UNIVERSITÀ DEGLI STUDI ROMA TRE

DOTTORATO DI RICERCA IN MATEMATICA
XXXII CICLO

Boundary correlation functions for non exactly solvable critical Ising models

Candidata:
Giulia Rosalba Cava

Relatore di tesi:
Prof. Alessandro Giuliani

Coordinatore:
Prof. Angelo Felice Lopez

Contents

1 Introduction **5**

1.1 Motivations . . . 5
1.2 The model and the main results . . . 8
1.3 Outline of the proof . . . 10
1.4 Perspectives . . . 12
1.5 Summary . . . 12

2 The Grassmann representation on a cylinder **15**

2.1 The unperturbed partition function . . . 16
2.1.1 The multipolygon representation . . . 17
2.1.2 The dimer representation . . . 18
2.1.3 The Grassmann integration rules . . . 19
2.1.4 The Grassmann representation . . . 20
2.2 The unperturbed generating function . . . 21
2.2.1 The unperturbed bulk and edge energy correlations . . . 21
2.2.2 The unperturbed edge spin correlations . . . 23
2.2.3 The expression in terms of unperturbed Grassmann correlations . 27
2.3 The perturbed generating function . . . 28

3 Massive and massless variables **33**

3.1 Diagonalization of the free action . . . 35
3.2 Critical propagators . . . 38
3.2.1 The massive and massless propagators . . . 38
3.2.2 Asymptotic behaviour . . . 42
3.2.3 Multiscale decomposition . . . 43
3.2.4 Bulk-edge decomposition . . . 44
3.2.5 Decay bounds and Gram decomposition . . . 44
3.3 Effective potential of the generating function . . . 46

4 Multiscale analysis and tree expansions **49**

4.1 Multiscale analysis . . . 49
4.1.1 The massive integration . . . 50
4.1.2 The massless integration . . . 53

4.1.3 Kernels expansion and recursive definitions . . . 54
4.1.4 The Pfaffian formula . . . 56
4.2 Localization and interpolation . . . 57
4.2.1 Localization for the sourceless kernels . . . 61
4.2.2 Localization in presence of spin sources . . . 68
4.2.3 Norms for the kernels of the effective potential . . . 70
4.3 Trees and tree expansions . . . 72
4.3.1 The contributions of the effective potential . . . 72
4.3.2 Tree expansion of the effective potential . . . 77

5 Bounds on the kernels of the effective potential 87
5.1 Bounds on the bulk and edge kernels . . . 88
5.2 Bounds on the spin source kernels . . . 99
5.3 The Beta functions . . . 102

6 The two-point edge spin correlation 105
6.1 The tree expansion . . . 107
6.2 Contributions from the trees with $n = 0$ . . . 108
6.3 The contribution from the trees with $n > 0$ . . . 110

Appendices 114

A The Kasteleyn matrix 115

B The massless propagators in real space 117

C The scaling limit of the massless propagators 123

References 127

Chapter 1

Introduction

1.1 Motivations

The two dimensional Ising model is an oversimplified description of a planar magnet with interactions restricted to neighboring spins. It was the first model to indicate that a microscopic short range interaction produces a phase transition at a critical point, becoming one of the most widely studied classical spin systems [52, 53, 62, 63, 76]. Onsager first provided the exact solution of this model in the absence of an external field, which was then reproduced in many different independent ways [43, 47, 53, 67, 69]. The integrability, i.e. the exact solvability, of the Ising model implies that this spin system can be exactly mapped into a system of free fermions [43, 57, 67]. As a consequence there are explicit computations of the thermodynamic properties of the model, exact results for spin-spin and energy-energy correlation functions [6, 47, 60, 61, 72] and some multi spin correlations [50], as well as for their asymptotic behaviour for large distances. The critical exponents of the model, which characterize the asymptotic behaviour at the critical point, are different from those predicted by the Curie-Weiss theory, hence the Ising model belongs to a different *universality class*. A universality class is a collection of models which display the same critical behaviour at a second order phase transition even if they are defined in terms of different microscopic Hamiltonians, and possibly describe different physical systems. The concept of universality class and the analysis of the critical behaviour were understood on the basis of Renormalization Group (RG) methods [17, 23, 49, 71, 73–75]. In particular, at the critical point the scaling limit is independent of microscopic details, such as the underlying lattice, and it is invariant under uniform changes of length scale. Conformal Field Theory (CFT) generalizes this scale invariance to an invariance under a larger group of *conformal transformations* [7, 25, 26, 26, 27, 29, 31, 46, 48, 65]. The conformal invariance allows one to investigate the nature of the critical regime of many models, including the Ising model, and to explicitly evaluate the finite size effects for the scaling limit of correlations at the critical point [1, 7, 12, 18, 31]. Further progress has been made by reformulating the conformal invariance of the scaling limit in terms of a suitable convergence for conformal invariant interfaces [19, 55, 68]. In this perspective, using techniques based on combinatorics, probability and discrete analysis, several CFT predictions have been rigorously proved for different two dimensional lattice model, including critical site percolation on triangular lattice [70], dimer models [54], and Ising models [11, 20–22, 28, 56, 66]. Unfortunately, most of the results obtained from these recent developments concern exactly solvable models, most notably Ising and dimers. Technically these results are based on discrete holomorphicity, which is an instance of the integrability of the lattice models. Such a restriction is a severe limitation of the method, conceptually quite unsatisfactory: the scaling limit, on the basis of the RG philosophy, should be robust under "irrelevant" perturbations of the microscopic integrable models.

From a physical point of view, there is no special reason to consider only interactions among neighbouring spins. It is natural to consider a perturbed, non-integrable, Ising model

$$H_\lambda = H_0 + \lambda W \,, \tag{1.1}$$

where W is an even finite range interaction among spins and H_0 is the nearest neighbour (n.n.) Hamiltonian.
A method to study a wide class of two dimensional classical spin systems, obtained as perturbations of the Ising model, was first introduced by [64] and [58]. The method is based on the exact mapping of the $\lambda \neq 0$ spin model into a model of interacting fermions in $1+1$ dimensions [44, 69] and on the implementation of the constructive fermionic RG method due to Benfatto, Gallavotti, Mastropietro and collaborators [8–10, 13, 35]. It seems natural to derive the relation between the microscopic non exactly solvable models and the CFT description via RG techniques, as a first step towards the development of rigorous methods which are completely independent of any exact solution of the microscopic model. In this framework, Giuliani, Greenblatt and Mastropietro [37, 39] verified some of the CFT predictions via constructive RG methods for a non integrable Ising model on a torus, which is a translational invariant system. However, in order to prove the full conformal invariance of the scaling limit, a control of the *boundary terms* is also required: it is then necessary to adapt the RG methods to a model on finite domains, which are no longer translational invariant systems. A first attempt was made by Antinucci [3] for interacting fermions on the half line, and later Antinucci, Giuliani and Greenblatt [4] derived the methods for the study of a non integrable Ising model on a cylinder, where the translational invariance is lost in the vertical direction. In their work, they construct the scaling limit of the bulk energy multipoint correlations. In a different framework, via random current representation and critical percolation techniques, the authors of [2] proved the asymptotic emergence of the Pfaffian structure in the multipoint correlations for spins located at the boundary, which is consistent with the expected picture of universality in critical phenomena, for all ferromagnetic pair spin interactions, beyond the weak coupling regime.

In order to state our result more precisely let us recall a few basic properties of the integrable Ising model. If $\lambda = 0$, Onsager's exact solution shows that the free energy is analytic for $\beta \neq \beta_c$, where $\beta_c = J^{-1} \tanh^{-1}(\sqrt{2} - 1)$ is the inverse critical temperature. For $\beta < \beta_c$, the "high temperature state", the spontaneous magnetization is vanishing and the multipoint spin correlations are exponentially decaying as the separation between spins is sent to infinity. For $\beta > \beta_c$, the "low temperature states", the spontaneous magnetization is non zero and the multipoint *truncated* spin correlations are exponentially decaying as the separation between spins is sent to infinity. As $\beta \to \beta_c$, the rate of the exponential decay goes to zero proportionally to $|\beta - \beta_c|$ and at $\beta = \beta_c$ the correlation functions decay polynomially to zero, with specific critical exponents. We can derive the critical exponents in terms of the local operator dimension values. Far away from the boundary the dimension is 1/8 for the spin field operator σ and is 1 for the energy field operator ε, while at the boundary the dimension is 1/2 for the spin field operator $\tilde{\sigma}$ and is 2 for the energy field operator $\tilde{\varepsilon}$. These dimension values are in correspondence with the power law decay of the critical correlations. For instance the m-point truncated spin correlation, if the spins are far away from the boundary, decays as $\langle \sigma_{\mathbf{x}_1}; \cdots ; \sigma_{\mathbf{x}_m} \rangle \sim (\text{const.}) \cdot d^{-m/8}$ asymptotically as the distance d between the spins increases to infinity, while, if the spin are at the boundary, decays as $\langle \tilde{\sigma}_{\mathbf{x}_1}; \cdots ; \tilde{\sigma}_{\mathbf{x}_m} \rangle \sim (\text{const.}) \cdot d^{-m/2}$. In the same way we can obtain the critical exponents for the energy correlation functions, as well as for the mixed correlation functions. Moreover the free fermions representation allows us to derive also the explicit exact expression for the multipoint energy correlations in a closed form, which we can immediately use to control the asymptotic behaviour [24]. On the contrary the determination of an explicit expression of the spin correlation is very involved if far away from the boundary: it is expressed as a determinant of a large matrix, of size increasing with the distance between spins, whose asymptotic behavior along special directions can be obtained via the use of Szego's lemma [60] or via the use of the analyticity structure of a set of exact quadratic difference equations that the correlation satisfies exactly at the lattice level [59]. Instead, if we consider spins at the boundary the explicit expression of the spin correlation is drastically simplified: actually it is possible to evaluate it at any temperature by Pfaffians of the corresponding two point function as first pointed out by [41] and derived in several works thereafter (e.g. see [42]).

It is natural to ask whether these features survive the presence of the perturbation $\lambda \neq 0$. Since a universality property was conjectured for the Ising model, the critical indices should remain unchanged by adding the finite range interaction. In fact, in this thesis we prove that, if $\lambda \neq 0$, the scaling limit of the two-point correlation for spins at the boundary has the same critical indices than the unperturbed scaling limit. Moreover, we prove that the perturbed scaling limit can be identified with the unperturbed scaling limit, up to finite renormalizations: the wave functions and the critical temperature are renormalized by the finite range interaction, while the critical exponents are protected against renormalization.

1.2 The model and the main results

The interacting Ising model on a cylinder

We consider perturbations of the n.n. Ising model where the spins interact via a finite range interaction of the form:

$$H_\lambda(\underline{\sigma}, \mathbf{J}) = H_0(\underline{\sigma}, \mathbf{J}) + \lambda W(\underline{\sigma}) = -\sum_{\mathbf{x}\in\Lambda}\sum_{i=1,2} J_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_i}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i} - \lambda\sum_{X\subset\Lambda} V(X)\prod_{\mathbf{x}\in X}\sigma_{\mathbf{x}}\,, \tag{1.2}$$

where $\Lambda := a\Lambda_{LM}$ with $\Lambda_{LM} = \mathbb{Z}/\big(L\mathbb{Z}\big)\times\big(\mathbb{Z}\cap[1,M]\big)$, $L\in 2\mathbb{N}, M\in\mathbb{N}$, is a cylindrical lattice, a is the lattice spacing, $\sigma_{\mathbf{x}}\in\{\pm1\}$ is a spin variable associated to each $\mathbf{x}=(x^{(1)},x^{(2)})\in\Lambda$, $\sigma_{(x^{(1)},aM+a)}=0$ for the open boundary condition in the vertical direction, $\mathbf{J}=\{J_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_i}\}_{\mathbf{x}\in\Lambda}^{i=1,2}$ is the set of the couplings between nearest neighbour spins and $\hat{\mathbf{e}}_1=(1,0)$, $\hat{\mathbf{e}}_2=(0,1)$ are the two unit coordinate vectors of the lattice; $V(X)$ is a finite range, translationally invariant, even interaction, such that $V(\{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_i\})=0$; the sum in the second term of Eq. (1.2) is over all unordered pairs of sites in Λ and λ is the strength of the interaction, which can be of either signs and, for most of the discussion below, the reader can think of as being small, compared to $J_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_i}$, but independent of the system size.

Observables

Note that in the infinite volume limit $L,M\to\infty$ the cylindrical lattice Λ tends to the discrete upper half-plane $\mathbb{H}_a=\{\mathbf{x}=(x^{(1)},x^{(2)})\in a\mathbb{Z}^2 : x^{(2)}\geq a\}$. If we think of $\mathbb{H}_a$ as the lattice covering the continuum half-plane $\mathbb{H}=\{\mathbf{x}=(x^{(1)},x^{(2)})\in\mathbb{R}^2 : x^{(2)}\geq 0\}$, each point of $\mathbb{H}$ is at distance at most $a/\sqrt{2}$ from a lattice site of $\mathbb{H}_a$. For convenience we choose the convention of associating the continuous coordinate $\mathbf{x}\in\mathbb{H}$ to the nearest lattice site at the top right, namely $\lceil\mathbf{x}\rceil := \lceil\mathbf{x}\rceil^a = (\lceil x^{(1)}\rceil^a, \lceil x^{(2)}\rceil^a)$ and $\lceil x^{(i)}\rceil^a := \min_{n\in a\mathbb{Z}}\{n : n\geq x^{(i)}\}$, $i=1,2$.

We can express the observables on the lattice sites of $\mathbb{H}_a$ in terms of the continuous coordinates of $\mathbb{H}$. Since we are interested in obtaining the scaling limit of the multipoint correlation functions, we must appositely rescale the field operators based on their dimension values. For an interior point $\mathbf{x}\in\mathbb{H}^\circ := \{\mathbf{x}\in\mathbb{R}^2 : x^{(2)}>0\}$, we define the *bulk spin* as $\sigma^{(a)}(\mathbf{x}) := a^{-\frac{1}{8}}\sigma_{\lceil\mathbf{x}\rceil}$ and the *bulk energy density* as $\varepsilon^{(a)}_j(\mathbf{x}) := a^{-1}\sigma_{\lceil\mathbf{x}\rceil}\sigma_{\lceil\mathbf{x}\rceil+a\hat{\mathbf{e}}_j}$, $j=1,2$. For a point at the boundary $\mathbf{x}\in\partial\mathbb{H} := \{\mathbf{x} : x^{(1)}\in\mathbb{R}, x^{(2)}=0\}$, we define the *edge spin* as $\tilde{\sigma}^{(a)}(\mathbf{x}) := \tilde{\sigma}^{(a)}(x^{(1)}) = a^{-\frac{1}{2}}\sigma_{\lceil\mathbf{x}\rceil}$ and the *edge energy density* as $\tilde{\varepsilon}^{(a)}_j(\mathbf{x}) := \tilde{\varepsilon}^{(a)}_j(x^{(1)}) := a^{-2}\sigma_{\lceil\mathbf{x}\rceil}\sigma_{\lceil\mathbf{x}\rceil+a\hat{\mathbf{e}}_j}$, $j=1,2$. As we will see, these are the suitable definitions for having the finite scaling limit of the correlation functions.

Correlation functions

The thermodynamical properties of the system described by (1.2) can be obtained by averaging with respect to the Gibbs measure at inverse temperature β: given $m\geq 1$

observables $F_1(\underline{\sigma}), \cdots, F_m(\underline{\sigma})$, the $m-$point expectation is defined as

$$\langle F_1(\underline{\sigma}) \cdots F_m(\underline{\sigma}) \rangle_{\beta,\Lambda} = \frac{\sum_{\underline{\sigma} \in \Omega_{LM}} F_1(\underline{\sigma}) \dots F_m(\underline{\sigma}) \, e^{-\beta H_\lambda(\underline{\sigma}, \mathbf{J})}}{\sum_{\underline{\sigma} \in \Omega_{LM}} e^{-\beta H_\lambda(\underline{\sigma}, \mathbf{J})}} \,, \tag{1.3}$$

where $\underline{\sigma} = \{\sigma_{\mathbf{x}}\}_{\mathbf{x} \in \Lambda}$ is a spin configuration and $\Omega_{LM} = \{\pm 1\}^{LM}$ is the spin configuration space. By introducing m auxiliary sources $\mathbf{Q} = \{Q_1, \cdots, Q_m\}$, the expectation in (1.3) can be expressed as

$$\langle F_1(\underline{\sigma}) \cdots F_m(\underline{\sigma}) \rangle_{\beta,\Lambda} = \frac{1}{\Xi_\lambda(\mathbf{Q})} \frac{\partial^m}{\partial Q_1 \cdots \partial Q_m} \Xi_\lambda(\mathbf{Q}) \Big|_{\mathbf{Q}=\mathbf{0}} \,, \tag{1.4}$$

where $\Xi_\lambda(\mathbf{Q})$ is the *generating function*, defined as

$$\Xi_\lambda(\mathbf{Q}) := \sum_{\underline{\sigma} \in \Omega_{LM}} e^{-\beta H_\lambda(\underline{\sigma}, \mathbf{J}) + Q_1 F_1(\underline{\sigma}) + \cdots + Q_m F_m(\underline{\sigma})} \,. \tag{1.5}$$

Moreover we let

$$\langle F_1(\underline{\sigma}); \dots; F_m(\underline{\sigma}) \rangle_{\beta,\Lambda} = \frac{\partial^m}{\partial Q_1 \cdots \partial Q_m} \log \Xi_\lambda(\mathbf{Q}) \Big|_{\mathbf{Q}=\mathbf{0}} \,, \tag{1.6}$$

be the *truncated* expectation. Note that we can always write the truncated expectations in terms of simple expectations, for instance if $m = 2$ we get

$$\langle F_1(\underline{\sigma}); F_2(\underline{\sigma}) \rangle_{\beta,\Lambda} = \langle F_1(\underline{\sigma}) F_2(\underline{\sigma}) \rangle_{\beta,\Lambda} - \langle F_1(\underline{\sigma}) \rangle_{\beta,\Lambda} \langle F_2(\underline{\sigma}) \rangle_{\beta,\Lambda} \,. \tag{1.7}$$

In absence of the perturbation $\langle \cdot \rangle_{\beta,\Lambda} |_{\lambda=0} := \langle \cdot \rangle^0_{\beta,\Lambda}$ denotes the unperturbed expectation. Finally, in the infinite volume limit, we let $\lim_{L,M\to\infty} \langle \cdot \rangle_{\beta,\Lambda} := \langle \cdot \rangle_\beta$ and $\lim_{L,M\to\infty} \langle \cdot \rangle^0_{\beta,\Lambda} := \langle \cdot \rangle^0_\beta$ be the perturbed and the unperturbed expectations on the upper half-plane, provided these limits exist. The expectations we are interested in are the expectations of observables at m distinct points on the upper half-plane, which we call the $m-$point correlation functions.

Results

Our main result is summarized in the following theorem.

Theorem 1.1 (The two-point edge spin correlation)**.** *There exists* $\lambda_0 > 0$ *such that, if* $|\lambda| \le \lambda_0$*, the critical temperature* $\beta_c(\lambda)$ *of the Ising model in Eq.* (1.2) *and the renormalization* $Z_\sigma(\lambda) \in \mathbb{R}$ *are analytic functions of* λ*; furthermore* $Z_\sigma(\lambda)$ *is analytically close to* 1 *and independent of* a*. Then, for* $\mathbf{x}_1, \mathbf{x}_2 \in \partial\mathbb{H}$ *distinct boundary points, for all* $\theta \in (0,1)$ *and a suitable constant* $C_\theta > 0$*, the two-point edge spin correlation on the discrete upper half-plane is given by*

$$\langle \tilde{\sigma}^{(a)}(\mathbf{x}_1) \tilde{\sigma}^{(a)}(\mathbf{x}_2) \rangle_{\beta_c(\lambda)} = Z^2_\sigma(\lambda) \, \langle \tilde{\sigma}^{(a)}(\mathbf{x}_1); \tilde{\sigma}^{(a)}(\mathbf{x}_2) \rangle^0_{\beta_c(0)} + R^{(a)}_\sigma(\mathbf{x}_1, \mathbf{x}_2) \,, \tag{1.8}$$

where $\langle \tilde\sigma^{(a)}(\mathbf{x}_1)\tilde\sigma^{(a)}(\mathbf{x}_2)\rangle^0_{\beta_c(0)}$ *is the unperturbed two-point edge spin correlation function and* $R^{(a)}_\sigma$ *is the correction term, which can be bounded as*

$$|R^{(a)}_\sigma(\mathbf{x}_1,\mathbf{x}_2)| \le C_\theta \frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\left(\frac{a}{|\mathbf{x}_1-\mathbf{x}_2|}\right)^\theta . \tag{1.9}$$

As a consequence of Thm. 1.1 we can explicitly compute the scaling limit of the two-point edge spin correlations, obtaining the correlations on the continuous upper half-plane. In particular we have a constructive procedure to compute the amplitude of the correlations and the subdominant corrections, that vanish in the scaling limit in which the lattice spacing a is sent to zero. We can immediately verify that in the scaling limit the finite range interaction changes only the critical temperature and the amplitude of the correlations: the critical exponents and the explicit form of the correlation function are the same as in the integrable case, as stated in the following corollary.

Corollary 1.1 (Correlation in the scaling limit)**.** *Given* $\lambda, \beta_c(\lambda), Z_\sigma(\lambda)$ *and* $\mathbf{x}_1,\mathbf{x}_2 \in \partial\mathbb{H}$ *as in the statement of Theorem 1.1, there exists a truncated correlation function*

$$\langle \tilde\sigma(\mathbf{x}_1);\tilde\sigma(\mathbf{x}_2)\rangle^0_{\beta_c(0)} \in \mathbb{R}\,,$$

such that in the scaling limit we get

$$\lim_{a\to 0}\langle \tilde\sigma^{(a)}(\mathbf{x}_1);\tilde\sigma^{(a)}(\mathbf{x}_2)\rangle_{\beta_c(\lambda)} = Z^2_\sigma(\lambda)\,\langle \tilde\sigma(\mathbf{x}_1);\tilde\sigma(\mathbf{x}_2)\rangle^0_{\beta_c(0)}\ .$$

Note that the explicit form of the limit function $\langle \tilde\sigma(\mathbf{x}_1);\tilde\sigma(\mathbf{x}_2)\rangle^0_{\beta_c(0)}$ is a well known expression (see e.g. [42]):

$$\langle \tilde\sigma(\mathbf{x}_1);\tilde\sigma(\mathbf{x}_2)\rangle^0_{\beta_c(0)} = \left(\frac{2}{\pi(\sqrt{2}-1)}\right)^2\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\,. \tag{1.10}$$

1.3 Outline of the proof

To derive the two-point edge spin correlations we use the expectations in (1.6) and the generating function in (1.5) with $\mathbf{Q}$ proper spin sources. We start by expressing the generating function $\Xi_\lambda(\mathbf{Q})$ as a Grassmann functional integral. The resulting structure is very similar to the one in (1.5), but expressed in Grassmann variables: $\Xi_\lambda(\mathbf{Q})$ will be a Grassmann integral of an exponential function such as $\exp(\mathcal{V}_\lambda(X,\mathbf{Q}))$, where $\mathcal{V}_\lambda(X,\mathbf{Q})$, which called effective potential, is a polynomial in the Grassmann variables X and in the source variables $\mathbf{Q}$ whose coefficients are called effective kernels.

We start from the well know representation of the Ising model partition function as a *Pfaffian* [60], which can be written as a Grassmann functional integral [45,67]. If $\lambda = 0$, $\Xi_0(\mathbf{0})$ is a *Gaussian* Grassmann functional integral, i.e. $\mathcal{V}_0(X)$ contains only quadratic monomials of Grassmann variables. If $\lambda \neq 0$, $\Xi_\lambda(\mathbf{Q})$ is a non-Gaussian Grassmann functional integral, i.e. in addition to the quadratic monomials, in $\mathcal{V}_\lambda(X, Q)$ there are monomials of arbitrary order in the Grassmann variables X and monomials involving the sources $\mathbf{Q}$, with effective kernels analytic in λ. If we expand the exponential of $\mathcal{V}_\lambda$ in Taylor series in λ and then naively integrate the Grassmann monomials using the Wick rule, we get non convergent perturbative series at the critical point. A multiscale analysis provide the non trivial resummations required to get the convergence. By introducing a multiscale decomposition, we illustrate an iterative procedure for the integration of the Grassmann variables such that, after each integration step we get an effective potential, which exhibits the same structure as the starting one up to modify some coupling constants. In particular, the effective kernels satisfy proper dimensional bounds at each iteration and the critical temperature and the renormalizations will be modified through the iterative scheme.

The presence of the lower boundary in the half-plane breaks the translation invariance of the model in the vertical direction: this causes several difficulties in using the constructive Renormalization Group methods, which are highly dependent on translation invariance. The authors of [4] provided the first rigorous treatment of the finite volume effects in a critical theory, studying how the boundaries affect the structure of correlation functions in the scaling limit. In particular, they showed that in the presence of a boundary, any contribution to the generating function of correlations of observables located at points in the interior of the domain, can be decomposed into a bulk part (which is defined in a straightforward way based on its infinite plane counterpart), plus a remainder, which it is called the *edge* part. The bulk contributions are translation invariant, so they can be treated as already discussed in [37]: by iterating the above mentioned procedure, we get a flow of running coupling constants which is controlled by proper dimensional bounds, so that we get the explicitly convergent series. The edge contributions, which are translationally invariant only in one direction, must be treated in a distinguished way, in order to reduce them to local edge terms (depending on boundary running coupling *constants* rather than running coupling functions, as a naive definition would produce) plus irrelevant terms. The right procedure for the edge contributions was already discussed in [4]: a key ingredient to get the explicit convergence is a cancellation property of the fermionic fields exactly at the boundary, which imply the necessary dimensional improvements. In this manner, the apparent divergences can be resummed by rigorous Renormalization Group methods: the outcome is a renormalized expansion for $\mathcal{V}_\lambda$ that is convergent for λ small enough, and then an explicit derivation of the correlation functions.
Here we adapt the procedure in presence of spin source variables at the boundary: we derive the generating function describing a perturbed Ising model which also includes the edge spins, we obtain the renormalized expansion of the effective potential as a function of the edge spin sources and the necessary bounds on the effective kernels, treating an effective theory with apparent divergences different from those derived in [4]. As a result we get the $2-$point edge spin correlation functions with valid estimates at the boundary and we derive the explicit exact expressions in the scaling limit.

1.4 Perspectives

The result in this thesis paves the way to derive more general edge observables correlation functions, such as the multipoint edge spin correlations, the multipoint edge energy correlations and the mixed multipoint spin and energy correlations.

In particular, the derivation of the $2n-$point edge spin correlations is an immediate extension of our result: in fact, in the following chapters 2-5 we introduce the Grassmann representation and we prove the convergence of the expansion for the effective potential taking into account an arbitrary number of edge spin sources. However, in chapter 6 we state our main result for the two-point edge spin correlation, in order to provide a more readable discussion.
In [2], the authors proved that, for ferromagnetic pair interactions, the scaling limit of the $2n-$point edge spin correlations at the critical point is a Pfaffian: therefore, combining their result with ours we find that, in their setting (FM pair interactions, i.e. in r.h.s. of (1.2) the only non vanishing interactions are $\lambda V(X) > 0$ for $|X| = 2$) the scaling limit of the $2n-$point edge spin correlation is the Pfaffian of the $2-$point function in (1.10). For more general interactions, the scaling limit of the $2n-$point edge spin correlation could be obtained by combining the localization procedure and the bounds for the effective kernels here introduced, with the techniques discussed in [4, 37] for the multipoint bulk energy correlations. Moreover, the same can be done for the generalized multipoint correlations involving both edge spins and bulk energies, it just requires more involved expressions with no significant differences.
Instead, deriving the edge energies correlation functions requires some significant modifications. In presence of the edge energies, the localization procedure seems to introduce a significant number of additional running coupling constants. However, we can expect that exploiting the symmetries of propagators will reduce this number: a starting point should be the study of the horizontal edge energies, which are defined in terms of nearest neighbor edge spins.

In this perspective we organize the thesis as follows.

1.5 Summary

- In Chapter 2 we introduce the Grassmann representation for the non integrable Ising model on a cylindrical lattice. Then we express the perturbed generating function as a functional integral of Grassmann variables and we obtain the unperturbed correlations in terms of Grassmann fields. Here we consider the generating function depending on the edge spin sources and the energy sources: so that, on the one hand, we can exploit the procedures already derived for the energy correlations to obtain the spin correlations, on the other hand, we can introduce the more general setting that will be useful to future extensions of the result.

- In Chapter 3 we perform a suitable change of Grassmann variables, so we get the representation in massive and massless variables of the generating function on the upper half-plane and we study the critical propagators. In particular, we derive the explicit expression of the critical propagators, we introduce the bulk-edge decomposition and the multiscale decomposition and derive the decay bounds.

- In Chapter 4 we illustrate how to derive a convergent expansion of the effective potential. We introduce a multiscale analysis and a localization procedure that allows to identify the possibly divergent contributions in the effective potential. From here on, we focus only on the edge spin sources and we describe how to get convergent expansions for effective kernels associated in presence of the edge spin sources.

- In Chapter 5 we derive the estimates for the kernels of the expansion of the effective potential in presence of the edge spin sources and we obtain an explicit solution for the flow equation of the running coupling constants associated with the spins.

- In Chapter 6 we prove the main result for the two-point edge spin correlations: we illustrate how to adapt the convergent expansion derived for the effective potentials to expand the correlation functions and, moreover, we prove that the explicit expression of the two-point edge spin correlation can be given in terms of a dominant term, which is exactly one of the massless critical propagator, and a subdominant term that vanishes in the scaling limit.

Chapter 2

The Grassmann representation on a cylinder

In this chapter we derive the Grassmann representation of the truncated correlations on a cylindrical lattice. Despite in our main result we are interested only in the two-point edge spin correlations, we derive the representation of the more general correlations which include an arbitrary number of edge spins, bulk energies and edge energies.
As mentioned above, we are interested in a scaling limit in the infinite volume limit, in which the cylindrical lattice tends to the continuous upper half-plane. When we perform the limit, we want to distinguish the edge observables from the bulk observables: we introduce $\delta > 0$, independent of the lattice spacing a, and we let $\Lambda^\circ := \{ \mathbf{x} \in \Lambda : x^{(2)} > \delta \}$ be the set of the lattice sites that will end up away from the the edge and we let $\partial\Lambda := \{ \mathbf{x} \in \Lambda : x^{(2)} = a \}$ be the set of those that will end up exactly on the edge of the half-plane. For $\mathbf{x} \in \partial\Lambda$, we define the *edge spin* operator as $\tilde{\sigma}_{\mathbf{x}} = a^{-1/2}\sigma_{\mathbf{x}}$ and the *edge energy* operator as $\tilde{\varepsilon}_{\mathbf{x},i} := a^{-2}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}$, $i = 1, 2$; for $\mathbf{x} \in \Lambda^\circ$, we define the *bulk energy* operator as $\varepsilon_{\mathbf{x},i} := a^{-1}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}$, $i = 1, 2$. Moreover, we introduce the sets of auxiliary sources: $\mathbf{\Psi} := \{ \Psi_{\mathbf{x}} \}_{\mathbf{x}\in\partial\Lambda}$ for the edge spin correlations, $\tilde{\mathbf{A}} := \{ \tilde{A}_{\mathbf{x},1}, \tilde{A}_{\mathbf{x},2} \}_{\mathbf{x}\in\partial\Lambda}$ for the edge energy correlations and $\mathbf{A} := \{ A_{\mathbf{x},1}, A_{\mathbf{x},2} \}_{\mathbf{x}\in\Lambda^\circ}$ for the bulk energy correlations. By using Eq.(1.6), we obtain the unperturbed truncated correlation functions as

$$\langle\tilde{\sigma}_{\mathbf{x}_1};\cdots;\tilde{\sigma}_{\mathbf{x}_{n_\sigma}};\varepsilon_{\mathbf{y}_1,j_1};\cdots;\varepsilon_{\mathbf{y}_m,j_m};\tilde{\varepsilon}_{\mathbf{z}_1,j'_1};\cdots;\tilde{\varepsilon}_{\mathbf{z}_{m'},j'_{m'}}\rangle_{\beta,\Lambda} =$$
$$= \frac{\partial^{n_\sigma}}{\partial\Psi_{\mathbf{x}_1}\cdots\partial\Psi_{\mathbf{x}_{n_\sigma}}}\frac{\partial^{m}}{\partial A_{\mathbf{y}_1,j_1}\cdots\partial A_{\mathbf{y}_m,j_m}}\frac{\partial^{m'}}{\partial \tilde{A}_{\mathbf{z}_1,j'_1}\cdots\partial \tilde{A}_{\mathbf{z}_{m'},j'_{m'}}}\log\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})\bigg|_{\mathbf{\Psi}=\mathbf{A}=\tilde{\mathbf{A}}=\mathbf{0}}, \tag{2.1}$$

where $\{ \mathbf{x}_i \in \partial\Lambda \}_{i=1}^{n_\sigma}$, $\{ \mathbf{y}_i \in \Lambda^\circ \}_{i=1}^{m}$ and $\{ \mathbf{z}_i \in \partial\Lambda \}_{i=1}^{m'}$ are sets of distinct points, and

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = \sum_{\underline{\sigma}\in\Omega_{LM}} e^{-\beta H_\lambda(\underline{\sigma},\mathbf{J})+\sum_{j=1}^2\left(\sum_{\mathbf{x}\in\Lambda^\circ}A_{\mathbf{x},j}\varepsilon_{\mathbf{x},j}+\sum_{\mathbf{x}\in\partial\Lambda}\tilde{A}_{\mathbf{x},j}\tilde{\varepsilon}_{\mathbf{x},j}\right)+\sum_{\mathbf{x}\in\partial\Lambda}\Psi_{\mathbf{x}}\tilde{\sigma}_{\mathbf{x}}}, \tag{2.2}$$

is the generating function of interest. In the rest of this chapter we illustrate how to derive the Grassmann representation of the generating function (2.2). As a consequence

of the Grassmann representation of $\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$, we will be able to derive the expressions of the truncated expectations in (2.1) in terms of truncated expectations of Grassmann fields, which will be explicitly evaluated in the next chapter 3.
Here we proceed as follows:

- in Sec. 2.1 we derive the Grassmann representation of $\Xi_0(\mathbf{0}, \mathbf{0}, \mathbf{0})$, which is the partition function of the unperturbed Ising model: despite being a well know result, we briefly review each necessary step, such as the multipolygon representation and the dimer representation;
- in Sec. 2.2 we derive the Grassmann representation of $\Xi_0(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$, which is the unperturbed generating function, by adapting the previously reviewed steps: in particular, the edge spin correlators will initially be defined on a modified cylindrical lattice, which has some additional bonds below the lower edge so we have to reproduce the multipolygon representation and the dimer representation on this modified lattice; moreover we derive the expression of the unperturbed correlations in terms of the Grassmann correlations;
- in Sec. 2.3 we introduce the perturbation $\lambda \neq 0$ and derive the Grassmann representation of $\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$, which is the desired partition functions of the correlations in (2.1).

The derivation of the unperturbed partition function is mainly based on [60] while the derivation of the perturbed energy correlation functions on the cylindrical lattice is mainly based on [4, 37]): the original contribution of this chapter is to provide the Grassmann representation of generating function with the edge spin sources and derive the corresponding perturbed edge spin correlation functions.

2.1 The unperturbed partition function

In this section we illustrate how to derive the Grassmann representation of $\Xi_0(\mathbf{0}, \mathbf{0}, \mathbf{0}) = Z_0(\mathbf{J})$, which is the partition function of the exactly solvable Ising model on a cylindrical lattice. Such model is described by the Hamiltonian in (1.2) with $\lambda = 0$, so that the partition function at inverse temperature β is given by

$$Z_0(\mathbf{J}) = \sum_{\underline{\sigma} \in \Omega_{LM}} e^{-\beta H_0(\underline{\sigma}, \mathbf{J})} = \sum_{\underline{\sigma} \in \Omega_{LM}} e^{\beta \sum_{\mathbf{x} \in \Lambda} \sum_{i=1}^{2} J_{\mathbf{x},i} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}}, \tag{2.3}$$

where $J_{\mathbf{x},i} := J_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_i}, i = 1, 2$ denote the interaction energies between nearest neighbor spins. The boundary conditions are free in the vertical direction, $\sigma_{(x^{(1)},aM+a)} = 0$ for all $x^{(1)} \in [-aL, aL]$, and periodic in the horizontal direction $\sigma_{(aL+a,x^{(2)})} = \sigma_{(-aL,x^{(2)})}$ for all $x^{(2)} \in [a, aM]$. First of all, in Subsec. 2.1.1 we illustrate how to represent $Z_0(\mathbf{J})$ as a sum over the multipolygons and in Subsec. 2.1.2 we illustrate how to relate this representation as a Pfaffian of the Kasteleyn matrix. In Subsec. 2.1.3 we introduce the Grassmann variables, in terms of which we can represent the Pfaffian of the Kasteleyn matrix as a Grassmann functional integral, as derived in Subsec. 2.1.4.

2.1.1 The multipolygon representation

For each site of Λ we let $b_{\mathbf{x},i} := (\mathbf{x}, \mathbf{x} + a\hat{\mathbf{e}}_i)$, $i = 1, 2$, be the horizontal and vertical bond, and we let $\sigma_{b_{\mathbf{x},i}} := \sigma_{\mathbf{x}} \cdot \sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}$ be the bond spin associated with $b_{\mathbf{x},i}$. Then, the partition function (2.3) can be rewritten as

$$Z_0(\mathbf{J}) = \sum_{\underline{\sigma} \in \Omega_{LM}} e^{\beta \sum_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} J_{\mathbf{x},i} \sigma_{b_{\mathbf{x},i}}} , \tag{2.4}$$

where $\mathcal{B}_\Lambda = \{ b_{\mathbf{x},i} \}_{\mathbf{x} \in \Lambda}^{i=1,2}$ is the set of nearest neighbor bonds in Λ. By expanding the exponential in power series ($\sigma_{\mathbf{b_{x,i}}} = \pm 1$) we get

$$\begin{aligned} Z_0(\mathbf{J}) &= \sum_{\underline{\sigma} \in \Omega_{LM}} \prod_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} \left(\cosh \beta J_{\mathbf{x},i} + \sigma_{b_{\mathbf{x},i}} \sinh \beta J_{\mathbf{x},i} \right) = \\ &= \prod_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} \cosh \beta J_{\mathbf{x},i} \sum_{\underline{\sigma} \in \Omega_{LM}} \prod_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} (1 + \sigma_{b_{\mathbf{x},i}} \tanh \beta J_{\mathbf{x},i}) \, . \end{aligned} \tag{2.5}$$

Developing the last product in (2.5), we are led to a sum of terms of the type

$$\sigma_{b_1} \tanh \beta J_1 \cdots \sigma_{b_s} \tanh \beta J_s \, , \tag{2.6}$$

and we can conveniently describe them through the geometric set of lines $b_1, \cdots, b_s \in \mathcal{B}_\Lambda$. If we perform the summation over the spin configurations $\underline{\sigma}$, many terms of the form (2.6) give vanishing contributions: the only terms which survive are those in which the vertices of the geometric figure $b_1 \cup b_2 \cup \cdots \cup b_s$ belong to an even number (0,2 or 4) of bonds. These terms are such that $\sigma_{b_1} \cdots \sigma_{b_s} = 1$. So we are considering only s sides figures, called *multipolygons*, as in Fig. 2.1. The multipolygons are closed figures, which

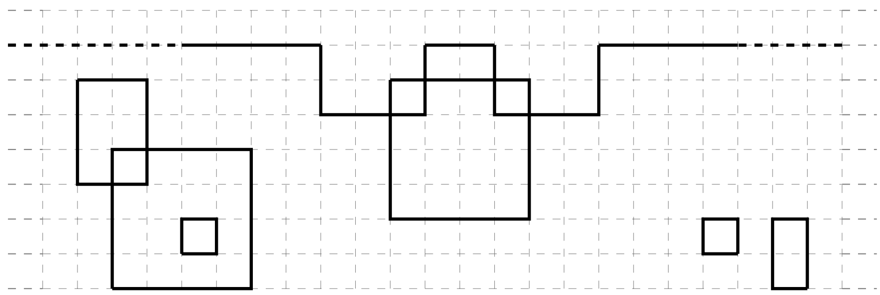

Figure 2.1: The multipolygons allowed on a cylindrical lattice. The lattice is periodic in the horizontal direction.

are allowed to wind up the lattice in the horizontal direction but not in the vertical direction; moreover, these figures can intersect, with the restrictions that no two sides may overlap. Let $P_s(\Lambda) \subset \mathcal{B}_\Lambda$ be the set of such multipolygons with s sides on the lattice Λ. Then the problem of evaluating (2.5) can be reduced to that of finding the partition function for the multipolygons on the lattice, namely

$$Z_0(\mathbf{J}) = 2^{LM} \prod_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} \cosh \beta J_{\mathbf{x},i} \sum_{s \geq 0} \sum_{P_s(\Lambda) \subset \mathcal{B}_\Lambda} \prod_{b_{\mathbf{x},i} \in P_s(\Lambda)} t_{\mathbf{x},i} \, , \tag{2.7}$$

where $t_{\mathbf{x},i} := \tanh \beta J_{\mathbf{x},i}$, $i = 1, 2$. It is well know that the sum over the multipolygons in (2.7) can be expressed in terms of an appropriate Pfaffian, but before we have to relate the Ising model on Λ to a combinatorial problem involving dimers.

2.1.2 The dimer representation

To relate the Ising model to a problem of closest-packed dimers, by following the construction originally derived in [30], we replace each Ising vertex $\mathbf{x} \in \Lambda$ by a cluster of six new vertices $\{\bar{H}_\mathbf{x}, H_\mathbf{x}, \bar{V}_\mathbf{x}, V_\mathbf{x}, \bar{T}_\mathbf{x}, T_\mathbf{x}\}$, as in Fig. 2.2. By this site replacement, in

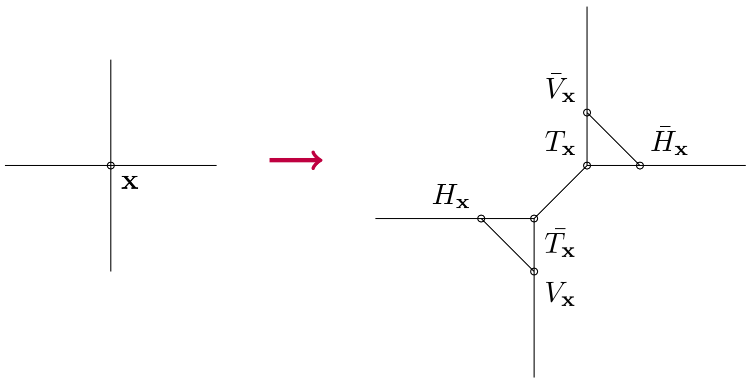

Figure 2.2: Original vertex $\mathbf{x}$ (on the left) corresponds to six new vertices $\{\bar{H}_\mathbf{x}, H_\mathbf{x}, \bar{V}_\mathbf{x}, V_\mathbf{x}, \bar{T}_\mathbf{x}, T_\mathbf{x}\}$ (on the right).

addition to horizontal and vertical bonds between nearest neighbor clusters there are now also bonds within a cluster, as shown in Fig. 2.3. A dimer is a figure that may be drawn on the lattice and that covers a bond and its endpoint sites. By filling the lattice with dimers, with the constraint that each lattice site is covered by one and only one dimer, we obtain a closest-packed configuration. The sum over the multipolygons in (2.7) corresponds to the partition function for the closest-packed dimer configurations on Λ, which can be evaluated by the Pfaffian of the *Kasteleyn matrix* [51]. The Kasteleyn matrix on Λ is an antisymmetric $6LM \times 6LM$ matrix whose entries, properly labelled by the lattice sites, are in correspondence with specific bond weights. Up to proper signs,

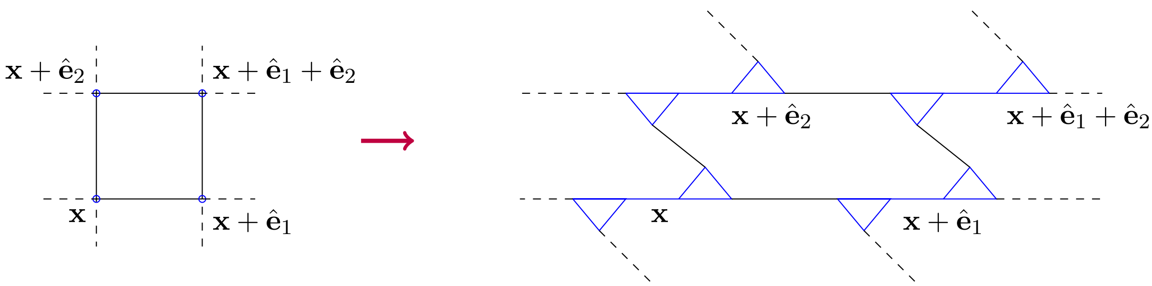

Figure 2.3: Bonds between nearest neighbour sites (on the left) correspond to bonds between nearest neighbour clusters (on the right). In addition there are seven bonds within a cluster (blue) for each lattice vertex.

the bonds between clusters have a weight of $t_{\mathbf{x},1}$ or $t_{\mathbf{x},2}$, if horizontal or vertical, and the bonds within a cluster have a weight of 1. The signs are assigned by a *clock-wise odd* orientation of the lattice (see Appendix A for details).
Then the partition function (2.7) can be rewritten as

$$Z_0(\mathbf{J}) = 2^{LM} \prod_{b_{\mathbf{x}_i} \in \mathcal{B}_\Lambda} \cosh \beta J_{\mathbf{x},i} \; Pf \mathcal{A}' , \tag{2.8}$$

where the definition of the Pfaffian of an $2N \times 2N$ antisymmetric matrix is

$$Pf \mathcal{A}' = \frac{1}{2^N N!} \sum_\pi (-1)^\pi \prod_{i=1}^N \mathcal{A}'_{\pi(2i-1),\pi(2i)} , \tag{2.9}$$

where π is a permutation of $\{1, \dots, 2N\}$ and $(-1)^\pi$ is its signature. One of the properties of the Pfaffian is that $(Pf \mathcal{A}')^2 = \det \mathcal{A}'$. The Pfaffian can be immediately expressed as a suitable Grassmann integral, which we are now going to introduce.

2.1.3 The Grassmann integration rules

A finite dimensional *Grassmann algebra*, is a set of *Grassmann variables* ψ_α, with α an index belonging to some finite set $\{ 1, \dots, 2n \}$, which are anticommuting variables, that is

$$\{ \psi_\alpha, \psi_{\alpha'} \} = \psi_\alpha \psi_{\alpha'} + \psi_{\alpha'} \psi_\alpha = 0 , \qquad \forall \alpha, \alpha' \in \{ 1, \dots, 2n \} , \tag{2.10}$$

and in particular

$$(\psi_\alpha)^2 = 0 , \qquad \forall \alpha \in \{ 1, \dots, 2n \} . \tag{2.11}$$

Let us introduce another set of Grassmann variables $d\psi_\alpha$, anticommuting with ψ_α. The *Grassmann integration* is a linear operation defined by

$$\int d\psi_\alpha = 0, \qquad \int d\psi_\alpha \psi_\alpha = 1, \qquad \alpha \in \{ 1, \dots, 2n \} . \tag{2.12}$$

If $F(\psi)$ is any analytic function in the Grassmann variables ψ, the Grassmann integral

$$\int \Big[\prod_{\alpha=1}^{2n} d\psi_\alpha \Big] F(\psi) \tag{2.13}$$

is simply defined by iteratively applying (2.12) and taking into account the anticommutation rules (2.10). For instance

$$\int d\psi_\alpha e^{\psi_\alpha} = \int d\psi_\alpha (1 + \psi_\alpha) = 1 , \tag{2.14}$$

analogously for all $\alpha_1, \alpha_2 \in \{ 1, \dots, 2n \}$ and $c \in \mathbb{C}$

$$\int d\psi_{\alpha_1} d\psi_{\alpha_2} \, e^{-\psi_{\alpha_1} c \psi_{\alpha_2}} = c . \tag{2.15}$$

When more than one pair of Grassmann variables is involved, if $A = 2 \times 2$ matrix, i.e. $A_{11} = A_{22} = 0$ and $A_{12} = -A_{21}$,

$$\int d\psi_{\alpha_1} d\psi_{\alpha_2}\, e^{-\sum_{i,j=1}^{2} \psi_{\alpha_i} A_{ij} \psi_{\alpha_j}} = A_{12}\,, \tag{2.16}$$

and by noticing that $Pf\,A = A_{12}$ (see (2.9)), we can easily generalize the above expression for any $2n \times 2n$ antisymmetric matrix A,

$$\int \Big[\prod_{\alpha=1}^{2n} d\psi_\alpha\Big] e^{-\frac{1}{2}\sum_{\alpha,\beta=1}^{2n} \psi_\alpha A_{\alpha,\beta} \psi_\beta} = Pf\,A\,. \tag{2.17}$$

A list of useful algebraic properties of the Grassmann integration can be found in several works, e.g. [34, Chap. 4] or [40, Prop. 1]. However, when necessary, we will recall other properties in the following sections.

2.1.4 The Grassmann representation

The Pfaffian of the Kasteleyn matrix in (2.8), using the equality in (2.17), can be expressed as a Grassmann functional integral, so we obtain the Grassmann representation of the partition function

$$Z_0(\mathbf{J}) = 2^{LM} \prod_{b_{\mathbf{x}_i} \in \mathcal{B}_\Lambda} \cosh \beta J_{\mathbf{x},i} \int d\bar{H}_{\mathbf{x}} dH_{\mathbf{x}} d\bar{V}_{\mathbf{x}} dV_{\mathbf{x}} d\bar{T}_{\mathbf{x}} dT_{\mathbf{x}}\; e^{S_{\mathbf{t}}(H,V,T)}\,, \tag{2.18}$$

where $\mathbf{t} \equiv \mathbf{t}(\beta) := \{\, t_{\mathbf{x},i} \,\}_{\mathbf{x}\in\Lambda}^{i=1,2}$, $t_{\mathbf{x},i} := \tanh \beta J_{\mathbf{x},i}$, $i = 1, 2$, and

$$S_{\mathbf{t}}(H, V, T) = \sum_{\mathbf{x}\in\Lambda} \left\{ \sum_{i=1}^{2} t_{\mathbf{x},i} E_{\mathbf{x},i} - \bar{H}_{\mathbf{x}} \bar{V}_{\mathbf{x}} - H_{\mathbf{x}} V_{\mathbf{x}} + \bar{H}_{\mathbf{x}} T_{\mathbf{x}} + H_{\mathbf{x}} \bar{T}_{\mathbf{x}} - \bar{V}_{\mathbf{x}} T_{\mathbf{x}} - V_{\mathbf{x}} \bar{T}_{\mathbf{x}} + \bar{T}_{\mathbf{x}} T_{\mathbf{x}} \right\}, \tag{2.19}$$

with $E_{\mathbf{x},1} := \bar{H}_{\mathbf{x}} H_{\mathbf{x}+a\hat{\mathbf{e}}_1}$, $E_{\mathbf{x},2} := \bar{V}_{\mathbf{x}} V_{\mathbf{x}+a\hat{\mathbf{e}}_2}$. The boundary conditions are $H_{(aL+a,x^{(2)})} = (-1)^{L+1} H_{(-aL,x^{(2)})}$, for all $x^{(2)} \in [a, aM]$, and $V_{(x^{(1)},aM+a)} = 0$, for all $x^{(1)} \in [-aL, aL]$. The $T-$fields appear only in the diagonal elements of (2.19) and they can be easily integrated out (see [38, Appendix A]), and we obtain the partition function as

$$Z_0(\mathbf{J}) = C_{0,\Lambda} \int \mathcal{D}X\; e^{S_{\mathbf{t}}(X)}\,, \tag{2.20}$$

where

$$C_{0,\Lambda} := C_{0,\Lambda}(\mathbf{J}) = (-2)^{LM} \prod_{b_{\mathbf{x},i} \in \mathcal{B}_\Lambda} \cosh \beta J_{\mathbf{x},i}\,, \tag{2.21}$$

$X = \{\, (\bar{H}_{\mathbf{x}}, H_{\mathbf{x}}, \bar{V}_{\mathbf{x}}, V_{\mathbf{x}}) \,\}_{\mathbf{x}\in\Lambda}$ is a collection of $4LM \times 4LM$ Grassmann variables,

$$\int \mathcal{D}X := \int \prod_{\mathbf{x}\in\Lambda} d\bar{H}_{\mathbf{x}} dH_{\mathbf{x}} d\bar{V}_{\mathbf{x}} dV_{\mathbf{x}}\,,$$

is a shorthand for the Grassmann integration and

$$S_{\mathbf{t}}(X) = \sum_{\mathbf{x}\in\Lambda} \left\{ t_{\mathbf{x},1}E_{\mathbf{x},1} + t_{\mathbf{x},2}E_{\mathbf{x},2} - \bar{H}_{\mathbf{x}}\bar{V}_{\mathbf{x}} - H_{\mathbf{x}}V_{\mathbf{x}} + \bar{H}_{\mathbf{x}}H_{\mathbf{x}} + \bar{V}_{\mathbf{x}}V_{\mathbf{x}} + V_{\mathbf{x}}\bar{H}_{\mathbf{x}} + H_{\mathbf{x}}\bar{V}_{\mathbf{x}} \right\} , \tag{2.22}$$

is the "quadratic action" of the unperturbed Ising model.

2.2 The unperturbed generating function

In this section we derive the Grassmann representation of $\Xi_0(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$ and we explicitly derive the multipoint unperturbed correlations for the edge spins and the bulk and edge energies in terms of Grassmann variables unperturbed correlations.

First of all we define the unperturbed expectation of Grassmann variables. By comparing the initial definition in (2.3) with the resulting representation in (2.20), we can easily extend the definition in (1.3) with $\lambda = 0$: given m monomials of Grassmann variables $F_1(X), \cdots, F_m(X)$, their unperturbed expectation is given by

$$\langle F_1(X) \cdots F_m(X) \rangle^0_{\mathbf{t}(\beta),\Lambda} := \frac{\int \mathcal{D}X \, F_1(X) \cdots F_m(X) \, e^{S_{\mathbf{t}}(X)}}{\int \mathcal{D}X \; e^{S_{\mathbf{t}}(X)}} , \tag{2.23}$$

where $\langle \cdot \rangle^0_{\mathbf{t}(\beta),\Lambda}$ is an average with respect to the "Gaussian Grassmann measure" $\mathcal{D}X e^{S_{\mathbf{t}}(X)}$: we use the term "Gaussian" since $S_{\mathbf{t}}(X)$ is quadratic. To each monomial $F_i(X)$ of even (resp. odd) degree in the Grassmann variables we can associate a commuting (resp. anticommuting) variable Q_i as auxiliary source. Then, the unperturbed simple and truncated expectations can be expressed as in (1.4) and (1.6) with $\lambda = 0$, where now $\Xi_0(\mathbf{Q})$ is the unperturbed generating function in Grassmann variables defined as

$$\Xi_0(\mathbf{Q}) = \int \mathcal{D}X \; e^{S_{\mathbf{t}}(X) + Q_1 F_1(X) + \cdots + Q_m F_m(X)} . \tag{2.24}$$

As a first step, in 2.2.1 we derive the Grassmann representation only of the energies sources, and then in 2.2.2 we derive the Grassmann representation of the edge spin sources.

2.2.1 The unperturbed bulk and edge energy correlations

Lemma 2.1. *If* $\{ \mathbf{y}_i \in \Lambda^\circ \}_{i=1}^m$ *and* $\{ \mathbf{z}_i \in \partial\Lambda \}_{i=1}^{m'}$ *are sets of distinct points, the multipoint energy correlation functions are given by*

$$\begin{aligned} &\langle \varepsilon_{\mathbf{y}_1,j_1}; \cdots; \varepsilon_{\mathbf{y}_m,j_m}; \tilde{\varepsilon}_{\mathbf{y}_1,j'_1}; \cdots; \tilde{\varepsilon}_{\mathbf{y}_{m'},j'_{m'}} \rangle^0_{\beta,\Lambda} = \\ &= \frac{\partial^m}{\partial A_{\mathbf{y}_1,j_1} \cdots \partial A_{\mathbf{y}_m,j_m}} \frac{\partial^{m'}}{\partial \tilde{A}_{\mathbf{y}_1,m_1} \cdots \partial \tilde{A}_{\mathbf{y}_{m'},j_{m'}}} \log \Xi_0(\mathbf{A}, \tilde{\mathbf{A}}) \bigg|_{\mathbf{A}=\tilde{\mathbf{A}}=\mathbf{0}} , \end{aligned} \tag{2.25}$$

where $\Xi_0(\mathbf{A}, \tilde{\mathbf{A}})$ *is the unperturbed Grassmann generating functional*

$$\Xi_0(\mathbf{A}, \tilde{\mathbf{A}}) := C_{0,\Lambda} \int \mathcal{D}X \; e^{S_{\mathbf{t}}(X)+(\mathbf{E},\mathbf{A})+(\mathbf{E},\tilde{\mathbf{A}})} \tag{2.26}$$

with $C_{0,\Lambda}$ *in* (2.21), $S_{\mathbf{t}}(X)$ *in* (2.22) *and*

$$(\mathbf{E}, \mathbf{A}) = a^{-1} \sum_{i=1}^{2} \sum_{\mathbf{x}\in\Lambda^\circ} (1 - t_{\mathbf{x},i}^2) A_{\mathbf{x},i} E_{\mathbf{x},i}, \qquad (\mathbf{E}, \tilde{\mathbf{A}}) = a^{-2} \sum_{i=1}^{2} \sum_{\mathbf{x}\in\partial\Lambda} (1 - t_{\mathbf{x},i}^2) \tilde{A}_{\mathbf{x},i} E_{\mathbf{x},i}\,, \tag{2.27}$$

are the source terms for the bulk and edge energies.

Proof of Lemma 2.1. By recalling the definitions $\varepsilon_{\mathbf{x},i} = \frac{\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}}{a}$ and $\tilde{\varepsilon}_{\mathbf{x},i} = \frac{\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+a\hat{\mathbf{e}}_i}}{a^2}$, of the bulk and edge energy observables, we can express the Hamiltonian in (1.2) with $\lambda = 0$ in terms of energy variables

$$H_0(\underline{\sigma}, \mathbf{J}) = -\sum_{i=1}^{2} \big(a \sum_{\mathbf{x}\in\Lambda^\circ} J_{\mathbf{x},i}\varepsilon_{\mathbf{x},i} + a^2 \sum_{\mathbf{x}\in\partial\Lambda} J_{\mathbf{x},i}\tilde{\varepsilon}_{\mathbf{x},i}\big)\,, \tag{2.28}$$

so the generating function given in (2.2) with $\mathbf{\Psi} = \mathbf{0}$, can be rewritten as

$$\Xi_0(\mathbf{A}, \tilde{\mathbf{A}}) = \sum_{\underline{\sigma}\in\Omega_{LM}} e^{\sum_{i=1}^{2} \left(a\beta \sum_{\mathbf{x}\in\Lambda^\circ} J_{\mathbf{x},i}\varepsilon_{\mathbf{x},i} + a^2\beta \sum_{\mathbf{x}\in\partial\Lambda} J_{\mathbf{x},i}\tilde{\varepsilon}_{\mathbf{x},i} + \sum_{\mathbf{x}\in\Lambda^\circ} A_{\mathbf{x},i}\varepsilon_{\mathbf{x},i} + \sum_{\mathbf{x}\in\partial\Lambda} \tilde{A}_{\mathbf{x},i}\tilde{\varepsilon}_{\mathbf{x},i}\right)}\,. \tag{2.29}$$

By expanding the exponential in power series ($a\varepsilon_{\mathbf{x},i} = \pm 1, a^2\tilde{\varepsilon}_{\mathbf{y},i} = \pm 1$) we get

$$\begin{aligned}\Xi_0(\mathbf{A}, \tilde{\mathbf{A}}) =& \big[\prod_{\mathbf{x}\in\Lambda^\circ} \prod_{i=1,2} \cosh(\beta J_{\mathbf{x},i} + a^{-1} A_{\mathbf{x},i})\big] \cdot \big[\prod_{\mathbf{x}\in\partial\Lambda} \prod_{i=1,2} \cosh(\beta J_{\mathbf{x},i} + a^{-2} \tilde{A}_{\mathbf{x},i})\big] \cdot \\ &\cdot \sum_{\underline{\sigma}\in\Omega_{LM}} \big[\prod_{\mathbf{x}\in\Lambda^\circ} \prod_{i=1,2} (1 + a^{-1}\varepsilon_{\mathbf{x},i} \tanh(\beta J_{\mathbf{x},i} + a^{1} A_{\mathbf{x},i}))\big] \cdot \\ &\cdot \big[\prod_{\mathbf{x}\in\partial\Lambda} \prod_{i=1,2} (1 + a^2\tilde{\varepsilon}_{\mathbf{x},i} \tanh(\beta J_{\mathbf{x},i} + a^{-2} \tilde{A}_{\mathbf{x},i}))\big]\,,\end{aligned} \tag{2.30}$$

and we can proceed as illustrated after (2.5), obtaining the partition function for the multipolygons

$$\begin{aligned}\Xi_0(\mathbf{A}, \tilde{\mathbf{A}}) = 2^{LM} &\big[\prod_{\mathbf{x}\in\Lambda^\circ} \prod_{i=1,2} \cosh(\beta J_{\mathbf{x},i} + a^{-1} A_{\mathbf{x},i})\big] \cdot \big[\prod_{\mathbf{x}\in\partial\Lambda} \prod_{i=1,2} \cosh(\beta J_{\mathbf{x},i} + a^{-2} \tilde{A}_{\mathbf{x},i})\big] \cdot \\ &\cdot \sum_{s\geq 0} \sum_{P_s(\Lambda)\subset\mathcal{B}_\Lambda} \big[\prod_{b\in P_s(\Lambda^\circ)} t_{\mathbf{x},i}(A)\big] \big[\prod_{b\in P_s(\partial\Lambda)} t_{\mathbf{x},i}(\tilde{A})\big]\end{aligned} \tag{2.31}$$

where $t_{\mathbf{x},i}(A) := \tanh(\beta J_{\mathbf{x},i} + a^{-1}A_{\mathbf{x},i})$, $t_{\mathbf{x},i}(\tilde{A}) := \tanh(\beta J_{\mathbf{x},i} + a^{-2}\tilde{A}_{\mathbf{x},i})$. The sum in the r.h.s. can be written (2.1.2-2.1.4) as a Grassmann functional integral:

$$\begin{aligned}\Xi_0(\mathbf{A},\tilde{\mathbf{A}}) = (-2)^{LM} \cdot \Big[\prod_{\mathbf{x}\in\Lambda^\circ} \prod_{i=1,2} \cosh(\beta J_{\mathbf{x},i} + a^{-1}A_{\mathbf{x},i})\Big] \cdot \\ \cdot \Big[\prod_{\mathbf{x}\in\partial\Lambda} \prod_{j=1,2} \cosh(\beta J_{\mathbf{x},j} + a^{-2}\tilde{A}_{\mathbf{x},j})\Big] \int \mathcal{D}X e^{S_{\mathbf{t}}(X;\mathbf{A},\tilde{\mathbf{A}})} ,\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}S_{\mathbf{t}}(X;\mathbf{A},\tilde{\mathbf{A}}) = \sum_{j=1}^{2}\sum_{\mathbf{x}\in\Lambda^\circ} t_{\mathbf{x},j}(A)E_{\mathbf{x},j} + \sum_{j=1}^{2}\sum_{\mathbf{x}\in\partial\Lambda} t_{\mathbf{x},j}(\tilde{A})E_{\mathbf{x},j} + \\ + \sum_{\mathbf{x}\in\Lambda}\big(- \bar{H}_{\mathbf{x}}\bar{V}_{\mathbf{x}} - H_{\mathbf{x}}V_{\mathbf{x}} + \bar{H}_{\mathbf{x}}H_{\mathbf{x}} + \bar{V}_{\mathbf{x}}V_{\mathbf{x}} + V_{\mathbf{x}}\bar{H}_{\mathbf{x}} + H_{\mathbf{x}}\bar{V}_{\mathbf{x}}\big) .\end{aligned} \tag{2.33}$$

We conclude the proof by noticing that when we take the following derivatives

$$\frac{\partial^m}{\partial A_{\mathbf{y}_1,j_1}\cdots\partial A_{\mathbf{y}_m,j_m}} \frac{\partial^{m'}}{\partial \tilde{A}_{\mathbf{y}_1,m_1}\cdots\partial \tilde{A}_{\mathbf{y}_{m'},j_{m'}}} \log \Xi_0(\mathbf{A},\tilde{\mathbf{A}})\Big|_{\mathbf{A}=\tilde{\mathbf{A}}=\mathbf{0}} \tag{2.34}$$

with $\Xi_0(\mathbf{A},\tilde{\mathbf{A}})$ expressed as in (2.32) or as in (2.26) in the statement of Lemma 2.1, we get the same result. ☐

2.2.2 The unperturbed edge spin correlations

Lemma 2.2. *If* $(\mathbf{x}_1,\ldots,\mathbf{x}_m)$, $\mathbf{x}_i \in \partial\Lambda$, $i = 1,\ldots,m$, *is an ordered sequence of* m *distinct boundary points, the edge spin correlation functions are given by*

$$\langle \tilde{\sigma}_{\mathbf{x}_1};\cdots;\tilde{\sigma}_{\mathbf{x}_m}\rangle^0_{\beta,\Lambda} = \frac{\partial^m}{\partial\Psi_{\mathbf{x}_1}\cdots\partial\Psi_{\mathbf{x}_m}} \log \Xi_0(\mathbf{\Psi})\Big|_{\mathbf{\Psi}=\mathbf{0}} \tag{2.35}$$

where $\Xi_0(\mathbf{\Psi})$ *is the Grassmann generating functional*

$$\Xi_0(\mathbf{\Psi}) := C_{0,\Lambda} \int \mathcal{D}X \; e^{S_{\mathbf{t}}(X)+(\mathbf{V},\mathbf{\Psi})} , \tag{2.36}$$

with $C_{0,\Lambda}$ *in* (2.21), $S_{\mathbf{t}}(X)$ *in* (2.22) *and*

$$(\mathbf{V},\mathbf{\Psi}) = a^{-1/2} \sum_{\mathbf{x}\in\partial\Lambda} \Psi_{\mathbf{x}}V_{\mathbf{x}} , \tag{2.37}$$

is the source term for the edge spins.

Proof of Lemma 2.2. To obtain the $2n-$point edge spin correlations, we consider a lattice $\Lambda_n = \Lambda \cup \{\tilde{b}_1,\ldots,\tilde{b}_n\}$, consisting of the cylindrical lattice Λ with n additional fixed bonds below the lower edge of Λ, as in Fig. 2.4. We consider $2n$ distinct ordered sites

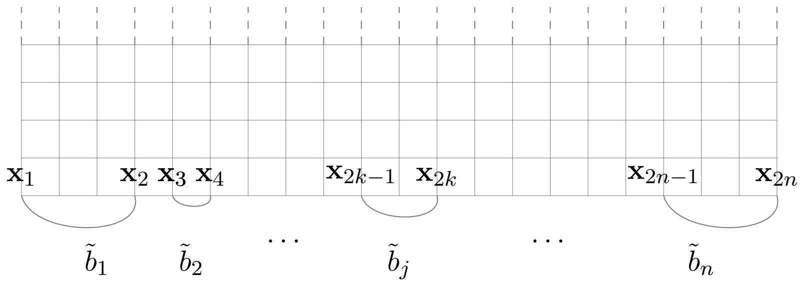

Figure 2.4: The lattice Λ_n obtained by fixing $\tilde{b}_1, \ldots, \tilde{b}_n$ additional bonds on the lower edge of the cylindrical lattice Λ.

on the lower edge of Λ, i.e. the set

$$\partial\tilde{\Lambda} := \{\, \mathbf{x}_j = (x_j^{(1)}, a) : -aL \le x_j^{(1)} < x_{j+1}^{(1)} \le aL \,\}_{j=1}^{2n} \subset \partial\Lambda\,,$$

and we connect each pair of sites $\mathbf{x}_{2k-1}, \mathbf{x}_{2k} \in \partial\tilde{\Lambda}$, $k = 1, \ldots, n$, with an additional bond $\tilde{b}_k := (\mathbf{x}_{2k-1}, \mathbf{x}_{2k})$, so that the n additional bonds are disjoint and non overlapping. The pair of edge spins located at the endpoints of $\tilde{b}_k$ interacts with energy $\tilde{J}_k$ for all $k = 1, \ldots, n$ and $\tilde{\mathbf{J}} := \{\, \tilde{J}_k \,\}_{k=1}^{n}$. The partition function of the n.n. Ising model on Λ_n is

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = \sum_{\underline{\sigma} \in \Omega_{LM}} e^{-\beta H_0(\underline{\sigma}, \mathbf{J}) + \beta \sum_{k=1}^{n} \tilde{J}_k \sigma_{\mathbf{x}_{2k-1}} \sigma_{\mathbf{x}_{2k}}}\,, \tag{2.38}$$

where $H_0(\underline{\sigma}, \mathbf{J})$ is the Hamiltonian of the n.n. Ising model on Λ. It should be noted that if we let $\beta\tilde{\mathbf{J}} = \{\, \beta\tilde{J}_1, \ldots, \beta\tilde{J}_n \,\}$ be a set of the auxiliary sources, so that we are dealing with the additional bond interactions as if they were auxiliary source terms, and let $\Xi_0(\beta\tilde{\mathbf{J}}) := Z_0(\mathbf{J}, \tilde{\mathbf{J}})$ be the unperturbed generating function, we obtain the $2n-$point pairwise edge spins correlation function on the original lattice Λ

$$\langle \sigma_{\mathbf{x}_1} \sigma_{\mathbf{x}_2} \cdots \sigma_{\mathbf{x}_{2n-1}} \sigma_{\mathbf{x}_{2n}} \rangle_{\beta, \Lambda; 0} := \frac{1}{\Xi_0(\beta\tilde{\mathbf{J}})} \frac{\partial^n}{\partial\beta\tilde{J}_1 \cdots \partial\beta\tilde{J}_n}\, \Xi_0(\beta\tilde{\mathbf{J}}) \Big|_{\beta\tilde{J}_1 = \cdots = \beta\tilde{J}_n = 0}\,. \tag{2.39}$$

Then deriving the Grassmann representation for the partition function of the n.n. Ising model on Λ_n we get the Grassmann representation for the unperturbed generating function for correlations of pairs of edge spins on Λ.
Let $\mathcal{B}_{\Lambda_n} := \mathcal{B}_\Lambda \cup \{\tilde{b}_1, \cdots, \tilde{b}_n\}$ be the set of bond variables of Λ_n, $\mathcal{B}_{\Lambda_0} = \mathcal{B}_\Lambda$. Let $b_{\mathbf{xy}}$ denote a generic bond of $\mathcal{B}_{\Lambda_n}$. We introduce the bond spin variables $\sigma_{b_{\mathbf{xy}}}$, given by the product over the spin variables over the two extremes of $b_{\mathbf{xy}}$, so that the unperturbed partition function (2.38) can be rewritten as

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = \sum_{\underline{\sigma} \in \Omega_{LM}} e^{\beta \sum_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda_n}} J_{\mathbf{xy}} \sigma_{b_{\mathbf{xy}}}}\,, \tag{2.40}$$

where $J_{\mathbf{xy}} := J_{b_{\mathbf{xy}}}$. By expanding the exponential in power series we get

$$\begin{aligned} Z_0(\mathbf{J}, \tilde{\mathbf{J}}) &= \sum_{\underline{\sigma} \in \Omega_{LM}} \prod_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda^n}} \left(\cosh \beta J_{\mathbf{xy}} + \sigma_{b_{\mathbf{xy}}} \sinh \beta J_{\mathbf{xy}} \right) = \\ &= \prod_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda^n}} \cosh \beta J_{\mathbf{xy}} \sum_{\underline{\sigma} \in \Omega_{LM}} \prod_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda^n}} (1 + \sigma_{b_{\mathbf{xy}}} \tanh \beta J_{\mathbf{xy}}) \, . \end{aligned} \tag{2.41}$$

As in 2.1.1, the problem to evaluate the last sum in (2.41) can be reduced to that of finding the partition function for the multipolygons on Λ_n. The sides of such multipolygons now may also include the additional bonds $\tilde{b}_1, \ldots, \tilde{b}_n$, as shown in Fig. 2.5. Let

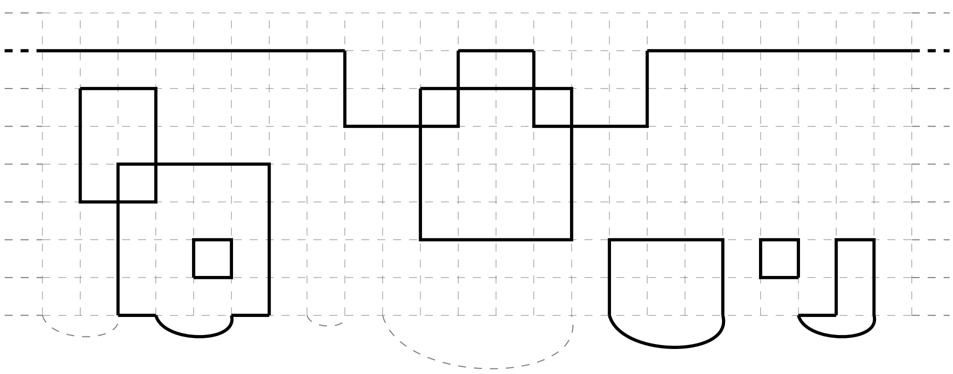

Figure 2.5: The multipolygons allowed on a cylindrical lattice with $\tilde{b}_1, \ldots, \tilde{b}_n$ additional bonds on the lower edge.

$P_s(\Lambda_n) \subset \mathcal{B}_{\Lambda_n}$ be the set of multipolygons with s sides in Λ_n, then we get

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = 2^{LM} \prod_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda^n}} \cosh \beta J_{\mathbf{xy}} \sum_{s \geq 0} \sum^{*}_{P_s(\Lambda_n) \subset \Lambda_n} \prod_{b_{\mathbf{xy}} \in P_s(\Lambda_n)} t_{\mathbf{xy}} \, , \tag{2.42}$$

where $t_{\mathbf{xy}} := \tanh \beta J_{\mathbf{xy}}$. Again, the sum over the multipolygons in (2.42) may be rewritten as a Pfaffian. We can proceed analogously to 2.1.2. To define the Kasteleyn matrix on Λ_n, we perform the same site replacement as in Fig. 2.2: with respect to the resulting the bonds depicted in Fig. 2.3 now we allow an additional class of bonds between different clusters: the additional bonds $\tilde{b}_1, \ldots, \tilde{b}_n$, which have a weight of $\tilde{t}_k := \tanh \beta \tilde{J}_k, j = 1 \cdots, n$. As in Fig. 2.6, an additional bond $\tilde{b}_k$ connects a Grassmann variable $V_{\mathbf{x}_{2k-1}}$ to a Grassmann variable $V_{\mathbf{x}_{2k}}$, with $\mathbf{x}_{2k-1}, \mathbf{x}_{2k} \in \partial \tilde{\Lambda}$. The Kasteleyn matrix is then the $6LM \times 6LM$ antisymmetric matrix $\tilde{A}'$, whose non vanishing entries

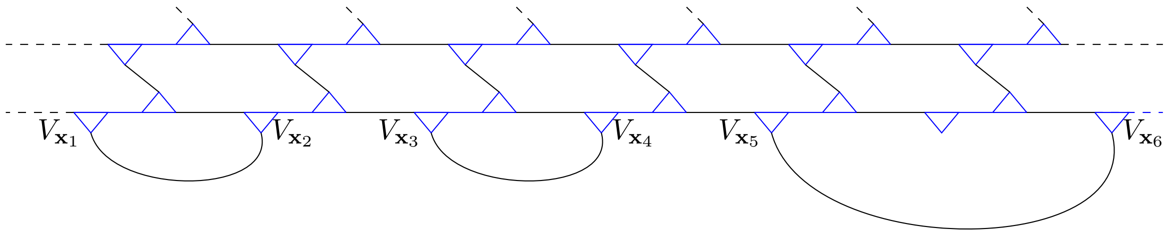

Figure 2.6: The resulting bonds on the lattice Λ_n after the site replacement in Fig. 2.2: bonds beyond adjacent site clusters are now allowed.

are the same as in (A.1)-(A.4) plus the additional entries given by

$$\mathcal{A}'_{\mathbf{x}_{2k-1},\mathbf{x}_{2k}} = -\mathcal{A}'_{\mathbf{x}_{2k},\mathbf{x}_{2k-1}} = (-1)^{d_k+1} \begin{array}{c} \\ \bar{H} \\ H \\ \bar{V} \\ V \\ \bar{T} \\ T \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{H} & H & \bar{V} & V & \bar{T} & T \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{t}_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{2.43}$$

if $\mathbf{x}_{2k-1}, \mathbf{x}_{2k} \in \partial\tilde{\Lambda}$, and where $d_k := |(x_{2k-1})_1 - (x_{2k})_1|$ is such that the factor $(-1)^{d_k+1}$ provides the clockwise odd orientation of the lattice Λ_n (see the discussion about the analogous sign in (A.3)). The partition function can be rewritten as

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = 2^{LM} \prod_{b_{\mathbf{xy}} \in \mathcal{B}_{\Lambda^n}} \cosh \beta J_{\mathbf{xy}} \cdot Pf\tilde{\mathcal{A}}', \tag{2.44}$$

and, proceeding as in 2.1.4, we can express the Pfaffian as a Grassmann functional integral so we get

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = C_{0,\Lambda} \prod_{k=1}^{n} \cosh \beta \tilde{J}_k \int d\bar{H}_\mathbf{x} dH_\mathbf{x} d\bar{V}_\mathbf{x} dV_\mathbf{x} d\bar{T}_\mathbf{x} dT_\mathbf{x} e^{S_\mathbf{t}(H,V,T) + \tilde{t}_1 V_{\mathbf{x}_1} V_{\mathbf{x}_2} + \ldots + \tilde{t}_n V_{\mathbf{x}_{2n-1}} V_{\mathbf{x}_{2n}}}, \tag{2.45}$$

with $S_\mathbf{t}(H, V, T)$ as in (2.19) and $C_{0,\Lambda}$ as in (2.21). We perform the T and $\bar{T}$ fields integration finally obtaining

$$Z_0(\mathbf{J}, \tilde{\mathbf{J}}) = C_{0,\Lambda} \cdot \prod_{k=1}^{n} \cosh \beta \tilde{J}_k \int \mathcal{D}X \; e^{S_\mathbf{t}(X, \beta\tilde{\mathbf{J}})}, \tag{2.46}$$

where

$$S_\mathbf{t}(X, \beta\tilde{\mathbf{J}}) = S_\mathbf{t}(X) + \tilde{t}_1 V_{\mathbf{x}_1} V_{\mathbf{x}_2} + \ldots + \tilde{t}_n V_{\mathbf{x}_{2n-1}} V_{\mathbf{x}_{2n}} \tag{2.47}$$

with $S_{\mathbf{t}}(X)$ as in (2.22). Then, by using (2.39) with $\Xi_0(\beta\tilde{\mathbf{J}}) = Z_0(\mathbf{J}, \tilde{\mathbf{J}})$ in (2.46), we get the following expression of the $2n-$point edge spin correlation function on Λ:

$$\langle \sigma_{\mathbf{x}_1}\sigma_{\mathbf{x}_2}\cdots\sigma_{\mathbf{x}_{2n-1}}\sigma_{\mathbf{x}_{2n}}\rangle^0_{\beta,\Lambda} = \langle V_{\mathbf{x}_1}V_{\mathbf{x}_2}\cdots V_{\mathbf{x}_{2n-1}}V_{\mathbf{x}_{2n}}\rangle^0_{\mathbf{t}(\beta),\Lambda}\ , \tag{2.48}$$

where the expectation in the r.h.s. is defined in (2.23). The sum over the spin configuration in definition (1.3) implies that the expectation value of an odd number of spin variables is vanishing. Similarly the expectations of an odd number of Grassmann variables are vanishing, since they are anti-commuting variables. Then the Grassmann representation in (2.48) can be extended to the $m-$point edge spin correlations:

$$\langle \sigma_{\mathbf{x}_1}\cdots\sigma_{\mathbf{x}_m}\rangle^0_{\beta,\Lambda} = \langle V_{\mathbf{x}_1}\cdots V_{\mathbf{x}_m}\rangle^0_{\mathbf{t}(\beta),\Lambda}\ , \tag{2.49}$$

where m can be both even and odd. Recall that the Grassmann variables in the r.h.s. of (2.50) are anti-commuting variables so that the equality holds for $(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ ordered sequence of m distinct points. By recalling the very definition of the edge spin observable $\tilde{\sigma}_{\mathbf{x}} = \frac{\sigma_{\mathbf{x}}}{\sqrt{a}}$, we obtain

$$\langle \tilde{\sigma}_{\mathbf{x}_1}\cdots\tilde{\sigma}_{\mathbf{x}_m}\rangle^0_{\beta,\Lambda} = a^{-m/2}\,\langle V_{\mathbf{x}_1}\cdots V_{\mathbf{x}_m}\rangle^0_{\mathbf{t}(\beta),\Lambda}\ . \tag{2.50}$$

The equality in (2.50) allows us to introduce the following sets of anticommuting variables $\{\,\Psi_{\mathbf{x}}\,\}_{\mathbf{x}\in\partial\Lambda}$ and $\{\,\frac{\partial}{\partial\Psi_{\mathbf{x}}}\,\}_{\mathbf{x}\in\partial\Lambda}$ and to derive unperturbed $m-$point edge spin correlation as

$$\langle \tilde{\sigma}_{\mathbf{x}_1}\cdots\tilde{\sigma}_{\mathbf{x}_m}\rangle^0_{\beta,\Lambda} = \frac{1}{\Xi_0(\mathbf{\Psi})}\frac{\partial^m}{\partial\Psi_{\mathbf{x}_1}\cdots\partial\Psi_{\mathbf{x}_m}}\,\Xi_0(\mathbf{\Psi})\Big|_{\mathbf{\Psi}=0}\ , \tag{2.51}$$

with the definitions in (2.36) and (2.37). Finally, the reconstruction of the truncated expectation values from the simple expectation values concludes the proof. □

2.2.3 The expression in terms of unperturbed Grassmann correlations

The generating function $\Xi_0(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$, which is given in (2.2) with $\lambda = 0$, by the results of lemma 2.1 and lemma 2.2, can be rewritten with the following Grassmann representation:

$$\Xi_0(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}}) = C_{0,\Lambda}\int \mathcal{D}X e^{S_{\mathbf{t}}(X)+(\mathbf{E},\mathbf{A})+(\mathbf{E},\tilde{\mathbf{A}})+(\mathbf{V},\mathbf{\Psi})}\ , \tag{2.52}$$

where $C_{0,\Lambda}$ is defined in (2.21), the quadratic action $S_{\mathbf{t}}(X)$ is defined in (2.22), the energy source terms $(\mathbf{E}, \mathbf{A}), (\mathbf{E}, \tilde{\mathbf{A}})$ are defined in (2.27) and the edge spin source term $(\mathbf{V}, \mathbf{\Psi})$ is defined in (2.37).

As a consequence of definition (2.52), the Grassmann representation is particularly convenient for computing the multipoint edge spin and energy correlation functions. By using (2.1), we can easily verify the following:

$$\langle \tilde{\sigma}_{\mathbf{x}_1}; \cdots ; \tilde{\sigma}_{\mathbf{x}_{n_\sigma}}; \varepsilon_{\mathbf{y}_1,j_1}; \cdots ; \varepsilon_{\mathbf{y}_m,j_m}; \tilde{\varepsilon}_{\mathbf{z}_1,j'_1}; \cdots ; \tilde{\varepsilon}_{\mathbf{z}_{m'},j'_{m'}} \rangle^0_{\beta,\Lambda} = a^{-\alpha} \prod_{i=1}^{m} (1 - t^2_{\mathbf{y}_i,j_i}) \cdot \\ \cdot \prod_{i'=1}^{m'} (1 - t^2_{\mathbf{z}_{i'},j_{i'}}) \, \langle V_{\mathbf{x}_1}; \cdots ; V_{\mathbf{x}_{n_\sigma}}; E_{\mathbf{y}_1,j_1}; \cdots ; E_{\mathbf{y}_m,j_m}; E_{\mathbf{z}_1,j'_1}; \cdots ; E_{\mathbf{z}_{m'},j'_{m'}} \rangle^0_{\mathbf{t}(\beta),\Lambda} \,, \tag{2.53}$$

where $\alpha := \alpha(n_\sigma, m, m') = \frac{n_\sigma}{2} + m + 2m'$ and $\langle \cdot \rangle^0_{\mathbf{t}(\beta),\Lambda}$ is the unperturbed average with respect to the Gaussiann Grassmann measure $\int \mathcal{D}X e^{S_{\mathbf{t}}(X)}$ (see after (2.23)). Therefore, the r.h.s. of Eq.(2.53) can be computed via the fermionic Wick rule of the Grassmann variables in (3.2), leading to a sum over all the Feynman graphs obtained by pairing ("contracting") the Grassmann fields involved; truncation means that only connected Feynman diagrams should be considered. In the infinite volume limit, these multipoint correlation functions are

$$\lim_{L,M\to\infty} \langle \tilde{\sigma}_{\mathbf{x}_1}; \cdots ; \tilde{\sigma}_{\mathbf{x}_{n_\sigma}}; \varepsilon_{\mathbf{y}_1,j_1}; \cdots ; \varepsilon_{\mathbf{y}_m,j_m}; \tilde{\varepsilon}_{\mathbf{z}_1,j'_1}; \cdots ; \tilde{\varepsilon}_{\mathbf{z}_{m'},j'_{m'}} \rangle^0_{\beta,\Lambda} = a^{-\alpha} \prod_{i=1}^{m} (1 - t^2_{\mathbf{y}_i,j_i}) \cdot \\ \cdot \prod_{i'=1}^{m'} (1 - t^2_{\mathbf{z}_{i'},j'_{i'}}) \, \langle V_{\mathbf{x}_1}; \cdots ; V_{\mathbf{x}_{n_\sigma}}; E_{\mathbf{y}_1,j_1}; \cdots ; E_{\mathbf{y}_m,j_m}; E_{\mathbf{z}_1,j'_1}; \cdots ; E_{\mathbf{z}_{m'},j'_{m'}} \rangle^0_{\mathbf{t}(\beta)} \,, \tag{2.54}$$

where $\langle \cdot \rangle^0_{\mathbf{t}(\beta)}$ denotes the $\Lambda \to \infty$ limit of $\langle \cdot \rangle^0_{\mathbf{t}(\beta),\Lambda}$.
In the next section we illustrate how to extend the results obtained so far to the presence of a perturbation $\lambda \neq 0$.

2.3 The perturbed generating function

As reviewed in the previous sections, in the unperturbed Ising model, the generating function for the correlation functions can be represented in terms of a Gaussian Grassmann integral. A similar representation in terms of a non-Gaussian Grassmann integral is valid also in the perturbed $\lambda \neq 0$ model, as stated in the following Lemma 2.3. In the proof we review the results derived in [4, 37] introducing also the edge observables, and then, by proving that the setting is not modified by the presence of the edge observables, we derive the Grassmann generating functional with edge spin sources, bulk energy sources and edge energy sources.

Lemma 2.3. *There exists $\lambda_0 > 0$ such that, if $|\lambda| \leq \lambda_0$, then for any ordered sequence of distinct boundary points $(\mathbf{x}_1, \ldots, \mathbf{x}_{n_\sigma})$, $\mathbf{x}_i \in \partial\Lambda$, $i = 1, \ldots, n_\sigma$ and for any sets of distinct pairs $\{ (\mathbf{y}_i, j_i) : \mathbf{y}_i \in \Lambda^\circ, j_i = 1, 2 \}_{i=1}^{m}$ and $\{ (\mathbf{z}_i, j'_i) : \mathbf{z}_i \in \partial\Lambda, j'_i = 1, 2 \}_{i=1}^{m'}$, the multipoint* *generating function is*

$$\langle \tilde\sigma_{\mathbf{x}_1};\cdots;\tilde\sigma_{\mathbf{x}_{n_\sigma}};\varepsilon_{\mathbf{y}_1,j_1};\cdots;\varepsilon_{\mathbf{y}_m,j_m};\tilde\varepsilon_{\mathbf{z}_1,j_1};\cdots;\tilde\varepsilon_{\mathbf{z}_{m'},j_{m'}}\rangle_\Lambda^\lambda =$$
$$= \frac{\partial^{n_\sigma}}{\partial\Psi_{\mathbf{x}_1}\cdots\partial\Psi_{\mathbf{x}_{n_\sigma}}}\frac{\partial^m}{\partial A_{\mathbf{y}_1,j_1}\cdots\partial A_{\mathbf{y}_m,j_m}}\frac{\partial^{m'}}{\partial\tilde A_{\mathbf{z}_1,j'_1}\cdots\partial\tilde A_{\mathbf{z}_{m'},j'_{m'}}}\log\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})\Bigg|_{\mathbf{\Psi}=\mathbf{A}=\tilde{\mathbf{A}}=\mathbf{0}}, \tag{2.55}$$

where $\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ *is the Grassmann generating functional*

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = C_{\lambda,\Lambda}\cdot e^{\tilde V_\Lambda(\lambda)}\int \mathcal{D}X\; e^{S_{\mathbf{t}}(X)+V(X;\mathbf{A},\tilde{\mathbf{A}})+(\mathbf{E},\mathbf{A})+(\mathbf{E},\tilde{\mathbf{A}})+(\mathbf{V},\mathbf{\Psi})}, \tag{2.56}$$

where

- $$C_{\lambda,\Lambda} := C_{0,\Lambda}\cdot\prod_{\{\mathbf{x},\mathbf{y}\}}\cosh^2\left(\frac{\beta\lambda}{2}V(\{\mathbf{x},\mathbf{y}\})\right), \tag{2.57}$$
  with $C_{0,\Lambda}$ *in* (2.21)*;*
- $\tilde V_\Lambda(\lambda)$ *is an analytic function of* λ*, independent of lattice spacing* a *and satisfying the bound* $|V_\Lambda(\lambda)|\le C|\lambda|LM$*, for a suitable* $C>0$*;*
- $S_{\mathbf{t}}(X)$ *is the unperturbed quadratic action in* (2.22) *and* $(\mathbf{E},\mathbf{A})$*,* $(\mathbf{E},\tilde{\mathbf{A}})$*,* $(\mathbf{V},\mathbf{\Psi})$ *are the source terms in* (2.27)*,*(2.37)*;*
- $V(X,\mathbf{A},\tilde{\mathbf{A}})$ *is the interaction term, which is given by the following polynomial expression*

$$\begin{aligned} V(X,\mathbf{A},\tilde{\mathbf{A}}) = &\sum_{\substack{p,q\ge 0\\ p+q\ge 1}}\sum_{\underline{k},\underline{k'},\underline{j},\underline{j'}}\int d\underline{\mathbf{x}}d\underline{\mathbf{x}}'d\underline{\mathbf{y}}d\underline{\mathbf{y}}' \\ &\cdot W_{p,q}(\underline{\mathbf{x}};\underline{\mathbf{x}}';\underline{\mathbf{y}};\underline{\mathbf{y}}')\prod_{i=1}^{n}E_{\mathbf{x}_i,k_i}\prod_{i=1}^{n'}E_{\mathbf{x}'_i,k'_i}\prod_{i=1}^{m}A_{\mathbf{y}_i,j_i}\prod_{i=1}^{m'}\tilde A_{\mathbf{y}'_i,j'_i}\end{aligned} \tag{2.58}$$

where $p=n+n', q=m+m'$*,* $\underline{k}=(k_1,\dots,k_n)$*,* $\underline{k'}=(k'_1,\dots,k'_{n'})$*,* $\underline{j}=(j_1,\dots,j_m)$*,* $\underline{j'}=(j'_1,\dots,j'_{m'})$*, and* $\underline{\mathbf{x}}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$*,* $\underline{\mathbf{y}}=(\mathbf{y}_1,\dots,\mathbf{y}_m)\in\Lambda^\circ$*,* $\underline{\mathbf{x}'}=(\mathbf{x}'_1,\dots,\mathbf{x}'_{n'})$ *and* $\underline{\mathbf{y}}'=(\mathbf{y}'_1,\dots,\mathbf{y}'_{m'})\in\partial\Lambda$*. Note that in the expression* (2.58) *are involved only the Grassmann variables in* $E_{\mathbf{x},1}:=\bar H_{\mathbf{x}}H_{\mathbf{x}+a\hat{\mathbf{e}}_1}, E_{\mathbf{x},2}:=\bar V_{\mathbf{x}}V_{\mathbf{x}+a\hat{\mathbf{e}}_2}$ *and the source variables* $\mathbf{A}_{\mathbf{x}}$ *and* $\tilde{\mathbf{A}}_{\mathbf{x}}$*: they are combined in polynomial whose coefficients* $W_{p,q}(\underline{\mathbf{x}};\underline{\mathbf{x}}';\underline{\mathbf{y}};\underline{\mathbf{y}}')$ *satisfy the following bound*

$$|W_{p,q}(\underline{\mathbf{x}};\underline{\mathbf{x}}';\underline{\mathbf{y}};\underline{\mathbf{y}}')|\le C^{p+q}(\beta|\lambda|)^{\max\{1,c(p+q)\}}a^{-(n+m+2n'+2m')}e^{-\kappa\,\delta(\underline{\mathbf{x}},\underline{\mathbf{x}}',\underline{\mathbf{y}},\underline{\mathbf{y}}')/a} \tag{2.59}$$

for suitable constants $C,c,\kappa>0$*; here* $\delta(\underline{\mathbf{x}},\underline{\mathbf{x}}',\underline{\mathbf{y}},\underline{\mathbf{y}}')$ *is the length of the shortest tree graph composed of bonds on* Λ *which connects all the elements of the set* $\{\underline{\mathbf{x}},\underline{\mathbf{x}}',\underline{\mathbf{y}},\underline{\mathbf{y}}'\}$*.*

Note that the factor $(\beta|\lambda|)^{\max\{1,c(p+q)\}}$ measures the smallness in λ of the coefficient W, while the factor $a^{-(n+m+2n'+2m')}$ will be associated with the *scaling dimension*; moreover we will refer to a coefficient W as a *kernel* of the interaction. Since in the interaction defined by the polynomial expression in (2.58) do not appear the edge spin source variables, the edge spin source are not dressed by the interaction, unlike the energy sources; moreover in the following proof we can simply adapt the proof of [37, Prop.1]: allowing the presence of the spin sources doesn't affect their result, so we obtain an interaction independent of $\mathbf{\Psi}$.

Proof of lemma.2.3. We start by considering the set $\mathcal{B}_{\Lambda_n} = \mathcal{B}_\Lambda \cup \tilde{\mathcal{B}}$ of bond variables of Λ_n where $\mathcal{B}_\Lambda$ is the set of n.n. bonds in Λ and $\tilde{\mathcal{B}} := \{\,\tilde{b}_1, \dots, \tilde{b}_n\,\}$ denotes the set of the additional bonds (the ones below the lower edge in Fig. 2.4). In $\mathcal{B}_\Lambda$ we want to distinguish in the bulk bonds and the edge bonds: $\mathcal{B}_\Lambda = \mathcal{B}_{\Lambda^\circ} \cup \mathcal{B}_{\partial\Lambda}$, $\mathcal{B}_{\Lambda^\circ} := \{\, b_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_\mathbf{i}}\,\}^{i=1,2}_{\mathbf{x}\in\Lambda^\circ}$ and $\mathcal{B}_{\partial\Lambda} := \{\, b_{\mathbf{x},\mathbf{x}+a\hat{\mathbf{e}}_\mathbf{i}}\,\}^{i=1,2}_{\mathbf{x}\in\partial\Lambda}$. For notational convenience, given any bulk bond $b \in \mathcal{B}_\Lambda$, ε_b, E_b, A_b denote the corresponding bond operators, given any edge bond $b' \in \mathcal{B}_{\partial\Lambda}$, $\tilde{\varepsilon}_{b'}$, $E_{b'}$, $\tilde{A}_{b'}$ denote the corresponding operators, and given any additional bond $\tilde{b} \in \tilde{\mathcal{B}}$, $\sigma_{\tilde{b}}$ denotes the bond spin operator, i.e. is the product of the edge spins at the endpoint of the additional bond $\tilde{b}$, and $\tilde{J}_{\tilde{b}}$ its energy interaction.

Now we combine the unperturbed generating function in (2.29) with the one in (2.40), so that we are considering all the source variables expressed in terms of bonds variables and we introduce the perturbation as the $\lambda-$dependent term of (1.2): then we get

$$\begin{aligned}\Xi_\lambda(\mathbf{A}, \tilde{\mathbf{A}}, \beta\tilde{\mathbf{J}}) = \sum_{\underline{\sigma}\in\Omega_{LM}} & e^{\sum_{b\in\mathcal{B}_{\Lambda^\circ}}(\beta J_b + a^{-1}A_b)a\varepsilon_b + \sum_{b'\in\partial\mathcal{B}_\Lambda}(\beta J_{b'} + a^{-2}\tilde{A}_{b'})a^2\tilde{\varepsilon}_{b'}} \cdot \\ & \cdot e^{\sum_{\tilde{b}\in\tilde{\mathcal{B}}}\tilde{J}_{\tilde{b}}\sigma_{\tilde{b}}}\, e^{\beta\lambda\sum_{X\subset\Lambda}V(X)\prod_{\mathbf{x}\in X}\sigma_\mathbf{x}}\,,\end{aligned} \tag{2.60}$$

where in $e^{\beta\lambda\sum_{X\subset\Lambda}V(X)\prod_{\mathbf{x}\in X}\sigma_\mathbf{x}}$ the product is over sites in Λ since to get Λ_n we added only new bonds and no new sites. The key to the first step in the proof is the remark is that if $b_1, \dots, b_m$ are m distinct bulk bonds, $b'_1, \dots, b'_{m'}$ are m' distinct edge bonds and $\tilde{b}_1, \dots, \tilde{b}_n$ are n distinct additional bonds, then the Grassmann representation for $\Xi_0(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$ induces the following correspondence:

$$\begin{aligned}\sum_{\underline{\sigma}\in\Omega_{LM}} & e^{\sum_{b\in\mathcal{B}^\circ_\Lambda}(\beta J_b + a^{-1}A_b)a\varepsilon_b + \sum_{b\in\mathcal{B}_{\partial\Lambda}}(\beta J_{b'} + a^{-2}\tilde{A}_{b'})a^2\tilde{\varepsilon}_{b'} + \sum_{\tilde{b}\in\tilde{\mathcal{B}}}\tilde{J}_{\tilde{b}}\sigma_{\tilde{b}}}\varepsilon_{b_1}\cdots\varepsilon_{b_m}\tilde{\varepsilon}_{b'_1}\cdots \\ & \cdots\tilde{\varepsilon}_{b'_{m'}}\sigma_{\tilde{b}_1}\cdots\sigma_{\tilde{b}_n} = \frac{\partial^m}{\partial A_{b_1}\cdots\partial A_{b_m}}\frac{\partial^{m'}}{\partial\tilde{A}_{b'_1}\cdots\partial\tilde{A}_{b'_{m'}}}\frac{\partial^n}{\partial\beta\tilde{J}_1\cdots\partial\beta\tilde{J}_n}\Xi_0(\mathbf{A}, \tilde{\mathbf{A}}, \beta\tilde{\mathbf{J}}) = \\ & = \frac{(-2)^{LM}}{a^{(m+2m')}}\Big[\prod_{b\in\mathcal{B}_{\Lambda^\circ}}\cosh(\beta J_b + a^{-1}A_b)\Big]\Big[\prod_{b'\in\mathcal{B}_{\partial\Lambda}}\cosh(\beta J_{b'} + \tilde{A}_{b'})\Big]\Big[\prod_{\tilde{b}\in\tilde{\mathcal{B}}}\cosh(\beta\tilde{J}_{\tilde{b}})\Big]\cdot \\ & \cdot\int\mathcal{D}X\,\big(t_{b_1}(A) + (1 - t^2_{b_1}(A))E_{b_1}\big)\cdots\big(t_{b_{m'}}(\tilde{A}) + (1 - t^2_{b_{m'}}(\tilde{A}))E_{b_{m'}}\big)\cdot \\ & \cdot\big(t_{\tilde{b}_1} + (1 - t^2_{\tilde{b}_1})E_{\tilde{b}_1}\big)\cdots\big(t_{\tilde{b}_n} + (1 - t^2_{\tilde{b}_n})E_{\tilde{b}_n}\big)e^{S_\mathbf{t}(X;\mathbf{A},\tilde{\mathbf{A}};\beta\tilde{\mathbf{J}})}\,,\end{aligned} \tag{2.61}$$

where $t_b(A) = \tanh(\beta J_b + a^{-1}A_b)$, $t_b(\tilde{A}) = \tanh(\beta J_b + \tilde{A}_b)$, $E_{\tilde{b}_k} := V_{\mathbf{x}_{2k-1}} V_{\mathbf{x}_{2k}}$ and $S_{\mathbf{t}}(X; \mathbf{A}, \tilde{\mathbf{A}}, \beta\tilde{\mathbf{J}})$ is obtained by combining (2.33) and (2.47). This correspondence is invalid for repeated bond variables: note that $[t_b + (1 - t_b^2)E_b]^2 = t_b^2 + 2t_b(1 - t_b^2)E_b$, while $\varepsilon_b^2 = a^{-2}$, $\tilde{\varepsilon}_b^2 = a^{-4}$ and $\sigma_{\tilde{b}}^2 = 1$. This last observation can be used to remove repeated bond operators from any expression; therefore, in order to derive a Grassmann representation for $\Xi_\lambda(\mathbf{A}, \tilde{\mathbf{A}}, \beta\tilde{\mathbf{J}})$, it is enough to express the interaction term (i.e. the λ-dependent term) in (2.60) as sum of products of *distinct* bond operators. Since we are interested in the representation on the lattice Λ and not on Λ_n, we consider only bond operators on Λ, i.e. only ε_b and $\tilde{\varepsilon}_b$. Next we can replace every bulk bond operator ε_b by $a^{-1}(t_b(A) + (1 - t_b^2(A))E_b)$ and every edge bond operator $\tilde{\varepsilon}_{b'}$ by $a^{-2}(t_{b'}(\tilde{A}) + (1 - t_{b'}^2(\tilde{A}))E_{b'})$, in the sense explained above, and finally we can re-exponentiate the big sum of products of Grassmann variables, so obtaining the desired Grassmann representation of $\Xi_\lambda(\mathbf{A}, \tilde{\mathbf{A}}, \beta\tilde{\mathbf{J}})$.

This can be implemented as follows. Any even interaction of the form $V(X)\prod_{\mathbf{x}\in X}\sigma_{\mathbf{x}}$ can always be rewritten in terms of products of $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$, with $\mathbf{x}, \mathbf{y} \in X \subset \Lambda$. The pair of sites $(\mathbf{x}, \mathbf{y})$ can be connected via an "up" path $\mathcal{C}_U(\mathbf{x}, \mathbf{y})$ and a "down" path $\mathcal{C}_D(\mathbf{x}, \mathbf{y})$ on Λ, as described in Fig. 2.7. Then $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$ can be rewritten in terms of a product of energy

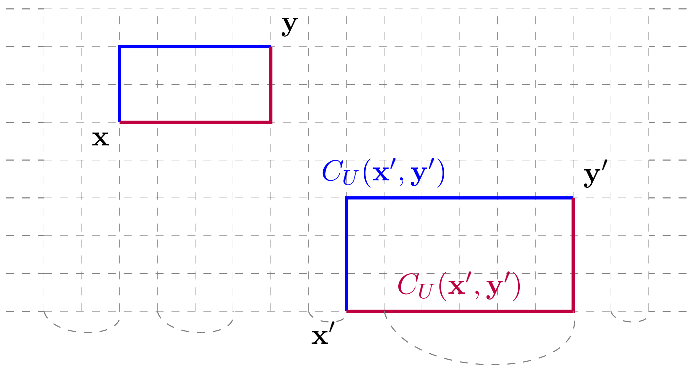

Figure 2.7: An example. The blue paths correspond to C_U, the purple paths corresponds to C_D. The two bulk points $\mathbf{x}$ and $\mathbf{y}$ are connected by two paths which will be both formed by 6 bulk bonds. The boundary point $\mathbf{x}'$ and the bulk point $\mathbf{y}'$ are connected by a path $C_U(\mathbf{x}', \mathbf{y}')$ (blue) formed by 1 edge bond and 8 bulk bonds, and by a path $C_D(\mathbf{x}', \mathbf{y}')$ (purple) formed by 7 edge bonds and 2 bulk bonds. Note that we can always avoid involving the additional bonds.

density operators associated with the bonds along blue and red paths: we will consider an operator ε_b for each bulk bond traveled by the path and an operator $\tilde{\varepsilon}_b$ for each edge bond traveled by the path. Then we get $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} = \frac{1}{2}U_{\mathbf{x},\mathbf{y}} + \frac{1}{2}D_{\mathbf{x},\mathbf{y}}$, where

$$U_{\mathbf{x},\mathbf{y}} = \Big[\prod_{b\in\mathcal{C}_U(\mathbf{x},\mathbf{y})\cap\mathcal{B}_{\Lambda^\circ}} a\varepsilon_b\Big]\Big[\prod_{b\in\mathcal{C}_U(\mathbf{x},\mathbf{y})\cap\mathcal{B}_{\partial\Lambda}} a^2\tilde{\varepsilon}_b\Big]\,,$$

and

$$D_{\mathbf{x},\mathbf{y}} = \Big[\prod_{b\in\mathcal{C}_D(\mathbf{x},\mathbf{y})\cap\mathcal{B}_{\Lambda^\circ}} a\varepsilon_b\Big]\Big[\prod_{b\in\mathcal{C}_D(\mathbf{x},\mathbf{y})\cap\mathcal{B}_{\partial\Lambda}} a^2\tilde{\varepsilon}_b\Big]\,.$$

Note that we choose paths in this way in order to be sure that we have an expression which manifestly retains the rotation and reflection symmetries of the original interaction. Since we are able to avoid the additional bonds we can consider $\beta\tilde{\mathbf{J}} = \mathbf{0}$ and we can proceed exactly as in [37], taking into account also the presence of the edge energies: they must be treated like the bulk energy operator but entail a different scaling dimension. Combining the proof in [37], which allows to write $\Xi_\lambda(\mathbf{A}, \tilde{\mathbf{A}})$ as in (2.56) with $\mathbf{\Psi} = \mathbf{0}$, with the proof of lemma 2.2, which allows to write as a Grassmann functional integral the $\mathbf{\Psi} \neq 0$ contribution, we conclude the proof of 2.3. □

Finally, we note that the polynomial representation of $V(X, \mathbf{A}, \tilde{\mathbf{A}})$ in (2.58), which contains monomials not only quadratic but of arbitrary order in the Grassmann variables, implies that if $\lambda \neq 0$ the expectation values will be averages with respect to a "non Gaussian" Grassmann measure (see (2.23)). However in the following chapters we will derive a procedure which allows to express these perturbed expectations in terms of the unperturbed ones, up to proper renormalizations of the critical temperature and of the Grassmann fields, which will include the entire effect of the perturbation.

Chapter 3

Massive and massless variables

In this chapter we want to rewrite the perturbed generating function $\Xi_\lambda(\boldsymbol{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$ in what will be the most convenient form for deriving the correlation functions. Moreover we want to introduce the *critical propagators* in terms of which will be derived the Grassmann correlations.

What we are really interested in discussing are the correlations on the upper half-plane. So, from now on, we will consider only the theory obtained by systematically replacing each finite volume quantity with corresponding formal thermodynamical limit. A rigorous approach would require studying the estimates at finite L, M first and then taking the infinite volume limit. However, here a less rigorous approach is preferred, which guarantees a clearer and more readable presentation of the results. On the other hand, the exchange of limits does not involve substantial modifications of the theory: the reader can find in [4] all the technical details needed for a rigorous justification of this choice. In particular, this simplification will allow us to obtain less elaborate expressions for the critical propagators (in this chapter) and for the localization procedures (in the next chapter).

In conclusion, we simplify the exposition by directly considering the discrete upper half plane $\mathbb{H}_1$, which is the infinite volume limit of the cylindrical lattice with $a = 1$, and by considering the interaction energies between nearest neighbour spins as independent of the lattice positions $\mathbf{J} := \{ J_1, J_2 \}$ and $\mathbf{t} = \{ t_1, t_2 \}$, with $t_i = \tanh \beta J_i, i = 1, 2$.

As a first step we will perform a change of Grassmann variables, obtaining the perturbed generating function in terms of the so-called *massive* and *massless* variables. To choose which new variables to use we go back to the Grassmann representation of $\Xi_0(\mathbf{0}, \mathbf{0}, \mathbf{0})$, which is the unperturbed partition function in (2.20), with the quadratic action in the upper half plane that can be expressed as $S_{\mathbf{t}}(X) := \frac{1}{2} \sum_{\mathbf{x},\mathbf{y} \in \mathbb{H}_a} X^T_{\mathbf{x}} A_{\mathbf{xy}} X_{\mathbf{y}}$ where $X^T_{\mathbf{x}} := (\bar{H}^T_{\mathbf{x}}, H^T_{\mathbf{x}}, \bar{V}^T_{\mathbf{x}}, V^T_{\mathbf{x}})$ and A is an infinite antisymmetric matrix. We can evaluate the unperturbed partition function as the Pfaffian of this matrix, namely

$$\int \mathcal{D}X e^{\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{H}_a} X_{\mathbf{x}}^T A_{\mathbf{xy}} X_{\mathbf{y}}} = Pf A\,, \tag{3.1}$$

and the averages of m Grassmann monomials $\tilde{X}_{\mathbf{x}_1},\ldots,\tilde{X}_{\mathbf{x}_m}$ with respect to the Grassmann Gaussian measure can be computed in terms of the fermionic Wick rule:

$$\langle \tilde{X}_{\mathbf{x}_1}\cdots\tilde{X}_{\mathbf{x}_m}\rangle_\beta^0 = \frac{1}{Pf A}\int \mathcal{D}X\ \tilde{X}_{\mathbf{x}_1}\cdots\tilde{X}_{\mathbf{x}_m}\ e^{\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{H}_a} X_{\mathbf{x}}^T A_{\mathbf{xy}} X_{\mathbf{y}}} = Pf G\,, \tag{3.2}$$

where G is $m\times m$ matrix (if m is even) with entries given by

$$G_{jk} = \langle X_{\mathbf{x}_j} X_{\mathbf{x}_k}\rangle_\beta^0 = -[A^{-1}]_{\mathbf{x}_j,\mathbf{x}_k}\,. \tag{3.3}$$

Then, we choose as new Grassmann variables the ones in terms of which A is diagonal; moreover the entries in Eq. (3.3) will associated to the massless and massive propagators, which are also diagonal. In the rest of the chapter we proceed as follows:

- in Sec. 3.1, we derive the change of the Grassmann variables; to invert the antisymmetric matrix A associated to the quadratic action $S_{\mathbf{t}}(X)$, we exploit the translation invariance in the horizontal direction to get a 'block diagonalization' procedure, which naturally leads to a separation in massive and massless degrees of freedom;

- in Sec. 3.2 we derive the explicit expression for the propagators by using the analogue of Eq. (3.3), the main properties, the limit in the continuous upper plane, as well the multiscale and the bulk-edge decomposition; moreover we derive the correspondent bounds and, as we will see, the massive propagators will exhibit an exponential decay with the distance, while the massless propagators have a singularity for a critical value of the temperatures, which will be changed by the perturbation;

- in Sec. 3.3 we finally derive the expression $\Xi_\lambda(\boldsymbol{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ in terms of the massless and the massive variables: we will perform some suitable rearrangement in order to obtain the best expression to derive with respect the source fields to get the correlations.

With this new structure of the Grassmann representation of $\Xi_\lambda(\boldsymbol{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ we will be able, in the next chapter 4, to introduce the iterative procedures needed to obtain convergent expansion of correlations.

The results we will talk about in this chapter, such as diagonalization and explicit form of quadratic action and decompositions and the bounds of propagators, were already introduced in [4] for the model on the cylindrical lattice. Keeping in mind that in the next chapters we will be interested in correlations for the observables at the lower edge of the half-plane, we will briefly review how to derive these results for the model on the upper half plane: compared to the finite volume expressions we will deal with simplified expressions that are on the one hand more readable and on the other less rigorous (but we will always provide references to [4] to follow the technical details we are neglecting).

3.1 Diagonalization of the free action

We start by recalling that, on $\mathbb{H}_1$, the quadratic action (2.22) can be rewritten as $S_{\mathbf{t}}(X) := \frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{H}_1} X_{\mathbf{x}}^T\, A_{\mathbf{x}\mathbf{y}}\, X_{\mathbf{y}}$ where $X_{\mathbf{x}}^T := (\bar{H}_{\mathbf{x}}^T, H_{\mathbf{x}}^T, \bar{V}_{\mathbf{x}}^T, V_{\mathbf{x}}^T)$ and $A_{\mathbf{x}\mathbf{y}} := A_{\mathbf{x}\mathbf{y}}(\mathbf{J})$ is the antisymmetric infinite matrix given by

$$A_{\mathbf{x}\mathbf{y}} = \begin{pmatrix} 0 & \mathbb{Z}_H & -\mathbb{1} & -\mathbb{1} \\ -\mathbb{Z}_H^T & 0 & \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -\mathbb{1} & 0 & \mathbb{Z}_V \\ \mathbb{1} & \mathbb{1} & -\mathbb{Z}_V^T & 0 \end{pmatrix}_{\mathbf{x}\mathbf{y}}, \tag{3.4}$$

where $\mathbb{1} = \mathbb{1}_1 \otimes \mathbb{1}_2$, $\mathbb{Z}_H = \mathbb{Z}_1 \otimes \mathbb{1}_2$, $\mathbb{Z}_V = \mathbb{1}_1 \otimes \mathbb{Z}_2$, and, for $j = 1, 2$ the matrices $\mathbb{1}_j$ and $\mathbb{Z}_j$ have entries given by

$$(\mathbb{1}_j)_{x^{(j)}y^{(j)}} = \begin{cases} 1 & \text{if } \ y^{(j)} = x^{(j)}\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad (\mathbb{Z}_j)_{x^{(j)}y^{(j)}} = \begin{cases} 1 & \text{if } \ y^{(j)} = x^{(j)}\,, \\ t_j & \text{if } \ y^{(j)} = x^{(j)} + 1\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.5}$$

where $t_1 := \tanh \beta J_1$ and $t_2 := \tanh \beta J_2$. We perform the Fourier transform in the horizontal direction, which is translation invariant: we let $k := k_1$ and for each $x^{(2)}$ we define

$$\begin{aligned} \hat{H}_{x^{(2)}}(k) &= \frac{1}{\sqrt{2}} \sum_{x^{(1)}=-\infty}^{+\infty} e^{ikx^{(1)}} H_{(x^{(1)},x^{(2)})}\,, \\ \hat{V}_{x^{(2)}}(k) &= \frac{1}{\sqrt{2}} \sum_{x^{(1)}=-\infty}^{+\infty} e^{ikx^{(1)}} V_{(x^{(1)},x^{(2)})}\,, \\ \hat{\bar{H}}_{x^{(2)}}(k) &= \frac{1}{\sqrt{2}} \sum_{x^{(1)}=-\infty}^{+\infty} e^{ikx^{(1)}} \bar{H}_{(x^{(1)},x^{(2)})}\,, \\ \hat{\bar{V}}_{x^{(2)}}(k) &= \frac{1}{\sqrt{2}} \sum_{x^{(1)}=-\infty}^{+\infty} e^{ikx^{(1)}} \bar{V}_{(x^{(1)},x^{(2)})}\,. \end{aligned} \tag{3.6}$$

Let $\hat{X} = \{\, (\hat{H}_{x^{(2)}}(k), \hat{H}_{x^{(2)}}(k), \hat{H}_{x^{(2)}}(k), \hat{H}_{x^{(2)}}(k)) : x^{(2)} \in [1, \infty)\,, k \in [-\pi, \pi]\,\}$, the quadratic action can be rewritten as

$$S_{\mathbf{t}}(\hat{X}) = \frac{1}{2} \sum_{x^{(2)},y^{(2)}=1}^{\infty} \int_{-\pi}^{\pi} dk \hat{X}_{x^{(2)}}^T(-k)\, \hat{A}_{x^{(2)}y^{(2)}}(k)\, \hat{X}_{y^{(2)}}(k)\,, \tag{3.7}$$

where $\hat{X}_{x^{(2)}}^T(k) := (\hat{\bar{H}}_{x^{(2)}}^T(k), \hat{H}_{x^{(2)}}^T(k), \hat{\bar{V}}_{x^{(2)}}^T(k), \hat{V}_{x^{(2)}}^T(k))$ and

$$\hat{A}_{x^{(2)}y^{(2)}}(k) = \begin{pmatrix} 0 & z_1 \cdot \mathbb{1}_2 & -\mathbb{1}_2 & -\mathbb{1}_2 \\ -\bar{z}_1 \cdot \mathbb{1}_2 & 0 & \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 & 0 & \mathbb{Z}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 & -\mathbb{Z}_2^T & 0 \end{pmatrix}_{x^{(2)}y^{(2)}}, \tag{3.8}$$

where $z_1 := z_1(k) = (1 + t_1 e^{-ik})$ and $\bar{z}_1$ denotes the complex conjugate. If we consider the blocks

$$\mathbb{A}_m = \begin{pmatrix} 0 & z_1 \cdot \mathbb{1}_2 \\ -\bar{z}_1 \cdot \mathbb{1}_2 & 0 \end{pmatrix}, \ \mathbb{A}_n = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ -\mathbb{Z}_2^T & 0 \end{pmatrix}, \ \mathbb{B} = \begin{pmatrix} -\mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}, \tag{3.9}$$

we can perform a Schur reduction ($\mathbb{A}_1$ is invertible), so we obtain

$$\hat{A}_{x^{(2)}y^{(2)}}(k) = \mathbb{L}^\dagger(k)\mathbb{M}(k)\mathbb{L}(k)\,, \tag{3.10}$$

where

$$\mathbb{L}(k) := \begin{pmatrix} \mathbb{1}_2 & \mathbb{A}_m^{-1}\mathbb{B} \\ 0 & \mathbb{1}_2 \end{pmatrix}, \qquad \mathbb{L}^\dagger(k) = \mathbb{L}^T(-k)\,, \qquad \mathbb{M}(k) := \begin{pmatrix} \mathbb{A}_m & 0 \\ 0 & \mathbb{A}_c \end{pmatrix}, \tag{3.11}$$

with

$$\mathbb{A}_m^{-1}\mathbb{B} = \begin{pmatrix} -\bar{z}_1^{-1}\mathbb{1}_2 & \bar{z}_1^{-1}\mathbb{1}_2 \\ -z_1^{-1}\mathbb{1}_2 & -z_1^{-1}\mathbb{1}_2 \end{pmatrix}, \qquad \mathbb{A}_c := \mathbb{A}_n + \mathbb{B}^T\mathbb{A}_m^{-1}\mathbb{B} = \begin{pmatrix} \alpha_1\mathbb{1}_2 & \tilde{\mathbb{Z}}_2 \\ -\tilde{\mathbb{Z}}_2^T & -\alpha_1\mathbb{1}_2 \end{pmatrix}, \tag{3.12}$$

where the matrix $\tilde{\mathbb{Z}}_2$ has the entries given by

$$\left(\tilde{\mathbb{Z}}_2\right)_{x^{(2)}y^{(2)}} = \begin{cases} \beta_1 & \text{if } x^{(2)} = y^{(2)}\,, \\ t_2 & \text{if } y^{(2)} = x^{(2)} + 1\,, \\ 0 & \text{otherwise}\ , \end{cases} \tag{3.13}$$

and

$$\alpha_1 := \alpha_1(k) = \frac{-2it_1 \sin k}{|1 + t_1 e^{ik}|^2}\,, \qquad \beta_1 := \beta_1(k) = \frac{t_1^2 - 1}{|1 + t_1 e^{ik}|^2}\,. \tag{3.14}$$

By substituting the Schur reduction in Eq. (3.10) in the action in (3.7), if we let $\hat{\Phi}_{y^{(2)}}(k) := \mathbb{L}(k)\hat{X}_{y^{(2)}}(k)$ be the new Grassmann variables given by

$$\begin{aligned} \hat{\xi}_{+,y^{(2)}}(k) &= \hat{\bar{H}}_{y^{(2)}}(k) - \bar{z}_1^{-1}\left(\hat{\bar{V}}_{y^{(2)}}(k) - \hat{V}_{y^{(2)}}(k)\right)\,, \\ \hat{\xi}_{-,y^{(2)}}(k) &= \hat{H}_{y^{(2)}}(k) - z_1^{-1}\left(\hat{\bar{V}}_{y^{(2)}}(k) + \hat{V}_{y^{(2)}}(k)\right)\,, \\ \hat{\varphi}_{+,y^{(2)}}(k) &= \hat{\bar{V}}_{y^{(2)}}(k)\,, \\ \hat{\varphi}_{-,y^{(2)}}(k) &= \hat{V}_{y^{(2)}}(k)\,, \end{aligned} \tag{3.15}$$

we get

$$S_{\mathbf{t}}(\hat{X}) = \frac{1}{2}\sum_{x^{(2)},y^{(2)}=1}^{\infty} \int_{-\pi}^{\pi} dk\ \hat{\Phi}^T_{x^{(2)}}(-k)\mathbb{M}(k)\hat{\Phi}_{y^{(2)}}(k) =: S(\hat{\Phi})\,, \tag{3.16}$$

where $\hat{\Phi} := \{ (\hat{\xi}_{+,y^{(2)}}(k), \hat{\xi}_{-,y^{(2)}}(k), \hat{\varphi}_{+,y^{(2)}}(k), \hat{\varphi}_{-,y^{(2)}}(k)) : y^{(2)} \in [1,\infty) , k \in [-\pi,\pi] \}$ and $\mathbb{M}(k)$ is defined in (3.11). We define

$$\begin{aligned}
\xi_{+,\mathbf{x}} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ikx^{(1)}} \hat{\xi}_{+,x^{(2)}}(k)\,, \\
\xi_{-,\mathbf{x}} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ikx^{(1)}} \hat{\xi}_{-,x^{(2)}}(k)\,, \\
\varphi_{+,\mathbf{x}} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ikx^{(1)}} \hat{\varphi}_{+,x^{(2)}}(k)\,, \\
\varphi_{-,\mathbf{x}} &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{-ikx^{(1)}} \hat{\varphi}_{-,x^{(2)}}(k)\,,
\end{aligned} \tag{3.17}$$

so in the real space we get the original Grassmann variables in terms of the new ones as

$$\begin{aligned}
\bar{H}_{\mathbf{x}} &= \xi_{+,\mathbf{x}} + \sum_{y^{(1)}\in a\mathbb{Z}} s_+(x^{(1)}-y^{(1)}) * \left(\varphi_{+,(y^{(1)},x^{(2)})} - \varphi_{-,(y^{(1)},x^{(2)})}\right)\,, \\
H_{\mathbf{x}} &= \xi_{-,\mathbf{x}} + \sum_{y^{(1)}\in a\mathbb{Z}} s_-(x^{(1)}-y^{(1)}) * \left(\varphi_{+,(y^{(1)},x^{(2)})} + \varphi_{-,(y^{(1)},x^{(2)})}\right)\,, \\
\bar{V}_{\mathbf{x}} &= \varphi_{+,\mathbf{x}}\,, \\
V_{\mathbf{x}} &= \varphi_{-,\mathbf{x}}\,,
\end{aligned} \tag{3.18}$$

where

$$s_\pm(x^{(1)}-y^{(1)}) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{e^{-ik(x^{(1)}-y^{(1)})}}{1+t_1 e^{\pm ik}}\,. \tag{3.19}$$

By a Paley-Wiener argument $s_\pm(x^{(1)}-y^{(1)})$ decays exponentially in the $|x^{(1)}-y^{(1)}|$ distance, with a rate arbitrarily close to $\log^{-1} t_1$. Note that the fields $\bar{H}$ and H undergo a transformation which is local only in the $x^{(2)}$ direction, while the $\bar{V}$ and V fields undergo a local transformation in both directions.

Now we can separate the *massive* and *massless* degrees of freedom of the theory, meaning that we can rewrite the quadratic action in (3.16) as

$$S(\hat{\Phi}) := \frac{1}{2}\sum_{x^{(2)},y^{(2)}=1}^{\infty} \int_{-\pi}^{\pi} dk \hat{\xi}_{x^{(2)}}(-k)\mathbb{A}_m \hat{\xi}_{y^{(2)}}(k) + \frac{1}{2}\sum_{x^{(2)},y^{(2)}=1}^{\infty} \int_{-\pi}^{\pi} dk \hat{\varphi}_{x^{(2)}}(-k)\mathbb{A}_c \hat{\varphi}_{y^{(2)}}(k)\,, \tag{3.20}$$

which is a sum of two terms, one describing massive fields ξ_+ and ξ_- and the other describing massless fields φ_+ and φ_-. We introduce the following definitions of Grassmann Gaussian "measures"

$$\begin{aligned}
P(d\xi) &:= \frac{1}{N_\xi}\Big[\prod_{x^{(2)}\in\mathbb{H}_1}\prod_{k\in\mathbb{Z}}\prod_{\omega=\pm} d\xi_{\omega,x^{(2)}}(k)\Big] e^{\frac{1}{2}\sum_{x^{(2)},y^{(2)}=1}^{\infty}\int_{-\pi}^{\pi} dk \hat{\xi}_{x^{(2)}}(-k)\mathbb{A}_m\hat{\xi}_{y^{(2)}}(k)}\,, \\
P(d\varphi) &:= \frac{1}{N_\varphi}\Big[\prod_{x^{(2)}\in\mathbb{H}_1}\prod_{k\in\mathbb{Z}}\prod_{\omega=\pm} d\varphi_{\omega,x^{(2)}}(k)\Big] e^{\frac{1}{2}\sum_{x^{(2)},y^{(2)}=1}^{\infty}\int_{-\pi}^{\pi} dk \hat{\varphi}_{x^{(2)}}(-k)\mathbb{A}_c\hat{\varphi}_{y^{(2)}}(k)}\,,
\end{aligned} \tag{3.21}$$

where N_ξ, N_φ are two normalization constants fixed in such a way that $\int P(d\xi) = \int P(d\varphi) = 1$. Then, we can express the unperturbed expectation of m monomials of massive and massless Grassmann variables $F_1(\xi,\varphi)\cdots F_m(\xi,\varphi)$ as

$$\langle F_1(\xi,\varphi)\cdots F_m(\xi,\varphi)\rangle^0_\beta , := \int P(d\xi)P(d\varphi)\, F_1(\xi,\varphi)\cdots F_m(\xi,\varphi)\,, \tag{3.22}$$

where $\langle\cdot\rangle^0_\beta$ is an average with respect the Gaussian Grassmann measures $P(d\xi)$ and $P(d\varphi)$: it is simply the rewriting of the unperturbed expectation in (2.23), where we dropped the subscript Λ meaning that we are considering the infinite limit (the upper half-plane) and we replaced the subscript $\mathbf{t}(\beta)$ by β.

3.2 Critical propagators

In this section we derive the main results on the massive and massless critical propagator, which will be used to explicitly evaluate the correlation functions in terms of the Grassmann fields, such as the right sides of (2.53), (2.54). Moreover we derive the decompositions and the bounds which will be the main ingredients of the multiscale analysis discussed in chapter 4.
We proceed as follows:

- in sub. 3.2.1 we derive the explicit expression of the massive and massless propagators on the discrete upper half-plane;
- in sub. 3.2.2 we derive the correspondent expression on the continuous upper half-plane: this allows us to verify an important cancellation at $x^{(2)} = 0$, that will play an important role in the following chapters;
- in sub. 3.2.3 we introduce the multiscale decomposition of the massless propagators;
- in sub. 3.2.4 we introduce the bulk edge decomposition of the propagators, which arises from the natural requirement that the propagators be translational invariant away from the boundary;
- in sub. 3.2.5 we state the decay bounds for the propagators.

3.2.1 The massive and massless propagators

We define the massive and massless propagators as

$$\begin{aligned} \hat g_\xi^{\omega\omega'} &:= \hat g_\xi^{\omega\omega'}(k_1; x^{(2)}, y^{(2)}) = \langle \xi_{\omega,x^{(2)}}(-k_1)\xi_{\omega',y^{(2)}}(k_1)\rangle^0_\beta\,, \\ \hat g_\varphi^{\omega\omega'} &:= \hat g_\varphi^{\omega\omega'}(k_1; x^{(2)}, y^{(2)}) = \langle \varphi_{\omega,x^{(2)}}(-k_1)\varphi_{\omega',y^{(2)}}(k_1)\rangle^0_\beta\,, \end{aligned} \tag{3.23}$$

with $\omega,\omega' \in \{\pm\}$ and with the unperturbed expectations $\langle\cdot\rangle^0_\beta$ defined as in (3.22). Note that, from now on, we write k_1 where we previously wrote k (see after (3.5)).

By exploiting the properties of Grassmann variables in (3.2)-(3.3) with the structure of the quadratic action in (3.20), the massive and the massless propagators in (3.23) can be explicitly evaluated in terms of elements of $\mathbb{A}_m^{-1}$ and $\mathbb{A}_c^{-1}$ respectively. Moreover, the propagators will be the covariances of the Gaussian Grassmann measures in (3.21).

By inverting the antisymmetric matrix $\mathbb{A}_m$ in (3.9), we get the massive propagators:

$$\mathbb{A}_m^{-1} = \begin{pmatrix} \hat{g}_\xi^{++} & \hat{g}_\xi^{+-} \\ \hat{g}_\xi^{-+} & \hat{g}_\xi^{--} \end{pmatrix} = \begin{pmatrix} 0 & -(1+t_1 e^{-ik_1})^{-1} \cdot \mathbb{1}_2 \\ (1+t_1 e^{ik_1})^{-1} \cdot \mathbb{1}_2 & 0 \end{pmatrix} . \tag{3.24}$$

In the real space we get

$$g_\xi^{\omega,-\omega}(\mathbf{x},\mathbf{y}) = (-\omega) s_{-\omega}(x^{(1)} - y^{(1)}) \delta_{x^{(2)},y^{(2)}} , \tag{3.25}$$

where $s_{-\omega}(x^{(1)} - y^{(1)})$ was defined in (3.19): the exponential decay in the distance $|x^{(1)} - y^{(1)}|$ justifies the name *massive*.

Similarly, by inverting the matrix $\mathbb{A}_c$, we get the massless propagators. However, first we need to diagonalize $\mathbb{A}_c$, by using a transformation which can be thought of as a Fourier sine transformation with modified frequencies in the vertical direction. Starting from the definition of the matrix $\mathbb{A}_c$ in (3.12), we consider the following eigenvalue problem

$$\begin{pmatrix} \alpha_1 \mathbb{1}_2 & \tilde{\mathbb{Z}}_2 \\ -\tilde{\mathbb{Z}}_2^T & -\alpha_1 \mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} , \tag{3.26}$$

which leads to

$$\begin{cases} (\lambda - \alpha_1)^{-1} \tilde{\mathbb{Z}}_2 v = \mathbb{1}_2 u \\ (\tilde{\mathbb{Z}}_2^T \tilde{\mathbb{Z}}_2 - \alpha_1^2 \mathbb{1}_2) v = -\lambda^2 \mathbb{1}_2 v . \end{cases} \tag{3.27}$$

We start by solving

$$\tilde{C}(k_1) v = -\lambda^2 v , \tag{3.28}$$

where $\tilde{C}(k_1) := (\tilde{\mathbb{Z}}_2^T \tilde{\mathbb{Z}}_2 - \alpha_1^2 \mathbb{1}_2)$ is a tridiagonal matrix whose entries are given by

$$\Big(\tilde{C}(k_1)\Big)_{x^{(2)} y^{(2)}} = \begin{cases} \tilde{d} - t_2^2 & \text{if } y^{(2)} = x^{(2)} = 1 , \\ \tilde{d} & \text{if } y^{(2)} = x^{(2)} \geq 2 , \\ \tilde{b} & \text{if } y^{(2)} = x^{(2)} \pm 1 , \\ 0 & \text{otherwise} , \end{cases} \tag{3.29}$$

with

$$\tilde{d} := \tilde{d}(k_1) = \beta_1^2(k_1) + t_2^2 - \alpha_1^2(k_1), \qquad \tilde{b} := \tilde{b}(k_1) = t_2 \beta_1(k_1) = \frac{t_2(t_1^2 - 1)}{|1 + t_1 e^{ik_1}|^2} , \tag{3.30}$$

$\alpha_1(k_1)$ and $\beta_1(k_1)$ were defined in (3.14) and $t_2 = \tanh \beta J_2$. Then (3.28) leads to the following equations

$$\begin{aligned} &(\tilde{d} - t_2^2) v(1) + \tilde{b} v(2) = -\lambda^2 v(1) \\ &\tilde{b} v(x^{(2)} - 1) + \tilde{d} v(x^{(2)}) + \tilde{b} v(x^{(2)} + 1) = -\lambda^2 v(x^{(2)}) , \qquad x^{(2)} \geq 2 , \end{aligned} \tag{3.31}$$

which recall a discrete Laplacian with peculiar open boundary conditions: it suggests the following ansatz for the eigenvectors

$$v(x^{(2)}) = c \sin\left(k_2 x^{(2)} + \theta_{k_2}\right), \tag{3.32}$$

where c is a constant. The (3.32) is solution of (3.28) if $\theta_{k_2} = -\arctan \frac{\sin k_2}{\frac{\beta_1}{t_2} + \cos k_2}$ and if the eigenvalues are $\lambda_\pm = \pm i\sqrt{\epsilon(k_1, k_2)}$, with

$$\epsilon(k_1, k_2) := 2t_2\beta_1(k_1)\cos k_2 + \beta_1^2(k_1) + t_2^2 + \alpha_1^2(k_1)\,. \tag{3.33}$$

By plugging (3.32) in the first equation of (3.27) we get

$$u^\pm(x^{(2)}) = \pm ic\left(\sqrt{\epsilon(k_1,k_2)} \mp \alpha_1(k_1)\right)^{-1}\sqrt{\epsilon(k_1,k_2) - \alpha_1^2(k_1)}\,\sin\left(x^{(2)}k_2\right). \tag{3.34}$$

Then, we let

$$\mathbb{E} := \sqrt{\epsilon(k_1,k_2)} \cdot \mathbb{1}_2\,, \qquad \mathbb{U} := \begin{pmatrix} \underline{\bar{u}}^+ & \underline{v}^+ \\ \underline{\bar{u}}^- & \underline{v}^- \end{pmatrix}, \tag{3.35}$$

where $\epsilon(k_1, k_2)$ is defined in (3.33) and

$$\underline{u}^\pm := \frac{1}{\mathcal{N}^\pm}\begin{pmatrix} u^\pm(x^{(2)} = 1) \\ \vdots \end{pmatrix}, \qquad \underline{v}^\pm := \frac{1}{\mathcal{N}^\pm}\begin{pmatrix} v(x^{(2)} = 1) \\ \vdots \end{pmatrix}, \tag{3.36}$$

with normalization constants given by

$$\mathcal{N}^\pm := c\sqrt{\frac{2\sqrt{\epsilon(k_1,k_2)}}{\sqrt{\epsilon(k_1,k_2)} \mp \alpha_1(k_1)}}\,. \tag{3.37}$$

Then we can rewrite $\mathbb{A}_c$ as

$$\mathbb{A}_c = \mathbb{U}^\dagger \begin{pmatrix} i\mathbb{E} & 0 \\ 0 & -i\mathbb{E} \end{pmatrix} \mathbb{U}\,, \tag{3.38}$$

and we can finally evaluate its inverse, obtaining

$$\mathbb{A}_c^{-1} = \begin{pmatrix} \hat{g}_\varphi^{++}(k_1,k_2) & \hat{g}_\varphi^{+-}(k_1,k_2) \\ \hat{g}_\varphi^{-+}(k_1,k_2) & \hat{g}_\varphi^{--}(k_1,k_2) \end{pmatrix} = \left(i\sqrt{\epsilon(k_1,k_2)}\right)^{-1}\begin{pmatrix} \underline{u}^+\underline{\bar{u}}^+ - \underline{u}^-\underline{\bar{u}}^- & \underline{u}^+\underline{v}^+ - \underline{u}^-\underline{v}^- \\ \underline{v}^+\underline{\bar{u}}^+ - \underline{v}^-\underline{\bar{u}}^- & \underline{v}^+\underline{v}^+ - \underline{v}^-\underline{v}^- \end{pmatrix}. \tag{3.39}$$

After lengthy but elementary calculations (reported in Appendix B), we get the explicit expression of the massless propagators in the real space for the critical case, i.e. with t_1 and t_2 such that $t_2 = \frac{1-t_1}{1+t_1}$:

$$\begin{aligned}\begin{pmatrix} g_\varphi^{++}(\mathbf{x},\mathbf{y}) & g_\varphi^{+-}(\mathbf{x},\mathbf{y}) \\ g_\varphi^{-+}(\mathbf{x},\mathbf{y}) & g_\varphi^{--}(\mathbf{x},\mathbf{y}) \end{pmatrix} &= \frac{1}{2}\int\limits_{[-\pi,\pi]^2} dk_1 dk_2\; e^{-i\mathbf{k}\delta(\mathbf{x},\mathbf{y})}\begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2) \end{pmatrix} + \\ &\quad -\frac{1}{2}\int\limits_{[-\pi,\pi]^2} dk_1 dk_2\; e^{-i\mathbf{k}\delta_E(\mathbf{x},\mathbf{y})}\begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,-k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2)e^{-2i\theta_{k_2}} \end{pmatrix},\end{aligned} \tag{3.40}$$

where $\delta(\mathbf{x},\mathbf{y}) := |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|$, $\delta_E(\mathbf{x},\mathbf{y}) := |x^{(1)} - y^{(1)}| + |x^{(2)} + y^{(2)}|$,

$$\hat{g}_1(\pm k_1, k_2) := \frac{\pm i t_1 \sin k_1}{d(k_1,k_2)}\,, \qquad \hat{g}_2(k_1, \pm k_2) := -\frac{(t_1^2-1) + t_2|1+t_1 e^{ik_1}|^2 e^{\mp i k_2}}{d(k_1,k_2)}\,, \tag{3.41}$$

$$d(k_1,k_2) := 2(1-t_2)^2(1-\cos k_1) + 2(1-t_1)^2(1-\cos k_2)\,, \tag{3.42}$$

and

$$e^{-2i\theta_{k_2}} = p(k_1,k_2) = \frac{(t_1^2-1) + t_2(1+t_1^2+2t_1\cos k_1)e^{ik_2}}{(t_1^2-1) + t_2(1+t_1^2+2t_1\cos k_1)e^{-ik_2}}\,. \tag{3.43}$$

We notice two important things:

1. the massless propagators in (3.40) are singular at $t_1 = t_c(0) = \sqrt{2}-1$, $t_2 = \frac{1-t_c(0)}{1+t_c(0)}$ if $(k_1,k_2) = (0,0)$, but the location of the singularity changes in the presence of the interaction moving to $t_c(\lambda) = \tanh(\beta_c(\lambda)J)$ as derived below;
2. the propagators are expressed as the sum of two terms: the term in the first line of (3.40) is translation invariant in both directions, while the term in the second line is not invariant in the vertical direction: this will imply the *bulk-edge* decomposition of the massless propagators discussed below (see Subsec. 3.2.4).

If we let $\hat{g}_{\omega\omega}(k_1,k_2) := \hat{g}_1(-\omega k_1, k_2)$, $\hat{g}_{\omega-\omega}(k_1,k_2) := -\omega\hat{g}_2(k_1,\omega k_2)$ (with the definitions in (3.41)), it is easy to verify the following identities:

$$\begin{aligned}&\hat{g}_{++}(k_1,k_2) = \hat{g}_{++}(k_1,-k_2) = -\hat{g}_{++}(-k_1,k_2) = \hat{g}_{--}(-k_1,k_2)\,,\\ &\hat{g}_{+-}(k_1,k_2) = \hat{g}_{+-}(-k_1,k_2) = -\hat{g}_{-+}(k_1,-k_2)\,.\end{aligned} \tag{3.44}$$

Furthermore, we introduce

$$\sin(2k_2+\theta_{k_2}) = \left(2\cos(k_2) - t_2\frac{|1+t_1e^{ik_1}|^2}{1-t_1^2}\right)\sin(k_2+\theta_{k_2})\,, \tag{3.45}$$

which is an alternative form for the quantization of k_2 with respect to the one after (3.32): it is equivalent to the following identity involving the off-diagonal propagators

$$\hat{g}_{+-}(k_1,k_2) = -e^{2i\theta_{k_2}}\hat{g}_{-+}(k_1,k_2)\,. \tag{3.46}$$

The relationships in (3.44)-(3.46) are the analogue of the ones in [4, Rmk.2.1], and are closely related to the symmetries of the original Ising model.

We can extend the definition of the massless propagator in (3.40) to all $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$: in particular we have an important cancellation at the sites with $x^{(2)} = y^{(2)} = 0$:

$$g_\varphi^{++}(\mathbf{x},\mathbf{y})|_{x^{(2)}=0} = g_\varphi^{++}(\mathbf{x},\mathbf{y})|_{y^{(2)}=0} = g_\varphi^{+-}(\mathbf{x},\mathbf{y})|_{x^{(2)}=0} = g_\varphi^{-+}(\mathbf{x},\mathbf{y})|_{y^{(2)}=0} = 0\,, \tag{3.47}$$

for all $\mathbf{x},\mathbf{y}$. This mean that any contraction of a massless $\varphi_{+,\mathbf{x}}$ field at a site with $x^{(2)} = 0$ vanishes identically, irrespective of which other field it is contracted with. Note that, the site with $x^{(2)} = 0$ is "immediately below" the lower edge of the discrete upper half-plane $\mathbb{H}_1$ (or $\mathbb{H}_a$). The property in (3.47) is a consequence of the symmetry properties in (3.44), (3.45) and (3.46). Notice, however, that these reflection symmetries do not imply that the propagator in the half-plane can be written via the image method as a linear combination of infinite-plane propagators. Such a property would significantly simplify the proof of the bounds on the edge propagator discussed in Subsec. 3.2.5. It would also open the way to computing the scaling limit of the model in more general domains (e.g., rectangles with open boundary conditions). It remains to be seen whether there exists a different definition of massless field whose propagator satisfies an exact image rule at finite volume.

3.2.2 Asymptotic behaviour

The definitions in (3.40) hold for $\mathbf{x}, \mathbf{y} \in \mathbb{H}_1$: if we consider a generic lattice spacing a instead of 1, by properly rescaling the positions we can take the following scaling of the propagators. Since the massive propagators are vanishing in the scaling limit, $\lim_{a\to 0} \frac{1}{a} g_\xi^{\omega\omega'}(\mathbf{x}/a, \mathbf{y}/a) = 0$, we let

$$g_{sl}^{\omega\omega'}(\mathbf{x}, \mathbf{y}) := \lim_{a\to 0} \frac{1}{a} g_\varphi^{\omega\omega'}\left(\frac{\mathbf{x}}{a}, \frac{\mathbf{y}}{a}\right),$$

which describes the behaviour of the propagators on the continuous upper half plane $\mathbb{H}$ and can be expressed as

$$\begin{pmatrix} g_{sl}^{++}(\mathbf{x},\mathbf{y}) & g_{sl}^{+-}(\mathbf{x},\mathbf{y}) \\ g_{sl}^{-+}(\mathbf{x},\mathbf{y}) & g_{sl}^{--}(\mathbf{x},\mathbf{y}) \end{pmatrix} = C2\pi i \begin{pmatrix} \frac{\Re(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} - \frac{\Re(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\bar{\mathbf{y}}|^2} & \frac{\Im(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} + \frac{\Im(\mathbf{x}-\bar{\mathbf{y}})}{|\mathbf{x}-\bar{\mathbf{y}}|^2} \\ \frac{\Im(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} - \frac{\Im(\mathbf{x}-\bar{\mathbf{y}})}{|\mathbf{x}-\bar{\mathbf{y}}|^2} & -\frac{\Re(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} - \frac{\Re(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\bar{\mathbf{y}}|^2} \end{pmatrix}. \tag{3.48}$$

See Appendix C for a proof of (3.48). By letting $g_{\varphi,\mathbb{Z}_a^2}^{\omega\omega'}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a})$ be the whole plan version of $g_{\varphi,\mathbb{H}_a^2}^{\omega\omega'}(\frac{\mathbf{x}}{a}, \frac{\mathbf{y}}{a})$, defined by

$$\begin{pmatrix} g_{\varphi,\mathbb{Z}_a^2}^{++}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a}) & g_{\varphi,\mathbb{Z}_a^2}^{+-}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a}) \\ g_{\varphi,\mathbb{Z}_a^2}^{-+}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a}) & g_{\varphi,\mathbb{Z}_a^2}^{--}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a}) \end{pmatrix} = \int_{[-\pi,\pi]^2} d\mathbf{k} e^{-i\mathbf{k}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a})} \begin{pmatrix} \hat{g}_1(-k_1, k_2) & -\hat{g}_2(k_1, -k_2) \\ \hat{g}_2(k_1, k_2) & \hat{g}_1(k_1, k_2) \end{pmatrix}, \tag{3.49}$$

we let $g_{\mathbb{R}^2}^{\omega\omega'}(\mathbf{x} - \mathbf{y}) := g_{\mathbb{R}^2}^{\omega\omega'}(\mathbf{x}, \mathbf{y})$ be the whole plane version of $g_{sl}^{\omega\omega'}(\mathbf{x}, \mathbf{y})$: it is given by

$$g_{\mathbb{R}^2}^{\omega\omega'}(\mathbf{x} - \mathbf{y}) = \lim_{a\to 0} \frac{1}{a} g_{\varphi,\mathbb{Z}_a^2}^{\omega\omega'}(\frac{\mathbf{x}}{a} - \frac{\mathbf{y}}{a}).$$

By the definitions in (3.41) is easy to verify that the following relations hold:

$$\begin{aligned} g_{sl}^{++}(\mathbf{x}, \mathbf{y}) &= g_{\mathbb{R}^2}^{++}(\mathbf{x} - \mathbf{y}) - g_{\mathbb{R}^2}^{++}(\mathbf{x} - \bar{\mathbf{y}}), \\ g_{sl}^{+-}(\mathbf{x}, \mathbf{y}) &= g_{\mathbb{R}^2}^{+-}(\mathbf{x} - \mathbf{y}) - g_{\mathbb{R}^2}^{+-}(\bar{\mathbf{x}} - \mathbf{y}), \\ g_{sl}^{-+}(\mathbf{x}, \mathbf{y}) &= g_{\mathbb{R}^2}^{-+}(\mathbf{x} - \mathbf{y}) - g_{\mathbb{R}^2}^{-+}(\mathbf{x} - \bar{\mathbf{y}}), \\ g_{sl}^{--}(\mathbf{x}, \mathbf{y}) &= g_{\mathbb{R}^2}^{--}(\mathbf{x} - \mathbf{y}) + g_{\mathbb{R}^2}^{++}(\mathbf{x} - \bar{\mathbf{y}}), \end{aligned} \tag{3.50}$$

so that we can easily verify the important cancellation at the lower edge of the continuum half plane:

$$g_{sl}^{++}(\mathbf{x},\mathbf{y})|_{x^{(2)}=0} = g_{sl}^{++}(\mathbf{x},\mathbf{y})|_{y^{(2)}=0} = g_{sl}^{+-}(\mathbf{x},\mathbf{y})|_{x^{(2)}=0} = g_{sl}^{-+}(\mathbf{x},\mathbf{y})|_{y^{(2)}=0} = 0\,, \quad (3.51)$$

for all $\mathbf{x},\mathbf{y} \in \mathbb{H}$. We state the following result for the decay of the critical propagator in the scaling limit (see [4, Prop.2.9] for the proof).

Lemma 3.1. *Let* $\delta(\mathbf{x}-\mathbf{y}) := |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|$*, if* $a\delta^{-1}(\mathbf{x},\mathbf{y})$ *is sufficiently small,*

$$g_{\varphi}^{\omega\omega'}(\mathbf{x},\mathbf{y}) = g_{sl}^{\omega\omega'}(\mathbf{x},\mathbf{y}) + R_a(\mathbf{x},\mathbf{y})\,, \quad (3.52)$$

and

$$||R_a(\mathbf{x},\mathbf{y})|| \leq Ca\delta^{-2}(\mathbf{x},\mathbf{y})\,. \quad (3.53)$$

3.2.3 Multiscale decomposition

Let $f_\alpha(k_1,k_2) := e^{-\alpha d(k_1,k_2)} \cdot d(k_1,k_2)$, with $d(k_1,k_2)$ in (3.42) and note that

$$\int_0^\infty f_\alpha(k_1,k_2)d\alpha = 1\,,$$

independently of k_1, k_2. Note also that, for large α, $f_\alpha(k_1,k_2)$ is peaked in a region where $d(k_1,k_2)$ is of the order α^{-1}, and so k_1, k_2 are of order $\alpha^{-1/2}$. We obtain the following multiscale decomposition for the massless propagators in (3.40):

$$g_{\varphi}^{\omega\omega'}(\mathbf{x},\mathbf{y}) = g_{\omega\omega'}^{(\leq h)}(\mathbf{x},\mathbf{y}) + \sum_{h'=h+1}^{0} g_{\omega\omega'}^{(h')}(\mathbf{x},\mathbf{y})\,, \quad (3.54)$$

where

$$g_{\omega\omega'}^{(\leq h)}(\mathbf{x},\mathbf{y}) := \int_{4^{-h-1}}^{\infty} d\alpha\; g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y})\,, \quad (3.55)$$

$$g_{\omega\omega'}^{(h')}(\mathbf{x},\mathbf{y}) := \int_{4^{-h'-1}}^{4^{-h'}} d\alpha\; g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y}) \qquad \text{for } h' < 0\,, \quad (3.56)$$

$$g_{\omega\omega'}^{(0)}(\mathbf{x},\mathbf{y}) := \int_0^1 d\alpha\; g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y})\,, \quad (3.57)$$

and

$$\begin{pmatrix} g_\alpha^{++}(\mathbf{x},\mathbf{y}) & g_\alpha^{+-}(\mathbf{x},\mathbf{y}) \\ g_\alpha^{-+}(\mathbf{x},\mathbf{y}) & g_\alpha^{--}(\mathbf{x},\mathbf{y}) \end{pmatrix} = \frac{1}{2}\int\limits_{[-\pi,\pi]^2} d\mathbf{k}\; e^{-i\mathbf{k}\delta(\mathbf{x},\mathbf{y})} f_\alpha(\mathbf{k}) \begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2) \end{pmatrix} +$$
$$- \frac{1}{2}\int\limits_{[-\pi,\pi]^2} d\mathbf{k}\; e^{-i\mathbf{k}\delta_E(\mathbf{x},\mathbf{y})} f_\alpha(\mathbf{k}) \begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,-k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2)e^{-2i\theta_{k_2}} \end{pmatrix}\,, \quad (3.58)$$

with the same definitions listed after (3.40). We will discuss below that such single scale massless propagators exhibits an exponential decay in an appropriate distance. Moreover, we let $g_{\xi}^{\omega\omega'}(\mathbf{x},\mathbf{y}) := g^{(1)}_{\omega\omega'}(\mathbf{x},\mathbf{y})$, which is defined in (3.25), where we already described that exhibits an exponential decay in the distance $|x^{(1)} - y^{(1)}|$.

In analogy with the multiscale decomposition of the massless propagators in (3.54), we can introduce a multiscale decomposition of the massless field,

$$\varphi = \varphi^{(\leq 0)} := \sum_{h\leq 0} \varphi^{(h)}\,, \tag{3.59}$$

such that each single scale propagator is given by $g^{(h)}_{\omega\omega'} := \langle \varphi^{(h)}_{\omega,\mathbf{x}}\varphi^{(h)}_{\omega,\mathbf{y}}\rangle$.

3.2.4 Bulk-edge decomposition

We decompose the cutoff propagator $g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y})$ into a 'bulk' part $g^{\omega\omega'}_{\alpha,B}(\mathbf{x},\mathbf{y})$ which is minimally sensitive to the presence of the lower boundary of the half-plane, plus a remainder, $g^{\omega\omega'}_{\alpha,E}(\mathbf{x},\mathbf{y})$ which we call the 'edge' part. The bulk part is simply chosen to be the restriction of $g^{\omega\omega'}_{\varphi,\mathbb{Z}^2_a}$, defined in (3.49), to the upper half plane $\mathbb{H}_a$; the edge part is, by definition, the difference between the full cutoff propagator and its bulk part

$$g^{\omega\omega'}_{\alpha,E}(\mathbf{x},\mathbf{y}) := g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y}) - g^{\omega\omega'}_{\alpha,B}(\mathbf{x},\mathbf{y})\,. \tag{3.60}$$

If we replace the cutoff propagator $g_{\alpha}^{\omega\omega'}(\mathbf{x},\mathbf{y})$ respectively with $g^{\omega\omega'}_{\alpha,B}(\mathbf{x},\mathbf{y})$ and $g^{\omega\omega'}_{\alpha,E}(\mathbf{x},\mathbf{y})$ in (3.56), we obtain the bulk and edge single scale propagators: $g^{(h')}_{\omega\omega';B}$ and $g^{(h')}_{\omega\omega';E}$; as a consequence, for any $h^* \leq h < 0$, we let

$$g_{\varphi}^{\omega\omega'}(\mathbf{x},\mathbf{y}) = g^{(\leq h)}_{\omega\omega'}(\mathbf{x},\mathbf{y}) + \sum_{h'=h+1}^{0} \left(g^{(h')}_{\omega\omega';B} + g^{(h')}_{\omega\omega';E} \right)\,. \tag{3.61}$$

Analogous definitions can be adapted for the massive propagators, by letting $g^{(1)}_{\omega\omega';B}(\mathbf{x},\mathbf{y})$ be the massive bulk propagator and $g^{(1)}_{\omega\omega';E}(\mathbf{x},\mathbf{y}) := g^{(1)}_{\omega\omega'}(\mathbf{x},\mathbf{y}) - g^{(1)}_{\omega\omega';B}(\mathbf{x},\mathbf{y})$ be the massive edge propagator. See [4, Eqs.(2.2.10)-(2.2.14)] for the details about the finite volume analog.

3.2.5 Decay bounds and Gram decomposition

We introduce a compact notation to denote all the possible discrete derivatives acting on a propagator: we let

$$\mathbf{d} := (\mathbf{d}_{\mathbf{x}},\mathbf{d}_{\mathbf{y}})\,,\quad \mathbf{d}_{\mathbf{x}} := (d_x^{(1)}, d_x^{(2)})\,,\quad \mathbf{d}_{\mathbf{y}} := (d_y^{(1)}, d_y^{(2)})\,, \tag{3.62}$$

$d_x^{(i)}, d_y^{(i)} \in \mathbb{Z}^+$ with $i = 1,2$, $\partial^{\mathbf{d}} := \partial^{d_x^{(1)}}_{x^{(1)}}\partial^{d_x^{(2)}}_{x^{(2)}}\partial^{d_y^{(1)}}_{y^{(1)}}\partial^{d_y^{(2)}}_{y^{(2)}}$. We define also $d := |\mathbf{d}| := d_x^{(1)} + d_x^{(2)} + d_y^{(1)} + d_y^{(2)}$ and $\mathbf{d}! := \prod_{i=1}^2 d_x^{(i)} \prod_{i=1}^2 d_y^{(i)}$. Moreover we let

$$||\partial^{\mathbf{d}} g^{(h)}(\mathbf{x},\mathbf{y})|| := \max_{\omega\omega'} |\partial^{\mathbf{d}} g^{(h)}_{\omega\omega'}(\mathbf{x},\mathbf{y})|\,, \tag{3.63}$$

be the norm for a propagator in (3.40), and analogous definitions hold for the norm of the single scale bulk and edge propagators. In analogy with [4, Prop.2.3], we state the following results:

Proposition 3.1 (Single scale decay bounds). *There exists constants c, C such that, for any integer $h \leq 1$, any $\mathbf{d}$ defined in (3.62) and any $\mathbf{x}, \mathbf{y} \in \mathbb{H}_1$,*

$$||\partial^{\mathbf{d}} g^{(h)}(\mathbf{x}, \mathbf{y})|| \leq C^{1+d} \mathbf{d}! 2^{(1+d)h} e^{-c2^h \delta(\mathbf{x}, \mathbf{y})} \tag{3.64}$$

where $\delta(\mathbf{x}, \mathbf{y}) := |x^{(1)} - y^{(1)}| + |x^{(2)} - y^{(2)}|$. Moreover

$$||\partial^{\mathbf{d}} g_E^{(h)}|| \leq C^{1+d} \mathbf{d}! 2^{(1+d)h} e^{-c2^h \delta_E(\mathbf{x}, \mathbf{y})} \tag{3.65}$$

where $\delta_E(\mathbf{x} - \mathbf{y}) := |x^{(1)} - y^{(1)}| + |x^{(2)} + y^{(2)}|$. Finally, since $g^{(h)}_{\omega\omega';B} := g^{(h)}_{\omega\omega'} - g^{(h)}_{\omega\omega';E}$ also the single scale propagator $g_B^{(h)}$ satisfies

$$||\partial^{\mathbf{d}} g_B^{(h)}|| \leq C^{1+d} \mathbf{d}! 2^{(1+d)h} e^{-c2^h \delta(\mathbf{x}, \mathbf{y})}\,. \tag{3.66}$$

Proposition 3.2 (Gram decompostition). *There exists a Hilbert space with inner product $(\cdot, \cdot)$ including elements $\gamma^{(h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}$, $\tilde{\gamma}^{(h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}$ $\gamma^{(\leq h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}$, $\tilde{\gamma}^{(\leq h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}$ with $\mathbf{d}_\mathbf{x}$ and $\mathbf{d}_\mathbf{y}$ defined in* (3.62)*, such that whenever $h \leq 1$*

$$\partial^{\mathbf{d}} g^{(h)}_{\omega\omega'}(\mathbf{x}, \mathbf{y}) \equiv (\tilde{\gamma}^{(h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}, \gamma^{(h)}_{\omega',\mathbf{y},\mathbf{d}_\mathbf{y}})\,, \qquad \partial^{\mathbf{d}} g^{(\leq h)}_{\omega\omega'}(\mathbf{x}, \mathbf{y}) \equiv (\tilde{\gamma}^{(\leq h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}, \gamma^{(\leq h)}_{\omega',\mathbf{y},\mathbf{d}_\mathbf{y}})\,, \tag{3.67}$$

and

$$|\gamma^{(h)}_{\omega,\mathbf{x},\mathbf{d}_x x}|^2, |\tilde{\gamma}^{(h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}|^2, |\gamma^{(\leq h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}|^2, |\gamma^{(\leq h)}_{\omega,\mathbf{x},\mathbf{d}_\mathbf{x}}|^2 \leq C^{1+|d_\mathbf{x}|} \mathbf{d}_\mathbf{x}! 2^{(1+2|\mathbf{d}_x|)h} \tag{3.68}$$

where $|\cdot|$ is the norm generated by the inner product $(\cdot, \cdot)$. Moreover, combining (3.67) *and* (3.68)*, we get*

$$||\partial^{\mathbf{d}} g^{(\leq h)}(\mathbf{x}, \mathbf{y})|| \leq C^{1+d} \mathbf{d}! 2^{(1+d)h} \tag{3.69}$$

Note that we can extend the definitions in (3.54)-(3.58) to obtain the multiscale decomposition of the propagator $g^{\omega\omega'}_{\alpha,\mathbb{Z}^2_a}$ in (3.49): $g^{(h)}_{\omega\omega',\mathbb{Z}^2_a}(\mathbf{x} - \mathbf{y})$ and $g^{(\leq h)}_{\omega\omega',\mathbb{Z}^2_a}(\mathbf{x} - \mathbf{y})$ will satisfy the bounds in Prop. 3.1-Prop. 3.2.

Since $d(k_1, k_2)$ is exactly the denominator of the massless propagator in (3.40), it is easily seen that the functions integrated in (3.58) is an entire function of t_1. All of the bounds in Prop. 3.1-Prop. 3.2 are obtained by writing the relevant quantities as absolutely convergent integrals or sums in k_1, k_2, and α; since these bounds are locally uniform in t_1 as long as it is bounded away from 0 and 1, we also see that all of the propagators are analytic functions of t_1 with all other arguments held fixed.

3.3 Effective potential of the generating function

In this section we rewrite the generating function resulting from the Grassmann representation derived in Lemma 2.3 with the new Grassmann variables ξ and φ and we will perform some necessary rearrangement to introduce the so called effective potential.

Before replacing the original Grassmann variables in (2.56) with the massive and massless variables in (3.18), we rewrite $\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}})$ in a different expression suitable for obtaining the correlation functions.

Back to the quadratic action $S_{\mathbf{t}}(X)$ in (2.22), with $t_i := t_{\mathbf{x},i}$ for all $\mathbf{x} \in \mathbb{H}_1$ and $i = 1, 2$: it can be rewritten as $S_{\mathbf{t}}(X) := t_1 S_1(X) + t_2 S_2(X) + S_0(X)$, where

$$\begin{aligned} S_1(X) &:= \sum_{\mathbf{x}\in\mathbb{H}_1} E_{\mathbf{x},1}\,, \qquad S_2(X) := \sum_{\mathbf{x}\in\mathbb{H}_1} E_{\mathbf{x},2}\,, \\ S_0(X) &:= \sum_{\mathbf{x}\in\mathbb{H}_1} \left\{ -\bar{H}_{\mathbf{x}}\bar{V}_{\mathbf{x}} - H_{\mathbf{x}}V_{\mathbf{x}} + \bar{H}_{\mathbf{x}}H_{\mathbf{x}} + \bar{V}_{\mathbf{x}}V_{\mathbf{x}} + V_{\mathbf{x}}\bar{H}_{\mathbf{x}} + H_{\mathbf{x}}\bar{V}_{\mathbf{x}} \right\}\,. \end{aligned} \tag{3.70}$$

Then, we introduce Z, which will later play the role of massless wave function renormalization, and we let $S_{t_c}(X) := Zt_c S_1(X) + Z\frac{1-t_c}{1+t_c} S_2(X) + ZS_0(X)$ be the *critical* quadratic action and $S_{\mathbf{t},t_c}(X) := S_{\mathbf{t}}(X) - S_{t_c}(X)$ be the *difference* quadratic action. Note that $S_{t_c}(X)$ is depending on the critical temperature $t_1 = t_c$, $t_2 = \frac{1-t_c}{1+t_c}$ and on Z, obtained by rescaling each Grassmann field by $\sqrt{Z}$ (each monomial in (3.70) is quadratic). For any $Z \neq 0$, if we rescale by $\sqrt{Z}$ also all the other Grassmann fields in (2.56), we get

$$\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}}) = C \int \mathcal{D}X\; e^{S_{t_c}(X) + S_{\mathbf{t},t_c}(X) + V(Z^{-1/2}X;\mathbf{A},\tilde{\mathbf{A}}) + Z^{-1}(\mathbf{E},\mathbf{A}) + Z^{-1}(\mathbf{E},\tilde{\mathbf{A}}) + Z^{-1/2}(\mathbf{V},\mathbf{\Psi})}\,, \tag{3.71}$$

where C is a generic constant in the Grassmann fields which contain all the contributions independent of the source variables, so that do not affect the derivation of correlations. Now we perform the change of Grassmann variables: we move from the old Grassmann variables $X = \{\, (\bar{H}_{\mathbf{x}}, H_{\mathbf{x}}, \bar{V}_{\mathbf{x}}, V_{\mathbf{x}})\,\}_{\mathbf{x}\in\mathbb{H}_1}$ to the new Grassmann variables $\Phi = \{\, (\xi_{+,\mathbf{x}}, \xi_{-,\mathbf{x}}, \varphi_{+,\mathbf{x}}, \varphi_{-,\mathbf{x}}\,\}_{\mathbf{x}\in\mathbb{H}_1}$ by using transformations in (3.18) evaluated at $t_1 = t_c$ (see (3.19) for the dependence on t_1), then

$$\mathcal{D}X\; e^{S_{t_c}(X)} \propto P_c(d\xi) P_c(d\varphi)\,, \tag{3.72}$$

where $P_c(d\xi)$ and $P_c(d\varphi)$ are the measures defined in (3.21): $P_c(d\xi)$ is the Gaussian Grassmann integration with covariance g_ξ defined in (3.25) with t_1 replaced by t_c and $P_c(d\varphi)$ is the one with covariance g_φ in (3.40) with t_1 and t_2 replaced by t_c and $\frac{1-t_c}{1+t_c}$ respectively. Now, the perturbed generating function in (3.71) can be rewritten as

$$\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}}) = C \int P_c(d\Phi)\, e^{S_{\mathbf{t},t_c}(\Phi) + V(\Phi) + B_\epsilon(\Phi;\mathbf{A},\tilde{\mathbf{A}}) + B_\sigma(\varphi;\mathbf{\Psi})}\,, \tag{3.73}$$

where

- $$S_{\mathbf{t},t_c}(\Phi) := S(\mathbf{t},t_c;\Phi) = (\frac{1}{Z}-1)S_0(\Phi)+(\frac{t_1}{Z}-t_c)S_1(\Phi)+(\frac{t_2}{Z}-\frac{1-t_c}{1+t_c})S_2(\Phi)\,, \quad (3.74)$$

  is the rewriting of the difference quadratic action $S_{\mathbf{t},t_c}(X)$ in terms of the massive and massless variables;

- if we express the $V(Z^{-1/2}X,\mathbf{A},\tilde{\mathbf{A}})$ in (2.58) in terms of the massive and massless variables we obtain $V(\Phi)+V_\epsilon(\Phi;\mathbf{A},\tilde{\mathbf{A}})$, where $V(\Phi)$ is a polynomial in the massive and massless fields only and $V_\epsilon(\Phi;\mathbf{A},\tilde{\mathbf{A}})$ is a polynomial involving the fields $A_{\mathbf{x}}$ and $\tilde{A}_{\mathbf{x}}$ (possibly combined with the massive and massless fields): their kernels will satisfy the same decay estimates as in (2.59) with $m=m'=0$ and $m+m'\geq 1$ respectively;

- if we express the $Z^{-1/2}(X,\mathbf{A})$ and $(X,\tilde{\mathbf{A}})$ in the new variables we obtain $(\Phi,\mathbf{A})$ and $(\Phi,\tilde{\mathbf{A}})$, then we can define

  $$B_\epsilon(\Phi;\mathbf{A},\tilde{\mathbf{A}}) := (\Phi,\mathbf{A})+(\Phi,\tilde{\mathbf{A}})+V_\epsilon(\Phi;\mathbf{A},\tilde{\mathbf{A}})\,; \quad (3.75)$$

- $B_\sigma(\varphi;\mathbf{\Psi})$ is the rewriting of $Z^{-1/2}(\mathbf{V},\mathbf{\Psi})$ defined in (2.37) in terms of the massless variables: by the last line of (3.18) we obtain

  $$B_\sigma(\varphi;\mathbf{\Psi}) := \sum_{\mathbf{x}\in\partial\mathbb{H}} \Psi_{\mathbf{x}}\varphi_{-,\mathbf{x}}\,. \quad (3.76)$$

Then, the quadratic action in (3.74) can be rearranged as

$$S_{\mathbf{t},t_c}(\Phi) = \nu_1 S_\nu(\Phi)+\zeta_1 S_\zeta(\Phi)+\eta_1 S_\eta(\Phi) =: S_{\underline{v}_1}(\Phi)\,, \quad (3.77)$$

where, given the following matrix of coefficients

$$a := \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1}\end{pmatrix} = \begin{pmatrix} \frac{t_c^2-2t_c-1}{(1+t_c)^2} & \frac{2t_c^2}{(1+t_c)^3} \\ \frac{2}{(1+t_c)^2} & \frac{1-t_c}{(1+t_c)^3}\end{pmatrix}\,, \quad (3.78)$$

we let

$$\begin{aligned} \nu_1 &:= a_{0,0}\big(\frac{1}{Z}-1\big)+a_{1,0}\big(\frac{t_1}{Z}-t_c\big)+\big(\frac{t_2}{Z}-\frac{1-t_c}{1+t_c}\big)\,,\\ \zeta_1 &:= a_{0,1}\big(\frac{1}{Z}-1\big)+a_{1,1}\big(\frac{t_1}{Z}-t_c\big)\,,\\ \eta_1 &:= \big(\frac{t_2}{Z}-\frac{1-t_c}{1+t_c}\big)\,, \end{aligned} \quad (3.79)$$

and

$$\begin{aligned} \tilde{S}_\nu(\Phi) &:= \frac{a_{1,1}}{\det a}\tilde{S}_0(\Phi)-\frac{a_{0,1}}{\det a}\tilde{S}_1(\Phi)\,,\\ \tilde{S}_\zeta(\Phi) &:= -\frac{a_{1,0}}{\det a}\tilde{S}_0(\Phi)+\frac{a_{0,0}}{\det a}\tilde{S}_1(\Phi)\,,\\ \tilde{S}_\eta(\Phi) &:= \tilde{S}_2(\Phi)-\tilde{S}_\nu(\Phi)\,. \end{aligned} \quad (3.80)$$

Finally we let $\underline{v}_1 := \{ \nu_1, \zeta_1, \eta_1 \}$ and $S_{\underline{v}_1}(\Phi) := S[\underline{v}_1](\Phi)$ (so we obtain the expression in (3.77)) and we write the perturbed generating function as

$$\Xi_\lambda(\mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}}) = C \int P_c(d\Phi)\; e^{\mathcal{V}(\Phi;\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,, \tag{3.81}$$

where $\mathcal{V}(\Phi; \mathbf{A}, \tilde{\mathbf{A}}, \mathbf{\Psi}) := S_{\underline{v}_1}(\Phi) + V(\Phi) + B_\epsilon(\Phi; \mathbf{A}, \tilde{\mathbf{A}}) + B_\sigma(\varphi; \mathbf{\Psi})$ is the so called effective potential. A naive expansion in the perturbation parameter λ leads to a not convergent expansion for the effective potential, as a consequence of the divergent massless at the critical point in (3.40). Then we have to introduce a multiscale analysis which leads to a convergent expansion, as described in the next chapter.

Chapter 4

Multiscale analysis and tree expansions

In this chapter we illustrate the iterative procedure needed to obtain a convergent expansion for the effective potential of the perturbed generating function. In particular, we introduce a multiscale analysis which allows us to identify each contribution of the effective potential, so that we can isolate the ones that may be source of divergences, and we derive its expression in terms of a convergent tree expansion. The convergence will be proved in the next chapter 5, here we introduce the necessary algebraic properties and representation, by proceeding as follows:

- in Sec. 4.1 we illustrate the multiscale integration of the perturbed generating function: we introduce the notation suitable to identify on each scale the contributions of the effective potential which may diverge and we derive a recursive relation between the effective potentials on two subsequent scales;
- in Sec. 4.2 we introduce the *localization* procedures, which isolate the apparent divergent terms to bound, in order to get a convergent expansion for the effective potential: the presence or absence of spin source fields in a contribution to the effective potential, requires defining different localizations, so that we will introduce the source, the bulk and the edge localizations, as well as the the corresponding norms;
- in Sec. 4.3 we explain how to rewrite the resulting effective potential with a structure similar to the starting one, with new effective action, new effective interaction and new effective source terms and how to represent it as convergent tree expansion.

4.1 Multiscale analysis

Here we want to perform the integration of the Grassmann fields in the perturbed generating function (3.81).

By performing the massless field decomposition in (3.59), $\varphi_{\omega,\mathbf{x}} = \sum_{h\le 0} \varphi^{(h)}_{\omega,\mathbf{x}}$, and by considering the massive fields as fields on the first scale, $\xi_{\omega,\mathbf{x}} = \varphi^{(1)}_{\omega,\mathbf{x}}$, we will use the properties of the Grassmann variables to proceed with a step by step integration. In particular we will use the following Grassmann integral properties:

- the addition principle, so that given any two integrations $P(d\psi_1)$ and $P(d\psi_2)$, with covariance g_{ψ_1} and g_{ψ_2}, respectively, then, for any function V which can be written as a sum over monomials of Grassmann variables, i.e. $V = V(\psi)$, with $\psi = \psi_1+\psi_2$, one has
$$\int P(d\psi)V(\psi) = \int P(d\psi_1)\int P(d\psi_2)V(\psi_1+\psi_2)\,; \tag{4.1}$$
- the invariance of exponentials, so that integrating an exponential expression one still gets an exponential expression, whose argument is expressed by the sum of truncated expectations, namely
$$\int P(d\psi_1)e^{V(\psi_1+\psi_2)} = e^{V'(\psi_2)}\,, \qquad V'(\psi_2) := \sum_{n\ge 0}\frac{1}{n!}\underbrace{\langle V(\cdot+\psi_2);\cdots;V(\cdot+\psi_2)\rangle}_{n \text{ times}}\,. \tag{4.2}$$

For a detailed review of the Grassmann properties see the references mentioned in Subsec.2.1.3.

As a first step, in Subsec. 4.1.1, we illustrate how to perform the massive integration, then, in Subsec. 4.1.2, we illustrate how to perform, on each scale, the massless integrations. In Subsec. 4.1.3 we express the resulting effective potential as a sum of truncated expectations: it will be a polynomial expression with coefficients that we will call 'kernels' of the effective potential and in Subsec. 4.1.4 we illustrate how to recursively derive the kernels.

4.1.1 The massive integration

To highlight the dependence on the massive fields, we associate a superscript label (1) to each $\xi-$dependent function in the generating partition function (3.81), namely

$$\Xi_\lambda(\mathbf{A},\tilde{\mathbf{A}},\Psi) = C\int P_c(d\varphi)\; e^{B_\sigma(\varphi;\mathbf{\Psi})}\int P_c(d\xi)\; e^{S^{(1)}_{\underline{v}}(\xi,\varphi)+V^{(1)}(\xi,\varphi)+B^{(1)}_e(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})}\,. \tag{4.3}$$

We define the *effective potential* on scale 1,

$$\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}}) := S^{(1)}_{\underline{v}}(\xi,\varphi) + V^{(1)}(\xi,\varphi) + B^{(1)}_e(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})\,, \tag{4.4}$$

where each term in the r.h.s. consists of the sum of the monomials involving the ξ fields (also combined with the massless fields and/or the source fields). We can refer to $S^{(1)}_{\underline{v}}(\xi,\varphi)$ as *effective quadratic action*, to $V^{(1)}(\xi,\varphi)$ as *effective interaction* and to $B^{(1)}_e(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})$ as *effective energy source term* on scale 1. By using the Grassmann property in (4.2), which allows us to rewrite the integral of an exponential expression as an exponential expression, we can integrate out the massive fields and get

$$C \int P_c(d\xi)\ e^{\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})} = e^{\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}})}\,. \tag{4.5}$$

where $\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}})$ will be an expression only dependent on the massless and source fields that can be decompose into the following sum

$$\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) := \tilde{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) + \tilde{E}_0 + \mathcal{F}_0(\mathbf{A},\tilde{\mathbf{A}})\,, \tag{4.6}$$

where

- $\tilde{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}})$ consists of the sum of the monomials involving at least one φ field ;
- $\tilde{E}_0$ collects the contributions that are constant both in the Grassmann and source fields;
- $\mathcal{F}_0(\mathbf{A},\tilde{\mathbf{A}})$ collects the contributions that depend on the source fields and are independent of the Grassmann fields.

The above contributions are normalized in such a way that $\mathcal{F}_0(\mathbf{0},\mathbf{0}) = \tilde{\mathcal{V}}^{(0)}(0;\mathbf{0},\mathbf{0}) = 0$. Then we can define the effective potential on scale 0 as $\mathcal{V}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) := \tilde{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) + B_\sigma(\varphi;\mathbf{\Psi})$, so it is the sum of all monomials with at least one field φ. In fact, $B_\sigma(\varphi;\mathbf{\Psi})$, which is the spin source term defined in (3.76), does not depend on the massive field but is sum of monomials which are linear in φ: we have to introduce it in the effective potential on scale 0. Next, we can rewrite (4.3) as

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = e^{E_0+\mathcal{F}_0(\mathbf{A},\tilde{\mathbf{A}})} \int P(d\varphi) e^{\mathcal{V}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,, \tag{4.7}$$

where $E_0 := \tilde{E}_0 - \log C$. Moreover, $\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}})$ can be expressed by the following sum of truncated expectations:

$$\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) - \log C = \sum_{s\geq 1} \frac{1}{s!} \langle \underbrace{\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}});\cdots;\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})}_{s\text{ times}} \rangle^{\xi}\,, \tag{4.8}$$

where we used (4.2) for $\bar{\mathcal{V}}^{(0)}(\varphi;\mathbf{A},\tilde{\mathbf{A}}) - \log C = \log \int P_c(d\xi) e^{\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})}$. In (4.8) each truncated expectation $\langle\cdot;\cdot\rangle^{\xi}$ is with respect the Gaussian Grassmann integration $P_c(d\xi)$ (see (3.72)) and each $\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{A},\tilde{\mathbf{A}})$ is a sum of different monomials (as mentioned after (4.4)). So far, we have referred to the Grassmann fields just with the ξ and φ symbols: we recall that each Grassmann field has a position $\mathbf{x}$ and a ω index, i.e. $\xi = \xi_{\omega,\mathbf{x}}$ and $\varphi = \varphi_{\omega,\mathbf{x}}$. To properly identify the Grassmann fields that are integrated on scale (1) in the truncated expectation, and to distinguish between them and the *external fields* (i.e. the fields that are not integrated on that scale), we introduce some additional notations. Let

- $M_0^{(1)}$ be the set of the tuples

$$\underline{f} \equiv (\underline{\omega}, \underline{\mathbf{x}}) := ((\omega_1, \mathbf{x}_1), (\omega_2, \mathbf{x}_2), \dots),$$

which are tuples of pairs of Grassmann field variables, $\omega_i \in \{\pm, \pm i\}$ and $\mathbf{x}_i \in \mathbb{H}$, $i = 1, \dots, |\underline{f}|$, where the tuple length is $|\underline{f}| \in 2\mathbb{N}$;

- M_E be the set of the tuples

$$\underline{e} \equiv (\underline{\mathbf{y}}, \underline{j}) := ((\mathbf{y}_1, j_1), (\mathbf{y}_2, j_2), \dots),$$

which are tuples of pairs of energy source field variables, $\mathbf{y}_k \in \mathbb{H}$, $j_k \in \{1, 2\}$, $k = 1, \dots, |\underline{e}|$, where the tuple length is $|\underline{e}| \in \mathbb{N}_0$.

Each effective potential $\mathcal{V}^{(1)}$ in (4.8) can then be expanded as

$$\mathcal{V}^{(1)}(\xi, \varphi; \mathbf{A}, \tilde{\mathbf{A}}) = \sum_{\underline{f} \in M_0^{(1)}} \sum_{\underline{e} \in M_E} W^{(1)}(\underline{f}, \underline{e})\, \varphi(\underline{f}) F(\underline{e}), \tag{4.9}$$

where $W^{(1)}(\underline{f}, \underline{e})$ are the kernels of the effective potential on scale 1 and $\varphi(\underline{f})F(\underline{e})$ are the monomials in the Grassmann and source fields, which are defined as follows:

- $\varphi(\underline{f})$ is the products of $|\underline{f}|$ Grassmann fields, namely

$$\varphi(\underline{f}) \equiv \varphi(\underline{\omega}, \underline{\mathbf{x}}) := \prod_{k=1}^{|\underline{f}|} \varphi_{\omega_k, \mathbf{x}_k}, \tag{4.10}$$

with

$$\varphi_{\omega_k, \mathbf{x}_k} := \begin{cases} \xi_{\pm, \mathbf{x}_k} & \text{if } \omega_k = \pm i, \\ \varphi_{\pm, \mathbf{x}_k} & \text{if } \omega_k = \pm. \end{cases} \tag{4.11}$$

Note that the total number of Grassmann fields $|\underline{f}|$ is given by the sum $|\underline{f}| = n^{(1)}(\underline{f}) + n(\underline{f})$, where $n^{(1)}(\underline{f}) := |\{\underline{f} : \omega_k \in \pm i, k = 1, \dots, |\underline{f}|\}|$, which is the number of the massive fields in the tuple $\underline{f}$ and $n(\underline{f}) := |\{\underline{f} : \omega_k \in \pm, k = 1, \dots, |\underline{f}|\}|$, which is the number of the massless fields in the tuple.

- $F(\underline{e})$ is the products of $|\underline{e}|$ source fields, namely

$$F(\underline{e}) \equiv F(\underline{\mathbf{y}}, \underline{j}) = \prod_{i=1}^{|\underline{e}|} F_{\mathbf{y}_i, j_i}, \tag{4.12}$$

with

$$F_{\mathbf{y}_i, j_i} = \begin{cases} A_{\mathbf{y}_i, j_i} & \text{if } \mathbf{y}_i \in \mathbb{H}^\circ, \\ \tilde{A}_{\mathbf{y}_i, j_i} & \text{if } \mathbf{y}_i \in \partial\mathbb{H}. \end{cases} \tag{4.13}$$

Note that, if $\underline{e} = \emptyset$, we have $F(\emptyset) = 1$.

Moreover, the kernels of the effective potential on scale 1 can be decomposed as

$$W^{(1)}(\underline{f},\underline{e}) = W^{(1)}_{\lambda=0}(\underline{f},\underline{e}) + W^{(1)}_{\lambda\neq 0}(\underline{f},\underline{e})\,, \tag{4.14}$$

where

- $W^{(1)}_{\lambda=0}(\underline{f},\underline{e})$ are the 'free', i.e. non interacting, kernels, which are non-zero only if $|\underline{f}| = 2$ and $|\underline{e}| = 1$, in which case

  $$\sum_{\omega_1,\omega_2\in\{\,\pm i,\pm\,\}}\ \sum_{j_1\in\{\,1,2\,\}}\ \sum_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1\in\mathbb{H}} W^{(1)}_{\lambda=0}((\omega_1,\omega_2,\mathbf{x}_1,\mathbf{x}_2),(\mathbf{y}_1,j_1))\varphi_{\omega_1,\mathbf{x}_1}\varphi_{\omega_2,\mathbf{x}_2}F_{\mathbf{y}_1,j_1}$$

  is the rewriting in terms of the massive and massless variables of $(\mathbf{E},\mathbf{A}) + (\mathbf{E},\tilde{\mathbf{A}})$, which are the energy source terms defined in (2.27);

- $W^{(1)}_{\lambda\neq 0}(\underline{f},\underline{e})$ are the interacting kernels, which satisfy the analogue bound of (2.59), namely

  $$\sup_{\underline{\omega}\in\{\,\pm,\pm i\,\}^n}\ \sup_{\mathbf{x}_1\in\mathbb{H}}\ \sum_{\mathbf{x}_2,\dots,\mathbf{x}_n\in\mathbb{H}}\ \sum_{\underline{j}\in\{\,1,2\,\}^N}\ \sum_{\mathbf{y}_1,\dots,\mathbf{y}_N\in\mathbb{H}} e^{c\delta(\underline{\mathbf{x}},\underline{\mathbf{y}})}|W^{(1)}_{\lambda\neq 0}(\underline{f};\underline{e})| \le C^{n+N}|\lambda|^{\max 1,\kappa(n+N)}\,, \tag{4.15}$$

  where $n := |\underline{f}|$ and $N := |\underline{e}|$.

4.1.2 The massless integration

To perform the massless integration, we can use the multiscale decomposition of the massless fields in (3.59), that is $\varphi = \varphi^{(\le 0)} := \sum_{h\le 0}\varphi^{(h)}$, so we can rewrite (4.7) as

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = e^{E_0+\mathcal{F}_0(\mathbf{A},\tilde{\mathbf{A}})}\int P_c(d\varphi^{(\le 0)})\ e^{\mathcal{V}^{(0)}(\varphi^{(\le 0)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,. \tag{4.16}$$

Next, noticing that $\varphi^{(\le 0)} = \varphi^{(0)} + \varphi^{(\le -1)}$, with $\varphi^{(\le -1)} := \sum_{h\le -1}\varphi^{(h)}$, and by recalling the addition principle for the Grassmann integral (4.1), we get

$$\int P_c(d\varphi^{(\le 0)})\ e^{\mathcal{V}^{(0)}(\varphi^{(\le 0)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})} = \int P_c(d\varphi^{(\le -1)})\int P_c(d\varphi^{(0)})\ e^{\mathcal{V}^{(0)}(\varphi^{(0)}+\varphi^{(\le -1)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,. \tag{4.17}$$

By proceeding as for the massive integration (4.5), we can perform the integration of $\varphi^{(0)}$, which is the first single scale massless field, as

$$\int P_c(d\varphi^{(0)})\ e^{\mathcal{V}^{(0)}(\varphi^{(0)}+\varphi^{(\le -1)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})} = e^{\bar{\mathcal{V}}^{(-1)}(\varphi^{(\le -1)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,. \tag{4.18}$$

Then, $\bar{\mathcal{V}}^{(-1)}(\varphi^{(\le -1)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})$, which is a sum of monomials of any degree, including the constant terms, can be decomposed as we did in (4.6), so we get

$$\bar{\mathcal{V}}^{(-1)}(\varphi^{(\le -1)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = \mathcal{V}^{(-1)}(\varphi^{(\le -1)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi}) + \bar{E}_{-1} + \bar{\mathcal{F}}_{-1}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})\,, \tag{4.19}$$

where $\mathcal{V}^{(-1)}(\varphi;\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})$ consists of the sum of monomials involving at least one field $\varphi^{(-1)}$, $\bar{E}_{-1}$ collects the constant contributions and $\bar{\mathcal{F}}_{-1}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ collects the contributions depending only on the source variables. As in (4.6), the contributions in the r.h.s. of (4.19) are normalized in such a way that $\bar{\mathcal{F}}_{-1}(\mathbf{0},\mathbf{0},\mathbf{0}) = \mathcal{V}^{(-1)}(0;\mathbf{0},\mathbf{0},\mathbf{0}) = 0$. Then, we can rewrite the perturbed generating function in (4.16) as

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = e^{E_{-1}+\mathcal{F}_{-1}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}\int P_c(d\varphi^{(\le -1)})e^{\mathcal{V}^{(-1)}(\varphi;\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,, \tag{4.20}$$

and $E_{-1} := E_0 + \bar{E}_{-1}$, $\mathcal{F}_{-1}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) := \mathcal{F}_0(\mathbf{A},\tilde{\mathbf{A}}) + \bar{\mathcal{F}}_{-1}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})$, are the effective constant terms on scale -1. Note that $\mathcal{V}^{(-1)}(\varphi;\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})$ is the effective potential on scale -1, which is defined as the sum of all monomials involving at least one $\varphi^{(-1)}$: after that on scale 0 we included 'by hand' the spin source term, now we are automatically taking into account all the monomials involving the spin source fields. Iterating the procedure, for any scale label $h \le -2$, we have the massless field decomposition $\varphi^{(\le h+1)} = \varphi^{(h+1)} + \varphi^{(\le h)}$, so we can integrate out the fields on scale $h+1$, obtaining

$$\int P_c(d\varphi^{(h+1)})\; e^{\mathcal{V}^{(h+1)}(\varphi^{(h+1)}+\varphi^{(\le h)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})} = e^{\bar{\mathcal{V}}^{(h)}(\varphi^{(\le h)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})}\,, \tag{4.21}$$

and we can decompose $\bar{\mathcal{V}}^{(h)}(\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ as

$$\bar{\mathcal{V}}^{(h)}(\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) := \mathcal{V}^{(h)}(\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) + \bar{E}_h + \bar{\mathcal{F}}_h(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})\,, \tag{4.22}$$

where the contributions are defined as after (4.19) (of course by replacing -1 with h). Then, on each scale we can rewrite the perturbed generating function with the same structure of the one on previous scale, namely

$$\Xi_\lambda(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = e^{E_h+\mathcal{F}_h(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}\int P(d\varphi^{(\le h)})e^{\mathcal{V}^{(h)}(\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}\,, \tag{4.23}$$

where, by replacing -1 with h the definitions are the same of the ones listed after (4.20). When we reach the last scale, $h = h^*$, we get

$$\Xi_\lambda(\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi}) = e^{E_{h^*}+\mathcal{F}_{h^*}(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}\int P(d\varphi^{(\le h^*)})e^{\mathcal{V}^{(h^*)}(\varphi^{(\le h^*)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}\,, \tag{4.24}$$

where $P(d\varphi^{(\le h^*)})$ is the Gaussian Grassmann measure with propagator $g^{(\le h^*)}$, defined in (3.55), evaluated at the critical point.

4.1.3 Kernels expansion and recursive definitions

As we have seen in (4.8) for the scale 0, for any scale $h \le -1$ we can derive an expansion of the effective potential on scale h as the following sum of truncated expectations:

$$\begin{aligned}\bar{\mathcal{V}}^{(h)}(\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) &= \log\int P_c(d\varphi^{(h+1)})e^{\mathcal{V}^{(h+1)}(\varphi^{(h+1)}+\varphi^{(\le h)};\mathbf{A},\tilde{\mathbf{A}},\mathbf{\Psi})} = \\ &= \sum_{s\ge 1}\frac{1}{s!}\langle\underbrace{\mathcal{V}^{(h+1)}(\varphi^{(h+1)}+\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}});\cdots;\mathcal{V}^{(h+1)}(\varphi^{(h+1)}+\varphi^{(\le h)};\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})}_{s\text{ times}}\rangle^{\varphi^{(h+1)}}\,.\end{aligned} \tag{4.25}$$

In (4.25) each truncated expectation $\langle\cdot;\cdot\rangle^{\varphi^{(h+1)}}$ is with respect the Gaussian Grassmann integration $P_c(d\varphi^{(h+1)})$ with propagator $g^{(h+1)}$. To properly identify the massless fields that are integrated on scale $(h+1)$ in the truncated expectations, each of the effective potential $\mathcal{V}^{(h+1)}$ in a truncated expectation in (4.25) can be expanded as we did in (4.9), namely

$$\mathcal{V}^{(h+1)}(\varphi;\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})=\sum_{\underline{f}\in M_0}\sum_{\underline{e}\in M_S}W^{(h+1)}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e})\,, \tag{4.26}$$

where the definitions are the same as listed after (4.9) with the following differences:

- $M_0 \subset M_0^{(1)}$ is the subset of $M_0^{(1)}$ where the ω_i indices are restricted to those of the massless fields, i.e. $\omega_k \in \{\pm\}$, $k=1,\dots,|\underline{f}|$, (because we already integrated out the massive fields);
- $M_S \supset M_E$ is the superset of M_E where the j_i indices are extended to also include the spin sources: if $\mathbf{y}_i \in \partial\mathbb{H}$, $j_i \in \{0,1,2\}$, so that

$$F_{\mathbf{y}_i,j_i}=\begin{cases}A_{\mathbf{y}_i,j_i} & \text{if } \mathbf{y}_i\in\mathbb{H}^\circ, j_i=1,2\,,\\ \tilde{A}_{\mathbf{y}_i,j_i} & \text{if } \mathbf{y}_i\in\partial\mathbb{H}, j_i=1,2\,,\\ \Psi_{\mathbf{y}_i} & \text{if } \mathbf{y}_i\in\partial\mathbb{H}, j_i=0\,.\end{cases} \tag{4.27}$$

If we let $n_\sigma(\underline{e}) := |\{\underline{e} : j_k=0, k=1,\dots,|\underline{e}|\}|$ be the number of spin sources, in addition to the conditions $|\underline{e}|\in\mathbb{N}_0$ and $|\underline{f}|\in 2\mathbb{N}$ now we have also $n_\sigma(\underline{e})+n(\underline{f})\in 2\mathbb{N}$ (see after (4.11)). Moreover we let $N_E(\underline{e}) := |\{\underline{e} : j_k=1,2, k=1,\dots,|\underline{e}|\}|$ denote the number of energy sources, so that $|\underline{e}| = N_E(\underline{e})+n_\sigma(\underline{e})$.

Recursive relations between the single scale kernels of the effective potential

By deriving the expression of the effective potential on scale h in terms of the effective potentials on the previous scale as in (4.25) and by expanding each effective potential as in (4.26), we get the following recursive definition for the kernels of the effective potential on scale $h\le -1$:

$$\begin{aligned}W^{(h)}(\underline{f};\underline{e}) = &\sum_{s=1}^{\infty}\frac{1}{s!}\sum^{*}_{\underline{f}_1,\dots,\underline{f}_s\in M_0}\ \sum^{*}_{\underline{e}_1,\dots,\underline{e}_s\in M_S}\left(\prod_{j=1}^{s}W^{(h+1)}(\underline{f}_j,\underline{e}_j)\right)\cdot\\ &\cdot\alpha(\underline{f};\underline{f}_1,\dots,\underline{f}_s)\cdot\langle\varphi(\underline{f}_1^c);\dots;\varphi(\underline{f}_s^c)\rangle^{\varphi^{(h+1)}}\,,\end{aligned} \tag{4.28}$$

where the field variables $\underline{f}_j$, $j=1,\dots,s$, are associated with the fields on scale $h+1$, the field variables $\underline{f}$ are associated with the fields on scale h, and we let $\underline{f}_i^c := \underline{f}_i\setminus\underline{f}$ be the variables associated with the fields that are contracted on scale $h+1$. The superscripts $*$ in (4.28) denote that the sums are performed with the constraint that $\cup_{i=1}^{s}\underline{f}_i^u \supset \underline{f}^u$, where $\underline{f}_i^u$ is the unordered set underlying the tuple $\underline{f}_i$ and$\underline{f}^u$ is the unordered set underlying the set $\underline{f}$, and the analogous constraint for the source field variables. Finally $\alpha(\underline{f};\underline{f}_1,\ldots,\underline{f}_s)$ is the sign of the permutation from $\underline{f}_1\oplus\cdots\oplus\underline{f}_s$ to $\underline{f}\oplus\underline{f}_1^c\oplus\cdots\oplus\underline{f}_s^c$.

Of course, also for $h=0$ holds a recursive expression similar to the one in (4.28), namely

$$\begin{aligned} W^{(0)}(\underline{f}) = &\sum_{s\ge 1}\frac{1}{s!}\sum_{\underline{f}_1,\ldots,\underline{f}_s\in M_0^{(1)}}\ \sum^{*}_{\underline{e}_1,\ldots,\underline{e}_s\in M_S}\left(\prod_{j=1}^{s}W^{(1)}(\underline{f}_j,\underline{e}_j)\right)\cdot \\ &\cdot\alpha(\underline{f};\underline{f}_1,\ldots,\underline{f}_s)\cdot\langle\varphi(\underline{f}_1^c);\ldots;\varphi(\underline{f}_s^c)\rangle^{\xi}\ , \end{aligned} \tag{4.29}$$

where the only difference with respect to (4.28) is that the sums are over the Grassmann field variables in the set $M_0^{(1)}$, including the massive fields, and over the source field variables in the set $M_E\subset M_S$, which is without the spin source fields (see the definitions above (4.9)). For $h\le -1$, also $\mathcal{F}_h(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}})$ admits an expansion similar to the one in (4.26), namely

$$\mathcal{F}_h(\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = \sum_{\underline{e}\in M_S}W^{(h)}(\underline{e})F(\underline{e})\,, \tag{4.30}$$

where the kernels (which depends only on the source field variables) admits a recursive representation similar to the one in (4.28), namely

$$W^{(h)}(\underline{e}) = \sum_{s=1}^{\infty}\frac{1}{s!}\sum^{*}_{\underline{e}_1,\ldots,\underline{e}_s\in M_S}\left(\prod_{j=1}^{s}W^{(h+1)}(\underline{e}_j)\right)\ , \tag{4.31}$$

where we use the same notation introduced after (4.28). Note that if $h=0$, in the r.h.s. we have to perform the sum only over the source field variables in $M_E\subset M_S$.

4.1.4 The Pfaffian formula

We illustrate how to evaluate the truncated expectation of the Grassmann monomials $\varphi(\underline{f}_1^c),\ldots,\varphi(\underline{f}_s^c)$ in the r.h.s. of (4.28) (resp. (4.29)) with respect to the Gaussian Grassmann integration with propagator $g^{(h+1)}$ (resp. g_ξ). In terms of the following Pfaffian formula, originally due to Battle, Brydges and Federbush [14, 15], later improved and simplified [5, 16] and rederived in several review papers [34, 36], see e.g. [40, Lemma 3]: if $\underline{f}_i^c\neq\emptyset$, for all $i=1,\ldots,s$, we can write the explicit expression of the truncated expectations as

$$\langle\varphi(\underline{f}_1^c);\ldots;\varphi(\underline{f}_s^c)\rangle^{\varphi^{(h+1)}} = \sum_{T\in\mathcal{S}(\underline{f}_1^c,\ldots,\underline{f}_s^c)}\mathcal{G}_T^{(h+1)}(\underline{f}_1^c,\ldots,\underline{f}_s^c)\,, \tag{4.32}$$

with

$$\mathcal{G}_T^{(h+1)}(\underline{f}_1^c,\ldots,\underline{f}_s^c) = \alpha_T(\underline{f}_1^c,\ldots,\underline{f}_s^c)\left(\prod_{\ell\in T}g_\ell^{(h+1)}\right)\int P_T(dU)\,Pf\,G^{(h+1)}_{T,\underline{f}_1^c,\ldots,\underline{f}_s^c}(U)\,, \tag{4.33}$$

where:

- $\mathcal{S}(\underline{f}_1^c,\dots,\underline{f}_s^c)$ is the set of 'spanning trees' on $\underline{f}_1^c,\dots,\underline{f}_s^c$: we can construct T, a spanning tree on $\underline{f}_1^c,\dots,\underline{f}_s^c$, as a tree graph formed by s vertices connected by $s-1$ lines $\ell := ((\omega,\mathbf{x}),(\omega',\mathbf{x}'))$, with $(\omega,\mathbf{x})\in\underline{f}_i^c$, $(\omega',\mathbf{x}')\in\underline{f}_j^c$ and $i<j$;

- $\alpha_T(\underline{f}_1^c,\dots,\underline{f}_s^c)$ is the sign of the permutation from $\underline{f}_1^c\oplus\cdots\oplus\underline{f}_s^c$ to $T\oplus\underline{f}_1^c\setminus T\oplus\cdots\oplus\underline{f}_1^c\setminus T$;

- if $\ell\in T$, $g_\ell^{(h+1)} := g_{\omega\omega'}^{(h+1)}(\mathbf{x},\mathbf{x}')$ which can be either the single scale massless propagator in (3.56) if $h\le -1$ or the massive propagator in (3.25) if $h=0$ ($g_\ell^{(1)} := g^{\xi}_{\omega\omega'}(\mathbf{x},\mathbf{x}')$);

- $U := \{u_{i,i'}\in[0,1], 1\le i,i'\le s\}$ and $P_T(dU)$ is a probability measure with support on a set of U such that $u_{i,i'} = \mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm;

- if $2q=\sum_{i=1}^s|f_i^c|$ is the number of contracted fields and $2(s-1)$ is the number of fields that belong to the lines of T, then $G^{(h+1)}_{T,\underline{f}_1^c,\dots,\underline{f}_s^c}(U)$ is an antisymmetric $(2q-2(s-1))\times(2q-2(s-1))$ matrix, whose elements are given by the propagators formed by the contracted fields that do not belong to the lines of T: $(G^{(h+1)}_{T,\underline{f}_1^c,\dots,\underline{f}_s^c})_{i,i'} = u_{i,i'}\cdot g^{(h+1)}_{\omega,\omega'}(\mathbf{x},\mathbf{x}')$, with $i,i'\in\{1,\dots,s\}$ such that $(\omega,\mathbf{x})\in\underline{f}_i^c\setminus T$ and $(\omega',\mathbf{x}')\in\underline{f}_{i'}^c\setminus T$;

Note that, if $s=1$, then $T=\emptyset$ and $\langle\varphi(\underline{f}_1^c)\rangle^{\varphi^{(h+1)}} = Pf\,G^{(h+1)}_{\emptyset,\underline{f}_1^c}$, where $\left(G^{(h+1)}_{\emptyset,\underline{f}_1^c}\right)_{i,i'} = g^{(h+1)}_{\omega,\omega'}(\mathbf{x},\mathbf{x}')$, with $(\omega,\mathbf{x}),(\omega',\mathbf{x}')\in\underline{f}_1^c$.

4.2 Localization and interpolation

In this section, we illustrate the procedure to follow to identify among all the contributions of $\mathcal{V}^{(h)}$ those that may diverge and to keep track of them during the iterative construction. In particular, two operators are introduced, $\mathcal{L}$ and $\mathcal{R}$, such that $\mathcal{I}=\mathcal{L}+\mathcal{R}$, which act on each $W^{(h)}$ dividing it into a *local* part, $\mathcal{L}W^{(h)}$, and a remainder $\mathcal{R}W^{(h)}$. This division of the effective kernel $W^{(h)}$ will correspond to a division of the effective potential $\mathcal{V}^{(h)}$ into $\mathcal{L}\mathcal{V}^{(h)}$ and $\mathcal{R}\mathcal{V}^{(h)}$.

The goal will be to describe $\mathcal{L}\mathcal{V}^{(h)}$ using as few couplings as possible, and to rewrite $\mathcal{R}\mathcal{V}^{(h)}$ with an appropriate interpolation: to do this, we will have to group some Grassmann fields in discrete derivatives

$$\partial_j\varphi_{\omega,\mathbf{y}} := \varphi_{\omega,\mathbf{y}+\hat{\mathbf{e}}_j} - \varphi_{\omega,\mathbf{y}}\,, \tag{4.34}$$

with $j=1,2$. Consequently, we will be interested in putting labels on each effective kernel that specify how and when the fields were derived. Let's take an example to explain what we mean. Consider a simple contribution to the effective potential: the one associated with the monomial $\varphi_{\omega,\mathbf{y}}\varphi_{\omega',\mathbf{z}}$ and its kernel $W((\omega,\mathbf{y}),(\omega',\mathbf{z}))$, where $\mathbf{y} = (y^{(1)}, y^{(2)})$ and $\mathbf{z} = (z^{(1)}, y^{(2)})$, namely

$$\sum_{\omega,\omega'\in\{\pm\}} \sum_{y^{(1)},z^{(1)},y^{(2)}\in\mathbb{H}} W((\omega,(y^{(1)},y^{(2)})),(\omega',(z^{(1)},y^{(2)})))\varphi_{\omega,(y^{(1)},y^{(2)})}\varphi_{\omega',(z^{(1)},y^{(2)})}\,. \tag{4.35}$$

Since the two fields have the same vertical coordinate, from now on we will only write the dependence on their horizontal coordinates $y^{(1)}$ and $z^{(1)}$. If we add and subtract $\varphi_{\omega,y^{(1)}}\varphi_{\omega',y^{(1)}}$, we can rewrite (4.35) as

$$\begin{aligned}&\sum_{\omega,\omega'\in\{\pm\}} \sum_{y^{(1)},z^{(1)}\in\mathbb{H}} W((\omega,y^{(1)}),(\omega',z^{(1)}))\varphi_{\omega,y^{(1)}}\varphi_{\omega',y^{(1)}}+\\ &+\sum_{\omega,,\omega'\in\{\pm\}} \sum_{y^{(1)},z^{(1)}\in\mathbb{H}} W((\omega,y^{(1)}),(\omega',z^{(1)}))\varphi_{\omega,y^{(1)}}(\varphi_{\omega',z^{(1)}}-\varphi_{\omega',y^{(1)}})\,,\end{aligned} \tag{4.36}$$

where the term in the first line is the local contribution and the term in the last line is the remainder. Now let's focus on the latter term: if we interpolate between the position $z^{(1)}$ and the position $y^{(1)}$, we can rewrite the difference $(\varphi_{\omega',z^{(1)}}-\varphi_{\omega',y^{(1)}})$ as the sum of the differences between adjacent fields, namely

$$\sum_{\omega,\omega'\in\{\pm\}} \sum_{y^{(1)},z^{(1)}\in\mathbb{H}} W((\omega,y^{(1)}),(\omega',z^{(1)}))\varphi_{\omega,y^{(1)}}(\varphi_{\omega',z^{(1)}}-\varphi_{\omega',z^{(1)}-1}+\cdots+\varphi_{\omega',y^{(1)}+1}-\varphi_{\omega',y^{(1)}}), \tag{4.37}$$

which, by using (4.34), can be rewritten as

$$\sum_{\omega,\omega'\in\{\pm\}} \sum_{y^{(1)},z^{(1)}\in\mathbb{H}} W((\omega,y^{(1)}),(\omega',z^{(1)}))\varphi_{\omega,y^{(1)}} \sum_{k=0}^{K-1} \partial_1\varphi_{\omega',t_k^{(1)}}\,, \tag{4.38}$$

where $K := |y^{(1)} - z^{(1)}| \in \mathbb{Z}$, $t_0^{(1)} := y^{(1)}$ and $t_k^{(1)} := z^{(1)}$.

This is a particularly simplified example but we will see that this same interpolation mechanism will appear later in more general contribution to the effective potential. However, comparing (4.35) with (4.38) shows why effective kernels must also be labeled with derivative indices. This discussion is rigorously justified in [4, Sec.4.1], where is provided a complete description of the kernel equivalence classes. Here, in order not to complicate the presentation, we prefer to simply rely on the conceptual motivation provided by the example above and introduce the following useful notation.

The derivative kernel indices

If $h = 1$, we rewrite the expansion of the effective potential in (4.9) as

$$\mathcal{V}^{(1)}(\xi,\varphi;\mathbf{\Psi},\mathbf{A},\tilde{\mathbf{A}}) = \sum_{\underline{f}\in M^{(1)}} \sum_{\underline{e}\in M_E} W^{(1)}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e})\,, \tag{4.39}$$

where M_E is defined after (4.9) and $M^{(1)}$ is the set of the tuples

$$\underline{f} \equiv (\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}}) := ((\omega_1, \mathbf{y}_1, \mathbf{d}_1), (\omega_2, \mathbf{y}_2, \mathbf{d}_2), \ldots),$$

where for each $k = 1, \ldots, |\underline{f}|$, $|\underline{f}| \in 2\mathbb{N}$, we have $\omega_k \in \{\pm, \pm i\}$, $\mathbf{y}_k \in \mathbb{H}$, $\mathbf{d}_k := (d_k^{(1)}, d_k^{(2)})$, $d_k^{(j)} \in \{0, 1, 2\}$ for $j = 1, 2$. With respect to $M_0^{(1)}$ defined after (4.9), we are now including the derivatives acting on the Grassmann fields ($M_0^{(1)} \subset M^{(1)}$), so that $\varphi(\underline{f})$ is now defined as the product of derivative Grassmann fields

$$\varphi(\underline{f}) \equiv \varphi(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}}) := \prod_{k=1}^{|\underline{f}|} \partial^{\mathbf{d}_k} \varphi_{\omega_k, \mathbf{y}_k}, \tag{4.40}$$

with $\partial^{\mathbf{d}_k} := \partial_1^{d_k^{(1)}} \partial_2^{d_k^{(2)}}$ and $\varphi_{\omega_k, \mathbf{x}_i}$ defined in (4.11). To evaluate the discrete derivative we use the definition in (4.34) and, for $j = 1, 2$, the following notation holds: $\partial_j^0 \varphi_{\omega_k, \mathbf{x}_i} = \varphi_{\omega_i, \mathbf{x}_i}$, $\partial_j^1 \varphi_{\omega_k, \mathbf{y}_k} = \partial_j \varphi_{\omega_k, \mathbf{y}_k}$, $\partial_j^2 \varphi_{\omega_k, \mathbf{y}_k} = \partial_j \varphi_{\omega_k, \mathbf{y}_k + \hat{\mathbf{e}}_j} - \partial_j \varphi_{\omega_k, \mathbf{y}_k}$. Analogously, for any $h \leq -1$, we can rewrite (4.26) as

$$\mathcal{V}^{(h+1)}(\varphi; \mathbf{\Psi}, \mathbf{A}, \tilde{\mathbf{A}}) = \sum_{\underline{f} \in M} \sum_{\underline{e} \in M_S} W^{(h+1)}(\underline{f}; \underline{e}) \varphi(\underline{f}) F(\underline{e}), \tag{4.41}$$

where M_S is defined after (4.26) and the M is the set that, with respect to the set M_0 defined after (4.26), includes the derivatives of the massless Grassmann fields ($M_0 \subset M$): the definitions are the same as the ones after (4.39), with the only difference that now $\omega_k \in \{\pm\}$ for all $k = 1, \ldots, |\underline{f}|$.

By recalling the definitions after (4.11), on scale 1 we can identify each kernel of the effective potential in the r.h.s. of (4.39) by defining

$$W^{(1)}_{(n+n^{(1)}, D+D^{(1)}), N_E}(\underline{f}; \underline{e}) := W^{(1)}(\underline{f}; \underline{e}) \Big|^{*}_{n(\underline{f})=n; n^{(1)}(\underline{f})=n^{(1)}; |\underline{e}|=N_E}, \tag{4.42}$$

where $*$ denotes the constraint that $\sum_{k=1}^{n} |\mathbf{d}_k| = D$ and $\sum_{k=1}^{n^{(1)}} |\mathbf{d}_k| = D^{(1)}$, $D, D^{(1)} \in \mathbb{N}$ which are the global order of the derivatives acting (respectively) on the massless and on the massive fields. Analogously, if $h \leq -1$, we can identify each kernel of the effective potential in the r.h.s. of (4.41) by defining

$$W^{(h+1)}_{(n,D), N_E + n_\sigma}(\underline{f}; \underline{e}) := W^{(h+1)}(\underline{f}; \underline{e}) \Big|^{*}_{n(\underline{f})=n; m_\epsilon(\underline{e})=m_\epsilon; n_\sigma(\underline{e})=n_\sigma}, \tag{4.43}$$

where $*$ denotes the constraint that $\sum_{i=1}^{n} ||\mathbf{d}_i||_1 = D$, $D \in \mathbb{N}$, which is the total order of the derivatives acting on the massless fields.

The scaling dimension

With the notations introduced so far for the kernels of the effective potential we are able to introduce the localization and renormalization operators, $\mathcal{L}$ and $\mathcal{R}$, which act on the kernels in (4.42) and (4.43). More precisely when $h = 1$, we get

$$W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E} = \mathcal{L}W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E} + \mathcal{R}W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E}\,, \tag{4.44}$$

where the kernel $W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E}$ is defined in (4.42); when $h \le 0$, we get

$$W^{(h)}_{(n,D),N_E+n_\sigma} = \mathcal{L}W^{(h)}_{(n,D),N_E+n_\sigma} + \mathcal{R}W^{(h)}_{(n,D),N_E+n_\sigma}\,, \tag{4.45}$$

where the kernel $W^{(h)}_{(n,D),N}$ is defined in (4.43). Note that in (4.44) and in (4.45), as we will do in the following, we use $=$ to denote an equivalence among the kernels.

At this point, it is useful to introduce the *scaling dimension*:

$$d_S := 2 - \frac{n}{2} - D - \frac{n_\sigma}{2} - N_E\,, \tag{4.46}$$

where we recall that n is the number of the massless fields, D global number of derivatives acting on them, n_σ is the number of the edge spin sources and N_E is the number of the energy sources (see after (4.42) and (4.43)). The scaling dimension in (4.46) can be used to identify the 'nature' of each contribution of the effective potential according to the indices $(n, D), N_E + n_\sigma$ of its kernels. We can distinguish between *relevant*, *marginal* and *irrelevant* contributions, corresponding to kernels with indices $(n, D), N_E + n_\sigma$ such that d_S is, respectively, positive, vanishing or negative. During the multiscale procedure, the relevant and marginal contributions tend 'to expand' under iterations: these terms will be selected and traced by the action of the $\mathcal{L}$ operator. The irrelevant terms, which are not the source of any divergence, will be bounded on each scale of the iteration by the action of $\mathcal{R}$ (in next section 4.2.3 we derive several estimates related to $\mathcal{R}$). As in (4.46) does not appear the indices $n^{(1)}$ and $D^{(1)}$ (the number of the massive fields and the global derivatives acting on them), we will consider the contributions of the kernels with $n^{(1)} > 0$ and $D^{(1)} \ge 0$ as irrelevant terms of the effective potential on scale 1.
The values of the kernel indices $(n, D), N_E + n_\sigma$ such that $d_S \ge 0$, are the following:

- if there are no source fields, $N_E = n_\sigma = 0$,

$$((n, D), 0) \in \{\, ((2, 0), 0); ((2, 1), 0); ((4, 0), 0) \,\}\ ; \tag{4.47}$$

- if there are only edge spin source fields, $N_E = 0$,

$$((n, D), n_\sigma) \in \{\, ((1, 0), 1); ((1, 1), 1); ((3, 0), 1); ((2, 0), 2) \,\}\ ; \tag{4.48}$$

- if there are only energy source fields, $n_\sigma = 0$,

$$((n, D), N_E) = ((2, 0), 1)\,, \tag{4.49}$$

  where $N_E = 1$ can be associated either to one bulk energy source field or to one edge energy source field.

If the kernel indices $(n, D), N_E + n_\sigma$ are different from those in (4.47)-(4.49), the scaling dimension in (4.46) is negative and we let

$$\mathcal{L}W^{(h)}_{(n,D),N_E+n_\sigma} := 0\,, \qquad W^{(h)}_{(n,D),N_E+n_\sigma} = \mathcal{R}W^{(h)}_{(n,D),N_E+n_\sigma}\,, \tag{4.50}$$

so that the local part of the irrelevant terms vanishes. If $h = 1$ (so that $n_\sigma = 0$) and $n^{(1)} = 0$ (so that also $D^{(1)} = 0$), we let $W^{(1)}_{(n,D),N_E} := W^{(1)}_{(n+0,D+0),N_E+0}$, so that (4.50) holds for any $h \le 1$. If $h = 1$ and $n^{(1)} > 0$ (and $D^{(1)} \ge 0$) we let

$$\mathcal{L}W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E} := 0\,, \qquad W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E} = \mathcal{R}W^{(1)}_{(n+n^{(1)},D+D^{(1)}),N_E}\,, \tag{4.51}$$

so that the kernels with any positive number of massive fields behave as irrelevant terms.

In order to derive the two-point edge spin correlations, we are especially interested in defining the action of $\mathcal{L}$ and $\mathcal{R}$ on the kernels with $n_\sigma > 0$ and $N_E = 0$. Then, from now on, we will present only results for the effective potential with the spin source ($\mathbf{\Psi} \neq \mathbf{0}$) but not the energy sources ($\mathbf{A} = \tilde{\mathbf{A}} = \mathbf{0}$), so that we can focus on the kernels depending only on spin source index $n_\sigma \ge 0$, $N_E = 0$. In fact, even if $\mathbf{\Psi} \neq \mathbf{0}$, in the effective potential there will be both kernels with $n_\sigma = 0$ and $n_\sigma > 0$. Then, as a first step it is convenient to illustrate the action of $\mathcal{L}$ and $\mathcal{R}$ on the kernels without any source fields ($n_\sigma = 0$ and $N_E = 0$), which we will see in Subsec. 4.2.1. Then, in Subsec. 4.2.2 we will derive the action of the operators on the kernels with only spin source fields, i.e. with $n_\sigma > 0$ and $N_E = 0$. The definitions of the operators acting on kernels with $n_\sigma = N_E = 0$, as well of the ones acting on kernels with only bulk energies ($N_E = 1$ associated with a bulk source), were already derived in [37, Subsec.3.3] and in [4, Subsec.4.1.1] for the infinite lattice, and in [4, Subsec.4.2.1] for the cylindrical lattice. The definitions for the operators acting on kernels with only edge energies ($N_E = 1$ associated with an edge source) have not yet been derived: they are not studied here, but the procedures introduced so far should be extended in the perspective of a generalization of our result for the edge spin sources. Moreover, in Subsec. 4.2.3 we will introduce the norms adapted to each type of kernels.

4.2.1 Localization for the sourceless kernels

If $n_\sigma = 0$, the bulk-edge decomposition of the propagators in 3.2.4 induces the bulk-edge decomposition of the kernels, namely

$$W^{(h)}_{(n,D),0} = W^{(h)}_{B;(n,D),0} + W^{(h)}_{E;(n,D),0}\,, \tag{4.52}$$

because $W^{(h)}_{(n,D),0}$ involves convolutions of propagators, as is evident from (4.28) and from the expressions in Subsec. 4.1.4. We let $W^{(h)}_B$ be the kernel involving only bulk propagators and $W^{(h)}_E := W^{(h)}_B - W^{(h)}$ be the edge kernel. The bulk-edge decomposition in (4.52) holds for each $h \leq 1$ (by this we mean that if $h = 1$, $n^{(1)} = 0$); if $h = 1$, $n^{(1)} > 0$ and $D^{(1)} \geq 0$, we get

$$W^{(1)}_{(n+n^{(1)},D+D^{(1)}),0} = W^{(1)}_{B;(n+n^{(1)},D+D^{(1)}),0} + W^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0}\,, \tag{4.53}$$

where $n^{(1)}$ and $D^{(1)}$ are referred to as the massive fields (see after (4.39)).

Next, to obtain the decomposition in (4.45) when $N = n_\sigma = 0$, we need to define the bulk operators $\mathcal{L}_B$ and $\mathcal{R}_B$, acting on the bulk kernels, and the edge operators $\mathcal{L}_E$ and $\mathcal{R}_E$, acting on the edge kernels.

Localization and interpolation for the bulk kernels

For the bulk kernels in the r.h.s. of (4.52), we introduce the $\mathcal{L}_B$ and $\mathcal{R}_B$ operators, so that

$$W^{(h)}_{B;(n,D),0} = \mathcal{L}_B W^{(h)}_{B;(n,D),0} + \mathcal{R}_B W^{(h)}_{B;(n,D),0}\,. \tag{4.54}$$

As previously mentioned (see (4.50)), for each $h \leq 1$ we let $\mathcal{L}_B W^{(h)}_{B;(n,D),0} := 0\,,$ if $n^{(1)} = 0$ and the massless indices (n, D) are such that $d_S < 0$; for $h = 1$ we let $\mathcal{L}_B W^{(h)}_{B;(n+n^{(1)},D+D^{(1)}),0} := 0\,,$ if $n^{(1)} \geq 1$. To define the action of the bulk localization operator $\mathcal{L}_B$ on the kernels with the Grassmann indices $(n, D) : d_S \geq 0\,,$ we need to define other basic operators: let $\mathcal{O}$ be the operator that antisymmetrizes with respect to permutations of the Grassmann field variables and symmetrizes with respect to reflections in the horizontal direction, and $\tilde{\mathcal{L}}_B$ be the operator which 'localizes' all the Grassmann field positions at the same site:

$$\begin{aligned}&\tilde{\mathcal{L}}_B W^{(h)}_{B;(n,D),0}((\omega_1, \mathbf{y}_1, \mathbf{d}_1), (\omega_2, \mathbf{y}_2, \mathbf{0}), \ldots, (\omega_n, \mathbf{y}_n, \mathbf{d}_n))\\ &:= \prod_{j=2}^{n} \delta_{\mathbf{y}_j,\mathbf{y}_1} \sum_{\mathbf{z}_2,\ldots,\mathbf{z}_n \in \mathbb{H}} W^{(h)}_{B;(n,D),0}((\omega_1, \mathbf{y}_1, \mathbf{d}_1), (\omega_2, \mathbf{z}_2, \mathbf{d}_2), \ldots, (\omega_n, \mathbf{z}_n, \mathbf{d}_n))\,,\end{aligned} \tag{4.55}$$

where we (arbitrary) chose the Grassmann field position $\mathbf{y}_1$ as localization site. The action of $\tilde{\mathcal{L}}_B$ preserves the kernel indices, i.e. $\tilde{\mathcal{L}}_B(W^{(h)}_{B;(n,D),0}) = (\tilde{\mathcal{L}}_B W^{(h)}_B)_{(n,D),0} = \tilde{\mathcal{L}}_B W^{(h)}_{B;(n,D),0}$. With these two operators we are able to localize the marginal terms: if $(n, D) = (2, 1), (4, 0)$, we let

$$\mathcal{L}_B(W^{(h)}_{B;(2,1),0}) := \mathcal{O}(\tilde{\mathcal{L}}_B W^{(h)}_{B;(2,1),0})\,, \qquad \mathcal{L}_B(W^{(h)}_{B;(4,0),0}) := \mathcal{O}(\tilde{\mathcal{L}}_B W^{(h)}_{B;(4,0),0})\,. \tag{4.56}$$

Now we consider the $n = 4$ term in (4.56) and we look at

$$\sum_{\underline{f} \in M} \tilde{\mathcal{L}}_B W^{(h)}_{B;(4,0),0}(\underline{f})\varphi(\underline{f})\,, \tag{4.57}$$

which is a contribution to the marginal part of the effective potential up to the action of $\mathcal{O}$. By using the definition of the $\tilde{\mathcal{L}}_B$ given in (4.55), one can easily check that (4.57) can be rewritten as

$$\sum_{\omega_1,\dots,\omega_4\in\{\pm\}}\sum_{\mathbf{y}_1,\dots,\mathbf{y}_4\in\mathbb{H}} W^{(h)}_{B;(4,0),0}((\omega_1,\mathbf{y}_1),\dots,(\omega_4,\mathbf{y}_4))\varphi_{\omega_1,\mathbf{y}_1}\varphi_{\omega_2,\mathbf{y}_1}\varphi_{\omega_3,\mathbf{y}_1}\varphi_{\omega_4,\mathbf{y}_1}\,, \tag{4.58}$$

which vanishes because ω_i can assume only two values (see (2.11)) so that in (4.56) we can consider $\mathcal{L}_B(W^{(h)}_{B;(4,0),0})=0$, a cancellation that will play an important role in the following. To localize the relevant terms ($d_S>0$), we introduce one more operator: let $\tilde{\mathcal{R}}_B$ be the operator which 'interpolates' between the localization site and the original position of the Grassmann field, so that $\tilde{\mathcal{R}}_B(W^{(h)}_{B;(n,D),0})$ is equivalent to $W^{(h)}_{B;(n,D),0}-\tilde{\mathcal{L}}_B W^{(h)}_{B;(n,D),0}$. The action of $\tilde{\mathcal{R}}_B$ does not preserve the kernel indices: it acts by increasing the value D by one, so that $\tilde{\mathcal{R}}_B(W^{(h)}_{B;(n,D),0})=(\tilde{\mathcal{R}}_B W^{(h)}_B)_{(n,D+1),0}$. Then, if $(n,D)=(2,0)$, we can start by considering

$$\begin{aligned}&\sum_{\underline{f}\in M}\{W^{(h)}_{B;(2,0),0}(\underline{f})-\tilde{\mathcal{L}}W^{(h)}_{B;(2,0),0}(\underline{f})\}\varphi(\underline{f})=\\&\quad=\sum_{\omega_1,\omega_2\in\{\pm\}}\sum_{\mathbf{y}_1,\mathbf{y}_2\in\mathbb{H}} W^{(h)}_B((\omega_1,\mathbf{y}_1,\mathbf{0}),(\omega_2,\mathbf{y}_2,\mathbf{0}))\,\varphi_{\omega_1,\mathbf{y}_1}\{\varphi_{\omega_2,\mathbf{y}_2}-\varphi_{\omega_2,\mathbf{y}_1}\}\,,\end{aligned} \tag{4.59}$$

and we rewrite $\{\varphi_{\omega_2,\mathbf{y}_2}-\varphi_{\omega_2,\mathbf{y}_1}\}$ as a sum of differences of fields located at adjacent positions (as seen in the easy example after (4.36)). Let $\gamma(\mathbf{y}_1,\mathbf{y}_2)$ be the shortest path obtained by going first horizontally and then vertically from $\mathbf{y}_1$ to $\mathbf{y}_2$, so that, by recalling the definition of derivative in (4.34), we can rewrite $\{\varphi_{\omega_2,\mathbf{y}_2}-\varphi_{\omega_2,\mathbf{y}_1}\}$ as

$$\{\sigma_0\partial_{j_0}\varphi_{\omega_2,\mathbf{y}'_0}+\sigma_1\partial_{j_1}\varphi_{\omega_2,\mathbf{y}'_1}+\cdots+\sigma_k\partial_{j_k}\varphi_{\omega_2,\mathbf{y}'_k}\}\,,$$

with $\mathbf{y}'_0,\dots,\mathbf{y}'_k\in\gamma(\mathbf{y}_1,\mathbf{y}_2)$ such that $\mathbf{y}'_0=\mathbf{y}_1$, $\mathbf{y}'_1=\mathbf{y}_1+\hat{\mathbf{e}}_{j_0}$, $\mathbf{y}'_{r+1}=\mathbf{y}'_r+\hat{\mathbf{e}}_{j_r}$, for $r=1,\dots,k-1$ and $\mathbf{y}_2=\mathbf{y}'_k+\hat{\mathbf{e}}_{j_k}$, with $j_0,j_1,\dots,j_k\in\{1,2\}$. Each σ_i, $i=1,\dots,k$, is a sign such that $\sigma_i=+$ if $\mathbf{y}'_i$ precedes $\mathbf{y}'_i+\hat{\mathbf{e}}_{j_i}$, $\sigma_i=-$ otherwise. Then we can rewrite the r.h.s. of (4.59) as

$$\sum_{\omega_1,\omega_2\in\{\pm\}}\sum_{\mathbf{y}_1,\mathbf{y}_2\in\mathbb{H}} W^{(h)}_B((\omega_1,\mathbf{y}_1,\mathbf{0}),(\omega_2,\mathbf{y}_2,\mathbf{0}))\sum_{\sigma,(\mathbf{y}_1,\mathbf{0}),(\mathbf{y}',\mathbf{d}')\in I(\mathbf{y}_1,\mathbf{y}_2)}\sigma\,\varphi_{\omega_1,\mathbf{y}_1}\partial^{\mathbf{d}'}\varphi_{\omega_2,\mathbf{y}'}\,, \tag{4.60}$$

where $I(\mathbf{y}_1,\mathbf{y}_2)$ is the set of $(\sigma,(\mathbf{y}_1,\mathbf{0}),(\mathbf{y}',\mathbf{d}'))$, with $\sigma\in\{\pm\}$, $\mathbf{y}_1,\mathbf{y}'\in\mathbb{H}$ and $\mathbf{d}'=(d'_1,d'_2)\in\{0,1\}^2$ with $|\mathbf{d}'|=|d'_1|+|d'_2|=1$, such that

$$\mathbf{d}'=\begin{cases}(1,0) & \text{if } \mathbf{y}',\mathbf{y}'+\hat{\mathbf{e}}_1\in\gamma(\mathbf{y}_1,\mathbf{y}_2)\,,\\(0,1) & \text{if } \mathbf{y}',\mathbf{y}'+\hat{\mathbf{e}}_2\in\gamma(\mathbf{y}_1,\mathbf{y}_2)\,,\end{cases} \tag{4.61}$$

and $I(\mathbf{y}_1,\mathbf{y}_2)=\emptyset$ either if $\mathbf{y}_1=\mathbf{y}_2$ or if $\mathbf{y}',\mathbf{y}'+\hat{\mathbf{e}}_j\notin\gamma(\mathbf{x}_1,\mathbf{x}_2)$. By exchanging the sums in (4.60) we finally get

$$\sum_{\omega_1,\omega_2\in\{\pm\}}\sum_{\mathbf{y}_1,\mathbf{y}'\in\mathbb{H}}\sum_{\mathbf{d}':|\mathbf{d}'|=1}(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,1)}((\omega_1,\mathbf{y}_1,\mathbf{0}),(\omega_2,\mathbf{y}',\mathbf{d}'))\ \varphi_{\omega_1,\mathbf{y}_1}\partial^{\mathbf{d}'}\varphi_{\omega_2,\mathbf{y}'}\,, \tag{4.62}$$

where

$$\begin{aligned}(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,1)}&((\omega_1,\mathbf{y}_1,\mathbf{0}),(\omega_2,\mathbf{y}',\mathbf{d}')):=\\ &=\sum_{\substack{\sigma,\mathbf{y}_1,\mathbf{y}_2:\\ \sigma,(\mathbf{y}_1,\mathbf{0}),(\mathbf{y}',\mathbf{d}')\in I(\mathbf{y}_1,\mathbf{y}_2)}}\sigma W^{(h)}_{B;(2,0)}((\omega_1,\mathbf{y}_1,\mathbf{0}),(\omega_2,\mathbf{y}_2,\mathbf{0}))\,.\end{aligned} \tag{4.63}$$

Then we can finally localize the relevant term: if $\mathcal{L}_B$ acts on a kernel with $(n,D)=(2,0)$, we let

$$\mathcal{L}_B(W^{(h)}_{B;(2,0),0})=\mathcal{O}(\tilde{\mathcal{L}}_B W^{(h)}_{B;(2,0),0}+\tilde{\mathcal{L}}_B(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,1),0})\,. \tag{4.64}$$

Note that in (4.64) the local part is defined in terms of the $\tilde{\mathcal{R}}_B$ operator, which does not preserve the kernel indices, while in (4.56) the kernel indices are preserved: then we let

$$\mathcal{L}_B W^{(h)}_{B;(n,D),0}:=\begin{cases}\mathcal{O}(\tilde{\mathcal{L}}_B W^{(h)}_{B;(2,0),0}) & \text{if } (n,D)=(2,0)\,,\\ \mathcal{O}(\tilde{\mathcal{L}}_B W^{(h)}_{B;(2,1),0}+\tilde{\mathcal{L}}_B(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,1),0}) & \text{if } (n,D)=(2,1)\,,\\ 0 & \text{otherwise.}\end{cases} \tag{4.65}$$

Now we want to define $\mathcal{R}_B W^{(h)}_{B;(n,D),0}$, which is the remainder term of the bulk kernels, directly in terms of the its indices (not of the kernel indices on which $\mathcal{R}_B$ acts). For any $h\le 1$, if $n^{(1)}=0$ and the massless indices (n,D) are such that $d_S\ge 0$, we let $\mathcal{R}_B W^{(h)}_{B;(n,D),0}:=0$; if $n^{(1)}>0$ (regardless of the massless indices (n,D)) we let

$$\mathcal{R}_B W^{(h)}_{B;(n+n^{(1)},D+D^{(1)}),0}=W^{(h)}_{B;(n+n^{(1)},D+D^{(1)}),0}\,; \tag{4.66}$$

if $n^{(1)}=0$ and the massless indices are such that $d_S<0$ with the constraint that $((n,D),0)\ \notin\ \{((2,2),0),((4,1),0)\}$, we let

$$\mathcal{R}_B W^{(h)}_{B;(n,D),0}=W^{(h)}_{B;(n,D),0}\,; \tag{4.67}$$

if $n^{(1)}=0$ and $((n,D),0)\in\{((2,2),0),((4,1),0)\}$ (so that $d_S<0$) we let

$$\begin{aligned}\mathcal{R}_B W^{(h)}_{B;(2,2),0}&=\mathcal{O}(W^{(h)}_{B;(2,2),0}+(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,2),0}+(\tilde{\mathcal{R}}_B(\tilde{\mathcal{R}}_B W_B^{(h)}))_{(2,2),0})\,,\\ \mathcal{R}_B W^{(h)}_{B;(4,1),0}&=\mathcal{O}(W^{(h)}_{B;(4,1),0}+(\tilde{\mathcal{R}}_B W_B^{(h)})_{(4,1),0})\,,\end{aligned} \tag{4.68}$$

where the $\tilde{\mathcal{R}}_B$ are defined similarly to (4.63). More precisely, if $(n,D)=(2,2)$ the $\tilde{\mathcal{R}}_B$ operator in (4.68) acts on a kernel with $(n,D)=(2,1)$, so that

$$\begin{aligned}(\tilde{\mathcal{R}}_B W_B^{(h)})_{(2,2)}&((\omega_1,\mathbf{y}_1,\mathbf{d}_1),(\omega_2,\mathbf{y}',\mathbf{d}_2+\mathbf{d}')):=\\ &=\sum_{\substack{\sigma,\mathbf{y}_1,\mathbf{y}_2:\\ (\sigma,(\mathbf{y}_1,\mathbf{d}_1),(\mathbf{y}',\mathbf{d}_2+\mathbf{d}'))\in I(\mathbf{y}_1,\mathbf{y}_2)}}\sigma W^{(h)}_{B;(2,1)}((\omega_1,\mathbf{y}_1,\mathbf{d}_1),(\omega_2,\mathbf{y}_2,\mathbf{d}_2))\,,\end{aligned} \tag{4.69}$$

where $\mathbf{d}'$ is defined after (4.63), so that $||\mathbf{d}'||_1 = 1$ and $||\mathbf{d}_1||_1 + ||\mathbf{d}_2||_1 + ||\mathbf{d}'||_1 = 2$. To define the action of the $\tilde{\mathcal{R}}_B$ operator in (4.68) if $(n, D) = (4, 0)$, we start by considering the following contribution:

$$\sum_{\underline{f}\in M} \{W^{(h)}_{B;(4,0)}(\underline{f}) - \tilde{\mathcal{L}} W^{(h)}_{B;(4,0)}(\underline{f})\}\varphi(\underline{f}) = \\ = \sum_{\underline{\omega}\in\{\pm\}} \sum_{\underline{\mathbf{y}}\in\mathbb{H}} W^{(h)}_{B;(4,0)}(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{0}})\, \varphi_{\omega_1,\mathbf{y}_1}\{\varphi_{\omega_2,\mathbf{y}_2}\varphi_{\omega_3,\mathbf{y}_3}\varphi_{\omega_4,\mathbf{y}_4} - \varphi_{\omega_2,\mathbf{y}_1}\varphi_{\omega_3,\mathbf{y}_1}\varphi_{\omega_4,\mathbf{y}_1}\}\,, \tag{4.70}$$

where we rewrite $\varphi_{\omega_1,\mathbf{y}_1}\{\varphi_{\omega_2,\mathbf{y}_2}\varphi_{\omega_3,\mathbf{y}_3}\varphi_{\omega_4,\mathbf{y}_4} - \varphi_{\omega_2,\mathbf{y}_1}\varphi_{\omega_3,\mathbf{y}_1}\varphi_{\omega_4,\mathbf{y}_1}\}$ as

$$\varphi_{\omega_1,\mathbf{y}_1}\{\varphi_{\omega_2,\mathbf{y}_1}\varphi_{\omega_3,\mathbf{y}_1}(\varphi_{\omega_4,\mathbf{y}_4} - \varphi_{\omega_4,\mathbf{y}_1}) + \varphi_{\omega_2,\mathbf{y}_1}(\varphi_{\omega_3,\mathbf{y}_3} - \varphi_{\omega_3,\mathbf{y}_1})\varphi_{\omega_4,\mathbf{y}_4} + \\ + (\varphi_{\omega_2,\mathbf{y}_2} - \varphi_{\omega_2,\mathbf{y}_1})\varphi_{\omega_3,\mathbf{y}_3}\varphi_{\omega_4,\mathbf{y}_4}\}\,, \tag{4.71}$$

or as a similar expression. Next, the first term in (4.71) vanishes (see (2.11)); the other terms, with the differences $(\varphi_{\omega_3,\mathbf{y}_3} - \varphi_{\omega_3,\mathbf{y}_1})$ and $(\varphi_{\omega_2,\mathbf{y}_2} - \varphi_{\omega_2,\mathbf{y}_1})$, can be rewritten as described after (4.59), so that (4.71) equals

$$\sum_{\sigma,(\mathbf{y}'_1,\mathbf{d}'_1),\dots,(\mathbf{y}'_4,\mathbf{d}'_4)\in I(\mathbf{y}_1,\dots,\mathbf{y}_4)} \sigma\partial^{\mathbf{d}'_1}\varphi_{\omega_1,\mathbf{y}'_1}\partial^{\mathbf{d}'_2}\varphi_{\omega_2,\mathbf{y}'_2}\partial^{\mathbf{d}'_3}\varphi_{\omega_3,\mathbf{y}'_3}\partial^{\mathbf{d}'_4}\varphi_{\omega_4,\mathbf{y}'_4}\,, \tag{4.72}$$

where $I(\mathbf{y}_1, \dots, \mathbf{y}_4)$ is the set of $(\sigma, (\mathbf{y}'_1, \mathbf{d}'_1), \dots, (\mathbf{y}'_4, \mathbf{d}'_4))$, with $\sigma \in \{\pm\}$, $\mathbf{y}'_1, \dots, \mathbf{y}'_4 \in \mathbb{H}$ and $\mathbf{d}'_1, \dots, \mathbf{d}'_4 \in \{0, 1\}^2$ with $|\mathbf{d}'_1| + \dots + |\mathbf{d}'_4| = 1$ such that: either $\mathbf{y}'_1 = \mathbf{y}'_2 = \mathbf{y}_1$, $\mathbf{y}'_4 = \mathbf{y}_4$, $\mathbf{d}'_1 = \mathbf{d}'_2 = \mathbf{d}'_4 = \mathbf{0}$ and $(\sigma, (\mathbf{y}_1, \mathbf{0}), (\mathbf{y}'_3, \mathbf{d}'_3)) \in I(\mathbf{y}_1, \mathbf{y}_3)$; or $\mathbf{y}'_1 = \mathbf{y}_1$, $\mathbf{y}'_3 = \mathbf{y}_3$, $\mathbf{y}'_4 = \mathbf{y}_4$, $\mathbf{d}'_1 = \mathbf{d}'_3 = \mathbf{d}'_4 = \mathbf{0}$ and $(\sigma, (\mathbf{y}_1, \mathbf{0}), (\mathbf{y}'_2, \mathbf{d}'_2)) \in I(\mathbf{y}_1, \mathbf{y}_2)$, so that in (4.70) we get

$$\sum_{\underline{\omega}\in\{\pm\}} \sum_{\underline{\mathbf{y}}\in\mathbb{H}} \sum_{\substack{\mathbf{d}'_1,\dots,\mathbf{d}'_4:\\ |\mathbf{d}'_1|+\dots+|\mathbf{d}'_4|=1}} (\tilde{\mathcal{R}}_B W^{(h)}_B)_{(4,1)}(\underline{\omega}, \underline{\mathbf{y}}', \underline{\mathbf{d}}')\partial^{\mathbf{d}'_1}\varphi_{\omega_1,\mathbf{y}'_1}\partial^{\mathbf{d}'_2}\varphi_{\omega_2,\mathbf{y}'_2}\partial^{\mathbf{d}'_3}\varphi_{\omega_3,\mathbf{y}'_3}\partial^{\mathbf{d}'_4}\varphi_{\omega_4,\mathbf{y}'_4}\,, \tag{4.73}$$

where

$$(\tilde{\mathcal{R}}_B W^{(h)}_B)_{(4,1)}(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}}') = (\tilde{\mathcal{R}}_B W^{(h)}_B)_{(4,1)}((\omega_1, \mathbf{y}'_1, \mathbf{d}'_1), \dots, (\omega_4, \mathbf{y}'_4, \mathbf{d}'_4)) := \\ = \sum_{\substack{\sigma,\mathbf{y}_1,\dots,\mathbf{y}_4:\\ \sigma,(\mathbf{y}'_1,\mathbf{d}'_1),\dots,(\mathbf{y}'_4,\mathbf{d}'_4)\in I(\mathbf{y}_1,\dots,\mathbf{y}_4)}} \sigma\, W^{(h)}_{B;(4,0)}((\omega_1, \mathbf{y}_1, \mathbf{0}), \dots, (\omega_4, \mathbf{y}_4, \mathbf{0}))\,, \tag{4.74}$$

and more in general we can define

$$(\tilde{\mathcal{R}}_B W^{(h)}_B)_{(n,D+1),0}(\underline{\omega}, \underline{\mathbf{y}}', \underline{\mathbf{d}}') := \sum_{\substack{\sigma,\underline{\mathbf{y}}:\\ \sigma,(\underline{\mathbf{y}}',\underline{\mathbf{d}}')\in I(\underline{\mathbf{y}})}} W^{(h)}_{B;(n,D),0}(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}})\,, \tag{4.75}$$

where $|\underline{\omega}| = |\underline{\mathbf{y}}| = |\underline{\mathbf{y}}'| = n$, $\sum_{i=1}^n |\underline{\mathbf{d}}| = D$ and $\sum_{i=1}^n |\underline{\mathbf{d}}'| = D + 1$.
Finally, we observe that the infinite plane kernel is basically the same as the bulk kernel,

i.e. $W^{(h)}_{B;(n,D),0} = W^{(h)}_{\infty;(n,D),0}$ and $W^{(1)}_{B;(n+n^{(1)},D+D^{(1)}),0} = W^{(1)}_{\infty;(n+n^{(1)},D+D^{(1)}),0}$. The definitions used so far for the half plane kernels are analogues to the ones on the infinite plane, except the field positions are supported on the infinite plane: $\mathbf{x}_i \in \mathbb{Z}^2$ in the set M and $M^{(1)}$ introduced before (see after (4.41) and (4.39), respectively). Also the definitions for the infinite plane operators $\mathcal{L}_\infty$ and $\mathcal{R}_\infty$, acting on the kernels of $W^{(h)}_\infty$, are almost identical, except for the sums over $\mathbb{Z}^2$ in (4.55), in definition of $I(\mathbf{y}_1, \mathbf{y}_2)$ after (4.60) and so on. To use the results already obtained for the relevant and marginal contributions in the infinite plane ([37, Subsec.3.3], [4, Subsec.4.1.1]), if $h \le 1$, $n^{(1)} = 0$ and (n, D) are such that $\mathcal{L}_B W^{(h)}_{B;(n,D),0} \neq 0$ we can rewrite the local part in (4.54) as

$$\mathcal{L}_B W^{(h)}_{B;(n,D),0} = \mathcal{L}_\infty W^{(h)}_{\infty;(n,D),0} + \mathcal{L}_* W^{(h)}_{B;(n,D),0}\,, \tag{4.76}$$

so that, by using (4.54) and (4.76) in (4.52) we get

$$W^{(h)}_{(n,D),0} = \mathcal{L}_\infty W^{(h)}_{\infty;(n,D),0} + \mathcal{R}_B W^{(h)}_{B;(n,D),0} + \bar{W}^{(h)}_{E;(n,D),0}\,, \tag{4.77}$$

with

$$\bar{W}^{(h)}_{E;(n,D),0} := W^{(h)}_{E;(n,D),0} + \mathcal{L}_* W^{(h)}_{B;(n,D),0}\,, \tag{4.78}$$

where we added the term $\mathcal{L}_* W^{(h)}_{B;(n,D),0}$ to the edge kernels, as it is given by the difference between the infinite plane local part and the bulk local part. In (4.77), the infinite plane local part is given by the analogues of (4.65) (as explained above, provided that the sums are over the infinite plane instead over the half plane, we can use the same expressions) and the bulk remainder part is given by (4.67)-(4.68). Finally, we notice that (4.78) also holds for $h = 1$, $n^{(1)} = 0$ and $n = 2$. Moreover, we use the same notation for any $h \le 1$ and any kernel indices, namely

$$\bar{W}^{(h)}_{E;(n,D),0} := \begin{cases} W^{(h)}_{E;(n,D),0} + \mathcal{L}_* W^{(h)}_{B;(n,D),0} & \text{if } h \le 1, n^{(1)} = 0 \text{ and } n = 2\,, \\ W^{(h)}_{E;(n,D),0} & \text{if } h \le 1, n^{(1)} = 0 \text{ and } n > 2\,, \\ W^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0} & \text{if } h = 1 \text{ and } n^{(1)} > 0\,. \end{cases} \tag{4.79}$$

Now we introduce the operators $\mathcal{L}_E$ and $\mathcal{R}_E$ acting on the kernels $\bar{W}^{(h)}_{E;(n,D),0}$.

Localization and interpolation for the edge kernels

For the edge kernels $\bar{W}^{(h)}_{E;(n,D),0}$ in (4.78), we introduce the $\mathcal{L}_E$ and $\mathcal{R}_E$ operators, so that

$$\bar{W}^{(h)}_{E;(n,D),0} = \mathcal{L}_E \bar{W}^{(h)}_{E;(n,D),0} + \mathcal{R}_E \bar{W}^{(h)}_{E;(n,D),0}\,. \tag{4.80}$$

With respect to the bulk contributions, the edge contributions will have a slightly different behavior, due to the cancellations in (3.47): the relevant terms are the only possible source of divergences. Consequently, for any $h \le 1$, if the Grassmann indices (n, D) are such that $d_S \le 0$, we let $\mathcal{L}_E \bar{W}^{(h)}_{E;(n,D),0} := 0$. For $h = 1$, if $n^{(1)} > 0$, we let $\mathcal{L}_E \bar{W}^{(1)}_{E;(n+n^{(1)},D),0} := 0$. Then $\mathcal{L}_E$ acts not trivially only on the edge kernels with $n^{(1)} = 0$ and $(n, D) = (2, 0)$: by proceeding as discussed for the bulk operators, we let

$$\tilde{\mathcal{L}}_E \bar{W}^{(h)}_{E;(2,0),0}((\omega_1, \mathbf{y}_1, \mathbf{0}), (\omega_2, \mathbf{y}_2, \mathbf{0})) := \\ = \prod_{j=1}^{2} \delta_{\mathbf{y}_j,(y_1^{(1)},0)} \sum_{z_1^{(2)},\mathbf{z}_2 \in \mathbb{H}} \bar{W}^{(h)}_{E;(2,0),0}((\omega_1, (y_1^{(1)}, z_1^{(2)}), \mathbf{0}), (\omega_2, \mathbf{z}_2, \mathbf{0}))\,, \tag{4.81}$$

where with respect to (4.55), now we arbitrarily choose to localize the Grassmann field positions at $(y_1^{(1)}, 0)$, where $y_1^{(1)}$ is the first site horizontal coordinate and 0 is the vertical coordinate immediately below the lower edge $\partial\mathbb{H}_1$. Next, by using (4.81) and the operator $\mathcal{O}$ introduced after (4.54), we let

$$\mathcal{L}_E(\bar{W}^{(h)}_{E;(2,0),0}) := \mathcal{O}(\tilde{\mathcal{L}}_E \bar{W}^{(h)}_{E;(2,0),0})\,. \tag{4.82}$$

An important remark is that $\sum_{\underline{f}\in M} \tilde{\mathcal{L}}_E \bar{W}^{(h)}_{E;(2,0),0}(\underline{f})\varphi(\underline{f})$, which is the contribution to the effective potential associated to (4.82) up to the action of $\mathcal{O}$, is given by

$$\sum_{\omega_1,\omega_2\in\{\pm\}} \sum_{\mathbf{y}_1,\mathbf{y}_2\in\mathbb{H}} \bar{W}^{(h)}_{E;(2,0),0}((\omega_1, \mathbf{y}_1), (\omega_2, \mathbf{y}_2))\varphi_{\omega_1,(y_1^{(1)},0)}\varphi_{\omega_2,(y_1^{(1)},0)}\,, \tag{4.83}$$

which is vanishing either because $\omega_1 = \omega_2$, (by using (2.11), $\varphi_{\omega_1,(y_1^{(1)},0)}\varphi_{\omega_1,(y_1^{(1)},0)} = 0$), or because, if $\omega_1 \neq \omega_2$, there has to be a contraction with $\varphi_{+,(y_1^{(1)},0)}$, which gives a vanishing contribution (see the cancellation in (3.47)). Then we can conclude that, for any $h \leq 1$, the local part of the edge kernels vanishes for any value of the kernel indices, namely

$$\mathcal{L}_E(\bar{W}^{(h)}_{E;(n,D),0}) = 0\,. \tag{4.84}$$

Next, by recalling the definition of $\bar{W}^{(h)}_{E;(n,D),0}$ in (4.79), we define the edge remainder parts as

$$\mathcal{R}_E \bar{W}^{(h)}_{E;(n,D),0} := \begin{cases} 0 & \text{if } (n, D) = (2, 0)\,, \\ \mathcal{O}(\bar{W}^{(h)}_{E;(2,1),0} + \tilde{\mathcal{R}}_E \bar{W}^{(h)}_{E;(2,1),0}) & \text{if } (n, D) = (2, 1)\,, \\ W^{(h)}_{E;(n,D),0} & \text{otherwise}\,, \end{cases} \tag{4.85}$$

where $(\tilde{\mathcal{R}}_E \bar{W}^{(h)}_E)_{(2,1)}$ is defined as in (4.63) with $\mathbf{y}_1 = (y_1^{(1)}, 0)$. Moreover, the definitions in (4.85) are valid for any $h \leq 1$, where if $h = 1$ we are considering $n^{(1)} = 0$. If $h = 1$ and $n^{(1)} > 0$, for any values of the massless indices (n, D), we let $\mathcal{R}_E \bar{W}^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0} := \bar{W}^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0}$. In conclusion, if $N = n_\sigma = 0$, the kernel in (4.77) can be rewritten as

$$W_{(n,D),0} = \mathcal{L}_\infty W_{\infty;(n,D),0} + \mathcal{R}_B W_{B;(n,D),0} + \mathcal{R}_E \bar{W}_{E;(n,D),0}\,, \tag{4.86}$$

where we used the edge local part cancellation in (4.84) and the edge remainder part is defined in (4.85).

4.2.2 Localization in presence of spin sources

For $h \leq 0$, if $N = n_\sigma \neq 0$, we do not use the bulk-edge decomposition of the kernels, because we are dealing with spin sources located at the boundary. In presence of the edge spin sources the scaling dimension in (4.46) is non negative for the values in (4.48): here we localize only the kernels with the Grassmann indices in (4.48) and $n_\sigma = 1$. We will not localize the kernels with indices $n = n_\sigma = 2$ and $D = 0$, so that we allow some terms with $d_S = 0$, which will be treated differently in the last chapter. Then, for the kernels with n_σ edge spin sources, for any $h \leq 0$, we introduce the $\mathcal{L}_\sigma$ and $\mathcal{R}_\sigma$ operators, so that

$$W^{(h)}_{(n,D),n_\sigma} = \mathcal{L}_\sigma W^{(h)}_{(n,D),n_\sigma} + \mathcal{R}_\sigma W^{(h)}_{(n,D),n_\sigma} . \tag{4.87}$$

If $n_\sigma \geq 2$, the scaling dimension d_S in (4.46) is always negative except that for the choice of indices $n_\sigma = 2$ and $(n, D) = (2, 0)$: for these irrelevant and marginal terms we let $\mathcal{L}_\sigma W^{(h)}_{(n,D),n_\sigma \geq 2} := 0$, so that $\mathcal{R}_\sigma W^{(h)}_{(n,D),n_\sigma \geq 2} = W^{(h)}_{(n,D),n_\sigma \geq 2}$. Then, the action of $\mathcal{L}_\sigma$ is not trivial only for the values in (4.48) with $n_\sigma = 1$.
If $n_\sigma = 1$, we let $\tilde{\mathcal{L}}_\sigma$ be the operator which localizes the Grassmann field positions at the edge source site:

$$\begin{aligned}\tilde{\mathcal{L}}_\sigma W^{(h)}_{(n,D),1}((\omega_1, \mathbf{y}_1, \mathbf{d}_1), (\omega_2, \mathbf{y}_2, \mathbf{0}), \ldots, (\omega_n, \mathbf{y}_n, \mathbf{d}_n); \mathbf{x}_1) :=& \\ = \prod_{j=1}^{n} \delta_{\mathbf{y}_j,\mathbf{x}_1} \sum_{\mathbf{z}_1,\ldots,\mathbf{z}_n \in \mathbb{H}} W^{(h)}_{(n,D),1}((\omega_1, \mathbf{z}_1, \mathbf{d}_1), (\omega_2, \mathbf{z}_2, \mathbf{d}_2), \ldots, (\omega_n, \mathbf{z}_n, \mathbf{d}_n); \mathbf{x}_1) ,&\end{aligned} \tag{4.88}$$

where the edge source position is $\mathbf{x}_1 \in \partial\mathbb{H}$ (recall that in (4.55) we localized at the position of an arbitrary Grassmann field and in (4.81) we localized at the position of edge Grassmann field).
To localize the marginal terms with $n_\sigma = 1$, if $((n, D), 1) \in \{ ((1, 1), 1), ((3, 0), 1) \}$ (so that $d_S = 0$), we let

$$\mathcal{L}_\sigma(W^{(h)}_{(1,1),1}) := \mathcal{O}(\tilde{\mathcal{L}}_\sigma W^{(h)}_{(1,1),1}) , \qquad \mathcal{L}_\sigma(W^{(h)}_{(3,0),1}) := \mathcal{O}(\tilde{\mathcal{L}}_\sigma W^{(h)}_{(3,0),1}) , \tag{4.89}$$

where we used the operator $\mathcal{O}$ introduced after (4.54) and the fact that $\tilde{\mathcal{L}}_\sigma$ does not change the kernel indices.
An important remark is that $\sum_{\underline{f}\in M} \sum_{\underline{e}\in M_S} \tilde{\mathcal{L}}_\sigma W^{(h)}_{(3,0),1}(\underline{f}; \underline{e})\varphi(\underline{f})F(\underline{e})$ equals

$$\sum_{\omega_1,\omega_2,\omega_3 \in \{\pm\}} \sum_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{x}_1 \in \mathbb{H}} W^{(h)}_{(3,0),1}((\omega_1, \mathbf{y}_1), \ldots, (\omega_3, \mathbf{y}_3); \mathbf{x}_1)\varphi_{\omega_1,\mathbf{x}_1}\varphi_{\omega_2,\mathbf{x}_1}\varphi_{\omega_3,\mathbf{x}_1}\Psi_{\mathbf{x}_1} , \tag{4.90}$$

which vanishes because ω_i can assume only two values (see (2.11)). So, as we did for (4.57), we can consider $\mathcal{L}_\sigma(W^{(h)}_{(3,0),1}) = 0$ in (4.89), a cancellation similar to the one for the kernels associated to 4 Grassmann fields in (4.56).
To localize the relevant terms ($d_S > 1$), let $\tilde{\mathcal{R}}_\sigma$ be the operator such that $\tilde{\mathcal{R}}_\sigma(W^{(h)}_{(n,D),1}) =$

$\tilde{\mathcal{R}}_\sigma W^{(h)}_{(n,D+1),1}$ is equivalent to $W^{(h)}_{(n,D),1} - \tilde{\mathcal{L}}_\sigma W^{(h)}_{(n,D),1}$. With the same procedure described in (4.59)-(4.63), if $(n,D),1 = (1,0),1$, we get

$$\begin{aligned}
&\sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S}\{W^{(h)}_{(1,0),1}(\underline{f};\underline{e}) - \tilde{\mathcal{L}}_\sigma W^{(h)}_{(1,0),1}(\underline{f};\underline{e})\}\varphi(\underline{f})F(\underline{e})\\
&= \sum_{\omega_1\in\{\pm\}}\sum_{\mathbf{y}_1,\mathbf{x}_1\in\mathbb{H}} W_{(1,0),1}((\omega_1,\mathbf{y}_1,\mathbf{0});\mathbf{x}_1)\{\varphi_{\omega_1,\mathbf{y}_1} - \varphi_{\omega_1,\mathbf{x}_1}\}\Psi_{\mathbf{x}_1}\\
&= \sum_{\omega_1\in\{\pm\}}\sum_{\mathbf{y}',\mathbf{x}_1\in\mathbb{H}}\sum_{\mathbf{d}':|\mathbf{d}'|=1}(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(1,1),1}((\omega_1,\mathbf{y}',\mathbf{d}');\mathbf{x}_1)\partial^{\mathbf{d}'}\varphi_{\omega_1,\mathbf{y}'}\Psi_{\mathbf{x}_1}\,,
\end{aligned} \tag{4.91}$$

with

$$(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(1,1),1}((\omega_1,\mathbf{y}',\mathbf{d}');\mathbf{x}_1) := \sum_{\substack{\sigma,\mathbf{y}_1:\\ \sigma,(\mathbf{y}',\mathbf{d}')\in I(\mathbf{x}_1,\mathbf{y}_1)}} \sigma W^{(h)}_{(1,0),1}((\omega_1,\mathbf{y}_1,\mathbf{0});\mathbf{x}_1)\,, \tag{4.92}$$

where $I(\mathbf{x}_1,\mathbf{y}_1)$ is the set of $(\sigma,(\mathbf{y}',\mathbf{d}'))$ such that $(\mathbf{y}',\mathbf{d}')\in\gamma(\mathbf{x}_1,\mathbf{y}_1)$, with the same definitions after (4.60).

Then we let

$$\mathcal{L}_\sigma(W^{(h)}_{(1,0),1}) = \mathcal{O}(\tilde{\mathcal{L}}_\sigma W^{(h)}_{(1,0),1} + \tilde{\mathcal{L}}_\sigma(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(1,1),1})\,, \tag{4.93}$$

and summarizing the results in terms of the source local part indices we get

$$\mathcal{L}_\sigma W^{(h)}_{(n,D),1} = \begin{cases} \mathcal{O}(\tilde{\mathcal{L}}_\sigma W^{(h)}_{(1,0),1}) & \text{if } (n,D),1 = (1,0),1\,,\\ \mathcal{O}(\tilde{\mathcal{L}}_\sigma W^{(h)}_{(1,1),1} + \tilde{\mathcal{L}}_\sigma(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(1,1),1}) & \text{if } (n,D),1 = (1,1),1\,,\\ 0 & \text{otherwise.}\end{cases} \tag{4.94}$$

Finally we define the $\mathcal{R}_\sigma$ operator if $n_\sigma = 1$: if the Grassmann indices are such that $d_S \geq 0$, i.e. $((n,D),n_\sigma) \in \{((1,0),1),((1,1),1),((3,0),1)\}$, we let $\mathcal{R}_\sigma W^{(h)}_{(n,D),1} := 0$; if $((n,D),1) \notin \{((1,2),1),((3,1),1)\}$ are such that $d_S < 0$ we let

$$\mathcal{R}_\sigma W^{(h)}_{(n,D),1} = W^{(h)}_{(n,D),1}\,; \tag{4.95}$$

if $((n,D),1)\in\{((1,2),1),((3,1),1)\}$ (so that $d_S<0$) we let

$$\begin{aligned}
\mathcal{R}_\sigma W^{(h)}_{(1,2),1} &= \mathcal{O}(W^{(h)}_{(1,2),1} + (\tilde{\mathcal{R}}_\sigma W^{(h)})_{(1,2),1} + (\tilde{\mathcal{R}}_\sigma(\tilde{\mathcal{R}}_\sigma W^{(h)}))_{(1,2,1)})\,,\\
\mathcal{R}_\sigma W^{(h)}_{(3,1),1} &= \mathcal{O}(W^{(h)}_{(3,1),1} + (\tilde{\mathcal{R}}_\sigma W^{(h)})_{(3,1),1})\,.
\end{aligned} \tag{4.96}$$

The operator $\tilde{\mathcal{R}}_\sigma$ does not preserve the kernel indices: $\tilde{\mathcal{R}}_\sigma(W^{(h)}_{(n,D),1}) := (\tilde{\mathcal{R}}_\sigma W^{(h)})_{(n,D+1),1}$ and $(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(n,D+1),1}$ is defined with a similar expression to the one in (4.75), namely

$$(\tilde{\mathcal{R}}_\sigma W^{(h)})_{(n,D+1),1}(\underline{\omega},\underline{\mathbf{y}}',\underline{\mathbf{d}}') := \sum_{\substack{\sigma,\underline{\mathbf{y}}:\\ \sigma,(\underline{\mathbf{y}}',\underline{\mathbf{d}}')\in I(\mathbf{x}_1,\underline{\mathbf{y}})}} W^{(h)}_{(n,D),1}(\underline{\omega},\underline{\mathbf{y}},\underline{\mathbf{d}})\,, \tag{4.97}$$

where if $n = 1$, $I(\mathbf{y}_1, \mathbf{x}_1)$ is defines as in (4.92) (or better the analogue of (4.69)), while, if $n = 3$, $I(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{x}_1)$ is the set of $(\sigma, (\mathbf{y}'_1, \mathbf{d}'_1), (\mathbf{y}'_2, \mathbf{d}'_2), (\mathbf{y}'_3, \mathbf{d}'_3))$, with $\sigma \in \{\pm\}$, $\mathbf{y}'_1, \ldots, \mathbf{y}'_3 \in \mathbb{H}$ and $\mathbf{d}'_1, \ldots, \mathbf{d}'_3 \in \{0, 1\}^2$ with $|\mathbf{d}'_1| + \ldots + |\mathbf{d}'_3| = 1$ such that: either $\mathbf{y}'_1 = \mathbf{y}'_2 = \mathbf{x}_1$, $\mathbf{d}'_1 = \mathbf{d}'_2 = \mathbf{0}$ and $(\sigma, (\mathbf{y}'_3, \mathbf{d}'_3)) \in I(\mathbf{x}_1, \mathbf{y}_3)$; or $\mathbf{y}'_1 = \mathbf{x}_1$, $\mathbf{y}'_3 = \mathbf{y}_3$, $\mathbf{d}'_1 = \mathbf{d}'_3 = \mathbf{0}$ and $(\sigma, (\mathbf{y}'_2, \mathbf{d}'_2)) \in I(\mathbf{x}_1, \mathbf{y}_2)$; or $\mathbf{y}'_2 = \mathbf{y}_2$, $\mathbf{y}'_3 = \mathbf{y}_3$, $\mathbf{d}'_2 = \mathbf{d}'_3 = \mathbf{0}$ and $(\sigma, (\mathbf{y}'_1, \mathbf{d}'_1)) \in I(\mathbf{x}_1, \mathbf{y}_1)$.

In conclusion, in (4.45) if $N = n_\sigma = 0$ we get (4.86), if $N = n_\sigma \geq 1$ we get

$$W^{(h)}_{(n,D),n_\sigma \geq 1} = \mathcal{L}W^{(h)}_{(n,D),1} + \mathcal{R}W^{(h)}_{(n,D),1} + W^{(h)}_{(n,D),n_\sigma}\Big|_{n_\sigma > 1}, \tag{4.98}$$

where the local part of the kernel with $n_\sigma = 1$ is given by (4.94), the remainder by (4.95)-(4.96) and the last term is given by the trivial action of the source localization when $n_\sigma \geq 2$, see after (4.87). Finally, by substituting (4.86) and (4.98) in (4.45), for any $h \leq 0$ we get

$$W^{(h)}_{(n,D),n_\sigma} = \mathcal{L}_\infty W^{(h)}_{\infty;(n,D),0} + \mathcal{L}_\sigma W^{(h)}_{(n,D),1} + \mathcal{R}_B W^{(h)}_{B;(n,D),0} + \mathcal{R}_E \bar{W}^{(h)}_{E;(n,D),0} + \mathcal{R}_\sigma W^{(h)}_\sigma, \tag{4.99}$$

where

$$\mathcal{R}_\sigma W^{(h)}_\sigma := \mathcal{R}_\sigma W^{(h)}_{(n,D),1} + W^{(h)}_{(n,D),n_\sigma}\Big|_{n_\sigma > 1}, \tag{4.100}$$

which is vanishing if $h = 0$, as it is the first scale for the spin source fields. For $h = 1$ we get

$$\begin{aligned} W^{(1)}_{(n+n^{(1)},D+D^{(1)}),0} =& \mathcal{L}_\infty W^{(1)}_{\infty;(n,D),0} + \mathcal{R}_B W^{(1)}_{B;(n,D),0} + \mathcal{R}_E \bar{W}^{(1)}_{E;(n,D),0} + \\ &+ W^{(1)}_{B;(n+n^{(1)},D+D^{(1)}),0}\Big|_{n^{(1)} \geq 1} + W^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0}\Big|_{n^{(1)} \geq 1}, \end{aligned} \tag{4.101}$$

where we recall that when $n^{(1)} \geq 1$ we are considering $D^{(1)} \geq 0$.

4.2.3 Norms for the kernels of the effective potential

Depending on the kernel indices we introduce three different types of norms: the *bulk and edge norms* for the kernels of the effective potential with $n_\sigma = 0$ and the *spin source norm* or simply *source norm* for the kernels of the effective potential with $n_\sigma > 0$: note that we are no longer considering any energy sources but only the spin sources so there is no ambiguity about the source fields.

Definition of bulk, edge and source norms

If $n_\sigma = 0$ for the bulk kernels we define the *bulk norm* as

$$||W^{(h)}_{B;(n,D),0}||^B_{(c_B 2^h)} := \sum_{\mathbf{y}_2,\ldots,\mathbf{y}_n \in \mathbb{H}} e^{c_B 2^h \delta(\underline{\mathbf{y}})} \sup_{\underline{\omega}} \sup_{\underline{\mathbf{d}}:|\underline{\mathbf{d}}|=D} |W^{(h)}_{B;(n,D),0}(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}})|, \tag{4.102}$$

where $c_B > 0$, $\delta(\underline{\mathbf{y}}) := \delta(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ is the tree distance between the n Grassmann field positions; for the edge kernels we define the *edge norm* as

$$||W^{(h)}_{E;(n,D),0}||^{E}_{(c_E 2^h)} := \sum_{y_1^{(2)}, \mathbf{y}_2, \ldots, \mathbf{y}_n \in \mathbb{H}} e^{c_E 2^h \delta_E(\underline{\mathbf{y}})} \sup_{\underline{\omega}} \sup_{\underline{\mathbf{d}}: |\underline{\mathbf{d}}| = D} |W^{(h)}_{E;(n,D),0}(\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}})| \,, \quad (4.103)$$

where $c_E > 0$ and with respect to (4.102) now we are fixing the coordinate $y_1^{(1)}$ and we introduce the edge distance $\delta_E(\underline{\mathbf{y}}) := \delta(\underline{\mathbf{y}}) + \delta(\underline{\mathbf{y}}, \partial\mathbb{H})$, where $\delta(\underline{\mathbf{y}}, \partial\mathbb{H})$ is the distance between the $\underline{\mathbf{x}}$ and the edge of the half plane. From now on, we drop the label subscript "B" and "E" from the kernels $W^{(h)}$ when these labels coincide with those appearing on the norms and on the operators.

If $n_\sigma = 1$ for the kernels with one edge spin source we define the *source norm* as

$$||W^{(h)}_{(n,D),n_\sigma}||^{\sigma}_{(c_\sigma 2^h)} := \sum_{\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathbb{H}} e^{c_\sigma 2^h \delta(\underline{\mathbf{y}}, \mathbf{x}_1)} \sup_{\underline{\omega}} \sup_{\underline{\mathbf{d}}: |\underline{\mathbf{d}}| = D} |W^{(h)}_{(n,D),n_\sigma}((\underline{\omega}, \underline{\mathbf{y}}, \underline{\mathbf{d}}); \mathbf{x}_1)| \,, \quad (4.104)$$

where $c_\sigma > 0$ and with respect to (4.102) and (4.103) we are now summing over all the Grassmann field positions because we are fixing the spin source position.

Bounds on the remainder terms

If $n_\sigma = 0$, the bulk norms of the kernels in (4.68) are bounded by

$$\begin{aligned} ||(\mathcal{R}_B W^{(h)})_{(2,2),0}||^B_{(2^h)} &\le ||W^{(h)}_{(2,2),0}||^B_{(2^h)} + 2^{-h} ||W^{(h)}_{(2,1),0}||^B_{(2^{h+1})} + c \cdot 2^{-2h} ||W^{(h)}_{(2,0),0}||^B_{(2^{h+1})} \,, \\ ||(\mathcal{R}_B W^{(h)})_{(4,1),0}||^B_{(2^h)} &\le ||W^{(h)}_{(4,1),0}||^B_{(2^h)} + c' \cdot 2^{-h} ||W^{(h)}_{(4,0),0}||^B_{(2^{h+1})} \,, \end{aligned} \quad (4.105)$$

where c and c' are positive constants and the bulk norm is the one introduced in (4.102). Analogously, the edge norm of the kernel in the second line of (4.85) is bounded by

$$||(\mathcal{R}_E W^{(h)})_{(2,1),0}||^E_{(2^h)} \le ||W^{(h)}_{(2,1),0}||^E_{(2^h)} + c \cdot 2^{-h} ||W^{(h)}_{(2,0),0}||^E_{(2^{h+1})} \,, \quad (4.106)$$

where c is a positive constant and the edge norm is the one defined in (4.103). If $n_\sigma = 1$, the source norms of the kernels in (4.96) are bounded by

$$\begin{aligned} ||(\mathcal{R}_\sigma W^{(h)})_{(1,2),1}||^\sigma_{(2^h)} &\le ||W^{(h)}_{(1,2),1}||^\sigma_{(2^h)} + 2^{-h} ||W^{(h)}_{(1,1),1}||^\sigma_{(2^{h+1})} + 4 \cdot 2^{-2h} ||W^{(h)}_{(1,0),1}||^\sigma_{(2^{h+1})} \,, \\ ||(\mathcal{R}_\sigma W^{(h)})_{(3,1),1}||^\sigma_{(2^h)} &\le ||W^{(h)}_{(3,1),1}||^\sigma_{(2^h)} + 3 \cdot 2^{-h} ||W^{(h)}_{(3,0),1}||^\sigma_{(2^{h+1})} \,, \end{aligned} \quad (4.107)$$

where the source norm is defined as in (4.104). See [4, Eqs.(4.1.41)-(4.1.43)] for the proof of these bounds.

In conclusion, for the purpose of an upper bound we can summarize the results in (4.105),(4.106) and (4.107) as follows: if $h \le h'$,

$$||(\mathcal{R}_\sharp W^{(h')})_{(n,D),n_\sigma}||^\sharp_{c_\sharp(2^h)} \le 2^{-h'(D-D')} ||W^{(h')}_{(n,D'),n_\sigma}||^\sharp_{c_\sharp(2^{h'})} \,, \quad (4.108)$$

where $\sharp = B, E, \sigma$ and $D \ge D'$.

4.3 Trees and tree expansions

Here we derive a convergent expansion of the resulting effective potential on each scale, the so called tree expansion, based on the tree representation of its kernels. The bounds to prove the convergence will be derived in the next chapter, where depending on the kernel indices will be used one of the norms introduced in Subsec. 4.2.3. To explain how to derive the representation of the tree kernels, first of all in Subsec. 4.3.1 we rewrite the resulting effective potential with a structure similar to the starting one in (3.81): there will be new effective parameters, a new effective action, a new effective interaction and a new effective source term. Next, in Subsec. 4.3.2 we derive the graphical representation of each kernel in terms of the trees and we obtain the entire effective potential polynomial as a sum over all the trees compatible with the construction.

4.3.1 The contributions of the effective potential

With the kernel indices introduced in (4.42) and (4.43) we are able to identify the different contributions to the effective potential and, in particular, to rewrite the effective potential with a structure similar to the starting one, taking into account the marginal and relevant contributions as well of the source, bulk and edge contributions. More precisely, we will rewrite, on each scale h, the local part of the effective potential as in (4.134): it is an explicit expression in terms of a finite number of coupling constants, which exhibits the a structure similar to the one on the first scale. Moreover, we will rewrite, on each scale h, the reminder term as in (4.135): it will be not source of any divergences in the perturbative expansion.

The effective potential on scale 1

Going back to the effective potential on scale 1 given in (4.4), if we expand the l.h.s. as (4.39) we get

$$\sum_{\underline{f}\in M^{(1)}} W^{(1)}(\underline{f})\varphi(\underline{f}) = S^{(1)}_{\underline{v}}(\xi,\varphi) + V^{(1)}(\xi,\varphi)\,, \tag{4.109}$$

where we considered $\mathbf{A} = \tilde{\mathbf{A}} = \mathbf{0}$ (recall that on scale 1 $\mathbf{\Psi} = 0$) so that $B^{(1)}_\epsilon = 0$ in (4.4). Now we want to use the kernel indices to identify the different contributions to the effective action and to the effective interaction, which were introduced after (4.4). In the l.h.s. of (4.109) the kernels $W^{(1)}_{(n+n^{(1)},D+D^{(1)}),0}$ with:

- $n + n^{(1)} = 2$, contribute to the effective action $S^{(1)}_{\underline{v}}(\xi,\varphi) := S^{(1)}[\underline{v}_1](\xi,\varphi)$, which is quadratic in the Grassmann fields and is a function of the constants $\underline{v}_1 := \{\,\nu_1, \zeta_1, \eta_1\,\}$ defined in (3.79);
- $n + n^{(1)} > 2$, with $n + n^{(1)} \in 2\mathbb{N}$, contribute to the effective interaction $V^{(1)}(\xi,\varphi)$, which is monomial with degree $n + n^{(1)}$ in the Grassmann fields.

Then, we can decompose the kernels with $n + n^{(1)} = 2$ as in (4.101), obtaining the following decomposition of the effective action in (4.109):

$$S^{(1)}_{\underline{v}}(\xi,\varphi) = S^{(1)}_{\underline{v};B}(\xi,\varphi) + S^{(1)}_{\underline{v};E}(\xi,\varphi) \tag{4.110}$$

where the bulk effective action is defined as

$$S^{(1)}_{\underline{v};B}(\xi,\varphi) := S^{(1)}_{c;B}(\varphi) + S^{(1)}_{i;B}(\xi,\varphi)\,, \tag{4.111}$$

with

$$S^{(1)}_{c;B}(\varphi) := \sum_{\substack{\underline{f}\in M^{(1)}_\infty:\\ n^{(1)}=0}} \mathcal{L}_\infty W^{(1)}_{\infty,(2,D),0}(\underline{f})\varphi(\underline{f})\,, \tag{4.112}$$

and

$$S^{(1)}_{i;B}(\xi,\varphi) := \sum_{\substack{\underline{f}\in M^{(1)}:\\ n^{(1)}=0}} \mathcal{R}_B W^{(1)}_{B;(2,D),0}(\underline{f})\varphi(\underline{f}) + \sum_{\substack{\underline{f}\in M^{(1)}:\\ n^{(1)}>0}} W^{(1)}_{B;(2,D+D^{(1)}),0}(\underline{f})\varphi(\underline{f})\,, \tag{4.113}$$

and the edge effective action is defined as

$$S^{(1)}_{\underline{v};E}(\xi,\varphi) = S^{(1)}_{c;E}(\varphi) + S^{(1)}_{i;E}(\xi,\varphi)\,, \tag{4.114}$$

with

$$S^{(1)}_{c;E}(\varphi) = \sum_{\substack{\underline{f}\in M^{(1)}:\\ n^{(1)}=0}} \mathcal{R}_E(\mathcal{L}_* W^{(h)}_{B;(2,D),0})\varphi(\underline{f})\,, \tag{4.115}$$

and

$$S^{(1)}_{i;E}(\xi,\varphi) = \sum_{\substack{\underline{f}\in M^{(1)}:\\ n^{(1)}=0}} \mathcal{R}_E W^{(1)}_{E;(2,D),0}(\underline{f})\varphi(\underline{f}) + \sum_{\substack{\underline{f}\in M^{(1)}:\\ n^{(1)}>0}} W^{(1)}_{E;(2,D+D^{(1)}),0}(\underline{f})\varphi(\underline{f})\,, \tag{4.116}$$

where to define (4.115) and (4.116) we used (4.79) in the edge remainder term of (4.101).

The only non vanishing local contribution is $S^{(1)}_{c;B}(\varphi)$ in (4.112) and it is given by a sum over the derivative indices of the following terms:

$$\begin{aligned}
&\sum_{\underline{f}\in M_\infty} \mathcal{L}_\infty W^{(1)}_{\infty;(2,0),0}(\underline{f})\varphi(\underline{f}) = 2\nu_1 \sum_{\mathbf{y}\in\mathbb{Z}^2} \varphi^+_{\mathbf{y}}\varphi^-_{\mathbf{y}}\,,\\
&\sum_{\substack{\underline{f}\in M_\infty:\\ |d^{(1)}_1|+|d^{(1)}_2|=1}} \mathcal{L}_\infty W^{(1)}_{\infty;(2,1),0}(\underline{f})\varphi(\underline{f}) = \zeta_1 \sum_{\mathbf{y}\in\mathbb{Z}^2} (\varphi^+_{\mathbf{y}} d_1\varphi^+_{\mathbf{y}} - \varphi^-_{\mathbf{y}} d_1\varphi^-_{\mathbf{y}})\,,\\
&\sum_{\substack{\underline{f}\in M_\infty:\\ |d^{(2)}_1|+|d^{(2)}_2|=1}} \mathcal{L}_\infty W^{(1)}_{\infty;(2,1),0}(\underline{f})\varphi(\underline{f}) = \eta_1 \sum_{\mathbf{y}\in\mathbb{Z}^2} \varphi^+_{\mathbf{y}} d_2\varphi^-_{\mathbf{y}}\,,
\end{aligned} \tag{4.117}$$

where ν_1, ζ_1 and η_1 are the constants in (3.79) and d_j, $j = 1, 2$, is the symmetric discrete derivative, related to the discrete derivative in (4.34) by $d_j \varphi_{\omega,\mathbf{y}} = \frac{1}{2}(\partial_j \varphi_{\omega,\mathbf{y}} + \partial_j \varphi_{\omega,\mathbf{y}-\hat{\mathbf{e}}_j})$. Also $\mathcal{L}_* W^{(h)}_{B;(2,D),0}$ in (4.115) is given by a sum of terms similar to the one in (4.117): the main difference is that, instead of depending on the constants ν_1, ζ_1 and η_1, it depends on $\tilde{\nu}_1(y^{(2)})$, $\tilde{\zeta}_1(y^{(2)})$ and $\tilde{\eta}_1(y^{(2)})$, which are functions of the vertical coordinates of the Grassmann fields (see [4, Eqs.(4.2.7)-(4.2.9)]).

We can decompose also the kernels with $n + n^{(1)} > 2$ as in (4.101), obtaining the decomposition of the effective interaction in (4.109) as

$$V^{(1)}(\xi, \varphi) = V^{(1)}_B(\xi, \varphi) + V^{(1)}_E(\xi, \varphi)\,, \tag{4.118}$$

where the bulk effective interaction is given by

$$V^{(1)}_B(\xi, \varphi) = \sum_{\substack{\underline{f} \in M^{(1)}:\\ n>2}} \mathcal{R}_B W^{(1)}_{B;(n,D),0}(\underline{f})\varphi(\underline{f}) + \sum_{\substack{\underline{f} \in M^{(1)}:\\ n^{(1)} \geq 1}} W^{(1)}_{B;(n+n^{(1)},D+D^{(1)}),0}(\underline{f})\varphi(\underline{f})\,, \tag{4.119}$$

and the edge effective interaction by

$$V^{(1)}_E(\xi, \varphi) = \sum_{\substack{\underline{f} \in M^{(1)}:\\ n>2}} \mathcal{R}_E \bar{W}^{(1)}_{E;(n,D),0}(\underline{f})\varphi(\underline{f}) + \sum_{\substack{\underline{f} \in M^{(1)}:\\ n^{(1)} \geq 1}} W^{(1)}_{E;(n+n^{(1)},D+D^{(1)}),0}(\underline{f})\varphi(\underline{f})\,. \tag{4.120}$$

The effective potential on scale $h \leq 0$

The effective potential on scale $h = 0$, defined in (4.6), expanded as in (4.41), when $\mathbf{A} = \tilde{\mathbf{A}} = \mathbf{0}$ and $\mathbf{\Psi} \neq \mathbf{0}$, can be expressed with a similar structure to the one in (4.4), namely

$$\sum_{\underline{f} \in M} \sum_{\underline{e} \in M_S} W^{(0)}(\underline{f}; \underline{e})\varphi(\underline{f})F(\underline{e}) = S^{(0)}_{\underline{v}}(\varphi) + V^{(0)}(\varphi) + B^{(0)}_\sigma(\varphi; \mathbf{\Psi})\,, \tag{4.121}$$

where, with respect to (4.109), we add the edge spin source contribution $B^{(0)}_\sigma(\varphi; \mathbf{\Psi})$. Analogously to the procedure illustrated above, in the l.h.s. of (4.121) the kernels $W^{(0)}_{(n,D),n_\sigma}$ with:

- $n = 2$ and $n_\sigma = 0$, contribute to $S^{(0)}_{\underline{v}}(\varphi) := S^{(0)}[\underline{v}_0](\varphi)$, which is the effective action on scale 0, it is quadratic in the massless fields and it is a function of the constants $\underline{v}_0 := \{\, \nu_0, \zeta_0, \eta_0 \,\}$ defined below;
- with $n > 2$ and $n_\sigma = 0$, contribute to $V^{(0)}(\varphi; \mathbf{\Psi})$, which is the effective interaction on scale 0;
- with $n = n_\sigma = 1$, contribute to $B^{(0)}_\sigma(\varphi; \mathbf{\Psi}) := B^{(0)}[\tilde{Z}_0](\varphi; \mathbf{\Psi})$, which is the source term defined in (3.76), where we add the superscript (0) for convenience and we think of it as function of $\tilde{Z}_0$, which is a constant that will be introduced below.

Then, we can use the decomposition in (4.99), obtaining the expressions of local part and remainder term of the effective potential on scale 0. By the kernels with $n = 2$ and $n_\sigma = 0$ in (4.99) we get the expression of the effective action in (4.121) similar the one in (4.110), and in particular its local part has the same form as the one in (4.112), namely

$$S^{(0)}_{c;B}(\varphi) = \sum_{\substack{\underline{f}\in M_\infty:\\ D\le 1}} \mathcal{L}_\infty W^{(0)}_{\infty;(2,D),0}(\underline{f})\varphi(\underline{f}) \,, \tag{4.122}$$

which is given by a sum of terms similar to the ones in (4.117) provided that $2\nu_1, \zeta_1$ and η_1 are replaced by ν_0, ζ_0 and η_0 respectively. By the kernels with $n_\sigma \ge 1$ in (4.99) (and by recalling that the remainder term in (4.100) vanishes if $h = 0$), we get that the only non vanishing contribution with $n_\sigma \ne 0$ is given by

$$B^{(0)}_\sigma(\varphi;\mathbf{\Psi}) = \sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(0)}_{(1,0),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) \,, \tag{4.123}$$

which is exactly the spin source in (3.76), as we introduced it as already 'local': the constant $\tilde{Z}_0$ is then equal to 1. Then we let

$$\mathcal{L}\mathcal{V}^{(0)}(\varphi) := S^{(0)}_{c;B}(\varphi) + B^{(0)}_\sigma(\varphi;\mathbf{\Psi}) \,, \tag{4.124}$$

be the local part of the effective potential on scale 0, and

$$\mathcal{R}\mathcal{V}^{(0)}(\varphi) := S^{(0)}_{c;E}(\varphi) + V^{(0)}_{B;i}(\varphi) + V^{(0)}_{E;i}(\varphi) \,, \tag{4.125}$$

be the remainder term. In (4.125), $S^{(0)}_{c;E}(\varphi)$ is defined as in (4.115), and

$$V^{(0)}_{B;i}(\varphi) := \sum_{\underline{f}\in M} \mathcal{R}_B W^{(0)}_{B;(n,D),0}(\underline{f})\varphi(\underline{f}) \,, \quad V^{(0)}_{E;i}(\varphi) := \sum_{\underline{f}\in M} \mathcal{R}_E \bar{W}^{(0)}_{E;(n,D),0}(\underline{f})\varphi(\underline{f}) \,, \tag{4.126}$$

where now we do not distinguish the irrelevant contributions to the effective action or to the effective interaction, i.e. we sum the contribution that on scale 1 was in (4.113) with the one in (4.119) (now $n^{(1)} = 0$) and, analogously, the contribution in (4.116) with the one in (4.120). With this procedure, on a generic scale h, we get

$$\sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S} W^{(h)}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) = S^{(h)}_{\underline{v}}(\varphi) + B^{(h)}_\sigma(\varphi;\mathbf{\Psi}) + V^{(h)}(\varphi;\mathbf{\Psi}) \,, \tag{4.127}$$

where the kernels with $n_\sigma = 0$ contribute to the effective action and the effective interaction as the ones on scale 0, while the kernels on scale h with $n_\sigma \ne 0$ contribute differently with respect the ones on scale 0. The local term in the massless fields ($n_\sigma = 0$) is the same as in (4.122), namely

$$S^{(h)}_{c;B}(\varphi) = \sum_{\substack{\underline{f}\in M_\infty:\\ D\le 1}} \mathcal{L}_\infty W^{(h)}_{\infty;(2,D),0}(\underline{f})\varphi(\underline{f}) \,, \tag{4.128}$$

where $S^{(h)}_{c;B}(\varphi) := S^{(h)}_{c;B}[\underline{v}_h](\varphi)$, which is a function some constants $\underline{v}_h = \{\,\nu_h, \zeta_h, \eta_h\,\}$ such that (4.128) is given by the sum of the following relevant and marginal contributions, which are a similar for the ones in (4.117):

$$\begin{aligned}
&\sum_{\underline{f}\in M_\infty} \mathcal{L}_\infty W^{(h)}_{\infty;(2,0),0}(\underline{f})\varphi(\underline{f}) = 2^h\nu_h \sum_{\mathbf{y}\in\mathbb{Z}^2} \varphi^+_{\mathbf{y}}\varphi^-_{\mathbf{y}}\,,\\
&\sum_{\substack{\underline{f}\in M_\infty:\\ |d_1^{(1)}|+|d_2^{(1)}|=1}} \mathcal{L}_\infty W^{(h)}_{\infty;(2,1),0}(\underline{f})\varphi(\underline{f}) = \zeta_h \sum_{\mathbf{y}\in\mathbb{Z}^2} (\varphi^+_{\mathbf{y}} d_1\varphi^+_{\mathbf{y}} - \varphi^-_{\mathbf{y}} d_1\varphi^-_{\mathbf{y}})\,,\\
&\sum_{\substack{\underline{f}\in M_\infty:\\ |d_1^{(2)}|+|d_2^{(2)}|=1}} \mathcal{L}_\infty W^{(h)}_{\infty;(2,1),0}(\underline{f})\varphi(\underline{f}) = \eta_h \sum_{\mathbf{y}\in\mathbb{Z}^2} \varphi^+_{\mathbf{y}} d_2\varphi^-_{\mathbf{y}}\,.
\end{aligned} \tag{4.129}$$

The local part with $n_\sigma = 1$ is given by

$$B^{(h)}_\sigma(\varphi;\mathbf{\Psi}) := \sum_{\substack{\underline{f}\in M:\\ D\le 1}} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,D),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e})\,, \tag{4.130}$$

where $B^{(h)}_\sigma(\varphi;\mathbf{\Psi}) := B^{(h)}[\underline{s}_h](\varphi;\mathbf{\Psi})$, which is a function of the spin running coupling constants $\underline{s}_h := \{\tilde{Z}_h, \tilde{Z}_{1;h}, \tilde{Z}_{2;h}\}$ such that (4.130) is given by the sum of the following relevant and marginal contributions:

- if $D = 0$ in (4.130) we get

$$\sum_{\underline{f}\in M} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,0),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) = \tilde{Z}_h \sum_{\mathbf{x}\in\partial\mathbb{H}} \Psi_{\mathbf{x}}\varphi_{-,\mathbf{x}}\,, \tag{4.131}$$

  with $\mathcal{L}_\sigma W^{(h)}_{(1,0),1}(\underline{f};\underline{e})$ given in the first line of (4.94);

- if $D = 1$ in (4.130) we get

$$\sum_{\substack{\underline{f}\in M:\\ |d_1^{(1)}|+|d_2^{(1)}|=1}} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,1),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) = \tilde{Z}_{1;h} \sum_{\mathbf{x}\in\partial\mathbb{H}} \Psi_{\mathbf{x}} d_1\varphi_{-,\mathbf{x}}\,, \tag{4.132}$$

  and

$$\sum_{\substack{\underline{f}\in M:\\ |d_1^{(2)}|+|d_2^{(2)}|=1}} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,1),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) = \tilde{Z}_{2;h} \sum_{\mathbf{x}\in\partial\mathbb{H}} \Psi_{\mathbf{x}} d_2\varphi_{-,\mathbf{x}}\,, \tag{4.133}$$

  with $\mathcal{L}_\sigma W^{(h)}_{(1,1),1}(\underline{f};\underline{e})$ given in the second line of (4.94).

Note that the expression in (4.131) is the same as the one with $h = 0$ given in (4.123). Moreover, the spin running coupling constants $\underline{s}_h$, here introduced for the first time, will be studied in Subsec.5.3.

Summarizing, the local part of the effective potential on scale h is given by

$$\mathcal{L}\mathcal{V}^{(h)}(\varphi;\mathbf{\Psi}) = S^{(h)}_{B;c}(\varphi) + B^{(h)}_{\sigma}(\varphi;\mathbf{\Psi})\,, \tag{4.134}$$

and the remainder term is now given by

$$\mathcal{R}\mathcal{V}^{(h)}(\varphi;\mathbf{\Psi}) := S^{(h)}_{E;c}(\varphi) + V^{(h)}_{B;i}(\varphi) + V^{(h)}_{E;i}(\varphi) + V^{(h)}_{\sigma}(\varphi;\mathbf{\Psi})\,, \tag{4.135}$$

with $S^{(h)}_{E;c}$, $V^{(h)}_{B;i}$ and $V^{(h)}_{E;i}$ which are the same remainder terms in (4.125), namely

$$\begin{aligned} S^{(h)}_{E;c}(\varphi) &= \sum_{\substack{\underline{f}\in M:\\ n^{(1)}=0}} \mathcal{R}_E(\mathcal{L}_* W^{(h)}_{B;(2,D),0})(\underline{f})\varphi(\underline{f})\,,\\ V^{(h)}_{B;i}(\varphi) &= \sum_{\underline{f}\in M} \mathcal{R}_B W^{(h)}_{B;(n,D),0}(\underline{f})\varphi(\underline{f})\,,\\ V^{(h)}_{E;i}(\varphi) &= \sum_{\underline{f}\in M} \mathcal{R}_E \bar{W}^{(h)}_{E;(n,D),0}(\underline{f})\varphi(\underline{f})\,, \end{aligned} \tag{4.136}$$

and $V^{(h)}_{\sigma}$ which is the remainder depending on the spin source fields, given by

$$V^{(h)}_{\sigma}(\varphi;\mathbf{\Psi}) := \sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S} \mathcal{R}_\sigma W^{(h)}_{(1,D),1}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e}) + \sum_{\underline{f}\in M}\sum_{\substack{\underline{e}\in M_E:\\ n_\sigma>1}} W^{(h)}_{(n,D),n_\sigma}(\underline{f};\underline{e})\varphi(\underline{f})F(\underline{e})\,, \tag{4.137}$$

which are the non local terms with $n_\sigma \geq 1$ in (4.99). Note that, as mentioned after (4.117), $\mathcal{L}_* W^{(h)}_{B;(2,D),0}$ in the first line of (4.137) is given by a sum of terms as (4.129), with functions $\tilde{\nu}_h(y^{(2)})$, $\tilde{\zeta}_h(y^{(2)})$ and $\tilde{\eta}_h(y^{(2)})$, which depend on the vertical coordinate of the Grassmann fields.

4.3.2 Tree expansion of the effective potential

If we consider the recursive definition of $W^{(0)}$ as given in (4.29), we can graphical represent the contributions associated to each $W^{(1)}$ in the r.h.s. of (4.29) as the tree graphs in Fig. (4.1). All such tree graphs are rooted in v_0, which is the leftmost vertex on scale is 1. The degree of v_0 corresponds to s, which is the number of branches extending from v_0. The rightmost vertices, whose scale is 2, are the endpoints: each one corresponds to a kernel $W^{(1)}(\underline{f}_j)$, $j = 1,\ldots,s$. Depending on the kernel indices, as we illustrated in (4.109) and the following formula, we identify each contribution and represent it as symbol on scale 2:

$$\begin{aligned} &\blacklozenge \quad \text{if it is associated to } \mathcal{K}(S^{(1)}_{\underline{v};B})\,, \\ &\lozenge \quad \text{if it is associated to } \mathcal{K}(S^{(1)}_{\underline{v};E})\,, \\ &\bullet \quad \text{if it is associated to } \mathcal{K}(V^{(1)}_{B})\,, \\ &\circ \quad \text{if it is associated to } \mathcal{K}(V^{(1)}_{E})\,, \end{aligned} \tag{4.138}$$

where $K(S^{(1)}_{\underline{v};B})$ denotes the kernel of $S^{(1)}_{\underline{v};B}$ in (4.111), $K(S^{(1)}_{\underline{v};E})$ denotes the kernel of $S^{(1)}_{\underline{v};E}$ in (4.114), $K(V^{(1)}_{B})$ denotes the kernel of $V^{(1)}_{B}$ in (4.119) and $K(V^{(1)}_{E})$ denotes the kernel of $V^{(1)}_{E}$ in (4.120). Note that, in (4.138), we used the same symbol with a different color, black or white, to identify the bulk and the edge contributions. The presence of a dot

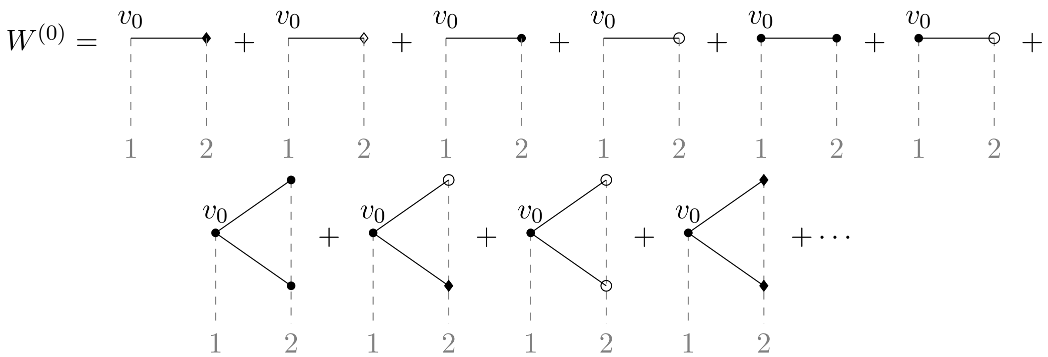

Figure 4.1: The graphical representation of some terms appearing in Eq. (4.29).

on v_0 suggests that $\underline{f}^c_1, \ldots, \underline{f}^c_s \neq \emptyset$, i.e. that there are contracted Grassmann fields on scale 1. Recall that on scale 1 all and only the massive fields are contracted, so that if $s = 1$ and there is only one endpoint on scale 2, v_0 may or may not be dotted: it is dotted if the endpoint is associated to a kernel with $n^{(1)} \geq 2$, otherwise it is not dotted. In particular, since $\blacklozenge$ and $\lozenge$ are associated with the local part with $n^{(1)} = 0$, respectively, in (4.112) and in (4.115), v_0 is not dotted. If $s > 1$ and there is more than one endpoint on scale 2, v_0 is always dotted.

In order to iterate the scheme, we decompose the kernels of $\mathcal{V}^{(0)}$ as in (4.121) and then as a local part plus a remainder obtaining (4.124) and (4.125). Next, to graphical represent the different contributions on scale 0 we introduce the following symbols on scale 1:

$$\begin{aligned} &\blacklozenge \quad \text{if it is associated to } \mathcal{K}(S^{(0)}_{B;c})\,, \\ &\lozenge \quad \text{if it is associated to } \mathcal{K}(S^{(0)}_{E;c})\,, \\ &\blacktriangle \quad \text{if it is associated to } \mathcal{K}(B^{(0)}_{\sigma})\,, \\ &\bullet \quad \text{if it is associated to } \mathcal{K}(V^{(0)}_{B;i} + V^{(0)}_{E;i})\,, \end{aligned} \tag{4.139}$$

where $\mathcal{K}(S^{(0)}_{B;c})$ is the kernel of $S^{(0)}_{B;c}$ in (4.122), $\mathcal{K}(S^{(0)}_{E;c})$ is the kernel of $S^{(0)}_{E;c}$ in (4.125), $\mathcal{K}(B^{(0)}_{\sigma})$ is the kernel of $B^{(0)}_{\sigma}$ in (4.123) and $\mathcal{K}(V^{(0)}_{B;i} + V^{(0)}_{E;i})$ is the kernel of the sum of $V^{(0)}_{B;i}$ and $V^{(0)}_{E;i}$ given in (4.126). Note that in (4.139) we used the black color for the bulk contributions, the white color for the edge contribution and the gray color for the kernels $\mathcal{K}(B^{(0)}_{\sigma})$ and $\mathcal{K}(V^{(0)}_{B;i} + V^{(0)}_{E;i})$, which are contributions irrespective of the bulk-edge decomposition. Moreover, ♦ , ◊ and ▲ in (4.139) are the endpoints on scale 1, while ● can be expanded as shown in Fig. 4.2, where the kernel $\mathcal{K}(V^{(0)}_{B;i} + V^{(0)}_{E;i})$ is expanded as described in Fig. 4.1.

Now we can graphically represent $W^{(-1)}$, the kernel of (4.19), expanded in terms

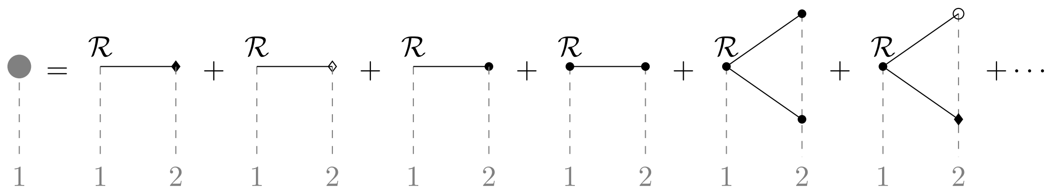

Figure 4.2: The graphical representation of the operator $\mathcal{R}W^{(0)}$ defined in Eq. (4.86), with the operator $\mathcal{R}$ acting on $W^{(0)}$ as in Fig. 4.1. Note that we are representing $\mathcal{R}W^{(0)}$ as a total contribution, i.e. without the bulk-edge decomposition.

of $W^{(0)}$, as in (4.28): by using the conventions of (4.139), we get the representation illustrated in Fig.4.3.

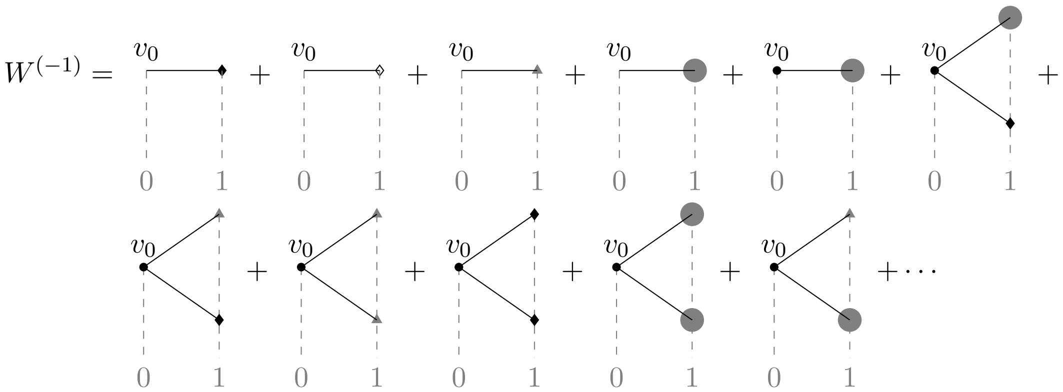

Figure 4.3: The graphical representation of $W^{(-1)}$ in terms of $W^{(0)}$, i.e. in terms of the contributions in (4.139). Note that now on scale 0 can be contracted also the Grassmann field in the spin source term.

Next, by using Fig. 4.2, we expand the big dots ● in Fig. 4.3 and we get the graphical representation of $W^{(-1)}$ in terms of $W^{(1)}$ in Fig. 4.4.

To iterate the scheme, we decompose the kernels of $\mathcal{V}^{(-1)}$ as in (4.127) and the into the local parts plus a remainder obtainig (4.134) and (4.135) (with $h = -1$) and we represent each contribution on scale 0 as

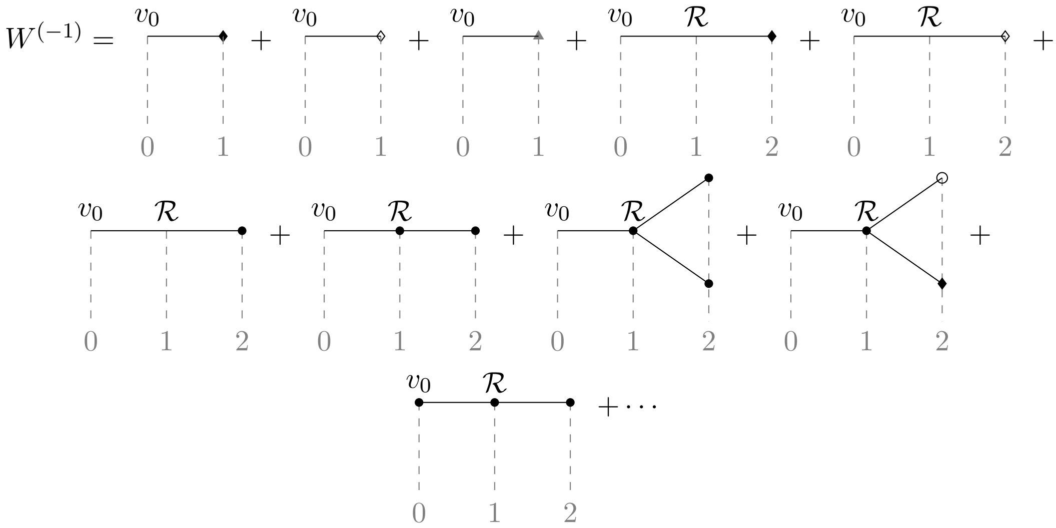

Figure 4.4: The graphical representation of $W^{(-1)}$ in terms of $W^{(1)}$, i.e. in terms of the endpoints in (4.138) and in (4.139), obtained by using Fig.4.2 in Fig.4.3.

$$
\begin{aligned}
&\blacklozenge \quad \text{if it is associated to } \mathcal{K}(S^{(-1)}_{B;c})\,,\\
&\lozenge \quad \text{if it is associated to } \mathcal{K}(S^{(-1)}_{E;c})\,,\\
&\blacktriangle \quad \text{if it is associated to } \mathcal{K}(B^{(-1)}_{\sigma})\,,\\
&\CIRCLE \quad \text{if it is associated to } \mathcal{K}(V^{(-1)}_{B;i}+V^{(-1)}_{E;i}+V^{(-1)}_{\sigma})\,,
\end{aligned}
\tag{4.140}
$$

where $\mathcal{K}(S^{(-1)}_{B;c})$ in the kernel of $S^{(-1)}_{B;c}$ in (4.128), $\mathcal{K}(S^{(-1)}_{E;c})$ in the kernel of $S^{(-1)}_{E;c}$ in (4.135), $\mathcal{K}(B^{(-1)}_{\sigma})$ is the kernel of $B^{(-1)}_{\sigma}$ in (4.130) and $\mathcal{K}(V^{(-1)}_{B;i}+V^{(-1)}_{E;i}+V^{(-1)}_{\sigma})$ is the kernel of $V^{(-1)}_{B;i}+V^{(-1)}_{E;i}+V^{(-1)}_{\sigma}$ in (4.135). The $\CIRCLE$ can be expanded until the endpoints are reached: on scale 1 the endpoints are $\blacklozenge$, $\lozenge$ and $\blacktriangle$ (the endpoints in (4.139)) and on scale 2 the endpoints are $\blacklozenge$, $\lozenge$, $\bullet$ and $\circ$ (the ones in (4.138)). With this procedure on a generic scale $h \le -1$, by iterating the graphical equations analogues to the one in Fig. 4.3 and by expanding the $\CIRCLE$ vertices until the endpoints are reached, we find that $W^{(h)}$ can be graphically expanded in terms of trees of the kind depicted in Fig. 4.5.

For each tree with root on scale $h+1$, as the example in Fig. 4.5, we have endpoints on scale 2 of type $\blacklozenge$, $\lozenge$, $\bullet$ and $\circ$ and endpoints on scale $h \le 1$ of type $\blacklozenge$, $\lozenge$ and $\blacktriangle$. Note that now the gray color is associated only to the presence of the edge spin sources, which effectively are not decomposed in bulk and edge part. In order to not overwhelm the figures, we prefer not to indicate explicitly the labels $\mathcal{R}$ at the intersections of the branches with the dashed vertical lines.

For $h \le 0$, let $\mathcal{T}^{(h)}_{n,n_\sigma}$ be the set of these trees, which are called 'GN trees' [32–34], where the superscript h indicates that the root v_0 is on scale $h+1$, and the subscripts

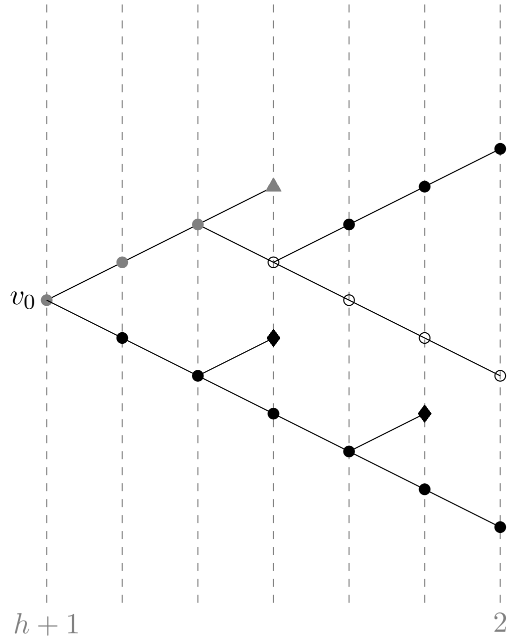

Figure 4.5: Example of a tree $\tau \in \mathcal{T}_{5,2}^{(h)}$ with root on scale $h+1$ that appears in the tree expansion of the kernels of the effective potential on scale h.

n and n_σ indicate the number of endpoints of type (4.138) and of type spin source respectively. A couple of important features of the GN trees:

- the root v_0 is on scale $h+1$: it is the unique leftmost vertex of the tree, its degree is $s \geq 1$, i.e. v_0 cannot be an endpoint, and, as we already mentioned, it may or may not be dotted;
- n is the number of endpoints of Grassmann fields, i.e. the endpoints ♦ , ◊ , ● , ○: the endpoints ● and ○ are necessarily on scale $h = 2$, the endpoints ♦ and ◊ can be on all scales $h \leq 2$;
- n_σ is the number of endpoints of spin sources, i.e. the endpoints ▲ , which can be on all scales $h \leq 0$;
- vertices, other than the root, with exactly one immediate successor, are called 'trivial vertices', otherwise they are called 'branching vertices';
- when either an endpoint ♦ or ◊ is on a scale $h < 2$, or an endpoint ▲ is on scale $h < 0$, it must be connected to a branching vertex on scale $h - 1$.

Tree expansion of the effective potential

In terms of these trees, we will write the expansion for $W^{(h)}$, thought of as a function of $\underline{v} := \{(\nu_h, \zeta_h, \eta_h)\}_{h\leq 1}$ and $\underline{\tilde{Z}} := \{(\tilde{Z}_h, \tilde{Z}_{1;h}, \tilde{Z}_{2;h})\}_{h\leq 0}$, as

$$W^{(h)}[\underline{v}, \underline{\tilde{Z}}] = \sum_{\tau\in\mathcal{T}^{(h)}_{n,n_\sigma}} W[\tau]\,, \tag{4.141}$$

where $W[\tau] := W[\underline{v}, \underline{\tilde{Z}}; \tau]$ is the 'tree kernel', for which we will soon give an explicit inductive definition. For this purpose, we first need to specify some additional notations and conventions about the GN trees:

- let $\mathcal{T}_{n,n_\sigma} := \cup_{h\leq 0}\mathcal{T}^{(h)}_{n,n_\sigma}$ and $\tau \in \mathcal{T}_{n,n_\sigma}$ denotes an element;
- let $V_0(\tau)$ the set of the dotted vertices and $V_e(\tau)$ be the set of the endpoints of τ, so that $V(\tau) := V_e(\tau) \cup V_0(\tau) \cup \{ v_0 \}$ is the set of all the vertices of the tree τ;
- given $v \in V(\tau)$ we let h_v be its scale;
- $w \geq v$ or 'w is a successor of v' means that the (unique) path in the tree from w to v_0 passes through v;
- $w \triangleright v$ or 'w is an immediate successor of v' means that $w > v$ and w and v are directly connected;
- let $S_v := \{ w \in V(\tau) : w \triangleright v \}$ be the set of the vertices w that are immediate successors of v and $|S_v| := s_v$ denotes its cardinality;
- for any $v > v_0$, u denotes the unique vertex such that $v \triangleright u$;
- for each $v \in V_0(\tau)$, let $\tau_v \in \mathcal{T}^{(h_v-1)}_{n',n'_\sigma}$ be the subtree consisting of root v and $w \geq v$ vertices ($n' \leq n, n'_\sigma \leq n_s$).

Next, we need to attach labels to the tree vertices, in order to distinguish the various contributions in the kernels, because we can chose different sets $\underline{f}_j, \underline{e}_j$, in (4.28) and in (4.29). In particular, with each $v \in V(\tau)$ is associated the number n_v of Grassmann fields and the number n^σ_v of spin sources. Let P_v be the set of Grassmann field labels: it consists of the field positions and the ω indices of the Grassmann fields that are external to the v vertex, $i = 1, \dots, n_v$. Moreover, for each $v \in V^*(\tau)$ we let $I^*_v =\in \{ 0, 1 \}$ be an index such that if $I^*_v = 0$, v is black and it is associated to a bulk contribution, if $I^*_v = 1$, v is white and it is associated to an edge contribution. The family $\underline{P} := \{ P_v \}_{v\in V(\tau)}$ is characterized by the following properties:

- if we consider an endpoint, $v_k \in V_e(\tau)$, then $P_{v_k} := \{ (p_{v_k})_1, \dots, (p_{v_k})_{n_{v_k}} \}$, $(p_{v_k})_i = (\omega_i, (k, i))$, with (k, i) which is the position of the Grassmann field (i is the field position and k is the position of v within an ordered list which they belong to), and with $\omega_i \in \{ \pm i, \pm \}$ if $h_{v_k} = 2$, $\omega_i \in \{ \pm \}$ if $h_{v_k} < 2$, $i = 1, \dots, n_{v_k}$;

- if we consider a vertex which is not an endpoint, $v \in V(\tau) \setminus V_e(\tau)$, we let $P_v \subset \cup_{w \in S_v} P_w$.

Let $\mathcal{P}(\tau)$ be the set of allowed $\underline{P}$. We can introduce the set of the *spanning trees*, formed by the contracted Grassmann fields. First of all, if $v \in V(\tau) \setminus V_e(\tau)$ we define $P_v^c := \cup_{w \in S_v} P_w \setminus P_v$ as the set of the contracted Grassmann fields, such that $P_v^c = \emptyset$ only if $v = v_0$ and v_0 is not dotted. If $h_v = 1$, $P_v^c = \cup_{w \in S_v} \{ (p_w)_i = (\omega_i, (j,i)) \in P_w : \omega_i \in \{ \pm i \} \}$, i.e. all and only the massive fields are contracted on scale 1. Finally, given v and w, two different vertices in $V(\tau)$ such that $P_v \cap P_w = \emptyset$, we note that $P_v^c \cap P_w^c = \emptyset$. Next, given $\underline{P} \in \mathcal{P}(\tau)$, for all $v \in V(\tau) \setminus V_e(\tau)$, we let $s := |S_v|$ and we define the set T_v

$$T_v = \{ (p_1, p_2), \dots, (p_{2s-3}, p_{2s-2}) \} \subset (P_v^c)^2 \,, \tag{4.142}$$

which is called the spanning tree associated with v. If $p \in P_v^c$, and $w(p)$ denotes the unique $w \in S_v$ such that $p \in P_w$, if $(p, p') \in T_v$, $w(p)$ and $w(p')$ are two distinct vertices whose positions are 'ordered', i.e. $(i,j) < (i',j')$ $((i,j) \in p, (i',j') \in p')$. The set $\{ (w(p_1), w(p_2)), \dots, (w(p_{2s-3}), w(p_{2s})) \}$ is the edge set of a tree whose vertex set is S_v. Finally, we let $\mathcal{S}(\tau; \underline{P})$ be the set of allowed $\underline{T} = \{ T_v \}_{v \in V(\tau) \setminus V_e(\tau)}$.

Each $v \in V(\tau)$ can be associated to a bulk or an edge contribution, i.e. it can be black or white, when it is without edge spin source fields. Then, we let $V^*(\tau) := \{ v \in V(\tau) : n_v^\sigma = 0 \}$, which is the set of vertices with only Grassmann fields, and $\bar{V}(\tau) := V(\tau) \setminus V^*(\tau)$. Next, we introduce the set of the black vertices, which are associated to the bulk contributions, $V_B^*(\tau) := \{ v \in V^*(\tau) : I_v^* = 0 \}$, and the set of the white vertices, which are associated to the edge contributions, $V_E^*(\tau) := \{ v \in V^*(\tau) : I_v^* = 1 \}$, so that $V^*(\tau) = V_B^*(\tau) \cup V_E^*(\tau)$. Consequently, we can consider $\bar{V}(\tau)$ as the set of the gray vertices.

Finally we give the notation such that the tree kernels can be expressed with the kernel indices introduced in (4.42) and (4.43). For each $v \in V(\tau)$, $\mathbf{d}_v$ denotes the map $\mathbf{d}_v : P_v \to \{ 0, 1, 2 \}^2$. If $(p_v)_i \in P_v$ we let $(\mathbf{d}_v)_i := \mathbf{d}_v((p_v)_i)$, and the reader should think that a derivative operator $\partial^{(\mathbf{d}_v)_i}$ acts on the Grassmann field labelled by $(p_v)_i$. We define the families of maps $\underline{D} := \{ \mathbf{d}_v \}_{v \in V(\tau)}$ and let $\underline{\mathbf{d}}_v$ be the tuple of components $(\mathbf{d}_v)_i$, $i = 1, \dots, n_v$ and $D_v := ||\underline{\mathbf{d}}_v||_1$. Given any subset of Grassmann indices $P_v' \subset P_v$, we define $\underline{\mathbf{d}}_v|_{P_v'}$ as the tuple whose non vanishing components are given by $(\mathbf{d}_v)_i \in \underline{\mathbf{d}}_v$ if $(p_v)_i \in P_v'$, and zero otherwise. We let $\mathbf{d}_{v,P_v}$ be the map such that $\underline{\mathbf{d}}_{v,P_v} := \bigoplus_{w \in S_v} \underline{\mathbf{d}}_w|_{P_v}$ and

$$D_{v,P_v} := ||\underline{\mathbf{d}}_{v,P_v}||_1 = \sum_{w \in S_v} || \, \underline{\mathbf{d}}_w|_{P_v} \, ||_1 \,, \tag{4.143}$$

so that D_v and D_{v,P_v} will play the role of the derivative indices for the tree kernels. We let $\mathcal{D}(\underline{P}, \tau)$ be the set of 'allowed' $\underline{D}$, i.e. such that $W[\tau; \underline{P}, \underline{T}, \underline{D}] \neq 0$. If $\underline{D}$ is allowed, then it must satisfy a number of constraints. In particular, if $v \in V_0(\tau)$, $w \in V_e(\tau)$ and $p \in P_v \cap P_w$, then $D_v(p) \geq D_w(p)$. Moreover, letting for any $v \in V_0(\tau)$, $R_v := D_v - D_{v,P_v}$

(see (4.143)), one has

$$R_v = \begin{cases} 2\,, & n_v + n_v^\sigma = 2, I_v = 0 \text{ and } D_{v,P_v} = 0\,, \\ 1\,, & n_v + n_v^\sigma = 2, I_v = 0 \text{ and } D_{v,P_v} = 1\,, \\ 1\,, & n_v + n_v^\sigma = 4, I_v = 0 \text{ and } D_{v,P_v} = 0\,, \\ 1\,, & n_v = 2, n_v^\sigma = 0, I_v = 1 \text{ and } D_{v,P_v} = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.144}$$

where, in order to derive a more compact notation, we assign a label I_v also to the vertices $v \in \bar{V}(\tau)$ (also if there is not a bulk-edge decomposition for such vertices) and then we let $I_v = 0$ for any $v \in V(\tau)$ such that $n_v^\sigma = 0$. Note that, for the root v_0, as well $D_{v_0,P_{v_0}} = D_{v_0}$, $R_{v_0} = 0$.
Finally, we assign a map $\mathbf{y} : P_v \to \mathbb{H}$, $\underline{\mathbf{y}}_v$ denotes the tuple of components $\mathbf{y}((p_v)_i)$ and $\underline{\omega}_v$ the tuple of components $\omega((p_v)_i)$, $(p_v)_i \in P_v$, $i = 1, \dots, n_v$ and we define $\underline{f}_v := (\underline{\omega}_v, \underline{\mathbf{y}}_v, \underline{\mathbf{d}}_v)$. If $v \in \bar{V}(\tau)$ we also define $\underline{e}_v := (\underline{\mathbf{x}}_v, \underline{j}_v)$, where $\underline{\mathbf{x}}_v$ and $\underline{j}_v$ are the tuple of the edge spin source variables.

Inductive definition of the tree kernels

In terms of the definitions introduced so far, we write $W[\tau]$ in the right side of (4.141) as

$$W[\tau] = \sum_{\underline{P} \in \mathcal{P}(\tau)} \sum_{\underline{T} \in \mathcal{S}(\tau;\underline{P})} \sum_{\underline{D} \in \mathcal{D}(\tau;\underline{P})} W[\tau; \underline{P}, \underline{T}, \underline{D}]\,, \tag{4.145}$$

where $W[\tau; \underline{P}, \underline{T}, \underline{D}]$ is the kernel inductively defined as follows: if $s_{v_0} := |S_{v_0}|$, for all $i = 1, \dots, s_{v_0}$, we let $\underline{f}^c_{v_i} := \underline{f}_{v_i} \setminus \underline{f}_{v_0}$ be Grassmann fields contracted on scale h_{v_0} and $\underline{f}^c := \underline{f}^c_{v_1}, \dots, \underline{f}^c_{v_{s_{v_0}}}$, we get

$$W[\tau; \underline{P}, \underline{T}, \underline{D}](\underline{f}_0, \underline{e}_0) = \frac{\chi_{v_0}\alpha_{v_0}}{s_{v_0}!} \sum_{\substack{\mathbf{x}: P_{S_{v_0}} \to \mathbb{H}: \\ \underline{\mathbf{x}}_0 = \underline{\mathbf{x}}_{v_0}}} \mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c) \prod_{v \in S_{v_0}} K_v(\underline{f}_v, \underline{e}_v) \tag{4.146}$$

where $P_{S_{v_0}} := \cup_{v \in S_{v_0}} P_v$, $\chi_{v_0} := \chi(\underline{\omega}_0 = \underline{\omega}_{v_0})\chi(\underline{\mathbf{d}}_0 = \underline{\mathbf{d}}_{v_0} = \underline{\mathbf{d}}'_{v_0})\chi(\underline{\mathbf{y}}_0 = \underline{\mathbf{y}}_{v_0})$, α_{v_0} and $\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)$ are defined as listed after (4.32), and

1. if $h_{v_0} = 1$, v is necessarily an endpoint and the kernel is non vanishing only if $n^\sigma_{v_0} = 0$, $\underline{\mathbf{d}}_{v_0} = \underline{\mathbf{d}}_{v_0} = \underline{\mathbf{0}}$ and

$$K_v(\underline{f}_v) = \begin{cases} \mathcal{K}(S^{(1)}_{\underline{v};B})(\underline{f}) \vee \mathcal{K}(S^{(1)}_{\underline{v};E})(\underline{f}) & \text{if } v \text{ is an endpoint } \blacklozenge \vee \lozenge\ , \\ \mathcal{K}(V^{(1)}_B)(\underline{f}_v) \vee \mathcal{K}(V^{(1)}_E)(\underline{f}_v) & \text{if } v \text{ is an endpoint } \bullet \vee \circ\ , \end{cases} \tag{4.147}$$

   where $\mathcal{K}(S^{(1)}_{\underline{v};B})$ is the kernel of $S^{(1)}_{\underline{v};B}$ in (4.111), $\mathcal{K}(S^{(1)}_{\underline{v};E})$ is the kernel of $S^{(1)}_{\underline{v};E}$ in in (4.111), $\mathcal{K}(V^{(1)}_B)$ is the kernel of $V^{(1)}_B$ in (4.119) and $\mathcal{K}(V^{(1)}_E)$ is the kernel of $V^{(1)}_E$ in (4.120);

2. if $h_{v_0} < 1$ and v is an endpoint

$$K_v(\underline{f}_v, \underline{e}_v) = \begin{cases} \mathcal{K}(S^{(h_{v_0})}_{c;B})(\underline{f}_v) \vee \mathcal{K}(S^{(h_{v_0})}_{c;E})(\underline{f}_v) & \text{if } v \text{ is } \blacklozenge \vee \lozenge \text{ and } h_v = h_{v_0}+1\,, \\ \mathcal{K}(S^{(1)}_{i;B})(\underline{f}) \vee \mathcal{K}(S^{(1)}_{i;E})(\underline{f})(\underline{f}) & \text{if } v \text{ is } \blacklozenge \vee \lozenge \text{ and } h_v = 2\,, \\ \mathcal{K}(V^{(1)}_B)(\underline{f}_v) \vee \mathcal{K}(V^{(1)}_E)(\underline{f}_v) & \text{if } v \text{ is } \bullet \vee \circ \text{ and } h_v = 2\,, \\ \mathcal{K}(B^{(h_{v_0})}_\sigma)(\underline{f}_v, \underline{e}_v) & \text{if } v \text{ is } \blacktriangle \text{ and } h_v = h_{v_0}+1\,, \\ \mathcal{K}(B^{(0)}_\sigma)(\underline{f}_v, \underline{e}_v) & \text{if } v \text{ is } \blacktriangle \text{ and } h_v = 1\,, \end{cases} \tag{4.148}$$

where $\mathcal{K}(S^{(h_{v_0})}_{c;B})$ is one of the kernels in (4.128) (see (4.129)), $\mathcal{K}(S^{(h_{v_0})}_{c;E})$ is one of the kernels of $S^{(h_{v_0})}_{c;E}$ in the first line of (4.137) (see after (4.137)), $\mathcal{K}(B^{(h)}_\sigma)$ is one of the kernels of $B^{(h)}_\sigma$ in (4.130) (see (4.131)-(**??**)), $\mathcal{K}(B^{(0)}_\sigma)$ is the kernel of $B^{(0)}_\sigma$ in (4.123) and the other kernels were introduced after (4.147).

3. if $h_{v_0} < 1$ and v is not an endpoint we introduce $\underline{P}_v, \underline{T}_v$ and $\underline{D}_v$ as the restrictions to the subtree τ_v of $\underline{P}, \underline{T}$ and $\underline{D}$ respectively; then we introduce

$$\underline{\tilde{D}}_v := \{\, \mathbf{d}_{v,P_v} \,\} \cup \{\, \mathbf{d}_w \,\}_{w \in V(\tau): w > v_0}\,, \tag{4.149}$$

and

$$K_v(\underline{f}_v, \underline{e}_v) = \begin{cases} \mathcal{R}_B W_B[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v)\Big|_{\underline{\mathbf{d}}_v} & \text{if } v \text{ is } \bullet\,, \\ \mathcal{R}_E W_E[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v)\Big|_{\underline{\mathbf{d}}_v} & \text{if } v \text{ is } \circ\,, \\ \mathcal{R}_\sigma W_\sigma[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v, \underline{e}_v)\Big|_{\underline{\mathbf{d}}_v} & \text{if } v \text{ is } \bullet\,, \end{cases} \tag{4.150}$$

where the remainder terms where defined in (4.68),(4.85) and (4.100) and $\mathcal{R}W|_{\underline{\mathbf{d}}_v}$ denotes the restriction of $\mathcal{R}W$ to the specific choice of $\underline{\mathbf{d}}_v = ((\mathbf{d}_v)_1, \ldots, (\mathbf{d}_v)_n)$ with $||\underline{\mathbf{d}}_v||_1 = D_v$. Note that in (4.145) if $s = 1$, then $T_{v_0} = \emptyset$: if $\underline{f}_{v_1} \neq \underline{f}_{v_0}$ v_0 is dotted while if $\underline{f}_{v_1} = \underline{f}_{v_0}$ there are no contracted fields and v_0 is not dotted, moreover $\underline{f}^c_{v_1} = \emptyset$ and $\mathcal{G}^{(1)}_\emptyset(\emptyset)$ has to be interpreted as 1.

Chapter 5

Bounds on the kernels of the effective potential

In this chapter we state and prove the bounds on kernels $W[\tau;\underline{P},\underline{T},\underline{D}]$, which are inductively defined after (4.145). These bounds will be the key ingredients to obtain a convergent tree expansion of the effective potential and then of the two-point spin correlations, as we will prove in the next chapter 6. In order to measure the size of $W[\tau;\underline{P},\underline{T},\underline{D}]$, we use one of the norms introduced in 4.2.3, depending on the presence or absence of spin source terms. Then we let $||\cdot||^{\sharp}_{(c_\sharp 2^{h_{v_0}})}$, with $\sharp = B, E, \sigma$ be the norm introduced, respectively, in (4.102),(4.103),(4.104) with $c_B = \frac{c}{2}$, $c_E = c_\sigma = \frac{c}{8}$ and $c > 0$, and we state the following result:

Proposition 5.1 (Bounds on the kernels of the effective potential)**.** *Let $W[\tau;\underline{P},\underline{T},\underline{D}]$ be inductively defined as in (4.145). There exist $C,\kappa,\lambda_0 > 0$ such that, for any $\tau \in \mathcal{T}^{(h)}_{n,n_\sigma}$, $\underline{P} \in \mathcal{P}(\tau)$, $\underline{T} \in \mathcal{S}(\tau,\underline{P})$, $\underline{D} \in \mathcal{D}(\tau,\underline{P})$ and $|\lambda| \le \lambda_0$,*

$$
\begin{aligned}
||W[\tau;\underline{P},\underline{T},\underline{D}]||^{\sharp}_{(c_\sharp 2^{h_{v_0}})} \le{}& C^{n^{\sigma}_{v_0}+n^{e}_{\tau}} \frac{1}{s_{v_0}!} 2^{h_{v_0} d_S(v_0)} 2^{-h_{v_0}\delta_{v_0,1}} \cdot \\
&\cdot \left(\prod_{v\in V(\tau)\setminus\{v_0\}} \frac{1}{s_v!} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})} \right) \left(\prod_{\substack{v\in \bar V(\tau):\\ s_v=\bar s_v>1}} 2^{2(s_v-1)h_v} e^{-\frac{c}{24}2^{h_v}\delta(\underline{\mathbf{x}}_v)} \right) \cdot \\
&\cdot \left(\prod_{v\in V_e^*(\tau)} |\lambda|^{\max\{1,\kappa n_v\}} 2^{\theta h_v} \right) \left(\prod_{v\in \bar V_e(\tau)} 2^{-h_v d_S(v)} \right),
\end{aligned}
\tag{5.1}
$$

where $n^e_\tau := \sum_{v\in V_e(\tau)} n_v$, $d_S(v) := 2 - \frac{n_v}{2} - D_v - \frac{n^\sigma_v}{2}$ is the scaling dimension in (4.46) associated with a vertex $v \in V(\tau)$, $\delta_{v,1} = 1$ if $v \in V(\tau)$ is such that $I_v = 1$ and is 0 otherwise, $\bar s_v := |\{ v \in S_v \cap \bar V(\tau) \}|$ and $\theta \in (0,1)$.

Note that the product in the second factor in the second line is over $v \in \bar V(\tau)$ such *that* $s_v = \bar{s}_v > 1$, *i.e. over the gray branching points that are followed only by* $v_1, \dots, v_{\bar{s}_v}$ *gray vertices and that* $\mathbf{x}_v \in \partial\mathbb{H}$ *so that* $\delta(\underline{\mathbf{x}}_v) = |x^{(1)}_{v_1} - x^{(1)}_{v_2}| + \cdots |x^{(1)}_{v_{\bar{s}_v - 1}} - x^{(1)}_{v_{\bar{s}_v}}|$.

The rest of the chapter will be devoted to the proof of Prop. 5.1. We will start by deriving the bounds for the kernels without spin sources in Lemma 5.2, which where already derived in [4, Prop. 4.20], and then in Lemma 5.3 we derive the estimates for the kernels in presence of the spin sources. The main differences due to the presence of the spin sources are:

1. the dimensional gains associated with the gray branching points (the last factor in the second line of (5.1)), which are related to the fact that we are not integrating over the source positions;
2. the different factors associated with the gray endpoints (the last factor in the last line of (5.1)), which are related to the fact that the endpoints with two Grassmann fields are different if the two fields are both massless or one massless and one spin source.

By recalling the tree expansion derived in (4.141), we see that the kernels $W[\tau; \underline{P}, \underline{T}, \underline{D}]$ actually depend on sequences of $h-$dependent parameters, $\underline{v} = \{\, \nu_h, \zeta_h, \eta_h \,\}_{h \leq 1}$ and $\underline{s} = \{\, \tilde{Z}_h, \tilde{Z}_{1;h}, \tilde{Z}_{2;h} \,\}_{h \leq 0}$. First of all, we will derive the bounds in terms of the explicit expression of their endpoints (and then depending on $\underline{v}$ and $\underline{s}$) and next, by using the bounds on the flow of $\underline{v}$ and $\underline{s}$ and the short memory property, we will derive the bounds as stated in Prop. 5.1. We will proceed as follows:

- in Sec. 5.1 we consider the trees without spin source endpoints and derive the bounds on the kernels depending on $\underline{v}$;
- in Sec. 5.2 we consider the trees with spin sources endpoints and derive the bounds on the kernels depending on $\underline{v}$ and of $\underline{s}$;
- in Sec. 5.3 we show that, by appropriately choosing the parameters on the first scale, the whole sequences $\underline{v}$ and $\underline{s}$ remain bounded, uniformly in h^*.

Under these conditions, in the next chapter, we will be able to show that the resulting expansions for two-point spin correlations are convergent, uniformly in in h^*.

5.1 Bounds on the bulk and edge kernels

In this section we derive the bounds on $||W[\tau; \underline{P}, \underline{T}, \underline{D}]||^B_{(c_B 2^{h_{v_0}})}$ and on $||W[\tau; \underline{P}, \underline{T}, \underline{D}]||^E_{(c_E 2^{h_{v_0}})}$. In Lemma 5.1 we state the bound on the bulk norms, which act on kernels associated with trees without source endpoints and with all the vertices associated with bulk contributions, i.e. trees of only black vertices. In Lemma 5.29, we derive the bound on the edge norms, which act on kernels associated with trees with at least one vertex associated with edge contributions, i.e. trees with white vertices. We start by deriving the bounds on the kernels of trees without edge spin sources endpoints.

Lemma 5.1 (Bound on the bulk kernels)**.** *For any $\tau \in \mathcal{T}_{n,0}$ with $I_{v_0} = 0$, the bound in* (5.1) *is given by*

$$||W[\tau; \underline{P}, \underline{T}, \underline{D}]||^B_{(\frac{c}{2} 2^{h_{v_0}})} \leq C^{n^e_\tau} \frac{1}{s_{v_0}!} 2^{h_{v_0} d_S(v_0)} \left(\prod_{v \in V(\tau) \setminus \{v_0\}} \frac{1}{s_v!} 2^{(h_v - h_u) d_S(v)} \right) \cdot \\ \cdot \prod_{v \in V_e(\tau)} \begin{cases} v^M_{h_v - 1} & \text{if } v \text{ is } \blacklozenge \ , \\ |\lambda|^{\max\{1, \kappa n_v\}} & \text{if } v \text{ is } \bullet \ , \end{cases} \tag{5.2}$$

where $v^M_{h_v-1} := \max\{|\nu_{h_v-1}|, |\zeta_{h_v-1}|, |\eta_{h_v-1}|\}$.

Note that the presence of $v^M_{h_v-1}$ in the bound encode the dependence on $\underline{v}$.

Proof of Lemma 5.1. If $n^\sigma_{v_0} = 0$, $\bar{V}(\tau) = \emptyset$ and $V(\tau) = V^*(\tau)$, i.e. there are not gray vertices in the tree. Moreover, if $I_{v_0} = 0$ all the vertices $v \in V(\tau)$ are black. We let $\tau_B \in \mathcal{T}^{(h)}_{n,0}$ be the tree with only black vertices, such that

$$\sum_{\tau_B \in \mathcal{T}^{(h)}_{n,0}} W[\tau_B] = W^{(h)}_{B;(n,D),0} \,, \tag{5.3}$$

where $W^{(h)}_{B;(n,D),0}$ is the bulk kernel in (4.52), and, as already mentioned, $W^{(h)}_{B;(n,D),0} = W^{(h)}_{\infty;(n,D),0}$ so that we can define

$$W[\tau_B; \underline{P}, \underline{T}, \underline{D}](\underline{f}, \underline{0}) := W_\infty[\tau; \underline{P}, \underline{T}, \underline{D}](\underline{f}) \,, \tag{5.4}$$

where

$$W_\infty[\tau; \underline{P}, \underline{T}, \underline{D}](\underline{f}_0) = \frac{\chi_{v_0} \alpha_{v_0}}{s_{v_0}!} \sum_{\substack{\mathbf{y}: \cup_{v \in S_{v_0}} \underline{P}_v \to \mathbb{Z}^2: \\ \underline{\mathbf{y}}_0 = \underline{\mathbf{y}}_{v_0}}} \mathcal{G}^{(h_{v_0})}_{T_{v_0}, \infty}(\underline{f}^c) \prod_{v \in S_{v_0}} K_v(\underline{f}_v) \,, \tag{5.5}$$

with the same definitions listed after (4.146) (without the source coordinates). By using (3.67)-(3.69), $\mathcal{G}_{T_{v_0},\infty}$ can be bounded as

$$|\mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}(\underline{f}^c)| \leq \left(\prod_{p \in \cup_{i=1}^s \underline{f}^c_{v_i} \setminus T_{v_0}} |\tilde{\gamma}^{(h_{v_0})}_{\omega(p), \mathbf{y}(p), \mathbf{d}_\mathbf{y}(p)}| \cdot |\gamma^{(h_{v_0})}_{\omega'(p), \mathbf{y}(p), \mathbf{d}_\mathbf{y}(p)}| \right)^{1/2} \left(\prod_{(p,p') \in T_{v_0}} |g^{(h_{v_0})}_\ell| \right) \\ \leq \left(C 2^{h_{v_0}} \right)^{\frac{1}{2} \sum_{i=1}^s |\underline{f}^c_{v_i}|} 2^{h_{v_0} \sum_{p \in \cup_{i=1}^s \underline{f}^c_{v_i}} ||\mathbf{d}(p)||_1} e^{-c 2^{h_{v_0}} \delta(T_{v_0})} \,, \tag{5.6}$$

where $\delta(T_{v_0}) := \sum_{(p,p') \in T_{v_0}} ||\mathbf{y}(p) - \mathbf{y}(p')||_1$. In (5.5) the $K_v(\underline{f}_v)$ are associated with the bulk contributions in (4.147), (4.148) and (4.150), i.e. v is a black vertex. The bulk-norm, as defined in (4.102), is bounded by

$$||W_\infty[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^B_{(\frac{c}{2}2^{h_{v_0}})} \leq \frac{1}{s_{v_0}!} \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{Z}^2:\\ \mathbf{y}(p_0)\ \text{fixed}}} e^{\frac{c}{2}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_{v_0})}|\mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}(\underline{f}^c)| \prod_{v\in S_{v_0}} |K_v(\underline{f}_v)|\,, \tag{5.7}$$

where we perform the sum over the coordinates keeping fixed a first coordinate $\mathbf{y}(p_0)$, with $p_0 \in P_{v_0}$. By using (5.6) and

$$\delta(\underline{\mathbf{y}}_{v_0}) \leq \delta(T_{v_0}) + \prod_{v\in S_{v_0}} \delta(\underline{\mathbf{y}}_v)\,, \tag{5.8}$$

in (5.7), we get the bound

$$\begin{aligned} ||W_\infty[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^B_{(\frac{c}{2}2^{h_{v_0}})} &\leq \frac{1}{s_{v_0}!}(C2^{h_{v_0}})^{\frac{|\underline{f}^c_{v_0}|}{2}} 2^{h_{v_0}D_{v_0,\underline{f}^c_{v_0}}} \cdot \\ &\cdot \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{Z}^2:\\ \mathbf{y}(p_0)\ \text{fixed}}} e^{-\frac{c}{2}2^{h_{v_0}}\delta(T_{v_0})} \left(\prod_{v\in S_{v_0}} e^{\frac{c}{2}2^{h_v}\delta(\underline{\mathbf{y}}_{v_0})}|K_v(\underline{f}_v)|\right). \end{aligned} \tag{5.9}$$

The sum over the coordinates in the last line, by recalling that $|T_{v_0}| = s_{v_0} - 1$, can be bounded as

$$\begin{aligned} &\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{Z}^2:\\ \mathbf{y}(p_0)\ \text{fixed}}} e^{-\frac{c}{2}2^{h_{v_0}}\delta(T_{v_0})} \left(\prod_{v\in S_{v_0}} e^{\frac{c}{2}2^{h_v}\delta(\underline{\mathbf{y}}_{v_0})}|K_v(\underline{f}_v)|\right) \leq \\ &\leq \left(\sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-\frac{c}{2}2^{h_{v_0}}||\mathbf{y}||_1}\right)^{s_{v_0}-1} \left(\prod_{v\in S_{v_0}} \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{Z}^2:\\ \mathbf{y}(p_0)\ \text{fixed}}} e^{\frac{c}{2}2^{h_v}\delta(\underline{\mathbf{y}}_{v_0})}|K_v(\underline{f}_v)|\right), \end{aligned} \tag{5.10}$$

where we can recognize that the last factor is

$$\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{Z}^2:\\ \mathbf{y}(p_0)\ \text{fixed}}} e^{\frac{c}{2}2^{h_v}\delta(\underline{\mathbf{y}}_{v_0})}|K_v(\underline{f}_v)| = ||K_v(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})}\,, \tag{5.11}$$

see the definition of bulk norm in (4.102). In (5.9) we finally get

$$||W_\infty[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^B_{h_{v_0}} \leq C^{n^e_\tau}\frac{1}{s_{v_0}!} 2^{h_{v_0}(\frac{|\underline{f}^c_{v_0}|}{2}+D_{v_0,\underline{f}^c_{v_0}}-2(s_{v_0}-1))} \prod_{v\in S_{v_0}} ||K_v(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})}\,, \tag{5.12}$$

where $n^e_\tau := \sum_{v\in V_e} n_v$. To obtain (5.12) we used that

$$\sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-c2^{h_{v_0}}||\mathbf{y}||_1} \le C\, 2^{-2h_{v_0}}\,, \tag{5.13}$$

and that $C^{\frac{|\underline{f}^c_{v_0}|}{2}+s_{v_0}-1} \le C^{\frac{|\underline{f}^c_{v_0}|}{2}+s_{v_0}} \le C^{\sum_{v\in V_0}(\frac{|\underline{f}^c_v|}{2}+s_v)} \le C'^{\sum_{v\in V_e} n_v}$.

The bounds on $||K_v(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})}$ are the following:

- if $h_{v_0}=1$, K_v is given by kernels of the black endpoints in (4.147): if v is ♦ we get

$$||\underline{v}_1\mathcal{K}(S^{(1)}_{\underline{v};B})(\underline{f}_v)||^B_{(c)} \le Cv^M_1\,, \tag{5.14}$$

  where $v^M_1 := \max\{|\nu_1|,|\zeta_1|,|\eta_1|\}$; if v is ● we get

$$||\mathcal{K}(V^{(1)}_B)(\underline{f}_v)||^B_{(c)} \le C^{n_v}|\lambda|^{\max\{1,\kappa n_v\}}\,; \tag{5.15}$$

- if $h_{v_0}<1$ and v is an endpoint, K_v is given by the kernels of the black endpoints in (4.148): if v is ♦ and $h_v = h_{v_0}+1$ we get

$$\begin{aligned}||\mathcal{K}(S^{(h_{v_0})}_{c;B})(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})} &\le C\begin{cases}2^{h_{v_0}}|\nu_{h_{v_0}}| & \text{if } D_v=0\\ \max\{|\zeta_{h_{v_0}}|,|\eta_{h_{v_0}}|\} & \text{if } D_v=1\end{cases}\\ &\le C'2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}v^M_{h_v-1}\,,\end{aligned} \tag{5.16}$$

  where $v^M_{h_v-1} := \max\{|\nu_{h_v-1}|,|\zeta_{h_v-1}|,|\eta_{h_v-1}|\}$, and we used that $n_v=2$ and $h_{v_0} = h_v-1$; if v is ♦ and $h_v=2$ we get

$$\begin{aligned}||\mathcal{K}(S^{(1)}_{i;B})(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})} \le& ||\mathcal{K}(S^{(1)}_{i;B})(\underline{f}_v)||^B_{(\frac{c}{2})}\\ \le& Cv_1 \le C2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}v_1\,;\end{aligned} \tag{5.17}$$

  if v is ● (and $h_v=2$) we get

$$||\mathcal{K}(V^{(1)}_B)(\underline{f}_v)||^B_{(\frac{c}{2}2^{h_{v_0}})} \le ||\mathcal{K}(V^{(1)}_B)(\underline{f}_v)||^B_{(\frac{c}{2})} \le C^{n_v}|\lambda|^{\max\{1,\kappa n_v\}}\,; \tag{5.18}$$

- if $h_{v_0}<1$ and v is not an endpoint, K_v is given by the kernel of the vertex ● in (4.150), so we get

$$||\,\mathcal{R}_BW_B[\tau_v;\underline{P}_v,\underline{T}_v,\underline{\tilde{D}}_v](\underline{f}_v)\Big|_{\underline{\mathbf{d}}_v}||^B_{(\frac{c}{2}2^{h_{v_0}})} \le 2^{-h_vR_v}||\mathcal{R}W_i[\tau_v;\underline{P}_v,\underline{0},\underline{T}_v,\underline{\tilde{D}}_v](\underline{f}_v)||^B_{(\frac{c}{2}2^{h_v})}\,, \tag{5.19}$$

  where we used the bounds in (4.105) and the R_v defined in (4.144).

By iterating (5.19), until the endpoints are reached, we can use the bounds (5.14)-(5.18) in (5.12), so we get

$$||W_\infty[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^B_{(\frac{c}{2}2^{h_{v_0}})} \leq C^{n^e_\tau}\left(\prod_{v\in V_0(\tau)\cup\{v_0\}}\frac{1}{s_v!}2^{h_v(\frac{|\underline{f}^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)}\right)\cdot \\ \cdot\prod_{v\in V_e(\tau)}\begin{cases}2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge\ ,\\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet\ ,\end{cases} \tag{5.20}$$

where, we used that by definition $R_{v_0}=0$ (see (4.144)). Next, we rewrite the product in the first line of (5.20) as

$$\prod_{v\in V_0(\tau)\cup\{v_0\}}2^{h_v(\frac{|\underline{f}^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)} = 2^{h_{v_0}\sum_{v\in V_0(\tau)\cup\{v_0\}}(\frac{|\underline{f}^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)}\cdot \\ \cdot\, 2^{\sum_{v\in V_0(\tau)\cup\{v_0\}}(h_v-h_{v_0})(\frac{|\underline{f}^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)}\,, \tag{5.21}$$

where, for all $v\in V_0(\tau)\cup\{v_0\}$, we recall the number of contracted fields, $|\underline{f}^c_v| = \sum_{w\in S_v}n_w - n_v$, and the definition of R_v in (4.144), $R_v = D_v - D_{v,P_v}$, with D_{v,P_v} defined in (4.143). We can use the equalities

$$\sum_{v\in V_0(\tau)\cup\{v_0\}}\Big(\sum_{w\in S_v}n_w - n_v\Big) = \sum_{v\in V_e(\tau)}n_v - n_{v_0}\,, \\ \sum_{v\in V_0(\tau)\cup\{v_0\}}\Big(D_{v,\underline{f}^c_v}+D_{v,P_v}+D_v\Big) = \sum_{v\in V_0(\tau)\cup\{v_0\}}\Big(\sum_{w\in S_v}D_w + D_v\Big) = \sum_{v\in V_e(\tau)}D_v - D_{v_0}\,, \\ \sum_{v\in V_0(\tau)\cup\{v_0\}}(s_v-1) = \sum_{v\in V_e(\tau)}n_v - 1\,, \tag{5.22}$$

to rewrite the first factor in the r.h.s. of (5.21) as

$$2^{h_{v_0}\sum_{v\in V_0(\tau)\cup\{v_0\}}(\frac{|\underline{f}^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)} = 2^{h_{v_0}(2-\frac{n_{v_0}}{2}-D_{v_0})}\prod_{v\in V_e(\tau)}2^{-h_{v_0}(2-\frac{n_v}{2}-D_v)}\,. \tag{5.23}$$

Next, let $\tau_{v'}$ be the subtree rooted in v' and u' is the vertex immediately preceding v', we get

$$\begin{aligned}
&\sum_{v\in V_0(\tau)\cup\{v_0\}}(h_v-h_{v_0})(\frac{|\underline{f}_v^c|}{2}+D_{v,\underline{f}_v^c}-2(s_v-1)-R_v)=\\
&=\sum_{v\in V_0(\tau)\cup\{v_0\}}\Big(\sum_{\substack{v'\in V_0(\tau):\\ v'\le v}}(h_{v'}-h_{u'})\Big)(\frac{|\underline{f}_v^c|}{2}+D_{v,\underline{f}_v^c}-2(s_v-1)-R_v)=\\
&=\sum_{v'\in V_0(\tau)}(h_{v'}-h_{u'})\sum_{v\in V_0(\tau_{v'})\cup\{v'\}}(\frac{|\underline{f}_v^c|}{2}+D_{v,\underline{f}_v^c}-2(s_v-1)-R_v)=\\
&=\sum_{v'\in V_0(\tau)}(h_{v'}-h_{u'})\big(2-\frac{n_{v'}}{2}-D_{v'}-\sum_{v\in V_e(\tau_{v'})}(2-\frac{n_v}{2}-D_v)\big)\,,
\end{aligned}\tag{5.24}$$

which can be used to rewrite the last factor in the r.h.s. of (5.21) as

$$\begin{aligned}
&2^{\sum_{v\in V_0(\tau)\cup\{v_0\}}(h_v-h_{v_0})(\frac{|\underline{f}_v^c|}{2}+D_{v,\underline{f}_v^c}-2(s_v-1)-R_v)}=\\
&=\Big(\prod_{v\in V_0(\tau)}2^{(h_v-h_u)(2-\frac{n_v}{2}-D_v)}\Big)\Big(\prod_{v\in V_e(\tau)}2^{-\sum_{v'\in V_0(\tau)}^{v'<v}(h_{v'}-h_{u'})(2-\frac{n_v}{2}-D_v)}\Big)\,.
\end{aligned}\tag{5.25}$$

With the results of (5.23) and of (5.25), the factor $\prod_{v\in V_0(\tau)\cup\{v_0\}}2^{h_v(\frac{|\underline{f}_v^c|}{2}+D_{v,\underline{f}_v^c}-2(s_v-1)-R_v)}$ in (5.21) can be rewritten as

$$2^{h_{v_0}(2-\frac{n_{v_0}}{2}-D_{v_0})}\left(\prod_{v\in V_0(\tau)}2^{(h_v-h_u)(2-\frac{n_v}{2}-D_v)}\right)\left(\prod_{v\in V_e(\tau)}2^{-h_u(2-\frac{n_v}{2}-D_v)}\right)\,,\tag{5.26}$$

where we used that $h_{v_0}+\sum_{v'\in V_0(\tau)}^{v'<v}(h_{v'}-h_{u'})=h_u$. Finally (5.20) can be rewritten as

$$\begin{aligned}
&||W_\infty[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^B_{h_{v_0}}\le C^{n^e_\tau}\frac{1}{s_{v_0}!}2^{h_{v_0}(2-\frac{n_{v_0}}{2}-D_{v_0})}\left(\prod_{v\in V_0(\tau)}\frac{1}{s_v!}2^{(h_v-h_u)(2-\frac{n_v}{2}-D_v)}\right)\cdot\\
&\qquad\cdot\left(\prod_{v\in V_e(\tau)}2^{-h_u(2-\frac{n_v}{2}-D_v)}\right)\prod_{v\in V_e(\tau)}\begin{cases}2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge\,,\\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet\,,\end{cases}
\end{aligned}\tag{5.27}$$

and the last line can be bounded as

$$\begin{aligned}
&\left(\prod_{v\in V_e(\tau)}2^{-h_u(2-\frac{n_v}{2}-D_v)}\right)\cdot\begin{cases}2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge\\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet\end{cases}\\
&\le\prod_{v\in V_e(\tau)}2^{(h_v-h_u)(2-\frac{n_v}{2}-D_v)}\cdot\begin{cases}v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge\\ 2^{n_v}|\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet\end{cases}
\end{aligned}\tag{5.28}$$

where we used $2^{h_v(2-\frac{n_v}{2}-D_v)} \geq 2^{-n_v}$ to bound the $\bullet$ endpoints. Finally, by recalling that $d_S(v) := 2 - \frac{n_v}{2} - D_v$ if $n_v^\sigma = 0$ for all $v \in V(\tau)$, we find (5.2). $\square$

Lemma 5.2 (Bound on the edge kernels). *For any $\tau \in \mathcal{T}_{n,0}^{(h)}$, with $I_{v_0} = 1$, the bound in* (5.1) *is given by*

$$||W[\tau;\underline{P},\underline{T},\underline{D}]||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq C^{n^e_\tau} \frac{1}{s_{v_0}!} 2^{h_{v_0}d_S(v_0)} 2^{-h_{v_0}} \left(\prod_{v\in V(\tau)\setminus\{v_0\}} \frac{1}{s_v!} 2^{(h_v-h_u)d_S(v)} 2^{-(h_v-h_u)\delta_{v,1}} \right) \cdot \prod_{v\in V_e(\tau)} \begin{cases} v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge, \\ v^M_{h_v-1}(y^{(2)}) & \text{if } v \text{ is } \lozenge, \\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet \text{ or } \circ, \end{cases} \tag{5.29}$$

where $\delta_{I_v,1} = 1$ if $I_v = 1$, vanishes if $I_v = 0$, $v^M_{h_v-1}$ was introduced after (5.2) *and* $v^M_{h_v-1}(y^{(2)}) := \max\{|\nu_{h_v-1}(y^{(2)})|, |\zeta_{h_v-1}(y^{(2)})|, |\eta_{h_v-1}(y^{(2)})|\}$.

Note that, with respect the result in (5.2), now there is a factor $2^{-h_{v_0}}$, which is related to the white v_0 (i.e. $I_{v_0} = 1$) and there is a factor $2^{-(h_v-h_u)}$ for each white $v \in V(\tau)$ (i.e. the vertices with $I_v = 1$).

Proof of Lemma 5.2. If $n^\sigma_{v_0} = 0$, $V(\tau) = V^*(\tau)$, i.e. there are not gray vertices in the tree, and if $I_{v_0} = 1$ there is at least one white vertices in $V(\tau)$. We start by considering the case when the root v_0 is the only white vertex of τ: $I_{v_0} = 1$ and $I_v = 0$ for all $v \in V(\tau) \setminus \{v_0\}$. We let $\tau_E^{v_0} \in \mathcal{T}^{(h)}_{n,0}$ be the tree with the root only white vertex, and we let

$$W[\tau_E^{v_0};\underline{P},\underline{T},\underline{D}](\underline{f}_0,\underline{0}) := \frac{\chi_{v_0}\alpha_{v_0}}{s_{v_0}!} \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}: \\ \underline{\mathbf{y}}_0=\underline{\mathbf{y}}_{v_0}}} \left(\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c) - \mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}(\underline{f}^c)\right) \prod_{v\in S_{v_0}} K_v(\underline{f}_v,\underline{0}), \tag{5.30}$$

be the difference between the kernel in (4.146) and the kernel in (5.5) evaluated for the tree obtained by changing from white to black the color of v_0 in $\tau_E^{v_0}$. Since $I_v = 0$ for each $v \in V(\tau) \setminus \{v_0\}$ each $K_v(\underline{f}_v)$ in (5.30) is associated with a bulk contribution in (4.147), (4.148) and (4.150). Let $|\mathcal{G}^{(h_{v_0})}_{T_{v_0}} - \mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}| := |\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c) - \mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}(\underline{f}^c)|$, by using (5.6) and the analogous bound for $\mathcal{G}^{(h_{v_0})}_{T_{v_0}}$, the bound for the difference is given by

$$|\mathcal{G}^{(h_{v_0})}_{T_{v_0}} - \mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}| \leq (C2^{h_{v_0}})^{\frac{1}{2}\sum_{i=1}^s |\underline{f}^c_{v_i}|} 2^{h_{v_0}\sum_{p\in\cup_{i=1}^s \underline{f}^c_{v_i}} ||\mathbf{d}(p)||_1} e^{-c2^{h_{v_0}}(\delta(T_{v_0})+\delta(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}))}, \tag{5.31}$$

where $\underline{\mathbf{y}}(\underline{f}^c_{v_0})$ is the tuple with elements $\mathbf{y}(p)$, $p \in \underline{f}^c_{v_0}$, $\delta(T_{v_0})$ was defined after (5.6) and $\delta(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}) := \mathrm{dist}(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H})$ (see [4, Lemma 4.16] for the details). The edge-norm, as defined in (4.103), of the kernel in (5.30) is bounded by

$$||W[\tau_E^{v_0};\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq \frac{1}{s_{v_0}!}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\ \text{fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_{v_0})}|\mathcal{G}^{(h_{v_0})}_{T_{v_0}} - \mathcal{G}^{(h_{v_0})}_{T_{v_0},\infty}| \prod_{v\in S_{v_0}} |K_v(\underline{f}_v)|\,, \tag{5.32}$$

where in the sum over the coordinates we fix only $x^{(1)}_{p_0}$, which is the horizontal component of $\mathbf{y}(p_0)$, with $p_0 \in \underline{f}^c_{v_0}$ such that $\mathrm{dist}(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}) = \mathrm{dist}(\mathbf{y}(p_0),\partial\mathbb{H}) = y^{(2)}_{p_0}$. By using (5.31) and

$$\delta_E(\underline{\mathbf{y}}_{v_0}) \leq \delta(T_{v_0}) + \delta(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}) + \sum_{v\in S_{v_0}}\delta(\underline{\mathbf{y}}_v)\,, \tag{5.33}$$

in (5.32), we get the bound

$$\begin{aligned}||W[\tau_E^{v_0};\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} &\leq \frac{1}{s_{v_0}!}(C2^{h_{v_0}})^{\frac{|\underline{f}^c_{v_0}|}{2}}2^{h_{v_0}D_{v_0,\underline{f}^c_{v_0}}}\cdot\\ &\cdot\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\ \text{fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}(\delta(T_{v_0})+\delta(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}))}\left(\prod_{v\in S_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right).\end{aligned} \tag{5.34}$$

To bound the sum over the coordinates in the last line, and proceed as in (5.9) we have to introduce some special field labels. We let $\tilde{p}_{w_0} := p_0$ with $w_0 \in S_{v_0}$ such that $p_0 \in P_{w_0}$. By recalling the definition of the set spanning tree in (4.142), we start by considering all the pairs of field indices $(p,p') \in T_{v_0}$ with $p \in P_{w_0}$: each field label p' will belong to a set P_{w_i}, $w_i \in V(\tau)\setminus\{w_0\}$, $i = 1,\dots,n_0$, $n_0 := |S_{w_0}\cap T_{v_0}|$ and we can define $\tilde{p}_{w_i} := p'$. Then we consider all the pairs of field indices $(p,p') \in T_{v_0}$ with $p \in P_{w_i}$: this time each field label p' will belong to a set P_{w_j}, $w_j \in V(\tau)\setminus(\{w_0\}\cup\{\,w_i\,\}_{i=1}^{n_0})$, $j = n_0+1,\dots,n_1$, $n_1 := |\cup_{i=1}^{n_0} S_{w_i}\cap T_{v_0}|$, and we define $\tilde{p}_{w_j} := p'$. Iterating these definition until all the field labels of the spanning tree are covered, we obtain the field labels $\tilde{p}_{w_0},\tilde{p}_{w_1},\dots,\tilde{p}_{w_{|T_{v_0}|}}$, which will be the coordinates to fix in order to exchange the summation in (5.34). Then, we get the bound

$$\begin{aligned}&\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\ \text{fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}(\delta(T_{v_0})+\delta(\underline{\mathbf{y}}(\underline{f}^c_{v_0}),\partial\mathbb{H}))}\left(\prod_{v\in S_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right) \leq\\ &\leq \left(\sum_{\mathbf{y}\in\mathbb{H}} e^{-\frac{7}{8}c2^{h_{v_0}}||\mathbf{y}||_1}\right)^{s_{v_0}-1}\left(\sum_{y^{(2)}=1}^{\infty} e^{-\frac{7}{8}c2^{h_{v_0}}y^{(2)}}\right)\left(\prod_{v\in S_{v_0}}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \mathbf{y}(\tilde{p}_v)\ \text{fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right),\end{aligned} \tag{5.35}$$

where we used that $|T_{v_0}| = s_{v_0} - 1$. Finally we recognize that the last factor is

$$\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \mathbf{y}(\tilde{p}_v)\ \text{fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v,\underline{0})| = ||K_v(\underline{f}_v,\underline{0})||^B_{(\frac{c}{8}2^{h_{v_0}})}\,, \tag{5.36}$$

which is similar to the one in (5.11), with the only difference that now the sum is over the coordinates on the half plane: however, it can be bounded exactly as in (5.14)-(5.18). With these results in (5.34) we get

$$||W[\tau_E^{v_0};\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} \le C^{n^e_\tau}2^{-h_{v_0}}\left(\prod_{v\in V_0(\tau)\cup\{v_0\}}\frac{1}{s_v!}2^{h_v(\frac{|f^c_v|}{2}+D_{v,\underline{f}^c_v}-2(s_v-1)-R_v)}\right)\cdot$$
$$\cdot\prod_{v\in V_e(\tau)}\begin{cases}2^{(h_v-1)(2-\frac{n_v}{2}-D_v)}\upsilon_{h_v-1} & \text{if } v \text{ is } \blacklozenge\,,\\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet\,,\end{cases} \tag{5.37}$$

where we used (5.13), namely

$$\sum_{y^{(2)}=1}^{\infty} e^{-\frac{7}{8}c2^{h_{v_0}}y^{(2)}} \le C2^{-h_{v_0}}\,, \tag{5.38}$$

to get the factor $2^{-h_{v_0}}$ in (5.51): this is the main difference with respect to (5.2), and it is due to the withe color of v_0.

Now we consider the case when the root v_0 is not the only white vertex of τ: $I_{v_0} = 1$ and $I_v = 1$ for some $v \in V(\tau)\setminus\{v_0\}$. We let $\tau_E \in \mathcal{T}^{(h)}_{n,0}$ be the tree with more than one white vertex, and we let

$$W[\tau_E;\underline{P},\underline{T},\underline{D}](\underline{f}_0) = \frac{\chi_{v_0}\alpha_{v_0}}{s_{v_0}!}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \underline{\mathbf{y}}_0=\underline{\mathbf{y}}_{v_0}}}\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)\prod_{v\in S_{v_0}}K_v(\underline{f}_v)\,, \tag{5.39}$$

with the same definitions listed after (4.146). Now, each $v \in S_{v_0}$ can be either black or white: we let $S^B_{v_0} := \{\, v\in S_{v_0} : I_v = 0\,\}$ and $S^E_{v_0} := \{\, v\in S_{v_0} : I_v = 1\,\}$, be the subsets of black and white vertices of S_{v_0}, so that $S_{v_0} = S^B_{v_0}\cup S^E_{v_0}$ and $S^B_{v_0}\cap S^E_{v_0} = \emptyset$. By the definition in (4.103), the edge-norm of the kernel in (5.39) is bounded by

$$||W[\tau_E;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} \le \frac{1}{s_{v_0}!}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\ \text{fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_{v_0})}|\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)|\prod_{v\in S_{v_0}}|K_v(\underline{f}_v)|\,, \tag{5.40}$$

where in the sum over the coordinates we fix $y^{(1)}_{p_0}$, which is the horizontal component of $\mathbf{y}(p_0)$, with $p_0 \in P_{w_0}$ and $w_0 \in S^E_{v_0}$.By using (3.67)-(3.69), $|\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)|$ can be bounded as in (5.6), with the difference that the coordinates are in the half-plane: with this result and the following inequality

$$\delta_E(\underline{\mathbf{y}}_{v_0}) \leq \delta(T_{v_0}) + \sum_{v\in S^B_{v_0}} \delta(\underline{\mathbf{y}}_v) + \sum_{v\in S^E_{v_0}} \delta_E(\underline{\mathbf{y}}_v)\,, \tag{5.41}$$

in (5.40), we get the the bound

$$\begin{aligned} ||W[\tau_E;\underline{P},\underline{T},\underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} &\leq \frac{1}{s_{v_0}!}(C2^{h_{v_0}})^{\frac{|\underline{f}^c_{v_0}|}{2}} 2^{h_{v_0}D_{v_0,\underline{f}^c_{v_0}}}\cdot \\ &\cdot \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\text{ fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}\delta(T_{v_0})}\left(\prod_{v\in S^B_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right)\left(\prod_{v\in S^E_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right). \end{aligned} \tag{5.42}$$

To bound the sum in the last line we proceed as after (5.34), but now $p_0 \in P_{w_0}$, $w_0 \in S^E_{v_0}$ and each field label in $\tilde{p}_{w_1},\ldots,\tilde{p}_{w_{|T_{v_0}|}}$ can be associated with a black or white vertex, i.e. $w_i \in S^B_{v_0}$ or $w_i \in S^E_{v_0}$, $i = 1,\ldots,|T_{v_0}|$. Then, we get the bound

$$\begin{aligned} &\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\text{ fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}\delta(T_{v_0})}\left(\prod_{v\in S^B_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right)\left(\prod_{v\in S^E_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right) \leq \\ &\leq \Big(\sum_{\mathbf{y}\in\mathbb{H}} e^{-\frac{7}{8}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{|S^B_{v_0}|}\Big(\sum_{\substack{\mathbf{y}\in\mathbb{H}:\\ y^{(2)}\text{ fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{|S^E_{v_0}|-1}\cdot \\ &\cdot\left(\prod_{v\in S^B_{v_0}}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \mathbf{y}(\tilde{p}_v)\text{fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right)\left(\prod_{v\in S^E_{v_0}}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{\tilde{p}_v}\text{ fixed}}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right), \end{aligned} \tag{5.43}$$

where we used that $|T_{v_0}| = |S^B_{v_0}| + |S^E_{v_0}| - 1$. Then, by recalling the definitions in (4.102) and (4.103) and by using the bounds (5.13) and (5.38), in (5.43) we get the bound

$$\begin{aligned} &\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{p_0}\text{ fixed}}} e^{-\frac{7}{8}c2^{h_{v_0}}\delta(T_{v_0})}\left(\prod_{v\in S^B_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right)\left(\prod_{v\in S^E_{v_0}} e^{\frac{c}{8}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\right) \\ &\qquad\leq (C2^{-2h_{v_0}})^{|S^B_{v_0}|}(C2^{-h_{v_0}})^{(|S^E_{v_0}|-1)}\left(\prod_{v\in S^B_{v_0}}||K_v||^B_{(\frac{c}{8}2^{h_{v_0}})}\right)\left(\prod_{v\in S^E_{v_0}}||K_v||^E_{(\frac{c}{8}2^{h_{v_0}})}\right). \end{aligned} \tag{5.44}$$

The bound on $||K_v||^B_{(\frac{c}{8}2^{h_{v_0}})}$ are the same as the ones in (5.14)-(5.18) and the bounds on $||K_v||^E_{(\frac{c}{8}2^{h_{v_0}})}$ are the following:

- if $h_{v_0} = 1$, K_v is given by the kernels of the white endpoints in (4.147): if v is $\diamond$ we get
$$||\underline{v}_1 \mathcal{K}(S^{(1)}_{\underline{v};E})(\underline{f}_v)||^E_{(\frac{c}{4})} \leq C v^M_1(y^{(2)})\,, \tag{5.45}$$
where $v^M_1(y^{(2)}) := \max\{|\nu_1(y^{(2)})|, |\zeta_1(y^{(2)})|, |\eta_1(y^{(2)})|\}$; if v is $\circ$ we get
$$||\mathcal{K}(V^{(1)}_E)(\underline{f}_v)||^E_{(\frac{c}{4})} \leq C^{n_v}|\lambda|^{\max\{1,\kappa n_v\}}\,; \tag{5.46}$$

- if $h_{v_0} < 1$ and v is an endpoint, K_v is given by the kernels of the white endpoints in (4.148): if v is $\diamond$ and $h_v = h_{v_0} + 1$ we get
$$\underline{v}_{h_{v_0}} \mathcal{K}(S^{(h_{v_0})}_{c;E})(\underline{f}_v)||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq C' 2^{(h_v-1)(2-\frac{n_v}{2}-D_v)} \tilde{v}^M_{h_v-1}(y^{(2)})\,, \tag{5.47}$$
where $\tilde{v}^M_{h_v-1}(y^{(2)}) := \max\{|\nu_{h_v-1}(y^{(2)})|, |\zeta_{h_v-1}(y^{(2)})|, |\eta_{h_v-1}(y^{(2)})|\}$; if v is $\diamond$ and $h_v = 2$ we get
$$||\underline{v}_1 \mathcal{K}(S^{(h_{v_0})}_{i;E})(\underline{f}_v)||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq ||\underline{v}_1 \mathcal{K}(S^{(h_{v_0})}_{i;E})(\underline{f}_v)||^E_{(\frac{c}{8})} \leq C\tilde{v}^M_1(y^{(2)})\,, \tag{5.48}$$
where $\tilde{v}^M_1(y^{(2)}) := \max\{|\nu_1(y^{(2)})|, |\zeta_1(y^{(2)})|, |\eta_{h_v-1}(y^{(2)})|\}$; if v is $\circ$ (and $h_v = 2$) we get
$$||\mathcal{K}(V^{(1)}_E)(\underline{f}_v)||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq ||\mathcal{K}(V^{(1)}_E)(\underline{f}_v)||^E_{(\frac{c}{4})} \leq C^{n_v}|\lambda|^{\max\{1,\kappa n_v\}}\,; \tag{5.49}$$

- if $h_{v_0} < 1$ and v is not an endpoint, K_v is given by the kernel of the vertex $\circ$ in (4.150), so we get
$$||\, \mathcal{R}_E W_E[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v)\Big|_{\underline{\mathbf{d}}_v} ||^E_{(\frac{c}{8}2^{h_{v_0}})} \leq 2^{-h_v R_v} ||W_{E;i}[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v)||^E_{(\frac{c}{8}2^{h_v})}\,. \tag{5.50}$$

By iterating (5.50), until the endpoints are reached, we can use the bounds in (5.45)-(5.49) in (5.44) for the edge norms. Moreover if we rewrite $(C2^{-2h_{v_0}})^{|S^B_{v_0}|}(C2^{-h_{v_0}})^{(|S^E_{v_0}|-1)}$ in (5.44) as $C^{s_{v_0}} 2^{-2h_{v_0}(s_{v_0}-1)} 2^{-h_{v_0}} 2^{h_{v_0}|S^E_{v_0}|}$, we get the bound

$$\begin{aligned}
||W[\tau^{v_0}_E; \underline{P}, \underline{T}, \underline{D}](\underline{f}_0)||^E_{(\frac{c}{8}2^{h_{v_0}})} &\leq C^{n^e_\tau} 2^{-h_{v_0}} 2^{|S^E_{v_0}|h_{v_0}} \cdot \\
&\cdot \left(\prod_{v \in V_0(\tau) \cup \{v_0\}} \frac{1}{s_v!} 2^{h_v(\frac{|f^c_v|}{2} + D_{v,\underline{f}^c_v} - 2(s_v-1) - R_v)} \right) \cdot \\
&\cdot \prod_{v \in V_e(\tau)} 2^{-\delta_{I_v,1} h_v} \begin{cases} 2^{(h_v-1)(2-\frac{n_v}{2}-D_v)} v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge, \\ 2^{(h_v-1)(2-\frac{n_v}{2}-D_v)} \tilde{v}^M_{h_v-1}(y^{(2)}) & \text{if } v \text{ is } \diamond, \\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet \text{ or } \circ, \end{cases}
\end{aligned} \tag{5.51}$$

which can be rewritten by using the results in (5.21)-(5.26) and in (5.28) and the definition of scaling dimension $d_S(v)$ after (5.1), thus obtaining (5.29). $\square$

5.2 Bounds on the spin source kernels

We are now able to derive the bounds on $||W[\tau;\underline{P},\underline{T},\underline{D}]||^{\sigma}_{(c_\sigma 2^{h_{v_0}})}$, which are the bounds on the source norms that acts on kernels associated with trees with edge spin source endpoints, i.e. trees with gray vertices.

Lemma 5.3 (Bounds on the source kernels)**.** *For any* $\tau \in \mathcal{T}^{(h)}_{n,n_\sigma}$*, with* $n_\sigma > 0$*, the bound in* (5.1) *is given by*

$$||W[\tau;\underline{P},\underline{T},\underline{D}]||^{\sigma}_{(\frac{c}{8}2^{h_{v_0}})} \leq C^{n^{\sigma}_{v_0}+n^{e}_{\tau}} \frac{1}{s_{v_0}!}\, 2^{h_{v_0} d_S(v_0)} \left(\prod_{v\in V(\tau)\setminus\{v_0\}} \frac{1}{s_v!}\, 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})}\right) \cdot$$
$$\cdot \left(\prod_{\substack{v\in \bar{V}(\tau):\\ s_v=\bar{s}_v>1}} 2^{2(s_v-1)h_v} e^{-\frac{c}{24}2^{h_v}\delta(\underline{\mathbf{x}}_v)}\right) \prod_{v\in V_e(\tau)} \begin{cases} v^M_{h_v-1} & \text{if } v \text{ is } \blacklozenge,\\ v^M_{h_v-1}(y^{(2)}) & \text{if } v \text{ is } \lozenge,\\ |\lambda|^{\max\{1,\kappa n_v\}} & \text{if } v \text{ is } \bullet \text{ or } \circ,\\ 2^{-(h_v-1)d_S(v)} s^M_{h_v-1} & \text{if } v \text{ is } \blacktriangle, \end{cases} \tag{5.52}$$

where $s^M_{h_v-1} := \max\{\ |\ \tilde{Z}_{h_v-1}|, |\tilde{Z}_{1;h_v-1}|, |\tilde{Z}_{2;h_v-1}|\,\}$.

Note that in the second line of (5.52) there is a factor $2^{2(s_v-1)h_v} e^{-\frac{c}{24}2^{h_v}\delta(\underline{\mathbf{x}}_v)}$ for each gray vertex v which is a branching vertex followed by $\bar{s}_v$ gray vertices: it is related to the fact that there are not integrations over the spin source coordinates $\mathbf{x}_1,\dots,\mathbf{x}_{\bar{s}_v}$. Moreover, by recalling that the spin sources are located at the edge of the half-plane we get that the distance $\mathrm{dist}(\mathbf{x}_1,\dots,\mathbf{x}_{\bar{s}_v})$ simply equals $\mathrm{dist}(x^{(1)}_1,\dots,x^{(1)}_{\bar{s}_v})$.
The bound in (5.52) differs form the similar bounds derived in [9] and in [34, Cap.12], for the two-point correlation in infinite domains: in fact here we take into account the presence of the edge of the domain, which allows us to derive an extra gain factor for each white vertex (when $\delta_{v,1} = 1$).

Proof of Lemma 5.3. If $n^{\sigma}_{v_0} \neq 0$, $V(\tau) = \bar{V}(\tau) \cup V^*(\tau)$ and $v_0 \in \bar{V}(\tau)$, i.e. if there is any gray vertex in the tree, the root has to be gray. The tree $\tau \in \mathcal{T}^{(h)}_{n,n^\sigma}$ is than a tree with gray, white and black vertices, as the one in Fig. 4.5, whose kernel is defined in (4.146), namely

$$W[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0,\underline{e}_0) = \frac{\chi_{v_0}\alpha_{v_0}}{s_{v_0}!} \sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \underline{\mathbf{y}}_0=\underline{\mathbf{y}}_{v_0}}} \mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c) \prod_{v\in S_{v_0}} K_v(\underline{f}_v,\underline{e}_v)\,. \tag{5.53}$$

The source-norm of the kernel in (5.53) can be bounded by

$$||W[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0,\underline{e}_0)||^{\sigma}_{(\frac{c}{8}2^{h}_{v_0})} \leq \frac{1}{s_{v_0}!} \sum_{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}} 2^{\frac{c}{8}h_{v_0}\delta(\underline{\mathbf{y}}_{v_0},\underline{\mathbf{x}}_{v_0})} |\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)| \prod_{v\in S_{v_0}} |K_v(\underline{f}_v,\underline{e}_v)|\,, \tag{5.54}$$

where, by recalling the definition of the source norm in (4.104), the sum over the Grassmann positions is without any fixed coordinate. By using in (5.54) the bound for $|\mathcal{G}^{(h_{v_0})}_{T_{v_0}}(\underline{f}^c)|$ similar to the one in (5.6) and the following inequality

$$\delta(\underline{\mathbf{y}}_{v_0},\underline{\mathbf{x}}_{v_0}) \le -\frac{1}{3}\delta(\underline{\mathbf{x}}_{v_0})+\frac{4}{3}\big(\delta(T_{v_0})+\sum_{v\in\bar{S}_{v_0}}\delta(\underline{\mathbf{y}}_v,\underline{\mathbf{x}}_v)+\sum_{v\in S^B_{v_0}}\delta(\underline{\mathbf{y}}_v)+\sum_{v\in S^E_{v_0}}\delta_E(\underline{\mathbf{y}}_v)\big)\,, \tag{5.55}$$

we get the bound

$$\begin{aligned}||W[\tau;\underline{P},\underline{T},\underline{D}](\underline{f}_0,\underline{e}_0)||^{\sigma}_{(\frac{c}{8}2^{h}_{v_0})} \le{}& \frac{1}{s_{v_0}!}(C2^{h_{v_0}})^{\frac{|\underline{f}^c_{v_0}|}{2}}2^{h_{v_0}D_{v_0,\underline{f}^c_{v_0}}}e^{-\frac{c}{24}\delta(\underline{\mathbf{x}}_{v_0})}\cdot\\ &\cdot\Bigg\{\sum_{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}}e^{-\frac{5}{6}c2^{h_{v_0}}\delta(T_{v_0})}\Bigg(\prod_{v\in\bar{S}_{v_0}}e^{\frac{c}{6}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v,\underline{\mathbf{x}}_v)}|K_v(\underline{f}_v,\underline{e}_v)|\Bigg)\cdot\\ &\cdot\Bigg(\prod_{v\in S^B_{v_0}}e^{\frac{c}{6}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\Bigg)\Bigg(\prod_{v\in S^E_{v_0}}e^{\frac{c}{6}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\Bigg)\Bigg\}\,.\end{aligned} \tag{5.56}$$

To bound the sum over the Grassmann positions in the curly brackets, we proceed to chose the field labels $\tilde{p}$ as illustrated after (5.34) and after (5.42). Now $p_0 \in P_{w_0}$, $w_0 \in \bar{S}_{v_0}$ and each field label in $\tilde{p}_{w_1},\dots,\tilde{p}_{w_{|T_{v_0}|}}$ can be associated either to a black, or a white or a gray vertex, i.e. $w_i \in S^B_{v_0}$ or $w_i \in S^E_{v_0}$ or $w_i \in \bar{S}_{v_0}$, $i=1,\dots,|T_{v_0}|$. Note that $S_{v_0}=S^B_{v_0}\cup S^E_{v_0}\cup\bar{S}_{v_0}$ and that $S^B_{v_0}\cap S^E_{v_0}\cap\bar{S}_{v_0}=\emptyset$, so that $|T_{v_0}|=|S^B_{v_0}|+|S^E_{v_0}|+\bar{s}_v-1$ ($\bar{s}_v$ was introduced after (5.52)). Then we get the following bound for the expression in the curly brackets of (5.56):

$$\begin{aligned}&\Big(\sum_{\mathbf{y}\in\mathbb{H}}e^{-\frac{5}{6}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{|S^B_{v_0}|}\Big(\sum_{\substack{\mathbf{y}\in\mathbb{H}:\\ y^{(2)}\ \text{fixed}}}e^{-\frac{5}{6}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{|S^E_{v_0}|}\cdot\Big(\sum_{\mathbf{y}\in\mathbb{H}}e^{-\frac{5}{6}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{\bar{s}_v-1}\cdot\\ &\cdot\Bigg(\prod_{v\in S^B_{v_0}}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ \mathbf{y}(\tilde{p}_v)\text{fixed}}}e^{\frac{c}{6}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\Bigg)\Bigg(\prod_{v\in S^E_{v_0}}\sum_{\substack{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}:\\ y^{(1)}_{\tilde{p}_v}\text{fixed}}}e^{\frac{c}{6}2^{h_{v_0}}\delta_E(\underline{\mathbf{y}}_v)}|K_v(\underline{f}_v)|\Bigg)\cdot\\ &\Bigg(\prod_{v\in\bar{S}_{v_0}}\sum_{\mathbf{y}:P_{S_{v_0}}\to\mathbb{H}}e^{\frac{c}{6}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v,\underline{\mathbf{x}}_v)}|K_v(\underline{f}_v,\underline{e}_v)|\Bigg)\,.\end{aligned} \tag{5.57}$$

The first line can be bounded as $(C2^{-2h_{v_0}})^{|S^B_{v_0}|}(C2^{-h_{v_0}})^{(|S^E_{v_0}|)}$, because, for the purpose of an upper bound, we replace $\Big(\sum_{\mathbf{y}\in\mathbb{H}}e^{-\frac{5}{6}c2^{h_{v_0}}||\mathbf{y}||_1}\Big)^{\bar{s}_v-1}$ by the factor 1, i.e. we separate the spanning tree T_{v_0} into $T_{v_1},\dots,T_{v_{\bar{s}_v}}$, and each T_{v_i} contains exactly one vertex v with

$n_v^\sigma > 0$. Moreover $(C2^{-2h_{v_0}})^{|S_{v_0}^B|}(C2^{-h_{v_0}})^{(|S_{v_0}^E|)}$ can be rewritten as

$$C^{s_{v_0}} 2^{-2h_{v_0}(s_{v_0}-\bar{s}_v)} 2^{h_{v_0}|S_{v_0}^E|} \,. \tag{5.58}$$

In the second line of (5.57) we recognize the bulk and edge norms, namely $||K_v||^B_{(\frac{c}{6}2^{h_{v_0}})}$ and $||K_v||^E_{(\frac{c}{6}2^{h_{v_0}})}$, which can be bounded respectively as in (5.14)-(5.18) and in (5.45)-(5.50). Finally, in the last line of (5.57), we recognize in the definition of source norm in (4.104):

$$\sum_{\mathbf{x}:\cup_{v\in S_{v_0}} P_v \to \mathbb{H}} e^{\frac{c}{6}2^{h_{v_0}}\delta(\underline{\mathbf{y}}_v,\underline{\mathbf{x}}_v)} |K_v(\underline{f}_v,\underline{e}_v)| = ||K_v(\underline{f}_v,\underline{e}_v)||^\sigma_{(\frac{c}{6}2^{h_{v_0}})} \,, \tag{5.59}$$

which can be bounded as follows:

- if $h_{v_0} = 0$, $h_v = 1$ and v is an endpoint ▲ , K_v is given by the kernel in the last line of (4.148), and we get

$$||\mathcal{K}(B_\sigma^{(0)})(\underline{f}_v,\underline{e}_v)||^\sigma_{(\frac{c}{6}2^{h_{v_0}})} \le |||\mathcal{K}(B_\sigma^{(0)})(\underline{f}_v,\underline{e}_v)||^\sigma_{(\frac{c}{6})} \le C|s_0^M| \,, \tag{5.60}$$

  where $s_0^M \equiv |\tilde{Z}_0|$;

- if $h_{v_0} < 0$ and v is an endpoint, K_v is given by one of the kernels of ▲ on $h_v = h_{v_0}+1$ in (4.148) and we get

$$\begin{aligned} ||\mathcal{K}(B_{\underline{s}}^{(h_{v_0})})(\underline{f}_v,\underline{e}_v)||^\sigma_{(\frac{c}{6}2^{h_{v_0}})} &\le C \begin{cases} |\tilde{Z}^\sigma_{h_{v_0}}| & \text{if } D_v = 0 \\ \max\{|\tilde{Z}^1_{h_{v_0}}|, |\tilde{Z}^2_{h_{v_0}}|\} & \text{if } D_v = 1 \end{cases} \\ &\le C' 2^{(h_v-1)(2-\frac{n_v+n_v^\sigma}{2}-D_v)} 2^{-(h_v-1)(2-\frac{n_v+n_v^\sigma}{2}-D_v)} s^M_{h_v-1} \,, \end{aligned} \tag{5.61}$$

  with $s^M_{h_v-1} := \max\{|\tilde{Z}_{h_v-1}|, |\tilde{Z}^1_{h_v-1}|, |\tilde{Z}^2_{h_v-1}|\}$, and note that in the last inequality we used the fact that $n_v + n_v^\sigma = 2$ and $h_{v_0} = h_v - 1$;

- finally if $h_v < 0$ and v is not an endpoint, K_v is given by the kernel of the vertex ● in (4.150) and we get

$$\begin{aligned} &|| \,\mathcal{R}W_\sigma[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v,\underline{e}_v)\Big|_{\underline{\mathbf{d}}_v} ||^\sigma_{(\frac{c}{6}2^{h_{v_0}})} \le \\ &\qquad \le 2^{-h_v R_v} ||\mathcal{R}W_\sigma[\tau_v; \underline{P}_v, \underline{T}_v, \underline{\tilde{D}}_v](\underline{f}_v,\underline{e}_v)||^\sigma_{(\frac{c}{6}2^{h_v})} \,. \end{aligned} \tag{5.62}$$

By iterating (5.62), until the endpoints are reached, we can use the bounds in (5.60)-(5.61), as well as the results mentioned after (5.57), so that we can proceed as in proves of Lemma 5.1 and of Lemma 5.2 (in particular by using the results in (5.21)-(5.26) and in (5.28) and the definition of scaling dimension after (5.1)), so we get (5.52). □

5.3 The Beta functions

To prove Prop. 5.1 we are left with derive the bounds on v_h^M, $\tilde{v}_h^M(y^{(2)})$ and s_h^M in the r.h.s. of (5.52). The proof of the bounds on v_h^M and $\tilde{v}_h^M(y^{(2)})$, which are related respectively to the endpoints ♦ and ◊ , was already derived in [4], so here we illustrate only how to derive the bounds on s_h^M.

We start by recalling the definition of $\underline{s}$, which is such that the local part of the effective spin source in (4.130) is given by the sum of the relevant term in (4.131) and the marginal term in (4.133): the GN tree expansion for the kernels of (4.130) imply the following equations:

$$\sum_{\tau\in\mathcal{T}_{n,1}^{(-1)}} \mathcal{L}W[\underline{s},\underline{v};\tau] =\tilde{Z}_{-1} \sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(-1)}_{(1,0),1}(\underline{f};\underline{e})\,, \tag{5.63}$$

and, for all $h \le -2$,

$$\sum_{\tau\in\mathcal{T}_{n,1}^{(h)}} \mathcal{L}W[\underline{s},\underline{v};\tau] = \tilde{Z}_h \sum_{\underline{f}\in M}\sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,0),1}(\underline{f};\underline{e}) + \tilde{Z}_{1;h} \sum_{\substack{\underline{f}\in M:\\ |d_1^{(1)}|+|d_2^{(1)}|=1}} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,1),1}(\underline{f};\underline{e}) + \tilde{Z}_{2;h} \sum_{\substack{\underline{f}\in M:\\ |d_1^{(2)}|+|d_2^{(2)}|=1}} \sum_{\underline{e}\in M_S} \mathcal{L}_\sigma W^{(h)}_{(1,1),1}(\underline{f};\underline{e})\,, \tag{5.64}$$

where in the left sides of (5.63) and of (5.64) the sums are only over the trees with exactly one spin source endpoint and $n \ge 0$ Grassmann field endpoints. Note that in (5.63) we have only the kernel associated with the relevant contribution while in (5.64) we have also the marginal contributions. The left sides of these equations can be naturally thought of as functions of $\underline{s}$ or, better, of the restriction of $\underline{s}$ to the scales larger than h. Therefore, these are recursive equations for the components of $\underline{s}$: given $\tilde{Z}_0$, $\tilde{Z}_{1;-1}, \tilde{Z}_{2;-1}$, one can in principle construct the whole sequence $\underline{s}$. The same discussion can be done for the sequence $\underline{v}$, defined by (4.128) and (4.129): the beta function equations and the bounds were derived in [4, Prop.(4.11-4.12], here we just present the bound we are interested in: for a particular choice of the counterterms ν_1, ζ_1, η_1 introduced in (4.117), for $\theta \in (0, 1)$ and $C_\theta > 0$ holds

$$|v_h^M| \le C_\theta|\lambda|2^{\theta h}\,, \tag{5.65}$$

and an analogous bound holds for $\tilde{v}^M_{h_v-1}(y^{(2)})$ (see [4, Eq.(4.1.9)]).

To derive the same bound on s_h^M, we will use the fact that, by the definitions of the tree expansion (in particular the definition of 'allowed' derivatives before Eq. (4.144)), the sum over the GN trees in (5.64) is absolutely convergent for all $h \le 0$. Moreover, the constants on the first scale $\tilde{Z}_0, \tilde{Z}_{1;-1}$ and $\tilde{Z}_{2;-1}$ are real analytic in λ and bounded as $|\tilde{Z}_0| \le C_0$, $|\tilde{Z}_{j;-1}| \le C_0$, with $j = 1, 2$, for all $|\lambda| \le \lambda_0$ and some $C_0 > 0$. Then, we distinguish the trees involved in the sums in the left sides of (5.63) and of (5.64): if $h < -1$ the contribution from the trees in $\mathcal{T}_{n,1}^{(h)}$ equals

- $\tilde{Z}_{h+1}, \tilde{Z}_{1;h+1}, \tilde{Z}_{2;h+1}$, if $\tau \in \mathcal{T}^{(h)}_{0,1}$, i.e. if τ is the tree with exactly one endpoint ▲ on scale $h_v = h+2$;
- $\beta^{\sigma}_{h+1}[\underline{s},\underline{v}], \beta^{\sigma}_{1;h+1}[\underline{s},\underline{v}], \beta^{\sigma}_{2;h+1}[\underline{s},\underline{v}]$, if $\tau \in \mathcal{T}^{(h)}_{n\geq 1,1}$, i.e. if τ is the tree with one endpoint ▲ and at least one Grassmann endpoint of type ♦ , ◊ , ● or ○ .

Therefore, for all $h < -2$, we can write the recursive equations

$$\tilde{Z}_{h+1} + \beta^{\sigma}_{h+1}[\underline{s},\underline{v}] = \tilde{Z}_h\,, \qquad \tilde{Z}_{j;h+1} + \beta^{\sigma}_{j;h+1}[\underline{s},\underline{v}] = \tilde{Z}_{j;h}\,, \tag{5.66}$$

for $j = 1, 2$. Then, by using (5.65), we get that, for $\theta \in (0,1)$ and for all $h \leq -1$

$$|\beta^{\sigma}_h[\underline{s},\underline{v}]| \leq C_\theta |\lambda| 2^{\theta h} \max_{h'\geq h}\{|\tilde{Z}_{h'}|\}\,, \qquad |\beta^{\sigma}_{j;h}[\underline{s},\underline{v}]| \leq C_\theta |\lambda| 2^{\theta h} \max_{h'\geq h}\{|\tilde{Z}_{j;h'}|\}\,, \tag{5.67}$$

with $j = 1, 2$. If $h = 0$, we get $\tilde{Z}_0 = 1$, $|\beta^{\sigma}_0[\underline{s},\underline{v}]| \leq C|\lambda|$ and $\tilde{Z}_{j;0} = 0$ for $j = 1, 2$ (as we can easily check by (4.123)). Eqs. (5.66) and (5.67) immediately imply that $\{\,\tilde{Z}_h\,\}_{h\leq -1}$ and $\{\,\tilde{Z}_{j;h}\,\}_{h\leq -2}$ with $j = 1, 2$, are Cauchy sequences, whose elements are real analytic in λ for $|\lambda| \leq \lambda_0$. Then we let

$$\tilde{Z}_{-\infty} = \tilde{Z}_{-\infty}(\lambda) := \lim_{h\to-\infty} \tilde{Z}_h\,, \qquad \tilde{Z}_{j;-\infty} = \tilde{Z}_{j;-\infty}(\lambda) := \lim_{h\to-\infty} \tilde{Z}_{j;h}\,, j = 1, 2\,, \tag{5.68}$$

which are real analytic in λ, and we get

$$|\tilde{Z}_h - \tilde{Z}_{-\infty}| \leq C|\lambda| 2^{\theta h}\,, \qquad |\tilde{Z}_{j;h} - \tilde{Z}_{j;-\infty}| \leq C|\lambda| 2^{\theta h}\,,\ j = 1, 2\,, \tag{5.69}$$

with (say) $\theta = 3/4$. Note that if $h = 0$ in (5.69) we get $|\tilde{Z}_{-\infty}| \leq 1 + C|\lambda|$ and $|\tilde{Z}_{j;-\infty}| \leq C|\lambda|$; moreover combining the bounds obtained above and recalling the definition $s^M_h := \max\{\,\tilde{Z}_h, \tilde{Z}_{1;h}, \tilde{Z}_{2;h}\,\}$ we get that

$$s^M_h \leq C\,. \tag{5.70}$$

To conclude the proof of (5.1), in the product over the endpoints of (5.52) we can use the bound (5.70) for each $s^M_{h_v-1}$ and the bound (5.65) for each $v^M_{h_v-1}$ and $\tilde{v}^M_{h_v-1}(y^{(2)})$.

Chapter 6

The two-point edge spin correlation

In this chapter we derive the explicit expression of the two-point edge spin correlation, proving the main result in Thm. 1.1. In particular, we will see that the explicit expression of the dominant contribution of $\langle \tilde{\sigma}_{\mathbf{x}_1} \tilde{\sigma}_{\mathbf{x}_2} \rangle_{\beta_c(\lambda)}$ can be derived by the propagator of the massless fields $\varphi^-_{\mathbf{x}_1}$ and $\varphi^-_{\mathbf{x}_2}$, which are exactly the massless fields that do not vanish at the boundary of the half-plane (see (3.47)). We will derive also the correction term of $\langle \tilde{\sigma}_{\mathbf{x}_1} \tilde{\sigma}_{\mathbf{x}_2} \rangle_{\beta_c(\lambda)}$ by the tree expansion for the kernels of the correlation function: it will be very similar to the tree expansion of the effective potential introduced in Chap. 4 and will admit bound analogous to the ones for the tree kernels of the effective potential of Chap. 5.

Given any ordered pair $(\mathbf{x}_1, \mathbf{x}_2)$ of distinct points at the edge of the half-plane, the two-point edge spin correlation is given by the following derivatives

$$\langle \tilde{\sigma}_{\mathbf{x}_1} \tilde{\sigma}_{\mathbf{x}_2} \rangle_{\beta_c(\lambda)} = \frac{\partial^2}{\partial \Psi_{\mathbf{x}_1} \partial \Psi_{\mathbf{x}_2}} \log \Xi_\lambda(\mathbf{\Psi}) \Big|_{\mathbf{\Psi}=\mathbf{0}} , \tag{6.1}$$

as derived in Lemma 2.3. Moreover, by recalling the definition of the perturbed generating function in (3.81) with $\mathbf{A} = \tilde{\mathbf{A}} = \mathbf{0}$, namely

$$\Xi_\lambda(\mathbf{\Psi}) = C \int P_c(d\varphi) P_c(d\xi) \; e^{S_{\underline{v}}(\Phi) + V(\Phi) + \Psi_{\mathbf{x}_1} \varphi_{-,\mathbf{x}_1} + \cdots + \Psi_{\mathbf{x}_{n_\sigma}} \varphi_{-,\mathbf{x}_{n_\sigma}}} , \tag{6.2}$$

we can easily check that (6.1) implies the following identity

$$\langle \tilde{\sigma}_{\mathbf{x}_1} \tilde{\sigma}_{\mathbf{x}_2} \rangle_{\beta_c(\lambda)} = \langle \varphi_{-,\mathbf{x}_1} \varphi_{-,\mathbf{x}_2} \rangle_{\beta_c(\lambda)} , \tag{6.3}$$

where the $\langle \cdot \rangle_{\beta_c(\lambda)}$ is with respect the Grassmann measure in (3.72).
Note that in (6.2) we wrote the spin source $B_\sigma(\varphi; \mathbf{\Psi})$, defined in (3.76), as $\Psi_{\mathbf{x}_1} \varphi_{-,\mathbf{x}_1} + \cdots + \Psi_{\mathbf{x}_{n_\sigma}} \varphi_{-,\mathbf{x}_{n_\sigma}}$, with $\mathbf{x}_1, \ldots, \mathbf{x}_{n_\sigma} \in \partial\mathbb{H}$ and $n_\sigma \in 2\mathbb{N}$, since in (6.1) we are interested

in some specific derivatives at $\mathbf{\Psi} = \mathbf{0}$.

In the following Prop. 6.1, we will derive the explicit expression of $\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle_{\beta_c(\lambda)}$ and decompose it into a dominant and subdominant contributions. These contributions, by using (6.3), will be related to the two-points edge spin correlation of Thm. 1.1.
For the correlations in the r.h.s. of (6.3) we can state the following result:

Proposition 6.1 (Correlations of the Grassmann fields at the boundary)**.** *Given any ordered pair* $(\mathbf{x}_1, \mathbf{x}_2)$ *of distinct points* $\mathbf{x}_1, \mathbf{x}_2 \in \partial\mathbb{H}$, *we can write the perturbed correlation* $\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle_{\beta_c(\lambda)}$ *as*

$$\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle_{\beta_c(\lambda)} = (\tilde{Z}_{-\infty})^2 \langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle^0_{\beta_c(0)} + R_\sigma(\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.4}$$

where $\tilde{Z}_{-\infty}\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle^0_{\beta_c(0)}$ *is the dominant contribution and* $R_\sigma(\mathbf{x}_1, \mathbf{x}_2)$ *is the subdominant contribution. Moreover,* $\tilde{Z}_{-\infty}$ *is defined in* (5.68)*, the two-point unperturbed correlation is given by* $\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle^0_{\beta_c(0)} := \sum_{h_\sigma=h^*-1}^{0} g^{(h_\sigma)}_{--}(\mathbf{x}_1, \mathbf{x}_2)$, *where* $g^{(h_\sigma)}_{--}(\mathbf{x}_1, \mathbf{x}_2)$ *is the single scale propagator in* (3.56)*, and the subdominant contribution is given by* $R_\sigma(\mathbf{x}_1, \mathbf{x}_2) := \sum_{h_\sigma=h^*-1}^{0} R^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$*, where the single scale correction* $|R^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)|$ *can be bounded as*

$$|R^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)| \le C|\lambda| 2^{h_\sigma(1+\theta)} e^{-c2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)}\,, \tag{6.5}$$

where $\delta(\mathbf{x}_1, \mathbf{x}_2) := |\mathbf{x}_1 - \mathbf{x}_2|$.

Note that, since $g^{(h_\sigma)}_{--}(\mathbf{x}_1, \mathbf{x}_2)$ admits the following bound

$$|g^{(h_\sigma)}_{--}(\mathbf{x}_1, \mathbf{x}_2)| \le C 2^{h_\sigma} e^{-c2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)}\,, \tag{6.6}$$

and a similar matching lower bound could be obtained along the lines of Subsections 3.2.2 and 3.2.3 (and Appendix C). Note that in the bound (6.5) there is a factor $2^{h_\sigma(1+\theta)}$ and in the bound (6.6) there is a factor 2^{h_σ}: in the following, by comparing the asymptotic behavior of the sum over the scales, it will be clear why $R_\sigma(\mathbf{x}_1, \mathbf{x}_2)$ is called "subdominant contribution".

The result in Prop. 6.1 for the two points Grassmann correlations at the boundary is similar to the one obtained for the two points Grassmann correlations on the infinite plane in [9] e reviewed in [34, Chap.12], where is also studied the emergence of new critical indices in the Schwinger functions. The original contribution of this chapter is to include the edge contributions, which are originated from the presence of a boundary and require to be treated with the methods introduced in [4] for the energy correlations.

Moreover, with the results in Prop. 6.1 we are able to prove the main result in Thm. 1.1: first of all, by using the correspondence in (6.3), we can rewrite (6.4) as

$$\langle \tilde{\sigma}_{\mathbf{x}_1}\tilde{\sigma}_{\mathbf{x}_2}\rangle_{\beta_c(\lambda)} = (\tilde{Z}_{-\infty})^2 \langle \tilde{\sigma}_{\mathbf{x}_1}\tilde{\sigma}_{\mathbf{x}_2}\rangle^0_{\beta_c(0)} + R_\sigma(\mathbf{x}_1, \mathbf{x}_2)\,. \tag{6.7}$$

Then, we can bound $R_\sigma(\mathbf{x}_1, \mathbf{x}_2)$ as

$$|R_\sigma(\mathbf{x}_1, \mathbf{x}_2)| \leq C_\theta \left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \right)^{(1+\theta)} , \tag{6.8}$$

where to bound the sum over h_σ of the single scale correction in (6.5), we used the fact that, for any $\alpha, \delta > 0$,

$$\sum_{h_\sigma \in \mathbb{Z}} 2^{\alpha h_\sigma} e^{-2^{h_\sigma}\delta} \leq 2^\alpha \int_{-\infty}^{\infty} dt\, 2^{\alpha t} e^{-2^t \delta} = \frac{2^\alpha \Gamma(\alpha)}{\log 2} \left(\frac{1}{\delta} \right)^\alpha . \tag{6.9}$$

Finally, in order to obtain the same result of Eq. (1.8) we chose $\tilde{Z}_\sigma(\lambda) = \frac{\tilde{Z}_{-\infty}(\lambda)}{Z}$ ($\tilde{Z}_{-\infty}(\lambda)$ was defined in (5.68) and Z was introduced after (3.71)) and we write out the dependence from the lattice spacing a: we properly rescale the distances by a and the edge spin observables by $a^{-1/2}$ (see (2.50)).
To prove Prop. 6.1 we do the following:

- in Sec. 6.1, we introduce the tree expansion for $\log \Xi_\lambda(\mathbf{\Psi})$ in (6.1), which is similar to the one of the effective potential illustrated in 4.3.2, and by using the derivatives with respect the spin sources $\Psi_{\mathbf{x}_1}$ and $\Psi_{\mathbf{x}_2}$ we identify in the expansion the tree kernels we are interested in;
- in Sec. 6.2, we focus on the contributions associated with the trees with exactly two spin source endpoints: we extract the dominant term and we use the estimates in Prop. 3.1 for the propagators and in (5.69) for the coupling constants to bound the subdominant terms;
- in Sec. 6.3 we consider the contributions associated with the trees with also the Grassmann endpoints: they will be related to the subdominant terms and will be bounded by using the results derived in Prop. 5.1 for the effective potential.

6.1 The tree expansion

By using the multiscale procedure introduced in Sec. 4.1 for the effective potential, we can write $\log \Xi_\lambda(\mathbf{\Psi})$ in the r.h.s. of (6.1) as

$$\log \Xi_\lambda(\mathbf{\Psi}) = \sum_{h_\sigma = h^*-1}^{0} \sum_{n_\sigma \geq 1} S_{n_\sigma}^{(h_\sigma)}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_\sigma}) \prod_{i=1}^{n_\sigma} \Psi_{\mathbf{x}_i} , \tag{6.10}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_{n_\sigma} \in \partial\mathbb{H}$. By deriving (6.10) with respect to the spin source fields $\Psi_{\mathbf{x}_1}$ and $\Psi_{\mathbf{x}_2}$ we get

$$\langle \varphi_{-,\mathbf{x}_1} \varphi_{-,\mathbf{x}_2} \rangle_{\beta_c(\lambda)} = \frac{\partial^2}{\partial \Psi_{\mathbf{x}_1} \partial \Psi_{\mathbf{x}_2}} \log \Xi_\lambda(\mathbf{\Psi}) \Big|_{\Psi_{\mathbf{x}_1} = \Psi_{\mathbf{x}_2} = 0} = \sum_{h_\sigma = h^*-1}^{0} S_2^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) , \tag{6.11}$$

where $S_2^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$ is the single scale kernel of the two point correlations (and it is similar to the one of the effective potential as in(4.31)).Now we can use for the kernels $S_2^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$ a tree expansion analogue to the one in (4.141), so we get

$$S_2^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{h=h^*-1}^{h_\sigma-1} \sum_{n\geq 0} \sum_{\tau\in\mathcal{T}_{n,2}^{(h),(h_\sigma)}} S_{h_\sigma,h}[\tau](\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.12}$$

where the sum is over $\tau \in \mathcal{T}_{n,2}^{(h),(h_\sigma)}$, which are the trees with

- $n+2$ endpoints:
  - 2 spin source endpoints: $\bar{v}_1$ on scale h_1 and $\bar{v}_2$ on scale h_2, with $h_1, h_2 < 2$;
  - n Grassmann endpoints: $v_1, \dots, v_n$ on scales $h_{v_i} \leq 2$;
- the root v_0 on scale $h+1$, which has $n_{v_0}^\sigma = 2$ and $n_{v_0} = 0$, i.e. in the root there are not external Grassmann fields;
- the vertex $\bar{v}_\sigma$ on scale h_σ, which is the only gray branching vertex, i.e. it is the only gray vertex followed by 2 gray vertices ($\bar{s}_{\bar{v}_\sigma} = n_\sigma = 2$).

Moreover, each spin source endpoint $\bar{v}_i$, $i = 1, 2$, is related to $\mathcal{K}(B_{\underline{s}}^{(h_i-1)})$, which is either one of the kernels in (4.131)-(4.133) if $h_i < 1$, or it is the kernel of $B_\sigma^{(0)}$ in (4.123) if $h_i = 1$. By proceeding as in (4.145), we can write the kernel in (6.12) as

$$S_{h_\sigma,h}[\tau](\mathbf{x}_1, \mathbf{x}_2) = \sum_{\underline{P}\in\mathcal{P}(\tau)} \sum_{\underline{T}\in\mathcal{S}(\tau;\underline{P})} \sum_{\underline{D}\in\mathcal{D}(\tau;\underline{P})} S_{h_\sigma,h}[\tau; \underline{P}, \underline{T}, \underline{D}](\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.13}$$

with the tree kernel $S_{h_\sigma,h}[\tau; \underline{P}, \underline{T}, \underline{D}](\mathbf{x}_1, \mathbf{x}_2)$ defined as in (4.146) with $\underline{f}_0 = \underline{0}$. By using (6.12) and (6.13) in (6.11), we can identify the dominant and the subdominant contributions. As already mentioned, the dominant contribution will be related to the trees with only $\bar{v}_1$ and $\bar{v}_2$ endpoints (so that $n = 0$). In particular, to obtain the dominant contribution $\bar{v}_1$ and $\bar{v}_2$ have to be both associated with the kernel in (4.131) or to the kernel in (4.123). Then, there will be subdominant contributions related to trees with $n = 0$ and at least one spin source endpoint associated with a kernel in (4.133); other subdominant contributions will be related to trees with 2 source endpoints and $n \geq 1$ Grassmann endpoints.

6.2 Contributions from the trees with $n = 0$

We start by considering the sum over n in (6.12): if $n = 0$ we get a sum over the trees τ_2 in $\mathcal{T}_{0,2}^{(h),(h_\sigma)}$, where we let τ_2 be a generic tree in $\mathcal{T}_{0,2}^{(h),(h_\sigma)}$. These trees can be represented as in Fig. 6.1: as already mentioned, if $h_e = 1$ both the ▲ endpoints are associated with the kernel in (4.123) and there is a unique $\tau_2 \in \mathcal{T}_{0,2}^{(-1),(0)}$; if $h_e < 1$ each ▲ can be

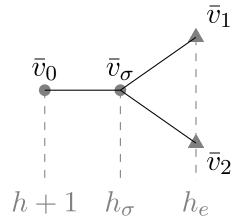

Figure 6.1: A generic tree $\tau_2 \in \mathcal{T}_{0,2}^{(h),(h_\sigma)}$: the spin source endpoints are on scale $h_1 = h_2 =: h_e$.

associated with one of the three kernels in (4.131)-(4.133), so there are 9 different trees $\tau_2 \in \mathcal{T}_{0,2}^{(h),(h_\sigma)}$, with $h < -1$. Then, we can rewrite (6.12) as

$$S_2^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{h=h^*-1}^{h_\sigma - 1} S_{h_\sigma,h}[\tau_2](\mathbf{x}_1, \mathbf{x}_2) + R_{n>0}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.14}$$

where

$$R_{n>0}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) := \sum_{h=h^*-1}^{h_\sigma-1} \sum_{n\geq 1} \sum_{\tau \in \mathcal{T}_{n,2}^{(h),(h_\sigma)}} S_{h_\sigma,h}[\tau](\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.15}$$

contains all the trees with $n \geq 1$ Grassmann endpoints, which will contribute to the subdominant term in (6.5). Moreover, $S_{h_\sigma,h}[\tau_2](\mathbf{x}_1, \mathbf{x}_2)$ in the r.h.s. of (6.14), depending on the kernels $\blacktriangle$ is associated with, can be decomposed as

$$S_{h_\sigma,h}[\tau_2](\mathbf{x}_1, \mathbf{x}_2) = (\tilde{Z}_{h_\sigma})^2 g_{--}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) + R_\partial^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)\,, \tag{6.16}$$

where

- $g_{--}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$ is the single scale propagator defined in (3.56), and it is related to the tree τ_2 with both endpoints $\blacktriangle$ associated with the kernel (4.131) if $h_e < 1$ or to the kernel (4.123) if $h_e = 1$;
- $R_\partial^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$ is related to the trees τ_2 whose endpoints $\blacktriangle$ are associated with one of the kernels in (4.133).

If $h_e = 1$ (and then $h_\sigma = 0$), $R_\partial^{(0)}(\mathbf{x}_1, \mathbf{x}_2) = 0$; if $h_e < 1$, $R_\partial^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)$ is given by

$$\begin{aligned} R_\partial^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) :=& 2\sum_{i=1}^{2} \tilde{Z}_{h_\sigma} \tilde{Z}_{i;h_\sigma} \sum_{\mathbf{d}_{\mathbf{x}_i}:|\mathbf{d}_{\mathbf{x}_i}|=1} \partial^{\mathbf{d}_{\mathbf{x}_i}} g_{--}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2) + \\ &+ \sum_{i,j=1}^{2} \tilde{Z}_{i;h_\sigma} \tilde{Z}_{j;h_\sigma} \sum_{\mathbf{d}_{\mathbf{x}_i},\mathbf{d}_{\mathbf{x}_j}:|\mathbf{d}_{\mathbf{x}_i}|+|\mathbf{d}_{\mathbf{x}_j}|=2} \partial^{\mathbf{d}_{\mathbf{x}_i}} \partial^{\mathbf{d}_{\mathbf{x}_j}} g_{--}^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)\,, \end{aligned} \tag{6.17}$$

which can be bounded as

$$|R_\partial^{(h_\sigma)}(\mathbf{x}_1, \mathbf{x}_2)| \leq C 2^{2h_\sigma} e^{-c 2^{h_\sigma} \delta(\mathbf{x}_1, \mathbf{x}_2)}\,, \tag{6.18}$$

where we used the bound on the derivative of the propagator given in (3.64).
Next, we rewrite the first term in the r.h.s. of (6.16) as

$$(\tilde{Z}_{h_\sigma})^2 g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) = (\tilde{Z}_{-\infty})^2 g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) + R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.19}$$

with

$$R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) := \{(\tilde{Z}_{h_\sigma})^2 - (\tilde{Z}_{-\infty})^2\} g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,. \tag{6.20}$$

By using the bound in (5.69), $R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ can be bounded bounded as

$$|R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)| \le C|\lambda| 2^{(\theta+1)h_\sigma} e^{-c2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)}\,. \tag{6.21}$$

Plugging (6.19) in (6.16) we get

$$S_{h_\sigma,h}[\tau_2](\mathbf{x}_1,\mathbf{x}_2) = (\tilde{Z}_{-\infty})^2 g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) + R_{n=0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.22}$$

where

$$R_{n=0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) := R_\partial^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) + R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.23}$$

with $R_\partial^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ defined in (6.17) and $R_Z^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ defined in (6.20).
We can easily see that, with respect to the bound for the propagator $g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ given in (6.6), the bounds in (6.18) and in (6.21) are subdominants, then we can say that $R_{n=0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ is a subdominant contribution. Moreover, by using (6.22) in (6.14), we can identify the dominant contribution as the one given by

$$\langle \varphi_{-,\mathbf{x}_1}\varphi_{-,\mathbf{x}_2}\rangle^0 = \sum_{h_\sigma=h^*-1}^{0} (\tilde{Z}_{-\infty})^2 g_{--}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.24}$$

which is exactly the dominant contribution in (6.4). Next, we define

$$R^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) := R_{n=0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) + R_{n>0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.25}$$

which is the subdominant contribution in (6.4). To prove (6.5), as we already proved that $R_{n=0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ is a subdominant contribution, we need to bound $R_{n>0}^{(h_\sigma)}$.

6.3 The contribution from the trees with $n > 0$

We recall the definition in (6.15), namely

$$R_{n>0}^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2) := \sum_{h=h^*-1}^{h_\sigma-1} \sum_{n\ge1} \sum_{\tau\in\mathcal{T}_{n,2}^{(h),(h_\sigma)}} S_{h_\sigma,h}[\tau](\mathbf{x}_1,\mathbf{x}_2)\,, \tag{6.26}$$

where the sum $\tau \in \mathcal{T}_{n,2}^{(h),(h_\sigma)}$ is over the trees with 2 endpoints ▲ and $n \ge 1$ endpoints of type ♦ , ◊ , ● , ○ , e.g. see the tree in Fig. 6.2 with $n = 5$.

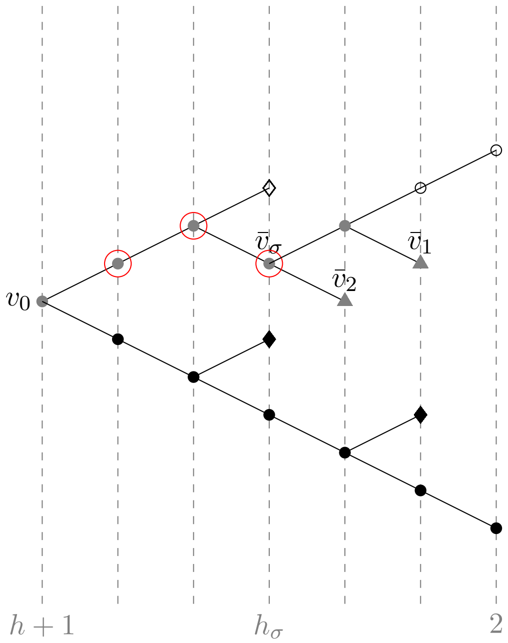

Figure 6.2: An example of $\tau \in \mathcal{T}^{(h),(h_\sigma)}_{n>0,2}$ with $n = 5$ Grassmann endpoints. The gray vertices in the red circles have $n^\sigma_v = 2$.

The kernel $S_{h_\sigma,h}[\tau](\mathbf{x}_1,\mathbf{x}_2)$ in (6.26) can be bounded as

$$|S_{h_\sigma,h}[\tau](\mathbf{x}_1,\mathbf{x}_2)| \le e^{-\frac{c}{8}2^h\delta(\mathbf{x}_1,\mathbf{x}_2)} \sum_{\substack{\underline{P}\in\mathcal{P}(\tau):\\ n_{v_0}=0}} \sum_{\underline{T}\in\mathcal{S}(\tau;\underline{P})} \sum_{\substack{\underline{D}\in\mathcal{D}(\tau;\underline{P}):\\ D_{v_0}=0}} ||S_{h_\sigma,h}[\tau;\underline{P},\underline{T},\underline{D}](\mathbf{x}_1,\mathbf{x}_2)||^\sigma_{\frac{c}{8}2^h}\,, \tag{6.27}$$

with the source norm $||S_{h_\sigma,h}[\tau;\underline{P},\underline{T},\underline{D}](\mathbf{x}_1,\mathbf{x}_2)||^\sigma_{\frac{c}{8}2^h}$ that can be bounded as in (5.1) with: $n^\sigma_{v_0} = 2$, $n_{v_0} = 0$, $D_{v_0} = 0$, $d_S(v_0) = 1$ and $h_{v_0} = h+1$; moreover if $\bar{v}_i \in \bar{V}_e(\tau)$, $d_S(\bar{v}_i) = 1$, $h_{v_i} := h_i$, for $i = 1, 2$ and there is only one vertex in the product over $v \in \bar{V}(\tau) : \bar{s}_v > 1$, which is $\bar{v}_\sigma$ on scale h_σ, with $\bar{s}_{\bar{v}_\sigma} = 2$, namely

$$\begin{aligned} ||S_{h_\sigma,h}[\tau;\underline{P},\underline{T},\underline{D}](\mathbf{x}_1,\mathbf{x}_2)||^\sigma_{\frac{c}{8}2^h} \le C^{2+n^e_\tau}\frac{1}{s_{v_0}!}2^{h+1}2^{2h_\sigma}e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)}2^{-h_1-h_2}\cdot \\ \cdot\left(\prod_{v\in V(\tau)\setminus\{v_0\}}\frac{1}{s_v!}2^{(h_v-h_u)(d_S(v)-\delta_{v,1})}\right)\left(\prod_{v\in V^*_e(\tau)}|\lambda|^{\max\{1,\kappa n_v\}}2^{\theta h_v}\right)\,. \end{aligned} \tag{6.28}$$

Next, by using (6.28) and $2^{-h_1-h_2} \le 2^{-2h_\sigma}$ and $e^{-\frac{c}{8}2^h\delta(\mathbf{x}_1,\mathbf{x}_2)} < e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)}$ in (6.27), we get the following bound

$$
\begin{aligned}
|S_{h_\sigma,h}[\tau](\mathbf{x}_1,\mathbf{x}_2)| \le C e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)} 2^{h+1} \sum_{\substack{\underline{P}\in\mathcal{P}(\tau):\\ n_{v_0}=0}} \sum_{\underline{T}\in\mathcal{S}(\tau;\underline{P})} \sum_{\substack{\underline{D}\in\mathcal{D}(\tau;\underline{P}):\\ D_{v_0}=0}} \frac{C^{n_\tau^e}}{s_{v_0}!}\cdot \\
\cdot \left(\prod_{v\in V(\tau)\setminus\{v_0\}} \frac{1}{s_v!} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})} \right) \left(\prod_{v\in V_e^*(\tau)} |\lambda|^{\max\{1,\kappa n_v\}} 2^{\theta h_v} \right),
\end{aligned}
\tag{6.29}
$$

where we use C to denote a constant which may differs from the previous one. Now, we look at the first product in the second line of (6.29): by the definition of the allowed $\underline{D}$, for any v with $n_v^\sigma \le 1$ the factor $(d_S(v)-\delta_{v,1})$ is always negative, so that $2^{(h_v-h_u)(d_S(v)-\delta_{v,1})}$ is exponentially small in $h_v - h_u$. If v is such that $n_v^\sigma = 2$ (and then $\delta_{v,1}=0$) we have that $d_S(v)$ can vanish if $n_v = 2$ and $D_v = 0$. This is a consequence of the localization procedure described in Subsec. 4.2.2, where we introduced the $\mathcal{L}$ operators only for the kernels with $n_\sigma \in \{0,1\}$, and we have not treated the kernels with $(n,D), n_\sigma = (2,0), 2$. Note that to have $n_v^\sigma = 2$, v has to be a gray vertex on scale h_v such that $h+1 < h_v \le h_\sigma$: in fact, by definition, $\bar{v}_\sigma$, which is on scale h_σ, is the rightmost vertex with $n_v^\sigma = 2$ and v_0, which is on scale $h+1$, has $n_{v_0} = 0$. An example of such vertices are the ones in the red circles in Fig. 6.2. To associate an exponential decay with the vertices v such that $n_v^\sigma = 2$ we multiply $\left(\prod_{v\in V(\tau)\setminus\{v_0\}} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})}\right)$ in (6.29) by the factor $2^{\theta'(h-h_\sigma)}$ with $\theta' \in (\theta,1)$, obtaining a factor

$$
2^{\theta'(h_\sigma-h-1)} \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n_v^\sigma<2}} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})} \Big) \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n_v^\sigma=2}} 2^{(h_v-h_u)(d_S(v)-\theta')} \Big), \tag{6.30}
$$

so that if $d_S(v) = 0$ in the last product we get $2^{-(h_v-h_u)\theta'}$, which is exponentially small in $h_v - h_u$. We can compare the bound for $S_{h_\sigma,h}[\tau](\mathbf{x}_1,\mathbf{x}_2)$ with the bound for the effective potential with two external Grassmann fields. As an example, we can consider the bound for the kernel $||W[\tau;\underline{P},\underline{T},\underline{D}](\mathbf{y}_1,\mathbf{y}_2)||^B_{\frac{c}{8}2^h}$, where τ are the trees with two ♦ endpoints, which is given by (5.1) with $n_{v_0} = 2$, $n_{v_0}^\sigma = 0$ and $\bar{V}(\tau) = \emptyset$, namely

$$
\begin{aligned}
||W[\tau;\underline{P},\underline{T},\underline{D}](\mathbf{y}_1,\mathbf{y}_2)||^B_{\frac{c}{8}2^h} \le C^{n_\tau^e} \frac{1}{s_{v_0}!} 2^{h+1} \Big(\prod_{v\in V(\tau)\setminus\{v_0\}} \frac{1}{s_v!} 2^{(h_v-h_u)d_S(v)} \Big)\cdot \\
\cdot \Big(\prod_{v\in V_e^*(\tau)} |\lambda|^{\max\{1,\kappa n_v\}} 2^{\theta h_v} \Big).
\end{aligned}
\tag{6.31}
$$

With respect (6.31), in the bound in (6.28) there are the following additional factors: a factor 2^{2h_σ} for the non integration over $\mathbf{x}_1$ and $\mathbf{x}_2$, a factor 2^{-h_i} for each spin source endpoints (see (5.61)) and the factor $2^{\theta'(h_\sigma-h-1)}$ for the presence of the gray vertices v with $h+1 < h_v \le h_\sigma$.

Coming back to the bound of (6.26):

$$
\begin{aligned}
|R^{(h_\sigma)}_{n>0}(\mathbf{x}_1,\mathbf{x}_2)| \le\ & 2^{\theta' h_\sigma} e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)} \sum_{h=h^*-1}^{h_\sigma-1} 2^{(h+1)(1-\theta')} \sum_{n\ge1} \sum_{\tau\in\mathcal{T}^{(h),(h_\sigma)}_{n,2}} \sum_{\substack{\underline{P}\in\mathcal{P}(\tau):\\ n_{v_0}=0}} \sum_{\underline{T}\in\mathcal{S}(\tau;\underline{P})} \cdot \\
& \cdot \sum_{\substack{\underline{D}\in\mathcal{D}(\tau;\underline{P}):\\ D_{v_0}=0}} \frac{C^{n^e_\tau}}{s_{v_0}!} \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n^\sigma_v<2}} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})} \Big) \cdot \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n^\sigma_v=2}} 2^{(h_v-h_u)(d_S(v)-\theta')} \Big) \cdot \\
& \cdot \Big(\prod_{v\in V^*_e(\tau)} |\lambda|^{\max\{1,\kappa n_v\}} 2^{\theta h_v} \Big) ,
\end{aligned}
\tag{6.32}
$$

where we recall that $0<\theta<\theta'<1$. Next, we can bound the sum over the trees as

$$
\begin{aligned}
& \sum_{\tau\in\mathcal{T}^{(h),(h_\sigma)}_{n,2}} \sum_{\substack{\underline{P}\in\mathcal{P}(\tau):\\ n_{v_0}=0}} \sum_{\underline{T}\in\mathcal{S}(\tau;\underline{P})} \sum_{\substack{\underline{D}\in\mathcal{D}(\tau;\underline{P}):\\ D_{v_0}=0}} \frac{C^{n^e_\tau}}{s_{v_0}!} \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n^\sigma_v<2}} 2^{(h_v-h_u)(d_S(v)-\delta_{v,1})} \Big) \cdot \\
& \cdot \Big(\prod_{\substack{v\in V(\tau)\setminus\{v_0\}:\\ n^\sigma_v=2}} 2^{(h_v-h_u)(d_S(v)-\theta')} \Big) \Big(\prod_{v\in V^*_e(\tau)} |\lambda|^{\max\{1,\kappa n_v\}} 2^{\theta h_v} \Big) \le C^n_\theta |\lambda|^n 2^{\theta h} ,
\end{aligned}
\tag{6.33}
$$

where, as discussed after (6.30), the factors in the l.h.s. associated with v with $n^\sigma_v = 2$ now behave like the ones with $n^\sigma_v < 2$, so that we can use the proof in [4, Lemma 4.9] to prove (6.33).

Finally, by using (6.33) in (6.32), we get

$$
|R^{(h_\sigma)}_{n>0}(\mathbf{x}_1,\mathbf{x}_2)| \le 2^{\theta' h_\sigma} e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)} \sum_{h=h^*-1}^{h_\sigma-1} 2^{(h+1)(1-\theta'+\theta)} \sum_{n\ge1} C^n_\theta |\lambda|^n , \tag{6.34}
$$

whose sum over $n\ge1$ and over $h < h_\sigma - 1$ can be bounded as

$$
|R^{(h_\sigma)}_{n>0}(\mathbf{x}_1,\mathbf{x}_2)| \le C|\lambda| 2^{h_\sigma(\theta+1)} e^{-\frac{c}{24}2^{h_\sigma}\delta(\mathbf{x}_1,\mathbf{x}_2)} , \tag{6.35}
$$

which is a subdominant contribution with respect to the bound of (6.24). By recalling the definition of $R^{(h_\sigma)}(\mathbf{x}_1,\mathbf{x}_2)$ in (6.25): we can use the bounds in (6.18), in (6.21) and in (6.35) to obtain the bound in (6.5) and than to conclude the proof of Prop. 6.1.

Appendix A

The Kasteleyn matrix

We recall here the Kasteleyn matrix entries, first introduced in [51]: it is a well known result that we report here for convenience in order to easily compare the dimer representation on Λ with the one in Λ_n, the modified lattice in Proof of Lemma 2.2. First of all, we provide a labeling of the lattice sites by the number of matrix column $x^{(1)}$ and the number of matrix row $x^{(2)}$. Then we provide an orientation to the lattice bonds in order to obtain the clockwise odd orientation, i.e. each elementary lattice face, including the "external face" Λ^c, has a number of bonds oriented in the clock wise direction. Then, the $\mathcal{A}' = 6LM \times 6LM$ antisymmetric matrix has the following entries:

$$\mathcal{A}'_{\mathbf{x}\mathbf{x}} = \begin{array}{c} \\ \bar{H}_{\mathbf{x}} \\ H_{\mathbf{x}} \\ \bar{V}_{\mathbf{x}} \\ V_{\mathbf{x}} \\ \bar{T}_{\mathbf{x}} \\ T_{\mathbf{x}} \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{H}_{\mathbf{x}} & H_{\mathbf{x}} & \bar{V}_{\mathbf{x}} & V_{\mathbf{x}} & \bar{T}_{\mathbf{x}} & T_{\mathbf{x}} \end{array} \\ \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \end{array}, \qquad \text{(A.1)}$$

with $1 \le x^{(1)} \le L\,,\, 1 \le x^{(2)} \le M\,;$

$$\mathcal{A}'_{\mathbf{x}\mathbf{x}+\hat{\mathbf{e}}_1} = \begin{array}{c} \\ \bar{H}_{\mathbf{x}} \\ H_{\mathbf{x}} \\ \bar{V}_{\mathbf{x}} \\ V_{\mathbf{x}} \\ \bar{T}_{\mathbf{x}} \\ T_{\mathbf{x}} \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{H}_{\mathbf{x}+\hat{\mathbf{e}}_1} & H_{\mathbf{x}+\hat{\mathbf{e}}_1} & \bar{V}_{\mathbf{x}+\hat{\mathbf{e}}_1} & V_{\mathbf{x}+\hat{\mathbf{e}}_1} & \bar{T}_{\mathbf{x}+\hat{\mathbf{e}}_1} & T_{\mathbf{x}+\hat{\mathbf{e}}_1} \end{array} \\ \begin{pmatrix} 0 & t_{\mathbf{x},1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad \text{(A.2)}$$

with $1 \leq x^{(1)} \leq L-1\,,\; 1 \leq x^{(2)} \leq M\,;$

$$\mathcal{A}'_{(L,x^{(2)};1,x^{(2)})} = (-1)^{L+1} \begin{array}{c} \\ \bar{H} \\ H \\ \bar{V} \\ V \\ \bar{T} \\ T \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{H} & H & \bar{V} & V & \bar{T} & T \end{array} \\ \begin{pmatrix} 0 & t_{\mathbf{x},1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{A.3}$$

with $1 \leq x^{(2)} \leq M\,;$

$$\mathcal{A}'_{\mathbf{x}\mathbf{x}+\hat{\mathbf{e}}_2} = \begin{array}{c} \\ \bar{H}_{\mathbf{x}} \\ H_{\mathbf{x}} \\ \bar{V}_{\mathbf{x}} \\ V_{\mathbf{x}} \\ \bar{T}_{\mathbf{x}} \\ T_{\mathbf{x}} \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{H}_{\mathbf{x}+\hat{\mathbf{e}}_2} & H_{\mathbf{x}+\hat{\mathbf{e}}_2} & \bar{V}_{\mathbf{x}+\hat{\mathbf{e}}_2} & V_{\mathbf{x}+\hat{\mathbf{e}}_2} & \bar{T}_{\mathbf{x}+\hat{\mathbf{e}}_2} & T_{\mathbf{x}+\hat{\mathbf{e}}_2} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{\mathbf{x},2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{A.4}$$

with $1 \leq x^{(1)} \leq L\,,\; 1 \leq x^{(2)} \leq M-1\,;$ all the other elements of $\mathcal{A}'$ are vanishing. Note that in (A.3) the factor $(-1)^{L+1}$ takes into account an an extra minus sign introduced to orientate in the 'extra bond' between the last column and the first column if L is even: in this way the face Λ^c is clockwise odd. Finally, by using $\sum_{s\geq 0} \sum^*_{P_s(\Lambda)\subset\Lambda} \prod_{b_{\mathbf{x},i}\in P_s(\Lambda)} t_{\mathbf{x},i} = Pf\,\mathcal{A}'$ in (2.7) we get

$$Z_0(\mathbf{J}) = 2^{LM} \prod_{i=1}^{2} \prod_{b_{\mathbf{x}_i}\in\mathcal{B}_\Lambda} \cosh\beta J_{\mathbf{x},i}\;\; Pf\,\mathcal{A}'\,. \tag{A.5}$$

Appendix B

The massless propagators in real space

We derive the expression of the massless propagators in the real space, starting from (3.39). From now on, we let $g^{\varphi}_{\omega\omega'} := g^{\varphi}_{\omega\omega'}(\mathbf{x},\mathbf{y})$ for any $\omega,\omega' \in \{\pm\}$ and $\epsilon := \epsilon(k_1,k_2)$, which is given in (3.33).
If $\omega = \omega' = +$, we consider

$$\begin{aligned} g^{\varphi}_{++} &= \int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})} \left(i\sqrt{\epsilon}\right)^{-1} \left(\underline{u}^+(x^{(2)})\underline{\bar{u}}^+(y^{(2)}) - \underline{u}^-(x^{(2)})\underline{\bar{u}}^-(y^{(2)})\right) = \\ &= \int dk_1 dk_2 \frac{e^{-ik_1(x^{(1)}-y^{(1)})}}{i\sqrt{\epsilon}} \left(\left(\frac{1}{\mathcal{N}^+}\right)^2 u^+(x^{(2)})\bar{u}^+(y^{(2)}) - \left(\frac{1}{\mathcal{N}^-}\right)^2 u^-(x^{(2)})\bar{u}^-(y^{(2)})\right), \end{aligned} \tag{B.1}$$

where in the last line we used the definitions in (3.36). By recalling the definitions of $u^{\pm}$ in (3.34) and of $\mathcal{N}^{\pm}$ in (3.37), we rewrite the factor in braces as

$$\begin{aligned} &c^2\left(\left(\frac{(\sqrt{\epsilon}-\alpha_1)^{-1}}{\mathcal{N}^+}\right)^2 - \left(\frac{(\sqrt{\epsilon}+\alpha_1)^{-1}}{\mathcal{N}^-}\right)^2\right)(\epsilon-\alpha_1^2)\sin(x^{(2)}k_2)\sin(y^{(2)}k_2) = \\ &= c^2\left(\frac{(\sqrt{\epsilon}-\alpha_1)^{-1}-(\sqrt{\epsilon}+\alpha_1)^{-1}}{2\sqrt{\epsilon}}\right)(\epsilon-\alpha_1^2)\sin(x^{(2)}k_2)\sin(y^{(2)}k_2) = \\ &= c^2\left(\frac{2\alpha_1}{2\sqrt{\epsilon}(\epsilon-\alpha_1^2)}\right)(\epsilon-\alpha_1^2)\sin(x^{(2)}k_2)\sin(y^{(2)}k_2), \end{aligned} \tag{B.2}$$

where α_1 and β_1 are those defined in (3.14). So we get

$$\begin{aligned} g^{\varphi}_{++} &= c^2 \int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})} \frac{\alpha_1}{i\epsilon}\left(\frac{e^{ik_2x^{(2)}}-e^{-ik_2x^{(2)}}}{2i}\right)\left(\frac{e^{ik_2y^{(2)}}-e^{-ik_2y^{(2)}}}{2i}\right) = \\ &= c^2 \int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})} \frac{\alpha_1}{i\epsilon}\frac{1}{(-4)}\left(2e^{-ik_2(x^{(2)}+y^{(2)})} - 2e^{-ik_2(x^{(2)}-y^{(2)})}\right), \end{aligned} \tag{B.3}$$

where we used that $\epsilon(k_1,-k_2)=\epsilon(k_1,k_2)$.
If $\omega=-\omega'=+$, we consider

$$\begin{aligned} g^{\varphi}_{+-} &= \int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}(i\sqrt{\epsilon})^{-1}(\underline{\bar{u}}^{+}(x^{(2)})\underline{v}^{+}(y^{(2)})-\underline{\bar{u}}^{-}(x^{(2)})\underline{v}^{-}(x^{(2)})) = \\ &= \int dk_1 dk_2 \frac{e^{-ik_1(x^{(1)}-y^{(1)})}}{i\sqrt{\epsilon}}\left(\left(\frac{1}{\mathcal{N}^{+}}\right)^2 \bar{u}^{+}(x^{(2)})v(y^{(2)})-\left(\frac{1}{\mathcal{N}^{-}}\right)^2 \bar{u}^{-}(x^{(2)})v(y^{(2)})\right), \end{aligned} \tag{B.4}$$

where we used the definition in (3.36). By recalling the definitions of $u^{\pm}$ in (3.34), of v in (3.32) and of $\mathcal{N}^{\pm}$ in(3.37), we rewrite the factor in braces as

$$\begin{aligned} &-c^2\left(\frac{i(\sqrt{\epsilon}-\alpha_1)^{-1}}{\frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}-\alpha_1}}-\frac{-i(\sqrt{\epsilon}+\alpha_1)^{-1}}{\frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}+\alpha_1}}\right)\sqrt{\epsilon-\alpha_1^2}\sin(x^{(2)}k_2)\sin(y^{(2)}k_2+\theta_{k_2}) = \\ &= -c^2\left(\frac{i}{2\sqrt{\epsilon}}-\frac{-i}{2\sqrt{\epsilon}}\right)\sqrt{\epsilon-\alpha_1^2}\sin(x^{(2)}k_2)\sin(y^{(2)}k_2+\theta_{k_2}) = \\ &= -c^2 i\sqrt{\epsilon}\frac{\sqrt{\epsilon-\alpha_1^2}}{\epsilon}\sin(x^{(2)}k_2)\sin(y^{(2)}k_2+\theta_{k_2}), \end{aligned} \tag{B.5}$$

so we get

$$\begin{aligned} g^{\varphi}_{+-} = \frac{c^2}{4}\int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}\frac{\sqrt{\epsilon-\alpha_1^2}}{\epsilon}\cdot \\ \cdot\left(2e^{-ik_2(x^{(2)}+y^{(2)})-i\theta_{k_2}}-2e^{-ik_2(x^{(2)}-y^{(2)})+i\theta_{k_2}}\right), \end{aligned} \tag{B.6}$$

where we used $\epsilon(k_1,-k_2)=\epsilon(k_1,k_2)$ and $\theta_{-k_2}=-\theta_{k_2}$.
If $\omega=-\omega'=-$, we consider

$$\begin{aligned} g^{\varphi}_{-+} &= \int_{-\pi}^{\pi} dk_1 e^{-ik_1(x^{(1)}-y^{(1)})}(i\sqrt{\epsilon})^{-1}(\underline{v}^{+}(x^{(2)})\underline{u}^{+}(y^{(2)})-\underline{v}^{-}(x^{(2)})\underline{u}^{-}(y^{(2)})) = \\ &= \int dk_1 dk_2 \frac{e^{-ik_1(x^{(1)}-y^{(1)})}}{i\sqrt{\epsilon}}\left(\left(\frac{1}{\mathcal{N}^{+}}\right)^2 v(x^{(2)})u(y^{(2)})^{+}-\left(\frac{1}{\mathcal{N}^{-}}\right)^2 v(x^{(2)})u^{-}(y^{(2)})\right), \end{aligned} \tag{B.7}$$

where we used the definition in (3.36). By recalling the definitions of $u^{\pm}$ in (3.34), of v in (3.32) and of $\mathcal{N}^{\pm}$ in(3.37), we rewrite the factor in braces as

$$\begin{aligned} &-c^2\left(\frac{-i(\sqrt{\epsilon}-\alpha_1)^{-1}\sqrt{\epsilon-\alpha_1^2}}{\frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}-\alpha_1}}-\frac{i(\sqrt{\epsilon}+\alpha_1)^{-1}\sqrt{\epsilon-\alpha_1^2}}{\frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}+\alpha_1}}\right)\sin(x^{(2)}k_2+\theta_{k_2})\sin(y^{(2)}k_2) = \\ &= -c^2\left(\frac{-i}{2\sqrt{\epsilon}}-\frac{i}{2\sqrt{\epsilon}}\right)\sqrt{\epsilon-\alpha_1^2}\sin(x^{(2)}k_2+\theta_{k_2})\sin(y^{(2)}k_2), \end{aligned} \tag{B.8}$$

so we get

$$g^{\varphi}_{-+} = -\frac{c^2}{4}\int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}\frac{\sqrt{\epsilon-\alpha_1^2}}{\epsilon}\cdot \\ \cdot\left(2e^{-ik_2(x^{(2)}+y^{(2)})}e^{-i\theta_{k_2}} - 2e^{-ik_2(x^{(2)}-y^{(2)})}e^{-i\theta_{k_2}}\right), \tag{B.9}$$

where we used $\epsilon(k_1,-k_2)=\epsilon(k_1,k_2)$ and $\theta_{-k_2}=-\theta_{k_2}$.
If $\omega=\omega'=-$, we consider

$$g^{\varphi}_{--} = \int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}(i\sqrt{\epsilon})^{-1}(\underline{v}^+(x^{(2)})\underline{v}^+(y^{(2)}) - \underline{v}^-(x^{(2)})\underline{v}^-(y^{(2)})) = \\ = \int dk_1 dk_2 \frac{e^{-ik_1(x^{(1)}-y^{(1)})}}{i\sqrt{\epsilon}}\left(\left(\frac{1}{\mathcal{N}^+}\right)^2 v(x^{(2)})v(y^{(2)}) - \left(\frac{1}{\mathcal{N}^-}\right)^2 v(x^{(2)})v(y^{(2)})\right), \tag{B.10}$$

where we used the definition in (3.36). By recalling the definition of v in (3.32) and of $\mathcal{N}^\pm$ in(3.37), we rewrite the factor in braces as

$$c^2\left(\frac{-2\alpha_1}{2\sqrt{\epsilon}}\right)\sin(x^{(2)}k_2+\theta_{k_2})\sin(y^{(2)}k_2+\theta_{k_2}) = \\ = c^2\left(\frac{-\alpha_1}{\sqrt{\epsilon}}\right)\left(\frac{e^{ik_2x^{(2)}}e^{i\theta_{k_2}} - e^{-ik_2x^{(2)}}e^{-i\theta_{k_2}}}{2i}\right)\left(\frac{e^{ik_2y^{(2)}}e^{i\theta_{k_2}} - e^{-ik_2y^{(2)}}e^{-i\theta_{k_2}}}{2i}\right), \tag{B.11}$$

so we get

$$g^{\varphi}_{--} = = c^2\int dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}\frac{\alpha_1}{-i\epsilon}\frac{1}{(-4)}\left(2e^{-ik_2(x^{(2)}+y^{(2)})}e^{-2i\theta_{k_2}} - 2e^{-ik_2(x^{(2)}-y^{(2)})}\right), \tag{B.12}$$

where again we used $\epsilon(k_1,-k_2)=\epsilon(k_1,k_2)$ and $\theta_{-k_2}=-\theta_{k_2}$.

To obtain the expressions in (3.41), with (3.42) and (3.43) we have to do some manipulations. First of all, we prove that the diagonal coefficients, which are the coefficients appearing in (B.3) and in (B.12), can be rewritten as

$$\frac{\pm i\alpha_1(k_1)}{\epsilon(k_1,k_2)} = \frac{\pm 2it_1\sin k_1}{2(1-t_2)^2(1-\cos k_1)+2(1-t_1)^2(1-\cos k_2)}. \tag{B.13}$$

By recalling the definitions of α_1 in (3.14) and of ϵ in (3.33), we can rewrite

$$\frac{\pm i\alpha_1(k_1)}{\epsilon(k_1,k_2)} = \frac{\pm 2it_1\sin k_1}{|1+t_1e^{ik_1}|^2\epsilon(k_1,k_2)}. \tag{B.14}$$

Now, we let $D := |1+t_1e^{ik_1}|^2\epsilon(k_1,k_2)$ and we proceed with the following manipulations

$$D = |1+t_1e^{ik_1}|^2\left(\frac{2t_2(t_1^2-1)\cos k_2}{|1+t_1e^{ik_1}|^2} + \frac{(t_1^2-1)^2}{|1+t_1e^{ik_1}|^4} + t_2^2 + \frac{(2t_1\sin k_1)^2}{|1+t_1e^{ik_1}|^4}\right) = \\ = \frac{(t_1^2-1)^2+(2t_1\sin k_1)^2}{|1+te^{ik_1}|^2} + t_2^2|1+t_1e^{ik_1}|^2 + 2t_2(t_1^2-1)\cos k_2\,; \tag{B.15}$$

the first term in the last line of (B.15) can be rewritten as

$$\begin{aligned}
&\frac{(t_1^2-1)^2+4t_1^2\sin^2 k_1}{|1+t_1e^{ik_1}|^2}=\frac{(t_1^2-1)^2+4t_1^2\sin^2 k_1}{1+t_1^2+2t_1\cos k_1}=\frac{(t_1^2-1)^2+4t_1^2(1-\cos^2 k_1)}{1+t_1^2+2t_1\cos k_1}=\\
&=\frac{1+t_1^4-2t_1^2+4t_1^2-4t_1^2\cos^2 k_1}{1+t_1^2+2t_1\cos k_1}=\frac{(1+t_1^2)^2-4t_1^2\cos^2 k_1}{1+t_1^2+2t_1\cos k_1}=\\
&=\frac{((1+t_1^2)-2t_1\cos k_1)((1+t_1^2)+2t_1\cos k_1)}{1+t_1^2+2t_1\cos k_1}=1+t_1^2-2t_1\cos k_1\,,
\end{aligned} \tag{B.16}$$

by substituting (B.16) in (B.15), we get

$$\begin{aligned}
D&=((1+t_1^2)-2t_1\cos k_1)+t_2^2(1+t_1^2+2t_1\cos k_1)+2t_2(t_1^2-1)\cos k_2=\\
&=(1+t_1^2)(1+t_2^2)+2t_1(t_2^2-1)\cos k_1+2t_2(t_1^2-1)\cos k_2\,.
\end{aligned} \tag{B.17}$$

By taking $t_2=\frac{1-t_1}{1+t_1}$ in (B.17) we get

$$\begin{aligned}
D&=(1+t_1^2)(1+t_2^2)+2\frac{1-t_2}{1+t_2}(t_2^2-1)\cos k_1+2\frac{1-t_1}{1+t_1}(t_1^2-1)\cos k_2=\\
&=(1+t_1^2)(1+t_2^2)-2(1-t_2)^2\cos k_1-2(1-t_1)^2\cos k_2=\\
&=(1+t_1^2)(1+t_2^2)-2(1-t_2)^2-2(1-t_1)^2-2(1-t_2)^2(\cos k_1-1)+\\
&-2(1-t_1)^2(\cos k_2-1)=-2(1-t_2)^2(\cos k_1-1)-2(1-t_1)^2(\cos k_2-1)=\\
&=2(1-t_2)^2(1-\cos k_1)+2(1-t_1)^2(1-\cos k_2)\,,
\end{aligned} \tag{B.18}$$

which is the denominator of (B.13).
Now we prove that the off-diagonal coefficients, which are the coefficients appearing in (B.6) and in (B.9), can be rewritten as

$$\left(\frac{\sqrt{\epsilon-\alpha_1^2}}{\epsilon}\right)e^{\pm i\theta_{k_2}}=-\frac{(t_1^2-1)+t_2|1+t_1e^{ik_1}|^2e^{\mp ik_2}}{2(1-t_2)^2(1-\cos k_1)+2(1-t_1)^2(1-\cos k_2)}\,. \tag{B.19}$$

We consider

$$\left(\beta_1+t_2e^{\pm ik_2}\right)=\tilde{\rho}e^{\pm i\tilde{\theta}}=\left(\sqrt{\epsilon-\alpha_1^2}\right)e^{\pm i(-\theta_{k_2}+\pi)}\,, \tag{B.20}$$

so that

$$\left(\frac{\sqrt{\epsilon-\alpha_1^2}}{\epsilon}\right)e^{\pm i\theta_{k_2}}=-\frac{\left(\beta_1+t_2e^{\mp ik_2}\right)}{\epsilon}\,, \tag{B.21}$$

and the r.h.s. of (B.21), by recalling the definition of β_1 in (3.14), can be rewritten as

$$-\frac{\frac{(t_1^2-1)}{|1+t_1e^{ik_1}|^2}+t_2e^{\mp ik_2}}{\epsilon}=-\frac{(t_1^2-1)+t_2|1+t_1e^{ik_1}|^2e^{\mp ik_2}}{|1+t_1e^{ik_1}|^2\epsilon}\,. \tag{B.22}$$

Finally, by recognizing $|1+t_1e^{ik_1}|^2\epsilon = D$, we can use (B.18) to obtain the expression in (B.19).
To conclude, we to prove that

$$e^{-2i\theta_{k_2}} = p(k_1,k_2) = \frac{(t_1^2-1)+t_2(1+t_1^2+2t_1\cos k_1)e^{ik_2}}{(t_1^2-1)+t_2(1+t_1^2+2t_1\cos k_1)e^{-ik_2}}\,, \tag{B.23}$$

which is the expression in (3.43) for the coefficient appearing in (B.12). We start by rewriting

$$\begin{aligned} y =& e^{-2i\theta_{k_2}} = \exp\left(-2i\arctan\frac{-\sin k_2}{\frac{\beta_1}{t_2}+\cos k_2}\right) = \\ &= \exp\left(\arctan 2i\frac{\sin k_2}{\frac{\beta_1}{t_2}+\cos k_2}\right) = \exp\left(2i\arctan x\right), \end{aligned} \tag{B.24}$$

so that $\arctan x = \frac{\log y}{2i}$, and

$$x = \tan\frac{\log y}{2i} = \frac{2}{2i}\frac{e^{2i\frac{\log y}{2i}}-1}{e^{2i\frac{\log y}{2i}}+1} = -i\frac{y-1}{y+1}\,, \tag{B.25}$$

and, by inverting the last expression, we get

$$\begin{aligned} e^{-2i\theta_{k_2}} &= \frac{i-x}{i+x} = \frac{i-\frac{\sin k_2}{\frac{\beta_1}{t_2}+\cos k_2}}{i+\frac{\sin k_2}{\frac{\beta_1}{t_2}+\cos k_2}} = \frac{i\frac{\beta_1}{t_2}+i\cos k_2-\sin k_2}{i\frac{\beta_1}{t_2}+i\cos k_2+\sin k_2} = \frac{i\frac{(t_1^2-1)}{t_2|1+t_1e^{ik_1}|^2}+i\cos k_2-\sin k_2}{i\frac{(t_1^2-1)}{t_2|1+t_1e^{ik_1}|^2}+i\cos k_2+\sin k_2} = \\ &= \frac{i(t_1^2-1)+it_2|1+t_1e^{ik_1}|^2(\cos k_2+i\sin k_2)}{i(t_1^2-1)+it_2|1+t_1e^{ik_1}|^2(\cos k_2-i\sin k_2)} = \frac{i(t_1^2-1)+it_2|1+t_1e^{ik_1}|^2e^{ik_2}}{i(t_1^2-1)+it_2|1+t_1e^{ik_1}|^2e^{-ik_2}}\,, \end{aligned} \tag{B.26}$$

which is the desired expression in (B.23).

Appendix C

The scaling limit of the massless propagators

By recalling the definition in (3.40) we are interested in the following expression

$$
\begin{aligned}
&\frac{1}{a}\begin{pmatrix} g_\varphi^{++}(\frac{\mathbf{x}}{a},\frac{\mathbf{y}}{a}) & g_\varphi^{+-}(\frac{\mathbf{x}}{a},\frac{\mathbf{y}}{a}) \\ g_\varphi^{-+}(\frac{\mathbf{x}}{a},\frac{\mathbf{y}}{a}) & g_\varphi^{--}(\frac{\mathbf{x}}{a},\frac{\mathbf{y}}{a}) \end{pmatrix} = \\
&\quad = \frac{1}{a}\frac{1}{2}\int\limits_{[-\pi,\pi]^2} dk_1 dk_2\; e^{-ik_1\frac{(x^{(1)}-y^{(1)})}{a}} e^{-ik_2\frac{(x^{(2)}-y^{(2)})}{a}} \begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2) \end{pmatrix} + \\
&\quad - \frac{1}{a}\frac{1}{2}\int\limits_{[-\pi,\pi]^2} dk_1 dk_2\; e^{-ik_1\frac{(x^{(1)}-y^{(1)})}{a}} e^{-ik_2\frac{(x^{(2)}+y^{(2)})}{a}} \begin{pmatrix} \hat{g}_1(-k_1,k_2) & -\hat{g}_2(k_1,-k_2) \\ \hat{g}_2(k_1,-k_2) & \hat{g}_1(k_1,k_2)e^{-2i\theta_{k_2}} \end{pmatrix} = \\
&\quad = \frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1 dk_2\; e^{-ik_1(x^{(1)}-y^{(1)})} e^{-ik_2(x^{(2)}-y^{(2)})} \begin{pmatrix} \hat{g}_1(-ak_1,ak_2) & -\hat{g}_2(ak_1,ak_2) \\ \hat{g}_2(ak_1,-ak_2) & \hat{g}_1(ak_1,ak_2) \end{pmatrix} + \\
&\quad - \frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1 dk_2\; e^{-ik_1(x^{(1)}-y^{(1)})} e^{-ik_2(x^{(2)}+y^{(2)})} \begin{pmatrix} \hat{g}_1(-ak_1,ak_2) & -\hat{g}_2(ak_1,-ak_2) \\ \hat{g}_2(ak_1,-ak_2) & \hat{g}_1(ak_1,ak_2)e^{-2i\theta_{ak_2}} \end{pmatrix} .
\end{aligned}
\tag{C.1}
$$

When we consider the scaling limit at the critical point we take $ak_1 \sim 0, ak_2 \sim 0, t_2 = \frac{1-t_1}{1+t_1}$ and we consider the following expressions for the propagators in (3.41)

$$
\begin{aligned}
&\hat{g}_1(\pm ak_1, ak_2) = \frac{\pm it_1 \sin ak_1}{2(1-t_2)^2(1-\cos ak_1)+2(1-t_1)^2(1-\cos ak_2)} \simeq \\
&\quad \simeq \frac{\pm it_1 ak_1}{a^2(1-t_2)^2k_1^2 + a^2(1-t_1)^2k_2^2} = \frac{1}{a}\frac{\pm it_1 k_1}{(1-t_2)^2k_1^2+(1-t_1)^2k_2^2} ,
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
&\hat{g}_2(ak_1, \pm ak_2) = -\frac{(t_1^2-1)+t_2(1+t_1^2+2t_1\cos ak_1)e^{\mp iak_2}}{2(1-t_2)^2(1-\cos ak_1)+2(1-t_1)^2(1-\cos ak_2)} \simeq \\
&\quad \simeq \pm\frac{1}{a}\frac{it_2(1+t_1)^2k_2}{(1-t_2)^2k_1^2+(1-t_1)^2k_2^2} .
\end{aligned}
\tag{C.3}
$$

When $t_2 = \frac{1-t_1}{1+t_1}$ we rewrite (B.23) as

$$\begin{aligned}
&\frac{(t_1-1)(t_1+1) - \frac{t_1-1}{1+t_1}(1+t_1^2+2t_1\cos k_1)e^{ik_2}}{(t_1-1)(t_1+1) - \frac{t_1-1}{1+t_1}(1+t_1^2+2t_1\cos k_1)e^{-ik_2}} = \\
&= \frac{(t_1-1)(t_1+1)^2 - (t_1-1)(1+t_1^2+2t_1\cos k_1)e^{ik_2}}{(t_1-1)(t_1+1)^2 - (t_1-1)(1+t_1^2+2t_1\cos k_1)e^{-ik_2}} = \\
&= \frac{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{ik_2}}{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{-ik_2}} - 1 + 1 = \\
&= -1 + \frac{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{ik_2} + (t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{-ik_2}}{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{-ik_2}} = \\
&= -1 + 2\frac{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)\cos k_2}{(t_1+1)^2 - (1+t_1^2+2t_1\cos k_1)e^{-ik_2}} = \\
&= -1 + 2\frac{(t_1^2+1)(1-\cos k_2) + 2t_1(1-\cos k_1\cos k_2)}{(t_1^2+1)(1-\cos k_2) + 2t_1(1-\cos k_1 e^{-ik_2})} =: -1 + \tilde{p}(k_1,k_2)
\end{aligned} \tag{C.4}$$

so that

$$e^{-2i\theta_{ak_2}} = -1 + \tilde{p}(ak_1, ak_2)\,, \tag{C.5}$$

which can be evaluated when $ak_1 \sim 0, ak_2 \sim 0$ as

$$\begin{aligned}
e^{-2i\theta_{(ak_2)}} &= -1 + 2\frac{(t_1^2+1)(1-\cos ak_2) + 2t_1(1-\cos ak_1\cos ak_2)}{(t_1^2+1)(1-\cos ak_2) + 2t_1(1-\cos ak_1 e^{-iak_2})} \simeq \\
&\simeq -1 + \frac{a\left((t_1^2+1)k_2^2 + 2t_1k_1^2 + 2t_1k_2^2\right)}{i2t_1k_2}\,.
\end{aligned} \tag{C.6}$$

Taking the limit we get the massless propagators in the scaling limit $g_{sl}^{\omega\omega'}(\mathbf{x},\mathbf{y}) := \lim_{a\to 0}\frac{1}{a}g_{\varphi}^{\omega\omega'}(\frac{\mathbf{x}}{a},\frac{\mathbf{y}}{a})$, namely

$$\begin{aligned}
g_{sl}^{++}(\mathbf{x},\mathbf{y}) := \lim_{a\to 0}\frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1dk_2\hat{g}_1(-ak_1,ak_2)e^{-ik_1(x^{(1)}-y^{(1)})}. \\
\cdot\left(e^{-ik_2(x^{(2)}-y^{(2)})} - e^{-ik_2(x^{(2)}+y^{(2)})}\right)\,,
\end{aligned} \tag{C.7}$$

with $\hat{g}_1(-ak_1,ak_2)$ as in (C.2) we get

$$\begin{aligned}
g_{sl}^{++}(\mathbf{x},\mathbf{y}) = -\pi t_1\left(\frac{(1-t_1)(x^{(1)}-y^{(1)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2 + (1-t_2)^2(x^{(2)}-y^{(2)})^2}\right) + \\
+\pi t_1\left(\frac{(1-t_1)(x^{(1)}-y^{(1)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2 + (1-t_2)^2(x^{(2)}+y^{(2)})^2}\right)\,.
\end{aligned} \tag{C.8}$$

By defining

$$g_{sl}^{+-}(\mathbf{x},\mathbf{y}) := \lim_{a\to 0}\frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1 dk_2 e^{-ik_1(x^{(1)}-y^{(1)})}\cdot \cdot\left(-\hat{g}_2(ak_1,ak_2)e^{-ik_2(x^{(2)}-y^{(2)})} + -\hat{g}_2(ak_1,-ak_2)e^{-ik_2(x^{(2)}+y^{(2)})}\right), \tag{C.9}$$

with $\hat{g}_2(ak_1,\pm ak_2)$ as in (C.3) we get

$$\begin{aligned} g_{sl}^{+-}(\mathbf{x},\mathbf{y}) = -\pi t_2(1+t_1)^2 &\left(\frac{(1-t_2)(x^{(2)}-y^{(2)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}-y^{(2)})^2}\right) + \\ -\pi t_2(1+t_1)^2 &\left(\frac{(1-t_2)(x^{(2)}+y^{(2)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}+y^{(2)})^2}\right); \end{aligned} \tag{C.10}$$

$$g_{sl}^{-+}(\mathbf{x},\mathbf{y}) := \lim_{a\to 0}\frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1 dk_2 \hat{g}_2(ak_1,-ak_2)e^{-ik_1(x^{(1)}-y^{(1)})}\cdot \left(e^{-ik_2(x^{(2)}-y^{(2)})} - e^{-ik_2(x^{(2)}+y^{(2)})}\right), \tag{C.11}$$

with $\hat{g}_2(ak_1,-ak_2)$ as in (C.3) we get

$$\begin{aligned} g_{sl}^{-+}(\mathbf{x},\mathbf{y}) = \pi t_2(1+t_1)^2 &\left(\frac{(1-t_2)(x^{(2)}-y^{(2)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}-y^{(2)})^2}\right) + \\ -\pi t_2(1+t_1)^2 &\left(\frac{(1-t_2)(x^{(2)}+y^{(2)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}+y^{(2)})^2}\right); \end{aligned} \tag{C.12}$$

finally

$$g_{sl}^{--}(\mathbf{x},\mathbf{y}) := \lim_{a\to 0}\frac{a}{2}\int\limits_{[\frac{-\pi}{a},\frac{\pi}{a}]^2} dk_1 dk_2 \hat{g}_1(ak_1,ak_2)e^{-ik_1(x^{(1)}-y^{(1)})}\cdot \left(e^{-ik_2(x^{(2)}-y^{(2)})} - e^{-2i\theta_{ak_2}}e^{-ik_2(x^{(2)}+y^{(2)})}\right), \tag{C.13}$$

with $\hat{g}_1(ak_1,ak_2)$ as in (C.2) and $e^{-2i\theta_{ak_2}}$ as in (C.5) we get

$$\begin{aligned} g_{sl}^{--}(\mathbf{x},\mathbf{y}) = \pi t_1 &\left(\frac{(1-t_1)(x^{(1)}-y^{(1)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}-y^{(2)})^2}\right) + \\ +\pi t_1 &\left(\frac{(1-t_1)(x^{(1)}-y^{(1)})}{(1-t_1)^2(x^{(1)}-y^{(1)})^2+(1-t_2)^2(x^{(2)}+y^{(2)})^2}\right). \end{aligned} \tag{C.14}$$

A straightforward computation leads to (3.48).

Bibliography

[1] I. Affleck. Universal term in the free energy at a critical point and the conformal anomaly. In *Current Physics–Sources and Comments*, volume 2, pages 347–349. Elsevier, 1988.

[2] M. Aizenman et al. Emergent planarity in two-dimensional Ising models with finite-range interactions. *Inventiones Mathematicae*, 216(3): 661–743, 2019.

[3] G. Antinucci. Interacting fermions on the half-line: boundary counterterms and boundary corrections. *ArXiv preprint* arXiv:1701.05356, 2017.

[4] G. Antinucci, A. Giuliani, and R.L. Greenblatt. Energy correlations of non-integrable Ising models: The scaling limit in the cylinder. *ArXiv preprint* arXiv:2006.04458, 2020.

[5] A. Abdesselam and V. Rivasseau. Explicit fermionic tree expansions. *Letters in Mathematical Physics*, 44: 77–88, 1998.

[6] E. Barouch, B.M. McCoy, and T. T. Wu. Zero-field susceptibility of the two-dimensional Ising model near T_c. *Physical Review Letters*, 31(23):1409, 1973.

[7] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333–380, 1984.

[8] G. Benfatto and G. Gallavotti. Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one-dimensional systems. *Journal of Statistical Physics*, 59(3-4):541–664, 1990.

[9] G. Benfatto, G. Gallavotti, A. Procacci, and B. Scoppola. Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface. *Communications in Mathematical Physics*, 160(1):93–171, 1994.

[10] G. Benfatto and V. Mastropietro. Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model. *Reviews in Mathematical Physics*, 13(11):1323–1435, 2001.

[11] S. Benoist, H. Duminil-Copin, C. Hongler, et al. Conformal invariance of crossing probabilities for the Ising model with free boundary conditions. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(4):1784–1798, 2016.

[12] H. Blöte, J.L. Cardy, and M. Nightingale. Conformal invariance, the central charge, and universal finite-size amplitudes at criticality. *Physical review letters*, 56(7):742, 1986.

[13] F. Bonetto and V. Mastropietro. Beta function and anomaly of the Fermi surface for a $d = 1$ system of interacting fermions in a periodic potential. *Communications in mathematical physics*, 172(1):57–93, 1995.

[14] D. Brydges. A Short Course on Cluster Expansions. *Phénomènes critiques, systèmes aléatoires, théories de jauge*, Ed. by K. Osterwalder and R. Stora. Les Houches summer school session 43, North-Holland: 129–183, 1986.

[15] D. Brydges and P. Federbush. A new form of the Mayer expansion in classical statistical mechanics. Journal of Mathematical Physics, 19:2064, 1978.

[16] D.C. Brydges and T. Kennedy. Mayer expansions and the Hamilton-Jacobi equation. Journal of Statistical Physics, 48:19–49, 1987.

[17] C.G. Callan Jr. Broken scale invariance in scalar field theory. *Physical Review D*, 2(8):1541, 1970.

[18] J.L. Cardy. Conformal invariance and surface critical behavior. *Nuclear Physics B*, 240(4):514–532, 1984.

[19] J.L. Cardy. Critical percolation in finite geometries. *Journal of Physics A: Mathematical and General*, 25(4):L201, 1992.

[20] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov. Convergence of Ising interfaces to Schramm's SLE curves. *Comptes Rendus Mathematique*, 352(2):157–161, 2014.

[21] D. Chelkak, C. Hongler, and K. Izyurov. Conformal invariance of spin correlations in the planar Ising model. *Annals of mathematics*, pages 1087–1138, 2015.

[22] D. Chelkak and S. Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Inventiones mathematicae*, 189(3):515–580, 2012.

[23] C. Di Castro and G. Jona-Lasinio. On the microscopic foundation of scaling laws. *Physics Letters A*, 29(6):322–323, 1969.

[24] P. Di Francesco, H. Saleur, and J.-B. Zuber. Relations between the Coulomb gas picture and conformal invariance of two-dimensional critical models. *Journal of statistical physics*, 49(1-2):57–79, 1987.

[25] V.S. Dotsenko and V.A. Fateev. Operator algebra of two-dimensional conformal theories with central charge $c \leq 1$. *Physics Letters B*, 154(4):291–295, 1985.

[26] V.S. Dotsenko and V.A. Fateev. Conformal algebra and multipoint correlation functions in 2D statistical models. *Nuclear Physics B*, 240(3):312–348, 1984.

[27] V. S. Dotsenko and V.A. Fateev. Four-point correlation functions and the operator algebra in the 2D conformal invariant theories with the central charge $c \leq 1$. *Nuclear Physics B*, 251:691–734, 1985.

[28] H. Duminil-Copin and S. Smirnov. Conformal invariance of lattice models. *Probability and statistical physics in two and more dimensions*, 15:213–276, 2012.

[29] S. Ferrara, A. Grillo, and R. Gatto. Conformal algebra in two space-time dimensions and the Thirring model. *Il Nuovo Cimento A (1965-1970)*, 12(4):959–968, 1972.

[30] M.E. Fisher. On the dimer solution of planar Ising models. *Journal of Mathematical Physics*, 7(10), 1776–1781, 1966.

[31] D. Friedan, Z. Qiu, and S. Shenker. Conformal invariance, unitarity, and critical exponents in two dimensions. *Physical Review Letters*, 52(18):1575, 1984.

[32] G. Gallavotti. Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods. *Reviews of Modern Physics*, 57: 471–562, 1985.

[33] G. Gallavotti and F. Nicolò. Renormalization theory in four-dimensional scalar fields, part I. *Communications in Mathematical Physics*,100:545–590, 1985.

[34] G. Gentile and V. Mastropietro. Renormalization group for one-dimensional fermions. A review on mathematical results. *Physics Reports*, 352(4-6): 273–438, 2001.

[35] G. Gentile and B. Scoppola. Renormalization group and the ultraviolet problem in the Luttinger model. *Communications in mathematical physics*, 154(1):135–179, 1993.

[36] A. Giuliani. The ground state construction of the two-dimensional Hubbard model on the honeycomb lattice. *Quantum Theory from Small to Large Scales*, Lecture Notes of the Les Houches Summer School (95), Ed. by J. Fröhlich et al. Oxford University Press, 2010.

[37] A. Giuliani, R.L. Greenblatt, and V. Mastropietro. The scaling limit of the energy correlations in non-integrable Ising models. *Journal of Mathematical Physics*, 53(9):095214, 2012.

[38] A. Giuliani and V. Mastropietro. Anomalous universality in the anisotropic Ashkin–Teller model. *Communications in mathematical physics*, 256(3):681–735, 2005.

[39] A. Giuliani and V. Mastropietro. Universal finite size corrections and the central charge in non-solvable Ising models. *Communications in Mathematical Physics*, 324(1):179–214, 2013.

[40] A. Giuliani, V. Mastropietro, and F.L. Toninelli. Height fluctuations in interacting dimers. *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, 53: 98–168, 2017.

[41] J. Groeneveld, R.J. Boel, and P.W. Kasteleyn. Corrrelation-function identities for general planar Ising systems. *Physica*, 93A: 138–154, 1978.

[42] C. Hongler. Conformal invariance of Ising model correlations. In *XVIIth International Congress on Mathematical Physics*, pages 326–335. World Scientific, 2014.

[43] C.A. Hurst. New approach to the ising problem. *Journal of Mathematical Physics*, 7(2):305–310, 1966.

[44] C.A. Hurst and H.S. Green. New solution of the Ising problem for a rectangular lattice. *The Journal of Chemical Physics*, 33(4):1059–1062, 1960.

[45] C. Itzykson and J.-M. Drouffe. *Statistical field theory: Volume 2, strong coupling, Monte Carlo methods, conformal field theory and random systems*, volume 2. Cambridge University Press, 1991.

[46] M. Jacob. *Dual theory*, volume 1. North Holland, 1974.

[47] M. Kac and J.C. Ward. A combinatorial solution of the two-dimensional Ising model. *Physical Review*, 88(6):1332, 1952.

[48] V. Kac. Group theoretical methods in physics. In *Lecture Notes in Physics*, volume 94, pages 441–445. Springer Berlin, 1979.

[49] L.P. Kadanoff. Scaling laws for Ising models near T_c. *Physics Physique Fizika*, 2(6):263, 1966.

[50] L.P. Kadanoff. Correlations along a line in the two-dimensional Ising model. *Physical Review*, 188(2):859, 1969.

[51] P.W. Kasteleyn. Dimer statistics and phase transitions. *Journal of Mathematical Physics*, 4(2), 287–293, 1963.

[52] B. Kaufman. Crystal statistics. II. Partition function evaluated by spinor analysis. *Physical Review*, 76(8):1232, 1949.

[53] B. Kaufman and L. Onsager. Crystal statistics. III. Short-range order in a binary Ising lattice. *Physical Review*, 76(8):1244, 1949.

[54] R. Kenyon. Dominos and the gaussian free field. *Annals of probability*, pages 1128–1137, 2001.

[55] R. Langlands, P. Pouliot, and Y. Saint-Aubin. Conformal invariance in two-dimensional percolation. *arXiv preprint math/9401222*, 1994.

[56] G.F. Lawler, O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. In *Selected Works of Oded Schramm*, pages 931–987. Springer, 2011.

[57] E. Lieb, T. Schultz, and D. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16(3):407–466, 1961.

[58] V. Mastropietro. Ising models with four spin interaction at criticality. *Communications in Mathematical Physics*, 244(3):595–642, 2004.

[59] B.M. McCoy, J.H. Perk, and T.T. Wu. Ising field theory: quadratic difference equations for the n-point Green's functions on the lattice. *Physical Review Letters*, 46(12):757, 1981.

[60] B.M. McCoy and T.T. Wu. *The two-dimensional Ising model.* Courier Corporation, 2014.

[61] E.W. Montroll, R.B. Potts, and J.C. Ward. Correlations and Spontaneous Magnetization of the Two-Dimensional Ising Model. *Journal of Mathematical Physics*, 4(2):308–322, 1963.

[62] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Physical Review*, 65(3-4):117, 1944.

[63] R. Peierls. On Ising's model of ferromagnetism. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 477–481. Cambridge University Press, 1936.

[64] H. Pinson and T. Spencer. Universality and the two dimensional Ising model. Unpublished preprint.

[65] A.M. Polyakov. Conformal symmetry of critical fluctuations (originally published in russian volume 12, number 11). *Soviet Journal of Experimental and Theoretical Physics Letters*, 12:381, 1970.

[66] V. Riva and J. Cardy. Holomorphic parafermions in the Potts model and stochastic Loewner evolution. *Journal of Statistical Mechanics: Theory and Experiment*, 2006(12):P12001, 19, 2006.

[67] S. Samuel. The use of anticommuting variable integrals in statistical mechanics. I. The computation of partition functions. *Journal of Mathematical Physics*, 21(12):2806–2814, 1980.

[68] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. In *Selected Works of Oded Schramm*, pages 791–858. Springer, 2011.

[69] T.D. Schultz, D.C. Mattis, and E.H. Lieb. Two-dimensional Ising model as a soluble problem of many fermions. *Reviews of Modern Physics*, 36(3):856, 1964.

[70] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 333(3):239–244, 2001.

[71] K. Symanzik. Small distance behaviour in field theory and power counting. *Communications in Mathematical Physics*, 18(3):227–246, 1970.

[72] C.A. Tracy and B.M. McCoy. Neutron scattering and the correlation functions of the Ising model near T_c. *Physical Review Letters*, 31(25):1500, 1973.

[73] K.G. Wilson. Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. *Physical review B*, 4(9):3174, 1971.

[74] K.G. Wilson. Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior. *Physical Review B*, 4(9):3184, 1971.

[75] K.G. Wilson and M.E. Fisher. Critical exponents in 3.99 dimensions. *Physical Review Letters*, 28(4):240, 1972.

[76] C.N. Yang. The spontaneous magnetization of a two-dimensional Ising model. *Physical Review*, 85(5):808, 1952.