ theory cannot be studied, however, using such a scalar toy model. Hence we will study the non-perturbative renormalizability of a classical dynamically equivalent model, that is a generic Brans-Dicke theory with $\omega = 0$ in the local potential approximation. Besides the relation to $f(R)$ theory, the study of the strongly coupled regime of the Brans-Dicke theory is interesting by itself. Scalar-tensor theories are in fact an example of dilaton gravity, i.e. a theory in which the gravitational interaction is mediated by the metric field and a supplementary scalar field, and are one of the oldest modifications of general relativity. Furthermore, they find several applications in cosmology and quantum gravity. Using the functional renormalization group, and in particular the exact RG equation, we will then derive a flow equation for the effective potential on a flat spacetime, integrating all the degrees of freedom of the theory. In particular, the flow equation will be evaluated for a more general class of scalar-tensor theories with ω left as a free parameter. For the sake of consistency, we will evaluate the RG equation employing two different gauges, namely a Landau and a Feynman gauge. We will then focus on the fixed point structure for $\omega = 0$, in light of the classical equivalence between the Brans-Dicke and the $f(R)$ theory. Thus we analyze the results and discuss the implications of the approximations we have taken into account, as well as the breaking of the equivalence of the two theories at the quantum level.

Finally, we will investigate the high energy behaviour of Hořava-Lifshitz gravity [11, 12], and examine whether the model features asymptotic freedom or not. In Hořava-Lifshitz gravity a scale anisotropy between time and space emerges at the Planck scale. Lorentz invariance, one of the pillars of quantum field theory, is then lost in the ultraviolet and supposed to be recovered in the low energy regime. At the price of losing general covariance, anyway, the model gains explicit unitarity and perturbative renormalizability. In the presence of anisotropy we can have higher-order spatial derivatives while keeping the number of time derivatives unchanged. Therefore, for a sufficiently high order of spatial derivatives the model features perturbative renormalizability, being now the Newton constant a marginal coupling, while the absence of higher time derivatives avoids the presence of unphysical poles in the propagator.

Although the renormalizability is the most appealing feature of this model, it is also its less studied property. One of the reasons is the technical difficulty of working on curved anisotropic backgrounds, and the large number of terms that the model contains in its most general action in 3+1 dimensions. We will thus try to give a first answer about the asymptotic freedom of the model by taking in consideration an easier case, namely the Hořava-Lifshitz gravity in 2+1 dimensions. For this dimensionality, in fact, we have the simplification of having no gravitons and a smaller number of invariants in the action. What we will investigate is then the high energy regime of a scalar toy model obtained by a conformal reduction of the theory. We expect indeed this toy model to be a good approximations of the full theory, being the scalar the only physical degree of freedom for this dimensionality. We will evaluate the β-function for the coupling at one-loop and examine the renormalization group flow of the interaction. Hence we will interpret our results in order to give a first answer about the ultraviolet behaviour of the higher-dimensional model.

- This Ph.D. thesis is structured in the following way.

We will summarize in the **first chapter** the basics of the renormalization group. In particular, we will introduce perturbative and non-perturbative techniques, concentrating on the functional renormalization group.

We will focus in the **second chapter** on the applications of the renormalization group to quantum gravity and the basics of the quantization of gravitational theories (e.g. the background field method, Ward identities, etc.). We introduce then the asymptotic safety scenario for gravity and the Hořava-Lifshitz quantum gravity, and list few well enstablished results.

We will investigate in the **third chapter** the high energy regime of the Einstein-Hilbert action employing a proper time RG scheme in the context of the functional renormalization group. Hence, we will study the RG flow of a scalar toy model, i.e. a conformally reduced Einstein-Hilbert action, on a spherical and flat topology. We will then study this scalar toy model in a local potential approximation in flat spacetime, and investigate the role of non-local operators in having a broken phase in the infrared regime of the theory. This chapter is bases on the publication A. Bonanno, F. Guarnieri, *Universality and Symmetry Breaking in Conformally Reduced Quantum Gravity*, arxiv: 1206.6531, published on Physical Review D, DOI: 10.1103/PhysRevD.86.105027.

We will study in the **fourth chapter** a generic Brans-Dicke theory in the local potential approximation within the context of the functional renormalization group. We will derive a renormalization group equation for the Brans-Dicke potential keeping the parameter ω arbitrary and using two different gauge choices. We will search then for fixed point solutions fixing $\omega = 0$, in light of the equivalence with a $f(R)$ theory. Hence we will compare the results obtained in the two gauges. This chapter is based on the publication D. Benedetti, F. Guarnieri, *Brans-Dicke theory in the local potential approximation*, arxiv: 1311.1081, accepted for publication on New Journal of Physics.

We will investigate in the **fifth chapter** the asymptotic freedom of the conformal reduction of projectable Hořava-Lifshitz quantum gravity in 2+1 dimensions without detailed balance. We present the evaluation of the one loop correction to the bare action and solve the β-functions of the couplings. We will then investigate the high energy behaviour of such a toy model and discuss their implications for the full theory. This chapter is based on the publication D. Benedetti, F. Guarnieri, *One-loop renormalization in a toy model of Horava-Lifshitz gravity*, arxiv: 1311.6253, published on Journal of High Energy Physics, 10.1007/JHEP03(2014)078.

Chapter 1

Basics of renormalization group

In physics the notion of *universality* is referred to the characteristic of different theories to posses the same power law behaviour of the correlation functions in the long range regime (despite of the microscopic details of the theory). If theories with different microscopic interactions posses the same set of power law behaviours (i.e. the critical exponents) it is said that they belong to the same *universality class*. One historical example of universality class is given by the set of power law behaviours of the thermodynamical quantities in second order phase transitions, like the paramagnetic-ferromagnetic transition.

Finding applications in different contexts, the concept of universality assumes in modern physics various facets. While on the one hand it is applied to low energy physics (e.g. critical phenomena and condensed matter) to classify all possible second order phase transitions, it is, on the other hand, employed in high energy physics to take the continuous limit of theories on a lattice.

This diversification in the application of the same concept is due to the historical parallel (but detached) understanding, in both statistical mechanics and particle physics, of the relevance of scale invariance in field theory. On the one hand, Feynman, Dyson, Tomonaga and Schwinger where developing in the 40s perturbative techniques to regularize and renormalize the divergencies appearing in the Feynman diagrams of quantum electrodynamics (QED). Applying those techniques they noted that the observable parameters of the theory where changing with the energy scale. That is, scale invariance is generally not a property of a quantum theory. On the other hand, the study of those perturbative techniques led Wilson, later in the 70s, to grasp that the key to understand the universal behaviour in phase transitions was given by a non-perturbative scale analysis of the partition function. He hence discovered that universality is a property of the linearized system around the scale invariant theory.

Besides the fact that both the perturbative and non-perturbative techniques developed so far find still nowadays disjoined applications, their ensemble goes nonetheless under the same name, the renormalization group (RG), from the landmark paper of Stueckelberg and Petermann [13].

1.1 Gell-Mann Low equation

The idea behind the perturbative renormalization group in particle physics is to express the n-point correlation functions (which are functions of the physical, i.e observed, parameters) by renormalizing the divergencies appearing when expanding the correlation functions in powers of a bare perturbative parameter. The divergencies are regularized by means of a regulator[1], which is removed after the renormalization. The n-point correlation functions in Fourier space can then be defined in terms of the bare one as

$$G_R^{(n)}(p_1, \cdots, p_n; g_R, \mu) = \left(\sqrt{Z}\right)^{\frac{n}{2}} G_B^{(n)}(p_1, \cdots, p_n; g_B, \Lambda)\,, \tag{1.1}$$

where $p_1, \cdots, p_n$ are external momenta, g_R and g_B are, respectively, the sets of renormalized and bare couplings, μ is a renormalization (observational) scale, Λ is an ultraviolet cutoff and Z a renormalization function. The masses are here treated as coupling constants.

The perturbative renormalization group equation can be built stating the independence of the n-point renormalized correlation function from the regulator, or even the independence of the bare function from the renormalization scale. Following the latter prescription leads to a renormalization group flow equation which reads

$$\mu\,\partial_\mu\, G_B^{(n)} = \mu\,\partial_\mu\, \left(Z(g_R, \mu)^{\frac{n}{2}}\, G_R^{(n)} \right) = 0\,. \tag{1.2}$$

Equation (1.2) can be recasted in terms of the flow equation for the i-th bare coupling $g_{B,i}$, called β-function, by defining

$$g_{B,i}(\Lambda) = g_{R,i}(\mu)\ \mathcal{Z}_i(g_b, \mu, \Lambda)\,, \tag{1.3}$$

where the calligraphic $\mathcal{Z}_i$ is now a renormalization factor which can be constructed by summing counterterms coming from loop corrections, i.e.

$$\mathcal{Z}_i(g_B, \mu, \Lambda) = \left(1 + \sum_{k=1}^{\infty} f_k(g_B, \Lambda, \mu) \right)\,. \tag{1.4}$$

The functions f_k in (1.4) contains polynomial or at least logarithmic divergencies in Λ. A renormalization group equation can be achieved from equation (1.3) by assuming that the bare coupling does not depend on the renormalization scale μ, obtaining

$$\begin{aligned} \mu\,\partial_\mu\, g_{B,i}(\Lambda) &= 0 \\ &= \mu\,\partial_\mu\,(g_{R,i}(\mu)\,\mathcal{Z}_i(g_B, \mu, \Lambda)) \\ &= (\mu\,\partial_\mu\, g_{R,i}(\mu))\,\mathcal{Z}_i(g_B, \mu, \Lambda) + (\mu\,\partial_\mu\,\mathcal{Z}_i(g_B, \mu, \Lambda))\, g_{R,i}(\mu)\,, \end{aligned} \tag{1.5}$$

[1] We will employ here a sharp momentum cutoff for the sake of simplicity. No other regularization scheme is introduced (Pauli-Villars, ζ-function, etc.), except for the dimensional regularization, treated in the subsection 2.2.1.

which, introducing the so-called anomalous dimension

$$\eta = -\frac{\mu\, \partial_\mu\, \mathcal{Z}}{\mathcal{Z}}\,, \tag{1.6}$$

and dividing both members of (1.5) by $\mathcal{Z}$, leads to the elegant flow equation

$$\beta_{g_i} = \mu\, \partial_\mu\, g_{R,i} = \eta\, g_{R,i}\,, \tag{1.7}$$

where the bare couplings in $\mathcal{Z}$ have been rewritten in terms of the renormalized one, and which takes the name of Gell-Mann Low equation [14]. Note that the β-function in (1.7) holds for a dimensionless coupling, for which the running is controlled by the sole anomalous dimension η. In the case of a dimensionful parameter of classical mass dimension d_g the flow is, instead, dominated in the perturbative regime by a term proportional to its classical dimension[2].

Since the β-function (1.7) does not depend explicitly on the regulator anymore it is then possible to remove it, i.e. take the limit $\Lambda \to \infty$, and solve the differential equation for the i-th renormalized coupling $g_{R,i}$, thus obtaining the running of the renormalized coupling varying with the observational scale μ.

Typically, the β-function for a dimensionless coupling can assume three generic behaviours, depicted in Fig. 1.1, depending on the sign of loop corrections and the non-linear dependence on the coupling of the anomalous dimension η.

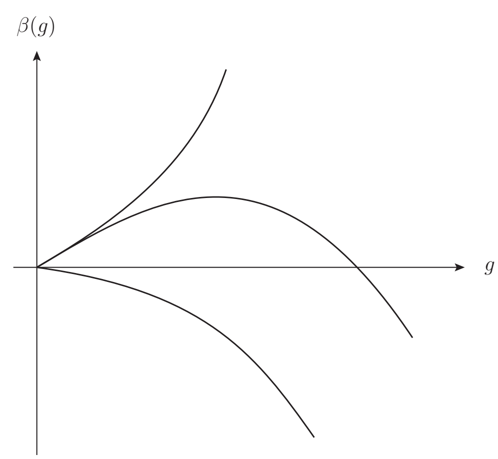

Figure 1.1: Three general behaviours for the beta function of a coupling.

In particular, in a perturbative regime ($g \ll 1$) the sign of η is defined by that of the one loop correction, being it the dominant contribution. Consequently, for $\eta > 0$ the

[2] The renormalization of a dimensionful coupling will be treated more in detail in chapter 2

β-function is positive (upper curve in Fig. 1.1) and the coupling strength increases at higher energies (as is the case of QED and the scalar field theory) reaching, however, a Landau pole at a certain energy scale $\mu \sim \mu^*$, thus being the perturbative approximation not reliable anymore. For a decreasing β-function (lower curve in Fig. 1.1) the interaction runs to zero, leading in the UV to a non-interacting theory. In this case, for example in non-abelian SU(N) gauge QFTs, the theory is perturbatively renormalizable and said to be asymptotically free.

The remaining case depicted in Fig. 1.1, that is a β-function with a non-trivial zero, emerges whenever subsists a competition between loop orders. Apart from particular cases in which the zero is defined for a small coupling, so that perturbation theory can be applied (e.g. the Caswell-Banks-Zaks fixed point for SU(N) theory in the $N \to \infty$ limit [15, 16]), non-trivial fixed points often live in a non-perturbative regime which cannot be studied with the RG scheme so far defined, which takes the name of minimal substraction (MS) scheme[3]. The flow of the above scheme is, moreover, not accurate for theories with IR instabilities (i.e. UV massless theories) for which arises an ulterior constraint $m^2_{dyn} \ll p^2$, being m^2_{dyn} a mass dynamically generated by loop corrections, since the presence of a mass influences consistently the flow already in the intermediate regime $p^2 \sim m^2_{dyn}$.

In the case of a non-trivial fixed point non-perturbative techniques need to be employed, since they take in consideration the contribution of operators that are irrelevant in the perturbative regime but can become relevant at the non-trivial fixed point.

1.2 Functional renormalization group

The philosophy behind the development of non-perturbative RG techniques is based on the renowned study of blocking transformations on a lattice which led to the Wilson's renormalization group [17]. What Wilson noted was that whenever we perform a coarse graining of a theory on a lattice, i.e. we average the field on blocks of a finite size, the effective low energy action for the averaged field contains the original operators but for renormalized couplings, as well as new irrelevant (and relevant) operators generated by the coarse graining procedure. The reiteration of the blocking transformation leads then to a non-perturbative flow in which are generated (and run) all the operators compatible with the symmetries of the action. Following the above idea, a consistent strategy to follow in order to construct a non-perturbative RG scheme is to translate for continuos fields the coarse graining procedure defined on a lattice.

To do this, we start by considering the definition of an averaged action on a lattice of step Λ^{-1} for a block size of length k^{-1}

$$e^{-S_k[\Psi]} = \int D[\psi] \prod_x \delta\big(\psi_k - \Psi\big)\, e^{-S_\Lambda[\psi]}\,, \tag{1.8}$$

[3] Where *minimal* entails that in (1.4) has not been subtracted the finite part of loop corrections.

where $\psi(x)$ is a generic field (or superfield), $\Psi(x)$ an effective field at scale k, S_Λ and S_k respectively the bare action at lattice scale Λ and an effective action at scale k, and $\psi_k(x)$ is an averaged field on the lattice given by

$$\psi_k(x) = \sum_y \psi(y)\, \rho_k(y,x)\,, \tag{1.9}$$

being $\rho_k(x,y)$ an arbitrary smearing function. The generalization of the definition of average field on continuos spacetime is obtained by simply substituting the sum over lattice sites with an integral

$$\psi_k(x) = \int d^d y\, \psi(y)\, \rho_k(y,x)\,, \tag{1.10}$$

where the smearing function satisfies the properties of being

- i) symmetric: $\rho_k(x,y) \equiv \rho_k(y,x)$,
- ii) normalized: $\int d^d y\, \rho_k(y,x) = 1$,
- iii) idempotent: $\int d^d y\, \psi_k(y)\, \rho_k(y,x) = \psi_k(x)$.

The last relation simply implies that the average of an average field is again an average field. The smearing function also satisfies a composition rule (here in Fourier space)

$$\rho_{k_1}(q)\, \rho_{k_2}(q) = \rho_k(q)\,, \tag{1.11}$$

where $k \equiv k(k_1,k_2) < \min(k_1,k_2)$. The effective action defined in (1.8) contains however a product over lattice sites $x \in \mathbb{R}^d$ which makes sense only on a lattice because of the presence of the δ-functions. A continuos version [18] can then be defined by substituting the product of deltas with a constraint operator $P_k[\psi_k,\Psi]$, so that (1.8) now reads

$$e^{-S_k[\Psi]} = \int D[\psi]\, P_k[\psi,\Psi]\, e^{-S_\Lambda[\psi]}\,. \tag{1.12}$$

The constraint operator should be defined in such a way to satisfy a composition rule at different k, that reads

$$\int D[\tilde\psi]\, P_{k_1}[\psi,\tilde\psi]\, P_{k_2}[\tilde\psi,\Psi] = P_k[\psi,\Psi]\,, \tag{1.13}$$

where k comes from (1.11), and to be renormalized, i.e.

$$\int D[\Psi]\, P_k[\psi,\Psi] = 1\,, \qquad \frac{1}{\xi}\int_{-\infty}^{\infty} D[\Psi_n]\, P_{k,n}[\psi_n,\Psi_n] = 1\,, \tag{1.14}$$

being ξ a renormalization factor and $P_k[\psi,\Psi] = \prod_n P_{k,n}[\psi_n,\Psi_n]$, with n the Fourier number of the n-th mode. Since the bare field ψ is univocally defined by the field

ψ_k (up to a q^2-dependent wavefunction renormalization) by means of the averaging procedure (1.9), in the case the operator P_k acts as an identity between ψ_k and Ψ we would then end up with a simple variable change and no coarse graining would be actually performed. We ask then the two fields to differ for a certain quantity. A choice is to employ a Gaussian operator such that in the $q \ll k$ regime, being q the Fourier mode of the bare field, it behaves like

$$P_k \approx C\, e^{-\nu \int d^d x\, |\psi_k(x)-\Psi(x)|^2}\,, \tag{1.15}$$

being ν the mean deviation between $\psi_k(x)$ and $\Psi(x)$ and $C \equiv C(\xi, \nu, \Omega)$ a constant which depends on the renormalization ξ, the mean deviation ν and the volume Ω of the system. The Wilsonian action S_k can then be considered as the expectation value of the constraint operator

$$S_k[\Psi] = -\log\left(\langle\, P_k \,\rangle\, Z\right), \tag{1.16}$$

or it can otherwise be considered as the partition function of a non-local constrained action

$$S_{const}[\psi;\Psi] = S_\Lambda[\psi] - \log P_k[\psi_k,\Psi]\,, \tag{1.17}$$

defined in terms of a background field Ψ, and well defined for all the modes of the field since the constraint operator P_k is strictly positive for every q.

Although the latter is, however, obviously difficult to implement in parameter space it makes feasible the application of the steepest-descend method to the calculation of S_k. For a correct evaluation of the Wilsonian action it is in fact necessary to be sure that we are integrating the fluctuation around the correct ground state of the theory. This is a well-known problem for theories in the broken phase, since loop contributions are not well defined in the inner zone of a double-well bare potential, being the logarithm not well defined for negative values (the inner zone of the potential).

An example is given by the Coleman-Weinberg potential [19] in the broken phase for an N-component scalar field theory [20], which is known to be not well defined in that region[4]. The problem is solved by computing the average potential at a certain scale k and noticing that in the inner zone the action is minimized by a spin-wave solution (i.e. a field configuration with a preferred momentum direction) instead of an homogenous field, and the integration of the fluctuations around that non-trivial ground state leads to a globally analytic average potential. As a consequence of the coarse graining, however, the Poincaré symmetry is broken down to the Lorentz symmetry and restored just in the limit $k \to 0^+$. The loss of symmetries, both extern and/or internal, is a common drawback of the implementation of the coarse graining. As we will see for the in the section (1.2.2) for the so-called exact renormalization group equation (ERGe) the explicit introduction of an infrared cutoff in momentum space breaks the gauge

[4] Since we assume to be in the broken phase the second derivative of the potential in the inner region $\phi_{inn} \approx 0$ takes negative values. The one loop correction, however, contains a logarithm of the second derivative, which is not defined for negative values. Therefore the one loop correction in the inner region is ill-defined.

invariance of the system and modified Ward identities are required in order to ensure BRST symmetry at the level of the effective action at $k = 0$.

Going back to definition (1.16), the Wilsonian action so far defined is not the functional generator of 1PI connected correlation functions we are interested in, so that a flow equation constructed from (1.16) would not furnish a connection with the perturbative RG schemes. To construct a functional generator from the action S_k we can start by noting that for a generic observable $O(x)$ holds [18]

$$O[\psi] = \int D[\Psi]\, O_k[\Psi] P_k[\psi, \Psi]\,, \tag{1.18}$$

that is, $O[\psi]$ and $O_k[\Psi]$ have the same expectation value. This translates in terms of the fields Ψ and ψ_k as

$$\begin{aligned} \langle\, \Psi \,\rangle = \int D[\Psi(x)]\, e^{-S_k[\Psi(x)]}\, \Psi(x) = \\ \int D[\Psi(x)]\, D[\psi(x)]\, P_k\left[\psi_k(x), \Psi(x)\right]\; e^{-S_\Lambda[\psi(x)]}\, \Psi = \\ \int D[\psi(x)]\, e^{-S_\Lambda[\psi(x)]}\, \psi_k(x) = \langle\, \psi_k(x) \,\rangle\,, \end{aligned} \tag{1.19}$$

where now the expectation value $\langle\, \Psi \,\rangle$ is defined by means of a functional derivative of the generator of connected one particle correlation functions W_k as

$$\langle\, \Psi \,\rangle = \frac{\partial W_k[J]}{\partial J(x)}\bigg|_{J=0}\,, \tag{1.20}$$

being $J(x)$ an auxiliary external source, so that in the end

$$W_k[J] = \int d^d x\, \{J(x)\, \Psi(x) - S_k[\Psi]\}\,. \tag{1.21}$$

The correlation functions at the observed scale k can then be evaluated by means of functional derivatives respect to the external sources. Finally, to obtain the β-function of the couplings we need to invert the source as a function of the expectation value $\langle\, \Psi \,\rangle \equiv \tilde{\Psi}$, that is $J \equiv J(x, \tilde{\Psi}(x))$, and perform a Legendre transform of the functional W_k, obtaining

$$\Gamma_k[\tilde{\Psi}] = -W_k[J] + \int d^d x\, \tilde{\Psi}(x)\, J(x, \tilde{\Psi}(x))\,. \tag{1.22}$$

Expanding the action Γ_k in a basis of local operators $O_i(x)$ built from the field $\tilde{\Psi}$ and its derivatives, the β-function for the dimensionful i-th running coupling $g_{i,k}$ is obtained as

$$\beta_{g_i} = k\, \partial_k\, g_{i,k} = k\, \partial_k\, \frac{\delta\, \Gamma_k}{\delta\, O_i}\bigg|_{\tilde{\Psi}=0}\,. \tag{1.23}$$

The comparison between the perturbative and non-perturbative schemes until now described (that is, equations (1.7) and (1.23)) is however not so straightforward. While the β-functions in the MS scheme show a simple dependence on the relevant parameters, the contributions to the running of the coupling in the non-perturbative approach come from all the operators present in the truncation of the effective action, and an infinite number of operators are required[5]. The β-functions in the fRG schemes, $\beta_{g,fRG}(k)$, and the β-function in the MS, $\beta_{g,MS}(\mu)$, have then to be compared around $k \approx \mu$ where it holds

$$\beta_{fRG}(k \approx \mu) = \beta_{MS}(k) + \mathcal{M}\,(g_j, \mu)\; g^2_{MS}(\mu) + \mathcal{O}(g^3_{MS})\,, \tag{1.24}$$

where $\mathcal{M}\,(g_j, \mu)$ is some function depending on the other coupling and the renormalization scale μ.

Moreover, differently from the perturbative schemes in which we start from finite-valued bare correlation functions and construct the renormalized functions by regularizing the divergent contributions coming from loop corrections, in this scheme the correlation functions at $k = 0$ are assumed to be finite functions of the "dressed" couplings by definition[6], and no divergences explicitly appear during the flow (as, after all, also in the Wilson's RG). The absence of ultraviolet singularities can then be re-casted in the statement that the flow equation, starting from $k = 0$, automatically flows in the ultraviolet limit $k \to \infty$ into an ultraviolet attractive fixed point of the renormalization group equation. This fixed point, however, does not identifies the bare action, which is indeed defined at a ultraviolet scale Λ, since there is no scale definition for a scale invariant action.

The construction in parameter space until now proposed is mostly explicative, and instrumental to the introduction of the different schemes, since it evidently does not lead to any scheme directly applicable. Practical schemes can be constructed in momentum space by a proper definition of a smearing function. We will introduce in the next sections the schemes most used in literature, starting with the sharp cutoff scheme.

1.2.1 Sharp cutoff equation

A sharp cutoff flow equation can be constructed by employing in the definition of the average field (1.9) a sharp smearing function in momentum space [23], i.e. by considering

$$\psi_k(x) = \int \frac{d^d q}{(2\pi)^d}\, e^{i\, q_\mu\, x^\mu}\, \rho_k(q)\, \psi(q)\,, \tag{1.25}$$

being $\rho_k(q)$ a sharp cutoff function

$$\rho_k(q) = \Theta(k - q)\,. \tag{1.26}$$

[5] For a more detailed comparison between RG schemes see [21, 22].

[6] In a non-perturbative RG context the effective action (i.e. the generator of renormalized Green functions) is a point in theory space that is finite-valued by definition for physical theories. Consequently, also the Green functions are finite functions of the infrared couplings.

As a consequence of the use of such a cutoff the original field gets decomposed in fast and slow modes

$$\psi(x) = \bar{\psi}(x) + \xi(x)\,, \tag{1.27}$$

where the field $\bar{\psi}$ belongs to the space $\mathcal{F}_k$ of functionals with non-vanishing Fourier modes $0 < p < k$, and ξ belongs to the space $\mathcal{F}_\Lambda/\mathcal{F}_k$, i.e. contains just the fast modes $k < p < \Lambda$. The evaluation of the Wilsonian action $S_k[\bar{\psi}]$ assumes in this case a rather simple expression, since we now need to integrate out only the fast modes (that is, integrating out just ξ), which reads

$$e^{-S_k[\bar{\psi}]} = \int D[\xi]\, e^{-S_\Lambda[\bar{\psi}+\xi]}\,. \tag{1.28}$$

A renormalization group equation can be constructed by reiterating the field decomposition (1.27) and integrating out only an infinitesimal momentum shell δk of the modes of $\bar{\psi}$, obtaining

$$e^{-S_{k-\delta k}[\bar{\psi}']} = \int D[\zeta]\, e^{-S_k[\bar{\phi}'+\zeta]}, \tag{1.29}$$

where we used the decomposition $\bar{\psi}(x) = \bar{\psi}'(x) + \zeta(x)$, being $\bar{\psi}'(x) \in \mathcal{F}_{k-\Delta k}$ and $\zeta(x) \in \mathcal{F}_k/\mathcal{F}_{k-\Delta k}$. Since the flow equation is given by the derivative of the Wilsonian action respect to the blocking scale k, our aim is then to evaluate the limit $\delta k \to 0$ of (1.29). Before we do that we expand the right hand term of (1.29) in a Taylor series around $\bar{\psi}'$, retaining up to the quadratic (one loop) contribution

$$e^{-S_{k-\delta k}[\bar{\psi}']} = \int D[\zeta]\, e^{-\left\{S_k[\bar{\psi}']+\int d^dx\, S_k^{(1)}[\bar{\psi}']\,\zeta(x)+\frac{1}{2}\int d^dy\,\zeta(y)\, S_k^{(2)}[\bar{\psi}']\,\zeta(x)+\mathcal{O}(\zeta^4)\right\}}\,, \tag{1.30}$$

where $S_k^{(2)}[\bar{\psi}']$ is the second functional derivative of the blocked action, id est

$$S_k^{(1)}[\bar{\psi}'] = \left(\frac{\delta\, S_k[\bar{\psi}'+\zeta]}{\delta\zeta(x)}\right)_{\zeta=0}, \qquad S_k^{(2)}[\bar{\psi}'] = \left(\frac{\delta^2\, S_k[\bar{\psi}'+\zeta]}{\delta\zeta(x)\delta\zeta(y)}\right)_{\zeta=0}. \tag{1.31}$$

Inasmuch as the one loop term is quadratic in the fields it can then be evaluated by means of a Gaussian integral. Taking the logarithm of both sides, equation (1.32) reads

$$S_{k-\delta k}[\bar{\psi}'] = S_k[\bar{\psi}'] + \frac{1}{2}\,\mathrm{STr'}\,(\log(S_k^{(2)}[\bar{\psi}'])) + \frac{1}{2}\int' dy\, S_k^{(1)}(S_k^{(2)})^{-1}S_k^{(1)} + ..., \tag{1.32}$$

where we have used the relation $\log(\det(S^{(2)})) = \mathrm{STr}(\log(S^{(2)}))$, being STr a functional supertrace over the field content of ψ and where the prime means that both the trace and the integral are performed in Fourier space integrating only the modes between k and $k - \delta k$. Taking now the limit $\delta k \to 0$ it can be seen that appears a new term $\delta k/k$ which plays the role of a small dimensionless parameter. In particular, since every loop integration gives a volume term δk, every higher n-loop contribution will be suppressed (when taking the limit) by a factor $(\delta k/k)^{n-1}$, leading then to an exact one loop renormalization group equation

$$\begin{aligned} k\,\partial_k\,S_k[\bar{\psi}'] = -k\lim_{\delta k\to 0}(S_{k-\delta k}[\bar{\psi}'] - S_k[\bar{\psi}']) = \\ -\lim_{\delta k\to 0}\frac{k}{\delta k}\left\{\frac{1}{2}\mathrm{STr'}(\log(S_k^{(2)}[\bar{\psi}'])) + \frac{1}{2}\int' d^d y\, S_k^{(1)}\,(S_k^{(2)})^{-1}\,S_k^{(1)}\right\}, \end{aligned} \tag{1.33}$$

which takes the name of Wegner-Houghton equation [24].

Contrariwise to the perturbative scheme (1.7) which furnishes a set of coupled ordinary differential equations (ODE), the flow equation (1.33) is a partial differential equation (PDE) which is intended to be solved for the whole Wilsonian action S_k, i.e. is a functional renormalization group equation (fRGe).

As any other fRG equation, however, the scheme (1.33) consists in an integro-differential equation which cannot be solved exactly but only through the use of truncations of the coarse grained action in the parameter space. The truncation, however, can highly affect the calculation of universal quantities whenever we neglect the contributions of the relevant operators in the ultraviolet regime. Of course the choice of the truncation depends on the model under investigation and, for example, if the theory is at or out equilibrium. In problems of equilibrium critical phenomena (which is one of historical targets of application of the non-perturbative renormalization group) the dependence of the vertices from the external momentum can be neglected and it can be considered as a good approximation of the full exact solution to solve the flow equation employing an homogeneous field configuration, that it to solve it for the sole potential. The latter is the leading order approximation, the so-called *local potential approximation* (LPA), of a derivative expansion of the average action, i.e. an expansion in powers of derivatives of the field.

Considering, for example, a scalar field theory with Z_2 symmetry, which belongs to the same universality class of the Ising model, the leading order of the derivative expansion is defined by the action

$$S_k[\phi] = \int d^d x\left\{\frac{1}{2}\,\partial_\alpha\phi\,\partial^\alpha\phi + U_k(\phi)\right\}, \tag{1.34}$$

where $\phi(x)$ is a real scalar field. In (1.34) the radiative corrections to the wavefunction renormalization have been discarded and the only running function is the potential, and we used as convention squared parentheses for functionals and rounded for functions. In the next-to-leading order the running of the wavefunction renormalization is taken in consideration and is promoted to a running function of the field, i.e. $Z \equiv Z_k(\phi)$, so that we end up with a coupled system of partial differential equations. In the next order we consider the running of the field-dependent coupling of the operator $\partial^4\phi$, and so on.

The accuracy of such a hierarchy of truncations has shown a surprisingly precision in the evaluation of the critical exponents (in particular the anomalous dimensions of the operators ϕ and ϕ^2, that is the critical exponents η and ν) already at the leading order

(1.34) for the scalar action in $d = 3$. The sharp RG flow equation for such a theory can be obtained inserting (1.34) in (1.33). Then, choosing a constant field $\phi(x) = \Phi =$ const we note that only term which contributes to the running of the potential is the logarithmic one [25]. Hence, we obtain a flow equation for the sole potential

$$k\,\partial_k\,U_k(\Phi) = -\lim_{\delta k\to 0}\frac{k}{\delta k}\left\{\frac{1}{2}\mathrm{Tr}'(\log(p^2 + U_k^{(2)}(\Phi)))\right\}\,, \tag{1.35}$$

which, performing the integral and taking the limit leads to the flow equation

$$k\,\partial_k\,U_k(\Phi) = -\frac{\Omega_d}{2}k^d\log\left(k^2 + U_k^{(2)}(\Phi)\right)\,, \tag{1.36}$$

where $\Omega_d = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)}$ is a d-dimensional solid angle.

The solution of the RG equation (1.36), besides being the lower order approximation in the derivative expansion, leads to a qualitatively correct phase diagram and a precise numerical evaluation of the universality class of the Ising model [26, 27, 28]. Being an approximated solution it shows however a certain dependence from the shape of the cutoff function, affecting then the numerical value of the critical exponents. The latter will converge (with a certain speed, depending on the cutoff function chosen) to their exact values as we reduce the entity of the approximation.

The matter is that a reduction of the entity of the approximation leads often to cumbersome calculations which makes technically difficult to improve the precision in the calculation of the universal quantities. The sharp cutoff equation itself, despite its exactness and clear structure, does not show an elevated speed of convergency and moreover the use of a sharp cutoff often brings technical difficulties which makes it not favorable as a scheme for technical purposes. For example, equation (1.36) fails in the reproduction of the discontinuity in the second derivative of the effective potential in the broken phase (that is, integrating the dimensionful equation down to $k \to 0$) which characterizes the magnetic susceptibility at first order transitions belonging to the Ising universality class (as for example the gas-vapor transition) [29]. For those reasons has not been widely used in the study of physical systems.

The property of the exact solution to be independent on the choice of the cutoff function, however, translated in the freedom to employ a generic smooth cutoff function. On the one hand this grants us the possibility to select a cutoff function featured with a convergence rate higher than that of the sharp cutoff, allowing us to work with leading order approximations, while on the other one gives us the freedom to construct the cutoff in such a way that the computation of the functional traces is simplified. Those schemes where developed mostly in the '90 and the most common scheme nowadays used in literature is the so-called exact renormalization group (ERG).

1.2.2 Exact equation

The exact renormalization group equation (ERGe) [30, 31] is a functional RG scheme in which the coarse graining procedure is implemented by cutting the modes of the field by introducing a local cutoff operator $\Delta_k S$ in the partition function, i.e. we evaluate the Wilsonian action for a constrained action of the type

$$S_{const}[\psi] = S_\Lambda[\psi] - \Delta_k S[\psi]\,, \tag{1.37}$$

being $S_\Lambda[\psi]$ the bare action and $\Delta_k S[\psi]$ a cutoff operator quadratic in the field which reads

$$\Delta_k S = \frac{1}{2}\int d^d x\, \psi(x)\, R_k[-\Box]\, \psi(x) \;=\; \int \frac{d^d p}{(2\pi)^d}\, \psi(p)\, R_k(p^2)\, \psi(p)\,. \tag{1.38}$$

This procedure resembles the definition of the constrained action in (1.17), besides in this case we do not have a non local parameter space implementation of the coarse graining due to the introduction of a background field. The coarse graining is in fact here introduced in Fourier space by means of the substitution $p^2 \to p^2 + R_k(p^2/k^2)$ in the propagator, where the dimensionful cutoff function R_k reads

$$R_k(p^2) = k^2\, R^{(0)}\left(p^2/k^2\right)\,, \tag{1.39}$$

being $R^{(0)}$ an arbitrary dimensionless cutoff function. The integration of the only modes with eigenvalue $p^2 > k^2$ is ensured by choosing the cutoff function in such a way that interpolates between 0 and 1 respectively for $p^2/k^2 > 1$ and $p^2/k^2 < 1$, so that infrared modes are suppressed with a constant mass term k^2 and small wavelength modes are unaffected.

Starting with the action (1.37) it is then possible to define an average functional generator of n-point connected Green functions using the standard Schwinger formalism, id est

$$Z_k[J] = e^{W_k[J]} = \int D[\psi]\, e^{-S_\Lambda[\psi] - \Delta_k S[\psi] + \int d^d x\, J(x)\, \psi(x)}\,. \tag{1.40}$$

where J(x) is an auxiliary external source. The effective average action (1.22) (EAA) can then be obtained by performing a Legendre transform of the functional $W_k[J]$, by introducing an effective field

$$\Psi = \langle\, \psi\, \rangle_k = \frac{\delta W_k[J]}{\delta J(x)}\,, \tag{1.41}$$

and solving (1.41) for the source in terms of the effective field Ψ(x), that is $J \equiv J_k(x, \Psi(x))$, and then taking the transform

$$\Gamma_k[\Psi] + \Delta_k S[\Psi] = -W[J_k[x,\Psi]] + \int d^d x\, J_k[x,\Psi]\cdot \Psi(x)\,. \tag{1.42}$$

Note that now holds

$$\begin{aligned}\frac{\delta\Gamma_k[\Psi]}{\delta\Psi}+\frac{\delta\Delta_k S[\Psi]}{\delta\Psi}&=-\frac{\delta W[J_k]}{\delta\Psi}+\int d^dx\,\frac{\delta(J_k[x]\cdot\Psi(x))}{\delta\Psi}\\&=-\frac{\delta W[J_k]}{\delta J}\frac{\delta J}{\delta\Psi}+J(x)+\int d^dx\,\frac{\delta J_k[x]}{\delta\Psi}\Psi=J(x)\,.\end{aligned}\tag{1.43}$$

Because of the presence of the cutoff operator in the partition function the Legendre transform of the generator $W_k[J]$ is $\Gamma_k[\Psi]+\Delta_k S[\Psi]$, thus the expression of the effective average action Γ_k reads

$$\Gamma_k[\Psi]=-W[J_k[x,\Psi]]-\Delta_k S[\Psi]+\int d^dx\,J_k[x,\Psi]\cdot\Psi(x)\,.\tag{1.44}$$

Moreover, not being a Legendre transform, (1.44) needs not to be a convex functional of the field, as it also happens for the Wilsonian action.

In order to build a functional renormalization group equation for the effective average action (1.44) it is convenient to rewrite the action in its functional form

$$e^{-\Gamma_k[\Psi]}=\int D[\chi]\,e^{-S[\Psi+\chi]-\Delta_k S[\Psi+\chi]+\Delta_k S[\Psi]+\int d^dx\left(\frac{\delta\Gamma_k[\Psi]}{\delta\Psi}+\frac{\delta\Delta_k S[\Psi]}{\delta\Psi}\right)\chi}\,,\,\langle\chi\rangle=0\,,\tag{1.45}$$

where we used (1.43) to rewrite the source, in combination with the procedure presented in Appendix A for the standard effective action. Using the feature of the operator $\Delta_k S$ to be quadratic in the field we can expand it in powers of the field around $\psi=\Psi$, obtaining

$$\begin{aligned}&-\Delta_k S[\Psi+\chi]+\Delta_k S[\Psi]+\int d^dx\frac{\delta\Delta_k S[\Psi]}{\delta\Psi}\chi\\=&\left(-\Delta_k S[\Psi]-\frac{\delta\Delta_k S[\Psi]}{\delta\Psi}\chi-\frac{\delta^2\Delta_k S[\Psi]}{\delta\Psi\,\delta\Psi}\chi^2\right)+\Delta_k S[\Psi]+\int d^dx\frac{\delta\Delta_k S[\Psi]}{\delta\Psi}\chi\\&=-\frac{1}{2}\frac{\delta^2\Delta_k S[\Psi]}{\delta\Psi\,\delta\Psi}\chi^2=-\Delta_k S[\chi]\,,\end{aligned}\tag{1.46}$$

so that (1.45) is rewritten as

$$e^{-\Gamma_k[\Psi]}=\int D[\chi]\,e^{-S[\Psi+\chi]-\Delta_k S[\chi]+\int d^dx\,\frac{\delta\Gamma_k[\Psi]}{\delta\Psi}\chi}\,,\,\langle\chi\rangle=0\,,\tag{1.47}$$

Taking the derivative of (1.47) respect to the RG time $t=\log(k/k_0)$ we obtain

$$\begin{aligned}&\partial_t\,e^{-\Gamma_k[\Psi]}=-(\partial_t\,\Gamma_k[\Psi])\,e^{-\Gamma_k[\Psi]}\\=&\int D[\chi]\,\left(-\partial_t\,\Delta_k S[\chi]+\int d^dx\,\partial_t\,\frac{\delta\Gamma_k[\Psi]}{\delta\Psi}\chi\right)\,e^{-S[\Psi+\chi]-\Delta_k S[\chi]+\int d^dx\,\frac{\delta\Gamma_k[\Psi]}{\delta\Psi}\chi}\,,\end{aligned}\tag{1.48}$$

which, using (1.40), can be written in terms of expectation values as

$$\partial_t\,\Gamma_k[\Psi]=\langle\partial_t\Delta_k S[\chi]\rangle=\frac{1}{2}\int d^dx\sqrt{g}\,(\partial_t R_{k,AB})\,\langle\chi_A\chi_B\rangle\,,\tag{1.49}$$

where in (1.49) we used the fact that $\langle \chi \rangle = 0$. Taking now the functional derivative of (1.41) and (1.43) respectively to J and Ψ we can now note that the twice derivated generator $W_k[J]$ is the inverse of the second functional derivative of its Legendre transform $\Gamma_k[\Psi] + \Delta_k S[\Psi]$, that is

$$\left(\frac{\delta^2 W_k[J]}{\delta J \, \delta J} \right) = \left(\frac{\delta^2 (\Gamma_k[\Psi] + \Delta_k S[\Psi])}{\delta \Psi \, \delta \Psi} \right)^{-1} , \tag{1.50}$$

so that in the end the two point correlation function of the field χ can be written as

$$\langle \chi_A \chi_B \rangle = \left(\frac{\delta^2 (\Gamma_k[\Psi] + \Delta_k S[\Psi])}{\delta \Psi \, \delta \Psi} \right)^{-1} = \left(\frac{\delta^2 \Gamma_k[\Psi]}{\delta \Psi \, \delta \Psi} + R_k \right)^{-1} , \tag{1.51}$$

which leads to the exact equation

$$\partial_t \, \Gamma_k[\Psi] = \frac{1}{2} \, \mathrm{STr} \left[(\partial_t \, R_{k,AB}) \left(\frac{\delta^2 \Gamma_k[\Psi]}{\delta \Psi_A \, \delta \Psi_B} + R_{k,AB} \right)^{-1} \right] . \tag{1.52}$$

The above equation, like also the Wegner-Houghton equation (1.33), is said to be exact in the sense that no approximations have been used in its derivation. Being exact, it reproduces the effective action in the limit $k \to 0^+$, i.e. that in the infrared regime all the fluctuation are correctly integrated out, besides their structure resembles that of a simple one loop correction. Driven by this one loop structure a diagrammatic representation of equation (1.52) is presented in Fig. 1.2.

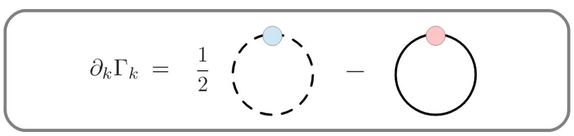

Figure 1.2: Diagrammatic representation of the exact renormalization group equation. The two loops characterize the bosonic (dashed) and fermionic (solid) content of the field ψ, with a - 2 term for coming from the trace over complex Grassmannian variables. The blue and red dots are respectively the cutoff insertions for bosonic and fermionic operators.

By means of field derivatives, equation (1.52) can be recasted in terms of a infinite hierarchy of flow equations for the n-point correlation functions, of which the first terms are

$$\begin{aligned} k \, \partial_k \, \Gamma_k &= \frac{1}{2} \, \mathrm{STr} \Big[(k \, \partial_k \, R_k) \, \mathcal{G}_k \Big] , \\ k \, \partial_k \, \Gamma^{(1)}_{k,x} &= -\frac{1}{2} \, \mathrm{STr} \Big[(k \, \partial_k \, R_k) \, \mathcal{G}_k \, \Gamma^{(3)}_k \, \mathcal{G}_k \Big] , \\ k \, \partial_k \, \Gamma^{(2)}_{k,y} &= \mathrm{STr} \Big[(k \, \partial_k \, R_k) \, \mathcal{G}_k \, \Gamma^{(3)}_k \, \mathcal{G}_k \, \Gamma^{(3)}_k \, \mathcal{G}_k \Big] - \frac{1}{2} \, \mathrm{STr} \Big[(k \, \partial_k \, R_k) \, \mathcal{G}_k \, \Gamma^{(4)}_k \, \mathcal{G}_k \Big] , \end{aligned} \tag{1.53}$$

where $\mathcal{G}_k$ is the modified inverse propagator

$$\mathcal{G}_k = (\Gamma_k^{(2)} + R_k)^{-1} \,. \tag{1.54}$$

The solution for the two point function permits to solve the flow equation for generic external momenta, and results to be a more appropriate framework in the applications to problems in dynamical critical phenomena. The hierarchy of n-point flow equations needs, however, to be truncated by means of some approximation in the momentum dependence of the higher order vertices $\Gamma_k^{(3)}$ and $\Gamma_k^{(4)}$ [32, 33, 34]. An other interesting application of the two point flow equation comes in the so-called LPA' approximation [35], that is an optimization of the standard local potential approximation in which the radiative correction to the renormalization function are taken in consideration in the proper vertices, without requiring then to promote the renormalization function to a function of the field how it happens in the next to leading order of the derivative expansion.

There is a considerable freedom in the choice of the cutoff function. The most used in literature are the mass-like, the exponential and the so-called optimized [36], which read respectively

$$\begin{aligned} R_k(z)^{mass} &= k^2 \,, \\ R_k(z)^{exp} &= \frac{z}{e^{\frac{z}{k^2}} - 1} \,, \\ R_k(z)^{opt} &= (k^2 - z^2)\,\Theta(k^2 - z^2) \,, \end{aligned} \tag{1.55}$$

being z the eigenvalue of the operator we want to cut the modes and where the Θ is the Heaviside function. Since the β-functions are solved most of the time using numerical techniques, it is convenient to employ a cutoff function which does not lead to a complicated expression of the flow equation. The optimized cutoff (that is the one we will use in chapter (4)) has been widely used for his well-know feature to simplify considerably the functional trace. Since the cutoff operator is defined in terms of a step function the derivative with respect to the RG scale returns

$$k\,\partial_k\, k^2 R_k(z)^{opt} = 2\,k^2\,\Theta\left(k^2 - z^2\right) + 2\,\frac{z}{k^2}\,(k^2 - z^2)\,\delta\left(k^2 - z^2\right)\,, \tag{1.56}$$

which reduces to the sole Heaviside function using the property of the distributional product of the delta function with its argument to be zero. Because of the step function, the trace reduces to a momentum integral between 0 and k, thus automatically rendering the functional traces UV finite.

The ERGe has been applied with success to many branches in modern physics, among which gauge theories [37], fermionic systems [38], superconductivity [39], statistical mechanics [40] and quantum gravity, as we will see in the next chapter. Aside of the broad application of the ERGe, the freedom of the cutoff function to do not lend to this scheme, anyway, a considerable speed of convergency of the universal quantities to their exact counterpart.

The last equation that we propose in this work is a renormalization group scheme in which the coarse graining is not implemented by means of a cut of modes of the field in Fourier space but by regulating the proper time integral of the heat kernel operator associated to the inverse propagator of the theory. Such a scheme has shown in literature the highest speed convergency, although it is not an exact scheme.

1.2.3 Proper time equation

The idea behind the construction of a proper time renormalization group flow consists in the implementation of the coarse graining procedure not as a cut of modes in Fourier space (that is, by regularizing IR divergencies by means of a momentum cutoff) but at the level of proper time representation of the heat kernel of the inverse propagator, regularizing then the proper time integral (the heat kernel itself is divergences-free operator).

A RG scheme can be built from both the momentum space schemes introduced in this chapter, namely the Wegner-Houghton equation (1.33) and the ERG equation (1.52), but for for the sake of discussion we will construct it from the proper time representation of the sharp cutoff equation and propose a comparison with proper time ERG schemes afterward.

We start by introducing the notion of heat kernel operator associated to a generic differential operator $\mathcal{A}$ as the operator $\mathcal{H}$ such that

$$\mathcal{H}(x, x', s; \mathcal{A}) = \langle\, x \,|\, e^{-s\,\mathcal{A}} \,|\, x' \,\rangle\,, \tag{1.57}$$

where s is a proper time variable and $\mathcal{H}$ satisfies the generalized heat diffusion equation

$$(\partial_s + \mathcal{A}_x)\,\mathcal{H}(x, x', s; \mathcal{A}) = 0\,, \tag{1.58}$$

with boundary condition

$$lim_{s\to 0^+}\,\mathcal{H}(x, x', s; \mathcal{A}) = \delta^d(x - x')\,, \tag{1.59}$$

being d the dimension of the space (statistical mechanics) or spacetime (quantum field theory). The heat kernel comes in help since we can rewrite functions of the operator $\mathcal{A}$, i.e. powers or logarithms, in terms of their integral representation. Since we are interested in a logarithmic one loop structure lets then take in consideration the representation

$$\ln(\mathcal{A}) = -\int_0^\infty \frac{ds}{s}\, e^{-s\,\mathcal{A}} = -\int_0^\infty \frac{ds}{s}\, \mathcal{H}(s; \mathcal{A})\,, \tag{1.60}$$

where the above definition holds up to a diverging quantity. The trace of the left hand term on (1.60) can then be calculated by evaluating the trace of the heat kernel operator and then performing the proper time integral associated to its representation. Since the trace of $\mathcal{H}$ is well defined (because of its exponential structure, UV and IR divergencies are mapped respectively on the eigenvalues 0 and 1 of the heat kernel) the divergencies coming from the trace of the logarithm are translated in divergencies in the proper time integral, which needs now to be regularized. The easier way to regularize it is by means of a sharp cutoff, which reads

$$\mathrm{Tr}[\,\ln(\mathcal{A})\,]_{reg} = -\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s}\,\mathrm{Tr}\,\mathcal{H}(s;\mathcal{A})\,, \tag{1.61}$$

being Λ^2 and k^2 respectively an ultraviolet and infrared momentum cutoff. The sharp cutoff for a certain class of operators actually corresponds to a sharp momentum cutoff, but as we will see afterward this is not generally true. As it happens in momentum space, the regularization of divergencies of the proper time integral can be obtained by using a generic smooth or sharp cutoff function $\rho(s,k,\Lambda)$ such that

$$\begin{aligned}
&\lim_{s\to\infty}\ \rho_{k\neq 0,\Lambda}(s) = 0\,,\\
&\lim_{k\to\Lambda}\ \rho_{k,\Lambda}(s) = 0\,,\\
&\lim_{\Lambda\to\infty}\ \rho_{k=0,\Lambda}(s) = 1\,,
\end{aligned} \tag{1.62}$$

and such that $\lim_{s\to 0}\rho_{k=0,\Lambda}(s) = 1$ in order to ensure that the UV behavior remains unaffected by the introduction of the cutoff. The regularized trace of (1.60) now reads

$$\mathrm{Tr}[\,\ln(\mathcal{A})\,]_{reg} = -\int_0^\infty \frac{ds}{s}\,\rho_{k,\Lambda}(s)\,\mathrm{Tr}\,\mathcal{H}(s;\mathcal{A})\,. \tag{1.63}$$

Since we are interested in the definition of an RG equation, i.e. we want to obtain the variation respect to an infinitesimal change $k \to k + \delta k$, we consider the derivative respect to the RG scale of the regularized trace which translates in the derivative of the only k-dependent object present in the right hand term of (1.63), i.e. the derivative of the cutoff function

$$k\,\partial_k\,\mathrm{Tr}[\,\ln(\mathcal{A})\,]_{reg} = -\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho_k(s))\,\mathrm{Tr}\,\mathcal{H}(s;\mathcal{A})\,, \tag{1.64}$$

where, since the dependence from the ultraviolet cutoff is lost, we can now send $\Lambda \to \infty$. Following the philosophy of the Wegner-Houghton equation (1.36) we can now use (1.64) to construct a one loop flow equation for the Wilsonian action S_k as

$$k\,\partial_k\,S_k[\psi] = \frac{1}{2}\,k\,\partial_k\,\mathrm{STr}[\,\ln(S_k^{(2)})\,]_{reg} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho_k(s))\,\mathrm{STr}\,\mathcal{H}(s;S_k^{(2)}(\psi))\,. \tag{1.65}$$

The above equation may seem not rigorous since in (1.64) we discarded a derivative respect to the operator $\mathcal{A}$, which is now a scale dependent operator, $\mathcal{A} \equiv S_k^{(2)}$. The latter has to be considered in terms of an optimization, that is we operated the substitution $S_\Lambda \to S_\kappa$ in the one loop term as to provide a partial resummation of the perturbation expansion. In spite of that it can be proved that by means of a proper choice of the cutoff function $\rho_k(s)$ the proper time equation (1.65) can be mapped on the Wegner-Houghton equation in the local potential approximation. To show it consider the sharp cutoff equation for the scalar field theory (1.36), in which we subtract a vacuum term in order to have a well defined proper time integral representation

$$k\,\partial_k\,U_k(\Phi) = -\frac{\Omega_d}{2}k^d \log\left(\frac{k^2+U_k^{(2)}(\Phi)}{k^2+U_k^{(2)}(\Phi_0)}\right) = \frac{\Omega_d}{2}\,k^d\int_0^\infty \frac{ds}{s}\,e^{-s\frac{k^2+U_k^{(2)}(\Phi))}{k^2+U_k^{(2)}(\Phi_0)}}\,, \tag{1.66}$$

where Φ_0 is the absolute minimum of the bare potential $U_\Lambda(\Phi)$. The vacuum contribution can be removed once performed the proper time integral, obtaining the standard sharp cutoff expression. We can now compare (1.66) to (1.65), being $S_k^{(2)}[\Phi] = \int d^dx\,\{-\Box + U_k^{(2)}(\Phi)\}$, obtaining then

$$\begin{aligned}&\frac{\Omega_d}{2}\,k^d\int_0^\infty \frac{ds}{s}\,e^{-s\,k^2}\,\left(e^{-s\,U_k^{(2)}(\Phi)} - e^{-s\,U_k^{(2)}(\Phi_0)}\right)\\ &= -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho_k(s))\,\left(e^{-sU_k^{(2)}(\Phi)} - e^{-s\,U_k^{(2)}(\Phi_0)}\right)\,\mathrm{Tr}\,\mathcal{H}(s;-\Box)\,.\end{aligned} \tag{1.67}$$

Now, since the trace of the heat kernel in the right hand term of (1.67) gives

$$\begin{aligned}&\mathrm{Tr}\,\mathcal{H}(s;-\Box) = \mathrm{Tr}_x\langle\,x\,|\,e^{-s(-\Box)}\,|\,x\,\rangle\\ &= \int d^dx\int_0^\infty dp\,p^{d-1}\,\frac{2}{(4\,\pi)^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)}\,e^{-s\,p^2} = \int d^dx\,\frac{1}{(4\,\pi\,s)^{\frac{d}{2}}}\,,\end{aligned} \tag{1.68}$$

we can insert (1.68) in (1.67) and solve respect to $k\,\partial_k\,\rho_k$ obtaining the expression

$$k\,\partial_k\,\rho_k(s) = -\frac{2}{\Gamma\left(\frac{d}{2}\right)}(s\,k^2)^{\frac{d}{2}}\,e^{-s\,k^2}\,. \tag{1.69}$$

We can now integrate (1.69) over k obtaining the proper time representation of the sharp momentum cutoff, which reads

$$\rho_{k,\Lambda}(s) = \frac{\Gamma\left(\frac{d}{2};s\,k^2\right) - \Gamma\left(\frac{d}{2};s\,\Lambda^2\right)}{\Gamma\left(\frac{d}{2}\right)}\,, \tag{1.70}$$

being $\Gamma(a;b)$ the incomplete Euler Gamma function. Note that the proper time regulating function (1.70) leads to the Wegner-Houghton equation just in the local potential approximation, so does not map to an exact sharp equation (1.33) . As already stated, since the sharp equation does not exhibit an high precision in the evaluation of critical exponents we can take advantage of the freedom to choose a smooth functions and use (1.70) as a starting point to define a one parameter family of cutoff functions parametrized by a cutoff parameter n, which reads

$$\rho_k(s,n) = \frac{\Gamma\left(\frac{d}{2}+n; s\,k^2\right) - \Gamma\left(\frac{d}{2}+n; s\,\Lambda^2\right)}{\Gamma\left(\frac{d}{2}+n\right)}\,, \tag{1.71}$$

where $n \in [0,\infty)$. The flow equation for the above family of cutoff functions can be computed taking the derivative of (1.71), namely

$$k\,\partial_k\,\rho_k(s,n) = -\frac{2}{\Gamma\left(\frac{d}{2}+n\right)}(s\,k^2)^{\frac{d}{2}+n}\,e^{-s\,k^2}\,, \tag{1.72}$$

which, inserting (1.72) in (1.65), leads to

$$k\,\partial_k\,U_k(\Phi) = -\frac{\Omega_d}{2}\,k^d\,\Gamma\left(\frac{d}{2}\right)\,\frac{\Gamma\,(n)}{\Gamma\left(n+\frac{d}{2}\right)}\,\left(\frac{1}{1+U_k^{(2)}/k^2}\right)^n\,. \tag{1.73}$$

The above equation is however ill-defined for large n because of the presence of the cutoff parameter in the argument of the gamma functions. Taking the limit $n\to\infty$ we see in fact that the ratio of gamma functions approaches a term

$$\lim_{n\to\infty}\frac{\Gamma\,(n)}{\Gamma\left(n+\frac{d}{2}\right)} = n^{-\frac{d}{2}}\,, \tag{1.74}$$

which suppressed the flow equation for large n. A finite limit can be obtained by considering a reparametrization of the RG scale, i.e. by inserting

$$k^2 \to n\,k^2\,, \tag{1.75}$$

in (1.73), which furnishes a factor $n^{d/2}$ that ensures the finiteness of the flow equation. The RG equation for the reparametrized equation reads then

$$k\,\partial_k\,U_k(\Phi) = -\frac{\Omega_d}{2}\,k^d\,\Gamma\left(\frac{d}{2}\right)\,\frac{\Gamma\,(n)\,n^{\frac{d}{2}}}{\Gamma\left(n+\frac{d}{2}\right)}\,\left(\frac{1}{1+U_k^{(2)}/(n\,k^2)}\right)^n\,, \tag{1.76}$$

and the limit for large n of equation (1.76) can now be taken noting that the limit of the propagator appearing in (1.76) is a well known limit

$$\lim_{n\to\infty}\left(1+\frac{U_k^{(2)}}{k^2\,n}\right)^{-n} = e^{-\frac{U_k^{(2)}}{k^2}}\,, \tag{1.77}$$

so that we end up with the exponential expression

$$\lim_{n\to\infty}\,k\,\partial_k\,U_k(\Phi) = -\frac{\Omega_d}{2}\,k^d\,\Gamma\left(\frac{d}{2}\right)\,e^{-\frac{U_k^{(2)}}{k^2}}\,. \tag{1.78}$$

Equation (1.78) is nonetheless the RG equation obtained by employing a sharp proper time integral (analogous to (1.61)).

The exponential fRG equation has shown so far the highest precision in the evaluation of the critical exponents for the scalar field theory [41, 42], besides, how we will see afterwards in this work, the large n limit is not always the regime in which the proper time cutoff results to be more accurate. Furthermore, it has been applied in gauge field theories [43, 44].

A well known issue of the proper time scheme constructed from the cutoff (1.71) is, however, its exactness. i.e. the reproduction of the correct effective action while integrating down to $k \to 0$. The use of the proper time regulator (1.71) in perturbation theory leads to correct results at two loop only when employing a linear dependence on the propagator (that is, $n = 1$ in (1.76)), while a logarithmic or non linear dependence on the propagator gives wrong combinatorial factors in diagram summation. The linearity in the propagator as an essential ingredient in the construction of an exact equation is moreover already expressed on the linear dependence of the ERG equation (1.49) from the 2-point correlation function. The relation between the scheme (1.65) and the exact equation (1.52) is made clear by rewriting the denominator of (1.52) in its proper time representation

$$\frac{1}{2}\mathrm{STr}\left[\frac{k\,\partial_k\,R_k}{\Gamma_k^{(2)}+R_k}\right] = -\frac{1}{2}\,\mathrm{STr}\left[(k\,\partial_k\,R_k)\,\int_0^\infty ds\,e^{-s\,(\Gamma_k^{(2)}+R_k)}\right], \tag{1.79}$$

and then de-entangling the cutoff function R_k from the heat kernel of $\Gamma_k^{(2)}$ by means of a Baker-Campbell-Hausdorff expansion. In particular, using the Zassenhaus formula

$$e^{-s\,(\mathcal{A}+\mathcal{B})} = e^{-s\,\mathcal{A}}\,e^{-s\,\mathcal{B}}\,\prod_{i=2}^{\infty} e^{-s^i\,\mathcal{C}_i}\,, \tag{1.80}$$

being $\mathcal{C}_i$ operators made by linear combination of i commutators of the generic non-commuting operators $\mathcal{A}$ and $\mathcal{B}$, we obtain

$$e^{-s\,(\Gamma_k^{(2)}+R_k)} = e^{-s\,\Gamma_k^{(2)}}\,e^{-s\,R_k}\,\left\{1-\frac{s}{2}[\Gamma_k^{(2)},R_k]+\mathcal{O}(s^2)\right\} \equiv e^{-s\,\Gamma_k^{(2)}}\,F_k[s\,R_k;s\,\Gamma_k^{(2)}], \tag{1.81}$$

where F_k is now a proper time cutoff function built from the ERG cutoff function R_k. The exact renormalization group equation reads now

$$k\,\partial_k\,\Gamma_k = -\frac{1}{2}\mathrm{STr}\left[\int_0^\infty \frac{ds}{s}\,\{s\,F_k[s\,R_k;s\,\Gamma_k^{(2)}]\,(k\,\partial_k\,R_k)\}\,e^{-s\,\Gamma_k^{(2)}}\right], \tag{1.82}$$

and the comparison between the proper time flow equation and the ERG equation is then straightforwardly characterized by the identity

$$\partial_t\,\rho_{k,\Lambda}(s) \;=\; s\,F_k[s\,R_k;s\,\Gamma_k^{(2)}]\,(k\,\partial_k\,R_k)\,. \tag{1.83}$$

Generally, there is no momentum independent cutoff equation $\rho_{k,\Lambda}(s)$ which satisfies equation (1.83), since the dependence on $\Gamma_k^{(2)}$ of F_k encodes the correct loop summation. Anyway, it is still possible that the two functions coincide under certain approximations, like it happens for the sharp cutoff equation. An other possibility is to use a spectrally adjusted cutoff, i.e. to build a cutoff operator R_k which cuts the modes of the whole effective action, $R_k \equiv R_k[\Gamma_k^{(2)}]$, so that the commutators appearing in (1.80) disappear. It can be demonstrated that using such a cutoff in the background field method the proper time equation for $n = 1$ actually corresponds to the ERGe obtained using the optimized cutoff (for more details see [45]).

Concluding, we briefly introduced in this chapter the perturbative and non-perturbative renormalization group schemes which we will employ of in the next chapters, although the introduction of the Wegner-Houghton equation was here mostly instrumental (to introduce the proper time equation) and nonetheless historical, since the renormalization group equation was obtained independently by Wegner and Wilson [46].

In particular, after an introducing in chapter 2 the application of the renormalization group to quantum gravity, we will work in chapter 3 in the proper time framework to investigate the scheme-dependence of the conformally reduced Hilbert-Einstein action, then we will apply the ERG equation to the quantization of the Brans-Dicke theory in chapter 4 and finally we will employ the MS scheme in the study of the reduced model of Hořava-Lifshitz gravity in chapter 5.

Chapter 2

Renormalization group in quantum gravity

The identification of a theory of quantum gravity is surely one of the most important unsolved puzzles in modern physics. Now that the landmarking success of quantum field theory to unify the strong, weak and electromagnetic interactions in a gauge field theory, that is the Standard Model, has been acclaimed by the recent discovery of a scalar particle at the LHC [47, 48], the identification of a quantum theory for gravitation assumes new facets, since it has now to satisfy the constraints coming from the Higgs mass and the disposal of minimal supersimmetric models.

While such a theory is still missing, it is not clear whenever it will be a string theory or a quantum field theory, if it will feature a unification of all the four fundamental interactions or it will instead describe separately quantum gravity. It is, however, very probable that particle physics at very high energy is profoundly altered by quantum gravitational effects.

Various approaches to quantum gravity, for example, suggest the existence of compactified dimensions already at the TeV scale, or that the spacetime exhibits some sort of quantum fuzziness at Planckian energies, featured in loop quantum gravity by the quantization of the area operator (which has a discrete spectrum) and in string theory by the presence of a generalized uncertainty principle, characterized by coordinates non commutativity. We expect some of these non-trivial properties to be part of the correct theory of quantum gravity.

Many insights are recently coming from different approaches to the quantization of gravity (among which asymptotic safety [1, 2], CDT [49], Hořava-Lifshitz [11] and loop quantum gravity [50]) which are making progress towards the definition of a consistent theory. Interestingly, some of them (besides being based on different philosophies and techniques) are converging towards analogous results[1], raising the possibility of being

[1]One of the most known results we refer here to is the value of the microscopic spectral dimension, which has been found to be equal to 2 in many different approaches of quantum gravity, among which asymptotic safety [51], CDT [52], Hořava-Lifshitz gravity [53], loop quantum gravity [54] and double special relativity [55].

grasping different details of the same microscopic theory.

Hence, before give up on quantum field theory as a possible candidate for a ultraviolet completion of general relativity it is indeed necessary to completely grasp how the gravitational fluctuations interact; also while assuming that at a certain energy scale gravity is defined by some other more fundamental theory, it is, in fact, still reasonable to consider QFT as a possible intermediate state, that is, an effective quantum theory.

2.1 Renormalizability and unitarity

The fulfillment of the requests of unitarity and renormalizability is, however, a necessary but not sufficient condition for the identification of the correct theory. The issues of renormalizability, in particular, is the historical problem which arose since the first attempt to quantize Einstein's theory. By employing a simple dimensional analysis, the method of the power counting, it was shown that Einstein's theory was likely to be not renormalizable, being the Newton's constant a negative dimensional parameter.

According to the method of power counting the perturbative renormalizability of a theory can be deduced by the scaling dimension of the couplings. Whenever a coupling a has a scale dimension d_a, the β-function of the dimensionless coupling $\tilde{a}$, $\tilde{a} = \mu^{-d_a}\, a$, takes at tree level a term proportional to its scale dimension, i.e. being μ a mass

$$\mu\, \partial_\mu\, \tilde{a}_\mu = \mu\, \partial_\mu \left(\mu^{-d_a}\, a_\mu\right) = -d_a\, \tilde{a}_\mu + \mathcal{O}(\hbar)\,. \tag{2.1}$$

Equation (2.1) is the generalization of the Gell-Mann Low equation (1.7) for a dimensionful parameter, where the term $\mathcal{O}(\hbar)$ is proportional to the anomalous dimension (containing thus loop corrections).

Now, whenever we are in a perturbative regime and the dimensionality of the coupling is positive the leading term of the β-function leads the coupling to zero in the high energy limit, realizing asymptotic freedom. The coupling in this case is said to be super renormalizable or relevant. For $d_a = 0$, that is the standard case of gauge field theories, the running of the coupling is dominated entirely by the loop corrections and such a coupling is said to be strictly renormalizable or marginal. The last case is the one with $d_a < 0$ which leads to a positive β-function, so that the coupling is catalogued as non renormalizable.

The non renormalizability of theories with negative dimensional interaction parameters can also be understood in terms of perturbation theory, since the divergences appearing in the diagrams of the n-th order in the perturbative expansion have now to compensate the negative dimension of the coupling (elevated to n). Hence, in order to built counterterms to absorb the divergencies in the bare couplings we need to add to the bare action invariant of positive mass dimension $d_a\, n$, for each order n. We end up then with an infinite number of higher dimension operators in the bare action, and since we now need to fine tune an infinite number of parameters the theory loses its predictable value.

Let us focus now on the Einstein-Hilbert action

$$S[g_{\mu\nu}] = -\frac{1}{16\,\pi\,G}\int d^dx\,\sqrt{g}\,\{R(g) - 2\,\Lambda\}\,, \tag{2.2}$$

where G is the Newton constant, $R(g)$ is the Ricci scalar constructed from the metric g and Λ is the cosmological constant. Being d the dimensionality of the volume element, the inverse of the coupling associated to the operator $\sqrt{g}\,R$, i.e. the Newton constant G, has mass dimension $2-d$; the gravitational interaction is then marginal (at tree level) in two dimensions (where the theory is however topological) and non renormalizable for $d > 2$. We expect then to receive divergent contributions from loop corrections.

The perturbative renormalizability has been investigated at a quantum level by 't Hooft and Veltman in '74 [56], whom discovered that at one loop the divergencies (which are proportional to the squared Ricci scalar and two contracted Ricci tensors) vanish on shell, granting renormalizability to the theory. The hope that divergencies would cancel also at higher orders disappeared when Goroff and Sagnotti proved in '85 that the theory contains non-vanishing divergencies already at two loops [57].

Hence, one of the first historical proposal to cure the UV behavior of general relativity was to employ higher derivative operators. Using the power counting it can be seen that operators built contracting two Riemann tensors (or tensors of the same dimension, like the Weyl tensor) are marginal operators, since their couplings are dimensionless in $d = 4$. There were then many attempts to define an UV completion of gravity by regularizing the high energy behavior of diagrams by inserting in the bare action marginal operators built from the Ricci tensor, like the Weyl-Eddington action

$$S[g_{\mu\nu}]_{hd} = -\frac{1}{16\,\pi\,G}\int d^dx\,\sqrt{g}\,\{R - \gamma\,R^2 - \beta\,R_{\mu\nu}\,R^{\mu\nu} - 2\Lambda\}\,, \tag{2.3}$$

where γ and β are respectively the couplings of the squared Ricci scalar and contracted Ricci tensor. However, an other serious issue emerged, that is the loss of unitarity [58].

The higher derivative action (2.3), in fact, leads to an effective propagator for the spin-2 modes

$$\frac{1}{p^2 - \beta\,G\,p^4}\,, \tag{2.4}$$

that runs to zero fast enough for $p \to \infty$ to ensure the absence of divergencies in the Feynman diagrams but contains an unphysical pole (the so-called poltergeist) at $p^2 = (\beta\,G)^{-1}$ which violates the Kallen-Lehmann theorem and spoils the unitarity of the theory.

It is, however, an old debate whether the ghost really implies a loss of unitarity or not. It can happen that employing an RG improvement in the perturbative expansion the unphysical poles move to complex values and cancel their contributions from the S matrix [59], restoring the unitarity of graviton-graviton amplitudes, or that gravity is strongly coupled in the high energy regime and that the contributions of irrelevant (higher derivative) operators push the pole to infinity, saving the physical value of the theory. The latter assumption, and in particular the assumption of non-perturbative renormalizability of gravity, has been conjectured by Weinberg in the '70 and goes under the name of the asymptotict safety scenario for quantum gravity [1, 2, 3, 4, 5].

2.2 Asymptotic safety scenario

The insight that gravity flows in the ultraviolet to a non-perturbative regime is a simple consequence of the β-function (2.1) for the Newton constant G, which in $d = 4$ reads

$$k\,\partial_k\,\tilde{G}(k) = 2\,\tilde{G}(k) + \mathcal{O}(\hbar)\,, \tag{2.5}$$

where $\tilde{G}$ is the dimensionless Newton constant and k a renormalization scale[2]. The solution of the tree term of (2.5), that is $\tilde{G}(k) = k^2\,G$, entails that, starting from a perturbative regime $\tilde{G} \ll 1$, the Newton constant grows for large k towards a non-perturbative regime, but, unfortunately, in a divergent way so that also the correlation functions diverge. In that regime, however, loop corrections can became of the same order of the tree term and the contribution of irrelevant interaction can become relevant and not anymore negligible. It can then happen that the fluctuations take control of the UV flow of the theory and keep correlation functions finite at all the scales. Hence, the theory is said to be asymptotically safe.

In terms of β-function the last definition can be casted in the following way. Lets consider an expansion of the microscopic gravitational action in local operators $O_i(x)$, that is

$$S[g_{\mu\nu}] = \int d^dx\sqrt{g}\,\sum_{i=1}^{N=\infty}\,a_i\,O_i(x,g)\,. \tag{2.6}$$

The scale dependence of the above action is encoded in a system of coupled β-functions for the dimensionless couplings $\tilde{a}_i$, namely

$$\begin{cases} k\,\partial_k\,\tilde{a}_1(k) = \beta_1(\tilde{a}_1, \tilde{a}_2, \cdots) \\ k\,\partial_k\,\tilde{a}_2(k) = \beta_2(\tilde{a}_1, \tilde{a}_2, \cdots) \\ \qquad\qquad \cdots \end{cases}, \tag{2.7}$$

that of course can be solved only if we truncate the action (2.6) to a finite N. The requirement of asymptotic freedom translates then in the existence of a trivial fixed point of the system (2.7), i.e. a common zero of the β-functions at vanishing couplings (except for the redundant couplings). The theory is, instead, said to be asymptotically

[2]Note that we changed notation from μ to k for the renormalization parameter, since we will work now in a non-perturbative framework. The identification of the two renormalization scales has already been done in the chapter 1 and it will be used also in the others chapters.

safe whenever the system (2.7) admits a non-trivial fixed point solution, thus a common zero of the β-functions

$$\begin{cases} \beta_1(\tilde{a}_1^*, \tilde{a}_2^*, \cdots) = 0 \\ \beta_2(\tilde{a}_1^*, \tilde{a}_2^*, \cdots) = 0 \\ \qquad \cdots \end{cases}, \tag{2.8}$$

for a set of non-zero couplings $\{a_j^*\}$. The latter fixed point is also said non-Gaussian (NGFP), as opposite to the trivial Gaussian fixed point, since it identifies a scale invariant interacting field theory. Hence, the assumption we make is that such a fixed point is strongly interactive and attractive in the UV direction, so that the flow of the Newton's constant is controlled, once in the high energy regime, by the presence of the fixed point and in the UV limit we get

$$\lim_{k\to\infty} \tilde{G}(k) = \tilde{G}^*\,, \tag{2.9}$$

being $\tilde{G}^*$ the dimensionless Newton's constant at the fixed point. Consequently, since the couplings stay finite along all the flow, the correlation functions are also finite and the theory is renormalizable in a non-perturbative sense.

To investigate the existence of such a non-perturbative non-Gaussian fixed point, however, perturbative RG techniques like the MS scheme are not appropriate, since a large number of loops would be eventually required to reproduce the non-trivial zero of the β-function. In the ultraviolet regime, in fact, fluctuations can be dominant respect to tree contributions so that irrelevant parameters can become relevant, and vice versa. It is thus necessary to employ non-perturbative techniques.

As mentioned in chapter 1, an appropriate non-perturbative framework is the Wilson's renormalization group, since the coarse graining automatically generates the coupled flow of an infinite number of parameters. The search for a non-Gaussian solution of the systems (2.8) translates then in the search for a non-trivial solution of the RG equation for the whole effective action, i.e.

$$k\,\partial_k\,\Gamma_k[g_{\mu\nu}] = 0\,, \tag{2.10}$$

that interpolates between the full gravitational effective action $\Gamma[g] \equiv \Gamma_{k=0}[g]$ and the Einstein-Hilbert action $S_\Lambda[g] \equiv \Gamma_{k=\Lambda}[g]$ at our energy scale (depicted in Fig. 2.1). The statement of renormalizability can then be formulated as the existence of an RG trajectory that, taking the limit $\Lambda \to \infty$, connects the bare action $S_\Lambda \equiv \Gamma_{k=\Lambda}$ to a scale invariant ultraviolet action $S^* \equiv \Gamma_{k=\infty}$, that is the fixed point we want to investigate about.

Although the theory space is now infinite dimensional, that is we need to fine tune an infinite number of couplings in order to select a trajectory in parameter space, we assume that we need to fine tune just a finite number of relevant parameters in the correlation functions, that is to say, the ultraviolet critical surface is finite dimensional

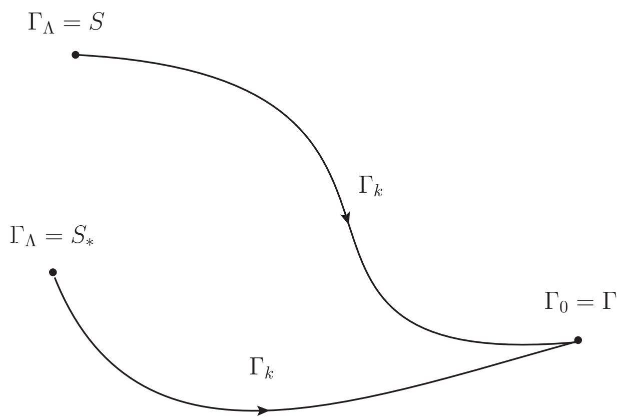

Figure 2.1: Renormalization group trajectories in theory space.

(illustrated in Fig. 2.2). Moreover, since the correlation functions depend just on the tuning of a finite set of couplings, the theory (besides in the form of an effective field theory) has still predictable value.

The dimensionality of the ultraviolet critical surface depends itself on the number of attractive directions in the linearized flow around the fixed point. Being a_i^* the fixed point value of the i-th coupling, the linearized running of the coupling in a neighborhood of the fixed point can be written as

$$a_i(k) = a_i^* + \sum_A k^{\lambda_A}\, C_A\, V_i^A\,, \tag{2.11}$$

where A is an index over the parameter space (like i), C_A are arbitrary coefficients, λ_A are the critical exponents and V_i are perturbation vectors. Attractive directions are then characterized by negative critical exponents λ_A, since in the limit $k \to \infty$ the perturbation goes to zero and the coupling reaches the fixed point, $a_i(k) \to a_i^*$, while for $\lambda_A > 0$ it runs away from a_i^*. The spectra of critical exponents λ_A is simply the spectra of eigenvalues of the stability matrix $\mathcal{B}$, being defined as

$$\mathcal{B}_{ij} = \frac{\partial\,\beta_i}{\partial\,a_j}\bigg|_{\{a_i\}=\{a_i^*\}}\,, \tag{2.12}$$

where the β-functions in (2.12) has been defined in (2.7). The eigenvectors of this matrix are the vectors V_i in (2.11), for which it holds

$$\sum_j B_{ij} V_j = \lambda_i\, V_i\,. \tag{2.13}$$

The dimensionality of the ultraviolet critical surface is then simply given by the number of negative eigenvalues of the stability matrix, which are the only parameters we need to fine tune in the correlation functions, assumed that the physical theory is described by a trajectory which lies on the critical surface.

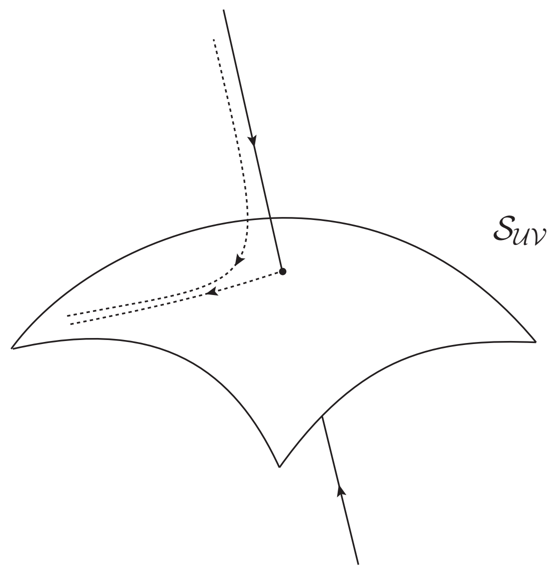

Figure 2.2: Trajectories out of the critical surface characterize non renormalizable theories since in the limit $k \to \infty$ they get pushed away from the fixed point by diverging contributions of irrelevant operators.

Despite the fact that the expression of the β-functions depends on the regulator (or scheme) used, the spectra of the $\mathcal{B}$ matrix is scheme-independent, since it encodes the informations about the linear behaviour or the flow around the fixed point, that is, the universality class of the theory.

Before applying the functional RG to an effective theory of gravity it is still, however, intriguing to investigate the existence of a non-Gaussian fixed point in a perturbative regime, that is when the Newton's constant is circa marginal, $d_G \approx 0$. As it often happens in statistical mechanics, the non-trivial fixed point collapses on the Gaussian for $d \to d_c$, where d_c is the critical dimension[3] of the coupling, $d_c = 2$ in the case of Newton's constant. We can then start by studying the phase diagram of gravity in $2+\epsilon$ dimensions, being ϵ an infinitesimal parameter, employing the ϵ-expansion developed by Wilson and Kogut [46].

2.2.1 Gravity in $2 + \epsilon$ dimensions

To study the quantization of Einstein's gravity in $2+\epsilon$ dimensions we employ here the dimensional regularization. Let us consider a bare dimensionless coupling $\tilde{a}_i(D)$, where D is the dimensionality of spacetime, and rewrite it in terms of a renormalized coupling

[3] The critical dimension d_c is defined by means of the Ginzburg criterion as the dimension at which the critical exponents agree with mean field theory, and the coupling becomes marginal.

$\tilde{a}_i(k, D)$ plus counterterms as

$$\tilde{a}_i(D) = a_i(D)\, k^{-d_i(D)} = \tilde{a}_i(k, D) + \sum_s \sum_{\nu=1}^{\infty} \frac{1}{(D - D_s)^\nu}\, b^{(s)}_{\nu,i}(\tilde{a}(k, D))\,, \tag{2.14}$$

where d_i is the scale dimension of the coupling $a_i(D)$, k is a renormalization scale, and, as usual in dimensional regularization, divergencies are parametrized by inverse powers of the parameter $\epsilon = D - D_s$. The parameters s and ν characterize respectively the set of critical dimensions and the order of the loop expansion, and $b_{\nu,i}$ contains the quantum corrections. The idea behind the dimensional regularization is to regularize divergencies by means of analytic continuation of the spacetime dimension D, where the scaling dimension of the coupling is considered to be some linear function

$$d_i(D) = \sigma_i + \rho_i\, D\,, \tag{2.15}$$

being σ_i and ρ_i two coefficients which depend on the theory.

The i-th β-function is then simply obtained stating the independence of the dimensionless bare coupling from the renormalization scale k, which leads to

$$\beta_i(\tilde{a}, D) = \beta_i(\tilde{a}(k), D) + \sum_j \sum_s \sum_{\nu=1}^{\infty} \frac{1}{(D - D_s)^\nu}\, b^{(s)}_{\nu,i,j}\, \beta_j(\tilde{a}(k), D) = 0\,, \tag{2.16}$$

where the coefficients $b^{(s)}_{\nu,i,j}$ are

$$b^{(s)}_{\nu,i,j}(a) = \frac{\partial\, b^{(s)}_{\nu,i}}{\partial\, a_j}\,. \tag{2.17}$$

The left hand term of (2.16) gets a contribution only from the dimension of the coupling, so that we have

$$\beta_i(\tilde{a}, D) = -d_i\, \tilde{a}_i\,. \tag{2.18}$$

Hence, we can put (2.15) and (2.14) in (2.18) and rewrite the bare coupling in term of renormalized coupling and counterterms, and separate the contributions linear in D from those divergent (i.e. given by counterterms with $\nu > 1$). Since we expect the divergent part of both left and right side of (2.16) to cancel out, we end up with an analytic β-function linear in D, in which just the one loop contribution is present, i.e.

$$\beta_i(\tilde{a}, D) = -\rho_i\, D\tilde{a}_i - \sigma_i\, \tilde{a}_i - \sum_s b^{(s)}_{1\,i}(\tilde{a})\rho_i + \sum_{sj} b^{(s)}_{1ij}(\tilde{a})\, \rho_j\, g_j\,. \tag{2.19}$$

In the gravitational case we are interested to study the running of the sole Newtons's constant, so that we have $D_s = 2$ and $D = 2 + \epsilon$. The mass dimension of the Newton's constant G is then $d_G = -\epsilon$, which means $\sigma = -2$ and $\rho = 1$. In terms of the renormalized parameter the bare coupling reads

$$\tilde{G} = G(\epsilon)\, k^\epsilon = \tilde{G}(k) + \sum_{\nu=1}^{\infty} \frac{1}{\epsilon^\nu}\, b_\nu(\tilde{G}(k))\,, \tag{2.20}$$

and the b-function (2.19) for the Newton's constant reads

$$\beta(\tilde{G}, \epsilon) = \epsilon\, \tilde{G} + b_1(\tilde{G}) - \tilde{G}\frac{\partial\, b_1}{\partial\, \tilde{G}}\,. \tag{2.21}$$

Since we are interested in a non-trivial fixed point nearly degenerate with the Gaussian one, we can considered $\tilde{G}$ to be really small. In this approximation we can expect the loop correction b_1 to be proportional to the squared of the Newton's constant, that is

$$b_1(\tilde{G}) = b\, \tilde{G}^2 + \mathcal{O}(\tilde{G}^3)\,, \tag{2.22}$$

being b a certain coefficient depending on the matter field coupled to gravity, so that putting (2.22) in (2.21) we obtain a β-function

$$\beta(\tilde{G}, \epsilon) = \epsilon\, \tilde{G} - b\, \tilde{G}^2 + \mathcal{O}(\tilde{G}^3)\,. \tag{2.23}$$

The spectra of fixed point depends now on the sign of b. For $b < 0$ we find only the Gaussian fixed point, $\tilde{G}^* = 0$, while assuming $b > 0$ the beta function (2.23) admits a non-trivial solution of the form

$$\tilde{G}^* = \frac{\epsilon}{b} + \mathcal{O}(\epsilon^2)\,. \tag{2.24}$$

The calculation of the parameter b has been performed by many group [60] [61] and leads to

$$b = \frac{38}{3} + 4\, N_V - \frac{1}{3}\, N_F - \frac{2}{3}\, N_S\,, \tag{2.25}$$

where N_V, N_F and N_S are respectively the number of vectorial, fermionic and scalar fields coupled to gravity. In particular, as it can be seen in (2.25), b is positive unless we add too many fermionic fields; the latter is however not a pathology of the theory since it is well known that also non abelian gauge field theories are not asymptotically free anymore after a certain number of fermionic fields.

Proven that there exist a non-Gaussian fixed point for an infinitesimal ϵ the question is if the fixed point survives while taking the limit $\epsilon \to 2$. Generally, it happens that the non-Gaussian fixed point moves aways from the gaussian for $D > D_r$ but disappears at a certain upper value of the dimension. In the Gross-Neveu model, which is an example of theory with a non-Gaussian fixed point in $2 + \epsilon$ dimensions, the non-trivial fixed point evolves when varying D but disappear at $D = 4$, hence the theory is non renormalizable.

The ϵ-expansion is anyway not the correct tool to use in order to investigate the existence of a non-trivial fixed point in $d = 4$ gravity. To study gravity in 4 dimensions we need then to employ the non-perturbative renormalization group, whose implementation is, however, not so straightforward. The concept of coarse graining is in fact not a priori defined in a gravitational context, since the length of the block we intend to use to average the field is defined by the field itself, that is the metric.

The implementation of the concept of coarse graining must also satisfy another of the key features of general relativity, that is, background independence. We expect in fact the coarse graining to be independent on the choice of the background we adopt, since otherwise we would end up with a set of β-functions which will depend on the field itself; quite an undesirable situation.

One way to solve this situation, and realize a coarse graining procedure which is consistent with the requirement of background independence, is by employing the background field method, which we will use as a general framework for all the calculations in this work.

2.2.2 Background field method

The background field method is a technique often employed in QFT to quantize the theory without losing gauge invariance [62].

The main idea behind this formalism consists in a decomposition of the field[4] in a classical background plus a quantum fluctuation, $\psi(x) = \tilde{\psi}(x) + B(x)$, being $B(x)$ the classical background field. The expression of $B(x)$ is never fixed in the calculations, nor the fluctuation $\tilde{\psi}(x)$ is intended to be such, so that it is not required to be small.

The partition function for a theory with bare action $S[\psi]$ in presence of an background field reads

$$\tilde{Z}[J; B] = e^{i\,\tilde{W}[J;B]} = \int \mathcal{D}[\tilde{\psi}]\; e^{i\,S[\tilde{\psi}(x)+B(x)] + i\int d^dx\, J(x)\,\tilde{\psi}(x)}\,, \tag{2.26}$$

and the expectation value of the fluctuating field $\tilde{\psi}$ in presence of the background B and of the external source J is defined in the standard way as

$$\langle\, \tilde{\psi}(x)\,\rangle = \frac{\delta\,\tilde{W}[J;B]}{\delta\,J(x)} = -i\,\frac{\delta}{\delta J(x)}\,\ln \tilde{Z}[J;B]\,. \tag{2.27}$$

The effective action in presence of the background field is obtained then by defining the variable $\tilde{\Psi} = \langle\tilde{\psi}\rangle_J$ and taking a Legendre transform of the functional $W[J;B]$ as

$$\tilde{\Gamma}[\tilde{\Psi}, B] \;=\; -\tilde{W}\Big[J[\tilde{\Psi}, B]; B\Big] \;+\; \int d^dx J[\tilde{\Psi}, B]\,\tilde{\Psi}\,. \tag{2.28}$$

Expressing now the fluctuation $\tilde{\psi}(x)$ in terms of the original field $\psi(x)$, i.e. $\tilde{\psi}(x) = \psi(x) - B(x)$, equation (2.26) now reads

$$\tilde{Z}[J; B] = e^{i\,\tilde{W}[J;B]} = \int \mathcal{D}[\psi]\; e^{i\,S[\psi] + i\int d^dx\, J(x)\,\psi(x)}\; e^{-\,i\int d^dx\, J(x)\,B(x)}\,, \tag{2.29}$$

[4]We will continue to call the field $\psi(x)$ following the notation used in the chapter 1.

which can be recasted as

$$\tilde{Z}[J;B] = Z[J]\; e^{-i\int d^dx\, J(x)\cdot B(x)} = e^{i\,W[J]}\; e^{-i\int d^dx\, J(x)\cdot B(x)}. \tag{2.30}$$

where $Z[J]$ is the partition function for the original action $S[\psi]$ in absence of a background field. Equation (2.30) entails that

$$\tilde{W}[J;B] \,=\, W[J] \,-\, \int d^dx\, J(x)\, B(x)\,, \tag{2.31}$$

which, using (2.27), leads to the relation between the expectation value of the field ψ with and without background field. Taking indeed a derivative of (2.31) respect to $J(x)$ we obtain

$$\tilde{\Psi}(x) = \frac{\delta\,\tilde{W}[J,B]}{\delta\,J(x)} = \frac{\delta\,W[J]}{\delta\,J(x)} - \frac{\delta\,(\int d^dx\, J(x)\cdot B(x))}{\delta\,J(x)} = \Psi(x) - B(x)\,, \tag{2.32}$$

where $\Psi(x) = \langle\psi(x)\rangle_J$. Using now (2.31) and (2.32) in (2.28) it can hence be shown that

$$\begin{aligned}\tilde{\Gamma}[\tilde{\Psi},B] &= -\tilde{W}[J,B] + \int d^dx\, J(x)\cdot\tilde{\Psi}(x)\\ &= -(W[J] - \int d^dx\, J(x)\cdot B(x)) + \int d^dx\, J(x)\cdot(\Psi(x) - B(x))\\ &= -W[J] + \int d^dx\, J(x)\cdot\Psi(x) \,=\, \Gamma[\Psi]\,,\end{aligned} \tag{2.33}$$

which leads to

$$\Gamma[\Psi] = \Gamma[\tilde{\Psi} + B] = \tilde{\Gamma}[\tilde{\Psi}, B]\,. \tag{2.34}$$

Equation (2.34) signifies that the effective action $\tilde{\Gamma}[\tilde{\Psi},B]$ in presence of a background field is equivalent (intended as a generator of correlation functions) to the standard effective action $\Gamma[\Psi]$ but where the effective field has been decomposed as $\Psi(x) = \tilde{\Psi}(x) + B(x)$. In particular, setting $\tilde{\Psi}(x) = 0$ we obtain the identity

$$\tilde{\Gamma}[0,B] = \Gamma[B]\,. \tag{2.35}$$

Equation (2.35) now states that the 1PI connected Green functions can be calculated by summing the vacuum diagrams of the effective action $\tilde{\Gamma}[0,B]$ in presence of the background field. This features is particularly useful for gauge field theories. In the latter case the gauge-fixing for the fluctuating field is built using the background field in such a way to preserve invariance under simultaneous transformations of the background and fluctuation fields. Upon the identification (2.35), such invariance is translated in the gauge invariance of the standard effective action $\Gamma[B]$.

Now, in order to quantize gravity in the background field formalism we decompose the microscopic metric tensor $\gamma_{\mu\nu}$ (we will use $g_{\mu\nu}$ for the effective field) as

$\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, being $\bar{g}_{\mu\nu}$ the classical background, $\bar{g}_{\mu\nu} = \langle \bar{g}_{\mu\nu} \rangle$, and $h_{\mu\nu}$ the fluctuation. Furthermore, we will work in Euclidean signature in order to avoid the complications present in Lorentzian signature. Although there is no Wick-rotation on a general background, we expect the β-functions to be independent of the signature. The partition function for gravity reads then

$$\tilde{Z}[J_{\mu\nu}; \bar{g}_{\mu\nu}] = e^{-\tilde{W}[J_{\mu\nu};\bar{g}_{\mu\nu}]} = \int \mathcal{D}[h_{\mu\nu}]\; e^{-S[\bar{g}_{\mu\nu}+h_{\mu\nu}] + \int d^dx \sqrt{\bar{g}}\, J^{\mu\nu}\, h_{\mu\nu}}\,, \tag{2.36}$$

where the expectation value of the fluctuation is obtained as

$$\bar{h}_{\mu\nu}(x) = \langle\, h_{\mu\nu}(x)\, \rangle_J = \frac{\partial\, \tilde{W}[J_{\mu\nu}; \bar{g}_{\mu\nu}]}{\partial\, J^{\mu\nu}(x)}\,. \tag{2.37}$$

The 1PI n-point correlation functions can then be evaluated by taking functional derivatives of the effective action $\tilde{\Gamma}[\bar{h}_{\mu\nu}; \bar{g}_{\mu\nu}]$ respect to the fluctuation $\bar{h}_{\mu\nu}$ and then removing external legs from diagrams by fixing $\bar{h}_{\mu\nu} = 0$, since it holds

$$\Gamma[\bar{g}_{\mu\nu}] = \tilde{\Gamma}[0; \bar{g}_{\mu\nu}]\,. \tag{2.38}$$

Consequently, the background field method is an appropriate framework for the formulation of coarse graining on gravity: we can in fact use the background metric $\bar{g}$ to construct the renormalization scale k and average the fluctuations of the field $h_{\mu\nu}$ with $\bar{p}^2 < k^2$, being $\bar{p}^2$ the eigenvalue of the mode of the Laplacian operator $\bar{\nabla}^2$ constructed employing the background metric $\bar{g}$, i.e. $\bar{\nabla}^2 \equiv \nabla(\bar{g})^2$. Since the fluctuation nor the background are fixed during the calculation the requirement of background independence is automatically satisfied.

To perform consistently the path integral we need now to define a gauge fixing (and the associated ghost sector) for the partition function (2.36).

2.2.3 Gauge fixing and ghosts

The gauge fixing is introduced following the standard Popov-Fadeev techniques, which consists in the removing of the gauge degrees of freedom from the path integral by constraining it with a generic gauge fixing condition $\mathcal{F}_\mu = C_\mu$, being $C_\mu(x)$ an auxiliary operator. The constraint is introduced as δ-function in the path integral so that, since $C_\mu(x)$ is a gauge invariant function, we can then integrate the δ-function over the configurations of $C_\mu(x)$ as

$$\int \mathcal{D}[C_\mu]\, \delta(\mathcal{F}_\mu - C_\mu)\, e^{-\frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\,\mathrm{Tr}\,(C_\mu\, C^\mu)} = e^{-\frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\,\mathrm{Tr}\,(\mathcal{F}_\mu\, \mathcal{F}^\mu)}\,. \tag{2.39}$$

The gauge fixing constraint takes then the form of an action term in the path integral, and in the background field method it reads

$$S_{gf}[h; \bar{g}] = \frac{1}{2\,\alpha} \int d^dx \sqrt{\bar{g}}\, \mathcal{F}_\mu(h; \bar{g})\, \mathcal{F}^\mu(h; \bar{g})\,, \tag{2.40}$$

that is by fixing the gauge freedom of the fluctuating field $h_{\mu\nu}$ on the background field $\bar{g}_{\mu\nu}$, being α a gauge fixing parameter. The different gauge choices can be taken by fixing the value of α after field rescaling. In particular, the Landau gauge can be obtained in the limit $\alpha \to 0$ while the Feynman for $\alpha \to 1$. The gauge condition $\mathcal{F}_\mu$ is constructed in the BFM as a function of the fluctuating field $h_{\mu\nu}$ and is convenient to consider it linear in the field. Hence it takes the general form

$$\mathcal{F}_\mu(h;\bar{g}) = \sqrt{2}\,\mathcal{F}_\mu^{\alpha\beta}(\bar{g})\ h_{\alpha\beta}\,, \tag{2.41}$$

where $\mathcal{F}_\mu^{\alpha\beta}[\bar{g}]$ is an operator built using the background metric. A common choice is the harmonic (de Donder) gauge fixing, for which it holds

$$\mathcal{F}_\mu^{\alpha\beta}[\bar{g}] = \delta_\mu^\beta\,\bar{g}^{\alpha\gamma}\,\bar{\nabla}_\gamma - \frac{1}{2}\,\bar{g}^{\alpha\beta}\,\bar{\nabla}_\mu\,, \tag{2.42}$$

being $\bar{\nabla}$ the covariant derivative built using the background metric. The introduction of the gauge fixing term in the path integral can be balanced by noting that the variation of the gauge condition $\mathcal{F}_\mu$ under an infinitesimal gauge transformation, in the gravitational case the coordinate reparameterization $x^\mu \to x^\mu + \xi^\mu$, can be rewritten as an operator $\mathcal{M}(h)_\mu{}^\nu\,\xi_\nu$, being $\mathcal{M}$ the Fadeev-Popov operator. The integration of the δ-function over the gauge group gives then the inverse of the determinant of the operator $\mathcal{M}$, i.e.

$$\int d\xi\,\delta(\mathcal{F}_\mu(h;\bar{g})) = \int d\xi\,\delta(\mathcal{M}_\mu{}^\nu\,\xi_\nu) = \frac{1}{\det\mathcal{M}}\,. \tag{2.43}$$

Since in the background field method the variation of the background field is null, and the function $C(x)$ is invariant, then we have

$$\delta_\xi(\mathcal{F}_\mu(\bar{g},h) - C) = \mathcal{F}_\mu(\bar{g},\delta_\xi h) = \mathcal{F}_\mu^{\alpha\beta}(\bar{g})\,\delta_\xi h_{\alpha\beta}\,, \tag{2.44}$$

where it holds for the variation of the fluctuation

$$\delta_\xi h_{\mu\nu} = \delta(\gamma_{\mu\nu} - \bar{g}_{\mu\nu}) = \mathcal{L}_\xi\,\gamma_{\mu\nu} = \xi^\alpha\,\partial_\alpha\,\gamma_{\mu\nu} + \gamma_{\mu\alpha}\,\partial_\nu\,\xi^\alpha + \gamma_{\nu\alpha}\,\partial_\mu\,\xi^\alpha = \mathcal{Q}(\gamma)^\alpha_{\mu\nu}\,\xi_\alpha\,, \tag{2.45}$$

being $\mathcal{L}_\xi$ the covariant Lie derivative respect to the vector ξ^μ. The variation of the gauge fixing condition then reads

$$\delta_\xi\mathcal{F}_\mu(\bar{g},h) = \mathcal{F}_\mu^{\alpha\beta}(\bar{g})\,\mathcal{Q}(\gamma)^\nu_{\alpha\beta}\,\xi_\nu\,, \tag{2.46}$$

so that the Popov-Fadeev operator reads $\mathcal{M}_\mu{}^\nu(\gamma,\bar{g}) = \mathcal{F}_\mu^{\alpha\beta}(\bar{g})\,\mathcal{Q}(\gamma)^\nu_{\alpha\beta}$. The determinant of $\mathcal{M}$ can then expressed in terms of a Gaussian integral of an action term quadratic in auxiliary complex anticommuting variables C_μ ad $\bar{C}_\mu$, i.e. the ghost fields, which reads

$$S_{gh} = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\ C_\mu\,\mathcal{M}_\nu{}^\mu\,\bar{C}^\nu\,, \tag{2.47}$$

where the explicit expression of the operator $\mathcal{M}_\nu{}^\mu$ in the gauge (2.42) is

$$\mathcal{M}_\nu{}^\mu(\gamma,\bar{g}) = \bar{g}^{\mu\rho}\,\bar{g}^{\sigma\lambda}\,\bar{\nabla}_\lambda\,(g_{\rho\nu}\,\nabla_\sigma + g_{\sigma\nu}\,\nabla_\rho) - \bar{g}^{\rho\sigma}\,\bar{g}^{\mu\lambda}\,\bar{\nabla}_\lambda\,g_{\sigma\nu}\,\nabla_\rho\,. \tag{2.48}$$

Inserting the gauge fixing (2.40) and ghost action (2.47) in the path integral, the partition function (2.36) now reads

$$Z[J;\bar{g}] = \int \mathcal{D}[h_{\mu\nu}]\mathcal{D}[C^{\mu}]\mathcal{D}[\bar{C}_{\mu}]\; e^{-S[\bar{g}+h]-S_{gf}[h;\bar{g}]-S_{gh}[C,\bar{C},h;\bar{g}]+\int d^dx\sqrt{\bar{g}}\,\{t^{\mu\nu}\,h_{\mu\nu}+\bar{\sigma}_{\mu}\,C^{\mu}+\sigma^{\mu}\,\bar{C}_{\mu}\}}, \tag{2.49}$$

being σ_μ and $\bar{\sigma}_\mu$ the sources associated respectively to $\bar{C}^\mu$ and C^μ. The total action in (2.49) is invariant under the local transformations

$$\begin{aligned} \delta_s h_{\mu\nu} &= \epsilon\,\mathcal{L}_C\,\gamma_{\mu\nu}\,, & \delta_s\,\bar{g}_{\mu\nu} &= 0\,, \\ \delta_s\,C^{\mu} &= \epsilon\,C^{\nu}\,\delta_{\nu}\,C^{\mu}\,, & \delta_s\,\bar{C}_{\mu} &= \frac{\epsilon}{\alpha}\,\mathcal{F}_{\mu}\,, \end{aligned} \tag{2.50}$$

were $\mathcal{L}_C$ is the Lie derivative respect to the ghost field C^μ, ϵ is an anticommuting parameter and s is the charge associated to the invariance of the action under (2.50), which takes the name of BRST symmetry [63]. In particular, the invariance under the BRST symmetry group leads to Ward identities for the correlation functions which are called Slavnov-Taylor identities and read

$$\int d^dx \frac{1}{\sqrt{\bar{g}}}\Big\{\frac{\delta\,\Gamma'}{\delta\,\bar{h}_{\mu\nu}}\,\frac{\delta\,\Gamma'}{\delta\,\beta^{\mu\nu}} + \frac{\delta\,\Gamma'}{\delta\,\xi_{\mu}}\,\frac{\delta\,\Gamma'}{\delta\,\tau^{\mu}}\Big\} = 0\,, \tag{2.51}$$

being $\Gamma'_k \equiv \Gamma_k - S_{gf}[\bar{h},\bar{g}]$ and $\beta^{\mu\nu}$ and τ_μ respectively the sources associated to the BRST variations $\delta_s h_{\mu\nu}$ and $\delta_s C_\mu$.

Provided a consistent path integral formulation for the quantization of gravity, it is then possible to build a non-perturbative renormalization group equation by defining a proper cutoff operator.

2.2.4 Exact equation for gravity

We want now to write an exact RG equation for gravity in which the cutoff operator $\Delta_k S[h;\bar{g}]$ is built in such a way to perform the coarse graining in a background independent way. In the background field formalism such an operator is quadratic in the fluctuation $h_{\mu\nu}$ and in its generic form reads

$$\Delta_k\,S[h,C,\bar{C};\bar{g}] = \frac{1}{16\,\pi\,G_k}\int d^dx\sqrt{\bar{g}}\;h_{\mu\nu}\,\mathcal{R}_k^{grav}[\bar{g}]^{\mu\nu\rho\sigma}\,h_{\rho\sigma} + \sqrt{2}\int d^dx\sqrt{\bar{g}}\;\bar{C}_{\mu}\,\mathcal{R}_k^{gh}[\bar{g}]\,C^{\mu}\,, \tag{2.52}$$

where $\mathcal{R}^{grav}$ and $\mathcal{R}^{gh}$ are respectively matrices in field space for the metric tensor and ghosts. The cutoff R_k is then constructed in such a way to cut the modes of the fluctuating fields (h and the ghosts) with eigenvalues of the Laplacian $\bar{p}^2 < k^2$, being $\bar{p}^2$ the momentum built from the background metric $\nabla^2(\bar{g})$. Its generic expression reads

$$\mathrm{R}_k[\bar{g}] = \mathcal{Z}_k\,k^2\,\mathcal{R}^{(0)}\Big(-\frac{\bar{\nabla}^2}{k^2}\Big)\,, \tag{2.53}$$

where $\mathcal{Z}_k$ is a renormalization function in field space, and where the dimensionless function $\mathcal{R}^{(0)}$ interpolates between $\mathcal{R}^{(0)}(0) = 1$ and $\mathcal{R}^{(0)}(\infty) = 0$, as already introduced in the subsection 1.2.2.

By defining the field $\psi = \{h_{\mu\nu}, C_\mu, \bar{C}_\mu\}$ and source $J = \{t^{\mu\nu}, \bar{\sigma}^\mu, \sigma^\mu\}$ the average partition function at the scale k can then be written as

$$Z_k[J;\bar{g}] = e^{-W_k[J;\bar{g}]} =$$
$$\int \mathcal{D}[\psi] \exp\Big\{-S[\bar{g}+h] - S_{gf}[h;\bar{g}] - S_{gh}[\psi;\bar{g}] - \Delta_k\, S[\psi;\bar{g}] + \int d^dx\sqrt{\bar{g}}\,\psi(x)J(x)\Big\}\,, \tag{2.54}$$

where $\mathcal{D}[\psi] = \mathcal{D}[h_{\mu\nu}]\mathcal{D}[C^\mu]\mathcal{D}[\bar{C}^\mu]$. The effective average action $\Gamma_k[h;\bar{g}]$ is then obtained in the standard way, by introducing $\Psi(x) = \langle\, \psi \,\rangle_J$, being $\Psi = \{\bar{h}_{\mu\nu}, \bar{\xi}^\mu, \xi^\mu\}$ and

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}}, \qquad \bar{\xi}^\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta \bar{\sigma}_\mu}, \qquad \bar{\xi}_\mu = \frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta \sigma^\mu}\,, \tag{2.55}$$

and by performing a Legendre transform

$$\Gamma_k[\Psi;\bar{g}] = -W_k[J;\bar{g}] + \int d^dx\sqrt{\bar{g}}\; J\cdot\Psi - \Delta_k\, S[\Psi;\bar{g}]\,. \tag{2.56}$$

The exact renormalization equation (1.52) can be casted in its generic form for the effective action (2.56) as

$$k\,\partial_k\,\Gamma_k[\Psi] = \frac{1}{2}\,\mathrm{STr}\left[\frac{k\,\partial_k\,R_k}{\Gamma_k^{(2)} + R_k}\right], \tag{2.57}$$

where now Str identifies a functional supertrace over the metric fluctuation and the ghost fields, and the cutoff operator R_k follows from (2.52). Because of the presence of the cutoff operator modified Ward identities which read

$$\int d^dx \frac{1}{\sqrt{\bar{g}}}\left\{\frac{\delta\,\Gamma'}{\delta\,\bar{h}_{\mu\nu}}\,\frac{\delta\,\Gamma'}{\delta\,\beta^{\mu\nu}} + \frac{\delta\,\Gamma'}{\delta\,\xi_\mu}\,\frac{\delta\,\Gamma'}{\delta\,\tau^\mu}\right\} = Y_k\,, \tag{2.58}$$

being $\Gamma'_k \equiv \Gamma_k - S_{gf}[\bar{h},\bar{g}]$ and Y_k an integral operator whose expression is reported in [64]. Since the term Y_k takes contribution from operators proportional to the cutoff function, and since it comes from the same exact equation that leads to the effective action in the limit $k\to 0$, it entails that Y_k also runs to zero in that limit, recovering then the standard Slavnov-Taylor equation (2.51). Being this a property of the exact solution, it does not hold however for approximated solutions. As already mentioned, equation (2.57) cannot be solved exactly but just by means of approximation, so that a consistent strategy in the definition of a hierarchy of truncations needs then to be employed. As a first approximation we can neglect the running of the ghosts[5], thus considering

$$\Gamma_k[\bar{h};\bar{g}] \equiv \Gamma_k[\bar{h},0,0;\bar{g}]\,. \tag{2.59}$$

[5]After having evaluated the flow equation, since their contribution to the second variation survives and affects the RG flow of the other couplings.

A second approximation consists in separating in the action the contributions coming from the operators depending on $g = \bar{h} + \bar{g}$ from those which shows a separate dependence on the fluctuation and the background, i.e. to consider

$$\Gamma_k[\bar{h};\bar{g}] = \bar{\Gamma}_k[g = \bar{h} + \bar{g}] + \hat{\Gamma}_k[\bar{h};\bar{g}] + S_{gf}[\bar{h};\bar{g}]\,, \tag{2.60}$$

where $\bar{\Gamma}$ encodes the contributions from $g = \bar{g}$ and $\hat{\Gamma}$ the deviations $g \neq \bar{g}$. We can then take in consideration to fix $\hat{\Gamma}_k = 0$ along all the flow, identifying then

$$\Gamma_k[g] = \Gamma_k[h = g - \bar{g} = 0; \bar{g}]\,, \tag{2.61}$$

To justify the assumption (2.61) we can note that putting (2.60) in (2.58) we find he action (2.61) to satisfy the standard Ward identities. and that Y_k gets contributions only from $\hat{\Gamma}_k$. The approximation (2.61) entails then that we neglect Y_k, which is acceptable since Y_k is an higher loop term. The RG equation for the action (2.61) is a non-perturbative functional integro-differential equation that is however still difficult to solve. A common approximation is thus to restrict our parameter space to a finite dimensional one, that is, to take in consideration a polynomial truncation of the effective action.

2.2.5 Polynomial truncations

The simplest polynomial truncation which can be taken in consideration is the Einstein-Hilbert truncation, which reads

$$\Gamma_k[g] = -\frac{1}{16\,\pi\,G_k} \int d^dx\,\sqrt{g}\,\{R(g) - 2\Lambda_k\}\,, \tag{2.62}$$

being G_k the renormalized Newton constant, and such that $G_{k=\Lambda} = \bar{G}$ with $\bar{G}$ the bare coupling. Putting (2.62) in (2.57) we have on the left hand side

$$k\,\partial_k\,\Gamma_k[g] = -\frac{1}{16\,\pi} \int d^dx\sqrt{g}\,\left\{\left(k\,\partial_k \frac{1}{G_k}\right)\,R(g) - k\,\partial_k\,\left(\frac{\Lambda_k}{G_k}\right)\right\}\,, \tag{2.63}$$

while on the right hand side the supertrace splits in a trace over the metric degrees of freedom and the ghosts,

$$\frac{1}{2}\,\mathrm{STr}\left[\frac{k\,\partial_k\,R_k}{\Gamma_k^{(2)} + R_k}\right] = \frac{1}{2}\,\mathrm{Tr}\left[\frac{(k\,\partial_k\,\hat{\mathcal{R}}_k)_{\bar{h}\bar{h}}}{(\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)_{\bar{h}\bar{h}}}\right] - \mathrm{Tr}\left[\frac{(k\,\partial_k\,\hat{\mathcal{R}}_k)_{\bar{\xi}\xi}}{(\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)_{\bar{\xi}\xi}}\right]\,, \tag{2.64}$$

where the latter term takes a factor -2 from the trace over Grassmannian complex structure. The trace over the ghost sector has a trace over a vector space, and

$$(\Gamma_k^{(2)} + \hat{\mathcal{R}}_k)_{\bar{\xi}\xi} = -\mathcal{M}[g,\bar{g}] + R_k^{gh}[\bar{g}]\,, \tag{2.65}$$

being $\mathcal{M}[g,\bar{g}]$ the Fadeev-Popov operator (2.48). The trace over the metric degrees of freedom can be evaluated by means of a traceless decomposition of the fluctuation

$$\bar{h}_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\,\bar{g}_{\mu\nu}\,\bar{h}\,, \tag{2.66}$$

being $\bar{h}$ the trace of $\bar{h}_{\mu\nu}$, namely $\bar{h} = \bar{h}_{\mu\nu}\,\bar{g}^{\mu\nu}$, and where $\hat{h}_{\mu\nu}$ is a traceless tensor, so that it satisfies $\hat{h}_{\mu\nu}\,\bar{g}^{\mu\nu} = 0$. In term of this field decomposition the second functional derivative reads

$$\begin{aligned}(\Gamma_k^{(2)})_{\bar{h}\bar{h}} = \frac{1}{16\,\pi\,G_k}\int d^dx\sqrt{\bar{g}}\Bigg\{\frac{1}{2}\,\hat{h}_{\mu\nu}\left(-\bar{\nabla}^2 - 2\,\bar{\Lambda}_k + \bar{R}\right)\hat{h}^{\mu\nu}\\ -\left(\frac{d-2}{4\,d}\right)\bar{h}\left(-\bar{\nabla}^2 - 2\,\bar{\Lambda}_k + \left(\frac{d-4}{d}\right)\bar{R}\right)\bar{h} - \bar{R}_{\mu\nu}\,\hat{h}^{\nu\rho}\,\hat{h}^{\mu}_{\rho}\\ +\bar{R}_{\alpha\beta\nu\mu}\,\hat{h}^{\beta\nu}\,\hat{h}^{\alpha\mu} + \left(\frac{d-4}{d}\right)\bar{h}\,\bar{R}_{\mu\nu}\,\hat{h}^{\mu\nu}\Bigg\}\,.\end{aligned} \tag{2.67}$$

The evaluation of the trace and the expression of the resulting beta functions depends on the choice of the cutoff operator (see [65] for the exponential operator and [51] for the optimized) and the spectra can be summed using heat kernel techniques on curved spacetime, i.e. collecting Seeley-Gilkey coefficients for the operators present in (2.67). The technical details about the calculation of the trace are however here omitted, since they will be discussed in detail in chapter 3.

Once evaluated the traces in (2.64) it is possible to expand the result in powers of the Ricci scalar and discard terms $\mathcal{O}(\bar{R}^2)$, so to match the operators present in (2.63). Writing then both sides of the equation in terms of dimensionless couplings $G_k = g_k\,k^{2-d}$, $\Lambda_k = \lambda_k\,k^2$, a set of coupled β-functions for the two couplings can be obtained as

$$\begin{cases}\beta_g(g,\lambda) = k\,\partial_k\,g_k\\ \beta_\lambda(g,\lambda) = k\,\partial_k\,\lambda_k\end{cases}. \tag{2.68}$$

Scale invariant theories can then be searched by looking for common zeroes of the system (2.68).

Although the results show a certain dependence on the shape of the cutoff function both sets of β-functions show a trivial repulsive Gaussian fixed point, $\{\lambda^* = 0, g^* = 0\}$, and a non-Gaussian fixed point for non-zero values of the couplings.

The ultraviolet stability of the NGFP, that is, the dimensionality of the ultraviolet critical surface, can be investigated by analyzing the spectra of eigenvalues of the $\mathcal{B}$ matrix, which reads

$$\mathcal{B} = \left.\begin{pmatrix}\partial_g\,\beta_g(g,\lambda) & \partial_\lambda\,\beta_g(g,\lambda)\\ \partial_g\,\beta_\lambda(g,\lambda) & \partial_\lambda\,\beta_\lambda(g,\lambda)\end{pmatrix}\right|_{\{g,\lambda\}=\{g^*,\lambda^*\}}. \tag{2.69}$$

The matrix (2.69) has, for the NGFP, a couple of complex conjugated eigenvalues $\theta_\pm = -\theta_1 \pm \theta_2$, being the physical critical exponents respectively minus the real part and the imaginary part of the eigenvalue, and we have in $d = 4$ for both cutoff functions

	g^*	λ^*	$g^*\lambda^*$	θ_1	θ_2
exp	0.272	0.359	0.098	1.422	4.307
opt	1.178	0.250	0.294	1.667	4.308

(2.70)

As it can be seen in (2.70) both RG schemes lead to positive critical exponents, entailing the UV-attractive behaviour of the non-Gaussian fixed point and assuring the dimensionality of the critical surface to be at least equal to two.

The system of equation (2.68) is a coupled pair of non linear ordinary differential equation (ODE) which can be integrated from a certain initial scale k_0 just by giving an initial conditions $g_{k_0} = g_0$ and $\lambda_{k_0} = \lambda_0$. In Fig. 2.3 is depicted the results of an integration for an interesting range in the parameter space.

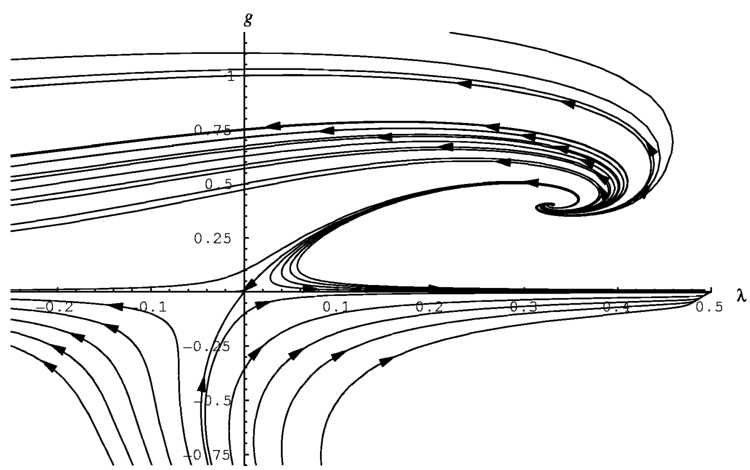

Figure 2.3: Phase diagram of quantum gravity in the Einstein-Hilbert truncation.

The spiral behaviour of the trajectories is caused by the complex character of the critical exponents. We can identify in Fig. 2.3 two kind of interesting trajectories emanating from the fixed point. One bundle goes to negative values of the cosmological constant, while the other one has a slow transient near the Gaussian fixed point and is then pushed away from it by the relevant direction of the cosmological constant. The latter bundle of trajectories, besides having a good semiclassical regime, encounters anyway a singularity in the infrared regime at $\lambda = 1/2$, attributed so far to the Einstein-Hilbert truncation.

Interestingly, not only the UV fixed point survives from $d = 2$ up to $d = 4$, but it happens to exist also for higher dimensions, where the critical value of the spacetime dimension at which the fixed point disappears shows for this truncation a strong dependence on the cutoff used. Moreover, reducing the dimensionality to $d = 2+\epsilon$ the ERGe gives for the β-function of the Newton's constant the same results (2.25) obtained by the ϵ-expansion, for all number of matter fields equal to zero.

It is then relevant to ask whenever the fixed point is an artifact of the truncation or whenever it survives enlarging the parameter space. In literature it has been proposed a polynomial truncation in power of the Ricci scalar, that reads

$$\Gamma_k[g_{\mu\nu}] = \int d^dx\,\sqrt{g}\,\sum_{i=0}^{N} g_i(k)\,R(g)^i\,, \tag{2.71}$$

being $g_0(k) = \Lambda_k/G_k$ and $g_1(k) = G_k$. The present truncation is not a truncation in powers of the field, i.e. the metric, which enters non polynomially in the curvature, but is still a quite natural way to organize a truncation in invariants of the symmetry.

The powers of the Ricci scalars in the polynomial ansatz have been increased in the program of a systematic study, $N = 2$ [66, 67], $N = 6$ [68, 69], $N = 8$ [70], $N = 10$ [71] and $N = 35$ [72], showing all the other directions to be irrelevant, and critical exponents to quickly converge after $N = 3$ to their exact values. In [73, 74] the RG flow has been studied adding a squared Weyl tensor $C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau}$ to the $\mathcal{O}(R^2)$ truncation. A particularity of the flow equation in the case of Weyl tensor is that the critical exponent become real, losing then the characteristic spiral of the linearized behavior around the fixed point.

2.2.6 Non-polynomial truncations

One of the drawbacks of using a polynomial truncations is anyway the appearance of spurious solutions, i.e. unphysical zeroes of the β-functions which are artifacts of the approximation. Altough those artifacts can be present for both infinite and finite dimensional truncations, the artifacts obtained solving the β-functions often disappear when we extend the dimension of the parameter space from finite to infinite.

It is then legitimate to wonder whether the NGFP for gravity is an artifact of the polynomial truncation (2.71) or not, which automatically translates in asking if it exists a non-trivial solution in an infinite dimensional parameter space truncation for the for the gravitation action.

One possibility is to investigate the existence of a non-trivial solution for the analogous for the Ricci scalar of the local potential approximation defined for the scalar field theory (1.34), which reads

$$\Gamma_k[g] = \frac{1}{16\,\pi\,G_k}\int d^dx\,\sqrt{g}\,f(R)\,. \tag{2.72}$$

The latter can in fact be considered as the simpler functional truncation of the effective action since it contains just one degree of freedom more then general relativity, that is a quartic derivative of the metric trace, which in general relativity is not a dynamic degree of freedom. Sixth or higher derivatives for the traceless sector of the Hessian arise just from derivatives of the Ricci tensor, like $R\,\nabla^2 R$ or $R_{\mu\nu}\nabla^2 R^{\mu\nu}$.

Different versions of the flow equation for the action (2.72) have been obtained in [69, 70, 75] for a spherical topology and further studied in [76, 77] for $d = 4$ and in [78] for $d = 3$. The identification of a NGFP fixed point consist then in the resolution of a partial differential equation

$$k\,\partial_k\,\tilde{f}_k(\tilde{R}) = 0\,, \tag{2.73}$$

where $\tilde{f}_k(\tilde{R})$ is a dimensionless function defined as

$$\tilde{f}_k(\tilde{R}) = k^{-d}\,f_k(k^2\tilde{R})\,, \tag{2.74}$$

being $\tilde{R}$ the dimensionless Ricci scalar. However, the identification of a non-trivial solution $\tilde{f}^*$ is itself not trivial. A functional RG flow equation introduces in fact a certain number of complications not present in the β-function approach and which make difficult to find global solutions (a detailed discussion about the strategy to solve (2.73) is left for the chapter 4).

The RG equation (2.73) is a partial differential equation of third order in the function $\tilde{f}(\tilde{R})$, so that it can be rewritten in normal form as

$$f'''(\tilde{R}) = \frac{\mathcal{N}(\tilde{f},\tilde{f}',\tilde{f}'',\tilde{R})}{\mathcal{P}(\tilde{R})\,\mathcal{D}(\tilde{f},\tilde{f}',\tilde{R})}\,, \tag{2.75}$$

where $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{D}$ are polynomials in their arguments, and we assume that $\mathcal{D}$ has no zeroes in R for generic f and f'. A fixed point solution is then a global solution of equation (2.75), which can be found by studying the solution in the large $\tilde{R}$ behavior and integrating it back to $\tilde{R} = 0$. Being a third order equation, equation (2.75) normally accept a 3-parameter family of solutions parametrized by the three initial conditions. However, most of the solutions will encounter a fixed or movable singularity and diverge, so that just a discrete set of initial conditions will lead to global solutions. Movable singularities are singularities of the solutions of (2.75) that occur at some value $R = R_c$ which depends on the initial conditions, and they are due to the non-linearity of the equation. Fixed singularities are instead zeroes of the polynomial $\mathcal{P}(\tilde{R})$, and they entail that the space of solution is constrained by supplementary analyticity conditions. Those conditions, however, reduce the number of free global degrees of freedom, hence reducing the dimensionality of the space of solution. Nonetheless, equation (2.75) posses a continuos set of solutions.

Although the result in [75, 77] is encouraging, it is however not definitive. A physical solution, to be such, should be a global solution in the whole $\tilde{R}$ field space, which means that should be prolongable to the $R < 0$ region without encountering a singularity; we expect then just a discrete set of physical solutions to satisfy this condition. A flow equation for hyperbolic geometries, however, has still not been obtained.

An important extension of the result in [75, 77] would be, for example, to consider the contribution of a generic function of the Weyl tensor $C_{\mu\nu\rho\sigma}$, that is $f(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$, since in [73] it has been shown how the contribution of such a tensor influences largely the universality class of the theory (in particular the critical exponents, that now become real) and the unitarity, since the massive pole is supposed to move to infinity in the UV regime.

2.3 Hořava-Lifshitz gravity

Although there is a strong evidence that gravitation can be renormalizable at a non-Gaussian fixed point, it is however not easy to give a definitive statement about its unitarity. Nonetheless, it is also possible that in the higher energy regime gravitation develops some kind of mechanism which shows explicitly the unitary of the microscopic theory; for example the emergence of a spacetime anisotropy. The incurrence of a scaling anisotropy between space and time, in fact, would lift the power of spatial derivatives, granting then the perturbative renormalizability typical of the higher derivative theories, while keeping the maximum number of time derivatives to two, hence avoiding the presence of ghost poles in the effective propagator. The price to pay, however, is to lose Lorentz invariance, which thus emerges as an accidental symmetry in the infrared limit.

2.3.1 Spacetime anisotropy

The way the spacetime anisotropy is introduced in quantum gravity mimic the way it is used in critical phenomena. In the latter, in fact, anisotropy is a common feature of condensed matter systems, especially those which exhibit spatially modulated phases. The first application dates back to 1975 [79], where it was introduced to characterize the modulated phase in ferromagnetism by employing a Landau free energy density of the type

$$F(\phi) = c\,(\nabla_m\phi)^2 + (\nabla_m^2\phi)^2 + (\nabla_n\phi)^2 + r\,\phi^2 + \lambda\,\phi^4\,, \tag{2.76}$$

being ϕ a scalar field, c a coupling constant, and ∇_m and ∇_n are spatial gradients with dimensionality m and $n = d - m$, being d the dimension of the space. The modulated phase occurs when c becomes negative and it is the result of the stabilization that the higher derivative operator exerts on the unstable m-dimensional second derivative operator. As a consequence of this anisotropy the phase diagram present three phases which encounter at a multi-critical point which takes the name of Lifshitz critical point, and the degree of anisotropy is characterized by the dynamical critical exponent z ($z = 2$ in (2.76)).

In the same way is it possible to define a non-relativistic free field theory for a

Lifshitz scalar field in $d+1$ dimensions (that is, $n=1$ in (2.76)), whose action reads

$$S[\phi] = \int dt\, d^d\mathbf{x} \left\{ -\phi(t,\mathbf{x}) \left(\partial_t^2 - \partial_{\mathbf{x}}^{2\,z}\right) \phi(t,\mathbf{x}) \right\} . \tag{2.77}$$

The scaling of the action (2.77) is controlled by the critical exponent z so that space and time scale according to their mass (classical) dimensions

$$[\mathbf{x}] = -1, \qquad [t] = -z\,, \qquad\qquad [\partial_{\mathbf{x}}] = 1, \qquad [\partial_t] = z\,, \tag{2.78}$$

and the differential operator entering in (2.77) is marginal. For such an action with $z > 1$ the operator

$$c^2 \int dt\, d^d\mathbf{x}\, \phi(t,\mathbf{x})\, \partial_{\mathbf{x}}^2\, \phi(t,\mathbf{x})\,, \tag{2.79}$$

acts as a relevant deformation, being c^2 a dimensionful speed of light, which becomes important in the infrared limit, when the operator $\partial_{\mathbf{x}}^{2z}$ becomes irrelevant. The effective propagator of the theory, i.e.

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z}\,, \tag{2.80}$$

where $\mathbf{k}$ is the spatial momentum and G a coupling, can be considered in fact as a low energy perturbation of the ultraviolet propagator, that is

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z} = \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \frac{1}{\omega^2 - G(\mathbf{k}^2)^z}\, c^2\mathbf{k}^2\, \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \dots\,. \tag{2.81}$$

2.3.2 Anisotropic gravitational actions

An action for the ultraviolet gravitational theory for a generic critical exponent z can be built using the ADM decomposition [11, 12]. Employing this formalism the time and space components of the full metric tensor decompose in a clear way, i.e. the microscopic d+1 dimensional metric tensor $\gamma_{\mu\mu}$ is decomposed in a spatial metric tensor $g_{\mu\nu}$ plus a lapse function N and shift vector N^μ. The standard ADM splitting techniques is briefly summarized in the appendix B. For a generic z the scaling dimensions of the ADM variables read

$$[g_{ij}] = 0, \qquad [N_i] = z-1, \qquad [N] = 0\,, \tag{2.82}$$

and the diffeomorfisms group Diff($\mathcal{M}$) is broken and substituted with a foliation preserving group Diff$_{\mathcal{F}}$($\mathcal{M}$) which consists of the coordinate reparametrization

$$\tilde{x}^i = \tilde{x}^i(x^j, t), \qquad \tilde{t} = \tilde{t}(t)\,. \tag{2.83}$$

In the local coordinate set the generators of the algebra of Diff$_{\mathcal{F}}$($\mathcal{M}$) are given by

$$\delta x^i = \zeta^i(t,\mathbf{x}), \qquad \delta t = f(t)\,. \tag{2.84}$$

A natural spacetime topology in presence of a foliation is

$$\mathcal{M} = \mathcal{N} \times \Sigma \,, \tag{2.85}$$

where we will consider Σ to be S^1 in order to avoid problems coming from non compact manifolds, and where $\mathcal{N}$ is a generic spatial d-dimensional Riemannian manifold. The fields N and N^i can then be seen as Legendre multipliers related, respectively, to the time and space reparameterization in (2.84). Both fields have to be intended as spacetime dependent, i.e. $N \equiv N(t,\mathbf{x})$, $N^i \equiv N^i(t,\mathbf{x})$, altough this general dependence may introduce many difficulties in the quantization procedure. It makes then sense to restrict our interest, whenever necessary, to the study of a projectable scenario, where with the term projectable we mean that an operator takes the same value over all the leafs Σ_t (i.e. are function of the sole time variable). The projectable scenario is then simply defined taking a lapse function constant over the leaf, $N(t,\mathbf{x}) = N(t)$.

The ADM variables transform under the new symmetry group as

$$\begin{aligned} \delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f\, \dot{g}_{ij} \,, \\ \delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f}\, N_i + f\, \dot{N}_i \,, \\ \delta N &= \zeta^j \partial_j N + \dot{f}\, N + f\, \dot{N} \,. \end{aligned} \tag{2.86}$$

To define an action for the anisotropic model we need then to build invariants under the reparameterization (2.84). The kinetic term must be constructed using time derivative of the metric, and it can be proved that the only invariant can be built from the extrinsic curvature tensor K_{ij}, which reads

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - D_i N_j - D_j N_i \right) \,, \tag{2.87}$$

being D_i the spatial covariant derivative and where the dot stands for time derivative. The most generic kinetic action is then given by contractions of the extrinsic curvature, i.e.

$$S_K[N, N^i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x}\, \sqrt{g}\, N \, \left(K_{ij}K^{ij} - \lambda K^2 \right) \,, \tag{2.88}$$

being λ a dimensionless coupling and $\kappa = 32\,\pi\, G$, with G the Newton's constant. Both operators in (2.88) are separately invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, so that the value of λ is left free and acquires its own running due to quantum corrections, differently from general relativity in which symmetry imposes a value $\lambda = 1$. Since general relativity is expected to be recovered in the infrared, however, it must flow to $\lambda_{IR} = 1$. The presence of the parameter λ appears now also in the DeWitt's "metric on the space of metrics", which reads

$$G^{ijk\ell} = \frac{1}{2} \left(g^{ik} g^{j\ell} + g^{i\ell} g^{jk} \right) - \lambda g^{ij} g^{k\ell} \,. \tag{2.89}$$

Note that in the kinetic action (2.88) the critical exponent z does never enter in the expression of the operators but just in the dimension of the integration measure, that is

$$[dt\, d^d\mathbf{x}] = -d - z \,, \tag{2.90}$$

that is the key feature of the model, since now the Newton's constant has dimension

$$[\kappa] = \frac{z-d}{2} \,. \tag{2.91}$$

The coupling κ is then marginal for $z = d$ and the theory is power counting strictly renormalizable.

In the full action, however, we have to include all the terms that are invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ with dimension equal or less than the kinetic operator, that is $[K^2] = 2\mathrm{z}$. Although there are no other invariants that can be built from time derivatives, we can still take into account many spatial operator built contracting spatial Riemann tensors, and that define the interacting content of the theory, i.e. a potential term

$$S_V[N, N^i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x}\, \sqrt{g}\, N\, V(g_{ij})\,, \tag{2.92}$$

so that the full action reads

$$S[N, N^i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x}\, N\sqrt{g}\, \left\{K_{ij}K^{ij} - \lambda K^2 + V(g_{ij})\right\} \,. \tag{2.93}$$

For $d = 3$ and $z = 3$, for example, the action (2.92) will contain marginal operators like

$$R^3\,, \qquad R^{ij}\, R_{jk}\, R^k{}_i\,, \qquad R\, R_{ij}R^{ij}\,, \tag{2.94}$$

that being cubic in the curvature represent pure interaction terms, then additional relevant terms as

$$D^2R^2\,, \qquad D_kR\, D^iR\,, \qquad D_kR_ij\, D^kR^{ij}\,, \qquad D^4\, R\,, \tag{2.95}$$

and so on. The number of invariants, anyway, grows fast with the spacetime dimension, and already for $d = 3$ and $z = 3$ it leads to a scenario in which the quantization involves rather difficult calculations. One way to reduce the number of free parameters is to assume that the action, for example, satisfies a detailed balance condition.

2.3.3 Detailed balance condition

The detailed balance condition is a further symmetry shared by many systems in critical phenomena which simplifies the study of the renormalization properties of a system in higher dimensions. Normally, the renormalization of a d-dimensional system are simpler than those of a $d+1$-dimensional one, because of the higher number of relevant parameters. When the theory satisfies the detailed balance condition the renormalization of the $d+1$-dimensional system can be put in relation to the renormalization of the lower dimensional system. It the context of Lifshitz gravity it can be casted as the assumption that the potential action (2.92) can be written as

$$S_V[N, N^i, g_{ij}] = \frac{2}{\kappa^2} \int dt\, d^d\mathbf{x}\, \sqrt{g}N\, E^{ij}G_{ijk\ell}E^{k\ell}\,, \tag{2.96}$$

where the tensor E^{ij} comes from some variational principle

$$\sqrt{g}\, E^{ij} = \frac{\delta W[g_{k\ell}]}{\delta g_{ij}}\,, \tag{2.97}$$

being W some d-dimensional action. The detailed condition implies here that the renormalization of the coupling contained in the potential can be reduced to study the renormalization of the simpler action W, plus the quantum corrections coming from the kinetic operator (2.88). In $d = 3$ in order to have a $z = 3$ theory in the UV we need then the tensor E^{ij} to contain third order spatial derivatives, and the only tensor which satisfies all the symmetries is the Cotton tensor

$$C^{ij} = \varepsilon^{ik\ell}\nabla_k \left(R^j{}_\ell - \frac{1}{4}\, R\, \delta^j \ell \right)\,, \tag{2.98}$$

which exhibits many properties, as being

- i) symmetric and traceless, $C^{ij} = C^{ji}\,,\ g_{ij}C^{ij} = 0$,
- ii) transverse, $\nabla_i C^{ij} = 0$,
- iii) conformal, with conformal weight $-5/2$.

The latter entails that, under local spatial Weyl transformations

$$g_{ij} \to e^{\phi(\mathbf{x})}\, g_{ij}\,, \tag{2.99}$$

the Cotton tensor transforms as

$$C^{ij} \to e^{-\frac{5}{2}\phi(\mathbf{x})}\, C^{ij}, \tag{2.100}$$

with no terms containing derivatives of $\phi(\mathbf{x})$. The fact that the Cotton tensor is invariant under spatial Weyl rescaling can also suggest the full action to be classically covariant under some kind of engineered anisotropic Weyl transformation.

2.3.4 Anisotropic Weyl invariance

It can be proved[6] that the full action with $E_{ij} = C_{ij}$ and $\lambda = \frac{1}{3}$ is invariant under a scaling transformation of the spatial metric

$$g_{ij} \to e^{\phi(t,\mathbf{x})}\, g_{ij}\,, \tag{2.101}$$

where now $\phi(t, \mathbf{x})$ is a function of both time and space coordinates, and rescaling of shift and lapse as

$$N \to e^{\frac{3}{2}\phi(t,\mathbf{x})}\, N\,, \qquad N_i \to e^{\phi(t,\mathbf{x})}\, N_i\,. \tag{2.102}$$

[6] See appendix F for the proof.

As a consequence of (2.102), however, the anisotropic conformal invariance is not obtained in the projectable case, in which case the lapse function is space-independent, so that the conformal invariance reduces to classical scale symmetry, defined for $\phi = \text{const}$ and N gauged to one (that is, on a flat background). We expect anyway Weyl invariance to be violated by quantum corrections.

Restricting now our interest to the action (2.93), we can assume the ultraviolet behavior of the theory as described by three couplings, namely κ, λ and the (inverse) coupling of the marginal operator in the potential, that we will call w. Of those three couplings, however, just one, i.e. w, controls the interaction strength in the perturbative expansion. The asymptotically free limit of the theory consist then in taking the limit $w \to 0$, while keeping constant λ and the ratio

$$\gamma = \frac{w}{k} \,. \tag{2.103}$$

This defines a priori a two-dimensional manifold of free fixed points in parameter space, so that we expect the theory to flow from the interacting regime to a free theory parameterized by λ and γ. The identification of such a point depends of course on quantum corrections, since the Newton's constant is now a marginal operator.

The quantization of the theory follows the one proposed in the section 2.2 for the Lorentz-invariant theory, besides differs from it for the gauge fixing sector . The detailed description of the quantization procedure is anyway left for the chapter 5, where we will present the flow of couplings in a lower dimensional case.

Chapter 3

Asymptotic safety in conformal gravity

The strong evidence of the existence of a NGFP for gravity has led the community to focus more on the mechanism behind the presence of such a fixed point. For example, the authors of [80] have studied the separate contributions of paramagnetic (potential) and diamagnetic (kinetic) interactions to the RG flow (in the sense of Nielsen [81]), and they found the paramagnetic one to be the one responsible for the presence of a fixed point in $d = 4$. Interestingly, in [6, 7] it has also been found that the fixed point survives when neglecting the graviton's contribution to the RG flow, that is by considering the RG flow of the sole conformal sector.

Although the conformal degree of freedom of the metric does not propagate in general relativity, thus being a pure gauge degree of freedom, a renormalization group study of the flow equation for the sole conformal sector has led to a phase diagram that is qualitatively correct and describes surprisingly well the universality class of the full theory. The importance of the scalar sector can, moreover, be understood in the $f(R)$ theory [75], in which the scalar sector of the fRG equation is the only one containing $f'''(R)$, granting then the third order character to the PDE and the existence of a continuos set of fixed point solutions. As we will see in this chapter, the study of the conformally reduced Einstein-Hilbert action (CREH) has emphasized the importance of the requirement of background-independence in the quantization of gravitational theories. The scalar sector of the Einstein-Hilbert action assumes, in fact, the form of a kinetically unstable massive $\lambda\,\phi^4$ scalar field theory, being here $\phi(x)$ a dimensionless Weyl rescaling field. The non-perturbative quantization of the theory with the requirement of background-independence leads to a phase diagram which presents an attractive NGFP; hence quite a different scenario from that of the standard $\lambda\,\phi^4$ theory, which it is well known to have no non-trivial fixed points in $d = 4$.

The conformally reduced case can then be considered as a scalar toy model useful to investigate gravity in an easier context, since the field space is now one dimensional and the action can be projected on a flat background metric. The opportunity to work on a flat spacetime allows us to avoid the use of complicated heat kernel techniques, and, as we will see in chapter 4, this is also the reason why we will try to find $f(R)$

non-perturbative fixed point solutions by solving instead the flow equation for a simpler scalar-tensor model, namely the Brans-Dicke theory, though in this case we will quantize all the degrees of freedom of the theory.

In this chapter we will study the scalar toy model in the proper time RG scheme introduced in 1.2.3, and in particular the RG flow equation for the CREH action, focusing on the analysis of the dependence of the universality class from the variation of the cutoff parameter n. We will then extend our analysis to a non-polynomial truncation, i.e. a local potential approximation, integrating the Wilsonian potential down to $k \to 0^+$ (where it coincides with the effective potential) and study the possibility to have a broken phase in the infrared regime, that cannot be investigated within the CREH truncation[1].

In order to propose a comparison with the results obtained for the full gravitational action we propose here a study of the RG equation for the Einstein-Hilbert action using a proper time scheme, generalizing to arbitrary d dimensions what has been done in [82] for $d = 4$.

3.1 RG flow equation for the Einstein-Hilbert action

We start here directly from the RG equation in the proper time formalism, since a detailed description of the background field method and gauge fixing sector has already been presented in section 2.2. The proper time RG equation (1.65) reads for gravity

$$k\,\partial_k\, S_k[h;\bar{g}] = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho_k(s))\,\mathrm{STr}\,\mathcal{H}(s;S_k^{(2)}[h;\bar{g}])\,. \tag{3.1}$$

where $S_k[h;\bar{g}]$ is the Wilsonian gravitation action at scale k, being h the metric fluctuation and $\bar{g}$ the background metric and $\rho_k(s)$ a cutoff function. The heat kernel $\mathcal{H}(s;S_k^{(2)}[h;\bar{g}])$ has matrix elements

$$\mathcal{H}(s,x,x';S_k^{(2)}[h;\bar{g}]) = \langle x\,|\,e^{-s\,S_k^{(2)}[h;\bar{g}]}\,|\,x'\rangle\,, \tag{3.2}$$

where $S_k^{(2)}[h;\bar{g}] = \delta^2 S_k^{(2)}[h;\bar{g}]/(\delta h\,\delta h)$, being $\delta^2 S_k^{(2)}[h;\bar{g}]$ is the second variation of the action in the background field formalism. The action S_k in (3.1) is the full action, $S \equiv S_{EH} + S_{gf} + S_{gh}$, where S_{EH} is the Einstein-Hilbert action

$$S_{EH}[\bar{g}] = \frac{1}{16\,\pi\,G}\int d^dx\sqrt{\bar{g}}\,\{-R(\bar{g}) + 2\,\Lambda\}\,, \tag{3.3}$$

[1] As it has been mentioned in the subsection 2.2.4, the bundle of physically interesting RG trajectories in the Einstein-Hilbert truncation encounters in the infrared a singularity at $\lambda = 1/2$. Hence, the spontaneous breaking of diffeomorphism symmetry (i.e. a non-zero minimum of the potential at $k = 0$) cannot be studied in a Einstein-Hilbert context. The singularity at $\lambda = 1/2$, moreover, persists in the conformally reduced case.

and where the gauge fixing and ghost action are those introduced in 2.2.3. The supertrace in (3.1) is intended over the field content of the theory, so that employing the traceless decomposition (2.66) the flow equation is rewritten as

$$\begin{aligned} k\,\partial_k\, S_k[h;\bar g] = &-\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho)\,\mathrm{Tr}_T\,\mathcal{H}(s;S_k^{(2)}[h;\bar g])_{\hat h\,\hat h} \\ &-\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho)\,\mathrm{Tr}_S\,\mathcal{H}(s;S_k^{(2)}[h;\bar g])_{\bar h\,\bar h} \\ &+\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho)\,\mathrm{Tr}_V\,\mathcal{H}(s;S_k^{(2)}[h;\bar g])_{C\bar C}\,, \end{aligned} \tag{3.4}$$

where the appendices $\hat h$, $\bar h$ and C means that we take the heat kernels of respectively the traceless, trace and ghost part of the Hessian, and where the ghost term takes a term -2 from the trace over the Grasmannian complex variables. The various traces contain a trace over the field structure, yielding

$$\frac{1}{\mathcal{V}}\,\mathrm{Tr}\,\mathbb{1}_S = 1\,,\qquad \frac{1}{\mathcal{V}}\,\mathrm{Tr}\,\mathbb{1}_V = d\,,\qquad \frac{1}{\mathcal{V}}\,\mathrm{Tr}\,\mathbb{1}_T = \frac{(d-1)(d+2)}{2}\,, \tag{3.5}$$

where $\mathcal{V} = \int d^dx\sqrt{g}$ and where the diagonalized Hessians on a maximally symmetric background read

$$\begin{aligned} S_k^{(2)}[h;\bar g]_{\hat h\,\hat h} &= \tau_k\,\{-\bar\nabla^2 - 2\lambda + C_T\,R(\bar g)]\}\,, \\ S_k^{(2)}[h;\bar g]_{\bar h\,\bar h} &= \tau_k\;Z\{-\bar\nabla^2 - 2\lambda + C_S\,R(\bar g)\}\,, \\ S_k^{(2)}[h;\bar g]_{C\bar C} &= \tau_k\,\{-\bar\nabla^2 + C_V\,R(\bar g)\}\,, \end{aligned} \tag{3.6}$$

being $\tau_k = (-16\,\pi\,G_k)^{-1}$, $Z = -(d-2)/2$ and

$$C_T = \frac{(d-3)d+4}{(d-1)d}\,,\qquad C_S = \frac{d-4}{d}\,,\qquad C_V = -\frac{1}{d}\,. \tag{3.7}$$

The cutoff function $\rho_k(s)$ in (3.4) is a generalized version of the family of n-parameter cutoff functions (1.71) which reads

$$\rho_k(s) \equiv \rho_k^n(s,\mathcal{Z}) = \frac{\Gamma(n,s\,\mathcal{Z}\,n\,k^2) - \Gamma(n,s\,\mathcal{Z}\,n\,\Lambda^2)}{\Gamma(n)}\,, \tag{3.8}$$

being Λ an UV cutoff, and where $\mathcal{Z}$ is a constant which has to be adjusted to make sure that the eigenvalues of $-\nabla^2$ are cut off around $\sim k^2$ rather than $\sim k^2/\mathcal{Z}$. Looking at the Hessians (3.6) we can see that $\mathcal{Z} = (-16\,\pi\,G_k)^{-1}$ for the vector and tensor sectors and $\mathcal{Z} = Z\,(-16\,\pi\,G_k)^{-1}$ for the scalar sector. In (3.8) we already took in consideration the cutoff rescaling $k^2 \to n\,k^2$, introduced in the subsection 1.2.3, and for convenience we shift[2] $n \to n-d/2$ in the cutoff (1.71) so to have the first argument of the incomplete

[2]Note that n can be freely shifted without requiring to shift also the parameter n present in the rescaling $k^2 \to n\,k^2$.

Euler function in (3.8) directly equal to n. After the shift, then, we have that $n > d/2$ and the RG flow becomes logarithmic for $n \to d/2$. The derivative $k\,\partial_k\,\rho_k^n(s, \mathcal{Z})$ in (3.1) then explicitly reads

$$k\,\partial_k\,\rho_k^n(s) = -\frac{2}{n!}\,(\mathcal{Z}\,s\,k^2\,n)^n\,e^{-\mathcal{Z}\,s\,k^2\,n}\,. \tag{3.9}$$

The traces contained in (3.4) can be evaluated by using heat kernel techniques, that is by expanding the trace of the heat kernel operator in powers of the proper time s, i.e.

$$\mathrm{Tr}\,\mathcal{H}(s; S_k^{(2)}[h;\bar{g}]) = \sum_i s^i\,E_i\,, \tag{3.10}$$

where E_i are operators built from the invariants of the symmetries of the action $S_k[h,\bar{g}]$ (e.g. powers of R, contractions of Riemann tensors, etc.). A more detailed introduction to heat kernel techniques can be found in the appendix D.

Since the Hessians (3.6) contain a Laplacian operator which commutes with the remaining part it is possible to take the latter out of the trace, i.e.

$$\mathrm{Tr}\,\mathcal{H}\Big(s; S_k^{(2)}[h;\bar{g}] = \mathcal{Z}\,\{-\bar{\nabla}^2 + \mathcal{B}\}\Big) = e^{-s\,\mathcal{Z}\,\mathcal{B}}\,\mathrm{Tr}\,\mathcal{H}(s; -\mathcal{Z}\,\bar{\nabla}^2)\,, \tag{3.11}$$

where B is the non-derivative part of the Hessian and $\mathcal{Z}$ the coefficient of the Laplacian, and then use the well known heat kernel expansion for the operator $-\nabla^2$ on a generic metric g, (D.42), up to the first order in the Ricci scalar, i.e.

$$\mathrm{Tr}\,\mathcal{H}(s; -\nabla^2) = \int d^dx\sqrt{g}\,\frac{1}{(4\,\pi\,s)^{\frac{d}{2}}}\,\left\{1 + s\,\frac{R(g)}{6} + \mathcal{O}(R^2)\right\}\,, \tag{3.12}$$

so that the three traces written in (3.4) yield

$$\mathrm{Tr}_T\,\mathcal{H}(s; S_k^{(2)}[h;\bar{g}])_{\hat{h}\,\hat{h}} = \frac{(d-1)(d+2)}{2}\,\frac{1}{(4\,\pi\,\tau_k\,s)^{\frac{d}{2}}}\,\int d^dx\sqrt{\bar{g}}\,\left\{1 + s\,\tau_k\,\frac{R(\bar{g})}{6} + \mathcal{O}(R^2)\right\}\,e^{-s\,\tau_k\,(-2\lambda + C_T\,R(\bar{g}))}\,, \tag{3.13}$$

$$\mathrm{Tr}_S\,\mathcal{H}(s; S_k^{(2)}[h;\bar{g}])_{\bar{h}\,\bar{h}} = \frac{1}{(4\,\pi\,Z\,\tau_k\;s)^{\frac{d}{2}}}\,\int d^dx\sqrt{\bar{g}}\,\left\{1 + s\,Z\,\tau_k\,\frac{R(\bar{g})}{6} + \mathcal{O}(R^2)\right\}\,e^{-s\,Z\,\tau_k\,(-2\lambda + C_S\,R(\bar{g}))}\,, \tag{3.14}$$

$$\mathrm{Tr}_V\,\mathcal{H}(s; S_k^{(2)}[h;\bar{g}])_{C\bar{C}} = \frac{d}{(4\,\pi\,\tau_k\,s)^{\frac{d}{2}}}\,\int d^dx\sqrt{\bar{g}}\,\left\{1 + s\,\tau_k\,\frac{R(\bar{g})}{6} + \mathcal{O}(R^2)\right\}\,e^{-s\,\tau_k\,C_V\,R(\bar{g})}\,, \tag{3.15}$$

where the heat kernel expansion has been performed using the background metric $\bar{g}$. We can then perform the integral over the proper time and collect operators linear in R and of 0-th order in R. The β-functions can be obtained matching powers of R in the right and left hand side of the flow equation (3.1), where the left hand side reads

$$k\,\partial_k\,S_k[\bar{g}] = -\frac{1}{16\,\pi}\int d^dx\sqrt{\bar{g}}\,\left\{k\,\partial_k\,\frac{1}{G_k}\,R - 2\,k\,\partial_k\,\frac{\Lambda_k}{G_k}\right\}\,. \tag{3.16}$$

The fixed point structure can then be investigated by using dimensionless quantities, that is

$$g_k = k^{d-2}\,G_k\,, \qquad \lambda_k = \Lambda_k\,k^{-2}\,, \qquad \tilde{R} = R\,k^{-2}\,. \tag{3.17}$$

In particular, the β-function of the Newton's constant can be obtained by defining the anomalous dimension

$$\eta = k\,\partial_k\,\ln G_k\,, \tag{3.18}$$

so that we have

$$\beta_g(g,\lambda) \equiv k\,\partial_k\,g_k = \partial_k\,(G_k\,k^{d-2}) = (d-2+\eta)\,g_k\,. \tag{3.19}$$

The β-function of the dimensionless cosmological constant can be obtained from the dimensionful equation

$$G_k\,\left(k\,\partial_k\,\frac{\Lambda_k}{G_k}\right) = k\,\partial_k\,\Lambda_k + \Lambda_k\,\eta\, = k^2\,(k\,\partial_k\,\lambda_k + 2\,\lambda_k + \eta\,\lambda_k)\,. \tag{3.20}$$

The anomalous dimension can be evaluated by collecting the linear term in the Ricci scalar from the left hand term of (3.1) and reads

$$\eta(g,\lambda) = \frac{g_k\,\left(-d\,(5\,d-7)\,n^{n+1}\,(n-2\,\lambda_k)^{\frac{d}{2}-n-1} - 4\,(d+6)\,n^{d/2}\right)\,\Gamma\left(-\frac{d}{2}+n+1\right)}{3\,2^{d-2}\,\pi^{\frac{d}{2}-1}\,\Gamma(n+1)}\,, \tag{3.21}$$

using which we obtain the set of coupled β-functions, which in $d=4$ read

$$\begin{aligned}\beta_g &= g_k\,\left(\frac{g_k\,\left(-13\,n^n\,(n-2\,\lambda)^{1-n} - 10\,n\right)}{3\,\pi\,(n-1)} + 2\right)\,,\\ \beta_\lambda &= \frac{g_k\,(5\,n^n\,(n-2\,\lambda_k)^2 - 4\,n^2(n-2\,\lambda_k)^n)\,(n-2\lambda_k)^{-n}\Gamma(n-2)}{\pi\Gamma(n)}\\ &\quad+\lambda_k\,\left(\frac{g_k\,(-13\,n^n\,(n-2\,\lambda_k)^{1-n} - 10\,n)}{3\,\pi\,(n-1)} - 2\right)\,.\end{aligned} \tag{3.22}$$

The expression of the β-functions for a generic dimension d, for the cutoff function (3.8) and in the limits $n \to \infty$ and $n \to \frac{d}{2}$ are listed in the appendix C, together with a table of universal quantities at varying d and n.

As it can be seen by comparing the universal quantities evaluated in the proper time RG scheme and the ERG scheme, respectively Tab. 2.70 and Tab. C.1, the use of the proper time scheme leads to results in general agreement with the ERG calculation for all values of n, underling the robustness of the NGFP under RG scheme change.

3.2 Conformally reduced action

We will now restrict our attention to the quantization of the sole conformal degree of freedom, studying the reduced action as a scalar toy model of quantum gravity. To build such a conformally reduced model we start with the Einstein-Hilbert (EH) action

$$S_k^{\mathrm{EH}}[g_{\mu\nu}] = -\frac{1}{16\,\pi\,G_k}\int d^dx\sqrt{g}\,\big(R(g) - 2\,\Lambda_k\big)\,, \tag{3.23}$$

and we restrict the metric to belong to a class of conformal metrics, i.e. metrics which differ only by a conformal factor, i.e.

$$g_{\mu\nu} = \phi(x)^{2\,\nu(d)}\,\hat{g}_{\mu\nu}\,, \tag{3.24}$$

being $\hat{g}_{\mu\nu}$ a reference metric, and where $\nu(d)$ is a generic function of the spacetime dimension. The Ricci scalar for this class of metrics reads

$$\begin{aligned} R(g) = \phi(x)^{2\,\nu(d)}\Big\{R(\hat{g}) - 2\,\nu\,(d-1)\,\frac{1}{\phi(x)}\,\hat{\nabla}^2\,\phi(x) \\ + \big(2\,\nu\,(d-1) - \nu^2\,(d-1)\,(d-2)\big)\,\frac{1}{\phi(x)^2}\,\hat{g}_{\mu\nu}\,(\partial_\mu\phi(x))\,(\partial_\nu\phi(x))\Big\}\,, \end{aligned} \tag{3.25}$$

where $\widehat{R}$ is the Ricci scalar of the reference metric, $\widehat{R} \equiv R(\widehat{g})$, while for the determinant we have

$$\sqrt{g} = \phi(x)^{d\,\nu(d)}\sqrt{\hat{g}}\,, \tag{3.26}$$

so that for the operator $\sqrt{g}R$ we have

$$\int d^dx\,\sqrt{g}\,R = \int d^dx\,\sqrt{\hat{g}}\,\phi(x)^{(d-2)\,\nu}\Big\{R(\hat{g})+\nu^2\,(d-1)\,(d-2)\,\frac{1}{\phi(x)^2}\,\hat{g}^{\mu\nu}\,(\partial_\mu\phi(x))\,(\partial_\nu\phi(x))\Big\}\,. \tag{3.27}$$

We can set $\nu(d) = 2/(d-2)$ in order to fix the relative factor of the kinetic operator to the standard value $1/2$, so that, inserting (3.24) in (3.23), it yields the scalar action

$$S_{CREH}[\phi] = \int \mathrm{d}^dx\,\sqrt{\widehat{g}}\,\mathcal{Z}_k\left(\tfrac{1}{2}\,\widehat{g}^{\,\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}\,A(d)\,\widehat{R}\,\phi^2 - 2\,A(d)\,\Lambda_k\,\phi^{\frac{2d}{(d-2)}}\right)\,, \tag{3.28}$$

where

$$\mathcal{Z}_k = -\frac{1}{2\,\pi\,G_k}\frac{d-1}{d-2}\,,\qquad A(d) = \frac{d-2}{8\,(d-1)}\,. \tag{3.29}$$

For such a choice of $\nu(d)$ the conformally reduced Einstein-Hilbert (CREH) action (3.28) reads

$$S_{CREH} = -\frac{3}{4\,\pi\,G}\int d^4x\sqrt{\hat{g}}\,\Big\{\frac{1}{2}\,\hat{g}^{\mu\nu}(\partial_\mu\phi(x))\,(\partial_\nu\phi(x)) + \frac{1}{12}\,R(\hat{g})\,\phi(x)^2 - \frac{1}{6}\,\Lambda\,\phi(x)^4\Big\}\,, \tag{3.30}$$

that is a $\lambda\,\phi^4$ scalar field theory, but with a wrong kinetic sign which entails an instability of the action. This instability, however, is believed to be a drawback of the sole Einstein-Hilbert truncation and to be cured adding higher derivative operators to the bare action. In [83], has in fact been studied the stabilization mechanism of an higher derivative action of the type

$$S[\phi] = \int d^d x \,\{(\partial_\mu^2 \phi)^2 - (\partial_\mu \phi)^2 + V(\phi)\}\,, \tag{3.31}$$

which bare propagator contains unstable modes for $p^2 < 1$. Interestingly, it has been proved that once averaged the fluctuations the effective propagator does not contain unstable modes anymore, and the effective theory is well defined.

To quantize the scalar toy model in the background field formalism let us then consider a microscopic scalar field $\chi(x)$ (we will use $\phi(x)$ for the effective field) and decompose it as $\chi(x) = \chi_B(x) + f(x)$, being f a fluctuation and χ_B the background. The path integral of the CREH action (3.28) reads then

$$Z = \int \mathcal{D}[f]\, e^{-S[f+\chi_B]}\,, \tag{3.32}$$

and we will assume that no unstable mode are present in the effective theory.

The coarse graining for such a theory can be built in the same way introduced in section 1.2 but where the averaged field (1.9) now depends on a generic scalar background $\chi_B(x)$ which we will use to define the block length, i.e.

$$f_k(x) = \int d^d y\, \sqrt{\hat{g}}\, f(y)\, \rho_k(y, x; \chi_{\rm B})\,, \tag{3.33}$$

where the function $\rho_k(x, y; \chi_{\rm B})$ is a smearing kernel satisfying the same conditions of the function (1.10), but generalized on a curved background (see [84] for a general discussion on smearing kernels in Riemannian spaces). Its explicit expression in terms of $\chi_{\rm B}$ dependence does not need to be specified at this level.

In this formalism χ plays the same role of the microscopic metric $\gamma_{\mu\nu}$ in the full theory and the expectation values $\bar{f} \equiv \langle f \rangle$ and $\phi \equiv \langle \chi \rangle = \chi_{\rm B} + \bar{f}$ are the analogs of $\bar{h}_{\mu\nu} \equiv \langle h_{\mu\nu} \rangle$ and $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$ in the full theory, although $\bar{f}$ and $\bar{h}_{\mu\nu}$ cannot be directly compared because of the non linearity in ϕ in the Weyl rescaling (3.24).

The central idea of the conformal field quantization is to employ the background metric

$$\bar{g}_{\mu\nu} \equiv \chi_{\rm B}^{2\nu}\, \widehat{g}_{\mu\nu}\,, \tag{3.34}$$

in constructing the smearing function $\rho_{\bar{k}}(x, y) \equiv \rho_k(x, y; \chi_B)$ via the spectrum of $-\bar{\nabla}^2$, being $\bar{k}$ and $k \equiv \hat{k}$, respectively the momentum operators built with the background metric $\bar{g}_{\mu\nu}$ and the fixed metric $\widehat{g}_{\mu\nu}$. The reference metric $\widehat{g}_{\mu\nu}$ plays no dynamical role in this process but it is fixed to perform the actual calculation, while all the dynamical fields are spectrally decomposed using the basis of the $-\bar{\nabla}^2$ eigenfunctions whose eigenvalues satisfy

$$\bar{k}^2 = \chi_{\rm B}^{-2\,\nu}\, k^2\,, \tag{3.35}$$

in the case of a constant χ_{B}.

The proper time RG flow equation (3.1) for the toy model reads then

$$\partial_t S_k[\tilde{f};\chi_B] = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,(k\,\partial_k\,\rho_k)\,\mathrm{Tr}\,\mathcal{H}(s;S_k^{(2)}[\tilde{f};\chi_B])\,, \tag{3.36}$$

where we will use as cutoff function $\rho_k(s)$ the same family of cutoff function used in the gravitational case, i.e. (3.8), adapted to the background case, that is

$$\rho_k(s) = \rho_k^n(s,\mathcal{Z}) = \frac{\Gamma(n, s\,\mathcal{Z}\,n\,k^2\,\chi_{\mathrm{B}}^{2\nu}) - \Gamma(n, s\,\mathcal{Z}\,n\,\Lambda^2\,\chi_{\mathrm{B}}^{2\nu})}{\Gamma(n)}\,, \tag{3.37}$$

where we we used the relation (3.35).Therefore, the derivative $k\,\partial_k\,\rho_k^n$ in (3.36) explicitly reads

$$k\,\partial_k\,\rho_k^n = -\frac{2}{n!}\,(\mathcal{Z}\,s\,k^2\,n\,\chi_{\mathrm{B}}^{2\nu})^n\,e^{-\mathcal{Z}\,s\,k^2\,n\,\chi_{\mathrm{B}}^{2\nu}}\,, \tag{3.38}$$

with $n > d/2$. For $n = d/2$ the kernel (3.38) does not regulate completely the UV because the proper time integral requires a field independent (vacuum) contribution to be subtracted from the right-hand side of equation (3.36).

The important difference between the flow equation (3.36) and the flow equation for the standard scalar field theory (but also the gravitation theory) is that besides the action (3.28) is defined for a reference metric $\hat{g}$ as background, and not $\bar{g}$, the coarse graining still requires the modes $\bar{p}$, and not $p \equiv \hat{p}$, to be cut. Hence, the trace over the reference metric in (3.36) has to be here computed by means of the representation provided by the spectrum of the Laplacian[3] operator $-\nabla^2$, i.e. using the relation

$$-\,\hat{\nabla}^2 = -\bar{\nabla}^2\,\chi_{\mathrm{B}}^{2\,\nu}\,, \tag{3.39}$$

we have

$$\mathrm{Tr}\,\mathcal{H}(s;-\hat{\nabla}^2) \equiv \overline{\mathrm{Tr}}\,\mathcal{H}(s;-\bar{\nabla}^2\,\chi_{\mathrm{B}}^{2\,\nu})\,, \tag{3.40}$$

where

$$\overline{\mathrm{Tr}}\,\mathcal{H}(s;-\bar{\nabla}^2\,\chi_{\mathrm{B}}^{2\,\nu}) \equiv \int d^dx\,\sqrt{\hat{g}}\,\langle x|e^{-s(-\bar{\nabla}^2\,\chi_{\mathrm{B}}^{2\,\nu})}|x\rangle = \int d^dx\,\sqrt{\bar{g}}\,\chi_{\mathrm{B}}^{d\nu}\,\langle x|e^{-s(-\bar{\nabla}^2\,\chi_{\mathrm{B}}^{2\,\nu})}|x\rangle\,, \tag{3.41}$$

where we expressed the background volume in terms of reference volume using the relation $\sqrt{\bar{g}} = \sqrt{\hat{g}}\,\chi_{\mathrm{B}}^{d\nu}$.

3.3 Polynomial truncations

In this section we shall discuss the structure of the NGFP obtained by the flow equation (3.36) as a function of the cutoff parameter n for different reference topologies.

[3] Note that the action (3.28) contains an operator $\hat{\nabla}^2$ and not $\bar{\nabla}^2$.

It is important to remark that, at variance with the well-known definition of the path integral for quantum gravity based on the sum over all possible metric/topologies, in our case the use of different topologies is only a technical device to project an infinite-dimensional functional flow equation in a finite dimensional theory space where only the flow of the operators $\sqrt{g}R$ and $\sqrt{g}$ is considered. From this point of view the projection to different topologies has nothing to do with a calculation performed in the Gibbons-Hawking spirit. Neither are we expanding the graviton propagator in inverse powers of momentum/curvature. On the contrary the (unprojected) functional flow equation is, by construction, independent on the topology and the same property is shared by the flow equation for the conformal factor. However because the irrelevant operators of the NGFP have a different impact on the renormalized flow at the zeroth order of the gradient expansion (spherical projection) and at first order (flat projection), the universal quantities will show this residual scheme dependence.

Let us then assume that the field dependence of the Wilsonian action S_k is completely encoded in a relation of the type $\phi = \chi_{\rm B} + \tilde{f}$, i.e. the approximation (2.61), so that S_k is a local function of ϕ. In particular, the second functional derivative of (3.28) reads

$$S_k^{(2)}[\phi] = \mathcal{Z}_k \left(- \hat{\nabla}^2 + 2\, A(d)\, \widehat{R} - 2\, A(d)\, B(d)\, \Lambda_k\, \phi^{\frac{2d}{(d-2)}-2} \right), \tag{3.42}$$

being

$$B(d) = \frac{2d}{d-2} \left(\frac{2d}{d-2} - 1 \right) . \tag{3.43}$$

3.3.1 S^d topology

Let us first consider the topology of the d dimensional sphere S^d. In this case the curvature of the reference metric $\widehat{g}_{\mu\nu}$ is constant and the running of the dimensionless coupling $g_k = G_k\, k^{d-2}$ can be obtained from the ϕ^2 term projecting the flow equation on the background field, i.e. fixing $\tilde{f} = 0$ and $\chi_{\rm B}(x)$ equal to a constant in action (3.28), so that the kinetic term vanishes and we get

$$S_k^{S^d}[\chi_{\rm B}] = \int \mathrm{d}^d x\, \sqrt{\widehat{g}}\, \mathcal{Z}_k \left(\tfrac{1}{2}\, A(d)\, \widehat{R}\, \chi_{\rm B}^2 - 2\, A(d)\, \Lambda_k\, \chi_{\rm B}^{\frac{2d}{(d-2)}} \right) . \tag{3.44}$$

The evaluation of the trace can then be performed using the heat kernel techniques already introduced, combined with the background-independent trace condition (3.40), so that

$$\mathrm{Tr}\, \mathcal{H}(s; S_k^{(2)}[\tilde{f}; \chi_B]) = e^{-s \mathcal{Z}_k \left(2\, A(d)\, \widehat{R} - 2\, A(d)\, B(d)\, \Lambda_k\, \phi^{\frac{2d}{(d-2)}-2} \right)}\, \overline{\mathrm{Tr}}\, \mathcal{H}(s; -\mathcal{Z}_k\, \bar{\nabla}^2\, \chi_{\rm B}^{2\nu}) \,, \tag{3.45}$$

where

$$\overline{\mathrm{Tr}}\, \mathcal{H}(s; -\mathcal{Z}_k\, \bar{\nabla}^2\, \chi_{\rm B}^{2\nu}) = \int d^d x \sqrt{\hat{g}}\, \frac{\chi_{\rm B}^{d\,\nu}}{(4\, \pi\, \mathcal{Z}_k\, s\, \chi_{\rm B}^{2\nu})^{\frac{d}{2}}} \left\{ 1 + s\, \mathcal{Z}_k\, \chi_{\rm B}^{2\nu}\, \frac{\hat{R}}{6} + \mathcal{O}(\hat{R}^2) \right\} , \tag{3.46}$$

and where we used the heat kernel expansion (3.12) for the Laplacian operator.

We can then expand (3.45) in powers of $\hat{R}$ and discard terms $\mathcal{O}(\hat{R}^2)$, then insert the result in the flow equation (3.36) together with (3.38) and evaluate the integral over the proper time. The coefficients of the operators $\chi_{\rm B}^2$ and $\chi_{\rm B}^{2d/(d-2)}$ are then easily identified selecting the term proportional to $\sqrt{\widehat{g}_{\mu\nu}}\,R(\widehat{g}_{\mu\nu})$ and $\sqrt{\widehat{g}_{\mu\nu}}$ on the right hand side of the flow equation. At last the β-functions for the dimensionless running Newton constant g_k and the dimensionless cosmological constant $\lambda_k = \Lambda_k\, k^2$ can be obtained with the introduction of the anomalous dimension $\eta \equiv k\,\partial_k\,\ln G_k$, so that

$$\beta_g(g,\lambda) \equiv k\,\partial_k\, g_k = (d-2+\eta)\, g_k\,. \tag{3.47}$$

Given the anomalous dimension

$$\eta(g,\lambda) \equiv \eta_{pot}(g,\lambda) = -\frac{2^{2-d}\,(d-2)\,\pi^{1-\frac{d}{2}}\,g_k\,n^n\,\Gamma\left(-\frac{d}{2}+n+1\right)}{(d-1)\,\Gamma(n)\,\left(n-\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{n-\frac{d}{2}+1}}\,, \tag{3.48}$$

where the *pot* implies that we evaluated the running of the Newton's constant from the potential, in four dimensions we have

$$\beta_g = \quad g_k\left(2-\frac{g_k}{(n-2\,\lambda_k)^{n-1}}\frac{n^n\,\Gamma[n-1]}{6\,\pi\,\Gamma(n)}\right)\,, \tag{3.49a}$$

$$\begin{aligned}\beta_\lambda = \quad & \lambda_k\left(-2-\frac{g_k}{(n-2\,\lambda_k)^{n-1}}\frac{n^n\,\Gamma[n-1]}{6\,\pi\,\Gamma(n)}\right)+\\ & +\frac{g_k}{(n-2\,\lambda_k)^{n-2}}\frac{n^n\,\Gamma[n-2]}{2\,\pi\,\Gamma(n)}\,.\end{aligned} \tag{3.49b}$$

The expressions for the d dimensional β-functions are listed in the appendix C, together with the limiting cases $n \to d/2$ and $n \to \infty$.

3.3.2 $\mathbb{R}^d$ topology

In the case of a flat $\mathbb{R}^d$ topology the scalar curvature of the reference metric vanishes, constraining the operator $\chi_{\rm B}^2$ in the action (3.28) to be zero. In order to extract the β-functions from the flow equation (3.36) it is convenient to consider a general truncation of the type

$$S_k[\phi] = \int \mathrm{d}^d x\,\sqrt{\widehat{g}}\,\left(\tfrac{1}{2}\,\widehat{g}^{\mu\nu}\,\mathcal{Z}_k\,\partial_\mu\,\phi\,\partial_\nu\,\phi\,+V_k(\phi)\right)\,, \tag{3.50}$$

where $V_k(\phi) = \mathcal{Z}_k\,U_k(\phi)$, and employ a derivative expansion around an homogeneous background plus a fluctuation, so that $\phi(x) = \chi_{\rm B} + \tilde{f}(x)$. In this case we have

$$\begin{aligned}S_k[\phi] = \int d^d x \sqrt{\widehat{g}}\Big\{ & -\frac{1}{2}\,\mathcal{Z}_k\,\tilde{f}(x)\,\hat{\nabla}^2\,\tilde{f}(x)+V_k(\chi_B)+\\ & +V_k'(\chi_B)\,\tilde{f}(x)+\frac{1}{2}\,V_k''(\chi_B)\,\tilde{f}(x)^2+\mathcal{O}(\tilde{f}(x)^3)+\mathcal{O}(\partial^4\,\tilde{f})\Big\}\,.\end{aligned} \tag{3.51}$$

Therefore,

$$k\,\partial_k\, S_k[\tilde f(x)] = -\frac{1}{2}\int d^dx\,\sqrt{\hat g}\;\chi_B^{d\,\nu}\int \frac{ds}{s}\,(k\,\partial_k\rho_k^n)\,\langle\, x\,|\, e^{-s\,(K+\delta K)}\,|\, x\,\rangle\,, \tag{3.52}$$

where

$$K = -\mathcal{Z}_k\,\widehat{\Box} + V_k''(\chi_B)\,, \qquad \delta K = V_k'''(\chi_B)\tilde f(x) + \frac{1}{2}\,V_k''''(\chi_B)\,\tilde f(x)^2\,. \tag{3.53}$$

The trace in (3.52) can be evaluated in a background-independent way by means of an integration in momentum space over the eigenvalues $\bar p^2$ of the Laplacian built from the background metric $\bar g_{\mu\nu}$, inserting in (3.52) the identity $\int d^d\bar p\,|\,\bar p\,\rangle\langle\,\bar p\,| = \mathbb{1}\,(2\,\pi)^d$ and using in (3.53) the substitution $-\hat\nabla^2 \to -\bar\nabla^2\,\chi_{\mathrm{B}}^{2\,\nu}$. In order to disentangle the trace in (3.52) a Baker-Campbell-Hausdorff expansion of the heat kernel is performed, so that

$$k\,\partial_k\, S_k[\tilde f(x)] = -\frac{1}{2}\int d^dx\,\sqrt{\hat g}\;\chi_{\mathrm{B}}^{d\,\nu}\int\frac{d^d\bar p}{(2\,\pi)^d}\int \frac{ds}{s}\,(k\,\partial_k\rho_k^n)\,\langle\, x\,|\,\bar p\,\rangle\langle\,\bar p\,|\,e^{-s\,K}B|\,x\,\rangle\,, \tag{3.54}$$

being

$$B(K,\delta K) = \left(1 - s\,\delta K + \frac{s^2}{2!}\{[\delta K, K] + \delta K^2\} + \ldots\right)\,, \tag{3.55}$$

where the dots stand for the higher order terms in the s expansion of the exponential and

$$\langle\, x\,|\,\bar p\,\rangle = e^{-i\,\bar p\,x}\,. \tag{3.56}$$

The matrix elements of the expanded heat kernel can then be calculated ordering the operators by means of the commutation rule

$$[\bar p_\mu, \tilde f(x)] = -i\,\partial_\mu\,\tilde f(x)\,. \tag{3.57}$$

It is then straightforward to identify the coefficients of the V_k and $\mathcal{Z}_k$ terms, obtaining the following set of coupled equations:

$$k\,\partial_k\, V_k = M\,(k^2\,\chi_{\mathrm{B}}^{2\,\nu})^{\frac{d}{2}}\,\left(1 + \frac{V_k''(\chi_{\mathrm{B}})}{k^2\,n\,\mathcal{Z}_k\,\chi_{\mathrm{B}}^2}\right)^{\frac{d}{2}-n}\,, \tag{3.58a}$$

$$k\,\partial_k\,\mathcal{Z}_k = N\,(k^2\chi_{\mathrm{B}}^{2\,\nu})^{\frac{d}{2}-3}\,(V_k'''/\mathcal{Z}_k)^2\,\left(1 + \frac{V_k''(\chi_{\mathrm{B}})}{k^2\,n\,\mathcal{Z}_k\,\chi_{\mathrm{B}}^2}\right)^{\frac{d}{2}-3-n}\,, \tag{3.58b}$$

where

$$M = \left(\frac{n}{4\pi}\right)^{\frac{d}{2}}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\,, \tag{3.59a}$$

$$N = \frac{(d-2\,(n+1))(d-2\,(n+2))}{24\,d\,n^2}\,\left(\frac{n}{4\pi}\right)^{\frac{d}{2}}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\,. \tag{3.59b}$$

The $\boldsymbol{\beta}$-functions for the dimensionless couplings of the CREH truncation are then obtained introducing a polynomial ansatz for the dimensionful potential U_k of the type

$$U_k(\chi_{\rm B}) = -\, k^2 \,\frac{\lambda_k}{6}\, \chi_{\rm B}^4 \,, \tag{3.60}$$

so that using the anomalous dimension calculated from the kinetic term

$$\eta(g,\lambda) \equiv \eta_{kin}(g,\lambda) = -\frac{2^{2-d}\, d^2\,(d+2)^2\,\pi^{1-\frac{d}{2}}\, g_k\,\lambda_k^2\, n^n\,\Gamma\left(-\frac{d}{2}+n+3\right)}{3\,(d-2)^3\,(d-1)^3\,\Gamma(n)\,\left(n-\frac{d\left(\frac{2\,d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{\frac{1}{2}(-d+2\,n+6)}}\,, \tag{3.61}$$

one obtains in four dimensions the coupled set of equations

$$\beta_g = \quad g_k\left(2 - 2\,\frac{g_k\,\lambda_k^2}{(n-2\,\lambda_k)^{n-1}}\,\frac{\Gamma(n+1)\,n^n}{9\,\pi\,\Gamma(n)}\right)\,, \tag{3.62a}$$

$$\begin{aligned}\beta_\lambda = \quad & \frac{g_k}{(n-2\,\lambda_k)^{n-2}}\,\frac{\Gamma(n-2)\,n^n}{2\,\pi\,\Gamma(n)} + \\ & +\lambda_k\left(-2-2\,\frac{g_k\,\lambda_k^2}{(n-2\,\lambda_k)^{n-1}}\,\frac{\Gamma(n+1)\,n^n}{9\,\pi\,\Gamma(n)}\right)\,.\end{aligned} \tag{3.62b}$$

3.3.3 Fixed points and linearized flow

The $\boldsymbol{\beta}$-functions (3.49) and (3.62) vanish both at the Gaussian fixed point located at $\boldsymbol{\lambda}_* = \boldsymbol{g}_* = 0$, and at a NGFP defined at $\boldsymbol{\lambda}_* \neq 0$, $\boldsymbol{g}_* \neq 0$. The properties of the linearized flow around the NGFP are, as already said, determined by the stability matrix B which for the non trivial fixed point owns a pair of complex eigenvalues $\theta_{1,2} = -\boldsymbol{\theta'} \pm i\boldsymbol{\theta''}$. A negative real part of the eigenvalues, i.e. a positive $\boldsymbol{\theta'}$ (we will refer to it as the first Lyapunov exponent, following the standard notation used in dynamical systems), implies the stability of the fixed point, while the imaginary part characterizes the spiral shape near the fixed point. Our results $d=4$ and generic d dimensions are summarized, respectively, in Table C.1 and Table C.2 in the Appendix C.

It is clear from Table C.1 that also the theory defined by the CREH approximation is asymptotically safe, although the scaling properties are rather different from those obtained from the full Einstein-Hilbert action in the section 3.1. For instance, the critical exponents $\boldsymbol{\theta'}$ and $\boldsymbol{\theta''}$ display an n-dependence which is stronger in the case of the CREH action than for the non-reduced theory, although the quantity $\boldsymbol{\lambda}_*\,\boldsymbol{g}_*$ is rather stable in both cases.

We can quantify the impact of the Einstein-Hilbert conformal reduction with respect to the full EH theory by defining a χ^2-type of “distance" in the space of the “universal” quantities, by means of

$$\chi^2(n) = \frac{(\boldsymbol{\lambda}_*\boldsymbol{g}_*(C)-\boldsymbol{\lambda}_*\boldsymbol{g}_*(E))^2}{\boldsymbol{\lambda}_*\boldsymbol{g}_*(C)^2+\boldsymbol{\lambda}_*\boldsymbol{g}_*(E)^2} + \frac{(\boldsymbol{\theta'}(C)-\boldsymbol{\theta'}(E))^2}{\boldsymbol{\theta'}(C)^2+\boldsymbol{\theta'}(E)^2} + \frac{(\boldsymbol{\theta''}(C)-\boldsymbol{\theta''}(E))^2}{\boldsymbol{\theta''}(C)^2+\boldsymbol{\theta''}(E)^2}\,, \tag{3.63}$$

where “C" and “E" stands for the conformally reduced (CREH) and gravitational (EH) theories, respectively.

A plot of this quantity as a function of n is depicted in the upper panel of Fig.(3.1), for the S^4 projection (solid line) and the $\mathbb{R}^4$ projection (dashed line) where it is clear that the minimum is attained for $n = 4$ in both cases. On the other hand, in the case of the $\mathbb{R}^d$ projection the scaling properties are much less sensitive to the cutoff parameter n, and the $n = \infty$ limit is as good as the $n = 4$ case.

Of particular interest is the $n = \infty$ limit for the S^d topology, in which the first Lyapunov exponent vanishes. In this case the theory is still UV finite although not asymptotically safe anymore, since now the linearized system is defined by pure imaginary eigenvalues $\pm\boldsymbol{\theta''}$ and every perturbation of the NGFP will evolve in a cyclic trajectory.

It is also interesting to discuss the scaling properties of the theory in the S^d projection as the dimension is changed. This is shown in the middle panel of Fig.(3.1) for $n = 4$ for $\boldsymbol{\theta'}$, $\boldsymbol{\theta''}$ and for the dimensionless quantity $\tau_d \equiv \boldsymbol{\lambda_*}\, \boldsymbol{g_*}^{2/(d-2)} = \Lambda_k\, G_k^{2/(d-2)}$. The first Lyapunov exponent $\boldsymbol{\theta'}$ vanishes for a critical dimension value d_c so that the fixed point undergoes an Hopf bifurcation as the dimension d crosses d_c (represented in Fig.(3.2)).

As it is shown in Fig.(3.3), for $d \to 2$ the cycle collapses on the $\lambda = 0$ line. In this regime it shows a non homogeneous running due to the low transient of the trajectory near the Gaussian fixed point, while it becomes an homogenous slow transient around the NGFP in the limit $d \to d_c$.

Notice that the critical dimension is a function of n, $d_c \equiv d_c(n)$, and while for $n = \infty$ the critical dimension is $d_c = 4$, generally holds $d_c(n) < 4$ for a finite value of the parameter n. At $d = d_c$ the UV behavior is regulated by a limit cycle whose behavior resembles the one of the Van der Pol oscillator.

For $d < d_c$ (see left panel of Fig.(3.2)) the theory space is now divided in two regions. The first is the set of points in parameter space outside the cycle, which trajectories flow towards the UV to the limit cycle and hit in the IR the singularity $\lambda = n/2$ (or flow towards $\lambda = -\infty$). Those are the trajectories which survive for $d > d_c$ and that require higher-order operators in order to cure the IR sector. The second region is the set of points inside the cycle which flow towards it in the UV and towards the NGFP in the IR. The latter case leads to a new interesting scenario in which the UV and IR critical manifolds coincide and the EH truncation is finite at every energy scale.

For this scenario to be plausible we require the cyclic trajectory to be close enough to the Gaussian fixed point, so that it shows a semiclassical regime. Unfortunately, as can be seen from Fig.(3.3), in the best case ($d_c = 4$ for $n = \infty$) a limit cycle with a good semiclassical regime occurs only for $d \approx 3$. It is also important to stress that the limit cycle never approaches the singularity $\lambda = n/2$, where the EH truncation stops to work.

Since the Hopf bifurcation is not present in the $\mathbb{R}^d$ projection for the CREH action, also for small values of the dimension, we analyzed the behavior of the linearized flow

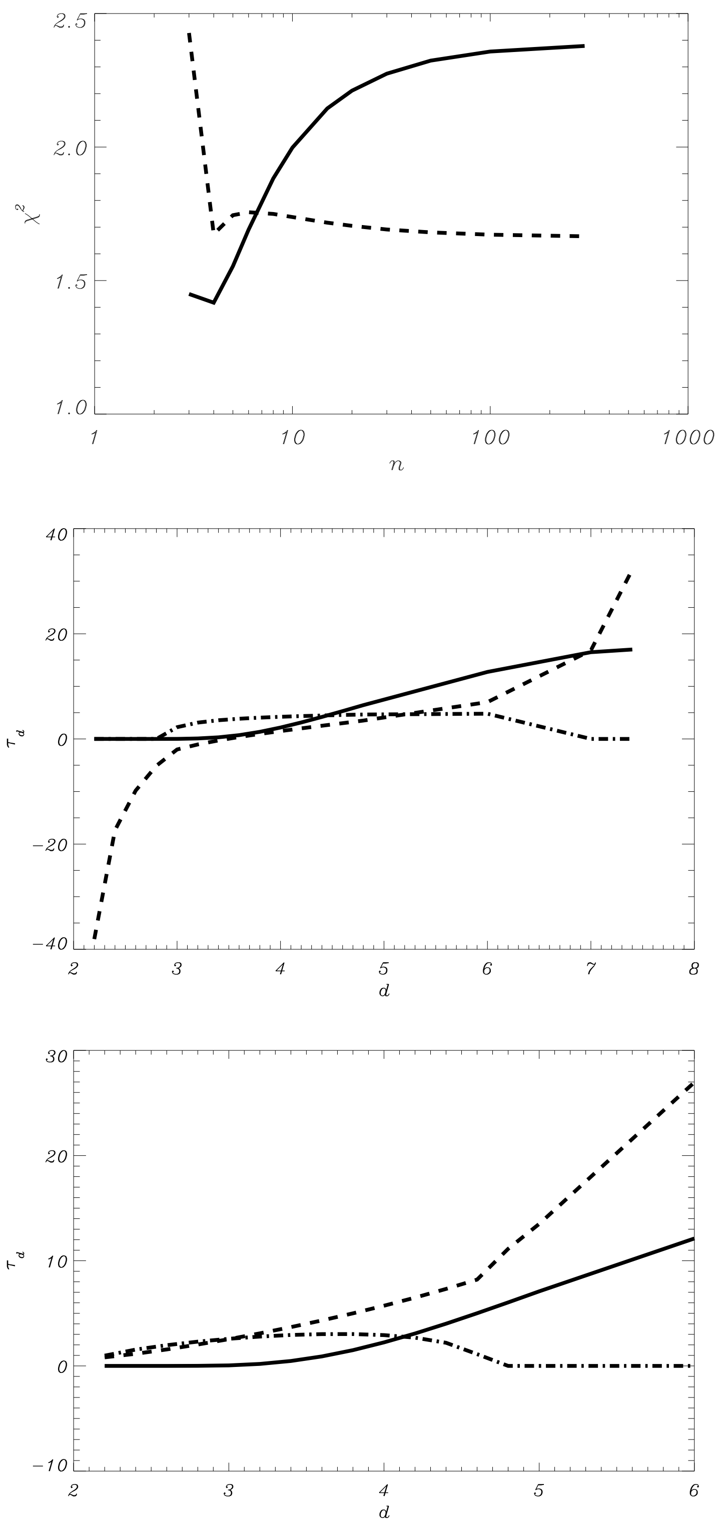

Figure 3.1: Top: the quantity $\chi^2(n)$ as a function of the cutoff parameter n in the case of S^4 projection (solid line) and $\mathbb{R}^4$ (dashed line). Middle and bottom: the quantity τ_d (solid line), $\boldsymbol{\theta'}$ (dashed line) and $\boldsymbol{\theta''}$ (dotted-dashed line) as a function of the dimension d for $n = 4$ in the case of S^4 (middle) and $\mathbb{R}^4$ (bottom) projections.

near the NGFP in the case of the full EH truncation, to verify if the Hopf bifurcation is still present in the S^d projection for some value of the parameter n. Numerical results and the β-functions are collected in Table C.2 in appendix C. As it can be seen from Table C.2 the full theory presents a stable NGFP in the whole $n-d$ plane, which means that the contribution of spin-2 degrees of freedom lower the value of the critical dimension under the "critical" value $d=2$.

Although such a non trivial behavior in the UV region seems to be a direct consequence of the strong dependence of the flow in the S^d projection on the cutoff parameter n, it is interesting to notice that recent investigations based on "tetrad only" theory spaces [85], and on the minisuperspace approximation of the EH truncation [86], also show the presence of limit cycles in the UV and IR limit, respectively. In the latter case, however, the limit cycle originates by an Hopf bifurcation of a specific cutoff parameter, while in our case the bifurcation is governed by the spacetime dimension, so that our limit cycle is UV and not IR.

The intriguing possibility of such a non trivial UV completion, however, was first pointed out by Wilson in a seminal paper (before the discovery of asymptotic freedom), in the context of QCD [87]. In particular it was argued that, at the experimental level, the presence of a limit cycle would show up in perpetual oscillations in the $e^+ - e^-$ total hadronic cross section in the limit of large momenta. In the case of gravity the natural arena to discuss this type of phenomenon is the physics of the early Universe, for which an effective Lagrangian $\mathcal{L}_{eff}(R)$ embodying the properties of the limit cycle can be determined by using the strategy outlined in [88]. In the case at hand we expect that $\mathcal{L}_{eff}(R) \propto \cos(R/\mu^2)$ where μ is a renormalization scale. On the other hand, discussing the detailed physical implications of this model is beyond the scope of this work.

As already mentioned in section 2.2.5 the infrared regime of both Einstein-Hilbert and CREH actions is plagued by a singularity[4] in the β-functions at $\lambda = \frac{1}{2}$. It is supposed that such a singular behavior can be cured by adding other irrelevant operators at the bare action, so that an appropriate framework for investigating the IR regime would be, for example, that of a local potential approximation.

3.4 Non-polynomial truncations

In this section we are interested in studing an RG equation for the scalar toy model in a local potential approximation, which, moreover, allows us to investigate the possibility of having a transition to a phase of broken diffeomorphism invariance at low energy (which has already been studied in an ERG context, see [7]).

We hence aim at solving numerically the equation (3.58a), that is a LPA equation for a generic potential $V_k(\phi)$ of the conformal factor for a flat spacetime, and study

[4] Note that the singularity occurs at $\lambda = \frac{n}{2}$ while using the reparameterized proper time cutoff.

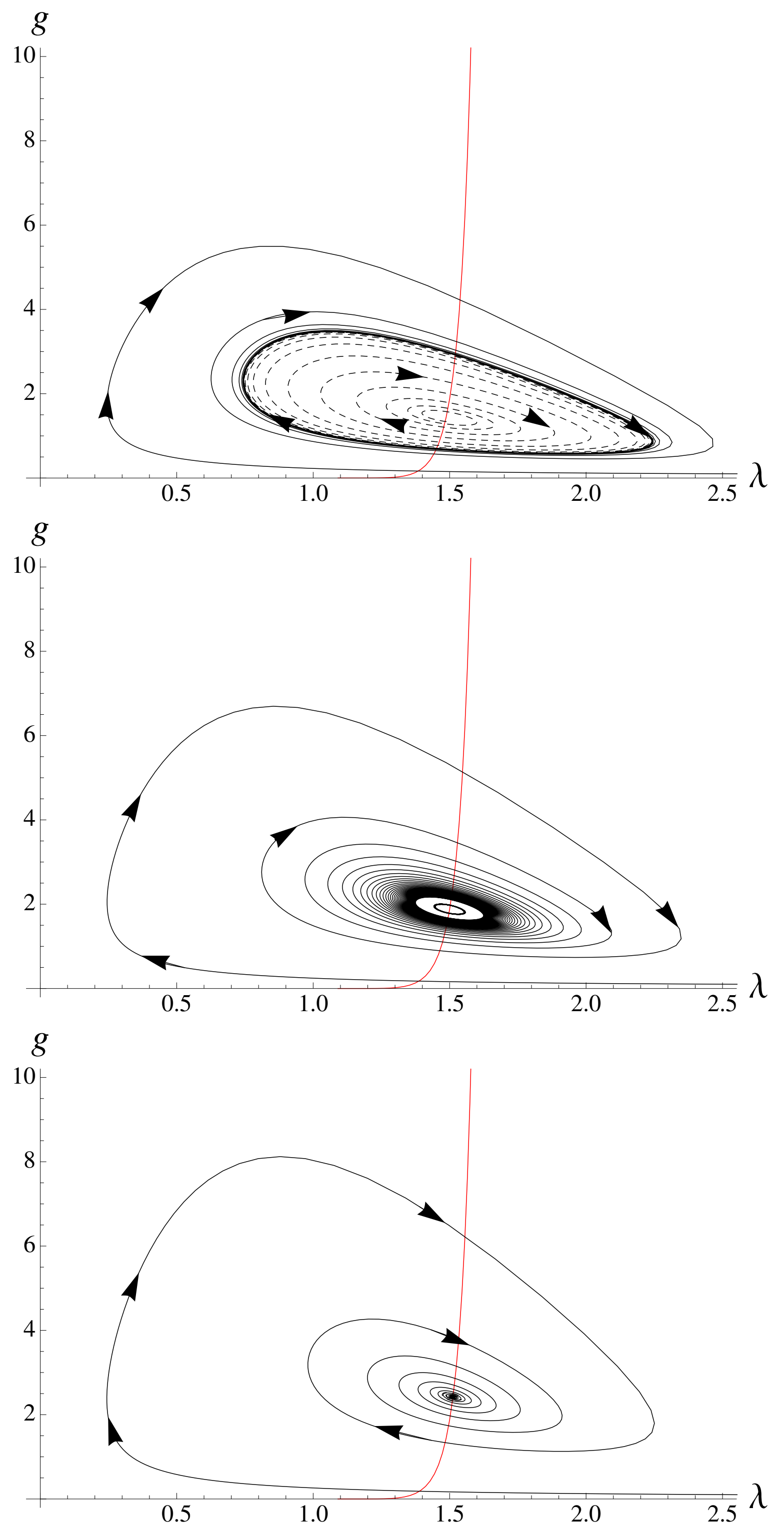

Figure 3.2: Flows in d dimension for the CREH S^d projection. Left: the limit cycle at $d < d_c$, the dashed line is the repulsive internal flow. Middle: the limit cycle at the critical dimension $d = d_c$. Right: the flow of the UV attractive NGFP at $d > d_c$. The red line is the location of the FP as a function of the dimension d. The plots have been obtained setting $n = \infty$ and, starting from the left, for $d = 3.9$, $d = 4$, $d = 4.1$.

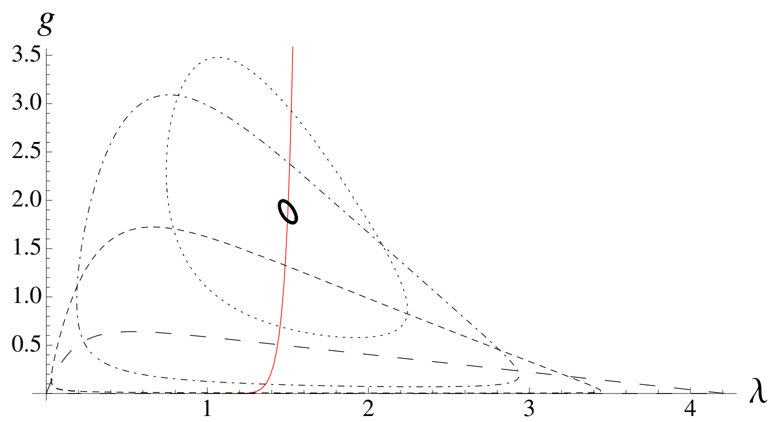

Figure 3.3: The Hopf bifurcation as a function of the dimension d for $n = \infty$ in the g-λ plane. The solid line is a cycle for an initial value near the FP at $d_c = 4$, the other cycles plotted are at $d = 3.9$ (dotted line), $d = 3.6$ (dotted-dashed line), $d = 3.3$ (dashed line) and $d = 2.9$ (long dashed line). The red line is the location of the FP as a function of the dimension d.

the evolution of an initial ultraviolet condition towards the infrared regime. Equation (3.58a), however, contains a dependence from the running renormalization function $\mathcal{Z}_k$ (the Newton's contant) which cannot be taken in account in an LPA context, but rather, for instance, in the so-called LPA' truncation, solving then the RG equation for the two point correlation function. We will however fix the running of the dimensionful coupling $\mathcal{Z}_k$ by hand; the solution of the coupled problem (3.58) is, in fact, beyond the aim of this work, and we will here just present a successful numerical strategy to deal with (3.58a) which we hope can eventually be extended to treat the coupled system (3.58a) and (3.58b) beyond the simple LPA truncation.

In particular, as we will see ahead, we shall investigate the role played by higher powers of volume operators of the type $\mathscr{V} = (\int d^dx\sqrt{g})$ in providing a transition to a phase of broken diffeomorphism invariance.

In order to carry out the numerical integration of (3.58a), it is useful to "linearize" the evolution equation for the potential by defining the quantity

$$W(\chi) = \chi^4 \left(1 + \frac{V''}{n\, k^2\, \mathcal{Z}\, \chi^2}\right)^{-\gamma}, \tag{3.64}$$

being $\chi \equiv \chi_{\mathrm{B}}$ and with $\gamma = n - 2 > 0$, that diverges at $+\infty$ as the "spinodal line" $n\, k^2\, \mathcal{Z}\, \chi_0^{2\nu} + V''(\chi_0) = 0$ is approached, but it behaves as a power law for large values of the field outside the "coexistence" region where $n\, k^2\, \mathcal{Z}\, \chi_0^2 + V''(\chi_0) < 0$. In terms of this new variable equation (3.58a)) reads

$$(2+\eta)\, n\, k^2\, \mathcal{Z}\, \chi^2 (W^{-\frac{1}{\gamma}}\, \chi^{\frac{4}{\gamma}} - 1) - n\, k^2\, \mathcal{Z}\, \chi^{\frac{4}{\gamma}+2} \gamma^{-1}\, W^{-\frac{1}{\gamma}-1}\, k\, \partial_k\, W = A_n\, k^4\, \partial^2_{xx}\, W\,, \tag{3.65}$$

where A_n is a volume element. The advantage of this manipulation is that Eq. (3.65) is now linear in the second derivative.

Ideally we would like to evolve an initial data W_{in} defined at the cutoff scale $k_{in} = \Lambda_{UV}$ along the RG direction towards the infrared. This is usually achieved by defining the RG time t via

$$\frac{k}{\Lambda_{UV}} = e^{-t}\,, \tag{3.66}$$

with $t > 0$. Since the PDE (3.65) is a boundary value problem the Cauchy problem is then fully determined once we give an initial condition at the UV scale, $W(\chi, t_{in} = 0)$, and we fix as a boundary condition the value of W for an asymptotic value χ_{out} of the field, i.e. $W(\chi_{\text{out}}, t)$, where $\chi_{\text{out}} \gg 1$ in actual calculations. However, if we intend to do so, we immediately run into the difficulty that as $\mathcal{Z} < 0$, equation (3.65) belongs to the restricted élite of the backward-parabolic equations, i.e. a class of diffusion-type partial differential equation with a negative diffusion constant. As it is well known, in this case the Cauchy problem is not “well-posed" and the existence of the solution for generic initial data is not guaranteed even for an infinitesimal time step.

To integrate the PDE it is therefore necessary to treat the question as a sort of inverse problem and to consider an integration in the UV direction instead of the IR, that is by defining the RG time as

$$\frac{k}{\Lambda_{UV}} = e^{t}\,, \tag{3.67}$$

with $t > 0$. Doing so in fact the equation becomes a backward-parabolic equation, and consequently the Cauchy problem is well posed and the solution is unique. Clearly, once the solution in the deep UV is found it is possible to ask whether that solution is an admissible initial data for a non singular IR flow or not. However, also in the case of the UV evolution, due to its strong nonlinearities, a proper numerical strategy is to implement a fully implicit predictor-corrector numerical scheme on an uniform spatial and temporal grid.

In solving the flow equation the predictor step is computed at times $t = (j + 1/2)\,\Delta t$ so that we can discretize (3.65) according to the scheme

$$\begin{aligned}\frac{1}{h^2}\,\delta_x^2\,W_{i,j+1/2} &= \frac{n\,\mathcal{Z}_{i,j}}{A_n}\,(2+\eta)\,e^{-2\,t}\,(ih)^2\,\left(W_{i,j}^{-\frac{1}{\gamma}}\,(ih)^{\frac{4}{\gamma}} - 1\right)\\ &-\frac{n\,\mathcal{Z}_{i,j}}{A_n\,\gamma}\,e^{-2\,t}(ih)^{\frac{4}{\gamma}+2}\,W_{i,j}^{-\frac{1}{\gamma}-1}\,\frac{2}{q}\,\left(W_{i,j+1/2} - W_{i,j}\right)\,,\end{aligned} \tag{3.68}$$

being $h = 1/\Delta\chi$ the spatial grid spacing, $q = 1/\Delta t$ the temporal grid spacing and, as usual, $\delta_x^2\,W_i = W_{i-1} - 2W_i + W_{i+1}$. The corrector step is instead given by

$$\begin{aligned}\frac{1}{2\,h^2}\,\delta_x^2\,\left[W_{i,j+1} + W_{i,j}\right] &= \frac{n\,\mathcal{Z}_{i,j+1/2}}{A_n}\,(2+\eta)\,e^{-2\,t}(ih)^2\,\left(W_{i,j+1/2}^{-\frac{1}{\gamma}}(ih)^{\frac{4}{\gamma}} - 1\right)\\ &-\frac{n\,\mathcal{Z}_{i,j+1/2}}{A_n\,\gamma}\,e^{-2\,t}(ih)^{\frac{4}{\gamma}+2}\,W_{i,j+1/2}^{-\frac{1}{\gamma}-1}\,\frac{1}{q}\,\left(W_{i,j+1} - W_{i,j}\right)\,,\end{aligned} \tag{3.69}$$

and the solution at $j+1/2$ in (3.69) is obtained from (3.68) from the solution of the linear tridiagonal system problem in the predictor step. As a consequence (3.69) also reduces to a linear problem for the $j+1$ time step which can be conveniently solved by standard tridiagonal solvers. The method is thus unconditionally stable and $\mathcal{O}(h^2+q^2)$ accurate.

About the boundary condition, at the inner boundary $0 < \chi_{\rm init} \ll 1$ we set a von Neumann type condition $\partial_x W(\chi_{init}, t) = 0$, and we have checked that our results are rather insensitive to the choice of $\chi_{\rm init}$ that could then be set arbitrarily close to zero in all calculations (note however that, strictly speaking, $\chi > 0$ always). At the outer boundary $\chi_{\rm out} \gg 1$ the function W is assumed to behave as a power low, like in the more familiar scalar field theory [29], which depends on the expression of the initial boundary condition W_{in}. A CREH truncation for the infrared potential, i.e.

$$V(\chi, 0) = \frac{\lambda}{6}\,\chi^4\,, \tag{3.70}$$

is however not appropriate, since it is an exact[5] polynomial truncation of the LPA equation (3.58a). Obviously the potential (3.70) cannot be used to characterize the emergence of a broken phase in the infrared regime, so that the problem cannot be investigated in a CREH context.

An assumption for the infrared potential can then be

$$V(\chi, 0) = \frac{\lambda}{6}\,\chi^4 + \sigma\,\chi^6 + \omega\,\chi^8\,, \tag{3.71}$$

where the bare values of λ, σ and ω have been chosen in order to display a non-zero minimum as an initial condition. Since a Weyl rescaling of powers of the Ricci scalar do not produce any operator in the potential that is not already present in the CREH action, looking at (3.26) it can be argued that such operators are generated by powers of the volume element $\int d^dx\sqrt{g}$.

In addition, we have also considered the coupling to be negative in order to have a real function W for large values of the field. In fact, unless $\omega = \sigma \equiv 0$ no consistent initial condition can be given in all the real line for the potential, as the threshold functions (the denominator in (3.58a)) become complex at a finite value of χ for $\omega > 0$; the infrared potential need to be lower unbounded in order to be a good initial condition. The latter is a simple consequence of the hyperbolic character of the equation: working in a "inverted picture" in which the instability of the kinetic sector is moved to the potential (the kinetic operator is stable and the CREH potential is unstable), the condition $\omega < 0$ is actually a requirement of global stability of the action.

[5]Here with exact we mean that employing a polynomial truncation of the LPA equation (3.58a) all the β-functions of higher powers of the field are equal to zero.

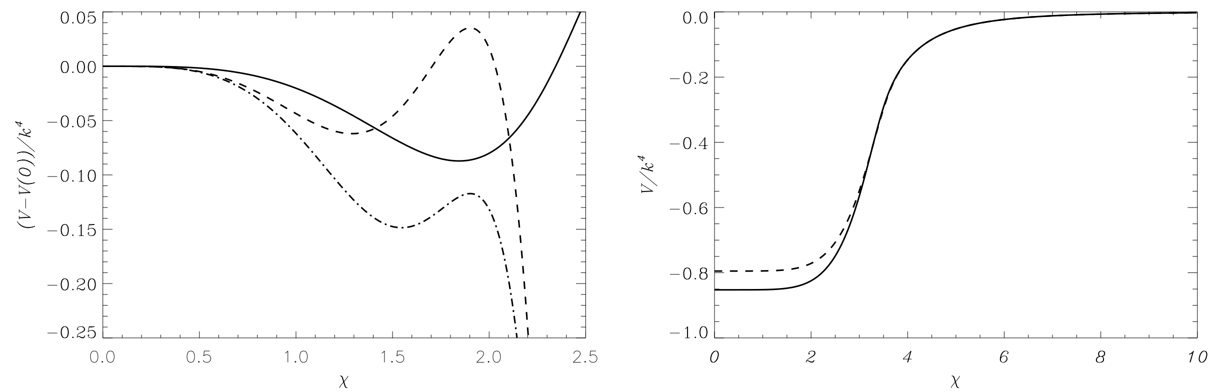

Figure 3.4: The dimensionless initial potential as a function of the RG time t for an initial condition with $\lambda = -1/2$, $\sigma = 0.05$ and $\omega = -0.00714$. Left: the early RG evolution is showed: $t = 0$ (dashed line), $t = 0.02$, (dotted-dashed line) and $t = 0.8$ (solid line). Right: the deep UV fixed point is presented: $t = 6$ (solid line) and $t = 8$ (dashed line). Further increase in t did not show significant changes in V.

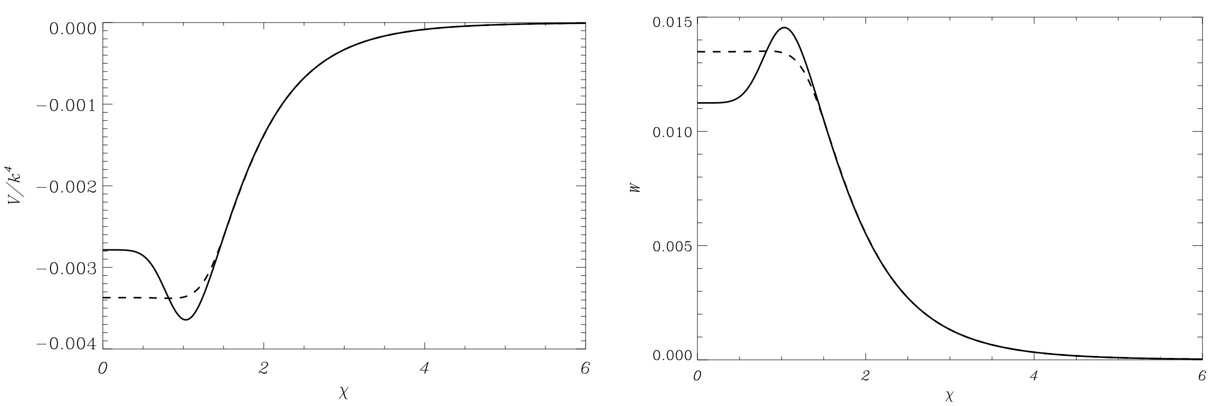

Figure 3.5: Left: dimensionless initial potential as a function of the RG time t for an initial condition with $\lambda = -6$, $\sigma = \omega = -1$ and $n = 5$: $t = 8$ (solid line), $t = 32$ (dashed line). Right: corresponding evolution of the function W is shown.

In solving (3.65) close to the NGFP, we have set $\eta \approx -2$ and $\mathcal{Z} \approx -k^2/g_*$ in $d = 4$ as we are interested in the UV evolution, but not changes has been observed while changing the value of η.

Our results are then summarized in Fig.(3.4): in the left panel a symmetry breaking initial state evolves towards a convex potential as the UV evolution is followed. The final, fixed point state, is then reached already for $t = 6$ as it can be seen in the right panel. Note the flat bottom of the potential and the almost exponential suppression at large values of the field in the final solution. We found that the appearance of a fixed point potential of the type shown in Fig.(3.4) seems to be quite generic if the initial condition is changed.

In Fig.(3.5) another example of the UV evolution is shown for $n = 5$ and for a different set of initial conditions. Note in particular that using instead the UV potential at $t = 32$ and integrating towards the IR, a symmetry breaking vacuum appears at low energy.

It is particularly interesting to note that the large field behavior of the fixed point potentials we found is characterized by an inverse power behavior, signaling then the presence of non local invariants in the fixed point potential. This result is however not surprising, since we expect the large field behavior of the potential to be defined by the sole quantum fluctuation (being the field dimensionless). The different initial conditions seems, moreover, to lead to slightly different fixed point solutions, suggesting than the existence for this toy model of a more complex fixed point structure than that found in the CREH truncation, and unaccessible using the standard $\boldsymbol{\beta}$-functions approach. Regarding the connection with the gravitational case, it has to be said that this class of fixed points solutions have no relation with the set of scale invariant $f(R)$ functions, since for the flat topology the $f(R)$ is identically null.

Chapter 4

Brans-Dicke theory in the LPA

As usual in quantum field theory and critical phenomena, the use of the functional renormalization group to study the fixed point structure of a theory is based on the definition of a consistent strategy in truncating the effective action. Results can then be trusted when, reducing systematically the entity of the truncation, little improvement of the results is gained by new refinements. Such a strategy often requires to employ a hierarchy of infinite dimensional truncations of the effective action in parameter space, inasmuch as the use of finite dimensional truncations can lead to spurious fixed point solutions. For the scalar field theory, for example, it consists in taking the derivative expansion, whose leading order is the local potential approximation (see equation (1.34)).

The definition of such a hierarchy of truncations for gravity is more subtle, since the theory has a more complicated structure. A very natural option is to organize the action as if it was an expansion around a maximally symmetric background. For the latter the only non-zero component of the Riemann tensor is the Ricci scalar R, which is constant: we have $\nabla_\mu R = S_{\mu\nu} = C_{\mu\nu\rho\sigma} = 0$, where $S_{\mu\nu}$ is the traceless Ricci tensor, and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. The analogue of the derivative expansion can then be an expansion in $S_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$ and their derivatives (by the Bianchi identity $\nabla_\mu R = \frac{2d}{d-2}\nabla^\nu S_{\mu\nu}$), with arbitrary dependence on R at each order. In the leading order of such an expansion we are left with an $f(R)$ theory, whose study in such spirit was begun in [75, 78, 76, 89, 77], and which has been briefly introduced in 2.2.6.

As compared to the LPA for scalar field theory, in the $f(R)$ approximation for gravity we face a number of additional technical complications, in particular a larger number of contributions to the functional renormalization group equation, with a more complicated dependence on the unknown function, and the challenge of evaluating functional traces on a curved background. The latter in particular introduces some subtleties related to the presence of zero modes in compact backgrounds and to the staircase nature of the results obtained for the traces when using cutoffs with step functions [75].

Also for these reasons, progresses from the recent results obtained for the f(R) has been slow in this direction, and it is desirable to find alternative ways to study the same problem. One way, already introduced in chapter 3, is to study reduced models, i.e. by neglecting the contributions of some degrees of freedom in the quantization procedure. The study of toy models, however, leads to approximated results which are physically interesting just in particular cases (as we will see in chapter 5).

One other possible way, which we will explore in this chapter, is to study dynamically equivalent formulations of the theory we are interested in.

4.1 Scalar-tensor theories

Two theories are dynamically equivalent (at a classical level) when by means of a variational principle they lead to the same equation of motion. It is a well known classical property of the $f(R)$ theory to be dynamically equivalent to a scalar-tensor theory (see [90, 91] for references).

Let us consider a (Euclidean) $f(R)$ theory with the metric field minimally coupled to a matter sector

$$S[g_{\mu\nu}, \psi] = \int d^d x \sqrt{g}\, f(R) + S_M[g_{\mu\nu}, \psi]\,, \tag{4.1}$$

being ψ is a generic matter field and where S_M defines the matter sector of the theory. An equivalent theory can be obtained by taking a Legendre transform of $f(R)$, that is by defining[1] a scalar field ϕ and a potential $V(\phi)$ such that

$$\phi = -f'(R)\,, \qquad V(\phi) = f(R(\phi)) + \phi\, R(\phi)\,, \tag{4.2}$$

where the prime in $f'(R)$ means derivative with respect to the Ricci scalar, and where as usual holds the regularity condition $f''(R) \neq 0$, so that we have

$$S[g_{\mu\nu}, \phi, \psi] = \int d^d x \sqrt{g}\, \{V(\phi) - \phi\, R\} + S_M[g_{\mu\nu}, \psi]\,, \tag{4.3}$$

that is a scalar-tensor theory corresponding to the Jordan frame of the action of a Brans-Dicke theory with $\omega = 0$ and generic potential $V(\phi)$.

The Brans-Dicke theory is one of the oldest modification of general relativity [92], and an example of dilaton gravity, i.e. a theory in which the gravitational interaction is mediated by the metric field and a supplementary scalar field $\phi(x)$, called scalaron or dilaton. The scalar field $\phi(x)$ can be though as a spacetime generalization of the Newton's constant G, now promoted to field, $\phi \equiv G^{-1}$, so that the Einstein action now reads

$$S_{BD}[g_{\mu\nu}, \phi] = \frac{1}{16\,\pi} \int d^d x\, \sqrt{g} \left\{ -\phi\, R + \frac{\omega}{\phi}\, (\partial_\mu\, \phi)(\partial^\mu\, \phi) \right\}\,, \tag{4.4}$$

where the kinetic term for ϕ (not present in the Einstein action) has been added so that the scalar field propagate according to the Klein-Gordon equation. The action (4.3)

[1] The signs in the Legendre transform are chosen in order to have a positive sign for the potential.

is then a simple generalization of the Brans-Dicke action (4.4) for $\omega = 0$ and with a general potential $V(\phi)$.

The equivalence of the scalar-tensor and $f(R)$ theory can be easily checked by looking at the equation of motion: the variation of (4.3) with respect to the metric leads to

$$\frac{\delta S}{\delta g^{\mu\nu}} = +\phi \, G_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, V(\phi) + \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\nabla^2\right)\phi - \kappa T_{\mu\nu} = 0\,, \tag{4.5}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\,\frac{\delta S_M}{\delta g^{\mu\nu}}\,, \tag{4.6}$$

while the variation with respect to ϕ gives

$$R = \frac{d\,V}{d\,\phi}\,. \tag{4.7}$$

Solving the latter for $\phi(R)$, substituting it in (4.5), and defining $f(R) = V(\phi(R)) - \phi(R)\,R$, leads to the equation of motion of an $f(R)$ theory. If on the contrary we wish to eliminate R, we can take the trace of (4.5), use (4.7), and obtain

$$3\,\nabla^2\phi + 2\,V(\phi) - \phi\frac{d\,V}{d\,\phi} = -\kappa\,T\,. \tag{4.8}$$

The scalar field acquires then a kinetic term from the coupling with the Ricci scalar (which contains derivatives of the metric tensor) also not having any kinetic sector in the action (4.3), and as usual with the trace of the energy momentum tensor acting as source. In particular, by performing a conformal transformation the Jordan frame can be rewritten in an equivalent frame, the so-called Einstein frame (actually an infinite number of classically equivalent conformal frames [93]), which contains a kinetic term, in countercheck that the absence of a kinetic operator in (4.3) does not imply the non propagation of ϕ.

Dynamically equivalent theories depends moreover on the variation chosen, since the same equations of motion for the $f(R)$ can be obtained by employing different variations starting from different actions. The $f(R)$ theory (4.1) is said to be a metric $f(R)$, since the variation of the action is performed respect to the sole metric tensor. Equations of motion can also be obtained in different formalisms, like for example the Palatini formalism [94], in which the metric and the connection are separate variables, so that the Riemann and Ricci tensor are built from the independent connection and the variation (called Palatini variation) is performed respect to the metric and the connection. In particular, the equivalent scalar-tensor theory for the Palatini $f(R)$ is not (4.3) but [95]

$$S[g_{\mu\nu}, \phi]_{Pal} = \int d^d x\,\sqrt{g}\left\{V(\phi) - \phi\,R + \frac{3}{2\,\phi}\,(\partial_\mu\,\phi)(\partial^\mu\,\phi)\right\}\,, \tag{4.9}$$

that is, a Brans-Dicke theory with generic potential and $\omega = \frac{3}{2}$.

From a RG perspective the advantage of working with a scalar-tensor formulation is then clear: we can study the running of $f(R)$ by investigating instead the running of the potential $V(\phi)$ by projecting the fRG equation on a flat background, thus sidestepping all the complications of curved backgrounds[2]. In particular, the projection on a flat background allows us to study such theory without truncating the potential to a polynomial form, thus performing an analysis similar to that of pure scalar theory [25, 99, 100, 101].

Of course, at a quantum level the two theories might be inequivalent. They are both perturbatively nonrenormalizable, and standard perturbative reasonings could only apply at an effective field theory level. When looking for a UV completion in the form of a nontrivial fixed point, we study the RG equations in two different theory spaces, and in the full theory the scalar field might couple to other geometric invariants, or acquire its own dynamical term. As a consequence, if fixed points were to be found in both formulations, they might describe different physics. It might also happen that one formulation admits an asymptotic safety scenario and the other does not.[3] However, we also do not know a priori whether the quantum theories are equivalent or not, and only a direct comparison (which we can at least do at the level of truncations) will allow to settle the question.

In any case, given that in asymptotic safety we are in principle allowing for extra degrees of freedom, there seems to be no reason to consider only pure metric theories of gravity, and the study of scalar-tensor theories is of interest in its own. The Brans-Dicke theory itself, together with its variations and generalizations, finds plenty of applications in cosmology [102], and in quantum gravity (e.g. [103, 104, 105]), and we will focus our interest toward the study a more general class of scalar-tensor theories, namely

$$S[g_{\mu\nu}, \phi] = \int d^d x \sqrt{g} \left\{ V(\phi) - \phi\, R + \frac{\omega}{\phi}\, \partial_\mu \phi\, \partial^\mu \phi \right\} . \tag{4.10}$$

[2]Note that in the context of asymptotic safety, Brans-Dicke theory with $\omega = 0$ was considered in [96] as a RG improvement of the Einstein-Hilbert truncation, in which the running gravitational and cosmological constants were promoted to fields as a result of an identification of scales with spacetime points. Clearly our work differs substantially from [96], as we study the RG equations directly for the Brans-Dicke theory. In a sense our work relates to [96] like the general $f(R)$ studies [75, 78, 76, 89, 77] relate to the $f(R)$ actions obtained by improvement of the Einstein-Hilbert truncation [88, 97, 98].

[3]In addition, we should also notice that often in the cosmology literature other "frames" are considered, in which a new metric field is defined via a conformal map, often together with a redefinition of the scalar field as well. Again, at the classical level these are all equivalent theories (although there has been some confusion on the issue in the past [93]), but they are probably inequivalent at the full quantum level. We will not study here those versions, having always in mind the original pure metric theory, whose metric we assume to define the coupling to ordinary matter.

Since in the end we are interested to invert our results to an $f(R)$ theory, we will keep to a large extent ω general, only to concentrate on the specific case $\omega = 0$ for our numerical analysis (studying the running of ω would require using a non-constant background, or looking at the 2-point function, which is beyond the aim of this work). Note that (4.10) differs from other scalar-tensor theories studied in the asymptotic safety literature [106, 107, 108] in two important aspects: it is not invariant under $\phi \to -\phi$ (and of course ϕ is not restricted to be positive), and the kinetic term (when present, that is, when $\omega \neq 0$) contains an inverse of the field.

4.2 Quantization procedure

In a fRG context we intend to study the action (4.10) as a local potential approximation for the effective action, i.e.

$$\bar{\Gamma}_k[g,\phi] = \int d^dx\sqrt{g}\left\{V_k(\phi) - \phi\, R + \frac{\omega}{\phi}\,\partial_\mu\phi\,\partial^\mu\phi\right\}\,, \tag{4.11}$$

where the potential V_k is the only running object, and that only to next order we would promote ω and the function coupled to R to general running functions of ϕ, i.e. $\omega \equiv \omega_k(\phi)$ and $Z \equiv Z_k(\phi)$, where we have set $Z = \phi$ in (4.11). Although, as we explained, we will then project the RG flow equation for (4.11) on a flat background and study only the running of the potential, we present however for future reference the results of variations and gauge fixing for a general maximally symmetric background metric and constant background scalar field.

The background field formalism is set up by introducing the background splitting

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon\, h_{\mu\nu}\,, \qquad \phi \to \phi + \epsilon\,\varphi\,, \tag{4.12}$$

being ϵ a perturbative parameter that will be set to unity afterwards. We make the usual approximation for the effective average action (see (2.60)) which reads

$$\Gamma_k[h,\varphi;g,\phi] = \bar{\Gamma}_k[g+\epsilon\, h,\phi+\epsilon\,\varphi] + S_{\rm gf}[h,\varphi;g,\phi] + S_{\rm gh}[\bar{C},C,h,\varphi;g,\phi]\,, \tag{4.13}$$

where $\bar{\Gamma}_k[g+\epsilon\, h,\phi+\epsilon\,\varphi]$ is the action (4.11) and we neglect the running of the gauge-fixing and ghost actions, $S_{\rm gf}$ and $S_{\rm gh}$.

For the fRG equation we will need the second variation of the effective average action, therefore we expand in powers of ϵ, i.e.

$$\bar{\Gamma}[g+\epsilon\, h,\phi+\epsilon\,\varphi] = \bar{\Gamma}[g,\phi] + \epsilon\,\delta^{(1)}\bar{\Gamma}[h,\varphi;g,\phi] + \epsilon^2\delta^{(2)}\bar{\Gamma}[h,\varphi;g,\phi] + O(\epsilon^3)\,, \tag{4.14}$$

and find (omitting from now on the field dependencies of the action functionals) for the second variation

$$\begin{aligned}\delta^{(2)}\bar{\Gamma}_k = \int d^dx\,\sqrt{g}\,\Big\{&\varphi\left(-\frac{\omega}{\phi}\nabla^2+\frac{1}{2}V_k''(\phi)\right)\varphi\\
&+\varphi\left(\nabla^2 h-\nabla^\mu\nabla^\nu h_{\mu\nu}+\frac{2-d}{2d}R\,h+\frac{1}{2}V_k'(\phi)\,h\right)+V_k(\phi)\left(\frac{1}{8}h^2-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\\
&+\frac{1}{8}\phi\,h_{\mu\nu}\Big(-(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-2g^{\mu\nu}g^{\rho\sigma})\nabla^2+4\,g^{\rho\mu}\nabla^\nu\nabla^\sigma-4\,g^{\rho\sigma}\nabla^\mu\nabla^\nu\Big)h_{\rho\sigma}\\
&+\phi\,R\,\Big(\frac{d\,(d-3)+4}{4\,d\,(d-1)}h^{\mu\nu}h_{\mu\nu}-\frac{d\,(d-5)+8}{8\,d\,(d-1)}h^2\Big)\Big\}\,.\end{aligned} \tag{4.15}$$

We can exploit then the gauge-fixing freedom to simplify the Hessian operator, adding to the original action the gauge-fixing term

$$S_{\text{gf}}=\frac{1}{2\,\alpha}\int d^dx\,\sqrt{g}\,\mathcal{F}_\mu G^{\mu\nu}\mathcal{F}_\nu\,, \tag{4.16}$$

for some choice of gauge-fixing constraint $\mathcal{F}_\mu$ and of a non-degenerate operator $G^{\mu\nu}$. Physical results should be independent of the gauge choice, however, it is well known that the off-shell effective action is not gauge independent, and furthermore, the approximations we employ in the fRG equation lead to additional gauge dependences. It is then important to test our analysis against different choices of gauge. We present in the following the two types of gauge which we will use in the forthcoming sections.

4.2.1 Feynman gauge

First we consider a Feynman de Donder-type gauge ($\alpha=1$) with

$$\mathcal{F}^{(F)}_\mu=\nabla^\nu\left(h_{\mu\nu}-\frac{1}{2}g_{\mu\nu}h\right)-\frac{1}{\phi}\,\nabla_\mu\varphi\,, \tag{4.17}$$

and

$$G^{(F)\mu\nu}=\phi\,g^{\mu\nu}\,. \tag{4.18}$$

Including the contribution of the gauge fixing The total quadratic action becomes

$$\begin{aligned}\delta^{(2)}\bar{\Gamma}_k+S^{(F)}_{\text{gf}}=\int d^dx\,\sqrt{g}\,\Big\{&\frac{1}{2}\varphi\Big(-\frac{1+2\,\omega}{\phi}\nabla^2+V_k''(\phi)\Big)\varphi\\
&+\frac{1}{2}\varphi\Big(\nabla^2+\frac{2-d}{d}R+V_k'(\phi)\Big)h\\
&-\frac{1}{8}h_{\mu\nu}\Big((g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma})(\phi\,\nabla^2+V_k(\phi))\Big)h_{\rho\sigma}\\
&+\phi\,R\,\Big(\frac{d\,(d-3)+4}{4\,d\,(d-1)}h^{\mu\nu}h_{\mu\nu}-\frac{d\,(d-5)+8}{8\,d\,(d-1)}h^2\Big)\Big\}\,,\end{aligned} \tag{4.19}$$

which, employing a traceless decomposing of the metric fluctuation, i.e. $h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d} g_{\mu\nu} h$, with $g^{\mu\nu}\hat{h}_{\mu\nu} = 0$, yields finally

$$\begin{aligned}\delta^{(2)}\bar{\Gamma}_k + S^{(F)}_{\rm gf} = \int d^dx \sqrt{g}\, \Big\{ & \frac{1}{2}\varphi\Big(-\frac{1+2\omega}{\phi}\nabla^2 + V''_k(\phi)\Big)\varphi \\ & + \frac{1}{2}\varphi\Big(\nabla^2 + \frac{2-d}{d}R + V'_k(\phi)\Big)h \\ & - \frac{1}{4}\hat{h}^{\mu\nu}\Big(\phi\,\nabla^2 - \frac{d\,(d-3)+4}{d\,(d-1)}\,\phi\, R + V_k(\phi)\Big)\hat{h}_{\mu\nu} \\ & + \frac{d-2}{8\,d} h\,\Big(\phi\,\nabla^2 - \frac{d-4}{d}\,\phi\, R + V_k(\phi)\Big)h\Big\} . \end{aligned} \tag{4.20}$$

We note that via the gauge-fixing procedure we have introduced a kinetic term for the auxiliary field φ even in the case $\omega = 0$. The kinetic term disappears for $\omega = -1/2$, which is a special value for the Brans-Dicke theory in this gauge.

For the gauge sector we employ a standard Fadeev-Popov determinant which we rewrite in terms of a quadratic integral over complex Grassmann fields C^μ and $\bar{C}^\mu$. For constant background scalar field, the ghost action reads

$$S_{gh}[C,\bar{C}] = \int d^dx \sqrt{g}\, \left\{ \bar{C}^\mu \left(\nabla^2 + \frac{R}{d}\right) C_\mu \right\} . \tag{4.21}$$

4.2.2 Landau gauge

As an alternative choice of gauge, we consider a Landau gauge ($\alpha = 0$) with

$$\mathcal{F}^{(L)}_\mu = \nabla^\nu \left(h_{\mu\nu} - \frac{1}{d}\, g_{\mu\nu}\, h \right) , \tag{4.22}$$

and

$$G^{(L)\mu\nu} = g^{\mu\nu} . \tag{4.23}$$

The interesting aspect of such gauge is that it does not modify the kinetic term of φ, and in particular it does not introduce one for $\omega = 0$.

In this case, in order to simplify the non-minimal operators that appear in the second variation, we use the transverse-traceless decomposition of the metric fluctuations, namely

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_\mu\, \xi_\nu + \nabla_\nu\, \xi_\mu + \nabla_\mu\, \nabla_\nu\, \sigma + \frac{1}{d}\, g_{\mu\nu}\, (h - \nabla^2\, \sigma) , \tag{4.24}$$

with the component fields satisfying

$$g^{\mu\nu}\, h^T_{\mu\nu} = 0 , \quad \nabla^\mu h^T_{\mu\nu} = 0 , \quad \nabla^\mu\, \xi_\mu = 0 , \quad h = g_{\mu\nu}\, h^{\mu\nu} . \tag{4.25}$$

In the $\alpha \to 0$ limit, the ξ_μ and σ field components decouple completely from the rest of the Hessian, and their contribution to the fRG equation cancels exactly with the ghost contribution, when properly implemented [109]. We thus write the second variation of the action directly omitting the contribution of the longitudinal components:

$$
\begin{aligned}
\delta^{(2)}\bar{\Gamma}_k + S^{(L)}_{\text{gf}} = \int d^dx\,\sqrt{g}\,\Big\{ & \frac{1}{2}\varphi\Big(-\frac{2\,\omega}{\phi}\nabla^2 + V_k''(\phi)\Big)\varphi \\
& + \varphi\Big(\frac{d-1}{d}\nabla^2 + \frac{2-d}{2\,d}R + \frac{1}{2}V_k'(\phi)\Big)h \\
& - \frac{1}{4}h^{T\mu\nu}\Big(\phi\,\nabla^2 - \frac{d\,(d-3)+4}{d\,(d-1)}\,\phi\,R + V_k(\phi)\Big)h^T_{\mu\nu} \\
& + \frac{d-2}{8\,d}h\,\Big(2\frac{d-1}{d}\,\phi\,\nabla^2 - \frac{d-4}{d}\,\phi\,R + V_k(\phi)\Big)h\Big\}\,.
\end{aligned}
\tag{4.26}
$$

Because of the change of variables (4.24), in this case there is also a Jacobian to keep track of, which we do by introducing auxiliary fields (see [109] for references). The Jacobian for the gravitational sector leads to the auxiliary action

$$
\begin{aligned}
S_{\text{aux}-\text{gr}} = \int d^dx\sqrt{g}\,\Big\{ & 2\,\bar{\chi}^{T\,\mu}\Big(\nabla^2 + \frac{R}{d}\Big)\chi^T_\mu + \Big(\frac{d-1}{d}\Big)\bar{\chi}\Big(\nabla^2 + \frac{R}{d-1}\Big)\nabla^2\chi \\
& + 2\,\zeta^{T\mu}\Big(\nabla^2 + \frac{R}{d}\Big)\zeta^T_\mu + \Big(\frac{d-1}{d}\Big)\zeta\Big(\nabla^2 + \frac{R}{d-1}\Big)\nabla^2\zeta\Big\}\,,
\end{aligned}
\tag{4.27}
$$

where the χ^T_μ and χ are complex Grassmann fields, while ζ^T_μ and ζ are real bosonic fields. The Jacobian for the transverse decomposition of the ghost action is given by

$$
S_{\text{aux}-\text{gh}} = \int d^dx\sqrt{g}\,\eta\,\nabla^2\,\eta\,,
\tag{4.28}
$$

with η a real scalar field.

4.3 The flow equation

We write here an RG flow equation employing the exact renormalization group (ERG) introduced in 1.2.2, which takes the generic form

$$
\partial_t\,\Gamma_k[\Psi] = \frac{1}{2}\text{STr}\left[\Big(\Gamma^{(2)}_k + \mathcal{R}_k\Big)^{-1}\,\partial_t\,\mathcal{R}_k\right]\,,
\tag{4.29}
$$

being

$$
\Gamma^{(2)}_k(x,y) = \frac{\delta^2\Gamma_k}{\delta\Psi_i(x)\,\delta\,\Psi_j(y)}\,,
\tag{4.30}
$$

and where Ψ is a superfield collecting all the fields involved in the quantum action, i.e. $\Psi \equiv \{\varphi, h_{\mu\nu}, \cdots\}$, $\mathcal{R}_k$ is a generic cutoff operator, $t \equiv \log(k)$ is the RG running scale and STr identifies a functional supertrace, carrying a factor 2 for complex fields and a factor -1 for Grassmann fields.

We will construct the cutoff operator in such a way to implement the substitution rule

$$-\nabla^2 \;\; \to \;\; P_k \equiv -\nabla^2 + k^2 r(-\nabla^2/k^2)\,, \tag{4.31}$$

being $r(z)$ a dimensionless smearing function. That is, we choose a cutoff of the form

$$\mathcal{R}_k = \Gamma_k^{(2)}|_{-\nabla^2 \to P_k} - \Gamma_k^{(2)}\,. \tag{4.32}$$

A convenient choice of smearing function, leading to a considerable simplification of the functional traces, and which we will therefore use here, is the so-called "optimized" cutoff [36] which reads

$$r(z) = (1-z)\Theta(1-z)\,, \tag{4.33}$$

where $\Theta(x)$ is a Heaviside step function.

4.3.1 Feynman gauge

The Hessian of the effective action is mostly diagonal in field space, with the only exception of the $\{h, \varphi\}$ sector, thus the supertrace in (4.29) can be easily decomposed into standard functional traces. In the Feynman gauge we obtain

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t \mathcal{R}_k\right]_{h,\varphi} + \frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t \mathcal{R}_k\right]_{h^T,h^T} - \mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t \mathcal{R}_k\right]_{\bar{C},C}\,, \tag{4.34}$$

where $\mathcal{H}_k$ is the modified inverse propagator, namely $\mathcal{H}_k = \Gamma_k^{(2)} + \mathcal{R}_k$. The evaluation of the first trace requires to invert the h-φ matrix, which is trivial since the matrix elements commute. The ghost term takes a factor of minus two with respect to the other terms, because of the complex Grassmannian nature of the ghost fields.

The trace over a generic Riemannian manifold can be evaluated by means of a heat kernel expansion, but since we are interested in projecting the flow equation on a flat background we can evaluate the trace over modes as a simple integral over momenta. The derivative of the cutoff operator with respect to the RG time returns

$$\partial_t\, k^2 r\left(\frac{p^2}{k^2}\right) = 2\,k^2\,\Theta\left(1-\frac{p^2}{k^2}\right) + 2\,\frac{p^2}{k^2}\,(k^2-p^2)\,\delta\left(1-\frac{p^2}{k^2}\right)\,, \tag{4.35}$$

which reduces to the sole Heaviside step function using the property that the distributional product of the delta function with its argument is zero. Because of the step function, moreover, the trace reduces to a momentum integral between 0 and k, thus automatically rendering the functional traces UV finite, a well-known feature of the FRGE. Performing the trace we obtain

$$\frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t \mathcal{R}_k\right]_{h,\varphi} = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)} k^{d+2}\frac{N_F}{D_F}\,, \tag{4.36}$$

$$\frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{h^T,h^T} = \left(\frac{d\,(d+1)}{2}-1\right)\frac{2^{1-d}\,\pi^{-\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)}k^{d+2}\,\frac{\phi}{(k^2\phi - V(\phi))}\,, \tag{4.37}$$

$$\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{\bar{C},C} = \frac{2^{2-d}\pi^{-\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\,k^d\,, \tag{4.38}$$

where

$$\begin{aligned} N_F &= \phi\left\{4\,k^2(d\,\omega + d - 2\,\omega - 1) + (d-2)\,\phi\,V''(\phi) - 2\,d\,V'(\phi)\right\} + \\ &\quad (2-d)(2\,\omega+1)V(\phi)\,, \\ D_F &= (2-d)V(\phi)(k^2\,(2\,\omega+1) + \phi\,V''(\phi)) + \phi\left\{k^2(2\,k^2(d\,\omega + d - 2\omega - 1) + \right. \\ &\quad \left.(d-2)\,\phi\,V''(\phi)) - 2\,d\,k^2 V'(\phi) + d\,V'(\phi)^2\right\}. \end{aligned}$$

The trace over the tensor structure gives the factor $d(d+1)/2-1$ for the $h^T_{\mu\nu}$ contribution and a factor d for the ghosts, counting the number of their independent components. Since we are working on a flat manifold and constant background field both the Ricci scalar and the kinetic operator vanish, so that equation (4.34) reduces to an RG flow equation for the dimensionful potential. We cast the equation in an autonomous form, i.e. with no explicit dependence on k, by introducing the dimensionless quantities

$$\tilde{\phi} = \phi\,k^{2-d}\,,\ \ \tilde{V}(\tilde{\phi}) = V(k^{d-2}\tilde{\phi})\,k^{-d}\,, \tag{4.39}$$

in terms of which we obtain

$$\partial_t\,\tilde{V}_k(\tilde{\phi}) = \mathcal{T}_{\mathrm{tree}} + \mathcal{T}^{(F)}_{\mathrm{quant}}\,, \tag{4.40}$$

where

$$\mathcal{T}_{\mathrm{tree}} = -d\,\tilde{V}(\tilde{\phi}) + (d-2)\,\tilde{\phi}\,\tilde{V}'(\tilde{\phi})\,, \tag{4.41}$$

is the classical part of the equation, which is linear in the potential, and

$$\mathcal{T}^{(F)}_{\mathrm{quant}} = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)}\left\{-2\,d + \frac{\left(\frac{1}{2}\,(d^2+d)-1\right)\,\tilde{\phi}}{\left(\tilde{\phi}-\tilde{V}(\tilde{\phi})\right)} + \frac{\widetilde{N}_F}{\widetilde{D}_F}\right\}\,, \tag{4.42}$$

with

$$\begin{aligned} \widetilde{N}_F &= \tilde{\phi}\left\{4\,(d\,\omega + d - 2\omega - 1) + (d-2)\,\tilde{\phi}\,\tilde{V}''(\tilde{\phi}) - 2\,d\,\tilde{V}'(\tilde{\phi})\right\} + \\ &\quad (2-d)(2\,\omega+1)\tilde{V}(\tilde{\phi})\,, \\ \widetilde{D}_F &= (2-d)\tilde{V}(\tilde{\phi})((2\,\omega+1) + \tilde{\phi}\tilde{V}''(\tilde{\phi})) + \tilde{\phi}\left\{(2\,(d\,\omega + d - 2\omega - 1) + \right. \\ &\quad \left.(d-2)\,\tilde{\phi}\,\tilde{V}''(\tilde{\phi})) - 2\,d\,\tilde{V}'(\tilde{\phi}) + d\,\tilde{V}'(\tilde{\phi})^2\right\}, \end{aligned} \tag{4.43}$$

is the quantum part, which contains all the loop contributions, and which is responsible for the nonlinear character of the equation. Note also that the second order character of the partial differential equation is given by the presence of second derivatives of the potential in the scalar $h-\varphi$ sector of the Hessian.

4.3.2 Landau gauge

Working in the Landau gauge the supertrace in (4.29) reads

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{h,\varphi}+\frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{h^{TT},h^{TT}}+\frac{1}{2}\mathrm{STr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{aux}, \tag{4.44}$$

where the contributions of ghosts and longitudinal modes have been omitted, since they exactly cancel each other as explained before.

After performing the integral over momenta we obtain

$$\frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{h,\varphi} = \frac{2^{2-d}\pi^{-\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)}k^{d+2}\frac{N_L}{D_L}\,, \tag{4.45}$$

$$\frac{1}{2}\mathrm{Tr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{h^{TT},h^{TT}} = \left(\frac{d\,(d+1)}{2}-d-1\right)\frac{2^{1-d}\pi^{-d/2}}{d\,\Gamma\left(\frac{d}{2}\right)}\,k^{d+2}\,\frac{\phi}{(k^2\phi-V(\phi))}\,, \tag{4.46}$$

$$\frac{1}{2}\mathrm{STr}\left[(\mathcal{H}_k)^{-1}\,\partial_t\,\mathcal{R}_k\right]_{aux} = -\frac{2^{1-d}\pi^{-d/2}}{\Gamma\left(\frac{d}{2}\right)}\,k^d\,, \tag{4.47}$$

being

$$\begin{aligned} N_L &= (d-1)\,\phi\left\{4\,k^2(d\,\omega+d-2\,\omega-1)+(d-2)\,\phi\,V''(\phi)-2\,d\,V'(\phi)\right\}+\\ &\quad (2-d)\,d\,\omega\,V(\phi)\,,\\ D_L &= \phi\,\{d^2\,V'(\phi)^2+2\,(d-1)\,k^2\left(2\,k^2(d\,\omega+d-2\,\omega-1)+(d-2)\,\phi\,V''(\phi)\right)+\\ &\quad -4\,(d-1)\,d\,k^2V'(\phi)\}+(2-d)\,d\,V(\phi)(2\,k^2\omega+\phi\,V''(\phi))\,. \end{aligned}$$

The RG flow equation for the dimensionless potential in such a gauge reads then

$$\partial_t\,\tilde{V}_k(\tilde{\phi}) = \mathcal{T}_{\text{tree}}+\mathcal{T}^{(L)}_{\text{quant}}\,, \tag{4.48}$$

where the classical part $\mathcal{T}_{\text{tree}}$ is the same as in (4.41), and the quantum part reads

$$\mathcal{T}^{(L)}_{\text{quant}} = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)}\left\{-d+\frac{\left(\frac{1}{2}\,(d^2+d)-d-1\right)\,\tilde{\phi}}{\left(\tilde{\phi}-\tilde{V}(\tilde{\phi})\right)}+2\,\frac{\widetilde{N}_L}{\widetilde{D}_L}\right\}\,, \tag{4.49}$$

with

$$\begin{aligned} \widetilde{N}_L &= (d-1)\,\tilde{\phi}\left\{4\,(d\,\omega+d-2\,\omega-1)+(d-2)\,\tilde{\phi}\,\tilde{V}''(\tilde{\phi})-2\,d\,\tilde{V}'(\tilde{\phi})\right\}+\\ &\quad (2-d)\,d\,\omega\,\tilde{V}(\tilde{\phi})\,, \qquad\qquad (4.50)\\ \widetilde{D}_L &= \tilde{\phi}\,\{d^2\,\tilde{V}'(\tilde{\phi})^2+2\,(d-1)\,\left(2\,(d\,\omega+d-2\,\omega-1)+(d-2)\,\tilde{\phi}\,\tilde{V}''(\tilde{\phi})\right)+\\ &\quad -4\,(d-1)\,d\,\tilde{V}'(\tilde{\phi})\}+(2-d)\,d\,\tilde{V}(\tilde{\phi})(2\,\omega+\tilde{\phi}\,\tilde{V}''(\tilde{\phi}))\,. \end{aligned}$$

4.4 Analytical study of the equation

We want now to search for fixed point solutions of equation (4.40) and (4.48), i.e. to search for scale invariant solutions $\tilde{V}_k^*$ such that $\partial_t \tilde{V}_k^* = 0$, requiring them to be globally analytic, i.e. defined for $\tilde{\phi} \in (-\infty, +\infty)$ [25, 110, 100]. The latter requirement has a well understood physical and mathematical justification, being a necessary condition for the existence of the average effective action at all values of k, and hence of the full effective action in the limit $k \to 0$ (which in $d > 2$ requires the existence of the solution for $\tilde{\phi} \to \pm\infty$, see (4.39)). In addition, the condition of global analyticity is expected to reduce the continuous set of solutions to a discrete subset of acceptable ones.

For $\partial_t \tilde{V}_k = 0$, both partial differential equations (4.40) and (4.48) reduce to second order ordinary differential equations, thus we expect 2-parameter families of local solutions, parametrized by the initial value conditions, $\tilde{V}(0)$ and $\tilde{V}'(0)$. Extending such local solutions to global ones, we generally have to impose constraints coming from the analyticity requirement and from the symmetries of the problem. In our case we do not have any constraints originating from symmetries (e.g. we have no $\tilde{\phi} \to -\tilde{\phi}$ symmetry, hence $\tilde{V}'(0) \neq 0$ in general), and we will have to study the equation on the full real line imposing asymptotic boundary conditions at $\tilde{\phi} \sim \pm\infty$. The latter, due to the non-linear nature of the equations, could contain less than two free parameters, implying that global solutions would also necessarily be parametrized by less than two degrees of freedom. Other explicit constraints can originate from fixed singularities of the equation, requiring analyticity conditions (e.g. [111]), and it is hoped that the equation does not have too many such fixed singularity, which would require an over constraining of the solutions [75, 76].

We will apply the following strategy to select solutions:

- i) we look for singularities of the equations, either fixed or movable, and study the behavior of the solution in a neighborhood of the singularity,
- ii) we study the large field asymptotic solutions of the equation and count the degrees of freedom of each class of solutions,
- iii) we numerically look for global solutions satisfying all the constraints.

The study of the large field asymptotic solutions is important also for other two reasons, namely, the derivation of the full effective action at the fixed point [75], and the relation to the $f(R)$ theory, as we will explain later.

We will present most of the analysis for the case $\omega = 0$, although occasionally we will refer to other values. As in the Landau gauge the $\omega = 0$ value is a critical value, analogous to the $\omega = -1/2$ value for the Feynman gauge, we will treat separately the two gauges, starting with the Feynman gauge. Most of our considerations apply to generic dimension $d > 2$, although we will most often specialize to $d = 4$. In appendix E.2 we will treat the special case $d = 2$.

4.4.1 Feynman gauge

Fixed singularities

In order to look for fixed singularities, we search for poles of the denominator of the scale invariant flow equation $\partial_t \tilde{V}_k = 0$ written in normal form, i.e.

$$\tilde{V}''(\tilde{\phi}) = \frac{\mathcal{N}(\tilde{V}, \tilde{V}', \tilde{\phi})}{\mathcal{D}(\tilde{V}, \tilde{V}', \tilde{\phi})}\,, \tag{4.51}$$

where $\mathcal{N}$ and $\mathcal{D}$ are polynomial functions obtained from (4.40). For $d > 2$ the only zero we find is at $\tilde{\phi} = 0$, while for $d = 2$ the equation reduces to a first order equation with no fixed singularities. To test the consequences of such singularity in $d > 2$ we impose analyticity, and study the equation in a Laurent expansion.

Locally, imposing analyticity means requiring the existence of a Taylor expansion of the solution, in other words we make the ansatz $\tilde{V}(\tilde{\phi}) = \sum_{n\geq 0} v_n \tilde{\phi}^n$, and after plugging it into the equation we expand the latter in a Laurent series centered at the origin. At leading order, the equation in the Feynman gauge reduces to

$$0 = (2\,\omega + 1)\left(\frac{2}{2^d\,\pi^{d/2}\,d^2\,v_0\,\Gamma(d/2) + 4\,d} - 1\right)\,, \tag{4.52}$$

which vanishes either restricting to $\omega = -1/2$ (the analogous case in Landau gauge will be $\omega = 0$, see 4.4.2), or fixing the potential in the origin to

$$v_0 \equiv \tilde{V}(0) = -\frac{2^{1-d}\,(2\,d-1)}{\pi^{d/2}\,d^2\,\Gamma(d/2)}\,. \tag{4.53}$$

As a consequence for $d > 2$ and $\omega \neq -1/2$ we have one constraint, thus reducing the number of degrees of freedom at the origin to one. For technical reasons, when integrating the equation numerically, we need to start from an arbitrary small value of the field ϵ. The boundary condition at ϵ can then be parametrized in terms of the derivative of the field in zero

$$\tilde{V}(\epsilon) = \tilde{V}(\epsilon; \tilde{V}(0), \tau), \qquad \tilde{V}'(\epsilon) = \tilde{V}'(\epsilon; \tilde{V}(0), \tau)\,, \tag{4.54}$$

being $\tau = \tilde{V}'(0)$ the free parameter, and evaluated by means of a MacLaurin series

$$\tilde{V}(\epsilon) = -\frac{2^{1-d}(2\,d-1)\pi^{-d/2}}{d^2\,\Gamma(d/2)} + \tau\,\epsilon + v_2(\tau)\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \tag{4.55}$$

where for example

$$v_2(\tau) = \frac{d\,\Gamma(d/2)\,\left\{d^2\left((d-1)\,d\,(2\,\omega+1) - 4\omega\right) - 2\,d^2\,\tau^2 - 4\,(d-2)\,(2\,d-1)(2\,\omega+1)\,\tau\right\}}{(4\,\pi^{\frac{1}{2}})\,8\,(d-2)\,(2d-1)}\,,$$

and higher order coefficients are likewise obtained solving the equation order by order in ϵ.

Movable singularities

The constrained differential equation admits now a one parameter family of local solutions parametrized by τ. Still, because of the non linearity of the equation, we expect most of the solutions to end at a movable singularity, i.e. at a singularity whose location depends on the initial condition. We want to study the behavior of solutions in the neighborhood of such singularities, in order to confirm analytically the existence of such singularities and be able to recognize them in the numerical integrations, as well as to discuss possible interpretations in the terms of the $f(R)$ theory. We will present in the next section the results of our search for a set of values of τ for which the singularity goes to infinity.

Hence, let $\tilde{\phi}_c$ be the value of the field at which the singularity occurs, and suppose that the singular behavior is such that there exists an $n_0 \geq 0$ such that $\tilde{V}^{(n)}(\tilde{\phi}_c) \sim \infty$ for every $n \geq n_0$. In order to understand what values of n_0 can occur for our equation, it is convenient to recast the equation (4.40) in the following form

$$- d\,\tilde{V}(\tilde{\phi}) + (d-2)\,\tilde{V}'(\tilde{\phi})\,\tilde{\phi} + \frac{1}{2^d\,d\,\pi^{d/2}\Gamma(d/2)}\,\frac{P_1(\tilde{V}'', \tilde{V}', \tilde{V}, \tilde{\phi})}{P_2(\tilde{V}'', \tilde{V}', \tilde{V}, \tilde{\phi})} = 0\,, \tag{4.56}$$

where the P_i are two polynomials containing the same monomials but with different coefficients. As the polynomials P_i have the same structure we deduce that for $\tilde{\phi} \to \tilde{\phi}_c$ their ratio will in general go to a constant for any value n_0. Special situations can arise when some cancellation occurs in P_2 which does not happen in P_1, and such cases will have to be discussed separately. As a consequence, in the general case the linear part of the equation cannot diverge, otherwise it could not be balanced by the rational part, i.e. both the potential and its first derivative do not diverge at the singularity, restricting the possible value of n_0 to $n_0 > 1$. At this stage, we can assume that in the neighborhood of $\tilde{\phi}_c$ the potential can be written as

$$\begin{aligned}\tilde{V}(\tilde{\phi}) =&(\tilde{\phi}-\tilde{\phi}_c)^\gamma \left\{A + A_1\,(\tilde{\phi}-\tilde{\phi}_c) + \mathcal{O}((\tilde{\phi}-\tilde{\phi}_c)^2)\right\}\\ &+ u_0 + u_1\,(\tilde{\phi}-\tilde{\phi}_c) + \mathcal{O}((\tilde{\phi}-\tilde{\phi}_c)^2)\,,\end{aligned} \tag{4.57}$$

and that $\gamma > 1$ (so that $n_0 > 1$), and we can try to determine the value of γ by means of the method of *dominant balance*.

In order to do so we can start with the guess that the second derivative is divergent at $\tilde{\phi}_c$, that is $1 < \gamma < 2$. In such case, by studying the balance of terms we arrive at the equation

$$\gamma - 1 = -\gamma + 2\,, \tag{4.58}$$

leading to

$$\gamma = 3/2\,, \tag{4.59}$$

in accordance with our guess. Plugging (4.57) with $\gamma = 3/2$ into (4.56), we can iteratively work out all the coefficients in the expansion as functions of the parameter u_0

and of the singular field value $\tilde{\phi}_c$. For example, in $d = 4$ we find

$$u_1(u_0) = \frac{4\,u_0\left(16\,\pi^2(u_0-\tilde{\phi}_c)+1\right)+\tilde{\phi}_c}{32\,\pi^2\,\tilde{\phi}_c\,(u_0-\tilde{\phi}_c)}, \tag{4.60}$$

$$A(u_0,u_1) = -\frac{\left(u_0^2(-(2\omega+1))+2u_0(2\omega+3)\tilde{\phi}_c+2u_1\tilde{\phi}_c\left(u_1\tilde{\phi}_c-2u_0\right)-(2\omega+3)\phi_c^2\right)^{\frac{1}{2}}}{6\pi\,\tilde{\phi}_c\,(u_0-\tilde{\phi}_c)\sqrt{2}},$$

for the leading order terms. The subleading corrections can be computed iteratively, and the next-to-leading are reported for both gauges in appendix E.

Other singular behaviors are possible if P_2 has a zero. Such situations are more easily uncovered by studying the equation written in normal form, (4.51). Assuming that the first derivative of the potential is divergent (or more divergent than the potential itself) at $\tilde{\phi} \sim \tilde{\phi}_c$, we obtain the equation

$$\tilde{V}''(\tilde{\phi}) \sim -2\frac{\tilde{V}'(\tilde{\phi})^2}{\tilde{\phi}_c - \tilde{V}(\tilde{\phi})}, \tag{4.61}$$

leading to a simple pole solution $\tilde{V}(\tilde{\phi}) \sim (\tilde{\phi}-\tilde{\phi}_c)^{-1}$, which is consistent with the assumption. Subleading corrections can be worked out, confirming the possibility that such type of singular behavior can appear in a solution of the fixed point equation.

Behavior at large field values

We apply here the method of dominant balance to study the large field regime of the differential equation (4.40).

We have already seen in (4.56) that whatever is the leading term (for $\tilde{\phi} \to \infty$ in this case) the quantum part of the equation in general goes as a constant plus subleading corrections, hence we have two possibilities: either the potential diverges at infinity, and the classical part of the equation defines the leading order, or the potential goes to a constant, and there must be some balance between linear and nonlinear part. In the first case, in the $\tilde{\phi} \to \infty$ limit the solution goes as

$$\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^{\frac{d}{d-2}} + \text{subleading terms}, \tag{4.62}$$

where A is a free parameter. Subleading terms can be calculated by solving iteratively the differential equation for an ansatz of the type

$$\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^{\frac{d}{d-2}}\left(1+\sum_{n>0} a_n(A)\,\tilde{\phi}^{-n}\right). \tag{4.63}$$

For $d = 4$, for example, the first few coefficients $a_n(A)$ are

$$a_1(A) = 0, \quad a_2(A) = -\frac{1}{16\,\pi^2}, \quad a_3(A) = \frac{-2\,\omega-61}{1152\,\pi^2 A}, \quad a_4(A) = \frac{-4\,\omega^2-4\,\omega-337}{9216\,\pi^2 A^2}. \tag{4.64}$$

The coefficients are all inversely proportional to the bare parameter A, so that this expansion cannot be continued to $A = 0$, and that case must be treated separately. The asymptotic solution so far constructed defines a one-parameter family of solutions parametrized by the variable A, but as the equation is second order, we can ask if the asymptotic solutions have more degrees of freedom. In order to answer such question (following [100, 76]) we perturb the flow equation in the neighborhood of the solution we just found, i.e. we introduce a perturbation to the potential,

$$\tilde{V}(\tilde{\phi}) \to \tilde{V}(\tilde{\phi}) + \epsilon\, \delta\tilde{V}(\tilde{\phi})\,, \tag{4.65}$$

substitute it into (4.40), and expand to linear order in ϵ. Replacing $\tilde{V}(\tilde{\phi})$ with (4.63), and keeping only the leading terms in the coefficients of the linear operator acting on the perturbation, in $d = 4$ we obtain the linear equation

$$\frac{(-2\,\omega - 1)\,\delta\tilde{V}''(\tilde{\phi})}{1152\,\pi^2\,A^2\,\tilde{\phi}} + 2\,\tilde{\phi}\,\delta\tilde{V}'(\tilde{\phi}) - 4\,\delta\tilde{V}(\tilde{\phi}) = 0\,, \tag{4.66}$$

which allows a solution which goes asymptotically like

$$\delta\tilde{V}(\tilde{\phi}) \sim B_1\,\tilde{\phi}^2 + B_2\,e^{\frac{768\,\pi^2\,A^2}{2\,\omega+1}\,\tilde{\phi}^3}\,, \tag{4.67}$$

where B_1 and B_2 are two integration constants. Note that eq. (4.66) seems to reduce to a first order equation for $\omega = -1/2$, but as we will see for the Landau gauge for $\omega = 0$ (which is the analogue of the case $\omega = -1/2$ in the Feynman gauge) for that critical value of ω we simply need to include the subleading correction of the coefficient of $\delta\tilde{V}''(\tilde{\phi})$.

Whereas the power-law solution in (4.67) merely shifts A in (4.63), the exponential solution would seem to be a new degree of freedom. However, for positive $\tilde{\phi}$ (and $\omega > -1/2$, otherwise the role of positive and negative $\tilde{\phi}$ are interchanged) it grows faster than the solution it is perturbing, contradicting our asymptotic analysis, hence it must be discarded. On the other hand, for negative $\tilde{\phi}$ it is an exponentially small perturbation, hence it is acceptable. As the perturbation is smaller than any power at large $\tilde{\phi}$, while the leading solution (4.63) contains only powers, it is not difficult to see that the full equation decomposes in a hierarchy of equations, according to powers of the exponential correction, that is, the exponential acts like an ϵ parameter and we can iteratively solve the equation to obtain

$$\tilde{V}(\tilde{\phi}) \sim \sum_{m\geq 0} \left(B\,e^{Z(\tilde{\phi},A,\omega)}\right)^m\,\tilde{V}_{[m]}(\tilde{\phi},A,\omega)\,, \tag{4.68}$$

where $\tilde{V}_{[0]}(\tilde{\phi},A,\omega)$ is the leading solution (4.63), while for $\omega = 0$ we find

$$Z(\tilde{\phi},A,0) = 768\,\pi^2 A^2 \tilde{\phi}^3 + 4224\,\pi^2 A\,\tilde{\phi}^2 + 64\left(24\,A + 769\,\pi^2\right)\tilde{\phi}\,, \tag{4.69}$$

$$\tilde{V}_{[1]}(\tilde{\phi},A,0) = \tilde{\phi}^{\frac{48\left(329\,A+5568\,\pi^2\right)}{A}}\left(1 - \frac{5712\,A^2 + 747937\,\pi^2 A + 8739072\,\pi^4}{6\,\pi^2 A^2}\tilde{\phi}^{-1} + \mathcal{O}\left(\tilde{\phi}^{-2}\right)\right)\,, \tag{4.70}$$

and so on, leaving A and B as free parameters.

The presence of a new degree of freedom at $\tilde{\phi} \sim -\infty$ creates an interesting situation, as we already know that we have an analyticity constraint at $\tilde{\phi} = 0$, hence if we had just one-parameter families of solutions at both plus and minus infinity it would be unlikely to have a global solution.[4]

There remains to consider the special case $A = 0$, which we now proceed to examine for $d = 4$ and $\omega = 0$. From the previous discussion of dominant balance we would expect in such case a solution that asymptotes to constant. Nevertheless, we should be careful as in that analysis we have excluded special cases leading to cancellations in the denominator of the quantum part of the equation. By plugging into the equation an ansatz of the type

$$\tilde{V}(\tilde{\phi}) \sim A_1\, \tilde{\phi} + \sum_{n\geq 0} b_n\, \tilde{\phi}^{-n}\,, \tag{4.71}$$

we find at leading order the equation $A_1 = 0$, in accordance with the previous analysis. However, a careful look at the higher orders of the expansion reveals the presence of poles at $A_1 = 1$ and $A_1 = 3/2$, meaning that for those values the general expansion is not valid, and a separate treatment is needed. In fact, we find that such special values of A_1 also lead to solutions that are solvable with an iterative algorithm.[5] In all three cases ($A_1 = 0$, 1 and 3/2) we find no free parameter in the expansion (4.71), but by studying the linear perturbations we discover the presence of exponentially small corrections at negative $\tilde{\phi}$ for $A_1 = 0$, exponentially small corrections at both positive and negative $\tilde{\phi}$ for $A_1 = 1$, and a non-integer power correction at negative $\tilde{\phi}$ for $A_1 = 3/2$. It is quite easy to see that exponentially small corrections always carry one new degree of freedom, while the analysis in the case of the non-integer power is slightly more tedious and we have not pushed it further (also because in our numerical analysis we saw no evidence of the $A_1 = 3/2$ asymptotic behavior for the Feynman gauge). Just as an example of the type of results, for $A_1 = 0$ we find that the coefficients in (4.71) read

$$b_0 = \frac{3}{128\,\pi^2}\,, \quad b_1 = \frac{7}{6144\,\pi^4}\,, \quad b_2 = \frac{985}{18874368\,\pi^6}\,, \quad b_3 = \frac{4793}{1811939328\,\pi^8}\,, \tag{4.72}$$

[4] Suppose that we start integrating at large positive $\tilde{\phi}$ with initial conditions dictated by (4.63), and that reaching $\tilde{\phi} = 0$ we compute $\tilde{V}(0)$ and $\tau_+ = \tilde{V}'(0)$ as functions of A. Upon imposition of the analyticity condition we expect to find a discrete set of solutions for A. As A was the only free parameter, τ is now completely fixed by it. If at this point we repeat the same procedure but starting from large negative $\tilde{\phi}$, and if also in this case the asymptotic solutions form a one-parameter family, we will end up with a new fixed value $\tau_- = \tilde{V}'(0)$. It is very unlikely to find that $\tau_+ = \tau_-$. On the contrary, if the asymptotic expansion at negative $\tilde{\phi}$ forms a two-parameter family, we could obtain in that case a continuum of values for τ_-, and chances would be higher to find a global solution, as that would only require that τ_+ be in the range of τ_-.

[5] For each of the special values of A_1 we find also poles in b_0, which however do not correspond to other solutions. Therefore we believe that we have exhausted the set of possible asymptotic solutions.

etc., and that the exponential perturbation at $\tilde{\phi} \sim -\infty$ leads to a solution of the form

$$\tilde{V}(\tilde{\phi}) \sim \sum_{m\geq 0} \left(B\, e^{192\,\pi^2\tilde{\phi}}\right)^m \tilde{V}_{[m]}(\tilde{\phi})\,, \tag{4.73}$$

where $\tilde{V}_{[0]}(\tilde{\phi})$ is the perturbed solution with coefficients (4.72), and

$$\tilde{V}_{[1]}(\tilde{\phi}) = \tilde{\phi}^8 \left(1 - \frac{233}{128\,\pi^2}\tilde{\phi}^{-1} + \mathcal{O}\left(\tilde{\phi}^{-2}\right)\right)\,, \tag{4.74}$$

$$\tilde{V}_{[2]}(\tilde{\phi}) = \tilde{\phi}^{17} \left(6144\,\pi^4 - \frac{463}{128\,\pi^2}\tilde{\phi}^{-1} + \mathcal{O}\left(\tilde{\phi}^{-2}\right)\right)\,, \tag{4.75}$$

and so on, leaving B as the only free parameter.

In conclusion, we found four isolated sets of solutions at $\tilde{\phi} \to \pm\infty$. As we will explain later, from the point of view of the $f(R)$ theory the most interesting solutions are those in the first class, i.e. (4.62), for which we have found the presence of two degrees of freedom at $\tilde{\phi} \to -\infty$ and one at $\tilde{\phi} \to +\infty$ (or the opposite for $\omega < -1/2$).

4.4.2 Landau gauge

Fixed singularities

We repeat here the analysis of the analyticity of the differential equation for the Landau gauge, starting with the study of the fixed singularity in $\tilde{\phi} = 0$. Following 4.4.1, we recast the differential equation in its normal form (4.51) and then we expand it in a Laurent series employing a Taylor expansion for the potential. In this gauge we find that at leading order the equation reduces to

$$0 = -4\,\omega\,\frac{2^d\,\pi^{d/2}\,d\,\tilde{V}(0)\Gamma(d/2+1) + d - 1}{d\,(2^d\,d\,\pi^{d/2}\,\tilde{V}(0)\,\Gamma(d/2) + 2)}\,, \tag{4.76}$$

which vanishes constraining the potential at the origin as

$$v_0 \equiv \tilde{V}(0) = -\frac{2^{-d}\,(d-1)\,\pi^{-d/2}}{d\,\Gamma(d/2+1)}\,, \tag{4.77}$$

or restricting to $\omega = 0$, which is the case we are interested in. Comparing (4.76) with (4.52) we note once more that the case $\omega = 0$ in the Landau gauge is analogous to the case $\omega = -1/2$ in the Feynman, so that the analytic properties of the equation in the two gauges are the same for those two particular values.

For $\omega = 0$ we have now an equation free of singularities. As a consequence, since the equation is unconstrained, we have (for $d > 2$) two degrees of freedom at the origin, $\tilde{V}(0)$ and $\tilde{V}'(0)$, and at least one at $\tilde{\phi} \pm \infty$, the parameter A of the asymptotic solution, so that it seems more likely to find global solutions. On the technical side, the absence of a singularity at $\tilde{\phi} = 0$ also means that in this case it is possible to integrate numerically from the origin without employing a MacLaurin expansion.

Movable singularities

As in the Feynman gauge we expect the non linearity of the equation to involve the presence of movable singularities. Since the polynomials P_i in equation (4.56) contain the same monomials in both gauges, the analysis carried out in the subsection 4.4.1 with the method of the dominant balance still holds and we find in general the singular behavior (4.57) with $\gamma = 3/2$. However, because of the gauge dependence of the off-shell effective action, we end up with different coefficients for both the analytic and divergent part. For example, for $d = 4$ and generic ω we obtain

$$u_1(u_0) = \frac{1}{64}\left(\frac{128\,u_0}{\tilde{\phi}_c} + \frac{1}{\pi^2}\left(\frac{5}{u_0 - \tilde{\phi}_c} + \frac{3}{2\,u_0 - 3\,\tilde{\phi}_c} + \frac{4}{\tilde{\phi}_c}\right)\right) \tag{4.78}$$

$$A(u_0, u_1) = -\frac{\left(-3\,\tilde{\phi}_c^2\,(6\,\omega - 4\,u_1^2 + 9) + 12\,u_0\,\tilde{\phi}_c\,(2\,\omega - 2\,u_1 + 3) - 8\,\omega\,u_0^2\right)^{\frac{1}{2}}}{6\,\pi\,\tilde{\phi}_c\,(3\,\tilde{\phi}_c - 2u_0)\,\sqrt{2}}\,, \tag{4.79}$$

et cetera (see appendix E.1 for next-to-leading coefficients.). Also similar to the Feynman gauge is the presence of simple pole singularities, with (4.61) replaced by

$$\tilde{V}''(\tilde{\phi}) \sim -4\,\frac{\tilde{V}'(\tilde{\phi})^2}{3\,\tilde{\phi}_c - 2\,\tilde{V}(\tilde{\phi})}\,. \tag{4.80}$$

Behavior at large field values

Since the method of the dominant balance leads to similar conclusions for both gauge choices, we expect also for the Landau gauge to find generically an asymptotic solutions of the form

$$\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^{\frac{d}{d-2}}\left(1 + \sum_{n>0} a_n(A)\,\tilde{\phi}^{-n}\right)\,. \tag{4.81}$$

We can iteratively solve the differential equation for this ansatz, obtaining in $d = 4$

$$a_1(A) = 0\,, \quad a_2(A) = -\frac{1}{32\,\pi^2}\,, \quad a_3(A) = \frac{-2\omega - 39}{1152\,\pi^2 A}\,, \quad a_4(A) = \frac{-4\,\omega^2 - 207}{9216\,\pi^2\,A^2}\,, \tag{4.82}$$

and so on. As for the other gauge, we see that the coefficients are inversely proportional to A, so that also in this gauge we have to treat separately that case. Before studying those other solutions we focus on the number of free parameters of (4.81), by introducing a perturbation $\delta\tilde{V}$. We then linearize the equation for the perturbation and study the leading terms, obtaining the equation

$$-\frac{\omega\,\delta\tilde{V}''(\tilde{\phi})}{576\,\pi^2\,A^2\,\tilde{\phi}} + 2\,\tilde{\phi}\,\delta\tilde{V}'(\tilde{\phi}) - 4\,\delta\tilde{V}(\tilde{\phi}) = 0\,. \tag{4.83}$$

For $\omega \neq 0$ the analysis is similar to the one we presented for the Feynman gauge. For $\omega = 0$ the coefficient of $\delta\tilde{V}''(\tilde{\phi})$ vanishes; hence we need to include the next order term in the coefficient of $\delta\tilde{V}''$ and consider instead the equation

$$\frac{\delta\tilde{V}''(\tilde{\phi})}{128\,\pi^2\,A^3\tilde{\phi}^2} + 2\,\tilde{\phi}\,\delta\tilde{V}'(\tilde{\phi}) - 4\,\delta\tilde{V}(\tilde{\phi}) = 0\,, \tag{4.84}$$

which admits solutions with the asymptotic behavior

$$\delta\tilde{V}(\tilde{\phi}) \sim \mathrm{B}_1\,\tilde{\phi}^2 + B_2\,e^{-64\,A^3\,\pi^2\,\tilde{\phi}^4}\,. \tag{4.85}$$

The novelty here is that the leading power in the exponent is fourth rather than third order (a consequence of the different power of $\tilde{\phi}$ in the coefficient of $\delta\tilde{V}''$ in (4.84) with respect to (4.83)), so that the solution does not discriminate positive from negative $\tilde{\phi}$, but rather leads to constraints on A. For $A < 0$, the solution (4.85) contains an exponential degree of freedom which grows faster then the perturbed function in both positive and negative field regimes, so that we must discard it. Interestingly such sector is the unphysical one, since negative A defines the asymptotic behavior of an unbounded potential. On the other hand, for $A > 0$ the perturbation is exponentially small both at positive and negative $\tilde{\phi}$, hence it is always acceptable, and we can work out the subleading corrections as done before for the Feynman case. The higher power in the exponent means that we have to solve more iteration steps before getting to the power-law corrections, but as we do not gain any qualitative insight from such analysis, we do not report further on that, the main message being that now we have two degrees of freedom at both plus and minus infinity.

Regarding the case $A = 0$, making the ansatz (4.71) we find again ($d = 4$ and $\omega = 0$) the same three special values $A_1 = 0$, 1 and 3/2, as in the Feynman gauge. The main difference appears in the case $A_1 = 3/2$, for which the expansion (4.71) now contains one degree of freedom, i.e. b_1 is a free parameter in terms of which all the other b_n are expressed:

$$b_0 = -\frac{3}{64\,\pi^2}\,, \quad b_2 = -\frac{b_1}{8\,\pi^2}\,, \quad b_3 = \frac{b_1(11 - 1024\,\pi^4 b_1)}{1024\,\pi^4}\,, \quad \text{etc.} \tag{4.86}$$

By perturbing around such solution we find that in order to discover new solutions we have to include at least the next-to-leading order coefficients for large $\tilde{\phi}$ in the linear equation, yielding

$$\begin{aligned}\left(\frac{64\,\pi^4 b_1 - 1}{2\,\pi^2 b_1}\tilde{\phi}^3 - \frac{\tilde{\phi}^4}{b_1}\right)\delta\tilde{V}''(\tilde{\phi}) + \left(-\frac{512\,\pi^4 b_1 - 3}{4\,\pi^2 b_1}\tilde{\phi}^2 - \frac{2\,\tilde{\phi}^3}{b_1}\right)\delta\tilde{V}'(\tilde{\phi}) \\ + \left(64\,\pi^2\tilde{\phi} + \frac{4}{3}\right)\delta\tilde{V}(\tilde{\phi}) = 0\,,\end{aligned} \tag{4.87}$$

whose asymptotic solutions are a superposition of a solution that simply perturbs (4.86), and a series of logarithmic corrections,

$$\delta\tilde{V}(\tilde{\phi}) \simeq c_1 \log\tilde{\phi}\left(\tilde{\phi}^{-1} - \frac{\tilde{\phi}^{-2}}{8\pi^2} + \mathcal{O}\left(\tilde{\phi}^{-3}\right)\right)\,, \tag{4.88}$$

that carries a second degree of freedom, namely the free parameter c_1.

4.5 Numerical results

In order to find global solutions we integrate out from $\tilde{\phi} = 0$ and search for a set of initial conditions τ such that the movable singularity goes to infinity in both the positive and negative field region. The numerical integrations has been performed implicit methods (backward differentiation formula) with maximal stepsize and precision fixed accordingly to avoid stiffness issues. We present here our analysis for both gauges for $\omega = 0$ and $d = 4$, starting with the Feynman gauge.

4.5.1 Feynman Gauge

We start a numerical integration at the origin (actually at $\tilde{\phi} = \pm\epsilon$, $\epsilon \sim 10^{-10}$, as explained in the subsection 4.4.1), and similarly to what done in [100], we plot the location at which we hit a singularity, as a function of the free parameter $\tau = \tilde{V}'(0)$. When we see a spike in such a plot, we interpret it as a hint of a possible global solution. Since spikes can occur as artifacts due to the scale of the plot, ending instead at a finite value, the next step is to show that such spike can be made arbitrarily long by increasing the numerical precision and by refining the mesh. In addition, in our case we have to produce such type of plots at both positive and negative $\tilde{\phi}$, looking for spikes that occur at the same value of τ in both ranges.

At negative $\tilde{\phi}$ the plot of the singularities looks like in Fig. 4.1. We apparently find a spike in the negative region for an initial condition $\tau \sim 1.638$, which however, when zooming in, reveals a richer fine structure, actually three peaks being present (only two of which are shown in the right panel of Fig. 4.1).

Such triple peak can be understood in terms of transition between different types of singular behavior. The most clear explanation is obtained in terms of the numerator and denominator of the normal equation, $\mathcal{N}$ and $\mathcal{D}$ in (4.51), which we plot in Fig. 4.2 for four representative cases. We find that for $\tau \lesssim 1.638534$ and $\tau \gtrsim 1.638597$ both $\mathcal{N}$ and $\mathcal{D}$ diverge, together with their ratio, at some $\tilde{\phi}_c$ thus signaling the pole type of singularity found in (4.61). In the range between those two values we find that $\mathcal{D}$ vanishes at some $\tilde{\phi}_c$, reaching zero with an infinite slope; at the same $\mathcal{N}$ reaches a finite value, and we deduce that we are hitting a singularity of the type (4.57) with $\gamma = 3/2$. The transitions between $\gamma = -1$ and $\gamma = 3/2$ coincide with two of the peaks observed in the fine structure of Fig. 4.1. We interpret the remaining spike at $\tau \sim 1.638591$ as signaling a transition (as τ increases) from a regime in which $\mathcal{N}$ is always positive, to one in which it changes sign twice before hitting hitting $\tilde{\phi}_c$. As seen in the zoomed plot in Fig. 4.1, spikes can be pushed farther away from the origin, however, high precision is needed and we have not tried to reach much beyond $\tilde{\phi}_c \sim -0.1$. In fact, it turns out

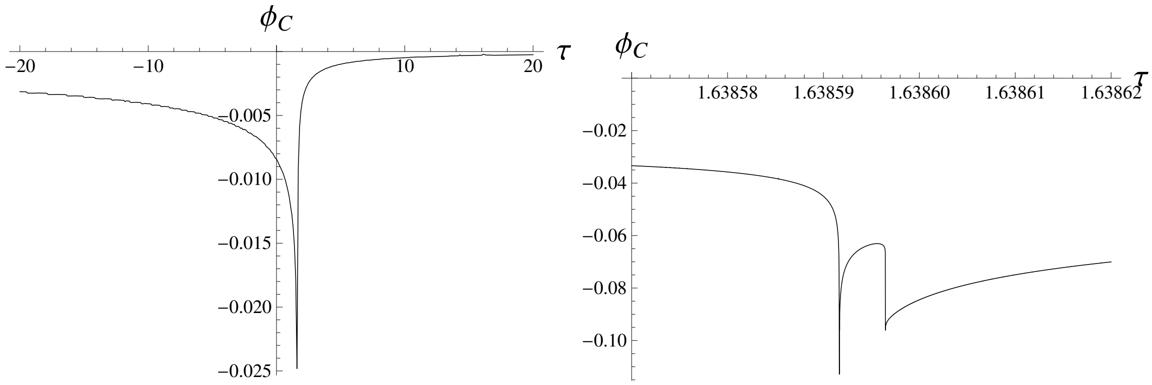

Figure 4.1: The critical field value ϕ_c in the negative domain, as a function of the initial condition $\tau = \tilde{V}'(0)$, for $d = 4$ and $\omega = 0$ in the Feynman gauge. In the right panel is a blow up of the spike, showing two spikes discussed in the text. A third spike at $\tau \sim 1.638534$ is not shown here.

that a more detailed investigation of the spikes is not worth, as the remaining part of the plot, for positive $\tilde{\phi}$, turns out to be quite disappointing. Integrating in the positive field region, including the neighborhood of $\tau \sim 1.638$, we encounter a singularity for any initial condition, as can be seen in Fig. 4.3, so that we would have not in any case a global solution. Only one type of singular behavior is found in the positive domain, a typical example of which is shown in Fig. 4.4, and from which we recognize a behavior consistent with (4.57) and $\gamma = 3/2$.

We did not find other spikes in both negative and positive region for other values of τ (outside the plot range in Fig. 4.3), so that in the end we conclude that there are no global solutions in $d = 4$ and $\omega = 0$ in the Feynman gauge.

4.5.2 Landau Gauge

The search of global solutions is more complicated in the Landau gauge since we have two degrees of freedom at the origin. In order to search for fixed points we adopted the following strategy: i) we integrate numerically from the origin (since there is no fixed singularity we can directly impose initial conditions at $\tilde{\phi} = 0$) for a fixed value of $\tilde{V}(0)$ varying the initial condition $\tau = \tilde{V}'(0)$, ii) we repeat the integration for a discrete set of positive and negative values of $\tilde{V}(0)$. As for the Feynman gauge we restrict our research to $\omega = 0$ and $d = 4$.

We start with $\tilde{V}(0) > 0$, for which we illustrate a representative outcome at negative $\tilde{\phi}$ in Fig. 4.5. In this case we find a spike at $\tau = 1.5$ and a continuum set of analytic solutions occurring for $\tau < \tau_c$, where τ_c is a critical value which depends on the initial

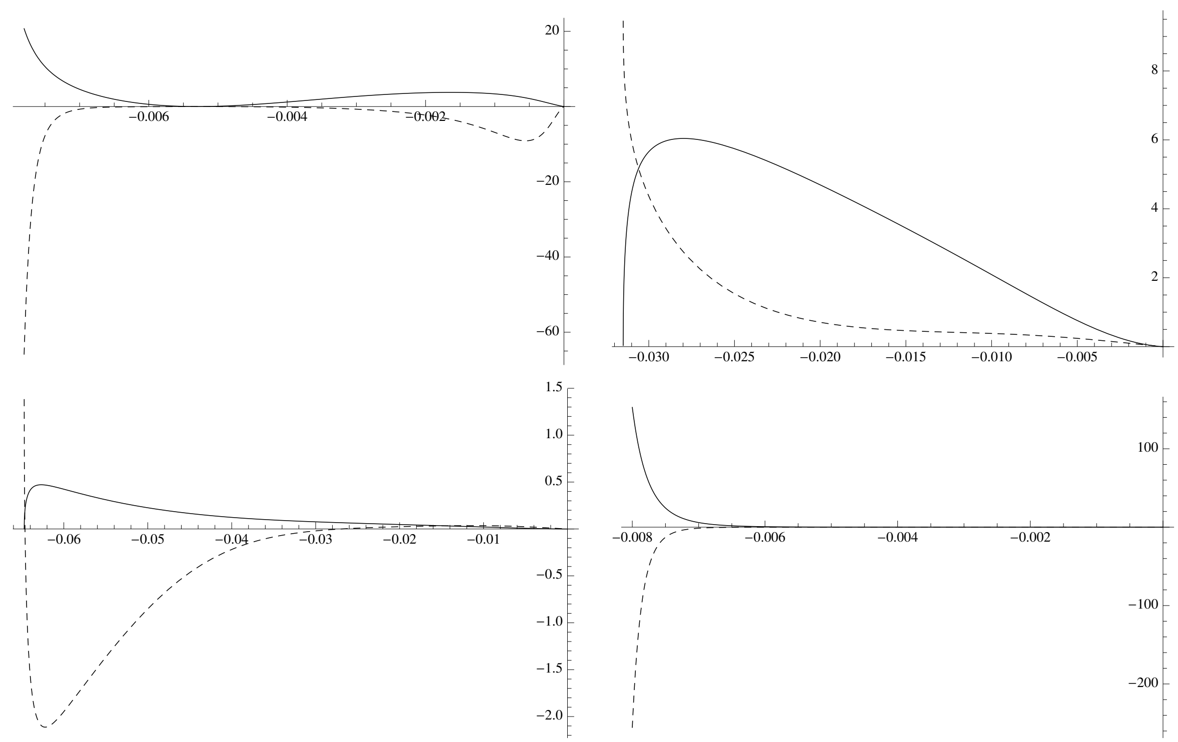

Figure 4.2: A table of plots of $\mathcal{N}$ (dashed) and $\mathcal{D}$ (solid) as functions of $\tilde{\phi}$, at four values of τ (from top to bottom, left to right, $\tau = -5$, $\tau = 1.63855$, $\tau = 1.6385965$ and $\tau = 1.7$) corresponding to the four different regimes we observed when integrating at negative $\tilde{\phi}$. Plots are not to scale, typically $\mathcal{D}$ is several orders of magnitude smaller than $\mathcal{N}$.

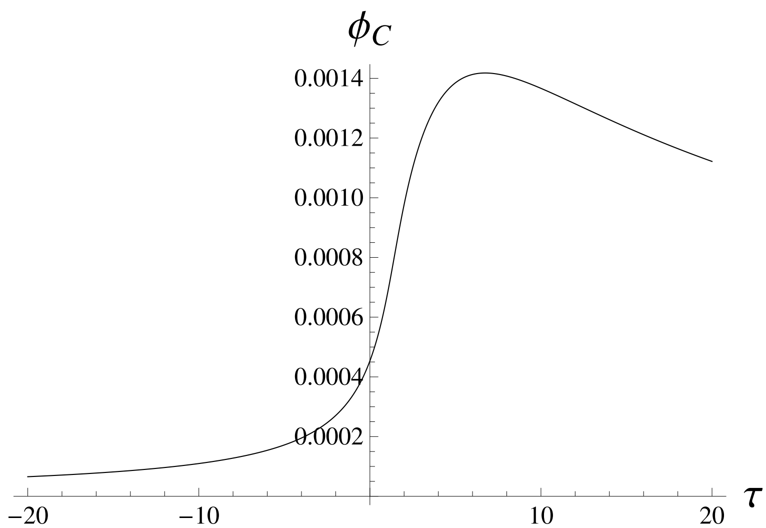

Figure 4.3: The critical field ϕ_c in the positive domain, as a function of the initial condition $\tau = \tilde{V}'(0)$, for $d = 4$ and $\omega = 0$ in the Feynman gauge.

condition $\tilde{V}(0)$, i.e. $\tau_c \equiv \tau_c(\tilde{V}(0))$. The peak at $\tau = 1.5$ actually corresponds to an exact solution of the differential equation in normal form, which for generic $d > 2$ is given by the simple linear function

$$\tilde{V}(\phi) = A + \frac{2\,(d-1)}{d}\tilde{\phi}\,, \tag{4.89}$$

being $A = \tilde{V}(0)$ a free parameter. However, we should be careful about such solution, as in the original equation it corresponds to a zero of both numerator and denominator of the h-φ trace, leading to an undetermined expression. The reason for the zero in the denominator is easily found by looking back at the second variation (4.26), and taking $\omega = 0$ and a linear function for $V(\phi)$: the φ-φ component immediately vanishes, while the h-φ component does so once we implement the rule (4.31) in combination with (4.33) and we choose the linear function as in (4.89) (the h-h component vanishes only for $A = 0$). As a consequence, the h-φ matrix is not invertible in such case. We also cannot use a limiting procedure to attribute to (4.89) the status of solution of the original equation, as perturbing the solution, i.e.

$$\tilde{V}(\phi) = A + \frac{2\,(d-1)}{d}\tilde{\phi} + \epsilon\, v(\tilde{\phi})\,, \tag{4.90}$$

and expanding in ϵ we find that the zeroth order term in ϵ does not vanish, leading instead to a nonlinear differential equation for $v(\tilde{\phi})$ (implying also that (4.89) does not admit linear perturbations). We are thus led to deem (4.89) unacceptable.

Regarding the continuum set at negative $\tilde{\phi}$, we find it for an initial conditions τ smaller then a critical value τ_c which, as we already mentioned, depends on the value of the initial condition $\tilde{V}(0)$. Varying $\tilde{V}(0)$ we observed the value of τ_c to oscillate

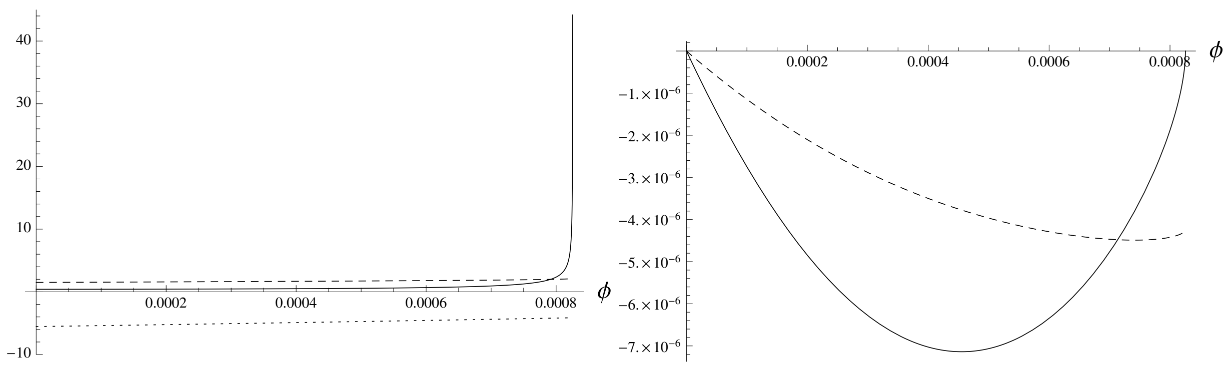

Figure 4.4: Typical plots of solutions hitting a singularity in the positive domain. We show here the case $d = 4$, $\omega = 0$ for Feynman gauge with $\tau = 1.5$. The left panel shows the potential (rescaled by a factor 10^3) together with its first and second derivative (rescaled by a factor 10^{-3}), respectively in dotted, dashed and continuous lines. The right panel shows the behavior of $\mathcal{N}$ (dashed) and $10^3 \times \mathcal{D}$.

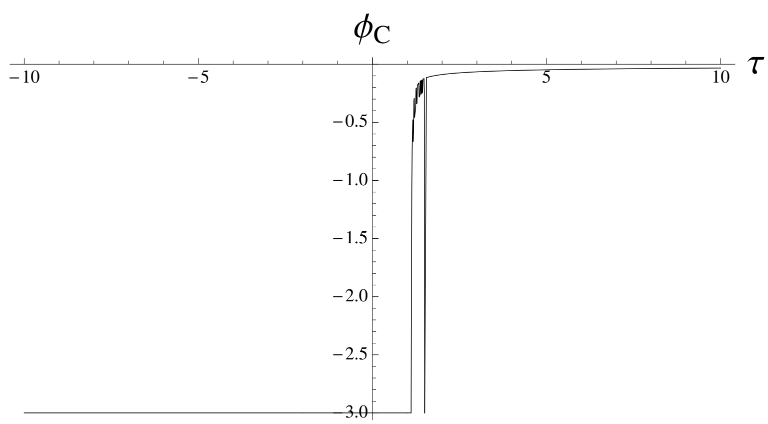

Figure 4.5: The critical field ϕ_c as a function of the initial condition $\tau = \tilde{V}'(0)$ for $\tilde{V}(0) = 0.1$, $d = 4$ and $\omega = 0$ in the Landau gauge.

between a minimum value $\tau_{min} \sim 0.96$ and a maximum $\tau_{max} \sim 1.12$. By increasing the numerical precision we were able to prolong at will the entire group of solutions and we found all of them to behave asymptotically as $A\,\tilde{\phi}^2$, being A a function of the initial conditions. A typical solution is illustrated in Fig. 4.6. The seemingly sharp edge in the second derivative is actually an optical artifact: working at high precision, and zooming around the edge one finds that the curve is smooth, as depicted in Fig. 4.7. We can understand the presence of such a short-scale transition as the rapid vanishing at large $\tilde{\phi}$ of the exponential part of the solutions we discussed in section 4.4.2 (it can be deduced from Fig. 4.6 that $A > 0$, hence the exponential corrections are possible).

As it can be seen in Fig. 4.5 all the numerical integrations performed using with initial conditions $\tau > \tau_c$ lead (with the exception of $\tau = 3/2$) to a singularity, which we found

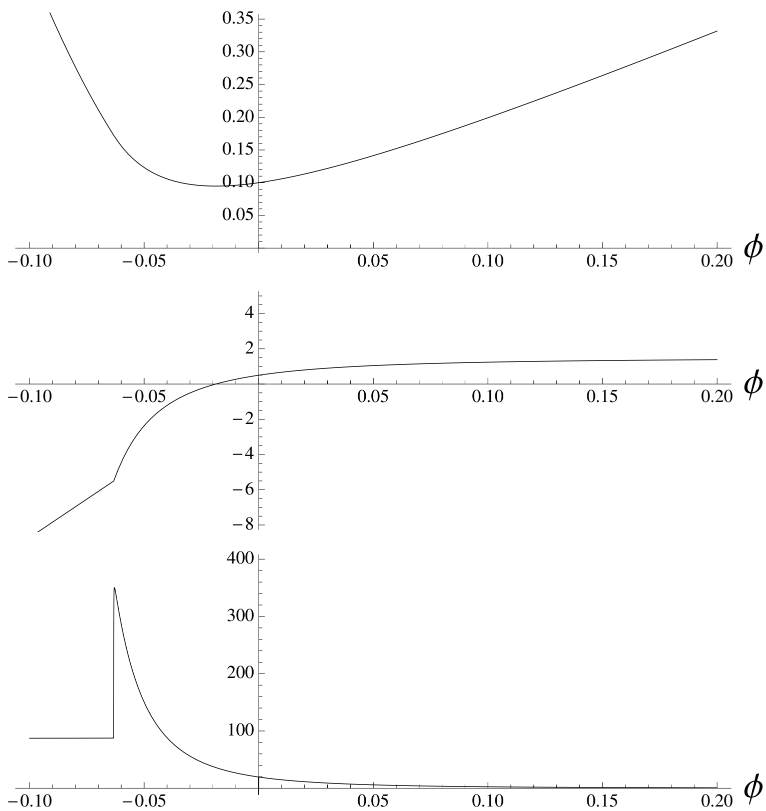

Figure 4.6: Plot of a typical global solution in the Landau gauge ($\tau = 0.5$, $\tilde{V}(0) = 0.1$, $d = 4$ and $\omega = 0$). In the upper, central and bottom panel are plotted respectively the potential, its first and its second derivative.

to be characterized by the exponent $\gamma = 3/2$. An acurate analysis reveals a transition in the way the solutions behave before reaching the movable singularity (i.e. the large field regime of the solution), from $\tilde{V}(\tilde{\phi}) \sim A\,\phi^2$ at $\tau \sim \tau_c$, to $\tilde{V}(\tilde{\phi}) \sim \frac{3}{2}\,\tilde{\phi}$ at $\tau \sim 3/2$. Such transition, together with the spurious solution (4.89), makes the equation particularly stiff around $\tau = 3/2$, as it can be seen from the noise in Fig. 4.5. However, because of the presence of a singularity we did not put much effort on a more precise numerical integration of this group of solutions.

Integrating towards positive $\tilde{\phi}$ we discover an interesting situation: for $\tilde{V}(0) > 0$ no solutions meet any singularity. We were able to push the integration to arbitrarily large $\tilde{\phi} > 0$ without encountering singularities for all values τ, and we found solutions with $\tau < 3/2$ to behave asymptotically like $\tilde{V}(\tilde{\phi}) \sim \frac{3}{2}\tilde{\phi}$, and solution with $\tau > 3/2$ to go as $\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^2$. Combining our findings for positive and negative $\tilde{\phi}$ we conclude that the solutions with $\tilde{V}(0) > 0$ and $\tau < \tau_c$ form a continuous set of global solutions.

At $\tilde{\phi} = 0$ and $\tilde{V}(0) = 0$ the equation is singular. Imposing an analyticity condition at the origin we find that $\tau = (1 \pm \sqrt{19})/4$. We did not study these special solutions in

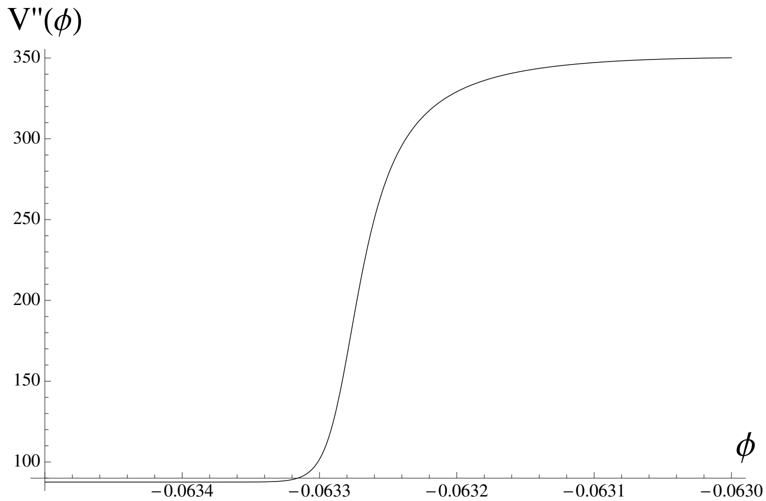

Figure 4.7: Plot of the second derivative of the potential in the range of the exponential transition to the asymptotic solution $\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^2$ in the Landau gauge ($\tau = 0.5$, $\tilde{V}(0) = 0.1$, $d = 4$ and $\omega = 0$).

detail.

For $\tilde{V}(0) < 0$ the typical situation is depicted in Fig. 4.8. All the singular solutions we found, for both positive and negative field values, diverge with exponent $\gamma = 3/2$. We found in the positive field region a continuum of solutions which do not end on a movable singularity for $\tau > 3/2$, while at negative $\tilde{\phi}$ we met no singularity for $\tau < 3/2$, in both cases with an asymptotic behavior $\tilde{V}(\tilde{\phi}) \sim \frac{3}{2}\tilde{\phi}$. The two sets have no overlap, hence there are no global solutions in this case.

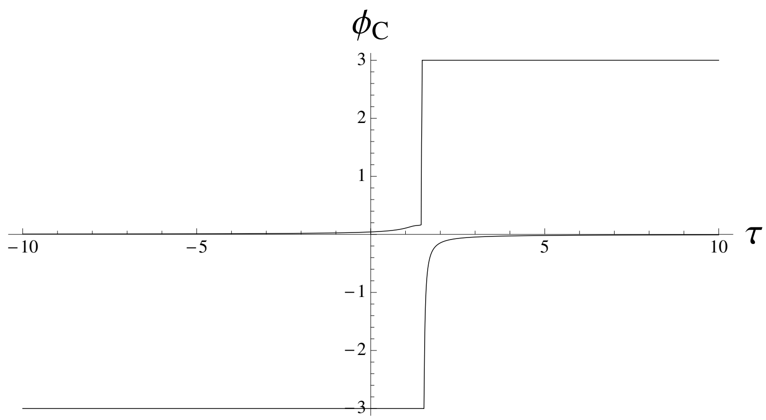

Figure 4.8: The critical field ϕ_c as a function of the initial condition $\tau = \tilde{V}'(0)$ for $\tilde{V}(0) = -0.1$, $d = 4$ and $\omega = 0$ in the Landau gauge.

In conclusion, in the Landau gauge in $d = 4$ and $\omega = 0$, we found a two parameter family of global solutions for $\tilde{V}(0) > 0$ and $\tau < \tau_c(\tilde{V}(0))$. Such result could have been expected to some extent, as in the Landau gauge we have no fixed singularity at the origin, and we have at least two classes of asymptotic behavior with two degrees of freedom each at both positive and negative $\tilde{\phi}$. The global solutions we found behave asymptotically as $\tilde{V}(\tilde{\phi}) \sim A\,\tilde{\phi}^2$ for $\tilde{\phi} \to -\infty$, and as $\tilde{V}(\tilde{\phi}) \sim \frac{3}{2}\,\tilde{\phi}$ for $\tilde{\phi} \to +\infty$. The latter is an indication of an unusual character of such solutions, as that type of asymptotic behavior is the result of a balance between the classical and quantum parts of the RG equation, to be contrasted to the usual situation, where for $k \to 0$ (i.e. the large field regime) only the classical part survives.

Comparing now the results obtained in the Feynman and Landau gauges we can thus argue whenever the dynamical equivalence with the metric $f(R)$ holds at quantum lever or not.

4.6 Quantum equivalence

While some gauge dependence was expected (due to the approximations employed and to the fact of working off-shell), we would have expected that the qualitative features of the fixed point structure, like the number of fixed points and the associated relevant directions, would be gauge independent (in principle together with any observable quantity, but in practice this property is expected to hold only approximately due to the approximations used). Being the results in our two gauges so different even at a qualitative level, we are led to infer some inconsistency of the model under consideration in the present approximation. Motivated by the relation to $f(R)$ gravity we did not analyze the case $\omega \neq 0$ in detail, and in particular we did not include its running, but we can identify the freezing of the Brans-Dicke parameter to $\omega = 0$ as the culprit of the inconsistent scenario we uncovered[6]. We expect the strong gauge dependence to be lifted once the Brans-Dicke parameter is promoted to a running coupling ω_k, in the sense that in any gauge there will be some critical value ω_c where something special happens (e.g. a discrete or continuous set of fixed points appears), the value of ω_c being gauge dependent, but not so the overall picture (i.e. the theory at criticality)[7]. For example, we already know that in the Feynman gauge the value $\omega = -1/2$ gives very similar results to the Landau gauge at $\omega = 0$, and it would be interesting to test whether such critical values correspond to fixed points of ω_k for the two gauges, reached either in the UV or in the IR.

In view of our results and of the possible solution we just outlined, we can draw an important conclusion: due to its renormalization group flow, the Brans-Dicke theory at

[6]Note that the inconsistency is given by having neglected the running of ω and not by setting it specifically to zero. Studying the case $\omega = -\frac{1}{2}$ (for which the theory is not equivalent to an $f(R)$), in fact, leads to the same issues.

[7]Our hypothesis is partially supported by the $d = 2$ case, in which the flow equations turn out to be ω-independent, and give similar results in the two gauges. See appendix E.2.

the quantum level needs a running coupling $\omega_k \neq 0$ in order to be consistent. Since in such case the equivalence with the $f(R)$ theory is broken, we conclude that Brans-Dicke theory and $f(R)$ gravity are inequivalent at the quantum level.

We should point out another aspect which also hints to a non-equivalence of Brans-Dicke theory and $f(R)$ gravity at the quantum level. As we explained, the condition for a solution of the FRGE to be a valid fixed point is that it should be a global solution. While it is quite clear from our analysis that, at least within the present approximation, no nontrivial fixed point can be found for the Brans-Dicke theory at $\omega = 0$ in the Feynman gauge, we should be careful in translating such statement back into $f(R)$ gravity. Due to the nonlinearity of the Legendre transform it could happen that a problematic singularity in one theory would turn into a harmless one in the other, or vice versa. We should indeed remember that the following relations hold (here in dimensionless variables):

$$\tilde{R} = \tilde{V}'(\tilde{\phi})\,, \qquad \tilde{f}'(\tilde{R}) = -\tilde{\phi}\,. \tag{4.91}$$

As a consequence, if a singular point $|\tilde{\phi}_c| < \infty$ is such that the first derivative of the potential is divergent, then in the $f(R)$ theory it simply means that $\tilde{\phi}_c$ is mapped to $\tilde{R}_c = \pm\infty$, depending on the sign of $\tilde{V}'(\tilde{\phi}_c)$. Although that would correspond to a strange situation in which $\tilde{f}'(\tilde{R})$ does not diverge at infinity (usually the asymptotic behavior is a power law dictated by the tree level part of the equation [75, 76, 89], implying that at infinity $\tilde{f}'(\tilde{R})$ diverges for any $d > 2$), that would not be something we can discard as unacceptable. This is precisely what happens in reverse for the Landau gauge: we found global solutions for $\tilde{V}(\tilde{\phi})$, but their first derivative is such that asymptotically $\tilde{V}'(\tilde{\phi}) \sim 3/2$ for $\tilde{\phi} \to +\infty$, and thus their transform would lead to an $f(R)$ theory valid only up to $\tilde{R}_c = 3/2$. On the other hand, if the potential is such that only its derivatives of order greater or equal to two are divergent, then the singular point is mapped to $|\tilde{R}_c| < \infty$, and thus also the transform of the potential is not a global function. The latter is precisely the case for the Feynman gauge, for which we saw that the singularities at positive $\tilde{\phi}$ are characterized by an exponent $\gamma = 3/2$, that is, they have a finite first derivative at the singular point.

Regardless of its connection to the $f(R)$ approximation, the study of Brans-Dicke theory is interesting in its own, as being a non renormalizable theory, and it is natural to wonder whether an asymptotic safety scenario applies to it. From such point of view, we should emphasize that what we have presented here is the result of the leading order in an approximation which should be systematically improved. The local potential approximation we employed can be considered, in fact, as a "double LPA" since we neglected both the renormalization of the coupling Z of the operator $\phi\, R$ (having set from the start $Z = 1$) and of the parameter ω. Both could be promoted to functions $Z(\phi)$ and $\omega(\phi)$, thus leading to a next-to-leading order approximation which could uncover an anomalous scaling of ϕ and the existence of nontrivial fixed points.

Chapter 5

Hořava-Lifshitz gravity in 2+1 dimensions

Up to now, the asymptotic safety program has obtained convincing results about the existence of a stable ultraviolet non-Gaussian fixed point. It is then conceivable that gravity could be strongly coupled and non-perturbatively renormalizable in the ultraviolet regime, despite the possible lack of unitarity. It is however still reasonable to think that perturbative renormalizability can be featured by some non-trivial property of the spacetime at the Planck-scale.

This is what happens for example in the Hořava-Lifshitz quantum gravity, which we have already introduced in section 2.3. In this theory Lorentz symmetry is broken at the Planck scale by the presence of a preferred foliation of the spacetime, which grants the theory with power counting renormalizability and explicit unitarity.

Despite the obvious drawback of lost Lorentz invariance, which in particular forces such models to face big observational challenges and fine tuning problems [112], the appealing feature of a renormalizable model of gravity in the usual sense has made Horava-Lifshitz gravity an intensely studied topic. Motivations are found, for example, in cosmology [113], or in the context of AdS/CFT as a candidate for the holographic dual description of non relativistic field theories. An other motivation comes from the relation to causal dynamical triangulations (CDT) [53, 114, 115], that is a quantum gravity approach in which the path integral is evaluated numerically summing over configurations which are obtained gluing tetrahedra related by a causality constraint. The phase diagram of CDT, in fact, shows a multicritical Lifshitz point whose universality class is conjectured to coincide with that of the model proposed by Hořava.

Oddly, the renormalization properties of Hořava-Lifshitz gravity, arguably their main motivation and the object under investigation in this chapter, are to date their least explored feature, with few important exceptions (see [116, 117, 118, 119]). Almost nothing is known about loop corrections to the Hořava-Lifshitz action, and a full proof of renormalizability is still missing. In particular, we do not know yet whether the theory is asymptotically free or if it suffers from triviality; neither do we know whether the theory flows towards general relativity in the infrared under the influence of relevant perturbations.

The reasons for the scarcity of results on the renormalization of Hořava-Lifshitz gravity are easily identifiable in the complexity of the calculations required, due to the lack of covariance (or equivalently the need to introduce a unit timelike vector [120]), as well as to the large number of terms present in the action of the most general model, i.e. the non-projectable model without detailed balance [121]. A very common strategy in trying to make progresses in similar situations is then to identify some essential features of the model we aim at, and study a simplified version of it in which such essential features are maintained while most of the complications are set aside.

One first simplification which can be made, and that we will adopt in this chapter, is to reduce the number of spacetime dimensions. In classical general relativity, four is the smallest number of dimensions in which the theory has propagating degrees of freedom, but three dimensional quantum gravity has nevertheless been a very active field of research, due to the fact that it shares many problematics with its higher-dimensional version [122]. In the case of Hořava-Lifshitz gravity, the three dimensional theory can be even more interesting than the isotropic one, because, while gravitons are still absent the new anisotropic scalar degree of freedom associated to the breaking of full diffeomorphism invariance is still present and propagates, contrariwise to the isotropic case in which is a pure gauge.

We can then study Hořava-Lifsthiz gravity in 2+1 dimensions, with dynamical exponent $z = 2$ in order to ensure power counting renormalizability, and focus on the quantization of the sole conformal degree of freedom, freeing ourself from the complications coming from taking in account the contributions of gravitons; and for this reason, in fact, lower dimensional models of Hořava-Lifshitz gravity have already received some attention [12, 123, 124, 125].

5.1 The action in 2+1 dimensions

As already mentioned in the subsection 2.3.2 there are two main versions of Hořava-Lifsthiz gravity, respectively known as projectable and non-projectable version, which differ respectively by selecting a spatially constant lapse function, i.e. $N = N(t)$, or a general space dependent function $N = N(t, \mathbf{x})$. We will here assume a spacetime topology $\mathbb{R} \times \Sigma$, with Σ a closed two-dimensional manifold, and we choose Euclidean signature for the spacetime metric, which we will decompose according to the standard ADM splitting, keeping the spacetime nomenclature despite the Euclidean signature.

In the non-projectable case the number of invariants quickly grows with the spatial dimensionality because of the spatial dependence of the lapse function, so that already in two dimensions we end up with twelve couplings [124]. In fact, the most general action in 2+1 dimensions and with $z = 2$ reads

$$S[N, N_i, g_{ij}] = \int dt\, d^2x N\sqrt{g}\left\{\frac{2}{\kappa^2}\left(\lambda K^2 - K_{ij}K^{ij} - 2\Lambda + c\,R + \gamma\,R^2\right) + c_1\,D^2\,R \right. \\ + c_2\,a_i\,a^i + c_3\,(a_i\,a^i)^2 + c_4\,R\,a_i\,a^i + c_5\,a_i\,a^i\,D^j\,a_j \\ \left. + c_6\,(D^j\,a_j)^2 + c_7\,(D_i\,a_j)(D^i\,a^j)\right\}, \tag{5.1}$$

where $a_i = D_i \ln N$ is the acceleration vector, being D_i the spatial covariant derivative, g is the determinant of the spatial metric, R its Ricci scalar, K_{ij} the extrinsic curvature of the leaves of the foliation and K its trace. The coupling κ^2 is proportional to Newton's constant, $\kappa^2 = 32\,\pi\,G$, and Λ is the cosmological constant, while λ and γ characterize the deviations from full diffeomorphisms invariance ($\lambda = 1$ and $\gamma = 0$ corresponding to general relativity in 2+1 dimensions[1]).

The value of λ defines a one-parameter family of deformed DeWitt supermetrics

$$\mathcal{G}^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda\,g^{ij}g^{kl}\,, \tag{5.2}$$

such that

$$G_{ijmn}\,\mathcal{G}^{mnkl} = \frac{1}{2}\left(\delta_i{}^{(k}\,\delta_j{}^{l)}\right)\,, \tag{5.3}$$

where round parenthesis identify commutation of indices with unitary weight and where

$$G_{ijkl} = \frac{1}{2}\left(g_{ik}\,g_{jl} + g_{il}\,g_{jk}\right) - \tilde{\lambda}\,g_{ij}\,g_{kl}\,, \qquad \tilde{\lambda} = \frac{\lambda}{2\,\lambda - 1}\,. \tag{5.4}$$

In terms of the DeWitt metric the kinetic action can be rewritten in the more elegant form

$$\int dt\, d^2\mathbf{x}\,\sqrt{g}N\,(K_{ij}\,K^{ij} - \lambda\,K^2) = \int dt\, d^2x\,\sqrt{g}N\,(K_{ij}\,\mathcal{G}^{ijkl}\,K_{kl})\,, \tag{5.5}$$

which leads to the standard case for $\lambda = 1$. For the particular value $\lambda = \frac{1}{2}$ the kinetic term becomes invariant under the anisotropic Weyl transformations (2.101) and (2.102) [12, 126] (see appendix F for the proof) so that we might already expect such value to play a special role in the RG flow of the theory.

The non-projectable case in 2+1 dimensions has already been studied with detailed balance [12], in which case the number of couplings is drastically reduced, since the action has no potential anymore. In fact, assuming the presence of detailed balance, the potential terms can be casted in the form (2.97), that is

$$V(g_{ij}) = E_{ij}\,\mathcal{G}^{ijkl}\,E_{kl}\,, \qquad E_{ij} = \frac{1}{\sqrt{g}}\,\frac{\delta\,W[g_{ij}]}{\delta\,g^{ij}}\,, \tag{5.6}$$

[1] Note that we have chosen the sign of the kinetic term in such a way that the quadratic action for the conformal mode has the correct sign for $\lambda = 1$, unlike in general relativity. This makes sense in $2 + 1$ dimensions because there are no gravitons.

where E_{ij} comes from the variation of a two-dimensional action $W[g_{ij}]$ which, in order to have $z = 2$, should contain only up to two derivatives of g_{ij}. The unique such action is the Einstein-Hilbert action, which however is topological in $d = 2$, and hence $E_{ij} = 0$, leading to no potential in the 2+1 dimensional case. We will then study here the more interesting case without detailed balance, which was also considered in [123, 124], reducing then to the more simple projectable case.

The projectable version of the theory in 2+1 dimensions and $z = 2$ action reads

$$S = \frac{2}{\kappa^2} \int dt\, d^2x N \sqrt{g} \left\{ \lambda K^2 - K_{ij} K^{ij} - 2\Lambda + c\, R + \gamma\, R^2 \right\} , \tag{5.7}$$

where, as a consequence of the Gauss-Bonnet theorem, we have also to take in account the further simplification

$$\int dt\, d^2x N \sqrt{g} R = \int dt\, N \int d^2x \sqrt{g}^{(2)} R = 4\,\pi\,\chi \int dt\, N\,, \tag{5.8}$$

with χ the Euler characteristic of the spatial manifold Σ.

However, it turns out that in order to study the running of all the couplings, even in three dimensions and for the simple $z = 2$ projectable model, some technical problems persist when evaluating the trace of the second variation of the anisotropic action (5.7).

In order to simplify the calculation as much as possible, and to get a glimpse over the questions about the renormalization of the theory, we will adopt one second main simplification, i.e. after having gauge-fixed lapse and shift, we will quantize only the conformal mode of the spatial metric, i.e. we will study a conformally reduced toy model analogous to that studied in chapter 3.

It is actually somewhat surprising that anything can be learned from such a reduction in the case of standard isotropic gravity, as in general relativity the scalar mode is not a propagating degree of freedom. Quite on the contrary, in the case of 2+1 dimensional Hořava-Lifshitz gravity, the scalar mode is the only physical degree of freedom, as gravitons are absent and the longitudinal modes of the spatial metric are killed by the constraints (as we will explain), and therefore we might expect the conformally reduced model to be much closer to the full theory.

5.2 Metric decomposition and gauge fixing

For the quantization of the action (5.7) we will make use of the background field method, which entails the linear splitting

$$g_{ij} \to g_{ij} + \epsilon\, h_{ij}\,; \quad N \to N + \epsilon\, n\,; \quad N_i \to N_i + \epsilon\, n_i\,, \tag{5.9}$$

where $\{h_{ij}, n, n_i\}$ are the quantum fluctuations, $\{g_{ij}, N, N_i\}$ the background fields and ϵ is a perturbative parameter which we will set at a later stage. The background fields are in principle generic and off-shell; however, for practical purposes it suffices to choose a background that will allow us to discern the invariants of interest. In our case, it will be enough to consider a generic spatial background g_{ij} and to restrict the background lapse and shift respectively to $N = 1$ and $N_i = 0$.

Concerning the fluctuating fields, it is convenient to use the trace-traceless decomposition for the spatial metric fluctuation, i.e.

$$h_{ij} = \hat{h}_{ij} + \frac{1}{2}\, g_{ij}\, h\,, \tag{5.10}$$

with $g^{ij}\hat{h}_{ij} = 0$. In general dimension, the traceless metric fluctuation $\hat{h}_{ij}$ can be further decomposed in transverse and longitudinal components, but in two dimensions it is well known that transverse traceless tensors form a finite dimensional vector space. In particular, on a closed manifold of genus $\mathfrak{g}$ there are precisely $(6\mathfrak{g} - 6)$ independent transverse traceless tensors for $\mathfrak{g} > 1$, just two for $\mathfrak{g} = 1$, and no such tensors for $\mathfrak{g} = 0$. In other words, we just recalled the well-known fact that any metric on a 2-dimensional manifold is conformal to a diffeomorphism-equivalent class of constant curvature metrics

$$g_{ij} = e^{2\,\phi(x)}\tilde{g}_{ij}\,, \tag{5.11}$$

where $\tilde{g}_{ij}$ is a reference metric of constant curvature, and the ensemble of such metrics modulo diffeomorphism is known as the moduli space of the manifold, which has the same dimension as the vector space discussed above, which actually is the cotangent space at $\tilde{g}_{ij}$ to the moduli space. Hence, once we fix the topology, the metric $\tilde{g}_{ij}$ carries only gauge degrees of freedom plus a finite number of global degrees of freedom. We can then choose a topology with genus $\mathfrak{g} = 0$ for the spatial slices, like a a spherical topology, and forget about the traceless components.

The two decompositions (5.9-5.10) and (5.11) obviously coincide at the linear level, upon the identification $\phi = h/4$, while at higher orders they lead to inessential differences in the off-shell effective action. The approximation we will employ in the following consists in discarding all the quantum fluctuations associated to the metric $\tilde{g}_{ij}$, which then will be treated as a background quantity, or equivalently, in discarding the traceless fluctuations $\hat{h}_{ij}$.

To do that, we can use a time-dependent diffeomorphism to gauge-fix $n = n_i = 0$, so that we remain in this case with a residual symmetry corresponding to time-independent spatial diffeomorpishms $\zeta^i = \zeta^i(\mathbf{x})$, which could be fixed by a de Donder-type gauge fixing on a single slice. A standard canonical analysis [12] shows that the constraints of the theory preserve such gauge fixing under time evolution, thus killing the longitudinal components of the metric fluctuations, and leaving us with only the scalar mode. However, in a correct path integral quantization, the longitudinal modes should be integrated over without restrictions (at most just imposing the single-slice gauge-fixing as in [127]). Our conformal reduction will consist in not performing such functional integration, thus freezing the longitudinal modes as if they had been eliminated by the constraints.

In order to implement the gauge condition we add the gauge-fixing action

$$S_{gf} = \frac{1}{2\,\alpha^2}\int dt \int d^2x\sqrt{g}\; n^2 + \frac{1}{2\,\beta^2}\int dt \int d^2x\sqrt{g}\; n_i\, n^i\,, \tag{5.12}$$

where we have already fixed $N = 1$, and take the limit $\alpha \to 0$ and $\beta \to 0$, which leads to a complete decoupling of n and n_i.

Since the fluctuations of lapse and shift transform linearly in the time derivative, the Fadeev-Popov operator simply reads $\mathcal{M} = \partial_t$. In order to avoid problems inherent to the non positivity of such an operator (that of course are solved taking its square) we employ for the ghost sector the square root of the determinant of the squared Fadeev-Popov operator, namely $\sqrt{\det(-\mathcal{M}^2)}$, which also leads to better properties under the RG flow (see [109]). The corresponding ghost action reads then

$$S_{gh} = \int dt\, N \int d^2x\sqrt{g}\left\{\bar{c}\,\partial_t^2\, c + \bar{c}_i\,\partial_t^2\, c^i + b\,\partial_t^2\, b + b_i\,\partial_t^2\, b^i\right\}, \tag{5.13}$$

being c_i and c Grassmannian complex fields and b_i and b real bosonic fields. The limit $\alpha \to 0$ and $\beta \to 0$ can be performed at the level of the second variation of the action, after the rescaling $n \to \alpha\, n$ and $n_i \to \beta\, n_i$. It is clear that in such limit the fields n, and n_i will only survive in the gauge-fixing term, and we can set them to zero when writing the variation of S. The gauge-fixing action is clearly non-dynamical and its integration in the path integral will only give an ultralocal contribution to the action (proportional to $\delta^{(3)}(0)$) which we do not keep track of. Concerning the ghosts, they will produce a determinant of the operator $-\partial_t^2$ to some power, which can only contribute to the renormalization of the cosmological constant term, which flow we are not interested to follow.

5.3 Setup of the one-loop calculation

Since we are interested to understand whenever the theory is asymptotically safe or not, we want to evaluate the β-functions of the dimensionless coupling κ, λ and γ, in order to study their renormalization group flow. Since we are in a perturbative framework we will limit us to a one-loop calculation and hence evaluate the one-loop effective action, which can be written as[2]

$$\Gamma[h_{ij}; g_{ij}] = S_{tot}[h_{ij}; g_{ij}] + \hbar\, S^{1-loop}[h_{ij}; g_{ij}] + \mathcal{O}(\hbar^2)\,, \tag{5.14}$$

being

$$S_{tot}[h_{ij}; g_{ij}] = S[g_{ij} + \epsilon\, h_{ij}] + S_{gf}[h_{ij}; g_{ij}] + S_{gh}[c, \bar{c}, b; g_{ij}]\,, \tag{5.15}$$

and where S^{1-loop} is the one-loop correction to the bare action, i.e.

$$S^{1-loop}[h; g_{ij}] = \frac{1}{2}\,\mathrm{STr}\ln S^{(2)}_{tot}[h; g_{ij}]\,, \tag{5.16}$$

[2] Occasionally we display Planck's constant $\hbar$ as a loop expansion parameter.

where $S^{(2)}$ indicates the second functional derivative respects to the fields and STr is a supertrace (which as usual includes a factor two for complex fields and a factor minus for Grassmann fields).

As generally happens in QFT, the one-loop correction S^{1-loop} will contain some UV divergences, which, being the theory power counting renormalizable, we will be able to absorb in a renormalization of the bare couplings. The dependence of the renormalized couplings upon the renormalization scale will determine the β-functions.

The first step of the one-loop calculation is the evaluation of the second functional derivative of the action. To that end, we use the splitting (5.9), under which the action decomposes as

$$S[g_{ij} + \epsilon\, h_{ij}] = S[g_{ij}] + \epsilon\, \delta S[g_{ij}; h_{ij}] + \epsilon^2\, \delta^2 S[g_{ij}; h_{ij}] + \mathcal{O}(\epsilon^3)\,. \tag{5.17}$$

The Hessian operator

$$S^{(2)}[g_{ij}] = \frac{\delta^{(2)} S}{\delta h_{kl}\, \delta h_{mn}}\Big|_{|h=0}\,, \tag{5.18}$$

can easily be read off from the second variation $\delta^2 S[g_{ij}; h_{ij}]$ by stripping off the fluctuation fields. As we already discussed, we will use the decomposition (5.10) and discard the traceless contributions $\hat{h}_{ij}$, thus having simply $h_{ij} = \frac{1}{2} g_{ij} h$. Expanding up to the second order in the fluctuations, we first note that in $d = 2$ the variation of the metric determinant

$$\sqrt{g} \to \sqrt{g}\, \left(1 + \epsilon\, \frac{1}{2}\, h + \mathcal{O}(\epsilon^3) \right)\,, \tag{5.19}$$

has no part which is quadratic in the trace mode, and thus the bare cosmological constant will not enter in the one-loop correction of the action. And due to (5.8), also the coupling c in (5.7) will not appear in S^{1-loop}.

Finally, as we are not interested here in discussing the renormalization of the cosmological constant, and as the gauge-fixing and ghost term can only contribute to that operator, we will forget both about the lapse and shift fluctuations as well as about the ghosts.[3] We are thus left with a second variation depending only on the trace mode, namely

$$\delta^2 S[g_{ij}; h_{ij}] = \frac{1}{2\kappa^2} \int dt\, d^2x\, \sqrt{g}\, \left\{ \left(\lambda - \frac{1}{2} \right)\, (\partial_t h)^2 + \gamma\, h\, (D^4 + 2\, R\, D^2 + R^2)\, h \right\}\,. \tag{5.20}$$

When perturbatively quantizing general relativity, the perturbative expansion parameter ϵ is chosen to be equal to κ, so that the kinetic term for the graviton will be canonically normalized. In the present case we see that such choice is not enough, as the operator in (5.20) depends on the two couplings λ and γ, and there is no choice by which we could remove both of them. We should notice however that from a canonical

[3] Note that this is not an approximation: we have discussed the gauge-fixing and ghosts in the section 5.2 precisely in order to show that they cannot contribute to the renormalization of the dimensionless couplings.

point of view what should be normalized to one half is really the coefficient of $(\partial_t h)^2$, all the rest being part of the potential. Restricting our analysis to the case $\lambda > \frac{1}{2}$ (for $\lambda < \frac{1}{2}$ the operator has the wrong sign, we should start again from (5.7) and flip the signs of the extrinsic curvature terms) we thus conclude that the effective perturbative coupling is

$$\epsilon = \frac{\kappa}{(\lambda - \frac{1}{2})^{\frac{1}{2}}} \,. \tag{5.21}$$

Absorbing ϵ into the second variation, and integrating by parts, equation (5.20) can now be rewritten as

$$\delta^2 S = \frac{1}{2}\int dt\, d^2x \sqrt{g}\, h\, \mathcal{D}\, h \,, \tag{5.22}$$

being

$$\mathcal{D} = -\frac{1}{\sqrt{g}}\,\partial_t \sqrt{g}\,\partial_t + \frac{\gamma}{\lambda - \frac{1}{2}}\,(D^2 + R)^2 \,. \tag{5.23}$$

5.4 Divergences and β-functions

The supertrace in (5.14) reduces in our case to a single trace over the conformal modes of the spatial metric, which we will evaluate by means of a heat kernel expansion. First, we regulate the trace of the logarithm by regulating its proper time representation, that is[4]

$$S^{1-loop} = \frac{1}{2}\,\mathrm{Tr}\ln(\mathcal{D}) = -\frac{1}{2}\int_{\frac{1}{\Lambda^4}}^{+\infty} \frac{ds}{s}\,\mathrm{Tr}\,\mathcal{H}(x,s;\mathcal{D}) \,, \tag{5.24}$$

being $\mathcal{H}(x,s;\mathcal{D})$ the diagonal part of the heat kernel operator

$$\mathcal{H}(x,x',s;\mathcal{D}) =< x|\, e^{-s\mathcal{D}}\, |x' > \,, \tag{5.25}$$

which satisfies the heat kernel equation (D.3) with boundary condition (D.4), and where $\mathcal{D}$ is the differential operator (5.23), s a proper time variable, and Λ a ultraviolet cutoff of mass dimension one (note that $[s] = -4$ due to the unusual mass-dimension of the time coordinate), not to be confused with the cosmological constant, which from now on will not appear anymore in our calculations. If the operator $\mathcal{D}$ has zero or negative modes, then expression (5.24) will need also an infrared cutoff $1/\mu^4$, being μ an IR mass, on the upper extreme of the proper time integration.

A well known feature of the heat kernel (see appendix D) is that it admits in the limit $s \to 0^+$ an expansion series in powers of s, which in the present case reads

$$\mathcal{H}(x,s;\mathcal{D}) = \sum_{n=0}^{\infty} s^{\frac{n}{2}-1}\, a_n(x;\mathcal{D}) \,, \tag{5.26}$$

[4] A more rigorous procedure for regularizing the functional trace would consist in using a ζ-function regularization [128], however, as the final result is the same, we stick here to this more simplistic regularization scheme.

the a_n coefficients being scalars built out of geometric tensors and their derivatives. Plugging (5.26) into (5.24), and exchanging sum and integral, we immediately find that for $n > 2$ we can safely take the $\Lambda \to \infty$ limit, and that all the UV divergences are contained in the first three terms of the expansion (since the heat kernel operator is dimensionless by definition). By simple dimensional analysis we expect the logarithmic divergences to be proportional to a_2, and we expect the latter to be a linear combination of the squares of the intrinsic and extrinsic curvatures of the spatial slices.

5.4.1 Heat kernel expansion

As a result of the heat kernel expansion, we write

$$\begin{aligned} \frac{1}{2}\,\mathrm{Tr}\ln(\mathcal{D}) = & -\frac{1}{2}\int_{\frac{1}{\Lambda^4}}^{\frac{1}{\mu^4}} \frac{ds}{s}\,\mathrm{Tr}\,\mathcal{H}(x,s;\mathcal{D}) = \\ & -\frac{1}{2}\int_{\frac{1}{\Lambda^4}}^{\frac{1}{\mu^4}} \frac{ds}{s^2}\int dt\,d^2x\sqrt{\hat{g}}\,\left\{a_0 + s^{\frac{1}{2}}\,a_1 + s\,a_2 + \mathcal{O}(s^{\frac{3}{2}})\right\}\,, \end{aligned} \tag{5.27}$$

where we have introduced also an IR cutoff μ on the proper time integral, which in the Wilsonian picture plays the role of a renormalization scale.

Whereas in the isotropic case the a_n coefficients of the corresponding heat kernel expansion have been worked out by many different means and for many different operators (in particular for higher derivative operators, see appendix D), very little is available about the anisotropic case. For the case at hand fortunately we can take advantage of the computations done in [129] for an anisotropic action in $d = 2$ with $z = 2$ of the type

$$S[\phi; g_{ij}] = \frac{1}{2}\int dt\,d^2x\,\sqrt{\hat{g}}\,N\phi\,\left\{-\frac{1}{N\sqrt{\hat{g}}}\,\partial_t\frac{1}{N}\,\sqrt{\hat{g}}\,\partial_t + D^4\right\}\,\phi\,, \tag{5.28}$$

where $\phi \equiv \phi(t, \mathbf{x})$ is a generic Lifshitz field. In fact, we can recognize that the action (5.22) is almost the same as the action (5.28), the only differences (beside our background choice $N = 1$ which is unimportant) being the replacement $D^2 \to D^2 + R$ and the presence of the coupling $\gamma/(\lambda - \frac{1}{2})$, both of which are easily taken care of.

Concerning the presence of the coupling, we can simply notice that it can be dealt with by introducing the auxiliary spatial metric

$$\hat{g}_{ij} = \left(\frac{\lambda - \frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} g_{ij}\,, \tag{5.29}$$

so that (5.22) now reads

$$\delta^2 S[h; g_{ij}] = \frac{1}{2}\left(\frac{\gamma}{\lambda - \frac{1}{2}}\right)^{\frac{1}{2}}\int dt\,d^2x\sqrt{\hat{g}}\,h\,\left\{-\frac{1}{\sqrt{\hat{g}}}\,\partial_t\sqrt{\hat{g}}\,\partial_t + (\hat{D}^2 + \hat{R})^2\right\}\,h\,, \tag{5.30}$$

where $\hat{D}$ is the spatial covariant derivate constructed from the auxiliary metric $\hat{g}_{ij}$, and $\hat{R}$ the associated curvature. The coefficient $(\gamma/(\lambda-\frac{1}{2}))^{1/2}$ in front of the integral decouples when taking the logarithm of the second functional derivative, giving an ultra-local contribution which can then be discarded. We thus are left with the operator

$$\hat{\mathcal{D}} = -\frac{1}{\sqrt{\hat{g}}}\,\partial_t\sqrt{\hat{g}}\,\partial_t + (\hat{D}^2+\hat{R})^2\,, \tag{5.31}$$

for which we can use the results of [129], in combination with the heat kernel results for a generalized second order partial differential operators reported in (D.50) (see [130] for more details).

From [129] we can directly borrow the extrinsic curvature terms in a_2, as the $\hat{R}$ term in (5.31) cannot contribute to those. For the terms depending only on the spatial Ricci scalar, we observe that the time derivatives cannot contribute to those and hence we can ad hoc choose a time-independent metric and use the standard results from [130]. Putting things together, we find

$$a_2 = -\frac{1}{64\,\pi}\left(\hat{K}_{ij}\,\hat{K}^{ij} - \frac{1}{2}\,\hat{K}^2\right). \tag{5.32}$$

The coefficient (5.32) does not contain powers of the expected $\hat{R}^2$ term since the order R^2 coefficient of the heat kernel expansion vanishes for any operator of the type $(D^2+X)^2$ in $d=2$, in agreement with the $X=0$ case of [129]. The vanishing of the spatial part of the coefficient a_2 can be checked by inserting $d=2$, $V_{\mu\nu}=2\,g_{\mu\nu}\,R$, $B_\mu=0$ and $X=R^2$ in the coefficient E_4 in (D.50) (see also [130]). As a consequence, we can deduce that no renormalization of the overall coupling of R^2 will take place. Similarly using (D.50) we can also obtain

$$a_0 = \frac{1}{16\,\pi}\,,\quad a_1 = \frac{7}{48\,\pi^{3/2}}\hat{R}\,. \tag{5.33}$$

Note that with respect to (D.50) the expansion coefficients get an extra factor $(4\pi)^{-1/2}$ because of the extra (time) dimension in the trace.

Plugging (5.32) into (5.27) and integrating over the proper time we find

$$\begin{aligned}\frac{1}{2}\,\hat{\mathrm{Tr}}\ln(\hat{\mathcal{D}}) = &-\frac{1}{2}\int dt\,d^2x\sqrt{\hat{g}}\,\Big\{(\Lambda^4-\mu^4)\,\frac{1}{16\,\pi} + (\Lambda^2-\mu^2)\,\frac{14}{48\,\pi^{3/2}}\hat{R}\\ &+\ln\left(\frac{\Lambda}{\mu}\right)\frac{1}{16\,\pi}\left\{-\hat{K}_{ij}\,\hat{K}^{ij}+\frac{1}{2}\,\hat{K}^2\right\}+\mathcal{O}\left(\frac{1}{\Lambda^2}\right)\Big\}\,.\end{aligned} \tag{5.34}$$

The only term of our interest is the logarithmic divergence, which we can now rewrite as

$$S^{1-loop}_{log} = \frac{1}{32\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}}\ln\left(\frac{\Lambda}{\mu}\right)\int dt\,d^2x\sqrt{g}\,\left\{K_{ij}\,K^{ij}-\frac{1}{2}\,K^2\right\}, \tag{5.35}$$

having used (5.29) to express it in terms of the original metric g_{ij}.

5.4.2 β-functions

Since we are in a perturbative setting we will evaluate the β-functions employing the MS scheme introduced in section 1.1. We can then reabsorb the logarithmic divergencies by rewriting the bare couplings as

$$g_{b,i} = g_{R,i} + \delta g_i \,, \tag{5.36}$$

being $g_{b,i}$ the bare coupling of the i-th local operator present in the action, δg_i a counterterm chosen so to cancel the divergences and $g_{R,i}$ the renormalized coupling. More specifically, we define the renormalized couplings as

$$\begin{aligned}
\frac{2}{\kappa_R^2} &= \frac{2}{\kappa^2} - \frac{1}{32\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\,,\\
\frac{2\,\lambda_R}{\kappa_R^2} &= \frac{2\,\lambda}{\kappa^2} - \frac{1}{64\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\,,\\
\frac{2\,\gamma_R}{\kappa_R^2} &= \frac{2\,\gamma}{\kappa^2}\,.
\end{aligned} \tag{5.37}$$

We can now solve the first of (5.37) obtaining the expression of the renormalized coupling κ_R^2, which reads

$$\kappa_R^2 = \frac{\kappa^2}{\left(1 - \frac{\kappa^2}{64\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\right)}\,, \tag{5.38}$$

that, once expanded in powers of $\hbar$ using the binomial expansion and discarding higher loop orders, leads to

$$\kappa_R^2 = \kappa^2\left(1 + \frac{\kappa^2}{64\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\right) + \mathcal{O}(\hbar^2)\,. \tag{5.39}$$

Hence, using (5.39) back in (5.37) we obtain

$$\begin{aligned}
\lambda_R &= \lambda + \frac{1}{64\,\pi}\,\frac{\kappa^2}{\gamma^{1/2}}\left(\lambda - \frac{1}{2}\right)^{\frac{3}{2}} \ln\left(\frac{\Lambda}{\mu}\right) + \mathcal{O}(\hbar^2)\,,\\
\gamma_R &= \gamma\left(1 + \frac{\kappa^2}{64\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}} \ln\left(\frac{\Lambda}{\mu}\right)\right) + \mathcal{O}(\hbar^2)\,.
\end{aligned} \tag{5.40}$$

Thus, the β-functions can be evaluated by stating the independence of the bare coupling from the renormalization scale μ, i.e. $\mu\,\partial_\mu\, g_b = \mu\,\partial_\mu\, g_R + \mu\,\partial_\mu\,\delta g = 0$, which leads to the system of β-functions

$$\begin{aligned} \beta_{\kappa^2} &= \mu\,\partial_\mu\,\kappa_R^2 = -\frac{\kappa^4}{64\,\pi}\left(\frac{\lambda-\frac{1}{2}}{\gamma}\right)^{\frac{1}{2}}\,, \\ \beta_\lambda &= \mu\,\partial_\mu\,\lambda_R = \frac{\left(\lambda-\frac{1}{2}\right)}{\kappa^2}\,\beta_{\kappa^2}\,, \\ \beta_\gamma &= \mu\,\partial_\mu\,\gamma_R = \frac{\gamma}{\kappa^2}\,\beta_{\kappa^2}\,. \end{aligned} \tag{5.41}$$

Since the right-hand side of (5.41) are $\mathcal{O}(\hbar)$ we can substitute the bare couplings with the renormalized one everywhere in the β-functions. Then we can use (5.41) to note that

$$\mu\,\partial_\mu\left(\frac{\lambda_R-\frac{1}{2}}{\gamma_R}\right) = \frac{1}{\gamma_R}\,\beta_\lambda - \frac{\lambda_R-\frac{1}{2}}{\gamma_R^2}\,\beta_\gamma = 0\,, \tag{5.42}$$

so that

$$\left(\frac{\lambda_R-\frac{1}{2}}{\gamma_R}\right) = b\,, \tag{5.43}$$

being b a constant. Inserting (5.43) in the first of (5.41) we can solve the differential equation for κ_R^2, obtaining the RG flow of the coupling κ_R^2, which reads

$$k_R^2(\mu) = \frac{64\,\pi}{b^{1/2}\,(\ln\frac{\mu}{\mu_0}+C)}\,, \tag{5.44}$$

where C is an integration constant fixed by the boundary condition at some initial scale $\mu=\mu_0$. Using (5.43) and (5.44) in (5.41) we can integrate the remaining two β-functions obtaining the flow of the renormalized couplings λ_R and γ_R, which respectively read

$$\lambda_R(\mu) = \frac{1}{2} + \frac{C_1}{\ln\frac{\mu}{\mu_0}+C}\,, \tag{5.45}$$

$$\gamma_R(\mu) = \frac{C_2}{\ln\frac{\mu}{\mu_0}+C}\,, \tag{5.46}$$

being C_1 and C_2 other two integration constants. Moreover, inserting (5.45) and (5.46) in (5.43) we can see that $b = C_1/C_2$.

We observe, then, that the running coupling (5.44) has the standard behavior of an asymptotically free coupling, running to zero for $\mu\to\infty$. However, we note that also $\lambda_R-\frac{1}{2}$ and γ_R have the same behavior, a fact which leads to a problem for the perturbative treatment of Hořava-Lifshitz gravity. We have argued before that the effective perturbative coupling is ϵ, and substituting (5.46) and (5.45) in (5.21), we find the renormalized coupling to be

$$\epsilon_R^2 = \frac{\kappa_R^2}{\lambda_R-\frac{1}{2}} = \frac{64\,\pi\,C_2^{1/2}}{C_1^{3/2}}\,, \tag{5.47}$$

so that it does not run to zero in the ultraviolet limit, but instead it remains constant along the renormalization group flow. That is, the coupling ϵ is marginal at one-loop order. Since the parameter ϵ characterizes the interaction strength of the theory, we are then in a situation in which the strength of the interaction remains finite at all scales, in particular meaning that the theory is not asymptotically free.

Looking back at (5.20), we can interpret the origin of such situation as a competition between the would-be asymptotic freedom of Newton's constant κ^2, and the strong coupling phenomenon that occurs when approaching $\lambda = 1/2$. The latter is indeed a singular limit, in which the scalar mode is non-propagating (since the kinetic term in the second variation vanishes for $\lambda = 1/2$. A similar strong-coupling phenomenon was pointed out in [131] in relation to the supposed IR limit $\lambda \to 1$ of the full Hořava-Lifshitz theory, and it can be generically expected that some form of strong coupling or discontinuity will be associated to the disappearance of degrees of freedom due to enhanced symmetry, as for example in the massless limit of gravitons [132, 133]. In our case, the enhanced symmetry could be traced back to the anisotropic version of Weyl invariance at $\lambda = 1/2$ and $\gamma = 0$ [12].

In analogy to the isotropic case, where scale invariance and unitarity of a quantum field theory imply conformal invariance (up to anomalies) in two dimensions (see [134]) and seemingly four dimensions[5] (see [138, 139]), we might expect to have anisotropic Weyl invariance at a fixed point of the renormalization group equations in Hořava-Lifsthiz gravity (but, also in this case, it holds up to anomalies, which we expect [140, 129, 126]), and we can thus conjecture that our conclusion will apply also to the full theory, at least for what concerns the running of the parameter λ.

As we have restricted our theory to the projectable case, however, Weyl invariance cannot be effectively realized since the transformations (2.102) would require the lapse function to be space-dependent, but anisotropic Weyl invariance could be still realized at a fixed point with $\gamma \neq 0$ for the more general non-projectable model.

An important point to emphasize in the analysis of our result is about the dimensionality of the fixed point structure. Besides in 2.3.4 we mentioned that a two-parameter family of free fixed points can be correctly identified, what we found here means that only one of them is reached by the interacting theory.

In order to better explain such point, it might be useful to look at a similar situation, by recalling what happens for a massless scalar field theory in four dimensional curved spacetime with non-minimal coupling $\xi\, R\, \phi^2$. Being quadratic in the scalar field, we could include the non-minimal coupling term in the free action, and as ξ is dimensionless we deduce that it defines a one-parameter family of fixed points. However, the β-function for the quartic self-interaction coupling g and the coupling ξ in the $\overline{MS}$-scheme

[5] With maybe the interesting exception of limit cycles, for whose the existence of a current associated to scale invariance does not imply a conserved current for conformal invariance. For an example in $4-\epsilon$ dimensions see [135, 136, 137].

read respectively [141, 142]

$$\beta_g = \frac{3\, g^2}{(4\,\pi)^2}\,, \qquad \beta_\xi = \frac{g}{(4\,\pi)^2}\left(\xi - \frac{1}{6}\right)\,, \tag{5.48}$$

and integrating them from a negative initial condition for the coupling g (so that it runs to zero in the ultraviolet limit, instead of hitting a Landau pole) we find that $\xi(\mu) \to 1/6$ for $\mu \to \infty$, independently on the initial value $g(\mu_0) < 0$. In this case $\xi = 1/6$ is the value at which the theory shows conformal invariance at the classical level, and so analogously to our situation it is a value which is preferred by the flow trajectories, being the only one among the line of Gaussian fixed points that can be reached by the interacting theory. Of course the analogy is limited to this observation, the scalar theory being truly asymptotically free (albeit unbounded from below), and not loosing any degree of freedom as a consequence of Weyl invariance.

For completeness, we should point out that whereas for the reasons just discussed we expect the one-loop approach to the anisotropic Weyl invariant action to be a feature that the full theory will share with our toy model, we have no argument to support an analogous situation with the approach being such that the effective perturbative coupling ϵ remains finite. Furthermore, even in our toy model, ϵ might cease to be marginal at two loops or beyond. Only an explicit calculation can of course tell us whether the additional degrees of freedom of the full higher dimensional model, or maybe the effects of higher loop corrections, might change the qualitative picture we found. The one-loop result obtained for the conformally reduced theory, however, is a first calculation which shows the existence of potential troubles associated to the strong coupling regime.

Conclusions

In this Ph.D. thesis we have investigated the renormalizability of quantum gravity approaches in the framework of the renormalization group (RG) employing both perturbative and non-perturbative schemes. In particular, we restricted our interest to the quantization of gravitational theories in which a central role is played by a scalar degree of freedom, since their RG flow is easier to analyze. We made use of scalar theories in a two-fold way: on the one hand we used scalar field theories as toy models of gravity, that is, conformal reductions of the full theories in which we neglect gravitons in the quantization procedure. On the other hand we have studied scalar-tensor models as dynamically equivalent theories of higher-derivative models.

The first approach we studied is Weinberg's asymptotic safety conjecture for gravity. In this approach it is suggested that the gravitational interaction flows in the high energy regime to a non-Gaussian fixed point (NGFP) which ensures the renormalizability of the theory and the finiteness of the n-point correlation functions. The existence of the NGFP has been widely investigated in the context of the functional renormalization group (fRG) revealing the presence of a non-trivial fixed point for numerous ansatz of the effective action, both finite- and infinite-dimensional in the parameter space. Since the fRG equation cannot be solved exactly, but just by means of an approximation of the effective action, the approximated solution shows then a dependence on the RG scheme. In order to check the robustness of the results obtained within a certain approximation it is then necessary to study the RG flow using different schemes.

For this reason we have investigated in the section 3.1 the existence of the NGFP for the simpler Einstein-Hilbert (EH) truncation employing a proper time RG scheme in which the coarse graining is introduced at the level of the proper time representation. For this truncation, in fact, the universality class has been analyzed only in the context of the exact renormalization group (ERG), using different cutoff functions [65, 51]. The results derived in the section 3.1 are a generalization for generic spacetime dimension of the results already obtained for $d = 4$ in [82]. Our calculation shows the existence of a NGFP with a universality class (e.g. the critical exponents θ' and θ'') in general agreement with that found in [65] for $d = 4$, and rather stable under the variation of the cutoff parameter n. Lowering the dimensionality d, however, we encounter a new situation in which the cosmological constant vanishes at $d = 3$, in coincidence with the vanishing of the critical exponent θ''. As a consequence, for $d < 3$ the NGFP is defined for a negative cosmological constant and by real critical exponents. The characteristic spiral behavior of the RG flow is then lost. If we continue to lower the dimensionality we observe the non-trivial fixed point to collapses on the Gaussian fixed point at $d = 2$, as expected by a dimensional analysis.

We employed in section 3.3 the proper time scheme to study the RG flow of a simplified scalar toy model of gravity, i.e. a conformal reduction of the Einstein-Hilbert action (CREH). It has in fact been proved [6, 7] that the quantization of the sole conformal degree of freedom of the metric leads to a flow qualitatively in agreement with that of the full theory. We have then evaluated the proper time RG flow equation for such a scalar toy model, projecting the flow on a spherical topology S^d and a flat one, $\mathbb{R}^d$. We found for both topologies and $d = 4$ a non-Gaussian fixed point in general agreement with [6] regarding the numerical values of the universal quantities.

We have then defined a $\chi^2(n)$ function in the "universal quantities space" to displays the distance, varying n, of the universality class of the full theory from that of the toy model for $d = 4$. We compared then the two χ^2 functions, one for each topology, looking for a minimum of both functions. Interestingly we found both minima to coincide at $n_{min} = 4$, which in the parametrization used in chapter 3 corresponds to a quadratic dependence on the propagator, thus near to the linear dependence owned by the ERG equation ($n = 3$). We are then tempted to interpret n_{min} as an optimized value, following the so-called principle of minimum sensitivity (PMS). For this value in fact the distance between the two universality classes is invariant under an infinitesimal transformation of the cutoff parameter. Hence, at the optimized value the RG flow dependence on the approximation is supposed to be minimized.

Interestingly, this optimized value of n emerges only when comparing the universality classes of the full and approximated theory. When looking at the sole universality class of the CREH action, in fact, the critical exponents grown monotonically by varying n, so that there is no minima. The principle of minimum sensitivity cannot thus be applied, and no optimized value of n can be estimated. This is also what happens for the standard scalar theory in $d = 3$ (see [41, 42]). In that case the critical exponents grow monotonically when increasing n, reaching their most precise value in the limit $n \to \infty$. The proper time scheme for $n = \infty$, interestingly, exhibits an high precision already at the leading order of the derivative expansion; the same level of precision is reached by the ERG equation only at the next order [143]. The surprising high precision shown by a non-exact RG scheme in the ultraviolet sector of the standard scalar field theory was indeed one of reasons which motivated us towards its use in gravity. What has been found for the standard scalar theory, however, is not in agreement with what we found here, where for $n = 4$ the use of the ERG equation is as good as the proper time equation, at least for $d = 4$.

For $d < 4$ the differences between the ERG and the proper time scheme are more consistent. Using a spherical topology, in fact, for $d < d_c(n)$, being d_c a critical dimension, we encounter a Hopf bifurcation which entails the emergence of an UV limit cycle. Although we are led to consider it as an artifact of the conformal reduction, it opens the intriguing possibility of having a limit cycle as an ultraviolet (or infrared) completion of the RG flow. Such a situation, perhaps atypical in quantum field theory, was however already suggested by Wilson for the ultraviolet regime of QCD [87].

In section 3.4, we have then extended our analysis from the simple CREH truncation to a more general non-polynomial effective action. We built then a functional RG flow equation for the conformal potential on a flat topology, i.e. we studied the conformally reduced toy model of gravity in the local potential approximation. Since we are working in a flat spacetime we expect all the contributions to the conformal potential that are coming from powers of the Ricci scalar to vanish. The only operators which contribute are then, for example, non-local operators of the type of powers of the spacetime volume. Starting from a symmetric phase at the UV fixed point we have numerically integrated the dimensionful equation back to $k = 0$, investigating the possibility of having a broken phase in the infrared regime. Studying the problem as an inverse problem (that is, integrating from $k = 0$ to $k = \infty$) we found not only that a broken phase in the infrared is possible, but also that the instability issue of the conformal factor is automatically cured, and that the ultraviolet fixed point structure is richer that that found in the CREH truncation. Those results, however, have been found by fixing the running of the Newton's constant along the flow, as normally happens in the LPA. We would then expect to gain a more precise determination of the fixed point structure taking in consideration the flow of Newton's constant, a work that we leave for the future.

The local potential approximation for the conformally reduced theory has been obtained, however, only on a flat topology. Consequently, this toy model cannot be used to investigate the existence of fixed point solutions for scalar approximations of more general infinite-dimensional truncations of the gravitation action, like a $f(R)$ theory. For this reason in chapter 4 we have quantized a dynamically equivalent theory to the $f(R)$ action, i.e. a Brans-Dicke theory with $\omega = 0$ in the local potential approximation. We have then derived a RG flow equation for a generic Brans-Dicke potential, working on a flat topology and keeping the parameter ω arbitrary. In order to test the self-consistency of our results, we quantized the theory using two different gauges, respectively a Landau and a Feynman gauge. Once obtained the RG equation we focused on the fixed point solution for $d = 4$ and for a fixed parameter $\omega = 0$, in light of the equivalence with the $f(R)$ theory. We employed then a consistent numerical strategy to look for scale-invariant solutions. That is, we counted the number of free parameters of the differential equation at the origin, finding one degree of freedom in the Feynman gauge and two in the Landau. Then we studied the asymptotic behaviour of the equations, finding in both gauges four different behaviours. Hence, we integrated numerically from the origin towards $\phi = \pm\infty$, by varying the free parameters at the origin. Because of the strong non-linearity of the equation we found all the solutions in the Feynman gauge to end at a movable singularity. We found instead a 2-parameter family of globally analytic solutions in the Landau gauge.

Although we expect a certain dependence of the universal quantities from the gauge choice, we also request the basic features of the fixed point structure (like its dimensionality) to be independent from it. Consequently, we deemed the local potential approximation as inconsistent, and that the running of the parameter ω has to be taken into account in order to correctly characterize the universality class of the theory.

The latter statement can also be understood by noting that for $\omega = -1/2$ we found the general picture to be inverted. For $\omega = -1/2$, in fact, we found a 2-dimensional family of global solutions for the Feynman gauge and no one in the Landau.

Including the running of ω we expect then the fixed point structure to be defined by a set of fixed point values ω^*, which sets the qualitative features of the universality class in a gauge-independent way. Still, we will expect the values of ω^* to show a dependence on the gauge choice.

An important point to discuss is the equivalence of the theories at a quantum level. The equations of motion for ϕ, in fact, are no longer algebraic once we take in consideration the running of ω, since in general $\omega_k \neq 0$; thus if we solve them we no longer obtain a simple $\phi(R)$ and the resulting theory is not a $f(R)$. We are then led to deem Brans-Dicke and $f(R)$ as theories not equivalent at a quantum level. Our result, however, should not surprise. At a non-perturbative level there is not way to say a priori whenever two theories would be equivalent at a quantum level or not. The flows of the two theories are, in fact, defined in two different theory spaces. It can happen that the scalar field couples with some invariant in one theory theory space and not in the other. Consequently, also if we find a class of fixed points in both theories they can describe different physics, and only a direct comparison between the universality classes would say if they are equivalent or not. Our does not want to be, however, a proof of the non equivalence, but only the logical interpretation of our results.

As already mentioned, all those important question cannot be answered within the local potential approximation we employed, but our calculation can be seen as the leading order of an approximation series that can be systematically improved. We leave the next-to-leading order for future work.

In chapter 5 we have investigated the asymptotic freedom of Hořava-Lifshitz quantum gravity. In this approach Lorentz invariance is lost at the Planck scale, by the emergence of a scale anisotropy between space and time. The microscopic action becomes thus invariant under a foliation-preserving diffeomorfism group and general covariance is supposed to be restored in the infrared limit. Interestingly, thanks to the loss of general covariance it is possible to obtain simultaneously explicit unitarity and power counting renormalizability. Because of the anisotropy, in fact, we can build a microscopic action containing operators with at most two time derivatives, while the spatial derivates are raised to higher orders. Consequently the propagator runs fast enough to zero at large momenta, thus ensuring perturbative renormalizability, while the absence of higher-order time derivatives avoids the presence of unphysical poles in the propagator.

Being the renormalizability the key feature of this model, it is also its less studied feature, because of the complexity of the calculations and the high number of invariants in the more general case, i.e. the non-projectable case without detailed balance.

We have then here studied a simplified case, i.e. the projectable case without detailed balance in 2+1 dimensions with $z = 2$, in order to give a first answer about the asymptotic freedom of the model. For this dimensionality, in fact, there are no gravitons, and the only physical degree of freedom is the scalar one. As a first step towards a more complete analysis, we simplified further our model, neglecting the longitudinal modes of the spatial metric and quantizing the sole conformal sector. The toy model we investigated is then a conformal reduction of the theory, analogous to that studied in chapter 3 for the asymptotic safety scenario. However, differently from the isotropic case in which the scalar degree of freedom is a pure gauge one, the conformal factor is a physical degree of freedom in the anisotropic case. We expect then the RG flow of the scalar toy model to be a good approximation of the full theory.

Using a heat kernel expansion for anisotropic higher-order operators we evaluated then the one-loop correction of the action. Hence, we obtained the one-loop order β-functions of the dimensionless parameter of our interest in the MS scheme. After calculating the running of Newton's constant, the anisotropic parameter λ and the R^2 coupling, we studied then the high energy regime of the theory. What we found is that while Newton's constant tends to flow to zero, realizing asymptotic freedom, the coupling λ tends to its Weyl invariant value $\lambda = 1/2$. The interaction coupling, however, is not defined by Newton's constant alone, but by its ratio with $\lambda - 1/2$, as a consequence of the anisotropic character of the model. Since we found λ and the Newton's constant to run with the same speed, we obtained the interaction strength to be constant along the flow, spoiling then the asymptotic freedom of the theory. As already said, being the scalar degree of freedom the only physical one we expect the one-loop marginal behaviour of the interaction strength to be a feature shared with the full model. This results, however, can in principle change at two loop, hence a higher-loop calculation is needed in order to answer about the true marginality of the interaction coupling.

Our result, being based on a one-loop correction of a toy model, should be seen just as a first step towards the understanding of the high energy regime of the full model. At least, perhaps, we gave an hint about the true fixed point structure of the theory. Assuming asymptotic freedom, in fact, it can be found a two-parameter family of fixed points characterized by λ and the dimensionless ratio of the Newton's constant and the R^2 coupling. What we found is that, starting from an interacting theory, just the conformal (Weyl) invariant point is reached, as a consequence of the running of the parameter λ. Such a situation is not unusual in QFT, since something similar happens for the standard scalar field theory non-minimally coupled to gravity. Also in that case, in fact, the request of conformal invariance reduces the number of UV fixed points to the sole conformal one. We expect the Weyl invariant point to be a fixed point also of the full model.

Appendix A

Functional representation of the effective action

We present in this appendix the procedure to rewrite the effective action $\Gamma[\Psi]$ in its functional formulation starting from its definition in terms of Legendre transform of the functional generator of connected correlation functions $W[J]$.

We start by briefly remind the standard definition of the effective action. Consider then a bare (or classical) action $S[\psi]$; the partition function is defined as

$$Z = \int D[\psi]\, e^{-S[\psi]}\,. \tag{A.1}$$

The functional generator $W[J]$ is defined by coupling the bare field with a fictitious source as

$$Z[J] = e^{W[J]} = \int D[\psi]\, e^{-S[\psi]+\int d^dx J(x)\,\psi(x)}\,, \tag{A.2}$$

thus being $W[J] = \log Z[J]$. Connected correlation function can then be obtained by functional differentiating $W[J]$. For example the expectation value of the field reads

$$\langle \psi(x) \rangle = Z[0]^{-1} \left(\frac{\delta Z[J]}{\delta J(x)} \right) \Big|_{J=0} = \frac{\delta W[J]}{\delta J(x)} \Big|_{J=0}\,, \tag{A.3}$$

while more generally for an n-point correlation function holds

$$\langle \psi(x_1)\, \cdots\, \psi(x_n) \rangle = \frac{\delta^n W[J]}{\delta J(x_n)\, \cdots \delta J(x_1)} \Big|_{J=0}\,. \tag{A.4}$$

We can now define the expectation value of the field in presence of the source $J(x)$ as

$$\langle \psi(x) \rangle_J = Z[J]^{-1} \left(\frac{\delta Z[J]}{\delta J(x)} \right) = \frac{\delta W[J]}{\delta J(x)}\,. \tag{A.5}$$

and define a new effective field $\Psi(x)_J = \langle \psi(x) \rangle_J$ as the Legendre conjugated of the source $J(x)$. We can then resolve (A.5) respect to J as a function of the effective variable Ψ_J, that is $J = J[x, \Psi_J]$, and perform the Legendre transform of the generator $W[J]$, i.e.

$$\Gamma[\Psi] = -W[J[x,\Psi]] + \int d^dx\, J[x,\Psi]\, \Psi(x)\,, \tag{A.6}$$

where $\Gamma[\Psi]$ is the effective action, that is the functional generator of 1PI connected correlation functions. Note that

$$\begin{aligned} \frac{\delta\Gamma[\Psi]}{\delta\Psi(y)} &= -\frac{\delta W[J]}{\delta\Psi(y)} + \int d^dx\, \frac{\delta(J[x,\Psi]\,\Psi(x))}{\delta\Psi(y)} \\ = -\int d^dx\, \frac{\delta W[J]}{\delta J(x)}\,\frac{\delta J(x)}{\delta\Psi(y)} &+ \int d^dx\, \frac{\delta J[x]}{\delta\Psi(y)}\,\Psi(x) + J[x,\Psi]\,\delta(x-y) \\ &= J(y)\,, \end{aligned} \tag{A.7}$$

where we used (A.5) in the last passage. To write the effective action in terms of a Schwinger functional we exponentiate both members of (A.6), obtaining

$$e^{-\Gamma[\Psi]} = e^{W[J]-\int d^dx\, J(x)\Psi(x)} = e^{W[J]-\int d^dx\, \frac{\delta\Gamma[\Psi]}{\delta\Psi}\,\Psi(x)}\,, \tag{A.8}$$

where in (A.8) we used (A.7) to rewrite the source in terms of the effective action. Since the two operators in the exponential on the right hand side commute, we can rewrite it as a product of two exponentials and using (A.2) we arrive to

$$e^{-\Gamma[\Psi]} = e^{-\int d^dx\, \frac{\delta\Gamma[\Psi]}{\delta\Psi}\,\Psi(x)} \int D[\psi]\, e^{-S[\psi]+\int d^dx J(x)\,\psi(x)}\,. \tag{A.9}$$

Since the path integral is defined over the configurations of the bare field $\psi(x)$ we can bring inside the first exponential obtaining

$$e^{-\Gamma[\Psi]} = \int D[\psi]\, e^{-S[\psi]+\int d^dx\, \frac{\delta\Gamma[\Psi]}{\delta\Psi}\,(\psi(x)-\Psi(x))} \tag{A.10}$$

where we used once more (A.7). Finally, we can perform a shift of the bare field as $\psi(x) = \chi(x) - \Psi(x)$ so that equation (A.10) is rewritten as

$$e^{-\Gamma[\Psi]} = \int D[\chi]\, e^{-S[\chi+\Psi]+\int d^dx\, \frac{\delta\Gamma[\Psi]}{\delta\Psi}\,\chi(x)} \tag{A.11}$$

being $\langle\chi(x)\rangle = 0$. It is easy to see then that the on shell partition function can be obtained from the generalized principle of minimum action, which reads

$$\left.\frac{\delta\Gamma[\Psi]}{\delta\Psi(x)}\right|_{\Psi=\Psi^*} = 0\,, \tag{A.12}$$

being $\Psi^*(x)$ the configuration which minimizes the effective action, so that equation (A.11) now reads

$$e^{-\Gamma[\Psi^*]} = \int D[\chi]\, e^{-S[\chi+\Psi^*]}\,. \tag{A.13}$$

That is to say that the effective action at the minimum simply furnishes the expression of the partition function without external source, i.e. $\Gamma[\Psi^*] = -\log Z[0]$.

Appendix B

Arnowitt-Deser-Misner splitting

We summarize in this appendix the standard ADM splitting techniques in $d+1$ dimensions. The convention here employed it taken from Carroll's book [144].

Let us consider a generic $d+1$ dimensional manifold $\mathcal{M}$ equipped with a spacetime metric $\gamma_{\mu\nu}$, and a signature $(\epsilon, +, +, +, ...)$, $\epsilon = \pm 1$. Greek indices will be used for the $d+1$ dimensional manifold

$$\mu, \nu, ... = 0, 1, ..., d\,, \tag{B.1}$$

and latin indices for the spatial one

$$i, j, ... = 1, ..., d\,. \tag{B.2}$$

The spacetime manifold is equipped with a Levi-Civita connection ∇_μ such that

$$[\nabla_\mu, \nabla_\nu]\mathcal{A}^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu}\,\mathcal{A}^\sigma\,, \tag{B.3}$$

being $\mathcal{A}^\mu$ a generic $d+1$ vector and where $\mathcal{R}^\alpha{}_{\mu\beta\nu}$ is the Riemann tensor built from the metric $\gamma_{\mu\nu}$. The Ricci tensor and the Ricci scalar are defined from the Riemann tensor as

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\rho{}_{\mu\rho\nu}\,, \qquad \mathcal{R} = \mathcal{R}_{\mu\nu}\,\gamma^{\mu\nu}\,. \tag{B.4}$$

We thus define a time function t and a time vector t^μ such that they satisfy the compatibility condition

$$t^\mu\,\nabla_\mu t = 1\,. \tag{B.5}$$

Hence we can define a foliation $\mathcal{F}$ over the manifold $\mathcal{M}$ which leafs are hypersurfaces Σ_t at constant time coordinate t. We can then define a unitary vector n^μ, i.e.

$$\gamma_{\mu\nu}\,n^\mu\,n^\nu = \epsilon\,, \tag{B.6}$$

which is othogonal to the slice Σ_t, that is

$$n^\mu A_\mu = 0\,, \tag{B.7}$$

where A^μ is a generic spatial vector $A^\mu \in T\Sigma_t$, being $T\Sigma_t$ the tangent vector space to the leaf Σ_t. Using the definition (B.6) the spacetime metric $\gamma_{\mu\nu}$ can be split in two terms

$$\gamma_{\mu\nu} = \epsilon\, n_\mu n_\nu + g_{\mu\nu}\,, \tag{B.8}$$

where $g_{\mu\nu}$ is an induced metric over the leaf, which reads in its contravariant form

$$g^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} . \tag{B.9}$$

Using the decomposition (B.8) we can split the time vector t^μ as

$$t^\mu = N\, n^\mu + N^\mu\,, \tag{B.10}$$

being N and N^μ respectively the lapse function and shift vector, for which hold

$$N = \epsilon\, \gamma_{\mu\nu}\, t^\mu\, n^\nu\,, \qquad N^\mu = g^\mu{}_\nu\, t^\nu\,. \tag{B.11}$$

The index of the shift vector is raised and lower using the full metric and it is by definition orthogonal to the vector n^μ, that is

$$N_\mu = g_{\mu\nu}\, N^\nu = \gamma_{\mu\nu} N^\nu\,, \qquad N^\mu\, n^\nu\, \gamma_{\mu\nu} = 0\,. \tag{B.12}$$

Since the contravariant vectors have components

$$t^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad N^\mu = \begin{pmatrix} 0 \\ N^i \end{pmatrix}, \quad n^\mu = \tfrac{1}{N} \begin{pmatrix} 1 \\ \epsilon N^i \end{pmatrix}, \tag{B.13}$$

using (B.13) and (B.9) in (B.8) we get for the contravariant spacetime metric

$$\gamma^{\mu\nu} = \begin{pmatrix} \dfrac{\epsilon}{N^2} & \dfrac{N^j}{N^2} \\ \dfrac{N^i}{N^2} & g^{ij} + \epsilon \dfrac{N^i N^j}{N^2} \end{pmatrix} . \tag{B.14}$$

To obtain the components of the covariant spacetime metric $\gamma_{\mu\nu}$ we can rewrite the various vectors in covariant form, i.e.

$$t_\mu = \begin{pmatrix} \epsilon N^2 + N^i\, N_i \\ N_i \end{pmatrix}, \qquad N_\mu = \begin{pmatrix} N^j\, N_j \\ g_{ij}\, N^j \end{pmatrix}, \qquad n_\mu = \begin{pmatrix} \epsilon N \\ 0 \end{pmatrix}, \tag{B.15}$$

which leads to

$$\gamma_{\mu\nu} = \begin{pmatrix} \epsilon N^2 + N^i\, N^j\, g_{ij} & g_{ij} N^i \\ g_{ij} N^j & g_{ij} \end{pmatrix}, \qquad g_{\mu\nu} = \begin{pmatrix} g_{ij} N^i N^j & g_{ij} N^i \\ g_{ij} N^j & g_{ij} \end{pmatrix} . \tag{B.16}$$

By using the induced metric $g_{\mu\nu}$ we can define a projector operator $\mathcal{P}$ which projects tensors on the leafs of the foliation, i.e.

$$\mathcal{P}(T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}) = g^{\mu_1}\alpha_1 \cdots g^{\mu_n}{}_{\alpha_n}\, g_{\nu_1}{}^{\beta_1} \cdots g_{\nu_m}{}^{\beta_m}\, T^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}\,, \tag{B.17}$$

that acts by contracting every free index with the induced metric $g_{\mu\nu}$. From (B.17) we can define the notion of induced covariant derivative as

$$\begin{aligned} D_\mu\, T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} &= \mathcal{P}(\nabla_\rho\, T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}) \\ &= g^{\mu_1}{}_{\alpha_1} \cdots g^{\mu_n}{}_{\alpha_n}\, g_{\nu_1}{}^{\beta_1} \cdots g_{\nu_m}{}^{\beta_m}\, (g^\sigma_\rho\, \nabla_\sigma\, T^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m})\,. \end{aligned} \tag{B.18}$$

and introduce the notion of spatial curvature as

$$[D_\mu, D_\nu]\, A^\rho = R^\rho{}_{\sigma\mu\nu}\, A^\sigma\,, \tag{B.19}$$

being $R^\rho{}_{\sigma\mu\nu}$ the spatial Riemann tensor and A^μ a spatial vector, $A^\mu\, n_\mu = 0$. The intrinsic curvature of the hypersurface Σ_t is given by the induced Ricci scalar, that is $R = g^{\mu\nu}\, R^\alpha{}_{\sigma\alpha\nu}$ and a notion of extrinsic curvature can be introduced by defining the acceleration vector a_μ as

$$a_\mu = n^\nu\, \nabla_\nu\, n_\mu\,, \tag{B.20}$$

and the extrinsic curvature tensor $K_{\mu\nu}$, which reads

$$K_{\mu\nu} = \frac{1}{2}\mathfrak{L}_n\, g_{\mu\nu} = \nabla_\mu\, n_\nu - \epsilon\, n_\mu\, a_\nu = \frac{1}{2N}(\dot{g}_{\mu\nu} - D_\mu\, N_\nu - D_\nu\, N_\mu)\,, \tag{B.21}$$

where $\mathfrak{L}_n$ is the covariant Lie derivative respect to the vector n^μ and the dot stands for the time derivate. The tensor (B.21) is a spatial tensor, i.e. $K_{\mu\nu}n^\nu = 0$, so that the extrinsic curvature is obtained contracting (B.21) with the induced or spacetime metric, i.e.

$$K = \gamma^{\mu\nu}\, K_{\mu\nu} = g^{\mu\nu}\, K_{\mu\nu}\,. \tag{B.22}$$

The spacetime Riemann tensor $\mathcal{R}$ is related to the spatial Riemann tensor R and the extrinsic curvature tensor by the Gauss-Codazzi equations

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= g^\rho_\alpha\, g^\beta_\rho\, g^\gamma_\mu\, g^\delta_\nu\, \mathcal{R}^\alpha{}_{\beta\gamma\delta} + \epsilon\, (K^\rho_\mu\, K_{\sigma\nu} - K^\rho_\nu\, K_{\sigma\mu})\,, \\ D_{[\mu}K^\mu_{\nu]} &= \frac{1}{2}\, g^\sigma_\nu\, \mathcal{R}_{\rho\sigma}\, n^\rho\,. \end{aligned} \tag{B.23}$$

Using (B.23) and (B.17) it can be verified that the following identities hold

$$R = \mathcal{R} + \epsilon\, (K^2 - K^{\mu\nu}\, K_{\mu\nu} - 2\, \mathcal{R}_{\mu\nu}\, n^\mu\, n^\nu)\,, \tag{B.24}$$

$$\mathcal{R}_{\mu\nu}\, n^\mu\, n^\nu = K^2 - K^{\mu\nu}\, K_{\mu\nu} + \nabla_\mu\, (a^\mu - n^\mu\, K)\,. \tag{B.25}$$

Appendix C

CREH: Critical exponents and β-functions

C.1 Critical exponents and universal quantities

Table C.1: The fixed point values and the critical exponents obtained in four dimensions from the $\boldsymbol{\beta}$-functions (3.49a) and (3.49b) for various values of the cutoff parameter n compared with those obtained from the full EH gravity, on the right.

	CREH-S^4					CREH-$\mathbb{R}^4$					Full EH-S^4				
n	$\boldsymbol{\lambda_*}$	$\boldsymbol{g_*}$	$\boldsymbol{\lambda_* g_*}$	$\boldsymbol{\theta'}$	$\boldsymbol{\theta''}$	$\boldsymbol{\lambda_*}$	$\boldsymbol{g_*}$	$\boldsymbol{\lambda_* g_*}$	$\boldsymbol{\theta'}$	$\boldsymbol{\theta''}$	$\boldsymbol{\lambda_*}$	$\boldsymbol{g_*}$	$\boldsymbol{\lambda_* g_*}$	$\boldsymbol{\theta'}$	$\boldsymbol{\theta''}$
3	1.125	1.571	1.767	3	4.795	0.800	2.084	1.670	8.580	0	0.355	0.388	0.138	1.835	1.300
4	1.2	1.810	2.171	1.5	4.213	0.837	2.666	2.234	5.721	2.928	0.265	0.472	0.125	1.770	1.081
5	1.25	1.885	2.356	1	3.873	0.867	2.914	2.528	5.000	3.428	0.230	0.517	0.119	1.750	1.000
6	1.285	1.914	2.461	0.75	3.665	0.889	3.041	2.706	4.578	3.627	0.211	0.546	0.115	1.742	0.959
8	1.333	1.930	2.574	0.5	3.427	0.921	3.159	2.910	4.102	3.788	0.191	0.582	0.111	1.734	0.916
10	1.364	1.931	2.633	0.375	3.295	0.941	3.209	3.023	3.839	3.850	0.181	0.603	0.109	1.731	0.894
15	1.406	1.923	2.703	0.230	3.129	0.971	3.253	3.161	3.511	3.903	0.169	0.630	0.106	1.727	0.868
20	1.428	1.914	2.735	0.167	3.050	0.987	3.265	3.225	3.356	3.919	0.163	0.644	0.105	1.725	0.856
30	1.451	1.904	2.764	0.107	2.974	1.004	3.271	3.285	3.206	3.929	0.158	0.658	0.104	1.723	0.846
50	1.470	1.894	2.785	0.062	2.914	1.018	3.271	3.331	3.089	3.933	0.154	0.668	0.103	1.722	0.837
100	1.485	1.886	2.800	0.030	2.871	1.028	3.269	3.364	3.003	3.934	0.152	0.676	0.102	1.721	0.831
300	1.495	1.880	2.810	0.010	2.842	1.036	3.266	3.385	2.948	3.934	0.150	0.682	0.102	1.721	0.828
∞	1.5	1.880	2.815	0	2.820	1.040	3.265	3.396	2.920	3.923	0.103	0.685	0.070	1.720	0.826

Table C.2: The fixed point values and the critical exponents obtained in for the cutoff parameter $n = 4$ from the $\boldsymbol{\beta}$-functions (3.49a) and (3.49b) for various values of the dimension compared with those obtained from the full EH gravity, on the right.

	CREH-S^4					CREH-$\mathbb{R}^4$					Full EH-S^4				
d	λ_*	g_*	$\lambda_* g_*$	θ'	θ''	λ_*	g_*	$\lambda_* g_*$	θ'	θ''	λ_*	g_*	$\lambda_* g_*$	θ'	θ''
2	0	0	0	$-\infty$	0	0	0	0	0	0	0	0	0	2	0
2.2	0.1825	4.07E-4	7.44E-5	-38.056	0	0.0381	0.0214	8.18E-4	0.8153	0.9756	-0.0176	0.0184	-3.25E-4	2.0025	0
2.4	0.3381	5.24E-3	1.77E-3	-17.356	0	0.0968	0.0632	6.12E-3	1.1495	1.5524	-0.0277	0.0428	-1.19E-3	1.9882	0
2.6	0.4748	0.0223	0.0106	-9.8196	0	0.1676	0.1318	0.0220	1.5641	1.9715	-0.0291	0.0742	-2.16E-3	1.9552	0
2.8	0.5980	0.0619	0.0370	-5.1014	0	0.2474	0.2369	0.0586	2.0317	2.3065	-0.0204	0.1132	-2.31E-3	1.8981	0
3	0.7111	0.1368	0.0973	-2.	2.2360	0.3344	0.3917	0.1309	2.5433	2.5771	0	0.16	0	1.8	0
3.2	0.8168	0.2637	0.2154	-1.05	3.1098	0.4270	0.6127	0.2616	3.0957	2.7887	0.0331	0.2140	7.09E-3	1.5031	0
3.4	0.9171	0.4639	0.4254	-0.2832	3.5584	0.5243	0.9210	0.4829	3.6884	2.9400	0.0791	0.2737	0.0216	1.5093	0.4259
3.6	1.0136	0.7643	0.7747	0.3727	3.8450	0.6256	1.3431	0.8402	4.3220	3.0241	0.1358	0.3373	0.0458	1.5738	0.6480
3.8	1.1075	1.1989	1.3278	0.9587	4.0516	0.7303	1.9115	1.3959	4.9986	3.0278	0.1994	0.4035	0.0804	1.6579	0.8617
4	1.2	1.8095	2.1714	1.5	4.2130	0.8379	2.6660	2.2341	5.7211	2.9284	0.2653	0.4724	0.1253	1.7690	1.0810
4.2	1.2918	2.6471	3.4197	2.0134	4.3464	0.9483	3.6549	3.4660	6.4931	2.6839	0.3299	0.5452	0.1799	1.9096	1.3047
4.4	1.3839	3.7724	5.2209	2.5111	4.4608	1.0611	4.9352	5.2368	7.3194	2.2018	0.3919	0.6238	0.2444	2.0789	1.5288
4.6	1.4771	5.2560	7.7639	3.0027	4.5609	1.1762	6.5736	7.7321	8.2057	1.1214	0.4508	0.7096	0.3199	2.2746	1.7504
4.8	1.5721	7.1786	11.285	3.4964	4.6491	1.2935	8.6465	11.185	11.129	0	0.5074	0.8036	0.4078	2.4945	1.9684

C.2 β-functions

In this section are listed the explicit expressions of the β-functions in d dimensions for different choices of the cutoff function. The β-functions obtained using the cutoff (3.37) will be referred as smooth proper time cutoff, where the limits $n \to d/2$ and $n \to \infty$ will be called respectively sharp momentum cutoff and sharp proper time cutoff.

C.2.1 The projection on S^d

- CREH - Smooth proper time cutoff

$$\beta_g = g_k \left(d - 2 - \frac{2^{2-d}\,(d-2)\,\pi^{1-\frac{d}{2}}\,g_k\,n^n\,\Gamma\left(-\frac{d}{2}+n+1\right)}{\left(n - \frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{n-\frac{d}{2}+1}(d-1)\,\Gamma(n)} \right), \tag{C.1}$$

$$\beta_\lambda = \frac{2^{3-d}\,\pi^{1-\frac{d}{2}}\,g_k\,n^n\,\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)\left(n - \frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{n-\frac{d}{2}}} + \lambda \left(-2 - \frac{2^{2-d}\,(d-2)\,\pi^{1-\frac{d}{2}}\,g_k\,n^n\,\Gamma\left(-\frac{d}{2}+n+1\right)}{\left(n - \frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{n-\frac{d}{2}+1}(d-1)\,\Gamma(n)} \right), \tag{C.2}$$

- CREH - Sharp proper time cutoff

$$\beta_g = g_k \left(d - 2 - \frac{2^{2-d}\,(d-2)\,\pi^{1-\frac{d}{2}}\,g_k}{d-1}\, e^{\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2(d-1)}} \right), \tag{C.3}$$

$$\beta_\lambda = 2^{3-d}\,\pi^{1-\frac{d}{2}}\,g_k\,e^{\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}} + \lambda_k \left(-2 - \frac{2^{2-d}\,(d-2)\,\pi^{1-\frac{d}{2}}\,g_k}{d-1}\, e^{\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2(d-1)}} \right), \tag{C.4}$$

- CREH - Sharp momentum cutoff

$$\beta_g = g_k \left(d - 2 + \frac{2^{3-\frac{3d}{2}}\,(d-2)\,\pi^{1-\frac{d}{2}}\,d^{d/2}\,g_k}{(d-1)\left(\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{d-1} - d\right)\Gamma\left(\frac{d}{2}\right)} \right), \tag{C.5}$$

$$\beta_\lambda = \left(-2 + \frac{2^{3-\frac{3d}{2}}\,(d-2)\,\pi^{1-\frac{d}{2}}\,d^{d/2}\,g_k}{(d-1)\left(\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{d-1} - d\right)\Gamma\left(\frac{d}{2}\right)} \right)\lambda_k - \frac{2^{3-\frac{3d}{2}}\,d^{d/2}\,g_k\,\ln\left(1-\frac{\left(\frac{2d}{d-2}-1\right)\lambda_k}{d-1}\right)}{\Gamma\left(\frac{d}{2}\right)\,\pi^{\frac{d}{2}-1}}\,, \tag{C.6}$$

- FULL EH - Smooth proper time cutoff

$$\beta_g = g_k\left(d-2+\frac{g_k\left(-d\,(5d-7)\,n^{n+1}\,(n-2\lambda_k)^{\frac{d}{2}-n-1} - 4\,(d+6)\,n^{d/2}\right)\Gamma\left(-\frac{d}{2}+n+1\right)}{3\,2^{d-1}\,\pi^{\frac{d}{2}-1}\,\Gamma(n+1)} \right)\,, \tag{C.7}$$

$$\begin{aligned}\beta_\lambda = {}& \frac{2^{2-d}\,d\,g_k\,(n-2\lambda_k)^{-n}\,\left((d+1)\,n^n\,(n-2\lambda_k)^{d/2} - 4\,n^{d/2}\,(n-2\lambda_k)^n\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)\,\pi^{\frac{d}{2}-1}} \\ &+ \left(-2+\frac{g_k\left(-d\,(5d-7)\,n^{n+1}\,(n-2\lambda_k)^{\frac{d}{2}-n-1} - 4\,(d+6)\,n^{d/2}\right)\Gamma\left(-\frac{d}{2}+n+1\right)}{3\,\pi^{\frac{d}{2}-1}\,2^{d-2}\,\Gamma(n+1)}\right)\lambda_k\,,\end{aligned} \tag{C.8}$$

- FULL EH - Sharp proper time cutoff

$$\beta_g = g_k\left(d-2-\frac{1}{3}\,2^{2-d}\,\pi^{1-\frac{d}{2}}\,g_k\,\left(d\,(5d-7)\,e^{2\lambda_k} + 4\,(d+6)\right)\right)\,, \tag{C.9}$$

$$\beta_\lambda = \frac{2^{2-d}\,d}{\pi^{\frac{d}{2}-1}}g_k\,\left((d+1)\,e^{2\lambda_k} - 4\right) + \left(-2-\frac{g_k 2^{2-d}}{3\pi^{\frac{d}{2}-1}}\,(d(5d-7)\,e^{2\lambda_k} + 4\,(d+6))\right)\lambda_k\,, \tag{C.10}$$

- FULL EH - Sharp momentum cutoff

$$\beta_g = g_k\left(d-2-\frac{2^{3-d}\,d^{d/2}\,\pi^{1-\frac{d}{2}}\,2^{-d/2}\,g_k\,\left(3\,d\,((d-1)\,d+4) - 2\,\left(d^2+d+24\right)\lambda_k\right)}{3\,(d-4\lambda_k)\,\Gamma\left(\frac{d}{2}+1\right)}\right) \tag{C.11}$$

$$\begin{aligned}\beta_\lambda = {}& \left(-2-\frac{d^{d/2}\,\pi^{1-\frac{d}{2}}\,2^{-d/2}\,g_k\,\left(3\,d\,((d-1)\,d+4) - 2\,\left(d^2+d+24\right)\lambda_k\right)}{3\,(d-4\lambda_k)\,\Gamma\left(\frac{d}{2}+1\right)\,2^{d-3}}\right)\lambda_k + \\ &+ \frac{-2^{2-d}\,(d+1)\,\pi^{1-\frac{d}{2}}\,2^{-d/2}\,d^{\frac{d}{2}+1}\,g_k\,\ln\left(1-\frac{4}{d}\lambda_k\right)}{\Gamma\left(\frac{d}{2}\right)}\,.\end{aligned} \tag{C.12}$$

C.2.2 The projection on $\mathbb{R}^d$

- CREH - Smooth proper time cutoff

$$\beta_g = g_k \left(d - 2 - \frac{2^{2-d}\, d^2\, (d+2)^2\, \pi^{1-\frac{d}{2}}\, g_k\, \lambda_k^2\, n^n\, \Gamma\left(-\frac{d}{2}+n+3\right)}{3\,(d-2)^3\,(d-1)^3\,\Gamma(n)\, \left(n - \frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{\frac{1}{2}(-d+2n+6)}} \right) \tag{C.13}$$

$$\beta_\lambda = \left(-2 - \frac{2^{2-d}\, d^2\, (d+2)^2\, \pi^{1-\frac{d}{2}}\, g_k\, \lambda_k^2\, n^n\, \Gamma\left(-\frac{d}{2}+n+3\right)}{3\,(d-2)^3\,(d-1)^3\,\Gamma(n)\, \left(n - \frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}\right)^{\frac{1}{2}(-d+2n+6)}} \right) \lambda_k + \frac{2^{3-d}\,\pi^{1-\frac{d}{2}}\, g_k\, \Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)\, \left(1 - \frac{d\,(d+2)\,\lambda_k}{2\,(d-2)\,(d-1)}\right)^{\frac{1}{2}(2n-d)}}\,, \tag{C.14}$$

- CREH - Sharp proper time cutoff

$$\beta_g = g_k \left(d - 2 - \frac{2^{2-d}\, d^2\, (d+2)^2\, \pi^{1-\frac{d}{2}}\, g_k\, \lambda_k^2}{3\,(d-2)^3\,(d-1)^3}\, e^{\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}} \right)\,, \tag{C.15}$$

$$\beta_\lambda = 2^{3-d}\,\pi^{1-\frac{d}{2}}\, g_k\, e^{\frac{d\,(d+2)\,\lambda_k}{2\,(d-2)\,(d-1)}} + \left(-2 - \frac{2^{2-d}\, d^2\, (d+2)^2\, \pi^{1-\frac{d}{2}}\, g_k\, \lambda_k^2}{3\,(d-2)^3\,(d-1)^3}\, e^{\frac{d\left(\frac{2d}{d-2}-1\right)\lambda_k}{2\,(d-1)}} \right) \lambda_k\,, \tag{C.16}$$

- CREH - Sharp momentum cutoff

$$\beta_g = g_k \left(d - 2 - \frac{d^{d/2} g_k\, \lambda_k^3 \left(3\,\left(d^2 - 3\,d + 2\right)^2 + (d+2)^2\,\lambda_k^2 - 3\,(d-2)\,(d-1)\,(d+2)\,\lambda_k\right)}{(d+2)^{-3}\pi^{\frac{d}{2}-1} 2^{\frac{3d}{2}-5} 3 (d-2)^3 (d-1)^3 \left(d^2 - d(\lambda+3) - 2\lambda + 2\right)^3 \Gamma\left(\frac{d}{2}+1\right)} \right)\,, \tag{C.17}$$

$$\beta_\lambda = \lambda_k \left(-2 - \frac{d^{d/2} g_k \lambda_k^3 \left(3\,\left(d^2 - 3\,d + 2\right)^2 + (d+2)^2\lambda_k^2 - 3(d-2)(d-1)(d+2)\lambda_k\right)}{2^{-5+\frac{3d}{2}}\,(d+2)^{-3}\pi^{\frac{d}{2}-1} 3\,(d-2)^3\,(d-1)^3\, \left(d^2 - d\,(\lambda_k+3) - 2\lambda_k + 2\right)^3\, \Gamma\left(\frac{d}{2}+1\right)} \right) - \frac{2^{3-\frac{3d}{2}}\,\pi^{1-\frac{d}{2}}\, d^{d/2}\, g_k\, \ln\left(1 - \frac{(d+2)\,\lambda_k}{(d-2)\,(d-1)}\right)}{\Gamma\left(\frac{d}{2}\right)}\,, \tag{C.18}$$

Appendix D

Heat kernel techniques

Heat kernel techniques play an important role in the evaluation of the effective action in quantum field theory, in particular in the case of gauge field theories and field theories on curved manifolds. The evaluation of loop corrections depends in fact on the calculation of functional traces of differential operators, that is the Hessians, and such traces have to be expanded on the basis of local operators of the action in order to have information about the divergencies of the theory, anomalies and various asymptotic behaviors of the effective action.

A useful way to evaluate such traces consists in rewriting the propagator in its integral representation

$$\frac{1}{S^{(2)}} = \int_0^\infty d^d s \, \mathcal{H}(x, s; S^{(2)}) \,, \tag{D.1}$$

where s is a proper time variable and $\mathcal{H}(x, s; S^{(2)})$ is an operator with matrix elements

$$\mathcal{H}(x, x', s; S^{(2)}) = \langle \, x \, | \, e^{-s \, S^{(2)}} \, | \, x' \, \rangle \,, \tag{D.2}$$

being $S^{(2)}$ the inverse propagator. The operator (D.2) is called heat kernel since it satisfies the heat equation

$$(\partial_s + S^{(2)}_x) \, \mathcal{H}(x, x', s; S^{(2)}) = 0 \,, \tag{D.3}$$

with the following boundary condition at $s \to 0^+$

$$\lim_{s \to 0^+} \mathcal{H}(x, x', s; S^{(2)}) = \delta^d(x - x') \,. \tag{D.4}$$

As noted first by Fock [145] and later by Schwinger [146], the study of the heat kernel, instead of the propagator itself, makes clearer many issues about the renormalizability and gauge invariance of the theory under investigation. The trace of the heat kernel (D.2), in fact, has many interesting properties. For example, for positive s it is divergencies-free, thanks of its exponential structure, and it admits a series expansion, which in the limit $s \to 0^+$ reads

$$\mathrm{Tr} \, \mathcal{H}(x, s; S^{(2)}) = \mathrm{Tr} \, \langle \, x \, | \, e^{-s \, S^{(2)}} \, | \, x \, \rangle = \sum_{n=0}^{\infty} E_n(x; S^{(2)}) \, s^{\frac{n-d}{\alpha}} \,, \tag{D.5}$$

where the operators E_n contain polynomials of the local invariants of the action, d is the dimensionality of spacetime and α is a parameter which depends on the order of the highest-derivative differential operator contained in the Hessian (that is the highest-derivative operator is built from α derivatives). In the case of curved background the coefficients E_n, which are usually called Seeley-Gilkey (occasionally Seeley-Gilkey-DeWitt) coefficients, contain invariants built from the Riemann, Weyl tensors et cetera, which means that we can obtain informations about the geometry of the background by summing the spectra of eigenvalues of differential operators defined on it.

The most known technique is the one introduced by DeWitt in [147], which is based on the resolution of the heat kernel equation on a generic curved manifold $\mathcal{M}$ using an ansatz of the type

$$\mathcal{H}(x, x', s; \mathcal{D}) = \mathcal{H}(x, x', s; \mathcal{D})_0 \sum_{n=0}^{\infty} a_n(x; \mathcal{D})\, s^n \,, \tag{D.6}$$

where $\mathcal{D}$ is now a second order non-minimal differential operator and $\mathcal{H}(x, x', s; \mathcal{D})_0$ is a straightforward generalization of the heat kernel for the simple Laplacian in flat space $\mathcal{M}_0 = \mathbb{R}^d$,

$$\mathcal{H}(x, x', s; -\partial^2)_0 = \frac{1}{(4\,\pi\, s)^{\frac{d}{2}}} e^{\frac{(x-x')^2}{4\,s}} \,, \tag{D.7}$$

where the squared euclidean distance (which is not a covariant quantity) is replaced by the Synge's biscalar of geodesic distance $\eta(x, x')$, which reads

$$\eta(x, x') = \frac{1}{2}\,(\tau'' - \tau') \int_{\tau'}^{\tau''} d\tau g^{\mu\nu}\, \frac{d\, x^\mu(\tau)}{d\,\tau}\, \frac{d\, x^\nu(\tau)}{d\,\tau} \,. \tag{D.8}$$

The coefficients a_n in (D.6) are then corrections containing invariants under diffeomorphisms built contracting Riemann tensors and their derivatives. To evaluate them, we can insert the ansatz (D.6) in the heat kernel equation (D.3), thus obtaining a hierarchy of iterative equations organized by the powers of the proper time which can be solved iteratively for the coefficients a_n.

Although this method has the advantage of being explicitly covariant, the ansatz (D.6) assumes a complicated expression in the case of non-minimal and higher-order differential operators which makes this approach not appropriate for general use (altough still possible, see [148, 149] for its higher derivative application). In the case of higher derivative operators, for example, the flat spacetime solution is a hypergeometric function, whose main drawback consists in its non analytic dependence on the proper time variable.

A more suitable approach is based on the method of the pseudodifferential operators, which does not fix the form of the ansatz but instead translate the evaluation of the Seeley-Gilkey coefficients to the evaluation of their pseudodifferential symbols. The drawback of this method, however, is the lack of invariance under general coordinate transformation and gauge choice, which issue has been solved in [130].

D.1 Pseudodifferential operators

We introduce here the basics of the pseudodifferential method, and later its application to the heat kernel. Let us consider a linear differential operator $\sigma(D)$ defined as

$$\sigma(D) = \sum_\alpha c_\alpha \, D^\alpha \,, \tag{D.9}$$

being D^α a derivative operator[1], $D^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_n}$, being α a multi-index on the tangent space, $\alpha = \{\alpha_1, \cdots, \alpha_n\}$, and c_α are matrix coefficients, so that

$$c_\alpha \, D^\alpha \equiv c_{\alpha_1 \cdots \alpha_n} \, \partial^{\alpha_1} \cdots \partial^{\alpha_n} \,. \tag{D.10}$$

Consider now the differential operator (D.9) acting on a local function $u(x)$, $x \in \mathcal{M}$, with compact support on $\mathbb{R}^d$. The way it acts can be defined in Fourier space as

$$\sigma(D)\, u(x) = \frac{1}{(2\pi)^d} \int \int e^{i\,(x-y)\,k}\, \sigma(k)\, u(y)\, dy\, dk \,, \tag{D.11}$$

where the polynomial $\sigma(k)$ is said symbol and reads

$$\sigma(k) = \sum_\alpha c_\alpha \, k^\alpha \,. \tag{D.12}$$

The operator (D.12) is related to (D.9) by an inverse Fourier transform, and in particular it holds

$$\sigma(D)_x \, e^{i\,(x-y)\,k} = e^{i\,(x-y)\,k}\, \sigma(k) \,. \tag{D.13}$$

A pseudodifferential operator $\sigma(x, D)$ is a generalization of the differential operator (D.9) such that the coefficients c_α are now non-homogeneous functions, $c_\alpha \equiv c_\alpha(x)$, so that the symbol is now coordinate-dependent, and (D.11) reads

$$\sigma(x, D)\, u(x) = \frac{1}{(2\pi)^d} \int \int e^{i\,(x-y)\,k}\, \sigma(x, k)\, u(y)\, dy\, dk \,. \tag{D.14}$$

The above definition can be generalized for compact manifolds equipped with a metric $g_{\mu\nu}$ by substituting the derivative in (D.9) with the covariant derivative ∇, and by choosing a covariant Fourier base, as we will see in a later stage. An example of pseudodifferential operator on a curved manifold, often used in QFT, is obtained by considering a second order operator with the metric tensor as coefficient of the principal part, so that the principal part is scalar (such operator is called minimal, we will not treat here non-minimal operators), i.e.

$$\sigma(x, \nabla) = -g^{\mu\nu}(x)\, \nabla_\mu \, \nabla_\nu - c^\mu(x)\, \nabla_\mu + b(x) \,, \tag{D.15}$$

where the covariant derivative ∇_μ contains not only the affine connections but also spin and gauge connections.

---

[1] We will take in consideration just partial differential operators, i.e. differential operators with integer exponent.

D.1.1 Heat kernel expansion

Given a pseudodifferential operator $\mathcal{D}$ we can now represent its matrix elements in terms of its symbol as

$$\langle\, x \,|\, \mathcal{D} \,|\, x' \,\rangle = \int \frac{d^d k}{(2\pi)^d \sqrt{g(x')}}\, e^{i\, k_\mu\, (x-x')^\mu}\, \sigma(x, x', k; \mathcal{D})\,, \tag{D.16}$$

For example, for the operator (D.15) we obtain on a flat space

$$\sigma(x, x', k; \mathcal{D}) = g^{\mu\nu}(x)\, k_\mu\, k_\nu + c^\mu(x)\, k_\mu + b(x)\,. \tag{D.17}$$

The symbol for the heat kernel of the operator $\mathcal{D}$ can be written by using the Cauchy representation of its exponential, i.e.

$$e^{-s\,\mathcal{D}} = -\oint_C \frac{d\Lambda}{2\pi i} \frac{e^{-s\,\Lambda}}{(\mathcal{D}-\Lambda)}\,, \tag{D.18}$$

where the contour path C encircles the spectrum of $\mathcal{D}$. Hence, using (D.16) and (D.18) the pseudodifferential symbol of the heat kernel operator $\mathcal{H}(x, x', s; \mathcal{D})$ is defined as

$$\begin{aligned} \mathcal{H}(x, x', s; \mathcal{D}) &= \langle\, x \,|\, e^{-s\,\mathcal{D}} \,|\, x' \,\rangle = \\ \int \frac{d^d k}{(2\pi)^d \sqrt{g(x')}} \oint_C \frac{d\Lambda}{2\pi i}\, e^{-s\Lambda}\, e^{i\, k_\mu\, (x-x')^\mu}\, &\sigma(x, x', k; (\mathcal{D}-\Lambda)^{-1})\,, \end{aligned} \tag{D.19}$$

where the operator $\sigma(x, x', k; (\mathcal{D}-\Lambda)^{-1})$ satisfies the Green equation

$$(\mathcal{D}_x - \Lambda)\left(\int \frac{d^d k}{(2\pi)^d \sqrt{g(x')}}\, e^{i\, k_\mu\, (x-x')^\mu}\, \sigma(x, x', k; (\mathcal{D}-\Lambda)^{-1})\right) = \frac{\delta^d(x-x')}{\sqrt{g(x')}}\,, \tag{D.20}$$

and where the expression of the δ-function reads

$$\frac{\delta^d(x-x')}{\sqrt{g(x')}} = \int \frac{d^d k}{(2\pi)^d \sqrt{g(x')}}\, e^{i\, k_\mu\, (x-x')^\mu}\,. \tag{D.21}$$

From (D.19) it is then clear that a strategy to calculate the Seeley-Gilkey coefficients involves the evaluation of the symbol $\sigma(x, x', k; (\mathcal{D}-\Lambda)^{-1})$.
The lack of general covariance of plane waves $e^{i\, k_\mu\, (x-x')^\mu}$ can be solved by the introduction of a covariant phase $l(x, x', k)$ which reduces to $k_\mu\, (x-x')^\mu$ in the limit of flat manifold. The connection with the DeWitt covariant method can then be established writing

$$l(x, x', k^\mu) = k_\mu\, \eta(x, x')^{;\mu}\,, \tag{D.22}$$

where $\eta(x, x')^{;\mu}$ is the covariant derivative of the Synge's biscalar (D.8), for which holds

$$\eta(x, x')_\mu\, \eta(x, x')^\mu = 2\, \eta(x, x')\,. \tag{D.23}$$

From (D.22) we can see that

$$\frac{\partial}{\partial k_\mu} l(x,x',k) = \eta(x,x')^{;\mu}\,, \tag{D.24}$$

that reduces to the Euclidean distance vector $(x-x')^\mu$ on a flat spacetime. As a consequence of the properties of the Synge's geodesic distance (see [147]) many features of the covariant phase can be understood, as its linearity in k_μ and x^μ once x'^μ has been fixed, that reads

$$\frac{\partial}{\partial x^\mu} l(x,x',k)\big|_{x=x'} = k_\mu \quad l(x,x',k)|_{x=x'} = 0\,, \tag{D.25}$$

and that the symmetrized covariant derivatives vanish in the coincidence limit $x=x'$ for $m \geqslant 2$, that is

$$\begin{aligned}&\{\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_m}\}\, l(x,x',k)|_{x=x'}] \equiv \\ &[\{\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_m}\}\, l(x,x',k)] = 0\,,\end{aligned} \tag{D.26}$$

where we used squared parenthesis to characterize the coincidence limits. Using (D.22) it is then possible to evaluated the coincidence limits for the non-symmetrized covariant derivatives starting from the coincidence limits of covariant derivatives of $\eta(x,x')$ [147], i.e.

$$\begin{aligned}&[l(x,x',k)] = 0\,,\\ &[\nabla_\mu\, l(x,x',k)] = k_\mu\,,\\ &[\nabla_\mu\nabla_\nu\, l(x,x',k)] = 0\,,\\ &[\nabla_\mu\nabla_\nu\nabla_\lambda\, l(x,x',k)] = -\frac{2}{3}\, k_\alpha\, R^\alpha{}_{(\lambda\mu\nu)}\,,\\ &\ldots\end{aligned} \tag{D.27}$$

and so on, where $R_{\alpha\lambda\mu\nu}$ is the Riemann tensor associated to the connection ∇_μ and the parenthesis stands for symmetrization with respect to outer indices with weight $\frac{1}{2}$.

The covariant generalization of equation (D.19) reads then

$$\begin{aligned}&\mathcal{H}(x,x',s;\mathcal{D}) = \langle\, x\,|\, e^{-s\,\mathcal{D}}\,|\, x'\,\rangle = \\ &\int \frac{d^dk}{(2\pi)^d\sqrt{g(x')}} \oint_C \frac{d\Lambda}{2\pi i}\, e^{-s\Lambda}\, e^{i\,l(x,x',k)}\, \sigma(x,x',k;(\mathcal{D}-\Lambda)^{-1})\,.\end{aligned} \tag{D.28}$$

In the case of manifold with a fibre bundle the Green equation (D.20) for the symbol $\sigma(x,x',k;(\mathcal{D}-\Lambda)^{-1})$ becomes

$$(\mathcal{D}(x,\nabla_x)-\Lambda)\left(e^{i\,l(x,x',k)}\,\sigma(x,x',k;(\mathcal{D}-\Lambda)^{-1})\right) = e^{i\,l(x,x',k)}\,\mathcal{I}(x,x') \tag{D.29}$$

where $\mathcal{I}(x,x')$ is the biscalar of parallel transport, which coincidence limits read

$$\begin{aligned}
&[\mathcal{I}(x,x')_{\alpha\beta}] = \mathbb{1}_{\alpha\beta}\,, \\
&[\nabla_\mu\,\mathcal{I}(x,x')] = 0\,, \\
&[\nabla_\mu\,\nabla_\nu\,\mathcal{I}(x,x')] = \frac{1}{2}\,W_{\mu\nu}\,, \\
&[\nabla_\mu\,\nabla_\nu\,\nabla_\lambda\,\mathcal{I}(x,x')] = -\frac{2}{3}\,\nabla_{(\mu}\,W_{\lambda\nu)}\,, \\
&\ldots
\end{aligned} \tag{D.30}$$

being α, β bundle indices and where W is the bundle curvature.

D.1.2 The Laplacian operators

As an example we present here the evaluation of the well known Seeley-Gilkey coefficients for the Laplacian operator acting on a scalar field; thus considering $\mathcal{D}\,\phi(x)$, where ϕ is a scalar field and $\mathcal{D}$ is the differential operator

$$\mathcal{D}(x,\nabla_\mu) = -g^{\mu\nu}(x)\,\nabla_\mu\,\nabla_\nu\,. \tag{D.31}$$

Inserting (D.31) in the Green equation (D.29), it now becomes

$$(-g^{\mu\nu}(x)\,\nabla_\mu\,\nabla_\nu - \Lambda)\,\left(e^{i\,l(x,x',k)}\,\sigma(x,x',k;\Lambda)\right) = e^{i\,l(x,x',k)}\,\mathcal{I}(x,x')\,, \tag{D.32}$$

where from now on for convenience we will rename $\sigma(x,x',k;\Lambda) \equiv \sigma(x,x',k;(\mathcal{D}-\Lambda)^{-1})$. By applying the covariant derivate on the phase, equation (D.32) can be rewritten as

$$(-(\nabla_\mu\ + i\,\nabla_\mu\,l)(\nabla^\mu + i\,\nabla^\mu\,l) - \Lambda)\ \sigma(x,x',k;\Lambda) = \mathcal{I}(x,x')\,. \tag{D.33}$$

To obtain the recursive relations it is then convenient to work in dimensionless quantities, keeping the proper time variable as an organizing mass dimension. Thus we rescale the covector k_μ in the expression (D.22) of the biscalar $l(x,x',k)$, and we will keep just the covariant derivatives with their mass dimensions, so that

$$l(x,x',k) \to l(x,x',k)\,s^{-\frac{1}{2}}\,, \tag{D.34}$$

where we remind that s has the dimension of the inverse of $\mathcal{D}$, $[s] = -[\mathcal{D}]$, since the heat kernel is dimensionless by definition. Hence, we expand the symbol $\sigma(x,x',k)$ in a series of powers in the proper time s (still, using it as a mass parameter), i.e.

$$\sigma(x,x',k;\Lambda) = \sum_{n=0}^{N=\infty}\ \sigma_n(x,x',k,\Lambda)\,s^{1+\frac{n}{\alpha}}\,, \tag{D.35}$$

where α is the order of the principal part of the operator $\mathcal{D}$ ($\alpha = 2$ for the Laplacian operator) and N is set to a finite value for actual calculations, and finally we rescale the resolvent Λ as

$$\Lambda \to \Lambda\, s^{-1} \,. \tag{D.36}$$

Isolating the terms with same powers of s the following equations are found

$$\begin{aligned} &((\nabla_\mu l)(\nabla^\mu l) - \Lambda)\ \sigma_0 = \mathcal{I}(x,x')\,, \\ &((\nabla_\mu l)(\nabla^\mu l) - \Lambda)\ \sigma_1 - i\ (\nabla_\mu \nabla^\mu l + 2\ (\nabla^\mu l)\,\nabla_\mu)\ \sigma_0 = 0\,, \\ &((\nabla_\mu l)(\nabla^\mu l) - \Lambda)\ \sigma_n - i\ (\nabla_\mu \nabla^\mu l + 2\ (\nabla^\mu l)\,\nabla_\mu)\ \sigma_{n-1} - \nabla^2\,\sigma_{n-2} = 0, \qquad n \geqslant 2\,, \end{aligned} \tag{D.37}$$

where ∇^2 is the Laplacian operator, $\nabla^2 \equiv g^{\mu\nu}\,\nabla_\mu \nabla_\nu$. Once solved consecutively the system of equations (D.37) respect to the symbols σ_n and have taken their coincidence limits $[\sigma_n]$ the Seeley-Gilkey coefficients $E_n(x;\mathcal{D})$ in (D.5) can be evaluated by means of the integral

$$E_n(x;\mathcal{D}) = \int \frac{d^d k}{(2\pi)^d\,[\sqrt{g(x')}]} \oint_C \frac{d\Lambda}{2\pi i}\, e^{-\Lambda}\,[\sigma_n(x,x,k;\Lambda)]\,, \tag{D.38}$$

which can be computed using the standard integral

$$\begin{aligned} &\int \frac{d^d k}{(2\pi)^d\,\sqrt{g(x')}}\, k_{\mu_1}\,k_{\mu_2}\cdots k_{\mu_{2Q}} \oint_C \frac{d\Lambda}{2\pi i}\, \frac{e^{-\Lambda}}{(k^2-\Lambda)^P} = \\ &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{1}{2^Q\,\Gamma(P)}\, g_{\{\mu_1\,\mu_2\cdots\mu_{2Q}\}} \end{aligned} \tag{D.39}$$

where Γ is the Euler gamma function and $g_{\{\mu_1\,\mu_2\cdots\mu_{2Q}\}}$ is the symmetrized product of metric tensors $g_{\mu_1\mu_2}\,\cdots\,g_{\mu_{2Q-1}\,\mu_{2Q}}$ with weight one, i.e.

$$g_{\{\mu\nu\rho\lambda\}} = g_{\mu\nu}\,g_{\lambda\rho} + g_{\mu\rho}\,g_{\lambda\nu} + g_{\mu\lambda}\,g_{\nu\rho}\,. \tag{D.40}$$

The integral (D.39) vanishes for an odd number of k^μ vectors so that the trace of heat kernel gets contributions only from the even terms

$$\mathcal{H}(x,x,s;-\nabla^2) = \sum_{n=0}^{\infty} E_{2n}(x;-\nabla^2)\,s^{n-\frac{d}{2}}\,. \tag{D.41}$$

The computation of the lower coefficients gives

$$\begin{aligned} E_0(x;-\nabla^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,, \\ E_2(x;-\nabla^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{R}{6}\,, \\ E_4(x;-\nabla^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\left\{\frac{R^2}{72} - \frac{1}{180}\,R_{\mu\nu}\,R^{\mu\nu} + \frac{1}{180}\,R_{\mu\nu\rho\lambda}\,R^{\mu\nu\rho\lambda} + \frac{1}{30}\,\nabla^2 R\right\}. \end{aligned} \tag{D.42}$$

The same calculation can be repeated for higher-order pseudodifferential operators without the drawbacks present in the DeWitt method.

D.1.3 The squared Laplacian operators

We briefly present here the evaluation of the heat kernel expansion for an higher-order operator, that is the squared Laplacian operator applying on a scalar field, where $\mathcal{D}$ reads in this case

$$\mathcal{D}(x, \nabla_x) = (-g^{\mu\nu}(x)\, \nabla_\mu\, \nabla_\nu)^2 \,. \tag{D.43}$$

Inserting (D.43) in (D.29) we obtain this time the Green equation

$$((\nabla_\mu + i\, \nabla_\mu\, l)(\nabla^\mu + i\, \nabla^\mu\, l)(\nabla_\nu + i\, \nabla_\nu\, l)(\nabla^\nu + i\, \nabla^\nu\, l) - \Lambda)\; \sigma(x, x', k; \Lambda) = \mathcal{I}(x, x') \,. \tag{D.44}$$

We then apply the rescaling $k_\mu \to k_\mu\, s^{-\frac{1}{4}}$ in the expression of the biscalar $l(x, x', k)$, and for the resolvent $\Lambda \to \Lambda\, s^{-1}$. Hence, after having performed an expansion in powers of s of the symbol, as done in (D.35) but with $\alpha = 4$, the recursive relations can be found by equating equations with the same powers of the proper time parameter. The number of terms contained in the recursive relations grows quickly with the order of the differential operators, so that for display purposes we present just the first two equations

$$\begin{aligned}
&\big((\nabla_\mu l\, \nabla^\mu l)^2 - \Lambda\big)\; \sigma_0 = \mathcal{I}(x, x') \,, \qquad \text{(D.45)}\\
&\big((\nabla_\mu l\, \nabla^\mu l)^2 - \Lambda\big)\; \sigma_1 - 2\, i\, \big(\nabla^2 l\, \nabla_\mu l\, \nabla^\mu l\, + 2\, \nabla^\mu l\, \nabla^\nu l\, \nabla_\mu \nabla_\nu l + 2\, \nabla^\mu l\, \nabla_\mu l\, \nabla^\nu l\, \nabla_\nu\big)\; \sigma_0 = 0 \,,\\
&\quad \ldots
\end{aligned}$$

The above calculation is explained more in details in [130]. Once the expressions of the symbols σ_n and their coincidence limits has been evaluated, the heat kernel expansion can be obtained setting $\alpha = 4$ in the general expression (D.5), so that

$$\mathcal{H}\left(x, x, s; (\nabla^2)^2\right) = \sum_{n=0}^{\infty} E_{2n}\left(x; (\nabla^2)^2\right)\; s^{\frac{n}{2} - \frac{d}{4}} \,. \tag{D.46}$$

The expression of the Seeley-Gilkey coefficients is still given by (D.38) while in the higher derivative case the actual calculation requires the use of the generalized integral for operators of order α, which reads

$$\begin{aligned}
&\int \frac{d^d k}{(2\pi)^d \sqrt{g(x')}}\, (k^2)^H\, k_{\mu_1}\, k_{\mu_2} \cdots k_{\mu_{2Q}} \oint_C \frac{d\Lambda}{2\pi i}\, \frac{e^{-\Lambda}}{(k^\alpha - \Lambda)^P} = \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\, \frac{\Gamma(\frac{d}{\alpha} + \frac{(Q+H)}{\alpha})}{2^Q\, \frac{\alpha}{2}\, \Gamma(P)\, \Gamma(\frac{d}{2} + Q)}\, g_{\{\mu_1\, \mu_2 \cdots \mu_{2Q}\}} \,.
\end{aligned} \tag{D.47}$$

Because of the form of integral (D.47) the coefficients will now exhibit a non trivial dependence on the dimensionality d of the manifold. The lower computed coefficients are

$$\begin{aligned}
E_0(x;(\nabla^2)^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(\frac{d}{4})}{2\,\Gamma(\frac{d}{2})}\,,\\
E_2(x;(\nabla^2)^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{d\,\Gamma(\frac{(d+2)}{4})}{2\,\Gamma(\frac{(d+2)}{2})}\,\frac{R}{6}\,,\\
E_4(x;(\nabla^2)^2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(\frac{(d+4)}{4})}{\Gamma(\frac{(d+2)}{2})}\,(d-2)\left\{\frac{R^2}{72}-\frac{1}{180}\,R_{\mu\nu}\,R^{\mu\nu}+\frac{1}{180}\,R_{\mu\nu\rho\lambda}\,R^{\mu\nu\rho\lambda}+\frac{\nabla R}{30}\right\}.
\end{aligned} \tag{D.48}$$

D.1.4 Other results

The method introduced in this section is a general covariant technique which can be easily applied to any differential operator which principal part is minimal (for non-minimal operators see [150, 151, 152]) and can moreover be generalized to manifold with torsion and fiber bundles. As for the DeWitt method, one of the most interesting features of this technique is its iterative structure, so that it can be implemented in a computer program (for example in C language). Seeley-Gilkey coefficients can then be systematically computed. The results reported above have been reproduced by implementing the iterative structure in a Mathematica notebook, using in particular the tensorial formalism package xAct (`www.xact.es`). Perhaps, the use of recursive relations has the drawback that the number of terms quickly grows with the order of the operator (in the recursive relations like also in the coincidence limits of the biscalar $l(x, x', k)$) so that, apart of simplifications, computation time and memory usage quickly overcome the machine capability. Hence, less brute-force methods are preferred (see in particular [153] and [154] for a general overview of heat kernel techniques). An other drawback of this class of methods is, ironically, their explicit covariance. In the case of manifolds with a foliation the use of a non-relativistic frameworks (like the ADM splitting for the metric) entails that the coincidence limits for the full dimensional geodesic distance (and then the biscalar $l(x, x', k)$) cannot be used anymore. In this case, however, there is no prescription given a priori to construct the Fourier phase $l(t, t', \mathbf{x}, \mathbf{x}', \omega, \mathbf{k})$, since we have now to define a time and space geodesic distance, respectively $\eta_t(t, t', \mathbf{x}, \mathbf{x}')$ and $\eta_{\mathbf{x}}(t, t', \mathbf{x}, \mathbf{x}')$. Also defining those two distances, however, we could not obtain the correct results for the coincidence limits of their mixed time and space derivatives. As a consequence of our analysis we concluded that the covariant DeWitt's method cannot be applied to the anisotropic case. In particular, since we have to rescale separately the time and space Fourier conjugated variable, ω and $\mathbf{k}_i$, to work in dimensionless quantities (in order to find in the n-th symbols σ_n the extrinsic and spatial curvature tensors according to their anisotropic dimensions), it is not possible nor convenient anymore to use a biscalar of phase argument, and we need instead to work directly with the two Synge's geodesic distances.

We conclude this appendix by reporting in this and the following pages few results obtained with the Mathematica notebook developed[2] using the package xAct that has been mentioned before. In particular, we have reproduced the first coefficients of the heat kernel expansion for respectively a second and fourth order minimal differential operators for a generic compact d dimensional Riemann manifolds. The coefficients that have been reproduced are those that have been used in chapters 3, 4 and 5.

To show the versatility and the potentialities of such notebook (and the potentialities of xAct) we present furthermore the first Seeley-Gilkey coefficients for a general minimal second order partial differential operator on a Cartan-Riemann manifold. One of the coefficients, E_4, is to the best of our knowledge not present in literature. The technique can be easily generalized to the evaluation of higher-order operators, but at the price of longer computational time. Because of the quickly growing number of invariants that can be built from the torsion tensor and Cartan-Riemann tensor (151 independent invariants of order $\mathcal{O}(R^2)$ in $d = 4$ [155], and probably in the order of thousands at the next order) the Seeley-Gilkey coefficients E_n with $n > 4$ are out of the reach of this program.

Riemannian manifold: $\mathcal{D}_1 = -g^{\mu\nu}(x)\,\nabla_\mu\,\nabla_\nu + B^\mu(x)\,\nabla_\mu + X(x)$

$$\begin{aligned}
E_0(x;\mathcal{D}_1) &= \tfrac{1}{(4\pi)^{\frac{d}{2}}} \quad , \\
E_2(x;\mathcal{D}_1) &= \tfrac{1}{(4\pi)^{\frac{d}{2}}} \quad \left\{-\tfrac{1}{4}B_\alpha B^\alpha + \tfrac{1}{6}R[\nabla] - X + \tfrac{1}{2}\nabla_\alpha B^\alpha\right\} , \\
E_4(x;\mathcal{D}_1) &= \tfrac{1}{(4\pi)^{\frac{d}{2}}} \quad \{\tfrac{1}{32}B_\alpha B^\alpha B_\beta B^\beta - \tfrac{1}{180}R[\nabla]_{\alpha\beta}R[\nabla]^{\alpha\beta} - \tfrac{1}{24}B_\alpha B^\alpha R[\nabla] \\
&\quad + \tfrac{1}{72}R[\nabla]^2 + \tfrac{1}{180}R[\nabla]_{\alpha\beta\gamma\delta}R[\nabla]^{\alpha\beta\gamma\delta} + \tfrac{1}{4}B_\alpha B^\alpha X - \tfrac{1}{6}R[\nabla]X \\
&\quad + \tfrac{1}{2}X^2 + \tfrac{1}{12}R[\nabla]\nabla_\alpha B^\alpha - \tfrac{1}{2}X\nabla_\alpha B^\alpha - \tfrac{1}{36}B^\alpha\nabla_\alpha R[\nabla] \\
&\quad + \tfrac{1}{30}\nabla_\alpha\nabla^\alpha R[\nabla] - \tfrac{1}{6}\nabla_\alpha\nabla^\alpha X \\
&\quad + \tfrac{1}{12}\nabla_\alpha\nabla_\beta\nabla^\beta B^\alpha - \tfrac{1}{8}B_\alpha B^\alpha\nabla_\beta B^\beta + \tfrac{1}{8}\nabla_\alpha B^\alpha\nabla_\beta B^\beta \\
&\quad + \tfrac{1}{18}B^\alpha\nabla_\beta R[\nabla]_\alpha{}^\beta - \tfrac{1}{12}\nabla_\beta\nabla_\alpha\nabla^\beta B^\alpha - \tfrac{1}{12}B^\alpha\nabla_\beta\nabla^\beta B_\alpha \\
&\quad + \tfrac{1}{12}\nabla_\beta\nabla^\beta\nabla_\alpha B^\alpha - \tfrac{1}{24}\nabla_\alpha B_\beta\nabla^\beta B^\alpha - \tfrac{1}{24}\nabla_\beta B_\alpha\nabla^\beta B^\alpha\} .
\end{aligned} \tag{D.49}$$

[2]We would like to sincerely thank Dr. Thomas Bäckdahl for the assistance with the use of the package xAct.

Riemannian manifold: $\mathcal{D}_2 = (-g^{\mu\nu}(x)\,\nabla_\mu\,\nabla_\nu)^2 + V^{\mu\nu}(x)\,\nabla_\mu\,\nabla_\nu + B^\mu(x)\,\nabla_\mu + X(x)$

$$
\begin{aligned}
E_0(x;\mathcal{D}_2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(\frac{d}{4})}{2\,\Gamma(\frac{d}{2})}\,,\\
E_2(x;\mathcal{D}_2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\big(\frac{1}{4}(2+d)\big)}{12\Gamma\big(1+\frac{1}{2}d\big)}\,\{dR[\nabla] + 3V_\alpha{}^\alpha\}\,, \qquad \text{(D.50)}\\
E_4(x;\mathcal{D}_2) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(1+\frac{1}{4}d)}{720\Gamma(2+\frac{1}{2}d)}\Big\{(-20+5d^2)R[\nabla]^2 + 2\big(-4+d^2\big)R[\nabla]_{\alpha\beta\gamma\delta}R[\nabla]^{\alpha\beta\gamma\delta}\\
&\quad +30dR[\nabla]V^\alpha{}_\alpha + 45V_{\alpha\beta}V^{\alpha\beta} - 2\big(2+d\big)R[\nabla]_{\alpha\beta}\big((-2+d)R[\nabla]^{\alpha\beta}\\
&\quad +30V^{\alpha\beta}\big) + 45V^{\alpha\beta}V_{\beta\alpha} + 45V^\alpha{}_\alpha V^\beta{}_\beta - 720\frac{\Gamma\big(2+\frac{1}{2}d\big)}{\Gamma(1+\frac{1}{4}d)}X + 360\nabla_\alpha B^\alpha\\
&\quad +180d\nabla_\alpha B^\alpha - 48\nabla_\alpha\nabla^\alpha R[\nabla] + 60R[\nabla]V^\alpha{}_\alpha - 60d\nabla_\beta\nabla_\alpha V^{\alpha\beta}\\
&\quad +12d^2\nabla_\alpha\nabla^\alpha R[\nabla] - 60\nabla_\alpha\nabla_\beta V^{\alpha\beta} - 60d\nabla_\alpha\nabla_\beta V^{\alpha\beta} - 60\nabla_\beta\nabla_\alpha V^{\alpha\beta}\\
&\quad +120\nabla_\beta\nabla^\beta V^\alpha{}_\alpha + 30d\nabla_\beta\nabla^\beta V^\alpha{}_\alpha\Big\}\,.
\end{aligned}
$$

Cartan-Riemannian manifold: $\mathcal{D}_1 = -g^{\mu\nu}(x)\,\blacktriangledown_\mu\,\blacktriangledown_\nu + B^\mu(x)\,\blacktriangledown_\mu + X(x)$

We report here the first coefficients for the most generic non-minimal second order partial differential operator in the case of Cartan-Riemann manifolds, without listing the coincidence limits of the biscalar function $l(x,x',k)$ which are long and not particularly interesting (the first coincidence limits can be found in [150]). In the following we use $\blacktriangledown_\mu$ for the Cartan-Riemann connection and respectively $T_{\mu\nu\rho}$ and $R_{\alpha\beta\gamma\delta}$ for the torsion and Cartan-Riemann tensor, being the torsion tensor defined as the non symmetric part of the Christoffel's symbol. We follow the convention used in [150].

$$
\begin{aligned}
E_0(x;\mathcal{D}_1) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\quad , \qquad \text{(D.51)}\\
E_2(x;\mathcal{D}_1) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\quad \{-\tfrac{1}{4}B_\alpha B^\alpha + \tfrac{1}{6}R[\blacktriangledown] - \tfrac{1}{24}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma} - \tfrac{1}{12}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma}\\
&\qquad -\tfrac{1}{12}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_{\beta\gamma} - X + \tfrac{1}{2}\blacktriangledown_\alpha B^\alpha - \tfrac{1}{6}\blacktriangledown_\beta T[\blacktriangledown]^\alpha{}_\alpha{}^\beta\}\,,\\
E_4(x;\mathcal{D}_1) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\quad \tilde{E}_4\,,
\end{aligned}
$$

where $\tilde{E}_4$ is a long expression which comprehends all the invariants of mass dimension four built contracting the Cartan-Riemann tensor and the torsion tensor, and reads

$$\tilde{E}_4 = \tfrac{1}{32}B_\alpha B^\alpha B_\beta B^\beta + \tfrac{7}{216}R[\blacktriangledown]_{\alpha\beta}R[\blacktriangledown]^{\alpha\beta} - \tfrac{1}{216}R[\blacktriangledown]^{\alpha\beta}R[\blacktriangledown]_{\beta\alpha} - \tfrac{1}{24}B_\alpha B^\alpha R[\blacktriangledown] + \tfrac{1}{72}R[\blacktriangledown]^2$$
$$-\tfrac{1}{36}R[\blacktriangledown]_{\alpha\beta\gamma\delta}R[\blacktriangledown]^{\alpha\beta\gamma\delta} + \tfrac{1}{27}R[\blacktriangledown]_{\alpha\gamma\beta\delta}R[\blacktriangledown]^{\alpha\beta\gamma\delta} - \tfrac{1}{54}R[\blacktriangledown]^{\alpha\beta\gamma\delta}R[\blacktriangledown]_{\gamma\delta\alpha\beta}$$
$$-\tfrac{1}{18}B^\alpha R[\blacktriangledown]^{\beta\gamma}T[\blacktriangledown]_{\alpha\beta\gamma} - \tfrac{1}{144}R[\blacktriangledown]T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma} - \tfrac{1}{18}B^\alpha R[\blacktriangledown]^{\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma}$$
$$-\tfrac{1}{72}R[\blacktriangledown]T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma} + \tfrac{1}{48}B^\alpha B^\beta T[\blacktriangledown]_\alpha{}^{\gamma\delta}T[\blacktriangledown]_{\beta\gamma\delta} - \tfrac{1}{72}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]_\alpha{}^{\gamma\delta}T[\blacktriangledown]_{\beta\gamma\delta}$$
$$+\tfrac{25}{432}R[\blacktriangledown]_{\alpha\gamma\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta} - \tfrac{53}{432}R[\blacktriangledown]_{\alpha\delta\gamma\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta}$$
$$-\tfrac{7}{144}R[\blacktriangledown]_{\gamma\delta\alpha\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta} - \tfrac{1}{144}R[\blacktriangledown]_{\delta\zeta\alpha\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta} - \tfrac{1}{72}B^\alpha R[\blacktriangledown]_{\alpha\beta\gamma\delta}T[\blacktriangledown]^{\beta\gamma\delta}$$
$$+\tfrac{1}{72}B^\alpha R[\blacktriangledown]_{\alpha\gamma\beta\delta}T[\blacktriangledown]^{\beta\gamma\delta} - \tfrac{5}{72}B^\alpha R[\blacktriangledown]_{\beta\gamma\alpha\delta}T[\blacktriangledown]^{\beta\gamma\delta} - \tfrac{5}{72}B^\alpha R[\blacktriangledown]_{\gamma\delta\alpha\beta}T[\blacktriangledown]^{\beta\gamma\delta}$$
$$+\tfrac{1}{96}B_\alpha B^\alpha T[\blacktriangledown]_{\beta\gamma\delta}T[\blacktriangledown]^{\beta\gamma\delta} - \tfrac{1}{72}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]_\beta{}^{\gamma\delta}T[\blacktriangledown]_{\gamma\alpha\delta} - \tfrac{1}{24}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]_\alpha{}^{\gamma\delta}T[\blacktriangledown]_{\gamma\beta\delta}$$
$$+\tfrac{1}{48}B_\alpha B^\alpha T[\blacktriangledown]^{\beta\gamma\delta}T[\blacktriangledown]_{\gamma\beta\delta} - \tfrac{1}{144}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\delta}{}^\eta T[\blacktriangledown]_{\gamma\zeta\eta} - \tfrac{1}{24}B^\alpha R[\blacktriangledown]_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_{\beta\gamma}$$
$$+\tfrac{1}{24}B^\alpha R[\blacktriangledown]^\beta{}_\alpha T[\blacktriangledown]^\gamma{}_{\beta\gamma} - \tfrac{1}{72}R[\blacktriangledown]T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_{\beta\gamma} - \tfrac{1}{144}R[\blacktriangledown]_{\beta\gamma\delta\zeta}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^{\gamma\delta\zeta}$$
$$-\tfrac{1}{144}R[\blacktriangledown]_{\beta\delta\gamma\zeta}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^{\gamma\delta\zeta} - \tfrac{5}{144}R[\blacktriangledown]_{\gamma\delta\beta\zeta}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^{\gamma\delta\zeta}$$
$$-\tfrac{1}{144}R[\blacktriangledown]_{\delta\zeta\beta\gamma}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^{\gamma\delta\zeta} - \tfrac{11}{144}B^\alpha T[\blacktriangledown]^\beta{}_\alpha{}^\gamma T[\blacktriangledown]_\gamma{}^{\delta\zeta}T[\blacktriangledown]_{\delta\beta\zeta}$$
$$-\tfrac{1}{18}B^\alpha T[\blacktriangledown]_\alpha{}^{\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta}T[\blacktriangledown]_{\delta\gamma\zeta} - \tfrac{1}{72}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^\eta T[\blacktriangledown]_{\delta\zeta\eta}$$
$$+\tfrac{83}{2160}T[\blacktriangledown]_{\alpha\beta}{}^\delta T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]_{\delta\zeta\eta} + \tfrac{83}{2160}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha}{}^\delta T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]_{\delta\zeta\eta}$$
$$+\tfrac{7}{720}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]^\gamma{}_\beta{}^\delta T[\blacktriangledown]_{\delta\zeta\eta} + \tfrac{1}{72}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]^\gamma{}_{\alpha\gamma}T[\blacktriangledown]^\delta{}_{\beta\delta}$$
$$+\tfrac{1}{36}R[\blacktriangledown]_{\alpha\gamma\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^\delta{}_\beta{}^\zeta + \tfrac{1}{24}R[\blacktriangledown]_{\alpha\delta\gamma\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^\delta{}_\beta{}^\zeta$$
$$+\tfrac{1}{36}R[\blacktriangledown]_{\alpha\zeta\gamma\delta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^\delta{}_\beta{}^\zeta + \tfrac{1}{72}R[\blacktriangledown]_{\gamma\zeta\alpha\delta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^\delta{}_\beta{}^\zeta$$
$$-\tfrac{1}{72}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]_{\alpha\beta}{}^\gamma T[\blacktriangledown]^\delta{}_{\gamma\delta} + \tfrac{1}{48}B_\alpha B^\alpha T[\blacktriangledown]^\beta{}_\beta{}^\gamma T[\blacktriangledown]^\delta{}_{\gamma\delta} - \tfrac{11}{216}R[\blacktriangledown]^{\alpha\beta}T[\blacktriangledown]^\gamma{}_{\alpha\beta}T[\blacktriangledown]^\delta{}_{\gamma\delta}$$
$$-\tfrac{1}{384}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\delta\zeta\eta}T[\blacktriangledown]^{\delta\zeta\eta} + \tfrac{1}{144}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_{\beta\gamma}T[\blacktriangledown]_{\delta\zeta\eta}T[\blacktriangledown]^{\delta\zeta\eta}$$
$$-\tfrac{161}{2160}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta}T[\blacktriangledown]_{\gamma\delta}{}^\eta T[\blacktriangledown]_{\zeta\alpha\eta} + \tfrac{53}{432}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^\delta T[\blacktriangledown]_\delta{}^{\zeta\eta}T[\blacktriangledown]_{\zeta\alpha\eta}$$
$$+\tfrac{1}{48}B^\alpha T[\blacktriangledown]^\beta{}_\alpha{}^\gamma T[\blacktriangledown]^\delta{}_\beta{}^\zeta T[\blacktriangledown]_{\zeta\gamma\delta} + \tfrac{67}{540}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\delta}{}^\eta T[\blacktriangledown]_{\zeta\gamma\eta}$$
$$+\tfrac{2}{45}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_\beta{}^\delta T[\blacktriangledown]_\delta{}^{\zeta\eta}T[\blacktriangledown]_{\zeta\gamma\eta} - \tfrac{4}{45}T[\blacktriangledown]_{\alpha\beta}{}^\delta T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]_{\zeta\delta\eta}$$
$$+\tfrac{1}{120}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha}{}^\delta T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]_{\zeta\delta\eta} - \tfrac{7}{720}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]_\beta{}^{\gamma\delta}T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]_{\zeta\delta\eta}$$
$$+\tfrac{1}{48}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]_\gamma{}^{\zeta\eta}T[\blacktriangledown]^\gamma{}_\beta{}^\delta T[\blacktriangledown]_{\zeta\delta\eta} - \tfrac{1}{288}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta\zeta\eta}T[\blacktriangledown]_{\zeta\delta\eta}$$
$$+\tfrac{1}{96}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_{\beta\gamma}T[\blacktriangledown]^{\delta\zeta\eta}T[\blacktriangledown]_{\zeta\delta\eta} + \tfrac{1}{24}B^\alpha T[\blacktriangledown]^\beta{}_\alpha{}^\gamma T[\blacktriangledown]^\delta{}_{\beta\delta}T[\blacktriangledown]^\zeta{}_{\gamma\zeta}$$
$$-\tfrac{1}{80}T[\blacktriangledown]_{\alpha\beta}{}^\delta T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\zeta\delta\eta}T[\blacktriangledown]^\zeta{}_\gamma{}^\eta + \tfrac{31}{720}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_\beta{}^\delta T[\blacktriangledown]_{\zeta\delta\eta}T[\blacktriangledown]^\zeta{}_\gamma{}^\eta$$
$$+\tfrac{1}{36}B^\alpha T[\blacktriangledown]_\alpha{}^{\beta\gamma}T[\blacktriangledown]^\delta{}_{\beta\gamma}T[\blacktriangledown]^\zeta{}_{\delta\zeta} - \tfrac{1}{36}B^\alpha T[\blacktriangledown]^\beta{}_\alpha{}^\gamma T[\blacktriangledown]^\delta{}_{\beta\gamma}T[\blacktriangledown]^\zeta{}_{\delta\zeta}$$
$$-\tfrac{37}{540}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_\beta{}^{\delta\zeta}T[\blacktriangledown]_{\delta\gamma}{}^\eta T[\blacktriangledown]_{\eta\alpha\zeta} + \tfrac{1}{36}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\delta}{}^\eta T[\blacktriangledown]_{\eta\gamma\zeta}$$
$$+\tfrac{1}{48}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^\eta T[\blacktriangledown]_{\eta\delta\zeta} - \tfrac{1}{90}T[\blacktriangledown]_{\alpha\beta}{}^\delta T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^\zeta{}_\gamma{}^\eta T[\blacktriangledown]_{\eta\delta\zeta}$$
$$-\tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha}{}^\delta T[\blacktriangledown]^\zeta{}_\gamma{}^\eta T[\blacktriangledown]_{\eta\delta\zeta} - \tfrac{29}{720}T[\blacktriangledown]^\alpha{}_\alpha{}^\beta T[\blacktriangledown]^\gamma{}_\beta{}^\delta T[\blacktriangledown]^\zeta{}_\gamma{}^\eta T[\blacktriangledown]_{\eta\delta\zeta}$$
$$-\tfrac{1}{192}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\eta\delta\zeta}T[\blacktriangledown]^\eta{}_{\beta\gamma} - \tfrac{1}{160}T[\blacktriangledown]_\alpha{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\eta\gamma\zeta}T[\blacktriangledown]^\eta{}_{\beta\delta}$$

$$-\tfrac{3}{80}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}T[\blacktriangledown]^{\eta}{}_{\zeta\eta} - \tfrac{7}{720}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}T[\blacktriangledown]^{\eta}{}_{\zeta\eta}$$
$$-\tfrac{7}{720}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}T[\blacktriangledown]_{\delta\gamma}{}^{\zeta}T[\blacktriangledown]^{\eta}{}_{\zeta\eta} + \tfrac{1}{288}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\delta}{}^{\zeta}T[\blacktriangledown]^{\eta}{}_{\zeta\eta}$$
$$-\tfrac{29}{2160}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}T[\blacktriangledown]^{\zeta}{}_{\gamma\delta}T[\blacktriangledown]^{\eta}{}_{\zeta\eta} + \tfrac{1}{4}B_{\alpha}B^{\alpha}X - \tfrac{1}{6}R[\blacktriangledown]X + \tfrac{1}{24}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}X$$
$$+\tfrac{1}{12}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma}X + \tfrac{1}{12}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}X + \tfrac{1}{2}X^{2} + \tfrac{1}{12}R[\blacktriangledown]\blacktriangledown_{\alpha}B^{\alpha} - \tfrac{1}{2}X\blacktriangledown_{\alpha}B^{\alpha}$$
$$+\tfrac{1}{12}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\alpha}B_{\beta} + \tfrac{1}{18}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\alpha}R[\blacktriangledown]_{\beta\gamma} + \tfrac{1}{72}B^{\alpha}T[\blacktriangledown]^{\beta\gamma\delta}\blacktriangledown_{\alpha}T[\blacktriangledown]_{\gamma\beta\delta}$$
$$+\tfrac{49}{2160}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_{\alpha}T[\blacktriangledown]_{\gamma\delta\zeta} - \tfrac{37}{360}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_{\alpha}T[\blacktriangledown]_{\gamma\delta\zeta}$$
$$-\tfrac{17}{216}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma} + \tfrac{77}{1080}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_{\alpha}T[\blacktriangledown]_{\delta\gamma\zeta} - \tfrac{1}{24}B^{\alpha}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\delta}{}_{\gamma\delta}$$
$$-\tfrac{1}{36}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\zeta}{}_{\delta\zeta} + \tfrac{1}{36}\blacktriangledown_{\alpha}\blacktriangledown^{\alpha}R[\blacktriangledown] - \tfrac{1}{6}\blacktriangledown_{\alpha}\blacktriangledown^{\alpha}X$$
$$-\tfrac{1}{36}\blacktriangledown_{\alpha}\blacktriangledown_{\beta}R[\blacktriangledown]^{\alpha\beta} + \tfrac{1}{12}\blacktriangledown_{\alpha}\blacktriangledown_{\beta}\blacktriangledown^{\beta}B^{\alpha} + \tfrac{1}{12}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\alpha}\blacktriangledown_{\gamma}B_{\beta}$$
$$+\tfrac{7}{90}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\alpha}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\delta} - \tfrac{1}{18}\blacktriangledown_{\alpha}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}T[\blacktriangledown]^{\alpha\beta\gamma} + \tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\alpha}\blacktriangledown_{\delta}T[\blacktriangledown]_{\beta\gamma}{}^{\delta}$$
$$-\tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\alpha}\blacktriangledown_{\delta}T[\blacktriangledown]^{\delta}{}_{\beta\gamma} - \tfrac{1}{12}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\beta}B_{\alpha} - \tfrac{1}{8}B_{\alpha}B^{\alpha}\blacktriangledown_{\beta}B^{\beta} + \tfrac{1}{8}\blacktriangledown_{\alpha}B^{\alpha}\blacktriangledown_{\beta}B^{\beta}$$
$$-\tfrac{1}{18}B^{\alpha}\blacktriangledown_{\beta}R[\blacktriangledown]_{\alpha}{}^{\beta} + \tfrac{1}{18}B^{\alpha}\blacktriangledown_{\beta}R[\blacktriangledown]^{\beta}{}_{\alpha} - \tfrac{1}{36}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\beta}R[\blacktriangledown] + \tfrac{7}{108}R[\blacktriangledown]^{\alpha\beta\gamma\delta}\blacktriangledown_{\beta}T[\blacktriangledown]_{\alpha\gamma\delta}$$
$$-\tfrac{1}{72}B^{\alpha}T[\blacktriangledown]^{\beta\gamma\delta}\blacktriangledown_{\beta}T[\blacktriangledown]_{\alpha\gamma\delta} - \tfrac{1}{36}R[\blacktriangledown]\blacktriangledown_{\beta}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta} + \tfrac{1}{6}X\blacktriangledown_{\beta}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}$$
$$-\tfrac{1}{72}B^{\alpha}T[\blacktriangledown]^{\beta\gamma\delta}\blacktriangledown_{\beta}T[\blacktriangledown]_{\gamma\alpha\delta} + \tfrac{1}{40}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_{\beta}T[\blacktriangledown]_{\gamma\delta\zeta} + \tfrac{5}{54}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\beta}T[\blacktriangledown]^{\gamma}{}_{\alpha\gamma}$$
$$+\tfrac{7}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_{\beta}T[\blacktriangledown]_{\delta\gamma\zeta} + \tfrac{1}{6}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\beta}X + \tfrac{1}{12}\blacktriangledown_{\beta}\blacktriangledown_{\alpha}R[\blacktriangledown]^{\alpha\beta}$$
$$-\tfrac{1}{12}\blacktriangledown_{\beta}\blacktriangledown_{\alpha}\blacktriangledown^{\beta}B^{\alpha} - \tfrac{1}{12}B^{\alpha}\blacktriangledown_{\beta}\blacktriangledown^{\beta}B_{\alpha} + \tfrac{1}{12}\blacktriangledown_{\beta}\blacktriangledown^{\beta}\blacktriangledown_{\alpha}B^{\alpha} - \tfrac{1}{12}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\beta}\blacktriangledown_{\gamma}B^{\gamma}$$
$$+\tfrac{1}{36}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\beta}\blacktriangledown_{\delta}T[\blacktriangledown]^{\gamma}{}_{\gamma}{}^{\delta} - \tfrac{1}{24}\blacktriangledown_{\alpha}B_{\beta}\blacktriangledown^{\beta}B^{\alpha} + \tfrac{1}{12}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}\blacktriangledown^{\beta}B^{\alpha}$$
$$-\tfrac{1}{24}\blacktriangledown_{\beta}B_{\alpha}\blacktriangledown^{\beta}B^{\alpha} - \tfrac{1}{12}\blacktriangledown_{\beta}T[\blacktriangledown]^{\gamma}{}_{\alpha\gamma}\blacktriangledown^{\beta}B^{\alpha} + \tfrac{1}{12}B^{\alpha}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}\blacktriangledown_{\gamma}B_{\alpha} - \tfrac{1}{12}B^{\alpha}T[\blacktriangledown]_{\alpha}{}^{\beta\gamma}\blacktriangledown_{\gamma}B_{\beta}$$
$$+\tfrac{1}{18}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}R[\blacktriangledown]_{\beta\alpha} - \tfrac{1}{72}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\gamma}R[\blacktriangledown]_{\beta}{}^{\gamma} - \tfrac{1}{24}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\gamma}R[\blacktriangledown]^{\gamma}{}_{\beta}$$
$$-\tfrac{1}{72}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\alpha\beta}{}^{\gamma} + \tfrac{43}{540}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\alpha\delta\zeta}$$
$$+\tfrac{1}{360}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\alpha\delta\zeta} - \tfrac{1}{30}T[\blacktriangledown]_{\alpha}{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\beta\delta\zeta}$$
$$+\tfrac{3}{80}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\beta\delta\zeta} + \tfrac{1}{24}B_{\alpha}B^{\alpha}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma} - \tfrac{1}{12}\blacktriangledown_{\alpha}B^{\alpha}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}$$
$$+\tfrac{7}{216}R[\blacktriangledown]^{\alpha\beta}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\gamma}{}_{\alpha\beta} + \tfrac{1}{6}\blacktriangledown^{\beta}B^{\alpha}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\gamma}{}_{\alpha\beta} + \tfrac{227}{2160}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\delta\alpha\zeta}$$
$$-\tfrac{1}{90}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_{\gamma}T[\blacktriangledown]_{\delta\beta\zeta} + \tfrac{1}{24}B^{\alpha}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\delta}{}_{\alpha\delta}$$
$$-\tfrac{1}{18}B^{\alpha}T[\blacktriangledown]_{\alpha}{}^{\beta\gamma}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\delta} + \tfrac{5}{72}B^{\alpha}T[\blacktriangledown]^{\beta}{}_{\alpha}{}^{\gamma}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\delta}$$
$$-\tfrac{1}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\zeta}{}_{\delta\zeta} - \tfrac{1}{18}B^{\alpha}\blacktriangledown_{\gamma}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}$$
$$-\tfrac{1}{30}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\delta}{}_{\beta\delta} - \tfrac{1}{36}\blacktriangledown_{\gamma}\blacktriangledown_{\alpha}\blacktriangledown_{\beta}T[\blacktriangledown]^{\alpha\beta\gamma} + \tfrac{1}{12}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}B_{\alpha}$$
$$+\tfrac{1}{12}B^{\alpha}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}T[\blacktriangledown]_{\alpha}{}^{\beta\gamma} + \tfrac{1}{18}B^{\alpha}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}T[\blacktriangledown]^{\beta}{}_{\alpha}{}^{\gamma} + \tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}T[\blacktriangledown]^{\delta}{}_{\alpha\delta}$$
$$-\tfrac{1}{36}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}\blacktriangledown_{\alpha}T[\blacktriangledown]^{\alpha\beta\gamma} + \tfrac{1}{36}\blacktriangledown_{\gamma}\blacktriangledown_{\beta}\blacktriangledown^{\gamma}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta} - \tfrac{1}{18}B^{\alpha}\blacktriangledown_{\gamma}\blacktriangledown^{\gamma}T[\blacktriangledown]^{\beta}{}_{\alpha\beta}$$
$$-\tfrac{1}{36}\blacktriangledown_{\gamma}\blacktriangledown^{\gamma}\blacktriangledown_{\beta}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta} - \tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\delta}T[\blacktriangledown]_{\alpha\beta}{}^{\delta} - \tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\delta}T[\blacktriangledown]_{\beta\alpha}{}^{\delta}$$
$$-\tfrac{1}{45}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_{\gamma}\blacktriangledown_{\delta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta} + \tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\gamma}\blacktriangledown_{\delta}T[\blacktriangledown]^{\delta}{}_{\alpha\beta}$$
$$+\tfrac{5}{108}\blacktriangledown_{\beta}T[\blacktriangledown]^{\delta}{}_{\gamma\delta}\blacktriangledown^{\gamma}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta} - \tfrac{31}{540}\blacktriangledown_{\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\delta}\blacktriangledown^{\gamma}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}$$
$$-\tfrac{1}{24}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\gamma}\blacktriangledown_{\delta}B_{\alpha} - \tfrac{1}{12}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}\blacktriangledown_{\delta}B_{\gamma} - \tfrac{1}{48}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_{\delta}B^{\delta}$$

$$\begin{aligned}
&-\tfrac{1}{24}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma}\blacktriangledown_\delta B^\delta-\tfrac{1}{24}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}\blacktriangledown_\delta B^\delta-\tfrac{5}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta R[\blacktriangledown]_{\alpha\beta\gamma}{}^{\delta}\\
&-\tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta R[\blacktriangledown]_{\alpha}{}^{\delta}{}_{\beta\gamma}-\tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta R[\blacktriangledown]_{\beta\gamma\alpha}{}^{\delta}-\tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta R[\blacktriangledown]_{\beta}{}^{\delta}{}_{\alpha\gamma}\\
&+\tfrac{1}{54}R[\blacktriangledown]^{\alpha\beta\gamma\delta}\blacktriangledown_\delta T[\blacktriangledown]_{\alpha\beta\gamma}+\tfrac{7}{72}B^\alpha T[\blacktriangledown]^{\beta\gamma\delta}\blacktriangledown_\delta T[\blacktriangledown]_{\alpha\beta\gamma}+\tfrac{1}{360}\blacktriangledown_\beta T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]_{\alpha\gamma}{}^{\delta}\\
&+\tfrac{1}{36}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_\delta T[\blacktriangledown]_{\alpha\gamma\zeta}-\tfrac{1}{120}\blacktriangledown_\alpha T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\\
&-\tfrac{1}{72}\blacktriangledown^\gamma T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta T[\blacktriangledown]_{\beta\gamma}{}^{\delta}-\tfrac{1}{12}R[\blacktriangledown]^{\alpha\beta\gamma\delta}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\alpha\beta}+\tfrac{1}{72}B^\alpha T[\blacktriangledown]^{\beta\gamma\delta}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\alpha\beta}\\
&-\tfrac{1}{18}B^\alpha T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\alpha}{}^{\delta}+\tfrac{1}{72}\blacktriangledown_\beta T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\alpha}{}^{\delta}\\
&+\tfrac{13}{360}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\alpha\zeta}-\tfrac{1}{72}B^\alpha T[\blacktriangledown]^{\beta}{}_{\alpha}{}^{\gamma}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\beta}{}^{\delta}\\
&+\tfrac{1}{180}\blacktriangledown^\gamma T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta T[\blacktriangledown]_{\gamma\beta}{}^{\delta}+\tfrac{1}{72}\blacktriangledown_\beta T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta T[\blacktriangledown]^{\gamma}{}_{\gamma}{}^{\delta}\\
&-\tfrac{1}{24}B^\alpha T[\blacktriangledown]^{\beta}{}_{\beta}{}^{\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\delta}{}_{\alpha\gamma}-\tfrac{1}{72}B^\alpha T[\blacktriangledown]_{\alpha}{}^{\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\delta}{}_{\beta\gamma}+\tfrac{1}{72}B^\alpha T[\blacktriangledown]^{\beta}{}_{\alpha}{}^{\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\delta}{}_{\beta\gamma}\\
&-\tfrac{1}{135}\blacktriangledown_\alpha T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\delta}{}_{\beta\gamma}-\tfrac{7}{540}\blacktriangledown^\gamma T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta T[\blacktriangledown]^{\delta}{}_{\beta\gamma}\\
&-\tfrac{1}{40}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\alpha\zeta}-\tfrac{17}{720}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\alpha\zeta}\\
&+\tfrac{1}{60}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\gamma}{}^{\delta}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\beta\zeta}-\tfrac{1}{180}T[\blacktriangledown]_{\alpha\beta}{}^{\delta}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\gamma\zeta}\\
&-\tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha}{}^{\delta}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\gamma\zeta}+\tfrac{19}{270}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\gamma\zeta}\\
&-\tfrac{13}{720}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\blacktriangledown_\delta T[\blacktriangledown]^{\zeta}{}_{\gamma\zeta}-\tfrac{1}{45}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown_\alpha T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\\
&-\tfrac{1}{180}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown_\alpha T[\blacktriangledown]^{\delta}{}_{\beta\gamma}+\tfrac{1}{20}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown_\gamma T[\blacktriangledown]_{\alpha\beta}{}^{\delta}\\
&+\tfrac{1}{36}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta\blacktriangledown_\gamma T[\blacktriangledown]_{\beta}{}^{\gamma\delta}+\tfrac{1}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta\blacktriangledown_\gamma T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\\
&-\tfrac{11}{120}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown_\gamma T[\blacktriangledown]^{\delta}{}_{\alpha\beta}-\tfrac{1}{30}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown^\delta T[\blacktriangledown]_{\alpha\beta\gamma}\\
&-\tfrac{17}{360}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\delta\blacktriangledown^\delta T[\blacktriangledown]_{\beta\alpha\gamma}+\tfrac{1}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}\blacktriangledown_\delta\blacktriangledown^\delta T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}\\
&-\tfrac{37}{540}\blacktriangledown_\alpha T[\blacktriangledown]_{\beta\gamma\delta}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}-\tfrac{1}{135}\blacktriangledown_\alpha T[\blacktriangledown]_{\delta\beta\gamma}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}\\
&-\tfrac{17}{270}\blacktriangledown_\gamma T[\blacktriangledown]_{\alpha\beta\delta}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}-\tfrac{13}{216}\blacktriangledown_\gamma T[\blacktriangledown]_{\beta\alpha\delta}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}\\
&+\tfrac{11}{540}\blacktriangledown_\delta T[\blacktriangledown]_{\alpha\beta\gamma}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}+\tfrac{17}{1080}\blacktriangledown_\delta T[\blacktriangledown]_{\beta\alpha\gamma}\blacktriangledown^\delta T[\blacktriangledown]^{\alpha\beta\gamma}\\
&+\tfrac{23}{135}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\alpha\gamma\delta}-\tfrac{13}{180}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\alpha\gamma\delta}\\
&-\tfrac{1}{60}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta\gamma}\blacktriangledown_\zeta T[\blacktriangledown]_{\alpha\delta}{}^{\zeta}+\tfrac{1}{180}T[\blacktriangledown]_{\alpha}{}^{\delta\zeta}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\zeta T[\blacktriangledown]_{\beta\gamma\delta}\\
&+\tfrac{7}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\beta\gamma\delta}+\tfrac{13}{720}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\gamma}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\beta\delta}{}^{\zeta}\\
&+\tfrac{37}{270}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\alpha\delta}+\tfrac{5}{144}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]^{\delta}{}_{\beta}{}^{\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\alpha\delta}\\
&+\tfrac{1}{60}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\beta\delta}+\tfrac{2}{45}T[\blacktriangledown]_{\alpha\beta}{}^{\delta}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}\\
&-\tfrac{23}{720}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}-\tfrac{1}{30}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}\\
&-\tfrac{1}{180}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\gamma\delta}{}^{\zeta}-\tfrac{257}{2160}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta}{}^{\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\delta\alpha\gamma}\\
&-\tfrac{1}{90}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\delta\alpha}{}^{\zeta}-\tfrac{1}{120}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma\delta\zeta}\blacktriangledown_\zeta T[\blacktriangledown]_{\delta\beta\gamma}\\
&-\tfrac{1}{40}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]_{\delta\gamma}{}^{\zeta}+\tfrac{1}{144}T[\blacktriangledown]_{\alpha\beta\gamma}T[\blacktriangledown]^{\alpha\beta\gamma}\blacktriangledown_\zeta T[\blacktriangledown]^{\delta}{}_{\delta}{}^{\zeta}\\
&+\tfrac{1}{72}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\alpha\gamma}\blacktriangledown_\zeta T[\blacktriangledown]^{\delta}{}_{\delta}{}^{\zeta}+\tfrac{1}{72}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta\gamma}\blacktriangledown_\zeta T[\blacktriangledown]^{\delta}{}_{\delta}{}^{\zeta}\\
&-\tfrac{11}{90}T[\blacktriangledown]^{\alpha\beta\gamma}T[\blacktriangledown]_{\beta\gamma}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]^{\zeta}{}_{\alpha\delta}+\tfrac{2}{135}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]_{\beta}{}^{\gamma\delta}\blacktriangledown_\zeta T[\blacktriangledown]^{\zeta}{}_{\gamma\delta}\\
&-\tfrac{7}{360}T[\blacktriangledown]^{\alpha}{}_{\alpha}{}^{\beta}T[\blacktriangledown]^{\gamma}{}_{\beta}{}^{\delta}\blacktriangledown_\zeta T[\blacktriangledown]^{\zeta}{}_{\gamma\delta}\,. \qquad\qquad \text{(D.52)}
\end{aligned}$$

Appendix E

Other results for the scalar-tensor model

E.1 Subleading corrections to singular behavior

We list here the coefficients A_1 and u_2 of the firsts subleading corrections in the Taylor expansion of the solution around the singularity, for $d = 4$ and generic ω.
In the Feynman gauge

$$
\begin{aligned}
&\mathrm{A}_1(A, u_0, u_1, u_2) = \\
&\{-864A^4\pi^2\tilde{\phi}_c^7 + 27A^4(128\pi^2u_0 + 1)\tilde{\phi}_c^6 - 18A^2(3A^2u_0(96\pi^2u_0 + 1) - \\
&4(2\omega + 3))\tilde{\phi}_c^5 + (27u_0^2(128\pi^2u_0 + 1)A^4 - 288(2\omega + 3)u_0A^2 + 16(2\omega + 3)^3)\tilde{\phi}_c^4 - \\
&8u_0(108\pi^2u_0^3A^4 - 9(10\omega + 11)u_0A^2 + 16(\omega + 1)(2\omega + 3)^2)\tilde{\phi}_c^3 - 16(2\omega + 1)u_0^2(9A^2u_0 - \\
&4(2\omega + 3)(3\omega + 4))\tilde{\phi}_c^2 - 64(2\omega + 1)^2(2\omega + 3)u_0^3\tilde{\phi}_c - 4(32\tilde{\phi}_c^3u_1^6 - 64\tilde{\phi}_c^2(2\tilde{\phi}_c + u_0)u_1^5 + \\
&16\tilde{\phi}_c((2\omega + 11)\tilde{\phi}_c^2 + 4(\tilde{\phi}_c - u_0)u_2\tilde{\phi}_c^2 + 20u_0\tilde{\phi}_c - (2\omega + 1)u_0^2)u_1^4 + \tilde{\phi}_c(-63A^2\tilde{\phi}_c^3 + \\
&(63A^2\tilde{\phi}_c - 256(\omega + 3))u_0\tilde{\phi}_c + 128(u_0 - \tilde{\phi}_c)(\tilde{\phi}_c + u_0)u_2\tilde{\phi}_c + 128(2\omega + 1)u_0^2)u_1^3 + ((33(3A^2\tilde{\phi}_c - 8) - \\
&32\omega(\omega + 7))\tilde{\phi}_c^3 + 32(2\omega + 3)(2\omega + 11)u_0\tilde{\phi}_c^2 - (99\tilde{\phi}_cA^2 + 32\omega(5\omega + 23) + 328)u_0^2\tilde{\phi}_c + \\
&32(u_0 - \tilde{\phi}_c)u_2((u_0 - \tilde{\phi}_c)u_2\tilde{\phi}_c^2 + u_0(-2\omega\tilde{\phi}_c - 13\tilde{\phi}_c + 2\omega u_0 + u_0))\tilde{\phi}_c + 16(2\omega + 1)^2u_0^3)u_1^2 + \\
&(32(2\omega + 3)^2\tilde{\phi}_c^3 + (-9(4\omega + 25)\tilde{\phi}_cA^2 - 16(2\omega + 3)(14\omega + 17))u_0\tilde{\phi}_c^2 + (9(8\omega + 27)\tilde{\phi}_cA^2 + \\
&128(4\omega(\omega + 2) + 3))u_0^2\tilde{\phi}_c - 2(u_0 - \tilde{\phi}_c)u_2(-9A^2\tilde{\phi}_c^3 + (9u_0A^2 + 64\omega + 96)\tilde{\phi}_c^2 - 96(2\omega + 3)u_0\tilde{\phi}_c + \\
&32u_0(u_0 - \tilde{\phi}_c)u_2\tilde{\phi}_c + 64(2\omega + 1)u_0^2)\tilde{\phi}_c - 6(2\omega + 1)(3\tilde{\phi}_cA^2 + 16\omega + 8)u_0^3)u_1 + \\
&2(u_0 - \tilde{\phi}_c)u_2((8(2\omega + 3)^2 - 9A^2u_0)\tilde{\phi}_c^3 + u_0(9A^2u_0 - 8(2\omega + 3)(6\omega + 7))\tilde{\phi}_c^2 + \\
&24(4\omega(\omega + 2) + 3)u_0^2\tilde{\phi}_c - 8(u_0 - \tilde{\phi}_c)((2\omega + 3)\tilde{\phi}_c^2 - 2(2\omega + 3)u_0\tilde{\phi}_c + (2\omega + 1)u_0^2)u_2\tilde{\phi}_c - \\
&8(2\omega + 1)^2u_0^3))\tilde{\phi}_c + 16(2\omega + 1)^3u_0^4\}/(45A\tilde{\phi}_c^2(u_0 - \tilde{\phi}_c)^2((-24A^2\pi^2\tilde{\phi}_c^2 + 2\omega + 3)\tilde{\phi}_c^2 + \\
&2(24A^2\pi^2\tilde{\phi}_c^2 - 2\omega - 3)u_0\tilde{\phi}_c - 2u_1(\tilde{\phi}_cu_1 - 2u_0)\tilde{\phi}_c + (-24A^2\pi^2\tilde{\phi}_c^2 + 2\omega + 1)u_0^2))\,.
\end{aligned}
\tag{E.1}
$$

$$
\begin{aligned}
&\mathrm{u}_2(A,u_0,u_1) = \\
&\{-4u_0\tilde{\phi}_c^2(9A^2\tilde{\phi}_c+8\omega(3\omega+8)+42)-u_1\tilde{\phi}_c(\tilde{\phi}_c^2(-9A^2u_0(192\pi^2u_0+1)+64\omega+96)+ \\
&96u_0\tilde{\phi}_c(6\pi^2A^2u_0^2-2\omega-3)+9A^2(192\pi^2u_0+1)\tilde{\phi}_c^3-576\pi^2A^2\tilde{\phi}_c^4+ \\
&16u_1(2u_1\tilde{\phi}_c(u_1\tilde{\phi}_c-2(u_0+\tilde{\phi}_c))-u_0(2\omega u_0-(2\omega+13)\tilde{\phi}_c+u_0))+64(2\omega+1)u_0^2)+ \\
&12u_0^2\tilde{\phi}_c(3A^2\tilde{\phi}_c+8\omega(\omega+2)+6)-8(2\omega+1)^2u_0^3+8(2\omega+3)^2\tilde{\phi}_c^3\}/\{16\tilde{\phi}_c(u_0- \\
&\tilde{\phi}_c)(2u_0\tilde{\phi}_c(36\pi^2A^2\tilde{\phi}_c^2-2\omega-3)+u_0^2(-36\pi^2A^2\tilde{\phi}_c^2+2\omega+1)+\tilde{\phi}_c^2(-36\pi^2A^2\tilde{\phi}_c^2+ \\
&2\omega+3)-2u_1\tilde{\phi}_c(u_1\tilde{\phi}_c-2u_0))\}\,,
\end{aligned}
\tag{E.2}
$$

In the Landau gauge

$$
\begin{aligned}
&\mathrm{u}_2(A,u_0,u_1) = \\
&\{16(\tilde{\phi}_c-u_0)^2(-27u_0\tilde{\phi}_c^2(3A^2\tilde{\phi}_c+8\omega^2+20\omega+12)+18u_0^2\tilde{\phi}_c(3A^2\tilde{\phi}_c+4\omega(2\omega+3))- \\
&32\omega^2u_0^3+27(2\omega+3)^2\tilde{\phi}_c^3)+3u_1\tilde{\phi}_c(2u_0(u_0(\tilde{\phi}_c^2(9(19A^2\tilde{\phi}_c(192\pi^2\tilde{\phi}_c+1)-480)-3392\omega)- \\
&2u_0(128u_0(6\pi^2A^2u_0\tilde{\phi}_c-39\pi^2A^2\tilde{\phi}_c^2+2\omega)+\tilde{\phi}_c(3A^2\tilde{\phi}_c(4288\pi^2\tilde{\phi}_c+11)-1088\omega-864)))+ \\
&9\tilde{\phi}_c^3(A^2(-\tilde{\phi}_c)(2304\pi^2\tilde{\phi}_c+11)+256\omega+384))+9\tilde{\phi}_c^4(3(A^2\tilde{\phi}_c(384\pi^2\tilde{\phi}_c-1)-64)-128\omega))- \\
&256u_0u_1^2\tilde{\phi}_c(\tilde{\phi}_c-u_0)^2(3(\omega+9)\tilde{\phi}_c-2\omega u_0)-768u_1^4\tilde{\phi}_c^3(\tilde{\phi}_c-u_0)^2+768u_1^3\tilde{\phi}_c^2(\tilde{\phi}_c-u_0)^2(2u_0+ \\
&3\tilde{\phi}_c)\}/\{32\tilde{\phi}_c(3\tilde{\phi}_c-2u_0)(\tilde{\phi}_c-u_0)^2(3\tilde{\phi}_c^2(108\pi^2A^2\tilde{\phi}_c^2-6\omega+4u_1^2-9)+12u_0\tilde{\phi}_c(-36\pi^2A^2\tilde{\phi}_c^2+ \\
&2\omega-2u_1+3)-8u_0^2(\omega-18\pi^2A^2\tilde{\phi}_c^2))\}\,,
\end{aligned}
\tag{E.3}
$$

$$
\begin{aligned}
&\mathrm{A}_1(A,u_0,u_1,u_2,u_3) = \\
&\Big\{\Big(-\frac{4(3\tilde{\phi}_c(2\omega+u_2\tilde{\phi}_c-2u_1+3)-2\omega u_0)(\tilde{\phi}_c(6\omega+6u_2\tilde{\phi}_c+4(u_1-3)u_1+9)-4u_0(\omega+u_2\tilde{\phi}_c))^2}{27\pi^2A^3\tilde{\phi}_c^3(3\tilde{\phi}_c-2u_0)^3}+ \\
&\frac{2(\tilde{\phi}_c(6\omega+6u_2\tilde{\phi}_c+4(u_1-3)u_1+9)-4u_0(\omega+u_2\tilde{\phi}_c))^3}{9\pi^2A^3\tilde{\phi}_c^2(3\tilde{\phi}_c-2u_0)^4}+\frac{3\tilde{\phi}_c(5A^2+4u_3(3\tilde{\phi}_c-2u_0)-16u_2)}{16\pi^2A(3\tilde{\phi}_c-2u_0)^2}+ \\
&\frac{2\omega u_1-9u_3\tilde{\phi}_c^2+12u_2\tilde{\phi}_c}{36\pi^2A\tilde{\phi}_c^2-24\pi^2Au_0\tilde{\phi}_c}-\frac{(7u_1-12)(\tilde{\phi}_c(6\omega+6u_2\tilde{\phi}_c+4(u_1-3)u_1+9)-4u_0(\omega+u_2\tilde{\phi}_c))}{2\pi^2A(3\tilde{\phi}_c-2u_0)^3}+ \\
&\frac{(7u_1-12)(3\tilde{\phi}_c(2\omega+u_2\tilde{\phi}_c-2u_1+3)-2\omega u_0)}{6\pi^2A\tilde{\phi}_c(3\tilde{\phi}_c-2u_0)^2}+\frac{4u_0(\omega+u_2\tilde{\phi}_c)-\tilde{\phi}_c(6\omega+6u_2\tilde{\phi}_c+4(u_1-3)u_1+9)}{\pi^2A\tilde{\phi}_c(3\tilde{\phi}_c-2u_0)^2}- \\
&\frac{5A\tilde{\phi}_c}{32\pi^2(\tilde{\phi}_c-u_0)^2}-\frac{8u_1(\omega-3u_2\tilde{\phi}_c)}{16\pi^2A(3\tilde{\phi}_c-2u_0)^2}4A\Big)\,24\pi^2A^2\tilde{\phi}_c(3\tilde{\phi}_c-2u_0)^2\Big\}/\{5(3\tilde{\phi}_c^2(72\pi^2A^2\tilde{\phi}_c^2-6\omega+ \\
&4u_1^2-9)+12u_0\tilde{\phi}_c(-24\pi^2A^2\tilde{\phi}_c^2+2\omega-2u_1+3)-8u_0^2(\omega-12\pi^2A^2\tilde{\phi}_c^2))\}\,.
\end{aligned}
\tag{E.4}
$$

E.2 The two-dimensional case

In two dimensions the fixed point equations in both gauges reduce to ω-independent first order equations. The analysis is thus quite different in this case, it is actually much easier, and we can proceed mostly by analytical means.

Explicitly, the equations in $d = 2$ reduce to

$$\frac{\tilde{V}(\tilde{\phi}) + (\tilde{\phi} - 2\tilde{V}(\tilde{\phi}))\tilde{V}'(\tilde{\phi})}{2\pi(\tilde{\phi} - \tilde{V}(\tilde{\phi}))\left(1 - \tilde{V}'(\tilde{\phi})\right)} - 2\tilde{V}(\tilde{\phi}) = 0\,, \tag{E.5}$$

for the Feynman gauge, and to

$$\frac{\tilde{V}'(\tilde{\phi})}{2\,\pi\,\left(1 - \tilde{V}'(\tilde{\phi})\right)} - 2\tilde{V}(\tilde{\phi}) = 0\,, \tag{E.6}$$

for the Landau gauge. Both equations can be easily integrated, leading to algebraic equations implicitly defining the solution $\tilde{V}(\tilde{\phi})$. As equation (E.5) is slightly more complicated to study than equation (E.6), but at the end it leads to very similar results, we will present the explicit analysis only for the Landau gauge.

Equation (E.6) can be integrated to give the algebraic relation

$$\tilde{V} - y_0 + \frac{1}{4\pi}\log\left(\tilde{V}/y_0\right) = \tilde{\phi} - \tilde{\phi}_0\,, \tag{E.7}$$

whose solution is by definition expressed in terms of the Lambert function W,

$$\tilde{V}(\tilde{\phi}) = \frac{W\left(4\,\pi\,e^{C+4\,\pi\,\tilde{\phi}}\right)}{4\,\pi}\,. \tag{E.8}$$

The constant of integration $C = 4\pi(v_0 - \tilde{\phi}_0) + \log y_0$ parametrizes a one-parameter family of global solutions, which hence are all acceptable fixed points. Some typical plots of such solutions are presented in Fig. E.1. The asymptotic behavior of the Lambert function is such that $\tilde{V}(\tilde{\phi}) \sim \tilde{\phi}$ for $\tilde{\phi} \to +\infty$, and $\tilde{V}(\tilde{\phi}) \sim e^{4\pi\tilde{\phi}+C}$ for $\tilde{\phi} \to -\infty$. We can study the linear perturbations around such fixed points, by writing as usual

$$\tilde{V}_k(\tilde{\phi}) = \tilde{V}(\tilde{\phi}) + \epsilon v(\tilde{\phi}) e^{-\lambda t}\,, \tag{E.9}$$

with $\tilde{V}(\tilde{\phi})$ given by (E.8). Expanding to first order in ϵ, we find the eigenvalue equation

$$(2 - \lambda)v(\tilde{\phi}) = \frac{\left(1 + W\left(4\pi e^{C+4\pi\tilde{\phi}}\right)\right)^2}{2\pi}\, v'(\tilde{\phi})\,, \tag{E.10}$$

whose solutions are

$$v(\tilde{\phi}) = A\left(\frac{W\left(4\pi e^{C+4\pi\tilde{\phi}}\right)}{W\left(4\pi e^{C+4\pi\tilde{\phi}}\right) + 1}\right)^{\frac{2-\lambda}{2}} = A\left(\frac{d}{d\tilde{\phi}}\tilde{V}(\tilde{\phi})\right)^{\frac{2-\lambda}{2}}\,, \tag{E.11}$$

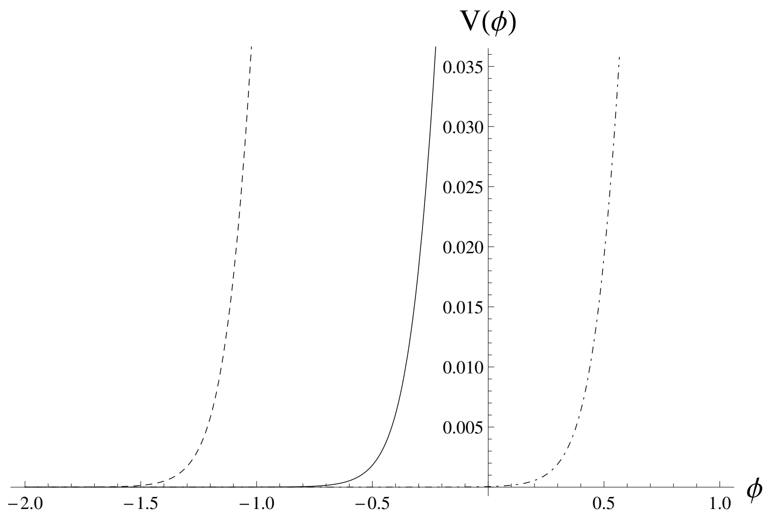

Figure E.1: Fixed point solutions (E.8) for $d = 2$ in the Landau gauge for $C = 10$ (dashed), $C = 0$ (solid) and $C = -10$ (dot-dashed).

A being an arbitrary normalization constant, which we can fix to one. Given the exponential fall-off at $\tilde{\phi} \sim -\infty$ of the fixed point solution $\tilde{V}(\tilde{\phi})$, we see that we must impose the constraint $\lambda \leq 2$ in order to avoid exponentially growing perturbations. Indeed the asymptotic behavior of the eigenperturbations is $v(\tilde{\phi}) \sim 1 - \frac{2-\lambda}{2}(4\pi\tilde{\phi})^{-1}$ for $\tilde{\phi} \to +\infty$, and $v(\tilde{\phi}) \sim (4\pi)^{\frac{2-\lambda}{2}} e^{\frac{2-\lambda}{2}(4\pi\tilde{\phi}+C)}$ for $\tilde{\phi} \to -\infty$. Apart from the upper bound on λ, we do not have other restrictions, hence the perturbations form a continuous spectrum.

Appendix F

Anisotropic Weyl invariance

Anisotropic Weyl invariance for a generic anisotropic theory in d dimensions and general dynamical critical exponent $z = d$ is defined as the invariance of the action under the set of anisotropic transformations ((2.101) and (2.102)) which read

$$g_{ij} \to e^{2\,\phi(t,\mathbf{x})}\, g_{ij}\,, \qquad N \to e^{z\,\phi(t,\mathbf{x})}\, N\,, \qquad N_i \to e^{2\,\phi(t,\mathbf{x})}\, N_i\,, \tag{F.1}$$

where g is the spatial metric, N is the lapse function and N_i the shift vector, and $\phi(t, \mathbf{x})$ an arbitrary local function. The transformations (F.1) defines a symmetry group $\mathrm{Weyl}_z(\mathcal{M}, \mathcal{F})$, being $\mathcal{M}$ the $d+1$ dimensional manifold and $\mathcal{F}$ the foliation, which extends the foliation-preserving diffeomorphisms group $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ into a semi-direct product

$$\mathrm{Weyl}_z(\mathcal{M}, \mathcal{F}) \rtimes \mathrm{Diff}_{\mathcal{F}}(\mathcal{M})\,, \tag{F.2}$$

which algebra is defined by the non commutation of the generators of the two groups group, respectively the generator of diffeomorphisms δ_{f,ζ^i}, being f and ζ^i the time and space reparameterization functions defined in (2.84), and the generator of anisotropic conformal transformation δ_ϕ, where the commutation leads to another infinitesimal Weyl transformation

$$[\delta_{f,\zeta^i}, \delta_\phi] = \delta_{f\,\partial_t\,\phi + \zeta^i\,\partial_i\,\phi}\,. \tag{F.3}$$

Since the transformations (F.1) requires the lapse function N to be space-dependent, Weyl invariance is not obtained in the projectable case, in which the lapse function is a constant function over the leaf.

It can be demonstrated that Weyl invariance requires the parameter λ in the kinetic part of the action to be equal to

$$\lambda = \frac{1}{d}\,. \tag{F.4}$$

To prove it, let us consider the kinetic action written in terms of the DeWitt metric

$$S_K[N, N_i, g_{ij}] = \int dt\, d^2x\, \sqrt{g} N\, (K_{ij}\, \mathcal{G}^{ijkl}\, K_{kl})\,, \tag{F.5}$$

where the extrinsic curvature tensor K_{ij} reads

$$K_{ij} = \frac{1}{N} \left(\partial_t \, g_{ij} + D_i \, N_j + D_j \, N_i \right) . \tag{F.6}$$

Using $\delta_\phi \, g_{ij} = 2 \, \phi \, g_{ij}$ we have for the time derivative of the spatial metric

$$\delta_\phi \left(\partial_t \, g_{ij} \right) = 2 \left(\partial_t \, \phi \right) g_{ij} + 2 \, \phi \, \partial_t \, g_{ij} \, . \tag{F.7}$$

Contracting the first operator in the right hand term of (F.7) with the DeWitt tensor (2.89) we obtain

$$2 \left(\partial_t \, \phi \right) g_{ij} \, \mathcal{G}^{ijk\ell} = 2 \left(\partial_t \, \phi \right) g_{ij} \, \left\{ \frac{1}{2} \left(g^{ik} g^{j\ell} + g^{i\ell} g^{jk} \right) - \lambda g^{ij} g^{k\ell} \right\} = 2 \left(\partial_t \, \phi \right) \left(1 - \lambda \, d \right) g^{k\ell} \, . \tag{F.8}$$

For the covariant derivative of the shift vector, using $\delta_\phi \, N_i = 2 \, \phi \, N_i$, we have

$$\begin{aligned} \delta_\phi \left(D_i \, N_j \right) &= \delta_\phi \, \left(\partial_i \, N_j + \frac{1}{2} \, N^k \left(\partial_j \, g_{ik} + \partial_i \, g_{jk} - \partial_k \, g_{ij} \right) \right) \\ &= 2 \, \phi \, D_i \, N_j + N_j \, \partial_i \, \phi - N_i \, \partial_j \, \phi + g_{ij} \, g^{kl} \, N_k \, \partial_l \, \phi \, , \end{aligned} \tag{F.9}$$

that contracting with the DeWitt metric leads to

$$\mathcal{G}^{ijk\ell} \, \delta_\phi \left(D_i \, N_j \right) = 2 \, \phi \, \mathcal{G}^{ijk\ell} D_i \, N_j + \left(1 - d \, \lambda \right) g^{k\ell} \, g^{ij} \, N_i \, \partial_j \phi \, . \tag{F.10}$$

It is then easy to see from (F.8) and (F.10) that for $\lambda = 1/d$ all the derivatives of the local function ϕ vanish, enhancing anisotropic conformal symmetry, at least at classical level. We expect in fact quantum contribution to break conformal symmetry, i.e. that the quantized theory contains an anisotropic Weyl anomaly which characterizes the non invariance under a Weyl transformation of the effective action. In the case of the Lifshitz scalar in $d = 2$ and $z = 2$ the Weyl anomaly has already been evaluated at one loop in [129].

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