Dipartimento di Matematica e Fisica

Response solutions for quasi-periodically forced systems with arbitrary nonlinearities and frequencies in the presence of strong dissipation

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Candidato:
Faenia Vaia Firma:__________________________

Relatore di tesi:
Prof. Guido Gentile Firma:__________________________

Coordinatore:
Prof. Angelo Felice Lopez Firma:__________________________

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Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre
Largo San Leonardo Murialdo 1, 00146, Roma

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Chapter 1

Introduction

Consider a one-dimensional system subject to a mechanical force g, in the presence of large dissipation and of a quasi-periodic forcing term f. The system is described by the following singular ordinary differential equation in $\mathbb{R}$:

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\boldsymbol{\omega} t), \tag{1.1}$$

where ε is a small real positive parameter and $\boldsymbol{\omega} \in \mathbb{R}^d$, $d \in \mathbb{N}$, is the frequency vector of the forcing. We are interested in systems with large dissipation: the inverse of the perturbation parameter $\gamma = 1/\varepsilon$ stands for the damping coefficient that is large if ε is supposed to be small. Therefore we can rewrite (1.1) as

$$\ddot{x} + \gamma\dot{x} + g(x) = f(\boldsymbol{\omega} t). \tag{1.2}$$

Contrary to [68], since ε is small, no smallness condition is assumed on the forces f and g: we just assume $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{T}^d \to \mathbb{R}$ to be real analytic functions with $t \to f(\boldsymbol{\omega} t)$ quasi-periodic in time. Denote by Σ_ξ the strip on $\mathbb{T}^d$ of width ξ where the quasi-periodic forcing term f is analytic. Here and in the following $\mathbb{T}^d$ denotes the d-dimensional torus, i.e. $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$.

Systems of the form (1.1) have a great relevance in at least four different contexts of classical mechanics and electronic engineering.

1. The dynamics of a point mass moving in one dimension subject to a nonlinear force, by taking suitable mass units, is described by the second order equation $\ddot{x}+g(x) = 0$; by adding a dissipation term proportional to the velocity and a forcing term, one obtains equation (1.1) – see [8, 20, 41, 47, 48, 73] and the references therein.

2. Equation (1.1) with $g(x) = x^\mu$, provided $\mu > 1$ and $x(t) > 0$ for all t, describes a simple electronic circuit known as the resistor-inductor-varactor circuit. The varactor is a particular type of diode, which is a nonlinear electronic device analogous to a nonlinear spring, where x is the extension and typically $\mu \in [1.5, 2.5]$. The

mechanical analogies of the resistor and the inductor are, respectively, a source of linear damping and a constant mass. The full model for this circuit, i.e. one in which the restriction $x > 0$ is removed, possesses a nonlinearity of a different form: $c_1 \exp(c_2|x|)$, c_1, c_2 constants, for $x \leq 0$ and has been extensively studied, see for instance [4, 5, 6, 61, 62, 69, 70].

3. Studies of ship roll and capsize motivated the investigations of the behaviour of the ODE $\ddot{x} + \gamma\dot{x} + x - x^2 = F\sin(\omega t)$, with $F > 0$, which is equation (1.1) with $f(\omega t) = F\sin(\omega t)$ and $g(x) = x - x^2$, see [23, 67].

4. Stationary wave solutions of a perturbed Korteweg-de Vries (kdV) equation are described by a special case of equation (1.2), see [15]. In fact, consider a perturbed (KdV) equation $u_\tau + cu_\xi + \beta u_{\xi\xi\xi} = f(u, \xi - V\tau)_\xi$, where $f(u, \xi - V\tau) = f_0 \cos(\omega(\xi - V\tau))_\xi$ and the subscripts refer to partial differentiation. Then by taking the standard transformation $\xi' \to \xi - V\tau$, $\tau' \to \tau$, one obtains $\beta u_{\xi'\xi'} - vu + u^2/2 = f_0 \cos\omega\xi' + C$ in the steady state ($u_{\tau'} = 0$), with $v = V \pm c$ and C a constant of integration. Finally, re-naming ξ' as t, we again obtain equation (1.2) with $f(\omega t) = (C + f_0\cos(\omega t))/\beta$, $\gamma = 0$ and $g(u) = (-vu + u^2/2)/\beta$.

We also refer to [3, 26, 32, 33, 40, 72] and references quoted therein for a more detailed physical motivation.

1.1 Hypotheses on the system

We want to prove the existence of *response solutions* for (1.1), i.e. quasi-periodic solutions with the same frequency vector as the forcing term. Notice that if $\varepsilon = 0$, any constant $c \in \mathbb{R}$ is a solution to (1.1). In particular we want to study whether it is possible to choose the constant c in such a way that, for small value of ε, response solutions exist and go to c as ε tends to zero. In general, this cannot be achieved without requiring some non-degeneracy condition involving the functions f and g.

Hypothesis 1 (Non-degeneracy condition)**.** *There exists $c \in \mathbb{R}$ such that $x = c$ is a zero of odd order $\mathfrak{n}$ of the equation*

$$g(x) = f_{\mathbf{0}}, \tag{1.3}$$

where $f_{\mathbf{0}}$ is the average of the function f on the d-dimensional torus. Equivalently, one has $g(c) = f_{\mathbf{0}}$, $\frac{d^{\mathfrak{n}} g(c)}{dx^{\mathfrak{n}}} \neq 0$ and $\frac{d^j g(c)}{dx^j} = 0$ for $j = 1, \ldots, \mathfrak{n} - 1$.

Remark 1.1. Hypothesis 1 is a necessary condition: indeed if this hypothesis is not satisfied, in general there is no response solution reducing to c as ε tends to zero, see Lemma B.1 in Appendix B for a detailed proof.

Hypothesis 2 (Non-resonance condition)**.** *The frequency vector $\boldsymbol{\omega}$ has rationally independent components, that is $\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0$ $\forall \boldsymbol{\nu} \in \mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{\mathbf{0}\}$.*

Remark 1.2. Hypothesis 2 is not restrictive: in fact if $\boldsymbol{\omega}$ is resonant, one can always reduce to the case of $\boldsymbol{\omega}$ with rationally independent components by changing the value of the dimension d, in the following way. If $\boldsymbol{\omega} \in \mathbb{R}^d$ is resonant with multiplicity $r < d$, i.e if there is a rank r subgroup G of $\mathbb{Z}^d$ such that $\boldsymbol{\omega} \cdot \boldsymbol{\nu} = 0$ for all $\boldsymbol{\nu} \in G$ and $\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d \setminus G$, then, after a transformation, $\boldsymbol{\omega}$ can be written as $\boldsymbol{\omega} = (\boldsymbol{\omega}', \mathbf{0})$, such that $\boldsymbol{\omega}' \in \mathbb{R}^{d'}$, with $d' = d - r$, has rationally independent components and the function f may be seen as a function on $\mathbb{T}^{d'}$ of the form $f(\boldsymbol{\omega} t) = \widetilde{f}(\boldsymbol{\omega}' t)$.

1.2 A brief review on the literature

The existence of response solutions to (1.1) are widely studied in literature when the frequency vector $\boldsymbol{\omega}$ satisfies a non-resonance condition stronger than Hypothesis 2. In order to clarify what follows, we recall the definitions of Diophantine vectors and Bryuno vectors (see [19]).

We say that $\boldsymbol{\omega} \in \mathbb{R}^d$ is a Diophantine vector if it satisfies the standard Diophantine condition $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq C_0 |\boldsymbol{\nu}|^{-\tau}$, where $|\boldsymbol{\nu}| \equiv |\boldsymbol{\nu}|_1 = |\nu_1| + \cdots + |\nu_d|$, for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ and for some positive constants C_0, τ with $\tau > d - 1$. Here and henceforth we denote by $\cdot$ the standard scalar product in $\mathbb{R}^d$.

Define

$$\mathfrak{B}(\boldsymbol{\omega}) := \sum_{k=0}^{\infty} \frac{1}{2^k} \log \frac{1}{\alpha_k(\boldsymbol{\omega})}, \quad \alpha_k(\boldsymbol{\omega}) := \inf\{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| : \boldsymbol{\nu} \in \mathbb{Z}^d \, \text{such that} \, 0 < |\boldsymbol{\nu}| \leq 2^k\}. \tag{1.4}$$

The vector $\boldsymbol{\omega} \in \mathbb{R}^d$ satisfies the Bryuno condition if $\mathfrak{B}(\boldsymbol{\omega}) < \infty$. Notice that the Diophantine condition is stronger than the Bryuno one: in fact if $\boldsymbol{\omega}$ is a Diophantine vector, then it also satisfies the Bryuno condition, whereas the converse is false, see Section 4.4 of [60] for an example of a Bryuno number which is not Diophantine.

In [41] a special case of (1.1), with $g(x) = x^2$, was considered. Response solutions were proved to exist in the case of analytic periodic forcing term ($d = 1$) by requiring only Hypothesis 1 with $\mathfrak{n} = 1$ and in the case of quasi-periodic forcing term by adding a Diophantine condition on $\boldsymbol{\omega}$ to deal with the small divisors problem. In [42] the same results were extended to equation (1.1) with g an analytic function satisfying Hypothesis 1 and in [9], by adding the constraint $g'(c) > 0$, it was proved that the solution describes a local attractor in the phase plane $(x, \dot{x})$, which implies that there is a unique response solution to (1.1) in a neighbourhood of $(c, 0)$. In [42] it was also proved that such solutions are Borel summable at the origin when the frequency vector is either any one-dimensional number (periodic case) or a two-dimensional vector satisfying the Diophantine condition with $\tau = 1$.

In [44, 46] response solutions were proved to exist under Hypothesis 1 with $\mathfrak{n} > 1$ and with $\boldsymbol{\omega}$ satisfying the Bryuno condition.

In [20, 29] the hypothesis on the frequency vector considered in [44, 46] was weakened further: define the sequence $\varepsilon_k(\boldsymbol{\omega})$ as $\varepsilon_k(\boldsymbol{\omega}) := \frac{1}{2^k}\log\frac{1}{\alpha_k(\boldsymbol{\omega})}$, with $\alpha_k(\boldsymbol{\omega})$ as in (1.4) and assume that $\varepsilon_k(\boldsymbol{\omega})$ converges to zero (as $k \to \infty$). In particular in [20] Hypothesis 1 with $\mathfrak{n} = 1$ was considered and response solutions were produced through fixed point methods. Under this mild non-resonance condition, the response solution is C^∞ in ε and analytic in a domain of the complex plane with boundary tangent to the origin, see [20, 28, 29].

If we consider Hypothesis 1 with $\mathfrak{n} = 1$, the existence of response solutions to (1.1) can be also obtained by only assuming Hypothesis 2 on the frequency vector $\boldsymbol{\omega}$, see [47] for the one dimensional case of (1.1), that is $x \in \mathbb{R}$, [48] for the extension of the results in [47] to the m-dimensional case $x \in \mathbb{R}^m$ and [73] for the cases of analytic, finitely differentiable and low regularity forcing (see below). Without any other condition on $\boldsymbol{\omega}$, only a continuous dependence of the solution on ε and analyticity in ε in a conical domain of the complex plane can be proved [20, 28, 47, 73].

In [47] the existence of response solutions was also demonstrated for more general equation

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon h(x, \boldsymbol{\omega}t) = 0 \tag{1.5}$$

with $h : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R}$ a real analytic function, by assuming only Hypothesis 2 on $\boldsymbol{\omega}$ and requiring $h_{\mathbf{0}}(x)$ to have a simple zero c, that is the equivalent version of Hypothesis 1 with $\mathfrak{n} = 1$; then this result was extended to the m-dimensional case in [48].

In [73] response solutions of (1.1) were proved to exist under weaker regularity assumptions on f and g: in the case of highly differentiable functions f and g, such as $f \in H^m$, with $m > d/2$, and $g \in C^{m+l}$, with $l = 1, 2, \dots$, existence of the response solution is obtained under very weak assumptions of regularity on the forcing and on the mechanical force, such as f in the L^2 space and g Lipschitz continuous. The solutions were produced through methods of fixed point theorem, which also requires a smallness assumption either for f or for the nonlinear part of g.

1.3 Main results

This work aims to be a generalisation of [47], where the case of $\mathfrak{n} = 1$ is treated; we want to show a similar result of existence for (1.1) under the weaker Hypothesis 1 with $\mathfrak{n} > 1$. First of all, we confine ourselves to the case of f a trigonometric polynomial and we prove the following Theorem.

Theorem 1. *Consider the ordinary differential equation* (1.1) *with f a trigonometric polynomial and assume Hypothesis 1 with $\mathfrak{n} > 1$. For any frequency vector $\boldsymbol{\omega} \in \mathbb{R}^d$ satisfying Hypothesis 2, there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there is at least one quasi-periodic solution $x(t) = c + X(\boldsymbol{\omega}t, \varepsilon)$ to* (1.1) *such that $X(\boldsymbol{\psi}, \varepsilon)$ is analytic in $\boldsymbol{\psi}$ and goes to zero as $\varepsilon \to 0$.*

The same result is already contained in [29], but our proof is slightly simpler. In [29] the solution of (1.1) is written as a power series in a suitable parameter different from ε. In order to bound the small divisors, a quantity α, depending on the order $\mathfrak{n}$ (see Hypothesis 1) and on the degree N of the trigonometric polynomial, is introduced: $\alpha := \min\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le (\mathfrak{n}+1)N\}$. Then the power series is proved to be convergent provided $|\varepsilon| < \varepsilon_0$, with ε_0 small enough proportional to α: this requires a careful estimate of the relation between end nodes and lines (which are defined in Chapter 2) with propagators which can be bounded proportionally to α (we refer to Section II of [29] for details). Here we write the solution as a series which is not a power series expansion and use inductive estimates to prove that it converges provided $|\varepsilon| < \varepsilon_0$, with ε_0 small enough. To do that, we use renormalization group ideas and multiscale decomposition techniques: with respect to [29] we weaken the dependence on the degree N of the parameter α to which ε_0 is proportional by requiring $\alpha := \min\{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| : 0 < |\boldsymbol{\nu}| \le \mathfrak{n}N\}$.

The generalisation of Theorem 1 to the case of f an analytic function is not as simple as one could think. In order to deal with the small divisors problem, we will confine ourselves to the two-dimensional case of frequency vectors and we will need the properties of continued fractions. We refer to [55, 57, 59, 63, 64] for a general overview on continued fractions. Without lost of generality, we will assume $\boldsymbol{\omega}$ to be of the form $\boldsymbol{\omega} := (1, \alpha)$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (according to Hypothesis 2). Let A be a set, we denote by $\overline{A}$ the closure of A.

Theorem 2. *Consider the ordinary differential equation* (1.1) *with* $\boldsymbol{\omega} = (1, \alpha)$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, *and assume Hypothesis 1 with* $\mathfrak{n} > 1$. *Let* p_n/q_n *be the convergents of* α. *Let* C_2 *be the fixed positive constant*

$$C_2 = \frac{\mathfrak{n}+1}{4(\mathfrak{n}^2 + 2\mathfrak{n} - 1)}\xi, \tag{1.6}$$

and let C_1 *be an arbitrary positive constant. Then there exist* $\varepsilon_0 \in \mathbb{R}$ *and* $N_1, N_2 \in \mathbb{N}$, *with* ε_0 *small and positive, such that*

$$\frac{1}{(C_1 q_{N_1})^{\mathfrak{n}+1}} \le \varepsilon_0, \tag{1.7}$$

$$\frac{1}{(C_1 q_n)^{\mathfrak{n}+1}} \ge e^{-C_2 q_n}, \qquad \text{for } n \ge N_2 \tag{1.8}$$

and, by defining $N := \max\{N_1, N_2\}$ *and* I_n *as the interval*

$$I_n := \left[e^{-C_2 q_n}, (C_1 q_n)^{-\mathfrak{n}-1}\right], \qquad \text{for } n \ge N, \tag{1.9}$$

for all $\varepsilon \in \overline{\cup_{n\ge N} I_n}$ *there is at least one quasi-periodic solution* $x(t) = c + X(\boldsymbol{\omega}t, \varepsilon)$ *to* (1.1), *such that* $X(\boldsymbol{\psi}, \varepsilon)$ *is analytic in* $\boldsymbol{\psi}$ *in the strip* $\Sigma_{\xi'}$ *(with* $\xi' < \xi/4$*), is continuous in the sense of Whitney in* ε, *and goes to zero as* $\varepsilon \to 0$.

We refer to [75] for the continuity –and differentiability– in the sense of Whitney, but in general if A is a closed set in the Euclidean space E and $f(x)$ is a function defined and continuous in A, then this function can be extended so as to be continuous throughout E; see also [31], §125. See also refs. [7, 37] for a similar use of the notion of continuity in the sense of Whitney in different contexts.

We will find a connection between the frequency and the perturbation parameter: the existence of response solutions can be proved for any frequency vector with some restriction in ε or for any ε but with some restriction on $\boldsymbol{\omega}$. As we can see from (1.9), once fixed N as in Theorem 2, the solution does not exist for all $|\varepsilon| < \varepsilon_0$: if we do not assume any condition on the frequency vector, there is in general no relation between $\frac{1}{(C_1 q_{n+1})^{\mathfrak{n}+1}}$ and $e^{-C_2 q_n}$, we refer to Chapter 4 for more details on the properties of q_n. It can happen that $\frac{1}{(C_1 q_{n+1})^{\mathfrak{n}+1}} < e^{-C_2 q_n}$. Therefore we might obtain response solutions to exist in a set with holes: the size of the holes depends on the frequency vector, in particular on the irrational number α and, as a consequence, on the convergents p_n/q_n.

The rest of the work is organised as follows: in Chapter 2 we set up the problem in the case of f a trigonometric polynomial and in Chapter 3 we prove the existence of response solutions to (1.1) under such an assumption on f. In Chapter 4 we highlight the main properties of continued fraction that we use to prove Theorem 2 in Chapters 5.1 and 5. For concreteness, a specific case of odd $\mathfrak{n}$ ($\mathfrak{n} = 3$) is treated in Appendix A.

1.4 Some related results

Existence of quasi-periodic and almost-periodic solutions to ordinary differential equations – we refer to [11, 24, 35] for an introduction to almost periodic functions – in problems where no hypothesis is made on the frequencies has not been studied in the literature as extensively as in the case in which one requires some non-resonance condition such as the Diophantine or the Bryuno condition. For instance, in the framework of KAM theory, it is well known that, in general, the invariant tori of the unperturbed system which are close to resonances break up when the perturbation is switched on, see for example [13, 38]. Therefore, if one is looking for results holding for all frequencies one has to consider either conservative systems away from the KAM regime – see for instance [10, 11, 12] – or non-conservative systems.

Typically response solutions arise by bifurcation. Bifurcation phenomena have been widely investigated in the literature – for instance we could mention [16, 17, 51, 52], based on the singularity theory method. The method has been applied to the study of stable quasi-periodic solutions for periodically and quasi-periodically forced systems, especially in the conservative case – for example see [18, 53, 54]. In particular, quasi-periodically forced Hamiltonian oscillators are considered in [18], where the persistence of quasi-periodic solutions is studied in the case of resonance between the Diophantine frequency vector of the forcing and the proper normal-internal frequency: first, the Hamiltonian of the oscillator is written as a perturbation of an integrable one, which describes a suitable one-dimensional system (backbone system) and a collection of rotators; then, under some non-degeneracy assumptions, the behaviour of the full system is investigated according to the bifurcations of the backbone system.

In order to guarantee the existence of response solutions, a non-degeneracy condition is generally assumed on the unperturbed solution, usually a condition of hyperbolicity or exponential dichotomy – see for instance [14, 27, 34, 77]. For example in [14] an invariant torus bifurcates from the equilibrium point under a suitable assumption of hyperbolicity. Our point of view is different since the unperturbed bifurcating solution is not given a priori and it does not satisfy any stability or hyperbolicity condition: in fact, any constant c is a solution to (1.1) when $\varepsilon = 0$ and the existence of response solutions is proved for a special value of the constant, which is the odd order zero of the equation $g(x) = f_0$.

In [22, 50] the authors proved the existence of an almost periodic solution for the differential equation

$$\dot{x} = f(t, x, y, \varepsilon), \qquad \varepsilon\dot{y} = g(t, x, y, \varepsilon), \tag{1.10}$$

with $\varepsilon > 0$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and f, g almost periodic in time, by requiring some assumptions on the unperturbed almost periodic solution of the unperturbed system

$$\dot{x} = f(t, x, y, 0), \qquad 0 = g(t, x, y, 0) \tag{1.11}$$

and by requiring that the linearised equation associated to the system possesses an exponential dichotomy. Notice that equation (1.1) is a particular case of (1.10), but the point of view considered in [22, 50] is different since the problem is of hyperbolic type and no small divisors appear. In both papers the vectors f, g with the respective Jacobian matrices are supposed to be almost-periodic in time, continuous in (x, y, ε) and bounded. Analogous results for equations of the form (1.10) are also provided in [66] always in the almost periodic context, and in [1, 34, 36, 71, 76], in the periodic context. The assumption of exponential dichotomy on the linearised system was also applied to the studies of almost periodic solutions to regularly perturbed linear and nonlinear systems; see for example [27, 34, 56, 77].

As we said in the previous section, we will use the continued fraction theory to bound the small divisors in the case of analytic f: the continued fractions are widely used when the frequency is a Liouvillean vector, see for instance [58, 74]. In [58] the authors look for a response solution to the equation

$$\ddot{x} + \lambda^2 x = \varepsilon F(\boldsymbol{\omega} t, x, \dot{x}), \quad x \in \mathbb{R}, \quad \boldsymbol{\omega} = (1, \alpha),\ \alpha \in \mathbb{R} \setminus \mathbb{Q}, \tag{1.12}$$

where ε, λ are positive parameters and $F : \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a real analytic function satisfying the reversibility condition

$$F(\boldsymbol{\psi}, x, y) = F(-\boldsymbol{\psi}, x, -y). \tag{1.13}$$

Without assuming $\boldsymbol{\omega}$ to satisfy any strong non-resonant (Diophantine or Bryuno) conditions, the authors proved that for any closed interval $\mathcal{O} \subset \mathbb{R} \setminus \{0\}$ and any sufficiently small $\gamma > 0$, there exist $\varepsilon_0 > 0$ and a Cantor set $\mathcal{O}_\gamma \subset \mathcal{O}$ with Lebesgue measure $meas(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$, such that if $0 < \varepsilon < \varepsilon_0$ and $\lambda \in \mathcal{O}_\gamma$, the quasi-periodically forced nonlinear oscillator described by (1.12) admits a response solution. The system considered in [74] is different with respect to (1.12) since the function F does not depend on $\dot{x}$. Both [58, 74] use the continued fractions theory and exploits the so-called CD-bridge between the denominators of the best rational approximation of a continued fraction, introduced by A. Avila, B. Fayad and R. Krikorian in [2]. With respect to [74], a modified KAM scheme is proposed in [58] to deal with the small divisors problem: in particular in [58], the authors impose the second Melnikov's condition for a Liouvillean frequency vector $\boldsymbol{\omega}$ by varying some parameters in $\mathcal{O}$. Notice that our problem is not the same as (1.12) since our system does not verify the reversibility condition in (1.13) and is a singular differential equation.

Chapter 2

Setting of the problem

Let us denote by $\Delta(c,\rho)$ the disk of center c and radius ρ in the complex plane.

By the assumption of analyticity on g, for any $c \in \mathbb{R}$ there exists $\rho_0 > 0$ such that $x \to g(x)$ is analytic in $\Delta(c,\rho_0)$. Then for all $\rho < \rho_0$, if we take c as in Hypothesis 1 and if we define $\Gamma := \max\{|g(x)| : x \in \Delta(c,\rho)\}$, one has

$$g(x) = g(c) + \sum_{p=\mathfrak{n}}^{\infty} g_p (x-c)^p, \quad g_p := \frac{1}{p!}\frac{\partial^p g}{\partial x^p}(c), \quad |g_p| \le \Gamma \rho^{-p}, \tag{2.1}$$

where we have used Cauchy's estimates for analytic functions.

Let us suppose that the quasi-periodic forcing term f is a trigonometric polynomial of degree N, that means $f_{\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$ such that $|\boldsymbol{\nu}| > N$. Hence if we define $\Phi := \max\{|f_{\boldsymbol{\nu}}| : |\boldsymbol{\nu}| \le N\}$, we have

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} f_{\boldsymbol{\nu}} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}, \qquad |f_{\boldsymbol{\nu}}| \le \Phi. \tag{2.2}$$

We shall see later how to modify the forthcoming analysis in order to discuss the case in which f is an analytic function containing all the harmonics.

Define

$$\alpha := \min_{0<|\boldsymbol{\nu}|\le \mathfrak{n}N} |\boldsymbol{\omega}\cdot\boldsymbol{\nu}|; \tag{2.3}$$

by the assumption of irrationality on $\boldsymbol{\omega}$, cf. Hypothesis 2, one has $\alpha > 0$.

We are interested in the existence of response solutions to (1.1), i.e. solutions of the form

$$x(t,\varepsilon) = c + \zeta + X(\boldsymbol{\omega}t;\varepsilon,\zeta,c), \tag{2.4}$$

where ζ is a value that has to be fixed and $\boldsymbol{\psi} \to X(\boldsymbol{\psi};\varepsilon,\zeta)$ is a zero-average function. Since we are looking for quasi-periodic solutions $X(\boldsymbol{\psi};\varepsilon,\zeta)$, analytic in $\boldsymbol{\psi}$, we can write (1.1) in Fourier space as

$$(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu})X_{\boldsymbol{\nu}} + \varepsilon[g(c+\zeta+X)]_{\boldsymbol{\nu}} = \varepsilon f_{\boldsymbol{\nu}} \qquad \boldsymbol{\nu} \neq \mathbf{0}, \tag{2.5}$$

with

$$X(\boldsymbol{\omega}t;\varepsilon,\zeta,c)=\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*}X_{\boldsymbol{\nu}}e^{\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}t}\qquad X_{\boldsymbol{\nu}}\equiv X_{\boldsymbol{\nu}}(\varepsilon,\zeta,c),$$

where $\mathbb{Z}^d_*:=\mathbb{Z}^d\setminus\{0\}$, provided for $\boldsymbol{\nu}=\mathbf{0}$ one has

$$[g(c+\zeta+X)]_{\mathbf{0}}=f_{\mathbf{0}}. \tag{2.6}$$

We call (2.5) the *range equation* and (2.6) the *bifurcation equation*, see also [25, 49, 65]. We first study the range equation looking for a solution to (2.5), depending on the parameter ζ that is supposed to be close enough to zero. Then we analyse the bifurcation equation (2.6) and we try to fix ζ in order to make this equation to be satisfied as well.

In order to write the perturbation expansion of the response solutions, we use some graphical objects, that we call *trees*, which we describe below.

2.1 Tree formalism

A *rooted tree* ϑ is a graph with no cycle, such that all the lines are oriented toward a unique point, that we call *root*, which has only one incident line, the *root line*. All the points in ϑ, except the root, are called *nodes*. The orientation of the lines in ϑ induces a partial ordering relation ($\preceq$) between the nodes. Given two nodes v and w, we shall write $w\prec v$ every time v is along the path (of lines) which connects w to the root; we shall write $w\prec\ell$ if $w\preceq v$, where v is the unique node that the line ℓ enters, see figure 2.1. For any node v, denote by p_v the number of lines entering v.

Given a rooted tree ϑ, we call *first node* the node the root line exits and we denote by $N(\vartheta)$ the set of nodes, by $E(\vartheta)$ the set of *end nodes*, i.e. nodes v with $p_v=0$, by $V(\vartheta)=N(\vartheta)\setminus E(\vartheta)$ the set of *internal nodes* and by $L(\vartheta)$ the set of lines. We impose the constraint $p_v\geq\mathfrak{n}$, $\forall v\in V(\vartheta)$. If for any discrete set A we denote by $|A|$ its cardinality, we define the *order* of ϑ as $k\equiv k(\vartheta):=|N(\vartheta)|$. See [45] for a general overview on the tree formalism.

Given a rooted tree ϑ, for any node $w\in N(\vartheta)$ the line exiting w can be considered as the root line of a tree θ_w formed by the nodes $v\in N(\vartheta)$ such that $v\preceq w$, by the lines which join such nodes and by the root line itself. Such a tree is called a *subtree* of ϑ with first node w.

We associate with each end node $v\in E(\vartheta)$ a *mode* label $\boldsymbol{\nu}_v\in\mathbb{Z}^d$ and we split $E(\vartheta)$ in two complementary sets: $E_0(\vartheta)=\{v\in E(\vartheta):\boldsymbol{\nu}_v=\mathbf{0}\}$ and $E_1(\vartheta)=\{v\in E(\vartheta):\boldsymbol{\nu}_v\neq\mathbf{0}\}$, in such a way that $E(\vartheta)=E_0(\vartheta)\sqcup E_1(\vartheta)$. With each line $\ell\in L(\vartheta)$ we associate a *momentum* $\boldsymbol{\nu}_\ell\in\mathbb{Z}^d$ with the *conservation law*

$$\boldsymbol{\nu}_\ell=\sum_{\substack{w\in E_1(\vartheta)\\ w\prec\ell}}\boldsymbol{\nu}_w, \tag{2.7}$$

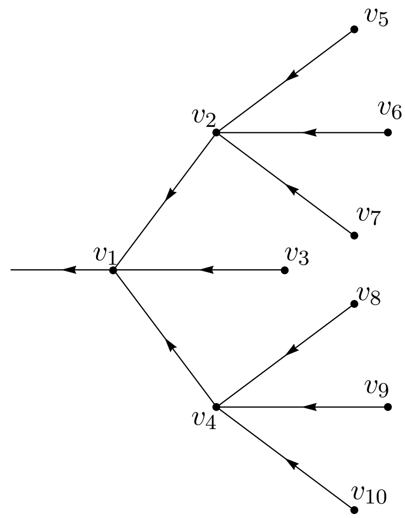

Figure 2.1: An example of tree with 10 vertices, 7 end nodes $v_3, v_5, v_6, v_7, v_8, v_9, v_{10}$ and 3 internal nodes v_1, v_2, v_4. According to the definition of *partial* ordering relation, $v_5 \prec v_2$ but there is no relation between v_5 and v_4.

i.e. the momentum of the line ℓ is the sum of the mode labels associated with the end nodes preceding ℓ. Equivalently, by induction, the momentum of the line ℓ is the sum of the momenta of the lines entering v, if v is the node the line ℓ exits.

With each line $\ell \in L(\vartheta)$ we also assign a *scale label* $\widetilde{n}_\ell \in \{0,1\}$. Generally one can consider the scale label as a number in $\mathbb{Z}$ (we refer to [45] for a general overview on multiscale analysis) but, for the problem we are studying, we just need the scale label to be either zero or one.

Definition 2.1 (Labelled trees)**.** *We call labelled rooted tree a rooted tree with the labels associated with $N(\vartheta)$ and $L(\vartheta)$.*

Definition 2.2 (Equivalent trees)**.** *We call equivalent two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other.*

In the following we shall consider only inequivalent labelled rooted trees and we denote by $\mathcal{T}_{k,\boldsymbol{\nu}}$ the set of non-equivalent trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line.

Definition 2.3 (Cluster)**.** *A cluster T on scale $\widetilde{n}$ is a maximal set of nodes and lines connecting them such that all the lines have scales $\widetilde{n}' \le \widetilde{n}$ and there is at least one line with scale $\widetilde{n}$. The lines entering the cluster T and the possible unique line coming out from it (if existing at all) are called the external lines of the cluster T. Given a cluster T on scale $\widetilde{n}$, we shall call $\widetilde{n}_T := \widetilde{n}$ the scale of the cluster. We call $V(T)$, $E(T)$ and $L(T)$ the set of internal nodes, of end nodes and of lines of the cluster T respectively,* *with the convention that the external lines of T do not belong to $L(T)$. In particular we denote with $E_1(T)$ and $E_0(T)$ the following sets: $E_1(T) := \{v \in E(T) : \boldsymbol{\nu}_v \neq \mathbf{0}\}$ and $E_0(T) := E(T) \setminus E_1(T)$.*

According to the definition of scale label, we have two possibilities for $\widetilde{n}_T$ (i.e. $\{0,1\}$): notice that we will only deal with clusters on scale 0, because the only possible cluster on scale 1 is the whole tree $\mathcal{T}_{k,\boldsymbol{\nu}}$ where there is at least one line on scale 1.

Definition 2.4 (Self-energy cluster and renormalized tree)**.** *We call self-energy cluster any cluster T (on scale 0) such that T has only one entering line that has the same momentum of exiting line. We denote by $\mathfrak{T}_{k,\boldsymbol{\nu}}$ the set of renormalized trees in $\mathcal{T}_{k,\boldsymbol{\nu}}$, i.e. of trees that do not contain any self-energy clusters and by $\mathfrak{R}_0$ the set of self-energy clusters (all on scale 0 by construction).*

Notice that, by Definition 2.4, the mode labels associated with the end nodes in a self-energy cluster T are such that

$$\sum_{v \in E(T)} \boldsymbol{\nu}_v = \mathbf{0}.$$

Let us introduce a *sharp partition of unity*: let χ and Ψ be functions defined on $\mathbb{R}_+$, such that

$$\chi(x) := \begin{cases} 1 & \text{for } x < \frac{\alpha}{2}, \\ 0 & \text{for } x \geq \frac{\alpha}{2}, \end{cases} \qquad \Psi(x) := \begin{cases} 0 & \text{for } x < \frac{\alpha}{2}, \\ 1 & \text{for } x \geq \frac{\alpha}{2}, \end{cases} \tag{2.8}$$

with α as in (2.3). Note that $\chi(x) + \Psi(x) = 1$, for all $x \in \mathbb{R}_+$.

We associate with each node $v \in N(\vartheta)$ a *node factor*

$$F_v := \begin{cases} -\varepsilon\, g_{p_v}, & v \in V(\vartheta), \\ \varepsilon\, f_{\boldsymbol{\nu}_v}, & v \in E_1(\vartheta), \\ \zeta, & v \in E_0(\vartheta), \end{cases} \tag{2.9}$$

and with each line $\ell \in L(\vartheta)$ a *propagator*

$$G_\ell \equiv G^{[\widetilde{n}_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \zeta, c), \quad \widetilde{n}_\ell \in \{0,1\}, \tag{2.10}$$

where the functions $G^{[\widetilde{n}_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \zeta, c)$ are defined as follows. For $\boldsymbol{\nu}_\ell \neq \mathbf{0}$

$$\begin{aligned} G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \zeta, c) &:= \frac{\Psi(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{\mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell (1 + \mathrm{i}\varepsilon \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)}, \\ G^{[1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \zeta, c) &:= \frac{\chi(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{\mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell (1 + \mathrm{i}\varepsilon \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) - \mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon, \zeta, c)}, \end{aligned} \tag{2.11}$$

with

$$\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\zeta,c) := \chi(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|)M(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\zeta,c), \tag{2.12a}$$

$$M(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\zeta,c) := \sum_{T\in\mathfrak{R}_0} \mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\zeta,c), \tag{2.12b}$$

$$\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon,\zeta,c) := \Bigg(\prod_{\ell\in L(T)} G_\ell\Bigg)\Bigg(\prod_{v\in V(T)} F_v\Bigg), \tag{2.12c}$$

where $\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon,\zeta,c)$ is called *the value of the self-energy cluster* T.

For $\boldsymbol{\nu}_\ell = \mathbf{0}$, we set $G_\ell = 1$ and for convenience we assign these lines the scale 0 only, that is $G^{[0]}(0;\varepsilon,\zeta,c) = G_\ell = 1$.

Remark 2.5. With the sharp partition considered above, a momentum $\boldsymbol{\nu}$ identifies uniquely the scale $\widetilde{n}$.

In order to simplify the notation we omit the dependence on the parameters ε,ζ,c in (2.10), (2.11), (2.12a), (2.12b) and (2.12c), hence, from now on, we will just write $G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, $G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, $M(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, $\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu})$.

Remark 2.6. Since we are considering only cluster on scale 0, the product over the lines in (2.12c) involves only propagators on scale 0, that means the formula of $\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ can be rewritten as

$$\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = \Bigg(\prod_{\ell\in L(T)} G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\Bigg)\Bigg(\prod_{v\in V(T)} F_v\Bigg).$$

Therefore the expressions of $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ in (2.12a) and of $\mathscr{V}(T,\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ in (2.12c) are well posed.

Now define *the value of the renormalized tree* ϑ as

$$\mathscr{V}(\vartheta) \equiv \mathscr{V}(\vartheta;\varepsilon,\zeta,c) := \Bigg(\prod_{\ell\in L(\vartheta)} G_\ell\Bigg)\Bigg(\prod_{v\in V(\vartheta)} F_v\Bigg) \tag{2.13}$$

and set

$$X^{[k]}_{\boldsymbol{\nu}} := \sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu}}} \mathscr{V}(\vartheta), \tag{2.14}$$

where $\mathfrak{T}_{k,\boldsymbol{\nu}}$ denotes the set of all renormalized trees of order k and momentum $\boldsymbol{\nu}$ associated with the root line, see Definition 2.4.

Finally we can define the *renormalized series* or *renormalized expansion* as

$$\bar{X}(\boldsymbol{\psi}) \equiv \bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c) := \sum_{\boldsymbol{\nu}\in Z^d_*} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\bar{X}_{\boldsymbol{\nu}}, \quad \bar{X}_{\boldsymbol{\nu}} := \sum_{k=1}^{\infty} X^{[k]}_{\boldsymbol{\nu}}. \tag{2.15}$$

Remark 2.7. In order to simplify the notation, we do not write the dependence on ε, c and ζ of the coefficients $X^{[k]}_{\boldsymbol{\nu}}$ in (2.15).

Our goal is to prove that the series $\bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ in (2.15) converges and solves the range equation (2.5), i.e.

$$\mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu}(1 + \mathrm{i}\boldsymbol{\omega} \cdot \boldsymbol{\nu})\bar{X}_{\boldsymbol{\nu}} = \varepsilon \, [f - g(c + \zeta + \bar{X})]_{\boldsymbol{\nu}}, \tag{2.16}$$

for all $\boldsymbol{\nu} \neq 0$.

Chapter 3

Proof of Theorem 1

Let α be as in (2.3) and set η as $\eta := \max\{|\varepsilon|, |\zeta|\} = \max\{\varepsilon, |\zeta|\}$, since ε is real and positive.

3.1 Bounds on the propagators

Lemma 3.1. *Given a tree ϑ, consider a self-energy cluster T in ϑ with momentum $\boldsymbol{\nu}$ associated with its external lines and with order k_T. Then one has*

$$|\mathscr{V}(T, \boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \rho\Gamma\, \bar{C}^{k_T}\, \varepsilon\, \eta^{k_T - 1}, \tag{3.1}$$

where $\bar{C}$ is defined as

$$\bar{C} := \rho^{-1} \max\Big\{\frac{2\Gamma}{\alpha}, \frac{2\Phi}{\alpha}, 1\Big\}, \tag{3.2}$$

with Γ, ρ as in (2.1) and Φ, α respectively as in (2.2), (2.3).

Proof. As we have discussed in Remark 2.6, we only deal with self-energy clusters on scale 0; hence we have

$$|\mathscr{V}(T, \boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \prod_{\ell \in L(T)} |G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| \prod_{v \in V(T)} |F_v|.$$

If $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ and $\widetilde{n}_\ell = 0$, by the sharp partition considered in (2.8), it follows that

$$|G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| \leq \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|} \leq \frac{2}{\alpha},$$

instead if $\boldsymbol{\nu}_\ell = \mathbf{0}$, one has $G^{[0]}(0) = 1$.

Therefore it follows that

$$\begin{aligned}|\mathscr{V}(T, \boldsymbol{\omega} \cdot \boldsymbol{\nu})| &\leq \Gamma^{|V(T)|}\Phi^{|E_1(T)|}\rho^{-(k_T-1)}\Big(\frac{2}{\alpha}\Big)^{|V(T)|-1+|E_1(T)|} \varepsilon\, \eta^{|V(T)|-1+|E_1(T)|+|E_0(T)|} \\ &\leq \rho\Gamma\bar{C}^{k_T}\, \varepsilon\, \eta^{k_T-1},\end{aligned}$$

with $\bar{C}$ as in (3.2). □

Remark 3.2. The lines exiting end nodes in $E_1(\vartheta)$ are on scale 0 since f is a trigonometric polynomial. In fact, if we consider $v \in E_1(\vartheta)$ and the corresponding exiting line ℓ, by the definition of node factor (2.9) it follows that $|\boldsymbol{\nu}_\ell| \leq N$. Therefore, by using the definition of α in (2.3) we see $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| \geq \alpha > \frac{\alpha}{2}$, that implies the line ℓ to be on scale 0. This can be generalised as long as $|\boldsymbol{\nu}_\ell| \leq \mathfrak{n}N$.

Lemma 3.3. *Given a tree ϑ, consider a self-energy cluster T in ϑ with momentum $\boldsymbol{\nu}$ associated with its external lines. For η small enough, one has*

$$|\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \geq A\,\varepsilon\,\eta^{\mathfrak{n}-1}, \tag{3.3}$$

with $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ defined in (2.12a) *and A a suitable positive constant depending on $\Phi, \Gamma, \rho, \alpha$, i.e. $A \equiv A(\Phi, \Gamma, \rho, \alpha)$, with Γ, ρ as in* (2.1) *and Φ, α respectively as in* (2.2), (2.3).

Proof. A cluster T must contain at least $\mathfrak{n}$ nodes, i.e. $k_T \geq \mathfrak{n}$. Indeed let us consider a tree with order $k \geq \mathfrak{n}+1$, in which the root line, ℓ_0, exits a node $v_0 \in V(\vartheta)$. By construction $p_{v_0} \geq \mathfrak{n}$, therefore a cluster T, if exists, must contain at least $\mathfrak{n}-1$ lines on scale 0 entering v_0 and hence $\mathfrak{n}-1$ nodes besides v_0.

In particular, by the analysis above, we notice that if a self-energy cluster has only $\mathfrak{n}$ nodes, then $\mathfrak{n}-1$ of such nodes are in $E(\vartheta)$ and the external lines of the cluster exit/enter the same node, see Figure 3.1.

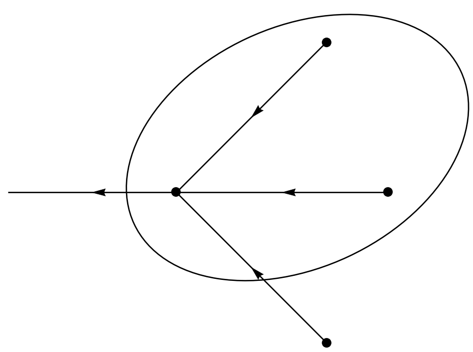

Figure 3.1: A cluster with $\mathfrak{n} = 3$ nodes.

We denote by $\mathcal{M}_\mathfrak{n}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$ the contributions to $\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$ corresponding to self-energy clusters with $\mathfrak{n}$ nodes and we denote by $\Delta\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$ the other summands of $\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$, see (2.12a) and (2.12b). Notice that $\mathcal{M}_\mathfrak{n}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) = \mathcal{M}_\mathfrak{n}(0)$, i.e. $\mathcal{M}_\mathfrak{n}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$ does not depend on $\boldsymbol{\nu}$ and is real. Hence we have

$$|\mathcal{M}_\mathfrak{n}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \geq A_0 \varepsilon \Big[(X^{[1]} + \zeta)^{\mathfrak{n}-1}\Big]_{\mathbf{0}}, \tag{3.4}$$

for a suitable positive constant A_0 depending on $\Phi, \Gamma, \rho, \alpha$. The right hand term of (3.4) represents the contribution of the self-energy clusters with $\mathfrak{n}$ nodes.

If $\zeta = o(\varepsilon)$, then
$$\left[(X^{[1]} + \zeta)^{\mathfrak{n}-1}\right]_{\mathbf{0}} \geq A_1 \varepsilon^{\mathfrak{n}-1},$$
for a suitable positive constant A_1 depending on Φ, α.

Analogously, if $\varepsilon = o(\zeta)$, then

$$\left[(X^{[1]} + \zeta)^{\mathfrak{n}-1}\right]_{\mathbf{0}} \geq A_2 \zeta^{\mathfrak{n}-1},$$

for a suitable positive constant A_2 depending on Φ, α.

If $\zeta = a\varepsilon + o(\varepsilon)$, with $a \neq 0$, then

$$\begin{aligned}\left[(X^{[1]} + \zeta)^{\mathfrak{n}-1}\right]_{\mathbf{0}} &= \left[(X^{[1]} + a\varepsilon + o(\varepsilon))^{\mathfrak{n}-1}\right]_{\mathbf{0}} \\ &\geq [(u^{[1]} + a)^{\mathfrak{n}-1}]_{\mathbf{0}}\, \varepsilon^{\mathfrak{n}-1} + o(\varepsilon^{\mathfrak{n}-1}) \\ &\geq A_3\, \varepsilon^{\mathfrak{n}-1},\end{aligned}$$

with A_3 positive constant and where we have defined the function $u^{[1]}_{\boldsymbol{\nu}}$ in such a way that $X^{[1]}_{\boldsymbol{\nu}} = \varepsilon u^{[1]}_{\boldsymbol{\nu}}$.

Hence we conclude that
$$|\mathcal{M}_{\mathfrak{n}}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \geq \widetilde{A}_0 \varepsilon \eta^{\mathfrak{n}-1}, \tag{3.5}$$
for a suitable positive constant $\widetilde{A}_0$ depending on A_0, A_1, A_2, A_3.

Therefore, from (3.5) and Lemma 3.1 applied to $\Delta\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$, we have:

$$|\Delta\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| < \widetilde{a}_1 \varepsilon \eta^{\mathfrak{n}} < \frac{|\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})|}{4}, \tag{3.6}$$

for a suitable positive constant $\widetilde{a}_1$ and for ε, η small enough.

Hence we obtain (3.3):

$$\begin{aligned}|\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| &\geq |\mathcal{M}_{\mathfrak{n}}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| - |\Delta\mathcal{M}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \\ &\geq \widetilde{a}_0 \varepsilon \eta^{\mathfrak{n}-1} - \widetilde{a}_1 \varepsilon \eta^{\mathfrak{n}} \\ &\geq A \varepsilon \eta^{\mathfrak{n}-1}\end{aligned}$$

for a suitable positive constants $\widetilde{a}_0$ and $\widetilde{a}_1$ depending on $\widetilde{A}_0$, ρ and $\bar{C}$ and $A = \widetilde{a}_0/2$. □

Lemma 3.4. *For any $\ell \in L(\vartheta)$, one has*

$$\begin{cases} |G^{[1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| \leq 4A^{-1}\varepsilon^{-1}\eta^{-\mathfrak{n}+1}, & \text{if } \boldsymbol{\nu}_\ell \neq \mathbf{0} \text{ and } \widetilde{n}_\ell = 1; \\ |G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| \leq \frac{2}{\alpha}, & \text{if } \boldsymbol{\nu}_\ell \neq \mathbf{0} \text{ and } \widetilde{n}_\ell = 0; \\ |G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)| = 1, & \text{if } \boldsymbol{\nu}_\ell = \mathbf{0}; \end{cases} \tag{3.7}$$

where A is as in Lemma 3.3.

Proof. If $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ and $\widetilde{n}_\ell = 1$, one has

$$|\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell(1+\mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) - \mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| \geq \frac{A\,\varepsilon\,\eta^{\mathfrak{n}-1}}{4}.$$

This can be proved as follows. Let $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}$. If $\varepsilon x^2 < \frac{|\mathcal{M}(x)|}{2}$, then

$$\begin{aligned} |\mathrm{i}x - (\varepsilon x^2 + \mathcal{M}(x))| &\geq |\mathrm{i}x - (\varepsilon x^2 + \mathcal{M}_{\mathfrak{n}}(0)) - \Delta\mathcal{M}(x))| \\ &\geq |\mathrm{i}x - (\varepsilon x^2 + \mathcal{M}_{\mathfrak{n}}(0))| - |\Delta\mathcal{M}(x))| \\ &\geq |\varepsilon x^2 + \mathcal{M}_{\mathfrak{n}}(0)| - |\Delta\mathcal{M}(x)| \\ &\geq \frac{|\mathcal{M}(x)|}{2} - \frac{|\mathcal{M}(x)|}{4} \geq \frac{A\,\varepsilon\,\eta^{\mathfrak{n}-1}}{4}, \end{aligned}$$

where we have used the bound (3.6) on $\Delta\mathcal{M}(x)$ and Lemma 3.3. If $\varepsilon x^2 > \frac{|\mathcal{M}(x)|}{2}$, again by using (3.6) and Lemma 3.3, one has

$$\begin{aligned} |\mathrm{i}x - (\varepsilon x^2 + \mathcal{M}(x))| &\geq |\mathrm{i}x - (\varepsilon x^2 + \mathcal{M}_{\mathfrak{n}}(0))| - |\Delta\mathcal{M}(x))| \\ &\geq |x| - |\Delta\mathcal{M}(x)| > \frac{\sqrt{A}}{4}\eta^{(\mathfrak{n}-1)/2} > \frac{A}{4}\varepsilon\eta^{\mathfrak{n}-1}, \end{aligned}$$

for η and ε small enough. Then the bound of $|G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)|$ follows directly from the definition (2.11).

If $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ and $\widetilde{n}_\ell = 0$, by the sharp decomposition considered in (2.8), it follows that

$$|G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| \leq \frac{1}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|} \leq \frac{2}{\alpha}.$$

If $\boldsymbol{\nu}_\ell = \mathbf{0}$, the last equation follows directly from the definition of $G^{[0]}(0)$. □

3.2 An estimate for $\mathscr{V}(\boldsymbol{\vartheta})$

Set

$$C := \rho^{-1}\max\left\{\frac{2\Gamma}{\alpha}, \frac{\Phi}{\alpha}, \frac{4\Gamma}{A}, 1\right\} \tag{3.8}$$

where ρ, Γ are as in (2.1), Φ is as in (2.2), α as in (2.3) and A is the constant defined in Lemma 3.3.

The following results provide an estimate for $\mathscr{V}(\vartheta)$, with ϑ a renormalized tree.

Lemma 3.5. *Let ϑ be a tree of order $k=1$, that is $\vartheta \in \mathfrak{T}_{1,\boldsymbol{\nu}}$. Then*

$$|\mathscr{V}(\vartheta)| \leq \rho C \begin{cases} \varepsilon, & \text{if } \boldsymbol{\nu} \neq \mathbf{0}; \\ |\zeta|, & \text{if } \boldsymbol{\nu} = \mathbf{0}. \end{cases} \tag{3.9}$$

Notice that when the order of the tree is 1, then $\mathcal{T}_{1,\boldsymbol{\nu}} \equiv \mathfrak{T}_{1,\boldsymbol{\nu}}$.

Figure 3.2: An example of trees with $k = 1$.

Proof. Let $k = 1$, that means there is only one node, v_0, that is in particular an end node. There are two possibilities: $v_0 \in E_0(\vartheta)$ or $v_0 \in E_1(\vartheta)$, see Figure 3.2.

Since we are in the situation described in Remark 3.2, there are no small divisors associated with the root line. Hence if $\boldsymbol{\nu} \neq \mathbf{0}$, the value of the tree is bounded by

$$|\mathscr{V}(\vartheta)| \leq \Phi\frac{\varepsilon}{\alpha} \leq \rho C\varepsilon, \tag{3.10}$$

while if $\boldsymbol{\nu} = 0$, one has

$$|\mathscr{V}(\vartheta)| = |\zeta| \leq \rho C|\zeta|, \tag{3.11}$$

with C as in (3.8).

$\square$

Remark 3.6. Since we are interested in studying the range equation, we require that the momentum of the root line is not zero. By construction, the momentum of every internal line is different from 0, as well.

Remark 3.7. In Remark 3.2, we have seen that if $|\boldsymbol{\nu}_\ell| \leq \mathfrak{n}N$, then $\widetilde{n}_\ell = 0$. This also means that if a line $\ell \in L(\vartheta)$ is on scale 1 then $|\boldsymbol{\nu}_\ell| > \mathfrak{n}N$. Therefore if we consider an internal line ℓ, that by construction is such that $\boldsymbol{\nu}_\ell \neq 0$, see Remark 3.6, these are the possible cases:

a) $|\boldsymbol{\nu}_\ell| \leq \mathfrak{n}N \quad (\Rightarrow \widetilde{n}_\ell = 0)$;

b) $|\boldsymbol{\nu}_\ell| > \mathfrak{n}N \quad \text{and } \widetilde{n}_\ell = 0$;

c) $|\boldsymbol{\nu}_\ell| > \mathfrak{n}N \quad \text{and } \widetilde{n}_\ell = 1$.

Lemma 3.8. *Let $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ be a renormalized tree of order $k \geq \mathfrak{n} + 1$ and momentum of the root line $\boldsymbol{\nu} \neq \mathbf{0}$. Then*

$$|\mathscr{V}(\vartheta)| \leq \rho C^k \eta^{\gamma k + \beta} \qquad \gamma := \frac{1}{2\mathfrak{n}+1}, \quad \beta := \frac{3\mathfrak{n}}{2\mathfrak{n}+1}, \tag{3.12}$$

with ρ and C as in (2.1) *and* (3.8) *respectively.*

Proof. Let $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ with $k \geq \mathfrak{n}+1$ and let v_0 be the first node, that is the node the root line exits. For any renormalized tree, we have the following structure:

- $\vartheta_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell_1}}, \dots, \vartheta_m \in \mathfrak{T}_{k_m,\boldsymbol{\nu}_{\ell_m}}$ are m renormalized trees of order $k_j > 1$ with $j = 1, \dots, m$ entering v_0;
- $\ell'_1, \dots, \ell'_s$ are s lines exiting end nodes in $E_1(\vartheta)$ and entering v_0;
- $\ell''_1, \dots, \ell''_r$ are r lines exiting end nodes in $E_0(\vartheta)$ and entering v_0;
- $m + r + s \geq \mathfrak{n}$;
- $m, r, s \geq 0$;
- $k = k_1 + \cdots + k_m + r + s + 1$;
- $\boldsymbol{\nu} = \sum_{j=1}^{m} \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^{s} \boldsymbol{\nu}_{\ell'_j}$.

Each ϑ_j, for $j = 1, \dots, m$, is represented by the graph element in Figure 3.3 as a line with label $\boldsymbol{\nu}_{\ell_j}$ exiting a ball with label (k_j).

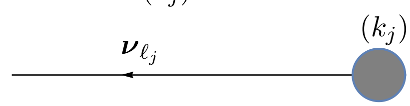

Figure 3.3

Then we can graphically represent the tree ϑ described above as depicted in Figure 3.4.

By induction, we assume that for any $1 < k' < k$, (3.12) holds with k' instead of k. Moreover we use the first of (3.9) to bound the value of any element in Figure 3.4 composed by an end node $v'_j \in E_1(\vartheta)$ and the respective exiting line ℓ'_j and the second of (3.9) to bound the value of any element in Figure 3.4 formed by an end node $v''_j \in E_0(\vartheta)$ and the respective exiting line ℓ''_j.

We want to prove the bound in (3.12) for every possible choice of the momentum of the root line, as described in Remark 3.7.

Let us study the case a) of Remark 3.7, i.e. the case in which the root line has momentum $|\boldsymbol{\nu}| \leq \mathfrak{n}N$.

By induction we have

$$
\begin{aligned}
|\mathscr{V}(\vartheta)| &\leq \rho^{m+r+s} C^{k-1} \frac{2\Gamma}{\alpha} \rho^{-(m+r+s)} \eta^{\frac{3m\mathfrak{n}+\sum_{i=1}^{m} k_i}{2\mathfrak{n}+1}} |\zeta|^r \varepsilon^{s+1} \\
&\leq 2\Gamma\alpha^{-1} C^{k-1} \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}} \eta^{\frac{-s-r-1+3(m-1)\mathfrak{n}}{2\mathfrak{n}+1}+r+s+1} \\
&\leq \rho C^k \widetilde{C}_0^{-1} \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}} \eta^{\frac{m\mathfrak{n}+2\mathfrak{n}(m+r+s)-\mathfrak{n}}{2\mathfrak{n}+1}},
\end{aligned}
\tag{3.13}
$$

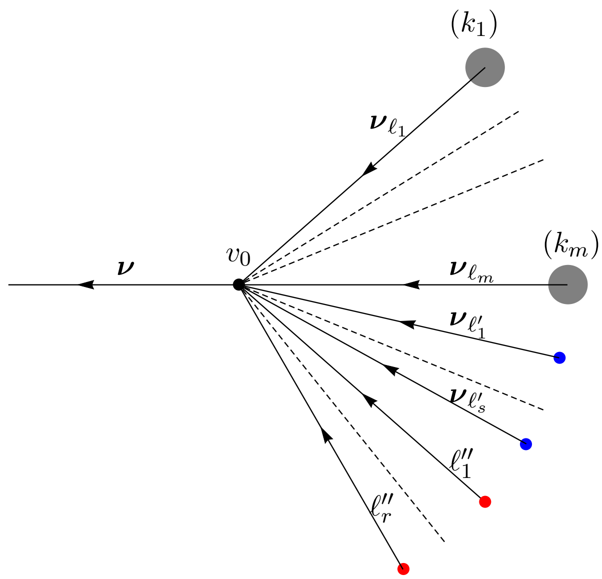

Figure 3.4: Tree of order k representing the described structure.

where we have used Lemma 3.4 to bound the propagator of the root line and where we have defined

$$\widetilde{C}_0 := \max\Big\{1, \frac{\Phi}{2\Gamma}, \frac{2\alpha}{A}, \frac{\alpha}{2\Gamma}\Big\}.$$

Notice that $\widetilde{C}_0^{-1} \le 1$.

As we can easily check, the bound in (3.12) holds, indeed:

$$\frac{m\mathfrak{n} + 2\mathfrak{n}(m+r+s) - \mathfrak{n}}{2\mathfrak{n}+1} \ge \frac{2\mathfrak{n}^2 - \mathfrak{n}}{2\mathfrak{n}+1} > 0.$$

The second case of Remark 3.7, b), is analogous to the case a), since the behaviour of the value of the tree is still described by (3.13).

In the case c), the root line is on scale 1. By induction and by Lemma 3.4 we have:

$$\begin{aligned} |\mathscr{V}(\vartheta)| &\le \rho^{m+r+s} C^{k-1} \frac{4\Gamma}{A\varepsilon\eta^{\mathfrak{n}-1}}\, \rho^{-(m+r+s)} \eta^{\frac{3m\mathfrak{n}+\sum_{i=1}^m k_i}{2\mathfrak{n}+1}} \,|\zeta|^r \,\varepsilon^{s+1} \\ &\le \rho C^k \,\widetilde{C}_1^{-1}\, \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}}\, \eta^{\Delta(m,r,s)}, \end{aligned} \tag{3.14}$$

where we have defined $\widetilde{C}_1$ and $\Delta(m,r,s)$ as

$$\begin{aligned} \widetilde{C}_1 &:= \max\Big\{\frac{A}{2\alpha}, \frac{\Phi A}{4\Gamma\alpha}, 1, \frac{A}{4\Gamma}\Big\} \ge 1, \\ \Delta(m,r,s) &:= \frac{m\mathfrak{n} + 2\mathfrak{n}(m+r+s) - 2\mathfrak{n}^2 - 2\mathfrak{n}}{2\mathfrak{n}+1}. \end{aligned} \tag{3.15}$$

If $m+r+s \geq \mathfrak{n}+1$ the thesis follows trivially as

$$\frac{m\mathfrak{n}+2\mathfrak{n}(m+r+s)-2\mathfrak{n}^2-2\mathfrak{n}}{2\mathfrak{n}+1} \geq \frac{m\mathfrak{n}}{2\mathfrak{n}+1} \geq 0.$$

If $m+r+s=\mathfrak{n}$ the argument is more delicate. In this case, $\Delta(m,r,s)$ reduces to

$$\Delta(m,r,s)=\frac{m\mathfrak{n}-2\mathfrak{n}}{2\mathfrak{n}+1},$$

where we have used $m+r+s=\mathfrak{n}$ in (3.15), so that the bound (3.12) trivially holds if $m \geq 2$, since $\Delta(m,r,s) \geq 0$ in such a case, whereas $\Delta(m,r,s)$ is negative if $m=0,1$. Since the root line is on scale 1, by construction $m \geq 1$. In fact, if $m=0$ and hence $s+r=\mathfrak{n}$, we have

$$|\boldsymbol{\nu}| \leq \sum_{j=1}^{s} |\boldsymbol{\nu}_{\ell'_j}| \leq sN \leq \mathfrak{n}N,$$

that implies the root line is on scale 0, but this is not the case we are considering.

Thus, we have to study the case $m=1$, i.e. when there is only one subtree $\vartheta_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell_1}}$, with $k_1>1$, entering v_0. If $m=1$, or equivalently $s+r=\mathfrak{n}-1$, the scale of the line exiting the tree ϑ_1 cannot be 1, i.e. $\widetilde{n}_{\ell_1}=0$, as one can show by reasoning as follows. If both ℓ_0, ℓ_1 are on scale 1, one has $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_0}|<\frac{\alpha}{2}$ and $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}|<\frac{\alpha}{2}$. Moreover we have $|\sum_{j=1}^{s}\boldsymbol{\nu}_{\ell'_j}| \leq \sum_{j=1}^{s}|\boldsymbol{\nu}_{\ell'_j}| \leq sN \leq \mathfrak{n}N$, that implies $|\boldsymbol{\omega}\cdot\big(\sum_{j=1}^{s}\boldsymbol{\nu}_{\ell'_j}\big)| \geq \alpha$. Besides by construction

$$\alpha \leq |\boldsymbol{\omega}\cdot\big(\sum_{j=1}^{s}\boldsymbol{\nu}_{\ell'_j}\big)| = |\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_{\ell_0}-\boldsymbol{\nu}_{\ell_1})| \leq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_0}|+|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}| < \frac{\alpha}{2}+\frac{\alpha}{2}=\alpha,$$

that is impossible.

The analysis of the first case a) and the previous observation suggest that the bound for ϑ_1 can be improved. Suppose m' trees and $s'+r'$ lines enter the first node of ϑ_1, that we call v_1. We denote by:

- $\vartheta'_1 \in \mathfrak{T}_{k'_1,\boldsymbol{\nu}_{\widetilde{\ell}_1}}, \ldots, \vartheta'_{m'} \in \mathfrak{T}_{k'_{m'},\boldsymbol{\nu}_{\widetilde{\ell}_{m'}}}$, with $k'_j>1$ for $j=1,\ldots,m'$, the m' renormalized trees entering v_1;
- $\widetilde{\ell}'_1,\ldots,\widetilde{\ell}'_{s'}$ the s' lines exiting end nodes in E_1 and entering v_1;
- $\widetilde{\ell}''_1,\ldots,\widetilde{\ell}''_{r'}$ the r' lines exiting end nodes in E_0 and entering v_1;

with the following constraints:

- $\boldsymbol{\nu}_{\ell_1}=\sum_{j=1}^{m'}\boldsymbol{\nu}_{\widetilde{\ell}_j}+\sum_{j=1}^{s}\boldsymbol{\nu}_{\widetilde{\ell}'_j} \neq \mathbf{0}$;
- $m'+r'+s' \geq \mathfrak{n}$;

- $m', r', s' \geq 0$;
- $k_1 = \sum_{i=1}^{m'} k_i' + s' + r' + 1$.

Then, by induction and by reasoning as in obtaining (3.13), it follows that

$$
\begin{aligned}
|\mathscr{V}(\vartheta_1)| &\leq \rho C^{k_1} \eta^{\frac{3m'\mathfrak{n} + \sum_{i=1}^{m'} k_i'}{2\mathfrak{n}+1}} |\zeta|^{r'} \varepsilon^{s'+1} \\
&\leq \rho C^{k_1} \eta^{\frac{k_1 + 3m'\mathfrak{n} + 2r'\mathfrak{n} + 2s'\mathfrak{n} + 2\mathfrak{n}}{2\mathfrak{n}+1}} \\
&\leq \rho C^{k_1} \eta^{\frac{k_1 + 2\mathfrak{n}^2 + 2\mathfrak{n}}{2\mathfrak{n}+1}},
\end{aligned}
$$

where in the last bound we have used that

$$
\frac{k_1 + m'\mathfrak{n} + 2\mathfrak{n}(r' + s' + m') + 2\mathfrak{n}}{2\mathfrak{n}+1} \geq \frac{k_1 + 2\mathfrak{n}^2 + 2\mathfrak{n}}{2\mathfrak{n}+1}.
$$

Then, by using that $k_1 = k - r - s - 1 = k - \mathfrak{n}$, for $m = 1$ (3.14) can be improved into

$$
\begin{aligned}
|\mathscr{V}(\vartheta)| &\leq \rho^{1+r+s} C^{k_1+r+s} \rho^{-(1+r+s)} \eta^{\frac{k_1 + 2\mathfrak{n}^2 + 2\mathfrak{n}}{2\mathfrak{n}+1}} |\zeta|^r \varepsilon^{s+1} \frac{4\Gamma}{A\varepsilon\eta^{\mathfrak{n}-1}} \\
&\leq \rho C^k \widetilde{C}_1^{-1} \eta^{\frac{k_1 + 2\mathfrak{n}^2 + 2\mathfrak{n}}{2\mathfrak{n}+1}} = \rho C^k \widetilde{C}_1^{-1} \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}} \eta^{\frac{2\mathfrak{n}^2 - 2\mathfrak{n}}{2\mathfrak{n}+1}}
\end{aligned}
$$

and the thesis follows since $\widetilde{C}_1^{-1} \eta^{\frac{2\mathfrak{n}^2 - 2\mathfrak{n}}{2\mathfrak{n}+1}} \leq 1$, for η small enough. □

3.3 Convergence of the renormalized expansion

Lemma 3.9. *For any $k \geq 1$ and $\boldsymbol{\nu} \in \mathbb{Z}_*^d$ one has*

$$
|X_{\boldsymbol{\nu}}^{[k]}| \leq \rho B^k \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}}, \tag{3.16}
$$

where $B \equiv B(\Gamma, \rho, \Phi, A, N, \alpha)$ is a positive constant proportional to C, with C defined as in (3.8).

Proof. To bound the coefficients $X_{\boldsymbol{\nu}}^{[k]}$ defined in (2.14), we use the estimate in (3.12) and we sum over all trees in $\mathfrak{T}_{k,\boldsymbol{\nu}}$. Then notice that the sum over the mode labels $\boldsymbol{\nu} \in \mathbb{Z}^d$ in (2.14) can be bounded by $(2N+1)^{dk}$ and the assertion follows. □

We have proved that the series described in (2.15) converges.

Corollary 3.10. *The function $\bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ is analytic in $\boldsymbol{\psi}$ in any strip Σ_ξ, provided η small enough (depending on ξ).*

Proof. If we consider the set of renormalized trees in $\mathfrak{T}_{k,\boldsymbol{\nu}}$, in particular we have that $|\boldsymbol{\nu}| \leq kN$. Hence we rewrite (2.15) as

$$\sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}| \leq kN} X_{\boldsymbol{\nu}}^{[k]} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}$$

and by using the bound on the Fourier coefficients given by Lemma 3.9, we have

$$\Big| \sum_{k=1}^{\infty} \sum_{|\boldsymbol{\nu}| \leq kN} X_{\boldsymbol{\nu}}^{[k]} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}} \Big| \leq \rho N \sum_{k=1}^{\infty} k B^k \eta^{\frac{k+3\mathfrak{n}}{2\mathfrak{n}+1}} e^{kN\xi},$$

that is convergent since η is supposed to be small. This also gives a lower bound of the radius of convergence. □

3.4 The renormalized expansion as a solution of (1.1)

After proving that the renormalized expansion $\bar{X} \equiv \bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$, defined as in (2.15), converges, we still have to check that it is a solution of (1.1), i.e. that (2.16) is satisfied.

Write

$$\bar{X}_{\boldsymbol{\nu}} = \sum_{\widetilde{n}=0}^{1} \bar{X}_{\boldsymbol{\nu},\widetilde{n}}, \qquad \bar{X}_{\boldsymbol{\nu},\widetilde{n}} = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu},\widetilde{n}}} \mathscr{V}(\vartheta), \tag{3.17}$$

where $\mathfrak{T}_{k,\boldsymbol{\nu},\widetilde{n}}$ is the subset of $\mathfrak{T}_{k,\boldsymbol{\nu}}$ of the renormalized trees with root line on scale $\widetilde{n}$.

Define

$$D_{\widetilde{n}}(x; \varepsilon, \zeta, c) := \mathrm{i}x(1 + \mathrm{i}\varepsilon x) - \mathcal{M}^{[\widetilde{n}-1]}(x; \varepsilon, \zeta, c), \qquad \text{for } \widetilde{n} = 0, 1, \tag{3.18a}$$

$$\mathcal{G}(x; \varepsilon, \zeta, c) := \frac{1}{\mathrm{i}x(1 + \mathrm{i}x)}, \tag{3.18b}$$

with the convention that $\mathcal{M}^{[-1]}(x; \varepsilon, \zeta, c) = 0$ and $\mathcal{M}^{[0]}(x; \varepsilon, \zeta, c) = \mathcal{M}(x; \varepsilon, \zeta, c)$, with $\mathcal{M}(x; \varepsilon, \zeta, c)$ defined as in (2.12a). Again, in order to simplify the notation we write just $D_{\widetilde{n}}(x)$, $\mathcal{M}^{[\widetilde{n}-1]}(x)$, $\mathcal{G}(x)$ instead of $D_{\widetilde{n}}(x; \varepsilon, \zeta, c)$, $\mathcal{M}^{[\widetilde{n}-1]}(x; \varepsilon, \zeta, c)$, $\mathcal{G}(x; \varepsilon, \zeta, c)$.

According to this notation $G^{[0]}(x) = \Psi(|x|)/D_0(x)$ and $G^{[1]}(x) = \chi(|x|)/D_1(x)$, where $G^{[0]}(x)$ and $G^{[1]}(x)$ are defined as in (2.11).

If we define

$$\Omega(\boldsymbol{\nu}; \varepsilon, \zeta, c) := \mathcal{G}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) \varepsilon [f - g(c + \zeta + \bar{X}(\cdot; \varepsilon, \zeta, c))]_{\boldsymbol{\nu}}, \tag{3.19}$$

then we have to prove that $\Omega(\boldsymbol{\nu}; \varepsilon, \zeta, c) = \bar{X}_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \neq 0$.

Now we write explicitly the right hand side of (3.19)

$$\begin{aligned}\mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\varepsilon[f-g(c+\zeta+\bar{X}(\cdot;\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} &= \\ &= \mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(\Psi(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|)+\chi(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|))\varepsilon[f-g(c+\zeta+\bar{X}(\cdot;\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} \\ &= \mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\big(D_0(\boldsymbol{\omega}\cdot\boldsymbol{\nu})G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})+ \\ &\quad + D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\big)\,\varepsilon\,[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}}.\end{aligned}\tag{3.20}$$

We now observe that

$$G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\varepsilon\,[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} = \sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},0}}\mathscr{V}(\vartheta),$$

$$G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\varepsilon\,[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} = \sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}^*_{k,\boldsymbol{\nu},1}}\mathscr{V}(\vartheta),$$

where $\mathfrak{T}^*_{k,\boldsymbol{\nu},1}$ differs from $\mathfrak{T}_{k,\boldsymbol{\nu},1}$ as it contains also trees which can have one renormalized self-energy cluster on scale 0 with exiting line ℓ_0, if ℓ_0 denotes the root line of ϑ. In fact if we analyse the contribution $G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\varepsilon\,[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}}$, we notice it differs from $\bar{X}_{\boldsymbol{\nu},1}$, in as much it also contains an additional contribution that represents the self-energy cluster on scale 0 mentioned above:

$$\begin{aligned}G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\,\varepsilon\,[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} &= \\ &= \sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},1}}\mathscr{V}(\vartheta)+G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})M(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},1}}\mathscr{V}(\vartheta).\end{aligned}$$

By inserting this expression in (3.20), we obtain

$$\begin{aligned}\Omega(\boldsymbol{\nu};\varepsilon,\zeta,c) &= \mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\varepsilon[f-g(c+\zeta+\bar{X}(\cdot,\varepsilon,\zeta,c))]_{\boldsymbol{\nu}} = \\ &= \mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big(D_0(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},0}}\mathscr{V}(\vartheta)+D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},1}}\mathscr{V}(\vartheta)+ \\ &\quad + D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})M(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{k=1}^{\infty}\varepsilon^k\sum_{\vartheta\in\mathfrak{T}_{k,\boldsymbol{\nu},1}}\mathscr{V}(\vartheta)\Big) = \\ &= \mathcal{G}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big(D_0(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\bar{X}_{\boldsymbol{\nu},0}+D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\bar{X}_{\boldsymbol{\nu},1}+D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})M(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\bar{X}_{\boldsymbol{\nu},1}\Big) = \\ &= D_0^{-1}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big(D_0(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\bar{X}_{\boldsymbol{\nu},0}+(D_1(\boldsymbol{\omega}\cdot\boldsymbol{\nu})+\chi(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|)M(\boldsymbol{\omega}\cdot\boldsymbol{\nu}))\bar{X}_{\boldsymbol{\nu},1}\Big) = \\ &= \bar{X}_{\boldsymbol{\nu},0}+\bar{X}_{\boldsymbol{\nu},1}=\bar{X}_{\boldsymbol{\nu}},\end{aligned}$$

so that (2.16) follows.

Remark 3.11. Note that at each step only absolutely converging series have been dealt with, so that the above analysis is rigorous and not only formal.

3.5 The bifurcation equation

In order to conclude our analysis, we have to solve the bifurcation equation, described by (2.6), which can be studied by using Hypothesis 1. We start by providing an estimate for the value of the trees describing the bifurcation equation. Define $\mathscr{V}^*(\vartheta)$ as $\mathscr{V}(\vartheta)$ with the only difference that the node factor of the first node v_0 is $F_{v_0} = g_{p_{v_0}}$ (without the factor $-\varepsilon$ appearing in (2.9)).

Lemma 3.12. *Let ϑ in $\mathfrak{T}_{k,\mathbf{0}}$, with $k \geq \mathfrak{n}+1$. Then*

$$|\mathscr{V}^*(\vartheta)| \leq \rho\Gamma C^k \eta^{\frac{k+2\mathfrak{n}^2-1}{2\mathfrak{n}+1}}, \tag{3.21}$$

with ρ, Γ as in (2.1) *and C as in* (3.8).

Proof. We refer to the proof of Lemma 3.8, in particular to Figures 3.3 and 3.4, for the construction and notations of the trees. Since the momentum of the root line is $\mathbf{0}$, by using Lemma 3.4 and Lemma 3.8, we bound the value of the tree as

$$\begin{aligned} |\mathscr{V}^*(\vartheta)| &\leq \Gamma\rho^{-(m+r+s)} C^{k-1} \rho^{m+r+s} \eta^{\frac{3m\mathfrak{n}+\sum_{i=1}^m k_i}{2\mathfrak{n}+1}} |\zeta|^r \varepsilon^s \\ &\leq \Gamma C^{-1} C^k \eta^{\frac{k-s-r-1+3m\mathfrak{n}}{2\mathfrak{n}+1}+r+s} \\ &\leq \rho\Gamma\hat{C}^{-1} C^k \eta^{\frac{k+m\mathfrak{n}+2\mathfrak{n}(m+r+s)-1}{2\mathfrak{n}+1}} \\ &\leq \rho\Gamma C^k \eta^{\frac{k+2\mathfrak{n}^2-1}{2\mathfrak{n}+1}}, \end{aligned}$$

with C as in (3.8) and Γ, ρ as in (2.1) and

$$\hat{C} := \rho C = \max\left\{\frac{2\Gamma}{\alpha}, \frac{\Phi}{\alpha}, \frac{4\Gamma}{A}, 1\right\} \geq 1.$$

□

If we define

$$\mathcal{H}(\varepsilon,\zeta) := [g(c+\zeta+\bar{X}(\cdot;\varepsilon,\zeta,c))]_{\mathbf{0}} - f_{\mathbf{0}}, \tag{3.22}$$

in such a way that the bifurcation equation becomes $\mathcal{H}(\varepsilon,\zeta)=0$, then the following results hold.

Lemma 3.13. *The function $\mathcal{H}(\varepsilon,\zeta)$ is $C^{\mathfrak{n}}$ with respect to ζ.*

Proof. By Hypothesis 1, one has $g(c) = f_{\mathbf{0}}$, $\frac{d^i g}{dx^i}(c) = 0$ for $i = 1, 2, \dots, \mathfrak{n}-1$ and $\frac{d^{\mathfrak{n}} g}{dx^{\mathfrak{n}}}(c) \neq 0$; recall that $\mathfrak{n}$ is odd. Hence

$$\mathcal{H}(\varepsilon,\zeta) = \sum_{p=\mathfrak{n}} g_p(c)[(\zeta+\bar{X}(\cdot;\varepsilon,\zeta,c))^p]_{\mathbf{0}} = \sum_{k=\mathfrak{n}+1}^{\infty} \sum_{\vartheta\in\mathfrak{T}_{k,\mathbf{0}}} \mathscr{V}^*(\vartheta).$$

Therefore it is sufficient to prove that the function $\mathscr{V}^*(\vartheta)$ is $C^{\mathfrak{n}}$ in ζ. In particular $\mathscr{V}^*(\vartheta)$ depends on ζ through the node factors and through the propagators associated with lines on scale 1, see (2.13) and (2.14), that means:

$$\partial_\zeta^j \mathscr{V}^*(\vartheta) = \partial_\zeta^j\Big(\prod_{v\in N(\vartheta)} F_v\Big)\cdot\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) + \Big(\prod_{v\in N(\vartheta)} F_v\Big)\cdot\partial_\zeta^j\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big), \quad 1\le j\le \mathfrak{n}.$$

If the derivative acts on the node factors, we have

$$\partial_\zeta\Big(\prod_{v\in N(\vartheta)} F_v\Big) = |E_0(\vartheta)|\zeta^{|E_0(\vartheta)|-1}\Big(\prod_{v\in N(\vartheta)\setminus E_0(\vartheta)} F_v\Big).$$

Then, by using (3.21), for $0\le j\le\mathfrak{n}$ one has

$$\begin{aligned}\Big|\partial_\zeta^j\Big(\prod_{v\in N(\vartheta)} F_v\Big)\cdot\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big)\Big| &\le |E_0(\vartheta)|(|E_0(\vartheta)|-1)\cdots(|E_0(\vartheta)|-j+1)\rho\Gamma C'^k\eta^{\frac{k+2\mathfrak{n}^2-1}{2\mathfrak{n}+1}-j}\\ &\le |E_0(\vartheta)|(|E_0(\vartheta)|-1)\cdots(|E_0(\vartheta)|-j+1)\rho\Gamma C'^k\eta^{\frac{k-\mathfrak{n}-1}{2\mathfrak{n}+1}},\end{aligned}$$

for a suitable positive constant C' depending on C, with C defined as in (3.8), that is bounded since η is supposed to be small and $k\ge\mathfrak{n}+1$.

If the derivatives act on the propagators the analysis is more delicate. We have to distinguish among two cases: the case in which the derivatives act on the same line and the case in which the derivatives act on different lines. The worst case is the first one. Denote $x=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$, with $\ell\in L(\vartheta)$, and suppose that $\mathfrak{n}$ derivatives act on the propagator $G^{[1]}(x)$, see (2.11). Then one has:

$$\begin{aligned}&\partial_\zeta G^{[1]}(x) = \chi(|x|)\frac{\partial_\zeta\mathcal{M}(x)}{(D_1(x))^2},\\ &\partial_\zeta^2 G^{[1]}(x) = \chi(|x|)\Big[\frac{2(\partial_\zeta\mathcal{M}(x))^2}{(D_1(x))^3}+\frac{\partial_\zeta^2\mathcal{M}(x)}{(D_1(x))^2}\Big],\\ &\partial_\zeta^j G^{[1]}(x) = \chi(|x|)\sum_{k=0}^{j-1}\sum_{i_1,\dots,i_{j-k}\in\mathcal{A}}\frac{a_{i_1,\dots,i_{j-k}}(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k+1}},\ 3\le j\le\mathfrak{n},\end{aligned} \tag{3.23}$$

for suitable constants $a_{i_1,\dots,i_{j-k}}$ and where

$$\mathcal{A} := \{i_1,\dots,i_{j-k}\in\mathbb{N}:\ i_1+\dots+i_{j-k}=j \text{ and } i_1\ge i_2\ge\dots\ge i_{j-k}\ge 1\}.$$

Before proceeding with the estimate of the derivatives, we have to provide a bound for $\partial_\zeta\mathcal{M}(x;\varepsilon,\zeta,c)$. Since the lines composing $\mathcal{M}(x;\varepsilon,\zeta,c)$ are on scale 0, see (2.12a), the derivatives act only on the node factor. By Lemma 3.1 and Lemma 3.3, it follows that

$$
\begin{aligned}
&|\partial_\zeta \mathcal{M}(x)| \le b_1\varepsilon(\zeta^{\mathfrak{n}-2}+\eta^{2\mathfrak{n}-2}) \le 2b_1\,\varepsilon\,\eta^{\mathfrak{n}-2},\\
&|\partial_\zeta^2 \mathcal{M}(x)| \le b_2\varepsilon(\zeta^{\mathfrak{n}-3}+\eta^{2\mathfrak{n}-3}) \le 2b_2\,\varepsilon\,\eta^{\mathfrak{n}-3},\\
&\dots\\
&|\partial_\zeta^{\mathfrak{n}-1} \mathcal{M}(x)| \le b_{\mathfrak{n}-1}\varepsilon(1+\eta^{\mathfrak{n}}) \le 2b_{\mathfrak{n}-1}\,\varepsilon,\\
&|\partial_\zeta^{\mathfrak{n}} \mathcal{M}(x)| \le b_{\mathfrak{n}}\varepsilon(1+\eta^{\mathfrak{n}-1}) \le 2b_{\mathfrak{n}}\,\varepsilon,
\end{aligned}
$$

with suitable positive constants $b_1,\dots,b_{\mathfrak{n}}$ and where the first contribution is from resonances containing only one internal node, while the second one is from all the contributions with at least two internal nodes. Hence, to sum up, we have

$$
|\partial_\zeta^j \mathcal{M}(x)| \le 2b_j\varepsilon\eta^{\mathfrak{n}-j-1}, \qquad \text{for } j=1,\dots,\mathfrak{n}-1, \tag{3.24}
$$

$$
|\partial_\zeta^{\mathfrak{n}} \mathcal{M}(x)| \le 2b_{\mathfrak{n}}\,\varepsilon. \tag{3.25}
$$

We can write the contributions of the derivatives acting on the same propagator as follows:

$$
\begin{aligned}
&\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) = \mathscr{V}^*(\vartheta)\frac{\partial_\zeta\mathcal{M}(x)}{D_1(x)}\\
&\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta^2\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) = \mathscr{V}^*(\vartheta)\Big[\frac{2(\partial_\zeta\mathcal{M}(x))^2}{(D_1(x))^2}+\frac{\partial_\zeta^2\mathcal{M}(x)}{D_1(x)}\Big]\\
&\qquad\qquad\dots\\
&\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta^{\mathfrak{n}}\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) =\\
&\qquad\qquad = \mathscr{V}^*(\vartheta)\sum_{k=0}^{\mathfrak{n}-1}\sum_{i_1,\dots,i_{\mathfrak{n}-k}\in\mathcal{A}}\frac{a_{i_1,\dots,i_{j-k}}(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{\mathfrak{n}-k}}\mathcal{M}(x))}{(D_1(x))^{\mathfrak{n}-k}},
\end{aligned}
$$

for $\mathfrak{n}\ge 3$. By using (3.24), (3.25) and Lemma 3.3, we have

$$
\begin{aligned}
\Big|\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k}}\Big| &\le \varepsilon^{j-k}\eta^{(\mathfrak{n}-1)(j-k)-\sum_{s=1}^{j-k}(i_s)}\,\varepsilon^{-j+k}\,\eta^{-(\mathfrak{n}-1)(j-k)}\\
&= \eta^{-j} \qquad \text{for } 1\le j\le\mathfrak{n}.
\end{aligned}
$$

Then, from Lemma 3.12, one has the bound

$$
\begin{aligned}
\Big|\mathscr{V}^*(\vartheta)\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k}}\Big| &\le \rho\Gamma C^k\eta^{\frac{k+2\mathfrak{n}^2-1}{2\mathfrak{n}+1}}\,\eta^{-j}\\
&= \rho\Gamma C^k\eta^{\frac{k+2\mathfrak{n}^2-1-j(2\mathfrak{n}+1)}{2\mathfrak{n}+1}} \qquad \text{for } 1\le j\le\mathfrak{n},
\end{aligned}
$$

and in particular

$$\left|\mathscr{V}^*(\vartheta)\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{\mathfrak{n}-k}}\mathcal{M}(x))}{(D_1(x))^{\mathfrak{n}-k}}\right|\le\rho\Gamma C^k\eta^{\frac{k-\mathfrak{n}-1}{2\mathfrak{n}+1}},$$

that is bounded since η is supposed to be small and $k\ge\mathfrak{n}+1$.

If the derivatives act on different lines, we have

$$\partial_\zeta^r\Big(\prod_{\ell\in L(\vartheta)}G_\ell\Big)=\partial_\zeta^{j_1}G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1})\partial_\zeta^{j_2}G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2})\cdots\partial_\zeta^{j_r}G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_r})\Big(\prod_{\ell\in L(\vartheta)\setminus L'(\vartheta)}G_\ell\Big),$$

with $L'(\vartheta)$ the set of the lines where the derivatives act, $j_1,j_2\neq0$, $j_3,\dots,j_r\ge0$ and $j_1+\dots+j_r=r$ for $1\le r\le\mathfrak{n}$. Then from the previous analysis, it can be easily seen that every $\mathfrak{n}$-th derivative of the function $\mathscr{V}^*(\vartheta)$ in the variable ζ is bounded.

Therefore we can conclude that $\mathcal{H}(\varepsilon,\zeta)$ is $C^{\mathfrak{n}}$ in ζ. □

Lemma 3.14. *There exists a neighbourhood $U\times V$ of $(\varepsilon,\zeta)=(0,0)$ such that for all $\varepsilon\in U$ there is at least one value $\zeta=\zeta(\varepsilon)\in V$, depending continuously on ε, for which one has $\mathcal{H}(\varepsilon,\zeta)=0$.*

Proof. By hypothesis 1, one has $g(c)=f_{\mathbf{0}}$, $\frac{d^ig}{dx^i}(c)=0$ for $i=1,2,\dots,\mathfrak{n}-1$ and $\frac{d^{\mathfrak{n}}g}{dx^{\mathfrak{n}}}(c)\neq0$, for odd $\mathfrak{n}$ such that $\mathfrak{n}>3$. Hence

$$\mathcal{H}(\varepsilon,\zeta)=\sum_{p=\mathfrak{n}}g_p(c)[(\zeta+\bar{X}(\cdot;\varepsilon,\zeta,c))^p]_{\mathbf{0}}.$$

We know that $\frac{d^{\mathfrak{n}}}{d\zeta^{\mathfrak{n}}}\mathcal{H}(0,0)=g_{\mathfrak{n}}(c)\neq0$.

We assume $g_{\mathfrak{n}}(c)>0$. The proof can be easily generalised also to the case $g_{\mathfrak{n}}(c)<0$. Call W and $V=[V_-,V_+]$ the neighbourhoods of $\varepsilon=0$ and $\zeta=0$, respectively. For any $\varepsilon\in W$, if we consider $\mathcal{H}(\varepsilon,\zeta)$ as a function of the variable ζ, then $(0,0)$ is a rising point of inflection of the function $\mathcal{H}(\varepsilon,\zeta)$. In particular, since $\mathcal{H}(0,0)=0$, then $\mathcal{H}(0,V_+)>0$ and $\mathcal{H}(0,V_-)<0$.

Now we look at $\mathcal{H}(\varepsilon,V_-)$ and $\mathcal{H}(\varepsilon,V_+)$ as functions of ε: they are both continuous functions in W, then by continuity, there exists $U\subset W$ neighbourhood of 0 such that $\mathcal{H}(\varepsilon,V_-)<0$ and $\mathcal{H}(\varepsilon,V_+)>0$ for all $\varepsilon\in U$.

Since for all $\varepsilon\in U$ the function $\mathcal{H}(\varepsilon,\zeta)$ is continuous and it is also increasing in ζ and $\mathcal{H}(\varepsilon,V_-)<0$ and $\mathcal{H}(\varepsilon,V_+)>0$, then there exists a continuous curve $\zeta\equiv\zeta(\varepsilon)\in V$ such that $\mathcal{H}(\varepsilon,\zeta(\varepsilon))=0$.

□

Chapter 4

A review on continued fractions

For this chapter, we refer to [55, 57, 59, 60, 63, 64] and the references contained therein. We denote by $x = [a_0, \dots, a_n]$ the expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} \tag{4.1}$$

with $a_0 \in \mathbb{R}$ and $a_1, \dots, a_n \in \mathbb{R}^+$.

Definition 4.1. *We call $[a_0, \dots, a_n]$ a* ***finite continued fraction*** *and the coefficients $a_0, \dots, a_n$ the* ***partial quotients*** *of the continued fraction.*

We call an ***infinite continued fraction*** *the limit for $n \to \infty$, if it exists, of $x_n = [a_0, \dots, a_n]$, i.e.*

$$x = \lim_{n\to\infty} [a_0, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{\ddots}}}}} \tag{4.2}$$

According to this definition, we say that if the limit exists the continued fraction converges.

Proposition 4.2. *A continued fraction is called* simple *if $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \geq 1$. Any irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ can be represented as a unique simple infinite continued fraction $[a_0, a_1, a_2, \dots]$.*

Proof. Denote by $[x]$ the integer part of a real number. Define the map

$$A:(0,1)\to[0,1],\qquad A(x):=\frac{1}{x}-\left[\frac{1}{x}\right]. \tag{4.3}$$

Then with each $x\in\mathbb{R}\setminus\mathbb{Q}$ we associate a continued fraction as follows:

$$\xi_0=x-[x], \tag{4.4}$$
$$a_0=[x], \tag{4.5}$$

so that we obviously have

$$x=a_0+\xi_0,\qquad \xi_0\in(0,1)$$

and define recursively for all $n\geq 1$

$$\xi_n=\frac{1}{\xi_{n-1}}-\left[\frac{1}{\xi_{n-1}}\right]=A(\xi_{n-1}), \tag{4.6}$$
$$a_n=\left[\frac{1}{\xi_{n-1}}\right]\geq 1, \tag{4.7}$$

in such a way that we can obtain recursively for $n\geq 1$

$$\xi_{n-1}^{-1}=a_n+\xi_n. \tag{4.8}$$

In this way, we have built the continued fraction expansion of the irrational number x:

$$x=a_0+\xi_0=a_0+\frac{1}{a_1+\xi_1}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\xi_2}}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}. \tag{4.9}$$

□

Proposition 4.3. *Given a simple continued fraction* $[a_0,a_1,a_2,\dots]$, *define*

$$p_0=a_0,\qquad p_1=a_1a_0+1,\qquad p_k=a_kp_{k-1}+p_{k-2},\qquad k\geq 2; \tag{4.10}$$
$$q_0=1,\qquad q_1=a_1,\qquad q_k=a_kq_{k-1}+q_{k-2},\qquad k\geq 2. \tag{4.11}$$

Then one has

$$\frac{p_k}{q_k}=[a_0,\dots,a_k],\qquad k\geq 0. \tag{4.12}$$

We call (4.12) *the* $k-th$ ***convergent*** *of the continued fraction.*

Proof. The proof is by induction. If $k = 0$ the statement is trivially true. Suppose that $\frac{p_k}{q_k} = [a_0, \ldots, a_k]$ for $k \le m$ and check that the thesis still holds for $k = m+1$. Indeed one has

$$[a_0, a_1, \ldots, a_m, a_{m+1}] = [a_0, a_1, \ldots, a_m + \frac{1}{a_{m+1}}] =$$
$$= \frac{(a_m + \frac{1}{a_{m+1}})p_{m-1} + p_{m-2}}{(a_m + \frac{1}{a_{m+1}})q_{m-1} + q_{m-2}} = \frac{p_m a_{m+1} + p_{m-1}}{q_m a_{m+1} + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}.$$

□

Remark 4.4. A continued fraction converges if and only if $\lim_{n\to\infty} \frac{p_n}{q_n}$ exists.

Proposition 4.5. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$x = \frac{p_{n+1} + p_n \xi_{n+1}}{q_{n+1} + q_n \xi_{n+1}}, \qquad \text{for } n \ge 0. \tag{4.13}$$

Proof. We proceed by induction on n. If $n = 0$, one has

$$\frac{p_1 + p_0 \xi_1}{q_1 + q_0 \xi_1} = \frac{a_0 a_1 + 1 + a_0 \xi_1}{a_1 + \xi_1} = a_0 + \frac{1}{a_1 + \xi_1} = a_0 + \xi_0 = x.$$

Now we assume that $x = \frac{p_{j+1} + p_j \xi_{j+1}}{q_{j+1} + q_j \xi_{j+1}}$ for $0 \le j \le n$ and prove that (4.13) still holds for $j = n+1$:

$$\begin{aligned} \frac{p_{n+2} + p_{n+1}\xi_{n+2}}{q_{n+2} + q_{n+1}\xi_{n+2}} &= \frac{a_{n+2}p_{n+1} + p_n + p_{n+1}\xi_{n+2}}{a_{n+2}q_{n+1} + q_n + q_{n+1}\xi_{n+2}} \\ &= \frac{(a_{n+2} + \xi_{n+2})p_{n+1} + p_n}{(a_{n+2} + \xi_{n+2})q_{n+1} + q_n} \\ &= \frac{p_n + p_{n+1}\xi_{n+1}^{-1}}{q_n + q_{n+1}\xi_{n+1}^{-1}} = \frac{p_n \xi_{n+1} + p_{n+1}}{q_n \xi_{n+1} + q_{n+1}} = x. \end{aligned}$$

□

Corollary 4.6. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $\xi_n = -\frac{xq_n - p_n}{xq_{n-1} - p_{n-1}}$ for $n \ge 1$.*

Proof. Again proof by induction. □

4.1 Some properties of the convergents

Proposition 4.7. *Let p_k, q_k be as in Proposition 4.3. Then for all $x \in \mathbb{R}$, one has:*

(1) $q_{k+1} > q_k > 0$ *for* $k \ge 1$*;*

(2) $p_k > 0\,[p_k < 0]$ *when* $x > 0\,[x < 0]$ *for* $k \geq 1$*;*

(3) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ *for* $k \geq 1$*;*

(4) $p_k q_{k-2} - p_{k-2} q_k = (-1)^k a_k$ *for* $k \geq 2$*.*

Proof. Parts (1) and (2) follow directly by definition of p_k, q_k in (4.10), (4.11). The proof of (3) and (4) can be easily obtained by induction.

Let us start by proving (3). If $k = 1$ then $p_1 q_0 - p_0 q_1 = 1$ and the thesis is satisfied. Let us suppose $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ for $1 \leq k \leq m$ and we study the case $k = m + 1$:

$$\begin{aligned} &p_{m+1} q_m - p_m q_{m+1} = (a_{m+1} p_m + p_{m-1}) q_m - p_m (a_{m+1} q_m + q_{m-1}) = \\ &= -(p_m q_{m-1} - p_{m-1} q_m) = -(-1)^{m-1} = (-1)^m, \end{aligned}$$

that implies (3).

Let us consider the case $k = 2$ in (4): the thesis trivially holds since $p_2 q_0 - p_0 q_2 = a_2$. Again we proceed by induction, by assuming (4) for $2 \leq k \leq m$:

$$\begin{aligned} &p_{m+1} q_{m-1} - p_{m-1} q_{m+1} = (a_{m+1} p_m + p_{m-1}) q_{m-1} - p_{m-1} (a_{m+1} q_m + q_{m-1}) = \\ &= a_{m+1} (p_m q_{m-1} - p_{m-1} q_m) = a_{m+1} (-1)^{m-1} = a_{m+1} (-1)^{m+1}, \end{aligned}$$

that implies (4). □

Proposition 4.8. *Let* $x_k = \frac{p_k}{q_k}$*,* $k \geq 0$*, be a sequence of convergents of* $x \in \mathbb{R}$*. Then*

(1) x_{2k} *is strictly increasing;*

(2) x_{2k+1} *is strictly decreasing;*

(3) $x_{2k} < x_{2k+1}$*;*

(4) $x_{2k} < x_n < x_{2m+1}$ *for all* $k, m \in \mathbb{N}$ *such that* $2k < n$ *and* $2m + 1 < n$*.*

Proof. (1) We have to show that if k is even then $x_k < x_{k+2}$. According to the definition of x_k and by using (4) in Proposition 4.7 we have

$$x_{k+2} - x_k = \frac{p_{k+2} q_k - p_k q_{k+2}}{q_{k+2} q_k} = \frac{(-1)^{k+2} a_{k+2}}{q_{k+2} q_k},$$

that is positive since k is even.

(2) Let us consider k odd. Again by using (4) in Proposition 4.7 we obtain the thesis:

$$x_k - x_{k+2} = \frac{-(p_k q_{k+2} - q_k p_{k+2})}{q_k q_{k+2}} = \frac{-(-1)^{k+2} a_{k+2}}{q_k q_{k+2}} = \frac{a_{k+2}}{q_k q_{k+2}} > 0.$$

(3) Let us suppose that k is odd (the case of even k is analogous). By using (3) in Proposition 4.7 we have

$$x_k - x_{k+1} = \frac{-(p_{k+1}qk - p_k q_{k+1})}{q_k q_{k+1}} = \frac{-(-1)^k}{q_k q_{k+1}} = \frac{1}{q_k q_{k+1}} > 0,$$

that implies $x_k > x_{k+1}$.

(4) If n is even, since x_{2k} is strictly increasing and $2k < n$, we have $x_{2k} < x_n$. By the previous point we also have $x_n < x_{2m+1}$.

If n is odd, since x_{2m+1} is strictly decreasing and $2m + 1 < n$, we have $x_{2m+1} > x_n$. By following the previous point we also have $x_n > x_{2k}$ since n is odd.

□

Corollary 4.9. *Given $x \in \mathbb{R} \setminus \mathbb{Q}$ and the convergents $\{x_k\}$, the following holds:*

$$x_{2k} < x < x_{2k+1} \quad \forall k \in \mathbb{N}, \tag{4.14}$$

see Figure 4.1.

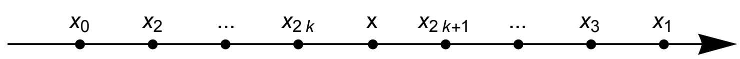

Figure 4.1

Proof. $\{x_{2k}\}$ is strictly increasing and $\{x_{2k+1}\}$ is strictly decreasing, hence

$$\sup_{k\in\mathbb{N}} x_{2k} = \lim_{k\to\infty} x_{2k} = x = \lim_{k\to\infty} x_{2k+1} = \inf_{k\in\mathbb{N}} x_{2k+1}.$$

In particular, since $x \in \mathbb{R} \setminus \mathbb{Q}$, we have $x \neq x_{2k}, x_{2k+1}$ for all k. Therefore we can conclude

$$x_{2k} < \sup_{k\in\mathbb{N}} x_{2k} = x = \inf_{k\in\mathbb{N}} x_{2k+1} < x_{2k+1}.$$

□

Proposition 4.10. *Let x be a positive irrational number and let $\{x_k\}$ be the convergents of the simple continued fraction representing x. Then*

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}. \tag{4.15}$$

Proof. Consider first the case of even k. We have to use the following inequalities:

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \tag{4.16}$$

for $\frac{a}{b}, \frac{c}{d}$ rational positive numbers such that $\frac{a}{b} < \frac{c}{d}$.

Therefore, since $x_k < x_{k+1}$ (k even), we have

$$\frac{p_k}{q_k} < \frac{p_k + p_{k+1}}{q_k + q_{k+1}} < \frac{p_k + 2p_{k+1}}{q_k + 2q_{k+1}} < \cdots < \frac{p_k + a_{k+2}p_{k+1}}{q_k + a_{k+2}q_{k+1}} = \frac{p_{k+2}}{q_{k+2}} < x.$$

Define $s_k := \frac{p_k+p_{k+1}}{q_k+q_{k+1}}$. By the previous inequality one has

$$x_k < s_k < x \Rightarrow x - x_k > s_k - x_k > 0,$$

that means

$$\begin{aligned} x - \frac{p_k}{q_k} > \frac{p_k + p_{k+1}}{q_k + q_{k+1}} - \frac{p_k}{q_k} &= \frac{q_k p_{k+1} - p_k q_{k+1}}{q_k(q_{k+1} + q_k)} = \\ &= \frac{(-1)^k}{q_k(q_{k+1} + q_k)} = \frac{1}{q_k(q_{k+1} + q_k)}. \end{aligned}$$

Besides, by Corollary 4.9 we have $x_k < x < x_{k+1}$, that implies $0 < x - x_k < x_{k+1} - x_k$. Hence we have the second inequality in (4.15):

$$x - \frac{p_k}{q_k} < \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k+1}} = \frac{1}{q_k q_{k+1}}.$$

If k is odd, by Corollary 4.9 one has $x_{k+1} < x < x_k$, hence we can reason as before by changing the roles of x_k and x_{k+1} in the following way:

$$\frac{p_k}{q_k} > \frac{p_{k+1} + p_k}{q_{k+1} + q_k} > \frac{2p_{k+1} + p_k}{2q_{k+1} + q_k} > \cdots > \frac{a_{k+2}p_{k+1} + p_k}{a_{k+2}q_{k+1} + q_k} = \frac{p_{k+2}}{q_{k+2}} > x.$$

By defining $s_k := \frac{p_k+p_{k+1}}{q_k+q_{k+1}}$, we obtain $0 > s_k - x_k > x - x_k$. Hence it follows that

$$-\frac{1}{q_k(q_{k+1} + q_k)} > x - x_k > x_{k+1} - x_k = -\frac{1}{q_k q_{k+1}}.$$

Finally the thesis follows by collecting together the two cases:

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.$$

□

Remark 4.11. The result in Proposition 4.10 still holds if we only assume x to be irrational. In fact, if $x < 0$ then also $p_k < 0$ for all $k \geq 0$, see Proposition 4.8. Then x and $\frac{p_k}{q_k}$ have the same sign and $x - \frac{p_k}{q_k}$ is still a difference, as in (4.15).

Proposition 4.12. *For any $x \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many p, q such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{4.17}$$

Proof. Consider the sequence of convergents of x, $\{p/q\}$. From (4.15) we have

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2} \quad \forall k \geq 1,$$

where the last inequality follows from Proposition 4.7. □

Remark 4.13. The result in Proposition 4.12 can be improved in the following way: for all $x \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many p, q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}. \tag{4.18}$$

Remark 4.14. The result in (4.18) cannot be improved, that is if $C > \sqrt{5}$, then for all $x \in \mathbb{R} \setminus \mathbb{Q}$, it is not possible to find infinite values of $p, q \in \mathbb{N}$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{Cq^2}$.

4.2 Best rational approximations

For reasons that will be clear later, we are interested in the continued fraction expansion of irrational numbers. Therefore, from now on, x is supposed to be an irrational number.

Definition 4.15. *A rational number p/q, with $q > 0$ and $GCD(p, q) = 1$, is called a* ***best rational approximation*** *of $x \in \mathbb{R} \setminus \mathbb{Q}$ if*

$$|nx - m| > |qx - p|, \qquad \forall m, n \in \mathbb{Z} : 0 < |n| \leq q \textit{ and } \frac{m}{n} \neq \frac{p}{q}. \tag{4.19}$$

Remark 4.16. $m, n \in \mathbb{N}$ if $x > 0$.

Proposition 4.17. *If a rational number $\frac{p}{q}$ is one of the best rational approximations of x then $\frac{p}{q}$ is a convergent of x.*

Proof. First of all, notice that if $\frac{p}{q}$ is one of the best rational approximations of x, then $\frac{p}{q} \geq a_0$. In fact, if this were not true, we would have a contradiction since $x > a_0$:

$$|x - a_0| < \left|x - \frac{p}{q}\right| \leq \Big|qx - p\Big|,$$

that means $\frac{p}{q}$ is not one of the best rational approximations.

Let us suppose that $\frac{p}{q}$ is not a convergent. Then by Corollary 4.9 we can assume $\frac{p}{q}$ to be either greater than $\frac{p_1}{q_1}$ or between two convergents x_{k-1}, x_{k+1} such that

$$\begin{cases} x < x_{k+1} < x_{k-1}, & \text{if } \; k \text{ is even;} \\ x_{k-1} < x_{k+1} < x, & \text{if } \; k \text{ is odd.} \end{cases}$$

I CASE: $\frac{p}{q} > \frac{p_1}{q_1}$.
We know that $\frac{p_1}{q_1} > x$. Then

$$\left|x - \frac{p}{q}\right| > \left|\frac{p_1}{q_1} - \frac{p}{q}\right| \geq \frac{1}{q_1 q} \Rightarrow |qx - p| > \frac{1}{q_1}.$$

By the definitions (4.10) and (4.11), $p_0 = a_0$ and $q_0 = 1$, so

$$|q_0 x - p_0| = |x - a_0| < \frac{1}{q_0}$$

and if we combine the two inequalities we obtain

$$|qx - p| > \frac{1}{q_1} > |x - a_0|,$$

which is a contradiction since $\frac{p}{q}$ is one of the best rational approximations.
II CASE: $\frac{p_{k-1}}{q_{k-1}} < \frac{p}{q} < \frac{p_{k+1}}{q_{k+1}}$.
If k is even we have $x > x_{k+1} > \frac{p}{q}$, while if k is odd we have $x < x_{k+1} < \frac{p}{q}$. Then, in both cases

$$\left|x - \frac{p}{q}\right| > \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p}{q}\right| > \frac{1}{q_k q_{k+1}}.$$

As $|q_k x - p_k| < \frac{1}{q_{k+1}}$, we obtain

$$|qx - p| > \frac{1}{q_{k+1}} > |q_k x - p_k|.$$

We have to show $q_k < q$ to obtain a contradiction: in fact, since $\frac{p}{q}$ is one of the best rational approximations, one has $|q_k x - p_k| > |qx - p|$ for $q_k < q$. We know that if k is odd one has $\frac{p_{k-1}}{q_{k-1}} < \frac{p}{q} < \frac{p_k}{q_k}$, while if k is even one has $\frac{p_k}{q_k} < \frac{p}{q} < \frac{p_{k-1}}{q_{k-1}}$. In both cases

$$\left|\frac{p}{q} - \frac{p_{k-1}}{q_{k-1}}\right| < \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \frac{1}{q_k q_{k-1}}.$$

Hence we can conclude $\frac{1}{q_k q_{k-1}} > \left|\frac{p}{q} - \frac{p_{k-1}}{q_{k-1}}\right| > \frac{1}{q_{k-1} q}$, that implies $q_k < q$. ☐

Proposition 4.18. *Given the sequence of convergents $\{p_n/q_n\}$ of x, the following inequalities hold:*

$$|q_0 x - p_0| > |q_1 x - p_1| > \cdots > |q_k x - p_k| > \ldots \tag{4.20}$$

Proof. By (4.15) it follows that $|q_k x - p_k| < \frac{1}{q_{k+1}}$ and also $|q_{k-1} x - p_{k-1}| > \frac{1}{q_{k-1} + q_k}$. Since $a_{k+1} \geq 1$, we have

$$\frac{1}{q_{k-1} + q_k} \geq \frac{1}{q_{k-1} + a_{k+1} q_k} = \frac{1}{q_{k+1}}$$

and then $|q_k x - p_k| < \frac{1}{q_{k+1}} \leq \frac{1}{q_{k-1} + q_k} < |q_{k-1} x - p_{k-1}|$. Hence $|q_k x - p_k| < |q_{k-1} x - p_{k-1}|$ for all $k \geq 1$. ☐

Proposition 4.19. *If $\{\frac{p_k}{q_k}\}$ is the sequence of convergents of $x > 0$, then $\frac{p_k}{q_k}$ is one of the best rational approximations, i.e. for all $k \geq 0$, $|qx - p| > |q_k x - p_k|$ for all $p, q \in \mathbb{N}$ such that $0 < q < q_{k+1}$ and $q \neq q_k$.*

Proof. Given $p/q \in \mathbb{Q}$, let us define μ, ν as the solutions of

$$\begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} \mu \\ \nu \end{pmatrix} = (-1)^k \begin{pmatrix} q_k & -p_k \\ -q_{k+1} & p_{k+1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

This means that $(\mu, \nu) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. If $\nu = 0$ and $\mu \neq 0$, i.e. $\mu \geq 1$, we have $q = \mu q_{k+1}$ that implies $q \geq q_{k+1}$, but by hypothesis $q < q_{k+1}$. Instead, if $\mu = 0$ and $\nu \neq 0$ we obtain $q = \nu q_k$ and since $q \neq q_k$, $\nu \geq 2$. This implies the thesis since

$$|qx - p| \geq 2|q_k x - p_k| > |q_k x - p_k|.$$

If $\mu, \nu \neq 0$, as $0 < q < q_{k+1}$, μ and ν must have opposite sign, as well as $q_{k+1}x - p_{k+1}$ and $q_k x - p_k$. Therefore $\mu(q_{k+1}x - p_{k+1})$ and $\nu(q_k x - p_k)$ have the same sign and since $|\mu|, |\nu| \geq 1$, we can conclude:

$$\begin{aligned} |qx - p| &= |(\mu q_{k+1} + \nu q_k)x - (\mu p_{k+1} + \nu p_k)| = \\ &= |\mu(q_{k+1}x - p_{k+1}) + \nu(q_k x - p_k)| = \\ &= |\mu(q_{k+1}x - p_{k+1})| + |\nu(q_k x - p_k)| > \\ &> |\nu(q_k x - p_k)| \geq |q_k x - p_k| \end{aligned}$$

that is the thesis. □

Remark 4.20. We can generalise Proposition 4.19 to all $x \in \mathbb{R}$ by considering $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $0 < |q| < q_{k+1}$ and $|q| \neq q_k$.

Corollary 4.21. *Any rational number $\frac{p}{q}$ is one of the best rational approximation if and only if $\frac{p}{q}$ is a convergent of x.*

Proof. It follows directly by Propositions 4.17 and 4.19. □

4.3 Diophantine vectors and continued fractions

Recall that a vector $\boldsymbol{\omega} \in \mathbb{R}^d$ is called Diophantine with exponent τ, if $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \gamma / |\boldsymbol{\nu}|^\tau$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$ and for a suitable positive constant γ (cf. Section 1.2).

Proposition 4.22. *Let $\boldsymbol{\omega} = (1, \alpha) \in \mathbb{R}^2$ be a Diophantine vector with exponent τ. Given the sequence of convergents $\{p_k/q_k\}$ of α, there exists a positive constant K_0 such that for all $k \in \mathbb{N}$*

$$q_{k+1} < K_0 q_k^\tau. \tag{4.21}$$

Proof. Define $\boldsymbol{\nu}_k := (-p_k, q_k)$.

As $\boldsymbol{\omega}$ is a Diophantine vector, we have $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_k| = |-p_k + \alpha q_k| > \frac{\gamma}{|\boldsymbol{\nu}_k|^\tau}$, with $|\boldsymbol{\nu}_k| = |\boldsymbol{\nu}_k|_2 = \sqrt{q_k^2 + p_k^2}$, for some positive constant γ.

By Proposition 4.10, it follows that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_k| < \frac{1}{q_{k+1}}$.

Moreover one has $q_k < |\boldsymbol{\nu}_k| \le q_k\sqrt{1+4\alpha^2}$. Indeed, by using Proposition 4.7, $q_{k+1} \ge a_2 a_1 + 1 \ge 2$ for any $k \ge 1$, that implies

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_k| < \frac{1}{2} \quad \text{and thus} \quad |p_k| \le |q_k\alpha| + \frac{1}{2} \le 2|q_k\alpha|.$$

Hence

$$q_k < |\boldsymbol{\nu}_k| \le \sqrt{q_k^2 + 4(q_k\alpha)^2} = q_k\sqrt{1+4\alpha^2}.$$

By summarising, we obtain

$$q_{k+1} < q^\tau \frac{(1+4\alpha^2)^{\tau/2}}{\gamma},$$

since

$$\frac{1}{q_{k+1}} > |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_k| > \frac{\gamma}{|\boldsymbol{\nu}_k|^\tau} > \frac{\gamma}{q_k^\tau(1+4\alpha^2)^{\tau/2}},$$

which yields (4.21) with $K_0 = \frac{(1+4\alpha^2)^{\tau/2}}{\gamma}$. □

4.4 The Bryuno function

Let

$$\beta_n := \prod_{j=0}^{n} \xi_i \quad \text{for} \quad n \ge 0 \quad \text{and} \quad \beta_{-1} = 1, \tag{4.22}$$

where $\{\xi_i\}_{i\ge 0}$ is the sequence described recursively by (4.4) and (4.6).

Proposition 4.23. *For any $x \in \mathbb{R}\setminus\mathbb{Q}$, one has:*

$$\beta_n = (-1)^n(q_n x - p_n) \quad \textit{for} \quad n \ge 0, \tag{4.23}$$

where p_n and q_n are defined respectively as in (4.10) and (4.11). In particular for $n \ge 0$ one has:

$$\xi_n = \frac{\beta_n}{\beta_{n-1}} \quad \textit{and} \quad \beta_{n-1} = a_{n+1}\beta_n + \beta_{n+1}. \tag{4.24}$$

Proof. From the definition of β_n in (4.22) and from Corollary 4.6, one has

$$\begin{aligned}\beta_n &= \xi_0\xi_1\cdots\xi_n = \\ &= (-1)^n\xi_0\,\frac{q_1x-p_1}{q_0x-p_0}\,\frac{q_2x-p_2}{q_1x-p_1}\cdots\frac{q_{n-1}x-p_{n-1}}{q_{n-2}x-p_{n-2}}\,\frac{q_nx-p_n}{q_{n-1}x-p_{n-1}} = \\ &= (-1)^n\xi_0\frac{q_nx-p_n}{q_0x-p_0} = (-1)^n\xi_0\frac{q_nx-p_n}{x-a_0} = \\ &= (-1)^n(q_nx-p_n).\end{aligned}$$

□

Proposition 4.24. *For all $x \in \mathbb{R}\setminus\mathbb{Q}$ and for all $n \geq 1$, one has*

$$\beta_n \leq \Big(\frac{\sqrt{5}-1}{2}\Big)^n, \tag{4.25}$$

$$\frac{1}{2} < \beta_n q_{n+1} < 1, \tag{4.26}$$

with β_n, q_n respectively as in (4.22) *and* (4.11).

Proof. We first prove (4.25). Call $g := \frac{\sqrt{5}-1}{2}$. One can have two possibilities: either $\xi_i \leq g$ for all $i = 0,\ldots,n$, or $\xi_i > g$ for some i.

If $\xi_i \leq g$ for all $i = 0,\ldots,n$, (4.25) follows trivially.

If $\xi_i > g$ for some i, then, according to (4.6), one has $\xi_{i+1} = \xi_i^{-1} - [\xi_i^{-1}]$. Notice that since $\xi_i > g$, then $[\xi_i^{-1}] \leq 1$ and by combining it with (4.7), one has $[\xi_i^{-1}] = 1$. Hence $\xi_{i+1} = \xi_i^{-1} - 1$ and $\xi_{i+1} < g^{-1} - 1 = g$, thus $\xi_i\xi_{i+1} = 1-\xi_i < 1-g = g^2$. Therefore, in the sequence $\beta_n = \xi_0\cdots\xi_n$, one can isolate the pairs $\xi_i\xi_{i+1}$ such that $\xi_i > g$ (since for each pair $\xi_i\xi_{i+1} < g^2$). The other terms in β_n are all smaller or equal to g except for possible $\xi_n < 1$ and (4.25) follows once again.

We now focus on (4.26). From (4.13), we have that

$$\begin{aligned}q_ix-p_i &= q_i\frac{p_{i+1}+p_i\xi_{i+1}}{q_{i+1}+q_i\xi_{i+1}} - p_i = \\ &= \frac{q_ip_{i+1}-p_iq_{i+1}}{q_{i+1}+q_i\xi_{i+1}} = \frac{(-1)^i}{q_{i+1}+q_i\xi_{i+1}},\end{aligned}$$

for all $i \geq 1$. Then

$$\beta_iq_{i+1} = (-1)^i(q_ix-p_i)q_{i+1} = \frac{q_{i+1}}{q_{i+1}+q_i\xi_{i+1}} = \frac{1}{1+\dfrac{q_i\xi_{i+1}}{q_{i+1}}} < 1.$$

Since the sequence $\{q_i\}$ is strictly increasing and $\xi_i \leq 1$, one has $q_i\xi_{i+1}/q_{i+1} < 1$. Hence

$$\beta_iq_{i+1} = \frac{1}{1+\dfrac{q_i\xi_{i+1}}{q_{i+1}}} > \frac{1}{2}$$

and we have proved (4.26). □

Corollary 4.25. *Let q_n be as in (4.11). Then for all $n \geq 1$ one has*

$$q_n > \frac{1}{2}\Big(\frac{2}{\sqrt{5}-1}\Big)^{n-1}. \tag{4.27}$$

Proof. (4.27) follows directly by combining (4.25) with (4.26). □

Definition 4.26. *The Bryuno function $B : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ is defined as*

$$B(x) = -\sum_{i=0}^{\infty} \beta_{i-1}(x) \log \xi_i, \tag{4.28}$$

where ξ_i follows the repeated iterations of the map $A(x)$ defined in (4.3) and β_i is as in (4.22).

Remark 4.27. The definition of Bryuno function can be extended to $x \in \mathbb{Q}$ by setting $B(x) = +\infty$.

Proposition 4.28. *The Bryuno function has the following properties:*

(a) $B(x) = B(x+1)$ *for all* $x \in \mathbb{R}$*;*

(b) For all $x \in (0, 1)$

$$B(x) = -\log x + xB\Big(\frac{1}{x}\Big); \tag{4.29}$$

(c) There exists a positive constant C such that for all $x \in \mathbb{R} \setminus \mathbb{Q}$ one has

$$\Big|B(x) - \sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j}\Big| \leq C, \tag{4.30}$$

with $\{q_j\}_{j\geq 0}$ as in (4.11).

Proof. (a) Given $x \in \mathbb{R} \setminus \mathbb{Q}$, the sequences $\{\xi_i\}_{i\geq 0}$ and $\{\beta_i\}_{i\geq 0}$ associated with x and $x + 1$ are the same.

(b) Let $y = \frac{1}{x}$. We denote by $\xi_{i,y}$, $a_{i,y}$ and $\xi_{i,x}$, $a_{i,x}$ the sequences for recurrence described in (4.6) and (4.7) associated respectively with y and x. We also call $\beta_{i,y}$, $\beta_{i,x}$ the sequences (4.22) associated with x and y.

If $x \in (0, 1)$, then $[x] = 0$. Hence, from (4.4), (4.5), (4.6) and (4.7) it follows that $\xi_{0,x} = x$, $a_{0,y} = a_{1,x}$, $\xi_{0,y} = \xi_{1,x}$. By induction, one has $\xi_{n,y} = \xi_{n+1,x}$ and $\beta_{n,y} = \beta_{n+1,x}/x$, for all $n \geq 0$. Thus

$$\begin{aligned} B(y) &= -\sum_{i=0}^{\infty} \beta_{i-1,y} \log \xi_{i,y} = -\log \xi_{0,y} - \sum_{i=1}^{\infty} \frac{1}{x}\beta_{i,x} \log \xi_{i+1,x} \\ &= -\frac{1}{x}\sum_{i=1}^{\infty} \beta_{i-1,x} \log \xi_{i,x} = \frac{1}{x}\big(B(x) + \log x\big). \end{aligned}$$

(c) We first notice that (4.23) implies

$$q_i\beta_{i-1} + q_{i-1}\beta_i = 1 \quad i \geq 1. \tag{4.31}$$

In fact,

$$\begin{aligned} q_i\beta_{i-1} + q_{i-1}\beta_i &= (-1)^{i-1}q_i(q_{i-1}x - p_{i-1}) + (-1)^i q_{i-1}(q_i x - p_i) \\ &= (-1)^i(q_i p_{i-1} - q_{i-1}p_i) = (-1)^{2i} = 1, \end{aligned}$$

where we have used Proposition 4.7.

Then, from (4.24), (4.28) and (4.31), one has

$$\begin{aligned} -B(x) + \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i} &= \sum_{i=0}^{\infty} \beta_{i-1} \log \frac{\beta_i}{\beta_{i-1}} + \sum_{i=0}^{\infty} \Big(\frac{q_i\beta_{i-1} + q_{i-1}\beta_i}{q_i}\Big) \log q_{i+1} \\ &= \sum_{i=0}^{\infty} \beta_{i-1} \log(\beta_i q_{i+1}) - \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_{i-1} + \sum_{i=0}^{\infty} \frac{q_{i-1}}{q_i} \beta_i \log q_{i+1}. \end{aligned}$$

From (4.26) and by combining the inequality $\log q_i \leq (2/e) q_i^{1/2}$ with (4.27), one has the following estimates:

$$\begin{aligned} \Big| \sum_{i=0}^{\infty} \beta_{i-1} \log(\beta_i q_{i+1}) \Big| &\leq 2 \sum_{i=0}^{\infty} \frac{\log 2}{q_i} \leq 2c_2, \\ \Big| \sum_{i=0}^{\infty} \beta_{i-1} \log \beta_{i-1} \Big| &\leq 2 \sum_{i=0}^{\infty} \frac{\log 2 + \log q_i}{q_i} \leq 2(c_1 + c_2), \\ \Big| \sum_{i=0}^{\infty} \frac{q_{i-1}}{q_i} \beta_i \log q_{i+1} \Big| &\leq 2 \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_{i+1}} \leq 2c_1, \end{aligned}$$

for suitable positive constants c_1, c_2. Then one has

$$\Big| B(x) - \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i} \Big| \leq C = 4(c_1 + c_2)$$

and we have proved (4.30).

□

Remark 4.29. Let $\boldsymbol{\omega} = (1, \alpha)$ and recall the definition of $\mathfrak{B}(\boldsymbol{\omega})$ in (1.4). Then notice that in general the three series

$$B(\alpha), \qquad \sum_{j=1}^{\infty} \frac{\log q_{j+1}}{q_j}, \qquad \mathfrak{B}(\boldsymbol{\omega}) = \sum_{j=0}^{\infty} \frac{1}{2^j} \log \frac{1}{\alpha_j(\boldsymbol{\omega})},$$

with $\alpha_j(\boldsymbol{\omega})$ as in (1.4), are different from each other, but each series converges if and only if the other two converge, see Proposition 4.28 and Lemma 1 of [43].

Definition 4.30. *In the light of Remark 4.29, the Bryuno condition can be expressed by*

$$\sum_{j=0}^{\infty} \frac{\log q_{j+1}}{q_j} < +\infty, \tag{4.32}$$

as well as by the condition $\mathcal{B}(\omega) < \infty$ *used on Section 1.2.*

Chapter 5

Proof of Theorem 2

In this Chapter we provide a proof of Theorem 2. In particular in Section 5.2 we give a bound for the value of any renormalized tree described in Section 5.1; in Section 5.3 we study the convergence of the series (2.15) and we also study the bifurcation equation. That completes the proof of Theorem 2. Finally in Section 5.4 we give some comments on the intervals I_n, see (1.9), where the solution exists. We recall that if $\mathfrak{n}$ is even, there is no response solution of the form (5.6), reducing to c as ε tends to zero, see Remark 1.1 and Appendix B.

5.1 Preliminaries for the proof of Theorem 2

We want to use the theory of continued fractions to deal with the small divisors problem appearing in. To this end, we shall restrict ourselves to two-dimensional frequency vectors: in the previous case of f a trigonometric polynomial, $\boldsymbol{\omega}$ was a vector of arbitrary dimension d with $d \geq 2$; instead from now on we need $\boldsymbol{\omega}$ to be a vector of dimension 2. Moreover, without any loss of generality, we assume $\boldsymbol{\omega}$ of the form $\boldsymbol{\omega} := (1, \alpha) \in \mathbb{R}^2$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The condition $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is required since we want $\boldsymbol{\omega}$ to have rationally independent components, see Hypothesis 2.

Let $\{p_n/q_n\}$ be the convergents of the infinite continued fraction representing α, which also are one of the best rational approximation of α, see Corollary 4.21.

Set $\boldsymbol{\nu} := (-\nu_1, \nu_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. By using the properties of the convergents, for all ν_1, ν_2 such that $0 < |\nu_2| < q_n$ and $|\nu_2| \neq q_{n-1}$, one has

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| = |\nu_2 \alpha - \nu_1| > |\alpha q_{n-1} - p_{n-1}| > \frac{1}{2q_n} \quad \forall n \geq 1. \tag{5.1}$$

Indeed, the first inequality follows from Proposition 4.19, while the second one follows from (4.15) :

$$|\alpha q_{n-1} - p_{n-1}| > \frac{1}{q_{n-1} + q_n} > \frac{1}{2q_n} \quad \forall n \geq 1,$$

where we have used $q_n > q_{n-1} > 0$, according to Proposition 4.7.

Let us denote by $\Delta(c, \rho)$ the disk of center c and radius ρ in the complex plane and by Σ_ξ the strip of width ξ on the 2-dimensional torus, that is

$$\Sigma_\xi := \{\boldsymbol{\psi} \in \mathbb{C}^2 : \Re(\boldsymbol{\psi}) \in \mathbb{T}^2, |\Im(\boldsymbol{\psi})| \leq \xi\}. \tag{5.2}$$

By the assumption on g, as in Section 2, for any $c \in \mathbb{R}$ there exists $\rho_0 > 0$ such that $g(x)$ is analytic in $\Delta(c, \rho_0)$. Then, if we take c as in Hypothesis 1, for all $\rho < \rho_0$, (2.1) still holds:

$$g(x) = g(c) + \sum_{p=\mathfrak{n}}^{\infty} g_p (x-c)^p, \quad g_p := \frac{1}{p!}\frac{\partial^p g}{\partial x^p}(c), \quad |g_p| \leq \Gamma \rho^{-p}, \tag{5.3}$$

with $\Gamma := \max\{|g(x)| : x \in \Delta(c, \rho)\}$.

By the analyticity assumption on the quasi-periodic forcing term f, for any $c \in \mathbb{R}$ there exists ξ_0 such that f is analytic in Σ_{ξ_0}. Then for all $\xi < \xi_0$, if we define $\Phi := \max\{|f(\boldsymbol{\psi})| : \boldsymbol{\psi} \in \Sigma_\xi\}$, one has:

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^2} f_{\boldsymbol{\nu}} e^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\psi}}, \qquad |f_{\boldsymbol{\nu}}| \leq \Phi e^{-\xi|\boldsymbol{\nu}|}. \tag{5.4}$$

Consider the ordinary differential equation (1.1) with $d = 2$ and assume Hypothesis 1. For any frequency vector $\boldsymbol{\omega} = (1, \alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we want to prove that, defined N and I_n as in Theorem 2 and for all $\varepsilon > 0$ such that

$$\varepsilon \in \overline{\bigcup_{n \geq N} I_n} := \overline{\bigcup_{n \geq N} \Big[e^{-C_2 q_n}, \frac{1}{(C_1 q_n)^{\mathfrak{n}+1}}\Big]}, \tag{5.5}$$

with C_1, C_2 positive constants with C_2 fixed as the value given at the end of the proof of Theorem 2 and C_1 an arbitrary constant, there exists at least one quasi-periodic solution of the form

$$x(t, \varepsilon) = c + \zeta + X(\boldsymbol{\omega} t; \varepsilon, \zeta), \tag{5.6}$$

where ζ is a value that has to be fixed and $\boldsymbol{\psi} \to X(\boldsymbol{\psi}; \varepsilon, \zeta)$ is a zero-average quasi-periodic function. Equation (5.5) can be rewritten as

$$\frac{1}{C_2} \log \frac{1}{\varepsilon} < q_n < \frac{1}{C_1 \varepsilon^{\frac{1}{\mathfrak{n}+1}}}. \tag{5.7}$$

Recall that one can write (1.1) in Fourier space as

$$(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\nu})(1 + \mathrm{i}\varepsilon\boldsymbol{\omega}\cdot\boldsymbol{\nu}) X_{\boldsymbol{\nu}} + \varepsilon[g(c + \zeta + X)]_{\boldsymbol{\nu}} = \varepsilon f_{\boldsymbol{\nu}} \qquad \boldsymbol{\nu} \neq \mathbf{0} \tag{5.8}$$

and, for $\boldsymbol{\nu} = \mathbf{0}$ as

$$[g(c + \zeta + X)]_{\mathbf{0}} = f_{\mathbf{0}}, \tag{5.9}$$

where (5.8) and (5.9) are called the *range equation* and the *bifurcation equation*, respectively. As in the case of f a trigonometric polynomial, we first study the range equation looking for a solution to (5.8), depending on the small parameter ζ. Then we analyse (5.9) and fix $\zeta = \zeta(\varepsilon)$ in order to make such an equation to be satisfied.

We find that the parameter ζ, in general, is no more than continuous in ε. In our case, this suffices, since all we need is to prove that the parameter goes to 0 as ε tends to to zero. Of course, in principle more regularity is possible, and the parameter could have different branches, as found in similar contexts when bifurcation phenomena occur. One could even think that a fractional power series may be constructed: for instance, this happens for both the Melnikov problem [30] and lower dimensional tori of codimension 1 [39], when the case of higher order zeroes is considered. For the problem under study the situation is more delicate, since already in the case of simple zeroes in general no more than continuity is found [47]. In any case, we do not exclude that stronger regularity results may be obtained, possibly with different methods; see also ref. [20, 28, 21] for some results about the form of the analyticity domains in dissipative perturbations of Hamiltonian systems.

In doing that we need to consider a slightly modified version of tree construction introduced on Section 2.1. Define $N(\vartheta), V(\vartheta), E(\vartheta), E_0(\vartheta), E_1(\vartheta)$ and $L(\vartheta)$ as previously and denote by $k \equiv k(\vartheta) = |N(\vartheta)|$ the order of ϑ. Again, with each line $\ell \in L(\vartheta)$ we assign a *scale label* $\widetilde{n}_\ell \in \{0, 1\}$ and we introduce the following partition of unity:

$$\chi(x) := \begin{cases} 1 & \text{for } x < \frac{C_1}{4}\varepsilon^{\frac{1}{\mathfrak{n}+1}}, \\ 0 & \text{for } x \geq \frac{C_1}{4}\varepsilon^{\frac{1}{\mathfrak{n}+1}}, \end{cases} \qquad \Psi(x) := \begin{cases} 1 & \text{for } x \geq \frac{C_1}{4}\varepsilon^{\frac{1}{\mathfrak{n}+1}}, \\ 0 & \text{for } x < \frac{C_1}{4}\varepsilon^{\frac{1}{\mathfrak{n}+1}}. \end{cases} \tag{5.10}$$

with C_1 be the same constant as (5.5).

We associate with each node $v \in N(\vartheta)$ a node factor and with each line $\ell \in L(\vartheta)$ a propagator, according to the rules of Section 2.1, see (2.9)–(2.12c) with $\chi(x)$, $\Psi(x)$ as in (5.10).

In order to simplify the analysis, we split the set $L(\vartheta)$ in two disjoint sets, $L_0(\vartheta) := \{\ell \in L(\vartheta) : \boldsymbol{\nu}_\ell = 0\}$ and $L_1(\vartheta) := \{\ell \in L(\vartheta) : \boldsymbol{\nu}_\ell \neq 0\} = L(\vartheta) \setminus L_0(\vartheta)$. Notice that if ℓ is an internal line, then $\ell \in L_1(\vartheta)$.

Remark 5.1. Let $\ell \in L_1(\vartheta)$. If $|\boldsymbol{\nu}_\ell| < q_n$, then also $|\nu_{\ell,2}| < q_n$. Hence we can apply (5.1) and see that the line ℓ has to be on scale 0: in fact we have $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| \geq \frac{C_1}{2}\varepsilon^{\frac{1}{\mathfrak{n}+1}} > \frac{C_1}{4}\varepsilon^{\frac{1}{\mathfrak{n}+1}}$ and, according to the sharp partition considered in (5.10), this implies that the line ℓ has to be on scale 0. The statement above can be rephrased as saying that if ℓ is on scale 1, then $|\boldsymbol{\nu}| \geq q_n$.

Therefore we have three different possibilities for a line $\ell \in L_1(\vartheta)$:

1. $|\boldsymbol{\nu}_\ell| < q_n$, that automatically implies scale 0,

2. $|\boldsymbol{\nu}_\ell| \geq q_n$ and scale $\widetilde{n}_\ell = 0$,

3. $|\boldsymbol{\nu}_\ell| \geq q_n$ and scale $\widetilde{n}_\ell = 1$.

In the first and in the second case, the propagator of the line ℓ is given by the expression of $G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)$ in (2.11), while a propagator satisfying the last condition is given by $G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)$ in the same reference, with $\Psi(x)$ and $\chi(x)$ as in (5.10).

We define $L_{<,0}(\vartheta)$, $L_{\geq,0}(\vartheta)$, $L_{\geq,1}(\vartheta)$ as follows:

- $L_{<,0}(\vartheta) := \{\ell \in L_1(\vartheta) : |\boldsymbol{\nu}_\ell| < q_n\}$,
- $L_{\geq,0}(\vartheta) := \{\ell \in L_1(\vartheta) : |\boldsymbol{\nu}_\ell| \geq q_n \text{ and } \widetilde{n}_\ell = 0\}$,
- $L_{\geq,1}(\vartheta) := \{\ell \in L_1(\vartheta) : |\boldsymbol{\nu}_\ell| \geq q_n \text{ and } \widetilde{n}_\ell = 1\}$.

By construction, one has $L_1(\vartheta) = L_{<,0}(\vartheta) \sqcup L_{\geq,0}(\vartheta) \sqcup L_{\geq,1}(\vartheta)$.

The definitions of cluster, self-energy-cluster, value of the tree and renormalized series are still as in Section 2.1. Note that if k_T is the order of a self-energy cluster, then $k_T \geq \mathfrak{n}$ by construction. Denote with $E_1(T)$ and $E_0(T)$ the following sets: $E_1(T) := \{v \in E(T) : \boldsymbol{\nu}_v \neq \mathbf{0}\}$ and $E_0(T) := E(T) \setminus E_1(T)$, where the sets $E(T)$ and $V(T)$ are as in Section 2.1.

Define

$$\eta := \max\{\varepsilon, |\zeta|\} \tag{5.11}$$

and look at Lemma 3.1 and Lemma 3.3: with the rules above we still have analogous results.

Lemma 5.2. *Given a tree ϑ, consider a self-energy cluster T in ϑ of order k_T. Then one has*

$$|\mathscr{V}(T)| \leq \Gamma\rho\bar{C}^{k_T}\varepsilon\eta^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(k_T-1)}, \tag{5.12}$$

with

$$\bar{C} := \rho^{-1}\max\left\{\frac{4\Phi}{C_1}, \frac{4\Gamma}{C_1}, 1\right\}, \tag{5.13}$$

with ρ, Γ as in (5.3) *and Φ as in* (5.4).

Proof. We recall the definition of the value of a self-energy cluster, that is

$$\mathscr{V}(T) = \Big(\prod_{v\in V(T)} F_v\Big)\Big(\prod_{\ell\in L(T)} G_\ell^{[0]}\Big).$$

Since every line in T is on scale 0, we bound the propagators as

$$|G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| \leq \frac{1}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell|} \leq \frac{4}{C_1}\varepsilon^{-\frac{1}{\mathfrak{n}+1}},$$

if $\boldsymbol{\nu}_\ell \neq 0$, otherwise

$$G^{[0]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell) = 1,$$

if $\boldsymbol{\nu}_\ell = \mathbf{0}$. Then, by using (5.3) and (5.4) to bound the node factors, we have

$$\begin{aligned}
|\mathscr{V}(T)| &\le \Gamma\rho^{-(k_T-1)}\Big(\frac{4\Phi}{C_1}\Big)^{|E_1(T)|}\Big(\frac{4\Gamma}{C_1}\Big)^{|V(T)|-1}\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(|V(T)|-1+|E_1(T)|)+1}|\zeta|^{|E_0(T)|} \\
&\le \Gamma\rho\bar{C}^{k_T}\varepsilon\,\eta^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(|V(T)|-1+|E_1(T)|+|E_0(T)|)+\frac{1}{\mathfrak{n}+1}|E_0(T)|} \\
&\le \Gamma\rho\bar{C}^{k_T}\varepsilon\,\eta^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(k_T-1)},
\end{aligned}$$

where we have used that $|V(T)| \ge 1$ by construction and defined $\bar{C}$ as in (5.13). □

Lemma 5.3. *Given a tree ϑ, consider a self-energy cluster T in ϑ with momentum of the external lines equal to $\boldsymbol{\nu}$. Define $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ as in* (2.12a)*. Then, for η small enough, one has*

$$|\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})| \ge A\varepsilon\eta^{\mathfrak{n}-1}, \tag{5.14}$$

with A positive constant depending on Γ, ρ, Φ, C_1, with Γ, ρ as in (5.3)*, Φ as in* (5.4) *and C_1 as in* (5.5)

Proof. The proof is the same as for Lemma 3.3. In particular, if we denote with $\mathcal{M}_{\mathfrak{n}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ the terms of $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ with $\mathfrak{n}$ nodes and with $\Delta\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ the other terms of $\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$, we have that $\mathcal{M}_{\mathfrak{n}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) = \mathcal{M}_{\mathfrak{n}}(0)$ and hence $\mathcal{M}_{\mathfrak{n}}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ is real. Therefore by combining the proof of Lemma 3.3 with Lemma 5.2, one has

$$|\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})| \ge |\mathcal{M}_{\mathfrak{n}}(0)| - |\Delta\mathcal{M}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})| \ge a\varepsilon\eta^{\mathfrak{n}-1} - b\varepsilon\eta^{\frac{\mathfrak{n}^2}{\mathfrak{n}+1}} \ge A\varepsilon\eta^{\mathfrak{n}-1},$$

for suitable positive constants a, b depending on Γ, ρ, Φ, C_1 and $A = a/2$, provide η is small enough. □

Remark 5.4. The constant A, considered in (5.14), is not the same constant as in Lemma 3.3, as it depends on f.

We want to prove that the series $\bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ described by (2.15) converges. In order to do that, we have to provide a bound for $X_{\boldsymbol{\nu}}^{[k]}$ in (2.14) and hence we need an estimate on the value of each tree, see equation (2.13). The latter will be the main object of the next chapter where we analyse the generic case of odd $\mathfrak{n} \ge 3$.

At the end, we will prove that the renormalized series solves the equation (1.1) and we will fix $\zeta = \zeta(\varepsilon)$ as a parameter that goes to zero as ε goes to zero in order to make the bifurcation equation satisfied.

5.2 A bound for $\mathscr{V}(\boldsymbol{\vartheta})$ in the case of a generic odd $\mathfrak{n} \ge 3$.

Here and henceforth, we consider $\mathfrak{n}$ as an odd number such that $\mathfrak{n} \ge 3$; the case $\mathfrak{n} = 1$ is already discussed in [47, 73]. If one is interested in the bound of the tree for a specific value of $\mathfrak{n}$, the case of $\mathfrak{n} = 3$ is treated in Appendix A.

Let C, $\widetilde{C}$ and $\widetilde{C}_1$ be defined as follows:

$$C := \rho^{-1} \max\left\{\frac{4\Phi}{C_1}, \frac{4\Phi}{A}, 1, \frac{4\Gamma}{A}, \frac{4\Gamma}{C_1}\right\}, \tag{5.15a}$$

$$\widetilde{C} := \max\left\{\frac{\Phi}{\Gamma}, \frac{C_1\Phi}{\Gamma A}, \frac{C_1}{4\Gamma}, \frac{C_1}{A}, 1\right\}, \tag{5.15b}$$

$$\widetilde{C}_1 := \max\left\{\frac{\Phi A}{C_1\Gamma}, \frac{\Phi}{\Gamma}, \frac{A}{4\Gamma}, \frac{A}{C_1}, 1\right\}, \tag{5.15c}$$

with the positive constants ρ, Γ, Φ, C_1 defined as in (5.3), (5.4), (5.5) respectively and A defined as in Lemma 5.3. Notice also that $\widetilde{C}, \widetilde{C}_1 \geq 1$.

Lemma 5.5. *Let ϑ be a tree of order $k = 1$ and let ℓ_0 be the root line. Then if $\varepsilon \in \cup_{n \geq N} I_n$, with I_n and N as in Theorem 2, one has*

$$|\mathscr{V}(\vartheta)| \leq \rho C \times \begin{cases} \varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} e^{-\xi|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{<,0}(\vartheta) \sqcup L_{\geq,0}(\vartheta), \\ \eta^{-\mathfrak{n}+1} e^{-\xi|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{\geq,1}(\vartheta), \\ |\zeta|, & \text{if } \ell_0 \in L_0(\vartheta), \end{cases} \tag{5.16}$$

with ρ, C and η respectively as in (5.3), (5.15a) *and* (5.11).

Proof. Let $\ell_0 \in L_{<,0}(\vartheta) \sqcup L_{\geq,0}(\vartheta)$, then the value of the tree is

$$\mathscr{V}(\vartheta) = \varepsilon f_{\boldsymbol{\nu}}\, G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}).$$

If we bound $|f_{\boldsymbol{\nu}}|$ according to (5.4) and if we estimate $|G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})|$ by using the sharp partition (5.10), i.e. $|G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \frac{4}{C_1} \varepsilon^{-\frac{1}{\mathfrak{n}+1}}$, we obtain the first equation in (5.16).

If $\ell_0 \in L_{\geq,1}(\vartheta)$, then the value of the tree is given by

$$\mathscr{V}(\vartheta) = \varepsilon f_{\boldsymbol{\nu}}\, G^{[1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$$

and we have to use the bound provided in Lemma 5.3 to obtain the second equation in (5.16) since $|G^{[1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \frac{4}{A\varepsilon} \eta^{-\mathfrak{n}+1}$.

If $\ell_0 \in L_0(\vartheta)$, then $\mathscr{V}(\vartheta) = \zeta$.

Hence the thesis holds by choosing C as in (5.15a). □

Lemma 5.6. *Let $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ be a tree of order $k \geq \mathfrak{n} + 1$ and momentum $\boldsymbol{\nu} \neq \mathbf{0}$ associated with the root line ℓ_0. Let C_2 be as in* (1.6) *and, for any fixed C_1 take $\varepsilon \in I_n$ for some $n \geq N$, where N satisfies* (1.8) *and I_n is as in* (1.9). *Then one has*

$$\mathscr{V}(\vartheta) = \bar{\mathscr{V}}(\vartheta) \prod_{v \in E_1(\vartheta)} e^{-\frac{\xi}{2}|\boldsymbol{\nu}_v|}, \tag{5.17}$$

where

$$|\bar{\mathscr{V}}(\vartheta)| \le \rho C^k \eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}} \times \begin{cases} e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{<,0}(\vartheta), \\ e^{-\frac{\xi}{4}q_n}, & \text{if } \ell_0 \in L_{\ge,0}(\vartheta), \\ \eta^{-\mathfrak{n}+1} e^{-\frac{\xi}{4}q_n}, & \text{if } \ell_0 \in L_{\ge,1}(\vartheta), \end{cases} \tag{5.18}$$

with ρ, C *and* η *as in* (5.3), (5.15a) *and* (5.11).

Proof. Let be $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ and denote by v_0 the first node, that is the node the root line exits. For any renormalized tree, one has the following structure:

- $\vartheta_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell_1}}, \dots, \vartheta_m \in \mathfrak{T}_{k_m,\boldsymbol{\nu}_{\ell_m}}$ enter the first node v_0 and the root lines $\ell_1, \dots, \ell_m$ are such that $|\boldsymbol{\nu}_{\ell_j}| < q_n$ for all $j = 1, \dots, m$, so that $\ell_1, \dots, \ell_m \in L_{<,0}(\vartheta)$;
- $\vartheta'_1 \in \mathfrak{T}_{k'_1,\boldsymbol{\nu}_{\ell'_1}}, \dots, \vartheta'_p \in \mathfrak{T}_{k'_p,\boldsymbol{\nu}_{\ell'_p}}$ enter the first node v_0 and the root lines $\ell'_1, \dots, \ell'_p$ are such that $|\boldsymbol{\nu}_{\ell'_j}| \ge q_n$ and $\widetilde{n}_{\ell'_j} = 0$ for all $j = 1, \dots, p$, so that $\ell'_1, \dots, \ell'_p \in L_{\ge,0}(\vartheta)$;
- $\vartheta''_1 \in \mathfrak{T}_{k''_1,\boldsymbol{\nu}_{\ell''_1}}, \dots, \vartheta''_l \in \mathfrak{T}_{k''_l,\boldsymbol{\nu}_{\ell''_l}}$ enter the first node v_0 and the root lines $\ell''_1, \dots, \ell''_l$ are such that $|\boldsymbol{\nu}_{\ell''_j}| \ge q_n$ and $\widetilde{n}_{\ell''_j} = 1$ for all $j = 1, \dots, l$, so that $\ell''_1, \dots, \ell''_l \in L_{\ge,1}(\vartheta)$;
- the lines $\widetilde{\ell}_1, \dots, \widetilde{\ell}_r$, entering the node v_0, exit the end nodes $\widetilde{v}_1, \dots, \widetilde{v}_r \in E_1(\vartheta)$ respectively and are such that $\widetilde{n}_{\widetilde{\ell}_j} = 0$ for all $j = 1, \dots, r$, so that $\widetilde{\ell}_j \in L_{<,0}(\vartheta) \sqcup L_{\ge,0}(\vartheta)$ for all $j = 1, \dots, r$;
- the lines $\widetilde{\ell'}_1, \dots, \widetilde{\ell'}_s$, entering the node v_0, exit the end nodes $\widetilde{v'}_1, \dots, \widetilde{v'}_s \in E_1(\vartheta)$ respectively and $\widetilde{\ell'}_j \in L_{\ge,1}$ for all $j = 1, \dots, s$;
- the lines $\bar{\ell}_1, \dots, \bar{\ell}_u$, entering the node v_0, exit the end nodes $\bar{v}_1, \dots, \bar{v}_u \in E_0(\vartheta)$;
- $m, p, l, r, s, u \ge 0$.

According to this construction, we have the following constraints:

- $k = k(\vartheta) = \sum_{j=1}^m k(\vartheta_j) + \sum_{j=1}^p k(\vartheta'_j) + \sum_{j=1}^l k(\vartheta''_j) + r + s + u + 1$;
- $m + p + l + r + s + u \ge \mathfrak{n}$;
- $\boldsymbol{\nu} = \sum_{j=1}^m \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^p \boldsymbol{\nu}_{\ell'_j} + \sum_{j=1}^l \boldsymbol{\nu}_{\ell''_j} + \sum_{j=1}^r \boldsymbol{\nu}_{\widetilde{\ell}_j} + \sum_{j=1}^s \boldsymbol{\nu}_{\widetilde{\ell'}_j}$;
- $\boldsymbol{\nu} \ne \mathbf{0}$.

In the expression of $\mathscr{V}(\vartheta)$ we collect a common factor $\prod_{v \in E_1(\vartheta)} e^{-\xi|\boldsymbol{\nu}_v|/2}$ and we verify by induction on the order of the tree the inequality in (5.18). In particular we will use (5.18) to bound subtrees of order k', with $1 < k' < k$ and (5.16) with ξ replaced with $\frac{\xi}{2}$ to bound the values of the subtrees of order 1 formed by an end node in $E_1(\vartheta)$ and the respective exiting line. Finally we use the last inequality in (5.16) to estimate the values of the subtrees of order 1 formed by end nodes in $E_0(\vartheta)$ and the respective exiting lines. See Section 2.1 for the definition of subtree.

I case: $\boldsymbol{\ell_0 \in L_{<,0}(\vartheta)}$.

We start by analysing the case in which the root line has momentum $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| < q_n$. In this case we have:

$$
\begin{aligned}
|\bar{\mathscr{V}}(\vartheta)| \leq & \frac{4\Gamma}{C_1}\rho^{-(m+p+l+r+s)}\rho^{m+p+l+r+s}C^{k-1}\eta^{\frac{k-r-s-u-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}(m+p+l)}\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(r+1)} \\
& \times \eta^{(-\mathfrak{n}+1)(l+s)}|\zeta|^u e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)} \\
& \leq \rho C^k \widetilde{C}^{-1}\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}}\eta^{\Delta_0(m,p,l,r,s,u)} \\
& \times e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)},
\end{aligned}
$$

with C and $\widetilde{C}$ as in (5.15a) and in (5.15b) respectively and

$$
\begin{aligned}
\Delta_0(m,p,l,r,s,u) := & \frac{(m+p)(\mathfrak{n}^3+\mathfrak{n}^2-\mathfrak{n}-1)+(l+r)(\mathfrak{n}^2-1)+u(\mathfrak{n}^2+\mathfrak{n}-1)}{\mathfrak{n}(\mathfrak{n}+1)}+ \\
& -\frac{s(\mathfrak{n}^3-\mathfrak{n}+1)-\mathfrak{n}+\mathfrak{n}^3}{\mathfrak{n}(\mathfrak{n}+1)}.
\end{aligned}
$$

Notice that, by the definition of $\widetilde{C}$, one has $\widetilde{C}^{-1} \leq 1$.

We want to verify the first bound in (5.18), that is equivalent to prove

$$
\widetilde{C}^{-1}\eta^{\Delta_0(m,p,l,r,s,u)}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}. \tag{5.19}
$$

We distinguish among three different cases:

(a) $p+l=1$;

(b) $p+l \geq 2$;

(c) $p+l=0$.

In the case (a), we have to prove

$$
\widetilde{C}^{-1}\eta^{\Delta_0(m,p,l,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|} \times e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)} \leq 1,
$$

since $e^{-\frac{\xi}{4}q_n(p+l)} = e^{-\frac{\xi}{4}q_n} < e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}$.

In doing that, we first analyse $p=1$ and $l=0$. The thesis is obviously satisfied when $s=0$, in fact

$$
\Delta_0(m,1,0,r,0,u) \geq \frac{(\mathfrak{n}^2-1)(m+r+u)+\mathfrak{n}^2-1}{\mathfrak{n}(\mathfrak{n}+1)} \geq \frac{\mathfrak{n}(\mathfrak{n}^2-1)}{\mathfrak{n}(\mathfrak{n}+1)} = \mathfrak{n}-1,
$$

so the thesis follows.

The thesis is satisfied for $s \geq 1$ as well. We start by noting that the function $\Delta_0(m,1,0,r,s,u)$ could be negative, so the conclusion is not as trivial as in the case $s=0$. We observe that

$$\Delta_0(m,1,0,r,s,u) \geq -\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}s,$$

so we have the following inequality

$$\begin{aligned}&\widetilde{C}^{-1}\eta^{\Delta_0(m,1,0,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|} \times e^{-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\\ &\leq \eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}s} \times e^{-\frac{\xi}{2}\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|}.\end{aligned}$$

Since $|\boldsymbol{\nu}_{\widetilde{\ell}'_j}| \geq q_n \geq C_2^{-1}\log\frac{1}{\varepsilon}$ for all $j=1,\dots,s$, we can deduce

$$\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}s} \times e^{-\frac{\xi}{2}\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|} \leq \left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}}\varepsilon^{\frac{\xi}{2C_2}}\right)^s \leq \left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}}\eta^{\frac{\xi}{2C_2}}\right)^s$$

and if we require $\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{2C_2}} \leq 1$, i.e. $C_2 \leq \frac{\mathfrak{n}(\mathfrak{n}+1)}{2(\mathfrak{n}^3-\mathfrak{n}+1)}\xi$, the thesis is satisfied once again.

We now analyse $p=0$ and $l=1$ by following the same approach: if $s=0$, the function $\Delta_0(m,0,1,r,0,u)$ is non-negative, as we can easily see since $m+r+u \geq \mathfrak{n}-1$

$$\Delta_0(m,0,1,r,0,u) \geq \frac{(\mathfrak{n}^2-1)(m+r+u)-\mathfrak{n}^3+\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)} \geq 0.$$

If $s \geq 1$, the function $\Delta_0(m,0,1,r,s,u)$ could be negative, but, as in the previous situation, we can use the decaying of the exponential part associated with the lines s. We have the following bound for the function $\Delta_0(m,0,1,r,s,u)$:

$$\Delta_0(m,0,1,r,s,u) \geq \frac{-\mathfrak{n}^3-\mathfrak{n}^2+\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}s = \frac{-\mathfrak{n}^2-\mathfrak{n}+1}{\mathfrak{n}+1}s,$$

so if we require $\eta^{\frac{-\mathfrak{n}^2-\mathfrak{n}+1}{\mathfrak{n}+1}+\frac{\xi}{2C_2}} \leq 1$, i.e. $C_2 \leq \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}\xi$, the thesis follows, by noticing that

$$\begin{aligned}&\widetilde{C}^{-1}\,\eta^{\Delta_0(m,0,1,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\\ &\leq \left(\eta^{\frac{-\mathfrak{n}^2-\mathfrak{n}+1}{\mathfrak{n}+1}}\varepsilon^{\frac{\xi}{2C_2}}\right)^s \leq 1.\end{aligned}$$

By comparing the bounds on C_2 we have

$$\begin{cases}C_2 \leq \frac{\mathfrak{n}(\mathfrak{n}+1)}{2(\mathfrak{n}^3-\mathfrak{n}+1)}\xi\\ C_2 \leq \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}\xi\end{cases} \qquad \text{so we require} \quad C_2 \leq \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}\xi. \tag{5.20}$$

An analogous result is obtained in the second case, (b), in which the function $\Delta_0(m,p,l,r,s,u)$ has the following lower bound:

$$\Delta_0(m,p,l,r,s,u) \geq -\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}s - \frac{(\mathfrak{n}-2)(\mathfrak{n}-1)}{\mathfrak{n}}.$$

Then we split $e^{-\frac{\xi}{4}q_n(p+l)}$ in the following way

$$e^{-\frac{\xi}{4}q_n(p+l)} = e^{-\frac{\xi}{4}q_n(p+l-1)}e^{-\frac{\xi}{4}q_n},$$

so we can use $e^{-\frac{\xi}{4}q_n}$ to erase the factor $e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}$ in (5.19), so that the thesis becomes

$$\widetilde{C}^{-1}\eta^{\Delta_0(m,p,l,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \times e^{-\frac{\xi}{4}q_n(p+l-1)} \leq 1.$$

Finally we can use $q_n \geq \frac{1}{C_2}\log\frac{1}{\varepsilon}$ and by requiring

$$\begin{cases} C_2 \leq \frac{\mathfrak{n}(\mathfrak{n}+1)}{2(\mathfrak{n}^3-\mathfrak{n}+1)}\xi, & \text{if } \mathfrak{n}=3, \\ C_2 \leq \frac{\mathfrak{n}}{4(\mathfrak{n}-2)(\mathfrak{n}-1)}\xi, & \text{if } \mathfrak{n}\geq 5 \text{ and } \mathfrak{n} \text{ odd}, \end{cases} \tag{5.21}$$

the thesis follows as

$$\begin{aligned} &\widetilde{C}^{-1}\eta^{\Delta_0(m,p,l,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \times e^{-\frac{\xi}{4}q_n(p+l-1)} \\ &\leq \eta^{-\frac{(\mathfrak{n}-2)(\mathfrak{n}-1)}{\mathfrak{n}}+\frac{\xi}{4C_2}} \times \left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{2C_2}}\right)^s \leq 1. \end{aligned}$$

Now we have to compare the conditions on C_2 in (5.21) with the one in (5.20):

$$\begin{cases} C_2 \leq \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}\xi, & \text{if } \mathfrak{n}\geq 3 \text{ and } \mathfrak{n} \text{ odd}, \\ C_2 \leq \frac{\mathfrak{n}(\mathfrak{n}+1)}{2(\mathfrak{n}^3-\mathfrak{n}+1)}\xi, & \text{if } \mathfrak{n}=3, \\ C_2 \leq \frac{\mathfrak{n}}{4(\mathfrak{n}-2)(\mathfrak{n}-1)}\xi, & \text{if } \mathfrak{n}\geq 5 \text{ and } \mathfrak{n} \text{ odd}. \end{cases} \tag{5.22}$$

If $\mathfrak{n}=3$, one has $\frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)} < \frac{\mathfrak{n}(\mathfrak{n}+1)}{2(\mathfrak{n}^3-\mathfrak{n}+1)}$; if $\mathfrak{n}=5$ one has $\frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)} < \frac{\mathfrak{n}}{4(\mathfrak{n}-2)(\mathfrak{n}-1)}$; if $\mathfrak{n}\geq 7$ one has $\frac{\mathfrak{n}}{4(\mathfrak{n}-2)(\mathfrak{n}-1)} < \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}$. Then, to summarize, the condition on C_2 becomes

$$\begin{cases} C_2 \leq \frac{\mathfrak{n}+1}{2(\mathfrak{n}^2+\mathfrak{n}-1)}\xi, & \text{if } \mathfrak{n}=3,5, \\ C_2 \leq \frac{\mathfrak{n}}{4(\mathfrak{n}-2)(\mathfrak{n}-1)}\xi, & \text{if } \mathfrak{n}\geq 7 \text{ and } \mathfrak{n} \text{ odd}. \end{cases} \tag{5.23}$$

In the third case, (c), by the definition of momentum $\boldsymbol{\nu}$, it follows that

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|},$$

so we have to prove that

$$\widetilde{C}^{-1}\eta^{\Delta_0(m,0,0,r,s,u)}e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\leq 1.$$

If $s=0$, that is $m+r+u\geq\mathfrak{n}$, the thesis follows trivially since $\Delta_0(m,0,0,r,0,u)\geq 0$.

If $s\geq 1$, once again we have that $\Delta_0(m,0,0,r,s,u)\geq -\frac{\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}+1}s$ and hence we have to require

$$\eta^{-\frac{\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}+1}}\varepsilon^{\frac{\xi}{4C_2}}\leq 1$$

that is $C_2\leq\frac{\mathfrak{n}+1}{4(\mathfrak{n}^2+\mathfrak{n}-1)}\xi$. This condition is sharper than the one in (5.23), so we require

$$C_2\leq\frac{\mathfrak{n}+1}{4(\mathfrak{n}^2+\mathfrak{n}-1)}\xi. \tag{5.24}$$

To sum up, if the root line is such that $|\boldsymbol{\nu}|<q_n$, the first bound in (5.18) holds by requiring (5.24).

II case: $\boldsymbol{\ell_0\in L_{\geq,0}(\vartheta)}$.

We consider now the case in which the root line has momentum $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}|\geq q_n$ and it is on scale 0. We can estimate the $|\bar{\mathscr{V}}(\vartheta)|$ in the following way:

$$\begin{aligned}|\bar{\mathscr{V}}(\vartheta)|\leq{}& C^{k-1}\frac{4\Gamma}{C_1}\rho^{-(m+p+l+r+s+u)}\rho^{m+p+l+r+s+u}\eta^{\frac{k-r-s-u-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}(m+p+l)}\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}(r+1)}\zeta^u\\ &\times\eta^{(-\mathfrak{n}+1)(l+s)}e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\\ &\leq\rho C^k\widetilde{C}^{-1}\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}}\eta^{\Delta_0(m,p,l,r,s,u)}\times\\ &\times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}e^{-\frac{\xi}{4}q_n(p+l)},\end{aligned}$$

with $C,\widetilde{C}$ as in (5.15a) and (5.15b) respectively and

$$\begin{aligned}\Delta_0(m,p,l,r,s,u):={}&\frac{(m+p)(\mathfrak{n}^3+\mathfrak{n}^2-\mathfrak{n}-1)+(l+r)(\mathfrak{n}^2-1)+u(\mathfrak{n}^2+\mathfrak{n}-1)}{\mathfrak{n}(\mathfrak{n}+1)}\\ &-\frac{s(\mathfrak{n}^3-\mathfrak{n}+1)+\mathfrak{n}^3-\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}.\end{aligned}$$

Note that the expression of the function $\Delta_0(m,p,l,r,s,u)$ is the same as in case (a); however, now, we want to verify the second bound in (5.18) by using that $|\boldsymbol{\nu}|\geq q_n$. The proof proceeds as in the case of $\ell_0\in L_{<,0}(\vartheta)$, since the function $\Delta_0(m,p,l,r,s,u)$ does not change with respect to the previous case. The only difference is that now we have to construct a factor $e^{-\frac{\xi}{4}q_n}$ in the bound of $|\bar{\mathscr{V}}(\vartheta)|$.

- If $p+l\geq 1$ it is trivial as we have the term $e^{-\frac{\xi}{4}q_n(p+l)}$.

- If $p+l=0$, then $\boldsymbol{\nu}=\sum_{j=1}^{m}\boldsymbol{\nu}_{\ell_j}+\sum_{j=1}^{r}\boldsymbol{\nu}_{\widetilde{\ell}_j}+\sum_{j=1}^{s}\boldsymbol{\nu}_{\widetilde{\ell}'_j}$, so we can factorise

$$e^{\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}=e^{\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}e^{\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}$$

  and we can use the fact that $|\boldsymbol{\nu}|\geq q_n$ in order to obtain the desired term:

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}\leq e^{-\frac{\xi}{4}q_n}.$$

Apart for that, the discussion proceeds as in the previous case.

III case: $\boldsymbol{\ell_0\in L_{\geq,1}(\vartheta)}$.

We analyse the third possibility, that is the root line on scale 1 and the momentum satisfying the inequality $|\boldsymbol{\nu}|\geq q_n$, i.e. $\ell_0\in L_{\geq,1}(\vartheta)$. We can bound the $|\bar{\mathscr{V}}(\vartheta)|$ as follows:

$$\begin{aligned}|\bar{\mathscr{V}}(\vartheta)|\leq{}& C^{k-1}\frac{4\Gamma}{A}\rho^{-(m+p+l+r+s+u)}\rho^{m+p+l+r+s+u}\eta^{\frac{k-r-s-u-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}(m+p+l)}\eta^{(-\mathfrak{n}+1)(l+s)}\times\\ &\times\eta^{-\mathfrak{n}+1}\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}r}\,|\zeta|^{u}\,e^{-\frac{\xi}{4}q_{\mathfrak{n}}(p+l)}\times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\\ &\leq\rho C^{k}\widetilde{C}_1^{-1}\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}}\,\eta^{\Delta_1(m,p,l,r,s,u)}\eta^{-\mathfrak{n}+1}\times\\ &\times e^{-\frac{\xi}{4}q_{\mathfrak{n}}(p+l)-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\end{aligned}$$

with C and $\widetilde{C}_1$ as in (5.15a) and (5.15c) respectively and $\Delta_1(m,p,l,r,s,u)$ defined as

$$\begin{aligned}\Delta_1(m,p,l,r,s,u):={}&\frac{(\mathfrak{n}^3+\mathfrak{n}^2-\mathfrak{n}-1)(m+p)+(\mathfrak{n}^2-1)(l+r)+u(\mathfrak{n}^2+\mathfrak{n}-1)}{\mathfrak{n}(\mathfrak{n}+1)}+\\ &-\frac{s(\mathfrak{n}^3-\mathfrak{n}+1)+\mathfrak{n}^3+\mathfrak{n}^2-\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}.\end{aligned}$$

Again we can distinguish among three different cases:

(a) $p+l=1$;

(b) $p+l\geq 2$;

(c) $p+l=0$.

In the first case, (a), we have to prove

$$\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)}\times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\leq 1$$

since $e^{-\frac{\xi}{4}q_n(p+l)}=e^{-\frac{\xi}{4}q_n}$.

We first focus on the case $p = 1$ and $l = 0$. If $s = 0$, that is $m + r + u \geq \mathfrak{n} - 1$, the function $\Delta_1(m, 1, 0, r, 0, u)$ is positive, since $\Delta_1(m, 1, 0, r, 0, u) \geq \frac{\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}+1} > 0$, hence the thesis follows.

If we consider $s \geq 1$, the function $\Delta_1(m, 1, 0, r, s, u)$ could be negative, so we have to take advantage of the decaying of the momentum of the lines $\widetilde{\ell}'_1, \dots, \widetilde{\ell}'_s$, since $|\boldsymbol{\nu}_{\widetilde{\ell}'_j}| \geq q_n \geq C_2^{-1} \log \frac{1}{\varepsilon}$. We also observe that

$$\Delta_1(m, 1, 0, r, s, u) \geq \frac{(m + r + u)(\mathfrak{n}^2 - 1) - s(\mathfrak{n}^3 - \mathfrak{n} + 1) - 1}{\mathfrak{n}(\mathfrak{n} + 1)} \geq -\frac{\mathfrak{n}^2 + \mathfrak{n} - 1}{\mathfrak{n} + 1} s$$

and the thesis follows once again as

$$\begin{aligned} &\widetilde{C}_1^{-1} \eta^{\Delta_1(m,p,l,r,s,u)} \times e^{-\frac{\xi}{4} \sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}| - \frac{\xi}{2} \left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}| + \sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}| \right)} \\ &\leq \left(\eta^{-\frac{\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}+1}} \varepsilon^{\frac{\xi}{2C_2}} \right)^s \leq \left(\eta^{-\frac{\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}+1} + \frac{\xi}{2C_2}} \right)^s < 1, \end{aligned}$$

where the last inequality holds if C_2 satisfies (5.24).

We now analyse $p = 0$ and $l = 1$. The case $s = 0$ is not as simple as the previous one since $\Delta_1(m, 0, 1, r, 0, u)$ is not positive; nevertheless it satisfies the lower bound

$$\Delta_1(m, 0, 1, r, 0, u) \geq -\frac{\mathfrak{n}}{\mathfrak{n} + 1}.$$

This case is possible only if $|\sum_{j=1}^m \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^r \boldsymbol{\nu}_{\widetilde{\ell}_j}| \geq q_n$. Indeed, if this is not the case, that is if $|\sum_{j=1}^m \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^r \boldsymbol{\nu}_{\widetilde{\ell}_j}| < q_n$, by using the properties of continued fractions, see (5.1), it follows that $|\boldsymbol{\omega} \cdot \big(\sum_{j=1}^m \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^r \boldsymbol{\nu}_{\widetilde{\ell}_j} \big)| \geq \frac{C_1}{2} \varepsilon^{\frac{1}{\mathfrak{n}+1}}$. By hypothesis the lines ℓ_0 and ℓ''_1 are both on scale 1, so we have $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| < \frac{C_1}{4} \varepsilon^{\frac{1}{\mathfrak{n}+1}}$ and $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell''_1}| < \frac{C_1}{4} \varepsilon^{\frac{1}{\mathfrak{n}+1}}$, but this is impossible as

$$\frac{C_1}{2} \varepsilon^{\frac{1}{\mathfrak{n}+1}} \leq |\boldsymbol{\omega} \cdot \big(\sum_{j=1}^m \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^r \boldsymbol{\nu}_{\widetilde{\ell}_j} \big)| = |\boldsymbol{\omega} \cdot (\boldsymbol{\nu} - \boldsymbol{\nu}_{\ell''_1})| \leq |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| + |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell''_1}| < \frac{C_1}{2} \varepsilon^{\frac{1}{\mathfrak{n}+1}}.$$

So we can obtain the thesis as follows:

$$\begin{aligned} \widetilde{C}_1^{-1} \eta^{\Delta_1(m,0,1,r,0,u)} e^{-\frac{\xi}{4} \left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}| + \sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}| \right) - \frac{\xi}{4} \sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|} &\leq \eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1}} e^{-\frac{\xi}{4} q_n} \\ &\leq \eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1}} \varepsilon^{\frac{\xi}{4C_2}} \leq \eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1} + \frac{\xi}{4C_2}} \leq \eta^{\mathfrak{n}-1} < 1 \end{aligned}$$

where we have used (5.24).

If $s \geq 1$, we notice that

$$\Delta_1(m, 0, 1, r, s, u) \geq -\frac{s(\mathfrak{n}^3 + \mathfrak{n}^2 - \mathfrak{n}) + \mathfrak{n}^2}{\mathfrak{n}(\mathfrak{n} + 1)} \geq -\frac{\mathfrak{n}^3 + 2\mathfrak{n}^2 - \mathfrak{n}}{\mathfrak{n}(\mathfrak{n} + 1)} s = -\frac{\mathfrak{n}^2 + 2\mathfrak{n} - 1}{\mathfrak{n} + 1} s,$$

hence we have the following estimate:

$$\begin{aligned}&\widetilde{C}_1^{-1}\eta^{\Delta_1(m,0,1,r,0,u)}\, e^{-\frac{\xi}{4}\left(\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)-\frac{\xi}{4}\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|}\\ &\le \left(\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}}\varepsilon^{\frac{\xi}{2C_2}}\right)^s \le (\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}+\frac{\xi}{2C_2}})^s\end{aligned}$$

and, by using (5.24), this bound implies the thesis as $\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}+\frac{\xi}{2C_2}} < \eta^{\mathfrak{n}-1} < 1$.

We now study the case (b), $p+l\ge 2$. We want to prove that

$$\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)}\times e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le e^{-\frac{\xi}{4}q_n}. \tag{5.25}$$

First of all notice that

$$e^{-\frac{\xi}{4}q_n(p+l)} = e^{-\frac{\xi}{4}q_n}e^{-\frac{\xi}{4}q_n(p+l-1)},$$

so we can use the first factor to erase the same factor on the right hand side of (5.25).

By using the fact that $p+l\ge 2$, we can bound $\Delta_1(m,p,l,r,s,u)$ in the following way:

$$\Delta_1(m,p,l,r,s,u)\ge -\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}s-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}$$

and we obtain

$$\begin{aligned}&\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)}\, e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}e^{-\frac{\xi}{4}q_n(p+l-1)}\\ &\le \eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}}\, e^{-\frac{\xi}{4}q_n}\left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}}e^{-\frac{\xi}{2}q_n}\right)^s\\ &\le \eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}}\,\varepsilon^{\frac{\xi}{4C_2}}\left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}}\varepsilon^{\frac{\xi}{2C_2}}\right)^s\\ &\le \eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{4C_2}}\left(\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{2C_2}}\right)^s.\end{aligned}$$

So if we require $\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{4C_2}} < 1$, that also implies $\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}+1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{2C_2}} < 1$, the thesis follows. In fact, we do not have to add any other condition on C_2, because if C_2 satisfies (5.24), then $\eta^{-\frac{\mathfrak{n}^3-\mathfrak{n}^2-\mathfrak{n}+2}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{4C_2}} < \eta^{\frac{2(\mathfrak{n}-1)}{\mathfrak{n}}}$, that is actually less than 1.

We now analyse the third case, (c), $p=l=0$. First of all we notice that

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}\le e^{-\frac{\xi}{4}q_n},$$

as $|\boldsymbol{\nu}|\ge q_n$. So we have to prove that

$$\widetilde{C}_1^{-1}\eta^{\Delta_1(m,0,0,r,s,u)}\times e^{-\frac{\xi}{4}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\le 1.$$

If $s=0$, one has $u+m+r\geq\mathfrak{n}$, that implies

$$\Delta_1(m,0,0,r,0,u)\geq\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u-\mathfrak{n}^2}{\mathfrak{n}(\mathfrak{n}+1)}.$$

Then if $m\geq 1$ the function $\Delta_1(m,0,0,r,0,u)$ is strictly positive, otherwise, if $m=0$, one has $\Delta_1(0,0,0,r,0,u)\geq-\frac{\mathfrak{n}}{\mathfrak{n}+1}$. Moreover, in such a case, $\boldsymbol{\nu}=\sum_{j=1}^{r}\boldsymbol{\nu}_{\widetilde{\ell}_j}$ and $|\boldsymbol{\nu}|\geq q_n\geq C_2^{-1}\log\frac{1}{\varepsilon}$, so, it follows that

$$\begin{aligned}\widetilde{C}^{-1}\eta^{\Delta_1(0,0,0,r,0,u)}e^{-\frac{\xi}{4}\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|}&\leq\eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1}}e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}\\&\leq\eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1}}\varepsilon^{\frac{\xi}{4C_2}}\leq\eta^{-\frac{\mathfrak{n}}{\mathfrak{n}+1}+\frac{\xi}{4C_2}}\\&<\eta^{\mathfrak{n}-1}<1\end{aligned}$$

and the thesis holds once again, provided C_2 is as in (5.24).

If $s\geq 1$, we notice again that $\Delta_1(m,0,0,r,s,u)\geq-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}s$ and the desired bound follows by requiring $\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}+\frac{\xi}{4C_2}}\leq 1$. Then

$$\begin{aligned}&\widetilde{C}_1^{-1}\eta^{\Delta_1(m,0,0,r,s,u)}\times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\\&\quad\leq\left(\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}}\varepsilon^{\frac{\xi}{4C_2}}\right)^s\leq\left(\eta^{-\frac{\mathfrak{n}^2+2\mathfrak{n}-1}{\mathfrak{n}+1}+\frac{\xi}{4C_2}}\right)^s\leq 1.\end{aligned}$$

Hence to sum up, the third inequality of (5.18) holds by requiring $C_2\leq\frac{\mathfrak{n}+1}{4(\mathfrak{n}^2+2\mathfrak{n}-1)}\xi$.

Therefore, if we replace the condition on C_2 in (5.24) with the last one, since

$$\begin{cases}C_2\leq\frac{(\mathfrak{n}+1)\xi}{4(\mathfrak{n}^2+\mathfrak{n}-1)}\\C_2\leq\frac{(\mathfrak{n}+1)\xi}{4(\mathfrak{n}^2+2\mathfrak{n}-1)}\end{cases}\qquad\Longrightarrow\qquad C_2\leq\frac{(\mathfrak{n}+1)\xi}{4(\mathfrak{n}^2+2\mathfrak{n}-1)},\tag{5.26}$$

that is, if we take C_2 as

$$C_2=\frac{\mathfrak{n}+1}{4(\mathfrak{n}^2+2\mathfrak{n}-1)}\xi,\tag{5.27}$$

both (5.17) and (5.18) are satisfied and the proof of Lemma 5.6 is completed. □

Corollary 5.7. *The bound in* (5.18) *can be simplified as follows:*

$$|\bar{\mathscr{V}}(\vartheta)|\leq\rho C^k\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}},\tag{5.28}$$

in such a way that (5.18) *becomes*

$$|\mathscr{V}(\vartheta)|\leq\rho C^k\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}}\prod_{v\in E_1(\vartheta)}e^{-\frac{\xi}{2}|\boldsymbol{\nu}_v|},\tag{5.29}$$

provided C_2 *as in* (5.27).

Proof. In order to obtain these expressions we have bounded the first term in (5.18) as $e^{-\frac{\xi}{4}|\boldsymbol{\nu}|} \le 1$, the second and the third alternatives as

$$e^{-\frac{\xi}{4}q_n} < \varepsilon^{\frac{\xi}{4C_2}} \le \varepsilon^{\frac{2\mathfrak{n}^3+\mathfrak{n}^2+1-2\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}} < 1,$$

$$\eta^{-\mathfrak{n}+1} e^{-\frac{\xi}{4}q_n} < \eta^{-\mathfrak{n}+1}\varepsilon^{\frac{\xi}{4C_2}} < \eta^{-\mathfrak{n}+1+\frac{\xi}{4C_2}} < \eta^{-\mathfrak{n}+1+\frac{2\mathfrak{n}^3+\mathfrak{n}^2+1-2\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}} \le \eta^{\frac{\mathfrak{n}^3+\mathfrak{n}^2+1-\mathfrak{n}}{\mathfrak{n}(\mathfrak{n}+1)}} < 1,$$

where we have used the lower bound of q_n in (5.7) with C_2 as in (5.27). □

Remark 5.8. The bound (5.18) has been used for the induction argument, but what we need in the following is the simpler bound (5.28).

5.3 Convergence of the Renormalized Expansion

We want to prove that the series described in (2.15) converges.

Lemma 5.9. *For any $k \ge 1$ and $\boldsymbol{\nu} \in \mathbb{Z}^2_*$ one has*

$$|X^{[k]}_{\boldsymbol{\nu}}| \le \rho B^k \eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}} e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}, \tag{5.30}$$

where B is a positive constant proportional to C, with C defined as in (5.15a)*, provided C_2 as in* (5.27) *and $\varepsilon \in \cup_{n\ge N} I_n$, with I_n and N as in Theorem 2.*

Proof. To bound the coefficients $X^{[k]}_{\boldsymbol{\nu}}$ defined as in (2.14), we use the estimate (5.29) and sum over all trees in $\mathfrak{T}_{k,\boldsymbol{\nu}}$.

The sum over the mode labels $\boldsymbol{\nu} \in \mathbb{Z}^2$ in (2.14) can be performed by using the factor $e^{-\frac{\xi}{2}|\boldsymbol{\nu}_v|}$ associated with end nodes in $E_1(\vartheta)$ and this gives a bound $B_1^{|E_1(\vartheta)|} e^{-\frac{\xi}{4}|\boldsymbol{\nu}_v|}$ for some positive constant B_1. The sum over the other labels produces a factor $B_2^{k(\vartheta)}$, with B_2 suitable positive constant. By taking $B = B_1 B_2 C$ the thesis follows. □

Corollary 5.10. *The function $\bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ is analytic in $\boldsymbol{\psi}$ in a strip $\Sigma_{\xi'}$, with $\xi' < \frac{\xi}{4}$.*

Proof. It follows from the bound on the Fourier coefficients given by Lemma 5.9. □

Lemma 5.11. *$\bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$, defined as in* (2.15)*, solves the range equation, i.e. is a solution to* (2.16)*.*

Proof. The proof is the same as in Section 3.4. □

We refer to [75] for the definition of continuity – and differentiability – in the sense of Whitney: let A be a closed set in euclidean space E and let $f(x)$ be a function defined and continuous in A. Then this function can be extended so as to be continuous throughout E; see also [31], §125.

Lemma 5.12. *$\bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ can be extended to a function $\widetilde{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$, defined for all $\varepsilon \in [0,\varepsilon_0)$, such that $\widetilde{X}(\boldsymbol{\psi};\varepsilon,\zeta,c) \equiv \bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ for $\varepsilon \in \overline{\cup_{n\geq N} I_n}$ and $\widetilde{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ is continuous in ε and $\widetilde{X}(\boldsymbol{\psi};\varepsilon,\zeta,c) \to 0$ as $\varepsilon \to 0$.*

Proof. Continuity of the function $\varepsilon \to \bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c)$ holds trivially for $\varepsilon > 0$ such that $\varepsilon \in \cup_{n\geq N} I_n$, with I_n defined as in (1.9). On the contrary continuity at $\varepsilon = 0$ needs some discussions. Set

$$\mathcal{F}(\varepsilon,\zeta) := ||\bar{X}(\cdot;\varepsilon,\zeta,c)||_\infty = \sup\{\bar{X}(\boldsymbol{\psi};\varepsilon,\zeta,c) : \boldsymbol{\psi} \in \Sigma_{\xi'}\}, \tag{5.31}$$

with $\xi' < \frac{\xi}{4}$ as in Corollary 5.10. Since $\mathcal{F}(0,\zeta) = 0$ by construction, see (5.8) and take $\varepsilon = 0$, we have to prove that $\mathcal{F}(\varepsilon,\zeta) \to 0$ as $\varepsilon \to 0$, that means for all $\iota > 0$ there exists $\delta > 0$ such that $0 < \varepsilon < \delta$ implies $|\mathcal{F}(\varepsilon,\zeta)| < \iota$.

We have

$$\mathcal{F}(\varepsilon,\zeta) \leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \left|X_{\boldsymbol{\nu}}^{[k]}\right| e^{\xi'|\boldsymbol{\nu}|}.$$

Then we bound the value of a tree with k nodes by using the bound (5.29) for $k-1$ nodes except one end node and by noticing that $E_1(\vartheta) \neq \emptyset$ (otherwise $\boldsymbol{\nu} = 0$). Thus, we have:

$$\begin{aligned}
\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \left|X_{\boldsymbol{\nu}}^{[k]}\right| e^{\xi'|\boldsymbol{\nu}|} &\leq \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \sum_{\vartheta\in\mathfrak{T}_{\boldsymbol{\nu},k}} \left|\mathscr{V}(\theta)\right| e^{\xi'|\boldsymbol{\nu}|} \\
&\leq \rho\, C^k \eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}} \left(\frac{\varepsilon}{\eta}\right)^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} \sum_{\vartheta\in\mathfrak{T}_{\boldsymbol{\nu},k}} \prod_{v\in E_1(\vartheta)} \sum_{\boldsymbol{\nu}_v\in\mathbb{Z}^d} e^{-\xi|\boldsymbol{\nu}_v|/4} \\
&\leq \rho B^k \eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}} \varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} e^{-\xi|\boldsymbol{\nu}|/4}.
\end{aligned}$$

Notice that we have used that an end node $w \in E_1(\vartheta)$ and the respective exiting line can be bounded or with $\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} e^{-\xi|\boldsymbol{\nu}_w|}$ or with $\eta^{-\mathfrak{n}+1}e^{-\xi|\boldsymbol{\nu}_w|}$. Hence we can collect a common factor $e^{-\frac{\xi}{2}|\boldsymbol{\nu}_w|}$ that multiplied by $e^{\xi'|\boldsymbol{\nu}|}\prod_{v\in E_1(\vartheta)\setminus\{w\}} e^{-\frac{\xi}{2}|\boldsymbol{\nu}_v|}$ gives the factor $\prod_{v\in E_1(\vartheta)} e^{-\frac{\xi}{4}|\boldsymbol{\nu}_v|}$. Therefore we are left with a contribution $\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} e^{-\frac{\xi}{2}|\boldsymbol{\nu}_w|}$ or $\eta^{-\mathfrak{n}+1}e^{-\frac{\xi}{2}|\boldsymbol{\nu}_w|}$: by using (5.7) and (5.26) we have

$$\eta^{-\mathfrak{n}+1}e^{-\frac{\xi}{2}|\boldsymbol{\nu}_w|} \leq \eta^{-\mathfrak{n}+1}e^{-\frac{\xi}{2}q_n} \leq \eta^{-\mathfrak{n}+1}\varepsilon^{\frac{2(\mathfrak{n}^2+2\mathfrak{n}-1)}{\mathfrak{n}+1}} \leq \varepsilon^{\frac{\mathfrak{n}^2+4\mathfrak{n}-1}{\mathfrak{n}+1}} \leq \varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}},$$

since $\eta \geq \varepsilon$.

Hence we obtain the following bound for $\mathcal{F}(\varepsilon,\zeta)$:

$$\mathcal{F}(\varepsilon,\zeta) \leq \rho\, \frac{\eta^{\frac{\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}}}{1-B\eta^{\frac{1}{\mathfrak{n}(\mathfrak{n}+1)}}}\, \varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} e^{-\xi|\boldsymbol{\nu}|/4} \tag{5.32}$$

and, for fixed $\iota > 0$, by choosing $\delta > 0$ suitably small and taking $0 < \varepsilon < \delta$ we have $\mathcal{F}(\varepsilon, \zeta) \le \iota$.

By reasoning ia a similar way, one proves that, for all $\varepsilon, \varepsilon' \in \cup_{n \ge N} I_n$, the function $\mathcal{F}$ satisfies the bound $|\mathcal{F}(\varepsilon, \zeta) - \mathcal{F}(\varepsilon', \zeta)| \le \omega(\varepsilon, \varepsilon')$ for a suitable modulus of continuity ω; the bounds above ensures that $\varepsilon \mapsto \mathcal{F}(\varepsilon, \zeta)$ is at least Hölder-continuous with exponent $\mathfrak{n}/(\mathfrak{n}+1)$

Therefore, in the light of (5.32), the function $\bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ can be extended in the sense of Whitney to a function $\widetilde{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$, defined for all $\varepsilon \in [0, \varepsilon_0)$, such that $\widetilde{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c) \equiv \bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ for $\varepsilon \in \cup_{n \ge N} I_n$. Therefore $\widetilde{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ represents the Whitney extension of $\bar{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c)$ to the interval $[0, \varepsilon_0]$ and is continuous in ε by construction, in particular $\widetilde{X}(\boldsymbol{\psi}; \varepsilon, \zeta, c) \to 0$ as $\varepsilon \to 0$. □

5.4 The bifurcation equation

Define the function $\mathcal{H}(\varepsilon, \zeta)$ as in (3.22), that is

$$\mathcal{H}(\varepsilon, \zeta) := [g(c + \zeta + \bar{X}(\cdot; \varepsilon, \zeta, c))]_{\mathbf{0}} - f_{\mathbf{0}}. \tag{5.33}$$

Hence the bifurcation equation in (5.9) becomes $\mathcal{H}(\varepsilon, \zeta) = 0$.

Lemma 5.13. *The function $\mathcal{H}(\varepsilon, \zeta)$ is $C^{\mathfrak{n}}$ with respect to ζ.*

Proof. By Hypothesis 1, one has $g(c) = f_{\mathbf{0}}$, $\frac{d^i g}{dx^i}(c) = 0$ for $i = 1, 2, \ldots, \mathfrak{n}-1$ and $\frac{d^{\mathfrak{n}} g}{dx^{\mathfrak{n}}}(c) \neq 0$, where $\mathfrak{n}$ is odd and such that $\mathfrak{n} \ge 3$. Hence

$$\mathcal{H}(\varepsilon, \zeta) = \sum_{p=\mathfrak{n}} g_p(c) [(\zeta + \bar{X}(\cdot; \varepsilon, \zeta, c))^p]_{\mathbf{0}} = \sum_{k=\mathfrak{n}+1}^{\infty} \sum_{\vartheta \in \mathfrak{T}_{k,\mathbf{0}}} \mathscr{V}^*(\vartheta),$$

$\mathscr{V}^*(\vartheta)$ is defined as $\mathscr{V}(\vartheta)$ with the only difference that the node factor of the first node v_0 is $F_{v_0} = g_{p_{v_0}}$ (without the factor $-\varepsilon$ appearing in (2.9)). Therefore it is sufficient to prove that the function $\mathscr{V}^*(\vartheta)$ is $C^{\mathfrak{n}}$ in ζ. We first provide a bound for the renormalized trees associated with the solution of the bifurcation equation. Take C as in (5.15a) and ρ, Γ as in (5.3).

We consider the same construction of Lemma 5.6; the only difference is that now, if we denote with $\boldsymbol{\nu}$ the momentum of the root line, one has $\boldsymbol{\nu} = 0$. The bound for $\mathscr{V}^*(\vartheta)$ can be easily obtained from the proof of Lemma 5.6 by noticing that the propagator of the root line is equal to one. Indeed we have:

$$
\begin{aligned}
|\mathscr{V}^*(\vartheta)| &\le \Gamma\rho^{-(m+p+l+r+s)}\rho^{m+p+l+r+s}C^{k-1}\,\eta^{\frac{k-r-s-u-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}^2-1}{\mathfrak{n}}(m+p+l)}\varepsilon^{\frac{\mathfrak{n}}{\mathfrak{n}+1}r}\\
&\quad\times\eta^{(-\mathfrak{n}+1)(l+s)}|\zeta|^u e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)}\\
&\le \rho\Gamma C^k\bar{C}^{-1}\eta^{\frac{k}{\mathfrak{n}(\mathfrak{n}+1)}+\Delta_2(m,p,l,r,s,u)}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\tilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|\right)}\\
&\le \rho\Gamma C^k\eta^{\frac{k+\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}}e^{-\frac{\xi}{4}q_n(p+l+s)},
\end{aligned}
\tag{5.34}
$$

with C as in (5.15a) and

$$\bar{C} := \rho C \ge 1$$

and where we have used that

$$
\begin{aligned}
\Delta_2(m,p,l,r,s,u) :=& \frac{(m+p+l+r+u)(\mathfrak{n}^2-1)+(m+p)(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}+\\
&-\frac{1+s(\mathfrak{n}^3-\mathfrak{n}+1)}{\mathfrak{n}(\mathfrak{n}+1)}\ge\\
&\ge\frac{\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}-\frac{s(\mathfrak{n}^2+\mathfrak{n}-1)}{\mathfrak{n}+1}
\end{aligned}
$$

and

$$\eta^{-s\frac{\mathfrak{n}^2+\mathfrak{n}-1}{\mathfrak{n}+1}}e^{-\frac{\xi}{4}\sum_{j=1}^s|\boldsymbol{\nu}_{\tilde{\ell}'_j}|}\le\left(\eta^{\frac{-\mathfrak{n}^2-\mathfrak{n}+1}{\mathfrak{n}+1}+\frac{\xi}{4C_2}}\right)^s\le 1,$$

since C_2 is as in (5.27).

In particular $\mathscr{V}^*(\vartheta)$ depends on ζ through the node factors and through the propagators associated with lines on scale 1, see (2.13) and (2.14). For $0\le j\le\mathfrak{n}$, this is:

$$\partial_\zeta^j\mathscr{V}^*(\vartheta)=\sum_{\vartheta\in\mathfrak{T}_{k,\mathbf{0}}}\left[\partial_\zeta^j\Big(\prod_{v\in N(\vartheta)}F_v\Big)\cdot\Big(\prod_{\ell\in L(\vartheta)}G_\ell\Big)+\Big(\prod_{v\in N(\vartheta)}F_v\Big)\cdot\partial_\zeta^j\Big(\prod_{\ell\in L(\vartheta)}G_\ell\Big)\right]$$

If the derivative acts on the node factors, we have

$$\partial_\zeta\Big(\prod_{v\in N(\vartheta)}F_v\Big)=|E_0(\vartheta)|\zeta^{|E_0(\vartheta)|-1}\Big(\prod_{v\in N(\vartheta)\setminus E_0(\vartheta)}F_v\Big).$$

Then, if we assume that $\mathfrak{n}$ derivatives act on the node factor, by using (5.34), one has

$$
\begin{aligned}
\left|\partial_\zeta^j\Big(\prod_{v\in N(\vartheta)}F_v\Big)\cdot\Big(\prod_{\ell\in L(\vartheta)}G_\ell\Big)\right|&\le|E_0(\vartheta)|(|E_0(\vartheta)|-1)\cdots(|E_0(\vartheta)|-j+1)\rho\Gamma C'^k\times\\
&\qquad\times\eta^{\frac{k+\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}-j}e^{-\frac{\xi}{4}q_n(p+l+s)}
\end{aligned}
\tag{5.35}
$$

for $0 \le j \le \mathfrak{n}$ and for a suitable positive constant C' depending on C, with C defined as in (5.15a).

If the derivatives act on the propagators the analysis is more delicate. We have to distinguish among two cases: the case in which the derivatives act on the same line and the case in which the derivatives act on different lines. The worst case is the first one. Denote $x = \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell$, with $\ell \in L(\vartheta)$, and suppose that $\mathfrak{n}$ derivatives act on the propagator $G^{[1]}(x)$, see (2.11). Then one has:

$$
\begin{aligned}
&\partial_\zeta G^{[1]}(x) = \chi(|x|)\frac{\partial_\zeta \mathcal{M}(x)}{(D_1(x))^2}, \\
&\partial_\zeta^2 G^{[1]}(x) = \chi(|x|)\Big[\frac{2(\partial_\zeta \mathcal{M}(x))^2}{(D_1(x))^3} + \frac{\partial_\zeta^2 \mathcal{M}(x)}{(D_1(x))^2}\Big], \\
&\partial_\zeta^j G^{[1]}(x) = \chi(|x|)\sum_{k=0}^{j-1}\sum_{i_1,\dots,i_{j-k}\in\mathcal{A}} a_{i_1,\dots,i_{j-k}} \frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k+1}}, \; 3 \le j \le \mathfrak{n},
\end{aligned}
\tag{5.36}
$$

where the index set in the last sum is

$$
\mathcal{A} := \{i_1,\dots,i_{j-k} \in \mathbb{N} : i_1 + \dots + i_{j-k} = j \text{ and } i_1 \ge i_2 \ge \dots \ge i_{j-k} \ge 1\}
$$

and for suitable constants $a_{i_1,\dots,i_{j-k}}$. Before proceeding with the estimate of the derivatives, we have to provide a bound for $\partial_\zeta \mathcal{M}(x;\varepsilon,\zeta,c)$. Since the lines composing $\mathcal{M}(x;\varepsilon,\zeta,c)$ are on scale 0, see (2.12a), the derivatives act only on the node factor. By Lemma 5.2 and Lemma 5.3, it follows that

$$
\begin{aligned}
&|\partial_\zeta \mathcal{M}(x)| \le b_1\varepsilon(\zeta^{\mathfrak{n}-2} + \eta^{\frac{2\mathfrak{n}^2-2\mathfrak{n}-1}{\mathfrak{n}+1}}) \le 2b_1\,\varepsilon\,\eta^{\mathfrak{n}-2}, \\
&|\partial_\zeta^2 \mathcal{M}(x)| \le b_2\varepsilon(\zeta^{\mathfrak{n}-3} + \eta^{\frac{2\mathfrak{n}^2-3\mathfrak{n}-2}{\mathfrak{n}+1}}) \le 2b_2\,\varepsilon\,\eta^{\mathfrak{n}-3}, \\
&\dots \\
&|\partial_\zeta^{\mathfrak{n}-1} \mathcal{M}(x)| \le b_{\mathfrak{n}-1}\varepsilon(1 + \eta^{\frac{\mathfrak{n}^2-\mathfrak{n}+1}{\mathfrak{n}+1}}) \le 2b_{\mathfrak{n}-1}\,\varepsilon, \\
&|\partial_\zeta^{\mathfrak{n}} \mathcal{M}(x)| \le b_{\mathfrak{n}}\varepsilon(1 + \eta^{\frac{\mathfrak{n}^2-2\mathfrak{n}}{\mathfrak{n}+1}}) \le 2b_{\mathfrak{n}}\,\varepsilon,
\end{aligned}
$$

with suitable positive constants $b_1,\dots,b_{\mathfrak{n}}$. The first contribution is from resonances that contain only one internal node, while the second one is from all the contributions of resonances with at least two internal nodes.

Hence, to sum up, we have

$$
|\partial_\zeta^j \mathcal{M}(x)| \le 2b_j\varepsilon\eta^{\mathfrak{n}-j-1}, \qquad \text{for } j = 1,\dots,\mathfrak{n}-1, \tag{5.37}
$$

$$
|\partial_\zeta^{\mathfrak{n}} \mathcal{M}(x)| \le 2b_{\mathfrak{n}}\,\varepsilon. \tag{5.38}
$$

We can write the contributions of the derivatives acting on the same propagator as follows:

$$\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) = \mathscr{V}(\vartheta)\frac{\partial_\zeta\mathcal{M}(x)}{D_1(x)}$$

$$\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta^2\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) = \mathscr{V}(\vartheta)\Big[\frac{2(\partial_\zeta\mathcal{M}(x))^2}{(D_1(x))^2}+\frac{\partial_\zeta^2\mathcal{M}(x)}{D_1(x)}\Big]$$

$$\dots$$

$$\Big(\prod_{v\in N(\vartheta)} F_v\Big)\partial_\zeta^{\mathfrak{n}}\Big(\prod_{\ell\in L(\vartheta)} G_\ell\Big) =$$

$$= \mathscr{V}(\vartheta)\sum_{k=0}^{\mathfrak{n}-1}\sum_{i_1,\dots,i_{\mathfrak{n}-k}\in\mathcal{A}} a_{i_1,\dots,i_{j-k}}\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{\mathfrak{n}-k}}\mathcal{M}(x))}{(D_1(x))^{\mathfrak{n}-k}},$$

for $\mathfrak{n}\geq 3$ and for suitable constants $a_{i_1,\dots,i_{j-k}}$.

By using Lemma 5.3 and (5.37), (5.38), up to a constant, we have

$$\Big|\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k}}\Big| \leq \varepsilon^{j-k}\eta^{(\mathfrak{n}-1)(j-k)-\sum_{s=1}^{j-k}(i_s)}\,\varepsilon^{-j+k}\,\eta^{-(\mathfrak{n}-1)(j-k)}$$
$$= \eta^{-j} \qquad \text{for } 1\leq j\leq \mathfrak{n}.$$

Then, from (5.34), one has the bound

$$\Big|\mathscr{V}^*(\vartheta)\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{j-k}}\mathcal{M}(x))}{(D_1(x))^{j-k}}\Big| \leq \rho\Gamma C^k\,\eta^{\frac{k+\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}-j}e^{-\frac{\xi}{4}q_n(p+l+s)}$$
$$\text{for } 1\leq j\leq\mathfrak{n},$$

and in particular the worst case is

$$\Big|\mathscr{V}^*(\vartheta)\frac{(\partial_\zeta^{i_1}\mathcal{M}(x))(\partial_\zeta^{i_2}\mathcal{M}(x))\dots(\partial_\zeta^{i_{\mathfrak{n}-k}}\mathcal{M}(x))}{(D_1(x))^{\mathfrak{n}-k}}\Big| \leq \rho\Gamma C^k\,\eta^{\frac{k+\mathfrak{n}^3-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}-\mathfrak{n}}e^{-\frac{\xi}{4}q_n(p+l+s)}$$
$$\leq \rho\Gamma C^k\,\eta^{\frac{k-\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}}e^{-\frac{\xi}{4}q_n(p+l+s)}. \tag{5.39}$$

Notice that we have to bound the same contribution as in (5.35).

If $p+l+s\geq 1$, by using (5.27), one has

$$\eta^{\frac{k-\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}}e^{-\frac{\xi}{4}q_n(p+l+s)} \leq \eta^{\frac{k-\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\xi}{4C_2}} \leq \eta^{\frac{k+\mathfrak{n}^3+\mathfrak{n}^2-2\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}},$$

that is bounded since η is supposed to be small.

If $p+l+s=0$, we have to distinguish among two different cases: $m \geq 1$ or $m=0$ and $r+u \geq \mathfrak{n}$. If $m \geq 1$, one has

$$\eta^{\frac{k-\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{m(\mathfrak{n}^3-\mathfrak{n})+\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}} \leq \eta^{\frac{k+\mathfrak{n}^3-\mathfrak{n}^2-2\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}},$$

that again is bounded since η is small, $\mathfrak{n} \geq 3$ and $k \geq \mathfrak{n}+1$.
If $m=0$ and $r+u \geq \mathfrak{n}$, we have to notice that in this case, the derivatives have to act on the node factor, since the propagators do not depend on ζ. Then if j is the number of the derivatives, one has the bound $j \leq u$. Hence, $u \geq \mathfrak{n}$ and one has

$$\eta^{\frac{k-\mathfrak{n}^2-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}+\frac{\mathfrak{n}u}{\mathfrak{n}(\mathfrak{n}+1)}} \leq \eta^{\frac{k-\mathfrak{n}-1}{\mathfrak{n}(\mathfrak{n}+1)}},$$

that is bounded since η is small and $k \geq \mathfrak{n}+1$.

If the derivatives act on different lines, it means

$$\partial_\zeta^r\Big(\prod_{\ell \in L(\vartheta)} G_\ell\Big) = \partial_\zeta^{j_1} G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1})\partial_\zeta^{j_2} G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_2})\cdots\partial_\zeta^{j_r} G^{[1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_r})\Big(\prod_{\ell \in L(\vartheta)\setminus L'(\vartheta)} G_\ell\Big),$$

with $L'(\vartheta)$ the set of the lines where the derivatives act, $j_1, j_2 \neq 0$, $j_3, \ldots, j_r \geq 0$ and $j_1 + \cdots + j_r = r$ for $1 \leq r \leq \mathfrak{n}$. Then from the previous analysis, it can be easily seen that every $\mathfrak{n}$-th derivative of the function $\mathscr{V}^*(\vartheta)$ in the variable ζ is bounded.

Therefore we can conclude that $\mathcal{H}(\varepsilon,\zeta)$ is $C^{\mathfrak{n}}$ in ζ. □

Lemma 5.14. *There exists a neighbourhood $U \times V$ of $(\varepsilon,\zeta)=(0,0)$ such that for all $\varepsilon \in U$ there is at least one value $\zeta=\zeta(\varepsilon) \in V$, depending continuously on ε, that solves the bifurcation equation.*

Proof. See the proof of Lemma 3.14 in the case of f a trigonometric polynomial. □

We summarize the previous results with the following Lemma.

Lemma 5.15. *Let ε_0 and N be as in Theorem 2 and $\zeta(\varepsilon)$ as in Lemma 5.14. Define C_2 as in* (5.27) *and I_n as in* (1.9)*. Then for all $\varepsilon \in \overline{\cup_{n \geq N} I_n}$, the function $x(t,\varepsilon) = c+\zeta(\varepsilon)+X(\boldsymbol{\omega}t;\varepsilon,\zeta(\varepsilon),c)$ solves* (1.1)*. Moreover $x(t,\varepsilon)$ is continuous in ε (in the sense of Whitney) and $x(t,\varepsilon) \to c$ as $\varepsilon \to 0$.*

5.5 Final comments

The intervals described by (5.5) give us the connection between the perturbation parameter and the frequency vector. In principle, one would expect to obtain the existence of response solutions for all frequency vectors provided $|\varepsilon| < \varepsilon_0$, with ε_0 small enough. To study if this really occurs, we need to analyse better the intervals where the solution exists. Once ε_0 and N have been fixed as in Theorem 2, if we define the intervals I_n as in (1.9), that is

$$I_n := \left[e^{-C_2 q_n}, (C_1 q_n)^{-\mathfrak{n}-1}\right], \qquad n \geq N,$$

the solution exists for all $|\varepsilon| < \varepsilon_0$ such that $\varepsilon \in \cup_{n \geq N} I_n$.

The intervals I_n are well defined for n large enough. However, they might be disjoint. From the theory of continued fractions, we know that the sequence q_n is increasing, so that

$$\frac{1}{(C_1 q_{n+1})^{\mathfrak{n}+1}} < \frac{1}{(C_1 q_n)^{\mathfrak{n}+1}}, \qquad e^{-C_2 q_{n+1}} < e^{-C_2 q_n}, \tag{5.40}$$

but there is no a priori relation between $e^{-C_2 q_n}$ and $(C_1 q_{n+1})^{-\mathfrak{n}-1}$. Therefore it may happen that $e^{-C_2 q_n} > (C_1 q_{n+1})^{-\mathfrak{n}-1}$. If that is the case, the intervals I_n and I_{n+1} are disjoint, as represented in Figure 5.1.

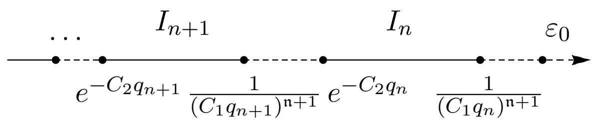

Figure 5.1

The overall measure of these intervals depends on the irrational number α, in particular on the convergents p_n/q_n, so we cannot say in general where the meas$(I_n) = (C_1 q_n)^{-\mathfrak{n}-1} - e^{-C_2 q_n}$ is either large or small.

Once ε_0 and $N > 0$ have been fixed as in Theorem 2, the problem to address is when it is possible to obtain the result of existence for all $|\varepsilon| < \varepsilon_0$. Of course, this is equivalent to require

$$e^{-C_2 q_n} \leq (C_1 q_{n+1})^{-\mathfrak{n}-1}, \qquad n \geq N, \tag{5.41}$$

so as to have the situation represented in Figure 5.2.

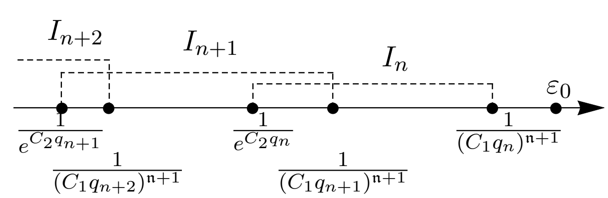

Figure 5.2

From the results available in the literature we know that, taken $\varepsilon_0 > 0$, if $\boldsymbol{\omega}$ is Diophantine or Bryuno, response solutions exist for all $|\varepsilon| < \varepsilon_0$ in a set without holes. Hence it is not surprising that (5.41) is satisfied when $\boldsymbol{\omega}$ is Diophantine or Bryuno.

Indeed, if $\boldsymbol{\omega}$ is Diophantine, condition (5.41) easily follows by using Proposition 4.22:

$$\frac{1}{(C_1 q_{n+1})^{\mathfrak{n}+1}} \geq \frac{1}{(K_0 C_1 q_n^{\tau})^{\mathfrak{n}+1}} > e^{-C_2 q_n}, \qquad n \geq N,$$

with K_0 as in the proof of Proposition 4.22, C_2 as in (5.27) and C_1 and N suitably chosen.

Recall that, defining

$$\varepsilon_n(\alpha) := \frac{1}{q_n} \log q_{n+1},$$

we say that $\boldsymbol{\omega} = (1, \alpha)$ is a Bryuno vector if $\varepsilon_n(\alpha)$ is summable, see Definition 4.30. If $\varepsilon_n(\alpha)$ is summable, (5.41) is satisfied:

$$\begin{aligned}\log(C_1 q_{n+1})^{\mathfrak{n}+1} &= (\mathfrak{n}+1)\log C_1 + (\mathfrak{n}+1)\log q_{\mathfrak{n}+1} = \\ &= (\mathfrak{n}+1)\log C_1 + (\mathfrak{n}+1) q_n \varepsilon_n(\alpha) < C_2 q_n\end{aligned}$$

with C_2 as in (5.27) and C_1 suitably chosen.

What is not true is the opposite: there are vectors which satisfy (5.41), but are not Bryuno vectors. In fact, it is sufficient that $\varepsilon_n(\alpha)$ be small enough so as to satisfy the bound above.

To summarize, for $\mathfrak{n} \geq 3$ and $d = 2$, if we do not require (5.41) on the convergents of α, we find that the response solutions exist in a set with holes; instead if we impose (5.41) on the frequency vectors, we have the existence of response solutions for all $|\varepsilon| < \varepsilon_0$ and for a class of frequency vectors which satisfy a condition that is weaker than Bryuno's and also weaker than the request that $\varepsilon_n(\alpha) \to 0$.

Notice also that, on the contrary, if $\mathfrak{n} = 1$, for any d, the response solutions exist in a set without holes, by only requiring ε to be small enough, see [47, 73].

Appendix A

Theorem 2: the case $\mathfrak{n} = 3$

In this Appendix we discuss explicitly the case $\mathfrak{n} = 3$ both, for concreteness, to discuss a case where all the constants are explicitly computed (in terms of the parameters) and as an explicit example where it is easier to draw pictures.

Let C, $\widetilde{C}$ and $\widetilde{C}_1$ be as in (5.15a) and define N and I_n as in Theorem 2 with $\mathfrak{n} = 3$.

Lemma A.1. *Let ϑ be a tree of order $k = 1$ and root line ℓ_0. Let us denote by $\boldsymbol{\nu}$ the momentum of the root line. Then if $\varepsilon \in \cup_{n \geq N} I_n$, one has*

$$|\mathscr{V}(\vartheta)| \leq \rho C \begin{cases} \varepsilon^{\frac{3}{4}} e^{-\xi|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{<,0}(\vartheta) \sqcup L_{\geq,0}(\vartheta); \\ \eta^{-2} e^{-\xi|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{\geq,1}(\vartheta). \\ |\zeta|, & \text{if } \ell_0 \in L_0(\vartheta). \end{cases} \tag{A.1}$$

Proof. If ϑ is a tree of order $k = 1$, then $\mathcal{T}_{1,\boldsymbol{\nu}} = \mathfrak{T}_{1,\boldsymbol{\nu}}$.

If $\ell_0 \in L_{<,0}(\vartheta)$, we have $|G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|} \leq \frac{2}{C_1} \varepsilon^{-\frac{1}{4}}$. Then

$$|\mathscr{V}(\vartheta)| \leq \frac{2\Phi}{C_1} \varepsilon^{\frac{3}{4}} e^{-\xi|\boldsymbol{\nu}|}.$$

If $\ell_0 \in L_{\geq,0}(\vartheta)$, by using the sharp partition (5.10) with $\mathfrak{n} = 3$, one has $|G^{[0]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \frac{1}{|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|} \leq \frac{4}{C_1} \varepsilon^{-\frac{1}{4}}$. Then

$$|\mathscr{V}(\vartheta)| \leq \frac{4\Phi}{C_1} \varepsilon^{\frac{3}{4}} e^{-\xi|\boldsymbol{\nu}|}.$$

If $\ell_0 \in L_{\geq,1}(\vartheta)$, from Lemma 5.3 we obtain $|G^{[1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})| \leq \frac{1}{A\,\varepsilon\,\eta^2}$. Then

$$|\mathscr{V}(\vartheta)| \leq \frac{\Phi}{A} \eta^{-2} e^{-\xi|\boldsymbol{\nu}|}.$$

If $\ell_0 \in L_0(\vartheta)$, then $|\mathscr{V}(\vartheta)| = |\zeta|$.

Hence if we choose C as in (5.15a), the thesis follows. $\square$

Lemma A.2. *Let $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$ be a renormalized tree of order $k \geq 4$ and momentum $\boldsymbol{\nu} \neq \mathbf{0}$ associated with the root line ℓ_0. Then*

$$\mathscr{V}(\vartheta) = \bar{\mathscr{V}}(\vartheta) \prod_{v \in E_1(\vartheta)} e^{-\frac{\xi}{2}|\boldsymbol{\nu}_v|} \tag{A.2}$$

where

$$|\bar{\mathscr{V}}(\vartheta)| \leq \rho C^k \eta^{\frac{k}{12}+\frac{8}{3}} \times \begin{cases} e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}, & \text{if } \ell_0 \in L_{<,0}(\vartheta), \\ e^{-\frac{\xi}{4}q_n}, & \text{if } \ell_0 \in L_{\geq,0}(\vartheta), \\ \eta^{-2} e^{-\frac{\xi}{4}q_n}, & \text{if } \ell_0 \in L_{\geq,1}(\vartheta), \end{cases} \tag{A.3}$$

with C as in (5.15a)*, C_2 as in* (1.6) *with $\mathfrak{n} = 3$ and $\varepsilon \in \cup_{n \geq N} I_n$, provided η small.*

Proof. Let us consider a renormalized tree, $\vartheta \in \mathfrak{T}_{k,\boldsymbol{\nu}}$, of order $k \geq 4$ and denote by v_0 the first node, that is the node the root line exits. For any renormalized tree, we have the following structure, see Figure A.1:

- $\vartheta_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell_1}}, \dots, \vartheta_m \in \mathfrak{T}_{k_m,\boldsymbol{\nu}_{\ell_m}}$ enter the first node v_0 and the respective root lines $\ell_1, \dots, \ell_m$ are such that $|\boldsymbol{\nu}_{\ell_j}| < q_n$ for all $j = 1, \dots, m$; equivalently, $\ell_1, \dots, \ell_m \in L_{<,0}(\vartheta)$;
- $\vartheta'_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell'_1}}, \dots, \vartheta'_p \in \mathfrak{T}_{k_p,\boldsymbol{\nu}_{\ell'_p}}$ enter the first node v_0 and the root lines $\ell'_1, \dots, \ell'_p$ are such that $|\boldsymbol{\nu}_{\ell'_j}| \geq q_n$ and $\widetilde{n}_{\ell'_j} = 0$ for all $j = 1, \dots, p$, so that $\ell'_1, \dots, \ell'_p \in L_{\geq,0}(\vartheta)$;
- $\vartheta''_1 \in \mathfrak{T}_{k_1,\boldsymbol{\nu}_{\ell''_1}}, \dots, \vartheta''_l \in \mathfrak{T}_{k_l,\boldsymbol{\nu}_{\ell''_l}}$ enter the first node v_0 and the root lines $\ell''_1, \dots, \ell''_l$ are such that $|\boldsymbol{\nu}_{\ell''_j}| \geq q_n$ and $\widetilde{n}_{\ell''_j} = 1$ for all $j = 1, \dots, l$, so that $\ell''_1, \dots, \ell''_l \in L_{\geq,1}(\vartheta)$;
- the lines $\widetilde{\ell}_1, \dots, \widetilde{\ell}_r$ enter the node v_0 and exit the end nodes $\widetilde{v}_1, \dots, \widetilde{v}_r \in E_1(\vartheta)$ respectively and are such that $\widetilde{n}_{\widetilde{\ell}_j} = 0$ for all $j = 1, \dots, r$; equivalently $\widetilde{\ell}_1, \dots, \widetilde{\ell}_r \in L_{<,0}(\vartheta) \sqcup L_{\geq,0}(\vartheta)$;
- the lines $\widetilde{\ell'}_1, \dots, \widetilde{\ell'_s}$ enter the node v_0 and exit the end nodes $\widetilde{v'}_1, \dots, \widetilde{v'}_s \in E_1(\vartheta)$ respectively and are such that $|\boldsymbol{\nu}_{\widetilde{\ell'}_j}| \geq q_n$ and $n_{\widetilde{\ell'}_j} = 1$ for all $j = 1, \dots, s$, so that $\widetilde{\ell'}_1, \dots, \widetilde{\ell'_s} \in L_{\geq,1}(\vartheta)$;
- the lines $\bar{\ell}_1, \dots, \bar{\ell}_u$ enter the node v_0 and exit the end nodes $\bar{v}_1, \dots, \bar{v}_u \in E_0(\vartheta)$.

According to this structure, we have the following constraints:

- $k = k(\vartheta) = \sum_{j=1}^{m} k(\vartheta_j) + \sum_{j=1}^{p} k(\vartheta'_j) + \sum_{j=1}^{l} k(\vartheta''_j) + r + s + u + 1$;
- $m + p + l + r + s + u \geq 3$;
- $m, p, l, r, s, u \geq 0$;

- $\boldsymbol{\nu} = \sum_{j=1}^{m} \boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^{p} \boldsymbol{\nu}_{\ell'_j} + \sum_{j=1}^{l} \boldsymbol{\nu}_{\ell''_j} + \sum_{j=1}^{r} \boldsymbol{\nu}_{\widetilde{\ell}_j} + \sum_{j=1}^{s} \boldsymbol{\nu}_{\widetilde{\ell'}_j}$;
- $\boldsymbol{\nu} \neq \mathbf{0}$.

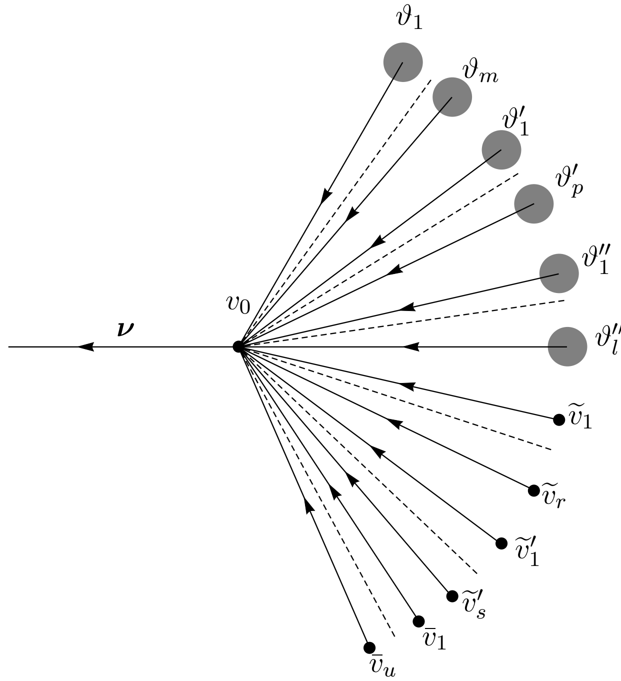

Figure A.1: Tree representation of the described structure.

We use (A.1) to bound the value of graph elements formed by an end node and the respective exiting line. Then we assume that any value of tree of order k', with $1 < k' < k$, follows (A.2) and in order to prove Lemma A.2 we proceed by induction.

By following the described structure, we can easily collect a common factor $\prod_{v\in E_1(\vartheta)} \exp(-\xi|\boldsymbol{\nu}_v|/2)$ in the expression of $\mathscr{V}(\vartheta)$, see (2.13). Hence we have to prove the second bound in Lemma A.2, and we will do it by using (A.3) for trees of order k' with $1 < k' < k$ and by estimating the value of the graph elements formed by an end node in $E_1(\vartheta)$ and the respective exiting line as in (A.1) with $\frac{\xi}{2}$ instead of ξ. We still use the last inequality in (A.1) to bound the value of the graph elements formed by an end node in $E_0(\vartheta)$ and the respective exiting line since the momentum of these lines is zero.

I case: $\boldsymbol{\ell_0 \in L_{<,0}(\vartheta)}$.
Let us start by taking into account a renormalized tree with root line $\ell_0 \in L_{<,0}(\vartheta)$. We want to prove

$$|\bar{\mathscr{V}}(\vartheta)| \leq \rho C^k \eta^{\frac{k}{12}+\frac{8}{3}} e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}.$$

Since $\eta := \max\{\varepsilon, |\zeta|\}$, by combining (5.3), (5.4) and by induction it follows that

$$
\begin{aligned}
|\bar{\mathscr{V}}(\vartheta)| &\le \frac{2\Gamma}{C_1}\, C^{k-1}\rho^{m+p+l+r+s+u}\rho^{-(m+p+l+r+s+u)}\eta^{\frac{k-r-s-u-1}{12}+\frac{8}{3}(m+p+l)} \\
&\quad \times \varepsilon^{\frac{3}{4}(r+1)}\,\eta^{-2l-2s}|\zeta|^{u}\; e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\
&\le \rho C^{k}\widetilde{C}^{-1}\eta^{\frac{k}{12}+\frac{8}{3}}\eta^{\frac{-r-s-u-1+32m+32p+32l-32-24l-24s+9(r+1)+12u}{12}}\times \\
&\quad \times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\
&\le \rho C^{k}\widetilde{C}^{-1}\eta^{\frac{k}{12}+\frac{8}{3}}\eta^{\frac{32m+32p+8l+8r-25s-24+11u}{12}}\times \\
&\quad \times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}.
\end{aligned}
$$

Since $\widetilde{C}^{-1} \le 1$, we have to prove that

$$
\eta^{\Delta_0(m,p,l,r,s,u)}e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}e^{-\frac{\xi}{4}q_n(p+l)} \le e^{-\frac{\xi}{4}|\boldsymbol{\nu}|},
$$

with

$$
\Delta_0(m,p,l,r,s,u) := \frac{32m+32p+8l+8r-25s-24+11u}{12}
$$

This can be seen by studying the following cases:

(1) $p+l=1$;

(2) $p+l\ge 2$;

(3) $p+l=0$.

If we are in the first case, (1), we have

$$
e^{-\frac{\xi}{4}q_n(p+l)} = e^{-\frac{\xi}{4}q_n} \le e^{-\frac{\xi}{4}|\boldsymbol{\nu}|},
$$

since the momentum of the root line is such that $|\boldsymbol{\nu}| < q_n$.
We have now two possibilities: $p=1$ and $l=0$ or $p=0$ and $l=1$. If $p=1$ and $l=0$, we have to prove that

$$
\eta^{\frac{32m+8r-25s+8+11u}{12}}\times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le 1.
$$

If $s=0$ this is trivially true, because

$$
\eta^{\frac{32m+8r+8+11u}{12}}\times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)} \le \eta^{\frac{8}{12}} \le 1.
$$

If $s \geq 1$, the function $\Delta_0(m,1,0,r,s,u)$ could be negative, but the lines $\widetilde{\ell'}_1, \ldots, \widetilde{\ell'}_s$ are such that $|\boldsymbol{\nu}_{\widetilde{\ell'}_j}| \geq q_n \geq \frac{\log \frac{1}{\varepsilon}}{C_2}$ $\forall j = 1, \ldots, s$. So we bound $\Delta_0(m,1,0,r,s,u)$ as follows

$$\Delta_0(m,1,0,r,s,u) \geq -\frac{25}{12}s$$

in such a way that

$$\begin{aligned}\eta^{\Delta_0(m,1,0,r,s,u)} & e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\ & \leq \eta^{-\frac{25}{12}s} e^{-\frac{\xi}{2}\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|} \leq (\eta^{-\frac{25}{12}} e^{-\frac{\xi}{2}q_n})^s \\ & \leq (\eta^{-\frac{25}{12}} e^{-\frac{\xi}{2C_2}\log\frac{1}{\varepsilon}})^s = (\eta^{-\frac{25}{12}} \varepsilon^{\frac{\xi}{2C_2}})^s\end{aligned}$$

and the thesis follows by choosing C_2 in such a way that $\eta^{-\frac{25}{12}} \varepsilon^{\frac{\xi}{2C_2}} \leq 1$. In particular $\eta^{-\frac{25}{12}} \varepsilon^{\frac{\xi}{2C_2}} \leq \eta^{-\frac{25}{12}+\frac{\xi}{2C_2}}$, so we have to require

$$-\frac{25}{12} + \frac{\xi}{2C_2} \leq 0 \quad \Rightarrow \quad C_2 \leq \frac{6}{25}\xi.$$

If $p = 0$ and $l = 1$, our statement becomes

$$\eta^{\frac{32m+8r-25s-16+11u}{12}} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq 1.$$

Again, if $s = 0$, that is $m + r + u \geq 2$, the thesis follows because

$$\begin{aligned}\eta^{\frac{32m+8r-16+11u}{12}} & \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)} \\ & \leq \eta^{\frac{24m+3u+8(m+r+u)-16}{12}} \leq \eta^{\frac{24m+3u}{12}} \leq 1.\end{aligned}$$

Instead, if $s \geq 1$, the function $\Delta_0(m,0,1,r,s,u)$ could be negative, but the thesis holds by noticing that

$$\Delta_0(m,0,1,r,s,u) \geq -\frac{11}{4}s$$

and by requiring $\eta^{-\frac{11}{4}} \varepsilon^{\frac{\xi}{2C_2}} \leq 1$. In particular $\eta^{-\frac{11}{4}} \varepsilon^{\frac{\xi}{2C_2}} \leq \eta^{-\frac{11}{4}+\frac{\xi}{2C_2}}$, hence we can choose C_2 such that

$$-\frac{11}{4} + \frac{\xi}{2C_2} \leq 0 \quad \Rightarrow \quad C_2 \leq \frac{2}{11}\xi.$$

Summarizing, the condition on C_2 becomes:

$$\begin{cases} C_2 \leq \frac{6}{25}\xi \\ C_2 \leq \frac{2}{11}\xi \end{cases} \quad \Rightarrow \quad C_2 \leq \frac{2}{11}\xi. \tag{A.4}$$

Let us now analyse the second case $p + l \geq 2$.

As we can see, we can rewrite the factor $e^{-\frac{\xi}{4}q_n(p+l)}$ as follows:

$$e^{-\frac{\xi}{4}q_n(p+l)} = e^{-\frac{\xi}{4}q_n}e^{-\frac{\xi}{4}q_n(p+l-1)}.$$

Since the momentum of the root line is such that $|\boldsymbol{\nu}| < q_n$ and since $q_n \geq \frac{\log\frac{1}{\varepsilon}}{C_2}$, by observing that $p+l-1 \geq 1$, it follows:

$$e^{-\frac{\xi}{4}q_n}e^{-\frac{\xi}{4}q_n(p+l-1)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}\varepsilon^{\frac{\xi}{4C_2}}.$$

The function $\Delta_0(m,p,l,r,s,u)$ can be bounded by using the condition $p+l \geq 2$:

$$\Delta_0(m,p,l,r,s,u) \geq \frac{32m+8r-25s+11u-8}{12} \geq -\frac{25}{12}s - \frac{2}{3}.$$

Then the thesis follows if

$$\begin{aligned}\eta^{\Delta_0(m,p,l,r,s,u)}e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}e^{-\frac{\xi}{4}q_n(p+l-1)} &\leq \\ \leq (\eta^{-\frac{2}{3}}\varepsilon^{\frac{\xi}{4C_2}})(\eta^{-\frac{25}{12}}\varepsilon^{\frac{\xi}{2C_2}})^s \leq (\eta^{-\frac{2}{3}+\frac{\xi}{4C_2}})(\eta^{-\frac{25}{12}+\frac{\xi}{2C_2}})^s &\leq 1,\end{aligned}$$

i.e. if we require $\eta^{-\frac{25}{12}+\frac{\xi}{2C_2}} \leq 1$ and $\eta^{-\frac{2}{3}+\frac{\xi}{4C_2}} \leq 1$. These inequalities are satisfied without requiring any other condition on C_2 since $C_2 \leq \frac{2}{11}\xi$, see equation (A.4). Indeed:

$$\eta^{-\frac{25}{12}+\frac{\xi}{2C_2}} \leq \eta^{\frac{2}{3}} \leq 1, \qquad \eta^{-\frac{2}{3}+\frac{\xi}{4C_2}} \leq \eta^{\frac{17}{24}} \leq 1.$$

Let us focus on the third case, (3), $p+l=0$, that is $m+r+s+u \geq 3$.
First of all notice that we can rewrite the exponential part as follows

$$\begin{aligned}&e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|\right)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} = \\ &e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}\end{aligned}$$

So, from the conservation law of $\boldsymbol{\nu}$, i.e. $\boldsymbol{\nu} = \sum_{j=1}^{m}\boldsymbol{\nu}_{\ell_j} + \sum_{j=1}^{r}\boldsymbol{\nu}_{\widetilde{\ell}_j} + \sum_{j=1}^{s}\boldsymbol{\nu}_{\widetilde{\ell'}_j}$, we have

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|}.$$

Hence we have to prove

$$\eta^{\frac{32m+8r-25s-24+11u}{12}} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq 1.$$

If $s=0$, the thesis follows since $\frac{32m+8r+11u-24}{12} \geq \frac{24m+3u}{12} \geq 0$. If $s \geq 1$ we can reason as before by noticing that

$$\Delta_0(m,0,0,r,s,u) \geq -\frac{11}{4}s$$

and the thesis is satisfied by requiring $\eta^{-\frac{11}{4}+\frac{\xi}{4C_2}} \leq 1$, i.e. $C_2 \leq \frac{1}{11}\xi$. By comparing this condition with the one in (A.4), it follows

$$\begin{cases} C_2 \leq \frac{1}{11}\xi \\ C_2 \leq \frac{2}{11}\xi \end{cases} \qquad \text{and thus} \quad C_2 \leq \frac{1}{11}\xi. \tag{A.5}$$

II case: $\boldsymbol{\ell_0 \in L_{\geq,0}(\vartheta)}$.

Let us now analyse the trees of order k with root line ℓ_0 such that $\ell_0 \in L_{\geq,0}(\vartheta)$. We want to prove:

$$|\bar{\mathscr{V}}(\vartheta)| \leq \rho C^k \eta^{\frac{k}{12}+\frac{8}{3}} e^{-\frac{\xi}{4}q_n}.$$

Since $\eta := \max\{\varepsilon, |\zeta|\}$, by combining (5.3), (5.4) and by induction it follows that

$$|\bar{\mathscr{V}}(\vartheta) \leq \rho C^k\, \widetilde{C}^{-1}\, \eta^{\frac{k}{12}+\frac{8}{3}}\, \eta^{\Delta(m,p,l,r,s,u)}\, e^{-\frac{\xi}{4}q_n(p+l)} e^{-\frac{\xi}{4}\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}| - \frac{\xi}{2}\left(\sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}| + \sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)},$$

with

$$\Delta_0(m,p,l,r,s,u) := \frac{32m+32p+8l+8r-25s-24+11u}{12}.$$

As before we can distinguish among three different cases:

(1) $p+l=1$,

(2) $p+l \geq 2$,

(3) $p+l=0$,

and since the function $\Delta_0(m,p,l,r,s,u)$ is the same as the previous case, the thesis can be checked by following the late argument. The only difference is that now the root line is such that $|\boldsymbol{\nu}| \geq q_n$, so a term $e^{-\frac{\xi}{4}q_n}$ has to appear in every listed case.

If $p+l \geq 2$, we have trivially $e^{-\frac{\xi}{4}q_n(p+l)} \leq e^{-\frac{\xi}{4}q_n}$.

If $p+l=0$, we extract the factor

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}| + \sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}| + \sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}$$

and we use the definition and the bound of $|\boldsymbol{\nu}|$ in the following way:

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^m |\boldsymbol{\nu}_{\ell_j}| + \sum_{j=1}^r |\boldsymbol{\nu}_{\widetilde{\ell}_j}| + \sum_{j=1}^s |\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|} \leq e^{-\frac{\xi}{4}q_n}.$$

III case: $\boldsymbol{\ell_0 \in L_{\geq,1}(\vartheta)}$.

Let us now consider a tree of order k with the root line ℓ_0 such that $\ell_0 \in L_{\geq,1}(\vartheta)$. We want to prove:

$$|\bar{\mathscr{V}}(\vartheta)| \leq \rho C^k \eta^{\frac{k}{12}+\frac{8}{3}} \eta^{-2} e^{-\frac{\xi}{4}q_n}.$$

By induction on the order of the tree, we know

$$|\bar{\mathscr{V}}(\vartheta)| \le \rho^{(m+p+l+r+s+u)}\, C^{k-1}\frac{\Gamma}{A}\rho^{-(m+p+l+r+s+u)}\eta^{\frac{k-r-s-u-1}{12}+\frac{8}{3}(m+p+l)}\eta^{-2l-2s-2}$$
$$\times\, \varepsilon^{\frac{3}{4}r}\,|\zeta|^u\, e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{4}q_n(p+l)}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}.$$

Now from the definition of η and $\widetilde{C}_1$ in (5.15a), our thesis becomes:

$$\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)}\times e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{4}q_n(p+l)-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le e^{-\frac{\xi}{4}q_n},$$

where the function $\Delta_1(m,p,l,r,s,u)$ is defined as

$$\Delta_1(m,p,l,r,s,u) := \frac{32m+32p+8l+8r-25s-33+11u}{12}.$$

We start again by analysing the case $p+l=1$. If $p=1$ and $l=0$, we have to prove that

$$\eta^{\frac{32m+8r-25s-1+11u}{12}}\times e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le 1,$$

since $\widetilde{C}_1^{-1}\le 1$. If $s=0$ this is trivially true since $m+r+u\ge 2$. If $s\ge 1$, the function $\Delta_1(m,1,0,r,s,u)$ could be negative and it has the following lower bound

$$\Delta_1(m,1,0,r,s,u) \ge -\frac{11}{4}s.$$

Then if we take C_2 as in (A.5), the thesis follows:

$$\eta^{\frac{32m+8r-25s-1+11u}{12}}\, e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le (\eta^{-\frac{11}{4}}e^{-\frac{\xi}{2}q_n})^s$$
$$\le (\eta^{-\frac{11}{4}}\varepsilon^{\frac{\xi}{2C_2}})^s \le (\eta^{-\frac{11}{4}+\frac{\xi}{2C_2}})^s \le \eta^{\frac{11}{4}s} \le 1.$$

If $p=0$ and $l=1$, the statement becomes:

$$\eta^{\frac{32m+8r-25s-25+11u}{12}}\times e^{-\frac{\xi}{4}\sum_{j=1}^m|\boldsymbol{\nu}_{\ell_j}|}e^{-\frac{\xi}{2}\left(\sum_{j=1}^r|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^s|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \le 1.$$

If $s\ge 1$ the thesis follows, since we can bound the function $\Delta_1(m,0,1,r,s,u)$ as $\Delta_1(m,0,1,r,s,u)\ge -\frac{7}{2}s$ and we can notice that $\eta^{-\frac{7}{2}+\frac{\xi}{2C_2}}\le 1$ because of (A.5). If $s=0$ the argument is more delicate.

This situation is possible only if $|\sum_{j=1}^m\boldsymbol{\nu}_{\ell_j}+\sum_{j=1}^r\boldsymbol{\nu}_{\widetilde{\ell}_j}|\ge q_n$. Indeed, if this is not the case, that is if $|\sum_{j=1}^m\boldsymbol{\nu}_{\ell_j}+\sum_{j=1}^r\boldsymbol{\nu}_{\widetilde{\ell}_j}|<q_n$, by using the properties of continued fractions, see (5.1), it follows that $|\boldsymbol{\omega}\cdot\big(\sum_{j=1}^m\boldsymbol{\nu}_{\ell_j}+\sum_{j=1}^r\boldsymbol{\nu}_{\widetilde{\ell}_j}\big)|\ge\frac{1}{2q_n}\ge\frac{C_1\sqrt[4]{\varepsilon}}{2}$. By construction we also know that $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|<\frac{C_1\sqrt[4]{\varepsilon}}{4}$ and $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_1}|<\frac{C_1\sqrt[4]{\varepsilon}}{4}$. Hence we have

$$\frac{C_1\sqrt[4]{\varepsilon}}{2}\le|\boldsymbol{\omega}\cdot\big(\sum_{j=1}^m\boldsymbol{\nu}_{\ell_j}+\sum_{j=1}^r\boldsymbol{\nu}_{\widetilde{\ell}_j}\big)| = |\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\ell'_1})|\le|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|+|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_1}|<\frac{C_1\sqrt[4]{\varepsilon}}{2}$$

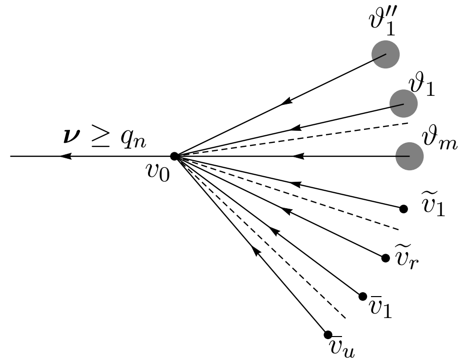

Figure A.2: The case of $p = 0$, $l = 1$ and $s = 0$.

that is impossible.

Then it follows that:

$$\begin{aligned}\eta^{\frac{32m+8r-25+11u}{12}} &\times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|} e^{-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)} \\ &\leq \eta^{-\frac{3}{4}} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|\right)} \\ &\leq \eta^{-\frac{3}{4}} e^{-\frac{\xi}{4}q_n} \leq \eta^{-\frac{3}{4}} \varepsilon^{\frac{\xi}{4C_2}} \\ &\leq \eta^{-\frac{3}{4}+\frac{\xi}{4C_2}} < \eta^2 < 1,\end{aligned}$$

where we have used (A.5).

The case $p + l \geq 2$ can be treated by using (5.5) and by verifying that

$$\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)} \times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{4}q_n(p+l-1)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq 1.$$

In fact

$$\begin{aligned}\widetilde{C}_1^{-1}\eta^{\Delta_1(m,p,l,r,s,u)} &\times e^{-\frac{\xi}{4}\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|-\frac{\xi}{4}q_n(p+l-1)-\frac{\xi}{2}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\ &\leq \eta^{-\frac{17}{12}} \varepsilon^{\frac{\xi}{4C_2}} (\eta^{-\frac{25}{12}} \varepsilon^{\frac{\xi}{2C_2}})^s \\ &\leq \eta^{-\frac{17}{12}+\frac{\xi}{4C_2}} (\eta^{-\frac{25}{12}+\frac{\xi}{2C_2}})^s \leq \eta^{\frac{4}{3}} \eta^{\frac{41}{12}s} \leq 1\end{aligned}$$

where we have used (A.5).

Now we look at the case $p + l = 0$, that is $m + r + s + u \geq 3$. First of all notice that we can bound the value of the tree in the following way:

$$\begin{aligned}|\bar{\mathscr{V}}(\vartheta)| \leq \rho\, C^k\, \widetilde{C}_1^{-1}\, \eta^{\frac{32m+8r-25s-33+11u}{12}} & e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\ \times\, e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)}&.\end{aligned}$$

By definition of $\boldsymbol{\nu}$ it follows that:

$$e^{-\frac{\xi}{4}\left(\sum_{j=1}^{m}|\boldsymbol{\nu}_{\ell_j}|+\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq e^{-\frac{\xi}{4}|\boldsymbol{\nu}|} \leq e^{-\frac{\xi}{4}q_n}.$$

So, since $\widetilde{C}_1^{-1} \leq 1$, we have to prove

$$\eta^{\frac{32m+8r-25s-33+11u}{12}} \times e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \leq 1.$$

This can be done in the following way: if $s = 0$, that implies $m + r + u \geq 3$, we have the following bound

$$\Delta_1(m,0,0,r,0,u) = \frac{32m + 8r - 33 + 11u}{12} \geq \frac{8m + u - 3}{4}.$$

Then, if $m \geq 1$ the thesis follows as

$$\eta^{\frac{32m+8r-33+11u}{12}} \times e^{-\frac{\xi}{4}\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|} \leq \eta^{\frac{5}{4}} \leq 1.$$

If $m = 0$, that means $r + u \geq 3$, and in particular $r \geq 1$ (otherwise it would be $\boldsymbol{\nu} = \mathbf{0}$) we have

$$\begin{aligned}\eta^{-\frac{3}{4}} \times e^{-\frac{\xi}{4}\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|} &\leq \eta^{-\frac{3}{4}}e^{-\frac{\xi}{4}|\boldsymbol{\nu}|} \leq \eta^{-\frac{3}{4}}e^{-\frac{\xi}{4}q_n} \\ &\leq \eta^{-\frac{3}{4}}\varepsilon^{\frac{\xi}{4C_2}} \leq \eta^{-\frac{3}{4}+\frac{\xi}{4C_2}} \leq \eta^2 \leq 1\end{aligned}$$

where we have used (A.5) and the thesis follows once again.

If $s \geq 1$, one has $\Delta_1(m,0,0,r,s,u) \geq -\frac{7}{2}s$. Then:

$$\begin{aligned}&\eta^{\frac{32m+8r-25s-33+11u}{12}} e^{-\frac{\xi}{4}\left(\sum_{j=1}^{r}|\boldsymbol{\nu}_{\widetilde{\ell}_j}|+\sum_{j=1}^{s}|\boldsymbol{\nu}_{\widetilde{\ell}'_j}|\right)} \\ &\qquad\qquad \leq (\eta^{-\frac{7}{2}}e^{-\frac{\xi}{4}q_n})^s \leq (\eta^{-\frac{7}{2}}\varepsilon^{\frac{\xi}{4C_2}})^s \leq (\eta^{-\frac{7}{2}+\frac{\xi}{4C_2}})^s\end{aligned}$$

and if we request $\frac{\xi}{4C_2} \geq \frac{7}{2}$, i.e. $C_2 \leq \frac{\xi}{14}$ we obtain the wished bound. By comparing this condition on C_2 with (A.5), we finally have:

$$\begin{cases} C_2 \leq \frac{\xi}{14} \\ C_2 \leq \frac{\xi}{11} \end{cases} \quad \Rightarrow \quad C_2 \leq \frac{\xi}{14}. \tag{A.6}$$

Then, if we take C_2 as

$$C_2 = \frac{\xi}{14}, \tag{A.7}$$

both (A.2) and (A.3) are satisfied and the proof of Lemma A.2 is completed. □

Appendix B

Zeroes of even order

In this section we discuss what we anticipated in Remark 1.1, that is what happens when the equation $g(x) = f_{\mathbf{0}}$ has zeroes of even order.

Lemma B.1. *Let c be a zero of even order of the equation* (1.3). *Then there is no quasi-periodic solution reducing to c when ε goes to* 0.

Proof. We focus on the bifurcation equation (2.6) in the case of c_0 a zero of even order of equation (1.3). As we have seen, we can rewrite the bifurcation equation as

$$[g(c_0 + \zeta + X(\cdot;\varepsilon,\zeta,c))]_{\mathbf{0}} - f_{\mathbf{0}} = g_{\mathfrak{n}}[(\zeta + X)^{\mathfrak{n}}]_{\mathbf{0}} + [O(\zeta + X)^{\mathfrak{n}+1}]_{\mathbf{0}} = 0, \tag{B.1}$$

where $g_{\mathfrak{n}}$ is as in (2.1) and $g_{\mathfrak{n}} \neq 0$. As usual, we denote by $[\cdot]_{\mathbf{0}}$ the Fourier component with label $\boldsymbol{\nu} = \mathbf{0}$.

Since $\mathfrak{n}$ is even, if $\varepsilon = O(\zeta)$ then we have $|g_{\mathfrak{n}}|[(\zeta + X)^{\mathfrak{n}}]_{\mathbf{0}} \geq C\varepsilon^{\mathfrak{n}}$, for some positive constant C and $O((\zeta + X)^{\mathfrak{n}+1}) = O(\varepsilon^{\mathfrak{n}+1})$, so that (B.1) cannot be satisfied for ε small enough.

If $\varepsilon = o(\zeta)$, then

$$[(\zeta + X)^{\mathfrak{n}}]_{\mathbf{0}} = \sum_{k=0}^{\mathfrak{n}} \binom{\mathfrak{n}}{k} \zeta^k [X^{\mathfrak{n}-k}]_{\mathbf{0}} = \zeta^{\mathfrak{n}} + o(\zeta^{\mathfrak{n}}), \tag{B.2}$$

for ε small enough. On the other hand, $O((\zeta + X)^{\mathfrak{n}+1}) = O(\zeta^{\mathfrak{n}+1})$ and hence once more there is no solution to (B.1) because of (B.2).

The case $\zeta = o(\varepsilon)$ can be discussed in a similar way. □

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