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Duality and asymptotic symmetries in gravisolitons and gauge theories

PhD thesis in Theoretical and Mathematical Physics

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Duality and asymptotic symmetries in gravisolitons and gauge theories

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Dedication

Knowledge begins with the recognition of one's ignorance; the realization that the search for knowledge is unending. The pearl stands for the world, the heavens, and an eye, representing the many forms of knowledge, never fully attainable. Even knowing that, the All-Knowing's hand grasps for it. But when Gideon, glimpsed into the will of Queen Marika, he shuddered in fear. At the end that should not be.

La conoscenza nasce dall'accettazione della propria ignoranza; dalla consapevolezza che la ricerca del sapere non trova mai compimento. La perla simboleggia il mondo, i cieli e un occhio, per rappresentare le diverse forme di una conoscenza sempre fuggevole. Pur consapevole di ciò, l'Onnisciente tenta comunque di afferrarla. Ma quando Gideon, scrutò nella volontà della regina Marika, fu scosso da brividi di terrore. Poichè scorse la fine che non si attendeva.

Federico Manzoni
2024/2025

Acknowledgments

Rather than using this space for a list of acknowledgments, I decided to use it for a reflection and, partly, a complaint.

During my PhD years I have seen and talked to many physicists. The thing that struck me most is that many of them have a truly elementary understanding of mathematics: almost nothing more than calculus. There are those who confuse Lie algebras with Lie groups, those who confuse motivations related to the differentiable structure with motivations related to the Riemannian structure, those who say: "What is topology for? In physics everything is differentiable", those who maintain that cohomology is only de Rham and everything reduces to "d-exact therefore also d-closed", those who talk about representations but do not know what a homomorphism is, those who maintain that the Dirac delta is an object in L^1, those who confuse invariance under homotopy classes with invariance under homeomorphism classes (perhaps not even knowing the meaning of the two terms), those who studies gauge theory but does not know what a bundle is (and I do not mean a principal bundle but even just a topological bundle) and so on and so forth.

Furthermore, when I ask: "Why? Why not use the tools that modern mathematics provides us to study physical phenomena and bring order to a chaos of proposals? Why not commit to understanding the problem in depth and tackling it from its foundations using the right tools (which must also be studied) instead of proceeding with explicit calculations or computer simulations?" the answers often leave me astonished: "You cannot be a student all your life!" or: "We are physicists not mathematicians" or again: "Mathematics often hides" and other answers of the same kind.

The story does not end here. The same people who gave some of the answers previously reported, identify themselves with the role of formal theoretical physicist. So my question arises: formal in what? I do not have an answer at the moment. I have even been teased by some of these people with phrases like: "According to Federico, you can not be a physicist if you do not know what a fiber bundle is but you can be one if you do not know field theory"; of course I do not think so, I think that a formal theoretical physicist should know both field theory and bundles and know how to put them together.

I understand that not every physicist needs to have a deep knowledge of modern mathematics, those who have chosen to be experimental physicists or phenomenological physicists have chosen different roles in which pure mathematics leaves more space for understanding, designing, analyzing or explaining a physical phenomena which has its roots in some experiment or in some which is experimentable. In contexts like this, I understand that models that are not entirely rigorous are used or that some rigor is left aside. In my opinion, it should be the role of the formal theoretical physicist to give a solid mathematical basis to the proposed models using the tools of modern mathematics. Instead, as far as I have seen, many of the formal theoretical physicists build theories and models that cannot be tested but at the same time do not have a solid mathematical basis that goes beyond calculus (more than other categories of physicists do, is this perhaps what makes them "formal"?): they do not use well defined mathematical structures or theorems, they do not use the appropriate terms and others are confused or deliberately interchanged.

Again, in my opinion formal theoretical physicists should be the bridge between the physicists realm and the mathematicians realm but, instead I have very rarely seen formal theoretical physicists working with mathematicians or mathematical physicists, at least in Italy and nearby countries. At this point, you might think that I am confusing the role of formal theoretical physicists with that of mathematical physicists. I admit that it is actually the closest role to what I consider the study of physics; however the mathematical physicists do not take many physics courses and certainly do not have an understanding of physical phenomena equal to that of any physicist.

The winning choice, in my opinion, is to study seriously pure and applied mathematics and, being aware of the gaps I have, I decided to get a master's degree in mathematics to fill them. I am slowly realizing how inappropriate is the university curriculum that should lead to be a formal theoretical physicist as I define it. Too little mathematics and that done is reduced to only calculus with some sketches (often confused or confusing) of group theory and differentiable manifold (how can you study what a differentiable manifold is if you do not know what a topological manifold is or what Lie group representation is if you do not know what an group homomorphism is? mystery of faith.) But the responsibility is not entirely and solely of the institutions. Each of us makes our own choices and perhaps very few are moved by an actual passion for the subject that leads them to enter the meanders of modern mathematics.

Ultimately, the formal theoretical physicist finds himself in a limbo: too mathematical for physicists (too many equations and abstractions according to some, too little experimentality according to others) and too physical for mathematicians (no need to repeat myself). So what is a formal theoretical physicist? This is the question with which I conclude my PhD, a question to which I do not have answer.

In any case, I thank those formal theoretical physicists who fight my same "Don Quixotian" battle and who try to use modern mathematics at their best.

The only thanks, which goes beyond the previous reflection, that I would like to make is to Marilù: thanks for the tie and the belt. You make the world a gentler place, I love you.

Federico Manzoni
2024/2025

Summary with results

This is the PhD thesis I argued in Roma Tre university in the academic year 24/25. The thesis deals with the theoretical, formal and mathematical study of gauge theories in the broadest sense of the term. In this realm, I studied asymptotic symmetries, asymptotic charges and duality of gauge theories together with the possible links and interconnections between them. The final perspective is to better understand the possible ways to a quantum theory of gravity.

The thesis has a hybrid point of view, both mathematical and physical which is, according to me, the right encounter point between Mathematics and Physics. The thesis starts with a general introduction in Chapter 1, followed by Chapter 2 where the background material is developed:

- in Paragraph 2.1 we discuss the essential algebraic and geometric tools to study fields and their representations. We discuss Young diagrams and tableaux, clarifying how they are related with the irreducible representations carried by tensor fields. We also discuss the variational bicomplex which is the framework where Noether theorem and asymptotic symmetries emerge in a formal way;
- in Paragraph 2.2 we discuss two prominent example in which asymptotic symmetries emerge: Maxwell and Einstein theories in $D = 4$. We discuss them in such a way that even the reader who is not aware of the variational bicomplex can follow in a manageable way;
- in Paragraph 2.3 we discuss how asymptotic symmetries can shed new light both on the IR structure of gauge theories and on the ways to a quantum theory of gravity. We discuss the remaining two corners of the IR triangle: soft theorems, which are fundamental in understanding the IR structure of gauge theories, and memory effects which are the observable effects associated to some asymptotic symmetries. Furthermore, after a brief introduction to the problem of quantization of gravity, we discuss two new paths towards a quantum theory of gravity both with roots in asymptotic symmetries: celestial holography and the corner proposal;
- in Paragraph 2.4 we focus on Einstein gravity studying gravitational solitons (gravisolitons). After a brief discussion of soliton solutions in general, we give a detailed overview on gravisolitons where we construct the solution, we study some of their main properties and we show that the Kerr black hole is a gravisoliton;
- in Paragraph 2.5 we discuss how tensor fields of arbitrary rank can be used as gauge field. We start with Higher-Spin discussing also some of their issues and we continue with alternating (forms) and mixed symmetry tensors developing a formal language useful to deal with mixed symmetry tensor fields. We also discuss the dual descriptions of these gauge theories focusing on some explicit examples.

In the same Chapter, I give some insight on possible future works, mainly on topics neighbors to the celestial holography, such as a possible topological and conformal explanation of the antipodal matching condition and a possible prototype of Celestial Conformal Field Theory under some specific requirements. Moreover, again in Chapter 2 I discuss very briefly some String Theory/holography works [310],[320] I have published during the PhD but which have no direct connection with the topics of the PhD project. Chapters 3, 4, 5 are based on published or in preparation works, two in collaboration with my supervisor Dario Francia and four with my individual signature.

Chapter 3 is based on [326] in collaboration with Dario Francia, and [330],[335] where p-form gauge theories duality and their asymptotic symmetries are studied; the main results are the following:

1. the understanding of the necessity of the logarithmic branch in the polyhomogeneous expansion to have asymptotic symmetries and of the fact that the logarithmic branches are consistent, also in presence of matter and with radiation fall-off for fields components, with the request of finite energy flux at null infinity;
2. the computation of asymptotic charges for any form degree p in any dimension D both for radiation and Coulomb fall-offs for the field components;
3. the proposal of duality maps, one linking the derived asymptotic electric-like charges of the dual descriptions and one leading to magnetic-like charges in the dual descriptions;
4. a proven theorem on the existence and uniqueness of the duality map linking electric-like asymptotic charges of the dual descriptions and a conjecture on the role of fall-off and gauge fixing choice in the perspective of the duality above;
5. a result on the existence and uniqueness of the solution of a particular class of PDEs on a Riemannian space form with positive sectional curvature.

Chapter 4 is based on the in preparation works [325], in collaboration with Dario Francia and on the work [334], where mixed symmetry tensor gauge theories duality and their asymptotic charges are studied; the main results, at the time of writing, are the following:

1. construction of a de Rham-like complex and its de Rham-like cohomology groups for the N-multi-form space;
2. developing of a new cohomology from differential manifolds with the proof of a Poincaré-like lemma for differential mixed symmetry tensors;
3. generalization of the theorem on the existence and uniqueness of the duality map to mixed symmetry tensors;
4. explicit study of the graviton-Curtrigth duality and their asymptotic symmetries in dimension $D = 5$ with the computation of the Noether current.

Chapter 5 is based on the work [329], where the asymptotic symmetries of axialgravisolitons are studied with an eye at the corner proposal; the main results are:

1. a systematic asymptotic expansion of the N-axialgravisolitons solution which play a similar role to Bondi or Fefferman-Graham expansions;
2. the computation of the leading order Killing vectors for the N-axialgravisolitons solution and their algebra, called $\mathfrak{agcsa}$, funding a subalgebra of the universal corner symmetry algebra making one of the first explicit test of the corner proposal;
3. the possibility, in the spirit of the corner proposal, to quantize the non-asymptotically flat and non-perturbative sector of gravity by studying the representations of $\mathfrak{agcsa}$;
4. the proposals of a new possible IR triangle whose corners are the axialgravisoliton asymptotic symmetries, the soft theorem on axialgravisoliton background and the axialgravisoliton memory effect and a new

possible declination of the holographic principle we refer as N-axialgravisoliton/$\mathfrak{agcsa}$-QFT correspondence.

In the end, in Chapter 6 there are the conclusions and some possible outlooks for future works.

In the following we propose a road map which should help the the reader to orient himself within the various chapters of the thesis. Starting from Chapter 1, the reader can go directly to Chapter 5 skipping Paragraph 2.5 or to Chapters 3 and 4 skipping Paragraphs 2.3 and 2.4.

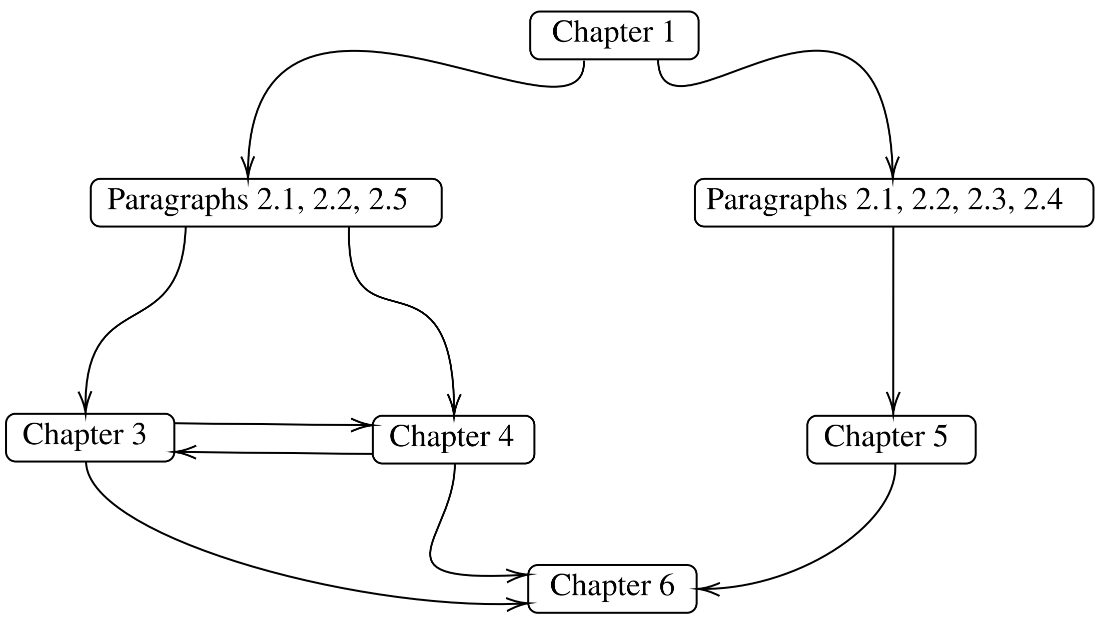


FIGURE 0.1 *Road map to navigate safely in the thesis.*

CONTENTS

LIST OF FIGURES

Notations, Terminology and Conventions

Symbol	Description
$\approx$	Equality when the equations of motion hold
$:=$	Definition of a quantity
$\simeq$	Equality when only the equations of motion of the traces hold
$\cong$	Isomorphism between quantities
$\sim$	Asymptotic behaviour
$\sim_{eq}$	Equivalence relation between quantities
$[\mathbf{\Phi}]_\phi$	Coordinate representation of $\mathbf{\Phi}$ through the chart ϕ
$\chi(M)$	Space of smooth vector fields on the manifold M
$\Omega^p(M)$	Space of differential forms of degree p on the manifold M
$\phi_{(\mu_1\ldots\mu_n)}$	Symmetrization of the indices in brackets without any weight
$\phi_{[\mu_1\ldots\mu_n]}$	Antisymmetrization of the indices in brackets without any weight
$X_{,y}\quad \partial_y X\quad \frac{\partial X}{\partial y}$	Ordinary derivative of X with respect to y
$X_{;y}\quad \nabla_y X$	Covariant derivative of X with respect to y
G	Group G
$\mathfrak{g}\quad Lie(G)$	Lie algebra of G
$\bar{z},\quad z^*$	Adjoint of z
D	Dimension of a manifold
$\mathcal{I}$	Identity matrix or operator
$\#A$	Cardinality of the set A
$\star,\quad *$	Hodge star operator
LHS, RHS	Left and right side of an equation
$c = G = k_{\mathrm{B}} = h = 1$	Choice of measure units

Chapter 1

General introduction

One of the greatest achievements of theoretical physics in the last eighty/ninety years is undoubtedly the use of gauge symmetries in order to describe fundamental interactions: from gravity to strong nuclear force. A peculiar feature of gauge theories emerges from the fact the Hamiltonian and the equations of motion are invariant under gauge transformations; therefore, due to the intrinsic redundancy introduced in the dynamical problem by the gauge symmetry, the time evolution is not well defined, even at the classical level, unless one introduces a gauge fixing condition that breaks local gauge invariance to a residual subset. This picture, along with the fact that Noether's theorems [4],[218] return on-shell vanishing charges associated to gauge symmetries, may lead to the conclusion that local gauge transformations are only a convenient artifact used to write down interesting and useful theories, but which does not carry any intrinsic physical relevance. While the first Noether theorem deals with global symmetries, that is symmetries associated to transformations applied simultaneously and equally at all points of spacetime, in order to construct Noether charge for gauge symmetry, defined as a local symmetry acting differently at different spacetime points, we need her second theorem. Indeed Noether first theorem gives a current associated to the local symmetry that necessarily vanishes on-shell, up to total derivatives. These total derivatives imply that Noether charges for gauge symmetries must be defined as surface integrals of $(D-2)$-forms, where D is the spacetime dimension: they are codimension two surface charges[1]. This is the core of Noether's second theorem but this result should not be surprising since, for instance, it is well known that the total electric charge of a system in electromagnetism is indeed defined as a surface integral, thanks to Gauss' law. In this context, the theory of asymptotic symmetries emerges and turns out to be a thriving and active area of investigation. It aims at understanding the real notion of observables in a physical system, especially in the presence of gauge symmetries. At first glance the name "asymptotic symmetries" seems to suggest these are symmetries that emerge when we study the behavior of a physical theory “at infinity” or, in a rigorous way, at the boundary of the spacetime. However this is a misunderstanding: the term "asymptotic" does not refer necessary to the boundary of spacetime but, more generally, to a boundary in the spacetime. Indeed the term "asymptotic" refers here to the fact that these conserved forms are computed near the boundary on which the charge surface integral is evaluated, for specific boundary conditions imposed on the fields. The mathematical foundation behind these results resides in the variational bicomplex [68],[69],[91],[98],[250] we are going to discuss later on. In a nutshell, the theory of asymptotic symmetries states that, given a choice of dynamics in the bulk of a spacetime with certain falloffs for the dynamical fields near the boundary, only certain gauge transformations, the so-called allowed one, preserve the falloffs of the fields. These allowed gauge symmetries can be separated in two classes at the boundary, by evaluating the associated surface charges. If the charge vanishes the associated symmetry is a so-called trivial transformations, representing, even in the presence of the boundary, a gauge redundancy of the theory. Contrarily, if the associated surface charge is non-vanishing then the associated symmetry is a physical transformation of the field space of the theory, mapping a field

[1] In physics literature they are also know simply as surface charges.

configuration into an inequivalent one. These are sometimes called non-trivial or large gauge symmetries. Non-vanishing surface charges then satisfy a potentially centrally extended charge algebra, since their algebra is a projective representation of the algebra of asymptotic gauge symmetries as brought to light by Brown and Henneaux [73],[74]. A crucial ingredient is thus the definition of charges. In the absence of fluxes, that is, in a non-dissipative system, charges are defined in a canonical way. Therefore the crucial object in asymptotic theory is the Asymptotic Symmetry Group (ASG) given by

$$\mathrm{ASG} := \frac{\text{allowed gauge transformations}}{\text{trivial gauge transformations}}; \tag{1.1}$$

as said before the allowed gauge transformations are those compatible with the boundary conditions, or fall-off conditions, of fields of the theory while the trivial one are those for which the associate surface charges are vanishing. This quotient group results in the gauge transformations that act not trivially on the phase space or, equivalently, those gauge symmetries for which the gauge parameters do not go to zero at the boundary but still respect the fall-off conditions we are considering.
One of the first example of asymptotic symmetry fits into General Relativity thanks to the pioneering work of Bondi, Metzner, Van der Burg and Sachs [24],[25],[26]. Their work was motivated by the will to understand the properties of gravitational waves far away from the strong gravity regime, i.e., at asymptotic infinity, that is the conformal boundary of Minkowski spacetime[27],[163],[164]. The fundamental result of the work of 1962 is that the a priori expectation that the Poincaré group is the asymptotic symmetry group of asymptotically flat spacetimes is not satisfied: the group of asymptotic physical symmetries, even on empty Minkowski spacetime, is not Poincaré but rather the infinite-dimensional Bondi-van der Burg-Metzner-Sachs (BMS) group at null conformal infinity that we will review in the main text in Chapter 2.

In recent days, more and more relaxed boundary conditions and asymptotic symmetry groups, that generalize the BMS group, have been found. The first series of papers in this direction are by Barnich and Troessaert [153],[154],[155],[159] where, for example, it is realized that one can consider conformal Killing vectors of the boundary structure that are only locally invertible; further generalizations followed by building bigger and bigger asymptotic symmetry groups [271],[275],[291],[309]. One of the most interesting application of the theory of asymptotic symmetry, still at the center of research today, is in the context of black holes horizons, which are null hypersurfaces, to solve the information paradox [47],[208],[279]. Haco, Hawking, Perry and Strominger proposed a possible solution to the paradox where information is stored in a specific BMS transformation, called soft hair, associated with the shift of the horizon caused by ingoing particles [223],[236],[254]. An important role in the proposal of Haco, Hawking, Perry and Strominger is played by the soft structure but this infrared structure has a more general relevance: the IR triangle. This is a triangular equivalence relation that governs the infrared dynamics of all physical theories with massless particles. The first corner is the subject of asymptotic symmetries we have already discuss in some detail; the second corner is the topic of soft theorems originated in the 30s in Quantum ElectroDynamics, significantly developed and generalized to gravity in 1965 by Weinberg [8],[15],[18],[19],[23],[29],[30]. Soft theorems characterize universal properties of Feynman diagrams and scattering amplitudes when a massless external particle becomes soft, i.e. its energy is taken to zero. These theorems tell us that an infinite number of soft particles are produced in any physical process; they are produced in a highly controlled manner that is central to the consistency of Quantum Field Theory. The last corner is the memory effect, investigated in the context of gravitational physics in 1974 by Zeldovich and Polnarev [45] and then developed [70],[77],[88],[90],[263]. This is a subtle direct current effect, in which the passage of gravitational waves produces a permanent shift in the relative positions of a pair of inertial detectors. This effect is osservable and could be detected at LIGO, LISA or via a pulsar timing array [156],[216],[225]. This memory corner of the triangle provides an important physical realization of the more abstractly formulated results of the other two corners, giving direct observational consequences of the infinite number of symmetries and conservation laws. It is interesting to note that the gauge theory analog appeared only recently[2] [214],[238],[240]. Moreover, the three vertices of the infrared triangle are linked by mathematical equivalence relations [196],[209],[229]. Soft theorems are statements about momentum space poles in scattering amplitudes, while memory effect concerns a shift in asymptotic data between late and early times. Since the Fourier transform of a pole in frequency space is a step function in time, these are essentially the same thing. Moreover, the step function can be understood as a domain wall connecting two inequivalent vacua that are related by an asymptotic symmetry. Hence the memory effect both physically manifests and directly measures the action of the asymptotic symmetries. The last connection is due to the fact that every symmetry has a Ward identity that equates scattering amplitudes of symmetry related states; these Ward identities are nothing but the soft theorems dressed. The infrared triangle is believed to appear in many and disparate fields of theoretical physics: from standard Quantum Field Theory to String Theory and black holes [182],[198],[205],[212],[213],[219],[221],[242], from Cosmology to supersymmetric Quantum Field Theory and Condensed Matter Physics [200],[204],[211],[233],[234],[270].

Asymptotic symmetries can also be investigated in more exotic gauge theories, for example those motivated by String Theory like Higher Spin theories [52],[53],[59],[128],[172],[184],[232],[248],[249],[272], p-form gauge theories [75],[190],[244],[274], and mixed symmetry gauge theories [71],[76],[123],[273]. In the interesting context of these exotic gauge theories, the works on which Chapter 3 and Chapter 4 are based, fit in [325],[326],[330],[334]; they are based on two main ingredients we are going to briefly discuss. On the one hand, the idea that asymptotic charges can store information about physical dual descriptions as shown recently in [251],[262] in the case of a 2-form gauge theory. On the other hand, the fact that gauge theories have a number of dual descriptions. These dual descriptions emerge as generalization and extension of electric-magnetic duality of Maxwell theory [6] and are implemented by a Young diagrams machinery duality [60],[81],[264]. The original electric-magnetic duality led to the hypothesis about magnetic monopoles in electromagnetism [14] while, its direct generalizations, led to the introduction of magnetic-type sources to gauge theories [40],[41],[46]. Less trivial generalizations of this duality emerged as weak-strong duality in Yang-Mills theories, called Montoner-Olive [50]; moreover, the duality principle was refined in supersymmetry [55] and was extended to $\mathcal{N} = 4$ supersymmetric gauge theories [56]. The $\mathcal{N} = 4$ Super Yang-Mills theory has an exact $\mathrm{SL}(2,\mathbb{Z})$ symmetry, typically referred to as S-duality [94] but also in the case of $\mathcal{N} = 1$ electric-magnetic

[2] Here we refer to gauge theories frequently used, for example in particle physics, such as QED and QCD.

duality plays a crucial role leading to the formulation of the so-called Seiberg duality [96]. Moreover, in the realm of String Theory, strong-weak duality exchanges the electrically charged string with the magnetically charged soliton in the heterotic String Theory [93],[97]. The electric-magnetic duality principle also passed through into gravitational theories, supergravity, Higher-Spin field theory, mixed symmetry tensor field theories, holographic dictionary, hypergravity, partially-massless and massive gravity on AdS and dS spacetimes [71],[75],[111],[113],[114],[115],[120],[135],[147],[150],[245],[246].
In Chapter 3 we apply the duality to p-form and $(q = D-2-p)$-form gauge theories, we discuss and compute the asymptotic charges for two different choices of the field components fall-offs and we establish two possibilities of duality map. The first possibility is a linking for the derived asymptotic charges of the dual descriptions while the second one is a map leading to magnetic-like charges in the dual description. In the same Chapter, in Paragraph 3.6, a more mathematical rigorous theorem on the map which link the derived asymptotic charges of the dual descriptions is proved. In Chapter 4 we discuss the applicability and the generalization of the aforementioned theorem to mixed symmetry tensors cases generalizing the de Rham complex using the N-multi-form space introduced by Hull and Medeiros [125],[128]. Moreover, in the same Chapter, we apply the duality to the graviton and the Curtright mixed symmetry tensor in $D = 5$, we discuss the computation of the Noether current with the goal to compute the asymptotic charges and linking them to the graviton asymptotic charges.

Asymptotic symmetries are the underlying symmetry structure of holography. It is interesting that the fact gravity is holographic is readily written in Noether's theorems: charges associated to gauge symmetries are codimension two quantities that can be reinterpreted as global symmetries of a theory living on the codimension one boundary. This could be interpreted as the strongest evidence that gravity and, in fact, all gauge theories are holographic [42],[314]. Note, however, that here "holography" is referred to the possibility of recasting gravity or gauge theory on a boundary theory; this does not imply that from one boundary and finitely many degrees of freedom one can exactly reconstruct the full bulk theory: this is a peculiarity of the AdS/CFT correspondence which is a gauge/gravity duality where the gravity side is in a D-dimensional ADS spacetime while the gauge theory side is a conformal field theory in one dimension less, i.e. on the boundary of the ADS spacetime which is in the same conformal class of the Minkowski spacetime. To this direction goes the program of flat holography, i.e. the attempt to extend holography to flat or asymptotically flat spaces, which essentially has two important declinations. The first one starts from the fact that in the asymptotically flat case the bulk asymptotic symmetry group is BMS; therefore the dual field theory should be a BMS-field theory. A crucial result is that the BMS group is isomorphic to a conformal Carroll group when the spatial non-degenerate metric is a codimension two sphere [89],[185],[186],[187]. The second approach is known as celestial holography [257],[299],[300],[301]. The central idea is to exploit the fact that the dual theory can be seen as a CFT living on the codimension two space at infinity: the so-called celestial sphere. This framework was built off the observation that gravity and gauge theories in asymptotically flat spacetimes are governed in the infrared by the already mentioned triangular equivalence. The celestial sphere is just a specific instance of a codimension two surface, on which Noether charges associated to gauge symmetries have support. Recently, there has been an appreciation of these surfaces from a purely geometric point of view, which led to the formulation of the corner proposal [222],[252],[255],[285],[305],[306],[314]. Here, the term "corner" indicates a generic codimension two surface on which surface charges are evaluated. The non-trivial gauge symmetry algebra is then an essential tool to reformulate geometric observables in an algebraic language, amenable to quantization. In this context, the idea is that the geometry is included into the charge algebra, the quantum kinematics into its representations and the quantum dynamics into fusion properties. It is thus a shift of paradigm that focuses on universal quantities that are expected to survive in a quantum theory of gravity [314]. This is the main underlying idea of the corner proposal: a refocusing on corners and charges, rather than spacetime and classical dynamics. The fundamental result [283],[289],[316] is the recently found maximal non-trivial asymptotic symmetry algebra at isolated corners, thus providing a universal corner symmetry group (UCS). Various subgroups of this group give rise to known asymptotic symmetry groups found in the literature, both for asymptotic corners, like the celestial sphere, and finite distance corners, such as the black holes horizons. In the context of the corner proposal and gravisoliton theory the basic work at the base of Chapter 5 is contextualised [329]. In general a so-called soliton is a particular wave solution emerging from the balance of non-linearity and dispersive effects which has incredible stability [136],[296], that can also be related to topological properties [130]. The first example came from the Kortveg-de Vries equation in hydrodynamics thanks to the 1967 pioneering work of Greene, Miura, Kruskal and Gardner [32] where the Inverse Scattering Transform method (IST) was first introduced. In the framework of Einstein gravity a soliton, also referred as gravitational soliton or gravisoliton, is not a wave but a spacetime solution of Einstein field equations coming from the same mathematical tools used to derive standard solitons solutions [51],[54],[57],[133]. The field of gravisolitons is still today the object of research as the IST can only be applied to very specific cases which, however, include almost all the physical situations of interest such as black holes or cosmological solutions. Moreover gravisolitons find applications also in dark matter [269], in the field of generalised symmetries and in the swampland program [297]. Here, in Chapter 5, we first compute the asymptotic behaviour of the axialsymmetric N-solitons solutions of Einstein gravity which could play similar role to the Bondi expansion for asymptotically flat spacetime and the Fefferman-Graham expansion for asymptotically AdS spacetime. We then study the asymptotic symmetries of the N-axialgravisolitons and we compute the leading order asymptotic Killing vectors algebra, called $\mathfrak{agcsa}$, finding a subalgebra of the $\mathfrak{ucs}$ algebra. This opens some interesting directions: the quantization of non-asymptotically flat and non-perturbative sector of pure gravity, a new declination of the holographic principle and a new copy of the IR triangle.

Chapter 2

Background material

2.1 Fields: algebra and geometry

Two of the most fruitful interplay between physics and mathematics are that concerning fields and Representation Theory leading to a deep understanding of Quantum Mechanics and Quantum Field Theory, and that dealing with fields and Differential Geometry, thanks to which, among others, General Relativity and Quantum Field Theory on curved space were able to have a solid structure. Here we build the fundamental structures with which to face what comes next.

2.1.1 Elementary particles as unitary irreducible representations

Let us consider a wave function ψ describing the possible states of a quantum-mechanical system; this is a vector in a Hilbert vector space $\mathcal{H}$ which, in general, is infinite-dimensional. The physical state is more precisely characterized by the associated projective Hilbert space $\mathbb{P}\mathcal{H}$, also called ray space. Two vectors that differ by a non-zero scalar factor correspond to the same physical quantum state represented by a ray in Hilbert space, which is an equivalence class in $\mathcal{H}$ and, under the natural projection map $\mathcal{H} \to \mathbb{P}\mathcal{H}$, an element of $\mathbb{P}\mathcal{H}$. In order to understand what states can exist, it is important to classify the symmetries of the system and their properties. For example, in a D dimensional relativistic quantum system on flat spacetime, the symmetry group is the Poincaré group $\mathbb{R}^{D-1,1} \rtimes \mathrm{O}(D-1,1)$, which is a $\frac{D(D+1)}{2}$ dimensional non-compact Lie group which plays the same role as the Euclidean group for Riemannian manifold in the case of Lorentzian manifold. Therefore, if the wave functions in question refers to a free particle and satisfy relativistic wave equations, a correspondence between the wave functions describing the same state in different frames there exists. This correspondence is made explicit by the elements of the Poincaré group $U(\Lambda, a)$ where Λ represents a Lorentz transformation and a represents a translation; they satisfy the group law

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = \pm U(\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1). \tag{2.1}$$

If ψ and ψ' are wave functions, respectively in the Lorentz frames $\mathcal{O}$ and $\mathcal{O}'$, then they are connected by

$$\psi' = U(\Lambda, a)\psi, \tag{2.2}$$

where $U(\Lambda, a)$ is a representation of the Poincaré transformation from $\mathcal{O}$ to $\mathcal{O}'$. Since all Lorentz frames are equivalent for the description of our system, it follows that both ψ and $U(\Lambda, a)\psi$ are possible states viewed from the original Lorentz frame $\mathcal{O}$. Thus, the vector space $\mathbb{P}\mathcal{H}$ contains, with every ψ, all states $U(\Lambda, a)\psi$, where $U(\Lambda, a)$ is any Poincaré transformation. We are using the wave functions in the Heisenberg picture, so that a given ψ represents the system for all times and may be chosen as the Schroedinger wave function at time $t = 0$

in a given Lorentz frame $\mathcal{O}$. To find ψ_{t_0}, the Schroedinger function at time t_0, one must therefore transform to a frame $\mathcal{O}'$ for which $t' = t - t_0$, while all other coordinates remain unchanged. Then $\psi_{t_0} = U(\Lambda, a)\psi$, where $U(\Lambda, a)$ represents transformation leading from $\mathcal{O}$ to $\mathcal{O}'$. A classification of all unitary representations of the Poincaré group, i.e. of all solution of (2.1), is therefore equivalent to a classification of all possible relativistic waves equations.

In order to study the irreducible representation of the Poincaré group, we follow the method of induced representations proposed by Wigner in 1939 [12]. As known from the Poincaré group algebra, the translation generators commute with each other, so it is natural to express physical states ψ in terms of eigenvectors of the translation generators P^μ. Let us introduce the label σ to denote all other degrees of freedom quantum numbers; therefore one has

$$P^\mu \psi_{p,\sigma} = p^\mu \psi_{p,\sigma}. \tag{2.3}$$

Moreover, acting on $\psi_{p,\sigma}$ with $U(\Lambda, 0)$ we produce an eigenvector of the translation generators with eigenvalue Λp, indeed

$$\begin{aligned} P^\mu U(\Lambda,0)\psi_{p,\sigma} &= U(\Lambda,0)U^{-1}(\Lambda,0)P^\mu U(\Lambda,0)\psi_{p,\sigma} = \\ &= U(\Lambda,0)[(\Lambda_\rho{}^\mu)^{-1}P^\rho]\psi_{p,\sigma} = \Lambda^\mu{}_\rho p^\rho U(\Lambda,0)\psi_{p,\sigma}. \end{aligned} \tag{2.4}$$

This means that $U(\Lambda, 0)\psi_{p,\sigma}$ must be a linear combination of states $\psi_{\Lambda p,\sigma}$, namely

$$U(\Lambda,0)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda, p)\psi_{\Lambda p,\sigma'}. \tag{2.5}$$

In general, it is possible by using suitable linear combinations of $\psi_{p,\sigma}$ to choose the σ labels in such a way that the coefficient matrix is block-diagonal; it is natural to identify the states of a specific particle type with the components of a representation of the Poincaré group which is irreducible, in the sense that it cannot be further decomposed in this way, namely the specific block cannot be decomposed in smaller blocks. The goal is to work out the structure of the coefficients $C_{\sigma\sigma'}(\Lambda, p)$ in irreducible representations of the Poincaré group, essentially the blocks. Note that from (2.5) we learn that all states $\psi_{p,\sigma}$ in an irreducible representation of the Poincaré group have momenta belonging to the same orbit of a single fixed momentum, that we call q^μ. Moreover, under a proper orthochronous Lorentz transformation $L^\mu{}_\nu$, quantities p^2 and $sign(p^0)$ are invariant[1]. Therefore, for each p^2 and $sign(p^0)$ we express any p^μ in the same orbit as

$$p^\mu = L^\mu{}_\nu(p) q^\nu. \tag{2.6}$$

We can then define the state of momentum p^μ as

$$\psi_{p,\sigma} := N(p)U(L(p),0)\psi_{q,\sigma}, \tag{2.7}$$

[1] The second one only if $p^2 \leq 0$.

where $N(p)$ is a normalizations; acting on it with Lorentz transformation $U(\Lambda, 0)$ and using (2.6) we find

$$U(\Lambda,0)\psi_{p,\sigma} = N(p)\underbrace{U(\Lambda L(p),0)}_{q^\mu\mapsto p^\mu\mapsto\Lambda p^\mu}\psi_{q,\sigma} = N(p)\underbrace{U(L(\Lambda p,0))}_{q^\mu\mapsto\Lambda p^\mu}\underbrace{U(L^{-1}(\Lambda p)\Lambda L(p),0)}_{q^\mu\mapsto p^\mu\mapsto\Lambda p^\mu\mapsto q^\mu}\psi_{q,\sigma}. \tag{2.8}$$

Note that the Lorentz transformation $L^{-1}(\Lambda p)\Lambda L(p)$ belongs to the subgroup of the Lorentz group consisting of those Lorentz transformations $W_\nu^{\ \mu}$ that leave q^μ invariant, namely $q^\mu = W_\nu^{\ \mu} q^\nu$. This subgroup is the stabilizer and it is called little group. As before in (2.5) we can write

$$U(W,0)\psi_{q,\sigma} = \sum_{\sigma'} D^q_{\sigma\sigma'}(W,q)\psi_{q,\sigma'} \tag{2.9}$$

and, applying a second transformation represented by $U(\bar{W},0)$, we find that

$$D^q_{\sigma'\sigma}(\bar{W}W,q) = \sum_{\sigma''} D^q_{\sigma'\sigma''}(\bar{W},q)D^q_{\sigma''\sigma}(W,q); \tag{2.10}$$

so, the coefficients $D^q_{\sigma'\sigma}(W,q)$ furnish a representation of the little group. In particular for $W_0 = L^{-1}(\Lambda p)\Lambda L(p)$ we can write, from relations (2.8) and (2.9) and definition (2.7),

$$U(\Lambda,0)\psi_{p,\sigma} = N(p)\sum_{\sigma'} D^q_{\sigma'\sigma}(W_0,q)U(L(\Lambda p),0)\psi_{q,\sigma'} = \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D^q_{\sigma'\sigma}(W_0,q)\psi_{\Lambda p,\sigma'}. \tag{2.11}$$

Apart from the question of normalization, the problem of determining the coefficients in the transformation rule (2.5) has been reduced to the problem of determining the coefficients in the transformation rule (2.9). In other words, the problem of determining all possible irreducible representations of the Poincaré group has been reduced to the problem of finding all possible irreducible representations of the little group, depending on the class of momentum to which q^μ belongs. This approach, of deriving representations of a semi-direct product group from the representations of the stability subgroup, is known as the method of induced representations [12],[282].

Parameterization	**Orbit**	**Stability subgroup**	**Classification**
$p^2 = -m^2$	Mass-shell	$\mathrm{SO}(D-1)$	Massive
$p^2 = 0$	Light-cone	$\mathrm{E}(D-2)$	Massless
$p^2 = +m^2$	Hyperboloid	$\mathrm{SO}(D-2,1)^+$	Tachyonic
$p^\mu = 0$	Origin	$\mathrm{SO}(D-1,1)^+$	Zero-momentum

FIGURE 2.1 *Unitary irreducible representations.*

Under Lorentz transformations, the orbit of a given non-vanishing vector q^μ of Minkowski spacetime $\mathbb{R}^{D-1,1}$ is, by definition, the hypersurface of constant momentum square p^2; geometrically speaking, it is a quadric of curvature radius $m > 0$. The unitary irreducible representations are classified according the above table.
In the case of massless representations we can retrace the same steps done previously to reduce the problem of studying irreducible representations of the euclidean group $\mathrm{E}(D-2)$ to the problem of studying irreducible representations of its stability subgroups. The classification is now given according to which class the eigenvectors π^a of the translation operators Π^a of the group $\mathrm{E}(D-2)$ belong. If $\pi^a = 0$ then the stability subgroup is $\mathrm{SO}(D-2)$, the orbit correspond to the origin and we call these irreducible representations "helicity"; they are the most famous representations of massless degrees of freedom. If $\pi^a \neq 0$ then the stability subgroup is $\mathrm{SO}(D-3)$, the orbit corresponds to sphere and we call these irreducible representations "infinite spin".

2.1.1.1 Young diagrams I: a mathematical perspective

To study the irreducible representations carried by elementary particles, a very useful tool are the combinatorial objects called Young diagram and Young tableau [81],[100]. These objects was introduced by Young [3] to study the representation theory of the symmetric group. Let us start with the following definition.

Definition 2.1.1 (Young diagram)**.** *A Young diagram is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order.*

Containment of one Young diagram in another bigger one defines a partial ordering on the set of all partitions, which is in fact a lattice structure[2], known as Young's lattice. This kind of structure is a tool we will employ to study the mixed symmetry tensors duality in Chapter 4 in order to extend result from p-form gauge theories to mixed symmetry tensors gauge theories. We can now define Young tableaux.

Definition 2.1.2 (Standard and semi-standard Young tableau)**.** *A Young tableau is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet, which is usually required to be a totally ordered set, for example a subset of integers. A Young tableau is called standard if the entries in each row and each column are increasing, while it is called semi-standard if the entries weakly increase along each row and strictly increase down each column.*

Among all the Young tableaux we can construct from a Young diagram there is a natural one particularly important that we call natural Young tableau

Definition 2.1.3 (Natural Young tableau)**.** *The natural Young tableau is the semi-standard Young tableau obtained by counting the boxes column by column.*

[2] This is meant in the sense of order theory: it is partially ordered set in which every pair of elements has a unique supremum and a unique infimum.

The point of the natural Young tableau is that listing the number of boxes in each column of a Young diagram gives a partition λ of a non-negative integer k where k is the total number of boxes of the diagram.
In a simplistic way the reason why Young diagram are useful in representation theory is the following. Let v_a represents a generic element of the cotangent space $T^*_P M$ of a D-dimensional manifold M. The action of non-singular linear operators on this space gives a D-dimensional irreducible representation $V \cong T^*_P M$ of the general linear group $\mathrm{GL}(D)$; indeed, this representation defines the group itself. The rank-two tensors with components T_{ab} carry a larger representation of $\mathrm{GL}(D)$ given by the tensor product $V \otimes V$, where the group elements act on the two indices simultaneously. The latter representation is, however, reducible: it decomposes into the subspace of symmetric and antisymmetric rank-two tensors of respective dimensions $\frac{D(D+1)}{2}$ and $\frac{D(D-1)}{2}$. Similarly, the tensor representation of rank k, denoted by $V^{\otimes k}$, decomposes into irreducible representations of $\mathrm{GL}(D)$ which are associated with the irreducible representations of the symmetric group $\mathfrak{S}_k$ acting on the indices, which in turn are labeled by the integer partitions of k, hence by natural Young tableaux. Let us clarify this point in a more rigorous way. By its definition, a natural Young tableau is in one-to-one correspondence with irreducible representations of the symmetric group over the complex numbers. This is so for the following standard theorem of representation theory of finite group

Theorem 2.1.1 (Non-isomorphic irreducible representations of a finite group over the complex numbers)**.** *In any finite group, the number of non-isomorphic irreducible representations over the complex numbers is precisely the number of conjugacy classes.*

Proof of the above theorem can be found in standard representation theory textbook such as [167]. The crucial point is that in the case of the symmetric group $\mathfrak{S}_k$ the conjugacy classes are labelled by partitions of k hence by natural Young tableaux. From Young tableau one can infer many facts about the corresponding representation such as dimension of a representation and restricted representations using, respectively, the hooke length formula and the branching rules.
However, we are interested in irreducible representation of the general linear group and its Lie subgroups[3]. To recycle the Young diagrams machinery we need a powerful theorem by Schur and Weyl

Theorem 2.1.2 (Schur–Weyl duality)**.** *Two algebras of operators, on the complex tensor space $V^{\otimes k}$ where V is a complex D-dimensional vector space, generated by the actions of* $\mathrm{GL}(D)$ *and* $\mathfrak{S}_k$ *are the full mutual centralizers*[4] *in the algebra of the endomorphisms* $\mathrm{End}_{\mathbb{C}}(V^{\otimes k})$.

The proof of the Schur–Weyl duality relies on the use of two algebraic lemmas but it goes beyond the scope of this Section and can be found on [100]. However, let us clarify the meaning of the theorem at least for the case of $V^{\otimes k} = (\mathbb{C}^D)^{\otimes k}$. Given an element $v_1 \otimes \ldots \otimes v_k \in (\mathbb{C}^D)^{\otimes k}$ where $v_i \in \mathbb{C}^D$ for every i, the symmetric group $\mathfrak{S}_k$ acts on this space (on the left) by permuting the factors, $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$;

[3]Recall that by Cartan's criterion every closed subgroup of a Lie group is a Lie subgroup.
[4]The centralizer of S in a set G endowed with a binary operation $\cdot$ is the set $Z_G(S) := \{g \in G \mid g \cdot s = s \cdot g \ \ \forall s \in S\}$.

while the general linear group GL(D) of invertible $n \times n$ matrices acts on it by the simultaneous matrix multiplication $g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k$ with $g \in \mathrm{GL}(D)$. These two actions commute and the Schur–Weyl duality asserts that under the joint action of the groups $\mathfrak{S}_k$ and GL(D), the tensor space decomposes into a direct sum of tensor products of irreducible modules (for these two groups) that actually determine each other

$$(\mathbb{C}^D)^{\otimes k} = \bigoplus_{|\lambda|=D} \pi_k^\lambda \otimes \rho_D^\lambda. \tag{2.12}$$

The summands are indexed by the natural Young tableaux λ with k boxes and at most D rows; representations π_k^λ of $\mathfrak{S}_k$ with different λ are mutually non-isomorphic, and the same is true for representations ρ_D^λ of GL(D). Therefore for each irreducible representation of $\mathfrak{S}_k$ there exist an irreducible representation of GL(D) which is in bijective correspondence with the natural Young tableau λ with at most D rows. Once we have the Young diagrams machinery for the general linear group, we can derive which natural Young tableaux label irreducible representations of its Lie subgroups thanks to the branching rules. Moreover, we can also determine the irreducible Lie algebra representations thanks to the following

Theorem 2.1.3 (Induced Lie algrebra representation from Lie group representation)**.** *Given a closed Lie group* G *and a continuous representation* $\rho : \mathrm{G} \to \mathrm{GL}(V)$ *there exists a unique Lie algebra representation* $d\rho : \mathfrak{g} := Lie(\mathrm{G}) \to \mathfrak{gl}(V)$ *such that*

$$e^{d\rho(x)} = \rho(g) \quad \forall x \in \mathfrak{g}, \forall g \in G \;\; s.t. \;\; g = e^x, \tag{2.13}$$

and the following diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\rho} & \mathfrak{gl}(V) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & \mathrm{GL}(V) \end{array}$$

commutes.

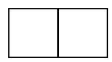
The proof of this theorem is a simple result which descends from the fact that every group homomorphism from ($\mathbb{R}$,+) and GL(V) can be written as the exponential of a unique element $X \in \mathfrak{gl}(V)$ which is a non-trivial statement whose proof relies on the local invertibility of the exponential map.

In what follow, for simplicity, we refer always to Young tableaux but we mean the natural Young tableaux[5] except where we are interested in information, such as the dimension or the restricted representations, about the irreducible representation labelled by the natural Young tableaux that require other types of Young tabelaux to be determined[6]. In the following, it will be useful to label Young tabelaux by the lengths of their columns; so that a Young tableau λ with columns of length $n_1, n_2, ..., n_\ell$ with $\ell \leq D$ will be called as $\lambda = \{n_1, n_2, ..., n_\ell\}$. Since we are mostly interested in helicity irreducible representation, let us focus on Young diagram machinery

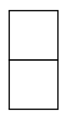
[5]Moreover, at graphical level we are going to always draw just the corresponding Young diagram.

[6]For example to compute the dimension of a irreducible representation we need to use the Young tableau constructed using the hook lengths.

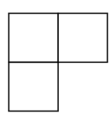
for O($D-2$). We can restrict ourselves to this case by considering contractions that are made by a metric tensor; so one is restricting the symmetry group of the problem from the general linear group to the subgroup that leaves the metric tensor invariant, that is the orthogonal group O(D), if the signature is definite, or a pseudo-orthogonal group O($D-i,i$), if the signature is indefinite. Given a Young tableau $\lambda = \{\lambda_1, ..., \lambda_\ell\}$ with $\ell \leq D$, it defines an irreducible representation of O($D-2$) if and only if $n_1^{(\lambda)} + n_2^{(\lambda)} \leq D-2$ where $n_1^{(\lambda)}$ and $n_2^{(\lambda)}$ are the number of boxes in the first two columns of the Young tableau while in the case of SO($D-2$) the condition is the same. In the case of pseudo-orthogonal and pseudo-special orthogonal groups the same holds.
For a deeper understanding of the mathematics behind Young tableaux we refer to [100],[207].
Before moving to the use in physics of Young diagrams and tableaux, let us fix the notation. Symmetrized indices are represented by boxes arranged in a row, so that a second rank symmetric tensor $h_{\mu\nu}$ (for example the graviton field) is represented by the Young diagram

 , (2.14)

while antisymmetrized indices are represented by boxes arranged in a column, so that a second rank antisymmetric tensor $B_{\mu\nu}$ (for example the 2-form field) is represented by the Young diagram

 , (2.15)

However this is not the whole story; indeed a general third rank tensor $E_{\mu\nu\rho}$ can be decomposed into a totally symmetric piece, a totally antisymmetric piece and remaining piece, which has a not unique symmetry structure of its indexes, represented by the Young diagram

 . (2.16)

The irreducible representation associated to the Young diagram (2.16) is an example of mixed symmetry tensor we will discuss in Paragraph 2.5.2

2.1.1.2 Young diagrams II: a physical perspective

As we have seen before, free massless particles in D-dimensional Minkowski space are classified by representations of the little group SO($D-2$) (if one takes into account only helicity representations) and so by a Young tableaux of the little group. Given a Young tableau λ of SO($D-2$), this corresponds to a field in the physical transverse gauge and its dynamical theory does not have the property of being Lorentz-covariant. In order to make the theory Lorentz-covariant we first have to find a covariant tensor with the same symmetry of

the physical gauge field. This step is easily work out by replacing the original physical gauge field, which is an irreducible representation of SO($D-2$) labelled by a Young tableau λ, with a field corresponding to the same irreducible representation labelled by the same Young tableau, now regarded as a Young tableau of GL(D) and such that it must transform under gauge symmetries that are sufficient to remove all negative-norm states and to allow the recovery of the physical gauge theory under a gauge fixing. Let us give an example: the graviton field. In fact, the graviton is represented in physical transverse gauge by a tensor h_{ij} (so traceless, $\delta^{ij}h_{ij} = 0$) where i, j take $D-2$ values and which corresponding to the Young tableau associated to the Young diagram (2.14) of SO($D-2$); the covariant formulation is the GL(D) irreducible representation with the same Young tableau associated to the same Young diagram (2.14), that is a symmetric tensor $h_{\mu\nu}$ with no constraints on its trace.
To a more detailed introduction on the use of Young diagrams and tableaux in physics, such as their employment in Feynman diagrammatic to compute the so-called group factors in particles scattering or in the classification of barions and mesons, see [81],[108].

2.1.2 Variational bicomplex, surface charges and asymptotic symmetries

Let $(M, \boldsymbol{g})$ be a Lorentzian manifold of dimension D with atlas (U_i, ϕ_i) and let us consider an arbitrary theory of matter fields denoted by $\boldsymbol{\Phi}_M = \{\boldsymbol{\Phi}^k_M\}_{k=1}^D$ such that $[\boldsymbol{\Phi}_M]_\phi = \Phi^I_M$ with a totally general index structure I. The theory is described by a Lagrangian density $\mathcal{L} := \mathcal{L}[\boldsymbol{\Phi}_M]$ whose associated action is $S := S[\boldsymbol{\Phi}_M] = \int_M \mathcal{L}[\boldsymbol{\Phi}_M]$. Semiclassically, it is always possible to couple this theory to gravity by introducing a non-flat metric $[\boldsymbol{g}]_\phi = g_{\mu\nu}$ in such a way that $\mathcal{L}[\boldsymbol{\Phi}_M] \mapsto \mathcal{L}[\boldsymbol{\Phi}_M, \boldsymbol{g}]$. The coupling is said to be minimal when it consists in merely replacing the Minkowski metric by a general non-flat metric, standard derivatives by covariant derivatives and, mutando mutandis, the induced change of the volume form $Vol_{\boldsymbol{\eta}} \mapsto Vol_{\boldsymbol{g}}$. After that, we can define a natural symmetric tensor whose components are

$$T^{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}, \tag{2.17}$$

where $g := det([\boldsymbol{g}]_\phi)$ and, with $\Psi^I \in \{\Phi^I, g_{\mu\nu}\}$,

$$\frac{\delta\mathcal{L}}{\delta\Psi^I} := \frac{\partial\mathcal{L}}{\partial\Psi^I} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^I)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\Psi^I)}\right) + ... + (-1)^n \underbrace{\partial_\mu...\partial_\nu}_{n}\left(\frac{\partial\mathcal{L}}{\partial(\underbrace{\partial_\mu...\partial_\nu}_{n}\Psi^I)}\right), \tag{2.18}$$

for theories of n-th order in derivatives. It is fundamental to observe that any relativistic field theory in Minkowski spacetime admits a symmetric stress tensor that is divergence-free when the equations of motion hold, $\nabla_\mu T^{\mu\nu} \approx 0$, known as Belinfante–Rosenfeld stress tensor [13] that matches with definition (2.17). The variation of the lagrangian density can be expressed as

$$\delta\mathcal{L} = \delta\Phi^I \frac{\partial\mathcal{L}}{\partial\Phi^I} + ... + \underbrace{(\partial_\mu ... \partial_\nu)}_{n}\delta\Phi^I \frac{\partial\mathcal{L}}{\partial(\underbrace{\partial_\mu ... \partial_\nu}_{n}\Phi^I)} = \delta\Phi^I \frac{\delta\mathcal{L}}{\delta\Phi^I} + \partial_\alpha\Theta^\alpha[\delta\Psi^I, \Psi^I]; \tag{2.19}$$

the first term contains the Euler-Lagrange equations of motion while the second one collects the remnants of the Leibniz rule, which was applied in order to factorize the variation of fields without any derivative acting on it. In other words, this second term is nothing else than a boundary term, expressed as a total derivative, or more precisely, the divergence of a vector field density $\Theta^\alpha[\delta\Psi^I, \Psi^I]$ called the bare presymplectic potential.
Let us now discuss how to construct the field space or jet space, which consists in the fields $\{\Phi^I, g_{\mu\nu}\}$ and a set of symmetrized derivatives of fields $\{\Phi^I, g_{\mu\nu}, \partial_\alpha\Phi^I, \partial_\alpha g_{\mu\nu}, \partial_\alpha\partial_\beta\Phi^I, \partial_\alpha\partial_\beta g_{\mu\nu}, ...\}$; they are defined, for example, as

$$\frac{\partial\partial_\alpha\partial_\beta\Phi^I}{\partial\partial_\sigma\partial_\rho\Phi^J} := \delta_\alpha^{(\sigma}\delta_\beta^{\rho)}\delta_J^I. \tag{2.20}$$

The variational operator δ is defined, in analogy with (2.19), as

$$\delta := \delta\Phi^I \frac{\partial}{\partial\Phi^I} + \delta(\partial_\mu\Phi^I)\frac{\partial}{\partial(\partial_\mu\Phi^I)} + ... + \delta(\partial_{(\mu}...\partial_{\nu)}\Phi^I)\frac{\partial}{\partial(\partial_{(\mu}...\partial_{\nu)}\Phi^I)}. \tag{2.21}$$

Using the convention according to which each field or derivative field variation is Grassmann odd, the variation operator satisfies $\delta\circ\delta = 0$; this means we can construct a cohomology with δ and every field or derivative field variation is a 1-form in jet space. It is fundamental to note that fields are abstract entities without dependence in the coordinates.

Focusing on spacetime (M, g) with atlas (U_i, ϕ_i), on each point $P \in M$, the tangent space $T_P M$ at P can be constructed and it contains vectors $v = v^\mu\partial_\mu$. The dual space of the tangent space is the so-called cotangent space $T_P^* M$ which contains 1-forms $w = w_\mu dx^\mu$ where, by definition, $dx^\mu(\partial_\nu) = \delta_\nu^\mu$. Both are endowed with the structure of vectore space. Thanks to the tangent space and its dual, we can build up the tensor product $(T_P M)^{\otimes_q^p} = T_P M \otimes ... \otimes T_P M \otimes T_P^* M \otimes ... \otimes T_P^* M$ whose objects that live inside it are the tensors. Two important subspaces are the space of alternating tensors and the space of symmetric tensors[7]. We denote with $\Omega^k(M) := (T_P M)^{\otimes_k^0}$ the space of k-forms, i.e. alternating tensors whose components have exactly k indexes[8]. We now define two important operators acting on spaces of forms. The first one is the interior product. Let us call $V \in \chi(M)$ the vector field extension of $v \in T_P M$; interior product ι_V is defined to be the map

$$\iota_V : \Omega^k(M) \to \Omega^{k-1}(M) \tag{2.22}$$

[7]In Paragraph 2.5, we will see that totally symmetric tensors are the main object of study of Higher Spin theory and we will see they can be described, in the free case, using twistor theory.

[8]Here we mean differential forms, i.e. forms whose components in every chart are differentiable functions.

which sends a k-form ω to a $(k-1)$-form $\iota_V\omega$ defined by the property that

$$(\iota_V\omega)\left(X_1,\dots,X_{k-1}\right)=\omega\left(V,X_1,\dots,X_{k-1}\right) \tag{2.23}$$

for any vector fields $X_1,\dots,X_{k-1}\in\chi(M)$. We have also at our disposal a differential operator d called the exterior derivative or de Rham differential that induces the de Rham complex[9] C^*_{dR}

$$0\to\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\Omega^2(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{D-1}(M)\xrightarrow{d}\Omega^D(M)\to 0. \tag{2.24}$$

We can now merge spacetime and the jet space together to give rise to the jet bundle and variational bicomplex [68],[69],[250]. In a sentence the variational bicomplex is a cochain complex of the differential graded algebra of exterior forms on jet manifolds of sections of a fiber bundle: lagrangians and Euler–Lagrange operators on a fiber bundle are defined as elements of this variational bicomplex while its cohomology leads to Noether's theorems. The jet bundle is locally a manifold with local coordinates $(x^\mu,\Phi^I,g_{\mu\nu},\partial_\alpha\Phi^I,\partial_\alpha g_{\mu\nu},\partial_\alpha\partial_\beta\Phi^I,\partial_\alpha\partial_\beta g_{\mu\nu},\dots)$; we refer to a (p,q)-form of this variational bicomplex if the expression on a coordinate base contains p times dx^μ and q times the fields variation and/or their derivative variation. The fields of the jet bundle are all fibers above the target spacetime manifold and taking a section of the fiber we obtain the coordinate dependent fields and their derivatives $(x^\mu,\Phi^I(x^\mu),g_{\mu\nu}(x^\mu),\partial_\alpha\Phi^I(x^\mu),\partial_\alpha g_{\mu\nu}(x^\mu),\partial_\alpha\partial_\beta\Phi^I(x^\mu),\partial_\alpha\partial_\beta g_{\mu\nu}(x^\mu),\dots)$.

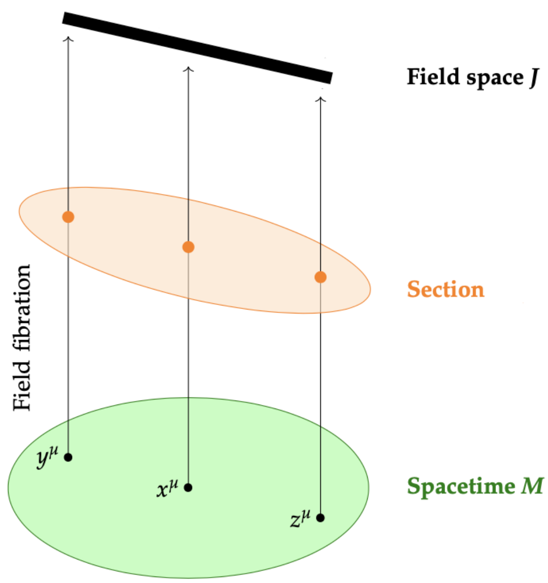


FIGURE 2.2 *Jet bundle schematic representation.*

Before moving on let us spend some time on Noether's theorems and the problems that arise in gauge theories. Noether's first theorem tells us that if we take any physical theory described by a Lagrangian density $\mathcal{L}$ defined on a spacetime manifold (M,g) that admits global and gauge symmetries then there exists a bijection between

[9] The reader not comfortable with such concepts can find information in [170] and [253].

the equivalence classes of global continuous symmetries of the theory and the equivalence classes of conserved vector fields called Noether currents. Therefore, in coordinates, two Noether currents j_1^μ and j_2^μ are equivalent if and only if they differ by a trivial current of the form

$$j_1^\mu = j_2^\mu + \partial_\mu k^{[\mu\nu]} + t^\mu \tag{2.25}$$

where $k^{[\mu\nu]}$ is an alternating tensor and $t^\mu \approx 0$. So $\partial_\mu j_1^\mu \approx \partial_\mu j_2^\mu$. Now, suppose we have a theory with only gauge symmetries, from Noether's first theorem, it exists only one equivalence class of conserved currents: the trivial one. We could, anyway, define a charge by integrating on a Cauchy slice Σ but, when the equations of motion hold, we get

$$Q = \int_\Sigma j_1^\mu d^{D-1}\Sigma_\mu \approx \int_{\partial\Sigma} k^{[\mu\nu]} d^{D-2}\sigma_{\mu\nu}; \tag{2.26}$$

therefore the charge would be manifestly completely arbitrary due to the fact that $k^{[\mu\nu]}$ is free form constraints. A solution to this issue is to define lower degree conserved charges. Indeed, let us imagine that we are able to define uniquely a closed $(D-2)$-form whose components are $k^{[\mu\nu]}d^{D-2}\sigma_{\mu\nu}$; then we can define the charge as before but now it will be conserved when we change surfaces without crossing any singularity[10] In 1995 Barnich, Brandt and Henneaux [110] proposed a generalized version of Noether theorem true for lower degree forms. The point is that given a physical theory described by a Lagrangian density $\mathcal{L}$ defined on a Lorentzian manifold (M, g) which admits global symmetries and gauge transformations, than there exists a bijection between the equivalence classes of gauge parameters that are field symmetries[11] and the equivalence classes of on-shell closed $(D-2)$-forms. In both cases the equivalence class is given by on-shell relations: in the first case on-shell equality while in the second one on-shell closure. This theorem is the core of asymptotic symmetries as we are going to discuss later on.

Now, let us came back to the variational bicomplex. Let us consider a field variation $\Phi_M \mapsto \Phi_M + \delta\Phi_M$, the lagrangian density variation can be always written as a term linear in the variations, and a total derivative

$$\delta\mathcal{L} = \mathrm{EOM}\delta\Phi_M + d\theta \tag{2.27}$$

where here $\mathcal{L}$ is considered as a top spacetime form. The first term give us the equation of motion (EOM) while in the second one θ is called pre-symplectic potential and it is a $(D-1,1)$-form. Now let be ∂M the boundary of M; obviously not all manifolds admit a boundary, in that cases we can conformally compactify M to get a conformal boundary ∂M_c and do the same with the compactified manifold M_c[12]. The variation of the action is given by

$$\delta S = \int_M \delta\mathcal{L} = \int_M (\mathrm{EOM}\delta\Phi_M + d\theta) \approx \int_{\partial M} \theta. \tag{2.28}$$

[10]An example is the source of the charge since in this case would consider a space region where charge would be produced.

[11]This means that the variations of all fields defined on the manifold vanish on-shell.

[12]The compactified manifold M_c has no the same pseudo-riemannian structure of M since lengths are modified by the conformal factor. However, M_c has the same causal structure since conformal transformations preserve angles.

The local pre-symplectic form[13] is defined as $\boldsymbol{\omega} := \delta\boldsymbol{\theta}$, which is a $(D-1,2)$-form; this local expression can be integrated on a Cauchy slice Σ to give the so-called pre-symplectic 2-form

$$\boldsymbol{\Omega} = \int_\Sigma \boldsymbol{\omega}, \tag{2.29}$$

and from its definition we have $\delta\boldsymbol{\Omega} = 0$. Let us consider a vector field $V \in \chi(J)$ such that the Lie derivative satisfies

$$L_V\boldsymbol{\omega} = 0; \tag{2.30}$$

using Cartan magic formula and the nilpotency of the differential we get

$$L_V\boldsymbol{\omega} = (i_V\delta + \delta i_V)\boldsymbol{\omega} = \delta(i_V\boldsymbol{\omega}) = 0 \tag{2.31}$$

where i_V is the analogue for the field space J of interior product (2.22). Equation (2.31) teaches us that $i_V\boldsymbol{\omega}$ is a cocycle. Therefore, assuming trivial cohomology[14] in the space of 1-forms on J

$$i_V\boldsymbol{\omega} = \delta\boldsymbol{j}_V \tag{2.32}$$

where the $(D-1,0)$-form $\boldsymbol{j}_V$ is the Noether current whose components are the Hodge dual of those we have been defined before. We can define a global functional as

$$Q_V := \int_\Sigma \boldsymbol{j}_V, \tag{2.33}$$

such that

$$i_V\boldsymbol{\Omega} = \delta Q_V. \tag{2.34}$$

The term "pre" is due to the fact that non-degeneracy is not guaranteed, and indeed there are in general zero modes, that is, non-vanishing symmetries that annihilate $\boldsymbol{\Omega}$. Once these modes are removed by quotienting these symmetries out, the symplectic 2-form carries a Poisson bracket representation of the charge algebra. Consider $V, W \in \chi(J)$, then we define the Poisson bracket between symplectomorphisms[15] as

$$\{Q_V, Q_W\} = L_W Q_V \tag{2.35}$$

and we can show that the symplectic 2-form determines the Poisson bracket of the theory, indeed

[13] Recall that a symplectic form is a closed non-degenerate differential 2-form. In our case is a 2-form on J.
[14] Said in other words we require exactness on the space of 1-forms on J.
[15] Vector fields V such that $L_V\boldsymbol{\Omega} = 0$ where L_V is the Lie derivative.

$$\{Q_W, Q_V\} = i_W i_V \mathbf{\Omega}. \tag{2.36}$$

In the case of a gauge theory with gauge symmetry, (2.33) becomes a codimension two conserved charge on-shell. Indeed, if the space-like hypersurface Σ has a codimension two surface $\sigma = \partial\Sigma$ at its boundary, we obtain

$$Q_V \approx \int_\Sigma d\boldsymbol{k}_V = \oint_\sigma \boldsymbol{k}_V \tag{2.37}$$

since $\boldsymbol{j}_V \approx d\boldsymbol{k}_V$. This is again called the Noether charge but for gauge symmetries; this is exactly the Noether charge (2.26). Given its codimension two nature, it is sometimes referred to as surface charge. The codimension two surface σ is generically called a corner of the theory, and so this charge is also called a corner charge.

A classical dynamical theory on a pseudo-riemannian manifold $(\boldsymbol{M}, \boldsymbol{g})$ with field space $\boldsymbol{J}$ is defined by giving a dynamics on the manifold, a set boundary conditions $\boldsymbol{J}|_{\partial M}$, i.e. the behaviours, also called fall-offs, of the bulk fields near the boundary and, in case of a gauge theory, a gauge fixing for the bulk fields. At this point we can compute the Noether charges and their algebra by considering the asymptotic symmetries; the crucial point is that the pre-symplectic 2-form becomes a true symplectic 2-form if restricted to the asymptotic symmetries only and the Poisson bracket of charges can be computed, and it gives rise to the algebra organizing the physical observables of the theory. The asymptotic symmetries are then defined to be

Definition 2.1.4. *The asymptotic symmetry transformations are defined as the quotient set given by*

$$\text{asymptotic symmetries transformations} = \frac{\text{residual gauge transformations}}{\text{trivial gauge transformations}}; \tag{2.38}$$

where the residual or allowed gauge transformations are the symplectomorphisms preserving fall-offs and gauge fixing while the trivial gauge transformations are those make vanishing the Noether charge and hence they are true redundancies of the theory.

Hence asymptotic symmetry transformations are those gauge transformations acting in a non-trivial way on the physical states. The trivial symmetries are also called proper or small gauge transformation. The point is that if the associated Noether charge is non-vanishing then the symmetry in question is an asymptotic symmetry, that is, a physical transformation that acts non-trivially on the field space and sends the system to a different inequivalent configuration. These are also called improper or large gauge transformations. It is interesting to note that quotient in Definition 2.1.4 has the structure of a group. We stress that the gauge fixing is a very subtle point: in principle, we should compute the conserved charges of the theory without gauge fixing, and show that the gauge symmetry needed to fix the gauge is indeed a trivial gauge symmetry transformation, hence with vanishing associated charges. A procedure in the middle between fixing a bulk gauge and not fixing the gauge in any way could be to fix an asymptotic gauge: this means to require a gauge fixing only in a neighborhood of where we are interested in computing the Noether charge.

As we are going to compute directly in the next Paragraph, asymptotic symmetry groups are typically bigger than the full bulk symmetry groups. The fundamental and historical example is Minkowski in $D = 4$, where the full bulk symmetries are given by the Poincaré group. However, the asymptotic symmetry group at null infinity has been found to be the infinite-dimensional BMS_4 group [24],[25],[26] where the four bulk translations are enhanced to the so-called supertranslations, which form an infinite dimensional Abelian algebra, while the rotation are enhanced to the so-called superrotations, which is again an infinite dimensional enhancement [154],[155],[159].

Anyway, at the end of the day, there is no guarantee that the final charges are well-defined. This is because, in general, the chosen boundary conditions or the gauge fixing procedure may lead to pathologies. A very conservative approach would define the final Noether charges associated to asymptotic symmetries to be well-defined if and only if the charges are: integrable, conserved and finite. Integrability means that equation (2.34),

$$i_V \boldsymbol{\Omega} = \delta \int_\sigma \boldsymbol{k}_V, \tag{2.39}$$

still holds; if the charge is integrable then it is called a canonical Noether charge. The second issue is the conservation, i.e.

$$Q_V|_{\Sigma(t)} - Q_V|_{\Sigma(t+dt)} = \int_{\Sigma(t)} \boldsymbol{j}_V - \int_{\Sigma(t+dt)} \boldsymbol{j}_V = \int_{\Sigma(t)} \boldsymbol{j}_V - \int_{\Sigma(t)} \boldsymbol{j}_V - \int_{\partial\Sigma(t)dt} \boldsymbol{j}_V = \int_{\sigma(t)dt} d\boldsymbol{j}_V = 0, \tag{2.40}$$

this is certainly true if $\boldsymbol{j}_V \approx d\boldsymbol{k}_V$ or in coordinates $j^\nu_V = \partial_\mu k^{[\mu\nu]}_V$, where $j^\mu_V = [\boldsymbol{j}_V]_\phi$ and $k^{[\mu\nu]}_V = [\boldsymbol{k}_V]_\phi$, modulo exact d-terms[16]. The last point is the finiteness, meaning that, as we approach the codimension two surface σ on which we integrate to build up the Noether charge, the integral does not diverge. Generally speaking finiteness is only guarantee if the codimension two surface on which we integration is located at finite distance.

If, at the end of the day, we get integrable, conserved, and finite charges, then the gauge fixing and the choosing of boundary fall-offs were non-pathological; the whole theory is well-defined. On the other hand, if charges do not satisfy these conditions then the gauge fixing and the choice of fall-offs should be modified. However, each of the features aforementioned, integrability, conservation and finiteness are questioned in every gauge theories[17], and understanding how to cure these pathologies is a very far-reaching and active area of research.

2.2 The case studies of Maxwell and Einstein theories in $D = 4$

We now discuss explicitly two elementary but important cases: Maxwell and Einstein theories. We are going to use a less formal approach in order to highlight better all the passages and allow those who are not familiar with the variational bicomplex to follow more easily. Moreover, here we are going to discuss these example in their easiest way; they can be enriched and discussed in more detail by assuming different expansions, different fall-offs, and other subtleties that may change the understanding of asymptotic symmetries in these theories.

[16]Therefore the Noether current $\boldsymbol{j}_V$ is a cohomology class on-shell.

[17]Including gravity.

2.2.1 Asymptotic charges in Maxwell theory

Let us consider electromagnetism in $D = 4$ Minkowski spacetime. As we want to study the behavior at infinity, we need to compactify Minkowski spacetime and by using Penrose conformal diagrams this is done easily. The point is that thanks to a conformal transformation (a point dependent scaling of the metric) we can take the infinitely far away "boundary" and bring it to a finite distance [27]. Obviously, conformal transformations do not preserve distances but they preserve angles and so the causal structure of Minkowski spacetime is not modified by this procedure.

In the Penrose diagram of Minkowski spacetime, Figure 2.3, we have five interesting asymptotic regions:

- i^0: this is spatial infinity, no matter can reach this point unless it is a tachion which moves faster than light;
- i^- this is past time like infinity where all massive worldlines start;
- i^+ this is future time like infinity where all massive worldlines end;
- $\mathcal{I}^-$ this is past light like, or null, infinity where all massless worldlines start;
- $\mathcal{I}^+$ this is future light like, or null, infinity where all massless worldlines end.

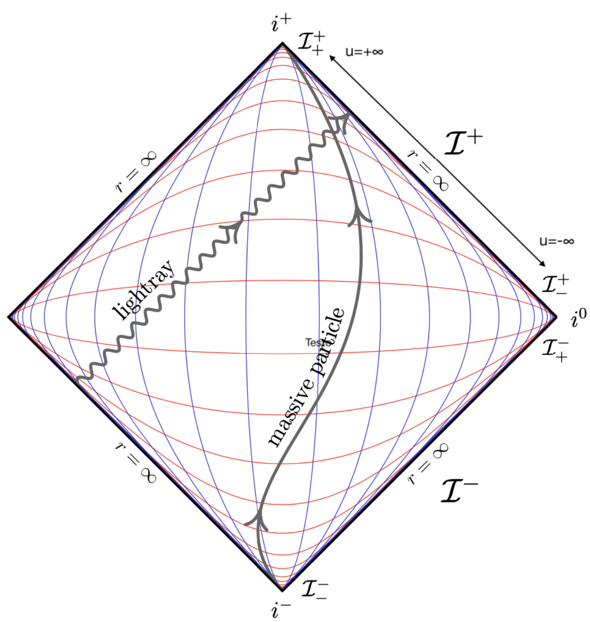


FIGURE 2.3 *Conformal Penrose diagram for Minkowski spacetime.*

Every left-right pair of points at the same (r, t) in this diagram corresponds to a 2-sphere with the exception of i^+, i^- and the straight line connecting them (this is the $r = 0$ region). For simplicity we will consider only massless particles and so we are interested only in $\mathcal{I}^+$ and $\mathcal{I}^-$; hence we introduce a coordinate systems useful to the description of these regions of Minkowski spacetime: Bondi coordinates [24],[26],[226]. Moreover, we focus on future null infinity $\mathcal{I}^+$ and so we need to introduce only retarded Bondi coordinate (advanced Bondi coordinate are useful to describe $\mathcal{I}^-$)

$$u = t - r, \quad r^2 = x_1^2 + x_2^2 + x_3^2, \quad z = \frac{x_1 + ix_2}{x_3 + r}, \quad \bar{z} = \frac{x_1 - ix_2}{x_3 + r}; \tag{2.41}$$

Minkowski line element reads

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{2.42}$$

Metric and inverse metric components in retarded Bondi coordinates are

$$g_{\mu\nu} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\gamma_{z\bar{z}} \\ 0 & 0 & r^2\gamma_{z\bar{z}} & 0 \end{bmatrix}; \quad g^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\gamma_{z\bar{z}}} \\ 0 & 0 & \frac{1}{r^2\gamma_{z\bar{z}}} & 0 \end{bmatrix}, \tag{2.43}$$

The line element (2.42) contains the 2-sphere thought as stereographic projection on the complex plane: $z = 0$ is the north pole, $z = \infty$ is the south pole and $\gamma_{z\bar{z}}$ is the unit sphere conformal metric. We underline that, despite we are describing flat spacetime these are curved coordinates, hence this metric has non-vanishing Christoffel symbols given by

$$\Gamma^z_{rz} = \frac{1}{r}, \quad \Gamma^z_{zz} = \frac{\partial_z\gamma_{z\bar{z}}}{\gamma_{z\bar{z}}}, \quad \Gamma^u_{z\bar{z}} = r\gamma_{z\bar{z}}, \quad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}}. \tag{2.44}$$

We are now ready to describe massless electromagnetism:

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right] = \int d^4x\mathcal{L} \tag{2.45}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and note that fields are not canonically normalized; this translates in e^2 factors in the conserved charge. Remember that electromagnetism has a $U(1)$ gauge symmetry given by $\delta_\epsilon A_\mu = \nabla_\mu\epsilon(u, r, z, \bar{z}) = \partial_\mu\epsilon(u, r, z, \bar{z})$. Varying the action with respect to A_μ gives us the equations of motion

$$\frac{\delta S}{\delta A^\nu} = 0 \Rightarrow \nabla^\mu F_{\mu\nu} = e^2 J_\nu \tag{2.46}$$

while varying the action with respect to $g_{\mu\nu}$ gives us the energy momentum tensor

$$\frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu} \Rightarrow T_{\mu\nu} = F_{\mu\alpha}F^\alpha_\nu - \frac{1}{4}g_{\mu\nu}(F^2 - 4e^2J^\alpha A_\alpha). \tag{2.47}$$

We now fix the so-called retarded radial gauge in which $A_r = 0$ and $A_u\Big|_{\mathcal{I}^+} = A_u(u, r = \infty, z, \bar{z}) = 0$ and we look at the energy stored at future null infinity

$$E(\mathcal{I}^+) = \int_{\mathcal{I}^+} \sqrt{g_{\mathcal{I}^+}} T_{\mu\nu} n^\mu t^\nu = \int_{\mathcal{I}^+} \sqrt{g_{\mathcal{I}^+}} T_{\mu\nu} g^{\mu\beta} \partial_\beta r \delta^\nu_u = \int_{\mathcal{I}^+} \sqrt{g_{\mathcal{I}^+}} T_{\mu u} g^{\mu r}; \tag{2.48}$$

inserting the energy momentum tensor we get

$$\begin{aligned} E(\mathcal{I}^+) &= \int_{\mathcal{I}^+} \sqrt{g_{\mathcal{I}^+}} [F_{\mu\alpha} F^\alpha_u g^{\mu r} \underbrace{- \frac{1}{4} g_{\mu u} (F^2 - 4e^2 J^\alpha A_\alpha) g^{\mu r}}_{-(...)(-1)(1)-(...)(-1)(-1) = 0}] = \int_{-\infty}^{\infty} du \int_{S^2} dz d\bar{z} r^2 \gamma_{z\bar{z}} F_{\mu\alpha} F^\alpha_u g^{\mu r} = \\ &= \int_{-\infty}^{\infty} du \int_{S^2} \gamma_{z\bar{z}} dz d\bar{z} \sum_{a=z,\bar{z}} \Big[-\partial_u A_a + \partial_a A_u + \partial_r A_a \Big] \Big[\partial_u A_{\bar{a}} - \partial_{\bar{a}} A_u \Big]. \end{aligned} \tag{2.49}$$

Now, the critical point: we have to choose the fall-off condition in a proper way so that our choice is physically sensible in relation at the infinite distance energy behavior and the gauge fixing relation; this is done making a simple power low asymptotic expansion of the general form

$$A_\mu(u, r, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{A^{(n)}_\mu (u, z, \bar{z})}{r^n}. \tag{2.50}$$

One possibility is

$$\begin{cases} A_z(u, r, z, \bar{z}) &= A^{(0)}_z (u, z, \bar{z}) + \frac{1}{r} A^{(1)}_z (u, r, z, \bar{z}) + ...; \\ A_{\bar{z}}(u, r, z, \bar{z}) &= A^{(0)}_{\bar{z}} (u, z, \bar{z}) + \frac{1}{r} A^{(1)}_{\bar{z}} (u, r, z, \bar{z}) + ...; \\ A_u(u, r, z, \bar{z}) &= \frac{1}{r} A^{(1)}_u (u, z, \bar{z}) + ...; \\ A_r(u, r, z, \bar{z}) &= 0; \end{cases} \tag{2.51}$$

hence we can expand the field strength taking into account only the leading order terms, we get

$$\begin{aligned} &F_{ur} = \nabla_u A_r - \nabla_r A_u = -\nabla_r \mathcal{A}_u = -\partial_u A_u - \Gamma^\alpha_{ru} A_\alpha = \frac{1}{r^2} A^{(1)}_u \equiv \frac{1}{r^2} F^{(2)}_{ur}; \\ &F_{z\bar{z}} = \partial_z A^{(0)}_{\bar{z}} - \partial_{\bar{z}} A^{(0)}_z \equiv F^{(0)}_{z\bar{z}}; \\ &F_{uz} = \partial_u A^{(0)}_z \equiv F^{(0)}_{uz}; \quad F_{u\bar{z}} = \partial_u A^{(0)}_{\bar{z}} \equiv F^{(0)}_{u\bar{z}}; \\ &F_{rz} = -\frac{1}{r^2} A^{(1)}_z \equiv -\frac{1}{r^2} F^{(2)}_{rz}; \quad F_{r\bar{z}} = -\frac{1}{r^2} A^{(1)}_{\bar{z}} \equiv -\frac{1}{r^2} F^{(2)}_{r\bar{z}}. \end{aligned} \tag{2.52}$$

Now, the gauge choice we made do not fix completely the arbitrariness of the gauge parameter; indeed the retarded gauge choice says nothing about the behavior of $\epsilon(u, r, z, \bar{z})$ in z and $\bar{z}$ variables. The gauge parameter is determined up to an arbitrary function $\eta(z, \bar{z}) := \epsilon(u, r, z, \bar{z})\Big|_{S^2_u}$ on the null infinity sphere coordinates: we have a residual gauge symmetry $\delta_\eta A_a(u, z, \bar{z}) = \partial_a \eta(z, \bar{z})$ where $a = z, \bar{z}$ and this is a gauge symmetry acting at null infinity since it contains only z and $\bar{z}$ variables. Let us compute the associated conserved charge in a fully covariant manner

$$Q^+_\eta = \int_{\mathcal{I}^+} j^\nu n_\nu = \int_{\mathcal{I}^+} j^r; \tag{2.53}$$

where $n_\nu = \nabla_\nu r = \delta^r_\nu$ since it is the normal vector to $\mathcal{I}^+$. So we need to compute the Noether current, let us implement the Noether procedure. On the one hand we have

$$\delta_\eta \mathcal{L} = -\frac{1}{e^2}\sqrt{-g}\underbrace{[(\nabla_\mu \delta_\eta A_\nu)F^{\mu\nu}]}_{(\nabla_\mu\nabla_\nu\eta)F^{\mu\nu}=0} - \sqrt{-g}J^\mu\delta_\eta A_\mu = -\sqrt{-g}J^\mu\nabla_\mu\eta = -\partial_\mu(\sqrt{-g}J^\mu\eta); \tag{2.54}$$

while, on the other hand we have

$$\begin{aligned}\delta_\eta \mathcal{L} &= -\frac{1}{e^2}[(\nabla_\mu \delta_\eta A_\nu)F^{\mu\nu}]\sqrt{-g} - \sqrt{-g}J^\mu\delta_\eta A_\mu = \\ &= -\frac{1}{e^2}\partial_\mu(\sqrt{-g}F^{\mu\nu}\delta_\eta A_\nu) + \frac{1}{e^2}\sqrt{-g}\underbrace{(\nabla_\mu F^{\mu\nu} - e^2 J^\nu)}_{\approx 0}\delta_\eta A_\nu \approx -\frac{1}{e^2}\partial_\mu(\sqrt{-g}F^{\mu\nu}\nabla_\nu\eta);\end{aligned} \tag{2.55}$$

where we have done extensive use of the identity $\sqrt{-g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{-g}V^\mu)$ and of the fact that the current is covariantly conserved. Now we have to equate those variations and we get

$$\begin{aligned}\partial_\mu\left[\frac{1}{e^2}\sqrt{-g}\Big(-F^{\mu\nu}\nabla_\nu\eta + e^2 J^\mu\eta\Big)\right] &\approx \partial_\nu\left[\frac{1}{e^2}\sqrt{-g}\Big(-F^{\nu\mu}\nabla_\mu\eta + \nabla_\mu F^{\mu\nu}\eta\Big)\right] = \\ &= \partial_\nu\left[\frac{1}{e^2}\sqrt{-g}\Big(F^{\mu\nu}\nabla_\mu\eta + \nabla_\mu F^{\mu\nu}\eta\Big)\right] = 0,\end{aligned} \tag{2.56}$$

from which we can read the Noether current

$$j^\nu \approx \frac{1}{e^2}\partial_\mu(\sqrt{-g}F^{\mu\nu}\eta). \tag{2.57}$$

We note that as we expected from Noether second theorem, which holds for gauge theories, that in this case of Maxwell electrodynamics, i.e. 1-form gauge theories, we get a Noehter current which is a total derivative. Plugging this into the Noether charge gives us

$$\begin{aligned}Q^+_\eta &= \int_{\mathcal{I}^+} j^r = \frac{1}{e^2}\int_{\mathcal{I}^+}\partial_\mu(\underbrace{\sqrt{-g}}_{r^2\gamma_{z\bar{z}}}F^{\mu r}\eta) = \frac{1}{e^2}\left(\int_{\mathcal{I}^+_+} - \int_{\mathcal{I}^+_-}\right)n_\mu r^2\gamma_{z\bar{z}}F^{\mu r}\eta = \\ &= \frac{1}{e^2}\left(\int_{\mathcal{I}^+_+} - \int_{\mathcal{I}^+_-}\right)r^2\gamma_{z\bar{z}}F^{ur}\eta = \frac{1}{e^2}\left(\int_{\mathcal{I}^+_+} - \int_{\mathcal{I}^+_-}\right)\gamma_{z\bar{z}}F^{(2)}_{ur}\eta\end{aligned} \tag{2.58}$$

where we used that the normal vector to $\mathcal{I}^+_+$ and $\mathcal{I}^+_-$ is $n_\mu = \delta^u_\mu$ and only the leading order term of the field strength is taking into account; we have also lowered the indexes using the inverse metric. We now assume that $F_{ur}|_{\mathcal{I}^+_+} = 0$ since there are no massive charged particles that can reach $\mathcal{I}^+_+$, and so

$$Q^+_\eta = -\frac{1}{e^2}\int_{\mathcal{I}^+_-} \gamma_{z\bar{z}} F^{(2)}_{ur}\eta; \tag{2.59}$$

but since $F^{(2)}_{ur} \equiv A^{(1)}_u$ we can determine it using the leading order equation of motion for $\nu = u$. To this scope let us define

$$J^{(2)}_u := \lim_{r\to\infty} r^2 J_u \tag{2.60}$$

which depends only on u, z and $\bar{z}$. It is, essentially, the second order term in the asymptotic expansion of J_u. The equation of motion with $\nu = u$ reads

$$\nabla^\alpha F_{\alpha u} = -\nabla^\alpha F_{u\alpha} = e^2 J_u, \tag{2.61}$$

therefore we get

$$-\nabla^z F_{uz} - \nabla^{\bar{z}} F_{u\bar{z}} - \nabla^r F_{ur} = e^2 J_u. \tag{2.62}$$

The LHS of the above equation can be made explicit as

$$\begin{aligned}
&-g^{\alpha z}\nabla_\alpha F_{uz} - g^{\alpha\bar{z}}\nabla_\alpha F_{u\bar{z}} - g^{\alpha r}\nabla_\alpha F_{ur} = -g^{\bar{z}z}\nabla_{\bar{z}} F_{uz} - g^{z\bar{z}}\nabla_z F_{u\bar{z}} - g^{ur}\nabla_u F_{ur} = \\
&= -\frac{1}{r^2\gamma_{z\bar{z}}}[\partial_{\bar{z}} F_{uz} - \Gamma^\alpha_{\bar{z}u} F_{\alpha z} - \Gamma^\alpha_{\bar{z}z} F_{u\alpha} + \partial_z F_{u\bar{z}} - \Gamma^\alpha_{zu} F_{\alpha\bar{z}} - \Gamma^\alpha_{z\bar{z}} F_{u\alpha}] - (-1)[\partial_u F_{ur} - \Gamma^\alpha_{uu} F_{\alpha r} - \Gamma^\alpha_{ur} F_{u\alpha}] = \\
&= -\frac{1}{r^2\gamma_{z\bar{z}}}[\partial_{\bar{z}} F_{uz} - \Gamma^u_{\bar{z}z} F_{uu} - \Gamma^r_{\bar{z}z} F_{ur} + \partial_z F_{u\bar{z}} - \Gamma^u_{z\bar{z}} F_{uu} - \Gamma^r_{z\bar{z}} F_{ur}] + \partial_u F_{ur} = \\
&= -\frac{1}{r^2\gamma_{z\bar{z}}}[\partial_{\bar{z}}\partial_u A^{(0)}_z + \underbrace{r\gamma_{z\bar{z}}\frac{A^{(1)}_u}{r^2}}_{\text{sub-leading}} + \partial_z\partial_u A^{(0)}_{\bar{z}} + \underbrace{r\gamma_{z\bar{z}}\frac{A^{(1)}_u}{r^2}}_{\text{sub-leading}}] + \frac{1}{r^2}\partial_u A^{(1)}_u
\end{aligned} \tag{2.63}$$

therefore we have

$$-\frac{1}{r^2\gamma_{z\bar{z}}}[\partial_{\bar{z}}\partial_u A^{(0)}_z + \underbrace{r\gamma_{z\bar{z}}\frac{A^{(1)}_u}{r^2}}_{\text{sub-leading}} + \partial_z\partial_u A^{(0)}_{\bar{z}} + \underbrace{r\gamma_{z\bar{z}}\frac{A^{(1)}_u}{r^2}}_{\text{sub-leading}}] + \frac{1}{r^2}\partial_u A^{(1)}_u = e^2 J_u; \tag{2.64}$$

which can be rewritten as

$$-[\partial_u(\partial_{\bar{z}} A^{(0)}_z + \partial_z A^{(0)}_{\bar{z}})] + \gamma_{z\bar{z}}\partial_u A^{(1)}_u = r^2\gamma_{z\bar{z}} e^2 J_u, \tag{2.65}$$

from which

$$\gamma_{z\bar{z}}\partial_u A_u^{(1)} = \partial_u(\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}) + \gamma_{z\bar{z}} e^2 J_u^{(2)} \Rightarrow A_u^{(1)} = \frac{\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}}{\gamma_{z\bar{z}}} + \int_{-\infty}^{+\infty} du e^2 J_u^{(2)}. \tag{2.66}$$

Hence plugging in, we get

$$\begin{aligned} Q_\eta^+ &= -\frac{1}{e^2}\int_{\mathcal{I}_-^+} \gamma_{z\bar{z}} A_u^{(1)} \eta \approx -\frac{1}{e^2}\int_{\mathcal{I}_-^+} \gamma_{z\bar{z}} \left(\frac{\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}}{\gamma_{z\bar{z}}} + \int_{-\infty}^{+\infty} du e^2 J_u^{(2)} \right) \eta = \\ &= -\frac{1}{e^2}\int_{\mathcal{I}^+} [\underbrace{\partial_u(\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)})}_{Q_S^+} + \underbrace{\gamma_{z\bar{z}} e^2 J_u^{(2)}}_{Q_H^+}]\eta. \end{aligned} \tag{2.67}$$

Q_H^+ is the hard charge, proportional to the matter electromagnetic current, and Q_S^+ is the soft term linear in the electromagnetic field tensor. If we now think at the electromagnetic field decomposed in oscillatory modes ("second quantization" procedure) we may note that the soft charge contains a term of the form

$$\int_{-\infty}^{\infty} \partial_u A_z^{(0)} = \int_{-\infty}^{\infty} F_{uz}^{(0)} = \lim_{\omega \to 0} \int_{-\infty}^{\infty} F_{uz}^{(0)} e^{i\omega u} \tag{2.68}$$

that is a term that contains soft photons. If $\eta = constant$ then the soft term collapses to zero since this would be an integral of a total derivative on a manifold without boundary and we recover the electric charge. However, η is an arbitrary function of z and $\bar{z}$, for example it can be any spherical harmonic, and then we have infinite many conserved charged associated to the asymptotic symmetries of electromagnetism. Obviously, we can repeat similar calculations for Q_η^-, the charges at past null infinity $\mathcal{I}^-$, obtaining the same answer with the change $u \mapsto v$, see [257]. It is interesting to note that this theory admits an antipodal matching condition according to which $F_{rt}(\vec{x})|_{\mathcal{I}_-^+} = F_{rt}(-\vec{x})|_{\mathcal{I}_+^-}$ (this condition can be derived starting from the Lienard-Wiechert solution as we see later on) which tells us that

$$Q_\eta^+ = Q_\eta^-; \tag{2.69}$$

hence, if $\eta = constant$ we recover conservation of electric charge since we get the sum of all charges at past null infinity equal to the sum of all charges at future null infinity.

This is the classical story, at the quantum level Q_η^+ and $Q_{\eta'}^+$ form an abelian charge algebra whose asymptotic charges are supposed to act on a *out* and *in* massless charged fields $\phi^\pm$ with charge q as usual

$$[Q_\eta^\pm, \phi^\pm] = q\eta\phi^\pm; \tag{2.70}$$

moreover, they are supposed to be exact symmetries of the S-matrix [180],[182]

$$\langle out|Q_\eta^+ \mathrm{S} - \mathrm{S} Q_\eta^- |in\rangle = 0; \tag{2.71}$$

note that the asymptotic charge are equal, due to antipodal matching condition, but we act with Q^+_η on the *out* states and with Q^-_η on the *in* states since they have are expressed in appropriate variables to act so. Equation (2.71) can be rewritten as

$$\langle out|Q^+_S \mathrm{S} - \mathrm{S} Q^-_S|in\rangle = -\langle out|Q^+_H \mathrm{S} - \mathrm{S} Q^-_H|in\rangle, \tag{2.72}$$

the LHS is a scattering amplitude with soft photons insertions while the RHS is the term that gives us back the soft factor multiplied by the scatting amplitude without soft photons insertions: that is the soft photon theorem [20],[30],[202],[257].

This discussion about Maxwell theory can be applied also in the case of its supersimmetric extension: $\mathcal{N} = 1$ SUSY QED. In supersymmetric theories, every particle has a superpartner; the partner of the photon is called the photino and we might expect a soft photino theorem as a superpartner of the soft photon theorem. In fact, this is indeed the case [287]. Since the soft theorem can be thought of as a relationship between two S-matrix amplitudes for every direction of the soft particle thanks to relation (2.72), the associated asymptotic symmetry should be infinite dimensional. The soft photino theorem is thus expected to give rise to an infinite number of fermionic symmetries of the S-matrix and their generators reside in the same supermultiplet as the physically non-trivial large gauge symmetries generators, associated with the soft photon theorem, do. It is an important and outstanding problem to extend this construction to other SUSY gauge theories and SUGRA theories. Moreover, recently [312], a first example of an asymptotic fermionic symmetry in a theory with no conventional supersymmetry has been found. Indeed, considering soft electrons in massless QED at tree-level, authors find that the emission amplitude at leading order in the soft electron energy factorizes in a way similar to the soft photon case. Furthermore, they recast the soft electron factorization formula as a Ward identity of an asymptotic charge suggesting that tree-level massless QED may possess an asymptotic supersymmetry algebra. We are going to discuss more soft theorems and the relation with Ward identities in the Paragraph 2.3.1.

2.2.2 Asymptotic Killing vectors in linearized Einstein theory and BMS_4 group

Let now focus on Einstein theory of gravity: General Relativity. Let us give the following preliminary definitions [129]

Definition 2.2.1 (Asymptotically simple and weakly asymptotically simple spacetime)**.** *A Lorentzian manifold* $(\boldsymbol{M}, \boldsymbol{g})$ *is asymptotically simple if it admits a conformal compactification* $\boldsymbol{M}_c$ *such that every null geodesic in* $\boldsymbol{M}$ *has future and past endpoints on the boundary of* $\boldsymbol{M}_c$*. Moreover, we defines a weakly asymptotically simple manifold as a manifold* $\boldsymbol{M}$ *with an open set* $\boldsymbol{U} \subset \boldsymbol{M}$ *which is isometric to a neighbourhood of the boundary of* $\boldsymbol{M}_c$*, where* $\boldsymbol{M}_c$ *is the conformal compactification of some asymptotically simple manifold.*

The following definition is central in our discussion

Definition 2.2.2 (Asymptotically flat spacetime)**.** *A manifold is asymptotically flat if it is weakly asymptotically simple and its curvature tensor vanishes in a neighbourhood of the boundary of* $\boldsymbol{M}_c$*.*

Roughly speaking, an asymptotically flat spacetime is a Lorentzian manifold in which the curvature vanishes at large distances from some region, so that at large distances, the geometry becomes indistinguishable from that of Minkowski spacetime.
Let us consider the $D = 4$ dimensional Lorentzian manifold with line element, in retarded Bondi coordinates $(u, r, z, \bar{z})$, given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \underbrace{-du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}}_{\text{Minkowski}} + \underbrace{+\frac{2m_B}{r}du^2 + rC_{zz}dz^2 + \left[D^zC_{zz} + \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}D_z(C_{zz}C^{zz})\right)\right]dudz + h.c.}_{\text{perturbation}} \tag{2.73}$$

this is similar to Fefferman-Graham expansion for asymptotically AdS spacetimes [72]. Here m_B is the Bondi mass aspect[18], N_i is the angular momentum aspect which describes the angular density of energy and momentum at future null infinity, while C_{ij} is symmetric and traceless and therefore it contains two polarizations and describes gravitational radiation around future null infinity; i and j are the coordinates of the 2-sphere and D^z is the covariant derivative with respect to the metric of the sphere. The triple $\{m_B, N_i, C_{ij}\}$ contains information about the physics at future null infinity, specifically C_{ij} is the radiative datum since it gives information on the gravitational radiation at future null infinity. Moreover, we can define $N_{ij} := \partial_u C_{ij}$ which is called Bondi news tensor: physically speaking it is the gravitational analogue of the Maxwell field strength since its square is proportional to the energy flux across null infinity.

Since we are interested in linearized gravity we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is the metric perturbation; hence at leading order, from (2.73), we have

$$h_{\mu\nu}dx^\mu dx^\nu = \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + D^zC_{zz}dudz + h.c.\,. \tag{2.74}$$

We ask ourselves what are the gauge transformations (in General Relativity case these are diffeomophisms) that preserve the form of the metric reported in the line element (2.74); therefore we need to look at the Killing equation

$$\delta h_{\mu\nu} = \xi_{(\mu;\nu)} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2\Gamma^\rho_{\mu\nu}\xi_\rho = 0. \tag{2.75}$$

Since we want to study asymptotic symmetries, we do not need the Killing equation to always be satisfied but it is sufficient that it is satisfied asymptotically; hence we can use Minkowski Christoffel symbols.

[18]This is a mass for asymptotically flat, not stationary spacetimes. In General Relativity there is no a unique way to define mass since the "gravitational field energy" is not a part of the energy–momentum tensor and what might be identified as the contribution of the gravitational field to a total energy is part of the Einstein tensor on the other side of Einstein's equation.

Let us consider a vector field $\xi_\mu = \xi_\mu(u, r, z, \bar{z})$ and let us start with the $(\mu, \nu) = (r, r)$ equation

$$\partial_r \xi_r + \partial_r \xi_r - 2 \underbrace{\Gamma^\rho_{rr}}_{= 0} \xi_\rho = 0 \Rightarrow \partial_r \xi_r = 0 \Rightarrow \xi_r = -T(z, \bar{z}) - uF(z, \bar{z}), \tag{2.76}$$

where $T(z, \bar{z})$ and $F(z, \bar{z})$ are arbitrary functions of the 2-sphere variables. Inserting this result in the $(\mu, \nu) = (u, r)$ equation and integrate it in r, we get

$$\partial_r \xi_u + \partial_u \xi_r - 2 \underbrace{\Gamma^\rho_{ur}}_{= 0} \xi_\rho = 0 \Rightarrow \int_r \partial_r \xi_u dr = - \int_r \partial_u(-T(z, \bar{z}) - uF(z, \bar{z}))dr \Rightarrow \xi_u = rF(z, \bar{z}) - S(u, z, \bar{z}), \tag{2.77}$$

where $S(u, z, \bar{z})$ is an arbitrary integration constant function in r. Let us move our attention to the $(\mu, \nu) = (z, r)$ equation using result (2.76),

$$\partial_z \xi_r + \partial_r \xi_z - 2 \underbrace{\Gamma^\rho_{zr}}_{= \Gamma^z_{zr} = \frac{1}{r}} \xi_\rho = 0 \Rightarrow \partial_z \xi_r + \partial_r \xi_z - \frac{2}{r}\xi_z = -D_z T(z, \bar{z}) - uD_z F(z, \bar{z}) + \partial_r \xi_z - \frac{2}{r}\xi_z = 0, \tag{2.78}$$

where since $T(z, \bar{z})$ and $F(z, \bar{z})$ are scalar function partial and covariant derivative are the same. We are looking for a solution of the type $\xi_z = r^a \psi_a(u, z, \bar{z})$:

$$\begin{aligned}
&a = 0 \Rightarrow \underbrace{\partial_r \psi_0}_{= 0} - \frac{2}{r}\psi_0 = D_z T + uD_z F \Rightarrow \psi_0 = -\frac{r}{2}(D_z T + uD_z F), \qquad \text{discarded due to } r\text{-dependence;} \\
&a = 1 \Rightarrow \underbrace{\partial_r(r\psi_1)}_{\psi_1} - \underbrace{\frac{2}{r}r = \psi_1}_{= 2\psi_1} = D_z T + uD_z F \Rightarrow \psi_1 = -D_z T - uD_z F, \qquad \text{acceptable;} \\
&a = 2 \Rightarrow \underbrace{\partial_r(r^2\psi_2)}_{= 2r\psi_2} - \underbrace{\frac{2}{r}r^2\psi_2}_{= 2r\psi_2} = D_z T + uD_z F \Rightarrow \psi_2 = Y_z(u, z, \bar{z}), \qquad \text{acceptable;} \\
&a > 2 \Rightarrow \psi_a = \psi_a(r), \qquad \text{discarded due to } r\text{-dependence.}
\end{aligned} \tag{2.79}$$

Hence possible solutions are only $\xi_z = r\psi_1(u, z, \bar{z}) = -rD_z T(z, \bar{z}) - ruD_z F(z, \bar{z})$ and $\xi_z = r^2\psi_2(u, z, \bar{z}) = r^2 Y_z(u, z, \bar{z})$. Since the equation is linear, superposition principle holds and so

$$\xi_z = -rD_z T(z, \bar{z}) - ruD_z F(z, \bar{z}) + r^2 Y_z(u, z, \bar{z}). \tag{2.80}$$

Using $(\mu, \nu) = (u, z)$ and $(\mu, \nu) = (z, z)$ equations we can show that $Y_z(u, z, \bar{z})$ is a function only of 2-sphere coordinates, i.e. $Y_z(z, \bar{z})$, and that it is a conformal Killing vector for the 2-sphere (for the metric γ_{ij} of the

2-sphere). Obviously similar consideration can be done for $(\mu, \nu) = (\bar{z}, r), (\mu, \nu) = (u, \bar{z})$ and $(\mu, \nu) = (\bar{z}, \bar{z})$ equations, obtaining a function $Y_{\bar{z}}(z, \bar{z})$ that is a confomal Killing vector field for the 2-sphere. Hence for the moment we have

$$\begin{cases} \xi_r &= -T(z, \bar{z}) - uF(z, \bar{z}); \\ \xi_u &= -S(z, \bar{z}) + rF(z, \bar{z}); \\ \xi_z &= -rD_zT(z, \bar{z}) - ruD_zF(z, \bar{z}) + r^2Y_z(z, \bar{z}); \\ \xi_{\bar{z}} &= -rD_{\bar{z}}T(z, \bar{z}) - ruD_{\bar{z}}F(z, \bar{z}) + r^2Y_{\bar{z}}(z, \bar{z}); \end{cases} \tag{2.81}$$

but using $(\mu, \nu) = (z, \bar{z})$ equation we can further constrain these objects. Indeed from the equation $(\mu, \nu) = (z, \bar{z})$, that is

$$\partial_z\xi_{\bar{z}} + \partial_{\bar{z}}\xi_z - 2\Gamma^{\rho}_{z\bar{z}}\xi_\rho = \partial_z\xi_{\bar{z}} + \partial_{\bar{z}}\xi_z - 2\gamma_{z\bar{z}}r(\xi_u - \xi_r) = 0, \tag{2.82}$$

we get, using that the only non-vanishing Christoffel symbols for the 2-sphere are the holomorphic and anti-holomorphic ones $\{\Gamma^z_{zz}, \Gamma^{\bar{z}}_{\bar{z}\bar{z}}\}$; that covariant derivatives acting on a scalar function commute since they are, in the end, ordinary derivative when act on a scalar function and that $D_i = \gamma_{ij}D^j$,

$$\partial_z\Big[-rD_{\bar{z}}T - ruD_{\bar{z}}F + r^2Y_{\bar{z}}\Big] + \partial_{\bar{z}}\Big[-rD_zT - ruD_zF + r^2Y_z\Big] - 2\gamma_{z\bar{z}}r(-S + rF + T + uF) = 0. \tag{2.83}$$

Reordering the above equation we can write it as

$$r^2\Big[D_zY_{\bar{z}} + D_{\bar{z}}Y_z - 2\gamma_{z\bar{z}}F\Big] - 2ur\Big[D_zD_{\bar{z}}F + \gamma_{z\bar{z}}F\Big] - 2r\gamma_{z\bar{z}}\Big[-S + T + D^zD_zT\Big] = 0; \tag{2.84}$$

In order to be satisfied, equation (2.84) needs that the coefficient of every monomial is zero, producing the following constrains

$$F(z, \bar{z}) = \frac{D_zY_{\bar{z}}(z, \bar{z}) + D_{\bar{z}}Y_z(z, \bar{z})}{2\gamma_{z\bar{z}}}, \quad S(u, z, \bar{z}) = T(z, \bar{z}) + D^zD_zT(z, \bar{z}); \tag{2.85}$$

hence the only independent functions are $T(z, \bar{z})$ and $Y_i(z, \bar{z})$. At the end of the day, Killing vectors that preserve the asymptotic flatness are

$$\begin{cases} \xi_r &= -T(z, \bar{z}) - u\left[\frac{D_zY_{\bar{z}}(z,\bar{z}) + D_{\bar{z}}Y_z(z,\bar{z})}{2\gamma_{z\bar{z}}}\right]; \\ \xi_u &= -T(z, \bar{z}) - D^zD_zT(z, \bar{z}) + r\left[\frac{D_zY_{\bar{z}}(z,\bar{z}) + D_{\bar{z}}Y_z(z,\bar{z})}{2\gamma_{z\bar{z}}}\right]; \\ \xi_z &= -rD_zT(z, \bar{z}) - ruD_z\left[\frac{D_zY_{\bar{z}}(z,\bar{z}) + D_{\bar{z}}Y_z(z,\bar{z})}{2\gamma_{z\bar{z}}}\right] + r^2Y_z(z, \bar{z}); \\ \xi_{\bar{z}} &= -rD_{\bar{z}}T(z, \bar{z}) - ruD_{\bar{z}}\left[\frac{D_zY_{\bar{z}}(z,\bar{z}) + D_{\bar{z}}Y_z(z,\bar{z})}{2\gamma_{z\bar{z}}}\right] + r^2Y_{\bar{z}}(z, \bar{z}). \end{cases} \tag{2.86}$$

Function $T(z,\bar{z})$ parameterizes the so-called supertranslations that are angles dependent translations; they constitute an infinite enhancement of the four global translations obtained for $T \in \{1, z, \bar{z}, z\bar{z}\}$. The infinite enhancement from translations to supertranslations was the surprising result of Bondi, van der Burg, Metzner and Sachs [24],[25],[26],: the asymptotic symmetry group of General Relativity in asymptotically flat spacetimes does not reduce to that of Special Relativity not even in linearized gravity. Functions $Y_i(z,\bar{z})$ are more tricky. They are conformal killing vectors but only six of them, given by $Y_z \in \{1, z, z^2, i, iz, iz^2\}$, are globally defined; these six functions parametrize Lorentz transformations. Also in this case we have an infinite dimensional enhancement called superrotations [155],[159].

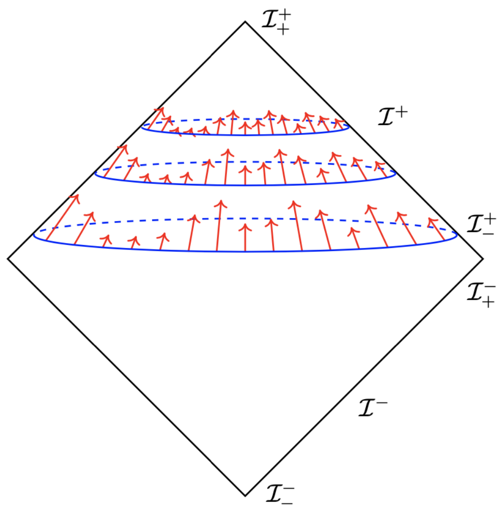


FIGURE 2.4 *Supertranslations representations; they act in a different way at any angle.*

In the end, the group of asymptotic symmetries of the asymptotic flat sector of General Relativity, even at the linearized level, is not the presumed Poincarè group but an infinite dimensional extension which, however, shares some similarities in the group structure like its semidirect product

$$\mathrm{BMS}_4 := superrotations \ltimes supertranslations. \tag{2.87}$$

This kind of analysis can be repeated for past null infinity to find another copy of BMS_4; therefore the complete asymptotic symmetry group of asymptotically flat four dimensional spacetime $\mathrm{G}_4^{\mathrm{AFS}}$ is given by

$$\mathrm{G}_4^{\mathrm{AFS}} = \mathrm{BMS}_4^+ \times \mathrm{BMS}_4^-, \tag{2.88}$$

where superscripts indicate respectively the copy at future and past null infinity. The BMS_4 group can be further generalized. On the one hand, we can consider not only the globally defined $Y_z(z,\bar{z})$ allowing for any meromorphic function on the sphere. The idea to consider also the non-globally defined vectors was proposed by Barnich and Troessaert in 2010 [152] following the path traced by Belavin, Polyakov, and Zamolodchikov [66] in the CFT realm more that twentyfive years before; this group is dubbed extended BMS_4 group. On the other hand, we can consider also Weyl rescaling generalizing the BMS_4 group to the Weyl BMS_4 (BMSW_4) group [291]. Moreover, authors show that after generalizing the Barnich-Troessaert bracket the Noether charges of the BMSW_4 group provide a centerless representation of the $\mathfrak{BMSW}_4$ Lie algebra at every cross section of null infinity. Another possible generalization goes in the direction of different signatures [137]; even if this has not direct applications to classical gravitational theories, in the study of quantum gravity spacetimes with signatures other than the usual Lorentzian one and complex spacetimes are frequently considered.

2.3 From corners to triangle through asymptotic symmetries

With the baggage of asymptotic symmetries and Noether charges seen in the previous paragraphs we can immerse ourselves into one of the most surprising and interesting ideas on the structure of field theories: the IR triangle. Moreover we are going to discuss two recent possibilities to deal with a theory of quantum gravity.

2.3.1 The infrared triangle

The IR triangle is a triangular equivalence relation that governs the IR dynamics of all theories with massless particles [257]. Each of the three corners of the triangle, reported in Figure 2.5, represents a central subject in physics that seems not to be related to the others. Instead, the tree corners are actually the same subject expressed in different notations and arrived at from very different starting points; moreover Figure 2.5 represents also the way to pass from one corner to the others.

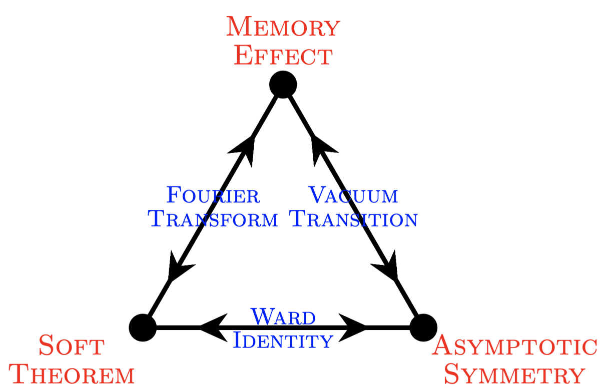


FIGURE 2.5 *The IR triangle in all its glory.*

Let us stress that the IR triangle has many and many copies throughout disparate areas of physics even if some of these triangles are not yet complete nowadays. It appears in Quantum ElectroDynamics and, more general, the contexts of Yang-Mills theories, gravity and also String Theory, SUSY QFTs and SUGRA theories [180],[189],[195],[210],[223],[227],[228],[241],[287]. Moreover the IR triangle emerges in any dimension and not only in Minkowski spacetime but also in the presence of a cosmological constant [224],[233],[234]. Probably, in the next future IR triangles will emerge also in non-maximally symmetric spacetimes and in Chapter 5 we give an example of this in the case of particular spacetimes called gravisolitons which we are going to introduce in Paragraph 2.4.

Generally speaking, the IR triangle is a classical result and, in some cases, it can be enhanced to a quantum triangle but in general it suffers anomalies [181],[188],[199],[220],[237]. An example in which it is possible to enhance the classical IR triangle to a quantum IR triangle is $\mathcal{N} = 8$ SUGRA [181]; indeed authors studied the five-point one-loop scattering amplitude and they found that the soft theorem introduced in [180] is not renormalized. However, in general the classical IR triangle cannot be enhanced and anomalies appears; these anomalies are linked to conformal anomaly. Indeed it was demonstrated in [191] that given the leading soft factor the sub-leading soft factor is determined by the conformal symmetry of tree-level amplitude. From this point of view, the loop corrections can be attributed to the fact that this symmetry can be anomalous at loop level [199],[220].

2.3.1.1 Soft theorems

Let us better study the three corners of the IR triangle. The corner of asymptotic symmetries has been discussed extensively in the previous paragraphs; let us focus first on the soft theorems corner. Soft theorems were investigated for the first time in QED in 1937 by Bloch and Nordsieck [9], were significantly developed in 1958 by Low and others [16],[17],[20] and were generalized to gravity in 1965 by Weinberg [29],[30]. Soft theorems characterize universal properties of Feynman diagrams and scattering amplitudes when a massless external particle becomes soft, i.e. its energy is taken to be zero. Despite soft theorems can be directly seen to hold at the level of S-matrix and Feynman diagrams, they can be often derived as a consequence of the invariance of the theory under a symmetry[19]. A continuous symmetry of the dynamics of a given theory is locally generated, according to Noether's theorem and in a canonical setup, by a conserved current $j^\nu(x^\mu)$. Therefore given a local operator $O(x_1^\mu, ..., x_n^\mu)$ its symmetry variation can be written as

$$\delta O(x_1^\mu, ..., x_n^\mu) = i \lim_{R\to+\infty} \int \chi_R[j^0(x^\mu), O(x_1^\mu, ..., x_n^\mu)] d\vec{x}, \tag{2.89}$$

where an event is of the form $y^\mu = (y^0, \vec{y})$ and $\chi_R[\cdot]$ is the mollification of a characteristic function of the interval $[0, R]$. In relativistic local field theories, the RHS converges by the requirement that the commutator vanishes when $x^\mu - y^\mu$ becomes space-like with respect to the region where $O(x^\mu)$ is localized; furthermore, it is also independent of time given the conservation of the Noether current. Taking the vacuum expectation value of (2.89), defining $\tilde{j}^0(\vec{k}, t)$ as the Fourier transform of the charge density, gives rise to Ward identities

$$\langle 0|\delta O(x_1^\mu, ..., x_n^\mu)|0\rangle = i \lim_{R\to+\infty} \int \chi_R \langle 0|[j^0(x^\mu), O(x_1^\mu, ..., x_n^\mu)]|0\rangle d\vec{x} = i \lim_{\vec{k}\to 0} \langle 0|[\tilde{j}^0(\vec{k}, t), O(x_1^\mu, ..., x_n^\mu)]|0\rangle. \tag{2.90}$$

[19] An example of this situation is that of the spontaneously broken (approximate) L-R symmetry of Quantum ChromoDynamics, which lies at the heart of soft-pion techniques.

Now, if the symmetry is unbroken[20], then there is actually a charge Q constructed from the Noether current such that

$$Q|0\rangle = 0 \tag{2.91}$$

and inserting it in (2.89) we get

$$\langle 0|\delta O(x_1^\mu, ..., x_n^\mu)|0\rangle = 0. \tag{2.92}$$

If the symmetry is spontaneously broken no charge Q exists and the vacuum state is not invariant under the symmetry, hence

$$\langle 0|\delta O(x_1^\mu, ..., x_n^\mu)|0\rangle \neq 0. \tag{2.93}$$

Looking at (2.93) and (2.89) we may note that they relate the vacuum expectation value of the symmetry transformation to the insertion of a soft operator. The intermediate states saturating the RHS of (2.89) must clearly have a gap-less dispersion relation, as $\vec{k} \to 0$, in order for it to be time-independent, as required by the Noether current conservation. Further inspection using the Källén-Lehmann spectral representation shows that they must be massless one-particle states: this is the Goldstone bosons theorem [21],[22],[298]. We can go further away and use LSZ reduction formulae gives rise to soft theorems if the symmetry is spontaneously broken and conservation of charge in scattering processes if the symmetry is not spontaneously broken.
Although the above considerations are in principle non-perturbative we must face the problem of taking into account possible corrections due to loop effects in order to extend their validity to the full quantum level. A possible way out is furnished by non-renormalization theorems that protect the commutators involving conserved currents from ultraviolet singularities corrections. Indeed, soft theorems have been investigated at loop level, displaying a certain degree of stability under renormalization [181]. Another important point that must be addressed in the discussion of loop-level results is the presence of infrared divergences. As known, for example, in $D = 4$ QED such infinities arise due to the masslessness of the photon and are canceled out when considering cross sections that sum over emissions/absorptions of photons below a certain energy threshold. Therefore, soft photons actually play a major role in the discussion of the infrared divergence problem.

One of the most interesting application of soft theorem are due to Weinberg [29], who showed that using Lorentz invariance, the pole structure of the S-matrix, together with masslessness and spins of the photon and of the graviton, it is possible to derive the conservation of electric charge and the universality of the gravitational coupling. Let us consider a process in which a massless particle is emitted with momentum $\vec{q}$ and integer helicity $\pm s$, limiting ourselves to integer spins. The transformation rules for S-matrix elements under Lorenz transformations Λ is

$$\mathrm{S}^{(\pm s)}(\vec{q}, p_1, ..., p_n) = \sqrt{\frac{|\vec{q}'|}{|\vec{q}|}} e^{\pm i s \alpha(\vec{q},\Lambda)} \mathrm{S}^{(\pm s)}(\vec{q}', p_1', ..., p_n') \tag{2.94}$$

where non-primed quantities are those before the Lorentz transformation while the primed ones are those after. Weinberg showed that it is always possible to write $\mathrm{S}^{(\pm s)}$ as a product of a polarization "tensor" and a symmetric

[20]Hence it admits a unitary operator U that implements the symmetry in the Hilbert space of the theory.

Lorentz tensor matrix element

$$\mathrm{S}^{(\pm s)}(\vec{q}, p_1, ..., p_n) = \frac{1}{\sqrt{2|\vec{q}|}}(\epsilon_{\pm}^{\mu_1}(\vec{q}))^* ...(\epsilon_{\pm}^{\mu_s}(\vec{q}))^* M^{(\pm)}_{\mu_1...\mu_s}(\vec{q}, p_1, ..., p_n). \tag{2.95}$$

Moreover, Weinberg showed that the single polarization "vector" transforms under a Lorentz transformation as

$$\left(\Lambda^{\mu}_{\ \nu} - \frac{q^{\mu}}{|\vec{q}|}\Lambda^{0}_{\ \nu}\right)\epsilon_{\pm}^{\nu}(\vec{q}') = e^{\pm i\alpha(\vec{q},\Lambda)}\epsilon_{\pm}^{\mu}(\vec{q}); \tag{2.96}$$

therefore it is not a true Lorentz vector, this is the meaning of the quotation marks. Using (2.95) for the transformed S-matrix and (2.96) we can rewrite (2.94) as

$$\mathrm{S}^{(\pm s)}(\vec{q}, p_1, ..., p_n) = \frac{e^{\pm is\alpha(\vec{q},\Lambda)}}{\sqrt{2|\vec{q}|}} T^{\mu_1}(\vec{q}', \Lambda)...T^{\mu_s}(\vec{q}', \Lambda) M^{(\pm)}_{\mu_1...\mu_s}(\vec{q}', p_1', ..., p_n'), \tag{2.97}$$

where

$$T^{\mu_i}(\vec{q}', \Lambda) = (\epsilon_{\pm}^{\mu_i}(\vec{q}'))^* - \frac{q'^{\mu_i}}{|\vec{q}|}\Lambda^{0}_{\ \nu}(\epsilon_{\pm}^{\nu}(\vec{q}'))^* \qquad \forall i \in [1, s]. \tag{2.98}$$

If we now consider an infinitesimal Lorentz transformation, we can linearize and make a comparison with (2.94). What we get is a condition that ensures the Lorentz invariance of $\mathrm{S}^{(\pm s)}(\vec{q}, p_1, ..., p_n)$, even if polarization "vectors" are not true vectors,

$$(\epsilon_{\pm}^{\mu_1}(\vec{q}))^* ...q^{\mu_i}...(\epsilon_{\pm}^{\mu_s}(\vec{q}))^* M^{(\pm)}_{\mu_1...\mu_s}(\vec{q}, p_1, ..., p_n) = 0 \qquad \forall i \in [1, s]; \tag{2.99}$$

therefore the S-matrix must vanish when one of the polarization "vectors" is replaced by the momentum of the massless particle. We now consider the vertex amplitude for a very low energy massless particle of integer helicity $\pm s$, emitted by the i-th particle of spin σ, mass $m^{(i)}$ (possibly vanishing) and momentum $p^{(i)}_{\mu} = (-E, \vec{p})$. The only tensor which can be used to form $M^{(\pm)}_{\mu_1...\mu_s}(\vec{q}, p_1, ..., p_n)$ is $p_{\mu_1}...p_{\mu_s}$ since terms involving the metric do not contribute to the S-matrix because of $(\epsilon^{\mu}_{\pm})^2 = 0$ and terms involving the soft momentum q^{μ} itself would either vanish by orthogonality with the polarization "vectors" or contribute to subleading orders. We define the coupling constant of a particle with the soft photon, i.e. its charge, as e and the coupling constant of a particle with the soft graviton, i.e. its gravitational charge, as f by demanding the vertex amplitudes for $s = 1$ and $s = 2$ are

$$\mathrm{S}^{(\pm 1)} = \frac{ie}{E\sqrt{4|\vec{q}|\pi}}\delta_{\sigma\sigma'} p^{(i)}_{\mu}(\epsilon^{\mu}_{\pm}(\vec{q}))^*, \qquad \mathrm{S}^{(\pm 2)} = \frac{2if}{E\sqrt{2|\vec{q}|}}\delta_{\sigma\sigma'}(p^{(i)}_{\mu}(\epsilon^{\mu}_{\pm}(\vec{q}))^*)^2. \tag{2.100}$$

Let us define $\mathrm{S}_{|in\rangle\langle out|}$ be the S-matrix for some process $|in\rangle \mapsto |out\rangle$, the states $|in\rangle$ and $|out\rangle$ consisting, respectively, in n and m charged and uncharged particles with four-momenta $p^{\text{in}}_1, ..., p^{\text{in}}_n$ and $p^{\text{out}}_1, ..., p^{\text{out}}_m$. The process is schematized in the Feynman diagram below, Figure 2.6:

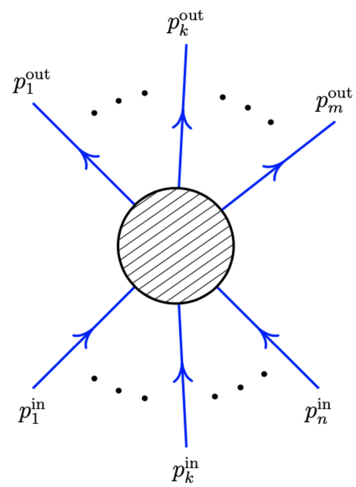


FIGURE 2.6 *Feynman diagram schematization for the process $|in\rangle \mapsto |out\rangle$.*

The same process can occur, as reported in Figure 2.7, with emission of a soft extra photon or graviton of four-momentum $q_\mu := q = (-E, \vec{q})$ and helicity ± 1, or ± 2, and we will dubbed the corresponding S-matrix element as $\mathrm{S}^{(\pm 1)}_{|in\rangle\langle out|}(\vec{q})$ and $\mathrm{S}^{(\pm 2)}_{|in\rangle\langle out|}(\vec{q})$.

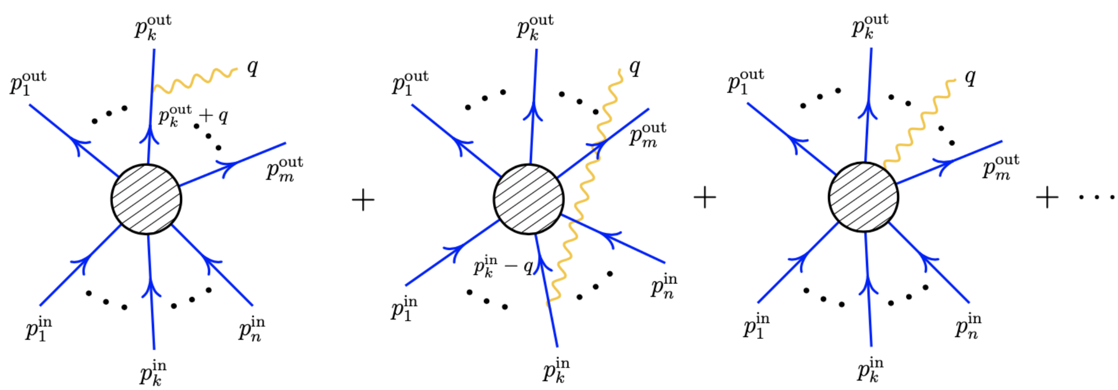


FIGURE 2.7 *Feynman diagram schematization for the process $|in\rangle \mapsto |out\rangle$ with the addition of a soft photon or graviton (wavy lines).*

The core of Weinberg soft theorems is that $\mathrm{S}^{(\pm 1)}_{|in\rangle\langle out|}(\vec{q})$ and $\mathrm{S}^{(\pm 2)}_{|in\rangle\langle out|}(\vec{q})$ factorize in $\mathrm{S}_{|in\rangle\langle out|}$ times a so-called soft factor depending from helicity and momentum of the soft particle emitted plus subleading corrections. Looking to Figure 2.7 Feynman diagrams in which the soft photon or graviton is emitted by one of the incoming or outgoing external particles, contains a propagator term of the form

$$P(p_k^{\text{in/out}}, m_k, q) = \frac{1}{(p_k^{\text{in/out}} \pm q)^2 + m_k^2} \approx \pm \frac{1}{2\left(p_k^{\text{in/out}}\right)^\mu \cdot q_\mu}, \tag{2.101}$$

where we are considering an emitting scalar particle with mass m_k, charge[21] $Q_k^{(\text{in/out})}$ and four-momentum $p_k^{\text{in/out}}$ depending on whether the particle is ingoing or outgoing in the process. Moreover, we are going on-shell since in a scattering amplitude all the external lines must be on-shell; hence $q^2 = 0$ and $(p_k^{\text{in/out}})^2 = -m_k^2$. These poles give the dominant contribution in the soft limit, while diagrams in which the soft propagator is attached to an internal line will give rise to subleading corrections. At the end of the day, in the soft limit we get

$$\begin{aligned}
S^{(\pm1)}_{|in\rangle\langle out|}(\vec{q}) &\approx \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2|\vec{q}|}}\left[\sum_{k=1}^{m} eQ_k^{(\text{out})}\frac{\left(p_k^{\text{out}}\right)_\mu(\epsilon^\mu_\pm(\vec{q}))^*}{\left(p_k^{\text{out}}\right)^\mu q_\mu} - \sum_{k=1}^{n} eQ_k^{(\text{in})}\frac{\left(p_k^{\text{in}}\right)_\mu(\epsilon^\mu_\pm(\vec{q}))^*}{\left(p_k^{\text{in}}\right)^\mu q_\mu}\right]S_{|in\rangle\langle out|};\\
S^{(\pm2)}_{|in\rangle\langle out|}(\vec{q}) &\approx \frac{(8\pi)^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\sqrt{2|\vec{q}|}}\left[\sum_{k=1}^{m} fQ_k^{(\text{out})}\frac{[\left(p_k^{\text{out}}\right)_\mu(\epsilon^\mu_\pm(\vec{q}))^*]^2}{\left(p_k^{\text{out}}\right)^\mu q_\mu} - \sum_{k=1}^{n} fQ_k^{(\text{in})}\frac{[\left(p_k^{\text{in}}\right)_\mu(\epsilon^\mu_\pm(\vec{q}))^*]^2}{\left(p_k^{\text{in}}\right)^\mu q_\mu}\right]S_{|in\rangle\langle out|}.
\end{aligned}\tag{2.102}$$

At this point, Lorentz invariance requires the vanishing of the vertex amplitude when a polarization is substituted with the corresponding four-momentum, therefore

$$\begin{aligned}
&\sum_{k=1}^{m} eQ_k^{(\text{out})} - \sum_{k=1}^{n} eQ_k^{(\text{in})} = 0, \qquad s = \pm1;\\
&\sum_{k=1}^{m} fQ_k^{(\text{out})}(p_k^{\text{out}})^\mu - \sum_{k=1}^{n} fQ_k^{(\text{in})}(p_k^{\text{in}})^\mu = 0, \qquad s = \pm2.
\end{aligned}\tag{2.103}$$

In Paragraph 2.5 we extend these relations to the case of Higher-Spins (HS). The first one is precisely the conservation of the electric charge, while the second one, when compared with the equation of total momentum conservation, i.e. $\sum_{k=1}^{m}(p_k^{\text{out}})^\mu - \sum_{k=1}^{n}(p_k^{\text{in}})^\mu = 0$, yields the universality of the gravitational coupling constant $fQ_k^{(\text{out})} = fQ_k^{(\text{in})} = 1$, for all k. This implies that, for any of the $m+n$ particles involved in the scattering with emission of soft gravitons, the gravitational mass $\tilde{m}_k$ is equal to the inertial mass m_k when the particle is at rest [29]. Indeed, by a direct calculation, the gravitational mass is given by the residue of the pole at $t = 0$ in the scattering of the particle by some other particle at rest,

$$\tilde{m}_k = \left(2E_k - \frac{m_k^2}{E_k}\right) fQ_k; \tag{2.104}$$

at rest we have $E_k = m_k$ so, using that $fQ_k = 1$, we get

$$\tilde{m}_k = 2m_k - \frac{m_k^2}{m_k} = m_k. \tag{2.105}$$

which is an explanation of the equivalence principle at the base of General Relativity.

A proven useful idea was to investigate the relation between soft theorems in gauge theories and asymptotic

[21] We mean as multiples of e if we are talking of electric coupling and as multiples of f is we are talking of gravitational coupling.

or large gauge symmetries. In paragraph 2.2.1 we discussed this point in the case of Maxwell theory; now we generalize that results. First of all, we identify the asymptotic symmetry and calculate the corresponding classical asymptotic charge Q given by (2.26). We then express the fact that the symmetry is also a symmetry of the S-matrix as

$$\langle out|Q^{+}\mathrm{S}-\mathrm{S}Q^{-}|in\rangle = 0, \tag{2.106}$$

where $Q^{\pm}$ is the charge computed at future and past null infinity $\mathcal{I}^{\pm}$. The piece of $Q^{\pm}$ that is responsible for the soft insertion is termed soft charge, $Q^{\pm}_{S}$, to be contrasted with the hard part, $Q^{\pm}_{H}$, involving the matter fields. Operators $Q^{\pm}$ are related by the antipodal matching condition; therefore, they must be evaluated by taking the limits of the same bulk symmetry transformation to the far future and far past. This condition was proposed by Strominger [257] looking at the Liénard-Wiechert solution. Indeed, Liénard-Wiechert solution has the following property: if we start at a point in the bulk, take the limit first to $\mathcal{I}^{+}$ and then to i^{0}, we get a different answer than if we take the limit to $\mathcal{I}^{-}$ and then to i^{0}. That is, the Liénard-Wiechert solution takes different values at fixed angles $\vec{x}$ on $\mathcal{I}^{-}_{+}$ and $\mathcal{I}^{+}_{-}$. Let us look more deeply to this fact; the set up is the Maxwell equation with a current given by n particles each one moving with constant velocity and with charge Q_k

$$j_{\nu}(x^{\mu}) = \sum_{k=1}^{n} Q_k \int d\tau (u_{\nu})_k \delta^{(4)}(x_{\alpha}-(u^{\alpha})_k), \tag{2.107}$$

with $\eta_{\mu\nu}(u^{\mu})_k(u^{\nu})_k = -1$ is the relativistic velocity and $\gamma_k = \frac{1}{1-\beta_k^2}$ is the Lorentz factor of the k-th particle. The electric field, computed in 1898 by Liénard and Wiechert, is

$$F_{rt}(\vec{x},t) = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k(r-t\hat{x}\cdot\vec{\beta}_k)}{|\gamma_k^2(t-r\hat{x}\cdot\vec{\beta}_k)^2-t^2+r^2|^{\frac{3}{2}}} \tag{2.108}$$

with $\vec{x}=r\hat{x}$. Using the appropriate retarded ($u=t-r$) or advanced ($v=t+r$) Bondi coordinates the limits to $\mathcal{I}^{-}_{+}$ and $\mathcal{I}^{+}_{-}$ are

$$\begin{cases} F_{rt}(\vec{x},t) &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1-\hat{x}\cdot\vec{\beta})^2} \quad \text{for } \mathcal{I}^{+}_{-};\\ F_{rt}(\vec{x},t) &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1+\hat{x}\cdot\vec{\beta})^2} \quad \text{for } \mathcal{I}^{-}_{+}; \end{cases} \tag{2.109}$$

we note that

$$\lim_{r\to+\infty} r^2 F_{rt}(\vec{x},u) = \lim_{r\to+\infty} r^2 F_{rt}(-\vec{x},v) \tag{2.110}$$

since performing u-limit and the v-limit is not necessary. Condition (2.110) is the so-called antipodal matching condition. For a general continuous and bounded operator $O(\vec{x},r,t) : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$ one can generalize the antipodal matching condition as

$$O(\vec{x},r,t)|_{\mathcal{I}^{+}_{-}} = O(-\vec{x},r,t)|_{\mathcal{I}^{-}_{+}}, \tag{2.111}$$

where the evaluations at $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ are taken in account by the r, u, v limits. This is a map from the celestial sphere at i^0 to itself. Now, the questions, not yet fully answered in the literature, are the following: "Why is there this antipodal matching condition? From what structure does it emerge? Why exactly i^0?". My proposal is that the antipodal matching condition arises from the topological structure[22] while the fact that this happens exactly in i^0 is due to the metric structure, specifically the conformal structure. The central result in this perspective is the Borsuk-Ulam theorem [142].

Theorem 2.3.1 (Borsuk-Ulam)**.** *Let $f : S^n \to \mathbb{R}^n$ be a continuous map then there exists an $x \in S^n$ such that $f(-x) = f(x)$.*

The proof of this theorem lies in algebraic topology frames and it goes beyond the scope of this Section; the interested reader can find it in [142]. For us the crucial point is the following: let us take continuous and bounded operator $O(\vec{x}, r, t)$, then $O^+(\vec{x}, r, t)|_{\mathcal{I}^+_-}$ is computed by using the u-coordinate and taking r, u limits, but the operator is continuous so

$$O(\vec{x}, r, u)|_{\mathcal{I}^+_-} = \lim_{(r,u)\to(\infty,-\infty)} O(\vec{x}, r, u) = O(\vec{x}, \infty, -\infty); \tag{2.112}$$

now $O(\vec{x}, \infty, -\infty)$ is a continuous map from $S^2_{i^0}$ to $\mathbb{R}^2$ so Borsuk-Ulam theorem applies to it. Therefore it exists $\vec{x} \in S^2_{i^0}$ such that $O(\vec{x}, \infty, -\infty) = O(-\vec{x}, \infty, -\infty)$. Hence we can write

$$O(\vec{x}, r, u)|_{\mathcal{I}^+_-} = O(-\vec{x}, r, u)|_{\mathcal{I}^+_-}; \tag{2.113}$$

if we now express the RHS in terms of the coordinate v we get exactly the antipodal matching condition (2.111). At this point we can ask why we need to take the restriction exactly at i^0; the answer can be given using the conformal structure of the problem. Indeed, looking also at Figure 2.8, we can note that compactified Minkowski spacetime can be embedded into the cylinder $S^3 \times \mathbb{R}$; it is wrapped on the cylinder and it is glued at i^0. Moreover, the null generators of $\mathcal{I}$ start at past null infinity $\mathcal{I}^-$ and pass through spatial infinity i^0 to future null infinity $\mathcal{I}^+$. Hence, it seems particularly suitable to apply Borsuk-Ulam theorem at this point.

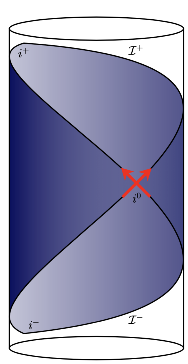


FIGURE 2.8 *Embedding of compactified Minkowski into the cylinder.*

[22]Note that topologically speaking $\mathbb{R}^{3,1} \cong \mathbb{R}^4$. Moreover, note that if $\vec{x} = \phi(\varphi, \theta)$ then $-\vec{x} = \phi(\varphi, \theta \pm \pi)$.

2.3.1.2 Memory effects

Memory effects are a class of observable phenomena that characterize the passage of radiation that interferes with a test charge and whose effect persists even when the radiation is extinguished. For example, a pair of test masses may undergo a non-vanishing relative displacement after the passage of gravitational radiation or a small electric charge initially at rest, may display a non-vanishing velocity after it is hit by electromagnetic radiation. These effects have also been proposed in Yang-Mills theories and in the context of the interaction of a 2-form with a test string and they can be also stored in the quantum states of superconducting electric and color condensates [241],[259],[260],[266]. Memory effects can be of two kind. Linear or ordinary memory effect is induced on a particle near null infinity by the radiation emitted by the movement of charged sources in the interior of the spacetime, while a non-linear or null memory effect is induced by the outflow of charged massless matter, which travels along null rays.

Let us discuss in an explicit way the $D = 4$ gravitational case. In this case, the memory effect characterizes a pair of inertial detectors stationed near $\mathcal{I}^+$ in a region with no Bondi news at both late and early times. At intermediate times, gravitational waves may pass causing oscillating distortions in their relative separations, which we denote $(s^z, s^{\bar{z}})$. In retarded Bondi coordinates, the equation of geodesic deviation, i.e. Jacobi equation, implies

$$r^2 \gamma_{z\bar{z}} \partial_u^2 s^{\bar{z}} = -R_{uzuz} s^z = -s^z \frac{r}{2} \partial_u^2 C_{zz}. \tag{2.114}$$

Integrating this equation reveals a DC effect since the initial and final separations, represented schematically in Figure 2.9, differ by

$$\Delta s^{\bar{z}} = \frac{\gamma^{z\bar{z}}}{r} s^z \Delta C_{zz}. \tag{2.115}$$

There are a numbers of proposals to measure the gravitational memory using, for example, LISA, even if the memory effect is harder to see than gravitational waves themselves since its observable effect is far smaller than that of a gravitational waves.

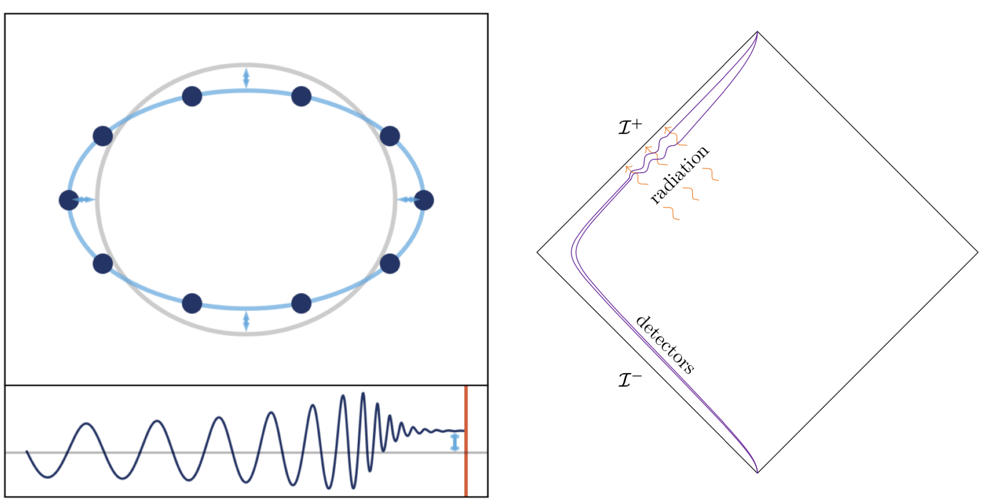


FIGURE 2.9 *Schematic representation of gravitational memory effect.*

Aside from being physically interesting in their own right, memory effects can be interpreted as observable effects associated to asymptotic symmetries. More precisely, we observe that the configurations of the system before and after the passage of radiation are mapped into one another by the action of a large gauge symmetry transformation; it is this transition between inequivalent radiative vacua that can be held responsible for a non-trivial memory effect. A simple case where this is evident is electromagnetism in radial gauge in $D = 4$, where we consider solutions to the Maxwell field equations subject to $A_r = 0$ and fall-offs at null infinity given by

$$A_u \sim \mathcal{O}\left(\frac{1}{r}\right), \quad A_z \sim A_{\bar{z}} \sim \mathcal{O}(r^0). \tag{2.116}$$

The asymptotic symmetries of this system are given by gauge parameters $\epsilon(z, \bar{z})$ that depend only on the angular coordinates; therefore they generate gauge transformations

$$\delta_\epsilon A_z = \partial_z \epsilon(z, \bar{z}), \qquad \delta_\epsilon A_{\bar{z}} = \partial_{\bar{z}} \epsilon(z, \bar{z}). \tag{2.117}$$

For a generic solution of the field equations, a test particle with unit electric charge which is initially at rest and a large radial distance from the origin, will feel a leading-order electric field given by

$$F_{uz} \sim \partial_u A_z, \qquad F_{u\bar{z}} \sim \partial_u A_{\bar{z}}, \qquad F_{ur} \sim \partial_r A_u. \tag{2.118}$$

Assuming now the test particle is subject to a radiation pulse with compact support between retarded times u_0 and u_1, it experiences a momentum kick

$$\Delta P_z = \int_{u_0}^{u_1} F_{uz} du = A_z\Big|_{u_1} - A_z\Big|_{u_0} + \mathcal{O}\left(\frac{1}{r}\right), \quad \Delta P_{\bar{z}} = \int_{u_0}^{u_1} F_{u\bar{z}} du = A_{\bar{z}}\Big|_{u_1} - A_{\bar{z}}\Big|_{u_0} + \mathcal{O}\left(\frac{1}{r}\right); \tag{2.119}$$

where we assumed that $|u_1 - u_0|$ is sufficiently small and allows us to neglect the contribution due to the magnetic field[23]. However, before and after the passage of radiation, the particle must be in a radiative vacuum configuration and hence there must exist a gauge parameter $\epsilon(z, \bar{z})$ such that

$$A_z\Big|_{u_1} - A_z\Big|_{u_0} = \partial_z \epsilon(z, \bar{z}), \qquad A_{\bar{z}}\Big|_{u_1} - A_{\bar{z}}\Big|_{u_0} = \partial_{\bar{z}} \epsilon(z, \bar{z}). \tag{2.120}$$

We thus finally see that the momentum kick memory effect can be interpreted as the action of a large gauge transformation on the underlying gauge field, which connects two different vacua of the theory. In a similar way but with some technical issues to fix, all memory effects can be interpreted as a vacuum transition where the two vacua are related by an asymptotic symmetry transformation. Moreover, soft theorems are a statement about momentum space poles in scattering amplitudes, while the memory effects concern a DC shift in asymptotic data between late and early times. These are in fact the same thing, because the Fourier transform of a pole in

[23]Note that magnetic field is further suppressed by the particle's velocity.

frequency space is a step function in time; this step function can be understood as a domain wall connecting two inequivalent vacua that are related by an asymptotic symmetry transformation.

2.3.2 Towards quantum gravity: celestial holography and the corner proposal

A theory of quantum gravity is one of the major issue in modern physics. Already at a classical level, gravity presents important problems, such as the impossibility of describing the energy of the gravitational field itself through a tensor object. This is, essentially, the main problem with gravity: the energy associated to the gravitational field is itself source of the gravitational field. This peculiarity of gravity leads to a perturbative non-renormalizable theory when we try to quantize it in the same manner we quantize other fundamental interactions described by the Standard Model of Particle Physics. The point is that a perturbative quantum theory must be characterized by a choice of finitely many parameters, which could, in principle, be set by experiments. In the case of perturbative quantum gravity we have infinitely many independent parameters needed to define the theory[24]. Let us explain better the point of non-renormalizability of gravity and, to fix ideas, let us consider the $D = 4$ dimensional case. In order to make dimensionless the Einstein-Hilbert action,

$$S_{\text{EH}} = -\frac{1}{2k}\int_M \sqrt{-g}R\, d^4x, \tag{2.121}$$

we need $\frac{1}{k}$ has mass dimension $[m^{-2}]$. If we now do a perturbative expansion around a flat background of the metric, we will encounter at each step more and more powers of $\frac{1}{k}$. Graphically speaking, this expansion is an expansion in numbers of loops in Feynman diagrams. At each step, i.e. at each loop level, the whole expression should be dimensionless hence at each step we need more and more powers of loop momentum (at each loop level two more powers, to be precise). In the end the final expressions become the more divergent the further away we go in the perturtabive expansion. In order to cancel these ever sickening divergences we have to introduce an infinite number of counterterms which which causes the theory to lose predictive power. Hence this theory is, by powercouting, non-renormalizable. To know more about the non-renormalizability of gravity we refer to [145].

For a given choice of those parameters we could make sense of the theory, but since it is impossible to conduct infinite experiments to fix the values of every parameter, this theory looses predictability. Despite that, when we look at low energy effects all but few of the infinite set of parameters are suppressed by huge energy scales[25] and hence can be neglected when computing low-energy effects. Therefore, perturbative quantum gravity is a predictive Quantum Field Theory if we treat it as an Effective Field Theory (EFT) [92]. Using perturbative quantum gravity, legitimate predictions can be made for quantum gravity such as the computation of the quantum gravitational corrections to classical black holes metrics [116],[121],[328] and to classical thermodynamic properties of black holes, most importantly the entropy. Moreover, an interesting relation, called double copy, exists and hypothesizes a perturbative duality between gauge theory and gravity. The double copy approach

[24] These are the counterterms we need to subtract all the infinities.

[25] In the case of gravity by the Planck energy scale, M_{PL}, that is $\mathcal{O}(10^{19})$ GeV.

[261],[302],[322] says that scattering amplitudes in non-Abelian gauge theories can be factorized such that replacement of the color factor by additional kinematic dependence factor, in a well-defined way, automatically leads to gravity scattering amplitudes. This duality can be used to make calculations of gravity scattering amplitudes simpler by instead calculating the Yang–Mills amplitude and following the double copy prescription. This technique has been used, for example, to calculate the shape of gravitational waves emitted by two merging black holes and this was proven to work at tree level and at higher orders until the fourth post-Minkowskian order. Recently the double copy was attacked from the Lagrangian point of view and authors in [324] built up a matching of the asymptotic symmetries of the two parts.

Beside the use of perturbative quantum gravity as an EFT, there is the possibility that gravity as a Quantum Field Theory could be non-perturbative renormalizable. This approach, originally proposed by Weinberg and known as asymptotic safety[26][48],[62]; has as its key ingredient the presence of a non-trivial fixed point of the theory's renormalization group flow which controls the behavior of the coupling constants in the UV regime and protect physical quantities from divergences. In an asymptotically safe theory, the running of the coupling constants, i.e. their scale dependence described by the renormalization group, is such that in its UV limit all their dimensionless combinations remain finite. This is enough to avoid unphysical divergences, for example in scattering amplitudes.

The standard approaches to a full theory of quantum gravity are essentially String Theory and Loop Quantum Gravity (LQG). Then most of the other proposed paths to quantum gravity are, in same sense, declinations of these two main approaches.

On the one hand, LQG [106],[179],[183],[194],[230] is a background independent approach and it seriously considers General Relativity's insight that spacetime is a dynamical field to be quantized and so a quantum object. The quantum state of spacetime is described in the theory by means of a mathematical structure called spin networks[27] which arise naturally from a non-perturbative quantization of General Relativity. The formally background independence of LQG means the equations of LQG are not embedded in, or dependent on, spacetime pseudo-riemannian structure but the issue of background independence in LQG still has some unresolved subtleties. In fact, some derivations require a fixed choice of the topology, while any consistent quantum theory of gravity should include topology change as a dynamical process. The second main idea of LQG is that the quantum discreteness that determines the particle-like behavior of other field theories also affects the structure of spacetime. Indeed, Rovelli and Smolin showed [95] that the operators representing the area and the volume of each surface or space regions have discrete spectra, for example if Σ is a two-dimensional surface

$$A_\Sigma = 8\pi \ell_{\mathrm{PL}}^2 \gamma \sum_i \sqrt{j_i(j_i+1)}, \tag{2.122}$$

[26]The idea of a non-trivial fixed point providing a possible UV completion can be applied also to other perturbatively non-renormalizable field theories.

[27]Formally speaking, a spin network may be defined as a directed graph whose edges are associated with irreducible representations of a compact Lie group and whose vertices are associated with intertwiners of the edge representations adjacent to it.

where ℓ_{PL} is $\mathcal{O}(10^{-35})$ m, γ is the Immirzi parameter[28] and j_i is the spin associated with the link i of the spin network. Thus, area and volume of any portion of space are also quantized and the quanta are elementary "particles" of space; it follows, then, that spacetime has a sort of granularity, an elementary quantum granular structure at the Planck scale which cuts off the UV infinities problem of Quantum Field Theory in a way very similar to the Lattice Quantum Field Theory case (LQFT). Interesting application of LQG inspired models can be found in the cosmological field with the Loop Quantum Cosmology (LQC) [157] or its toy-model version known as the Polymer Quantum Mechanics (PQM) [141],[176]. In these models, the unphysical Big Bang singularity is resolved in favor of a Big Bounce due to semiclassical or quantum geometry effects, see for example [280],[281],[333].

On the other hand, String Theory refashions the fundamental degrees of freedom [143],[144],[168],[169],[175]. The central idea is to replace the classical concept of a point particle in Quantum Field Theory with a quantum theory of one-dimensional extended objects: strings. At the energies reached in current experiments, these strings are indistinguishable from point-like particles but different modes of oscillation of strings appear as particles with different charges and properties. In String Theory there is always a mode corresponding to a graviton; then it reproduces, at least, perturbatively quantum gravity. However, for the theory to be consistent there must be six extra dimensions[29] that are compactified in particular geometric structures called Calabi-Yau varieties in order to preserve supersymmetry[30]. In this sense, String Theory promises to be a unified description of all particles and interactions. The presence of an undefined number of possible Calabi-Yau three-folds, i.e. complex three-dimensional Calabi-Yau varieties, in which compactifying the extra dimensions makes the number of string voids exorbitant. This leads to the so-called string landscale problem and countless efforts have been made to understand how to reduce this number of voids. However, recently, a completely antithetic new kind of approach called swampland program was proposed [268],[303]. Its purpose is to find the set of consistent-looking theories with no consistent UV completion with the addition of gravity in order to discard the latter as possible consistent quantum gravity theories. The way swampland program proceed is to delineate the theories of quantum gravity by identifying the universal principles shared among all theories compatible with gravitational UV completion; this universal principles are called swampland conjectures. The most important ones are the no global symmetry conjecture [158],[292], according to which higher-form global symmetries have no role in a consistent quantum gravity theory; the completeness of spectrum hypothesis [131],[293], that conjectures quantum gravity has the spectrum of charges under any gauge symmetry completely realized and the weak gravity conjecture [140], which states that gravity should be the weakest force with respect other to gauge forces in any consistent theory of quantum gravity.

One of the most interesting and breakthrough result of String Theory is the explicit realization, in the case

[28] This parameter is fundamental in LQG; it measures the size of the quantum of area in Planck units. Its value is currently fixed by matching the semiclassical black hole entropy and the counting of microstates in Loop Quantum Gravity.

[29] Extra dimensions arise also in other theories; for example the bosonic version of String Theory is coherent only in 26 dimensions, the enigmatic M-theory works in 11 dimension [101] and the F-theory in 12 dimensions [102]. Recently a tentative approach to a plane theory was done and seeems to be coherent only in 57 dimensions [311].

[30] Supersymmetry requirement is fundamental to ensure that String Theory is able to contain in its oscillation spectrum also fermionic states.

of Anti de Sitter (AdS) spacetime, of the holographic principle according to which a D-dimensional quantum gravity theory is equivalently described in terms of a $(D-1)$-dimensional Quantum Field Theory living at the boundary. This realization is known as AdS/CFT correspondence where the QFT side is a Conformal Field Theory (CFT) living at the boundary of AdS, i.e. conformal Minkowski [109]. Some of the swampland conjectures have been formulated or refined in light of the results of the holographic correspondence. Moreover, the correspondence provides new techniques to study non-perturbative regimes of QFTs and was applied to a lot of cases of physical interest[31]. With the entry in the game of the AdS/CFT correspondence, a new branch has emerged: the Geometrical Engineering of Quantum Field Theories where using extensions of AdS/CFT correspondence, we can built up QFTs, with or without supersymmetry and with or without conformal symmetry, whose description can be completely expressed by geometric/topological data. In this context, between the master's thesis [295] and the PhD, I proposed a new technique to compute important physical quantities, such as superconformal central charges, [310] and I showed how some IR duality emerging from the Geometrical Engineering of QFTs can be traced back to geometric/topological transitions typical of any quantum theory of gravity [320] using concepts and tools of algebraic geometry.

In the context of quantum gravity, asymptotic symmetries are the core of some new interesting approach to quantum gravity: the celestial holography program [257],[299],[300],[301] and the corner proposal [314], which we are going to review in the following. Anyway, asymptotic symmetries play also a subtler role in quantum gravity. The subtlety arises from the fact that they are, in principle, able to discern which quantum theory of gravity is the right one and are able to do so classically. In fact, the reformulations of General Relativity (and more generally of theories of modified and/or high derivative gravity) are classically equivalent but lead to different quantum formulations; this means that these formulations have different actions and different actions are associated, in principle, to different asymptotic symmetries with different memory effects. If we were able to experimentally verify gravitational memory effects we would know whether the gravitational theory to quantize is the metric formulation of gravity or the Barbero-Immirzi formulation; or again, whether to work in Jordan or Einstein frame.

2.3.2.1 Celestial holography

Celestial holography provides a new approach to quantum gravity in asymptotically flat spacetimes by seeking to establish a holographic correspondence similar to AdS/CFT [257],[300],[301],[323]. More specifically, celestial holography proposes a duality between gravitational scatterings in asymptotically flat spacetimes of dimension D and a Conformal Field Theory living on the celestial sphere S_u^{D-2}, known as Celestial CFT (CCFT).

[31] AdS/CFT correspondence has three main forms; here we refer to the so-called weak form. In this case, the gravitational side is fully classical while the QFT side is fully quantum with large coupling constant which does not allow for perturbation theory. Hence, thanks to the holographic dictionary we can compute observables of the strongly coupled QFT side directly in the classical gravitational side [197],[295]. Among the successes of this approach, there are the applications to QCD and QCD-like theories which focus on chiral symmetry breaking and phases of QCD, to out of equilibrium physics in which the evolution of a QFT away from thermodynamical equilibrium is mapped into the dynamics of classical fields in black hole backgrounds and to information theory where the Ryu-Takayanagi formula [138],[139] is used to compute the entanglement entropy of a QFT. To know more about we refer to [304].

In this Paragraph we are going to introduce it a bit. The central point is that when recast in a basis of boost eigenstates, scattering amplitudes transform as conformal correlators of primary operators in the dual Celestial CFT. These celestial correlators appear to have some, but not all, of the properties of standard CFT correlators.

Let us focus on the $D = 4$ case which is, currently, the most investigated case in literature. Given the Lorentz group O(3, 1), the proper orthochronous Lorentz group is the subgroup of the Lorentz group given by

$$L_+^\uparrow \equiv \mathrm{SO}^+(3,1) := \{\Lambda \in \mathrm{O}(3,1) | det(\Lambda) = 1, \Lambda^0_0 \geq 1\} \tag{2.123}$$

and it preserves the orientation of space and the direction of the arrow of time. This subgroup is isomorphic to the quotient group $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}$ thanks to the map

$$(M^\dagger)_{\alpha\dot{\beta}} x^\mu (\sigma_\mu)_{\gamma\dot{\beta}} (M)_{\gamma\dot{\delta}} = (\sigma_\mu)_{\alpha\dot{\delta}} \Lambda^\mu_\nu x^\nu, \tag{2.124}$$

where $(\sigma_\mu)_{\gamma\dot{\beta}} = (\mathbb{1}, \sigma^i)_{\gamma\dot{\beta}}$ with σ^i are the Pauli matrices, x^μ is a four-vector,

$$(M)_{\alpha\dot{\beta}} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \quad a, b, c, d \in \mathbb{C} \quad \text{s.t. } ab - cd = 1, \tag{2.125}$$

and

$$\Lambda^\mu_\nu = \frac{1}{2}\begin{bmatrix} a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} & a\bar{b} + b\bar{a} + d\bar{c} + c\bar{d} & i(a\bar{b} - b\bar{a} + c\bar{d} - d\bar{c}) & -a\bar{a} + b\bar{b} - c\bar{c} + d\bar{d} \\ a\bar{c} + c\bar{a} + b\bar{d} + d\bar{b} & a\bar{d} + d\bar{a} + b\bar{c} + c\bar{b} & i(a\bar{d} - d\bar{a} - b\bar{c} + c\bar{b}) & -a\bar{c} - c\bar{a} + b\bar{d} + d\bar{b} \\ i(-a\bar{c} + c\bar{a} - b\bar{d} + d\bar{b}) & i(-a\bar{d} + d\bar{a} - b\bar{c} + c\bar{b}) & a\bar{d} + d\bar{a} - b\bar{c} - c\bar{b} & i(a\bar{c} - c\bar{a} - b\bar{d} + d\bar{b}) \\ -a\bar{a} - b\bar{b} + c\bar{c} + d\bar{d} & -a\bar{b} - b\bar{a} + c\bar{d} + d\bar{c} & i(-a\bar{b} + b\bar{a} + c\bar{d} - d\bar{c}) & -a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d} \end{bmatrix} \tag{2.126}$$

is a Lorentz matrix in $\mathrm{SO}^+(3,1)$, i.e. it satisfies $det(\Lambda) = 1$ and $\Lambda^0_0 \geq 1$. The quotient is taken into account since (2.124) is invariant under $M \mapsto -M$. This is the Lorentzian metric analogue of the group isomorphism between SO(3) and $\frac{\mathrm{SU}(2)}{\mathbb{Z}_2}$.

To connect conformal transformations to the action of the Lorentz group on the celestial sphere we recast the transformation $x^\mu \mapsto \Lambda^\mu_\nu x^\nu$, with Λ^μ_ν expressed as in (2.126), in Bondi coordinates and take the large r limit. We find that

$$z \mapsto \frac{az + b}{cz + d} + \mathcal{O}\left(\frac{1}{r}\right), \tag{2.127}$$

hence Lorentz transformations coincide with conformal Moebius transformations of the celestial sphere.

Asymptotic plane wave solutions to the free wave equations have manifest translation symmetry but the Lorentz invariance, expressed in terms of $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}$, is not obvious since plane waves transform non-trivially into each other. Therefore, we would like to find a basis of asymptotic states that recasts the S-matrix in a $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}$ covariant form. Let us start with massless particles. A null vector is labelled by a point on the sphere up to an overall magnitude; therefore, there is a natural map from null vectors to points on the sphere towards which the null vector is directed. In fact, given a null momentum vector $k^\mu = \omega q^\mu$, with $k^0 = \omega$ and $q^\mu q_\mu = 0$, we can parameterize it as

$$k^\mu = \frac{\omega}{1+z\bar{z}}(1+z\bar{z}, z-\bar{z}, i(\bar{z}-z), 1-z\bar{z}), \tag{2.128}$$

where $(z,\bar{z})$ are the stereographic projection coordinates on the celestial sphere S^2_u which are related with the standard Archimede coordinates by

$$\begin{aligned} z &= tan\left(\frac{\theta}{2}\right)e^{i\phi}, \\ \bar{z} &= tan\left(\frac{\theta}{2}\right)e^{-i\phi}. \end{aligned} \tag{2.129}$$

A massless spin-s particle in $D=4$ is labelled by its null momentum four vector which defines a null cone of momentum directed at a point $(z,\bar{z})$ on the celestial sphere. Thus for massless particles we only have to trade ω for $\Delta = h+\bar{h}$[32]; this can be done using the Mellin transform in the energy

$$M(\cdot) := \int_0^{+\infty} \frac{d\omega}{\omega}\omega^\Delta(\cdot). \tag{2.130}$$

For example, the on-shell plane waves $e^{\pm i\omega q^\mu x_\mu}$ is mapped to

$$\phi^{(\pm)}_\Delta(x^\mu, z, \bar{z}) := \int_0^\infty \frac{d\omega}{\omega}\omega^\Delta e^{\pm i\omega q^\mu x_\mu} = \frac{(\pm i)^\Delta \Gamma(\Delta)}{\left(q^\mu x^{(\pm)}_\mu\right)^\Delta}, \tag{2.131}$$

where $x^{(\pm)}_\mu = x_\mu \pm i\epsilon(-1,0,0,0)$ is a convergence regulator. The massless scalar wave function (2.131) transforms as a $D=4$ spin $s=0$ field under Lorentz transformations expressed in terms of $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}$ and as $D=2$ conformal primary under Moebius transformations with conformal dimension Δ. This can be used as starting point in definition of conformal primary wave functions

Definition 2.3.1 (Conformal primary wave functions)**.** *Let $R_s(\Lambda)$ be the spin-s representation of the Lorentz algebra. Conformal primary wave functions are functions on $\mathbb{R}^{3,1}\times\bar{\mathbb{C}}$, depending on a spacetime vector $x^\mu \in \mathbb{R}^{3,1}$ and a point $(z,\bar{z}) \in \bar{\mathbb{C}}$, which transform under simultaneous Lorentz transformations of x^μ and Moebius transformation of $(z,\bar{z})$ as $D=2$ conformal primaries with conformal dimension Δ and spin J*

$$\Phi^s_{\Delta,J}\left(\Lambda^\mu_\nu x^\nu, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{\Delta+J}(\bar{c}\bar{z}+\bar{d})^{\Delta-J} R_s(\Lambda)\Phi^s_{\Delta,J}(x^\mu, z, \bar{z}). \tag{2.132}$$

There exist two types of conformal primary wave functions. Radiative conformal primary wave functions

[32]Recall that, in a CFT, h is the eigenvalue of Virasoro generator L_0 while $\bar{h}$ is the eigenvalue of Virasoro generator $\bar{L}_0$. Moreover, recall that the generic Virasoro generator satisfy $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$ where c is the central charge.

have $|J| = s$ and satisfy the equations of motion for massless spin-s fields in vacuum, while this condition is relaxed for generalized conformal primary wave functions which have $|J| \leq s$. In the following we are going to consider only radiative conformal primary wave functions. In order to find scattering solutions to the massless spin-s equation, we can start from the scalar solution to the massless Klein-Gordon equation given by the plane wave with null momentum, Mellin transform it and find the appropriate polarization spinor/vector/tensor to satisfy the considered equation. These radiative conformal primary wave functions have been shown [239] to form a complete distibution-like normalizable basis when their conformal dimension lies on the principal continuous series $\Delta \in \text{PCS} := \{k \in \mathbb{C} | \text{Re}(k) = 1\}$. Another basis[33] of spin-s radiative conformal primary wave functions is obtained from the so-called shadow transformation which maps a radiative conformal primary wave function to a radiative wave function of flipped and shifted conformal dimension and flipped spin

$$\tilde{\Phi}^s_{\Delta,J} = \tilde{\Phi}^s_{2-\Delta,-J}, \tag{2.133}$$

where the shadow transform is the operator defined by

$$\tilde{O}_{h,\bar{h}}(w,\bar{w}) := \frac{C_{1-h,1-\bar{h}}}{2\pi} \int dz d\bar{z} \frac{O_{1-h,1-\bar{h}}}{(w-z)^{2h}(\bar{w}-\bar{z})^{2\bar{h}}}, \tag{2.134}$$

with $C_{1-h,1-\bar{h}}$ a constant. Under the shadow transform, radiative conformal primary wave functions with $\Delta \in \text{PCS}$ get mapped to radiative conformal shadow primary wave functions with the same range of conformal dimension. We can show that radiative conformal primary wave functions defined by Mellin transform, diagonalize boost and obey the highest weight condition. Indeed, let us consider the scalar case and, for simplicity, the null vector $q^\mu = (1,0,0,1)$ pointing in the x^3 direction, i.e. corresponding to the North pole on the celestial sphere, and a regressive wave

$$\phi^{(+)}_\Delta(x^\mu, 0, 0) = \frac{(i)^\Delta \Gamma(\Delta)}{(x^3 - x^0)^\Delta}. \tag{2.135}$$

From the generators of the global conformal transformation in $D = 2$ we have

$$\begin{aligned} L_0 + \bar{L}_0 &= -iK_3 = x^0 \partial_{x^3} + x^3 \partial_{x^0}, \\ L_0 - \bar{L}_0 &= J_3 = i(x^1 \partial_{x^2} - x^2 \partial_{x^1}), \end{aligned} \tag{2.136}$$

so we can compute that

$$J_3 \phi^{(+)}_\Delta(x^\mu, 0, 0) = 0, \qquad -iK_3 \phi^{(+)}_\Delta(x^\mu, 0, 0) = \Delta \phi^{(+)}_\Delta(x^\mu, 0, 0); \tag{2.137}$$

hence, $\phi^{(+)}_\Delta(x^\mu, 0, 0)$ diagonalizes the boost in the x^3-direction.

[33] Recently yet another basis was found [327]. This base is discrete and associated to $\Delta \in \mathbb{Z}$.

At this point we can express the S-matrix for massless spin-s particles in a basis that diagonalizes boosts, dubbed $|in\rangle_{boost}$ and $|out\rangle_{boost}$, using Mellin transform; we stress that we have to apply it to each of the external particle energies. If the external particles are n, then

$$
{}_{boost}\langle out|\mathrm{S}|in\rangle_{boost} = \prod_{j=1}^{n} \int_0^\infty \frac{d\omega_j}{\omega_j} \omega_j^{\Delta_j} \langle out|\mathrm{S}|in\rangle = \left\langle \prod_{j=1}^{n} O^{(\pm)}_{\Delta_j, J_j}(z_j, \bar{z}_j) \right\rangle_{\mathrm{CCFT}}; \tag{2.138}
$$

where the correlator is computed in the Celestial CFT (CCFT). This is the celestial amplitude which defines the spin-s massless observable on the celestial sphere. At this point an interesting personal observation about the structure of the CCFT, which will be further investigated in future research works, is necessarily. If we assume that the celestial correlators are invariant under the antipodal map at i^0, which is so if every operator in the correlator satisfies the antipodal matching condition map, we could construct explicitly the CCFT. Indeed a very well known topological isomorphism comes in our aid[34] : $\frac{S^2}{\mathbb{Z}_2} \cong \mathbb{RP}^2$ where the quotient by $\mathbb{Z}_2$ is representing the action of the antipodal map on the sphere. Now, the point is that if we build up a theory on $\frac{S^2}{\mathbb{Z}_2}$ every operator valuated on $x \in S^2$ assume, by construction, the same value on $-x \in S^2$ since x and $-x$ are in the same equivalence class in the quotient: the antipodal matching condition is automatically implemented if $S^2 \equiv S^2_{i^0}$. Moreover, the antipodal matching condition is topological by nature thanks to the Borsuk-Ulam theorem, hence the homeomorphism between the 2-sphere modulo antipodal map and the real projective plane does not affect it. However, build up the theory on $\mathbb{RP}^2$ has its advantage since the real projective plane automatically implement scale invariance by definition. In the end, if we demand that a CCFT is a CFT where correlators satisfy the antipodal metching condition the right answer could be use $\mathbb{RP}^2$ as underlying manifold. The only missing point seems to be a very old standing issue of theoretical physics: it is not know in general when a scale invariant theory is also a conformal invariant theory. Under not always satisfied hypothesis we can prove that in $D = 2$ a scale invariant Quantum Field Theory necessarily possesses the enhanced conformal symmetry [192]. Moreover, recently, some advance in the study of Quantum Field Theory on real projective space have been done [278].

We briefly cover also the case of massive particles. In this case there is no direct way to relate the momenta of massive particles to points on the celestial sphere since they do not reach null infinity. Instead they are labelled by momenta $p^\mu = \pm m\hat{p}^\mu$ with $p^2 = -m^2$ and $p^0 > 1$. Hence we have to embed the hyperbolic space $\mathbb{H}_3$ into the upper branch of the unit hyperboloid in Minkowski spacetime using the unit time-like vector

$$
\hat{p}^\mu = \frac{1}{2y}(1 + y^2 + z\bar{z}, z + \bar{z}, i(\bar{z} - z), 1 - y^2 - z\bar{z}), \tag{2.139}
$$

which transform linearly under Lorentz transformations. Contrary to the massless case, in the massive case we

[34]The proof is simple; we give it for a generic dimension. Since $\frac{S^n}{\mathbb{Z}_2}$ is compact and $\mathbb{RP}^n$ is Hausdorff, it suffices to construct a continuous bijection between them since it will be automatically a homeomorphism. Let us consider $\pi \circ i : S^n \to \mathbb{RP}^n$ where $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is the projection and $i : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is the inclusion. Then $\pi \circ i$ is clearly a continuous surjection where $x_1, x_2 \in S^n$ have the same image if and only if $x_2 = \pm x_1$. Therefore the map goes to the quotient as a continuous bijection.

have to taken into account all points on the embedded hyperboloid through

$$M(\cdot) = \int_0^{+\infty} \frac{dy}{y^3} \int dz d\bar{z} G_\Delta(y, z, \bar{z}; w, \bar{w}), \tag{2.140}$$

which uses the scalar bulk-to-boundary propagator in $\mathbb{H}_3$ usually used in AdS/CFT context.

2.3.2.2 The corner proposal

As we saw before, contrarily to global symmetries, Noether charges for gauge symmetries have support on codimension two surfaces. These codimension two surfaces will be called a corner: corners are therefore the fundamental geometric ingredients of gauge theories. If we consider a gauge theory without global symmetries, then all the observables and physical degrees of freedom are localized on corners; thus they are, in some sense, the atomic constituents of gauge theories. A better understanding of corners properties is what led to the corner proposal, through which we have gained better control on symmetries in classical and quantum gauge theories, including a theory of quantum gravity [256],[276],[277],[284],[286],[288],[290],[307],[315]. In this context, a corner is a generic codimension two surface in the space where the theory lives; more deeply corners are the codimension two surfaces where non-trivial Noether charges due to the asymptotic symmetry group transformations take support. Here we focus on gravity and the first important result we need is that the Poisson bracket of charges represents the Lie bracket of symmetries projectively. To show this result we need the following preparatory lemma

Lemma 2.3.1. *Given two symplectomorphisms associated to a spacetime diffeomorphism V_ξ and V_ζ, the following identity holds:*

$$\delta\{Q_{V_\xi}, Q_{V_\zeta}\} = i_{V_{[\xi,\zeta]}}\mathbf{\Omega} \tag{2.141}$$

Proof. Starting from the identity $i_{[V_\xi, V_\zeta]} = [L_{V_\xi}, i_{V_\zeta}]$ we get

$$i_{V_{[\xi,\zeta]}}\mathbf{\Omega} = [L_{V_\xi}, i_{V_\zeta}]\mathbf{\Omega} = L_{V_\xi} i_{V_\zeta}\mathbf{\Omega} - i_{V_\zeta} L_{V_\xi}\mathbf{\Omega} = L_{V_\xi} i_{V_\zeta}\mathbf{\Omega}, \tag{2.142}$$

since for a symplectomorphisms the Lie derivative vanishes. Using Cartan magic formula we get

$$i_{V_{[\xi,\zeta]}}\mathbf{\Omega} = \delta i_{V_\xi} i_{V_\zeta}\mathbf{\Omega} + i_{V_\xi}\delta i_{V_\zeta}\mathbf{\Omega} = \delta i_{V_\xi} i_{V_\zeta}\mathbf{\Omega}, \tag{2.143}$$

since using $\delta\mathbf{\Omega} = 0$ we can write $i_{V_\xi}\delta i_{V_\zeta}\mathbf{\Omega} = i_{V_\xi} L_{V_\zeta}\mathbf{\Omega} = 0$ because also V_ζ is a symplectomorphisms. Therefore we get the thesis by equation (2.36). □

We can now prove the main statement

Theorem 2.3.2 (Relation between Poisson bracket of charges and Lie bracket of symmetries)**.** *The Poisson bracket of charges represents the Lie bracket of symmetries projectively*

$$\{Q_{V_\xi}, Q_{V_\zeta}\} = -Q_{[\xi,\zeta]} + \mathcal{K}_{\xi,\zeta}. \tag{2.144}$$

Proof. First we observe that the identity $i_{V_\xi}\mathbf{\Omega} = -\delta Q_{V_\xi}$ sets a cohomological correspondence between the field-space vector V_ξ and the functional Q_{V_ξ}. Indeed two charges Q_{V_ξ} and $Q_{V_\xi} + \mathcal{K}$ with $\delta\mathcal{K} = 0$ correspond to the same spacetime vector field V_ξ. From (2.141) and the very same argument we get that $\{Q_{V_\xi}, Q_{V_\zeta}\}$ and $\{Q_{V_\xi}, Q_{V_\zeta}\} + \mathcal{K}_{\xi,\zeta}\ \delta\mathcal{K}_{\xi,\zeta} = 0$ correspond to the same spacetime vector field $V_{[\xi,\zeta]}$. Hence, from (2.141) we have

$$\delta\{Q_{V_\xi}, Q_{V_\zeta}\} = -\delta Q_{[\xi,\zeta]}, \tag{2.145}$$

and (2.144) follows. □

As we have proven, the charge algebra projectively represents via Poisson brackets the symmetry algebra. In gravity, we are interested in the algebra of diffeomorphisms, given by the Lie bracket. Therefore, if we understand the Lie bracket algebra around corners, and we have a canonical covariant phase space theory at hand, then the Poisson bracket of charges gives the same algebra, modulo possible central extensions. Expanding the vector fields around the corner we find a maximal finitely generated sub-algebra of diffeomorphisms [314] which is

$$\mathfrak{ucs} = (\mathfrak{Diff}(S) \niplus \mathfrak{gl}(2,\mathbb{R})) \niplus \mathbb{R}^2, \tag{2.146}$$

where S is the corner under consideration. Algebra (2.146) is referred to as universal corner symmetry algebra while the associated group is known as Universal Corner Symmetry (UCS) group. Despite not all metric configurations or dynamics will have non-vanishing Noether charges associated to all generators of this algebra, the crucial claim is that this is the maximal one that can be sourced from a single corner. The existence of the UCS group and its Lie algebra, it is a promising feature. Indeed, studying this group, its representations and fusion rules, can lead to a guiding principle in quantum gravity given by the symmetries of observables. This is the central core of the corner proposal as we will see in a while.
In pure gravity, Noether charges are the only observables and we have seen that there are diffeomorphisms that are asymptotic symmetries, i.e. they have non-vanishing Noether charge; therefore, the symmetries associated are not redundancies but they physically act on the field space. The corner proposal focuses on these symmetries: they are believed to survive in quantum gravity and to become the organising principle for quantum observables. The main conjecture of the corner poposal is the following

Conjecture 2.3.1 (The corner proposal)**.** *A gravitational theory is described by a set of charges and their algebra at corners.*

At this point a couple of comments are in order. First of all, we eliminate trivial diffeomorphisms thanks to the quotient of the asymptotic symmetry group. Moreover, there is no classical metric to begin with, but only a set of charges at corners which can be promoted to quantum operators. This approach is closer to the approach of Quantum Mechanics, where the fundamental ingredients are the Hamiltonian and the complete set of commuting observables. Last but not least, corners can provide a discrete geometry in the normal planes. Indeed, given two corners and their charges, there is no a priori guarantee that the two corners are connected by a smooth bulk subspace. This is a peculiar discretization program, similar in spirit to LQG, only in the normal 2-dimensional space and it is promising due to the unique features that characterize 2-dimensional geometries[35]. It is interesting to note that putting aside the bulk in favour of a lower dimensional geometry is a primitive but general notion of holography.

2.4 The theory of solitons for Einstein gravity

Let us now focus, in this Paragraph, on a particularly interesting sector of gravity where the ideas of the corner proposal can be fruitful; specifically, we focus on particular solutions of the Einstein theory of gravity: gravitational solitons. The main goal is to make a bridge between the corner proposal and the theory of gravisolitons in order to test explicitly the corner proposal and better understanding the role of gravisolitons in classical and quantum gravity. We are going to reach this goal in Chapter 5 but here we introduce all the necessary ingredients, starting with standard solitonic solutions of non-linear PDEs.

Solitons were first observed in 1834 by the Scottish engineer John Scott Russell in the Union Canal; later, with the passage of time, the solitons were observed in other contexts such as, for example, in Physical Oceanography, Biological Sciences or Condensed Matter Physics. By soliton we mean a particular wave-like solution of non-linear differential equations in which the dispersion effect is exactly balanced by the non-linearity effects. With the intensification of research in the field of non-linear Mathematical Physics and in the theory of solitons, it has been discovered that this type of solution is present in many equations of great physical interest: from Fluid Mechanics to non-linear optics, from physics of superconductors to lattice defects in crystals, and in many other areas of physics and beyond[36]. In Fluid Mechanics, the Korteweg-de Vries (KdV) equation describes long surface waves[37] in the one dimensional approximation and it admits soliton solutions. In the context of non-linear optics the non-linear Schroedinger equation (NLS) derives from the Helmholtz equation once it is assumed that the variation of the refractive index of the material is small compared to the refractive index itself, that depends only on the intensity of the light and that there is paraxiality[38]. The NLS equation ad-

[35] An example is the Riemann uniformization theorem [178] which states that every Riemann surface admits a universal covering which is biholomorphic to one among the Riemann sphere, the entire complex plane and the unit complex disk. This result can be generalized to 3 dimensions and it is known as the Thurston's geometrization conjecture [64] proved by Grigori Perelman [119],[126],[127] who solves one of the millennial problems. The theorem says that that every 3-manifold can be cut into pieces that are geometrizable according to one or more among the eight Thurston geometries.

[36] As, for example, in the study of telecommunications and DNA.

[37] By long wave we mean that $hk << 1$, where h is the depth of the seabed and k the wave number of the water wave; example of long waves are the offshore tsunamis. If instead $hk >> 1$ then we talk of short waves.

[38] A paraxial ray is a ray which makes a small angle to the optical axis of the system.

mits solutions of solitonic character. In superconductor physics and crystallography, the sine-Gordon equation describes the dynamics, respectively, of the quantized magnetic flux in a Josephson junction[39] and of crystal dislocations; also the sine-Gordon equation has soliton-like solutions.
For our scopes, the main interest is in solitons that emerge as solution of Einstein field equations: gravisolitons. The term is used in an improper way since gravisolitons are not waves: they are spacetime manifolds and their name is merely due to the mathematics used which is very close to that used in standard solitonic context discussed above. For more information about standard solitons it is useful to consult [130],[136] while for gravisolitons the main reference is [133], however this Section is strongly based on the review [296].
The main method for the search of solitonic solutions, the spectral transform, will be discussed in Paragraph 2.4.1 together with the Darboux transform, another important mathematical tool for the theory of solitons that allows us to build solitonic solutions knowing a particular background solution of our problem. The extension to Einstein gravity of these methods is going to be discussed in Paragraph 2.4.2 where we construct the N-gravisoliton solution and we give the example of Kerr black hole as axially symmetric double gravisoliton.

2.4.1 Non-linear equations and soliton solutions

As already anticipated in the introduction, many non-linear equations of physical interest show soliton solutions. Therefore, let us concentrate on solving non-linear equations; the methods we are going to develop are the spectral transform and the Darboux transform, which we are going to discuss in their general features.

2.4.1.1 Spectral transform and Darboux transform

The spectral transform or Inverse Scattering Transform (IST) was introduced in 1967 by Gardner, Greene, Miura and Kruskal [32] to solve the problem at the initial values of the KdV and then extended to other situations of interest, for example as done by Zakharov and Shabat in 1972 [37] with the NLS equation. The method is based on the possibility of writing the equation under consideration as a condition of integrability of an associated operatorial system of two linear differential equations, the so-called Lax pair of the problem. This has the form

$$\begin{aligned}\psi_{,z} &= U\psi,\\ \psi_{,t} &= V\psi,\end{aligned} \tag{2.147}$$

where ψ, U and V depend on a parameter, $\lambda \in sp(U)$, called the spectral parameter and on the variables z and t. The spectral parameter satisfies the isospectrality condition $\lambda_{,t} = 0$. The condition of integrability is obtained as

$$\psi_{,t,z} = \psi_{,z,t} \Rightarrow V_{,z}\psi + V\psi_{,z} = U_{,t}\psi + U\psi_{,t} \Rightarrow V_{,z}\psi + VU\psi = U_{,t}\psi + UV\psi,$$

[39] The Josephson junction is formed by two superconducting plates interspersed by an insulating membrane and is based on the tunnel effect of Cooper pairs (bonded states of two electrons or holes).

from which follows the operatorial equation

$$V_{,z} + VU - U_{,t} - UV = 0 \Rightarrow V_{,z} - U_{,t} + [V, U] = 0, \tag{2.148}$$

called Lax equation and corresponding to the original equation of our Cauchy problem. Imagine that our non-linear differential equation determines the quantity $u(z,t)$ and that the initial condition $u(z,0)$ is given. Once the appropriate Lax pair has been determined, the problem is divided into two parts: the direct problem and the inverse problem. The direct problem consists in associating the spectral parameters, such as the transmission coefficient $T(\lambda)$, the reflection coefficient $R(\lambda)$ and the elements λ_n of the discrete spectrum, which correspond to the potential $u(z,0)$; these spectral parameters are denoted, as a whole, with $S(\lambda, 0)$. The inverse problem, on the other hand, consists in reconstructing the potential at a generic time, $u(z,t)$, given the spectral parameters at time t, obtained by solving the spectral problem given by the Lax pair. The evolved spectral parameters are indicated with $S(\lambda, t)$. In summary, the idea is, therefore, not to find the solution $u(z,t)$ directly from the starting equation because this evolution is often too complicated: it is preferable to solve the associated spectral problem, which gives us the evolution over time of the spectral parameters and from the evolved spectral parameters we can obtain the solution $u(z,t)$. The fact that this way is preferable derives from the linearity of the problem when looking at it from the point of view of the spectral problem; in fact, the evolution of the spectral parameters is ultimately given by the Lax pair, which is an operatorial but linear differential system. The spectral transform method is shown in the diagram 2.149 below.

$$\begin{tikzcd} u(z,0) \arrow[r, "\text{direct problem}"] \arrow[d, "\text{non-linear evolution}"'] & S(\lambda,0) \arrow[d, "\text{linear evolution}"] \\ u(z,t) & S(\lambda,t) \arrow[l, "\text{inverse problem}"'] \end{tikzcd} \tag{2.149}$$

From diagram 2.149 we note a certain degree of analogy with the Fourier transform scheme: in fact, the spectral transform method can be considered as a generalization to non-linear equations of the Fourier method for linear equations and it reduces to this in the small field limit, $|u(z,t)| << 1$, and purely continuous spectrum.

We are interested in solitonic solutions which emerge in the presence of a potential $u(z,0)$ corresponding to $R(\lambda) = 0$; in this case the inverse problem is considerably simplified since we pass from having to solve an integral equation, known as Gelfand-Levitan-Marchenko equation, to an algebraic system. The IST method shows us how solitons are intimately related to the discrete elements λ_n of the spectrum: for each of them a soliton emerges. Moreover each λ_n is a pole of the transmission coefficient and this implies that there are as many solitons as poles of $T(\lambda)$.

We now introduce another instrument of particular importance: the Darboux transform. First of all, it is important to underline that Darboux transform is an algebraic method and it consists in constructing new exact solutions for a non-linear integrable equation[40], starting from the knowledge of a simple exact solution called background solution, that is

$$\psi = \chi \psi_b, \tag{2.150}$$

where the χ is called the dressing operator. We assume a dressing matrix of the shape

$$\chi = \mathcal{I} + \sum_{k=1}^{N} \frac{O_k}{\lambda - \lambda_k} \tag{2.151}$$

in which the operator O does not depend on the spectral parameter; A N-poles dressing matrix can generate the N-solitons solution.

The Darboux and the spectral transforms are linked by the fact that, as can be explicitly seen in the relatively simple case of the KdV, the Darboux transformation adds a pole to the transmission coefficient and this involves the addition of a soliton.

2.4.2 Gravitational solitons

In this chapter we look at the soliton solutions of Einstein's equations in vacuum: the so-called gravitational solitons or gravisolitons. The application and extension of the IST technique to General Relativity begins at the end of the seventies thanks to the works, besides others, of Belinski, Zakharov and Maison [51],[54] and continues with the extension in the context of the Maxwell-Einstein equations by Alekseev [57] in the early eighties.

2.4.2.1 Generalization of the IST to gravity and the N-gravisolitons solution

Let us start by establishing the problem. Einstein field equations are

$$E_{\mu\nu} = 8\pi T_{\mu\nu}; \tag{2.152}$$

in vacuum the momentum energy tensor is identically zero and the field equations can be expressed as

$$R_{\mu\nu} = 0. \tag{2.153}$$

In $D = 4$, they are a system of partial differential equations for the 10 independent components of the metric, modulo gauge freedom, and they represent our starting problem.

Our goal is to generalize the IST method to gravity in vacuum, however, the IST method, defined in its general lines in Paragraph 2.4.1, is applicable only when the unknown function depends on two variables but, in general, four dimensional metrics depend on the three spatial coordinates and on the time coordinate. We therefore

[40]It is intended to be integrable with the IST method. There is therefore a Lax pair whose compatibility condition is the starting equation.

assume that spacetime admits two space like Killing vector fields that allow us to eliminate the dependence on two spatial variables[41]: x^1 and x^2. With this choice, the metric depends only on $x^0 = t$ and $x^3 = z$. To simplify the problem further we refer to the gauge symmetry of General Relativity; in four dimensional spacetime this symmetry allows us to impose four conditions on the metric without loss of generality: we will impose that

$$g_{00} = -g_{33}\ ,\ g_{03} = 0\ ,\ g_{0i} = 0, \tag{2.154}$$

where $i = 1, 2$.
Under these hypothesis, the spacetime interval is written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(dt^2 - dz^2) + g_{ij} dx^i dx^j + 2g_{i3} dx^i dz, \tag{2.155}$$

in which, from here to the end of paragraph 2.4.2.2, $i, j = 1, 2$ (later on we will introduce the index h, which also runs on 1 and 2). It is not yet known how to apply the IST to the (2.155) metric, hence further simplification is needed. Since we have exhausted our freedom to manipulate the metric without loss of generality, we must impose as a physical condition that $g_{i3} = 0$; this allows us to write

$$ds^2 = g_{00}(dt^2 - dz^2) + g_{ij} dx^i dx^j. \tag{2.156}$$

We note that if we had chosen the Killing fields, one space like and one time like, we would have obtained a metric similar to (2.156) but stationary, in which the independent variables would be both space like; this case will be considered in paragraph 2.4.2.3 to study axialsymmetric gravisolitons or axialgravisolitons.
From here to the end of paragraph 2.4.2.2, we are going to call $\hat{g}$ the 2×2 block g_{ij} and we set $det(\hat{g}) = \alpha^2$. Equations (2.153) for the metric (2.156) are divided into two sets which, passing to the light cone coordinates introduced by defining $\zeta := \frac{1}{2}(z + t)$ e $\eta := \frac{1}{2}(z - t)$, take the form

$$\begin{gathered}\left(\alpha \hat{g}_{,\zeta} \hat{g}^{-1}\right)_{,\eta} + \left(\alpha \hat{g}_{,\eta} \hat{g}^{-1}\right)_{,\zeta} = 0;\\ (ln(g_{00}))_{,\zeta}(ln(\alpha))_{,\zeta} = (ln(\alpha))_{,\zeta,\zeta} + \frac{A^2}{4\alpha^2}\ ,\ (ln(g_{00}))_{,\eta}(ln(\alpha))_{,\eta} = (ln(\alpha))_{,\eta,\eta} + \frac{B^2}{4\alpha^2},\end{gathered} \tag{2.157}$$

where $A := -\alpha \hat{g}_{,\zeta} \hat{g}^{-1}$ and $B := \alpha \hat{g}_{,\eta} \hat{g}^{-1}$. The first of the equations (2.157) gives us the 2×2 block and the second ones gives us g_{00}, once $\hat{g}$ is given. We also note that taking the trace of the first equation and making use of the fact that $det(\hat{g}) = \alpha^2$ yields

$$\alpha_{,\zeta,\eta} = 0; \tag{2.158}$$

[41]Given a certain vector field, $\vec{\xi}(x^\mu)$, it identifies an isometry if and only if it satisfies the Killing equation $(\mathcal{L}_\xi(g))_\phi = g_{\mu\nu,\rho}\xi^\rho + g_{\beta\nu}\xi^\beta_{,\mu} + g_{\mu\beta}\xi^\beta_{,\nu} = 0$. Let us consider that we have a time like Killing field, $\vec{\xi}$, then we could choose the coordinate such that the time-like basic vector, $\vec{e}_{(0)}$, is at any point aligned with the $\vec{\xi}$ field. In this reference we can write $\xi^\alpha = (1, 0, 0, 0)$; then the metric, due to the Killing equation, is independent of time and therefore stationary. An analogous reasoning can be made in the case of a space like or light like Killing field.

therefore, the root of the determinant satisfies the wave equation[42].

We note that the two equations (2.157) can be obtained directly from the definitions of A and B: for the former it is sufficient to replace the definitions in the first of the (2.157) while, the latter, is instead the condition of integrability between A and B with respect to $\hat{g}$; that is

$$\begin{aligned} &B_{,\zeta} - A_{,\eta} = 0; \\ &A_{,\eta} + B_{,\zeta} + \alpha^{-1}[A, B] - \alpha_{,\eta}\alpha^{-1}A - \alpha_{,\zeta}\alpha^{-1}B = 0. \end{aligned} \tag{2.159}$$

The (2.159) are just a rewriting of the (2.153) Einstein field equations.

The next step is the search for an appropriate Lax pair that defines the spectral problem and therefore depends on a complex parameter: the spectral parameter λ. For this purpose we define the operators

$$D_- := \partial_\zeta - \frac{2\alpha_{,\zeta}\lambda}{\lambda - \alpha}\partial_\lambda \ , \ D_+ := \partial_\eta + \frac{2\alpha_{,\eta}\lambda}{\lambda + \alpha}\partial_\lambda \tag{2.160}$$

and consider the following system of equations which provides the Lax pair

$$\begin{aligned} D_-\psi &= \frac{A}{\lambda - \alpha}\psi, \\ D_+\psi &= \frac{B}{\lambda + \alpha}\psi, \end{aligned} \tag{2.161}$$

where $\psi = \psi(\zeta, \eta, \lambda)$ is called generating matrix. When $\lambda = 0$ the system reduces to

$$\begin{aligned} &\psi_{,\zeta}(\zeta, \eta, 0) = -\frac{A}{\alpha}\psi(\zeta, \eta, 0) \Rightarrow A = -\alpha\psi_{,\zeta}(\zeta, \eta, 0)\psi^{-1}(\zeta, \eta, 0), \\ &\psi_{,\eta}(\zeta, \eta, 0) = \frac{B}{\alpha}\psi(\zeta, \eta, 0) \Rightarrow B = \alpha\psi_{,\eta}(\zeta, \eta, 0)\psi^{-1}(\zeta, \eta, 0), \end{aligned} \tag{2.162}$$

which, compared with the definitions of A and B, it allows us to conclude that the 2×2 block is nothing more than the generating matrix evaluated for $\lambda = 0$:

$$\hat{g}(\zeta, \eta) = \psi(\zeta, \eta, 0). \tag{2.163}$$

We know that for the integration procedure it is necessary to know a backgound solution[43] $\hat{g}_b$ from which, thanks to the definitions of A and B, we get the A_b and B_b matrices, which, replaced in the system (2.161), allows us to get the background generating matrix ψ_b. We therefore operate a Darboux transform of the form

$$\psi = \chi\psi_b, \tag{2.164}$$

[42]Given the definitions $\zeta := \frac{1}{2}(z+t)$ and $\eta := \frac{1}{2}(z-t)$ we have $\alpha_{,\zeta,\eta} = \frac{1}{4}[(\partial_z + \partial_t)(\partial_z - \partial_t)]\alpha = \frac{1}{4}(\partial_z^2 - \partial_t^2)\alpha = 0 \Rightarrow (\partial_z^2 - \partial_t^2)\alpha = 0$.

[43]Solution of the first of the equations (2.157).

which, replaced in the system (2.161), returns a system of equations for the dressing matrix

$$D_-\chi = \frac{(A\chi - \chi A_b)}{\lambda - \alpha};$$
$$D_+\chi = \frac{(B\chi - \chi B_b)}{\lambda + \alpha}. \tag{2.165}$$

Putting together (2.163) and (2.164) we see that

$$\hat{g}(\zeta,\eta) = \psi(\zeta,\eta,0) = \chi(\zeta,\eta,0)\psi_b(\zeta,\eta,0) = \chi(\zeta,\eta,0)\hat{g}_b(\zeta,\eta). \tag{2.166}$$

At this point it is important to remember that the metric is real and symmetric; these requirements implies some conditions on the dressing matrix[44] and on the generating matrix, that is

$$\chi^*(\zeta,\eta,\lambda^*) = \chi(\zeta,\eta,\lambda)\,, \qquad \psi^*(\zeta,\eta,\lambda^*) = \psi(\zeta,\eta,\lambda)\,, \qquad \hat{g} = \chi(\zeta,\eta,\lambda)\hat{g}_b\chi^T(\zeta,\eta,\alpha^2/\lambda). \tag{2.167}$$

We underline that, for the moment, condition $det(\hat{g}) = \alpha^2$ has not yet been imposed; the condition is imposed by requiring that

$$\hat{g}^{(ph)} = \alpha\frac{\hat{g}}{\sqrt{det(\hat{g})}}, \tag{2.168}$$

where the superscript underlines that this is the solution that respects all the physical properties of the metric: this is the physical solution. Obviously, (2.168) involves a transformation of the A and B matrices which results in new equations for the g_{00} component which will return the physical component, $g_{00}^{(ph)}$.

We now proceed by discussing the construction of the N-gravisolitons solution. This solution emerges by imposing a rational dependence of the dressing matrix on the spectral parameter with a finite number of simple poles

$$\chi = \mathcal{I} + \sum_{k=1}^{N}\frac{O_k}{\lambda - \lambda_k}, \tag{2.169}$$

where O_k does not depend on the spectral parameter; from the reality condition of the matrix $\hat{g}$, and therefore from the first of (2.167), it follows that the poles are either real or appear as pairs of conjugated complexes. The final solution is given by (2.166) so

$$\hat{g}(\zeta,\eta) = \left(\mathcal{I} - \sum_{k=1}^{N}\frac{O_k}{\lambda_k}\right)\hat{g}_b(\zeta,\eta); \tag{2.170}$$

it remains therefore to determine the matrix O_k. For this purpose, we insert (2.169) in system (2.165) and we evaluate the equation on the pole of the dressing matrix, i.e. for $\lambda = \lambda_k$. In this way we get two fundamental

[44]The normalization for the dressing matrix is chosen such that $\chi(\zeta,\eta,\lambda) = \mathcal{I}$ when $|\lambda| \to \infty$.

information about the structure of the problem.
The first information is obtained by noting that the operators D_- and D_+ contain derivations with respect to ζ, η and λ and that, when applied to the non-trivial part of the dressing matrix, they produce terms containing double poles, terms that are not present in the second member: it is necessary that the coefficients of the double poles cancel out. In fact, studying only the k-th term for linearity we have[45]

$$\begin{aligned} \left(\partial_\zeta - \frac{2\alpha_{,\zeta}\lambda}{\lambda-\alpha}\partial_\lambda\right)\frac{O_k}{\lambda-\lambda_k} &= \frac{O_{k,\zeta}}{\lambda-\lambda_k} + O_k\left(\frac{-\lambda_{,\zeta}}{(\lambda-\lambda_k)^2}\right) - \frac{2\alpha_{,\zeta}\lambda}{\lambda-\alpha}\frac{O_k}{(\lambda-\lambda_k)^2}; \\ \left(\partial_\eta + \frac{2\alpha_{,\eta}\lambda}{\lambda+\alpha}\partial_\lambda\right)\frac{O_k}{\lambda-\lambda_k} &= \frac{O_{k,\eta}}{\lambda-\lambda_k} + O_k\left(\frac{-\lambda_{,\eta}}{(\lambda-\lambda_k)^2}\right) + \frac{2\alpha_{,\eta}\lambda}{\lambda+\alpha}\frac{O_k}{(\lambda-\lambda_k)^2}; \end{aligned} \tag{2.171}$$

equating to zero the coefficients of the double poles calculated on the poles themselves we have

$$\begin{aligned} \left(-\lambda_{,\zeta} - \frac{2\alpha_{,\zeta}\lambda}{\lambda-\alpha}\right)\Big|_{\lambda=\lambda_k} &= 0 \Rightarrow \lambda_{k,\zeta} = \frac{2\alpha_{,\zeta}\lambda_k}{\alpha-\lambda_k}; \\ \left(-\lambda_{,\eta} + \frac{2\alpha_{,\eta}\lambda}{\lambda+\alpha}\right)\Big|_{\lambda=\lambda_k} &= 0 \Rightarrow \lambda_{k,\eta} = \frac{2\alpha_{,\eta}\lambda_k}{\alpha+\lambda_k}; \end{aligned} \tag{2.172}$$

equations (2.172) determine the trajectories of the poles which are not fixed but move in spacetime. The second information is obtained noting that O_k and $\chi^{-1}(\lambda_k)$ are degenerate matrices[46] , so we can write them as[47]

$$(O_k)_{ij} = n_i^{(k)} m_j^{(k)} \ , \ (\chi_k^{-1}(\lambda_k))_{ij} = q_i^{(k)} p_j^{(k)}. \tag{2.173}$$

Rewriting equations (2.165) multiplying them from the right by χ^{-1} we get

$$\frac{A}{\lambda-\alpha} = (D_-\chi)\chi^{-1} + \chi\frac{A_b}{\lambda-\alpha}\chi^{-1}, \qquad \frac{B}{\lambda+\alpha} = (D_+\chi)\chi^{-1} + \chi\frac{B_b}{\lambda+\alpha}\chi^{-1}, \tag{2.174}$$

and, since the LHSs of the equations (2.174) are regular at the poles $\lambda = \lambda_k$, it follows that the residues of the poles in RHSs of the equations must be zero, which leads to the equations

$$\begin{aligned} O_{k,\zeta}\chi^{-1}(\lambda_k) + O_k\frac{A_b}{\lambda_k-\alpha}\chi^{-1}(\lambda_k) &= 0, \\ O_{k,\eta}\chi^{-1}(\lambda_k) + O_k\frac{B_b}{\lambda_k+\alpha}\chi^{-1}(\lambda_k) &= 0. \end{aligned} \tag{2.175}$$

[45] Remember that O_k does not depend on λ.
[46] This conclusion can be seen starting from the relation $\chi\chi^{-1} = \mathcal{I}$. In fact the identity matrix has no poles while the left side gives rise to the term $\sum_{k=1}^{N} \frac{O_k\chi^{-1}}{\lambda-\lambda_k}$ which contains simple poles: the residue must vanishes and therefore $O_k\chi^{-1}(\lambda_k) = 0$, from which degeneration follows thanks to Binet theorem.
[47] This is a general property of dyadic product which can be shown simply calculating the determinant using Laplace method.

Inserting relations (2.173) into (2.175) equations we obtain the equations that determine the vectors $m_i^{(k)}$:

$$\begin{aligned}\left(m_{i,\zeta}^{(k)} + m_j^{(k)}\frac{(A_b)_{ij}}{\lambda_k - \alpha}\right)q_i^{(k)} = 0,\\ \left(m_{i,\eta}^{(k)} + m_j^{(k)}\frac{(B_b)_{ij}}{\lambda_k + \alpha}\right)q_i^{(k)} = 0,\end{aligned} \tag{2.176}$$

whose solution is given by

$$m_i^{(k)} = m_{0j}^{(k)}(\psi_b^{-1}(\lambda_k, \zeta, \eta))_{ji} \tag{2.177}$$

where the $m_{0j}^{(k)}$ are arbitrary complex constant vectors. To determine the $n_i^{(k)}$ the third condition of (2.167) is exploited, which allows, ultimately, to write the vectors $n_i^{(k)}$ as

$$n_i^{(k)} = \sum_{l=1}^{N}\frac{1}{\lambda_l}(\Pi)_{kl}L_i^{(l)}, \tag{2.178}$$

where $L_i^{(l)} := m_j^{(l)}(\hat{g}_b)_{ji}$ and $(\Pi)_{kl} := (\Gamma^{-1})_{kl}$ with $(\Gamma)_{kl} := -m_i^{(k)}m_j^{(l)}(\hat{g}_b)_{ij}(\alpha^2 - \lambda_k\lambda_l)^{-1}$.

From (2.170), using the first equation (2.173) and (2.178) gives[48]

$$\begin{aligned}(\hat{g})_{ij} &= (\hat{g}_b)_{ij} - \sum_{k=1}^{N}\frac{(O_k)_{ih}}{\lambda_k}(\hat{g}_b)_{hj} = (\hat{g}_b)_{ij} - \sum_{k=1}^{N}\frac{1}{\lambda_k}n_i^{(k)}m_h^{(k)}(\hat{g}_b)_{hj} =\\ &= (\hat{g}_b)_{ij} - \sum_{k=1}^{N}\sum_{l=1}^{N}\frac{1}{\lambda_k}\frac{1}{\lambda_l}(\Pi)_{kl}L_i^{(l)}\underbrace{m_h^{(k)}(\hat{g}_b)_{hj}}_{=L_j^{(k)}} = (\hat{g}_b)_{ij} - \sum_{k=1}^{N}\sum_{l=1}^{N}\frac{1}{\lambda_k}\frac{1}{\lambda_l}(\Pi)_{kl}L_i^{(l)}L_j^{(k)};\end{aligned} \tag{2.179}$$

it should be stressed, however, that the solution (2.179) is not the physical one: condition $det(\hat{g}) = \alpha^2$ is not satisfied. To obtain the physical solution we refer to (2.168); the computation of the determinant of $\hat{g}$ can be performed step by step: it is first computed for the $N = 1$ gravisoliton solution, then the physical gravisoliton solution is used as the background solution and the $N = 2$ gravisolitons solution is constructed, the determinant is computed and the $N = 2$ gravisolitons solution is used as the backgound solution to build up the $N = 3$ gravisolitons solution, the determinant is computed and so on. At the end of the day, the computation yields

$$det(\hat{g}) = \alpha^{2N+2}\prod_{k=1}^{N}\frac{1}{\lambda_k^2}. \tag{2.180}$$

As already mentioned, the calculation of the solution $\hat{g}^{(ph)}$ implies that the coefficient g_{00} must also be recalcu-

[48]Remember that the metric is symmetric.

lated to obtain the physical coefficient $g_{00}^{(ph)}$. In conclusion we get

$$
\begin{aligned}
\hat{g}^{(ph)} &= \frac{\hat{g}}{\alpha^N}\prod_{k=1}^{N}\lambda_k, \\
g_{00}^{(ph)} &= C g_{b00}\alpha^{-\frac{N^2}{2}}\left(\prod_{k=1}^{N}\lambda_k\right)^{N+1}\left(\prod_{k>l=1}^{N}\frac{1}{(\lambda_k-\lambda_l)^2}\right)det(\Gamma),
\end{aligned}
\tag{2.181}
$$

where C is an arbitrary constant to choose with appropriate sign in order to have the correct sign for $g_{00}^{(ph)}$; g_{b00} is the g_{00} background solution corresponding to $\hat{g}_b$ and $\hat{g}$ is given by (2.179).

The N-gravisolitons solution is therefore given by replacing (2.181) in the spacetime interval (2.156) and using (2.179)

Definition 2.4.1 (N-solitons solution in Einstein gravity or N-gravisolitons solution)**.** *The N-solitons solution in Einstein gravity, or N-gravisolitons solution, is the Lorentzian manifold (M,$\mathbf{g}$) with coordinate line element given by*

$$
ds^2 = g_{00}^{(ph)}(dt^2-dz^2)+g_{ij}^{(ph)}dx^i dx^j = \tag{2.182a}
$$

$$
= C g_{b00}\alpha^{-\frac{N^2}{2}}\left(\prod_{k=1}^{N}\lambda_k\right)^{N+1}\left(\prod_{k>l=1}^{N}\frac{1}{(\lambda_k-\lambda_l)^2}\right)det(\Gamma)(dt^2-dz^2)+\frac{\left((\hat{g}_b)_{ij}-\sum_{k,l=1}^{N}\lambda_k^{-1}\lambda_l^{-1}(\Pi)_{kl}L_i^{(k)}L_j^{(l)}\right)}{\alpha^N}\left(\prod_{k=1}^{N}\lambda_k\right)dx^i dx^j; \tag{2.182b}
$$

where

$$
\begin{aligned}
(\Pi)_{kl} &:= (\Gamma^{-1})_{kl}, \quad (\Gamma)_{kl} := -m_i^{(k)}m_j^{(l)}(\hat{g}_b)_{ij}(\alpha^2-\lambda_k\lambda_l)^{-1}, \\
L_i^{(l)} &:= m_j^{(l)}(\hat{g}_b)_{ji}, \quad m_i^{(k)} := m_{0j}^{(k)}(\psi_b^{-1}(\lambda_k,\zeta,\eta))_{ji}.
\end{aligned}
\tag{2.183}
$$

2.4.2.2 Some properties of gravitational solitons

Given the N-gravisolitons solution we now want to discuss some interesting properties of gravitational solitons. We explicitly report the $N=1$ gravisoliton solution; with reference to Definition 2.4.1 and considering that in this case Π and Γ are not matrices we have

$$
\begin{aligned}
(\hat{g}^{(ph,N=1)})_{ij} &= \frac{(\hat{g}_b)_{ij}}{\alpha}\lambda_1-\left(\frac{(\alpha^2-\lambda_1^2)L_iL_j\lambda_1}{-\alpha\lambda_1^2 m_i m_j}(\hat{g}_b^{-1})_{ij}\right) = \frac{(\hat{g}_b)_{ij}}{\alpha}\lambda_1+\frac{(\alpha^2-\lambda_1^2)L_iL_j}{\alpha\lambda_1 m_i m_j}(\hat{g}_b^{-1})_{ij}, \\
\hat{g}_{00}^{(ph,N=1)} &= C g_{b00}\alpha^{-\frac{1}{2}}\lambda_1^2 m_i m_j(\hat{g}_b)_{ij}(\alpha^2-\lambda_1^2)^{-1},
\end{aligned}
\tag{2.184}
$$

where the superscript indicates the physical one soliton solution. The trajectory of the pole λ_1 is given by the equations (2.172) with $k=1$; the solution of these equations is given by the roots of the quadratic equation

$$\lambda_1^2 + 2(\beta - w_1)\lambda_1 + \alpha^2, \tag{2.185}$$

where w_1 is an arbitrary constant and α and β are two independent solutions of the equation (2.158). The solutions of (2.185) are

$$\begin{aligned}
\lambda_1^{(in)} &= -(\beta - w_1) + \sqrt{(\beta - w_1)^2 - \alpha^2} = (w_1 - \beta)\left(1 - \sqrt{1 - \frac{\alpha^2}{(\beta - w_1)^2}}\right), \\
\lambda_1^{(out)} &= -(\beta - w_1) - \sqrt{(\beta - w_1)^2 - \alpha^2} = (w_1 - \beta)\left(1 + \sqrt{1 - \frac{\alpha^2}{(\beta - w_1)^2}}\right),
\end{aligned} \tag{2.186}$$

in which apexes underline that the solutions are inside or outside the circle generated by $|\lambda| = \alpha$; specifically $|\lambda_1^{(in)}| < \alpha$ and $|\lambda_1^{(out)}| > \alpha$. In the Figure 2.10below[49] we show the case where $\lambda_1^{(in)}$ and $\lambda_1^{(out)}$ are real with $w_1 = 3$ and $\alpha = 2$.

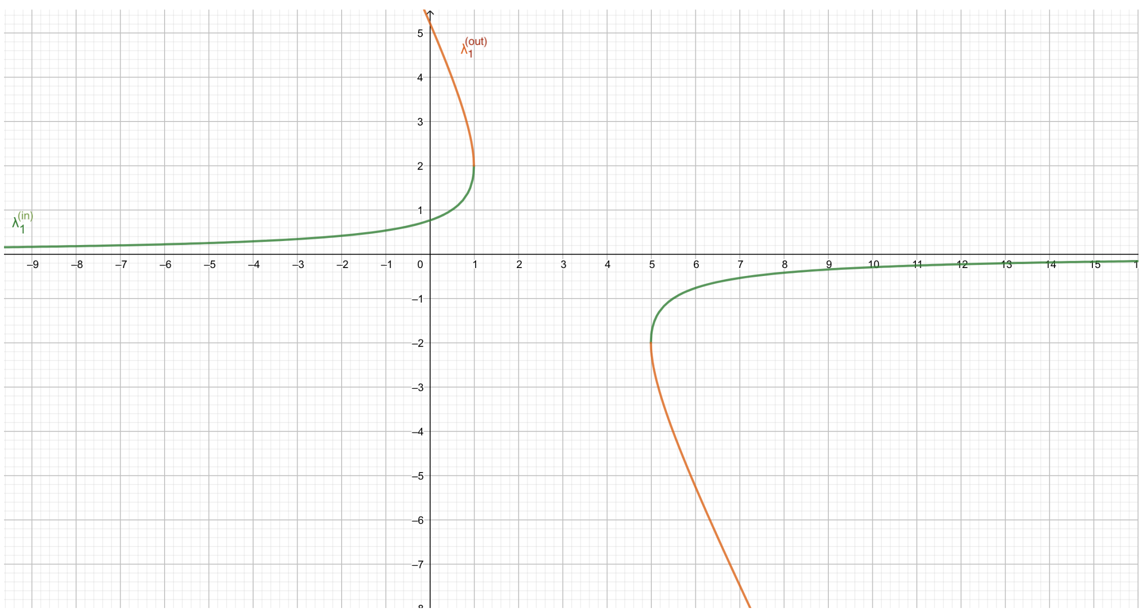


FIGURE 2.10 *Behaviour of $\lambda_1^{(in)}$ (green line) and $\lambda_1^{(out)}$ (orange line) as a function of β if they are real, with $w_1 = 3$ and $\alpha = 2$. Note that $|\lambda_1^{(in)}|$ is always less than $\alpha = 2$ while $|\lambda_1^{(out)}|$ is always greater than $\alpha = 2$; in general, $\lambda_1^{(in)}$ is always contained in the region $|\lambda| = \alpha$ while $\lambda_1^{(out)}$ is always outside this region.*

A first interesting property is the shock wave-like nature of gravisolitons. Let us consider the one soliton solution generated by a real pole: looking at equations (2.186) we see that the solutions, in the case of a real pole, are defined only in the region $(\beta - w_1)^2 \geq \alpha^2$, however the solution can be continued analytically in the

[49]The graph was created with GeoGebra.

$(\beta - w_1)^2 < \alpha^2$ region. When passing between the two regions, the λ_1 function becomes complex:

$$\lambda_1 = (w_1 - \beta)\left(1 \pm i\sqrt{\frac{\alpha^2}{(\beta - w_1)^2} - 1}\right); \tag{2.187}$$

since the complex poles of the dressing matrix can appear only in pairs, it follows that, necessarily, in the region $(\beta - w_1)^2 < \alpha^2$ there is a $N = 2$ gravisolitons solution generated by two conjugated complex poles λ_1 and $\bar{\lambda}_1$. The square modulus of these complex poles, however, is equal to α^2; indeed

$$|\lambda_1|^2 = |\lambda_1^*|^2 = (w_1 - \beta)^2 + (w_1 - \beta)^2\left(\frac{\alpha^2}{(\beta - w_1)^2} - 1\right) = \alpha^2 \tag{2.188}$$

and from (2.178), remembering the definitions of $L_i^{(l)}$, $(\Pi)_{kl}$ and $(\Gamma)_{kl}$, it follows that $n_i^{(k)}$ are null if $|\lambda_1|^2 = \alpha^2$ and therefore the dressing matrix is reduced to identity if the poles lie on the circumference generated by $|\lambda| = \alpha$. This means that the $N = 2$ gravisolitons solution actually reduces to the background solution in the region where the dressing matrix reduces to the identity and we note that this happens in the region $(\beta - w_1)^2 \leq \alpha^2$ where we have $|\lambda_1|^2 = \alpha^2$. Moreover, from (2.184) we see that the metric component $g_{00}^{(ph,N=1)}$ is singular when $|\lambda_1|^2 = \alpha^2$.
To recap: in the region $(\beta - w_1)^2 > \alpha^2$ we have the $N = 1$ gravisoliton solution while in the region $(\beta - w_1)^2 < \alpha^2$ the solution is the background one; moreover in the region defined by the relation $(\beta - w_1)^2 = \alpha^2$ the metric experiences a discontinuity due to the coefficient $g_{00}^{(ph,N=1)}$; in this sense the gravitational solitons can be considered as gravitational shock waves[50]. Similar considerations are possible for the case of the N-gravisolitons solution: there are regions in which we have the N-gravisolitons solution, regions in which we have the $(N-1)$-gravisolitons solution and so on; moreover, the metric is discontinuous in each interface between the various regions. However, it should be emphasized that this is true only for solutions generated by real poles; if the poles were all complexes conjugated in pairs there would be no regions of discontinuity and the whole spacetime would be perturbed with respect to the background metric.

A second interesting property is the so-called fusion of poles. Let us consider the $N = 2$ gravisolitons solution; suppose that the constants $m_{0i}^{(1)}$ and $m_{0i}^{(2)}$ depend, respectively, on w_1 and w_2 and that in the limit $w_2 \to w_1$ we have $m_{0i}^{(2)} \to m_{0i}^{(1)}$, consequently we have $\lambda_2 \to \lambda_1$. It is possible to show, from the $N = 2$ gravisolitons metric, that this does not vanish under the limit procedure just described but rather reduces to a $N = 1$ gravisoliton solution corresponding to a double pole. This conclusion is interesting and is a non-trivial and unexpected phenomenon: we started by considering only simple poles and we find ourselves with the possibility of double poles thanks to the fusion of poles. The same analysis can be done in the case of the N-gravisolitons solution and of the fusion of the N poles which leads to a $N = 1$ gravisoliton solution corresponding to a pole of order N. The fusion of the poles can occur in different ways, for example if the we

[50]Shock waves arise from the regularization of the solution of a certain model: a discontinuous but monodrome solution is preferred to a continuous but a polyhydrome solution; this is the shock wave.

consider the $N = 2n$ gravisolitons solution, hence we have $2n$ poles, it is possible to fuse poles separately to form two poles of order n, and therefore two $N = n$ gravisolitons solutions.

The last properties that we want to deal with, even if briefly, are the topological properties of gravitational solitons. Specifically, we want to focus on the fact that it is not clear when a gravitational soliton is a topological object and a topological charge can be associated with it. In general, a topological soliton is a solution that presents itself with a different topology from the vacuum topology of the problem; to these objects it is possible to associate quantities, called topological charges[51], which characterize the topology.

2.4.2.3 Kerr black hole as double gravitational soliton

In this paragraph we will consider an axialsymmetric stationary metric, which therefore admits two Killing fields, one space like and one time like. We will use the notation $(x^0, x^1, x^2, x^3) = (t, \phi, \rho, z)$ for the coordinates. With considerations similar to those made in paragraph 2.4.2.1 we can write the metric in the form[52]

$$ds^2 = g_{22}(d\rho^2 + dz^2) + g_{ij}dx^i dx^j, \tag{2.189}$$

where, from here to the end of the paragraph, $i, j \in \{0, 1\}$ and where g_{22} and g_{ij} depend only on $x^2 = \rho$ and $x^3 = z$ which are both space like variables. Also in this paragraph we are going to call $\hat{g}$ the 2×2 block g_{ij}. For stationary metrics of the type (2.189) it can be imposed, without loss of generality, that the 2×2 block $(\hat{g})_{ij}$ has $det(\hat{g}) = -\rho^2$. This is a slightly different choice compared to the case seen in paragraph 2.4.2.1, because in the case of non-stationary metrics the determinant of the 2×2 block could be space like or time like, while in the case of stationary metrics it can only be time like.
Einstein field equations in vacuum for the (2.189) metric can be rewritten as

$$\begin{aligned}
&(\rho\hat{g}_{,\rho}\hat{g}^{-1})_{,\rho} + (\rho\hat{g}_{,z}\hat{g}^{-1})_{,z} = 0, \\
&(ln(g_{22}))_{,\rho} = -\frac{1}{\rho} + \frac{1}{4\rho}(U^2 - V^2)\ ,\ (ln(g_{22}))_{,z} = \frac{1}{2\rho}UV;
\end{aligned} \tag{2.190}$$

where $U := \rho\hat{g}_{,\rho}\hat{g}^{-1}$ and $V := \rho\hat{g}_{,z}\hat{g}^{-1}$. With steps analogous to those described in paragraph 2.4.2.1, the N-axialgravisolitons solution is obtained in the case of the metric (2.189)

Definition 2.4.2 (N-axialsolitons solution in Einstein gravity or N-axialgravisolitons solution)**.** *In Einstein gravity the N-axialsolitons solution, or N-axialgravisolitons solution, is the Lorentzian manifold (M,**g**) with*

[51] Topological charges, are quantities that characterize the topology of the soliton. In general, topological charges are topological invariants (which means that the object is invariant under homeomorphisms) or homotopical invariance (which means that the object is invariant under homotopies). Examples could be the homotopy groups, the homology groups and the cohomology groups. A field of theoretical and mathematical physics in which homotopically invariant quantities are used is, for example, the field of generalized and/or non-invertible symmetries [206],[313],[321].

[52] We have set $g_{22} = g_{33}$, $g_{i2} = g_{i3} = g_{23} = 0$.

coordinate line element given by

$$ds^2 = g_{22}^{(ph)}(d\rho^2 + dz^2) + g_{ij}^{(ph)} dx^i dx^j = \tag{2.191a}$$

$$= \tilde{C} g_{b22} \rho^{-\frac{N^2}{2}} \left(\prod_{k=1}^{N} \lambda_k\right)^{N+1} \left(\prod_{k>l=1}^{N} \frac{1}{(\lambda_k - \lambda_l)^2}\right) det(\Gamma)(d\rho^2 + dz^2) \pm \frac{\left((\hat{g}_b)_{ij} - \sum_{k,l=1}^{N} (D)_{kl} \lambda_k^{-1} \lambda_l^{-1} L_i^{(k)} L_j^{(l)}\right)}{\rho^N} \left(\prod_{k=1}^{N} \lambda_k\right) dx^i dx^j \tag{2.191b}$$

where

$$\begin{aligned} (D)_{kl} &:= (\Gamma^{-1})_{kl}, \quad (\Gamma)_{kl} := m_i^{(k)} (\hat{g}_b)_{ij} m_j^{(l)} (\rho^2 + \lambda_k \lambda_l)^{-1}, \\ L_i^{(k)} &:= m_j^{(k)} (\hat{g}_b)_{ji}, \quad m_i^{(k)} := m_{2j}^{(k)} (\psi_b^{-1}(\lambda_k, \rho, z))_{ji}. \end{aligned} \tag{2.192}$$

In the above Definition 2.4.2, matrix $(\hat{g}_b)_{ij}$ is the background solution and g_{b22} is the solution g_{22} corresponding to $(\hat{g}_b)_{ij}$ given by (2.190) evaluated with $U_b := \rho \hat{g}_{b,\rho} \hat{g}_b^{-1}$ and $V_b := \rho \hat{g}_{b,z} \hat{g}_b^{-1}$; moreover $\psi^{-1}(\lambda, \rho, z)$ is the inverse of the solution generating matrix of the system[53]

$$\begin{aligned} \left(\partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda\right) \psi &= \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \psi, \\ \left(\partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2} \partial_\lambda\right) \psi &= \frac{\rho V + \lambda U}{\lambda^2 + \rho^2} \psi. \end{aligned} \tag{2.193}$$

Furthermore, $\tilde{C} := 16C$ and C is an arbitrary constant but with a sign such that we have $g_{22}^{(ph)} \geq 0$; the + or − sign in the line element must be chosen appropriately in order to have the correct signature of the metric.

The generating matrix satisfies the relation

$$g(\rho, z) = \psi(0, \rho, z); \tag{2.194}$$

therefore

$$\psi(\lambda, \rho, z) = g(\rho, z) + \sum_n \lambda^n F(\rho, z). \tag{2.195}$$

A first difference with respect to the general case treated in paragraph 2.4.2.1 is due to the equations that describe the trajectories of the poles, which in this case, are written as

$$\begin{aligned} \lambda_{k,z} &= \frac{-2\lambda_k^2}{\lambda_k^2 + \rho^2}, \\ \lambda_{k,\rho} &= \frac{2\lambda_k \rho}{\lambda_k^2 + \rho^2}; \end{aligned} \tag{2.196}$$

[53] To obtain the background generating matrix just replace the matrices U and V with their background counterpart, that is $U_b := \rho \hat{g}_{b,\rho} \hat{g}_b^{-1}$ and $V_b := \rho \hat{g}_{b,z} \hat{g}_b^{-1}$.

the solutions of equations (2.196) are

$$\lambda_k^{(\pm)} = (w_k - z) \pm \sqrt{(w_k - z)^2 + \rho^2}, \tag{2.197}$$

where the w_k are complex arbitrary constants. We can see the root is always positive for real poles (since w_k is real) and therefore the solution, in this case, will not experience discontinuity and it will be present in all spacetime, contrary to what we saw in paragraph 2.4.2.2. The second difference derives from the requirement that the condition $det(\hat{g}) = -\rho^2$, necessary to have a physical solution, must be satisfied. As discussed in paragraph 2.4.2.1, to obtain the physical solution, $\hat{g}^{(ph)}$, we need to compute the determinant of the solution $\hat{g}$; the calculation leads to

$$det(\hat{g}) = (-1)^N \rho^{2N} \left(\prod_{k=1}^{N} \frac{1}{\lambda_k^2} \right) det(\hat{g}_b). \tag{2.198}$$

The above equation (2.198) shows that if the background solution is such that $det(\hat{g}_b) = -\rho^2$ then the solution must necessarily be an even number of solitons, $N = 2n$ with $n \in \mathbb{N} \setminus \{\infty\}$, otherwise the sign of the determinant of $\hat{g}$ would change and this would lead to a non-physical metric.

We now come to the Kerr black hole. As mentioned earlier, in the case of axialsymmetric stationary metrics the simplest solution is the $N = 2$ axialgravisolitons solution and we can show the following

Theorem 2.4.1 (Kerr black hole as $N = 2$ axialgravisolitons solution)**.** *The Kerr black hole is a N-axialgravisoliton solution of Einstein gravity with $N = 2$ and flat background solution.*

Proof. We choose as the background metric the flat metric in cylindrical coordinates

$$ds^2 = -dt^2 + \rho^2 d\phi^2 + d\rho^2 + dz^2, \tag{2.199}$$

comparing with (2.189) we see that $g_{b22} = 1$ and that $\hat{g}_b = diag(-1, \rho^2)$ (which trivially satisfies $det(\hat{g}_b) = -\rho^2$). Matrices V_b and U_b are given by

$$V_b := \rho \hat{g}_{b,z} \hat{g}_b^{-1} = 0, \quad U_b := \rho \hat{g}_{b,\rho} \hat{g}_b^{-1} = -\frac{1}{\rho} \begin{bmatrix} 0 & 0 \\ 0 & 2\rho \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{\rho^2} \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{\rho^2} \end{bmatrix}; \tag{2.200}$$

that inserted in the (2.193) allow us to calculate the background generating matrix as solution of the following system

$$\begin{aligned}\left(\partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda\right)\begin{bmatrix} A & B \\ C & D\end{bmatrix} &= +\frac{\lambda}{\lambda^2+\rho^2}\begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{\rho^2}\end{bmatrix}\begin{bmatrix} A & B \\ C & D\end{bmatrix};\\ \left(\partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda\right)\begin{bmatrix} A & B \\ C & D\end{bmatrix} &= -\frac{\lambda}{\lambda^2+\rho^2}\begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{\rho^2}\end{bmatrix}\begin{bmatrix} A & B \\ C & D\end{bmatrix}.\end{aligned} \tag{2.201}$$

The solution is

$$\psi_b = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2 - 2z\lambda - \lambda^2\end{bmatrix}. \tag{2.202}$$

From the matrix ψ_b, remembering the definitions of $m_i^{(k)}$, of $L_i^{(k)}$ and of $(\hat{\Gamma})_{kl}$ and using (2.196), which gives us $\lambda_k^2 + 2z\lambda_k - \rho^2 = 2w_k\lambda_k$, we get

$$m_0^{(k)} = m_{2j}^{(k)}(\psi_b^{-1})_{j0} = m_{20}^{(k)}(\psi_b^{-1}(\lambda_k,\rho,z))_{00} + m_{21}^{(k)}(\psi_b^{-1}(\lambda_k,\rho,z))_{10} = -m_{20}^{(k)} := C_0^{(k)};$$

$$\begin{aligned} m_1^{(k)} &= m_{2j}^{(k)}(\psi_b^{-1})_{j1} = m_{20}^{(k)}(\psi_b^{-1}(\lambda_k,\rho,z))_{01} + m_{21}^{(k)}(\psi_b^{-1}(\lambda_k,\rho,z))_{11} = \\ &= \frac{m_{21}^{(k)}}{\rho^2 - 2z\lambda_k - \lambda_k^2} = \frac{m_{21}^{(k)}}{-2w_k\lambda_k} := C_1^{(k)}\lambda_k^{-1};\end{aligned}$$

$$L_0^{(k)} := m_j^{(k)}(\hat{g}_b)_{j0} = m_0^{(k)}(\hat{g}_b)_{00} + m_1^{(k)}(\hat{g}_b)_{10} = -m_0^{(k)} = -C_0^{(k)}; \tag{2.203}$$

$$L_1^{(k)} := m_j^{(k)}(\hat{g}_b)_{j1} = m_0^{(k)}(\hat{g}_b)_{01} + m_1^{(k)}(\hat{g}_b)_{11} = m_1^{(k)}\rho^2 = C_1^{(k)}\lambda_k^{-1}\rho^2;$$

$$\begin{aligned}(\hat{\Gamma})_{kl} &:= m_i^{(k)}(\hat{g}_b)_{ij}m_j^{(l)}(\rho^2+\lambda_k\lambda_l)^{-1} = \\ &= m_0^{(k)}(\hat{g}_b)_{00}m_0^{(l)}(\rho^2+\lambda_k\lambda_l)^{-1} + m_1^{(k)}(\hat{g}_b)_{11}m_1^{(l)}(\rho^2+\lambda_k\lambda_l)^{-1} = \\ &= (\rho^2+\lambda_k\lambda_l)^{-1}(-C_0^{(k)}C_0^{(l)} + C_1^{(k)}\lambda_k^{-1}C_1^{(l)}\lambda_l^{-1}\rho^2).\end{aligned}$$

Equations (2.203) are all we need, together with (2.196) for the pole values and the Definition 2.4.2 for the physical quantities, to build the $N = 2n$ axialgravisolitons solution with flat background metric.
Let us specialize for $n = 1$ and write the two constants w_1 and w_2 of the two poles as

$$w_1 = \sigma_1 + \sigma_2, \qquad w_2 = \sigma_1 - \sigma_2; \tag{2.204}$$

σ_1 is always taken real while σ_2 can be real (the poles will be real) or imaginary (the poles will be complex conjugated). We introduce two new variables θ and r defined by the relations[54]

$$\rho = sin(\theta)\sqrt{(r-m)^2-\sigma_2^2}, \qquad z = \sigma_1 + (r-m)cos(\theta), \tag{2.205}$$

where m is an arbitrary constant. By inserting the (2.204) in the (2.196), using the (2.205), we obtain

$$\lambda_1 = 2(r-m+\sigma_2)sin^2(\theta/2), \qquad \lambda_2 = 2(r-m-\sigma_2)sin^2(\theta/2); \tag{2.206}$$

in which the same sign has been chosen for the two poles. Without loss of generality we can fix

$$C_1^{(1)}C_0^{(2)} - C_0^{(1)}C_1^{(2)} = \sigma_2, \qquad C_1^{(1)}C_0^{(2)} + C_0^{(1)}C_1^{(2)} = -m \tag{2.207}$$

and we can define the new arbitrary constants

$$-b := C_1^{(1)}C_1^{(2)} - C_0^{(1)}C_0^{(2)}, \qquad a := C_1^{(1)}C_1^{(2)} + C_0^{(1)}C_0^{(2)}; \tag{2.208}$$

where $C_0^{(1)}$, $C_0^{(2)}$, $C_1^{(1)}$ and $C_1^{(2)}$ are the constants that appear in the first two equations (2.203); putting together (2.207) and (2.208) we get

$$\sigma_2^2 = m^2 - a^2 + b^2. \tag{2.209}$$

From Definition 2.4.2 thanks to (2.203) we can calculate the physical quantities $(\hat{g})_{ij}^{(ph)}$ and $g_{00}^{(ph)}$ which, substituted in the line element of Definition 2.4.2 expressed in the coordinates (t, ϕ, θ, r), allow us to obtain

$$ds^2 = -\frac{\tilde{C}\omega dr^2}{\Delta} - \tilde{C}\omega d\theta^2 - \frac{K_1[dt+2ad\phi]^2 + K_2[dt+2ad\phi]d\phi - K_3 d\phi^2}{\omega}, \tag{2.210}$$

in which the quantities

$$\begin{aligned}
\omega &:= r^2 + [b - acos(\theta)]^2, \qquad \Delta := r^2 - 2mr + a^2 - b^2, \\
K_1 &:= \Delta - a^2 sin^2(\theta), \\
K_2 &:= 4\Delta b cos(\theta) - 4a sin^2(\theta)[mr + b^2], \\
K_3 &:= \Delta[a sin^2(\theta) + 2b cos(\theta)]^2 - sin^2(\theta)[r^2 + b^2 + a^2]^2,
\end{aligned} \tag{2.211}$$

[54] The root is defined in such a way that in the limit $r \to \infty$ the dominant term returns the usual transformations between spherical and cylindrical coordinates.

have been defined. Setting $b = 0$, redefining the time coordinate as $t := t + 2a\phi$ and setting $\tilde{C} = -1$ we have

$$ds^2 = \frac{\omega dr^2 + \omega d\theta^2}{\Delta} - \frac{[\Delta - a^2 sin^2(\theta)]dt^2}{\omega} - \frac{[\Delta a^2 sin^4(\theta) - sin^2(\theta)(r^2 + a^2)^2]d\phi^2}{\omega}, \tag{2.212}$$

where, now, $\omega = r^2 + a^2 cos^2(\theta)$ e $\Delta = r^2 - 2mr + a^2$. We can rewrite (2.212) as

$$ds^2 = -dt^2 + \omega\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)sin^2(\theta)d\phi^2 + \frac{2mr[asin^2(\theta)d\phi - dt]^2}{\omega}; \tag{2.213}$$

we immediately notice the presence of two possible singularities placed at $\Delta = 0$ and $\omega = 0$. The singularity in $\omega = 0$ is a gravitational singularity[55] and it is a ring singularity, while the singularity in $\Delta = 0$ exists only if[56] $m^2 \geq a^2$; in this case it can be shown that, passing to the Kerr coordinates, the singularity disappears and it is therefore a coordinate singularity corresponding to an horizon.[57]. For Penrose cosmic censorship principle [35], a gravitational singularity must be hidden by an event horizon and this prompts us to choose $m^2 \geq a^2$ and therefore, with reference to (2.209) with $b = 0$, we must have σ_2 real; the consequence is that, from (2.204), the two poles of the dressing matrix must be real. With this choice of poles, line element (2.213) is the Kerr line element in the Boyer-Lindquist coordinates[58] which describes an uncharged rotating black hole with a ratio between mass and angular momentum equal to a. If this ratio were null it would mean that the black hole does not rotate, in fact for $a = 0$ we obtain the Schwarzschild metric. □

The last comment is reserved for the topological structure of the soliton solution leading to black hole metrics. As already mentioned, in gravisoliton theory it is not clear when a gravisoliton is a topological object with an associated topological charge. For the Kerr black hole case we could conclude that this is a topological solitons: Kerr black holes have a ring-shaped singularity and therefore there exist non-contractable laces, which means a non-trivial first homotopy group which can be regarded as the topological charge associated to the Kerr balk hole gravisoliton. On the other hand, the unperturbed space, i.e. flat spacetime, has trivial topology hence trivial first homotopy group.

2.5 Higher-Spin gauge theory and duality in gauge theories

One of the goals of Quantum Field Theory is to explore the landscape of consistent theories; since the free fields are characterized by spin and mass we can ask ourselves which sets of fields specified by their spins and masses admit consistent interactions. The massless fields case of spins $s \in \frac{1}{2}\mathbb{Z}$ being gauge fields are of

[55]The singularity is at $r = 0$, $\theta = \frac{\pi}{2}$.

[56]This is because the equation $\Delta = 0$ admits real solutions only if the discriminant is non-negative, i.e. only if $4m^2 - 4a^2 \geq 0 \Rightarrow m^2 \geq a^2$.

[57]In reality, since the equation is quadratic there are two event horizons, one internal and one external.

[58]They are a generalization of the spherical coordinates. The transition from a Cartesian triple (x^1, x^2, x^3) to a Boyer-Lindquist one (r, θ, ϕ) is given by $x^1 = \sqrt{r^2 + a^2} sin(\theta)cos(\phi)$, $x^2 = \sqrt{r^2 + a^2} sin(\theta)sin(\phi)$, $x^3 = rcos(\theta)$.

particular interest because their interactions are severely constrained by gauge symmetry. These kind of fields emerge quite naturally from String Theory whose spectrum contains infinitely many fields of all spins, mostly massive and highly degenerate in spin but which can be considered as massless in a suitable EFT. An interesting feature is that the finitude of the spectrum of a given field theory depends by a threshold value of spin, which is $s = 2$: if a theory consists of lower spins its spectrum can be finite while if the theory consists in higher spins the spectrum is necessarily infinite containing fields of all spins. In the following we refer as Lower-Spin (LS) if the theory contains at most $s = 2$ while we refer as Higher-Spin (HS) if the theory contains $s > 2$. In the Higher-Spin context, the Vasiliev Higher-Spin theory [84],[85],[86],[87]. can be considered as the missing link in the evolution from the field theories of Lower-Spin to String Theories; in fact, Vasiliev theory is the minimal theory whose spectrum contains Higher-Spin fields. Its spectrum consists of massless fields of all spins $s \in \mathbb{N}$ each appearing once; hence in this respect it is much simpler than String Theory. Moreover, tensor fields in exotic, i.e. mixed symmetric, Higher-Spin representations of the Lorentz group arise as massive modes in String Theory too and also in this case limits in which such fields might become massless are of particular interest since these would be exotic gauge fields. Such exotic gauge fields can also arise as dual representations of more familiar gauge theories as we will see in Paragraph 2.5.3.

2.5.1 Standard Higher-Spin theories and a glimpse of Vasiliev theory

It is commonly stated that the theory of HS fields originates from the works of Dirac, Fierz and Pauli [7],[10],[11]. The key result is an elegant description of the physical state conditions for symmetric tensors $\phi_{\mu_1\dots\mu_s}$ of arbitrary rank of spin s and for symmetric tensor–spinors $\psi_{\mu_1\dots\mu_s}$ of arbitrary rank of spin $s+\frac{1}{2}$ given by

$$\begin{cases} (\Box + m^2)\phi_{\mu_1\dots\mu_s} = 0, \\ \partial^{\mu_1}\phi_{\mu_1\dots\mu_s} = 0, \\ \eta^{\mu_1\mu_2}\phi_{\mu_1\mu_2\dots\mu_s} = 0; \end{cases} \qquad \begin{cases} (\not{\partial} m)\psi_{\mu_1\dots\mu_s} = 0, \\ \partial^{\mu_1}\psi_{\mu_1\dots\mu_s} = 0, \\ \gamma^{\mu_1}\psi_{\mu_1\dots\mu_s} = 0. \end{cases} \tag{2.214}$$

For any given s, the first equations define the mass–shell, the second eliminate unwanted "time" components and finally the last confine the available excitations to irreducible multiplets. During the seventies action principles for massive symmetric Bose and Fermi HS fields were found [43],[44]; these results were the starting point for the subsequent work by Fronsdal and Fang where gauge invariant actions were finally recovered in the massless limit [52],[53],[61]. Fronsdal arrived to a natural generalization of the Maxwell and linearized Einstein equations given by

$$\mathfrak{F}_{\mu_1\dots\mu_s} := \Box\phi_{\mu_1\dots\mu_s} - \partial_{(\mu_1}\partial\cdot\phi_{\mu_2\dots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\eta^{\nu_1\nu_2}\phi_{\nu_1\nu_2\mu_3\dots\mu_s)} = 0 \tag{2.215}$$

where $\mathfrak{F}_{\mu_1\dots\mu_s}$ is known as Fronsdal tensor and equations (2.215) are known as Fronsdal equations; their structure is quite natural but their behavior under gauge transformations $\delta\phi_{\mu_1\dots\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2\dots\mu_s)}$ entails a surprise, since its gauge variation is proportional to the derivatives of the gauge parameter trace. In other words, gauge invariance demands that $\eta^{\mu_1\mu_2}\epsilon_{\mu_1\mu_2...\mu_{s-1}} = 0$, so that the gauge parameter must be traceless. This condition first shows up for $s = 3$ and is dubbed Fronsdal's first constraint. Moreover in Fronsdal formulation of HS gauge theories there is one subtle constraint. The free Einstein-like lagrangian can be written, upon integration by parts, as

$$\mathcal{L} = \phi_{\mu_1...\mu_s}\left(\mathfrak{F}^{\mu_1...\mu_s} - \frac{1}{2}\eta^{\mu_1\mu_2}\eta_{\nu_1\nu_2}\mathfrak{F}^{\nu_1\nu_2\mu_3...\mu_s}\right) \tag{2.216}$$

and for it to be gauge invariant we need to require that $\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\phi_{\mu_1\mu_2\mu_3\mu_4...\mu_s} = 0$, i.e. the field must be doubly traceless; this condition, known as Fronsdal's second constraint, arise for the first time for $s = 4$. The need for these constraints is a sign that, in some sense, Fronsdal's formulation is incomplete, since it does not accommodate the natural expected gauge symmetry, despite the fact that algebraic constraints certainly do not raise doubts on the nature of the actual propagating degrees of freedom.

Higher-Spin theories have an interesting geometrical structure. For this purpose, it is convenient to focus on a variant of the usual Riemann curvature that is symmetric under interchanges within its two groups of indices, obviously this is a only a change of basis and it is merely a combination of usual Riemann tensor on account of the cyclic identity. Anyway, when proceeding to arbitrary values of s, this choice has the virtue of suggesting a recursive construction where the linearized Christoffel connection symbols $\Gamma_{\nu_1\mu_1\mu_2}$ opens the way to a tower of $s-1$ Christoffel-like connections symbols $\Gamma_{\nu_1\mu_1\mu_2...\mu_s}, ..., \Gamma_{\nu_1\nu_2...\nu_{s-1}\mu_1\mu_2...\mu_s}$. The generic one of these objects are defined, using notation of [124],[258], as

$$\Gamma_{\nu_1\nu_2...\nu_m\mu_1\mu_2...\mu_s} := \frac{1}{m+1}\sum_{k=0}^{m}\frac{(-1)^k}{\binom{m}{k}}\partial^{m-k}_{(\mu)}\partial^{k}_{(\nu)}\phi \qquad m \in [1, s] \tag{2.217}$$

where we write (μ) and (ν) in the subscript of ∂ to denote that $m-k$ derivatives carry one set of symmetric indices $(\mu_1, ..., \mu_s)$, whereas k derivatives carry the other set of symmetric indices $(\nu_1, ..., \nu_m)$. As an example if $s = 2$ and $m = 1$ we have

$$\Gamma_{\nu_1\mu_1\mu_2} = \frac{1}{2}\left(\partial_{\mu_1}\phi_{\mu_2\nu_1} + \partial_{\mu_2}\phi_{\mu_1\nu_1} - \partial_{\nu_1}\phi_{\mu_1\mu_2}\right) \tag{2.218}$$

which is exactly the linearized first connection as we know it from linearized General Relativity. Expressions (2.217) carry, by construction, $1, ..., s-1$ derivatives of the original field, and the next member of the sequence is a spin-s analogue of the linearized $s = 2$ curvature in its symmetric incarnation, a tensor $R_{\nu_1...\nu_s\mu_1...\mu_s}$ built from combinations of s derivatives of the original field that is gauge invariant under the gauge transformations we expect for a spin-s field. This systematic in HS gauge field was proposed by de Wit and Freedman [63] and one can recognize that the Fronsdal tensor $\mathfrak{F}_{\mu_1...\mu_s}$ is related to the second Christoffel-like connection according to

$$\mathfrak{F}_{\mu_1...\mu_s} \propto \eta^{\nu_1\nu_2}\Gamma_{\nu_1\nu_2\mu_1...\mu_s}. \tag{2.219}$$

What we have discussed here can be applied also in the case of Fermi fields carrying a set of totally symmetric vector labels, up to standard complications brought about by γ-matrices [52],[53],[61],[134],[146]. It is interesting to underline that free spin-s fields can be described in the realm of twistor theory [33],[39],[243]. Roughly speaking, twistor theory is a framework for encoding physical information on spacetime as geometric data on a complex projective space, known as a twistor space ($\mathbb{PT}$) in $D = 4$ and ambitwistor space ($\mathbb{PA}$) in $D > 4$ with a relationship which is non-local. The crucial result, which we do not prove, is the following [33],[34],[36]

Theorem 2.5.1 (Penrose transform)**.** *Given a* HS *gauge theory with spin-s gauge field $\phi_{\mu_1\dots\mu_s}$ on a complexified manifold $(M_{\mathbb{C}}, g)$ the following map*

$$\mathcal{PT} \;:\; \frac{\{\phi_{\mu_1\dots\mu_s}\}}{\{\phi_{\mu_1\dots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\dots\mu_s)}\}} \to H^{0,1}(\mathbb{X}, L^{s-1}) \tag{2.220}$$

where the quotient means we are taking into account only non-pure gauge field configurations, L^{s-1} is the $(s-1)$-power of the line boundle over $M_{\mathbb{C}}$ and $\mathbb{X}$ is $\mathbb{PT}$ in $D = 4$ and $\mathbb{PA}$ in $D > 4$ is an isomorphism.

Therefore Penrose transform tells us that every non-pure gauge field configuration corresponds to a Dolbeault cohomology class.

We can now turn on interactions in HS theories and it is interesting to note that same classic results of the 1960's and 1970's reveal unexpected difficulties in HS interactions [172].

We have already encountered the most astonishing one in Section 2.3.1.1 where we discussed soft theorems. In the realm of HS theory, the terms in square brackets in the RHS of (2.102) become two sums of polynomials in momenta which difference is ensure to vanish if and only if the coupling constants are zero for all k. Therefore, we have too many conservation laws that trivialise the scattering matrix and a generic amplitudes with a single massless HS particle would thus seem inconsistent, i.e. only non-interacting HSs seem possible to exist. However we can note [166],[173] that the problem presents itself only when the exchanged particles have spins lower than the external ones.

The next classic result, the Coleman-Mandula theorem [31], is a formal S-matrix argument that has had a wide impact. In a few words, considering a theory under some physical condition the theorem asserts that the symmetry group of this theory is necessarily a direct product of the Poincaré group and an internal symmetry group. In fact dropping some of the hypothesis of Coleman-Mandula theorem such as the mass gap and the fact that only bosonic symmetry generators are allowed we get, respectively, conformal and supersymmetric extensions of standard field theories. Moreover, one of its key hypothesis is the presence of finitely many excitations below any given mass level; this means also a finite number of HS modes and makes it not directly applicable to the HS constructions whose gauge algebras bring about infinitely many massless gauge fields, like the HS theories emerging from String Theory. The theorem provides also a reason for the failure of naive attempts involving finitely many HS gauge fields.

Another feature of HS is the loss of causality they tend to manifest in the presence of minimal couplings to uniform electromagnetic fields. In this perspective String Theory has proved very useful. Indeed, Argyres and Nappi [80] showed that the massive $s = 2$ mode of the open string, that in some respects behaves like a massless HS field, has a coupling to uniform electromagnetic fields free from such pathology. Recent extension of their analysis shows that all symmetric tensors of the first Regge trajectory[59] work exactly in the same way without any loss of causality [165]. Anyway, the minimal couplings of these states to electromagnetism vanishes when the string tension or the HS masses tend to zero. This suggests again that HS fields are not suitable for the description of long range interactions which are mediated by low spin particles.

The last classical result is the Aragone-Deser argument [58],[65]; authors try to construct a standard gravitational coupling for a HS field with $s = \frac{5}{2}$ without success except in $D = 3$[60]. Similar results hold for every HS field with $s > 2$. Nevertheless in 1986 Fradkin and Vasiliev [78],[79] showed that in the presence of a cosmological term Λ, conventional looking gravitational couplings are allowed for HS fields up to the cubic order. However, they are inevitably accompanied by higher derivative partners that, for dimensional reasons, bring along negative powers of Λ; hence the proposal of Fradkin and Vasiliev has a singular behavior in the limit of a flat background therefore not solving the Aragone-Deser problem completely. In subsequent years, research in HS theories focused on the direction of HS interactions in the presence of a cosmological term which are in some sense minimal. The main outcome of this research are the Vasiliev equations which are formally consistent gauge invariant non-linear equations whose linearization over a specific vacuum solution describes free massless HS fields [84],[85],[86],[87]. They are a mathematical realization of free differential algebras[61] with non-polynomial scalar couplings and give rise to classically consistent HS interactions. The Vasiliev theory is the minimal theory whose spectrum contains Higher-Spin fields of all spins: its spectrum consists of massless fields of all spins each appearing once. Therefore Vasiliev theory can be considered the missing link in the evolution from the low spin field theories to String Theory which is far more complicate since HS modes in String Theory are infinitely many for each spins. Since the Vasiliev equations are quite complicated there are few exact solutions known. Among these there are empty anti-de Sitter space, whose existence allows to interpret the linearized fluctuations as massless fields of all spins [112], and a family of solutions to the four dimensional Vasiliev equations that are interpreted as black holes, although the metric transforms under the Higher-Spin transformations and for that reason it is difficult to rely on the usual definition of the horizon [151],[160],[201]. However this situation seems similar to the soft hair proposed in the context of the information paradox [223],[236],[254].

Vasiliev theory plays an important role also in AdS/CFT correspondence contexts. The starting point is the Klebanov-Polyakov conjecture [118] according to which the free and critical $O(N)$ vector models, as Conformal Field Theories in three dimensions, should be dual to a theory in four dimensional anti-de Sitter space with infinite number of massless Higher-Spin gauge fields. The point is that there are some Conformal Field Theories that in addition to the stress tensor have an infinite number of conserved tensors, i.e. $\partial^c j_{ca_2 \dots a_s} = 0$ with $s \in \mathbb{N} \setminus \{\infty\}$; the stress tensor corresponds to the $s = 2$ case. According to the standard AdS/CFT dictionary, fields that are dual to conserved currents have to be gauge fields as in the stress tensor and spin-two graviton field case. A generic example of a conformal field theory with Higher-Spin currents is any free CFT; for instance,

[59] For a review of the Regge theory see, for example, [38].

[60] This kind of theories are called hypergavity and it seems possible to avoid inconsistencies by requiring infinite tower of higher spin fields thanks to Vasiliev works we briefly discuss in a while.

[61] For a review on free differential algebras see, for example, [99].

the free $O(N)$ model defined by the sigma model action

$$S = \frac{1}{2} \int d^{D-1}x\, \partial_m \phi^i \partial^m \phi^j \delta_{ij}, \qquad i, j = 1, ..., N, \tag{2.221}$$

where D are the spacetime dimensions in the AdS side. Therefore, HS theories are generic duals of free Conformal Field Theories; a theory that is dual to the free scalar CFT is called Type-A in the literature and the theory that is dual to the free fermion CFT is called Type-B. This observation laid the foundations for the so-called HS AdS/CFT correspondence.
It should be underlined that Vasiliev equations are classical equations and no Lagrangian is known that starts from canonical two derivative Fronsdal Lagrangian and is completed by interactions terms; so they provide an example of possible non-lagrangian theory. Moreover, it is not known in general how to extract the interaction vertices of the HS theory out of the equations. We refer to [103],[112],[132],[184] to a detailed discussion on Vasiliev theory.

2.5.2 Towards mixed symmetry tensor Higher-Spins

In 1980, in a series of seminal papers [59],[60],[71], Curtright proposed to use arbitrary irreducible tensor representation of the Lorentz group as gauge fields. In the previous Paragraph we discussed the case of totally symmetric tensors; now we are going to discuss the case of totally antisymmetric tensor and mixed symmetry ones. A primary interest in the covariant description of free particles carrying arbitrary irreducible representations of the Lorentz group is strong motivated by String Field Theories where these kind of fields emerge [122]; but also a field theoretical description of the massless representations based on the principle of gauge invariance itself would be an important achievement for theoretical and mathematical physics. The starting point to understand how to describe gauge field carrying arbitrary irreducible representation of the Lorentz group is the Young tableaux machinery we developed in Section 2.1.1.1. In the case of a standard spin-s HS field, which is totally symmetric, the Young tableaux is of the form

$$\underbrace{\boxed{}\boxed{}\cdots\boxed{}\boxed{}}_{s} \tag{2.222}$$

while the gauge transformation is written as the derivative of an $s-1$ indexes tensor, which plays the role of the gauge parameter, then completely symmetrized; we draw this as

$$\underbrace{\boxed{}\boxed{}\cdots\boxed{}}_{s-1}\boxed{\partial} \tag{2.223}$$

where the box with the derivative indicates that we need to eliminate a box to get a totally symmetric tensor with $s-1$ indexes and to take a derivative of this object and symmetrize over all the s indexes to get the gauge transformation. It is simple to check that the number of physical degrees of freedom that this theory describes are the ones of the corresponding representation of the little group; hence for a HS field the story ends here. Let us now look at the case of a totally antisymmetric gauge field with p indexes: a p-form gauge field. The Young tableaux for a p-form gauge field is simply

$$p\left\{\begin{array}{c}\boxed{}\\ \boxed{}\\ \vdots\\ \boxed{}\\ \boxed{}\end{array}\right. \tag{2.224}$$

while the gauge transformation is written as the derivative of an $p-1$ indexes tensor, which play the role of the gauge parameter, then completely antisymmetrized; we draw this as

$$\begin{array}{l} p-1\left\{\begin{array}{c}\boxed{}\\ \boxed{}\\ \vdots\\ \boxed{}\\ \boxed{}\end{array}\right. \\ \hphantom{p-1\{}\boxed{\partial}\end{array} \tag{2.225}$$

where the box with the derivative indicates that we need to eliminate a box to get a totally antisymmetric tensor with $p-1$ indexes and to take a derivative of this object and antisymmetrize over all the p indexes to get the gauge transformation. However, in this case the counting of physical degrees of freedom is not so straightforward as in the previous one. The reason is that, unlike the previous case of standard HS, there exist transformations of the gauge parameter that leave invariant the p-form gauge field. These transformations, called first gauge for gauge, are given by

$$\begin{array}{l} p-2\left\{\begin{array}{c}\boxed{}\\ \boxed{}\\ \vdots\\ \boxed{}\\ \boxed{}\end{array}\right. \\ \hphantom{p-2\{}\boxed{\partial}\end{array} \tag{2.226}$$

and are parameterized by the so-called first gauge for gauge parameter. Furthermore, the first gauge for gauge parameter also possesses gauge transformations that leave invariant the gauge parameter; these are called second gauge for gauge transformations, are parameterized by the so-called second gauge for gauge parameter and are given by

$$p-3\left\{\begin{ytableau} \ \\ \ \\ \vdots \\ \ \end{ytableau}\right.\hspace{-0.3em}\begin{ytableau} \none \\ \partial \end{ytableau}\ . \tag{2.227}$$

This chain of gauge for gauge redundancy go on until the end arriving to a scalar.

It is now the turn of the mixed symmetry tensors [76]. The first step is to better understand the principle of gauge invariance; from the previous examples we infer:

1. the covariant description can be formulated with a field whose spacetime indexes have the structure of the Young tableau corresponding to an irreducible representation of the general linear group, generalizing the ideas of HS theories;
2. this field is invariant under gauge transformations whose gauge parameters are fields with the structure of their indices corresponding to all the Young tableaux that one can make by removing one box from the original Young tableau;
3. there is a highly non-trivial degrees of freedom counting that leads to the introduction of more and more generation of gauge for gauge parameters.

Let us give an example to construct the full gauge for gauge family; let us consider the following Young tableau

$$\ydiagram{3,3,1} \tag{2.228}$$

whose corresponding field carries the $\lambda = (3, 2, 2)$ irreducible representation The first gauge for gauge parameters are given by

$$\ydiagram{3,3}\ ,\ \ydiagram{3,2,1}\ ; \tag{2.229}$$

the second gauge for gauge parameters are given by

$$\ydiagram{3,2}\ ,\ \ydiagram{3,2}\ ,\ \ydiagram{2,2,1}\ ; \tag{2.230}$$

the third gauge for gauge parameters are given by

$$\ydiagram{2,2}\,,\quad \ydiagram{2,2}\,,\quad \ydiagram{2,2}\,. \tag{2.231}$$

In the perspective of this gauge for gauge redundancy, we refer to the gauge transformation of the original field as the zeroth gauge for gauge transformation and at the gauge parameters of the original field as the zeroth gauge for gauge parameters. Hence in the description of a field carrying the $\lambda = (3, 2, 2)$ irreducible representation we have a total of 8 arbitrary tensor functions parameterizing all the levels of the gauge for gauge.

This example shows that, probably, to study mixed symmetry tensors gauge theories and their duality the right way are formal mathematical tools[62] instead of explicit computations.
In this spirit, we now review the so-called multi-form formalism we are going to use in Chapter 5 to generalize the theorem of Section 3.6. Let us start with the case of the bi-form formalism [125],[128]. First of all, the basic definition

Definition 2.5.1 (Bi-form space)**.** *The bi-form space on a D-dimensional differential manifold $(M, \mathcal{A})$ with metric $\mathbf{g}$, dubbed $\Omega^{p\otimes q}(M)$, is the tensor product space between the space of p-forms $\Omega^p(M)$ and the space of q-form $\Omega^q(M)$.*

In general the resulting tensor T can be written as

$$T = \frac{1}{p!q!} T_{[\mu_1 \dots \mu_p][\nu_1 \dots \nu_q]} (dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) \otimes (dx^{\nu_1} \wedge \dots \wedge dx^{\nu_q}), \tag{2.232}$$

and, always in general, carries a reducible representation of GL(D). To extract the irreducible representation labelled by the Young tableau $\lambda = (p, q)$ we make us of the Young projector $\Pi_{(p,q)} : \Omega^{p\otimes q}(M) \to \Omega^{p\otimes q}(M)$. In the biform space the left and right differentials are well defined

Definition 2.5.2 (Left and right differential)**.** *The left differential d_L and right differential d_R are the usual de Rham differential that act only on one of the spaces of the differential forms used to construct the space of the bi-forms. Therefore*

$$d_L : \Omega^{p\otimes q}(M) \to \Omega^{p+1\otimes q}(M), \qquad d_R : \Omega^{p\otimes q}(M) \to \Omega^{p\otimes q+1}(M). \tag{2.233}$$

The following result is easy to show

Proposition 2.5.1 (Left and right differential basic properties)**.** *The left and right differentials are such that $d_L \circ d_L = 0$, $d_R \circ d_R = 0$, moreover they commute $d_L \circ d_R = d_R \circ d_L$.*

[62] For example, homological and cohomological algebra tools and algebroids in order to extend and generalize theorems one can prove directly in HS or p-form gauge theories such as the one proved in Section 3.6.

Proof. It follows immediately from the nilpotency property of the standard de Rham differential and from the commutativity of derivatives since differential forms have smooth components. □

Since we are interested in the application of this formalism in gauge theories we can define the left and right field strength as follows.

Definition 2.5.3 (Left and right field strength)**.** *Given a gauge field B whose writing on a chart $(\phi, U) \in \mathcal{A}$ is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = (p, q)$, the left field strength H_L and the right field strength H_R are the Young projected bi-forms given by*

$$H_L := d_L B \in \Omega^{p+1 \otimes q}(M), \qquad H_R := d_R B \in \Omega^{p \otimes q+1}(M), \tag{2.234}$$

when they are meaningful, i.e. when the application of Young projector extracts a bi-form with a well defined associated Young tableau.

Roughly speaking, these are partial field strengths which are not completely gauge invariant but that are useful to write down lagrangians density; they correspond to the irreducible representation given by the Young tableaux $\lambda_L = (p + 1, q)$ and $\lambda_R = (p, q + 1)$. The full gauge invariant field strength can be constructed using the left and right differentials

Definition 2.5.4 (Field strength)**.** *Given a gauge field B whose writing on a chart $(\phi, U) \in \mathcal{A}$ is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = (p, q)$, the field strength H is the Young projected bi-forms given by*

$$H := d_L \circ d_R B = d_R \circ d_L B \in \Omega^{p+1 \otimes q+1}(M), \tag{2.235}$$

when they are meaningful, i.e. when the application of Young projector extracts a bi-form with a well defined associated Young tableau.

This object is by construction fully gauge invariant due to the properties proven in Proposition 2.5.1 and its writing in a chart corresponds to a mixed symmetry tensor which carries the irreducible representation associated to the Young tableau $\lambda = (p + 1, q + 1)$. In a similar way we can introduce the left and right Hodge morphism

Definition 2.5.5 (Left and right Hodge morphism)**.** *The left Hodge morphism $\star_L$ and right Hodge morphism $\star_R$ are the usual Hodge morphism that act only on one of the spaces of the differential forms used to construct the space of the bi-forms. Therefore*

$$\star_L : \Omega^{p \otimes q}(M) \to \Omega^{D-p \otimes q}(M), \qquad \star_R : \Omega^{p \otimes q}(M) \to \Omega^{p \otimes D-q}(M). \tag{2.236}$$

The full Hodge morphism is simply given by $\star := \star_L \circ \star_R = \star_R \circ \star_L$.

This formalism can be generalized to the case of more than two form spaces as follows

Definition 2.5.6 (N-multi-form space)**.** *The N-multi-form space on a D-dimensional differential manifold $(M, \mathcal{A})$ with metric $\mathbf{g}$, dubbed $\Omega^{p_1 \otimes \ldots \otimes p_N}(M)$, is the tensor product space between the spaces of p_i-forms $\Omega^{p_i}(M)$ with $i \in [1, N]$.*

An object living in the N-multi-form space is given, in local coordinates, by a mixed symmetry tensor $T_{[\mu_1^{(1)} \ldots \mu_{p_1}^{(1)}] \ldots [\mu_1^{(N)} \ldots \mu_{p_N}^{(N)}]}$ and to extract the irreducible representation labelled by the Young tableaux $\lambda = (p_1, ..., p_N)$ we need to introduce the Young projector $\Pi_{(p_1,..,p_N)} : \Omega^{p_1 \otimes \ldots \otimes p_N}(M) \to \Omega^{p_1 \otimes \ldots \otimes p_N}(M)$. In the N-multi-form space is defined the i-th differential

Definition 2.5.7 (i-th differential)**.** *The i-th differential $d^{(i)}$ is the usual de Rham differential that acts only on one of the spaces of the differential forms used to construct the space of the N-multi-forms. Therefore*

$$d^{(i)} : \Omega^{p_1 \otimes \ldots \otimes p_i \otimes \ldots \otimes p_N}(M) \to \Omega^{p_1 \otimes \ldots \otimes p_i + 1 \otimes \ldots \otimes p_N}(M). \tag{2.237}$$

Results of Proposition 2.5.1 generalize quite obviously to $d^{(i)} \circ d^{(i)} = 0 \quad \forall i \in [1, N]$ and $d^{(i)} \circ d^{(j)} = d^{(j)} \circ d^{(i)} \quad \forall i \neq j$. Thanks to these differentials we can build up the i-th field strength

Definition 2.5.8 (i-th field strength)**.** *Given a gauge field B whose writing on a chart $(\phi, U) \in \mathcal{A}$ is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = (p_1, ...p_N)$, the i-th field strength $H^{(i)}$ is the Young projected N-multi-forms given by*

$$H^{(i)} := d^{(i)} B \in \Omega^{p_1 \otimes \ldots \otimes p_i + 1 \otimes \ldots \otimes p_N}(M), \tag{2.238}$$

when it is meaningful, i.e. when the application of Young projector extracts a N-multi-form with a well defined associated Young tableau.

As before, these are not fully gauge invariant and their writing in a chart corresponds to a mixed symmetry tensor which carries the irreducible representation associated to the Young tableaux $\lambda^{(i)} = (p_1, ..., p_i + 1, ..., p_N)$. To define the field strength we introduce the following definition

Definition 2.5.9 (k-cumulative field strength)**.** *Given a gauge field B whose writing on a chart $(\phi, U) \in \mathcal{A}$ is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = (p_1, ...p_N)$, the k-cumulative field strength $H^{(k)}$ is the Young projected N-multi-forms given by*

$$H^{(k)} := d^{(k)} \circ d^{(k-1)} \circ \ldots \circ d^{(2)} \circ d^{(1)} B \in \Omega^{p_1 + 1 \otimes \ldots \otimes p_k + 1 \otimes p_{k+1} \otimes \ldots \otimes p_N}(M), \qquad k \leq N \tag{2.239}$$

when it is meaningful, i.e. when the application of Young projector extracts a N-multi-form with a well defined associated Young tableau.

These objects will be useful in the description of the duality of HS gauge theories in the next Paragraph. In the end we have the definition of the field strength

Definition 2.5.10 (Field strength)**.** *Given a gauge field B whose writing in a chart $(\phi, U) \in \mathcal{A}$ is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = (p_1, ...p_N)$, the field strength H is the Young projected N-multi-forms given by*

$$H := d^{(N)} \circ d^{(N-1)} \circ ... \circ d^{(2)} \circ d^{(1)} B \in \Omega^{p_1+1 \otimes ... \otimes p_i+1 \otimes ... \otimes p_N+1}(M), \tag{2.240}$$

when it is meaningful, i.e. when the application of Young projector extracts a bi-form with a well defined associated Young tableau.Equivalently, the field strength H is the N-cumulative field strength.

This object is by construction fully gauge invariant due to the properties generalized from Proposition 2.5.1 and its writing on a chart corresponds to a mixed symmetry tensor which carries the irreducible representation associated to the Young tableau $\lambda = (p_1 + 1, ..., p_N + 1)$. We can also introduce the i-th Hodge morphism as follows

Definition 2.5.11 (i-th Hodge morphism)**.** *The i-th Hodge morphism $\star^{(i)}$ is the usual Hodge morphism that acts only on one of the spaces of the differential forms used to construct the space of the N-multi-forms. Therefore*

$$\star^{(i)} : \Omega^{p_1 \otimes ... \otimes p_i \otimes ... \otimes p_N}(M) \to \Omega^{p_1 \otimes ... \otimes D-p_i \otimes ... \otimes p_N}(M). \tag{2.241}$$

The full Hodge morphism is simply given by $\star := \star^{(N)} \circ \star^{(N-1)} \circ ... \circ \star^{(2)} \circ \star^{(1)}$.

In the case $N = 1$, i.e. the tensor product is trivial and we have only a standard space of forms of some degree, all the i-th field strength degenerate to the only field strength H.

2.5.3 Duality in gauge theories

As we know from previous paragraphs, given a Young tableau $\lambda = \{\lambda_1, ..., \lambda_\ell\}$ with $\ell \leq D$, it defines an irreducible representation of O$(D-2)$ if and only if $n_1^{(\lambda)} + n_2^{(\lambda)} \leq D - 2$ where $n_1^{(\lambda)}$ and $n_2^{(\lambda)}$ are the number of boxes in the first two columns of the Young tableau while in the case of SO$(D-2)$ the condition is the same. In the case of pseudo-orthogonal and pseudo-special orthogonal groups the same holds. However, different Young tableaux can be linked by a duality: given λ and ξ such that $n_1^{(\lambda)} = D - 2 - n_1^{(\xi)}$ and $n_i^{(\lambda)} = n_i^{(\xi)}$ $\forall i > 1$ we say that λ and ξ are dual. Moreover, for SO$(D-2)$ these two Young tableaux define the same irreducible representation [81]. It is fundamental to underline that Young tabelaux λ and ξ are completely uncorrelated as Young tabelaux labelling different irreducible representations of GL(D). From the physical point of view this means that the particle carrying irreducible representations λ and ξ are not dual off-shell but they become propagating the same degrees of freedom when we go on-shell. Let us give some explicit examples.

Example 1: the scalar and the two form in $D = 4$

The scalar representation of SO(2) has the following trivial Young diagram λ:

$$\bullet\ . \tag{2.242}$$

If we now apply the rules given before we can construct a dual Young tableau ξ with $0 = 2 - n_1^{(\xi)}$ and $0 = n_i^{(\xi)}$, that is

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\ , \tag{2.243}$$

which defines the two form irreducible representation of SO(2). It is interesting to note again that in [251],[262] authors investigate the asymptotic symmetries of a 2-form gauge theory and they find a correspondence between the asymptotic symmetry charges of the two theories. These works motivated the ideas that dual theories, in the sense of Young tableaux machinery, can encode information about the asymptotic symmetries and charges of the original theory. They are, in some sense, the milestones for the works on which Chapters 3 and 4 are based.

Example 2: the p-form and the q-form in any D

Without loss of generality we can reduce to the case $p \leq \left[\frac{D-2}{2}\right]$ since the other cases are already taken into account by the dual description. The p-form representation of SO($D-2$) has the following Young tableau λ

$$p\left\{\begin{array}{|c|}\hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline \end{array}\right. \tag{2.244}$$

If we now apply the rules given before we can construct a dual Young tableau ξ with $p = D - 2 - n_1^{(\xi)}$ and $0 = n_i^{(\xi)}$, that is

$$q\left\{\begin{array}{|c|}\hline \\ \hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline \end{array}\right. \tag{2.245}$$

hence, for a q-form to be dual to a p-form we need $q = D - 2 - p \geq \frac{D-2}{2}$.

The gauge invariant dynamic of a p-form is described by the lagrangian density [75]

$$\mathcal{L} = -\frac{1}{(p+1)!} H_{\mu_1\cdots\mu_{p+1}} H^{\mu_1\cdots\mu_{p+1}}, \tag{2.246}$$

where

$$H_{\mu_1\cdots\mu_{p+1}} = \partial_{\mu_1} B_{\mu_2\cdots\mu_{p+1}} - \sum_{i=2}^{p+1} \partial_{\mu_i} B_{\mu_2\cdots\mu_{i-1}\mu_1\mu_{i+1}\cdots\mu_{p+1}} \tag{2.247}$$

is the $(p+1)$-field strength form of the p-form $B_{\mu_1\cdots\mu_p}$ and the equations of motion are simply

$$\partial_{\mu_1} H^{\mu_1\cdots\mu_{p+1}} = 0. \tag{2.248}$$

The description in a gauge invariant lagrangian framework of p-form fields was motivated by the possibility of a p-form electromagnetism, i.e. a gauge theory whose gauge field is a p-form. It is interesting to note [75] that the authors show that this extension of standard Maxwell theory is compatible with spacetime locality only if the gauge group is $U(1)$.

Example 3: the graviton and the Curtright field in D dimensions

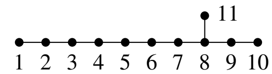
The graviton is the two indexes symmetric irreducible representation of $\mathrm{SO}(D-2)$ in D dimensions; its Young tableau λ is

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \;. \tag{2.249}$$

Applying the rules given above, we can build up the dual Young tableau ξ with $n_1^{(\xi)} = D-2-n_1^{(\lambda)} = D-3$, $n_2^{(\xi)} = n_2^{(\lambda)} = 1, 0 = n_j^{(\xi)}\ \forall j > 2$

$$D-3\left\{\begin{array}{l} \begin{array}{|c|c|}\hline & \\ \hline \end{array} \\ \begin{array}{|c|}\hline \\ \hline \end{array} \\ \;\vdots \\ \begin{array}{|c|}\hline \\ \hline \end{array} \end{array}\right. \tag{2.250}$$

which defines a mixed symmetry tensor, called Curtright hook field, carrying the irreducible representation $\xi = (D-3, 1)$ of $\mathrm{SO}(D-2)$. In the case of $D = 11$ dimensions, which are relevant to maximal supergravity and/or M-theory, the dual graviton defines a field with dual Young tableau $\xi = (8, 1)$; the interesting point is related to the conjectured $\mathfrak{e}_{11}$ algebra hidden symmetry of M-theory [148]. The $\mathfrak{e}_{11}$ algebra can be decomposed into $\mathfrak{gl}(11)$ subalgebras; the decomposition includes subalgebras whose vector subspace is an irreducible submodule corresponding to the irreducible representations of the graviton and its dual, i.e. the Curtright field. The infinite-dimensional lorentzian Kac–Moody algebra $\mathfrak{e}_{11}$ can be constructed from $\mathfrak{e}_8$ through extensions of $\mathfrak{e}_8$ Dynkin diagram with three extra nodes



, (2.251)

whose corresponding Cartan matrix is

$$C_{\mathrm{E}_{11}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}. \tag{2.252}$$

Turning back to arbitrary D, the lagrangian density of the the Curtright field $\phi_{[\mu_1\ldots\mu_{D-3}]\nu}$ can be expressed in terms of the so-called Curtright field strength[63]

$$H_{[\mu_1\ldots\mu_{D-2}]\nu} := \partial_{[\mu_1}\phi_{\mu_2\ldots\mu_{D-2}]\nu} \tag{2.253}$$

and its unique non-trivial trace

$$H_{[\mu_1\ldots\mu_{D-2}]} := \eta^{\alpha\beta} H_{[\mu_1\ldots\mu_{D-3}\alpha]\beta} \tag{2.254}$$

as

$$\mathcal{L} = -\frac{1}{(D-2)!}\left(H_{[\mu_1\ldots\mu_{D-2}]\nu}H^{[\mu_1\ldots\mu_{D-2}]\nu} - (D-2)H_{[\mu_1\ldots\mu_{D-2}]}H^{[\mu_1\ldots\mu_{D-2}]}\right), \tag{2.255}$$

which is invariant under the gauge transformation

$$\delta_{\Lambda,\chi}\phi_{[\mu_1\ldots\mu_{D-3}]\nu} = \partial_{[\mu_1}\chi_{\mu_2\ldots\mu_{D-3}]\nu} + \partial_{[\mu_1}\Lambda_{[\mu_2\ldots\mu_{D-3}]]\nu} - (-1)^{D-3}(D-3)\partial_\mu\chi_{\mu_1\ldots\mu_{D-3}}; \tag{2.256}$$

[63] More appropriately, since it is not gauge invariant and in the perspective of the definitions of Paragraph 2.5.2, it should be called Curtright 1-cumulative field strength (in N-multi-form formalism) or left field strength (in bi-form formalism) but in the literature, for historical reasons, it is called simply Curtright field strength.

the two gauge parameters are given by

$$D-3\left\{\begin{array}{|c|}\hline \\ \hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline\end{array}\right. , \qquad D-4\left\{\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{|c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right. , \tag{2.257}$$

where the first Young tableau is associated to the gauge parameter $\chi_{\mu_1 \dots \mu_{D-3}}$ while the second to the gauge parameter $\Lambda_{[\mu_2 \dots \mu_{D-3}]\nu}$. Obviously, this is only the zeroth level of the gauge for gauge and all the gauge for gauge levels have to be taken into account to describe correctly the physical degrees of freedom. Despite the lagrangian density is gauge invariant the Curtright field strength is not, indeed

$$\delta_{\Lambda,\chi} H_{[\mu_1 \dots \mu_{D-2}]\nu} = -(D-3)\partial_\nu \partial_{[\mu_1} \Lambda_{[\mu_2 \dots \mu_{D-2}]]}; \tag{2.258}$$

this was expected since the Curtright field strength is not the true field strength for the Curtright field: it carries the irreducible representation associated to the following Young tableau

$$D-2\left\{\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{|c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right. , \tag{2.259}$$

while the true field strength is given be the Young tableau

$$D-2\left\{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right. . \tag{2.260}$$

The Young tableaux duality we have introduced and explained with the previous three examples is the algebraic version and extension of the well known electric-magnetic duality in Maxwell theory. Indeed, starting from a vector gauge field A_{μ_1} whose Young tableau is simply

$$\begin{array}{|c|}\hline \\ \hline\end{array} , \tag{2.261}$$

in D dimensions, by dualizing, we get

$$D-3\left\{\begin{array}{|c|}\hline \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline \\ \hline \\ \hline \end{array}\right. , \tag{2.262}$$

which is a $(D-3)$-form $\tilde{A}_{\mu_1 \cdots \mu_{D-3}}$. Now both are gauge fields and both admit a field strength, i.e. a gauge curvature

$$F = dA, \qquad \tilde{F} = d\tilde{A}, \tag{2.263}$$

where we are now using the differential form notation which is clearer in this case. Now the interesting point is that the two field strengths are, respectively, a 2-form and a $(D-2)$-form and the Young tableaux duality became a elegant Hodge duality at the covariant level

$$\tilde{F} = \star F, \quad F = (-1)^{2(D-2)} s \star \tilde{F} \tag{2.264}$$

where s is the parity of the signature of the metric of the Lorentzian manifold, that is, the sign of the determinant of the in coordinates representation of the metric. Using mostly plus convention we have $s = -1$ for any D. Let us now consider the Maxwell equations in vacuum

$$dF = 0, \quad d \star F = 0; \tag{2.265}$$

using the duality we get

$$d\tilde{F} = 0, \quad d \star \tilde{F} = 0. \tag{2.266}$$

Hence, the equations are interchanged under the duality, moreover in $D = 4$ both F and $\tilde{F}$ are 2-forms and the Hodge duality preserves the form of the field equations: this is the electric-magnetic duality symmetry of the Maxwell equations which leads to the introduction of magnetic monopoles in order to preserve this duality in the presence of electric charges. Therefore, in general, we can compute not only the asymptotic charges $Q^{(e)}_{1,D}$ and $\tilde{Q}^{(m)}_{1,D}$ associated to the large gauge transformations of the electric-like photon A but also the asymptotic charges $Q^{(e)}_{D-3,D}$ and $\tilde{Q}^{(m)}_{D-3,D}$ associated to the large gauge transformations of the magnetic-like photon $\tilde{A}$ where the magnetic-like charges are computed dualizing the electric-like ones. In [228], Strominger showed how, in $D = 4$, electric-like and magnetic-like asymptotic charges are linked, under duality, as

$$Q^{(e)}_{1,4} = \tilde{Q}^{(m)}_{4-3,4}, \quad Q^{(e)}_{4-3,4} = -\tilde{Q}^{(m)}_{1,4}; \tag{2.267}$$

moreover, Strominger showed that they can be merged in a unique electromagnetic-like asymptotic charge and its dual

$$\begin{aligned} Q^{(em)}_{1,4} &:= Q^{(e)}_{1,4} + i\tilde{Q}^{(m)}_{1,4}, \\ Q^{(em)}_{4-3,4} &:= Q^{(e)}_{4-3,4} + i\tilde{Q}^{(m)}_{4-3,4}, \end{aligned} \tag{2.268}$$

which both generate a complexified $\mathfrak{u}_{\mathbf{C}}(1)$ symmetry algebra[64]. Under duality this complexified charge transforms as

$$\tilde{Q}^{(em)}_{4-3,4} = -iQ^{(em)}_{4-3,4}, \quad \tilde{Q}^{(em)}_{1,4} = iQ^{(em)}_{1,4} \tag{2.269}$$

It is interesting to note [228] that the electric-like and magnetic-like asymptotic charges acts in the same way, modulo an irrelevant multiplication factor, on the physical states. Indeed, using the Hodge duality for the field strengths components $(\mu, \nu) = (u, z)$ we have, at leading order, $\tilde{F}^{(0)}_{uz} = -F^{(0)}_{uz}$ so that the field strengths determine the asymptotic electric and magnetic vector potentials up to an angular dependent integration constant. It is natural to choose this constant so that

$$\tilde{A}^{(0)}_{z} = -A^{(0)}_{z}, \quad \tilde{A}^{(0)}_{\bar{z}} = A^{(0)}_{\bar{z}}. \tag{2.270}$$

It follows that magnetic-like charge acts on the electric-like photon in the same way as electric-like charge acts modulo possible multiplications by a complex number.

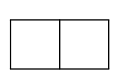
However, this is not the only Young tableaux duality we can build up; indeed we can dualize, in the same manner we have discussed above, more than one column: given λ and ξ such that $n^{(\lambda)}_i = D - 2 - n^{(\xi)}_i \; \forall i \in [1, \tilde{\ell}]$ where $\tilde{\ell} \leq \ell$ and $n^{(\lambda)}_j = n^{(\xi)}_j \; \forall j > \tilde{\ell}$ we say that λ and ξ are dual. Also in this case the two Young tableaux are completely uncorrelated as Young tableaux labelling different irreducible representations of GL(D) but they label the same irreducible representation for SO$(D-2)$. Therefore a general free gauge theory has a bunch of dual descriptions and they are many more the more complex is the Young tableau of the original theory. Hence, we have the following useful definition

Definition 2.1 (r-th dual description)**.** *A dual description of a gauge field is called r-th dual description if it is obtained by dualizing r columns of the Young tableau labelling the original field irreducible representation.*

Let us give again the example of the graviton in arbitrary D, the example of a field carrying the irreducible representation $\lambda = \{5, 4, 3\}$ in $D = 11$ dimensions and the example of the generic spin-s HS field. For more details on the electric-magnetic duality and its applications in HS theory and mixed symmetry tensors we refer for example to the review [265].

Example 4: all the dual descriptions of the graviton in D dimensions

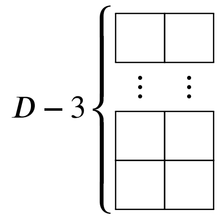
The Young tableau of the graviton is



. (2.271)

[64]The first one in the theory described by the electric-like photon while the second one in the theory described by the magnetic-like photon.

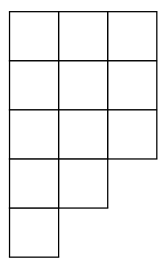
Dualizing only the first column we get the Curtright three indexes field whose Young tableau is given in (2.250) and this corresponds to the first dual description. Moreover, by dualizing both the columns we get a a field carrying the irreducible representation $\xi = (D-3, D-3)$ whose Young tableau is


, (2.272)

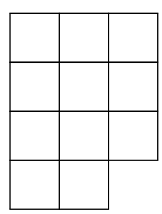
which is the second dual description of the graviton. In $D = 5$ the above Young tableau reduces to the window Young tableau and the dual field has the same symmetry of the Riemann tensor.

Example 5: all the dual descriptions of the field carrying the irreducible representation $\lambda = \{5, 4, 3\}$ in $D = 11$ dimensions

This particular field, which emerges for example in M-theory, has the following Young tableau

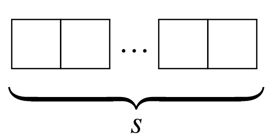

; (2.273)

we can dualize only the first column because otherwise we get something which is not even a Young diagram. The dual Young tableau is

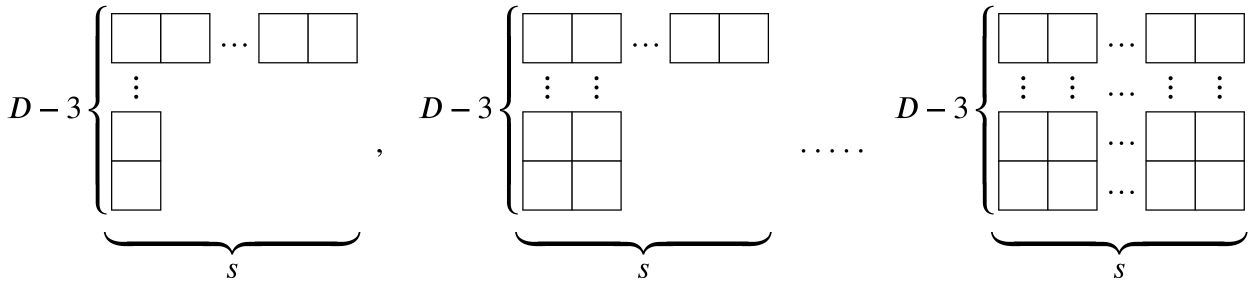

. (2.274)

Example 6: spin-s HS field in D dimensions

Spin-s HS field is given by the following Young tableau


; (2.275)

these kind of fields have a bunch of dual descriptions: a spin-s field has s dual descriptions. The dual descriptions are fields carrying the irreducible representations associated to the Young tableaux

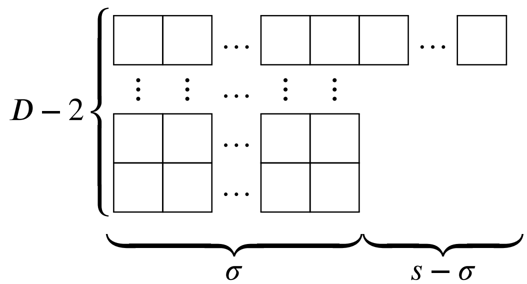


(2.276)

In a similar way we did for the case of the graviton, i.e. spin-2 HS field, we can write the dynamics of all these dual descriptions using the Curtright field strength; for the σ-th dual description with $\sigma \in [1, s]$ this is given by an object

$$H_{[\mu_1^{(1)}\ldots\mu_{D-2}^{(1)}]\ldots[\mu_1^{(\sigma)}\ldots\mu_{D-2}^{(\sigma)}]\nu_1\ldots\nu_{s-\sigma}}, \tag{2.277}$$

which carries the irreducible representation labeled by the following Young tableau



(2.278)

which corresponds to the σ-cumulative field strength. In the case of the first dual description the explicit form of the Curtright field strength is

$$H_{[\mu_1\ldots\mu_{D-2}]\nu_1\ldots\nu_{s-1}} := \partial_{[\mu_1}\phi_{\mu_2\ldots\mu_{D-2}]\nu_1\ldots\nu_{s-1}} \tag{2.279}$$

that possesses $s-1$ non-trivial traces

$$H^{(i)}_{[\mu_1\ldots\mu_{D-2}]\nu_1\ldots\nu_i\ldots\nu_{s-1}} := \eta^{\alpha\beta}H_{[\mu_1\ldots\mu_{D-3}\alpha]\nu_1\ldots\nu_{i-1}\beta\nu_{i+1}\ldots\nu_{s-1}}, \tag{2.280}$$

which are not all independent.

Chapter 3

Duality and asymptotic charges of p-form gauge theories in arbitrary dimension

3.1 Summary with results

This Chapter is based on the works [326] and [330] where the asymptotic symmetries, the duality and their interplay in p-form gauge theories are studied from theoretical, formal and mathematical point of view. Dynamical p-form fields emerge in many contexts of theoretical physics but, probably, the most relevant one is String Theory, where an infinite tower of p-form excitations are generated by the string oscillations. These fields are massive but in the tensionless limit of String Theory on flat background all the massive tower of states gets squeezed to a common zero mass level and the free theory is described by an infinite amount of massless free fields. Furthermore, p-form gauge theories display, in any dimension D, a systematic of duality; indeed a p-form gauge theory is dual, on-shell, to a q-form gauge theory with $q = D - 2 - p$. The crucial question here is if asymptotic symmetries and/or asymptotic charges computed in one theory have access to some information about the dual theory; hence, we are interested in study p-form gauge theories, their duality, their asymptotic symmetries and asymptotic charges. We first show that the polyhomogeneous expansion with leading order logarithm term in the field component $B_{ui_1\ldots i_{p-1}}$ for every p is justified also in the presence of matter such that the current has specific fall-offs. This happens, for example, considering the stringy inspired vertex of $\mathcal{N} = 0$ supergravity. Upon Lorenz-like gauge fixing we perform the asymptotic analysis and we compute the asymptotic electric-like charge $Q^{(e)}_{p,D}[S^{D-2}_u]$, for every form degree p and in any dimension D, both with radiation and Coulomb fall-offs for the field components. In the first case the charge is always finite while in the other case the radial behavior of the charge depends on the dimension D and on the form degree. Moreover, in the perspective of the Young tableaux duality between a p-form gauge theory and a q-form gauge theory we investigate how to link the two theories at the level of the asymptotic charges. We develop two possibilities: in the first we look for a map that sends $Q^{(e)}_{p,D}[S^{D-2}_u]$ to $Q^{(e)}_{q,D}[S^{D-2}_u]$ and vice versa while, in the second, we follow directly the Hodge duality equations to derive, from the electric-like charge in the q-form gauge theory, a magnetic-like charge in the p-form gauge theory and vice versa, getting $\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]$ and $\tilde{Q}^{(m)}_{q,D}[S^{D-2}_u]$. These possibilities are studied for charges with both radiation and Coulomb fall-offs on the field components. Moreover, we prove an existence and uniqueness theorem for the map which sends $Q^{(e)}_{p,D}[S^{D-2}_u]$ to $Q^{(e)}_{q,D}[S^{D-2}_u]$ and vice versa under the hypothesis of well defined charges, i.e. charges with radiation fall-off on the field components in any D and charges with Coulomb fall-off on the field components in the so-called critical dimension $D = 2p + 2$. The existence and uniqueness of the duality map can be retraced to the triviality of some cohomology groups which is surely true on Minkowski spacetime. Therefore the duality map is topological in nature. Moreover, we give a conjecture, based on the p-form gauge theory and its dual structure, on the link between trivial gauge transformations and fall-offs of the field components which generate a divergent asymptotic charge.

3.2 Introduction

Gauge theories play a crucial role in our Universe's understanding. The first modern gauge theory was introduced by Maxwell in his "A Treatise on Electricity and Magnetism" [1] but gauge theories gradually took their place and today we find them in all fields of physics, from Condensed Matter Physics to High Energy Physics [82],[174]. In the simplest case of electromagnetism, we use a vector representation of the Lorentz group as gauge field but in 1986 Henneaux and Teitelboim proposed a p-form electromagnetism where the gauge field is played by a p-form representation of the Lorentz group [75]. If we demand locality in such theories we must necessarily have an Abelian $U(1)$ gauge group; from this the name p-form electromagnetism [75],[171].
p-form fields emerge quite naturally in many physical contexts. In String Theory, Dp-branes can be introduced as sources carrying the charge of $(p+1)$-form and p-form fields are part of the string spectrum [169] which reduce to free p-form gauge fields in the tensionless limit of String Theory [122]. In supersymmetric theories, multiplets of supersymmetry algebra generically contain p-forms [203]. p-forms are also introduced after Kaluza-Klain reduction in supergravity and in the context of AdS/CFT correspondence [104] and they received recently interest in the context of celestial holography where a basis of conformal primary wave functions for p-form fields in any dimension has been constructed [317].

As a general feature of gauge theories, we can find some large gauge transformations that act in a non-trivial way on the physical states at infinity, where infinity means in the nearby of a boundary. One of the first examples is provided by the BMS group in the context of Einstein gravity [24],[25],[26].

The two fall-offs we are going to consider in this Chapter, which has physical interpretation, are the radiation one and the Coulomb one. The first is inferred by the radiation flux, which is known to have radial fall-off r^{-D+2}, which means the field yielding this radiation should fall-off as $r^{-\frac{D-2}{2}}$ [257]; the second one are the fall-offs behavior of fields generated by extended electric-like sources and for a p-form is given by $r^{-(D-p-2)}$ [193],[244],[274]. Choosing these fall-offs for the field components we compute the asymptotic electric-like charge in any dimension D and for any degree of the form p: (3.122) and (3.125). The first one is always finite and non-vanishing while the second one is finite and non-vanishing only in the critical dimension $D = 2p + 2$.

Moreover, it was shown that the asymptotic symmetries of the scalar a $D = 4$ can be interpreted as the asymptotic symmetries of a 2-form; hence the asymptotic charge of a description contains information about the dual one [251],[262]. This interpretation follow from representation theory; indeed there exist a duality between different Young tableaux labelling completely uncorrelated irreducible representations of GL(D), which label the same irreducible representation of SO($D-2$) [81]. Therefore the two fields propagate the same degrees of freedom. One of our goal is to discuss something similar in the case of p-form gauge theories and their duals

3.3 p-form gauge theories and their duality

We now study p-form gauge theories: in Paragraph 3.3.1, we consider the free case, we discuss the gauge for gauge redundancy, we introduce the Bondi coordinates and we make explicit the difference in the fall-offs with respect to cartesian ones; in Paragraph 3.3.2 we study the interacting case and the possibility to consider polyhomogeneous expansions for the field components with leading logarithmic terms; in Paragraph 3.3.3 we study the Lorentz-like gauge and we perform the asymptotic analysis for the residual gauge both for radiation and Coulomb fall-offs on the field components.

3.3.1 *p*-form gauge theories gauge invariant dynamics I: the free case

The gauge invariant dynamic of a p-form is described by the lagrangian [75]

$$\mathcal{L} = -\frac{1}{(p+1)!} H_{\mu_1 \dots \mu_{p+1}} H^{\mu_1 \dots \mu_{p+1}} \tag{3.1}$$

where

$$H_{\mu_1 \dots \mu_{p+1}} = \partial_{\mu_1} B_{\mu_2 \dots \mu_{p+1}} - \sum_{i=2}^{p+1} \partial_{\mu_i} B_{\mu_2 \dots \mu_{i-1} \mu_1 \mu_{i+1} \dots \mu_{p+1}} \tag{3.2}$$

is the $(p+1)$-field strength form of the p-form $B_{\mu_1 \dots \mu_p}$ and the equations of motion are

$$\partial_{\mu_1} H^{\mu_1 \dots \mu_{p+1}} = 0. \tag{3.3}$$

The gauge transformation of the p-form is parameterized by a $(p-1)$-form $\epsilon_{\mu_3 \dots \mu_{p+1}}$, indeed removing one box from the Young tableau of the p-form we get the Young tableau of a $(p-1)$-form. The gauge transformation is than

$$\delta_{gauge}(B_{\mu_2 \dots \mu_{p+1}}) = \partial_{\mu_2} \epsilon_{\mu_3 \dots \mu_{p+1}} - \sum_{i=3}^{p+1} \partial_{\mu_i} \epsilon_{\mu_3 \dots \mu_{i-1} \mu_2 \mu_{i+1} \dots \mu_{p+1}}; \tag{3.4}$$

this is the first level of the gauge for gauge redundancy typical of exotic gauge theories. However, we can eliminate one box from the Young tableau of the $(p-1)$-form gauge parameter $\epsilon_{\mu_3 \dots \mu_{p+1}}$, getting a $(p-2)$-form parameterizing the gauge transformation of $\epsilon_{\mu_3 \dots \mu_{p+1}}$. These steps repeat up to a gauge for gauge level parameterized by a 0-form, i.e a scalar, ϵ that parameterizes the gauge transformation of the $(p-(p-1))$-form $\epsilon_{\mu_{p+1}}$.

In order to discuss asymptotic symmetries at future null infinity we introduce Bondi coordinates $(u, r, \{x^i\})$ where $u = t - r$ and the set $\{x^i\}$ contains $D-2$ angular variables parameterizing the $(D-2)$-dimensional sphere S^{D-2} at null infinity. In the following we will refer to the angular variable indexes simply as $i, j, k, \dots$. Minkowski line element in these coordinates is

$$ds^2 = -du^2 - 2du dr + r^2 \gamma_{ij} dx^i dx^j \quad i, j = 1, \dots, D-2. \tag{3.5}$$

Metric components are

$$g_{\mu\nu} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \gamma_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2} \gamma_{ij}^{-1} \end{bmatrix}; \tag{3.6}$$

and the non-vanishing Christoffel symbols are

$$\Gamma^i_{jr} = \Gamma^i_{rj} = \frac{1}{r}\delta^i_j, \quad \Gamma^u_{ij} = -\Gamma^r_{ij} = r\gamma_{ij}, \quad \Gamma^k_{ij} = \frac{1}{2}\gamma^{kl}[-\partial_l\gamma_{ij} + \partial_j\gamma_{li} + \partial_i\gamma_{jl}]; \tag{3.7}$$

where we have assumed the Levi-Civita connection which always exists (and it is unique) thank to the Levi-Civita theorem as long as the metric is not degenerate. The proof can be found in every Riemannian and pseudo-Riemannian textbook, for example in [267]. The transformation from retarded Bondi to polar coordinates is given by

$$\{x^\alpha\} = (u, r, x^i) \mapsto \{x^{a'}\} = (u - r, r, rf_1(\{x^i\}), rf_2(\{x^i\}), ..., rf_{D-2}(\{x^i\})) \tag{3.8}$$

where $\{x^i\}$ is a set of $D-2$ angular variables. For future convenience we define $\mathbb{I}_{\{x^i\}} := \{\mu | x^\mu \in \{x^i\}\}$, which is the set of indexes such that the variable is an angular variable. In Bondi coordinates fields carry additional powers of r due to the jacobian of the coordinate transformation; indeed, a generic field component will transform as

$$B_{a_2...a_{p+1}}(x') = \frac{\partial x^{\mu_2}}{\partial x^{a'_2}}\frac{\partial x^{\mu_3}}{\partial x^{a'_3}}...\frac{\partial x^{\mu_{p+1}}}{\partial x^{a'_{p+1}}}\phi_{\mu_2...\mu_{p_1}}(x) \quad \Rightarrow \quad B_{\mu_2...\mu_{p_1}}(x) = \frac{\partial x^{a'_2}}{\partial x^{\mu_1}}\frac{\partial x^{a'_3}}{\partial x^{\mu_2}}...\frac{\partial x^{a'_{p+1}}}{\partial x^{\mu_{p+1}}}B_{a_2...a_{p+1}}(x'); \tag{3.9}$$

so, for example for the field component B_{uri}, the jacobian matrix produces an additional r with respect to the cartesian fall-offs. In general, for each angular index $\mu \in \mathbb{I}_{\{x^i\}}$ we have one r more than the cartesian component. For the radiation fall-offs, when $s \leq p$ indexes are angular one, we have

$$r^s\mathcal{O}\left(r^{-\frac{(D-2)}{2}}\right) = \mathcal{O}\left(r^{\frac{-(D-2-2s)}{2}}\right); \tag{3.10}$$

while for the Coulomb fall-offs, when $s \leq p$ indexes are angular one, we get

$$r^s\mathcal{O}\left(r^{-(D-p-2)}\right) = \mathcal{O}\left(r^{-(D-p-2-s)}\right). \tag{3.11}$$

Moreover, under the coordinates transformation, the magnetic-like field components in the two coordinates system are mapped in a linear combination ones to the others. In other words components without the index t when transformed they give life to components without the index u.

3.3.2 p-form gauge theories gauge invariant dynamics II: logarithmic fall-offs

In order to generalize the recent observation of Peraza on logarithmic fall-off [331] we propose a stringy inspired possibility but we first review the computation for $p = 1$, i.e. scalar QED. The equations of motion in Lorenz gauge are

$$\Box B_\mu = J_\mu, \tag{3.12}$$

where the current is $J_\mu = ie\phi D_\mu \phi$ and the behaviours

$$J_u \sim \frac{J_u^{(D-2)}}{r^{D-2}}, \quad J_r \sim \frac{J_r^{(D-1)}}{r^{D-1}}, \quad J_i \sim \frac{J_i^{(D-2)}}{r^{D-2}} \tag{3.13}$$

are inferred by the scalar behavior

$$\phi \sim \frac{\phi^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}}. \tag{3.14}$$

Maxwell equations can be solved as in Appendix A with the difference to consider that only the overleading orders with respect to the current fall-off have to satisfy a homogeneous equation while the subleading orders with respect to the current fall-off satisfy an unhomogeneous equation with source given by the current components. We assume the following polyhomogeneous expansions for the gauge field components, hence we have the presence of a logarithmic branch in the expansions

$$B_u \sim \frac{B_u^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}} + \frac{\tilde{B}_u^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} + ..., \quad B_r \sim \frac{B_r^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}} + \frac{B_r^{(\frac{D}{2})}}{r^{\frac{D}{2}}} + \frac{\tilde{B}_r^{(\frac{D}{2})} ln(r)}{r^{\frac{D}{2}}} + ..., \quad B_i \sim \frac{B_i^{(\frac{D-4}{2})}}{r^{\frac{D-4}{2}}} + \frac{\tilde{B}_i^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} + ..., \tag{3.15}$$

Looking at the $\mu = r$ equation for the logarithmic branch, i.e. equation (A.8c), we get for $l = \frac{D}{2}$

$$2\partial_u \tilde{B}_r^{(\frac{D}{2})} + (D-2)\tilde{B}_u^{(\frac{D-2}{2})} = 0; \tag{3.16}$$

while for the Lorenz gauge condition of the non-logarithmic branch, i.e. the second of equations (A.11), we get with $l = \frac{D-2}{2}$

$$\partial_u B_r^{(\frac{D-2}{2})} = 0. \tag{3.17}$$

Looking at the components of the field strength and using equations (3.16) and (3.17), we have

$$H_{iu} \sim \frac{\partial_i B_u^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}} + \frac{\partial_i \tilde{B}_u^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} - \frac{\partial_u B_i^{(\frac{D-4}{2})}}{r^{\frac{D-4}{2}}} - \frac{\partial_u \tilde{B}_i^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} \sim \mathcal{O}(r^{-\frac{D-4}{2}}); \tag{3.18a}$$

$$H_{ir} \sim \frac{\partial_i B_r^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}} + \frac{\partial_i B_r^{(\frac{D}{2})}}{r^{\frac{D}{2}}} + \frac{\partial_i \tilde{B}_r^{(\frac{D}{2})} ln(r)}{r^{\frac{D}{2}}} + \left(\frac{D-4}{2}\right)\frac{B_i^{(\frac{D-4}{2})}}{r^{\frac{D-2}{2}}} + \left(\frac{D-2}{2}\right)\frac{\tilde{B}_i^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D}{2}}} - \frac{\tilde{B}_i^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D}{2}}} \sim \mathcal{O}(r^{-\frac{D-2}{2}}); \tag{3.18b}$$

$$H_{ij} \sim \frac{\partial_i B_j^{(\frac{D-4}{2})}}{r^{\frac{D-4}{2}}} + \frac{\partial_i \tilde{B}_j^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} - \frac{\partial_j B_i^{(\frac{D-4}{2})}}{r^{\frac{D-4}{2}}} - \frac{\partial_j \tilde{B}_i^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D-2}{2}}} \sim \mathcal{O}(r^{-\frac{D-4}{2}}); \tag{3.18c}$$

$$H_{ru} \sim \frac{\tilde{B}_u^{(\frac{D-2}{2})}}{r^{\frac{D}{2}}} - \left(\frac{D-2}{2}\right)\frac{B_u^{(\frac{D-2}{2})}}{r^{\frac{D}{2}}} - \left(\frac{D-2}{2}\right)\frac{\tilde{B}_u^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D}{2}}} - \frac{\partial_u B_r^{(\frac{D-2}{2})}}{r^{\frac{D-2}{2}}} - \frac{\partial_u B_r^{(\frac{D}{2})}}{r^{\frac{D}{2}}} - \frac{\partial_u \tilde{B}_r^{(\frac{D-2}{2})} ln(r)}{r^{\frac{D}{2}}} \sim \mathcal{O}(r^{-\frac{D}{2}}); \tag{3.18d}$$

which are the right fall-offs to have a finite energy flux at null infinity and the right radial behavior for the long range electric field. Hence the presence of a current due to a scalar with radial behaviour (3.14) does not break the mechanism by which logarithmic terms do not come into play in the field strength component; hence not all the kind of scalar matter can be coupled to Maxwell field since the cancellation mechanism is intimately linked to the shape of the current (3.13). Each current with different fall-off in the J_r component breaks the cancellation mechanism.

3.3.2.1 The p-form case and currents from stringy inspired vertexes.

We can repeat the same analysis in the case of $p > 1$; let us consider to have equations of motion of the form

$$\Box B_{\mu_1 \ldots \mu_p} = J_{\mu_1 \ldots \mu_p} \tag{3.19}$$

with a current such that

$$J_{ri_1 \ldots i_{p-1}} \sim \frac{J_r^{(\frac{D-2p+2}{2})}}{r^{\frac{D-2p+2}{2}}}; \tag{3.20}$$

this kind of currents can arise, for example, by coupling p-form to a scalar field Φ with the $\mathcal{N} = 0$ supergravity vertex $\mathcal{V}_{\mathcal{N}=0} = e^{k\Phi} H_{\mu_1 \ldots \mu_{p+1}} H^{\mu_1 \ldots \mu_{p+1}}$. We can solve the equation as before: only the overleading orders with respect to the current have to satisfy a homogeneous equation while the subleading orders satisfy an unhomogeneous equation with source given by the current components. We assume a polyhomogeneous expansion for the gauge field components

$$\begin{aligned}
B_{ui_1 \ldots i_{p-1}} &\sim \frac{B_{ui_1 \ldots i_{p-1}}^{(\frac{D-2p}{2})}}{r^{\frac{D-2p}{2}}} + \frac{\tilde{B}_{ui_1 \ldots i_{p-1}}^{(\frac{D-2p}{2})} ln(r)}{r^{\frac{D-2p}{2}}} + \ldots, \\
B_{rui_1 \ldots i_{p-2}} &\sim \frac{B_{rui_1 \ldots i_{p-2}}^{(\frac{D-2p+2}{2})}}{r^{\frac{D-2p+2}{2}}} + \frac{B_{rui_1 \ldots i_{p-2}}^{(\frac{D-2p+4}{2})}}{r^{\frac{D-2p+4}{2}}} + \frac{\tilde{B}_{rui_i \ldots i_{p-2}}^{(\frac{D-2p+4}{2})} ln(r)}{r^{\frac{D-2p+4}{2}}} + \ldots, \\
B_{ri_1 \ldots i_{p-1}} &\sim \frac{B_{ri_1 \ldots i_{p-1}}^{(\frac{D-2p}{2})}}{r^{\frac{D-2p}{2}}} + \frac{B_{ri_1 \ldots i_{p-1}}^{(\frac{D-2p+2}{2})}}{r^{\frac{D-2p+2}{2}}} + \frac{\tilde{B}_{ri_1 \ldots i_{p-1}}^{(\frac{D-2p+2}{2})} ln(r)}{r^{\frac{D-2p+2}{2}}} + \ldots, \\
B_{i_i \ldots i_p} &\sim \frac{B_{i_i .. i_p}^{(\frac{D-2p-2}{2})}}{r^{\frac{D-2p-2}{2}}} + \frac{\tilde{B}_{i_1 \ldots i_p}^{(\frac{D-2p}{2})} ln(r)}{r^{\frac{D-2p}{2}}} + \ldots .
\end{aligned} \tag{3.21}$$

As for the $p = 1$ case, we have to taken into account both the Lorenz-like gauge fixing condition and the equations of motion. Let us start with the Lorenz-like gauge condition $\nabla^\nu B_{\nu\mu_1 \ldots \mu_{p-1}} = 0$ for $(\mu_1, \ldots, \mu_{p-1}) = (i_1, \ldots, i_{p-1})$, inserting a generic polyhomogeneous expansion for the gauge field components we get

$$\begin{aligned}
&\partial_u \tilde{B}^{(l)}_{ri_1...i_{p-1}} - G_{p,D}(\tilde{B}^{(l-1)}_{ui_1...i_{p-1}} - \tilde{B}^{(l-1)}_{ri_1...i_{p-1}}) - \nabla^i \tilde{B}^{(l-2)}_{ii_1...i_{p-1}} = 0;\\
&\partial_u B^{(l)}_{ri_1...i_{p-1}} - G_{p,D}(B^{(l-1)}_{ui_1...i_{p-1}} - B^{(l-1)}_{ri_1...i_{p-1}}) - \nabla^i B^{(l-2)}_{ii_1...i_{p-1}} + \tilde{B}^{(l-1)}_{ui_1...i_{p-1}} - \tilde{B}^{(l-1)}_{ri_1...i_{p-1}} = 0.
\end{aligned} \tag{3.22}$$

where $G_{p,D} := (l - D + 2p - 1)$; choosing $l = \frac{D-2p}{2}$ and considering the expansions (3.21) we get from the second one

$$\partial_u B^{(\frac{D-2p}{2})}_{ri_1...i_{p-1}} = 0. \tag{3.23}$$

Focusing now our attention on the homogeneous equations of motion, we choose $(\mu_1...\mu_p) = (r, i_1...i_{p-1})$; the explicit writing is

$$\mathcal{D}_{p,D} B_{ri_1...i_{p-1}} + \frac{E^{(1)}_{p,D}}{r^2} B_{ri_1...i_{p-1}} + \frac{E^{(2)}_{p,D}}{r^2} B_{ui_1...i_{p-1}} + \frac{\sum B}{r} - \frac{2}{r^3}\nabla^i B_{ii_1...i_{p-1}} = 0. \tag{3.24a}$$

where

$$\begin{aligned}
\mathcal{D}_{p,D} &:= \partial_r^2 - 2\partial_u\partial_r - \frac{(D-2p)}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2};\\
E^{(1)}_{p,D} &:= p^2 - 2p + 3 - D;\\
E^{(2)}_{p,D} &:= D - 2p;\\
\sum B &:= \sum_{s=1}^{p-1} (-1)^{s-1} \partial_{i_s} B_{rui_1...i_{s-1}i_{s+1}...i_{p-1}}.
\end{aligned} \tag{3.25}$$

Inserting a generic polyhomogeneous expansion, we get the order by order relations

$$[k^{(p)}_r(l) + \Delta]\tilde{B}^{(l-1)}_{ri_1...i_{p-1}} + A_1\partial_u \tilde{B}^{(l)}_{ri_1...i_{p-1}} + A_2 \tilde{B}^{(l-1)}_{ui_1...i_{p-1}} + \sum \tilde{B}^{(l-1)} - 2\nabla^i \tilde{B}^{(l-2)}_{ii_1...i_{p-1}} = 0; \tag{3.26a}$$

$$[k^{(p)}_r(l) + \Delta]B^{(l-1)}_{ri_1...i_{p-1}} + A_1\partial_u B^{(l)}_{ri_1...i_{p-1}} + A_2 B^{(l-1)}_{ui_1...i_{p-1}} + \sum B^{(l-1)} - 2\nabla^i B^{(l-2)}_{ii_1...i_{p-1}} + A_3 \tilde{B}^{(l-1)}_{ri_1...i_{p-1}} - 2\partial_u \tilde{B}^{(l)}_{ri_1...i_{p-1}} = 0; \tag{3.26b}$$

where

$$\begin{aligned}
k^{(p)}_r(l) &:= l(l - 1 - D + 2p) - 4p + p^2 + 3;\\
A_1 &:= 2l - D + 2p;\\
A_2 &:= D - 2p;\\
A_3 &:= D - 2l + 2p - 1;\\
\sum \tilde{B}^{(l-1)} &:= \sum_{s=1}^{p-1} (-1)^{s-1} \partial_{i_s} \tilde{B}^{(l-1)}_{rui_1...i_{s-1}i_{s+1}...i_{p-1}};\\
\sum B^{(l-1)} &:= \sum_{s=1}^{p-1} (-1)^{s-1} \partial_{i_s} B^{(l-1)}_{rui_1...i_{s-1}i_{s+1}...i_{p-1}}.
\end{aligned} \tag{3.27}$$

Now, since the current has well defined fall-offs, the first order that satisfies (3.26a) is $l = \frac{D-2p+2}{2}$ since we have to cancel out the order $\mathcal{O}(r^{-\frac{D-2p+2}{2}} ln(r))$. Therefore, inserting $l = \frac{D-2p+2}{2}$ in (3.26a) and using the expansions (3.21) we get

$$2\partial_u \tilde{B}^{(\frac{D-2p+2}{2})}_{ri_1...i_{p-1}} + (D-2p)\tilde{B}^{(\frac{D-2p}{2})}_{ui_1...i_{p-1}} = 0. \tag{3.28}$$

In the very same manner of the $p = 1$ case, there is a cancellation of the leading order in the field strength component $H_{rui_1...i_{p-1}}$ while the other components are not generated overleading pieces that require cancellations. In the case of a polyhomogeneous expansion for the gauge field components with Coulomb fall-offs, equations (3.23) and (3.28) are modified to

$$\begin{aligned} &\partial_u B^{(D-2p-1)}_{ri_1...i_{p-1}} = 0; \\ &(D-2p)\left[\partial_u \tilde{B}^{(D-2p)}_{ri_1...i_{p-1}} + \tilde{B}^{(D-2p-1)}_{ui_1...i_{p-1}}\right] = 0. \end{aligned} \tag{3.29}$$

3.3.3 Lorenz-like gauge fixing and residual gauge

Let us briefly discuss the Lorenz-like gauge fixing in the realm of p-form gauge theories. The equations of motion can be written as

$$\Box B_{\mu_2...\mu_{p+1}} - \sum_{i=2}^{p+1} \partial^{\mu_1}\partial_{\mu_i} B_{\mu_2...\mu_{i-1}\mu_1\mu_{i+1}...\mu_{p+1}} = 0; \tag{3.30}$$

fixing the a Lorenz-like gauge means to reduce the equation of motion to simply

$$\Box B_{\mu_2...\mu_{p+1}} = 0. \tag{3.31}$$

So, the condition required to reduce the equations of motion from (3.30) to (3.31) is

$$\sum_{i=2}^{p+1} \partial^{\mu_1}\partial_{\mu_i} B_{\mu_2...\mu_{i-1}\mu_1\mu_{i+1}...\mu_{p+1}} = 0 \tag{3.32}$$

which means, using complete antisymmetry of the p-form,

$$\partial^{\mu_1} B_{\mu_1\mu_3...\mu_{p+1}} = 0. \tag{3.33}$$

Inserting the gauge transformation (3.4), this condition is satisfied if the gauge parameter $\epsilon_{\mu_3...\mu_{p+1}}$ is such that

$$\Box\epsilon_{\mu_3...\mu_{p+1}} - \sum_{i=3}^{p+1} \partial^{\mu_1}\partial_{\mu_i}\epsilon_{\mu_3...\mu_{i-1}\mu_2\mu_{i+1}...\mu_{p+1}} = 0; \tag{3.34}$$

requiring

$$\sum_{i=3}^{p+1} \partial^{\mu_1}\partial_{\mu_i}\epsilon_{\mu_3\cdots\mu_{i-1}\mu_2\mu_{i+1}\dots\mu_{p+1}} = 0 \tag{3.35}$$

means, using complete antisymmetry of the $(p-1)$-form gauge parameter,

$$\partial^{\mu_1}\epsilon_{\mu_3\cdots\mu_{p+1}} = 0. \tag{3.36}$$

Inserting the gauge for gauge transformation of $\epsilon_{\mu_3\dots\mu_{p+1}}$, this condition is satisfied if the gauge for gauge parameter $\epsilon_{\mu_4\cdots\mu_{p+1}}$ is such that

$$\Box\epsilon_{\mu_4\cdots\mu_{p+1}} - \sum_{i=4}^{p+1} \partial^{\mu_1}\partial_{\mu_i}\epsilon_{\mu_4\cdots\mu_{i-1}\mu_3\mu_{i+1}\dots\mu_{p+1}} = 0; \tag{3.37}$$

requiring

$$\sum_{i=4}^{p+1} \partial^{\mu_1}\partial_{\mu_i}\epsilon_{\mu_4\cdots\mu_{i-1}\mu_3\mu_{i+1}\dots\mu_{p+1}} = 0 \tag{3.38}$$

means, using complete antisymmetry of the $(p-2)$-form gauge for gauge parameter,

$$\partial^{\mu_1}\epsilon_{\mu_4\cdots\mu_{p+1}} = 0. \tag{3.39}$$

This pattern repeats p times and in the end the Lorenz-like gauge fixes the gauge up to gauge transformation such that the gauge for gauge parameters satisfy the set of equations

$$\Box\epsilon_{\mu_3\cdots\mu_{p+1}} = 0, \quad \Box\epsilon_{\mu_4\cdots\mu_{p+1}} = 0, \quad \dots \quad \Box\epsilon_{\mu_{p+1}} = 0, \quad \Box\epsilon = 0. \tag{3.40}$$

3.3.3.1 Residual gauge for the inductive case $p = 1$

In this Paragraph we compute the leading order of the gauge parameter for a 1-form gauge theory in arbitrary D both assuming radiation and Coulomb fall-offs on the field components.

Radiation fall-offs

In the case of a 1-form the gauge parameter is a scalar and the covariantized gauge transformation reduces to

$$\delta_{gauge}(B_{\mu_2}) = \partial_{\mu_2}\epsilon \sim \begin{cases} \mathcal{O}\big(r^{-\frac{(D-2)}{2}}\big) & if\ \mu_2 \notin \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\big(r^{-\frac{(D-4)}{2}}\big) & if\ \mu_2 \in \mathbb{I}_{\{x^i\}}; \end{cases} \tag{3.41}$$

moreover we assume a polyhomogeneous expansion

$$\epsilon = \sum_{l \in \frac{1}{2}\mathbb{Z}} \frac{\epsilon^{(l)}(u, \{x^i\}) + \tilde{\epsilon}^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.42}$$

Inserting the expansion in the gauge transformation we get the relations

$$\partial_u \epsilon^{(l)} = \partial_u \tilde{\epsilon}^{(l)} = 0 \quad \forall l < \frac{D-2}{2}, \qquad \epsilon^{(l)} = \partial_i \epsilon^{(l)} = 0 \quad \forall l < \frac{D-4}{2}, \qquad \tilde{\epsilon}^{(l)} = \partial_i \tilde{\epsilon}^{(l)} = 0 \quad \forall l \leq \frac{D-4}{2}. \tag{3.43}$$

Therefore the gauge parameter has the form

$$\epsilon = \frac{\epsilon^{(\frac{D-4}{2})}(\{x_i\})}{r^{\frac{D-4}{2}}} + \frac{\epsilon^{(\frac{D-3}{2})}(\{x_i\}) + \tilde{\epsilon}^{(\frac{D-3}{2})}(\{x_i\}) ln(r)}{r^{\frac{D-3}{2}}} + \sum_{l \geq \frac{D-2}{2}} \frac{\epsilon^{(l)}(u, \{x_i\}) + \tilde{\epsilon}^{(l)}(u, \{x_i\}) ln(r)}{r^l}. \tag{3.44}$$

One may note that this gauge parameter diverges in $D = 3$; however in $D = 3$ also the field component B_i diverges. We could impose $D \geq 4$ but, even if in the case $p = 1$ this seems harmless, if we apply this approach to the generic p case we would cut out a whole series of duality between forms. We note that the presence of a u-dependent $\tilde{\epsilon}^{(\frac{D-2}{2})}(u, \{x_i\})$ gives rise to a field configuration with a fall-off $\mathcal{O}\big(r^{-\frac{(D-2)}{2}} ln(r)\big)$, however if we trivialize this u-dependence we would get a vanishing $\epsilon^{(\frac{D-4}{2})}(\{x_i\})$ due to the preserving Lorenz-gauge conditions (A.4) and we would not find any non-vanishing asymptotic charge. Therefore we have to slightly modify the fall-off of B_u from $\mathcal{O}\big(r^{-\frac{(D-2)}{2}}\big)$ to $\mathcal{O}\big(r^{-\frac{(D-2)}{2}} ln(r)\big)$. For this point we have two possible approaches. On the one hand, this term is pure gauge, meaning it does not play any role in physical quantities and no divergences can appear due to this order. On the other hand, in view of Paragraph 3.3.2, we could have required from the beginning that B_u had fall-off $\mathcal{O}\big(r^{-\frac{(D-2)}{2}} ln(r)\big)$ and we would have gotten exactly the same result

Coulomb fall-offs

In the case of a 1-form the gauge parameter is a scalar and the covariantized gauge transformation reduces to

$$\delta_{gauge}(B_{\mu_2}) = \partial_{\mu_2} \epsilon \sim \begin{cases} \mathcal{O}\big(r^{-(D-3)}\big) & if\ \mu_2 \notin \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\big(r^{-(D-4)}\big) & if\ \mu_2 \in \mathbb{I}_{\{x^i\}}; \end{cases} \tag{3.45}$$

moreover we assume a polyhomogeneous expansion

$$\epsilon = \sum_{l \in \mathbb{Z}} \frac{\epsilon^{(l)}(u, \{x^i\}) + \tilde{\epsilon}^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.46}$$

This fall-off gives us, with analogous computation as before, a gauge parameter of the form

$$\epsilon = \frac{\epsilon^{(D-4)}(\{x_i\})}{r^{D-4}} + \sum_{l \geq D-3} \frac{\epsilon^{(l)}(u,\{x_i\}) + \tilde{\epsilon}^{(l)}(u,\{x_i\}) ln(r)}{r^l}. \tag{3.47}$$

Similar considerations done previously apply here as well with the only difference that in the case of Coulomb fall-offs the equations of motion are not able to cancel out the logarithmic overleading terms in H_{ru} if we assume it from the beginning in the field component. Hence, in this case we can only require a logarithmic violation of the fall-offs by a pure gauge term which does not play any role in physical quantities.

In the end, we have showed that

Lemma 3.3.1 (Leading order of the gauge parameter ϵ in the 1-form gauge theory in any D)**.** *In the framework of* 1*-form gauge theory in any* D*, the leading order* X *of the polyhomogenoues expansion of the gauge parameter* ϵ *is*

$$\begin{cases} X = \frac{D-4}{2} & \text{for radiation fall-off;} \\ X = D-4 & \text{for Coulomb fall-off.} \end{cases} \tag{3.48}$$

Proof. The proof is contained in the computations of Paragraph 3.3.3.1. □

3.3.3.2 Residual gauge for the pivotal example of the 2-form

In this Paragraph we compute the leading order of the gauge parameter for a 2-form gauge theory in arbitrary D both assuming radiation and Coulomb fall-offs on the field components.

Radiation fall-offs

In the case of the 2-form in any D and in Bondi coordinates the covariantized gauge transformation is

$$\delta_{gauge}(B_{\mu_2\mu_3}) = \nabla_{\mu_2}\epsilon_{\mu_3} - \nabla_{\mu_3}\epsilon_{\mu_2} = \partial_{\mu_2}\epsilon_{\mu_3} - \partial_{\mu_3}\epsilon_{\mu_2} \sim \begin{cases} \mathcal{O}\big(r^{-\frac{(D-2)}{2}}\big) & if\ \mu_2,\mu_3 \notin \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\big(r^{-\frac{(D-4)}{2}}\big) & if\ \mu_2 \text{ or } \mu_3 \in \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\big(r^{-\frac{(D-6)}{2}}\big) & if\ \mu_2,\mu_3 \in \mathbb{I}_{\{x^i\}}; \end{cases} \tag{3.49}$$

let us study the three cases in a separate way assuming a polyhomogeneous expansion for the gauge parameter of the form[1]

$$\epsilon_\alpha = \sum_{l \in \frac{1}{2}\mathbb{Z}} \frac{\epsilon^{(l)}_\alpha(u,\{x^i\}) + \tilde{\epsilon}^{(l)}_\alpha(u,\{x^i\}) ln(r)}{r^l}. \tag{3.50}$$

[1]We will be interested in l of the form $\frac{D-n}{2}$ where $n \in \mathbb{Z}$; we have $\frac{D-n}{2} < \frac{D-m}{2}$ if and only if $m < n$ and $\frac{D-n}{2} > \frac{D-m}{2}$ if and only if $m > n$. Given $\frac{1}{r^{\frac{D-n}{2}}}$ the power $\frac{1}{r^{\frac{D-m}{2}}}$ is subleading if and only if $m < n$.

Case I

If $\mu_2, \mu_3 \notin \mathbb{I}_{\{x^i\}}$, due to the antisymmetry of the 2-form, then $\mu_2 = r, \mu_3 = u$ or $\mu_2 = u, \mu_3 = r$; however, again for the antisymmetry of the 2-form we can, without loss of generality, take into account only the case $\mu_1 = u, \mu_2 = r$. The gauge transformation is

$$\partial_u \epsilon_r - \partial_r \epsilon_u \sim \mathcal{O}\big(r^{-\frac{(D-2)}{2}}\big) \tag{3.51}$$

and inserting the expansions (3.50) we get

$$\begin{aligned} \epsilon_r &= \sum_{l<\frac{D-4}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-4}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \\ \epsilon_u &= \sum_{l<\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \end{aligned} \tag{3.52}$$

with

$$\begin{aligned} &\partial_u \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_u^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < \frac{D-4}{2}; \\ &\partial_u \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < \frac{D-4}{2}. \end{aligned} \tag{3.53}$$

However, the second condition impose us to change the fall-off in $\mathcal{O}\big(r^{-\frac{(D-2)}{2}} ln(r)\big)$ by a pure gauge term. Alternatively, we can impose that the second relation in (3.53) holds also for $l = \frac{D-4}{2}$ avoiding violations of the fall-off. The only reason for choosing to violate the fall-off would be an analogy with the case $p = 1$ however, in the case $p = 2$ this choice is not necessary to have finite and non-vanishing asymptotic charges.

Case II

If $\mu_2 \in \mathbb{I}_{\{x^i\}}$ and $\mu_3 \notin \mathbb{I}_{\{x^i\}}$ then $\mu_3 \in \{u, r\}$. In the first case we have the gauge transformation

$$\partial_i \epsilon_u - \partial_u \epsilon_i \sim \mathcal{O}\big(r^{-\frac{(D-4)}{2}}\big); \tag{3.54}$$

inserting the expansions of the gauge parameter we get

$$\begin{aligned} \epsilon_u &= \sum_{l<\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \\ \epsilon_i &= \sum_{l<\frac{D-4}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-4}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l}; \end{aligned} \tag{3.55}$$

with

$$\begin{aligned} &\partial_i \epsilon_u^{(l)}(u,\{x^i\}) = \partial_u \epsilon_i^{(l)}(u,\{x^i\}) \quad \forall l < \frac{D-4}{2}; \\ &\partial_i \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = \partial_u \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) \quad \forall l < \frac{D-4}{2}. \end{aligned} \tag{3.56}$$

However these conditions impose us to change the fall-off in $\mathcal{O}\big(r^{-\frac{(D-4)}{2}} ln(r)\big)$ by a pure gauge term. In a similar way to what happens for the case $p = 1$ and in the spirit of (3.3.2), we could have required from the beginning the fall-off $\mathcal{O}\big(r^{-\frac{(D-4)}{2}} ln(r)\big)$ for B_{iu} and we would have gotten exactly the same. In the case of the B_{iu} field component, unlike for the case of the B_{ur} field component, if we make vanishing the logarithmic leading order term, i.e. $\mathcal{O}\big(r^{-\frac{(D-4)}{2}} ln(r)\big)$, we would get no non-vanishing asymptotic charges. This mechanism resemble to what happens in the $p = 1$ case with the field component B_u.
In the second case the gauge transformation reads

$$\partial_i \epsilon_r - \partial_r \epsilon_i \sim \mathcal{O}\big(r^{-\frac{(D-4)}{2}}\big); \tag{3.57}$$

inserting the gauge parameter expansions we get

$$\begin{aligned} \epsilon_r &= \sum_{l<\frac{D-6}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-6}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \\ \epsilon_i &= \sum_{l<\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \end{aligned} \tag{3.58}$$

with

$$\begin{aligned} &\partial_i \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_i^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l < \frac{D-6}{2}; \\ &\partial_i \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l \leq \frac{D-6}{2}. \end{aligned} \tag{3.59}$$

Case III

In this case $\mu_2, \mu_3 \in \mathbb{I}_{\{x^i\}}$ and the gauge transformation is

$$\partial_i \epsilon_j - \partial_j \epsilon_i \sim \mathcal{O}\big(r^{-\frac{(D-6)}{2}}\big); \tag{3.60}$$

inserting the expansion of the gauge parameter we have

$$\epsilon_i = \sum_{l<\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \tag{3.61}$$

with

$$\begin{aligned} \epsilon_i^{(l)}(u,\{x^i\}) &= \partial_i f^{(l)}(u,\{x^i\}) \qquad \forall l < \frac{D-6}{2};\\ \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) &= \partial_i \tilde{f}^{(l)}(u,\{x^i\}) \qquad \forall l \leq \frac{D-6}{2}. \end{aligned} \tag{3.62}$$

Conclusive remarks

The residual gauge in the case of radiation fall-offs is then parameterized by

$$\begin{aligned} \epsilon_u &= \sum_{l<\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-4}{2}} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l},\\ \epsilon_r &= \sum_{l<\frac{D-6}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-6}{2}} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l},\\ \epsilon_i &= \sum_{l<\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq\frac{D-6}{2}} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \end{aligned} \tag{3.63}$$

with

$$\begin{aligned} &\partial_u \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_u^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \qquad \forall l < \frac{D-4}{2};\\ &\partial_u \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \qquad \forall l < \frac{D-4}{2};\\ &\partial_i \epsilon_u^{(l)}(u,\{x^i\}) = \partial_u \epsilon_i^{(l)}(u,\{x^i\}) \qquad \forall l < \frac{D-4}{2};\\ &\partial_i \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = \partial_u \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) \qquad \forall l < \frac{D-4}{2};\\ &\partial_i \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_i^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \qquad \forall l < \frac{D-6}{2};\\ &\partial_i \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \qquad \forall l \leq \frac{D-6}{2};\\ &\epsilon_i^{(l)}(u,\{x^i\}) = \partial_i f^{(l)}(u,\{x^i\}) \qquad \forall l < \frac{D-6}{2};\\ &\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = \partial_i \tilde{f}^{(l)}(u,\{x^i\}) \qquad \forall l \leq \frac{D-6}{2}. \end{aligned} \tag{3.64}$$

These parameters have to satisfy the conditions to not spoil the Lorenz-like gauge, $\Box\epsilon_\alpha = \Box\epsilon = \nabla^\mu\epsilon_\mu = 0$; we note that we can use gauge for gauge to fix

$$\tilde{\epsilon}_u^{(\frac{D}{2})} = 0, \qquad \epsilon_i^{(\frac{D-8}{2})} = 0, \qquad \tilde{\epsilon}_i^{(\frac{D-6}{2})} = 0. \tag{3.65}$$

We stress that the quantities in (3.65) are the gauge transformed gauge parameter components; from now on we do not use apexes for simplicity of notation. Looking to the gauge fixing equations (A.8) we get, for the gauge parameter we are interested in,

$$\epsilon_i = \frac{\epsilon_i^{(\frac{D-6}{2})}(\{x^i\})}{r^{\frac{D-6}{2}}} + \sum_{l \geq \frac{D-5}{2}} \frac{\epsilon_i^{(l)}(u, \{x^i\}) + \tilde{\epsilon}_i^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.66}$$

The point is therefore that, thanks to results in appendices A, B and C it is possible, in sufficiently high dimensions to start the expansion of the gauge parameter $\epsilon_i(\{x^i\})$ with overleading terms with respect to the standard and expected $\mathcal{O}(r^{-\frac{D-6}{2}})$ since they can be cancelled out with a gauge for gauge fixing. We expect the expansion of the gauge parameters ϵ_i start with a term of order $\mathcal{O}(r^{-\frac{D-6}{2}})$ since the field components B_{ij} have expansions that start with a term of order $\mathcal{O}(r^{-\frac{D-6}{2}})$.

Coulomb fall-offs

In the case of the 2-form in any D and in Bondi coordinates the covariantized gauge transformation is

$$\delta_{gauge}(B_{\mu_2\mu_3}) = \nabla_{\mu_2}\epsilon_{\mu_3} - \nabla_{\mu_3}\epsilon_{\mu_2} = \partial_{\mu_2}\epsilon_{\mu_3} - \partial_{\mu_3}\epsilon_{\mu_2} \sim \begin{cases} \mathcal{O}\left(r^{-(D-4)}\right) & if\ \mu_2, \mu_3 \notin \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\left(r^{-(D-5)}\right) & if\ \mu_2\ or\ \mu_3 \in \mathbb{I}_{\{x^i\}}; \\ \mathcal{O}\left(r^{-(D-6)}\right) & if\ \mu_2, \mu_3 \in \mathbb{I}_{\{x^i\}}; \end{cases} \tag{3.67}$$

let us study the three cases in a separate way assuming a polyhomogeneous expansion for the gauge parameter of the form[2]

$$\epsilon_\alpha = \sum_{l \in \mathbb{Z}} \frac{\epsilon_\alpha^{(l)}(u, \{x^i\}) + \tilde{\epsilon}_\alpha^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.68}$$

Case I

If $\mu_2, \mu_3 \notin \mathbb{I}_{\{x^i\}}$, due to the antisymmetry of the 2-form, then $\mu_2 = r, \mu_3 = u$ or $\mu_2 = u, \mu_3 = r$; however, again for the antisymmetry of the 2-form we can, without loss of generality, take into account only the case $\mu_1 = u, \mu_2 = r$. The gauge transformation is

$$\partial_u\epsilon_r - \partial_r\epsilon_u \sim \mathcal{O}\left(r^{(D-4)}\right); \tag{3.69}$$

[2]We will be interested in l of the form $D-n$ where $n \in \mathbb{Z}$; we have $D-n < D-m$ if and only if $m < n$ and $D-n > D-m$ if and only if $m > n$. Given $\frac{1}{r^{D-n}}$ the power $\frac{1}{r^{D-m}}$ is subleading if and only if $m < n$.

and inserting the expansions 3.50 we get

$$\begin{aligned}
\epsilon_r &= \sum_{l<D-5} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-5} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l},\\
\epsilon_u &= \sum_{l<D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l},
\end{aligned} \tag{3.70}$$

with

$$\begin{aligned}
&\partial_u \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_u^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-5;\\
&\partial_u \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-5.
\end{aligned} \tag{3.71}$$

However these conditions impose us to change the fall-off in $\mathcal{O}\big((r^{-(D-4)} ln(r)\big)$ by a pure gauge term. Alternatively, we can impose that the second relation in (3.71) holds also for $l = D-5$ avoiding violations of the fall-off. The only reason for choosing to violate the fall-off would be an analogy with the case $p = 1$ however, in the case $p = 2$ this choice is not necessary to have finite and non-vanishing asymptotic charges.

Case II

If $\mu_2 \in \mathbb{I}_{\{x^i\}}$ and $\mu_3 \notin \mathbb{I}_{\{x^i\}}$ then $\mu_3 \in \{u, r\}$. In the first case we have the gauge transformation

$$\partial_i \epsilon_u - \partial_u \epsilon_i \sim \mathcal{O}\big(r^{-(D-5)}\big); \tag{3.72}$$

inserting the expansion of the gauge parameter we get

$$\begin{aligned}
\epsilon_u &= \sum_{l<D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l},\\
\epsilon_i &= \sum_{l<D-5} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-5} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l},
\end{aligned} \tag{3.73}$$

with

$$\begin{aligned}
&\partial_i \epsilon_u^{(l)}(u,\{x^i\}) = \partial_u \epsilon_i^{(l)}(u,\{x^i\}) \quad \forall l < D-5;\\
&\partial_i \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = \partial_u \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) \quad \forall l < D-5.
\end{aligned} \tag{3.74}$$

However, these conditions impose us to change the fall-off in $\mathcal{O}\big((r^{-(D-5)} ln(r)\big)$ by a pure gauge term which play no role in physical quantities. Hence, no divergence can arise if we add the term $\mathcal{O}\big((r^{-(D-5)} ln(r)\big)$ as a pure gauge term. Nevertheless, at the time of writing and at least to the author, it is not yet fully clear how far one can go with this argument. In fact, it seems that we could violate the behavior of fields of any arbitrary power as long as it is produced by pure gauge terms. In this way it would always be possible to find asymptotic symmetries, i.e. non-trivial asymptotic charges, even if the original fall-offs preclude such asymptotic symmetries. This

observation could be at the basis of something deeper: pure gauge sectors, even though they have no dynamical effects since they have vanishing field strength, could still have kinematic relevance through the asymptotic symmetries of the theory that bring to light[3]. In the case of Coulomb fall-off, add a pure gauge term is the only possibility since, in this case, the equations of motion and the Lorenz-like gauge conditions are not able to cancel out the possible emerging of overleading unphysical[4] terms which would appear in the field strength if we admitted a logarithmic leading order, i.e. $\mathcal{O}\big((r^{-(D-5)}ln(r)\big)$, from the beginning.
In the second case the gauge transformation reads

$$\partial_i\epsilon_r - \partial_r\epsilon_i \sim \mathcal{O}\big(r^{-(D-5)}\big); \tag{3.75}$$

inserting the gauge parameter expansions we get

$$\begin{aligned}\epsilon_r &= \sum_{l<D-6}\frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\})ln(r)}{r^l} + \sum_{l\geq D-6}\frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\})ln(r)}{r^l},\\ \epsilon_i &= \sum_{l<D-6}\frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\})ln(r)}{r^l} + \sum_{l\geq D-6}\frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\})ln(r)}{r^l},\end{aligned} \tag{3.76}$$

with

$$\begin{aligned}&\partial_i\epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_i^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-6;\\ &\partial_i\tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l \leq D-6.\end{aligned} \tag{3.77}$$

Case III

In this case $\mu_2, \mu_3 \in \mathbb{I}_{\{x^i\}}$ and the gauge transformation is

$$\partial_i\epsilon_j - \partial_j\epsilon_i \sim \mathcal{O}\big(r^{-(D-6)}\big); \tag{3.78}$$

inserting the expansion of the gauge parameter we have

$$\epsilon_i = \sum_{l<D-6}\frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\})ln(r)}{r^l} + \sum_{l\geq D-6}\frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\})ln(r)}{r^l}, \tag{3.79}$$

with

$$\begin{aligned}\epsilon_i^{(l)}(u,\{x^i\}) &= \partial_i f^{(l)}(u,\{x^i\}) \quad \forall l < D-6;\\ \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) &= \partial_i \tilde{f}^{(l)}(u,\{x^i\}) \quad \forall l \leq D-6.\end{aligned} \tag{3.80}$$

[3]This note is no present in the case of radiation fall-offs simply because it exists a mechanism, presented in Paragraph 3.3.2 that is able to cancel out the overleading term also in the case of non-pure gauge. Moreover, see also [332] for considerations in this direction.

[4]In dimension grater than the critical one, this terms could lead to a divergence energy flux at null infinity.

Conclusive remarks

The residual gauge in the case of Coulomb fall-offs is then parameterized by

$$
\begin{aligned}
\epsilon_u &= \sum_{l<D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-5} \frac{\epsilon_u^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \\
\epsilon_r &= \sum_{l<D-6} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-6} \frac{\epsilon_r^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_r^{(l)}(u,\{x^i\}) ln(r)}{r^l}, \\
\epsilon_i &= \sum_{l<D-6} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l} + \sum_{l\geq D-6} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l},
\end{aligned}
\tag{3.81}
$$

with

$$
\begin{aligned}
&\partial_u \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_u^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-5; \\
&\partial_u \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-5; \\
&\partial_i \epsilon_u^{(l)}(u,\{x^i\}) = \partial_u \epsilon_i^{(l)}(u,\{x^i\}) \quad \forall l < D-5; \\
&\partial_i \tilde{\epsilon}_u^{(l)}(u,\{x^i\}) = \partial_u \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) \quad \forall l < D-5; \\
&\partial_i \epsilon_r^{(l+1)}(u,\{x^i\}) + l\epsilon_i^{(l)}(u,\{x^i\}) - \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l < D-6; \\
&\partial_i \tilde{\epsilon}_r^{(l+1)}(u,\{x^i\}) + l\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = 0 \quad \forall l \leq D-6; \\
&\epsilon_i^{(l)}(u,\{x^i\}) = \partial_i f^{(l)}(u,\{x^i\}) \quad \forall l < D-6; \\
&\tilde{\epsilon}_i^{(l)}(u,\{x^i\}) = \partial_i \tilde{f}^{(l)}(u,\{x^i\}) \quad \forall l \leq D-6.
\end{aligned}
\tag{3.82}
$$

These parameters have to satisfy the conditions to not spoil the Lorenz-like gauge, $\Box\epsilon_\alpha = \Box\epsilon = \nabla^\mu\epsilon_\mu = 0$; moreover, it is possible to perform a gauge for gauge fixing analog to the one used for the radiation fall-off. Looking to the gauge fixing equations (A.8) we get, for the gauge parameter we are interested in,

$$
\epsilon_i = \frac{\epsilon_i^{(D-6)}(\{x^i\})}{r^{D-6}} + \sum_{l\geq D-5} \frac{\epsilon_i^{(l)}(u,\{x^i\}) + \tilde{\epsilon}_i^{(l)}(u,\{x^i\}) ln(r)}{r^l}. \tag{3.83}
$$

The point is therefore that, thanks to results in Appendices A, B and C it is possible, in sufficiently high dimensions, to start the expansion of the gauge parameter $\epsilon_i(\{x^i\})$ with overleading terms with respect to the standard and expected[5] $\mathcal{O}(r^{-(D-6)})$ since they can be cancelled out with a gauge for gauge fixing.
Therefore, we have showed that

Lemma 3.3.2 (Leading order of the gauge parameter components ϵ_i in the 2-form gauge theory in any D)**.** *In the framework of* 2*-form gauge theory in any* D*, the leading order* X *of the polyhomogenoues expansion of the*

[5]We expect the expansion of the gauge parameters ϵ_i start with a term of order $\mathcal{O}(r^{-(D-6)}$ since the field components B_{ij} have expansions that start with a term of order $\mathcal{O}(r^{-(D-6)}$.

gauge parameter components ϵ_i is

$$\begin{cases} X = \frac{D-6}{2} & \text{for radiation fall-off;} \\ X = D-6 & \text{for Coulomb fall-off.} \end{cases} \tag{3.84}$$

Proof. The proof is contained in the computations of Paragraph 3.3.3.2 and Appendices A, B and C. □

3.3.3.3 Residual gauge in the general case

The case of the 2-form is a guiding example: also in the general case we have four equations to take into account. Since the dimension of the exterior algebra is given by $dim(\bigwedge^p \mathbb{R}^{D-1,1}) = \binom{D}{p}$, the existing condition for a p-form in D dimension is $D \geq p$; from this information we get the minimal number of angular variables we need to have non-trivial information from the gauge transformation of the p-form, indeed from the complete antisymmetry we get

$$\#\{x^i\} = D-2 \geq p-2 = \min_{\mathbb{N}}(\#\mathbb{I}_{\{x^i\}}). \tag{3.85}$$

Condition (3.85), i.e. the complete antisymmetry, tells us that only the following field components are to be taken into account:

$$B_{ru\underbrace{ij...k}_{p-2}}, \quad B_{r\underbrace{ij...k}_{p-1}}, \quad B_{u\underbrace{ij...k}_{p-1}}, \quad B_{\underbrace{ij...k}_{p}}; \tag{3.86}$$

all the others are trivially zero. However is not necessary to do all the computation for the generic case: the cases for $p=1$ and $p=2$ are enough. Indeed, we have the following

Theorem 3.3.1 (Leading order of the gauge parameter components $\epsilon_{i_1 \dots i_{p-1}}$ in the p-form gauge theory in any D)**.** *In the framework of p-form gauge theory in any D, the leading order X of the polyhomogenoues expansion of the gauge parameter components $\epsilon_{i_1 \dots i_{p-1}}$ is*

$$\begin{cases} X = \frac{D-(2p+2)}{2} & \text{for radiation fall-off;} \\ X = D-(2p+2) & \text{for Coulomb fall-off.} \end{cases} \tag{3.87}$$

Proof. Let us take for $\epsilon_{i_1 \dots i_{p-1}}$ a polyhomogeneous expansion of the form

$$\epsilon_{i_1 \dots i_{p-1}} = \sum_{l \in \frac{1}{2}\mathbb{Z}} \frac{\epsilon^{(l)}_{i_1 \dots i_{p-1}}(u, \{x^i\}) + \tilde{\epsilon}^{(l)}_{i_1 \dots i_{p-1}}(u, \{x^i\}) ln(r)}{r^l}, \tag{3.88}$$

Following $p=1$ and $p=2$ cases we can ask what is the leading order of the gauge parameter components expansion (3.88) that preserves the Lorenz-like gauge and the fall-offs of the fields. Let us call this order X: so would have

$$\epsilon_{i_1\dots i_{p-1}} = \frac{\epsilon^{(X)}_{i_1\dots i_{p-1}}(\{x^i\})}{r^X} + \sum_{l>X} \frac{\epsilon^{(l)}_{i_1\dots i_{p-1}}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}_{i_1\dots i_{p-1}}(u,\{x^i\}) ln(r)}{r^l}, \tag{3.89}$$

where $X := f(p, D)$ is a function of the dimension D and the form degree p. The derivatives acting on the gauge parameter to get the gauge transformation do not change the kind of function $f(p, D)$ is; therefore since we have assumed radiation or Coulomb fall-offs, which have a would be $f(p, D)$ linear both in D and in p, we can conclude that $f(p, D)$ for the gauge parameter components (3.88) must be linear both in D and in p. We know its value for $p = 1$ and $p = 2$ thanks to Lemmas 3.3.1 and 3.3.2; therefore, for radiation fall-offs we have

$$f(p, D) = a(D)p + b(D), \qquad \text{with} \quad f(1, D) = \frac{D-4}{2}, \quad f(2, D) = \frac{D-6}{2}, \tag{3.90}$$

while for Coulomb fall-offs we have

$$f(p, D) = a'(D)p + b'(D), \qquad \text{with} \quad f(1, D) = D-4, \quad f(2, D) = D-6. \tag{3.91}$$

From the above relations we get $a(D) = -1$ and $b(D) = \frac{D-2}{2}$ for radiation fall-offs while $a'(D) = -2$ and $b'(D) = D-2$ for Coulomb fall-offs. Therefore we have

$$\begin{aligned} X &:= f(p, D) = -p + \frac{D-2}{2} = \frac{D-(2p+2)}{2} && \text{radiation fall-off;} \\ X &:= f(p, D) = -2p + D - 2 = D-(2p+2) && \text{Coulomb fall-off;} \end{aligned} \tag{3.92}$$

which complete the proof. □

3.4 Asymptotic charges

In order to compute asymptotic charges associated to residual gauge transformations, the first step is to recall the Noether second theorem [4],[217]: classically, the charges arising from local symmetries vanish identically since these are redundancies of the theory and act trivially of physical states. However, charges can attain non-trivial values if associated to large gauge transformations. Indeed, for gauge symmetries identified by the set of gauge parameters $\mathcal{S}$, the conserved Noether current $j^\mu_{\mathcal{S}}$ is given by the divergence of an antisymmetric tensor, $j^\mu_{\mathcal{S}} = \partial_\nu k^{\mu\nu}_{\mathcal{S}}$ with $k^{\mu\nu}_{\mathcal{S}} = -k^{\nu\mu}_{\mathcal{S}}$. This property is usually referred to local Gauss law since it closely resembles the case of the Maxwell equations for electrodynamics. Any of such a Noether current is conserved as a consequence of the antisymmetry of $k^{\mu\nu}_{\mathcal{S}}$; the corresponding charges, obtained by integrating $j^\mu_{\mathcal{S}}$ on a space like hypersurface Σ, are in fact given by integrals over the boundary $\sigma = \partial\Sigma$ by Stokes' theorem

$$Q_{\mathcal{S}} := \int_{\Sigma} j^{\mu}_{\mathcal{S}} d\Sigma_{\mu} \approx \oint_{\partial\Sigma} k^{\mu\nu}_{\mathcal{S}} d\sigma_{\mu\nu}. \tag{3.93}$$

These kind of charges are therefore intrinsically related to the behavior of the theory near the boundary we are evaluating the integral. It is clear that $Q_{\mathcal{S}}$ is zero if $\mathcal{S}$ is a set of gauge parameters localized in a compact region. In the case of our interest we integrate on the S^{D-2}_u, located at null infinity, for fixed u and $r \to +\infty$:

$$Q_{p,D}[S^{D-2}_u] := \lim_{r\to+\infty} \oint_{S^{D-2}_u} k^{ur}_{p,D} r^{D-2} d\Omega. \tag{3.94}$$

where $k^{\mu\nu}_{p,D}$ is the Noether two form[6] for a p-form in dimension D. To compute the asymptotic charges we need to compute the charges of the residual gauge, hence we have to find the two form $k^{\mu\nu}_{p,D}$; for this scope let us start computing the Noether current.

The variation of the lagrangian (3.1) is given by

$$\begin{aligned}\delta\mathcal{L} &= -\frac{2}{(p+1)!}\delta(H_{\mu_1\cdots\mu_{p+1}})H^{\mu_1\cdots\mu_{p+1}} = -\frac{2}{(p+1)!}H^{\mu_1\cdots\mu_{p+1}}[\partial_{\mu_1}\delta(B_{\mu_2\cdots\mu_{p+1}}) - \sum_{i=2}^{p}\partial_{\mu_i}\delta(B_{\mu_2\cdots\mu_{i-1}\mu_1\mu_{i+1}\ldots\mu_{p+1}})] = \\ &= -\frac{2}{(p+1)!}\partial_{\mu_1}\delta(B_{\mu_2\cdots\mu_{p+1}})[(p+1)H^{\mu_1\cdots\mu_{p+1}}] = -\frac{2(p+1)}{(p+1)!}[\partial_{\mu_1}\delta(B_{\mu_2\cdots\mu_{p+1}})]H^{\mu_1\cdots\mu_{p+1}},\end{aligned} \tag{3.95}$$

where we used the complete antisymmetry property of the field strength. The Noether current, upon integration by parts and using the equations of motion, is identified with[7]

$$j^{\mu_1} \approx -\frac{2(p+1)}{(p+1)!}\delta(B_{\mu_2\cdots\mu_{p+1}})H^{\mu_1\cdots\mu_{p+1}} \qquad p > 0. \tag{3.96}$$

Imposing the gauge variation (3.4) we can write the current as

$$j^{\mu_1} \approx -\frac{2(p+1)p}{(p+1)!}(\partial_{\mu_2}\epsilon_{\mu_3\cdots\mu_{p+1}})H^{\mu_1\cdots\mu_{p+1}} \tag{3.97}$$

and, upon integration by parts and using the equations of motion, we get

$$j^{\mu_1} \approx -\frac{2(p+1)p}{(p+1)!}\partial_{\mu_2}(\epsilon_{\mu_3\cdots\mu_{p+1}}H^{\mu_1\cdots\mu_{p+1}}). \tag{3.98}$$

[6]This would be the generalization of the well-known Noether two form for electromagnetism to general case of p-form.

[7]The condition $p > 0$ is essential to derive the Noether's two forms since otherwise it will be not defined. This problem could be resolved if we analytically extend the factorial to the Euler $\Gamma(z)$ function since this function has simple poles for every $z = -n$ with $n \in \mathbb{N}$ [28].

Therefore we have found the Noether two form

$$k^{\mu_1\mu_2}_{p,D} := -\frac{2(p+1)p}{(p+1)!}\epsilon_{\mu_3\cdots\mu_{p+1}} H^{\mu_1\cdots\mu_{p+1}} \qquad p>0, \tag{3.99}$$

which is antisymmetric in μ_1 and μ_2 due to the total antisymmetry of $H^{\mu_1\cdots\mu_{p+1}}$. For $p=1$ it reduces to the well known Noether two form of electromagnetism.

Let us now compute the asymptotic charge for ($p>0$)-forms starting from the examples of the 1-form and of the 2-form. Moreover, for future future usefulness, let us give two important definitions

Definition 3.4.1 (Well defined charges)**.** *Well defined charges are those charges which are finite and non-vanishing in the limit $r \to \infty$ and that have the same radial behavior in the dual theories.*

and

Definition 3.4.2 (Power law divergent (vanishing) charges)**.** *Power law divergent (vanishing) charges are those charges which are divergent (vanishing) in the limit $r \to \infty$ and such that the degree of divergence (vanishing) is polynomial.*

These definitions, even if at the moment they seem decontextualized, will be useful later when we study what relationship exists between the charges of a description and those of the dual description. In fact, the discussion about the relationship strongly depends from what type of charge we have in the game.

3.4.1 The starting example of the 1-form

The asymptotic charge in the case of the 1-form is

$$\begin{aligned} Q_{1,D}[S^{D-2}_u] :=& \lim_{r\to+\infty}\oint_{S^{D-2}_u} k^{ur}_{1,D} r^{D-2} d\Omega = \\ =& -2\lim_{r\to+\infty}\oint_{S^{D-2}_u} \epsilon H^{ur} r^{D-2} d\Omega = \\ =& -2\lim_{r\to+\infty}\oint_{S^{D-2}_u} g^{\alpha u} g^{\beta r}\epsilon H_{\alpha\beta} r^{D-2} d\Omega = \\ =& -2\lim_{r\to+\infty}\oint_{S^{D-2}_u} \epsilon H_{ru} r^{D-2} d\Omega; \end{aligned} \tag{3.100}$$

inserting the definition of the covariantized field strength in the charge (3.100) we get

$$Q_{1,D}[S^{D-2}_u] = -2\lim_{r\to+\infty}\oint_{S^{D-2}_u} r^{D-2}\epsilon[\nabla_r B_u - \nabla_u B_r] d\Omega = -2\lim_{r\to+\infty}\oint_{S^{D-2}_u} r^{D-2}\epsilon\left[\partial_r B_u - \partial_u B_r\right] d\Omega. \tag{3.101}$$

3.4.1.1 Asymptotic charge in radiation fall-offs

The gauge parameter we are interested in is

$$\epsilon = \frac{\epsilon^{(\frac{D-4}{2})}(\{x^i\})}{r^{\frac{D-4}{2}}} + \sum_{l\geq\frac{D-3}{2}} \frac{\epsilon^{(l)}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}(u,\{x^i\})ln(r)}{r^l}. \tag{3.102}$$

Looking at the charge (3.101) we get that the leading order is given by

$$Q_{1,D}[S_u^{D-2}] \sim -2\oint_{S_u^{D-2}} \epsilon^{(\frac{D-4}{2})}(\{x^i\}) H_{ru}^{(\frac{D}{2})} \mathcal{O}(r^0) d\Omega. \tag{3.103}$$

where

$$H_{ru}^{(\frac{D}{2})} = -\left(\frac{D-2}{2}\right) B_u^{(\frac{D-2}{2})} - \partial_u B_r^{(\frac{D}{2})} + \tilde{B}_u^{(\frac{D-2}{2})}; \tag{3.104}$$

the term $\tilde{B}_u^{(\frac{D-2}{2})}$ is present only if we start with logarithmic terms in the expansion of the field component.

3.4.1.2 Asymptotic charge in Coulomb fall-offs

The gauge parameter we are interested in is

$$\epsilon = \frac{\epsilon^{(D-4)}(\{x^i\})}{r^{D-4}} + \sum_{l\geq D-3} \frac{\epsilon^{(l)}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}(u,\{x^i\})ln(r)}{r^l}. \tag{3.105}$$

Looking at the charge (3.101) we get that the leading order and the next to leading order are given by

$$Q_{1,D}[S_u^{D-2}] \sim -2\oint_{S_u^{D-2}} \epsilon^{(D-4)}(\{x^i\}) \left[H_{ru}^{(D-2)} + \tilde{H}_{ru}^{(D-2)} ln(r)\right] \mathcal{O}(r^{-(D-4)}) d\Omega, \tag{3.106}$$

where

$$\begin{aligned} H_{ru}^{(D-2)} &= -(D-3)B_u^{(D-3)} - \partial_u B_r^{(D-2)} + \tilde{B}_u^{(D-3)}; \\ \tilde{H}_{ru}^{(D-2)} &= -(D-3)\tilde{B}_u^{(D-3)} - \partial_u \tilde{B}_r^{(D-2)} = (D-4)\tilde{B}_u^{(D-3)} \end{aligned} \tag{3.107}$$

The logarithmic term is present only if we admit, from the beginning, a logarithmic leading order in the Coulomb fall-off expansion of the field component B_u, so if it is not produced by pure gauge[8]; this term is vanishing in critical dimension $D = 4$ since can be rewritten, for $D \neq 2$, using the second equation of (3.29) as

$$\tilde{H}_{ru}^{(D-2)} = (D-4)\tilde{B}_u^{(D-3)}. \tag{3.108}$$

[8]Recall that this choice could lead to unwanted unphysical divergences in the energy flux at null infinity.

In $D = 4$ the charge has the same leading order of charge (3.103) as should be and in $D = 3$ the first term of both $H^{(D-2)}_{ru}$ and $\tilde{H}^{(D-2)}_{ru}$ is vanishing. Moreover, we note that apart from the logarithmic term, which can be avoided considering it as a pure gauge term, this charge is power law vanishing in $D > 4$ while it has a power law divergence in $D < 4$. From a physical point of view, this faster decay in higher dimensions can be attributed to the greater number of "angular" dimensions in which the field strength can be diluted.

3.4.2 The pivotal example of the 2-form

The asymptotic charge in the case of the 2-form is

$$\begin{aligned}
Q_{2,D}[S^{D-2}_u] :=& \lim_{r\to+\infty} \oint_{S^{D-2}_u} k^{ur}_{2,D} r^{D-2} d\Omega = \\
=& -2 \lim_{r\to+\infty} \oint_{S^{D-2}_u} \epsilon_\alpha H^{ur\alpha} r^{D-2} d\Omega = \\
=& -2 \lim_{r\to+\infty} \oint_{S^{D-2}_u} g^{\alpha u} g^{\beta r} g^{i\tilde{i}} \epsilon_i H_{\alpha\beta\tilde{i}} r^{D-2} d\Omega = \\
=& -2 \lim_{r\to+\infty} \oint_{S^{D-2}_u} \frac{1}{r^2} \gamma^{i\tilde{i}} \epsilon_i H_{ru\tilde{i}} r^{D-2} d\Omega;
\end{aligned} \tag{3.109}$$

inserting the definition of the covariantized field strength in the charge (3.109) we get

$$\begin{aligned}
Q_{2,D}[S^{D-2}_u] =& -2 \lim_{r\to+\infty} \oint_{S^{D-2}_u} r^{D-4} \gamma^{i\tilde{i}} \epsilon_i [\nabla_r B_{u\tilde{i}} - \nabla_u B_{r\tilde{i}} - \nabla_{\tilde{i}} B_{ur}] d\Omega = \\
=& -2 \lim_{r\to+\infty} \oint_{S^{D-2}_u} r^{D-4} \gamma^{i\tilde{i}} \epsilon_i \left[\partial_r B_{u\tilde{i}} - \frac{1}{r} \delta^i_{\tilde{i}} B_{ui} - \partial_u B_{r\tilde{i}} - \partial_{\tilde{i}} B_{ur} + \frac{1}{r} \delta^i_{\tilde{i}} B_{ui} \right] d\Omega.
\end{aligned} \tag{3.110}$$

3.4.2.1 Asymptotic charge in radiation fall-offs

The gauge parameter components we are interested in are

$$\epsilon_i = \frac{\epsilon_i^{(\frac{D-6}{2})}(\{x^i\})}{r^{\frac{D-6}{2}}} + \sum_{l \geq \frac{D-5}{2}} \frac{\epsilon_i^{(l)}(u, \{x^i\}) + \tilde{\epsilon}_i^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.111}$$

Looking at the charge (3.110) and inserting the expansion for the gauge parameter components given above, we get that the leading order is given by

$$Q_{2,D}[S^{D-2}_u] \sim -2 \oint_{S^{D-2}_u} \gamma^{i\tilde{i}} \epsilon_i^{(\frac{D-6}{2})}(\{x^i\}) H_{ru\tilde{i}}^{(\frac{D-2}{2})} \mathcal{O}(r^0) d\Omega, \tag{3.112}$$

where

$$H_{ru\tilde{i}}^{(\frac{D-2}{2})} = -\left(\frac{D-4}{2}\right) B_{u\tilde{i}}^{(\frac{D-4}{2})} - \partial_{\tilde{i}} B_{ur}^{(\frac{D-2}{2})} - \partial_u B_{r\tilde{i}}^{(\frac{D-2}{2})} + \tilde{B}_{u\tilde{i}}^{(\frac{D-4}{2})}; \tag{3.113}$$

the term $\tilde{B}_{u\tilde{i}}^{(\frac{D-4}{2})}$ is present only if we start with logarithmic terms in the expansion of the field components.

3.4.2.2 Asymptotic charge in Coulomb fall-offs

The gauge parameter components we are interested in are

$$\epsilon_i = \frac{\epsilon_i^{(D-6)}(\{x^i\})}{r^{D-6}} + \sum_{l \geq D-5} \frac{\epsilon_i^{(l)}(u, \{x^i\}) + \tilde{\epsilon}_i^{(l)}(u, \{x^i\}) ln(r)}{r^l}. \tag{3.114}$$

Looking at the charge (3.110) we get that the leading order and the next to leading order are given by

$$Q_{2,D}[S_u^{D-2}] \sim -2 \oint_{S_u^{D-2}} \gamma^{i\tilde{i}} \epsilon_i^{(D-6)}(\{x^i\}) \left[H_{ru\tilde{i}}^{(D-4)} + \tilde{H}_{ru\tilde{i}}^{(D-4)} ln(r) \right] \mathcal{O}(r^{-(D-6)}) d\Omega, \tag{3.115}$$

where

$$\begin{aligned} H_{ru\tilde{i}}^{(D-4)} &= -(D-5) B_{u\tilde{i}}^{(D-5)} - \partial_{\tilde{i}} B_{ur}^{(D-4)} - \partial_u B_{r\tilde{i}}^{(D-4)} + \tilde{B}_{u\tilde{i}}^{(D-5)}; \\ \tilde{H}_{ru\tilde{i}}^{(D-4)} &= -(D-5) \tilde{B}_{u\tilde{i}}^{(D-5)} - \partial_u \tilde{B}_{r\tilde{i}}^{(D-4)}. \end{aligned} \tag{3.116}$$

The logarithmic term is present only if we admit, from the beginning, a logarithmic leading order in the Coulomb fall-off expansion of the field components $B_{u\tilde{i}}$, so if it is not produced by pure gauge[9]; this term is vanishing in critical dimension $D = 6$ since can be rewritten, for $D \neq 2$, using the second equation of (3.29) as

$$\tilde{H}_{ru\tilde{i}}^{(D-4)} = (D-6) \tilde{B}_{u\tilde{i}}^{(D-5)}. \tag{3.117}$$

In $D = 6$ the charge has the same leading order of charge (3.112) as should be and in $D = 5$ the first term of both $H_{ru\tilde{i}}^{(D-4)}$ and $\tilde{H}_{ru\tilde{i}}^{(D-4)}$ is vanishing. Moreover, we note that apart from the logarithmic term, which can be avoided by considering it as a pure gauge term, this charge is power law vanishing in $D > 6$ while it has a power law divergence in $D < 6$. From a physical point of view, this faster decay in higher dimensions can be attributed to the greater number of "angular" dimensions in which the field strength can be diluted. Moreover, with respect to the case of the 1-form, now the field strength has more "angular direction" since it is associated with a differential form of greater degree, therefore the dimension necessary to "dilute" the field strength becomes greater.

[9] Recall that this choice could lead to unwanted unphysical divergences in the energy flux at null infinity.

3.4.3 The general case

The general case is not so different from the previous ones apart from the heavier algebra. Defining $\Lambda_p := -\frac{2(p+1)p}{(p+1)!}$, the charge is given by

$$
\begin{aligned}
Q_{p,D}[S_u^{D-2}] :=& \lim_{r\to+\infty}\oint_{S_u^{D-2}} k^{ur}_{2,D} r^{D-2} d\Omega = \\
=&\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} \epsilon_{\mu_3\dots\mu_{p+1}} H^{ur\mu_3\dots\mu_{p+1}} r^{D-2} d\Omega = \\
=&\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} \epsilon_{i_1\dots i_{p-1}} H^{uri_1\dots i_{p-1}} r^{D-2} d\Omega = \\
=&\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} g^{\alpha u} g^{\beta r} g^{i_1\tilde{i}_1}\dots g^{i_{p-1}\tilde{i}_{p-1}} \epsilon_{i_1\dots i_{p+-1}} H_{\alpha\beta\tilde{i}_1\dots\tilde{i}_{p-1}} r^{D-2} d\Omega = \\
=&\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} \gamma^{i_1\tilde{i}_1}\dots\gamma^{i_{p-1}\tilde{i}_{p-1}} \epsilon_{i_1\dots i_{p-1}} H_{ru\tilde{i}_1\dots\tilde{i}_{p-1}} r^{D-2p} d\Omega;
\end{aligned}
\tag{3.118}
$$

we may note that for $D = 4$ and $p = 1$, i.e. standard four dimensional Maxwell theory, this charge is in agreement with the one in literature [257] while for $p = 2$ it reduces to charge (3.109). Inserting the definition of the covariantized field strength in the charge (3.118) we get

$$
Q_{p,D}[S_u^{D-2}] =\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} r^{D-2p}\gamma^{i_1\tilde{i}_1}\dots\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon_{i_1\dots i_{p-1}}\left[\nabla_r B_{u\tilde{i}_1\dots\tilde{i}_{p-1}} - \nabla_u B_{r\tilde{i}_1\dots\tilde{i}_{p-1}} - \sum_{j=1}^{p-1}\nabla_{\tilde{i}_j} B_{u\dots\tilde{i}_{j-1}r\tilde{i}_{j+1}\dots\tilde{i}_{p-1}}\right] d\Omega = \tag{3.119a}
$$

$$
=\Lambda_p \lim_{r\to+\infty}\oint_{S_u^{D-2}} r^{D-2p}\gamma^{i_1\tilde{i}_1}\dots\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon_{i_1\dots i_{p-1}}\left[\partial_r B_{u\tilde{i}_1\dots\tilde{i}_{p-1}} - \partial_u B_{r\tilde{i}_1\dots\tilde{i}_{p-1}} - \sum_{j=1}^{p-1}\partial_{\tilde{i}_j} B_{u\dots\tilde{i}_{j-1}r\tilde{i}_{j+1}\dots\tilde{i}_{p-1}}\right] d\Omega. \tag{3.119b}
$$

3.4.3.1 Asymptotic charge in radiation fall-offs

The gauge parameter components we are interested in are

$$
\epsilon_{i_1\dots i_{p-1}} = \frac{\epsilon^{(\frac{D-(2p+2)}{2})}_{i_1\dots i_{p-1}}(\{x^i\})}{r^{\frac{D-(2p+2)}{2}}} + \sum_{l\geq\frac{D-(2p+1)}{2}} \frac{\epsilon^{(l)}_{i_1\dots i_{p-1}}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}_{i_1\dots i_{p-1}}(u,\{x^i\}) ln(r)}{r^l}. \tag{3.120}
$$

Looking at the charge (3.119b), using

$$
\begin{aligned}
&\partial_u B^{(\frac{D-2p}{2})}_{r\tilde{i}_1\dots\tilde{i}_{p-1}} = \partial_u B^{(\frac{D-2p+1}{2})}_{r\tilde{i}_1\dots\tilde{i}_{p-1}} = \partial_u \tilde{B}^{(\frac{D-2p+1}{2})}_{r\tilde{i}_1\dots\tilde{i}_{p-1}} = 0, \\
&2\partial_u \tilde{B}^{(\frac{D-2p+2}{2})}_{r\tilde{i}_1\dots\tilde{i}_{p-1}} + (D-2p)\tilde{B}^{(\frac{D-2p}{2})}_{u\tilde{i}_1\dots\tilde{i}_{p-1}} = 0,
\end{aligned}
\tag{3.121}
$$

we get the leading order

$$Q_{p,D}[S_u^{D-2}] \sim \Lambda_p \oint_{S_u^{D-2}} \gamma^{i_1\tilde{i}_1}...\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon^{(\frac{D-(2p+2)}{2})}_{i_1...i_{p-1}}(\{x^i\})H^{(\frac{D-2p+2}{2})}_{ru\tilde{i}_1...\tilde{i}_{p-1}}\mathcal{O}(r^0)d\Omega, \tag{3.122}$$

where

$$H^{(\frac{D-2p+2}{2})}_{ru\tilde{i}_1...\tilde{i}_{p-1}} = -\left(\frac{D-2p}{2}\right)B^{(\frac{D-2p}{2})}_{u\tilde{i}_1...\tilde{i}_{p-1}} - \sum_{j=1}^{p-1}\partial_{\tilde{i}_j}B^{(\frac{D-2p+2}{2})}_{u...\tilde{i}_{j-1}r\tilde{i}_{j+1}...\tilde{i}_{p-1}} - \partial_u B^{(\frac{D-2p+2}{2})}_{r\tilde{i}_1...\tilde{i}_{p-1}} + \tilde{B}^{(\frac{D-2p}{2})}_{u\tilde{i}_1..\tilde{i}_{p-1}}. \tag{3.123}$$

This charge is well defined for every p in every D since if we formally operate the substitution $p \mapsto q$ we get again a finite non-vanishing charge. the term $B^{(\frac{D-2p}{2})}_{u\tilde{i}_1...\tilde{i}_{p-1}}$ is present only if we start with logarithmic terms in the expansion of the field components. It is worth emphasizing that the charge can be expressed only in terms of $\nabla^i B^{(\frac{D-2p}{2})}_{ii_1...i_{p-1}}$ modulo possible non-dynamical terms given by eigenfunction laplacian on the sphere.

3.4.3.2 Asymptotic charge in Coulomb fall-offs

The gauge parameter components we are interested in are

$$\epsilon_{i_1...i_{p-1}} = \frac{\epsilon^{(D-(2p+2))}_{i_1...i_{p-1}}(\{x^i\})}{r^{D-(2p+2)}} + \sum_{l\geq D-(2p+1)} \frac{\epsilon^{(l)}_{i_1...i_{p-1}}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}_{i_1...i_{p-1}}(u,\{x^i\})ln(r)}{r^l}. \tag{3.124}$$

Looking at the charge (3.119b) we get that the leading order and the next to leading order is given by

$$Q_{p,D}[S_u^{D-2}] \sim \Lambda_p \oint_{S_u^{D-2}} \gamma^{i_1\tilde{i}_1}...\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon^{(D-(2p+2))}_{i_1...i_{p-1}}(\{x^i\})\left[H^{(D-2p)}_{ru\tilde{i}_1...\tilde{i}_{p-1}} + \tilde{H}^{(D-2p)}_{ru\tilde{i}_1...\tilde{i}_{p-1}}ln(r)\right]\mathcal{O}(r^{-(D-2p-2)})d\Omega, \tag{3.125}$$

where

$$\begin{aligned} H^{(D-2p)}_{ru\tilde{i}_1...\tilde{i}_{p-1}} &= -\left(D-2p-1\right)B^{(D-2p-1)}_{u\tilde{i}_1...\tilde{i}_{p-1}} - \sum_{j=1}^{p-1}\partial_{\tilde{i}_j}B^{(D-2p)}_{u...\tilde{i}_{j-1}r\tilde{i}_{j+1}...\tilde{i}_{p-1}} - \partial_u B^{(D-2p)}_{r\tilde{i}_1...\tilde{i}_{p-1}} + \tilde{B}^{(D-2p-1)}_{u\tilde{i}_1..\tilde{i}_{p-1}}; \\ \tilde{H}^{(D-2p)}_{ru\tilde{i}_1...\tilde{i}_{p-1}} &= -\left(D-2p-1\right)\tilde{B}^{(D-2p-1)}_{u\tilde{i}_1...\tilde{i}_{p-1}} - \partial_u\tilde{B}^{(D-2p)}_{r\tilde{i}_1...\tilde{i}_{p-1}}. \end{aligned} \tag{3.126}$$

Similarly to the case of the 1-form and 2-form, the logarithmic term is present only if we admit, from the beginning, a logarithmic leading order in the Coulomb fall-off expansion of the field components $B_{u\tilde{i}_1...\tilde{i}_{p-1}}$, so if it is not produced by pure gauge[10]; this term is vanishing in critical dimension $D = 2p + 2$ since can be

[10]Recall that this choice could lead to unwanted unphysical divergences in the energy flux at null infinity.

rewritten, for $D \neq 2$, using the second equation (3.29) as

$$\tilde{H}^{(D-2p)}_{ru\tilde{i}_1\ldots\tilde{i}_{p-1}} = (D-2p-2)\tilde{B}^{(D-2p-1)}_{u\tilde{i}_1\ldots\tilde{i}_{p-1}}. \tag{3.127}$$

In $D = 2p+2$ the charge has the same leading order of charge (3.122) as should be[11] and in $D = 2p+1$ the first term of both $H^{(D-2p)}_{ru\tilde{i}_1\ldots\tilde{i}_{p-1}}$ and $\tilde{H}^{(D-2p)}_{ru\tilde{i}_1\ldots\tilde{i}_{p-1}}$ is vanishing. Moreover, we note that apart from the logarithmic term, which can be avoided by considering it as a pure gauge term, this charge is power law vanishing in $D > 2p+2$ while it has a power law divergence in $D < 2p+2$. Note that the charge obtained by the formal substitution $p \mapsto q$, which is the analogue electric-like charge but in the theory of the q-form, is power law vanishing in $D < 2p+2$ while it has a power law divergence in $D > 2p+2$.

3.5 The duality map

In this section we study the possibility of a duality map linking the asymptotic charges of p-forms and $(q = D-p-2)$-forms. We first look at the case of well defined charges, in which the charges of both dual descriptions are finite and non-vanishing in the $r \to +\infty$ limit: no restriction in the case of radiation fall-offs while we need to restrict to $D = 2p+2$ in the case of Coulomb fall-offs[12]. In Paragraph 3.5.1 we are going to consider only the case of radiation fall-off since the case of Coulomb fall-off is very similar apart for the quantities' orders which enter in the computations. Then, we are going to discuss the case of Coulomb fall-offs when $D \neq 2p+2$ where we have a power law divergent/vanishing charge duality.

3.5.1 Duality map for well defined charges

First of all, let us note that charges are surely zero if the field strength components in (3.122) and (3.125) are vanishing, however the equation of motion implies that

$$H^{(\frac{D-2p+2}{2})}_{rui_1\ldots i_{p-1}} \neq 0, \qquad H^{(2)}_{rui_1\ldots i_{p-1}} \neq 0. \tag{3.128}$$

Since we checked that the charges are not zero let us proceed first with an example and then with the general case.

3.5.1.1 1-form and 2-form in $D = 5$

The charges are

$$Q^{(e)}_{1,5}[S^3_u] \sim \Lambda_1 \oint_{S^3_u} \epsilon^{(\frac{1}{2})}(\{x^i\}) F^{(\frac{5}{2})}_{ru}\mathcal{O}(r^0)d\Omega, \qquad Q^{(e)}_{2,5}[S^3_u] \sim \Lambda_2 \oint_{S^3_u} \gamma^{i_1\tilde{i}_1}\epsilon^{(-\frac{1}{2})}_{i_1}(\{x^i\}) H^{(\frac{3}{2})}_{ru\tilde{i}_1}\mathcal{O}(r^0)d\Omega; \tag{3.129}$$

[11] This general result is in agreement to what is known in literature [244] but it is derived in a different way.

[12] Indeed, if $D = 2p+2$ we have that $D-2p-2 = 0$ and $D-2q-2 = D-2(D-p-2)-2 = -D+2p+2 = 0$.

where the superscripts underline these are the electric-like asymptotic charges associated to the two theories since are computed using a electric-like component of the field strength. The duality map can be derived using the physical information of the duality between the field strengths $dA = -\star dB$ where B is the 2-form and A is the 2-form. This equation in coordinates reads

$$F_{\alpha\beta} = -\frac{r^3}{6} g^{\rho\tilde{\rho}} g^{\nu\tilde{\nu}} g^{\mu\tilde{\mu}} H_{\mu\nu\rho} \varepsilon_{\tilde{\mu}\tilde{\nu}\tilde{\rho}\alpha\beta} \sqrt{det(\gamma^{ij})}; \tag{3.130}$$

Let us focus on the $(\alpha, \beta) = (l, s)$ equation

$$\partial_l A_s - \partial_s A_l = -\frac{1}{3} r \gamma^{ij} \varepsilon_{urils} H_{ruj} \sqrt{det(\gamma^{ij})}; \tag{3.131}$$

this is coherent order by order due to the presence of r in the RHS. However, the duality could be read on the other way around as $\star dA = dB$ which, in coordinates, reads as

$$H_{\mu\nu\rho} = \frac{r^3}{2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} \varepsilon_{\tilde{\alpha}\tilde{\beta}\mu\nu\rho} \sqrt{det(\gamma^{ij})}; \tag{3.132}$$

choosing the $(\mu, \nu, \rho) = (i, l, s)$ equation we get

$$(\partial_i B_{ls} + \partial_s B_{il} + \partial_l B_{si}) = r^3 (\partial_r A_u - \partial_u A_r) \varepsilon_{urils} \sqrt{det(\gamma^{ij})}, \tag{3.133}$$

which is coherent order by order. In the spirit of [228] and in order to link the 1-form leading order field components $A_i^{(\frac{1}{2})}$ and its dual 2-form leading order field components $B_{ij}^{(-\frac{1}{2})}$ we look at the Hodge duality equation (3.131) with $(\alpha, \beta) = (u, i)$, which gives

$$F_{ui}^{(\frac{1}{2})} = -\frac{\sqrt{det(\gamma^{ij})}}{3} \varepsilon_{urijk} \gamma^{ks} \gamma^{jl} H_{uls}^{(-\frac{1}{2})}. \tag{3.134}$$

Hence, the field strengths determine the asymptotic 1-form and 2-form fields up to a angle dependent integration constant which can be fixed in such a way $A_i^{(\frac{1}{2})}$ and $B_{ij}^{(-\frac{1}{2})}$ are proportional.

We have now two possibilities: use Hodge duality equations to derive the dual charge or looking for a map, derived by the Hodge duality and some other conditions, such that the two electric-like charges are mapped one into the other. A priori it is not guaranteed that the two paths will merge.

In the first case we use equation $dA = -\star dB$ to derive a charge in the 1-form theory associated to the charge $Q_{2,5}^{(e)}[S_u^3]$[13]. Inserting the field components expansion in equation (3.131) we get

$$\partial_l A_s^{(\frac{1}{2})} - \partial_s A_l^{(\frac{1}{2})} = -\frac{1}{3} \gamma^{ij} \varepsilon_{urils} H_{ruj}^{(\frac{3}{2})} \sqrt{det(\gamma^{ij})}; \tag{3.135}$$

[13] Equivalently, we can use the equation $dB = \star dA$ to derive a charge in the 2-form theory associated to $Q_{1,5}^{(e)}[S_u^3]$.

normalizing the Levi-Civita symbol as $\varepsilon_{ur123} = 1$ we have

$$\begin{aligned}
\partial_1 A_2^{(\frac{1}{2})} - \partial_2 A_1^{(\frac{1}{2})} &= -\frac{1}{3}\gamma^{33} H_{ru3}^{(\frac{3}{2})}\sqrt{det(\gamma^{ij})},\\
\partial_1 A_3^{(\frac{1}{2})} - \partial_3 A_1^{(\frac{1}{2})} &= \frac{1}{3}\gamma^{22} H_{ru2}^{(\frac{3}{2})}\sqrt{det(\gamma^{ij})},\\
\partial_2 A_3^{(\frac{1}{2})} - \partial_3 A_2^{(\frac{1}{2})} &= -\frac{1}{3}\gamma^{11} H_{ru1}^{(\frac{3}{2})}\sqrt{det(\gamma^{ij})}
\end{aligned} \tag{3.136}$$

and

$$\begin{aligned}
&\gamma^{i\tilde{i}}\epsilon_i^{(-\frac{1}{2})}(\{x^i\})H_{ru\tilde{i}}^{(\frac{3}{2})}\sqrt{det(\gamma^{ij})} =\\
&= \left[\epsilon_1^{(-\frac{1}{2})}(\{x^i\})\gamma^{11}H_{ru1}^{(\frac{3}{2})} + \epsilon_2^{(-\frac{1}{2})}(\{x^i\})\gamma^{22}H_{ru2}^{(\frac{3}{2})} + \epsilon_3^{(-\frac{1}{2})}(\{x^i\})\gamma^{33}H_{ru3}^{(\frac{3}{2})}\right]\sqrt{det(\gamma^{ij})} =\\
&= -3\epsilon_1^{(-\frac{1}{2})}(\{x^i\})F_{23}^{(\frac{1}{2})} + 3\epsilon_2^{(-\frac{1}{2})}(\{x^i\})F_{13}^{(\frac{1}{2})} - 3\epsilon_3^{(-\frac{1}{2})}(\{x^i\})F_{12}^{(\frac{1}{2})} =\\
&= -\frac{3}{2}\varepsilon^{ijk}\epsilon_i^{(-\frac{3}{2})}(\{x^i\})F_{jk}^{(\frac{1}{2})}.
\end{aligned} \tag{3.137}$$

Therefore, using (3.137) we get

$$Q_{2,5}^{(e)}[S_u^3] \sim \Lambda_2 \oint_{S_u^3} \gamma^{i\tilde{i}}\epsilon_i^{(-\frac{1}{2})}(\{x^i\})H_{ru\tilde{i}}^{(\frac{3}{2})}\mathcal{O}(r^0)d\Omega = -\frac{3\Lambda_2}{2}\oint_{S_u^3}\frac{\varepsilon^{ijk}\epsilon_i^{(-\frac{1}{2})}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}F_{jk}^{(\frac{1}{2})}\mathcal{O}(r^0)d\Omega. \tag{3.138}$$

This charge is written using the field strength of the 1-form gauge theory but the gauge parameter components involved are the ones of the 2-form gauge theory. We dub this charge $\tilde{Q}_{1,5}^{(m)}[S_u^3]$ where the tilde underlines the charge is computed using the duality equations and the superscript underlines this charge contains the magnetic-like components of the field strength

$$\tilde{Q}_{1,5}^{(m)}[S_u^3] \sim \frac{3\Lambda_2}{2}\oint_{S_u^3}\varepsilon^{ijk}\frac{\epsilon_i^{(-\frac{1}{2})}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}F_{jk}^{(\frac{1}{2})}\mathcal{O}(r^0)d\Omega. \tag{3.139}$$

Hence we can recognize the charge $\tilde{Q}_{1,5}^{(m)}[S_u^3]$ as a magnetic-like counterpart[14], in the dual theory, of the charge $Q_{2,5}^{(e)}[S_u^3]$. Moreover, could be interesting to note that we can reinterpret the gauge parameter components $\epsilon_i^{(-\frac{1}{2})}(\{x^i\})$ as a gauge parameter for the 1-form if we choose

$$\epsilon_i^{(-\frac{1}{2})}(\{x^i\}) = \epsilon^{(-\frac{1}{2})}(\{x^i\})(1,1,1), \tag{3.140}$$

[14]In fact, in self-dual cases, like 1-form gauge theory in $D = 4$, this is exactly the magnetic charge. However, in that cases, the problem of reinterpretation of the gauge parameter, that we have to discuss in a while, is not encountered.

therefore we can write

$$\tilde{Q}^{(m)}_{1,5}[S^3_u] \sim \frac{3\Lambda_2}{2}\oint_{S^3_u}\frac{\varepsilon^{ijk}\epsilon_i^{(-\frac{1}{2})}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}F_{jk}^{(\frac{1}{2})}\mathcal{O}(r^0)d\Omega \qquad \text{with} \quad \epsilon_i^{(-\frac{1}{2})}(\{x^i\}) = \epsilon^{(-\frac{1}{2})}(\{x^i\})(1,1,1). \tag{3.141}$$

We note, however, that the gauge parameter $\epsilon^{(-\frac{1}{2})}(\{x^i\})$ is overleading with respect to (3.44) and it does not preserve the fall-off of the field; so in our framework the charge (3.141) cannot be derived since the gauge parameter would be zero. In an analogue way we can use (3.133) to write down a magnetic-like charge in the 2-form gauge theory $\tilde{Q}^{(m)}_{2,5}[S^3_u]$ dual to $Q^{(e)}_{1,5}[S^3_u]$, that is

$$\tilde{Q}^{(m)}_{2,5}[S^3_u] \sim \frac{\Lambda_1}{18}\oint_{S^3_u}\frac{\varepsilon^{ijk}\epsilon^{(\frac{1}{2})}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}H_{ijk}^{(-\frac{1}{2})}\mathcal{O}(r^0)d\Omega; \tag{3.142}$$

this charge is written using the field strength of the 2-form gauge theory but the gauge parameter components involved are the ones of the 1-form gauge theory. Unlike before, it seems difficult to reinterpret $\epsilon^{(\frac{1}{2})}(\{x^i\})$ as a gauge parameter for the 2-form gauge theory. Anyway, in the end we have the relations

$$Q^{(e)}_{1,5}[S^3_u] = \tilde{Q}^{(m)}_{2,5}[S^3_u], \quad Q^{(e)}_{2,5}[S^3_u] = -\tilde{Q}^{(m)}_{1,5}[S^3_u]. \tag{3.143}$$

The dual magnetic-like charges seem to be just a rephrasing, in dual variables, of the electric-like charges, however the fact that the gauge parameter of the other formulation appears can lead to think that these are charges associated with hidden symmetries of the theory not directly related to the gauge symmetry of the original theory but to the gauge symmetry of the dual formulation. These charges could be combined to give rise to a unique charge associated to a complexified symmetry algebra as we discuss in general in the next example.

In the second case, under the requirement that the charge $Q^{(e)}_{2,5}[S^3_u]$ is mapped into $Q^{(e)}_{1,5}[S^3_u]$, or vice versa, we require

$$\Lambda_1\varepsilon^{urils}\epsilon(\{x^i\})H_{ils} \leftrightarrow \Lambda_2\varepsilon^{urils}\epsilon_i(\{x^i\})F_{ls} \tag{3.144}$$

which identifies the celestial sphere degrees of freedom only under the requirement that $Q_{2,5}[S^3_u] \leftrightarrow Q_{1,5}[S^3_u]$. Expanding (3.144) using (3.131) and (3.133) we get

$$\Lambda_2[\epsilon_1(\{x^i\})\gamma^{11}H_{ru1} + \epsilon_2(\{x^i\})\gamma^{22}H_{ru2} + \epsilon_3(\{x^i\})\gamma^{33}H_{ru3}] \leftrightarrow \Lambda_1\epsilon(\{x^i\})r^2F_{ru} \tag{3.145}$$

where we used $\varepsilon^{urils}\varepsilon_{urils} = 6$, $\varepsilon_{ur123} = 1$ and its complete antisymmetry. Inserting the fields expansion we get for the leading orders

$$\Lambda_2\epsilon_i^{(-\frac{1}{2})}(\{x^i\})\gamma^{i\tilde{i}}H_{ru\tilde{i}}^{(\frac{3}{2})} \leftrightarrow \Lambda_1\epsilon^{(\frac{1}{2})}(\{x^i\})F_{ru}^{(\frac{5}{2})}. \tag{3.146}$$

Looking at the electric-like charge for the 2-form we have

$$Q^{(e)}_{2,5}[S^3_u] \sim \Lambda_2 \oint_{S^3_u} \gamma^{i\tilde{i}} \epsilon_i^{(-\frac{1}{2})}(\{x^i\}) H^{(\frac{3}{2})}_{ru\tilde{i}} d\Omega \leftrightarrow \Lambda_1 \oint_{S^3_u} \epsilon^{(\frac{1}{2})}(\{x^i\}) F^{(\frac{5}{2})}_{ru} d\Omega \sim Q^{(e)}_{1,5}[S^3_u]; \tag{3.147}$$

hence, as wanted, $Q^{(e)}_{2,5}[S^3_u]$ is mapped to $Q^{(e)}_{1,5}[S^3_u]$ and vice versa.

3.5.1.2 *p*-form and $(q = D-2-p)$-form in arbitrary D with $p \in \left[1, \frac{D-2}{2}\right]$

Like the specific case above, we have the electric-like charges

$$\begin{aligned} Q^{(e)}_{p,D}[S^{D-2}_u] &\sim \Lambda_p \oint_{S^{D-2}_u} \gamma^{i_1\tilde{i}_1}...\gamma^{i_{p-1}\tilde{i}_{p-1}} \epsilon^{(\frac{D-(2p+2)}{2})}_{i_1...i_{p-1}}(\{x^i\}) H^{(\frac{D-2p+2}{2})}_{ru\tilde{i}_1...\tilde{i}_{p-1}} \mathcal{O}(r^0) d\Omega; \\ Q^{(e)}_{q,D}[S^{D-2}_u] &\sim \Lambda_q \oint_{S^{D-2}_u} \gamma^{i_1\tilde{i}_1}...\gamma^{i_{q-1}\tilde{i}_{q-1}} \epsilon^{(\frac{D-(2q+2)}{2})}_{i_1...i_{q-1}}(\{x^i\}) H^{(\frac{D-2q+2}{2})}_{ru\tilde{i}_1...\tilde{i}_{q-1}} \mathcal{O}(r^0) d\Omega; \end{aligned} \tag{3.148}$$

where $q = D-2-p$ and $H_{\mu_1...\mu_{p+1}}$ and $H_{\mu_1...\mu_{q+1}}$ are the field strengths of the *p*-form and $(q = D-2-p)$-form respectively. As before, the duality map can be derived starting from the physical information of the duality between the field strengths $dA = (-1)^{(p+1)(q+1)} s \star dB$ where B is the $(q = D-2-p)$-form, A is the *p*-form and s is the parity of the signature of the metric of the Lorentzian manifold. Moreover, the duality can be read on the other way around as $\star dA = dB$. In coordinates, the Hodge duality equations, with only angular indexes as free indexes, read

$$\begin{aligned} H_{k_1...k_{p+1}} &= (-1)^{(p+1)(q+1)} s \frac{(p+1)!}{(q+1)!} r^{D-2-2(q-1)} \gamma^{i_1j_1}...\gamma^{i_{q-1}j_{q-1}} \varepsilon_{uri_1...i_{q-1}k_1...k_{p+1}} H_{ruj_1...j_{q-1}} \sqrt{det(\gamma^{ij})}; \\ H_{k_1...k_{q+1}} &= \frac{(q+1)!}{(p+1)!} r^{D-2-2(p-1)} \gamma^{i_1j_1}...\gamma^{i_{p-1}j_{p-1}} \varepsilon_{uri_1...i_{p-1}k_1...k_{q+1}} H_{ruj_1...j_{p-1}} \sqrt{det(\gamma^{ij})}. \end{aligned} \tag{3.149}$$

Again in the spirit of [228] and in order to link the *p*-form leading order field components $B^{(\frac{D-2p-2}{2})}_{i_1...i_p}$ and its dual *q*-form leading order field components $B^{(\frac{D-2q-2}{2})}_{i_1...i_q}$, we look at the Hodge duality equations with $(\alpha,\beta) = (u,i)$ which gives

$$H^{(\frac{D-2p-2}{2})}_{ui_1...i_p} = (-1)^{(p+1)(q+1)} s \frac{(p+1)!}{(q+1)!} \varepsilon_{uri_1...i_pk_1...k_q} \gamma^{j_1k_1}...\gamma^{j_qk_q} H^{(\frac{D-2q-2}{2})}_{uj_1...j_q} \sqrt{det(\gamma^{ij})}, \tag{3.150}$$

so that the field strengths determine the asymptotic electric and magnetic form potentials up to a angle dependent integration constant which can be fixed in such a way $B^{(\frac{D-2p-2}{2})}_{i_1...i_p}$ and $B^{(\frac{D-2q-2}{2})}_{i_1...i_q}$ are proportional.

As in the explicit case above, we can use the equation $dA = (-1)^{(p+1)(q+1)} s \star dB$ to derive a charge in the

($q = D-2-p$)-form theory associated to the charge $Q^{(e)}_{p,D}[S^{D-2}_u]$[15]. Inserting the field components expansion in the equation above we get

$$H^{(\frac{D-(2p+2)}{2})}_{k_1...k_{p+1}} = (-1)^{(p+1)(q+1)} s\frac{(p+1)!}{(q+1)!}\gamma^{i_1j_1}...\gamma^{i_{q-1}j_{q-1}}\varepsilon_{uri_1...i_{q-1}k_1...k_{p+1}} H^{(\frac{D-2q+2}{2})}_{ruj_1...j_{q-1}}\sqrt{det(\gamma^{ij})}; \tag{3.151}$$

normalizing the Levi-Civita symbol as $\varepsilon_{ur1...D-2} = 1$ we have

$$\gamma^{i_1\tilde{i}_1}...\gamma^{i_{q-1}\tilde{i}_{q-1}}\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\})H^{(\frac{D-2q+2}{2})}_{ru\tilde{i}_1...\tilde{i}_{q-1}} = (-1)^{(p+1)(q+1)} s\frac{(q+1)!}{(p+1)!}\frac{\varepsilon^{i_1...i_{q-1}\tilde{i}_1...\tilde{i}_{p+1}}\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\})}{(p+1)!(q-1)!\sqrt{det(\gamma^{ij})}}H^{(\frac{D-(2p+2)}{2})}_{\tilde{i}_1...\tilde{i}_{p+1}}. \tag{3.152}$$

Therefore, using (3.152) we get

$$Q^{(e)}_{q,D}[S^{D-2}_u] \sim (-1)^{(p+1)(q+1)} s\tilde{\Lambda}_p \oint_{S^{D-2}_u} \frac{\varepsilon^{i_1...i_{q-1}\tilde{i}_1...\tilde{i}_{p+1}}\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}H^{(\frac{D-(2p+2)}{2})}_{\tilde{i}_1...\tilde{i}_{p+1}}\mathcal{O}(r^0)d\Omega, \tag{3.153}$$

where $\tilde{\Lambda}_p := \frac{(q+1)!}{(p+1)!}\frac{\Lambda_p}{(p+1)!(q-1)!}$. This charge is written using the field strength of the p-form gauge theory but the gauge parameter components involved are the ones of the q-form gauge theory. As before, we dub this charge $\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]$ where the tilde underlines the charge is computed using the duality equations and the superscript underlines this charge contains the magnetic-like component of the field strength

$$\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u] \sim \tilde{\Lambda}_p \oint_{S^{D-2}_u} \frac{\varepsilon^{i_1...i_{q-1}\tilde{i}_1...\tilde{i}_{p+1}}\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}H^{(\frac{D-(2p+2)}{2})}_{\tilde{i}_1...\tilde{i}_{p+1}}\mathcal{O}(r^0)d\Omega. \tag{3.154}$$

Hence we can recognize the charge $\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]$ as a magnetic-like counterpart[16], in the dual theory, of the charge $Q^{(e)}_{q,D}[S^{D-2}_u]$. Moreover, it is interesting to note that we can reinterpret the gauge parameter components $\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\})$ as gauge parameter components for the p-form if we choose

$$\epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{q-1}}(\{x^i\}) = \epsilon^{(\frac{D-(2q+2))}{2})}_{i_1...i_{p-1}}(\{x^i\}) \otimes \mathbb{J}^{(p+q)}_{i_p...i_{q-1}}, \tag{3.155}$$

where $\mathbb{J}^{(p+q)}_{i_p...i_{q-1}}$ is a completely antisymmetric ($q-p$)-rank tensor with only 1 as independent upper diagonals

[15] Equivalently, we can use the equation $dB = \star dA$ to derive a charge in the p-form theory associated to $Q^{(e)}_{q,D}[S^{D-2}_u]$.

[16] In fact, in self-dual cases, like 1-form gauge theory in $D = 4$, this is exactly the magnetic charge. However, in that cases, the problem of reinterpretation of the gauge parameter, that we have to discuss in a while, is not encountered.

entries and $\otimes$ is the Kronecker-like product of tensors[17]. We note the gauge parameter $\epsilon^{(\frac{D-(2q+2)}{2})}_{i_1\ldots i_{p-1}}(\{x^i\})$ is a overleading term in the p-form theory and it does not preserve the fall-off of the field; so in our framework the charge (3.154) with (3.155) cannot be derived since the gauge parameter would be zero. To get the magnetic-like charge $\tilde{Q}^{(m)}_{q,D}[S^{D-2}_u]$, dual to $Q^{(e)}_{p,D}[S^{D-2}_u]$, it is enough to operate the change $p \leftrightarrow q$ in (3.154) although in this case the problem arises of how to reinterpret the gauge parameter components. In the general case we have a generalization of relations (3.143) to

$$Q^{(e)}_{p,D}[S^{D-2}_u] = \tilde{Q}^{(m)}_{q,D}[S^{D-2}_u], \quad Q^{(e)}_{q,D}[S^{D-2}_u] = (-1)^{(p+1)(q+1)}s\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]. \tag{3.156}$$

The same comment as for the specific case discussed above can be done here. The dual magnetic-like charges seem to be just a rephrasing, in dual variables, of the electric-like charges, however the fact that the gauge parameter of the other formulation appears leads to reflection that these are charges associated with hidden symmetries of the theory not directly related to the gauge symmetry of the original theory but to the gauge symmetry of the dual formulation.

Following [228], we can merge electric-like and magnetic-like asymptotic charges in an electromagnetic-like one

$$\begin{aligned} Q^{(em)}_{p,D} &:= Q^{(e)}_{p,D} + i\tilde{Q}^{(m)}_{p,D}, \\ Q^{(em)}_{q,D} &:= Q^{(e)}_{q,D} + i\tilde{Q}^{(m)}_{q,D}, \end{aligned} \tag{3.157}$$

which, as in the case of electromagnetism in $D = 4$, generate a complexified $\mathfrak{u}_{\mathbb{C}}(1)$ symmetry algebra; the first one in the theory described by the p-form while the second one in the theory described by the dual q-form. Under duality these charges transform according to Moebius transformations which can be parameterized by the matrix $A \in \mathrm{PGL}(2, \mathbb{C})$ given by

$$A = \begin{bmatrix} 0 & 1 \\ (-1)^{(p+1)(q+1)}s & 0 \end{bmatrix}; \tag{3.158}$$

indeed, since $\mathbb{C} \cong \mathbb{R}^2$, we have

$$Q^{(em)}_{p,D} \cong \begin{bmatrix} Q^{(e)}_{p,D} \\ \tilde{Q}^{(m)}_{p,D} \end{bmatrix}, \quad Q^{(em)}_{q,D} \cong \begin{bmatrix} Q^{(e)}_{q,D} \\ \tilde{Q}^{(m)}_{q,D} \end{bmatrix}, \quad \tilde{Q}^{(em)}_{q,D} \cong \begin{bmatrix} \tilde{Q}^{(m)}_{q,D} \\ (-1)^{(p+1)(q+1)}sQ^{(e)}_{q,D} \end{bmatrix}, \quad \tilde{Q}^{(em)}_{p,D} \cong \begin{bmatrix} (-1)^{(p+1)(q+1)}s\tilde{Q}^{(m)}_{p,D} \\ Q^{(e)}_{p,D} \end{bmatrix}, \tag{3.159}$$

and so

$$\begin{bmatrix} \tilde{Q}^{(m)}_{q,D} \\ (-1)^{(p+1)(q+1)}sQ^{(e)}_{q,D} \end{bmatrix} = A \begin{bmatrix} Q^{(e)}_{q,D} \\ \tilde{Q}^{(m)}_{q,D} \end{bmatrix}, \quad \begin{bmatrix} (-1)^{(p+1)(q+1)}s\tilde{Q}^{(m)}_{p,D} \\ Q^{(e)}_{p,D} \end{bmatrix} = A^{-1} \begin{bmatrix} Q^{(e)}_{p,D} \\ \tilde{Q}^{(m)}_{p,D} \end{bmatrix}. \tag{3.160}$$

As shown in the Figure 3.1, according to the value of $(-1)^{(p+1)(q+1)}s$, the Moebius transformation is a reflection

[17]Meaning we have take elements of $\mathbb{J}^{(p+q)}_{i_p\ldots i_{q-1}}$ and multiply each of them for the entire tensor $\epsilon^{(\frac{D-(2q+2)}{2})}_{i_1\ldots i_{p-1}}(\{x^i\})$ to get a $(q-p)$-rank tensor whose components are $(p-1)$-rank tensors given by $\pm\epsilon^{(\frac{D-(2q+2)}{2})}_{i_1\ldots i_{p-1}}(\{x^i\})$.

with respect the bisector of the first quadrant if $(-1)^{(p+1)(q+1)}s = 1$ and a rotation of $\theta = \pm\frac{\pi}{2}$ (clockwise on $Q^{(em)}_{q,D}$ and counterclockwise on $Q^{(em)}_{p,D}$) if $(-1)^{(p+1)(q+1)}s = -1$.

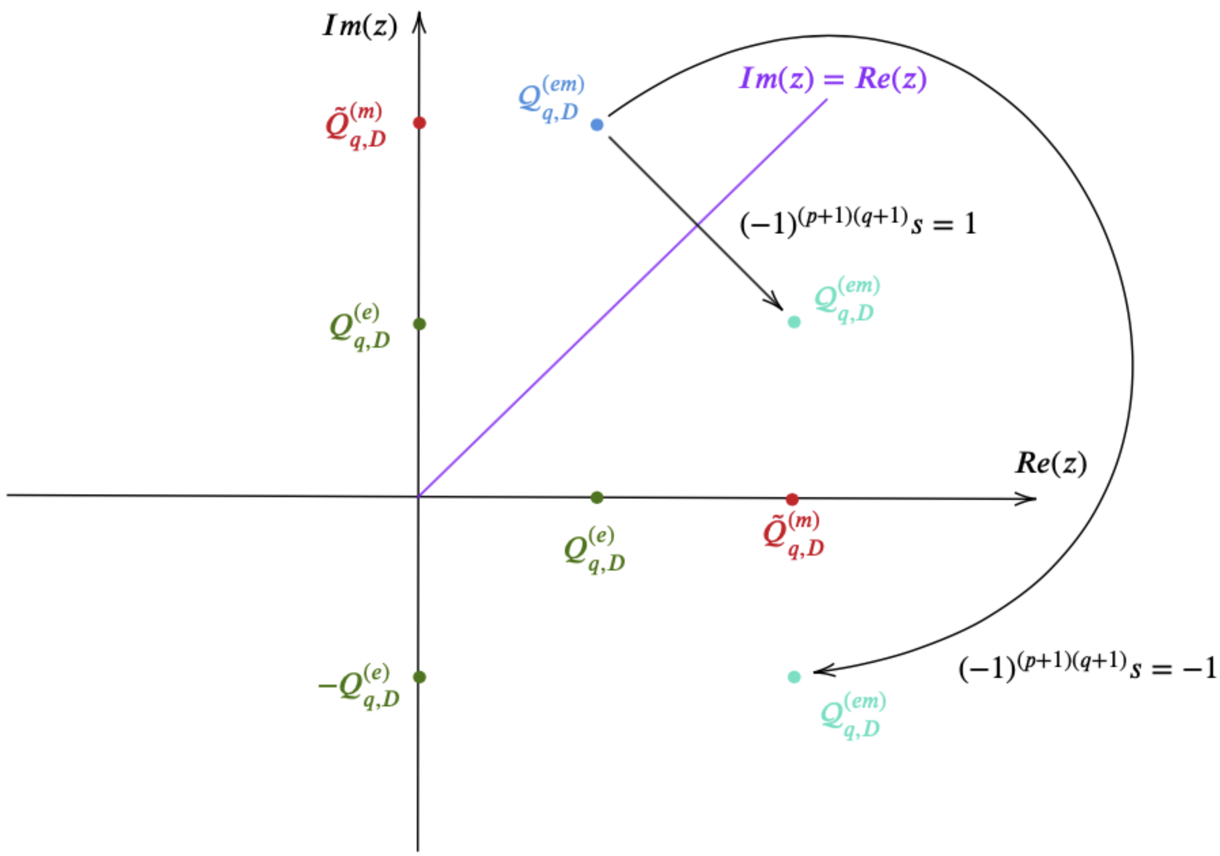


FIGURE 3.1 *Schematic picture of the action of the duality on the complex electromagnetic-like asymptotic charge $Q^{(em)}_{q,D}$. A similar scheme can be drawn for $Q^{(em)}_{p,D}$.*

The other approach is that, under the requirement that the charge $Q^{(e)}_{p,D}[S^{D-2}_u]$ is mapped into $Q^{(e)}_{q,D}[S^{D-2}_u]$ or vice versa, we require

$$\Lambda_p \varepsilon^{uri_1\ldots i_{p-1}j_1\ldots j_{q+1}} \epsilon_{i_1\ldots i_{p-1}} H_{j_1\ldots j_{q+1}} \leftrightarrow \Lambda_q \varepsilon^{uri_1\ldots i_{q-1}j_1\ldots j_{p+1}} \epsilon_{i_1\ldots i_{q-1}} H_{j_1\ldots j_{p+1}}. \tag{3.161}$$

Normalizing the Levi-Civita symbols as $\varepsilon_{ur1\ldots D-2} = 1$, using $\varepsilon_{ur1\ldots D-2}\varepsilon^{ur1\ldots D-2} = (D-2)! = (p+q)!$, using the Hodge duality equations and inserting the field components expansions we get, at leading order

$$\Lambda_q \epsilon^{(\frac{D-(2q+2)}{2})}_{i_1\ldots i_{q-1}}(\{x^i\})\gamma^{i_1j_1}\ldots\gamma^{i_{q-1}j_{q-1}} H^{(\frac{D-2q+2}{2})}_{ruj_1\ldots j_{q-1}} \leftrightarrow \Lambda_p \epsilon^{(\frac{D-(2p+2)}{2})}_{i_1\ldots i_{p-1}}(\{x^i\})\gamma^{i_1j_1}\ldots\gamma^{i_{p-1}j_{p-1}} H^{(\frac{D-2p+2}{2})}_{ruj_1\ldots j_{p-1}}; \tag{3.162}$$

looking at the charge of the $(q = D-2-p)$-form and using (3.162) we get

$$Q^{(e)}_{q,D}[S^{D-2}_u] \leftrightarrow Q^{(e)}_{p,D}[S^{D-2}_u]. \tag{3.163}$$

Hence, $Q^{(e)}_{q,D}[S^{D-2}_u]$ is a functions of $Q^{(e)}_{p,D}[S^{D-2}_u]$. In Paragraph 3.6 we are going to further explain this point.

3.5.2 The form of scalars

We now use the Hodge duality equations to derive a candidate of dual asymptotic charge for the scalar theory in any D, following the idea of [251].

3.5.2.1 Scalar and 1-form in $D = 3$

For a scalar Φ and a 1-form B in $D = 3$ the duality from the field strengths reads as $\star d\Phi = dB$ which in coordinates is

$$H_{\mu\nu} = \frac{1}{2} r \varepsilon_{\mu\nu\alpha} g^{\alpha\tilde{\alpha}} \partial_{\tilde{\alpha}} \Phi. \tag{3.164}$$

Choosing the $(\mu, \nu) = (r, u)$ equation we get, at leading order,

$$H^{(\frac{3}{2})}_{ru} = -\frac{1}{2} \partial_\phi \Phi^{(\frac{1}{2})} =: -\frac{1}{2} A^{(\frac{1}{2})}. \tag{3.165}$$

The asymptotic 1-form charge in $D = 3$ is given by

$$Q_{1,3}[S^1_u] \sim \Lambda_1 \int \epsilon^{(-\frac{1}{2})}(\{x^i\}) H^{(\frac{3}{2})}_{ru} d\phi \tag{3.166}$$

and using equation (3.165) we get

$$Q_{1,3}[S^1_u] \sim \frac{\Lambda_1}{2} \int A^{(\frac{1}{2})} \epsilon^{(-\frac{1}{2})}(\{x^i\}) d\Omega =: Q^{(D=3)}_{\text{scalar}}. \tag{3.167}$$

3.5.2.2 Scalar and 2-form in $D = 4$

For a scalar Φ and a 2-form B in $D = 4$ the duality from the field strengths reads as $\star d\Phi = dB$ which in coordinates is

$$H_{\mu\nu\rho} = \frac{1}{2} r^2 sin(\theta) \varepsilon_{\mu\nu\rho\alpha} g^{\alpha\tilde{\alpha}} \partial_{\tilde{\alpha}} \Phi. \tag{3.168}$$

Choosing the $(\mu, \nu, \rho) = (r, u, i)$ equation we get, at leading order,

$$\begin{aligned} H^{(1)}_{rui} &= \frac{sin(\theta)}{2} \varepsilon_{ruij} \gamma^{kj} \partial_k \Phi^{(1)} = -\frac{sin(\theta)}{2} [\varepsilon_{uri\theta} \gamma^{\theta\theta} \partial_\theta \Phi^{(1)} + \varepsilon_{uri\phi} \gamma^{\phi\phi} \partial_\phi \Phi^{(1)}] = \\ &= -\frac{sin(\theta)}{2} \left[\varepsilon_{uri\theta} \partial_\theta \Phi^{(1)} + \varepsilon_{uri\phi} \frac{\partial_\phi \Phi^{(1)}}{sin^2(\theta)} \right] =: -\frac{sin(\theta)}{2} A^{(1)}_i. \end{aligned} \tag{3.169}$$

The asymptotic 2-form charge in $D = 4$ is given by

$$Q_{2,4}[S_u^2] \sim \Lambda_2 \int \gamma^{ij} \epsilon_i^{(-1)}(\{x^i\}) H_{ruj}^{(1)} sin(\theta) d\theta d\phi \tag{3.170}$$

and using equation (3.169) we have

$$Q_{2,4}[S_u^2] \sim \frac{\Lambda_2}{2} \int \gamma^{ij} A_j^{(1)} \epsilon_i^{(-1)}(\{x^i\}) sin(\theta) d\Omega =: Q_{\text{scalar}}^{(D=4)}. \tag{3.171}$$

3.5.2.3 Scalar and 3-form in $D = 5$

For a scalar Φ and a 3-form B in $D = 5$ the duality from the field strengths reads as $\star d\Phi = dB$ which in coordinates is

$$H_{\mu\nu\rho\sigma} = \frac{1}{2} r^3 sin^2(\theta_1) sin(\theta_2) \varepsilon_{\mu\nu\rho\sigma\alpha} g^{\alpha\tilde{\alpha}} \partial_{\tilde{\alpha}} \Phi. \tag{3.172}$$

Choosing the $(\mu, \nu, \rho, \sigma) = (r, u, i, j)$ equation we get, at leading order,

$$\begin{aligned}
H_{ruij}^{(\frac{1}{2})} &= \frac{1}{2} sin^2(\theta_1) sin(\theta_2) \varepsilon_{ruijk} \gamma^{lk} \partial_l \Phi^{(\frac{3}{2})} = \\
&= -\frac{1}{2} sin^2(\theta_1) sin(\theta_2) [\varepsilon_{urij\theta_1} \gamma^{\theta_1\theta_1} \partial_{\theta_1} \Phi^{(\frac{3}{2})} + \varepsilon_{urij\theta_2} \gamma^{\theta_2\theta_2} \partial_{\theta_2} \Phi^{(\frac{3}{2})} + \varepsilon_{urij\phi} \gamma^{\phi\phi} \partial_\phi \Phi^{(\frac{3}{2})}] = \\
&= -\frac{1}{2} sin^2(\theta_1) sin(\theta_2) \left[\varepsilon_{urij\theta_1} \partial_{\theta_1} \Phi^{(\frac{3}{2})} + \varepsilon_{urij\theta_2} \frac{\partial_{\theta_2} \Phi^{(\frac{3}{2})}}{sin^2(\theta_1)} + \varepsilon_{urij\phi} \frac{\partial_\phi \Phi^{(\frac{3}{2})}}{sin^2(\theta_1) sin^2(\theta_2)} \right] =: \\
&=: -\frac{sin^2(\theta_1) sin(\theta_2)}{2} A_{ij}^{(\frac{3}{2})}.
\end{aligned} \tag{3.173}$$

The asymptotic 3-form charge in $D = 5$ is given by

$$Q_{3,5}[S_u^3] \sim \Lambda_3 \int \gamma^{il} \gamma^{js} \epsilon_{ls}^{(-\frac{3}{2})}(\{x^i\}) H_{ruij}^{(\frac{1}{2})} sin^2(\theta_1) sin(\theta_2) d\theta_1 d\theta_2 d\phi \tag{3.174}$$

and using the above (3.173) we have

$$Q_{3,5}[S_u^3] \sim \frac{\Lambda_3}{2} \int \gamma^{il} \gamma^{js} A_{ij}^{(\frac{3}{2})} \epsilon_{ls}^{(-\frac{3}{2})}(\{x^i\}) sin^2(\theta_1) sin(\theta_2) d\Omega =: Q_{\text{scalar}}^{(D=5)}. \tag{3.175}$$

3.5.2.4 Scalar and $(D-2)$-form in arbitrary D

For a scalar Φ and a $(D-2)$-form B in arbitrary D the duality from the field strengths reads as $\star d\Phi = dB$ which in coordinates is

$$H_{\mu_1 \dots \mu_{D-1}} = \frac{1}{2} r^{D-2} J \varepsilon_{\mu_1 \dots \mu_{D-1} \alpha} g^{\alpha\tilde{\alpha}} \partial_{\tilde{\alpha}} \Phi, \tag{3.176}$$

where

$$J := \prod_{i=1}^{D-3} sin^{D-3-(i-1)}(\theta_i). \tag{3.177}$$

Choosing the $(\mu_1, ..., \mu_{D-1}) = (r, u, i_1, ..., i_{D-3})$ equation we get, at leading order,

$$H^{(\frac{-D+6}{2})}_{rui_1...i_{p-1}} = -\frac{J}{2} A^{(\frac{D-2}{2})}_{i_1...i_{D-3}}; \tag{3.178}$$

where

$$A^{(\frac{D-2}{2})}_{i_1...i_{D-3}} := \varepsilon_{uri_1...i_{D-3}\phi} \frac{\partial_\phi \Phi^{(\frac{D-2}{2})}}{\prod_{i=1}^{D-3} sin^2(\theta_i)} + \sum_{i=1}^{D-3} \varepsilon_{uri_1...i_{D-3}\theta_i} \frac{\partial_{\theta_i} \Phi^{(\frac{D-2}{2})}}{\prod_{j=1}^{i-1} sin^2(\theta_j)}. \tag{3.179}$$

Defining

$$J_{d\theta_i} := \prod_{i=1}^{D-3} sin^{D-3-(i-1)}(\theta_i) d\theta_i, \tag{3.180}$$

the asymptotic $(D-2)$-form charge in arbitrary D is given by

$$Q_{D-2,D}[S_u^{D-2}] \sim \Lambda_{D-2} \int \gamma^{i_1 s_1} ... \gamma^{i_{D-3} s_{D-3}} \epsilon^{(-\frac{D-2}{2})}_{s_i...s_{D-3}}(\{x^i\}) H^{(\frac{-D+6}{2})}_{rui_1...i_{p-1}} J_{d\theta_i} d\phi \tag{3.181}$$

and using equation (3.178) we have

$$Q_{D-2,D}[S_u^{D-2}] \sim \frac{\Lambda_{D-2}}{2} \int \gamma^{i_1 s_1} ... \gamma^{i_{D-3} s_{D-3}} A^{(\frac{D-2}{2})}_{i_1...i_{D-3}} \epsilon^{(\frac{-D+2}{2})}_{s_i...s_{D-3}}(\{x^i\}) J d\Omega =: Q^{(D)}_{\text{scalar}}, \tag{3.182}$$

Charge (3.182) is a possible asymptotic charge for the scalar theory derived thanks to the duality with the $(D-2)$-form. Asymptotic scalar charges first appear in $D = 4$ with the interpretation of supertranslation-like diffeomorphisms [247]. With respect to this charge, the scalar charge (3.182) contains derivatives with respect to the angular variables and not with respect to u; this may indicate that the charge (3.182) could be the analogue of superrotations in the scalar case in any dimension.

3.5.3 Duality map for power law divergent/vanishing charges

In the case of charges with Coulomb fall-offs on the field components, charges (3.125), we can repeat the same passages and considerations with some precautions. First of all, we have different orders with respect to those in Paragraph 3.5.1.2 since Coulomb fall-offs have different behaviors with respect to radiation fall-off. Second, charges (3.125) have logarithmic terms if we admit leading order logarithmic terms in the field component $B_{ui_1...i_{p-1}}$, which are not cancel out in $D \neq 2p+2$. Hence the analogues of (3.151), (3.152) and (3.162) are duplicated: one for the logarithmic leading order and one for the first, non-logarithmic, subleading order of the field strength components with two indexes given by (r, u). Taken into account these two precautions, the magnetic-like charge, in the p-form theory, dual to $Q^{(e)}_{q,D}[S^{D-2}_u]$ is given by

$$\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u] \sim \tilde{\Lambda}_p \oint_{S^{D-2}_u} \frac{\varepsilon^{i_1...i_{q-1}\tilde{i}_1...\tilde{i}_{p+1}}\epsilon^{(D-(2q+2))}_{i_1...i_{q-1}}(\{x^i\})}{\sqrt{det(\gamma^{ij})}}[H^{(D-(2p+2))}_{\tilde{i}_1...\tilde{i}_{p+1}} + \tilde{H}^{(D-(2p+2))}_{\tilde{i}_1...\tilde{i}_{p+1}} ln(r)]\mathcal{O}(r^{D-2p-2})d\Omega, \tag{3.183}$$

where $\tilde{\Lambda}_p := \frac{(q+1)!}{(p+1)!}\frac{\Lambda_p}{(p+1)!(q-1)!}$. Apart from the logarithmic term[18], this charge is power law divergent in $D > 2p+2$ and power law vanishing in $D < 2p+2$; this is a mirror behavior with respect to the charge (3.125). Therefore, when electric-like charge (3.125) is power law vanishing the magnetic-like charge (3.183) is power law divergent and vice versa. In other words, the product between $\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]$ and $Q_{p,D}[S^{D-2}_u]$ goes like r^0, so if one diverges the other is vanishing.
As for the case of well defined charges, we can require that $Q^{(e)}_{p,D}[S^{D-2}_u]$ is mapped into $Q^{(e)}_{q,D}[S^{D-2}_u]$; however, in this case we need to consider a radial compensatory factor, R, since in the two theories the charges have different radial behaviors. In general, the compensatory factor is

$$\mathrm{R} := \frac{r^{-(D-2p-2)}}{r^{-(D-2(D-2-p)-2)}} = r^{-2D+4p+4} \tag{3.184}$$

to pass from the charge $Q^{(e)}_{q,D}[S^{D-2}_u]$ to the charge $Q^{(e)}_{p,D}[S^{D-2}_u]$ while it is its inverse for the specular case. Taking into account also the precaution of the compensatory factor (3.184), we get the analogue of (3.163) for charges with Coulomb fall-off for the field components. Therefore a subleading charge, with respect to the charge with radiation fall-off on the field components, in a theory is mapped to an overleading one in the dual description and vice versa. This compensatory factor could be used also in the scalar/$(D-2)$-form duality; for example in the case of 2-form and scalar in $D = 4$. In that case, for example, the overlading $\mathcal{O}(r^2)$ 2-form charge is mapped in a scalar charge of order $\mathcal{O}(r^{-2})$. Subleading scalar charges with this radial behavior appear, for example, in [235] where soft pion theorem is discussed.

3.6 A mathematical note on the duality map

Let us consider the duality map

$$Q^{(e)}_{q,D}[S^{D-2}_u] \leftrightarrow Q^{(e)}_{p,D}[S^{D-2}_u], \tag{3.185}$$

which link the electric-like charges we have computed in the two dual theories. In the case of well defined charges[19], the map does not have to take into account a compensatory radial factor. In the case of power law divergent or power law vanishing charges[20] we need the compensatory radial factor R given by (3.184).

Our main goal in this Paragraph, based of the work [330], is to give a solid mathematical basis for the duality

[18]This term would not be present if we choose to avoid the logarithmic term in charge (3.125) considering it as a pure gauge term.
[19]We stress again that this happens for charges with radiation fall-off on the field components in any D and for charges with Coulomb fall-off on the field components in $D = 2p+2$.
[20]We recall again that these are charges with Coulomb fall-off on the field components in any $D \neq 2p+2$.

map (3.185).

3.6.1 Duality map for well defined charges

For the rest of the Paragraph we define

$$\mathcal{R}^{(p)} := H^{(\frac{D-2p+2}{2})}_{rui_1 \ldots i_{p-1}}, \quad C^{(p)} := H^{(2)}_{ru\tilde{i}_1 \ldots \tilde{i}_{p-1}}; \tag{3.186}$$

we give also the following definition

Definition 3.6.1. *We define $\Omega^k_{\mathcal{R}^{(k)}\neq 0,0}(M)$ and $\Omega^k_{C^{(k)}\neq 0,0}(M)$ as the subspace of k-forms on the differential manifold $(M,\mathcal{A})$ with the condition that, respectively, $\mathcal{R}^{(k)} \neq 0$ and $C^{(k)} \neq 0$ except for the identically vanishing form. Moreover we define their dimensions as $n_k := dim(\Omega^k_{\mathcal{R}^{(k)}\neq 0,0}(M)$ and $m_k := dim(\Omega^k_{C^{(k)}\neq 0,0}(M)$.*

The existence and uniqueness of the duality map (3.185) is stated by the following theorem

Theorem 3.6.1 (Existence and uniqueness of the duality map for well defined charges)**.** *Let $(M_D, \boldsymbol{\eta})$ be the D-dimensional Minkowski spacetime and let $\Omega^p_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ and $\Omega^q_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$ with $p \in \left[1, \frac{D-2}{2}\right]$ and $q = D-p-2$ be as definition. Then a duality map $f \in \mathrm{GL}(n_p, \mathbb{C})$, such that the following diagram*

$$\begin{tikzcd}
\Omega^{p+1}(M_D) \arrow[r, "\star"] & \Omega^{q+1}(M_D) \\
\Omega^p_{\mathcal{R}^{(p)}\neq 0,0}(M_D) \arrow[u, "d"] \arrow[r, "\star_{D-2}"] \arrow[d, "\pi_1"'] & \Omega^q_{\mathcal{R}^{(q)}\neq 0,0}(M_D) \arrow[u, "d"'] \arrow[d, "\pi_2"] \\
\mathbb{C}^{n_p} \arrow[r, "f"] & \mathbb{C}^{n_q}
\end{tikzcd} \tag{3.187}$$

commutes, exists. Moreover f admits a unique restriction to a 1-dimensional subspace such that $f|_Q : Q^{(e)}_{p,D} \mapsto Q^{(e)}_{q,D}$ and $f^{-1}|_Q : Q^{(e)}_{q,D} \mapsto Q^{(e)}_{p,D}$.

Proof. First of all let us consider the exterior derivative d; in our case this operation is invertible since we are considering only forms such that they are exact and not pure gauge, i.e. they admit a non-zero field strength, by the requirement we are doing asymptotic symmetries. This point will be taken into consideration in a deeper way in the next Paragraph where a more general algebraic topology interpretation will be outlined.

Let us construct πs maps. Let us call, respectively, $Q^{(e,B)}_{p,D}$ and $Q^{(e,\tilde{B})}_{q,D}$ the electric-like asymptotic charges associated to forms in space $B \in \Omega^p_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ and $\tilde{B} \in \Omega^q_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$. The idea is that π_1 associates to each form $B \in \Omega^{p}_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ with charge $Q^{(e,B)}_{p,D}$, given by (3.122), a vector $v \in \mathbb{C}^{n_p}$ defined by

$$v := \begin{cases} \pi_1(B) = 0 \quad \text{if } B \equiv 0; \\ \pi_1(B) = Q^{(e,B)}_{p,D} e_1 + \sum_{k=2}^{n_p} b_k^{(B)} e_k \quad \text{otherwise,} \end{cases} \tag{3.188}$$

where $e_1, ..., e_{n_p}$ is an orthonormal base of $\mathbb{C}^{n_p}$ and $b_k^{(B)} \in \mathbb{R}$ are some of the independent entries of the in coordinate representation of the form chosen in such a way that different forms have a different string objects. Moreover, $\forall B, C \in \Omega^{p}_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ we have

$$\begin{aligned} \pi_1(B+C) &= Q^{(e,B+C)}_{p,D} e_1 + \sum_{k=2}^{n_p} (b_k^{(B)} + b_k^{(C)}) e_k, \\ \pi_1(\lambda B) &= Q^{(e,\lambda B)}_{p,D} e_1 + \sum_{k=2}^{n_p} (\lambda b_k^{(B)}) e_k, \quad \lambda \in \mathbb{C}, \end{aligned} \tag{3.189}$$

hence the map is linear. Furthermore, it is injective and surjective by construction, therefore it is bijective and hence invertible. Same consideration holds for π_2 map, which is defined similarly to π_1, and therefore for $\pi_2|_{\mathbb{C}^{n_p}}$ where $\mathbb{C}^{n_p} \in \mathbb{C}^{n_q}$ is a subspace with first component given by $Q^{(e,\tilde{B})}_{q,D}$ for some $\tilde{B} \in \Omega^{q}_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$. Since $\pi_2|_{\mathbb{C}^{n_p}}$ is a bijection the subspace $\mathbb{C}^{n_p} \in \mathbb{C}^{n_q}$ is mapped by $\pi_2^{-1}|_{\mathbb{C}^{n_p}}$ into a subspace of $\Omega^{q}_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$ of dimension n_p; the operator $\star_{D-2}$ maps $\Omega^{p}_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ into this subspace and it can be defined as

$$\star_{D-2} := d^{-1} \circ \star \circ d. \tag{3.190}$$

We know that d is a linear bijection because it is injective since we have removed forms with vanishing exterior derivative, i.e. pure gauge for which the field strength is vanishing, and it is surjective since we have removed field configurations which differ by pure gauge thanks to the quotient in the definition of asymptotic symmetries. Moreover, $\star$ is an isomorphism by definition and therefore $\star_{D-2} := d \circ \star \circ d^{-1}$ is a linear bijection from a subspace of $\Omega^{p}_{\mathcal{R}^{(p)}\neq 0,0}(M_D)$ to a subspace of $\Omega^{q}_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$. In this set up the existence and uniqueness of the duality map f is due to the "well behaviour" of $\star_{D-2}$, π_1 and π_2. Indeed since π functions were bijective they are invertible and the duality map can be defined as $f := \pi_2 \circ \star_{D-2} \circ \pi_1^{-1}$. f itself is a bijection since it is a composition of bijections. The duality map f is therefore realized as an automorphism of $\mathbb{C}^{n_p}$, therefore $f \in Aut(\mathbb{C}^{n_p}) = \mathrm{GL}(\mathbb{C}^{n_p}) \cong \mathrm{GL}(n_p, \mathbb{C})$.

Let us write the generic transformation f of $\mathrm{GL}(n_p,\mathbb{C})$ as

$$f=\begin{bmatrix}\eta & \vec{\alpha}\\ \vec{\beta}^{\mathrm{t}} & A\end{bmatrix}; \tag{3.191}$$

with $\eta\in\mathbb{C}\setminus\{0\}$ while $A\in\mathrm{GL}(n_p-1,\mathbb{C})$ and $\vec{\alpha},\vec{\beta}\in\mathbb{C}^{n_p-1}$. Applying the transformation to the vector $v=\pi_1(B)=Q^{(e,B)}_{p,D}e_1+\sum_{k=2}^{n}b^{(B)}_k e_k=(Q^{(B)}_{p,D},\vec{b}^{(B)})$ we get a vector

$$\tilde{v}=\pi_2(\tilde{B})=Q^{(e,\tilde{B})}_{q,D}e_1+\sum_{k=2}^{n}\tilde{b}^{(\tilde{B})}_k e_k=(\tilde{Q}^{(\tilde{B})}_{q,D},\vec{\tilde{b}}^{(\tilde{B})}), \tag{3.192}$$

with $\tilde{b}^{(\tilde{B})}_k\in\mathbb{R}$ and $\tilde{B}=\star_{D-2}B\in\Omega^q_{\mathcal{R}^{(q)}\neq 0,0}(M_D)$.
Therefore

$$\begin{aligned}\eta Q^{(e,B)}_{p,D}+\vec{b}^{(B)}\cdot\vec{\alpha}&=Q^{(e,\tilde{B})}_{q,D},\\ \vec{\beta}Q^{(e,B)}_{p,D}+A\cdot\vec{b}^{(B)}&=\vec{\tilde{b}}^{(\tilde{B})}.\end{aligned} \tag{3.193}$$

Since an invertible matrix is triangularizable if and only if its characteristic polynomial admits all roots in the underlying field and $\mathbb{C}$ is algebraically closed we can always choose, by a change of basis, f such that $\vec{\alpha}=0$ in (3.193). The complex parameter η, which depends from the dimension D and the degrees of the forms in the game, is simply given by

$$\eta=\frac{Q^{(e,\tilde{B})}_{q,D}}{Q^{(e,B)}_{p,D}}; \tag{3.194}$$

written in other way

$$Q^{(e,\tilde{B})}_{q,D}=f|_Q(Q^{(e,B)}_{p,D}) \tag{3.195}$$

where $f|_Q(\bullet)=\eta\bullet$. Without loss of generality we can fix $\eta=1$ by a change of basis.
Explicitly, knowing (3.122) and (3.194), we have

$$\Lambda_q\oint_{S^{D-2}_u}\gamma^{i_1\tilde{i}_1}\ldots\gamma^{i_{q-1}\tilde{i}_{q-1}}\epsilon^{(\frac{D-(2q+2)}{2})}_{i_1\ldots i_{q-1}}\tilde{\mathcal{R}}^{(q)}d\Omega=\Lambda_p\oint_{S^{D-2}_u}\gamma^{i_1\tilde{i}_1}\ldots\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon^{(\frac{D-(2p+2)}{2})}_{i_1\ldots i_{p-1}}\mathcal{R}^{(p)}d\Omega; \tag{3.196}$$

taking everything on one side we get a vanishing charge but it cannot be associated to the identically vanishing form since for $p\neq q$ we cannot isolate the difference $\tilde{\mathcal{R}}^{(q)}-\mathcal{R}^{(p)}$. Therefore we have to have, explicitly

$$\Lambda_q\gamma^{j_1\tilde{j}_1}\ldots\gamma^{j_{q-1}\tilde{j}_{q-1}}\epsilon^{(\frac{D-(2q+2)}{2})}_{j_1\ldots j_{q-1}}\tilde{\mathcal{R}}^{(q)}=\Lambda_p\gamma^{i_1\tilde{i}_1}\ldots\gamma^{i_{p-1}\tilde{i}_{p-1}}\epsilon^{(\frac{D-(2p+2)}{2})}_{i_1\ldots i_{p-1}}\mathcal{R}^{(p)}. \tag{3.197}$$

If $p = q$, i.e. $p = \frac{D-2}{2}$, the charges contain form of the same degree and we can isolate the sum which has to be the identically vanishing form, we get

$$\tilde{\mathcal{R}}^{(\frac{D-2}{2})} - \mathcal{R}^{(\frac{D-2}{2})} = 0; \tag{3.198}$$

which means that the charge is mapped to a multiple of itself, hence the form is self-dual. □

We observe that similar steps can be retraced in the case of Coulomb fall-off and $D = 2p + 2$, hence also in this case the theorem above will hold. Equation (3.197) is exactly (3.162) and we get (3.185).
One might wonder if the duality map found is consistent with the duality of Young tableaux introduced in Chapter 2 but it is so by construction. First of all, the automorphism f is given in terms of $\star_{D-2}$ which is constructed from the Hodge duality from the field strengths and this is the covariant interpretation of the Young tableaux duality at the level of the space of forms. Second, the unique restriction $f|_Q$ is solely determined by the charges associated to the forms we are considering. Therefore $f|_Q$ is the map induced on the space of charges by the duality of Young tableaux.

3.6.1.1 Algebraic topology interpretation

Theorem 3.6.1 can be interpreted in the realm of algebraic topology; the interesting point is the topological nature of the duality map. Let us consider two copies of the de Rham complex, one labelled by p, $C^{*(p)}_{\mathrm{dR}}$, and one labelled by $q = D - p - 2$, $C^{*(q)}_{\mathrm{dR}}$

$$\begin{tikzcd}
\cdots\ \Omega^{p-2} \arrow[r, "d_{p-2}"] \arrow[d, "\star_{D-6}"] & \Omega^{p-1} \arrow[r, "d_{p-1}"] \arrow[d, "\star_{D-4}"] & \Omega^{p} \arrow[r, "d_{p}"] \arrow[d, "\star_{D-2}"] & \Omega^{p+1} \arrow[r, "d_{p+1}"] \arrow[d, "\star_{D}"] & \Omega^{p+2}\ \cdots \arrow[d, "\star_{D+2}"] \\
\cdots\ \Omega^{q-2} \arrow[r, "d_{q-2}"] & \Omega^{q-1} \arrow[r, "d_{q-1}"] & \Omega^{q} \arrow[r, "d_{q}"] & \Omega^{q+1} \arrow[r, "d_{q+1}"] & \Omega^{q+2}\ \cdots
\end{tikzcd} \tag{3.199}$$

where every $\star_{D+2n}$ with $n \in \mathbb{Z} \setminus \{-\infty, +\infty\}$ is required to be a group homomorphism and $\star_D$ is the Hodge operator. Noting that $q - p = D - p - 2 - p = D - 2p - 2$, we have that

$$\star^* : C^{*(p)}_{\mathrm{dR}} \mapsto C^{*(q)}_{\mathrm{dR}} \quad [D - 2p - 2] \tag{3.200}$$

is an homotopy of cochain complexes; in critical dimension, i.e. $p = \frac{D-2}{2} = q$, the p-form gauge theory is self dual and the homotopy $\star$ is an isomorphism of cochain complexes since every space of forms is mapped in itself. Now, since we are interested in considering gauge field theories on Minkowski spacetime we can assume trivial topology, i.e. all the cohomology groups of the de Rham complex are trivial

$$H^n = \frac{Z_n := \{B \in \Omega^n | d_n B = 0\}}{B_n := \{d_{n-1}A \in \Omega^n | A \in \Omega^{n-1}\}} = 0; \tag{3.201}$$

this means that every cocycle is also a coboundary[21]. Therefore, the de Rham complexes in (3.199) are exact sequences of Abelian groups. Now, let us restrict to only one de Rham complex[22], $C^{*(p)}_{\mathrm{dR}}$

$$\dots\ \Omega^{p-2} \xrightarrow{d_{p-2}} \Omega^{p-1} \xrightarrow{d_{p-1}} \Omega^{p} \xrightarrow{d_{p}} \Omega^{p+1} \xrightarrow{d_{p+1}} \Omega^{p+2}\ \cdots, \tag{3.202}$$

and let us taken into account the fact we are interested in asymptotic symmetries. We start with a p-form gauge theory, with gauge field $B \in \Omega^p$ and the asymptotic charge is written in terms of the field strength $H = d_p B \in \Omega^{p+1}$ which we require to be non-vanishing[23]. Since $C^{*(p)}_{\mathrm{dR}}$ on Minkowski spacetime is exact we have $H = 0 \Leftrightarrow B = d_{p-1}A$ for some $A \in \Omega^{p-1}$; hence we need to throw away all those elements $B \in \Omega^p$ such that $B = d_{p-1}A$ for some $A \in \Omega^{p-1}$. Moreover, only the zero form can have vanishing field strength but, again for exactness $B = 0 \Leftrightarrow A = d_{p-2}C$ for some $C \in \Omega^{p-2}$. Therefore, for asymptotic symmetries scopes we can replace Ω^{p+1} with $\Omega^{p+1}_{\mathrm{AS}} := Im(d_p) \setminus \{ker(d_p) \setminus \{0\}\}$, Ω^p with $\Omega^p_{\mathrm{AS}} := \Omega^p \setminus \{Im(d_{p-1}) \setminus \{0\}\}$ and Ω^{p-1} with 0 to get the asymptotic symmetries de Rham complex $C^{*(p)}_{\mathrm{ASdR}}$

$$0 \xrightarrow{d_{p-1}} \Omega^p_{\mathrm{AS}} \xrightarrow{d_p} \Omega^{p+1}_{\mathrm{AS}} \xrightarrow{d_{p+1}} 0 \tag{3.203}$$

which is a short exact sequence. By general reasoning or by explicit computations follows that d_p is an isomorphism. Looking now at diagram (3.199) and reducing the de Rham complexes to the asymptotic symmetries de Rham complexes, Theorem (3.6.1) can be used to construct f and then the duality map since now diagram (3.199) reduces to the upper part of the diagram of Theorem (3.6.1). Therefore, the duality map is topological in nature and can be constructed if and only if

$$H^p = H^{p+1} = 0 = H^{q+1} = H^q. \tag{3.204}$$

Indeed the vanishing of these cohomology groups is sufficient to reduce the full de Rham complex to the asymptotic symmetries de Rham complex and it is also necessary since to construct the duality map we need that d_p and d_q are isomorphisms.

[21] In other words, every closed form is exact.

[22] The same considerations hold for $C^{*(q)}_{dR}$.

[23] Otherwise the asymptotic charge would be zero and would be associated to a trivial gauge transformation.

3.6.1.2 Physical interpretation

Let us spend some time on the result of Theorem (3.6.1) from the physical point of view. First of all, when the spacetime satisfies conditions (3.204), and so we can write $Q^{(e)}_{q,D} = \eta Q^{(e)}_{p,D}$, magnetic-like and electric-like charges act equivalently on the physical states. Indeed to promote charges to operators acting on the Hilbert vector space $\mathcal{H}$ we need continuous functional calculus and, in the end, electric-like and magnetic-like charges operators act different on a vector in the Hilbert space by the multiplicative factor[24]. However, they act in the same way on the projective Hilbert space, i.e. the state space or ray space, since it is defined by quotienting for the equivalence relation $|\psi\rangle \sim_{eq} \lambda|\psi\rangle$ with $|\psi\rangle \in \mathcal{H},\ \lambda \in \mathbb{C}$. This seems to be consistent with the fact that the two theories describe the same degrees of freedom. Indeed, we can construct electric-like and magnetic-like Hilbert vectors using, respectively, the p-form gauge field operator $B^{(p)} \in \Omega^p$ and its dual $\tilde{B}^{(q)} \in \Omega^q$

$$|e\rangle_{B^{(p)}}, \quad |m\rangle_{\tilde{B}^{(q)}}; \tag{3.205}$$

but we can also construct another pair of electric-like and magnetic-like Hilbert vectors using, respectively, the dual q-form gauge field operator $B^{(q)} \in \Omega^q$ and its dual $\tilde{B}^{(p)} \in \Omega^p$.

$$|e\rangle_{B^{(q)}}, \quad |m\rangle_{\tilde{B}^{(p)}}. \tag{3.206}$$

However, the dual field $\tilde{B}^{(q)}$ is, by definition, the field $B^{(q)}$, hence

$$|e\rangle_{B^{(q)}} = |m\rangle_{\tilde{B}^{(q)}} \tag{3.207}$$

and, by the same reasoning modulo the scalars needed for the inversion of the Hodge star operator, we have

$$|e\rangle_{B^{(p)}} = (-1)^{(p+1)(q+1)} s |m\rangle_{\tilde{B}^{(p)}}. \tag{3.208}$$

Moreover, since the duality[25] links $B^{(p)}$ with $\tilde{B}^{(q)}$ and $B^{(q)}$ with $\tilde{B}^{(p)}$, we have

$$|e\rangle_{B^{(p)}} = (-1)^{(p+1)(q+1)} s\alpha |m\rangle_{\tilde{B}^{(q)}} = (-1)^{(p+1)(q+1)} s\alpha |e\rangle_{B^{(q)}} \tag{3.209}$$

and

$$|e\rangle_{B^{(q)}} = (-1)^{(p+1)(q+1)} s\beta |m\rangle_{\tilde{B}^{(p)}} = \beta |e\rangle_{B^{(p)}} \tag{3.210}$$

where $\alpha \in \mathbb{C}$ and, by consistency,

$$\beta^{-1} = (-1)^{(p+1)(q+1)} s\alpha. \tag{3.211}$$

[24] This is η or $\frac{1}{\eta}$.

[25] Recall that our notation is such that $dB^{(q)} = \star dB^{(p)}$ and $dB^{(p)} = (-1)^{(p+1)(q+1)} s \star dB^{(q)}$.

Therefore, putting all the pieces together, we get

$$Q^{(e)}_{p,D}|e\rangle_{B^{(p)}} = Q^{(e)}_{p,D}(-1)^{(p+1)(q+1)}s\alpha|e\rangle_{B^{(q)}}, \tag{3.212}$$

but the right charge that should act on the vector $|e\rangle_{B^{(q)}}$ is $Q^{(e)}_{q,D}$, hence

$$Q^{(e)}_{q,D} = Q^{(e)}_{p,D}(-1)^{(p+1)(q+1)}s\alpha \Rightarrow Q^{(e)}_{q,D} = \eta Q^{(e)}_{p,D} \tag{3.213}$$

with $\eta := (-1)^{(p+1)(q+1)}s\alpha \in \mathbb{C}$. However, we know that electric-like and magnetic-like charges of the two dual descriptions are linked by (3.156) and to be consistent with them we need to fix $\eta = 1$.

In Chapter 4 we are going to discuss the case of mixed symmetry tensors thinking of them as Young projected elements into the space of N-multi-forms with the goal to extend Theorem 3.6.1 to mixed symmetry gauge theories.

3.6.2 Comments on the duality for power law divergent/vanishing charges

In this brief section we present some speculative results on the link between the asymptotic charges with Coulomb fall-off on the field components of dual formulations. In this section we are going to consider the logarithmic terms as pure gauge, hence not present in the charges.

The main difference with the case studied before is that we need to add the point at infinity in $\mathbb{C}^n$; we denote $\overline{\mathbb{C}^n} = \mathbb{C}^n \cup \{\infty\}$ the projectively, or Aleksandrov, compactification of $\mathbb{C}^n$, also known as Riemann sphere. Let us focus on the case $D < 2p+2$ since for $D > 2p+2$ we have the inverse picture. The duality map link the asymptotic charge of a p-form with those of a ($q = D-p-2$)-form; this means that when the asymptotic charge of the p-form is power law divergent those of the dual form is power law vanishing and vice versa[26]. Hence the duality maps a vanishing charge in a divergent one. To have a well define picture, first we associate the infinite charge to the point at infinity and, second, the vanishing one to the origin. The duality map can be recognized in the reciprocal map, $R(v_1, ..., v_n) = (\frac{1}{v_1}, ..., \frac{1}{v_n})$, since in the projectively extended space this is a total function[27]. Therefore every component of a vector in one copy of $\overline{\mathbb{C}^n}$ is mapped in its inverse in the other copy of $\overline{\mathbb{C}^n}$; if we partially follow the idea of Theorem 3.6.1's proof we can restrict the duality map to its first component in order to get the duality map between the charges

$$Q^{(e,\tilde{B})}_{q,D} = \frac{1}{Q^{(e,B)}_{p,D}}. \tag{3.214}$$

[26]Indeed the asymptotic charge of a ($q = D-p-2$)-form goes like $\mathcal{O}(r^{D-2p-2})$ while those of a p-form like $\mathcal{O}(r^{-(D-2p-2)})$.
[27]This structure allows for division by zero and presence of the infinite element.

This duality map is a special case of Moebius transformation[28] but the very interesting point is the fact that a power law vanishing charge, i.e. a gauge theory with still a trivial gauge transformation at null infinity, is mapped in a power law divergent charge, i.e. a gauge theory with a boundary condition too weak. Let us give the following definition

Definition 3.6.2 (Power law weak fall-offs)**.** *Let $(\boldsymbol{J}, \boldsymbol{M}, \boldsymbol{G})$ be an Abelian gauge theory with $(\boldsymbol{M}, \boldsymbol{g})$ a D-dimensional Lorentzian manifold with conformal boundary $\partial \boldsymbol{M}_c$, $\boldsymbol{J}$ a set of fields and $\boldsymbol{G}$ an Abelian group. The set of boundary conditions $\boldsymbol{J}|_{\partial \boldsymbol{M}_c}$ are said to be power law weak if the asymptotic charge computed on $\partial \boldsymbol{M}_c$ with field fall-off given by $\boldsymbol{J}|_{\partial \boldsymbol{M}_c}$ is power law divergent.*

Therefore a p-form gauge theory with Coulomb fall-offs has trivial gauge transformations at null infinity if and only if the dual q-form gauge theory has a boundary conditions power law weak. We can try to generalize this observation to generic Abelian gauge theories and their duals with the following

Conjecture 3.6.1 (Link between trivial gauge transformations and power law weak fall-offs)**.** *Let $(\boldsymbol{M}, \boldsymbol{g})$ be a D-dimensional Lorentzian manifold with conformal boundary $\partial \boldsymbol{M}_c$ and let $(\boldsymbol{J}, \boldsymbol{M}, \boldsymbol{G})$ be an Abelian gauge theory with gauge group $\boldsymbol{G}$ and fields boundary conditions $\boldsymbol{J}|_{\partial \boldsymbol{M}_c}$. The dual gauge theory $(\tilde{\boldsymbol{J}}, \boldsymbol{M}, \boldsymbol{G})$ has trivial gauge transformations at the conformal boundary $\partial \boldsymbol{M}_c$ if and only if $\boldsymbol{J}|_{\partial \boldsymbol{M}_c}$ are power law weak.*

3.7 Conclusions and outlook

In this Chapter, based on works [326] and [330], we discussed the asymptotic symmetries and the Young machinery duality in the realm of p-form gauge theories where the gauge field is a p-form $\boldsymbol{B}$. First of all, in Paragraph 3.3.2, we find that, for every $p \in \mathbb{N} \setminus \{0\}$ and for radiation fall-offs on the field components, we can consider a leading order logarithmic term, i.e. $\mathcal{O}(r^{-\frac{D-2p}{2}} ln(r))$, in the field components $B_{ui_1 \dots i_{p-1}}$ which do not generate unwanted unphysical divergences in the energy flux at null infinity even in the presence of matter which however must satisfy some specific fall-offs. The mechanism for this cancellation is due to both the equations of motion and the Lorenz-like gauge fixing condition. However, a fundamental piece of the mechanism is the necessary subleading logarithmic term, i.e. $\mathcal{O}(r^{-\frac{D-2p+2}{2}} ln(r))$, in the field components $B_{ri_1 \dots i_{p-1}}$. We stress also that this mechanism does the right job for every finite power of the logarithm. Nevertheless, the very same mechanism fails in the case of Coulomb fall-offs on the field components. As general feature, it would be interesting to understand better the role of matter in mechanisms such as the one presented above.
In order to compute the asymptotic charge, in which only the gauge parameter components $\epsilon_{i \dots i_{p-1}}$ appears, we assume a polyhomogeneous expansion for the gauge parameter and we find the leading order as stated by Theorem 3.3.1. The asymptotic charges for both radiation and Coulomb fall-offs are given by (3.122) and (3.125), respectively. The first one is finite, non-vanishing and fits in the Definition 3.4.1 of well defined charges.

[28] This is coherent with the fact that now, by a modification of Theorem 3.6.1, we have $f \in Aut(\overline{\mathbb{C}^n})$ and that Moebius transformations are the automorphisms of the Riemann sphere to itself.

The case of charge with Coulomb fall-offs on the field components is more subtle. Indeed, this charge presents non-trivial radial behavior and a logarithmic term, if we consider leading order logarithmic term in the field components $B_{ui_1\dots i_{p-1}}$, which is vanishing in critical dimension $D = 2p+2$. We can avoid the logarithmic term requiring that the violation of the standard Coulomb fall-offs is given by a pure gauge term which does not play any significant role in physical quantities like the charge. Doing so the charge fits in the Definition 3.4.2 of power law divergent (vanishing) charge.

Charges (3.122) and (3.125) are dubbed electric-like since only electric-like components of the field strengths enter in the game. In order to distinguish between electric-like charges for forms of different degrees, we call $Q^{(e)}_{p,D}[S^{D-2}_u]$ the electric-like charge for a p-form and $Q^{(e)}_{q,D}[S^{D-2}_u]$ the electric-like charge for a q-form. In the perspective of the duality between Young tableaux, and therefore of the on-shell duality which link p-forms to q-forms with $q = D - p - 2$, we studied, in Paragraph 3.5 the possibility of a duality map. Our proposals go in two different directions and we have to discuss in a separate way well defined charges and power law divergent (vanishing) ones.

Let us focus on well defined charges. A first proposal is, using the Hodge duality equations[29] to derive a magnetic-like charge in the dual theory. The dubbing "magnetic-like" is due to the presence of only magnetic-like components of the field strength. These magnetic-like charges can be derived starting from both $Q^{(e)}_{p,D}[S^{D-2}_u]$ and $Q^{(e)}_{q,D}[S^{D-2}_u]$, leading to $\tilde{Q}^{(m)}_{q,D}[S^{D-2}_u]$ and $\tilde{Q}^{(m)}_{p,D}[S^{D-2}_u]$, respectively. An interesting feature of the magnetic-like charges is that the gauge parameter components involved is the one of the original theory; so, for example $\tilde{Q}^{(m)}_{q,D}[S^{D-2}_u]$ contains the gauge parameter components of the p-form gauge theory. This set of four charges satisfy relations (3.156) which teach us that electric-like and magnetic-like charges act in the same way on the physical states since they differs by irrelevant numerical factors. Using these four charges we can built up two electromagnetic-like charges given by (3.157) which transform according to Moebius transformation under the duality; for a schematic representation see Figure 3.1. The electromagnetic-like charges generate a complexified $\mathfrak{u}_{\mathbb{C}}(1)$ symmetry algebra in analogy to what happens in $D = 4$ Maxwell theory. Hence, even if we can reinterpret[30] the gauge parameter components using for example (3.155), the presence of the gauge parameter of the original theory in the magnetic-like charge seems to point in the direction that the $\mathfrak{u}_{\mathbb{C}}(1)$ symmetry algebra contains both gauge parameters of the two dual descriptions: one acting on the theory in a standard way and the other acting on the theory written in the dual variables. It would be interesting to better understand the meaning of this complexified symmetry algebra and the relation of its transformation under the duality with celestial holography. For the case of power law divergent (vanishing) charges the same holds with some technical precautions discussed in Paragraph 3.5.3. The new feature is that when electric-like charge (3.125) is power law vanishing the magnetic-like charge (3.183) is power law divergent and vice versa. This partially motivate the Conjecture 3.6.1.

The second proposal, always focusing on well defined charges, is to find a map such that the electric-like charges of the two dual theories are mapped one in the other. In this direction, Theorem 3.6.1 asserts the existence and

[29] Recall that in a covariant way the duality between Young tableaux associated to the physical field is a Hodge duality between the field strengths.

[30] Recall we were able to do so only in one direction.

uniqueness of such a map, at least on Minkowski spacetime. However, the theorem has a more general validity since we interpreted it, in Paragraph 3.6.1.1, in the realm of algebraic topology. Indeed, under the requirement of triviality of some cohomology groups, i.e. relations (3.204), we get the same result. Therefore the duality map has a topological nature. In Paragraph 3.6.1.2, we give the physical interpretation working in the physical state space, i.e. ray space. In ray space electric-like and magnetic-like charges of both descriptions act in the same way since they are equal, modulo complex numbers, as operators in Hilbert space[31]. This seems to be consistent to the fact that the two theories are describing the same physics and the same degrees of freedom. For the case of power law divergent (vanishing) charges, the dual map can be recognized to be the Moebius transformation given by the inversion. Moreover, noting that in this case a divergent charge became a vanishing one, we give a speculative result, contained in Conjecture 3.6.1, on the interplay between trivial gauge transformations and fall-offs too weak, which make the asymptotic charge divergent. In Chapter 4 we are going to extend the algebraic topology interpretation of Theorem 3.6.1 to mixed symmetry tensor gauge theories.

[31]Here we are also using relations (3.156).

Chapter 4

Duality and asymptotic charges of mixed symmetry tensor gauge theories

4.1 Summary with results

This Chapter is based on the works [325] and [334] where the asymptotic symmetries, the duality and their interplay in mixed symmetry tensor gauge theories are studied from theoretical, formal and mathematical point of view. Massive mixed symmetry tensor fields arise in the spectrum of String Theory and in its tensionless limit on flat background all the massive fields gets squeezed to a common zero mass level. Hence, in this limit a plethora of mixed symmetry tensor gauge fields emerge. Moreover, without bringing String Theory into play, mixed symmetry tensor gauge fields are much less studied and understood than their more phenomenologically used Lower Spin gauge theories. Therefore, a deeper understanding of the asymptotic symmetries in mixed symmetry tensor gauge theories would be an interesting step forward for Theoretical and Mathematical Physics. Furthermore, in mixed symmetry tensor gauge theories there are a bunch of dual descriptions; the most known are the one related to the graviton field in arbitrary D: the Curtright hook field carrying the irreducible representation $\lambda = (D-3, 1)$ and the Riemann-like field carrying the irreducible representation $\lambda = (D-3, D-3)$. Once again, the crucial question here is if asymptotic symmetries and/or asymptotic charges computed in one theory have access to some information about the dual theories. In the first half of the Chapter, we develop new mathematical tools with the aim to extend results of Theorem 3.6.1 to mixed symmetry tensors. Specifically, we develop de Rham-like complexes, which play an analogue role of the de Rham complex, and we define their cohomology groups. The extension of Theorem 3.6.1 is fully abstract since we assume to have computed the asymptotic charges in all dual descriptions. In the other half of the Chapter, we approach the problem to better understand mixed symmetry tensor gauge theories form an explicit way. We study the case of the graviton and its dual descriptions in $D = 5$; in order to compute the asymptotic charge for the Curtright three indexes field we start computing the Noether current by varying the lagrangian density. Anyway, at the time of the writing of this PhD thesis, the computation of the Noether current for the Curtright three indexes field is the step did in the direction of an explicit computation of the asymptotic charge for this field. The continuation of this path is one of the items for the future.

4.2 Introduction

Since Maxwell's modern theory of electromagnetism [1] until today, gauge theories have played a major role in understanding and describing Nature. The invariance of physics under local transformations has led to the description of all fundamental interactions from General Relativity to the Standard Model of Particle Physics. However, in these theories only a very small subclass of representations of the Lorentz group are used as gauge fields and between the seventies and the eighties, efforts were directed towards the possibility of using arbitrarily high spin fields as gauge fields leading to the Higher-Spin theories [59],[128],[172],[184],[232],[248],[249],[272]. A systematic study of massless arbitrary spin fields was initiated by Fronsdal in 1978 [52],[53]. Usually, the spectrum of such theories contains the graviton as a massless spin-two field, moreover Higher-Spin theories are supposed to be consistent quantum theories and so to give examples of consistent quantum gravity theories. However, Higher-Spin theories, suffer of no-go theorems which, for example, trivialize interactions [29],[31],[58],[65],[80],[172].

More or less in the same years, Curtright showed how mixed symmetry tensor fields can be used as consistent gauge field, generalizing the concept of a gauge field to include higher rank Lorentz tensors which are neither totally symmetric nor totally antisymmetric [71]. The main interest is due to the fact that an infinite family of mixed symmetry gauge fields arises in the zero tension limit of String Theory [122]. The simplest of these mixed symmetry tensor gauge theories has as fundamental object a three indexes mixed symmetry tensor field called Curtright hooke field; moreover, its gauge-invariant dynamics is dual to that of the graviton in $D = 5$ dimensions. This is due to an underlying duality between the Young tableaux defining the two irreducible representations. This duality goes well beyond this simple example and a generic mixed symmetry tensor field possesses a bunch of on-shell dual descriptions [81]. Therefore, studying the Curtright hooke field in generic dimension can lead to new insights on the nature of gravity, at least on its perturbative sector since we are talking of the graviton, by revealing new asymptotic symmetries of linearized gravity as symmetries dual to the symmetries of the Curtright hooke field.

As a general feature of gauge theories, we can find some large gauge transformations that act in a non-trivial way on the physical states at infinity, where infinity means in the nearby of a boundary. One of the first examples is provided by the BMS group in the context of Einstein gravity [24],[25],[26].

Moreover, it was shown that the asymptotic symmetries of the scalar in $D = 4$ can be interpreted as the asymptotic symmetries of a 2-form; hence the asymptotic charge of a description contains information about the dual one [251],[262]. Furthermore, in [326],[330] we show that also in the realm of p-form gauge theories, asymptotic charges contain information on the dual formulation.

In this Chapter the first goal is to extend Theorem 3.6.1 [330] to the case of mixed symmetry tensor gauge fields in a full abstract way. The idea is to use the algebraic topology interpretation developing new mathematical tools, such as the de Rham-like complexes and the de Rham-like cohomology groups, continuing and developing the directions opened by the authors in [107],[117]. This is done in Paragraph 4.3. In Paragraph 4.4.1 we instead study the asymptotic charges of the Curtright hooke field in $D = 5$ but, at the time of the writing of this PhD thesis, just the computation of the Noether current has been completed.

4.3 Duality for well defined charges in mixed symmetry gauge theories

In this Paragraph we want to extend the Theorem 3.6.1 to the case of mixed symmetry tensors. We suppose we are dealing with a gauge field B whose writing in a chart is a mixed symmetry tensor carrying the irreducible representation corresponding to the Young tableau $\lambda = \{p_1, ..., p_N\}$. We proceed in fully general and abstract way, supposing that we have already calculated the asymptotic charges of both the original description and its r

dual descriptions and we assume these are well defined charges. We refer to these charges as $Q_0, ..., Q_r$ where Q_0 is the asymptotic charge of the original description while Q_r that of its r-th dual description. The point is to look at the mixed symmetry tensor as a Young projected element of the N-multi-form space using the Young projector $\Pi_{(p_1,...,p_N)}$. Doing so we can construct de Rham-like complexes for the N-multi-form space; moreover, requiring the vanishing of some de Rham-like cohomology groups we can reduce them to the asymptotic symmetries de Rham-like complexes and following the idea of Theorem 3.6.1's proof, the game is over.

4.3.1 The de Rham-like complexes for differential mixed symmetry tensors

We now generalize the de Rham complex where space of differential forms are replaced by space of differential N-multi-form. Specifically, we search for the generalization to differential mixed symmetry tensors where the term differential means that, in chart, components are C^∞ functions. Before moving on, let us give an useful definition

Definition 4.3.1 (Principal λ-subspace of the N-multi-form space)**.** *Let $\Omega^{p_1\otimes\ldots\otimes p_N}(M)$ be the N-multi-form space on a differential manifold $(M, \mathcal{A})$ of dimension D, its principal λ-subspace $\Omega_\lambda^{p_1\otimes\ldots\otimes p_N}(M)$ is the subspace of mixed symmetry tensors which carry the irreducible representation labelled by the Young tableau $\lambda = \{\lambda_1, ..., \lambda_N\}$ such that $\lambda_i \leq p_i \leq D \ \forall i \in [1, N]$.*

Hence, for example, the principal $\{1, 1\}$-subspace of $\Omega^{1\otimes 1}(M)$ is the subspace containing tensors which carry the irreducible representation labelled by the Young tableau $\lambda = \{1, 1\}$, i.e. tensors with the indexes symmetry of the graviton field. The Young projector Π_λ furnishes a natural projection from the N-multi-form space to its principal subspace

$$\Pi_\lambda : \Omega^{p_1\otimes\ldots\otimes p_N}(M) \to \Omega_\lambda^{p_1\otimes\ldots\otimes p_N}(M) \tag{4.1}$$

which sends a generic N-multi-form in a mixed symmetry tensor carrying the irreducible representation labelled by the Young tableau λ. In order to formulate a de Rham-like complex for the differential mixed symmetry tensors, we have at our disposal the i-th differentials $d^{(i)}$ with $i \in [1, N]$. The main idea is to follow the structure of the Young lattice, or more precisely, the Hasse diagram of the Young lattice we report in the following Figure 4.1.

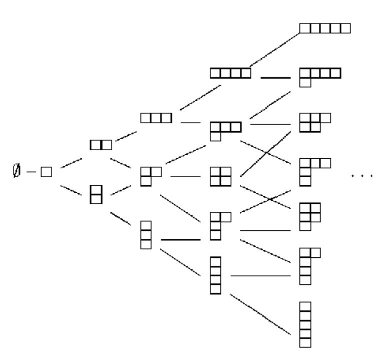


FIGURE 4.1 *Hasse diagram of Young's lattice*

In the following and until the end of the Paragraph 4.3.2 (and also in Appendix D), we are going to call the $\{p_1, ..., p_N\}$-subspace $\Omega^{p_1\otimes...\otimes p_N}_{\{p_1,...,p_N\}}(M)$ of $\Omega^{p_1\otimes...\otimes p_N}(M)$ simply with $\Omega^{p_1\otimes...\otimes p_N}$ unless explicitly specified. Looking at the Hasse diagram of the Young's lattice we can construct, for every fixed N, a N-complex where every square commutes. In the case $N = 2$, we can draw it as

$$\begin{tikzcd}
\Omega^{0\otimes 0} \arrow[d, "d^{(1)}"] \arrow[dr] & & & & & \\
\Omega^{1\otimes 0} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{1\otimes 1} \arrow[d, "d^{(1)}"] \arrow[dr] & & & & \\
\Omega^{2\otimes 0} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{2\otimes 1} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{2\otimes 2} \arrow[d, "d^{(1)}"] \arrow[dr] & & & \\
\Omega^{3\otimes 0} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{3\otimes 1} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{3\otimes 2} \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] \arrow[dr] & \Omega^{3\otimes 3} \arrow[d, "d^{(1)}"] \arrow[dr] & & \\
\dots \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] & \dots \arrow[r, "d^{(2)}"] \arrow[d, "d^{(1)}"] & \dots \arrow[r, "d^{(2)}"] & \dots \arrow[r, "d^{(2)}"] \arrow[dr] & \dots \arrow[dr] & \\
\Omega^{D\otimes 0} \arrow[r, "d^{(2)}"] & \Omega^{D\otimes 1} \arrow[r, "d^{(2)}"] & \dots \arrow[r, "d^{(2)}"] & \dots \arrow[r, "d^{(2)}"] & \Omega^{D\otimes D-1} \arrow[r, "d^{(2)}"] & \Omega^{D\otimes D}
\end{tikzcd}. \tag{4.2}$$

We note that the first column of this diagram is exactly the standard de Rham complex with differential $d^{(1)}$ thanks to the canonical isomorphism $\Omega^{p\otimes 0} \cong \Omega^p$ for every $p \leq D$.
However, for the general N case it is not so easy to draw the diagram, since we need N independent directions to taken into account all the ramifications; however, also in this case there is a standard de Rham complex with differential $d^{(1)}$ thanks to the canonical isomorphism $\Omega^{p\otimes 0\otimes...\otimes 0} \cong \Omega^p$ for every $p \leq D$.
In order to reproduce and extend Theorem 3.6.1, a necessary property we want to emulate of the de Rham complex is the fact that given a form $B \in \Omega^p$ for same p, the form $dB \in \Omega^{p+1}$ is its field strength. In other words, the space Ω^{p+1} contains the field strengths of all the forms in Ω^p. In the case of a N-multi-form whose writing in a chart is mixed symmetry tensor carrying the irreducible representation associated to the Young tableau λ, there is a unique unambiguous way to construct a field strength, that is, acting with the composition of all the i-th differential, i.e. Definition 2.5.10. Therefore we can define

Definition 4.3.2 (De Rham-like differential)**.** *Given the N-multi-form space $\Omega^{p_1\otimes...\otimes p_N}(M)$ and the i-th differential $d^{(i)}$ with $i \in [1, N]$ the de Rham-like differential d is given by the composition of all the i-th differentials*

$$\delta^{(N)} := d^{(N)} \circ d^{(N-1)} \circ ... \circ d^{(2)} \circ d^{(1)}. \tag{4.3}$$

We have the following

Proposition 4.3.1 (Fundamental property of the de Rham-like differential)**.** *The de Rham-like differential* $\delta^{(N)}$ *squares to zero*

$$\delta^{(N)} \circ \delta^{(N)} = 0. \tag{4.4}$$

Proof. Since the i-th differentials satisfy $d^{(i)} \circ d^{(i)} = 0 \;\; \forall i \in [1, N]$ and $d^{(i)} \circ d^{(j)} = d^{(j)} \circ d^{(i)} \;\; \forall i \neq j$ we get

$$\begin{aligned} \delta^{(N)} \circ \delta^{(N)} &= d^{(N)} \circ d^{(N-1)} \circ ... \circ d^{(2)} \circ d^{(1)} \circ d^{(N)} \circ d^{(N-1)} \circ ... \circ d^{(2)} \circ d^{(1)} = \\ &= d^{(N)} \circ d^{(N)} \circ d^{(N-1)} \circ d^{(N-1)} \circ ... d^{(1)} \circ d^{(1)} = 0. \end{aligned} \tag{4.5}$$

□

Let us focus on the case $N = 2$ where the de Rham-like differential is given by $\delta^{(2)} := d^{(2)} \circ d^{(1)}$. Therefore we have the 2-de Rham-like complex

Definition 4.3.3 (2-de Rham-like complex)**.** *A de Rham-like complex for the biform space on a differential manifold* $(M, \mathcal{A})$ *such that* $D = dim(M)$*, is the cochain complex with differential given by the de Rham-like differential* $\delta^{(2)}$.

We stress that in the perspective of this definition, the standard de Rham complex given by the first column of diagram 4.2 could be defined as a 1-de Rham-like complex with differential $d^{(1)} = \delta^{(1)}$.
In diagram 4.2 some 2-de Rham-like complexes are highlighted with colored arrows. However, we note that the 2-de Rham-like complex with blue arrows have length[1] D while the one with orange arrows $D - 1$ and so on. Hence, we are going to consider augmented cochain complexes in order to have cochain complexes all of the same length D and all starting from $\Omega^{0 \otimes 0}$. Therefore we have the following

Definition 4.3.4 (k-augmented 2-de Rham-like complex)**.** *We define the k-augmented 2-de Rham-like complex as the 2-de Rham-like complex with length* $D - k$ *augmented by the first k terms of the 1-de Rham-like complex with differential* $\delta^{(1)}$*, or equivalently, as the 2-de Rham-like complex with length* $D - k$ *augmented by the first k terms of the standard de Rham complex with differential* $d^{(1)}$.

To give an example of k-augmented 2-de Rham-like complex let us consider the 2-de Rham-like complex with orange arrows, it has length $D - 1$ for every fixed D and so we have the 1-augmented 2-de Rham-like complex

$$\Omega^{0 \otimes 0} \xrightarrow{\delta^{(1)}} \Omega^{1 \otimes 0} \xrightarrow{\delta^{(2)}} \Omega^{2 \otimes 1} \xrightarrow{\delta^{(2)}} \Omega^{3 \otimes 2} \xrightarrow{\delta^{(2)}} ... \xrightarrow{\delta^{(2)}} \Omega^{D \otimes D-1}; \tag{4.6}$$

[1]We mean the number of arrows between non-trivial modules.

in an analog way, considering the 2-de Rham-like complex with green arrows we have the 2-augmented 2-de Rham-like complex

$$\Omega^{0\otimes 0}\xrightarrow{\delta^{(1)}}\Omega^{1\otimes 0}\xrightarrow{\delta^{(1)}}\Omega^{2\otimes 0}\xrightarrow{\delta^{(2)}}\Omega^{3\otimes 1}\xrightarrow{\delta^{(2)}}\dots\xrightarrow{\delta^{(2)}}\Omega^{D\otimes D-2}. \tag{4.7}$$

At this point we can generalize our definitions to the general N case; therefore, we have the following

Definition 4.3.5 (N-de Rham-like complex)**.** *A de Rham-like complex for the N-multi-form space on a differential manifold $(M,\mathcal{A})$ such that $D = dim(M)$, is the cochain complex with differential given by the de Rham-like differential $\delta^{(N)}$.*

and, for the same reasons as before, their augmented cochain complexes

Definition 4.3.6 ($(k_1,...,k_{N-1})$-augmented N-de Rham-like complex)**.** *The $(k_1,...,k_{N-1})$-augmented N-de Rham-like complex is the N-de Rham-like complex with length $D-(k_1+...+k_{N-1})$ augmented by the first $k_1+...+k_{N-1}$ terms of the $(k_1,...,k_{N-2})$-augmented $(N-1)$-de Rham-like complex.*

Let us see consider the following clarifying example.

Example 1: the augmenting of the N-de Rham-like complex passing for $\Omega^{q_1\otimes...\otimes q_N}$

Let us consider the N-de Rham-like complex passing for $\Omega^{q_1\otimes...\otimes q_N}$, this is given by

$$\Omega^{q_1-q_N\otimes...\otimes q_{N-1}-q_N\otimes 0}\xrightarrow{\delta^{(N)}}\dots\xrightarrow{\delta^{(N)}}\Omega^{q_1\otimes...\otimes q_N}\xrightarrow{\delta^{(N)}}\dots\xrightarrow{\delta^{(N)}}\Omega^{D\otimes D-(q_1-q_2)\otimes...\otimes D-(q_1-q_N)}. \tag{4.8}$$

This complex has length $D-(q_1-q_N)$ and we want to construct the $(q_1-q_2,...,q_{N-1}-q_N)$-augmented N-de Rham-like complex. Therefore we need to add the first $\sum_{i=1}^{N-1} q_i - q_{i+1} = q_1 - q_N$ terms of the $(q_1 - q_2,...,q_{N-2}-q_{N-1})$-augmented $(N-1)$-de Rham-like complex. These first q_1-q_N terms are

$$\begin{array}{ccccc}
 & \Omega^{0\otimes...\otimes 0\otimes 0} & \xrightarrow{\delta^{(1)}} \dots \xrightarrow{\delta^{(1)}} & \Omega^{q_1-q_2\otimes 0\otimes...\otimes 0} & \\
 & & & \Big\downarrow \delta^{(2)} & \\
 & \Omega^{q_1-q_3\otimes q_2-q_3\otimes...\otimes 0} & \xleftarrow{\delta^{(2)}} \dots \xleftarrow{\delta^{(2)}} & \Omega^{q_1-q_2+1\otimes 1\otimes 0\otimes...\otimes 0} & \\
 & \Big\downarrow \delta^{(3)} & & & \\
 & \Omega^{q_1-q_3+1\otimes q_2-q_3+1\otimes 1\otimes 0\otimes...\otimes 0} & \xrightarrow{\delta^{(3)}} \dots \xrightarrow{\delta^{(3)}} & \Omega^{q_1-q_4\otimes q_2-q_4\otimes q_3-q_4\otimes...\otimes 0} & \\
 & \vdots & \vdots \quad \vdots \quad \vdots & \vdots & \\
 & \Omega^{q_1-q_N\otimes q_2-q_N\otimes...\otimes q_{N-1}-q_N\otimes 0} & \xleftarrow{\delta^{(N-1)}} \dots \xleftarrow{\delta^{(N-1)}} & \Omega^{q_1-q_{N-1}+1\otimes q_2-q_{N-1}+1\otimes...\otimes 1\otimes 0}. &
\end{array} \tag{4.9}$$

In the end, adding the augmented complex (4.9) to the original N-de Rham-like complex (4.8) we get the $(q_1 - q_2, ..., q_{N-1} - q_N)$-augmented N-de Rham-like complex.

For every N-de Rham complex we can define the de Rham-like cohomology groups as

Definition 4.3.7 (De Rham-like cohomology groups of a k-augmented N-de Rham-like complex)**.** [2] *Given a $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex its de Rham-like cohomology groups are*

$$H^{p_1,...,p_N} := \frac{Z^{p_1,...,p_N}}{B^{p_1,...,p_N}}, \tag{4.10}$$

where

$$\begin{aligned} Z^{p_1,...,p_N} &:= \{X \in \Omega^{p_1 \otimes ... \otimes p_N} | \delta^{(N)} X = 0\}, \\ B^{p_1,...,p_N} &:= \{X = \delta^{(N)} Y \in \Omega^{p_1 \otimes ... \otimes p_N} | Y \in \Omega^{p_1 - 1 \otimes ... \otimes p_N - 1}\}. \end{aligned} \tag{4.11}$$

if $p_1, ..., p_N > 0$ and

$$H^{p_1,...,p_i,0,...,0} := \frac{Z^{p_1,...,p_i,0,...,0}}{B^{p_1,...,p_i,0,...,0}}, \tag{4.12}$$

where

$$\begin{aligned} Z^{p_1,...,p_i,0,...,0} &:= \{X \in \Omega^{p_1 \otimes ... \otimes p_i \otimes 0 \otimes ... \otimes 0} | \delta^{(i+1)} X = 0\}, \\ B^{p_1,...,p_i,0,...,0} &:= \{X = \delta^{(i)} Y \in \Omega^{p_1 \otimes ... \otimes p_i \otimes 0 \otimes ... \otimes 0} | Y \in \Omega^{p_1 - 1 \otimes ... \otimes p_i - 1 \otimes 0 \otimes ... \otimes 0}\}. \end{aligned} \tag{4.13}$$

if only $p_1, ..., p_i > 0$

These groups can be considered, for differential manifolds, as the generalization of de Rham cohomology groups; however, to make fully meaningful this conclusion we should prove that de Rham-like cohomology groups are topological invariants[3]. In this perspective we first prove a Poincaré-like lemma for differential mixed symmetry tensors and then the main theorem about the isomorphism between de Rham-like cohomology groups and de Rham cohomology groups using abstract de Rham theorem (see Appendix D for a review).

Lemma 4.3.1 (Poincaré-like lemma)**.** *Let $U \subset \mathbb{R}^n$ a open polyinterval (product of n open intervals even unlimited of $\mathbb{R}$). For every $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex and $(p_1, ..., p_N) \neq (0, ..., 0)$ then every closed differential mixed symmetry tensor T is exact.*

Proof. Unless translations, it is not restrictive to assume that $0 \in U$. Let us proceed by induction on N. For $N = 1$ this is just the Poincaré lemma. Let us assume $N > 1$ and let us consider the case $p_1, ..., p_N > 0$ first.

[2] A proposal for more usable names is: manzonian differential, N-manzonian cochain complexes and manzonian cohomology groups. Hence we propose manzonian cohomology as a name for this new cohomology.

[3] To be more precise they are, as the de Rham cohomology groups, homotopy invariants.

Every mixed symmetry tensor $T \in \Omega^{p_1 \otimes \ldots \otimes p_N}$ can be written as

$$T = \sum_{I_1,\ldots,I_N} T_{I_1,\ldots,I_N} dx_{I_1} \otimes \ldots \otimes dx_{I_N} \qquad |I_1| = p_1, \ldots, |I_N| = p_N \tag{4.14}$$

We now define the subspace $\Omega^{p_1 \otimes \ldots \otimes p_N}_{m_1,\ldots,m_N}$ with $m_1, \ldots, m_N < n$ given by the mixed symmetry tensors (4.14) with $T_{I_1,\ldots,I_N} = 0$ if $I_i \not\subset \{1, \ldots, m_i\} \quad \forall i \in [1, N]$, hold simultaneously. We now proceed by induction on $(m_1, .., m_N)$. If $m_1 = \ldots = m_N = 0$ there are only vanishing mixed symmetry tensors and there is nothing to show. If $m_i = 0$ for some i then the I_i indexes do not appear and we are dealing with mixed symmetry tensors that belong to the $(N-1)$-multi-form space and by the induction hypothesis on N we get the result. If $m_1, \ldots, m_N > 0$ then we write T as

$$T = \sum_{I_1,\ldots,I_N} T_{I_1,\ldots,I_N} dx_{m_1} \wedge dx_{I_1} \otimes \ldots \otimes dx_{m_N} \wedge dx_{I_N} + \sum_{J_1,\ldots,J_N} \tilde{T}_{J_1,\ldots,J_N} dx_{J_1} \otimes \ldots \otimes dx_{J_N} \tag{4.15}$$

where $|I_i| = p_i - 1$, $|J_i| = p_i$ and $I_i, J_i \subset \{1, \ldots, m_i - 1\} \quad \forall i \in [1, N]$. From the condition $\delta^{(N)} T = 0$ we get that

$$\frac{\partial T_{I_1,\ldots,I_N}}{\partial x_{h_1} \ldots \partial x_{h_N}} = 0, \quad h_1 > m_1, \ldots, h_N > m_N. \tag{4.16}$$

At this point we construct the sequence of C^∞ functions (here we use that U is a polyinterval)

$$\begin{aligned} C^{(N)}_{I_1 \ldots I_N}(x_1, \ldots, x_n) &:= \int_0^{x_{m_N}} C^{(N-1)}_{I_1 \ldots I_N}(x_1, \ldots x_{m_N - 1}, t, x_{m_N + 1}, \ldots, x_n) dt, \\ C^{(1)}_{I_1 \ldots I_N}(x_1, \ldots, x_n) &:= \int_0^{x_{m_1}} T_{I_1 \ldots I_N}(x_1, \ldots, x_{m_1 - 1}, t, x_{m_1 + 1}, \ldots, x_n) dt \end{aligned} \tag{4.17}$$

such that

$$\frac{\partial C^{(N)}_{I_1,\ldots,I_N}}{\partial x_{m_1} \ldots \partial x_{m_N}} = T_{I_1,\ldots,I_N}; \qquad \frac{\partial C^{(N)}_{I_1,\ldots,I_N}}{\partial x_{h_1} \ldots \partial x_{h_N}} = 0, \quad h_1 > m_1, \ldots, h_N > m_N. \tag{4.18}$$

We then define

$$C = \sum_{I_1,\ldots,I_N} C^{(N)}_{I_1,\ldots,I_N} dx_{I_1} \otimes \ldots \otimes dx_{I_N}, \quad |I_i| = p_i - 1, \ I_i \subset \{1, \ldots, m_i - 1\} \quad \forall\, i \in [1, N]; \tag{4.19}$$

therefore

$$T - \delta^{(N)} C \in \Omega^{p_1 \otimes \ldots \otimes p_N}_{m_1 - 1, \ldots, m_N - 1} \tag{4.20}$$

since the differential of C cancels out the first addendum in (4.15). Since $\delta^{(N)}(T - \delta^{(N)} C) = 0$ by the induction hypothesis on $(m_1, ..., m_N)$ there exists a $S \in \Omega^{p_1-1\otimes...\otimes p_N-1}$ such that $\delta^{(N)}S = T - \delta^{(N)}C$ and so $T = \delta^{(N)}(S - C)$.

If only $p_1, .., p_j > 0$ and the condition is $\delta^{(j+1)}T = 0 \in \Omega^{p_1+1\otimes...\otimes p_j+1\otimes 1\otimes 0\otimes...\otimes 0}$ the same reasonings hold except that now it is enough to consider the sequence (4.17) up to its j-th element such that

$$\frac{\partial C^{(j)}_{I_1,...,I_j}}{\partial x_{m_1}...\partial x_{m_j}} = T_{I_1,...,I_j}; \qquad \frac{\partial C^{(j)}_{I_1,...,I_j}}{\partial x_{h_1}...\partial x_{h_j}} = 0, \quad h_1 > m_1, ..., h_j > m_j. \tag{4.21}$$

As before we define

$$C = \sum_{I_1,...,I_j} C^{(j)}_{I_1,...,I_j} dx_{I_1} \otimes ... \otimes dx_{I_j}, \quad |I_i| = p_i - 1, \ I_i \subset \{1, ..., m_i - 1\} \quad \forall\, i \in [1, j]; \tag{4.22}$$

therefore

$$T - \delta^{(j)}C \in \Omega^{p_1\otimes...\otimes p_j\otimes 0\otimes 0...\otimes 0}_{m_1-1,...,m_j-1} \tag{4.23}$$

Since $\delta^{(j+1)}(T - \delta^{(j)}C) = 0$ by the induction hypothesis on $(m_1, ..., m_j)$ there exists a $S \in \Omega^{p_1-1\otimes...\otimes p_j-1\otimes 0\otimes...\otimes 0}$ such that (in this case the condition $\delta^{(j+1)}(T - \delta^{(j)}C) = 0$ implies a weaker condition since some N-multi-form degrees are zero) $\delta^{(j)}S = T - \delta^{(j)}C$ and so $T = \delta^{(j)}(S - C)$.

□

We can now prove the main theorem

Theorem 4.3.1 (Isomorphism between de Rham-like cohomology groups and de Rham cohomology groups)**.** *Given a differential manifold $(M, \mathcal{A})$ and a $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex its de Rham-like cohomology groups are isomorphic to de Rham cohomology groups.*

Proof. The theorem follows as an application of abstract de Rham theorem. In fact, thanks to Poincaré and Poincaré-like lemmas, both the de Rham and every $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complexes are exact. Therefore they are all exact sequences of shaves. Moreover the atlas $\mathcal{A}$ induces a structure sheaf $\mathcal{E}$ and the elements of de Rham complex and of every $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complexes can be viewed as an $\mathcal{E}$-module, therefore they are all fine sheaves and hence acyclic. Since the underlying field is $\mathbb{R}$ we have different acyclic resolutions of the constant sheaf $\mathbb{R}_X$ and de Rham cohomology groups and de Rham-like cohomology groups of every $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complexes coincide with the Cech cohomology groups of the constant sheaf $\mathbb{R}_X$ and hence isomorphic one to the others. □

A first obvious observation that lead to this theorem is that for every fixed $(k_1, ..., k_{N-1})$-augmented N-de

Rham-like complex there are exactly D de Rham-like cohomology groups by construction, where D is the dimension of the differential manifold we are considering. Moreover, with reference to the example discussed above, the first $q_1 - q_2$ de Rham-like cohomology groups are exactly the standard de Rham cohomology groups since the first line of diagram (4.9) is the standard de Rham complex. Furthermore, Theorem 4.3.1 is motivated by the observation that, on a differential manifold with dimension D, we can construct countable infinity many N-multi-form spaces, that is, one for every $N \in \mathbb{N} \setminus \{0\}$. Any of them leads to de Rham-like cohomology groups and we end up with countable infinitely many of these groups. On the one hand, it is very unlike that only for $N = 1$, i.e. the de Rham complex, the de Rham-like cohomology groups furnishes interesting information about the topology of the manifold and, on the other hand, it is unlikely that all these countable infinity many groups give different information about the topology. Therefore, it seems quite natural that there exists an isomorphism between de Rham-like cohomology groups and de Rham cohomology groups.

4.3.2 The extension of Theorem 3.6.1 to mixed symmetry tensors

Once we have the generalization of de Rham complex for the N-multi-form space, we can extend the Theorem 3.6.1 to the mixed symmetry tensor cases using the algebraic topology interpretation of the theorem. The idea is essentially the same. Thanks to implications coming from the fact we are interested in asymptotic symmetries and the triviality of some de Rham-like cohomology groups we conclude that the de Rham-like differential is an isomorphism between the space of gauge fields and the space of their field strengths. Moreover, constructing an homotopy between specific $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complexes we can conclude the existence and uniqueness of a duality map in the case of well defined asymptotic charges computed in mixed symmetry tensor gauge theories which are duals.

Suppose we are interested in studying asymptotic symmetries in a gauge theory whose gauge field is a mixed symmetry tensor field $T \in \Omega^{q_1 \otimes ... \otimes q_N}$ and its field strength is $H := \delta^{(N)} T \in \Omega^{q_1+1 \otimes ... \otimes q_N+1}$. Therefore, let us consider the $(q_1 - q_2, ..., q_{N-1} - q_N)$-augmented N-de Rham-like complex and, in a specific way, the last terms, i.e.

$$\Omega^{q_1-q_N \otimes ... \otimes q_{N-1}-q_N \otimes 0} \xrightarrow{\delta^{(N)}} \cdots \xrightarrow{\delta^{(N)}} \Omega^{q_1 \otimes ... \otimes q_N} \xrightarrow{\delta^{(N)}} \cdots \xrightarrow{\delta^{(N)}} \Omega^{D \otimes D-(q_1-q_2) \otimes ... \otimes D-(q_1-q_N)}. \quad (4.24)$$

Requiring we are interested in asymptotic symmetries of a gauge theory whose gauge field is T and requiring the vanishing of $H^{q_1,...,q_N}$ and $H^{q_1+1,...,q_N+1}$ means, following the very same reasoning[4] of Paragraph 3.6.1.1, that the modules $\Omega^{q_1 \otimes ... \otimes q_N}$ and $\Omega^{q_1+1 \otimes ... \otimes q_N+1}$ can be replaced respectively by $\Omega_{\text{AS}}^{q_1 \otimes ... \otimes q_N} := \Omega^{q_1 \otimes ... \otimes q_N} \setminus \{Im(\delta^{(N)}) \setminus \{0\}\}$ and $\Omega_{\text{AS}}^{q_1+1 \otimes ... \otimes q_N+1} := Im(\delta^{(N)}) \setminus \{ker(\delta^{(N)}) \setminus \{0\}\}$. Moreover both $\Omega^{q_1-1 \otimes ... \otimes q_N-1}$ and

[4]That is, we have $H = 0 \Leftrightarrow T = \delta^{(N)} A$ for some $A \in \Omega^{q_1-1 \otimes ... \otimes q_N-1}$; hence we need to throw away all those elements $T \in \Omega^{q_1 \otimes ... \otimes q_N}$ such that $T = \delta^{(N)} A$ for some $A \in \Omega^{q_1-1 \otimes ... \otimes q_N-1}$. Moreover, only the zero form can have vanishing field strength but, again for exactness $B = 0 \Leftrightarrow A = \delta^{(N)} C$ for some $C \in \Omega^{q_1-2 \otimes ... \otimes q_N-2}$.

$\Omega^{q_1+2\otimes\ldots\otimes q_N+2}$ can be replaced by 0. Therefore we have the following short exact sequence

$$0 \xrightarrow{\delta^{(N)}} \Omega^{q_1\otimes\ldots\otimes q_N}_{\mathrm{AS}} \xrightarrow{\delta^{(N)}} \Omega^{q_1+1\otimes\ldots\otimes q_N+1}_{\mathrm{AS}} \xrightarrow{\delta^{(N)}} 0 \tag{4.25}$$

which teaches us that the differential $\delta^{(N)}$ between $\Omega^{q_1\otimes\ldots\otimes q_N}_{\mathrm{AS}}$ and $\Omega^{q_1+1\otimes\ldots\otimes q_N+1}_{\mathrm{AS}}$ is an isomorphism.
In order to extend Theorem 3.6.1, let us proceed with the specific case of the graviton in $D = 5$ and its dual descriptions: the Curtright three indexes field, or hooke field, and the Riemann-like field. Therefore, we need to consider two copies of the 0-augmented 2-de Rham-like complex (one for the graviton field and one for the Riemann-like field) and a copy of the 1-augmented 2-de Rham-like complex (for the Curtright hooke field). Let us suppose we have computed the asymptotic charges Q_0, Q_1, Q_2 of all descriptions; the diagram we consider to show the existence and uniqueness of maps between dual descriptions is the following

$$\begin{tikzcd}
0 & 0 & 0 \\
\Omega^{3\otimes 2}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] & \Omega^{2\otimes 2}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] \arrow[l, "\star_L"'] \arrow[r, "\star"] & \Omega^{3\otimes 3}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] \\
\Omega^{2\otimes 1}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] \arrow[dd, bend right, "\pi_1"'] & \Omega^{1\otimes 1}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] \arrow[l, "\star_L|_3"'] \arrow[r, "\star|_3"] \arrow[dd, bend left, "\pi_0"] & \Omega^{2\otimes 2}_{\mathrm{AS}} \arrow[u, "\delta^{(2)}"] \arrow[dd, bend left, "\pi_2"] \\
0 \arrow[u, "\delta^{(2)}"] & 0 \arrow[u, "\delta^{(2)}"] & 0 \arrow[u, "\delta^{(2)}"] \\
\mathbb{C}^{n_{2,1}} & \mathbb{C}^{n_{1,1}} \arrow[l, "f_1"'] \arrow[r, "f_2"] & \mathbb{C}^{n_{2,2}}
\end{tikzcd} \tag{4.26}$$

where $n_{2,1} = dim(\Omega^{2\otimes 1}_{\mathrm{AS}})$, $n_{1,1} = dim(\Omega^{1\otimes 1}_{\mathrm{AS}})$ and $n_{2,2} = dim(\Omega^{2\otimes 2}_{\mathrm{AS}})$. The first part of the proof is essentially given by the discussion around the short exact complex (4.25) since then both $\star_L|_3$ and $\star|_3$ are isomorphisms. The second part of the proof goes through constructing the maps π_0, π_1, π_2 and proving that they are isomorphisms. Both these points are essentially identical to the case of Theorem 3.6.1. Indeed the vector in the complex spaces can be constructed having the first component given by the charge while the others given by the independent components the mixed symmetry tensor in a coordinate chart. Hence to show the maps π_0, π_1, π_2 are isomorphisms we can follow, quite exactly, the steps given in the proof of Theorem 3.6.1.

In the general case we can prove the following

Theorem 4.3.2 (Existence and uniqueness of a set of duality maps for well defined charges)**.** *Let $(M, \mathcal{A})$ be a differential manifold of dimension D and let $\Omega^{n_1\otimes\ldots\otimes n_N}(M)$ be the N-multi-form space on it. Let $T \in \Omega^{p_1\otimes\ldots\otimes p_N}$ with $p_k \leq \left[\frac{D-2}{2}\right]$ $\forall k \in [1,N]$ be a mixed symmetry tensor carrying the irreducible representation associated to the Young tableau $\lambda_0 = \{p_1, ..., p_N\}$ having r dual descriptions and let $\lambda_1 = \{q_1 = D-2-p_1, p_2, ..., p_N\}, ..., \lambda_r = \{q_1 = D-2-p_1, ..., q_r = D-2-p_r, ..., q_N = D-2-p_N\}$ be the Young tableaux associated to the irreducible representation carried by the dual mixed symmetry tensors. Let $Q_0, ..., Q_r$ be the well defined charges of the original and of the dual descriptions and let be $n_{n_1,...,n_N} = dim(\Omega^{n_1\otimes\ldots\otimes n_N}_{\text{AS}})$. Then there exist a set of duality maps $\{f_1, ..., f_r\}$ such that the following diagram commute*

$$\begin{tikzcd}
{} & 0 & {} & {} \\
0 & \Omega_{\text{AS}}^{q_1+1\otimes\ldots\otimes q_i+1\otimes\ldots\otimes p_N+1} \arrow[u, "\delta^{(N)}"] & 0 & 0 \\
\Omega_{\text{AS}}^{q_1+1\otimes\ldots\otimes p_N+1} \arrow[u, "\delta^{(N)}"] & & \Omega_{\text{AS}}^{p_1+1\otimes\ldots\otimes p_N+1} \arrow[u, "\delta^{(N)}"] \arrow[ul, "\star^{(i)}"'] \arrow[ll, "\star^{(1)}"'] \arrow[r, "\star^{(r)}"] & \Omega_{\text{AS}}^{q_1+1\otimes\ldots\otimes q_r+1\otimes\ldots\otimes p_N+1} \arrow[u, "\delta^{(N)}"] \\
& \Omega_{\text{AS}}^{q_1\otimes\ldots\otimes q_i\otimes\ldots\otimes p_N} \arrow[uu, "\delta^{(N)}"] \arrow[dd, bend right, "\pi_i"'] & & \\
\Omega_{\text{AS}}^{q_1\otimes\ldots\otimes p_N} \arrow[uu, "\delta^{(N)}"] \arrow[dd, bend right, "\pi_1"'] & 0 \arrow[u, "\delta^{(N)}"] & \Omega_{\text{AS}}^{p_1\otimes\ldots\otimes p_N} \arrow[uu, "\delta^{(N)}"] \arrow[ul, "\star^{(i)}|_{D-2}"'] \arrow[ll, "\star^{(1)}|_{D-2}"'] \arrow[r, "\star^{(r)}|_{D-2}"] \arrow[dd, bend left, "\pi_0"] & \Omega_{\text{AS}}^{q_1\otimes\ldots\otimes q_r\otimes\ldots\otimes p_N} \arrow[uu, "\delta^{(N)}"] \arrow[dd, bend left, "\pi_r"] \\
0 \arrow[u, "\delta^{(N)}"] & \mathbb{C}^{n_{q_1,...,q_i,...,p_N}} & 0 \arrow[u, "\delta^{(N)}"] & 0 \arrow[u, "\delta^{(N)}"] \\
\mathbb{C}^{n_{q_1,...,p_N}} & & \mathbb{C}^{n_{p_1,...,p_N}} \arrow[ul, "f_i"'] \arrow[ll, "f_1"'] \arrow[r, "f_r"] & \mathbb{C}^{n_{q_1,...,q_r,...,p_N}}
\end{tikzcd} \tag{4.27}$$

Moreover, every $f_i \in \{f_1, ..., f_r\}$ admits a unique restriction to a 1-dimensional subspace such that

$$f_i|_{Q_i} : Q_0 \mapsto Q_i, \qquad f_i^{-1}|_{Q_i} : Q_i \mapsto Q_0. \tag{4.28}$$

Proof. First of all, from the exact short sequences we get that $\delta^{(N)}$ is an isomorphism from the space of gauge fields to the space of their field strengths. Since for every $i \in [1,r]$, $\star^{(i)}$ is an isomorphism we can define

$$\star|^{(i)}_{D-2} := (\delta^{(N)})^{-1} \circ \star^{(i)} \circ \delta^{(N)}, \tag{4.29}$$

which sends $\Omega_{\text{AS}}^{p_1 \otimes \ldots \otimes p_N}$ in a subspace of $\Omega_{\text{AS}}^{q_1 \otimes \ldots \otimes q_i \otimes \ldots \otimes p_N}$ of dimension $n_{p_1,\ldots,p_N}$ since it is an isomorphism. Let us construct the π_i maps as

$$v^{(i)} := \begin{cases} \pi_i(T_i) = 0 \quad \text{if} \quad T_i \equiv 0; \\ \pi_i(T_i) = Q_i^{(T_i)} e_1^{(i)} + \sum_{k=2}^{n_{p_1,\ldots,p_N}} b_k^{(T_i)} e_k^{(i)} \quad \text{otherwise,} \end{cases} \tag{4.30}$$

where $T_i \in \Omega_{\text{AS}}^{q_1 \otimes \ldots \otimes q_i \otimes \ldots \otimes p_N}$, $e_1^{(i)}, \ldots, e_{n_{p_1,\ldots,p_N}}^{(i)}$ is an orthonormal base of $\mathbb{C}^{n_{p_1,\ldots,p_N}}$ and $b_k^{(T_i)} \in \mathbb{R}$ are some of the independent entries of the in coordinate representation of the form chosen in such a way that different mixed symmetry tensors have different string objects. These maps are all linear bijections; therefore the f_i maps are well defined as $f_i := \pi_i \circ \star_{D-2}^{(i)} \circ \pi_0^{-1}$. Hence, $f_i \in Aut(\mathbb{C}^{n_{p_1,\ldots,p_N}}) = \text{GL}(\mathbb{C}^{n_{p_1,\ldots,p_N}}) \cong \text{GL}(n_{p_1,\ldots,p_N}, \mathbb{C})$. By a general theorem of linear algebra, every f_i can be chosen to be lower triangular; calling $\eta_i \in \mathbb{C} \setminus \{0\}$ the top left element of f_i, we have

$$\eta_i = \frac{Q_i^{(T_i)}}{Q_0^{(T_0)}}; \tag{4.31}$$

written in other way

$$Q_i^{(T_i)} = f_i|_{Q_i}(Q_0^{(T_0)}) \tag{4.32}$$

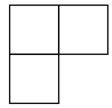
where $f_i|_{Q_i}(\bullet) = \eta_i \bullet$. □

4.4 Explicit case: the graviton and its dual descriptions in $D = 5$

In $D = 5$ the graviton field admits two dual descriptions that have been discussed in Paragraph 2.5.3 with explicit examples. Here, we focus on the Curtright hooke field which carries the irreducible representation associated to the Young tableau $\lambda = \{2, 1\}$. The study of the second dual description of the graviton in $D = 5$, i.e. the Riemann-like window field, could be approached in future work with the aim of understanding the role of the dual descriptions of gravity.

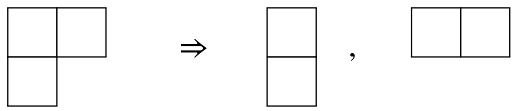
4.4.1 Generalities of the Curtright three indexes field

In order to describe gauge fields, the standard way is to use both totally symmetric and totally antisymmetric Lorentz tensors of arbitrary rank. Both types of gauge field tensors appear quite naturally in theoretical analyses of some interesting physical problems. However, in 1980 Thomas Curtright generalize the concept of a gauge field to include higher rank Lorentz tensors which are neither totally symmetric nor totally antisymmetric under spacetime index permutations [59],[60],[71]. The simplest example is the hooke field: a rank three tensor whose index permutation symmetry corresponds to the Young tableau $\lambda = \{2, 1\}$, i.e.



; (4.33)

we call this $\phi_{[\rho\sigma]\nu}$. From the Young tableau of the Curtright three indexes field $\phi_{[\rho\sigma]\nu}$ we can extract the gauge parameters removing one box so that we still have a Young tableau



; (4.34)

so we have one totally symmetric gauge parameter $\lambda_{(\mu\nu)}$ and one totally antisymmetric gauge parameter $\Lambda_{[\mu\nu]}$; the gauge for gauge parameter is then given by a vector field Θ_μ.
The gauge transformation for the Curtright three indexes field has to take in to account the irreducibility condition under GL$(D-1, 1)$, namely

$$\phi_{[\rho\sigma]\nu} + \phi_{[\nu\rho]\sigma} + \phi_{[\sigma\nu]\rho} = 0; \tag{4.35}$$

therefore we have

$$\delta_{gauge}(\phi_{[\rho\sigma]\nu}) = \partial_\rho \lambda_{(\sigma\nu)} - \partial_\sigma \lambda_{(\rho\nu)} + \partial_\rho \Lambda_{[\sigma\nu]} - \partial_\sigma \Lambda_{[\rho\nu]} + 2\partial_\nu \Lambda_{[\sigma\rho]}. \tag{4.36}$$

accompanied by the gauge for gauge transformations

$$\begin{aligned} \delta_{gauge}(\Lambda_{[\mu\nu]}) &= \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu, \\ \delta_{gauge}(\lambda_{(\mu\nu)}) &= 2\partial_\mu \Theta_\nu + 2\partial_\nu \Theta_\mu, \end{aligned} \tag{4.37}$$

In the following of the Section, we adopt the notations

$$\begin{aligned} \hat{\phi}^\nu &:= \phi_\alpha^{\ \alpha\nu} = \eta_{\alpha\beta}\phi^{[\alpha\beta]\nu}, \qquad \bar{\phi}^\rho := \phi_\alpha^{\ \rho\alpha} = \eta_{\alpha\beta}\phi^{[\alpha\rho]\beta}; \\ \hat{\partial}\cdot\phi^{\rho\nu} &:= \partial_\alpha \phi^{\alpha\rho\nu} = \eta_{\alpha\beta}\partial^\alpha \phi^{[\beta\rho]\nu}, \qquad \bar{\partial}\cdot\phi^{\rho\sigma} := \partial_\alpha \phi^{\rho\sigma\alpha} = \eta_{\alpha\beta}\partial^\alpha \phi^{[\rho\sigma]\beta}. \end{aligned} \tag{4.38}$$

The gauge invariant dynamics can be described by the Labastida-Fronsdal lagrangian density [83]

$$\mathcal{L} = \frac{1}{2}\phi_{[\rho\sigma]\nu}\left(F^{[\rho\sigma]\nu} - \frac{1}{2}\eta^{\rho\nu}\bar{F}^\sigma + \frac{1}{2}\eta^{\sigma\nu}\bar{F}^\rho\right) =: \frac{1}{2}\phi_{[\rho\sigma]\nu}G^{[\rho\sigma]\nu}, \tag{4.39}$$

where the Ricci-like tensor, or Labastida-Fronsdal tensor is given by

$$F^{[\rho\sigma]\nu} := \Box\phi^{[\rho\sigma]\nu} - \partial^\rho \hat{\partial}\cdot\phi^{\sigma\nu} + \partial^\sigma \hat{\partial}\cdot\phi^{\rho\nu} - \partial^\nu \bar{\partial}\cdot\phi^{\rho\sigma} + \partial^\rho\partial^\nu \bar{\phi}^\sigma - \partial^\sigma\partial^\nu \bar{\phi}^\rho; \tag{4.40}$$

while the traces are

$$\begin{aligned}\frac{1}{2}\eta^{\rho\nu}\bar{F}^{\sigma} &= \frac{1}{2}\eta^{\rho\nu}(2\Box\bar{\phi}^{\sigma} - 2\bar{\partial}\cdot\hat{\partial}\cdot\phi^{\sigma} + \bar{\partial}\cdot\partial^{\sigma}\hat{\phi});\\ \frac{1}{2}\eta^{\sigma\nu}\bar{F}^{\rho} &= \frac{1}{2}\eta^{\sigma\nu}(2\Box\bar{\phi}^{\rho} - 2\bar{\partial}\cdot\hat{\partial}\cdot\phi^{\rho} + \bar{\partial}\cdot\partial^{\rho}\hat{\phi}).\end{aligned} \tag{4.41}$$

From lagrangian density (4.39) we can derive the lagrangian equation of motion expressed using the Einstein-like tensor

$$G^{[\rho\sigma]\nu} := F^{[\rho\sigma]\nu} - \frac{1}{2}\eta^{\rho\nu}\bar{F}^{\sigma} + \frac{1}{2}\eta^{\sigma\nu}\bar{F}^{\rho} = 0; \tag{4.42}$$

Moreover, by construction the Einstein-like tensor is gauge invariant and satisfies Bianchi identities

$$\partial_{\rho}G^{[\rho\sigma]\nu} = \partial_{\sigma}G^{[\rho\sigma]\nu} = \partial_{\nu}G^{[\rho\sigma]\nu} = 0. \tag{4.43}$$

However, as in the gravity case in vacuum, the lagrangian equations of motion are equivalent to

$$F^{[\rho\sigma]\nu} = 0. \tag{4.44}$$

The Curtright three indexes field, has another presentation[5] which is more manageable to our goals due directly to Curtright. We define a field strength[6]

$$H_{[\alpha\beta\gamma]\mu} := \partial_{\alpha}\phi_{[\beta\gamma]\mu} + \partial_{\beta}\phi_{[\gamma\alpha]\nu} + \partial_{\gamma}\phi_{[\alpha\beta]\mu} \tag{4.45}$$

which satisfies

$$\delta_{\lambda,\Lambda}H_{[\alpha\beta\gamma]\mu} = -2\partial_{\nu}(\partial_{\alpha}\Lambda_{[\beta\gamma]} + \partial_{\beta}\Lambda_{[\gamma\alpha]} + \partial_{\gamma}\Lambda_{[\alpha\beta]}). \tag{4.46}$$

One can note that the combinations $H_{[\alpha\beta\gamma]\mu}H^{[\alpha\beta\gamma]\mu}$ and $H_{[\alpha\beta]}H^{[\alpha\beta]}$, where $H_{[\alpha\beta]} := \eta^{\gamma\mu}H_{[\alpha\beta\gamma]\mu}$, can be combined into a gauge invariant lagrangian density

$$\mathcal{L} = -\frac{1}{6}(H_{[\alpha\beta\gamma]\mu}H^{[\alpha\beta\gamma]\mu} - 3H_{[\alpha\beta]}H^{[\alpha\beta]}); \tag{4.47}$$

which is lagrangian density (2.255) with $D = 5$. The Euler-Lagrange equations obtained by varying the lagrangian density are

$$E_{[\alpha\beta]\gamma} + \frac{1}{2}(\eta_{\alpha\gamma}E_{\beta} - \eta_{\beta\gamma}E_{\alpha}) = 0, \tag{4.48}$$

where

$$E_{[\alpha\beta]\gamma} := \eta^{\mu\nu}\partial_{\mu}H_{[\nu\alpha\beta]\gamma} - \eta^{\mu\nu}\partial_{\gamma}H_{[\alpha\beta\mu]\nu}, \qquad E_{\alpha} = \eta^{\mu\nu}E_{[\alpha\mu]\nu} = 2\eta^{\mu\nu}\partial_{\nu}H_{[\mu\alpha]}. \tag{4.49}$$

[5]For a review on the usefulness of an action principle in computing surface charges see [149].

[6]Note, however, that this object is not really a field strength but it is a left field strength for the hooke field.

As before, these equations are equivalent to

$$E_{[\alpha\beta]\gamma} = \Box\phi_{[\alpha\beta]\gamma} + \partial_\gamma\partial_{[\alpha}\bar{\Phi}_{\beta]} + \partial_{[\beta}\hat{\partial}\cdot\phi_{\alpha]\gamma} - \partial_\gamma\bar{\partial}\cdot\phi_{[\alpha\beta]} = 0, \tag{4.50}$$

where we used definition analogue to (4.38). This presentation in equivalent, on-shell, to the Labastida-Fronsdal one but their lagrangian densities may differ in boundary terms that are crucial in determining the asymptotic symmetries. Here we will conduct the analysis in the Curtrigh presentation but it would be interesting to proceed also in the other presentation. The expectation is that different presentations of the Curtrigh field that differ only in boundary terms can be dual to different presentations of linearized gravity that differ in boundary terms.

In order to discuss asymptotic symmetries at future null infinity in $D = 5$, we introduce Bondi coordinates $(u, r, \{x^i\})$ where $u := t - r$ and $\{x^i\}$ is a set of 3 angular variables parameterizing the null infinity sphere S^3_u. Minkowski line element in these coordinates appears as

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{ij}dx^i dx^j \qquad i, j = 1, 2, 3; \tag{4.51}$$

metric and non vanishing Christoffel symbols are given by

$$g_{\mu\nu} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\gamma_{ij} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2}\gamma^{-1}_{ij} \end{bmatrix}; \tag{4.52}$$

$$\Gamma^i_{jr} = \Gamma^i_{rj} = \frac{1}{r}\delta^i_j, \quad \Gamma^u_{ij} = -\Gamma^r_{ij} = r\gamma_{ij}, \quad \Gamma^k_{ij} = \frac{1}{2}\gamma^{kl}[-\partial_l\gamma_{ij} + \partial_j\gamma_{li} + \partial_i\gamma_{jl}]. \tag{4.53}$$

The equations of motion are the same as before but now with covariant derivative instead of ordinary ones. In Bondi coordinates the gauge transformation and gauge for gauge transformation are expressed in terms of covariant derivatives as

$$\begin{aligned} &\delta_{gauge}(\phi_{[\rho\sigma]\nu}) = \nabla_\rho\lambda_{(\sigma\nu)} - \nabla_\sigma\lambda_{(\rho\nu)} + \nabla_\rho\Lambda_{[\sigma\nu]} - \nabla_\sigma\Lambda_{[\rho\nu]} + 2\nabla_\nu\Lambda_{[\sigma\rho]}; \\ &\delta_{gauge}(\Lambda_{[\mu\nu]}) = \nabla_\mu\Theta_\nu - \nabla_\nu\Theta_\mu; \\ &\delta_{gauge}(\lambda_{(\mu\nu)}) = 2\nabla_\mu\Theta_\nu + 2\nabla_\nu\Theta_\mu. \end{aligned} \tag{4.54}$$

4.4.2 The Lorenz-like gauge fixing

Let us use the Curtright presentation with the left field strength instead of the Labastida-Fronsdal tensor. In order to reduce the equations of motion (4.50) to a massless wave equation we have to require that

$$\mathcal{D}_{\beta\alpha} = \hat{\partial}\cdot\phi_{\beta\alpha} - \frac{1}{2}\partial_\alpha\bar{\Phi}_\beta = 0, \tag{4.55}$$

which is de Donder-like, or Lorenz-like. Indeed, calling $E^{(\not\Box)}_{[\alpha\beta]\gamma}$ the non-Box term in the equations of motion, we have

$$
\begin{aligned}
E^{(\not\Box)}_{[\alpha\beta]\gamma} &= \partial_\gamma(\partial_\alpha\bar{\Phi}_\beta - \partial_\beta\bar{\Phi}_\alpha) + \partial_\beta\hat{\partial}\cdot\phi_{\alpha\gamma} - \partial_\alpha\hat{\partial}\cdot\phi_{\beta\gamma} - \partial_\gamma\bar{\partial}\cdot\phi_{[\alpha\beta]} = \\
&= \partial_\gamma(\partial_\alpha\bar{\Phi}_\beta - \partial_\beta\bar{\Phi}_\alpha) + \frac{1}{2}\partial_\beta\partial_\gamma\bar{\Phi}_\alpha - \frac{1}{2}\partial_\alpha\partial_\gamma\bar{\Phi}_\beta - \partial_\gamma\bar{\partial}\cdot\phi_{[\alpha\beta]} = \\
&= \frac{1}{2}\partial_\alpha\partial_\gamma\bar{\Phi}_\beta - \frac{1}{2}\partial_\beta\partial_\gamma\bar{\Phi}_\alpha - \partial_\gamma\hat{\partial}\cdot(\phi_{\beta\alpha} - \phi_{\alpha\beta}) = 0,
\end{aligned}
\tag{4.56}
$$

where in the last line we used the irriducibility condition. More generally, the equations of motion (4.50) can be written as

$$
\Box\phi_{[\alpha\beta]\gamma} + \partial_\gamma(\mathcal{D}_{\alpha\beta} - \mathcal{D}_{\beta\alpha}) + \partial_\beta\mathcal{D}_{\alpha\gamma} - \partial_\alpha\mathcal{D}_{\beta\gamma} = 0. \tag{4.57}
$$

The de Donder-like gauge fixing leaves the possibility of further gauge fixing[7] such that

$$
\Box(\lambda_{(\beta\alpha)} + \Lambda_{[\beta\alpha]}) - \partial_\beta\partial\cdot(\lambda_\alpha + \Lambda_\alpha) - 3\partial_\alpha\partial\cdot\Lambda_\beta - \frac{1}{2}\partial_\alpha\partial\cdot(\lambda_\beta + \Lambda_\beta) + \frac{1}{2}\partial_\alpha\partial_\beta\lambda^\mu_\mu = 0, \tag{4.58}
$$

where $\lambda^\mu_\mu := \eta^{\mu\nu}\lambda_{\mu\nu}$ is the trace of the symmetric gauge parameter while the trace of $\Lambda_{[\mu\nu]}$ is identically vanishing thanks to its totally antisymmetry.

4.4.3 Asymptotic charges

In order to compute asymptotic charges associated to residual gauge transformations, the first step is to recall the Noether second theorem [4],[217]: classically, the charges arising from local symmetries vanish identically since these are redundancies of the theory and act trivially of physical states. However, charges can attain non-trivial values if associated to large gauge transformations. Indeed, for gauge symmetries identified by the set of gauge parameters $\mathcal{S}$, the conserved Noether current $j^\mu_{\mathcal{S}}$ is given by the divergence of an antisymmetric tensor, $j^\mu_{\mathcal{S}} = \partial_\nu k^{\mu\nu}_{\mathcal{S}}$ with $k^{\mu\nu}_{\mathcal{S}} = -k^{\nu\mu}_{\mathcal{S}}$. This property is usually referred to local Gauss law since it closely resembles the case of the Maxwell equations for electrodynamics. Any of such a Noether current is conserved as a consequence of the antisymmetry of $k^{\mu\nu}_{\mathcal{S}}$; the corresponding charges, obtained by integrating $j^\mu_{\mathcal{S}}$ on a space like hypersurface Σ, are in fact given by integrals over the boundary $\sigma = \partial\Sigma$ by Stokes' theorem

$$
Q_{\mathcal{S}} := \int_\Sigma j^\mu_{\mathcal{S}} d\Sigma_\mu \approx \oint_{\partial\Sigma} k^{\mu\nu}_{\mathcal{S}} d\sigma_{\mu\nu}. \tag{4.59}
$$

These kind of charges are therefore intrinsically related to the behavior of the theory near the boundary we are evaluating the integral. It is clear that $Q_{\mathcal{S}}$ is zero if $\mathcal{S}$ is a set of gauge parameters localized in a compact region. In the case of our interest we integrate on the S^3_u, located at null infinity, for fixed u and $r \to +\infty$:

$$
Q[S^3_u] := \lim_{r\to+\infty} \oint_{S^3_u} k^{ur}_{\lambda,\Lambda} r^3 d\Omega. \tag{4.60}
$$

[7] In analogy with the Lorenz-like gauge fixing of p-form gauge theories.

where $k^{\mu\nu}_{\lambda,\Lambda}$ is the Noether two form. To compute the asymptotic charges we need to compute the charges of the residual gauge, hence we have to find the two form $k^{\mu\nu}_{\lambda,\Lambda}$; for this scope let us start computing the Noether current. We perform the computation for flat Minkowski spacetime without tidal effect do to the Boindi coordinates. In fact, since we will be dealing with tensor equations, the results obtained will be valid for any Lorenzian manifold by replacing ordinary derivatives with covariant derivatives and the components of the Minkowski metric with those of the generic metric.

The variation of the lagrangian density (4.47) is

$$\delta\mathcal{L} = -\frac{1}{3}(\delta(H_{[\alpha\beta\gamma]\mu})H^{[\alpha\beta\gamma]\mu} - 3\delta(H_{[\alpha\beta]})H^{[\alpha\beta]}), \tag{4.61}$$

where

$$\begin{aligned}\delta(H_{[\alpha\beta\gamma]\mu}) &= \partial_\alpha\delta\phi_{[\beta\gamma]\mu} + \partial_\beta\delta\phi_{[\gamma\alpha]\nu} + \partial_\gamma\delta\phi_{[\alpha\beta]\mu} = \\ &= -3\partial_\mu(\partial_\alpha\Lambda_{[\beta\gamma]} + \partial_\beta\Lambda_{[\gamma\alpha]} + \partial_\gamma\Lambda_{[\alpha\beta]}) - \partial_\alpha\partial_\gamma\Lambda_{[\beta\mu]} - \partial_\alpha\partial_\beta\Lambda_{[\gamma\mu]} - \partial_\beta\partial_\gamma\Lambda_{[\alpha\mu]};\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}\delta(H_{[\alpha\beta]}) &= \eta^{\gamma\mu}\delta(H_{[\alpha\beta\gamma]\mu}) = \\ &= -3\partial^\gamma(\partial_\alpha\Lambda_{[\beta\gamma]} + \partial_\beta\Lambda_{[\gamma\alpha]} + \partial_\gamma\Lambda_{[\alpha\beta]}) - \partial^\mu[\partial_\alpha\Lambda_{[\beta\mu]} + \partial_\beta\Lambda_{[\alpha\mu]}].\end{aligned} \tag{4.63}$$

Inserting the above definitions in the variation of the lagrangian density, relabeling and integrating by parts we get

$$\delta\mathcal{L} \approx -\frac{1}{3}\partial_\alpha\left[\delta\phi_{[\gamma\beta]\mu}\big(- H^{[\alpha\beta\gamma]\mu} + 3\eta^{\gamma\mu}H^{[\alpha\beta]} + H^{[\beta\alpha\gamma]\mu} - 3\eta^{\gamma\mu}H^{[\beta\alpha]} + H^{[\gamma\beta\alpha]\mu} - 3\eta^{\alpha\mu}H^{[\gamma\beta]}\big)\right] \tag{4.64}$$

where the non-boundary term $\delta\mathcal{L}_{\text{NB}}$ is zero by the equations of motion, indeed

$$\delta\mathcal{L}_{\text{NB}} = \frac{1}{3}\delta\phi_{[\beta\gamma]\mu}\bigg(- 3\partial_\alpha H^{[\alpha\beta\gamma]\mu} + 6\eta^{\gamma\mu}\partial_\alpha H^{[\alpha\beta]} + 3\partial^\mu H^{[\beta\gamma]}\bigg) \approx 0 \tag{4.65}$$

since

$$\partial_\alpha H^{[\alpha\beta\gamma]\mu} - \partial^\mu H^{\beta\gamma} = 0, \qquad 2\partial_\alpha H^{[\alpha\beta]} = 0. \tag{4.66}$$

Therefore, the Noether current is

$$j^\alpha \approx \delta\phi_{[ba]c}\big(H^{[\alpha ab]c} - 2\eta^{bc}H^{[\alpha a]} + \eta^{\alpha c}H^{[ba]}\big) = \delta\phi_{[ab]c}\big(- H^{[\alpha ab]c} + 2\eta^{bc}H^{[\alpha a]} - \eta^{\alpha c}H^{[ba]}\big). \tag{4.67}$$

where we used the antisymmetry properties. Before moving on, let us show the following identities

Lemma 4.1 (Off-shell and on-shell identities of $H_{[\alpha ab]c}$)**.** *The left field strength $H_{[\alpha ab]c}$ satisfies the off-shell* *identity*

$$H_{[\alpha ab]c} - H_{[c\alpha a]b} + H_{[bc\alpha]a} - H_{[abc]\alpha} = 0; \tag{4.68}$$

and the on-shell identity

$$\partial^c (H_{[\alpha ab]c} - \eta_{bc} H_{[\alpha a]} + \eta_{ac} H_{[\alpha b]} + \eta_{\alpha c} H_{[ba]}) \approx 0. \tag{4.69}$$

Proof. The left field strength is given by

$$H_{[abc]d} = \partial_a \phi_{[bc]d} + \partial_b \phi_{[ca]d} + \partial_c \phi_{[ab]d}; \tag{4.70}$$

using the irriducibility condition of the fields $\phi_{[bc]d}, \phi_{[ca]d}, \phi_{[ab]d}$, namely

$$\phi_{[bc]d} + \phi_{[db]c} + \phi_{[cd]b} = 0, \qquad \phi_{[ca]d} + \phi_{[dc]a} + \phi_{[ad]c} = 0, \qquad \phi_{[ab]d} + \phi_{[bd]a} + \phi_{[da]b} = 0, \tag{4.71}$$

we can rewrite (4.70) as

$$H_{[abc]d} = \partial_a (-\phi_{[db]c} - \phi_{[cd]b}) + \partial_b (-\phi_{[dc]a} - \phi_{[ad]c}) + \partial_c (-\phi_{[bd]a} - \phi_{[da]b}). \tag{4.72}$$

Using the definition of $H_{[adb]c}$ and $H_{[acd]b}$ we have

$$H_{[abc]d} = -H_{[adb]c} - H_{[acd]b} + \partial_d (\phi_{[ba]c} + \phi_{[ac]b}) - \partial_b \phi_{[dc]a} - \partial_c \phi_{[bd]a}; \tag{4.73}$$

thanks to the irriducibility condition of $\phi_{[ba]c}$ and the definition of $H_{[dcb]a}$ we get

$$H_{[abc]d} = -H_{[adb]c} - H_{[acd]b} - H_{[dcb]a} = H_{[dab]c} - H_{[cda]b} + H_{[bcd]a}. \tag{4.74}$$

Relabelling $(a, b, c, d) \mapsto (\alpha, a, b, c)$ we get (4.68). Let us consider now the following expression

$$\partial^c H_{[\alpha ab]c} - \partial^c \eta_{bc} H_{[\alpha a]} + \partial^c \eta_{ac} H_{[\alpha b]} + \partial^c \eta_{\alpha c} H_{[ba]}, \tag{4.75}$$

using (4.68) we rewrite it as

$$\partial^c H_{[c\alpha a]b} - \partial^c H_{[bc\alpha]a} + \partial^c H_{[abc]\alpha} - \partial_b H_{[\alpha a]} + \partial_a H_{[\alpha b]} + \partial_\alpha H_{[ba]}, \tag{4.76}$$

which is vanishing using equations of motion (4.50). Therefore

$$\partial^c(H_{[c\alpha a]b} - H_{[bc\alpha]a} + H_{[abc]\alpha} - \eta_{bc}H_{[\alpha a]} + \eta_{ac}H_{[\alpha b]} + \eta_{\alpha c}H_{[ba]}) \approx 0 \tag{4.77}$$

and we get (4.69).

□

We are now ready to derive the Noether two form. Inserting the gauge variation in the Noether current we get

$$j^\alpha = (\partial_a\lambda_{(bc)} - \partial_b\lambda_{(ac)} + \partial_a\Lambda_{[bc]} - \partial_b\Lambda_{[ac]} + 2\partial_c\Lambda_{[ba]})\big[-H^{[\alpha ab]c} + 2\eta^{bc}H^{[\alpha a]} - \eta^{\alpha c}H^{[ba]}\big]; \tag{4.78}$$

integrating by parts, the boundary term is given by

$$\begin{aligned} j^\alpha \approx &+\partial_a\{(\lambda_{(bc)} + \Lambda_{[bc]})[-2H^{[\alpha ab]c} + 2\eta^{bc}H^{[\alpha a]} + 2\eta^{\alpha c}H^{[ab]} - 2\eta^{ac}H^{[\alpha b]}]\}+ \\ &+\partial_c[\Lambda_{[ba]}(-2H^{[\alpha ab]c} + 2\eta^{bc}H^{[\alpha a]} - 2\eta^{ac}H^{[\alpha b]} + 2\eta^{\alpha c}H^{[ab]})]; \end{aligned} \tag{4.79}$$

The non-boundary term is instead given by

$$\begin{aligned} j^\alpha_{\mathrm{NB}} = &+(\lambda_{(bc)} + \Lambda_{[bc]})[\partial_a H^{[\alpha ab]c} - 2\partial_a\eta^{bc}H^{[\alpha a]} + \partial_a\eta^{\alpha c}H^{[ba]}]+ \\ &+(\lambda_{(ac)} + \Lambda_{[ac]})[-\partial_b H^{[\alpha ab]c} + 2\partial_b\eta^{bc}H^{[\alpha a]} - \partial_b\eta^{\alpha c}H^{[ba]}]+ \\ &+2\Lambda_{[bc]}[\partial_c H^{[\alpha ab]c} - 2\partial_c\eta^{bc}H^{[\alpha a]} + \partial_c\eta^{\alpha c}H^{[ba]}]; \end{aligned} \tag{4.80}$$

using the equations of motion for the trace of the left field strength we can rewrite it as

$$\begin{aligned} j^\alpha_{\mathrm{NB}} := j^\alpha_{\mathrm{NB1}} + j^\alpha_{\mathrm{NB2}} \simeq &+(\lambda_{(bc)} + \Lambda_{[bc]})[\partial_a H^{[\alpha ab]c}] + (\lambda_{(ac)} + \Lambda_{[ac]})[-\partial_b H^{[\alpha ab]c} + 2\partial_b\eta^{bc}H^{[\alpha a]}]+ \\ &+2\Lambda_{[ba]}(\partial_c H^{[\alpha ab]c} - 2\partial_c\eta^{bc}H^{[\alpha a]} + \partial_c\eta^{\alpha c}H^{[ba]}). \end{aligned} \tag{4.81}$$

The first line term is

$$j^\alpha_{\mathrm{NB1}} \simeq 2(\lambda_{(ac)} + \Lambda_{[ac]})[\partial_b H^{[\alpha ba]c} + \partial^c H^{[\alpha a]}] \approx 0 \tag{4.82}$$

due to the equations of motion. The second line term is

$$j^\alpha_{\mathrm{NB2}} \simeq 2\Lambda_{[bc]}[\partial_c H^{[\alpha ab]c} - \partial_c\eta^{bc}H^{[\alpha a]} + \partial_c\eta^{ac}H^{[\alpha b]} + \partial_c\eta^{\alpha c}H^{[ba]}] \approx 0. \tag{4.83}$$

where we used identities of Lemma (4.1). In the end, making the appropriate substitutions, the Noether current con be written with the inverse Bondi metric components $g^{\mu\nu}$ and its Levi-civita connection covariant derivative

∇_μ. The Noether current satisfy $\nabla_\alpha j^\alpha \approx 0$ and at this point we can compute the asymptotic charge with

$$k^{\alpha a}_{\lambda,\Lambda} := k^{\alpha a}_{\lambda} + k^{\alpha a}_{\Lambda} \tag{4.84}$$

where

$$\begin{aligned} k^{\alpha a}_{\lambda} &:= \lambda_{bc}[-2H^{\alpha abc} + 2g^{bc}H^{\alpha a} + 2g^{\alpha c}H^{ab} - 2g^{ac}H^{\alpha b}]; \\ k^{\alpha a}_{\Lambda} &:= \Lambda_{bc}[-2H^{\alpha abc} + 2g^{bc}H^{\alpha a} + 2g^{\alpha c}H^{ab} - 2g^{ac}H^{\alpha b} - 2H^{\alpha cba} + 2g^{ba}H^{\alpha c} - 2g^{ca}H^{\alpha b} + 2g^{\alpha a}H^{cb}]. \end{aligned} \tag{4.85}$$

The component we are interested in is $k^{ur}_{\lambda,\Lambda}$, hence we need

$$k^{ur}_{\lambda} = 2\lambda^{\mu}_{\mu}H^{ur} + 4\lambda_{ur}H^{ur} - 2\lambda_{ij}H^{urij} - 2\lambda_{rr}H^{ur} - 2\lambda_{ri}[H^{urir} + H^{ri} + H^{ui}] - 2\lambda_{ui}[H^{uriu} - H^{ui}], \tag{4.86}$$

and

$$k^{ur}_{\Lambda} = 2\Lambda_{ri}[3H^{ui} + 3H^{ri} - H^{urir}] + 2\Lambda_{ij}[H^{ij} - H^{urij}] + 2\Lambda_{ui}[H^{uriu} - H^{ui}]. \tag{4.87}$$

At the time of writing this PhD thesis, what was written in this Paragraph is what we investigated in the direction of the links between asymptotic symmetries and duality in the case of the graviton in $D = 5$ up to now. The actual calculation of the asymptotic charge, the check that the gauge parameters appearing in the asymptotic charge are not zero due to the gauge preservation equations[8], together with the possibility of polyhomogeneous expansions and the link with the asymptotic charges of the graviton are the next agenda items[9].

4.5 Conclusions and outlook

In this Chapter, based on works [325] and [334], we discussed the asymptotic symmetries and the Young machinery duality in the realm of mixed symmetry tensor gauge theories where the gauge field is a mixed symmetry tensor T. In the first half of the Chapter, we discuss how to extend Theorem 3.6.1 to mixed symmetry tensors case. In Paragraph 4.3.1, we discuss and develop a generalization of the de Rham complex to the case of mixed symmetry tensors. This leads to the Definition 4.3.6 of $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex and to Definition 4.3.7 of de Rham-like cohomology groups which characterize the topology of the differential manifold under consideration thanks to Theorem 4.3.1. These abstract mathematical tools are employed, in Paragraph 4.3.2, to prove Theorem 4.3.2 on the existence and uniqueness of a set of duality maps for well defined charges. This theorem is the direct generalization of Theorem 3.6.1 and, once we have at our disposal the $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complexes, can be proven using the same ideas of Theorem 3.6.1. The physical meaning is that given a description and its asymptotic charge, this charge has access to physical information related to asymptotic charges of the dual formulations. This means that the symmetries of a gauge theory are intimately related to the symmetries of the dual formulations, and under

[8]As we saw this happens in p-form gauge theories.

[9]We should also repeat the same analysis for the case of the second dual description, i.e. the Riemann-like window tensor, of the graviton.

suitable topological conditions there exists a unique way to associate the charges of these symmetries with each other. Therefore, to understand the physics and the physical information associated with a particular gauge theory it is essential to know the mathematical properties of the space on which the theory is built and not only the content in fields and the properties of those fields. The property of being able to uniquely map the asymptotic charges of the dual descriptions of a specific formulation of a mixed symmetry tensor gauge theory could allow a mathematical classification of the spaces on which these theories are formulated. In this sense, it would be essential to construct particular cohomology classes that, when vanishing, make possible the existence and uniqueness of the duality map and, possibly, a vice versa.

In the second half of the Chapter we have discussed the specific case of the graviton and its dual descriptions. Specifically, the calculation of the Noether current for the Curtright hooke field in $D = 5$ was discussed in Paragraph 4.4.3. In this direction, the results of Lemma 4.1 are fundamental, since gives us a off-shell and a on-shell identity for the left field strength of the Curtright hooke field. The next point in the agenda is to perform the asymptotic analysis and compute the asymptotic charge. Moreover, having the asymptotic charge explicit, we could, using the Hodge duality equations, search for magnetic charges as done in the case of p-forms.

Chapter 5

Axialgravisolitons at infinite corners

5.1 Summary with results

This Chapter is based on the work [329] where the gravitational solitons with axial symmetry are studied. Gravitational solitons (gravisolitons) are particular exact solutions of Einstein field equations in vacuum built on a given background solution. Their interpretation in classical and quantum gravity is not yet fully clear but they contain many of the physically relevant solutions as low N-gravisolitons solutions, such as black holes and cosmological ones. However, a systematic study and characterization of gravisolitons solution for every N is lacking and their relevance in a theory of quantum gravity is not fully understood. The goal is to investigate and characterize some properties of N-axialgravisolitons solutions such as their asymptotically behaviour and asymptotic symmetries given minimal assumptions on the background metric. We develop an explicit systematic asymptotically expansion for the N-axialgravisolitons solution and we compute the leading order of the asymptotic Killing vectors and their algebra at the infinite distance corner. Moreover, to better understand the role of gravisolitons in quantum gravity we make a link, and one of the first explicit test, to the corner symmetry proposal deriving which subalgebra of the universal corner symmetry algebra is generated by the asymptotic Killing vectors of N-axialgravisolitons solution. We call this subalgebra the axialgravisoliton corner symmetry algebra ($\mathfrak{agcsa}$); the existence of this algebra opens some new possible scenarios. First, in the spirit of the corner proposal and taking into account that axialgravisolitons are exact solutions that do not have to preserve the possible asymptotic flatness of the background solution, $\mathfrak{agcsa}$ can be useful for the quantization of the non-asymptotically flat and non-perturbative sector of pure gravity through the study of its representations. Second, in the spirit of IR triangle, where there are asymptotic symmetries, two other physical phenomena emerge; the observable effects associated to the axialgravisolitons asymptotic symmetries should be an axialgravisolitons memory effect while the particle physics analogue should be a soft theorem on axialgravisolitons background. Third, in the spirit of holography, since $\mathfrak{agcsa}$ are the isometries of the "boundary" of the axialgravisoliton spacetime, we could construct a $\mathfrak{agcsa}$-QFT on this "boundary" which should be dual to some process in the axialgravisoliton spacetime: this would be a new holographic correspondance which can be called N-axialgravisolitons/$\mathfrak{agcsa}$-QFT correspondence.

5.2 Introduction

Solitons were first observed in 1834 by the Scottish engineer John Scott Russell in the Union Canal. After the discovery of soliton solutions in hydrodynamic equations, such as KdV equation, the study of these solutions led to the development of new powerful techniques with the aim of having systematic procedures for the construction of soliton solutions. One of these techniques is the spectral transform, or in a better way, the Inverse Scattering Method (ISM) that was introduced in 1967 by Gardner, Greene, Miura and Kruskal [32] to solve the problem at the initial values of the KdV and then extended to other situations of interest, for example as did by Zakharov and Shabat in 1972 [37] with the non-linear Schrödinger equation. The extension of the spectral transform in the realm of Einstein gravity happened in 1978 by Belinskii, Zakharov and Maison [51],[54] and two years later Alekseev proposed the extension to the Maxwell-Einstein theory [57].

Spectral transform method is essentially based on the possibility of writing the non-linear equation under consideration as a condition of integrability of an associated matrix system of linear differential equations, the so-called Lax pair of the problem. Once the appropriate Lax pair has been determined, the problem is divided into two parts: the direct problem and the inverse problem. Let us highlight the most important steps:

1. find the Lax pair, that is two linear operators L and M such that $Lv = \lambda v$ and $\partial_t v = Mv$. It is extremely important that the eigenvalue λ be independent of time (isospectrality). Necessary and sufficient conditions for this to occur is given by the so-called Lax equation, $\partial_t L + LM - ML = 0$. After finding the appropriate Lax pair it should be the case that Lax equation recovers the original non-linear PDE;
2. determine the time evolution of the eigenfunctions associated to each eigenvalue λ, the normalization constants and the reflection coefficient; these form the so-called scattering data. This time evolution is given by a system of linear ordinary differential equations and this step is called the direct problem;
3. solve the Gelfand–Levitan–Marchenko (GLM) integral equation, a linear integral equation, to obtain the final solution of the original non-linear PDE. All the scattering data is required in order to do this. If the reflection coefficient is zero, the process becomes much easier. This step is called the inverse problem.

The spectral transform method can be thought as a non-linear analogue, and in some sense generalization, of the Fourier transform.

Soliton solutions emerge when the reflection coefficient vanishes; in that case the GLM integral equation reduces to an algebraic system. The discrete elements λ_k of the spectrum are intimately related to solitons. Indeed given a background solution, i.e. the simplest one, of our problem we can define a new solution

$$\psi = \chi \psi_b \tag{5.1}$$

where ψ_b is called generating matrix and

$$\chi = \mathcal{I} + \sum_k \frac{O_k}{\lambda - \lambda_k} \tag{5.2}$$

is called the dressing matrix; O_k are operators that do not depend on the spectral parameter. Note that the dressing matrix has pole exactly on the value of the discrete eigenvalues and a N pole dressing matrix can generate a N-solitons solution.

In this work we focus on gravitational solitons, also known as gravisolitons. These are particular exact solution of the Einstein, or Einstein-Maxwell, field equations that can be constructed by the use of the Inverse Scattering Method known, respectively as Belinski–Zakharov (BZ) transform [54] and Belinski-Zakharov-Alekseev (BZA) transform [57]. Their interpretation is not fully clear but black holes are special cases of gravitational solitons. Despite the term 'soliton' being used to describe gravisolitons, their behavior is very different from other solitons; in particular, gravisolitons do not preserve their amplitude and shape in time and in scattering processes. Recently [269] has been proposed, with some consistency tests, that gravisolitons have many properties of dark matter, such as no interaction with electromagnetic field but a non-trivial action on matter via gravitation. Moreover the authors showed that the gravitational lensing effect of gravisolitons agreed with the lensing effect of usual matter and that they have the same equation of state $w = 0$ as matter has. Moreover, in [297], the author shows how gravitational solitons can be considered as topological in the sense of some non-trivial homotopy group. Hence, gravisolitons naturally carry charges beyond those of any local excitation of the quantum fields. Therefore, there are symmetries of the EFT which are broken as soon as the topology of spacetime is allowed to fluctuate, even without any additional UV degrees of freedom. The author finds that the effect of gravitational solitons is to break the non-invertible symmetry to the maximal group-like subsymmetry and if we further demand that remaining group-like symmetry is broken by additional degrees of freedom, we find a complete spectrum. Hence, the completeness hypothesis follows from the absence of group-like symmetries once gravitational solitons are taken into account. Therefore gravisolitons can play a fundamental role in a theory of quantum gravity but this role is mysterious at the moment and a more deep understanding of gravisolitons could shed new light on the issue. The interesting point is that the high energy spectrum of General Relativity is dominated by black holes and gravisolitons generalize these particular solutions. Moreover, gravisolitons can have topological charges associated and their quantization could be useful to better understand topological changes in quantum gravity.

The most recent developments in the theory of gravisolitons are mainly contained in [161],[308],[318],[319]; in the first the hamiltonian methods are applied to gravisolitons and it is proved that the transition matrix satisfies equations familiar from integrable PDEs while the others deal with the study of gravisolitons in Einstein theory with negative cosmological constant, constructing supersymmetric gravitational soliton solutions of five-dimensional gauged supergravity coupled to arbitrarily many vector multiplets and studying the behaviour of causal geodesics and thermodynamic properties of Eguchi-Hanson-AdS$_5$ gravisolitons. Therefore, motivated by the recent partially lacking, except for what is contained in [161],[308],[318],[319], of formal and mathematical characterizations of gravisoliton solutions and by the possible central role of them in a theory of quantum gravity, we are interested in study some of their properties, from a formal and mathematical point of view, focusing on gravisolitons with axial symmetry or axialgravisolitons. We first search and develop a systematic expansion for the N-axialgravisoliton metrics to better understand some of their asymptotic properties. This expansion can play, for N-axialgravisoliton spacetime, a similar role of Bondi expansion for asymptotically flat spacetime and Fefferman-Graham expansion for asymptotically AdS spacetime. As the N-axialgravisoliton solution strongly depends on the background solution also its asymptotic expansion depends on it and, more specifically, depends on its radial expansion which is divided into cases as we will see later. As first application of the aforementioned expansion, we investigate the asymptotic symmetries of the N-axialgravisoliton solution. The first example of asymptotic symmetry in gravity is due to the pioneering work of Bondi, Metzner, Van der Burg and Sachs [24],[25],[26],[250] which found that the group of asymptotic physical symmetries, even on empty Minkowski space- time, is not Poincaré but rather the infinite-dimensional Bondi-van der Burg-Metzner-Sachs (BMS) group. Recently it was shown how supertranslations at null infinity can be achieved in the convolutional double copy framework [324]. However, asymptotic symmetries find place not only in gravity but also in other gauge theories [238],[249],[251],[279] like standard Maxwell and Yang-Mills theories and more exotic gauge theories like p-form and mixed symmetry tensor gauge theories [75],[244],[274],[325],[326],[330],[334]. Thanks to the asymptotic expansion we developed, we compute, in Paragraph 5.3.2, the leading order Killing vectors and study their bracket and therefore their algebra. In the same Paragraph we make a link, and one of the first explicit test, to the corner proposal which is a novel way to deal with quantum gravity [252],[255],[283],[285],[305],[306],[314] where corners, i.e. codimension two embedded manifold and the symmetry algebra at corners, are lifted to be fundamental ingredients of any theory of quantum gravity. We find that for every N, the axialgravisoliton corner symmetry algebra ($\mathfrak{agcsa}$) is a subalgebra of the proposed universal corner symmetry algebra. This suggests a positive feedback to testing the corner proposal explicitly and opens the way to a possible quantization to the non-asymptotically flat and non-perturbative sector of gravity by studying the representations of $\mathfrak{agcsa}$. Moreover, the possibilities of a new N-axialgravisolitons/$\mathfrak{agcsa}$-QFT holographic correspondence and a new N-axialgravisolitons IR triangle are suggested.

5.3 Asymptotic behavior and asymptotic symmetries

Let us consider the axisymmetric gravisolitons metric of Definition 2.4.2, explicitly

$$\begin{aligned}
g^{(ph)}_{22} &= 16C(g_b)_{22}\rho^{-\frac{N^2}{2}}\left(\prod_{k=1}^{N}\lambda_k\right)^{N+1}\left(\prod_{k>l=1}^{N}\frac{1}{(\lambda_k-\lambda_l)^2}\right)\det(\Gamma);\\
g^{(ph)}_{00} &= \pm\frac{(g)_{00}}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right) = \pm\frac{\left((g_b)_{00}-\sum_{k,l=1}^{N}D_{kl}\lambda_k^{-1}\lambda_l^{-1}L_0^{(k)}L_0^{(l)}\right)}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right);\\
g^{(ph)}_{11} &= \pm\frac{(g)_{11}}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right) = \pm\frac{\left((g_b)_{11}-\sum_{k,l=1}^{N}D_{kl}\lambda_k^{-1}\lambda_l^{-1}L_1^{(k)}L_1^{(l)}\right)}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right);\\
g^{(ph)}_{10} &= g^{(ph)}_{01} = \pm\frac{(g)_{10}}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right) = \pm\frac{\left((g_b)_{10}-\sum_{k,l=1}^{N}D_{kl}\lambda_k^{-1}\lambda_l^{-1}L_1^{(k)}L_0^{(l)}\right)}{\rho^N}\left(\prod_{k=1}^{N}\lambda_k\right),
\end{aligned} \tag{5.3}$$

Our task is to develop an asymptotic expansion of this metric in order to study the subgroup of diffeomorphisms which leaves the asymptotic metric invariant. In cylindrical coordinates the asymptotic expansion is performed taking the limits

$$\rho\to\infty,\ |z|\to\infty \qquad \text{s.t.} \quad \frac{\rho}{z}\sim O(1); \tag{5.4}$$

however, it is simpler taking limits in spherical coordinates[1]

$$r\to\infty. \tag{5.5}$$

[1]The change of coordinates is given by $(\phi,\rho,z)\to(\phi=\phi, r=\sqrt{\rho^2+z^2}, \theta=acrtang\left(\frac{\rho}{z}\right))$, or in other terms $\rho=rsin(\theta), \phi=\phi, z=rcos(\theta)$.

It is therefore necessary to study the asymptotic behavior of the various factors appearing in (5.3).

We will not fix the background metric in order to work in full generality

$$ds^2 = (g_b)_{22}(d\rho^2 + dz^2) + (g_b)_{ij}dx^i dx^j \quad i,j = 0,1; \tag{5.6}$$

we only adopt the necessary requests for consistency on $(g_b)_{ij}$: a symmetric matrix with determinant given by $det(g_b) = -\rho^2$:

$$(g_b)_{ij} = \begin{bmatrix} (g_b)_{00} := A & (g_b)_{01} := B \\ (g_b)_{10} := B & (g_b)_{11} := C \end{bmatrix} \qquad \text{s.t.} \quad det((g_b)_{ij}) = AC - B^2 = -\rho^2. \tag{5.7}$$

Since we are interested in asymptotic behaviours, let us give an asymptotic expansion for elements A, B, C. We have assumed that background metric has a time-like Killing vector hence the metric is stationary; this implies we can define Komar-like quantities of the background metric. Although these quantities may not have a physical reasoning they are, however, conserved quantities associate to the two Killing vectors of the background metric; therefore, under the physical requirement that conserved quantities do not diverge we can restrict the possible choice of elements A, B, C.

Let us now switch to spherical coordinates $(t, \phi, r, \theta) \equiv (0, 1, 2, 3)$; the background metric reads

$$ds^2 = (g_b)_{22}(dr^2 + r^2 d\theta^2) + (g_b)_{ij}dx^i dx^j \quad i,j = 0,1; \tag{5.8}$$

and metric elements are now function of r and θ.

Komar-like quantities are proportional to the surface integral of Killing forms, hence

$$K_t \propto \int_{\partial S} * \, dk, \qquad K_\phi \propto \int_{\partial S} * \, dm; \tag{5.9}$$

as shown in Appendix E some combination of the background metric elements have to have a specific asymptotic behavior in order to make Komar-like quantities finite; specifically

$$\begin{aligned} \frac{C}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial A}{\partial r} &\sim r^{n_1} f_1(\theta); \\ \frac{B}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial C}{\partial r} &\sim r^{n_2} f_2(\theta); \\ \frac{C}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial B}{\partial r} &\sim r^{n_3} f_3(\theta); \end{aligned} \tag{5.10}$$

with

$$n_1, n_2, n_3 \leq 1 \tag{5.11}$$

and $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ arbitrary functions of azimuthal angle that could be expanded in Legendre polynomials.

We now assume a power law behavior[2] for the background metric elements when $r \to \infty$

$$A \sim r^{\alpha}\mathcal{A}(\theta), \quad B \sim r^{\beta}\mathcal{B}(\theta), \quad C \sim r^{\gamma}\mathcal{C}(\theta), \quad (g_b)_{22} \sim r^{\delta}\mathcal{D}(\theta); \tag{5.12}$$

the condition on the determinant can be made explicit in two ways as summarized in Table 5.1

1	$\alpha+\gamma=2, \quad \beta=1, \quad \mathcal{AC}(\theta)-\mathcal{B}^2(\theta)=-sin^2(\theta)$
2	$\alpha+\gamma=2, \quad \mathcal{AC}(\theta)=-sin^2(\theta), \quad \mathcal{B}^2(\theta)=0$

TABLE 5.1 *Possible explicit choices for the determinant condition.*

The possible choices of exponents, coherent with the condition on the determinant, are summarized in the following Table 5.2.

a	$\alpha \neq 0, \beta \neq 0, \gamma \neq 0$
b	$\alpha \neq 0, \beta \neq 0, \gamma = 0$
c	$\alpha = 0, \beta \neq 0, \gamma \neq 0$
e	$\alpha = 0, \beta = 0, \gamma \neq 0$
f	$\alpha \neq 0, \beta = 0, \gamma = 0$
g	$\alpha \neq 0, \beta = 0, \gamma \neq 0$

TABLE 5.2 *Possible coherent choices of exponents due to the determinant condition.*

These cases have to be combined with one of the two ways to made explicit the determinant condition, so, for example the c2 case means case c combined to the conditions 2 of the determinant condition. In the following we study in details the case a1 then proceeding with the other cases in a similar way.

Case a1

Plugging the asymptotic expression into (5.10) we get

$$\begin{aligned}
\alpha r^{\gamma+\frac{\gamma}{2}-\frac{\delta}{2}+\alpha-1} \sim r^{n_1} \quad &\Rightarrow \quad \frac{3\gamma}{2}-\frac{\delta}{2}+\alpha-1=n_1;\\
\gamma r^{\beta+\frac{\gamma}{2}-\frac{\delta}{2}+\gamma-1} \sim r^{n_2} \quad &\Rightarrow \quad \beta+\frac{3\gamma}{2}-\frac{\delta}{2}-1=n_2;\\
\beta r^{\gamma+\frac{\gamma}{2}-\frac{\delta}{2}+\beta-1} \sim r^{n_3} \quad &\Rightarrow \quad \frac{3\gamma}{2}-\frac{\delta}{2}+\beta-1=n_3.
\end{aligned} \tag{5.13}$$

[2]Recently also logarithmic terms are taken in account in these kind of expansions [326]; it would be interesting to study also this possibility.

We get the following complete matrix for our system

$$M|b = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & \frac{3}{2} & -\frac{1}{2} & n_1 - 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} & n_2 - 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} & n_3 - 1 \end{array}\right]. \tag{5.14}$$

From Rouché-Capelli theorem, a linear system admits solutions if and only if $Rk(M) = Rk(M|b)$ where M and $M|b$ are the complete and incomplete matrices associated to the linear system. In our case, both matrices have rank 4 if and only if $n_2 = n_3$; moreover the solution space is a subspace of dimension 4, hence the solution is unique. Therefore given the asymptotic data $\{n_1, n_2\}$ we get the unique radial asymptotic expansion of the metric components fixed by the unique set of solution

$$\{\alpha = n_1 - n_2 + 1, \beta = 1, \gamma = -n_1 + n_2 + 1, \delta = -3n_1 + n_2 + 7\}. \tag{5.15}$$

This solution can be expressed in terms of two parameters α and δ as

$$\{\alpha = n_1 - n_2 + 1, \beta = 1, \gamma = -\alpha + 2, \delta = -3n_1 + n_2 + 7\}; \tag{5.16}$$

we can note that, since $n_1, n_2 \leq 1$ we must have

$$\begin{cases} \alpha \leq 2 & if\ n_2 \geq 0; \\ \alpha > 2 & if\ n_2 < 0. \end{cases} \tag{5.17}$$

Moreover, we get also relations between the angular functions

$$f_2(\theta) = f_3(\theta) = \frac{\mathcal{B}C(\theta)}{sin^2(\theta)}\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}, \quad f_1(\theta) = \frac{\mathcal{A}C(\theta)}{sin^2(\theta)}\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}. \tag{5.18}$$

Case a2

Also in the a2 case we must have $n_2 = n_3$ to have a solution;

$$\{\alpha = -2n_1 + 8 - \delta, \beta = -n_1 + n_2 + \alpha, \gamma = 2n_1 - 6 + \delta, \delta\} \tag{5.19}$$

where δ is undetermined by the system. For the angular functions we get

$$f_2(\theta) = f_3(\theta) = 0, \quad f_1(\theta) = -\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}. \tag{5.20}$$

Case b1

We have

$$\{\alpha = 2, \beta = 1, \gamma = 0, \delta = -2n_3 = 2(1 - n_1)\}; \tag{5.21}$$

while for the angular functions

$$f_2(\theta) = \frac{\mathcal{B}C(\theta)}{sin^2(\theta)}\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}, \quad f_1(\theta) = \frac{\mathcal{A}C(\theta)}{sin^2(\theta)}\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}. \tag{5.22}$$

Case b2

For the case b2 we have

$$\{\alpha = 2, \beta = n_3 - n_1 + 2, \gamma = 0, \delta = 2(1 - n_1)\}; \tag{5.23}$$

the angular relations are

$$f_2(\theta) = 0, \quad f_1(\theta) = -\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}. \tag{5.24}$$

Case c1

In this case we get

$$\{\alpha = 0, \beta = 1, \gamma = 2, \delta = 2(3 - n_2) = 2(3 - n_3)\}; \tag{5.25}$$

while the angular relations read

$$f_2(\theta) = f_3(\theta) = \frac{\mathcal{B}C(\theta)}{sin^2(\theta)}\sqrt{\frac{C(\theta)}{\mathcal{D}(\theta)}}. \tag{5.26}$$

Case c2

We have

$$\{\alpha = 0, \beta = \delta/2 + n_2 - 2 = \delta/2 + n_3 - 2, \gamma = 2, \delta\}, \tag{5.27}$$

where δ is undetermined and the relations

$$f_2(\theta) = f_3(\theta) = 0. \tag{5.28}$$

Case e2

For this case we have
$$\{\alpha = 0, \beta = 0, \gamma = 2, \delta = 2(2 - n_2)\}, \tag{5.29}$$
moreover
$$f_2(\theta) = 0. \tag{5.30}$$

Case f2

We get
$$\{\alpha = 2, \beta = 0, \gamma = 0, \delta = 2(1 - n_1)\}, \tag{5.31}$$
and
$$f_1(\theta) = -\sqrt{\frac{C(\theta)}{D(\theta)}}. \tag{5.32}$$

Case g2

For the final case we get

$$\{\alpha = -2n_1 + 4 - \delta = -n_2/5 - \delta/10, \beta = 0, \gamma = -\alpha + 2, \delta\}, \tag{5.33}$$

where δ is undetermined and we have
$$f_1(\theta) = -\sqrt{\frac{C(\theta)}{D(\theta)}}. \tag{5.34}$$

5.3.1 Asymptotic behavior of the N-axialgravisolitons metric

In order to find the asymptotic expansion of the metric (5.3) we need to compute the asymptotic behaviour of

$$\begin{aligned}
(D)_{kl} &:= (\Gamma^{-1})_{kl};\\
(\Gamma)_{kl} &:= m_i^{(k)}(g_b)_{ij} m_j^{(l)}(\rho^2 + \lambda_k\lambda_l)^{-1};\\
L_i^{(k)} &:= m_j^{(k)}(g_b)_{ji};\\
m_i^{(k)} &:= m_{2j}^{(k)}(\psi_b^{-1}(\lambda_k, \rho, z))_{ji}.
\end{aligned} \tag{5.35}$$

Let us start with $\psi_b^{-1}(\lambda_k, \rho, z))_{ji}$, which is determined by the system of equations

$$\left(\partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda\right)\psi_b = \frac{\rho V_b - \lambda U_b}{\lambda^2 + \rho^2}\psi_b, \qquad \left(\partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda\right)\psi_b = \frac{\rho V_b + \lambda U_b}{\lambda^2 + \rho^2}\psi_b; \tag{5.36}$$

we can rewrite it in spherical coordinates[3] as

$$\begin{aligned}\left(cos(\theta)\frac{\partial}{\partial r}-sin(\theta)\frac{\partial}{\partial\theta}-\frac{2\lambda^2}{\lambda^2+r^2sin^2(\theta)}\partial_\lambda\right)\psi_b&=\frac{rsin(\theta)V_b-\lambda U_b}{\lambda^2+r^2sin^2(\theta)}\psi_b,\\ \left(sin(\theta)\frac{\partial}{\partial r}+cos(\theta)\frac{\partial}{\partial\theta}+\frac{2\lambda\rho}{\lambda^2+r^2sin^2(\theta)}\partial_\lambda\right)\psi_b&=\frac{rsin(\theta)V_b+\lambda U_b}{\lambda^2+r^2sin^2(\theta)}\psi_b;\end{aligned} \tag{5.37}$$

and expand them asymptotically, as done in Appendix F, to find a solution for the generating matrix. However the result is not handle and we prefer to follow another path. We have the following

Proposition 5.3.1 (Leading order of the background generating matrix)**.** *For every $\lambda \neq \lambda_k$ the background generating matrix and the background metric have the same leading order:*

$$\lim_{r\to\infty} g_b(r,\theta) = \lim_{r\to\infty} \psi_b(\lambda, r, \theta). \tag{5.38}$$

Proof. From the general theory about gravisolitons we have, in spherical coordinates, that

$$g_b(r,\theta) = \psi_b(0, r, \theta); \tag{5.39}$$

therefore

$$\psi_b(\lambda, r, \theta) = g_b(r,\theta) + \sum_{n=1}^{\infty} \lambda^n F_b^{(n)}(r,\theta) = \sum_{n=0}^{\infty} \lambda^n F_b^{(n)}(r,\theta), \tag{5.40}$$

where $F_b^{(0)}(r,\theta) := g_b(r,\theta)$. We assume a power law expansion in the radial coordinate

$$F_b^{(n)} = \sum_{k\in\mathbb{Z}} r^k f^{(k)}(\theta) \Rightarrow \psi_b(\lambda, r, \theta) = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} \lambda^n r^k f^{(k)}(\theta). \tag{5.41}$$

Now we note that due to a version of Moore-Osgood theorem we have

$$\lim_{r\to\infty}\left(\lim_{\lambda\to 0}\psi_b(\lambda, r, \theta)\right) = \lim_{r\to\infty} g_b(r,\theta) = \lim_{\lambda\to 0}\left(\lim_{r\to\infty}\psi_b(\lambda, r, \theta)\right); \tag{5.42}$$

since the generating matrix is well-behaved, i.e. the limit for $\lambda \to 0$ converges uniformly to $g_b(r,\theta)$ and from

[3] Derivatives become $\frac{\partial}{\partial\rho} = \frac{\partial\rho}{\partial r}\frac{\partial}{\partial r} + \frac{\partial\rho}{\partial\theta}\frac{\partial}{\partial\theta} = sin(\theta)\frac{\partial}{\partial r} + cos(\theta)\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z} = \frac{\partial z}{\partial r}\frac{\partial}{\partial r} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial\theta} = cos(\theta)\frac{\partial}{\partial r} - sin(\theta)\frac{\partial}{\partial\theta}$.

(5.39) we have $\lim_{r\to\infty} g_b(r,\theta) = \lim_{r\to\infty} \psi_b(0,r,\theta)$. From relation (5.42) and (5.41) we have

$$\begin{aligned}\lim_{r\to\infty} g_b(r,\theta) &= \lim_{\lambda\to 0}\left(\lim_{r\to\infty}\psi_b(\lambda,r,\theta)\right) = \lim_{\lambda\to 0}\left(\lim_{r\to\infty}\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\lambda^n r^k f^{(k)}(\theta)\right) = \\ &= \lim_{\lambda\to 0}\left(\lim_{r\to\infty}\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} r^n(\pm 1 - cos(\theta))^n r^k f^{(k)}(\theta)\right),\end{aligned} \tag{5.43}$$

therefore the $\lambda \to 0$ limit has to suppress all orders in r which are not the leading order of the background metric; for example if the leading order for the background metric is $\mathcal{O}(r^p)$ we must have

$$(\pm 1 - cos(\theta))^n f^{(k)}(\theta) = 0 \qquad \text{if} \quad n \neq p-k. \tag{5.44}$$

Now, if the leading order in r of $\psi_b(\lambda,r,\theta)$ was λ-dependent, in the $\lambda \to 0$ limit the leading order would be vanishing contradicting the non-vanishing leading order, solution of system (5.13), of $g_b(r,\theta)$. Then, (5.44) must hold also for general λ-dependence of $\psi_b(\lambda,r,\theta)$ and its leading order in r coincides with those of $g_b(r,\theta)$. □

However, it is a fact of life that in same cases, for example for Minkowski spacetime [296], when we compute $\psi_b(\lambda_k,r,\theta)$ the leading order of some elements of the generating matrix are different from those of the background matrix. This is a very peculiar structure due to the appearing in the generating matrix elements of a specific combination, that is

$$r^2 sin^2(\theta) - 2r cos(\theta)\lambda - \lambda^2; \tag{5.45}$$

indeed for $\lambda = \lambda_k$ this is the LHS of equation (2.197) and reduces to $-2w_k\lambda_k$. This particular feature is not captured by the general reasoning above and in order to compute the θ-dependent coefficients and the subleading orders in r, we need to solve order by order the set of equations (F.4) but this is not a simple task in a fully general setting. Case by case is easier since one knows the leading order of the background metric and can solve equations (F.4) to get the leading order of the background generating matrix. We therefore introduce new factors to parametrize the leading order behavior of the background generating matrix on the poles

$$(\psi_b)_{00}(\lambda_k,r,\theta) \sim r^{\epsilon_1}\mathcal{E}_1(\theta), \quad (\psi_b)_{10}(\lambda_k,r,\theta) \sim r^{\epsilon_2}\mathcal{E}_2(\theta), \quad (\psi_b)_{11}(\lambda_k,r,\theta) \sim r^{\epsilon_3}\mathcal{E}_3(\theta). \tag{5.46}$$

We can now expand the interesting quantities to compute the asymptotic N-axialgravisolitons metric. The result is the following

Theorem 5.3.1 (N-axialgravisolitons asymptotic expansion)**.** *The N-axialgravisolitons metric components,* (5.3)*, admit the following asymptotic expansion*

$$g_{22}^{(ph)} \sim \mathcal{M}_N^{(\#)}(\theta)\Bigg(\frac{\Big(\prod_{j=1}^{\#}\frac{1}{(w_k-w_l)_j^2}\Big)^{\frac{1}{N}}}{(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta)-r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2}\Bigg)^N r^{x^{(\#)}(N)}\sum_{\sigma\in S_N}\Big(\operatorname{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]\Big); \tag{5.47a}$$

$$g_{00}^{(ph)} \sim \pm\mathcal{N}_N(\theta)\Bigg[r^\alpha\mathcal{A}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\Bigg]; \tag{5.47b}$$

$$g_{11}^{(ph)} \sim \pm\mathcal{N}_N(\theta)\Bigg[r^\gamma\mathcal{C}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\Bigg]; \tag{5.47c}$$

$$g_{10}^{(ph)} = g_{01}^{(ph)} \sim \pm\mathcal{N}_N(\theta)\Bigg[r^\beta\mathcal{B}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\Bigg]; \tag{5.47d}$$

where

$$\mathcal{M}_N^{(\#)}(\theta) = 4^{N(N-1)-2\#+2}C\mathcal{D}(\theta)[(\pm 1-cos(\theta))]^{N^2+N}sin(\theta)^{-\frac{N^2}{2}}(2\mp 2cos(\theta))^{-N}; \qquad x^{(\#)}(N) := \delta-\frac{N^2}{2}+2N+2\#; \tag{5.48a}$$

$$\mathcal{K} := \frac{\sum_{\tilde\sigma\in S_{N-1}}\operatorname{sgn}(\tilde\sigma)\prod_{s=0}^{N-1}[A_{s\tilde\sigma_s}+B_{s\tilde\sigma_s}+C_{s\tilde\sigma_s}+D_{s\tilde\sigma_s}]}{\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]}; \qquad \mathcal{N}_N := (\theta)\left[\frac{(\pm 1-cos(\theta))}{sin(\theta)}\right]^N; \tag{5.48b}$$

and

$$\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta) := m_{20}^{(k)}[r^{\epsilon_3+\alpha}\mathcal{A}\mathcal{E}_3(\theta)-r^{\epsilon_2+\beta}\mathcal{B}\mathcal{E}_2(\theta)]+m_{21}^{(k)}[r^{\epsilon_1+\beta}\mathcal{B}\mathcal{E}_1(\theta)-r^{\epsilon_2+\alpha}\mathcal{A}\mathcal{E}_2(\theta)]; \tag{5.49a}$$

$$\mathcal{G}^{(k)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta) := m_{20}^{(k)}[r^{\epsilon_3+\beta}\mathcal{B}\mathcal{E}_3(\theta)-r^{\epsilon_2+\gamma}\mathcal{C}\mathcal{E}_2(\theta)]+m_{21}^{(k)}[r^{\epsilon_1+\gamma}\mathcal{C}\mathcal{E}_1(\theta)-r^{\epsilon_2+\beta}\mathcal{B}\mathcal{E}_2(\theta)]; \tag{5.49b}$$

and

$$A_{kl} := r^{2\epsilon_3+\alpha}\mathcal{A}(\theta)[m_{20}^{(k)}m_{20}^{(l)}\mathcal{E}_3^2(\theta)+m_{21}^{(k)}m_{21}^{(l)}\mathcal{E}_2^2(\theta)]-r^{\epsilon_2+\epsilon_3+\alpha}\mathcal{E}_2\mathcal{E}_3\mathcal{A}(\theta)[m_{20}^{(k)}m_{21}^{(l)}+m_{21}^{(k)}m_{20}^{(l)}]; \tag{5.50a}$$

$$B_{kl} := -2r^{\epsilon_2+\epsilon_3+\beta}\mathcal{E}_2\mathcal{E}_3\mathcal{B}(\theta)m_{20}^{(k)}m_{20}^{(l)}-2r^{\epsilon_1+\epsilon_2+\beta}\mathcal{E}_1\mathcal{E}_2\mathcal{B}(\theta)m_{21}^{(k)}m_{21}^{(l)}; \tag{5.50b}$$

$$C_{kl} := r^{\epsilon_1+\epsilon_3+\beta}\mathcal{E}_1\mathcal{E}_3\mathcal{B}(\theta)[m_{20}^{(k)}m_{21}^{(l)}+m_{21}^{(k)}m_{20}^{(l)}]+r^{2\epsilon_2+\beta}\mathcal{E}_2^2\mathcal{B}(\theta)[m_{20}^{(k)}m_{21}^{(l)}+m_{21}^{(k)}m_{20}^{(l)}]; \tag{5.50c}$$

$$D_{kl} := r^{2\epsilon_2+\gamma}\mathcal{C}(\theta)[m_{20}^{(k)}m_{20}^{(l)}\mathcal{E}_2^2(\theta)+m_{21}^{(k)}m_{21}^{(l)}\mathcal{E}_1^2(\theta)]-r^{\epsilon_1+\epsilon_2+\gamma}\mathcal{E}_1\mathcal{E}_2\mathcal{C}(\theta)[m_{20}^{(k)}m_{21}^{(l)}+m_{21}^{(k)}m_{20}^{(l)}]. \tag{5.50d}$$

Proof. Starting from (5.35) and taking into account (5.46) and (5.12), we have

$$m_0^{(k)} \sim \frac{m_{21}^{(k)} r^{\epsilon_2}\mathcal{E}_2(\theta) - m_{20}^{(k)} r^{\epsilon_3}\mathcal{E}_3(\theta)}{r^{\epsilon_1+\epsilon_3}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2}\mathcal{E}_2^2(\theta)} \qquad m_1^{(k)} \sim \frac{m_{21}^{(k)} r^{\epsilon_1}\mathcal{E}_1(\theta) - m_{20}^{(k)} r^{\epsilon_2}\mathcal{E}_2(\theta)}{r^{\epsilon_1+\epsilon_3}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2}\mathcal{E}_2^2(\theta)}, \tag{5.51}$$

from which

$$L_0^{(k)} \sim \frac{\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)}{r^{\epsilon_1+\epsilon_3}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2}\mathcal{E}_2^2(\theta)}; \qquad L_1^{(k)} \sim \frac{\mathcal{G}^{(k)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)}{r^{\epsilon_1+\epsilon_3}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2}\mathcal{E}_2^2(\theta)}; \tag{5.52}$$

where

$$\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta) := m_{20}^{(k)}[r^{\epsilon_3+\alpha}\mathcal{A}\mathcal{E}_3(\theta) - r^{\epsilon_2+\beta}\mathcal{B}\mathcal{E}_2(\theta)] + m_{21}^{(k)}[r^{\epsilon_1+\beta}\mathcal{B}\mathcal{E}_1(\theta) - r^{\epsilon_2+\alpha}\mathcal{A}\mathcal{E}_2(\theta)]; \tag{5.53a}$$

$$\mathcal{G}^{(k)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta) := m_{20}^{(k)}[r^{\epsilon_3+\beta}\mathcal{B}\mathcal{E}_3(\theta) - r^{\epsilon_2+\gamma}\mathcal{C}\mathcal{E}_2(\theta)] + m_{21}^{(k)}[r^{\epsilon_1+\gamma}\mathcal{C}\mathcal{E}_1(\theta) - r^{\epsilon_2+\beta}\mathcal{B}\mathcal{E}_2(\theta)]. \tag{5.53b}$$

Using these expansions we get

$$\Gamma_{kl} \sim \frac{A_{kl} + B_{kl} + C_{kl} + D_{kl}}{2(1 \mp cos(\theta))(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2}; \tag{5.54}$$

where

$$A_{kl} := r^{2\epsilon_3+\alpha}\mathcal{A}(\theta)[m_{20}^{(k)}m_{20}^{(l)}\mathcal{E}_3^2(\theta) + m_{21}^{(k)}m_{21}^{(l)}\mathcal{E}_2^2(\theta)] - r^{\epsilon_2+\epsilon_3+\alpha}\mathcal{E}_2\mathcal{E}_3\mathcal{A}(\theta)[m_{20}^{(k)}m_{21}^{(l)} + m_{21}^{(k)}m_{20}^{(l)}]; \tag{5.55a}$$

$$B_{kl} := -2r^{\epsilon_2+\epsilon_3+\beta}\mathcal{E}_2\mathcal{E}_3\mathcal{B}(\theta)m_{20}^{(k)}m_{20}^{(l)} - 2r^{\epsilon_1+\epsilon_2+\beta}\mathcal{E}_1\mathcal{E}_2\mathcal{B}(\theta)m_{21}^{(k)}m_{21}^{(l)}; \tag{5.55b}$$

$$C_{kl} := r^{\epsilon_1+\epsilon_3+\beta}\mathcal{E}_1\mathcal{E}_3\mathcal{B}(\theta)[m_{20}^{(k)}m_{21}^{(l)} + m_{21}^{(k)}m_{20}^{(l)}] + r^{2\epsilon_2+\beta}\mathcal{E}_2^2\mathcal{B}(\theta)[m_{20}^{(k)}m_{21}^{(l)} + m_{21}^{(k)}m_{20}^{(l)}]; \tag{5.55c}$$

$$D_{kl} := r^{2\epsilon_2+\gamma}\mathcal{C}(\theta)[m_{20}^{(k)}m_{20}^{(l)}\mathcal{E}_2^2(\theta) + m_{21}^{(k)}m_{21}^{(l)}\mathcal{E}_1^2(\theta)] - r^{\epsilon_1+\epsilon_2+\gamma}\mathcal{E}_1\mathcal{E}_2\mathcal{C}(\theta)[m_{20}^{(k)}m_{21}^{(l)} + m_{21}^{(k)}m_{20}^{(l)}]. \tag{5.55d}$$

To find the asymptotic behaviour of $(D)_{kl}$ we write it as

$$(D)_{kl} := (\Gamma^{-1})_{kl} = \frac{(Adj(\Gamma))_{kl}}{\det(\Gamma)} \tag{5.56}$$

and we have to taken into account that for an N-axialgravisolitons solution $(\Gamma)_{kl}$ is an $N \times N$ matrix. The determinant is given by

$$\det(\Gamma) = \sum_{\sigma\in S_N}\left(\operatorname{sgn}(\sigma)\prod_{s=0}^{N}(\Gamma)_{s,\sigma_s}\right); \tag{5.57}$$

asymptotically, the generic addend is

$$\mathrm{sgn}(\sigma)\prod_{s=0}^{N}(\Gamma)_{s,\sigma_s}\sim \mathrm{sgn}(\sigma)\left(\frac{1}{2(1\mp cos(\theta))(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta)-r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2}\right)^N\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]. \tag{5.58}$$

Using the expression above for the determinant we can also compute the asymptotic behaviour of the cofactor which is computed by a determinant of an $(N-1)\times(N-1)$ matrix; therefore we compute that

$$(D)_{kl}\sim[2(1\mp cos(\theta))(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta)-r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2]\frac{\sum_{\tilde\sigma\in S_{N-1}}\mathrm{sgn}(\tilde\sigma)\prod_{s=0}^{N-1}[A_{s\tilde\sigma_s}+B_{s\tilde\sigma_s}+C_{s\tilde\sigma_s}+D_{s\tilde\sigma_s}]}{\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]}; \tag{5.59}$$

where $s\neq k$ and $\tilde\sigma_s\neq l$. We may wonder and question the effective usefulness and ease of use of expansions (5.51), (5.52), (5.54) and (5.59); the point is that they are completely general but, case by case, one knows exponents $\{\alpha,\beta,\gamma,\epsilon_1,\epsilon_2,\epsilon_3\}$ and angular functions $\{\mathcal{A}(\theta),\mathcal{B}(\theta),\mathcal{C}(\theta),\mathcal{E}_1(\theta),\mathcal{E}_2(\theta),\mathcal{E}_3(\theta)\}$ and can extract the leading orders in a simple way.

We can finally expand the N-solitons metric; let us start with the combination appearing in the g_{ij} elements, for example in g_{00}

$$\sum_{k,l=1}^{N}(D)_{kl}\lambda_k^{-1}\lambda_l^{-1}L_0^{(k)}L_0^{(l)}\sim\sum_{k,l=1}^{N}K\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\frac{\sum_{\tilde\sigma\in S_{N-1}}\mathrm{sgn}(\tilde\sigma)\prod_{s=0}^{N-1}[A_{s\tilde\sigma}+B_{s\tilde\sigma}+C_{s\tilde\sigma}+D_{s\tilde\sigma}]}{\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\tilde\sigma_s}+B_{s\tilde\sigma_s}+C_{s\tilde\sigma_s}+D_{s\tilde\sigma_s}]}, \tag{5.60a}$$

where $K=2\frac{(1\mp cos(\theta))}{(\pm1-cos(\theta))^2}=\frac{2}{(1\mp cos(\theta))}$ since $(\pm1-cos(\theta))^2=(1\mp cos(\theta))^2$. Similar expansions hold for $g_{01}=g_{10}$ and g_{11}. Putting all together and defining

$$\begin{aligned}
&\mathcal{M}_N(\theta):=4^{N(N-1)+2}\mathcal{C}\mathcal{D}(\theta)[(\pm1-cos(\theta))]^{N^2+N}sin(\theta)^{-\frac{N^2}{2}}(2\mp2cos(\theta))^{-N};\\
&x(N):=\delta-\frac{N^2}{2}+2N;\\
&\mathcal{K}:=\frac{\sum_{\tilde\sigma\in S_{N-1}}\mathrm{sgn}(\tilde\sigma)\prod_{s=0}^{N-1}[A_{s\tilde\sigma_s}+B_{s\tilde\sigma_s}+C_{s\tilde\sigma_s}+D_{s\tilde\sigma_s}]}{\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]};\\
&\mathcal{N}_N:=(\theta)\left[\frac{(\pm1-cos(\theta))}{sin(\theta)}\right]^N;
\end{aligned} \tag{5.61}$$

the metric component of the N-gravisoliton solution have the following asymptotic expansion

$$
\begin{aligned}
g_{22}^{(ph)} &\sim \mathcal{M}_N(\theta)\left(\frac{1}{(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta)-r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2}\right)^N r^{x(N)}\sum_{\sigma\in S_N}\Big(\operatorname{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]\Big);\\
g_{00}^{(ph)} &\sim \pm\mathcal{N}_N(\theta)\left[r^{\alpha}\mathcal{A}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\right];\\
g_{11}^{(ph)} &\sim \pm\mathcal{N}_N(\theta)\left[r^{\gamma}\mathcal{C}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\right];\\
g_{10}^{(ph)} = g_{01}^{(ph)} &\sim \pm\mathcal{N}_N(\theta)\left[r^{\beta}\mathcal{B}(\theta)-\sum_{k,l=1}^{N}K\mathcal{K}\mathcal{G}^{(k)\alpha,\beta,\beta,\alpha}_{0(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\mathcal{G}^{(l)\beta,\gamma,\gamma,\beta}_{1(\epsilon_3,\epsilon_2,\epsilon_1\epsilon_2)}(r,\theta)\right].
\end{aligned}
\tag{5.62}
$$

We note that for the $g_{22}^{(ph)}$ component we have more choices for the asymptotic expansion according to the relative signs in front of the square roots in the definition of the poles due to the presence of the factors $\frac{1}{(\lambda_k-\lambda_l)^2}$ which give $\frac{1}{4r^2}$ if the signs disagree while give $\frac{1}{w_k-w_l}$ if signs agree. Therefore, defining

$$
\begin{aligned}
\mathcal{M}_N^{(\#)}(\theta) &:= 4^{N(N-1)-2\#+2}C\mathcal{D}(\theta)[(\pm1-cos(\theta))]^{N^2+N}sin(\theta)^{-\frac{N^2}{2}}(2\mp 2cos(\theta))^{-N};\\
x^{(\#)}(N) &:= \delta-\frac{N^2}{2}+N(N+1)-N(N-1)+2\#;
\end{aligned}
\tag{5.63}
$$

with # the number of poles couple with concordant root signs, the most general expansion for $g_{22}^{(ph)}$ is

$$
g_{22}^{(ph)} \sim \mathcal{M}_N^{(\#)}(\theta)\left(\frac{\left(\prod_{j=1}^{\#}\frac{1}{(w_k-w_l)_j^2}\right)^{\frac{1}{N}}}{(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta)-r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))^2}\right)^N r^{x^{(\#)}(N)}\sum_{\sigma\in S_N}\Big(\operatorname{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s}+B_{s\sigma_s}+C_{s\sigma_s}+D_{s\sigma_s}]\Big). \tag{5.64a}
$$

Note that $x^{(0)}(N)=x(N)$ and $\mathcal{M}_N^{(0)}(\theta)=\mathcal{M}_N(\theta)$. This completes the proof. □

We can check that choosing Minkowski metric as background and computing the asymptotic behaviour of the background generating matrix, the asymptotic behaviour of $m_0^{(k)}, m_1^{(k)}, L_0^{(k)}, l_1^{(k)}, \Gamma_{kl}, D_{kl}$ and of the metric component is exactly what was expected [133],[296]. We note that the physical information of how many solitons make up the solution is reached by all the metric components meaning again that non-diagonal solution can be generated even if the background solution was diagonal; this is the case of Kerr black hole 2-gravisolitons solution. Moreover, there are no reasons why the asymptotic behavior of the chosen background solution must be preserved by the N-gravisolitons solution and in general this is not the case. Indeed, let us consider the following example of a background metric with $\delta = 1$ and with $\epsilon_1=\epsilon_2=\epsilon_3=0$. This is not restrictive since the determination of the asymptotic behavior of the background generating matrix does not depend on δ. In this case the N-gravisolitons solution, with # = 0, gives us a $g_{22}^{(ph)}$ metric component with asymptotic behavior $r^{1-\frac{N^2}{2}+2N}r^{h(\alpha,\beta,\gamma)}$ where $h(\alpha,\beta,\gamma)\in\frac{1}{2}\mathbb{Z}$ has to be derived case by case depending on α,β and γ but then it is fixed. However, surely there exist a N for which $1-\frac{N^2}{2}+2N+h(\alpha,\beta,\gamma)\neq 0$. This shows that the N-gravisolitons solution does not have to preserve the asymptotic behavior of the background metric in general, at least in the assumption that Komar-like quantities are finite. Therefore it is a well posed question asking for asymptotic symmetries; the expectation is that these may depend on the number of gravisolitons that make up the solution.

5.3.2 Leading order of the asymptotic symmetries and the UCS group

In order to discuss asymptotic symmetries let us find solutions of the asymptotic Killing equation, i.e. Killing equation for the asymptotic N-gravisolitons metric. As known, a vector field ξ is a Killing field if the Lie derivative with respect to ξ of the metric vanishes

$$\mathcal{L}_\xi \boldsymbol{g}=0, \tag{5.65}$$

in terms of the Levi-Civita connection, we can write it as

$$\boldsymbol{g}\left(\nabla_Y\xi, Z\right)+\boldsymbol{g}\left(Y,\nabla_Z\xi\right)=0, \tag{5.66}$$

for all vector fields Y and Z. In local coordinates expressed by coordinate fields, Killing equation assumes its standard form

$$\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu=\xi_{(\mu;\nu)}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu-2\Gamma^\rho_{\mu\nu}\xi_\rho=0; \tag{5.67}$$

moreover, we assume a homogeneous expansion for the Killing vectors of the form

$$\xi_\mu=\sum_{l\in\mathbb{Z}}\frac{\bar{\xi}^{(l)}_\mu(\theta)}{r^l}\qquad \mu=0,1,2,3. \tag{5.68}$$

Our goal is to determine the first possible l of the expansion above; this information can be obtained looking at the Killing equation and requiring that it is satisfied only asymptotically, i.e. it is not really necessary for it to be exactly zero as long as it is satisfied within the limit of large radial distances. We focus on the Killing equation with $(\mu,\nu)=(3,3)$ which reduces, using results in Appendix G, to

$$2\partial_3\xi_3-2\Gamma^3_{33}\xi_3\sim\mathcal{O}(r^0). \tag{5.69}$$

For $N=0$ we have, inserting the Killing vectors expansion

$$2\sum_{l\in\mathbb{Z}}\frac{\bar{\xi}'^{(l)}_3(\theta)}{r^l}-\frac{\mathcal{D}'(\theta)}{\mathcal{D}(\theta)}\sum_{l\in\mathbb{Z}}\frac{\bar{\xi}^{(l)}_3(\theta)}{r^l}\sim\mathcal{O}(r^0), \tag{5.70}$$

which means that the leading order is $l = 0$ since for $l > 0$ we have subleading terms while for $l < 0$ we have overleading terms; moreover coefficients must satisfy the equation

$$2\bar{\xi}_3^{\prime(l)}(\theta) - \frac{\mathcal{D}'(\theta)}{\mathcal{D}(\theta)}\bar{\xi}_3^{(l)}(\theta) = 0 \quad \forall l \in (-\infty, 0], \tag{5.71}$$

Let us continue with the case of $N = 2$; in this case using the Christoffel symbols derived in Appendix G, the equation is

$$2\sum_{l\in\mathbb{Z}} \frac{\bar{\xi}_3^{\prime(l)}(\theta)}{r^l} - 2\left(\Gamma^3_{33}\right)_{N=2} \sum_{l\in\mathbb{Z}} \frac{\bar{\xi}_3^{(l)}(\theta)}{r^l} \sim \mathcal{O}(r^0), \tag{5.72}$$

since $\left(\Gamma^3_{33}\right)_{N=2}$ turns out to have a leading order independent of r then, again, the leading order of the Killing vector is $l = 0$; moreover coefficients satisfy

$$2\bar{\xi}_3^{\prime(l)}(\theta) - 2\left(\Gamma^3_{33}\right)_{N=2}\bar{\xi}_3^{(l)}(\theta) = 0 \quad \forall l \in (-\infty, 0]. \tag{5.73}$$

The same kind of analysis can be done in the general N case but we note that despite the leading order is the same for all N, the angular dependence can be different; this can be seen since the Christoffel symbols have different angular dependence depending on N since the N-gravisolitons metric has. Therefore, in the case of Killing vectors whose components fall-off with the same behaviour, the leading order is r^0 while the specific case coefficients have to be derived case by case but their angular dependence depend on the number of gravisolitons. The other components of the Killing equations give relations between expansion parameters, i.e. the exponents of r in the metric expansion, and l; knowing that the leading order is for $l = 0$, some of these relations may not be satisfied for the leading order and we find that not for all choices of expansion parameters we have a non-vanishing $\mathcal{O}(r^0)$ order. In these cases the Killing vectors decay too quickly and do not generate any symmetry algebra on the corner at infinite distance.

The point now is to compute the algebra by investigating the vector field bracket in order to make a link with the corner proposal. The corner proposal [314] is based on the fact that gauge symmetries are classical redundancies of the system that are not expected to survive in quantum gravity. This applies in particular to diffeomorphisms that are pure gauge; however, there are diffeomorphisms that are asymptotic symmetries which are not redundancy but they physically act on the field space. The corner proposal focuses on these symmetries, and asserts that they survive in quantum gravity. The central point is that a gravitational theory is described by a set of charges and their algebra at corners. In this direction we can prove the following

Theorem 5.3.2 (Axialgravisoliton corner symmetry algebra)**.** *The Lie algebra generated by the leading $\mathcal{O}(r^0)$ order asymptotic Killing vectors of the N-axialgravisoliton metric on the infinite distance corner S, is*

$$\mathfrak{agcsa} = \mathfrak{Diff}(S) \oplus \mathbb{R}^2; \tag{5.74}$$

which is a subalgebra of $\mathfrak{ucs}$.

Proof. The only Killing vectors that can generate a symmetry algebra on the infinite distance corner are the leading order ones; we refer to them simply as ξ^μ. We can expand a vector field in the coordinate basis on the corner obtained in the radial limit at fixed time ∂_i with $i = 1, 3$ and the "normal" coordinates ∂_a with $a = 0, 2$. Therefore

$$\xi^\mu \partial_\mu = \xi^i \partial_i + \xi^a \partial_a. \tag{5.75}$$

It is easy to see that these Killing vectors generate, as expected, a subalgebra of the $\mathfrak{ucs}$ algebra and therefore its exponential generate a subgroup of the UCS group. This is because the Lie bracket gives us, where expressed in coordinates

$$[\xi^i \partial_i + \xi^a \partial_a, \eta^i \partial_i + \eta^a \partial_a] = [\xi^i \partial_i, \eta^i \partial_i]^j \partial_j + (\xi^i \partial_i \eta^a - \eta^i \partial_i \xi^a)\partial_a; \tag{5.76}$$

the first term is $\mathfrak{Diff}(S)$ while the second one is the action of $\mathfrak{Diff}(S)$ on $\mathbb{R}^2$ where S is the infinite distance corner whose coordinates are the angles. Therefore the axialgravisoliton corner symmetry algebra ($\mathfrak{agcsa}$) generated by the leading order Killing vectors is

$$\mathfrak{agcsa} = \mathfrak{Diff}(S) \oplus\!\!\!\!+ \, \mathbb{R}^2. \tag{5.77}$$

□

5.4 Conclusions and outlook

In this Chapter, based on the work [329], we studied some formal and mathematical properties of a particular class of gravisolitons: axialgravisolitons. First of all we study the background metric compatible with the requests on the determinant of the metric, which is an essential condition to have physical metrics, and with the finiteness of associated Komar-like integrals, which are computed in Appendix E. These conditions force us to divide the asymptotic expansion of the background solution in cases summarized in Table 5.2. According to each case, the set of exponents $\{\alpha, \beta, \gamma, \delta\}$ is different but respects both the determinant condition and the Komar-like integrals finitness. Giving only these conditions on the background solution we find the expansion for any value of the soliton numbers N. According to the parameters $\{\alpha, \beta, \gamma, \delta, \epsilon_1, \epsilon_2, \epsilon_3\}$, the final N-soliton metric can develop different asymptotic behaviour with respect the background solution used to generate the solitons. This is reported in (5.47d), (5.48b). As technical note we underline that the asymptotic expansion of the component $g_{22}^{(ph)}$ depends strongly on the relative signs in front of the square roots in the definition of the poles. The asymptotic expansion developed can play, for N-axialgravisoliton spacetime, a similar role of Bondi expansion for asymptotically flat spacetime and Fefferman-Graham expansion for asymptotically AdS spacetime. We showed at the end of Paragraph 5.3.1 that, in general, the N-soliton metric does not have to preserve the asymptotically behaviour of the background metric even developing off-diagonal terms as in the case of Minkowski background and Kerr black hole (which is a 2-pole axialgravisoliton); therefore it is a well posed question asking for asymptotic symmetries and expecting that these may depend on the number of solitons that make up the solution. In order to compute the asymptotic Killing vectors we use the expansion derived to compute its Christoffel symbols in Appendix G. This is in fact the case since the leading order coefficients satisfy different differential equations according to the number of solitons N, however the radial leading order turn out to be the same in every case. The algebra generated by these Killing vectors can be computed looking to their bracket and turn out to be a subalgebra of the universal corner symmetry algebra given by $\mathfrak{agcsa} = \mathfrak{Diff}(S) \loplus \mathbb{R}^2$. This suggests a positive feedback for the corner proposal which is explicitly tested and opens the way to a possible quantization of the non-asymptotically flat sector of gravity by studying the representations of $\mathfrak{agcsa}$ since N-axialgravisoliton has not to be necessary asymptotically flat. Moreover, $\mathfrak{agcsa}$ could play a similar role of the conformal algebra for AdS spacetime or the $\mathfrak{BMS}$ algebra for asymptotically flat spacetime upon conformal compactification: we could construct a QFT at the boundary of the N-axialgravisoliton solution with symmetry algebra given by $\mathfrak{agcsa}$ and this could be dual to some gravitational process which take place in the bulk of the N-axialgravisoliton metric. This is a kind of holographic correspondence we can call N-axialgravisoliton/$\mathfrak{agcsa}$-QFT correspondence. The applicability and construction of this correspondence can be material for future work with the aims to better understand the role of gravisolitons in classical and quantum gravity.

Moreover, in the spirit of Strominger IR triangle, $\mathfrak{agcsa}$ is only one of the three corners. These axialgravisoliton asymptotic symmetries should be related, on the one hand, to some version or extension of the soft theorem on curved background[4] and, on the other hand, to an observable memory effect[5].

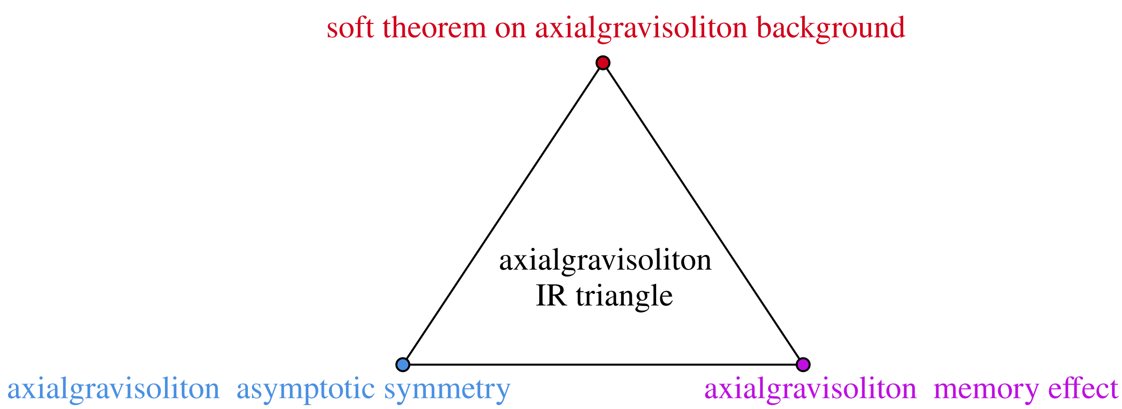


FIGURE 5.1 *Schematization of the axialgravisoliton IR triangle.*

Beyond the technical observational issues, the possible observation of this axialgravisoliton memory effect is complicated by the fact that we should be in the asymptotic region of a spacetime with a certain number of

[4] It is well-known that a global description of the S-matrix may not exist in an arbitrary curved spacetime. In [294], authors give a local construction of S-matrix in quantum field theory in curved spacetime using Riemann-normal coordinates which mimics the methods generally used in Minkowski spacetime.

[5] To distinguish this effect from the gravitational and spin-gravitational memory effects [227],[229] we call it axialgravisoliton memory effect.

axialsymmetric gravisolitons. The study and characterization of these other two corners of the IR triangle may be addressed in the near future.

As other future applications and extensions, it would be interesting to consider the case of higher derivative and $f(R)$ gravity. Both these cases are natural evolutions of Einstein gravity at UV regimes and the understanding, in these framework, of the gravisolitons role, of their properties and asymptotic symmetries could help in the road to quantum gravity. Other important cases of extension are those with spinors and supersymmetry, i.e. Einstein-spinors gravity and SUGRA theories. However, in all such contexts, the theory of gravisolitons, in the sense of ISM and Darboux transformation, is poorly developed. A first step in considering these extensions would be to develop a systematic metric valid for every number of solitons such as (5.3).

Chapter 6

General conclusions and outlook

In this conclusive Chapter, let us briefly summarize what we have been able to achieve and let us try to identify possible directions to be explored in the future.

The PhD thesis has dealt with gauge theories and has aimed at understanding them from a theoretical, formal and mathematical point of view. Specifically, the thesis had two main directions[1]. On the one hand in Chapters 3 and 4, we dealt with exotic gauge theories, i.e. whose gauge field carries p-forms or mixed symmetry tensors irreducible representation of the Lorentz group, of their asymptotic symmetries, i.e. gauge transformations acting non-trivially on physical states, and their duality emerging from representation theory and Young tableaux machinery. The main goal was to understand if the asymptotic charge of one description have access to the dual description information and if there exists a duality map between asymptotic charges of different dual descriptions. On the other hand, in Chapter 5, we studied gravity as gauge theory focusing on some particular solution of Einstein gravity in vacuum called gravisolitons; we studied their asymptotic expansion and their asymptotic symmetries with the ideas to emulate the BMS construction and to make a connection with the corner proposal.

In Chapter 3, based on works [326] and [330], we discussed the asymptotic symmetries and the Young machinery duality in the realm of p-form gauge theories. First of all, the issue of polyhomogeneous expansion with leading order logarithmic terms is addressed. We understood that, for every form degree and for radiation fall-offs on the field components, we should consider a leading order logarithmic term in the field components $B_{ui_1 \dots i_{p-1}}$ otherwise the asymptotic charges trivialize. This term does not generate unwanted unphysical divergences in the energy flux at null infinity even in the presence of matter if and only if matter current satisfies some specific fall-offs. Nevertheless, the very same mechanism fails in the case of Coulomb fall-offs on the field components. We stress also that this mechanism works for every finite power of the logarithm and would be interesting to understand if similar mechanisms happens for different functions. Moreover, it would be interesting to understand better the role of matter in mechanisms such as the one discussed above and in which physical contexts matter currents with the right fall-offs emerge in addition to the cited case of the $\mathcal{N} = 0$ supergravity vertex.

The asymptotic charges for both radiation and Coulomb fall-offs are computed and are dubbed electric-like since only electric-like components of the field strengths enter in the game. In the perspective of the duality between Young tableaux, and therefore of the on-shell duality which link p-forms to q-forms with $q = D - p - 2$, we studied the possibilities for a duality map.

On the one hand, we used the Hodge duality equations to derive a magnetic-like charge in the dual theory where the dubbing "magnetic-like" is due to the presence of only magnetic-like components of the field strength. An interesting feature of the magnetic-like charges is that the gauge parameter components involved is the one of the original theory. The meaning of this point is not yet fully clear and magnetic-like charges could be seen, in appropriate writing, as generalized charges since they are associated with fields whose sources are extended

[1] During the PhD, I investigate also topics concerning String Theory and the Geometrical Engineering of QFTs [310],[320].

and not point-like. Considering the possibility of generalized symmetries where the charged objects are not point-like in the framework of asymptotic symmetries could be a fruitful area that will be investigated in future works that could lead to a deeper understanding of the higher form symmetries in physical theories. Electric-like and magnetic-like can be merged to form electromagnetic-like charges which transform according to Moebius transformation under the duality. The electromagnetic-like charges generate a complexified $\mathfrak{u}_{\mathbb{C}}(1)$ symmetry algebra in analogy to what happens in $D = 4$ Maxwell theory. Hence, the presence of the gauge parameter of the original theory in the magnetic-like charge seems to point in the direction that the $\mathfrak{u}_{\mathbb{C}}(1)$ symmetry algebra contains both gauge parameters of the two dual descriptions: one acting on the theory in a standard way and the other acting on the theory written in the dual variables. This is similar to gravitational electric-magnetic duality in which electric and magnetic parts of the Weyl tensor contribute to different asymptotic charges. Moreover, it would be interesting to better understand the meaning of this complexified symmetry algebra and the relation of its transformation under the duality in the realm of celestial holography.
On the other hand, we proved the existence and uniqueness of a map, at least on Minkowski spacetime, which sends the electric-like charges of the dual descriptions one into an other. The theorem has a general validity; indeed, under the requirement of triviality of some cohomology groups of the differential manifold under consideration, we get the same result. Therefore the duality map has a topological nature. The physical interpretation can be achived by working in the physical state space, i.e. ray space. In ray space electric-like and magnetic-like charges of both descriptions act in the same way since they are equal, modulo complex numbers, as operators in Hilbert space. This seems to be consistent to the fact that the two theories are describing the same physics and the same degrees of freedom. Moreover, we proposed a speculation on the interplay of trivial gauge transformations and fall-offs too weak which make the asymptotic charge divergent. Hence, under topological conditions the asymptotic charge of a description inform us about the physics of the dual description. Last but not least, the issue of soft theorems: it would be interesting understand if and how these electric-like and magnetic-like charges enter in the business of soft theorems. Understanding this point will shed new light on this charges and could give some insights on soft theorems for p-form gauge theories.

In Chapter 4, based on works [325] and [334], we discussed the asymptotic symmetries and the Young machinery duality in the realm of mixed symmetry tensor gauge theories. We discussed how to develop a generalization of the de Rham complex to the case of mixed symmetry tensors. This leads to the definition of $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex and its cohomology-like groups which could characterize the topology of the differential manifold under consideration. These abstract mathematical tools can be useful to extend results proven in p-form gauge theories to mixed symmetry tensor gauge theories. Specifically, all results that are based on the underling cochain complex of p-form spaces can be extended looking to the $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex. Here, we employed it to prove a theorem, which extends the one proved for p-form gauge theories, on the existence and uniqueness of a set of duality maps for mixed symmetry tensor gauge theories. The physical meaning is that given a description and its asymptotic charge, this charge has access to physical information related to asymptotic charges of the dual formulations. This is expected since it happens for differential forms which are, in a certain sense, the building blocks of mixed symmetry tenors. A crucial point for the duality map result is to prove that there exist an isomorphism between de Rham-like cohomology groups and de Rham cohomology groups. This issue can be approched in three main ways. The first is to prove a Poincaré-like lemma for the differential mixed symmetry tensors in order to make the problem of exactness from a local one to a global one; in this way we have hints that the de Rham-like cohomology groups are not affected by the differential structure, that is of local nature, but only by the topological structure. The second is to use acyclic resolutions and abstract de Rham theorem to conclude that de Rham-like cohomology group are isomorphic to singular cohomology groups. The third one is to show that every $(k_1, ..., k_{N-1})$-augmented N-de Rham-like complex is homotopy equivalent to the standard de Rham complex in order to get an isomorphism between the de Rham-like cohomology groups and the standard de Rham cohomology groups. The reason for this pure mathematical effort is, on one side to understand the topological nature of the duality map and, on the other side, to better understand how strong are the hypothesis and which manifold satisfies them. The outcome is a theorem which ensures the equivalence between the de Rham-like cohomology groups and the standard de Rham cohomology groups.

From a more theoretical point of view, we are interested in computing the asymptotic charges of the dual description of the graviton in $D = 5$. In this direction we were able to compute the Noether current and the next points in the agenda are to perform the asymptotic analysis and compute the asymptotic charge. Moreover, from the explicit writing of the asymptotic charge, we could, using the Hodge duality equations, search for magnetic-like charges as done in the case of p-forms. Despite this project is at its dawn, its implications in understanding gravity, at least at the the linearized level, could be very deep. Indeed, if we are able to quantize dual formulations of gravity and if these dual formulation have access to all the physical information of the graviton formulation, we could get now insights on how to quantize gravity itself.

In Chapter 5, based on the work [329], we studied some formal and mathematical properties of a particular class of gravisolitons: axialgravisolitons. Giving minimal, but necessary, assumptions on the background solution metric we developed an asymptotic expansion for any value of the gravisolitons number N. The asymptotic expansion developed can play, for any N-axialgravisolitons spacetime, a similar role of Bondi expansion for asymptotically flat spacetime and Fefferman-Graham expansion for asymptotically AdS spacetime.

Since, in general, the N-gravisolitons metric does not have to preserve the asymptotically behaviour of the background metric, It is a well-posed question asking for asymptotic symmetries; the expectation is that these might depend on the number of gravisolitons. We computed the asymptotic Killing vectors and the algebra generated by these Killing vectors turned out to be a subalgebra of the universal corner symmetry algebra we called $\mathfrak{agcsa}$. This suggests a positive feedback for the corner proposal which is explicitly tested and opens the way to a possible quantization of the non-asymptotically flat sector of gravity by studying the representations of $\mathfrak{agcsa}$. Moreover, we could construct a QFT at the boundary of the N-axialgravisolitons solution with symmetry algebra given by $\mathfrak{agcsa}$ and this could be dual to some gravitational processes which take place in the bulk of the N-axialgravisolitons spacetime. This is a kind of holographic correspondence we called N-axialgravisolitons/$\mathfrak{agcsa}$-QFT correspondence. The applicability and construction of this correspondence can be material for future work with the aim of better understanding the role of gravisolitons in classical and quantum gravity. Moreover, in the spirit of Strominger IR triangle, $\mathfrak{agcsa}$ is only one of the three corners. These axialgravisoliton asymptotic symmetries should be related, on the one hand, to some version or extension of

the soft theorem on curved background and, on the other hand, to an observable memory effect. The study and characterization of these other two corners of the IR triangle may be addressed in the near future.
As other future applications and extensions, it would be interesting to consider the case of higher derivative and $f(R)$ gravity. Both these cases generalize the Einstein-Hilbert action adding terms which are suppressed in the low energy limits, so they are natural evolutions of Einstein gravity at UV regimes. The understanding, in these modified theories of gravity, of the gravisolitons role, of their properties and asymptotic symmetries could help in the road to quantum gravity. Other important cases of extension are those with spinors and supersymmetry, i.e. Einstein-spinors gravity and SUGRA theories. However, in all such contexts, the theory of gravisolitons, in the sense of ISM and Darboux transformation, is still poorly developed. The better understanding of integrability issues linked to gravisoliton solutions, in gravitational theories which could be effective low energy theories of a full quantum theory of gravity, can lead to new insights on the mathematical structure of gravity with the possibility of understanding the topology changes present in any quantum theory of gravity. Furthermore, gravitational solitons can also be studied on complex manifolds and it may be interesting to proceed to determine their asymptotic algebras on Calabi-Yau manifolds with the idea of quantizing the compactified trasverse geometry in string theories.

In the end, this PhD thesis provides new insights in many areas of interest of both physics and mathematics involving gauge theories and their asymptotic symmetries, the integrable structure of General Relativity and gravisolitons, dualities in gauge theory and their observables, quantum gravity and holography, the cohomology of differentiable manifolds and partial differential equations.
From a physical perspective works [325],[326],[330],[334] ensures that asymptotic charges of one description have access, under topological conditions for the underlining manifold, to physical information contained in the asymptotic charges of the other dual descriptions. This leads to a greater understanding of physical observables in gauge theories. From this point of view, the dual description of linearized gravity can have access to the physical information of asymptotic observables of linearized gravity itself, which makes the study of dual descriptions of linearized gravity and their quantization a tool to better understand the asymptotic nature of quantum gravity in the linearized sector. Indeed, the quantization of asymptotic observables of dual descriptions of linearized gravity could have access to physical information related to the quantum nature of the asymptotic observables of the linearized sector of quantum gravity. At the full non-linear level of gravity, work [329] sheds light on the integrable structure of General Relativity, providing its asymptotic structure by asymptotically expanding the gravisoliton solutions. The infinite corner algebra computed makes the ideas of the corner proposal applicable to the quantization of this sector of gravity and opens the way to new holographic correspondences that can give new insights into the holographic structure in general.
From a mathematical perspective works [326],[334] develop a new cohomological theory for differentiable manifolds bringing new tools for the development of a mathematics suitable for a theory of quantum gravity while work [335] gives a result on the solutions of a system of partial differential equations that is often encountered in the study of asymptotic symmetries in gauge theories.

"Ai posteri l'ardua sentenza".

Appendix A

Derivation of Lorenz-like gauge preserving condition for the $p = 1$ and $p = 2$ forms

The Laplace-Beltrami operator[1], in Bondi coordinates reads

$$\square := g^{\mu\nu}\nabla_\mu\nabla_\nu = -2\nabla_u\nabla_r + \nabla_r\nabla_r + \frac{1}{r^2}\gamma^{ij}\nabla_i\nabla_j. \tag{A.1}$$

Let us start with the scalar gauge parameter; the homogeneous wave equation reads

$$-2\nabla_u\nabla_r\epsilon + \nabla_r\nabla_r\epsilon + \frac{1}{r^2}\gamma^{ij}\nabla_i\nabla_j\epsilon = 0; \tag{A.2}$$

making explicit the covariant derivatives we get

$$\left[\partial_r^2 - 2\partial_u\partial_r - \frac{(D-2)}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right]\epsilon = 0, \tag{A.3}$$

where $\Delta := \Delta_{\mathbb{S}^{D-2}}$ is the Laplace-Beltrami operator on the $(D-2)$-sphere. Inserting a polyhomogeneous expansion gives us

$$\begin{aligned}
&[(l-1)(l-D+2)+\Delta]\tilde{\epsilon}^{(l-1)} + (2l-D+2)\partial_u\tilde{\epsilon}^{(l)} = 0;\\
&[(l-1)(l-D+2)+\Delta]\epsilon^{(l-1)} + (2l-D+2)\partial_u\epsilon^{(l)} - 2\partial_u\tilde{\epsilon}^{(l)} + (D-2l-1)\tilde{\epsilon}^{(l-1)} = 0;
\end{aligned} \tag{A.4}$$

for future convenience we define

$$k^{(p=1)}(l) := -(l-1)(l-D+2). \tag{A.5}$$

For the vector gauge parameter the procedure is very similar; the wave equation reads

$$-2\nabla_u\nabla_r\epsilon_\alpha + \nabla_r\nabla_r\epsilon_\alpha + \frac{1}{r^2}\gamma^{ij}\nabla_i\nabla_j\epsilon_\alpha = 0, \tag{A.6}$$

[1] This is a generalization of the Laplace operator to functions defined on Riemannian and pseudo-Riemannian manifolds. More generally, by Laplace-Beltrami operator, we mean the extension which operate on tensors as the divergence of the covariant derivative. More generally, one can define a Laplacian differential operator on sections of the bundle of differential forms on a pseudo-Riemannian manifold. On a Riemannian manifold it is an elliptic operator, while on a Lorentzian manifold it is hyperbolic. The Laplace–de Rham operator is defined by $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ where d is the exterior derivative or differential and δ is the codifferential. The Laplace–de Rham operator differs from the Laplace-Beltrami operator restricted to act on skew-symmetric tensors due to the Weitzenböck identity.

and making explicit the covariant derivatives we get

$$
\begin{aligned}
&\left[\partial_r^2 - 2\partial_u\partial_r - \frac{(D-2)}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right]\epsilon_u = 0,\\
&\left[\partial_r^2 - 2\partial_u\partial_r - \frac{(D-2)}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right]\epsilon_r + \frac{D-2}{r^2}(\epsilon_u - \epsilon_r) - \frac{2}{r^3}\nabla^i\epsilon_i = 0,\\
&\left[\partial_r^2 - 2\partial_u\partial_r - \frac{(D-4)}{r}(\partial_u - \partial_r) + \frac{\Delta}{r^2}\right]\epsilon_i - \frac{D-3}{r^3}\epsilon_i - \frac{2}{r}\nabla_i(\epsilon_u - \epsilon_r) = 0.
\end{aligned} \tag{A.7}
$$

First we note that the equation for ϵ_u is the same as the one for the scalar gauge parameter due to the lack of connection symbols with u as down index; then inserting the polyhomogeneous expansion for the other components of the vector gauge parameter we get

$$[(l-1)(l-D+2)+\Delta]\tilde{\epsilon}_u^{(l-1)} + (2l-D+2)\partial_u\tilde{\epsilon}_u^{(l)} = 0; \tag{A.8a}$$

$$[(l-1)(l-D+2)+\Delta]\epsilon_u^{(l-1)} + (2l-D+2)\partial_u\epsilon_u^{(l)} - 2\partial_u\tilde{\epsilon}_u^{(l)} + (D-2l-1)\tilde{\epsilon}_u^{(l-1)} = 0; \tag{A.8b}$$

$$[l(l+1-D)+\Delta]\tilde{\epsilon}_r^{(l-1)} + (2l-D+2)\partial_u\tilde{\epsilon}_r^{(l)} + (D-2)\tilde{\epsilon}_u^{(l-1)} - 2\nabla^i\tilde{\epsilon}_i^{(l-2)} = 0; \tag{A.8c}$$

$$[l(l+1-D)+\Delta]\epsilon_r^{(l-1)} + (2l-D+2)\partial_u\epsilon_r^{(l)} + (D-2)\epsilon_u^{(l-1)} - 2\nabla^i\epsilon_i^{(l-2)} + (D-2l-1)\tilde{\epsilon}_r^{(l-1)} - 2\partial_u\tilde{\epsilon}_r^{(l)} = 0; \tag{A.8d}$$

$$[l(l-D+3)-1+\Delta]\tilde{\epsilon}_i^{(l-1)} + (2l-D+4)\partial_u\tilde{\epsilon}_i^{(l)} + 2\nabla_i(\tilde{\epsilon}_r^{(l)} - \tilde{\epsilon}_u^{(l)}) = 0; \tag{A.8e}$$

$$[l(l-D+3)-1+\Delta]\epsilon_i^{(l-1)} + (2l-D+4)\partial_u\epsilon_i^{(l)} + 2\nabla_i(\epsilon_r^{(l)} - \epsilon_u^{(l)}) + (D-2l-3)\partial_u\tilde{\epsilon}_i^{(l-1)} - 2\partial_u\tilde{\epsilon}_i^{(l)} = 0; \tag{A.8f}$$

for future convenience we define

$$
\begin{aligned}
k_u^{(p=2)}(l) &:= -(l-1)(l-D+2);\\
k_r^{(p=2)}(l) &:= -l(l+1-D);\\
k_i^{(p=2)}(l) &:= -l(l-D+3).
\end{aligned} \tag{A.9}
$$

Together with these equations, we have to taken into account the divergence free of the gauge parameter:

$$\nabla^\mu\epsilon_\mu = -\partial_u\epsilon_r + \frac{1}{r}(r\partial_r + D - 2)(\epsilon_r - \epsilon_u) + \frac{1}{r^2}\nabla^i\epsilon_i = 0; \tag{A.10}$$

which inserting the polyhomogenoues expansion gives us

$$
\begin{aligned}
&-\partial_u\tilde{\epsilon}_r^{(l)} + (l-D+1)(\tilde{\epsilon}_u^{(l-1)} - \tilde{\epsilon}_r^{(l-1)}) + \nabla^i\tilde{\epsilon}_i^{(l-2)} = 0;\\
&-\partial_u\epsilon_r^{(l)} + (l-D+1)(\epsilon_u^{(l-1)} - \epsilon_r^{(l-1)}) + \nabla^i\epsilon_i^{(l-2)} - \tilde{\epsilon}_u^{(l-1)} + \tilde{\epsilon}_r^{(l-1)} = 0.
\end{aligned} \tag{A.11}
$$

Appendix B

The necessity of logarithms for 1-form and 2-form

For the 1-form, let us consider the Lorentz gauge fixing preserving conditions (A.4). Consider $l = \frac{D-2}{2}$; we get

$$\left[-k^{(p=1)}\left(\frac{D-2}{2}\right)+\Delta\right]\epsilon^{(\frac{D-4}{2})} = 2\partial_u\tilde{\epsilon}^{(\frac{D-2}{2})}. \tag{B.1}$$

If we suppress the logarithmic terms we would get

$$\left[-k^{(p=1)}\left(\frac{D-2}{2}\right)+\Delta\right]\epsilon^{(\frac{D-4}{2})} = 0 \quad \Rightarrow \quad \epsilon^{(\frac{D-4}{2})} = 0; \tag{B.2}$$

and we would not have, in sufficiently high dimension, a finite non-zero asymptotic charges. In the same manner, for the 2-form, let us consider the Lorentz gauge fixing preserving conditions (A.8) and the divergence free condition. Before proceeding on, let us consider how to fix the gauge for gauge fixing (3.65). The conditions to fix the gauge for gauge fixing are given by

$$\begin{aligned}
\tilde{\epsilon}_u^{(\frac{D}{2})} &= -\partial_u\tilde{\epsilon}^{(\frac{D}{2})};\\
\epsilon_i^{(\frac{D-8}{2})} &= -\partial_i\epsilon^{(\frac{D-8}{2})};\\
\tilde{\epsilon}_i^{(\frac{D-6}{2})} &= -\partial_i\tilde{\epsilon}^{(\frac{D-6}{2})}.
\end{aligned} \tag{B.3}$$

In order to impose these constraints we need to consider a 2-polyhomogeneous expansion for the gauge for gauge parameter of the form

$$\epsilon = \sum_{l\in\frac{1}{2}\mathbb{Z}} \frac{\epsilon^{(l)}(u,\{x^i\}) + \tilde{\epsilon}^{(l)}(u,\{x^i\})ln(r) + \tilde{\tilde{\epsilon}}^{(l)}(u,\{x^i\})ln^2(r)}{r^l}. \tag{B.4}$$

which inserted in $\Box\epsilon = 0$ gives us

$$[(l-1)(l-D+2)+\Delta]\tilde{\tilde{\epsilon}}^{(l-1)} + (2l-D+2)\partial_u\tilde{\tilde{\epsilon}}^{(l)} = 0; \tag{B.5a}$$

$$[(l-1)(l-D+2)+\Delta]\tilde{\epsilon}^{(l-1)} + (2l-D+2)\partial_u\tilde{\epsilon}^{(l)} - 4\partial_u\tilde{\tilde{\epsilon}}^{(l)} + 2(D-2l-1)\tilde{\tilde{\epsilon}}^{(l-1)} = 0; \tag{B.5b}$$

$$[(l-1)(l-D+2)+\Delta]\epsilon^{(l-1)} + (2l-D+2)\partial_u\epsilon^{(l)} + 2\tilde{\tilde{\epsilon}}^{(l-1)} - 2\partial_u\tilde{\epsilon}^{(l)} + (D-2l-1)\tilde{\epsilon}^{(l-1)} = 0. \tag{B.5c}$$

From equation (B.5a) with $l = \frac{D-2}{2}$ and proceeding recursively we find that, in sufficiently high dimension,

$$\tilde{\tilde{\epsilon}}^{(l)} = 0 \qquad \forall l \leq \frac{D-4}{2}; \tag{B.6}$$

moreover from equation (B.5b) with $l = \frac{D+2}{2}$, $l = \frac{D}{2}$, $l = \frac{D-2}{2}$ and $l = \frac{D-4}{2}$ we get respectively

$$\left[-k^{(p=1)}\left(\frac{D+2}{2}\right)+\Delta\right]\tilde{\epsilon}^{(\frac{D}{2})} = -4\partial_u\tilde{\epsilon}^{(\frac{D+2}{2})} + 4\partial_u\tilde{\tilde{\epsilon}}^{(\frac{D+2}{2})} + 6\tilde{\tilde{\epsilon}}^{(\frac{D}{2})}; \tag{B.7a}$$

$$\left[-k^{(p=1)}\left(\frac{D}{2}\right)+\Delta\right]\tilde{\epsilon}^{(\frac{D-2}{2})} = -2\partial_u\tilde{\epsilon}^{(\frac{D}{2})} + 4\partial_u\tilde{\tilde{\epsilon}}^{(\frac{D}{2})} + 2\tilde{\tilde{\epsilon}}^{(\frac{D-2}{2})}; \tag{B.7b}$$

$$\left[-k^{(p=1)}\left(\frac{D-2}{2}\right)+\Delta\right]\tilde{\epsilon}^{(\frac{D-4}{2})} = 4\partial_u\tilde{\tilde{\epsilon}}^{(\frac{D-2}{2})}; \tag{B.7c}$$

$$\left[-k^{(p=1)}\left(\frac{D-4}{2}\right)+\Delta\right]\tilde{\epsilon}^{(\frac{D-6}{2})} = 2\partial_u\tilde{\epsilon}^{(\frac{D-4}{2})}; \tag{B.7d}$$

while from (B.5c) with $l = \frac{D+2}{2}$, $l = \frac{D}{2}$, $l = \frac{D-2}{2}$, $l = \frac{D-4}{2}$, $l = \frac{D-6}{2}$ and $l = \frac{D-8}{2}$ we get respectively

$$\left[-k^{(p=1)}\left(\frac{D+2}{2}\right)+\Delta\right]\epsilon^{(\frac{D}{2})} = -4\partial_u\epsilon^{(\frac{D+2}{2})} - 2\tilde{\tilde{\epsilon}}^{(\frac{D}{2})} + 2\partial_u\tilde{\epsilon}^{(\frac{D+2}{2})} + 3\tilde{\epsilon}^{(\frac{D}{2})}; \tag{B.8a}$$

$$\left[-k^{(p=1)}\left(\frac{D}{2}\right)+\Delta\right]\epsilon^{(\frac{D-2}{2})} = -2\partial_u\epsilon^{(\frac{D}{2})} - 2\tilde{\tilde{\epsilon}}^{(\frac{D-2}{2})} + 2\partial_u\tilde{\epsilon}^{(\frac{D}{2})} + \tilde{\epsilon}^{(\frac{D-2}{2})}; \tag{B.8b}$$

$$\left[-k^{(p=1)}\left(\frac{D-2}{2}\right)+\Delta\right]\epsilon^{(\frac{D-4}{2})} = 2\partial_u\tilde{\epsilon}^{(\frac{D-2}{2})} - \tilde{\epsilon}^{(\frac{D-4}{2})}; \tag{B.8c}$$

$$\left[-k^{(p=1)}\left(\frac{D-4}{2}\right)+\Delta\right]\epsilon^{(\frac{D-6}{2})} = 2\partial_u\epsilon^{(\frac{D-4}{2})} + 2\partial_u\tilde{\epsilon}^{(\frac{D-4}{2})} - 3\tilde{\epsilon}^{(\frac{D-6}{2})}; \tag{B.8d}$$

$$\left[-k^{(p=1)}\left(\frac{D-6}{2}\right)+\Delta\right]\epsilon^{(\frac{D-8}{2})} = 4\partial_u\epsilon^{(\frac{D-6}{2})} + 2\partial_u\tilde{\epsilon}^{(\frac{D-6}{2})} - 5\tilde{\epsilon}^{(\frac{D-8}{2})}; \tag{B.8e}$$

$$\left[-k^{(p=1)}\left(\frac{D-8}{2}\right)+\Delta\right]\epsilon^{(\frac{D-10}{2})} = 6\partial_u\epsilon^{(\frac{D-8}{2})} + 2\partial_u\tilde{\epsilon}^{(\frac{D-8}{2})} - 7\tilde{\epsilon}^{(\frac{D-10}{2})}. \tag{B.8f}$$

The gauge for gauge fixing equation fix for us the functions $\tilde{\epsilon}^{(\frac{D}{2})}$, $\tilde{\epsilon}^{(\frac{D-6}{2})}$ and $\epsilon^{(\frac{D-8}{2})}$; then equations (B.7c) and (B.7d) fix the functions $\tilde{\tilde{\epsilon}}^{(\frac{D-2}{2})}$ and $\tilde{\epsilon}^{(\frac{D-4}{2})}$. Moreover, considering equation (B.5b) for $l \leq \frac{D-6}{2}$ we get that all the functions $\tilde{\epsilon}^{(l)}$ with $l \leq \frac{D-8}{2}$ are fixed. At this point, equations (B.8c), (B.8d), (B.8e) and (B.8f) together to (B.5c) for $l \leq \frac{D-10}{2}$ fix the functions

$\epsilon^{(\frac{D-6}{2})}$, $\epsilon^{(\frac{D-4}{2})}$, $\tilde{\epsilon}^{(\frac{D-2}{2})}$, and $\epsilon^{(l)}$ with $l \leq \frac{D-10}{2}$. However, from equations (B.7a), (B.7b), (B.8a) and (B.5b) we have the unfixed functions $\tilde{\tilde{\epsilon}}^{(\frac{D+2}{2})}$, $\tilde{\tilde{\epsilon}}^{(\frac{D}{2})}$, $\tilde{\epsilon}^{(\frac{D+2}{2})}$, $\tilde{\epsilon}^{(\frac{D-2}{2})}$, $\epsilon^{(\frac{D+2}{2})}$, $\epsilon^{(\frac{D}{2})}$ and $\epsilon^{(\frac{D-2}{2})}$ to stop the chain of fixing functions and be sure not to set gauge fixing with the same arbitrary functions.

After these considerations on the gauge fixing, we can proceed considering the preserving Lorentz gauge condition for the 2-form. Starting from (A.8a), we consider first $l = \frac{D}{2}$ getting

$$\left[-k_u^{(p=2)}\left(\frac{D}{2}\right)+\Delta\right]\tilde{\epsilon}_u^{(\frac{D-2}{2})} = 0 \quad\Rightarrow\quad \tilde{\epsilon}_u^{(\frac{D-2}{2})} = 0; \tag{B.9}$$

next, we consider $l = \frac{D-2}{2}$ to get

$$\left[-k_u^{(p=2)}\left(\frac{D-2}{2}\right)+\Delta\right]\tilde{\epsilon}_u^{(\frac{D-4}{2})} = 0 \quad\Rightarrow\quad \tilde{\epsilon}_u^{(\frac{D-4}{2})} = 0; \tag{B.10}$$

going on in this way we get, in sufficiently high dimension,

$$\tilde{\epsilon}_u^{(l)} = 0 \qquad \forall l \leq \frac{D}{2}, \tag{B.11}$$

where the equality comes from our gauge for gauge fixing. Now we take into account equation (A.8b) and we consider respectively $l = \frac{D-2}{2}$ and $l = \frac{D-4}{2}$:

$$\begin{aligned}
&\left[-k_u^{(p=2)}\left(\frac{D-2}{2}\right)+\Delta\right]\epsilon_u^{(\frac{D-4}{2})} = 0 \quad\Rightarrow\quad \epsilon_u^{(\frac{D-4}{2})} = 0;\\
&\left[-k_u^{(p=2)}\left(\frac{D-4}{2}\right)+\Delta\right]\epsilon_u^{(\frac{D-6}{2})} = 0 \quad\Rightarrow\quad \epsilon_u^{(\frac{D-6}{2})} = 0;
\end{aligned} \tag{B.12}$$

going on, we get, in sufficiently high dimension,

$$\epsilon_u^{(l)} = 0 \qquad \forall l \leq \frac{D-4}{2}, \tag{B.13}$$

It is now the turn of (A.8c); let us start with $l = \frac{D-2}{2}$, we get

$$\left[-k_r^{(p=2)}\left(\frac{D-2}{2}\right)+\Delta\right]\tilde{\epsilon}_r^{(\frac{D-4}{2})} = 0 \quad\Rightarrow\quad \tilde{\epsilon}_r^{(\frac{D-4}{2})} = 0; \tag{B.14}$$

if we now consider equation (A.8c) with $l = \frac{D-4}{2}$ and equation (A.8e) with $l = \frac{D-6}{2}$ we get

$$\begin{aligned} \left[-k_r^{(p=2)} \left(\frac{D-4}{2} \right) + \Delta \right] \tilde{\epsilon}_r^{(\frac{D-6}{2})} &= 2\nabla^i \tilde{\epsilon}_i^{(\frac{D-8}{2})}; \\ \left[-k_i^{(p=2)} \left(\frac{D-6}{2} \right) + \Delta \right] \tilde{\epsilon}_i^{(\frac{D-8}{2})} &= -2\nabla_i \tilde{\epsilon}_r^{(\frac{D-6}{2})}; \end{aligned} \tag{B.15}$$

and using results from Appendix C we find that

$$\tilde{\epsilon}_r^{(\frac{D-6}{2})} = \tilde{\epsilon}_i^{(\frac{D-8}{2})} = 0. \tag{B.16}$$

Taking now equation (A.8c) with $l = \frac{D-6}{2}$ and equation (A.8e) with $l = \frac{D-8}{2}$ we get

$$\begin{aligned} \left[-k_r^{(p=2)} \left(\frac{D-6}{2} \right) + \Delta \right] \tilde{\epsilon}_r^{(\frac{D-8}{2})} &= 2\nabla^i \tilde{\epsilon}_i^{(\frac{D-10}{2})}; \\ \left[-k_i^{(p=2)} \left(\frac{D-8}{2} \right) + \Delta \right] \tilde{\epsilon}_i^{(\frac{D-10}{2})} &= -2\nabla_i \tilde{\epsilon}_r^{(\frac{D-8}{2})}; \end{aligned} \tag{B.17}$$

and using again results from Appendix C we find that

$$\tilde{\epsilon}_r^{(\frac{D-8}{2})} = \tilde{\epsilon}_i^{(\frac{D-10}{2})} = 0. \tag{B.18}$$

Going on we get, in sufficiently high dimension,

$$\begin{aligned} \tilde{\epsilon}_r^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-4}{2}; \\ \tilde{\epsilon}_i^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-6}{2}. \end{aligned} \tag{B.19}$$

where the equality in the second equation take into account our gauge for gauge fixing. Before continuing with our discussion, let us take into account equations (3.64), i.e. the preserving fall-off conditions up to logarithmic terms. The first and the third conditions give us, using results from above,

$$\begin{aligned} \partial_u \epsilon_r^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-4}{2}; \\ \partial_u \epsilon_i^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-6}{2}; \end{aligned} \tag{B.20}$$

the fifth and the seventh are not useful now while the other are trivially solved. Now, considering the first equation (A.11) with $l = \frac{D-2}{2}$ we get

$$\partial_u \tilde{\epsilon}_r^{(\frac{D-2}{2})} = 0. \tag{B.21}$$

Let us consider equation (A.8d) with $l = \frac{D-4}{2}$ we have

$$\left[- k_r^{(p=2)} \left(\frac{D-4}{2} \right) + \Delta \right] \epsilon_r^{(\frac{D-6}{2})} = 0 \quad \Rightarrow \quad \epsilon_r^{(\frac{D-6}{2})} = 0; \tag{B.22}$$

considering, now equation (A.8d) with $l = \frac{D-6}{2}$ and equation (A.8f) with $l = \frac{D-8}{2}$ we get

$$\begin{aligned} \left[- k_r^{(p=2)} \left(\frac{D-6}{2} \right) + \Delta \right] \epsilon_r^{(\frac{D-8}{2})} &= 2\nabla^i \epsilon_i^{(\frac{D-10}{2})}; \\ \left[- k_i^{(p=2)} \left(\frac{D-8}{2} \right) + \Delta \right] \epsilon_i^{(\frac{D-10}{2})} &= -2\nabla_i \epsilon_r^{(\frac{D-8}{2})}; \end{aligned} \tag{B.23}$$

and using results from Appendix C we find that

$$\epsilon_r^{(\frac{D-8}{2})} = \epsilon_i^{(\frac{D-10}{2})} = 0; \tag{B.24}$$

Like the case of tilded functions above, we can go on using results of Appendix C; therefore, in sufficiently high dimension,

$$\begin{aligned} \epsilon_r^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-6}{2}; \\ \epsilon_i^{(l)} &= 0 \qquad \forall\, l \leq \frac{D-8}{2}. \end{aligned} \tag{B.25}$$

The fifth and the seventh equation of the preserving fall-off condition up to logarithm are now trivially solved.

Let us now show the necessity of logarithmic terms for the 2-form; as we will see in a while, in sufficently high dimension the presence of $\tilde{\epsilon}_i^{(\frac{D-4}{2})}$ is the lifesaver of asymptotic symmetry since it indirectly makes the asymptotic charges non-zero. Considering equation (A.8d) with $l = \frac{D-2}{2}$

and equation (A.8f) with $l = \frac{D-4}{2}$ we get, using constrain (B.21)

$$\begin{aligned}
&\left[-k_r^{(p=2)}\left(\frac{D-2}{2}\right)+\Delta\right]\epsilon_r^{(\frac{D-4}{2})} = 2\nabla^i\epsilon_i^{(\frac{D-6}{2})};\\
&\left[-k_i^{(p=2)}\left(\frac{D-4}{2}\right)+\Delta\right]\epsilon_i^{(\frac{D-6}{2})} = -2\nabla_i\epsilon_r^{(\frac{D-4}{2})} + 2\partial_u\tilde{\epsilon}_i^{(\frac{D-4}{2})};
\end{aligned} \tag{B.26}$$

and we would deduce that $\epsilon_r^{(\frac{D-4}{2})} = \epsilon_i^{(\frac{D-6}{2})} = 0$ due to results in Appendix C if logarithmic terms in the gauge parameter expansion would suppressed.

The very same analysis, with minimal modification, can be retraced in the case of Coulomb fall-offs leading to the same results.

Appendix C

On the solution of the harmonic-divgrad PDE system

Motivated by classical field theory, specifically by gauge theories where the gauge field is a p-form, we study a particular system of partial differential equations. Moreover, motivated by the study of asymptotic symmetries in gauge theories we consider function defined on a $(D-2)$-dimensional sphere known as the celestial sphere. By extension, we consider functions defined on a maximally symmetric space, mathematically known as space form. Physically speaking these spaces admits the maximal number of Killing vectors while mathematically speaking they are complete Riemannian manifolds of constant sectional curvature.
A powerful theorem of Riemannian geometry, the Killing–Hopf theorem [2],[5],[267], ensures that every space form admits a Riemanninan covering given by one of the sphere, the hyperbolic space or the Euclidean space depending on the sign and value of the sectional curvature. Thanks to this theorem and previous results on the biharmonic equation [215], we are able to prove that a particular system of partial differential equations, given in Definition C.2.1, on a positive curvature space form admits as only solution the trivial one [335].

C.1 Space form and Killing–Hopf theorem

Let us star with the definition of space form

Definition C.1.1 (Space form)**.** *A space form is a complete Riemannian manifold $(M, \boldsymbol{g})$ of constant sectional curvature K.*

The three most fundamental examples are Euclidean D-dimensional space, the D-dimensional sphere, and D-dimensional hyperbolic space, although a space form need not be simply connected. The fundamental theorem we are interested in is the following

Theorem C.1.1 (Killing-Hopf)**.** *Let $(M, \boldsymbol{g})$ be a complete D-dimensional Riemannian manifold of constant sectional curvature K, then the Riemannian universal cover $(\tilde{M}, \tilde{\boldsymbol{g}})$ of $(M, \boldsymbol{g})$ is*

- *the D-dimensional sphere if $K > 0$;*
- *the Euclidean D-dimensional space if $K = 0$;*
- *the D-dimensional hyperbolic space if $K < 0$.*

The proof of Killing-Hopf theorem lies in the use of Jacobi fields and standard lemmas of Riemannian geometry such as Gauss's lemma; the reader interested in the proof can find it, for

example, in [267]. The original references are [2],[5].
In a diagrammatic way the Killing-Hopf theorem can be viewed as

$$\begin{tikzcd} \tilde{M} \arrow[d, "\pi"'] \\ M \end{tikzcd} \tag{C.1}$$

where the pullback of the covering map is such that

$$\tilde{\boldsymbol{g}} = \pi^* \boldsymbol{g}. \tag{C.2}$$

Therefore, according to Killing-Hopf theorem, any complete Riemannian manifold of constant sectional curvature is the quotient of one of the three canonical examples above by a group (which is a subgroup of the corresponding isometry group) that acts freely and properly discontinuously. As a result, only very few smooth manifolds can admit a constant sectional curvature metric; despite this, these spaces are among the most widely used Riemannian manifolds in theoretical physics for example to model cosmological phenomena [231].

C.2 A result on the harmonic-divgrad PDEs system

Let us start with the definition of the harmonic-divgrad PDEs system

Definition C.2.1 (Harmonic-divgrad equations)**.** *Let* $(M, \boldsymbol{g})$ *a pseudo-Riemannian manifold with connection* ∇. *Let* $f, \{h_i\}_{i\in\mathbb{I}}$ *with* $\#\mathbb{I} < \#\mathbb{N}$ *be sufficiently smooth functions defined on* M *and let* k_1 *and* k_2 *be non-negative real numbers. The harmonic-divgrad equations are the system of partial differential equations*

$$[-k_1 + \Delta]f - 2\nabla^i h_i = 0; \tag{C.3a}$$

$$[-k_2 + \Delta]h_i + 2\nabla_i f = 0. \tag{C.3b}$$

This kind of system appears in same physical contexts such as p-form gauge theories [75], where its use is essential for the calculation of asymptotic charges in high dimensions or in the explicit computations of the asymptotic symmetries in mixed symmetry tensor gauge theories

[326],[330],[334].

We can prove the following result

Theorem C.2.1 (Trivialization of harmonic-divgrad equations on a positive sectional curvature space form)**.** *Let $(M, \boldsymbol{g})$ be a space form (with sectional curvature $K > 0$) of dimension $D-2$. Then the harmonic-divgrad equations admit as only solution the trivial solution, $f = 0$ and $h_i = 0 \quad \forall i \in \mathbb{I} := [1, D-2]$.*

Proof. By Killing–Hopf Theorem C.1.1 the Riemanninan covering of $(M, \boldsymbol{g})$ is a $(D-2)$-dimensional sphere whose components of the metric tensor will be called $\gamma_{\alpha\beta}$. Hence we first prove the theorem for a $(D-2)$-dimensional sphere.
Let us now taking the gradient of equation (C.3b) we get

$$2\Delta f - k_2 \nabla^i h_i + \nabla^i \nabla^j \nabla_j h_i = 0; \tag{C.4}$$

where, due to curvature effects, we cannot exchange covariant derivatives. However using the Riemann tensor components identity $[\nabla^i, \nabla^j]h_i = \nabla^i \nabla^j h_i - \nabla^j \nabla^i h_i = R^k_{iij} h_k = g^{ks} R_{siij} h_j$ we can go on. Indeed

$$\begin{aligned}
\nabla^i \nabla^j (\nabla_j h_i) &= \nabla^j \nabla^i (\nabla_j h_i) + R^j_{kij} (\nabla^k h^i) + R^i_{kij} (\nabla^j h^k) = \\
&= R^j_{kij} (\nabla^k h^i) + R^i_{kij} (\nabla^j h^k) + \nabla^j \nabla_j \nabla_i h^i + \nabla^j (R^i_{kij} h^k) = \\
&= \nabla^j \nabla_j \nabla_i h^i - R_{ki} (\nabla^k h^i) + R_{kj} (\nabla^j h^k) + \nabla^j (R_{kj} h^k) = \\
&= \nabla^j \nabla_j \nabla_i h^i + \nabla^j (R_{kj} h^k);
\end{aligned} \tag{C.5}$$

but the Riemann tensor components for a maximally symmetric space with sectional curvature $K > 0$ are given by

$$R_{\alpha\beta\gamma\delta} = K \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right) \tag{C.6}$$

from which, using the metric of the $(D-2)$-sphere $\gamma_{\alpha\beta}$ we get

$$R_{\beta\delta} = \gamma^{\alpha\gamma} R_{\alpha\beta\gamma\delta} = (D-2)\gamma_{\beta\delta} - \delta^\gamma_\delta \gamma_{\beta\gamma} = K(D-3)\gamma_{\beta\delta}. \tag{C.7}$$

Therefore

$$\nabla^i \nabla^j (\nabla_j h_i) = \nabla^j \nabla_j \nabla^i h_i + K(D-3) \nabla^i h_i. \tag{C.8}$$

Hence, returning to equation (C.4) we can write

$$2\Delta f + \left[\Delta + \left(K(D-3) - k_2\right)\right] \nabla^i h_i = 0; \tag{C.9}$$

from which

$$\Delta f = -\frac{1}{2}[\Delta + \bar{k}_2]\nabla^i h_i, \tag{C.10}$$

where we defined $\bar{k}_2 := \left(K(D-3) - k_2\right)$. Now, inserting the expression for Δf from above in equation (C.3a) we get

$$-\frac{1}{2}[\Delta + \bar{k}_2]\nabla^i h_i = 2\nabla^i h_i + k_1 f; \tag{C.11}$$

from which

$$f = -\frac{[\Delta + \bar{k}_2 + 4]}{2k_1}\nabla^i h_i, \tag{C.12}$$

and reinserting it back in the original equation we have

$$-[-k_1 + \Delta]\frac{[\Delta + \bar{k}_2 + 4]}{2k_1}\nabla^i h_i - 2\nabla^i h_i = 0 \tag{C.13a}$$

and rearranging

$$[\Delta^2 + (\bar{k}_2 - k_1 + 4)\Delta - k_1\bar{k}_2]\nabla^i h_i = 0. \tag{C.14}$$

However, in similar way, we can find from equation (C.3a)

$$\nabla^i h_i = \frac{1}{2}[-k_1 + \Delta]f; \tag{C.15}$$

and substituting it in equation (C.4) to get

$$2\Delta f = -\frac{1}{2}[\Delta + \bar{k}_2][-k_1 + \Delta]f \quad \Rightarrow \quad [\Delta^2 + (\bar{k}_2 - k_1 + 4)\Delta - k_1\bar{k}_2]f = 0. \tag{C.16}$$

Therefore functions $\nabla^i h_i$ and f satisfy the same differential equation; moreover the full bi-harmonic equation on the ball under sufficiently smooth hypothesis of the functions admits a unique solution [215]. To use this result, we extend to zero both f and $\nabla^i h_i$ in the interior of the $(D-2)$-dimensional sphere. Therefore we conclude that

$$f = \nabla^i h_i; \tag{C.17}$$

and using this information in equation (C.3a) we have

$$[-k_1 - 2 + \Delta]f = 0 \quad \Rightarrow \quad f = 0, \tag{C.18}$$

while using it in equation (C.3b) we get

$$[-k_2 + \Delta]h_i = 0 \quad \Rightarrow \quad h_i = 0 \quad \forall i \in \mathbb{I}. \tag{C.19}$$

In the steps above if k_1 is zero the equality (C.12) is not defined and an alternative way is needed. We can start again form equation (C.10) and (C.3a) with $k_1 = 0$ to get

$$[\Delta^2 + (\bar{k}_2 + 4)\Delta]f = 0; \tag{C.20}$$

while from the laplacian of (C.11) with $k_1 = 0$ we get

$$[\Delta^2 + (\bar{k}_2 + 4)\Delta]\nabla^i h_i = 0. \tag{C.21}$$

Hence, again

$$f = \nabla^i h_i, \tag{C.22}$$

and from (C.3a) with $k_1 = 0$ we deduce that

$$[-2 + \Delta]f = 0 \quad \Rightarrow \quad f = 0, \tag{C.23}$$

and using it in (C.3b) we get the same result of (C.19). If $k_2 = 0$ it is enough to use $\bar{k}_2 = K(D-3)$ in all steps.

The theorem on a general positive sectional curvature space form follows by composing with the Riemannian covering map π. Indeed, if now f and h_i are defined on $(\boldsymbol{M}, \boldsymbol{g})$ then $f \circ \pi$ and $h_i \circ \pi$ are defined on the $(D-2)$-dimensional sphere. Therefore by the validity of the theorem on the $(D-2)$-dimensional sphere we have

$$f \circ \pi = h_i \circ \pi = 0 \quad \forall i \in \mathbb{I}; \tag{C.24}$$

since the existence of a non-trivial Riemanninan covering map is insured by Killing-Hopf theorem, it follows that

$$f = h_i = 0 \quad \forall i \in \mathbb{I}. \tag{C.25}$$

□

Appendix D

Basic elements of shaves theory

We give a brief review of shaves theory and their cohomology with the aim of proving the abstract de Rham theorem [177]. In the following we consider X as a Hausdorff topological space where every open set is paracompact. An example are all the metrizable spaces.

D.1 Sheaves

Intuitively, a sheaf is a tool for systematically tracking data (such as sets, abelian groups, rings) attached to the open sets of a topological space and defined locally with regard to them. More formally

Definition D.1.1 (Sheaf)**.** *A sheaf of abelian groups $\mathcal{F}$ on X is the datum of an abelian group $\mathcal{F}(U)$ for every open set $U \subset X$ and a group homomorphism (called restrictions) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subset U$. The couple (U, s) with U open set of X and $s \in \mathcal{F}(U)$ is called a section. The datum has to satisfy the following conditions:*

- $\mathcal{F}(\emptyset) = \emptyset$;
- $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ *is the identity for every* U;
- *if* $W \subset V \subset U$ *then* $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$;
- *if* $U = \bigcup_i U_i$ *and* $s \in \mathcal{F}(U)$ *than* $s = 0$ *if and only if* $\rho_{UU_i}(s) = 0\ \forall U_i$;
- *given, for every i, a section* $s_i \in \mathcal{F}(U_i)$ *such that* $\rho_{U_i U_i \cap U_j}(s_i) = \rho_{U_j U_i \cap U_j}(s_j)$ *than there exists a section* $s \in \mathcal{F}(U)$ *such that* $\rho_{UU_i}(s) = s_i\ \forall i$.

Examples of sheaves are the sheaf of locally constant functions on a field $\mathbb{K}$, denoted by $\mathbb{K}_X$, and the sheaf of discontinues sections of a sheaf $\mathcal{F}$, denoted by $\mathcal{DF}$ and defined by $\mathcal{DF}(U) := \prod_{x \in U} \mathcal{F}_x$.

Definition D.1.2 (Germs and stalks of a sheaf)**.** *Given two sections (U, s) and (V, t) they are equivalent if there exist an open $W \subset U \cap V$ such that $\rho_{UW}(s) = \rho_{VW}(t)$. The equivalence class $[U, s]$ is called a germ and the abelian group $\mathcal{F}_x$ on the set of germs around the point $x \in X$ is called a stalk.*

Definition D.1.3 (Shaves morphism and its support)**.** *A shave morphism of abelian groups $f : \mathcal{F} \to \mathcal{F}$ is a family of group homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{F}(U)$ for every open $U \subset X$ which commutes with the restrictions. A shave morphism naturally induces a morphism on the stalks*

$f_x : \mathcal{F}_x \to \mathcal{F}_x$ *and the support of a shave morphism is given by*

$$Supp(f) = \{x \in X \mid f_x(x) \neq 0\} \tag{D.1}$$

Definition D.1.4 (Exact sequence of shaves)**.** *A sequence of shaves on X*

$$\cdots \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{E} \xrightarrow{h} \cdots \tag{D.2}$$

is called exact if $\forall x \in X$ the sequence of stalk

$$\cdots \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{E}_x \xrightarrow{h_x} \cdots \tag{D.3}$$

is exact.

Definition D.1.5 (Partition of the identity)**.** *Let $\mathcal{F}$ be a sheaf and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of X. A partition of the identity of $\mathcal{F}$ subordinated to $\mathcal{U}$ is a family of shaves morphisms f_i : $\mathcal{F} \to \mathcal{F}$ such that*

- $\overline{Supp(f_i)} \subset U_i \ \forall i$;
- $\{Supp(f_i)\}_{i\in I}$ *forms a locally finite covering of X;*
- $\sum_{i\in I} f_i = id_{\mathcal{F}}$.

Definition D.1.6 (fine sheaf)**.** *A sheaf $\mathcal{F}$ on X is fine if it admits a partition of the identity subordinate to every open covering $\mathcal{U}$ of X.*

D.2 Shaves cohomology

Čech cohomology is a cohomology theory based on the intersection properties of open covers of a topological space. Given an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ we call $U_{i_0\ldots i_q} = U_{i_0} \cap \ldots \cap U_{i_q}$.

Definition D.2.1 (Čech cochain and Čech differential)**.** *We define the Čech q-cochain (with $q > 0$) as an element of*

$$\check{C}^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0\ldots i_q}), \tag{D.4}$$

and Čech differential $\check{\delta} : \check{C}^q(\mathcal{U},\mathcal{F}) \to \check{C}^{q+1}(\mathcal{U},\mathcal{F})$ as

$$(\check{\delta}c)_{i_0 \dots i_q+1} := \sum_{k=0}^{q+1} (-1)^k c_{i_0 \dots i_{k-1} i_{k+1} \dots i_q+1} | U_{i_0 \dots i_q+1}. \tag{D.5}$$

It is a matter of computations to show that $\check{\delta} \circ \check{\delta} = 0$. Hence $(\check{C}^*, \check{\delta})$ is a cochain complex known as Čech complex and its cohomology is known as Čech or shaves cohomology

Definition D.2.2 (Čech cohomology)**.** *The q-th Čech cohomology group is*

$$\check{H}^q(\mathcal{U},\mathcal{F}) := \frac{\check{Z}^q(\mathcal{U},\mathcal{F})}{\check{B}^q(\mathcal{U},\mathcal{F})} = \frac{\{c \in \check{C}^q(\mathcal{U},\mathcal{F}) | \check{\delta}c = 0\}}{\{\check{\delta}c \in \check{C}^q(\mathcal{U},\mathcal{F}) | c \in \check{C}^{q-1}(\mathcal{U},\mathcal{F})\}}. \tag{D.6}$$

Thanks to a mathematical procedure known as colimit [105] we can eliminate the dependence on the open covering of X in the Čech complex (and hence also in its cohomology groups) by replacing it directly with the dependence on the topological space X.

Definition D.2.3 (Acyclic sheaf)**.** *A sheaf $\mathcal{F}$ on X is called acyclic if $\check{H}^q(X,\mathcal{F}) = 0 \; \forall q > 0$.*

Theorem D.2.1 (Every fine sheaf is acyclic)**.** *Let $\mathcal{F}$ be a fine sheaf on X then $\mathcal{F}$ is acyclic.*

Proof. We show a stronger fact from which the theorem follows easily. The stronger assertion we want to prove is that if $\mathcal{F}$ is a sheaf on X which admits a partition of the identity subordinated to an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X then $\check{H}^q(\mathcal{U},\mathcal{F}) = 0 \; \forall q > 0$. In fact, let us consider the shaves $\mathcal{F}_i$ by placing $\mathcal{F}_i(U) := \mathcal{F}(U_i \cap U)$ for every open $U \subset X$ and the shaves morphisms $g_i : \mathcal{F}_i \to \mathcal{F}$ defined, for every $s \in \mathcal{F}_i(U)$, by

$$g_i(s) := \begin{cases} f_i(s) & \mathit{on}\ U \cap U_i \\ 0 & \mathit{on}\ U \setminus \overline{Supp(f_i)} \end{cases} \quad ; \tag{D.7}$$

where the f_i are the shave morphisms of the partition of identity subordinated to $\mathcal{U}$. Choosing a cocycle $a \in \check{Z}^q(\mathcal{U},\mathcal{F})$ and defining $b \in \check{C}^{q-1}(\mathcal{U},\mathcal{F})$ as

$$b_{i_1 \dots i_n} := \sum_j g_j(a_{j i_1 \dots i_n}) \tag{D.8}$$

is easy to show that $\check{\delta}b = a$. The theorem now follows since a fine sheaf $\mathcal{F}$ admits a partition of the identity subordinated to every open covenging of X, hence passing to the colimit we get $\check{H}^q(X,\mathcal{F}) = 0\ \forall q > 0$. and so $\mathcal{F}$ is acyclic. $\square$

Theorem D.2.2 (Long exact sequence in Čech cohomology)**.** *For every short exact sequence of shaves*

$$0 \longrightarrow \mathcal{E} \xrightarrow{g} \mathcal{F} \xrightarrow{h} \mathcal{G} \longrightarrow 0 \tag{D.9}$$

induce a long exact sequence in Čech cohomology

$$\cdots \longrightarrow \check{H}^q(X,\mathcal{F}) \xrightarrow{h^*} \check{H}^q(X,\mathcal{G}) \longrightarrow \check{H}^{q+1}(X,\mathcal{E}) \xrightarrow{g^*} \check{H}^{q+1}(X,\mathcal{F}) \xrightarrow{h^*} \cdots \tag{D.10}$$

Proof. The short exact sequence of shaves induces for every open covering $\mathcal{U}$, by replacing $\mathcal{G}$ with the image $\tilde{\mathcal{G}}$ of f, a short exact sequence of Čech complexes

$$0 \longrightarrow \check{C}^*(\mathcal{U},\mathcal{E}) \xrightarrow{g^*} \check{C}^*(\mathcal{U},\mathcal{F}) \xrightarrow{h^*} \check{C}^*(\mathcal{U},\tilde{\mathcal{G}}) \longrightarrow 0, \tag{D.11}$$

which induces, by standard theorem of cohomological algebra, a long exact sequence in cohomology

$$\cdots \longrightarrow \check{H}^q(\mathcal{U},\mathcal{F}) \xrightarrow{h^*} \check{H}^q(\mathcal{U},\tilde{\mathcal{G}}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{E}) \xrightarrow{g^*} \check{H}^{q+1}(\mathcal{U},\mathcal{F}) \xrightarrow{h^*} \cdots \tag{D.12}$$

At this point is enough to pass to the colimit, which preserves exactness, to get

$$\cdots \longrightarrow \check{H}^q(X,\mathcal{F}) \xrightarrow{h^*} \check{H}^q(X,\mathcal{G}) \longrightarrow \check{H}^{q+1}(X,\mathcal{E}) \xrightarrow{g^*} \check{H}^{q+1}(X,\mathcal{F}) \xrightarrow{h^*} \cdots \tag{D.13}$$

and the only non-trivial step is to show that $\check{H}^q(X,\tilde{\mathcal{G}}) \cong \check{H}^q(X,\mathcal{G})$ that can be done thanks to the good properties of Čech cohomology under changing of refinement function. $\square$

D.3 The abstract de Rham theorem

Definition D.3.1 (Resolution of a sheaf)**.** *A resolution of a sheaf is a exact sequence of the form*

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \tag{D.14}$$

if every $\mathcal{E}^j$ is acyclic the resolution is called an acyclic resolution.

We note that if we discard the sheaf $\mathcal{F}$ and we consider the global sections $\mathcal{E}^j(X)$ of shaves $\mathcal{E}^j$ we can define their cohomology in the standard way getting the cohomology groups $H^q(\mathcal{E}^*(X))$. For example if the $\mathcal{E}^*(X)$ is the singular cochain complex then $H^q(\mathcal{E}^*(X))$ will be the singular cohomology groups.

Theorem D.3.1 (Abstract de Rham)**.** *Given an acyclic resolution of the sheaf $\mathcal{F}$*

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \tag{D.15}$$

there is an isomorphism

$$\check{H}^q(X, \mathcal{F}) \cong H^q(\mathcal{E}^*(X)) \tag{D.16}$$

Proof. We show that for every resolution of the sheaf $\mathcal{F}$ there exist homomorphisms

$$\alpha_q : H^q(\mathcal{E}^*(X)) \to \check{H}^q(X, \mathcal{F}) \qquad q \geq 0 \tag{D.17}$$

such that

- if $\check{H}^{q-i-1}(X, \mathcal{E}^i) = 0 \; \forall i \in [0, q-2]$ then α_q is injective;
- if $\check{H}^{q-i}(X, \mathcal{E}^i) = 0 \; \forall i \in [0, q-1]$ then α_q is surjective.

Hence the theorem follow since if the resolution is acyclic both the condition are satisfied and the α_q is an isomorphism for every q. The assertion can be shown by induction.
Let us start with $q = 0$. In this case, using the sheaf property and the definition of the Čech differential we can show that $\check{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$ (the global section of the sheaf). Hence we have the following exact sequence (due to the left exactness of global sections)

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{i} \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \cdots \tag{D.18}$$

which means that

$$\check{H}^0(X,\mathcal{F}) = \mathcal{F}(X) \cong Ker(d : \mathcal{E}^0(X) \to \mathcal{E}^1(X)) = H^0(\mathcal{E}^*(X)). \tag{D.19}$$

For the general $q > 0$ we first define $\mathcal{F}'$ as the kernel of the morphism $d : \mathcal{E}^1(X) \to \mathcal{E}^2(X)$. Therefore we have a resolution

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \cdots, \tag{D.20}$$

which provides the inductive step by which we have a map

$$\alpha_{q-1} : H^{q-1}(\mathcal{E}^{*-1}(X)) = H^q(\mathcal{E}^*(X)) \to \check{H}^{q-1}(X,\mathcal{F}'), \tag{D.21}$$

and a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{F}' \longrightarrow 0, \tag{D.22}$$

which gives us a long exact sequence in cohomology and composing with the map

$$\delta : \check{H}^{q-1}(X,\mathcal{F}') \to \check{H}^q(X,\mathcal{F}) \tag{D.23}$$

we get the map we are looking for. □

D.4 The sheaf of differential mixed symmetry tensors

Let us now set $(X,\mathcal{A})$ a differential manifold with differential maximal atlas $\mathcal{A}$. This atlas naturally furnishies a structure sheaf $\mathcal{E}$ (this is the sheaf associated to the manifold; for example for the case of $\mathbb{R}^n$ this is the sheaf of C^∞ functions). In the following definition $\chi(U)$ is the sheaf of differential vector field on U.

Definition D.4.1 (Differential mixed symmetry tensor sheaf)**.** *We define the sheaf $\mathcal{D}\Omega^{p_1\otimes\ldots\otimes p_N}$ as the sheaf associated to the map $x \mapsto \Omega^{p_1\otimes\ldots\otimes p_N}(X)$. Then there exist, for every open $U \subset X$, a map $\langle -; -,...,- \rangle : \mathcal{D}\Omega^{p_1\otimes\ldots\otimes p_N}(U) \times \chi^{p_1}(U) \times \ldots \times \chi^{p_N}(U) \to \mathcal{D}\mathbb{R}_X$ defined by*

$$\langle T; (\alpha_1,..,\alpha_{p_1}),...,(\beta_1,..,\beta_{p_N})\rangle(x) := (T(x); (\alpha_1(x),..,\alpha_{p_1}(x)),...,(\beta_1(x),..,\beta_{p_N}(x))) \tag{D.24}$$

which is nothing but the evaluation of the mixed symmetry tensor T on sets vector fields. A differential mixed symmetry tensor is an element $T \in \mathcal{D}\Omega^{p_1\otimes\ldots\otimes p_N}(U)$ such that for every $V \subset U$

$$\langle T;(\alpha_1,..,\alpha_{p_1}),...,(\beta_1,..,\beta_{p_N})\rangle|_V(x) \in \mathcal{E}(V). \tag{D.25}$$

As for the case of vector fields and differential forms, differential mixed symmetry tensors give rise to a sheaf and, with a little abuse of notation, we will still indicate with $\Omega^{p_1\otimes\ldots\otimes p_N}$ the sheaf of differential mixed symmetry tensors. We also stress that $\Omega^{p_1\otimes\ldots\otimes p_N}$ is also an $\mathcal{E}$-module under the pointwise multiplication.

Theorem D.4.1 (Fine shaves on differential manifold)**.** *Let $\mathcal{F}$ be an $\mathcal{E}$-module on X. If X is a differential manifold than $\mathcal{F}$ is a fine sheaf.*

Proof. Given an open covering $\mathcal{U}$, it is enough to consider as partition of the identity $\mathcal{F}$ subordinated to $\mathcal{U}$ the multiplication of the elements of $\mathcal{F}$ by the functions of the partition of the unity of X subordinated to $\mathcal{U}$. □

Appendix E

Evaluation of Komar-like quantities

In an explicit way metric and its inverse read

$$(g_b)_{\mu\nu}\begin{bmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & (g_b)_{rr} & 0 \\ 0 & 0 & 0 & r^2(g_b)_{rr} \end{bmatrix}, \qquad (g_b)^{\mu\nu}\begin{bmatrix} -\frac{C}{r^2 sin^2(\theta)} & \frac{B}{r^2 sin^2(\theta)} & 0 & 0 \\ \frac{B}{r^2 sin^2(\theta)} & -\frac{A}{r^2 sin^2(\theta)} & 0 & 0 \\ 0 & 0 & \frac{1}{(g_b)_{rr}} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2(g_b)_{rr}}. \end{bmatrix} \tag{E.1}$$

Let us start with the first Komar-like integral, we need $* \, dk$. $k^\mu = (1,0,0,0)$ and so $k_\mu = (g_b)_{\mu\nu}k^\nu = (g_b)_{\mu t}$; the form associated is $k = (g_b)_{\mu t}dx^\mu = Adt + Bd\phi$ and its exterior derivative reads

$$dk = \frac{\partial A}{\partial r}dr \wedge dt + \frac{\partial A}{\partial \theta}d\theta \wedge dt + \frac{\partial B}{\partial r}dr \wedge d\phi + \frac{\partial B}{\partial \theta}d\theta \wedge d\phi. \tag{E.2}$$

Taking the Hodge dual we get

$$\begin{aligned}(* \, dk)_{\mu\nu} &= \frac{\epsilon_{\mu\nu\rho\sigma}}{2}(g_b)^{\rho\rho'}(g_b)^{\sigma\sigma'}(dk)_{\rho'\sigma'} = \\ &= \frac{\epsilon_{\mu\nu\rho\sigma}}{2}[(g_b)^{\rho r}(g_b)^{\sigma t}(dk)_{rt} + (g_b)^{\rho\theta}(g_b)^{\sigma t}(dk)_{\theta t} + (g_b)^{\rho r}(g_b)^{\sigma\phi}(dk)_{r\phi} + (g_b)^{\rho\theta}(g_b)^{\sigma\phi}(dk)_{\theta\phi}].\end{aligned} \tag{E.3a}$$

A similar procedure can be done for computing $* \, dm$: $m = Bdt + Cd\phi$ and the final result is

$$\begin{aligned}(* \, dm)_{\mu\nu} &= \frac{\epsilon_{\mu\nu\rho\sigma}}{2}(g_b)^{\rho\rho'}(g_b)^{\sigma\sigma'}(dm)_{\rho'\sigma'} = \\ &= \frac{\epsilon_{\mu\nu\rho\sigma}}{2}[(g_b)^{\rho r}(g_b)^{\sigma t}(dm)_{rt} + (g_b)^{\rho\theta}(g_b)^{\sigma t}(dm)_{\theta t} + (g_b)^{\rho r}(g_b)^{\sigma\phi}(dm)_{r\phi} + (g_b)^{\rho\theta}(g_b)^{\sigma\phi}(dm)_{\theta\phi}],\end{aligned} \tag{E.4a}$$

with

$$dm = \frac{\partial B}{\partial r}dr \wedge dt + \frac{\partial B}{\partial \theta}d\theta \wedge dt + \frac{\partial C}{\partial r}dr \wedge d\phi + \frac{\partial C}{\partial \theta}d\theta \wedge d\phi. \tag{E.5}$$

To calculate the Komar quantities K_t and K_ϕ one needs to choose an appropriate boundary surface. As underlined in [67] and [162] a good choice is the boundary of a spatial three volume characterised by a constant r and $dt = -\frac{A}{B}d\phi$; In this way, infinitesimally close points are simultaneous events. Since r is constant the dual Killing forms reduce to

$$\begin{aligned}(*\, dk)_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[(g_b)^{\rho r}(g_b)^{\sigma t}(dk)_{rt} + (g_b)^{\rho r}(g_b)^{\sigma\phi}(dk)_{r\phi}] = \\ &= \frac{(g_b)^{rr}}{2}\{(dk)_{rt}[\epsilon_{\mu\nu rt}(g_b)^{tt} + \epsilon_{\mu\nu r\phi}(g_b)^{\phi t}] + (dk)_{r\phi}[\epsilon_{\mu\nu rt}(g_b)^{t\phi} + \epsilon_{\mu\nu r\phi}(g_b)^{\phi\phi}]\}; \\ (*\, dm)_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[(g_b)^{\rho r}(g_b)^{\sigma t}(dm)_{rt} + (g_b)^{\rho r}(g_b)^{\sigma\phi}(dm)_{r\phi}] = \\ &= \frac{(g_b)^{rr}}{2}\{(dm)_{rt}[\epsilon_{\mu\nu rt}(g_b)^{tt} + \epsilon_{\mu\nu r\phi}(g_b)^{\phi t}] + (dm)_{r\phi}[\epsilon_{\mu\nu rt}(g_b)^{t\phi} + \epsilon_{\mu\nu r\phi}(g_b)^{\phi\phi}]\}\end{aligned} \tag{E.6}$$

and the only non-vanishing elements are

$$\begin{aligned}(*\, dk)_{\theta\phi} &= \frac{(g_b)^{rr}}{2}\epsilon_{\theta\phi rt}[(g_b)^{tt}(dk)_{rt} + (g_b)^{t\phi}(dk)_{r\phi}]; \\ (*\, dk)_{\theta t} &= \frac{(g_b)^{rr}}{2}\epsilon_{\theta tr\phi}[(g_b)^{\phi t}(dk)_{rt} + (g_b)^{\phi\phi}(dk)_{r\phi}],\end{aligned} \tag{E.7}$$

for $*\, dk$ and

$$\begin{aligned}(*\, dm)_{\theta\phi} &= \frac{(g_b)^{rr}}{2}\epsilon_{\theta\phi rt}[(g_b)^{tt}(dm)_{rt} + (g_b)^{t\phi}(dm)_{r\phi}]; \\ (*\, dm)_{\theta t} &= \frac{(g_b)^{rr}}{2}\epsilon_{\theta tr\phi}[(g_b)^{\phi t}(dm)_{rt} + (g_b)^{\phi\phi}(dm)_{r\phi}],\end{aligned} \tag{E.8}$$

for $*\, dm$.

Since we are performing an integral over a surface of simultaneous events terms $(*\, dk)_{\theta t}$ and $(*\, dm)_{\theta t}$ do not have to be included since then their integrals should be subtracted because doing cycle along the compact direction ϕ would bring a shift in time [49],[67],[162].
In order to avoid divergences in the Komar-like quantities, $(*\, dk)_{\theta t}$ and $(*\, dm)_{\theta t}$ must have a good radial fall-off behavior, therefore

$$\begin{aligned}(g_b)^{rr}(g_b)^{tt}(dk)_{rt} &= -\frac{C}{r^2 sin^2(\theta)(g_b)_{rr}}\frac{\partial A}{\partial r} \Rightarrow \frac{C}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial A}{\partial r} \sim r^{n_1} f_1(\theta) \\ (g_b)^{rr}(g_b)^{t\phi}(dk)_{r\phi} &= \frac{B}{r^2 sin^2(\theta)(g_b)_{rr}}\frac{\partial C}{\partial r} \Rightarrow \frac{B}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial C}{\partial r} \sim r^{n_2} f_2(\theta) \\ (g_b)^{rr}(g_b)^{tt}(dm)_{rt} &= -\frac{C}{r^2 sin^2(\theta)(g_b)_{rr}}\frac{\partial B}{\partial r} \Rightarrow \frac{C}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial B}{\partial r} \sim r^{n_3} f_3(\theta) \\ (g_b)^{rr}(g_b)^{t\phi}(dm)_{r\phi} &= \frac{B}{r^2 sin^2(\theta)(g_b)_{rr}}\frac{\partial C}{\partial r} \Rightarrow \frac{B}{sin^2(\theta)}\sqrt{\frac{C}{(g_b)_{rr}}}\frac{\partial C}{\partial r} \sim r^{n_4} f_4(\theta)\end{aligned} \tag{E.9}$$

with

$$n_1, n_2, n_3, n_4 \leq 1 \tag{E.10}$$

and $f_1(\theta), f_2(\theta), f_3(\theta), f_3(\theta)$ arbitrary function of azimuthal angle; we have also taken into account the square root of the induced metric determinant that enters in the computation of the Komar-like integrals $\sqrt{|\gamma|} = r\sqrt{C(g_b)_{rr}}$.

Appendix F

Asymptotic generating matrix equations

Let us start with the asymptotic expansion for poles λ_k

$$
\begin{aligned}
\lambda_k &= (w_k - z) \pm \sqrt{(w_k - z)^2 + \rho^2} = \\
&= (w_k - rcos(\theta)) \pm \sqrt{(w_k - rcos(\theta))^2 + r^2 sin^2(\theta)} = \\
&= (w_k - rcos(\theta)) \pm \sqrt{(w_k^2 + r^2 + 2w_k cos(\theta))} \sim r(-cos(\theta) \pm 1);
\end{aligned} \tag{F.1}
$$

Assuming the asymptotic behaviour of the spectral parameter is the same of the poles[1] we have to expand the following coefficients:

$$
\begin{aligned}
\frac{2\lambda^2}{\lambda^2 + \rho^2} &\sim \frac{2(-rcos(\theta) \pm r)^2}{(-rcos(\theta) \pm r)^2 + r^2 sin^2(\theta)} = \frac{2[r^2cos^2(\theta) + r^2 \mp 2r^2cos(\theta)]}{2r^2 \mp 2r^2cos(\theta)} = \\
&= 1 \mp \frac{cos(\theta)}{1 \mp cos(\theta)} + \frac{cos^2(\theta)}{1 \mp cos(\theta)}; \\
\frac{2\lambda\rho}{\lambda^2 + \rho^2} &\sim \frac{2(-rcos(\theta) \pm r) r sin(\theta)}{2r^2 \mp 2r^2cos(\theta)} = -\frac{cos(\theta)sin(\theta)}{1 \mp cos(\theta)} \pm \frac{sin(\theta)}{1 \mp cos(\theta)}; \\
\frac{\rho}{\lambda^2 + \rho^2} &\sim \frac{r sin(\theta)}{2r^2 \mp 2r^2cos(\theta)} = \frac{1}{2r}\frac{sin(\theta)}{1 \mp cos(\theta)}; \\
\frac{\lambda}{\lambda^2 + \rho^2} &\sim \frac{-rcos(\theta) \pm r}{2r^2 \mp 2r^2cos(\theta)} = \frac{1}{2r}\frac{-cos(\theta) \pm 1}{1 \mp cos(\theta)}.
\end{aligned} \tag{F.2}
$$

We can sum and subtract equations in system (5.36) to get

$$
\begin{aligned}
&\left([c(\theta) + s(\theta)]\frac{\partial}{\partial r} + [c(\theta) - s(\theta)]\frac{\partial}{\partial \theta} + \frac{2\lambda\rho - 2\lambda^2}{\lambda^2 + r^2 s^2(\theta)}\partial_\lambda - \frac{2rs(\theta)V_b}{\lambda^2 + r^2 s^2(\theta)} \right)\psi_b = 0, \\
&\left([c(\theta) - s(\theta)]\frac{\partial}{\partial r} - [s(\theta) + c(\theta)]\frac{\partial}{\partial \theta} - \frac{2\lambda^2 + 2\lambda\rho}{\lambda^2 + r^2 s^2(\theta)}\partial_\lambda + \frac{2\lambda U_b}{\lambda^2 + r^2 s^2(\theta)} \right)\psi_b = 0;
\end{aligned} \tag{F.3}
$$

[1] This is suggested from the fact that the spectral parameter meets the poles λ_k along its trajectory; if the poles are considered on the asymptotic boundary also the whole trajectory has to be consider asymptotically and it has to meet poles on the asymptotic boundary.

where $c(\theta) := cos(\theta)$ and $s(\theta) := sin(\theta)$. Inserting the asymptotic expansions (F.2)

$$\begin{aligned} &\left([c(\theta)+s(\theta)]\frac{\partial}{\partial r}+[c(\theta)-s(\theta)]\frac{\partial}{\partial \theta}+\mathcal{F}_1^{\uparrow\downarrow}(\theta)\partial_\lambda\right)\psi_b=\frac{1}{r}\frac{s(\theta)}{1\mp c(\theta)}V_b\psi_b, \\ &\left([c(\theta)-s(\theta)]\frac{\partial}{\partial r}-[s(\theta)+c(\theta)]\frac{\partial}{\partial \theta}+\mathcal{F}_2^{\uparrow\downarrow}(\theta)\partial_\lambda\right)\psi_b=-\frac{1}{r}\frac{-c(\theta)\pm 1}{1\mp c(\theta)}U_b\psi_b; \end{aligned} \tag{F.4}$$

where

$$\begin{aligned} \mathcal{F}_1^{\uparrow\downarrow}(\theta)&=\frac{1\mp 2c(\theta)+c^2(\theta)+c(\theta)s(\theta)\mp s(\theta)}{1\mp c(\theta)}=\mp s(\theta)\mp c(\theta)+1; \\ \mathcal{F}_2^{\uparrow\downarrow}(\theta)&=\frac{1\mp 2c(\theta)+c^2(\theta)-c(\theta)s(\theta)\pm s(\theta)}{1\mp c(\theta)}=\pm s(\theta)\mp c(\theta)+1; \end{aligned} \tag{F.5}$$

where the apexes "$\uparrow\downarrow$" indicate respectively the up or down choice for the signs; matrices V_b and U_b could be expanded to get

$$\begin{aligned} &V_b=\rho(g_b)_{,z}(g_b)^{-1}\sim \\ &\sim\frac{1}{rsin(\theta)}\begin{bmatrix} r^\gamma \mathcal{C}sin(\theta)r^\alpha\mathcal{A}'-r^\beta\mathcal{B}r^\beta sin(\theta)\mathcal{B}' & -r^\beta\mathcal{B}sin(\theta)r^\alpha\mathcal{A}'+r^\alpha\mathcal{A}r^\beta sin(\theta)\mathcal{B}' \\ r^\beta\mathcal{B}'sin(\theta)r^\gamma\mathcal{C}-r^\beta\mathcal{B}r^\gamma sin(\theta)\mathcal{C}' & -r^\beta\mathcal{B}sin(\theta)r^\beta\mathcal{B}'+r^\alpha\mathcal{A}r^\gamma sin(\theta)\mathcal{C}' \end{bmatrix}; \end{aligned} \tag{F.6}$$

$$\begin{aligned} &U_b=\rho(g_b)_{,\rho}(g_b)^{-1}\sim \\ &\sim\frac{1}{rsin(\theta)}\begin{bmatrix} -r^\gamma \mathcal{C}cos(\theta)r^\alpha\mathcal{A}'+r^\beta\mathcal{B}r^\beta cos(\theta)\mathcal{B}' & r^\beta\mathcal{B}cos(\theta)r^\alpha\mathcal{A}'-r^\alpha\mathcal{A}r^\beta cos(\theta)\mathcal{B}' \\ -r^\beta\mathcal{B}'cos(\theta)r^\gamma\mathcal{C}+r^\beta\mathcal{B}r^\gamma cos(\theta)\mathcal{C}' & r^\beta\mathcal{B}cos(\theta)r^\beta\mathcal{B}'-r^\alpha\mathcal{A}r^\gamma cos(\theta)\mathcal{C}' \end{bmatrix}. \end{aligned} \tag{F.7}$$

Note that asymptotically we have $V_b \sim cotang(\theta)U_b$.

Appendix G

Christoffel symbols for the N-gravisolitons solutions

Let us derive here the non-vanishing Christoffel symbols for a metric like

$$ds^2 = g_{22}(dr^2 + r^2 d\theta^2) + g_{ij} dx^i dx^j \quad i, j = 0, 1; \tag{G.1}$$

which is of the same kind of the gravisolitons metric. The non-vanishing symbols are

$$\begin{aligned}
&\Gamma^2_{00} = -\frac{1}{2} g^{22} \partial_2 g_{00}; \qquad \Gamma^3_{00} = -\frac{1}{2r^2} g^{22} \partial_3 g_{00}; \\
&\Gamma^2_{01} = -\frac{1}{2} g^{22} \partial_2 g_{01}; \qquad \Gamma^3_{01} = -\frac{1}{2r^2} g^{22} \partial_3 g_{01}; \\
&\Gamma^0_{20} = \frac{1}{2} g^{00} \partial_2 g_{00} + \frac{1}{2} g^{10} \partial_2 g_{01}; \qquad \Gamma^1_{20} = \frac{1}{2} g^{01} \partial_2 g_{00} + \frac{1}{2} g^{11} \partial_2 g_{01}; \\
&\Gamma^0_{30} = \frac{1}{2} g^{00} \partial_3 g_{00} + \frac{1}{2} g^{10} \partial_3 g_{01}; \qquad \Gamma^1_{30} = \frac{1}{2} g^{01} \partial_3 g_{00} + \frac{1}{2} g^{11} \partial_3 g_{01}; \\
&\Gamma^2_{11} = -\frac{1}{2} g^{22} \partial_2 g_{11}; \qquad \Gamma^3_{11} = -\frac{1}{2r^2} g^{22} \partial_3 g_{11}; \\
&\Gamma^0_{22} = -\frac{1}{2} g^{00} \partial_2 g_{22}; \qquad \Gamma^2_{22} = -\frac{1}{2} g^{22} \partial_2 g_{22}; \\
&\Gamma^0_{12} = \frac{1}{2} g^{00} \partial_2 g_{01} + \frac{1}{2} g^{10} \partial_2 g_{11}; \qquad \Gamma^1_{12} = \frac{1}{2} g^{01} \partial_2 g_{11} + \frac{1}{2} g^{11} \partial_2 g_{11}; \\
&\Gamma^0_{13} = \frac{1}{2} g^{00} \partial_3 g_{10} + \frac{1}{2} g^{10} \partial_3 g_{11}; \qquad \Gamma^1_{13} = \frac{1}{2} g^{01} \partial_3 g_{11} + \frac{1}{2} g^{11} \partial_3 g_{11}; \\
&\Gamma^3_{23} = \frac{1}{2r^2} g^{22} \partial_2 g_{33}; \qquad \Gamma^2_{23} = \frac{1}{2} g^{22} \partial_3 g_{22}; \\
&\Gamma^3_{22} = -\frac{1}{2r^2} g^{22} \partial_3 g_{22}; \\
&\Gamma^2_{31} = -\frac{1}{2} g^{22} \partial_2 g_{33}; \\
&\Gamma^3_{33} = \frac{1}{2r^2} g^{22} \partial_3 g_{33}.
\end{aligned} \tag{G.2}$$

The Christoffel symbol we are interested in to deduce which is the leading order of the killing vectors is the last one and; for $N = 0$ we have

$$\left(\Gamma^3_{33}\right)_{N=0} = \frac{\mathcal{D}'(\theta)}{2\mathcal{D}(\theta)}; \tag{G.3}$$

while for the case $N = 2$ we get

$$
\begin{aligned}
\left(\Gamma^3_{33}\right)_{N=2} &= \mathcal{M}_2(\theta)\left[(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))\right]^{-4} r^{\delta+2+2\#}\left(\prod_{j=1}^{\#}\frac{1}{(w_k-w_l)^2_j}\right)\Xi_2 \times \\
&\times\left[\frac{\mathcal{M}_2'(\theta)}{\mathcal{M}_2(\theta)} - 4\frac{r^{\epsilon_1+\epsilon_3+1}(\mathcal{E}_1\mathcal{E}_3(\theta))' - r^{2\epsilon_2+1}(\mathcal{E}_2^2(\theta))'}{r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta)} + \frac{\Xi_2'}{\Xi_2}\right]\frac{1}{2g^{22}} = \\
&= \frac{1}{2}\left[\frac{\mathcal{M}_2'(\theta)}{\mathcal{M}_2(\theta)} - 4\frac{r^{\epsilon_1+\epsilon_3+1}(\mathcal{E}_1\mathcal{E}_3(\theta))' - r^{2\epsilon_2+1}(\mathcal{E}_2^2(\theta))'}{r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta)} + \frac{\Xi_2'}{\Xi_2}\right];
\end{aligned}
\tag{G.4}
$$

where $\Xi_2 := \sum_{\sigma\in S_2}\left(\operatorname{sgn}(\sigma)\prod_{s=0}^{2}[A_{s\sigma_s} + B_{s\sigma_s} + C_{s\sigma_s} + D_{s\sigma_s}]\right)$. The first term has no power of the radial cooordinate while the second and third terms are of the form

$$
\frac{r^a f'(\theta) + r^b g'(\theta)}{r^a f(\theta) + r^b g(\theta)},
\tag{G.5}
$$

where a and b are certain numbers; if r^a is dominant with respect to r^b we will get $\frac{f'(\theta)}{f(\theta)}$ and if r^b is dominant with respect to r^a we will get $\frac{g'(\theta)}{g(\theta)}$. Therefore, in any case, the leading order is independent of r but strongly depends on the initial data. For the general N case the story is no different; we get

$$
\begin{aligned}
\left(\Gamma^3_{33}\right)_{N} &= \mathcal{M}_N(\theta)\left[(r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta))\right]^{-2N} r^{\bar{\delta}}\left(\prod_{j=1}^{\#}\frac{1}{(w_k-w_l)^2_j}\right)\Xi_N \times \\
&\times\left[\frac{\mathcal{M}_N'(\theta)}{\mathcal{M}_N(\theta)} - 2N\frac{r^{\epsilon_1+\epsilon_3+1}(\mathcal{E}_1\mathcal{E}_3(\theta))' - r^{2\epsilon_2+1}(\mathcal{E}_2^2(\theta))'}{r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta)} + \frac{\Xi_N'}{\Xi_N}\right]\frac{1}{2g^{22}} = \\
&= \frac{1}{2}\left[\frac{\mathcal{M}_N'(\theta)}{\mathcal{M}_N(\theta)} - 2N\frac{r^{\epsilon_1+\epsilon_3+1}(\mathcal{E}_1\mathcal{E}_3(\theta))' - r^{2\epsilon_2+1}(\mathcal{E}_2^2(\theta))'}{r^{\epsilon_1+\epsilon_3+1}\mathcal{E}_1\mathcal{E}_3(\theta) - r^{2\epsilon_2+1}\mathcal{E}_2^2(\theta)} + \frac{\Xi_N'}{\Xi_N}\right];
\end{aligned}
\tag{G.6}
$$

where now $\Xi_N := \sum_{\sigma\in S_N}\left(\operatorname{sgn}(\sigma)\prod_{s=0}^{N}[A_{s\sigma_s} + B_{s\sigma_s} + C_{s\sigma_s} + D_{s\sigma_s}]\right)$, $\bar{\delta} := \delta - \frac{N^2}{2} + N(N+1) - N(N-1) + 2\#$ and the same consideration above for the case $N = 2$ can be applied.

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