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# GIT on Hilbert and Chow schemes of curves

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# Introduction

A classical problem in algebraic geometry is the construction of moduli spaces for geometric objects, like algebraic curves, surfaces, vector bundles, with fixed invariants. The most used method for constructing moduli spaces is based on Geometric Invariant Theory (GIT). This theory was developed by David Mumford and provides tools to construct the quotient of an algebraic variety $X$ with respect to the action of a reductive group $G$ in the context of algebraic geometry. In particular, if $X$ is a projective variety and the action admits a very ample linearization $L$, up to restricting to an open subset of $X$ we can construct a quotient as a projective scheme. More precisely the quotient is

$$X/\!\!/G := \operatorname{Proj}\Big(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G\Big),$$

and the open set where the quotient map is defined is the so-called *semistable locus* $X^{ss}$, which is the set of points $p$ in $X$ for which there is an invariant section $s \in H^0(X, L^{\otimes n})^G$ for some $n$ such that $s(p) \neq 0$. Thus the main goal of the so-called GIT analisys is to determine the semistable locus. We recall that a point $p$ in $X$ is said to be *polystable* if it is semistable and its orbit is closed in $X^{ss}$. Finally $p$ is said to be *stable* if it is polystable and its stabilizer is finite; the stable locus is denoted by $X^s$. Moreover we remind that $X/\!\!/G$ is a *categorical* quotient, i. e. the quotient map $\pi : X^{ss} \longrightarrow X/\!\!/G$ is universal with respect to $G$-invariant morphism, while the restriction of $\pi$ to $X^s$ is a *geometric* quotient, i. e. a categorical quotient such that $\pi$ induces a one-to-one correspondence between the orbits of $X^s$ and the geometric points of the image of $X^s$ inside $X/\!\!/G$.

The most famous application of GIT was the construction of the moduli space $M_g$ of smooth curves of genus $g \geq 2$ and its compactification $\overline{M}_g$ via *Deligne-Mumford stable* curves, carried out by Mumford (([Mum77])) and Gieseker (([Gie82])).

We recall the GIT construction of $\overline{M}_g$. Fix an integer $g \geq 2$. Given $d = n(2g-2)$ with $n$ sufficiently large, denote by $\mathrm{Hilb}_d$ the Hilbert scheme of curves of degree $d$ and arithmetic genus $g$ in $\mathbb{P}^{d-g}$ and by $\mathrm{Chow}_d$ the Chow scheme of 1-cycles of degree $d$ in

$\mathbb{P}^{d-g}$. Consider the Hilbert-Chow map

$$\mathrm{Ch} : \mathrm{Hilb}_d \to \mathrm{Chow}_d,$$

which sends a one dimensional subscheme $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ to its 1-cycle. The linear algebraic group $\mathrm{SL}_{\mathrm{d-g+1}}$ acts naturally on $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$ so that Ch is an equivariant map; moreover, these actions are naturally linearized, so it makes sense to talk about (GIT) (semi-,poly-)stability of a point in $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$. Given $g \geq 2$, one consider locally closed subsets of $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$, denoted by $\mathrm{Hilb}_d^{\mathrm{can}}$ and $\mathrm{Chow}_d^{\mathrm{can}}$ respectively, parametrizing the $n$-canonical smooth connected curves $X$ of genus $g$ in $\mathbb{P}^{d-g}$. A GIT analysis based on the Hilbert-Mumford numerical criterion shows that for $n$ sufficiently large each curve in $\mathrm{Hilb}_d^{\mathrm{can}}$ and $\mathrm{Chow}_d^{\mathrm{can}}$ is GIT stable with respect to the action of $\mathrm{SL}_{\mathrm{d-g+1}}$ (for details see [Mum77, Thm. 4.15] or [HM98, Chap. 4.B, Thm 4.34]), so that the quotient of both these subsets by the action of the linear group $\mathrm{SL}_{\mathrm{d-g+1}}$ is well defined and is the moduli space $M_g$ of the smooth curves of genus $g$, which is only a quasi-projective variety. If we would like to obtain a compactification of $M_g$, we have to consider the action of $\mathrm{SL}_{\mathrm{d-g+1}}$ over the projective closures $\overline{\mathrm{Hilb}_d^{\mathrm{can}}}$ and $\overline{\mathrm{Chow}_d^{\mathrm{can}}}$. In the end of the seventies of the last century Mumford and Gieseker proved that if $n$ is sufficiently large, then the semistable locus consists of only Deligne-Mumford stable curves (or *stable* for short), i.e. connected nodal projective curves with finite automorphism group. Moreover the GIT quotient is geometric (i.e. there are no strictly semistable objects), so that the GIT quotient is the moduli space $\overline{M}_g$ of stable curves. Indeed Mumford in [Mum77] works under the stronger assumption that $n \geq 5$ while Gieseker assumes only that $n \geq 10$, but later it was discovered that Gieseker's proof works also under the assumption $n \geq 5$ (see [Mor10, Sec. 3]).

We could ask ourself what it happens when $n < 5$. This question is very interesting because the quotient can be described always as a modular birational model of $\overline{M}_g$ and in these last years many authors have found connections of this problem with the so called Hassett-Keel program, whose ultimate goal is to find the minimal model of $M_g$. Let us sum up what it was discovered in these last years.

- $n = 3$. The description of the quotient is due to David Schubert in 1991 ([Sch91]). He proved that the semistable locus in the Chow scheme (and the same description holds for the Hilbert scheme) consists of pseudo-stable curves (or *p-stable curves* for short). These are connected projective curves with finite automorphism group, whose only singularities are nodes and cusps, and without elliptic tails (i.e. connected subcurves of arithmetic genus one meeting the rest of the curve in one point). Since the GIT quotient analyzed by Schubert is geometric, the GIT quotient is the moduli space of pseudo-stable curves of genus $g$. Later, Hassett-Hyeon constructed in [HH09] a modular map $T : \overline{M}_g \to \overline{M}_g^{\mathrm{p}}$ which on geometric points sends a stable curve onto the p-stable curve obtained by contracting all its elliptic tails to cusps. Moreover, the authors of loc. cit. identified the map $T$ with the first contraction in the Hassett-Keel program for $\overline{M}_g$.

- $n = 4$. This case was studied by Hyeon and Morrison in [HM10]. It is very interesting because although the quotients are again isomorphic to the moduli space of pseudo-stable curves, their properties are different: the Hilbert GIT quotient is geometric, while the Chow GIT quotient is only categorical (i. e. there are strictly semistable objects, some of which then get identified in the quotient). More precisely, the Hilbert semistable locus coincides with Schubert's, while the Chow semistable locus is strictly bigger and it consists of weakly-pseudo-stable curves (or *wp-stable curves* for short). These are connected projective curves with finite automorphism group, whose only singularities are nodes and cusps (and having possibly elliptic tails). This result can be reinterpreted as saying that the non-separated stack of wp-stable curves and its open and proper substack of p-stable curves have the same moduli space.

- $n = 2$. The quotients, which are not geometric but only categorical, were described by Hassett-Hyeon in [HH]. The Hilbert GIT quotient $\overline{M}_g^h$ and the Chow GIT quotient $\overline{M}_g^c$ (they are now different) are new compactification of $M_g$, respectively the moduli spaces of $h$-semistable and the moduli space of $c$-semistable curves. Moreover, they constructed a small contraction $\Psi : \overline{M}_g^{\mathrm{p}} \to \overline{M}_g^c$ and identified the natural map $\Psi^+ : \overline{M}_g^h \to \overline{M}_g^c$ as the flip of $\Psi$. These maps are then interpreted as further steps in the Hassett-Keel program for $\overline{M}_g$.

- $n = 1$. This case is still open. For some partial results on the GIT quotient for the Hilbert scheme of 1-canonically embedded curves, we refer the reader to the work of Alper, Fedorchuk and Smyth (see [AFS13]).

What I worked about in my PhD thesis is to construct modular compactifications of the universal jacobian

$$J_{d,g} = \{(X, L) \,|\, X \text{ is smooth of genus } g \text{ and } L \text{ is a line bundle of degree } d\}/\sim,$$

via GIT quotients of the locus of connected curves inside $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$. This problem was studied first by Lucia Caporaso in her PhD thesis. She proved that for $d \geq 10(2g-2)$ the semistable locus of $\mathrm{Hilb}_d$ consists of *quasi-stable* curves (i. e. Deligne-Mumford semistable curves with no chains of $\mathbb{P}^1$ of length greater than 1) such that the line bundle $L$ defining the embedding in $\mathbb{P}^{d-g}$ is *balanced*, i. e. satisfies an inequality that essentially says that the multidegree of $L$ does not differ too much from the multidegree of $\omega_X^{\otimes n}$ for some suitable $n$ (we will introduce the inequality later).

As before, one may ask what happens when $d < 10(2g-2)$. In my PhD thesis I studied the following

**Problem:** *Describe the GIT quotient for the Hilbert and Chow scheme of curves of genus $g$ and degree $d$ in $\mathbb{P}^{d-g}$, as $d$ decreases with respect to $g$.*

This PhD thesis consists of the paper *GIT for polarized curves*, which was written in collaboration with Gilberto Bini, Margarida Melo and Filippo Viviani. In this paper we characterize completely the (semi-,poly-)stable points $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ and of its image $\text{Ch}([X \subset \mathbb{P}^{d-g}]) \in \text{Chow}_d$ for $d > 2(2g-2)$. Before of stating the main results of the paper, we introduce some definitions.

A curve $X$ is said to be *quasi-stable* if it is obtained from a stable curve $Y$ by "blowing up" some of its nodes, i.e. by taking the partial normalization of $Y$ at some of its nodes and inserting a $\mathbb{P}^1$ connecting the two branches of each node. A curve $X$ is said to be *quasi-p-stable* (resp. *quasi-wp-stable*) if it is obtained from a p-stable curve (resp. a wp-stable curve) $Y$ by "blowing up" some of its nodes (as before) and "blowing up" some of its cusps, i.e. by taking the partial normalization of $Y$ at some of its cusps and inserting a $\mathbb{P}^1$ tangent to the branch point of each cusp (the singularity that one gets by blowing up a cusp is called *tacnode with a line*). Given a quasi-wp-stable curve $X$, we call the $\mathbb{P}^1$'s inserted by blowing up nodes or cusps of $Y$ the *exceptional components*, and we denote by $X_{\text{exc}} \subset X$ the union of all of them.

The following table summarize the previous definitions:

| SINGULARITIES | $\omega_X$ NEF + IRREDUCIBLE EXCEPTIONAL SUB-CURVES | $\omega_X$ AMPLE |
|---|---|---|
| nodes, cusps, tacnodes with a line | **quasi-wp-stable** | **wp-stable** |
| nodes, cusps, tacnodes with a line, without elliptic tails | **quasi-p-stable** | **p-stable** |
| nodes | **quasi-stable** | **stable** |

Table 1: Singular curves

A line bundle $L$ of degree $d$ on a quasi-wp-stable curve $X$ of genus $g$ is said to be *balanced* if for each subcurve $Z \subset X$ the following inequality (called the Basic Inequality) is satisfied

$$\left|\deg_Z L - \frac{d}{2g-2}\deg_Z(\omega_X)\right| \leq \frac{|Z \cap Z^c|}{2}, \tag{*}$$

where $|Z \cap Z^c|$ denotes the length of the 0-dimensional subscheme of $X$ obtained as the scheme-theoretic intersection of $Z$ with the complementary subcurve $Z^c := \overline{X \setminus Z}$. A balanced line bundle $L$ on $X$ is said to be *properly balanced* if the degree of $L$ on each exceptional component of $X$ is 1. Moreover, a properly balanced line bundle $L$ is said to be *strictly balanced* (resp. *stably balanced*) if the basic inequality (*) is strict except possibly for the subcurves $Z$ such that $Z \cap Z^c \subset X_{\rm exc}$ (resp. such that $Z$ or $Z^c$ is entirely contained in $X_{\rm exc}$).

For $d \geq 10(2g-2)$, Caporaso's description says that the semistability of a polarized curve $(X, L)$ depends only on the isomorphism class of $X$ and the multidegree of $L$. In the paper we prove that for $d < 10(2g-2)$ there are cases where our information on the multidegree of $L$ is not sufficient to determine if $(X, L)$ is semistable or not. More precisely this happens for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ when $X$ admits irreducible elliptic tails (i.e. irreducible components of $X$ of arithmetic genus one and meeting the rest of the curve in one point). Hence we need another definition that concerns the behavior of irreducible elliptic tails of $X$ with respect to a line bundle on $X$. Let $F$ be an irreducible elliptic tail of $X$ and let us denote by $p$ the intersection point between $F$ and $X \setminus F$. Given a line bundle $L$ on $X$, there exists a unique smooth point $q$ such that $L_{|F} = \mathcal{O}_F((d_F-1)p+q)$, where $d_F = \deg_F L$ is the degree of $L$ on $F$. The elliptic tail $F$ is said to be *special* with respect to $L$ (or simply special when the line bundle $L$ is clear from the context) when $q = p$ and *non-special* (with respect to $L$) otherwise. (Notice that $n$-canonical curves contain only special elliptic tails since $\omega_X^{\otimes n}{}_{|F} = \mathcal{O}_F(np)$.)

We can now state the main theorems that we prove in the paper. Our first main result extends the description of semistable (resp. polystable, resp. stable) points $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ given by Caporaso in [Cap94] to the case $d > 4(2g-2)$ and also to the Chow scheme.

**Theorem A** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d > 4(2g-2)$; assume moreover that $X$ is connected. Then the following conditions are equivalent:*

- *(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*
- *(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*
- *(iii)* $X$ *is quasi-stable and* $\mathcal{O}_X(1)$ *is balanced (resp. strictly balanced, resp. stably balanced).*

*In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.*

When $d = 4(2g-2)$, the description of the semistable locus in Theorem A breaks down because in [HM10] Hyeon and Morrison proved that there exist some semistable cuspidal curves. The semistable locus for $d = 4(2g-2)$ is described in the following theorem. (Notice that in this case the Hilbert and Chow semistable loci admit a different description.)

**Theorem B** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d = 4(2g-2)$ and $g \geq 3$; assume moreover that $X$ is connected. Then it holds that*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable if and only if $X$ is quasi-wp-stable without tacnodes nor special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced.*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable if and only if $X$ is quasi-wp-stable without tacnodes and $\mathcal{O}_X(1)$ is balanced.*

*In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.*

The next range where the Hilbert and Chow semistable loci coincide and stay constant is the interval $\frac{7}{2}(2g-2) < d < 4(2g-2)$, where we have the following description. (A simple way to enunciate the following theorem is to say that ordinary cusps appear in the semistable locus and special elliptic tails disappear.)

**Theorem C** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$; assume moreover that $X$ is connected. Then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*

*(iii) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced.*

*In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.*

When $d = \frac{7}{2}(2g-2)$, the description of the Hilbert or Chow semistable locus in Theorem C breaks down again because there exist semistable curves that contain tacnodes with lines. Similarly to the case $d = 4(2g-2)$, we get that the Hilbert and Chow semistable loci admit a different description.

**Theorem D** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d = \frac{7}{2}(2g-2)$ and $g \geq 3$; assume moreover that $X$ is connected. Then it holds that*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable if and only if $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is balanced.*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable if and only if* $X$ *is quasi-wp-stable without special elliptic tails (with respect to* $\mathcal{O}_X(1)$*) and* $\mathcal{O}_X(1)$ *is balanced (resp. strictly balanced, resp. stably balanced).*

*In each of the above cases,* $X \subset \mathbb{P}^{d-g}$ *is non-degenerate and linearly normal, and* $\mathcal{O}_X(1)$ *is non-special.*

The next range where the Hilbert and Chow semistable loci coincide and stay constant is the interval $2(2g-2) < d < \frac{7}{2}(2g-2)$, where we have the following description. (A simple way to enunciate the following theorem is to say that tacnodes with lines appear in the semistable locus and all elliptic tails disappear.)

**Theorem E** *Consider a point* $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ *with* $2(2g-2) < d < \frac{7}{2}(2g-2)$ *and* $g \geq 3$*; assume moreover that* $X$ *is connected. Then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*

*(iii)* $X$ *is quasi-p-stable and* $\mathcal{O}_X(1)$ *is balanced (resp. strictly balanced, resp. stably balanced).*

*In each of the above cases,* $X \subset \mathbb{P}^{d-g}$ *is non-degenerate and linearly normal, and* $\mathcal{O}_X(1)$ *is non-special.*

Note that Theorem E breaks down for $d = 2(2g-2)$ since, for this value of $d$, there are stable points $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ (hence semistable points $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$) with $X$ having arbitrary tacnodal singularities and not just tacnodes with a line (see [HH]).

Extending another result by Caporaso, for $d > 2(2g-2)$ we can say exactly when the Hilbert and Chow quotients are geometric, i. e. the semistable points are also stable: this happens if and only if $\gcd(d+1-g, 2g-2) = 1$. On the contrary, if $\gcd(d+1-g, 2g-2) \neq 1$ a combinatorial argument proves that for some semistable curve $[X \subset \mathbb{P}^{d-g}]$ there is a subcurve $Z$ for which the multidegree of $\mathcal{O}_X(1)$ achieves one of the extremes of the basic inequality; this fact implies that in the closure of the orbit of such $[X \subset \mathbb{P}^{d-g}]$ there is another semistable point $[X' \subset \mathbb{P}^{d-g}]$ such that $X'$ is obtained from $X$ by blowing up the nodes of $(Z \cap Z^c) \setminus X_{\mathrm{exc}}$, so that there is an identification of the two orbits in the quotient. Moreover these are the only orbits identifications that occur in the Hilbert or Chow GIT quotients outside of the critical values $d = \frac{7}{2}(2g-2)$ or $4(2g-2)$ (these last cases will be explained later).

A fondamental hypothesis that is present in all the above theorems is the connectivity of the curve $X$. Indeed, under the assumption that $d > 2(2g-2)$, the locus of connected curves in the Hilbert or Chow semistable locus is a connected and irreducible component. In the paper we prove that there are no other components in the Hilbert or Chow semistable locus if and only if $\gcd(d, g-1) = 1$. More generally, we prove

**Theorem F** *The Chow map* $\mathrm{Ch}: \mathrm{Hilb}_d^{ss} \longrightarrow \mathrm{Chow}_d^{ss}$ *induces a one-to-one correspondence between the connected components of* $\mathrm{Hilb}_d^{ss}$ *and the connected components of* $\mathrm{Chow}_d^{ss}$*. Moreover the cardinality of these sets is the number of integer partition of* $\gcd(d, g-1)$.

As an application of Theorems A, C and E, one gets three compactifications of the universal Jacobian stack $\mathcal{J}_{d,g}$, i.e. the moduli stack of pairs $(C, L)$ where $C$ is a smooth projective curve of genus $g$ and $L$ is a line bundle of degree $d$ on $C$, and of its coarse moduli space $J_{d,g}$.

Denote by $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$) the category fibered in groupoids over the category of $k$-schemes, whose fiber over a $k$-scheme $S$ is the groupoid of pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f : \mathcal{X} \to S$ is a family of quasi-stable curves (resp. quasi-p-stable curves) of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree $d$ over $S$ whose restriction to the geometric fibers of $f$ is properly balanced. Moreover, denote by $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ the category fibered in groupoids over the category of $k$-schemes, whose fiber over a $k$-scheme $S$ is the groupoid of pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f$ is a family of quasi-wp-stable curves of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree $d$ that is properly balanced on the geometric fibers of $f$ and such that the geometric fibers of $f$ do not contain tacnodes with a line nor special elliptic tails with respect to $\mathcal{L}$.

In the following theorem, we summarize the properties of $\overline{\mathcal{J}}_{d,g}$, $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ and $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ we prove in the paper.

**Theorem G** *Let* $g \geq 3$ *and* $d \in \mathbb{Z}$.

1. $\overline{\mathcal{J}}_{d,g}$ *(resp.* $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$, $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$*) is a smooth, irreducible and universally closed Artin stack of finite type over* $k$ *and of dimension* $4g-4$*, containing* $\mathcal{J}_{d,g}$ *as a dense open substack.*

2. $\overline{\mathcal{J}}_{d,g}$ *(resp.* $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$, $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$*) admits an adequate moduli space* $\overline{J}_{d,g}$ *(resp.* $\overline{J}_{d,g}^{\mathrm{wp}}$*, resp.* $\overline{J}_{d,g}^{\mathrm{ps}}$*), which is a normal integral projective variety of dimension* $4g-3$ *containing* $J_{d,g}$ *as a dense open subvariety.*
   *Moreover, if* $\mathrm{char}(k) = 0$*, then* $\overline{J}_{d,g}$ *(resp.* $\overline{J}_{d,g}^{\mathrm{wp}}$*, resp.* $\overline{J}_{d,g}^{\mathrm{ps}}$*) has rational singularities, hence it is Cohen-Macaulay.*

3. *Denote by* $\widetilde{H}_d$ *the main component of the semistable locus of* $\mathrm{Hilb}_d$*, i.e. the open subset of* $\mathrm{Hilb}_d$ *consisting of all the points* $[X \subset \mathbb{P}^{d-g}]$ *that are semistable and such that* $X$ *is connected. Then it holds:*

   (i) $\overline{\mathcal{J}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL_{d-g+1}}]$ *and* $\overline{J}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL_{d-g+1}}$ *if* $d > 4(2g-2)$,

   (ii) $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL_{d-g+1}}]$ *and* $\overline{J}^{\mathrm{wp}}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL_{d-g+1}}$ *if* $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$,

   (iii) $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL_{d-g+1}}]$ *and* $\overline{J}^{\mathrm{ps}}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL_{d-g+1}}$ *if* $2(2g-2) < d \leq \frac{7}{2}(2g-2)$.

4. *We have the following commutative diagrams*

$$
\begin{array}{ccc}
\overline{\mathcal{J}}_{d,g} & \longrightarrow & \overline{J}_{d,g} \\
\downarrow{\scriptstyle \Psi^{\mathrm{s}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{s}}} \\
\overline{\mathcal{M}}_g & \longrightarrow & \overline{M}_g
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} & \longrightarrow & \overline{J}^{\mathrm{wp}}_{d,g} \\
\downarrow{\scriptstyle \Psi^{\mathrm{wp}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{wp}}} \\
\overline{\mathcal{M}}^{\mathrm{wp}}_g & \longrightarrow & \overline{M}^{\mathrm{p}}_g
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} & \longrightarrow & \overline{J}^{\mathrm{ps}}_{d,g} \\
\downarrow{\scriptstyle \Psi^{\mathrm{ps}}} & & \downarrow{\scriptstyle \Phi^{\mathrm{ps}}} \\
\overline{\mathcal{M}}^{\mathrm{p}}_g & \longrightarrow & \overline{M}^{\mathrm{p}}_g
\end{array}
$$

   *where* $\Psi^{\mathrm{s}}$ *(resp.* $\Psi^{\mathrm{wp}}$, $\Psi^{\mathrm{ps}}$*) is universally closed and surjective and* $\Phi^{\mathrm{s}}$ *(resp.* $\Phi^{\mathrm{wp}}$*, resp.* $\Phi^{\mathrm{ps}}$*) is projective and surjective. Moreover:*

   (i) *The morphisms* $\Phi^{\mathrm{s}} : \overline{J}_{d,g} \to \overline{M}_g$ *and* $\Phi^{\mathrm{ps}} : \overline{J}^{\mathrm{ps}}_{d,g} \to \overline{M}^{\mathrm{p}}_g$ *have equidimensional fibers of dimension* $g$*; moreover, if* $\mathrm{char}(k) = 0$, $\Phi^{\mathrm{s}}$ *and* $\Phi^{\mathrm{ps}}$ *are flat over the smooth locus of* $\overline{M}_g$ *and* $\overline{M}^{\mathrm{p}}_g$*, respectively.*

   (ii) *The fiber of the morphism* $\Phi^{\mathrm{wp}} : \overline{J}^{\mathrm{wp}}_{d,g} \to \overline{M}^{\mathrm{p}}_g$ *over a p-stable curve* $X \in \overline{M}^{\mathrm{p}}_g$ *has dimension equal to the sum of* $g$ *with the number of cusps of* $X$*.*

5. *Let* $\overline{\mathcal{J}}^{\star}_{d,g}$ *be equal to either* $\overline{\mathcal{J}}_{d,g}$ *or* $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$ *or* $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$*. Denote by* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *the rigidification of* $\overline{\mathcal{J}}^{\star}_{d,g}$ *by* $\mathbb{G}_m$ *and by* $\widehat{\Psi}^{\star} : \overline{\mathcal{J}}^{\star}_{d,g} \to \overline{\mathcal{M}}^{\star}_g$ *the associated morphism, where* $\overline{\mathcal{M}}^{\star}_g$ *is equal to either* $\overline{\mathcal{M}}_g$ *or* $\overline{\mathcal{M}}^{\mathrm{wp}}_g$ *or* $\overline{\mathcal{M}}^{\mathrm{p}}_g$*. Then the following conditions are equivalent:*

   (i) $\gcd(d+1-g, 2g-2) = 1$*;*

   (ii) *The stack* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *is a DM-stack;*

   (iii) *The stack* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *is proper;*

   (iv) *The morphism* $\widehat{\Psi}^{\star} : \overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}^{\star}_g$ *is representable.*

6. *If* $\mathrm{char}(k) = 0$*, then it holds*

   (i) $(\Phi^{\mathrm{st}})^{-1}(X) \cong \overline{\mathrm{Jac_d}}(X)/\mathrm{Aut}(X)$ *for any* $X \in \overline{M}_g$, *(ii)* $(\Phi^{\rm ps})^{-1}(X) \cong \overline{\rm Jac_d}(X)/{\rm Aut}(X)$ *for any* $X \in \overline{M}_g^{\rm p}$,

*where* $\overline{\rm Jac_d}(X)$ *is the moduli space of of rank-1, torsion-free sheaves on* $X$ *of degree* $d$ *that are slope-semistable with respect to* $\omega_X$ *(and it is called the canonical compactified Jacobian of* $X$ *in degree* $d$*).*

My contribution to the paper was to solve the Problem for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$. Precisely I proved Theorem B, Theorem C, Theorem D, Theorem F and Theorem G for the case of the stack $\overline{\mathcal{J}}_{d,g}^{\rm wp}$.

Now let us make come comments about the proof strategy. The approach to the problem of determining the semistable locus is the same as that developed by Mumford, Gieseker and Caporaso: firstly we use Hilbert-Mumford numerical criterion in order to find necessary conditions for a point $[X \subset \mathbb{P}^{d-g}]$ in the Hilbert scheme to be semistable (see Fact 4.20, Corollary 9.4 and Corollary 9.7) and finally we characterize the entire semistable locus using combinatorial properties of the multidegree of $\mathcal{O}_X(1)$ and separateness property of suitable stacks of curves. For $d \geq 4(2g-2)$ and $2(2g-2) < d < \frac{7}{2}(2g-2)$ this strategy does work because the semistable locus consists only of quasi-stable and quasi-pseudo-stable curves respectively, thus in the second step it suffices to work with separated stacks like $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\rm p}$ respectively (for $\overline{\mathcal{M}}_g^{\rm p}$ it is necessary to suppose that $g \geq 3$, because $\overline{\mathcal{M}}_2^{\rm p}$ is not separated).

Unfortunately for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ it is not very hard to prove the existence of semistable curves admitting cusps and elliptic tails (see Remark 11.4 and Corollary 12.3), so that we have to work with the stack $\overline{\mathcal{M}}_g^{\rm wp}$ of weakly-pseudo-stable curves, which is not separated. For this reason it is necessary to use other techniques. A very naive idea is to apply again Hilbert-Mumford numerical criterion. We recall that the Hilbert-Mumford criterion states that given a curve $X \subset \mathbb{P}^{d-g}$

$$[X \subset \mathbb{P}^{d-g}] \text{ is semistable } \Longleftrightarrow \mu([X \subset \mathbb{P}^{d-g}], \rho) \geq 0 \text{ for each 1ps } \rho : \mathbb{G}_m \longrightarrow {\rm SL_{d-g+1}}$$

(see [Dol03] for the definition of $\mu([X \subset \mathbb{P}^{d-g}], \rho)$). "Unfortunately" this criterion is easier to apply when we would like to prove the instability of curves rather than the semistability. [1]

One way to solve this difficulty is to apply Tits' results about the parabolic group associated to a fixed one-parameter subgroup (see for more details [Dol03, Sec. 9.5] or [MFK94, Chap. 2, Sec. 2]). These results allowed G. Kempf to prove that if $[X \subset \mathbb{P}^{d-g}]$ is unstable, then there exists a unique one-parameter subgroup which in some sense

[1] Observe that recently Li and Wang in [LW] managed to give a different proof of Caporaso's description applying exclusively the Hilbert-Mumford numerical criterion, but their proof works only for $d \gg 0$.

is the main responsible for the instability of $[X \subset \mathbb{P}^{d-g}]$. The idea, hence, is to use the properties of the parabolic group to study the behaviour of curves having elliptic tails under the action of one parameter subgroups: we prove that, if $[X \subset \mathbb{P}^{d-g}]$ has an elliptic tail, i. e. $X$ is the union of an elliptic curve $F$ and another curve $C$ such that $F$ and $C$ intersect each other in one node, the GIT analysis can be restricted to 1ps $\rho : \mathbb{G}_m \longrightarrow \mathrm{SL}_{\mathrm{d-g+1}}$ diagonalized by bases of $\mathbb{P}^{d-g}$ that come out from the union of bases of the linear spans $\langle F \rangle$ and $\langle C \rangle$ in $\mathbb{P}^{d-g}$. In other words we can study the semistability of $X$ by analyzing the subcurves $F$ and $C$ in their linear spans separately. Essentially this is the content of the Criterion of stability of tails (see Proposition 8.3).

Motivated by this criterion, we study the behaviour of polarized elliptic curves $F \subset \mathbb{P}^r$ for some suitable $r$ under the action of one parameter subgroups and we prove that for $\frac{7}{2}(2g-2) < d < 4(2g-2)$ there are semistable curves $[X \subset \mathbb{P}^{d-g}]$ that admit non-special elliptic tails (see Remark 11.4) for all models of non-special elliptic tail (see Corollary 12.3).

The final part of the GIT analysis is based on a nice numerical trick. We will explain this trick briefly in the case $\frac{7}{2}(2g-2) < d < 4(2g-2)$. Given a quasi-wp-stable curve $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ like above with $F$ non-special, we define a new polarized curve $X'$ by replacing the polarized subcurve $F$ with a polarized smooth curve $Y$ of genus $g$ and degree $d - d_F$ in such a way that $Y$ and $C$ intersect again in one node. If we denote by $d'$ and $g'$ respectively the degree of the new line bundle $L'$ and the genus of $X'$, one can consider the Hilbert point $[X' \subset \mathbb{P}^{d'-g'}] \in \mathrm{Hilb}_{d'}$. It can be easily checked that

$$\frac{d'}{2g'-2} = \frac{d}{2g-2}$$

and

$$\mathcal{O}_X(1) \text{ is balanced } \Longleftrightarrow \mathcal{O}_{X'}(1) \text{ is balanced.}$$

Applying our criterion, one prove that

$$[X' \subset \mathbb{P}^{d'-g'}] \text{ is semistable } \Longrightarrow [X \subset \mathbb{P}^{d-g}] \text{ is semistable,}$$

so that the GIT analysis can be completed by an induction argument on the number of non-special elliptic tails of $X$. The proof of the base of induction requires the separateness of $\overline{\mathcal{M}}_g^{\mathrm{p}}$, so that we need to suppose again that $g \geq 3$.

Let us now comment on the origin of the two *critical values* $d = 4(2g-2)$ and $d = \frac{7}{2}(2g-2)$, at which the Hilbert and Chow semistable loci change.

The first critical value $d = 4(2g-2)$ is due to the presence of Chow semistable points $\mathrm{Ch}([X_0 \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$ such that $X_0$ has a cuspidal elliptic tail which is special with respect to $\mathcal{O}_{X_0}(1)$. Inside the group of automorphism of this polarized curve there is a non-trivial copy of the multiplicative group $\mathbb{G}_m$ that induces a one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL_{d-g+1}}$ whose image in $\mathrm{PGL_{d-g+1}}$ is contained in the stabilizer subgroup of $[X_0 \subset \mathbb{P}^{d-g}]$. The basins of attraction of $[X_0 \subset \mathbb{P}^{d-g}]$ with respect to $\rho$ and $\rho^{-1}$ are the ones depicted in Figure 1 below.

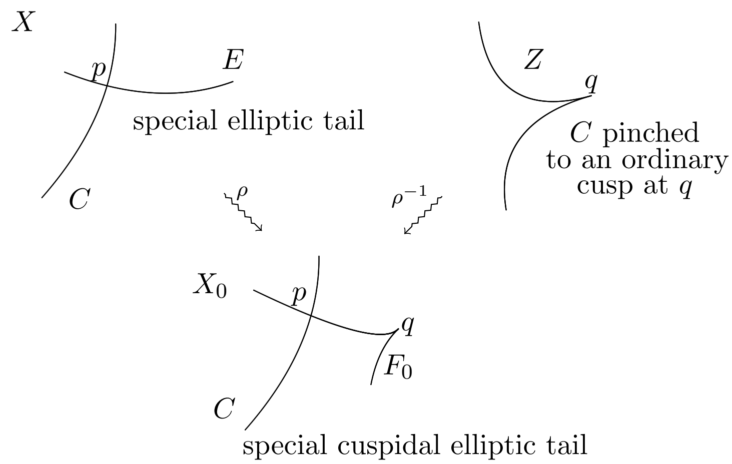


Figure 1: The basin of attraction of a curve $X_0$ with a special cuspidal elliptic tail $F_0$.

This implies that as $\frac{d}{2g-2}$ passes from $4+\epsilon$ to $4-\epsilon$ for a small $\epsilon$, special elliptic tails become (Hilbert or Chow) unstable and they get replaced by cusps. Moreover, Hilbert semistability for $d = 4(2g-2)$ behaves like Hilbert (or Chow) semistability for $\frac{7}{2}(2g-2) < d < 4(2g-2)$; hence Hilbert semistability is strictly stronger than Chow semistability for $d = 4(2g-2)$.

The second critical value $d = \frac{7}{2}(2g-2)$ is due to the presence of Chow semistable points $\mathrm{Ch}([X_0 \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$ such that $X_0$ has a tacnodal elliptic tail. As before, the presence of a non-trivial copy of the multiplicative group $\mathbb{G}_m$ inside the group of automorphism of this polarized curve induces a one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL_{d-g+1}}$ whose image in $\mathrm{PGL_{d-g+1}}$ is contained in the stabilizer subgroup of $[X_0 \subset \mathbb{P}^{d-g}]$. The basins of attraction of $[X_0 \subset \mathbb{P}^{d-g}]$ with respect to $\rho$ and $\rho^{-1}$ are the ones depicted in Figure 2 below.
This implies that as $\frac{d}{2g-2}$ passes from $\frac{7}{2}+\epsilon$ to $\frac{7}{2}-\epsilon$ for a small $\epsilon$ non-special elliptic tails become (Hilbert or Chow) unstable and they get replaced by tacnodes with a line. Moreover, Hilbert semistability for $d = \frac{7}{2}(2g-2)$ behaves like Hilbert (or Chow) semistability for $2(2g-2) < d < \frac{7}{2}(2g-2)$; hence Hilbert semistability is strictly stronger than Chow semistability for $d = \frac{7}{2}(2g-2)$.

Notice that the basins of attraction of Figure 1 were already considered by Hyeon-Morrison in [HM10] in order to determine the semistable locus of $\overline{\mathrm{Hilb}^{\mathrm{can}}_{4(2g-2)}}$ and $\overline{\mathrm{Chow}^{\mathrm{can}}_{4(2g-2)}}$. On the other hand, the basins of attraction of Figure 2 are clearly

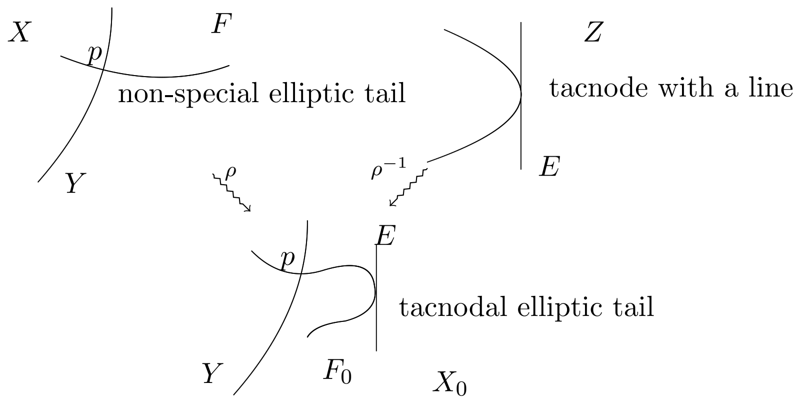


Figure 2: The basin of attraction of a curve $X_0$ with a tacnodal elliptic tail $F_0$.

not visible inside the pluricanonical locus (because they occur for a fractional value of $\frac{d}{2g-2}$).

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# GIT FOR POLARIZED CURVES

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Abstract. We investigate the GIT quotients of the Hilbert and Chow schemes of curves of degree $d$ and genus $g$ in a projective space of dimension $d-g$, as the degree $d$ decreases with respect to the genus $g$. We prove that the first three values of $d$ at which the GIT quotients change are given by $d = 4(2g-2)$, $d = \frac{7}{2}(2g-2)$ and $d = 2(2g-2)$. In the range $d > 4(2g-2)$, we show that the previous results of L. Caporaso hold true both for the Hilbert and Chow semistability. In the range $4(2g-2) < d < \frac{7}{2}(2g-2)$, the Hilbert semistable locus coincides with the Chow semistable locus and it maps to the moduli stack of weakly-pseudo-stable curves. In the range $2(2g-2) < d < \frac{7}{2}(2g-2)$, the Hilbert and Chow semistable loci coincide and they map to the moduli stack of pseudo-stable curves. We also analyze in detail the first two critical values $d = 4(2g-2)$ and $d = \frac{7}{2}(2g-2)$, where the Hilbert semistable locus is strictly smaller than the Chow semistable locus. As an application of our results, we get two new compactifications of the universal Jacobian over the moduli space of weakly-pseudo-stable and pseudo-stable curves, respectively.

## Contents



2010 *Mathematics Subject Classification.* 14L24, 14H10, 14H40, 14H20, 14C05, 14D23.

## 1. Introduction

1.1. **Motivation and previous related works.** One of the first successful applications of Geometric Invariant Theory (GIT for short), and perhaps one of the major motivation for its development by Mumford and his co-authors (see [MFK94]), was the construction of the moduli space $M_g$ of smooth curves of genus $g \geq 2$ and its compactification $\overline{M}_g$ via *stable curves* (i.e. connected nodal projective curves with finite automorphism group), carried out by Mumford ([Mum77]) and Gieseker ([Gie82]). Indeed, the moduli space of stable curves was constructed as a GIT quotient of a locally closed subset of a suitable Hilbert scheme (as in [Gie82]) or Chow scheme (as in [Mum77]) parametrizing $n$-canonically embedded curves, for $n$ sufficiently large. More precisely, Mumford in [Mum77] works under the assumption that $n \geq 5$ and Gieseker in [Gie82] requires the more restrictive assumption that $n \geq 10$. However, it was later discovered that Gieseker's approach can also be extended to the case $n \geq 5$ (see [HM98, Chap. 4, Sec. C] or [Mor10, Sec. 3]).

Recently, there has been a lot of interest in extending the above GIT analysis to smaller values of $n$, especially in connection with the so called Hassett-Keel program whose ultimate goal is to find the minimal model of $M_g$ via the successive constructions of modular birational models of $\overline{M}_g$ (see [FS13] and [AH12] for nice overviews).

The first work in this direction is due to Schubert, who described in [Sch91] the GIT quotient of the locus of 3-canonically embedded curves (of genus $g \geq 3$) in the Chow scheme as the coarse moduli space $\overline{M}_g^{\mathrm{p}}$ of pseudo-stable curves (or *p-stable curves* for short). These are connected projective curves with finite automorphism group, whose only singularities are nodes and cusps, and which have no elliptic tails (i.e. connected subcurves of arithmetic genus one meeting the rest of the curve in one point). Since the GIT quotient analyzed by Schubert is geometric (i.e. there are no strictly semistable objects), one gets exactly the same description working with 3-canonically embedded curves inside the Hilbert scheme (see [HH13, Prop. 3.13]). Later, Hassett-Hyeon constructed in [HH09] a modular map $T : \overline{M}_g \to \overline{M}_g^{\mathrm{p}}$ which on geometric points sends a stable curve onto the p-stable curve obtained by contracting all its elliptic tails to cusps. Moreover, the authors of loc. cit. identified the map $T$ with the first contraction in the Hassett-Keel program for $\overline{M}_g$.

The case of 4-canonical curves was worked out by Hyeon-Morrison in [HM10]. The Hilbert GIT-semistable points turn out to correspond again to p-stable curves, while the Chow GIT-semistable locus is strictly bigger and it consists of weakly-pseudo-stable curves (or *wp-stable curves* for short), which are connected projective curves with finite automorphism group, whose only singularities are nodes and cusps (and having possibly elliptic tails). However, Hyeon-Morrison also proved that the GIT quotient for the Chow scheme turns out to be again isomorphic to the moduli space $\overline{M}_g^{\mathrm{p}}$ of p-stable curves, a fact that can be reinterpreted as saying that the non-separated stack of wp-stable curves and its open and proper substack of p-stable curves have the same moduli space (see § 2.1 for more details).

Finally, the case of 2-canonical curves was studied by Hassett-Hyeon in [HH13], where the authors described the Hilbert GIT quotient $\overline{M}_g^h$ and the Chow GIT quotient $\overline{M}_g^c$ (they are now different), as moduli spaces of $h$-semistable (resp. $c$-semistable) curves; see loc. cit. for the precise description. Moreover, they constructed a small contraction $\Psi : \overline{M}_g^{\mathrm{p}} \to \overline{M}_g^c$ and identified the natural map
$\Psi^+ : \overline{M}_g^h \to \overline{M}_g^c$ as the flip of $\Psi$. These maps are then interpreted as further steps in the Hassett-Keel program for $\overline{M}_g$.

For some partial results on the GIT quotient for the Hilbert scheme of 1-canonically embedded curves, we refer the reader to the work of Alper, Fedorchuk and Smyth (see [AFS13]).

From the point of view of constructing new projective birational models of $\overline{M}_g$, it is of course natural to restrict the GIT analysis to the locally closed subset inside the Hilbert or Chow scheme parametrizing $n$-canonical embedded curves. However, the problem of describing the whole GIT quotient seems very natural and interesting too. The first result in this direction is the pioneering work of Caporaso [Cap94], where the author describes the GIT quotient of the Hilbert scheme of connected curves of genus $g \geq 3$ and degree $d \geq 10(2g-2)$ in $\mathbb{P}^{d-g}$. The GIT quotient obtained by Caporaso in loc. cit. is indeed a modular compactification of the universal Jacobian $J_{d,g}$, which is the moduli scheme parametrizing pairs $(C, L)$ where $C$ is a smooth curve of genus $g$ and $L$ is a line bundle on $C$ of degree $d$. Note that recently Li and Wang in [LW] have studied Chow (semi-)stability of polarized nodal curves of sufficiently high degree, giving in particular a different proof of Caporaso's result for $d \gg 0$[1].

Our work is motivated by the following

**Problem:** *Describe the GIT quotient for the Hilbert and Chow scheme of curves of genus $g$ and degree $d$ in $\mathbb{P}^{d-g}$, as $d$ decreases with respect to $g$.*

1.2. **Our results.** In order to describe our results, we need to introduce some notation. Fix an integer $g \geq 2$. For any natural number $d$, denote by $\mathrm{Hilb}_d$ the Hilbert scheme of curves of degree $d$ and arithmetic genus $g$ in $\mathbb{P}^{d-g}$; denote by $\mathrm{Chow}_d$ the Chow scheme of 1-cycles of degree $d$ in $\mathbb{P}^{d-g}$ and by

$$\mathrm{Ch} : \mathrm{Hilb}_d \to \mathrm{Chow}_d$$

the map sending a one dimensional subscheme $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ to its 1-cycle. The linear algebraic group $\mathrm{SL}_{d-g+1}$ acts naturally on $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$ so that Ch is an equivariant map; moreover, these actions are naturally linearized (see Section 4.1 for

[1]Notice that Li-Wang worked more generally with polarized pointed weighted nodal curves.

details [2]), so it makes sense to talk about (GIT) (semi-,poly-)stability of a point in $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$.

The aim of this work is to give a complete characterization of the (semi-,poly-)stable points $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ or of its image $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$, provided that $d > 2(2g-2)$. Our characterization of Hilbert or Chow (semi-, poly-)stability will require some conditions on the singularities of $X$ and some conditions on the multidegree of the line bundle $\mathcal{O}_X(1)$. Let us introduce the relevant definitions.

A curve $X$ is said to be *quasi-stable* if it is obtained from a stable curve $Y$ by "blowing up" some of its nodes, i.e. by taking the partial normalization of $Y$ at some of its nodes and inserting a $\mathbb{P}^1$ connecting the two branches of each node. A curve $X$ is said to be *quasi-p-stable* (resp. *quasi-wp-stable*) if it is obtained from a p-stable curve (resp. a wp-stable curve) $Y$ by "blowing up" some of its nodes (as before) and "blowing up" some of its cusps, i.e. by taking the partial normalization of $Y$ at some of its cusps and inserting a $\mathbb{P}^1$ tangent to the branch point of each cusp (the singularity that one gets by blowing up a cusp is called *tacnode with a line*). Note that quasi-stable and quasi-p-stable curves are special cases of quasi-wp-stable curves: the quasi-stable curves are exactly the quasi-wp-stable curves without cusps nor tacnodes with a line; the quasi-p-stable curves are exactly the quasi-wp-stable curves without elliptic tails. Given a quasi-wp-stable curve $X$, we call the $\mathbb{P}^1$'s inserted by blowing up nodes or cusps of $Y$ the *exceptional components*, and we denote by $X_{\mathrm{exc}} \subset X$ the union of all of them.

A line bundle $L$ of degree $d$ on a quasi-wp-stable curve $X$ of genus $g$ is said to be *balanced* if for each subcurve $Z \subset X$ the following inequality (called the basic inequality) is satisfied

$$(*) \qquad \left| \deg_Z L - \frac{d}{2g-2} \deg_Z(\omega_X) \right| \leq \frac{|Z \cap Z^c|}{2},$$

where $|Z \cap Z^c|$ denotes the length of the 0-dimensional subscheme of $X$ obtained as the scheme-theoretic intersection of $Z$ with the complementary subcurve $Z^c := \overline{X \setminus Z}$. A balanced line bundle $L$ on $X$ is said to be *properly balanced* if the degree of $L$ on each exceptional component of $X$ is 1. Moreover, a properly balanced line bundle $L$ is said to be *strictly balanced* (resp. *stably balanced*) if the basic inequality (*) is strict except possibly for the subcurves $Z$ such that $Z \cap Z^c \subset X_{\mathrm{exc}}$ (resp. such that $Z$ or $Z^c$ is entirely contained in $X_{\mathrm{exc}}$).

The last definition we need concerns the behavior of irreducible elliptic tails of $X$ (i.e. irreducible components of $X$ of arithmetic genus one and meeting the rest of the curve in one point) with respect to a line bundle on $X$. So, let $F$ be an irreducible elliptic tail of $X$ and let $p$ denote the intersection point between $F$ and the complementary subcurve. Given a line bundle $L$ on $X$, we can write $L_{|F} = \mathcal{O}_F((d_F - 1)p + q)$, where

[2] In particular, when working with $\mathrm{Hilb}_d$, we will always consider the $m$-linearization for $m \gg 0$; see Section 4.1 for details.

$d_F = \deg_F L$ denotes the degree of $L$ on $F$, for a uniquely determined smooth point $q$ of $F$. We say that $F$ is *special* with respect to $L$ (or simply special when the line bundle $L$ is clear from the context) and *non-special* (with respect to $L$) otherwise.

We can now state the main theorems that we prove in this paper. Our first main result extends the description of semistable (resp. polystable, resp. stable) points $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ given by Caporaso in [Cap94] to the case $d > 4(2g-2)$ and also to the Chow scheme.

**Theorem A.** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d > 4(2g-2)$; assume moreover that $X$ is connected. Then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*

*(iii)* $X$ *is quasi-stable and* $\mathcal{O}_X(1)$ *is balanced (resp. strictly balanced, resp. stably balanced).*

*In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.*

Theorem A follows by combining Theorem 11.1(1), Corollary 11.2(1) and Corollary 11.3(1).

When $d = 4(2g-2)$, the description of the semistable locus in Theorem A breaks down and we get that the Hilbert and Chow semistable loci admit a different description.

**Theorem B.** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $d = 4(2g-2)$ and $g \geq 3$; assume moreover that $X$ is connected. Then it holds that*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable if and only if $X$ is quasi-wp-stable without tacnodes nor special elliptic tails (with respect to $\mathcal{O}_X(1)$) and $\mathcal{O}_X(1)$ is balanced.*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable if and only if $X$ is quasi-wp-stable without tacnodes and $\mathcal{O}_X(1)$ is balanced.*

*In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.*

Theorem B follows from Theorem 13.5. For a description of the Hilbert or Chow polystable (resp. stable) locus, we refer the reader to Corollary 13.6 (resp. Corollary 13.7).

The next range where the Hilbert and Chow GIT-semistable loci coincide and stay constant is the interval $\frac{7}{2}(2g-2) < d < 4(2g-2)$, where we have the following description.

**Theorem C.** *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ with $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$; assume moreover that $X$ is connected. Then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*

*(iii)* $X$ *is quasi-wp-stable without tacnodes nor special elliptic tails (with respect to* $\mathcal{O}_X(1)$*) and* $\mathcal{O}_X(1)$ *is balanced.*

*In each of the above cases,* $X \subset \mathbb{P}^{d-g}$ *is non-degenerate and linearly normal, and* $\mathcal{O}_X(1)$ *is non-special.*

Theorem C follows by combining Theorem 13.2, Corollary 13.3 and Corollary 13.4.

When $d = \frac{7}{2}(2g-2)$, the description of the Hilbert or Chow semistable locus in Theorem C breaks down again and we get that the Hilbert and Chow semistable loci admit a different description, similarly to the case $d = 4(2g-2)$.

**Theorem D.** *Consider a point* $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ *with* $d = \frac{7}{2}(2g-2)$ *and* $g \geq 3$*; assume moreover that* $X$ *is connected. Then it holds that*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable if and only if* $X$ *is quasi-p-stable and* $\mathcal{O}_X(1)$ *is balanced.*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable if and only if* $X$ *is quasi-wp-stable without special elliptic tails (with respect to* $\mathcal{O}_X(1)$*) and* $\mathcal{O}_X(1)$ *is balanced.*

*In each of the above cases,* $X \subset \mathbb{P}^{d-g}$ *is non-degenerate and linearly normal, and* $\mathcal{O}_X(1)$ *is non-special.*

Theorem D follows from Theorem 11.5. For a description of the Hilbert or Chow polystable (resp. stable), we refer the reader to Corollary 11.6 (resp. Corollary 11.7).

The next range where the Hilbert and Chow semistable loci coincide and stay constant is the interval $2(2g-2) < d < \frac{7}{2}(2g-2)$, where we have the following description.

**Theorem E.** *Consider a point* $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ *with* $2(2g-2) < d < \frac{7}{2}(2g-2)$ *and* $g \geq 3$*; assume moreover that* $X$ *is connected. Then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^{d-g}]$ *is semistable (resp. polystable, resp. stable);*

*(ii)* $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}])$ *is semistable (resp. polystable, resp. stable);*

*(iii)* $X$ *is quasi-p-stable and* $\mathcal{O}_X(1)$ *is balanced (resp. strictly balanced, resp. stably balanced).*

*In each of the above cases,* $X \subset \mathbb{P}^{d-g}$ *is non-degenerate and linearly normal, and* $\mathcal{O}_X(1)$ *is non-special.*

The above Theorem E follows by combining Theorem 11.1(2), Corollary 11.2(2) and Corollary 11.3(2). Note that Theorem E breaks down for $d = 2(2g-2)$ since, for this value of $d$, there are stable points $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ (hence semistable points $\mathrm{Ch}([X \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$) with $X$ having arbitrary tacnodal singularities and not just tacnodes with a line (see Remark 5.3).

Let us now briefly comment on the assumptions of the above theorems. First of all, with the exception of Theorem A, the other four theorems require that $g \geq 3$. The reason for this assumption is that the moduli stack of p-stable curves of genus $g$ is not separated for $g = 2$ (see § 2.1) and this causes some extra-difficulties in the

GIT analysis. In particular, we use the hypothesis that $g \geq 3$ (whenever p-stable or wp-stable curves are involved) in a crucial way in Theorem 6.4, Propositions 10.5 and 10.8. Therefore, for simplicity, we restrict in this paper to the case $g \geq 3$ whenever dealing with p-stable or wp-stable curves (i.e. for $d \leq 4(2g-2)$); the GIT analysis for $g = 2$ and the missing values of $d$ (i.e. $d = 5, 6, 7, 8$) will be dealt with in a future work.

Another hypothesis that is present in all the above theorems is the connectivity of the curve $X$. Indeed, under the assumption that $d > 2(2g-2)$, the locus of connected curves in the Hilbert or Chow semistable locus is a connected and irreducible component (see the beginning of Section 10 and Corollary 14.7), that we call the main component (see Section 14). In Section 15, we prove that there are no other components in the Hilbert or Chow semistable locus if and only if $\gcd(d, g-1) = 1$. More generally, we prove in Theorem 15.4 that the number of connected components (which are also irreducible) of the Hilbert or Chow semistable locus is equal to the number of partitions of $\gcd(d, g-1)$.

Now let us make come comments about the proof strategy. The approach to the problem of determining the semistable locus is the same as that developed by Mumford, Gieseker and Caporaso: firstly we use Hilbert-Mumford numerical criterion in order to find necessary conditions for a point $[X \subset \mathbb{P}^{d-g}]$ in the Hilbert scheme to be semistable (see Fact 4.20, Corollary 9.4 and Corollary 9.7) and finally we characterize the entire semistable locus using combinatorial properties of the multidegree of $\mathcal{O}_X(1)$ and separateness property of suitable stacks of curves. For $d \geq 4(2g-2)$ and $2(2g-2) < d < \frac{7}{2}(2g-2)$ this strategy does work because the semistable locus consists only of quasi-stable and quasi-pseudo-stable curves respectively, thus in the second step it suffices to work with separated stacks like $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\mathrm{p}}$ respectively (for $\overline{\mathcal{M}}_g^{\mathrm{p}}$ it is necessary to suppose that $g \geq 3$, because $\overline{\mathcal{M}}_2^{\mathrm{p}}$ is not separated).

Unfortunately for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ it is not very hard to prove the existence of semistable curves admitting cusps and elliptic tails (see Remark 11.4 and Corollary 12.3), so that we have to work with the stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ of weakly-pseudo-stable curves, which is not separated. For this reason it is necessary to use other techniques. A very naive idea is to apply again Hilbert-Mumford numerical criterion. We recall that the Hilbert-Mumford criterion states that given a curve $X \subset \mathbb{P}^{d-g}$

$$[X \subset \mathbb{P}^{d-g}] \text{ is semistable } \iff \mu([X \subset \mathbb{P}^{d-g}], \rho) \geq 0 \text{ for each 1ps } \rho : \mathbb{G}_m \longrightarrow \mathrm{SL}_{d-g+1}$$

(see [Dol03] for the definition of $\mu([X \subset \mathbb{P}^{d-g}], \rho)$). “Unfortunately” this criterion is easier to apply when we would like to prove the instability of curves rather than the semistability.

One way to solve this difficulty is to apply Tits' results about the parabolic group associated to a fixed one-parameter subgroup (see for more details [Dol03, Sec. 9.5] or [MFK94, Chap. 2, Sec. 2]). These results allowed G. Kempf to prove that if $[X \subset \mathbb{P}^{d-g}]$ is unstable, then there exists a unique one-parameter subgroup which in some sense is the main responsible for the instability of $[X \subset \mathbb{P}^{d-g}]$. The idea, hence, is to use the properties of the parabolic group to study the behaviour of curves having elliptic tails under the action of one parameter subgroups: we prove that, if $[X \subset \mathbb{P}^{d-g}]$ has an elliptic tail, i. e. $X$ is the union of an elliptic curve $F$ and another curve $C$ such that $F$ and $C$ intersect each other in one node, the GIT analysis can be restricted to 1ps $\rho : \mathbb{G}_m \longrightarrow \mathrm{SL}_{d-g+1}$ diagonalized by bases of $\mathbb{P}^{d-g}$ that come out from the union of bases of the linear spans $\langle F \rangle$ and $\langle C \rangle$ in $\mathbb{P}^{d-g}$. In other words we can study the semistability of $X$ by analyzing the subcurves $F$ and $C$ in their linear spans separately. Essentially this is the content of the Criterion of stability of tails (see Proposition 8.3).

Motivated by this criterion, we study the behaviour of polarized elliptic curves $F \subset \mathbb{P}^r$ for some suitable $r$ under the action of one parameter subgroups and we prove that for $\dfrac{7}{2}(2g-2) < d < 4(2g-2)$ there are semistable curves $[X \subset \mathbb{P}^{d-g}]$ that admit non-special elliptic tails (see Remark 11.4) for all models of non-special elliptic tail (see Corollary 12.3).

The final part of the GIT analysis is based on a nice numerical trick. We will explain this trick briefly in the case $\dfrac{7}{2}(2g-2) < d < 4(2g-2)$. Given a quasi-wp-stable curve $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ like above with $F$ non-special, we define a new polarized curve $X'$ by replacing the polarized subcurve $F$ with a polarized smooth curve $Y$ of genus $g$ and degree $d - d_F$ in such a way $Y$ and $C$ intersect again in one node. If we denote by $d'$ and $g'$ respectively the degree of the new line bundle $L'$ and the genus of $X'$, one can consider the Hilbert point $[X' \subset \mathbb{P}^{d'-g'}] \in \mathrm{Hilb}_{d'}$. It can be easily checked that

$$\frac{d'}{2g'-2} = \frac{d}{2g-2}$$

and

$$\mathcal{O}_X(1) \text{ is balanced } \iff \mathcal{O}_{X'}(1) \text{ is balanced.}$$

Applying our criterion, one prove that

$$[X' \subset \mathbb{P}^{d'-g'}] \text{ is semistable } \implies [X \subset \mathbb{P}^{d-g}] \text{ is semistable,}$$

so that the GIT analysis can be completed by an induction argument on the number of non-special elliptic tails of $X$. The proof of the base of induction requires the separateness of $\overline{\mathcal{M}}_g^{\mathrm{p}}$, so that we need to suppose again that $g \geq 3$.

Let us now comment on the origin of the two *critical values* $d = 4(2g-2)$ and $d = \frac{7}{2}(2g-2)$, at which the Hilbert and Chow semistable loci change. It turns out that the existence of these two critical values is related to the presence in the Chow semistable locus of a point $\mathrm{Ch}([X \subset \mathbb{P}^r])$ whose stabilizer subgroup in $\mathrm{PGL}_{d-g+1}$ contains a copy of the multiplicative subgroup $\mathbb{G}_m$. This resembles very much what happens in the Hassett-Keel program for $\overline{\mathcal{M}}_g$ where the variations of the log canonical models of $\overline{\mathcal{M}}_g$ are expected to be accounted for by curves with a $\mathbb{G}_m$-automorphism; see [AFS1].

The first critical value $d = 4(2g-2)$ is due to the presence of Chow semistable points $\mathrm{Ch}([X_0 \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$ such that $X_0$ has a cuspidal elliptic tail which is special with respect to $\mathcal{O}_{X_0}(1)$. Such a point has a non-trivial copy of the multiplicative group $\mathbb{G}_m$ into its stabilizer subgroup inside $\mathrm{PGL}_{d-g+1}$ (see Lemma 6.1 and Theorem 6.4). With respect to a suitable one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}_{d-g+1}$ whose image in $\mathrm{PGL}_{d-g+1}$ is contained in the stabilizer subgroup of $[X_0 \subset \mathbb{P}^{d-g}]$ (as in the proof of Theorem 9.1), we prove in Theorem 9.2 that the basins of attraction of $[X_0 \subset \mathbb{P}^{d-g}]$ with respect to $\rho$ and $\rho^{-1}$ are the ones depicted in Figure 1 below.

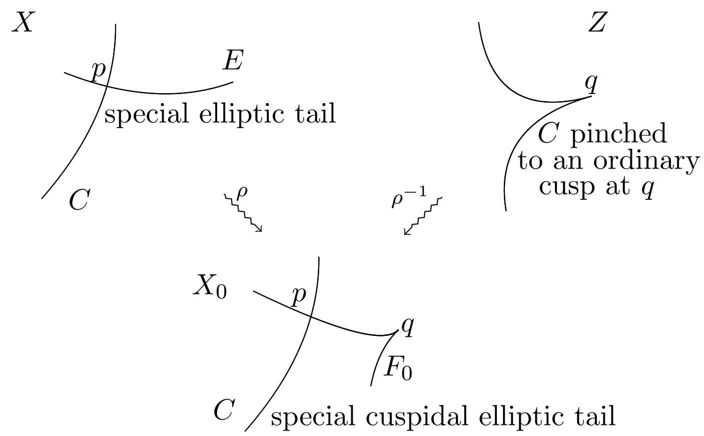


FIGURE 1. The basin of attraction of a curve $X_0$ with a special cuspidal elliptic tail $F_0$.

This implies that, in crossing the critical value $d = 4(2g-2)$ (i.e. as $\frac{d}{2g-2}$ passes from $4+\epsilon$ to $4-\epsilon$ for a small $\epsilon$), special elliptic tails become (Hilbert or Chow) unstable and they get replaced by cusps. Moreover, Hilbert semistability for $d = 4(2g-2)$ behaves like Hilbert (or Chow) semistability for $\frac{7}{2}(2g-2) < d < 4(2g-2)$; hence Hilbert semistability is strictly stronger than Chow semistability for $d = 4(2g-2)$.

The second critical value $d = \frac{7}{2}(2g-2)$ is due to the presence of Chow semistable points $\mathrm{Ch}([X_0 \subset \mathbb{P}^{d-g}]) \in \mathrm{Chow}_d$ such that $X_0$ has a tacnodal elliptic tail. Such a point has a non-trivial copy of the multiplicative group $\mathbb{G}_m$ into its stabilizer subgroup with respect to $\mathrm{PGL}_{d-g+1}$ (see Lemma 6.1 and Theorem 6.4). With respect to a suitable one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}_{d-g+1}$ whose image in $\mathrm{PGL}_{d-g+1}$ is contained in the stabilizer subgroup of $[X_0 \subset \mathbb{P}^{d-g}]$ (as in the proof of Theorem 9.6), the basins of attraction of $[X_0 \subset \mathbb{P}^{d-g}]$ with respect to $\rho$ and $\rho^{-1}$ are depicted in Figure 2 below (see Theorem 9.8 for the proof).
This implies that, in crossing the critical value $d = \frac{7}{2}(2g-2)$ (i.e. as $\frac{d}{2g-2}$ passes from $\frac{7}{2}+\epsilon$ to $\frac{7}{2}-\epsilon$ for a small $\epsilon$), non-special elliptic tails become (Hilbert or Chow) unstable and they get replaced by tacnodes with a line. Moreover, Hilbert semistability for $d = \frac{7}{2}(2g-2)$ behaves like Hilbert (or Chow) semistability for $2(2g-2) < d < \frac{7}{2}(2g-2)$; hence Hilbert semistability is strictly stronger than Chow semistability for $d = \frac{7}{2}(2g-2)$.

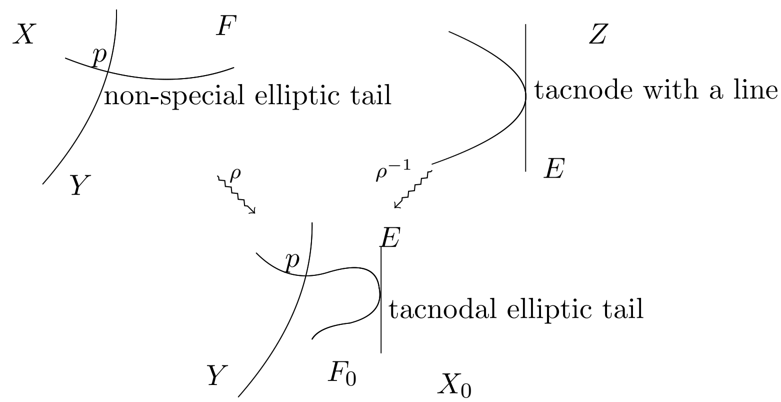


FIGURE 2. The basin of attraction of a curve $X_0$ with a tacnodal elliptic tail $F_0$.

To conclude, observe that the basins of attraction of Figure 1 are already visible in the 4-canonical locus inside $\mathrm{Hilb}_{4(2g-2)}$ or $\mathrm{Chow}_{4(2g-2)}$ (because all the elliptic tails are special with respect to the canonical line bundle!) and indeed they were already considered by Hyeon-Morrison in [HM10]; on the other hand, the basins of attraction of Figure 2 are clearly not visible inside the pluricanonical locus (because they occur for a fractional value of $\frac{d}{2g-2}$!).

Finally, one last comment on the orbits identifications that occur in the GIT quotient. It is well-known the GIT quotient of the (Hilbert or Chow) semistable locus parametrizes polystable orbits (i.e. semistable orbits that are closed inside the semistable locus) and each semistable orbit contains a unique polystable orbit in its closure. If $d > 2(2g-2)$ but $d \neq \frac{7}{2}(2g-2)$ or $4(2g-2)$, then Theorems A, C, E imply that the polystable orbits correspond to the orbits of Hilbert semistable points $[X \subset \mathbb{P}^{d-g}]$ such that moreover $\mathcal{O}_X(1)$ is strictly balanced (and similarly for Chow semistable points). Indeed, we prove in Section 7 that if a Hilbert semistable point $[X \subset \mathbb{P}^{d-g}]$ is such that $\mathcal{O}_X(1)$ achieves one of the extremes of the basic inequality at a subcurve $Z \subset X$ such that $Z \cap Z^c \subsetneq X_{\text{exc}}$, then there is an isotrivial specialization of $[X \subset \mathbb{P}^{d-g}]$ to a Hilbert semistable point $[X' \subset \mathbb{P}^{d-g}]$ such that $X'$ is obtained from $X$ by blowing up the nodes of $(Z \cap Z^c) \setminus X_{\text{exc}}$ (see Theorem 7.5); hence the orbit of $[X \subset \mathbb{P}^{d-g}]$ contains the orbit of $[X' \subset \mathbb{P}^{d-g}]$ in its closure. The same thing happens for Chow semistable points. Therefore, Theorems A, C and E say that these are the only orbits identifications that occur in the Hilbert or Chow GIT quotients outside of the critical values $d = \frac{7}{2}(2g-2)$ or $4(2g-2)$. Moreover, an easy combinatorial argument (see [Cap94, Lemma 6.3]) shows that the extreme of the basic inequalities can be achieved if and only if $\gcd(d+1-g, 2g-2) \neq 1$; therefore if $\gcd(d+1-g, 2g-2) = 1$ and $d \neq \frac{7}{2}(2g-2)$ or $4(2g-2)$ then the Hilbert or Chow GIT quotients that we get are geometric, i.e. semistable points are also stable.

On the other hand, if $d$ is equal to one of the two critical values $\frac{7}{2}(2g-2)$ or $4(2g-2)$, then the orbits identifications in the Hilbert and Chow GIT quotient are different. Indeed, while in the Hilbert GIT quotient $\overline{Q}_{d,g}^h$ it is still true that the unique orbits identifications are given by the isotrivial specializations described above, in the Chow GIT quotient $\overline{Q}_{d,g}^c$ there are new isotrivial specializations that correspond to the basins of attraction depicted in Figure 1 for $d = 4(2g-2)$ and Figure 2 for $d = \frac{7}{2}(2g-2)$. Note that there is a natural morphism $\Xi : \overline{Q}_{d,g}^h \to \overline{Q}_{d,g}^c$ from the Hilbert GIT quotient to the Chow GIT quotient (because a Hilbert semistable point is also Chow semistable) and we prove in Section 14 that $\Xi$ is an isomorphism if $d = \frac{7}{2}(2g-2)$ (see Proposition 14.5) while it is not an isomorphism if $d = 4(2g-2)$ (see Proposition 14.6).

1.3. **Application: compactifications of the universal Jacobian.** As an application of Theorems A, C and E, one gets three compactifications of the universal Jacobian stack $\mathcal{J}_{d,g}$, i.e. the moduli stack of pairs $(C, L)$ where $C$ is a smooth projective curve of genus $g$ and $L$ is a line bundle of degree $d$ on $C$, and of its coarse moduli space $J_{d,g}$.

To this aim, denote by $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$) the category fibered in groupoids over the category of $k$-schemes, whose fiber over a $k$-scheme $S$ is the groupoid of pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f : \mathcal{X} \to S$ is a family of quasi-stable curves (resp. quasi-p-stable curves) of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree $d$ over $S$ whose restriction to the geometric fibers of $f$ is properly balanced. Moreover, denote by $\overline{\mathcal{J}}_{d,g}^{\text{wp}}$ the category fibered in groupoids over the category of $k$-schemes, whose fiber over a $k$-scheme $S$ is the groupoid of pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f$ is a family of quasi-wp-stable curves of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree $d$ that is properly balanced on the geometric fibers of $f$ and such that the geometric fibers of $f$ do not contain tacnodes with a line nor special elliptic tails with respect to $\mathcal{L}$.

In the following theorem, we summarize the properties of $\overline{\mathcal{J}}_{d,g}$, $\overline{\mathcal{J}}_{d,g}^{\text{wp}}$ and $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$ that will be proved in Section 16.

**Theorem F.** *Let $g \geq 3$ and $d \in \mathbb{Z}$.*

*(1) $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\text{wp}}$, $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$) is a smooth, irreducible and universally closed Artin stack of finite type over $k$ and of dimension $4g-4$, containing $\mathcal{J}_{d,g}$ as a dense open substack.*

*(2) $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\text{wp}}$, $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$) admits an adequate moduli space $\overline{J}_{d,g}$ (resp. $\overline{J}_{d,g}^{\text{wp}}$, resp. $\overline{J}_{d,g}^{\text{ps}}$), which is a normal integral projective variety of dimension $4g-3$ containing $J_{d,g}$ as a dense open subvariety.*
*Moreover, if $\mathrm{char}(k) = 0$, then $\overline{J}_{d,g}$ (resp. $\overline{J}_{d,g}^{\text{wp}}$, resp. $\overline{J}_{d,g}^{\text{ps}}$) has rational singularities, hence it is Cohen-Macauly.*

*(3) Denote by $\widetilde{H}_d$ the main component of the semistable locus of $\mathrm{Hilb}_d$, i.e. the open subset of $\mathrm{Hilb}_d$ consisting of all the points $[X \subset \mathbb{P}^{d-g}]$ that are semistable and such that $X$ is connected. Then it holds:*
  *(i) $\overline{\mathcal{J}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL}_{d-g+1}]$ and $\overline{J}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL}_{d-g+1}$ if $d > 4(2g-2)$,*

*(ii)* $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL}_{d-g+1}]$ *and* $\overline{J}^{\mathrm{wp}}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL}_{d-g+1}$ *if* $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$,

*(iii)* $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL}_{d-g+1}]$ *and* $\overline{J}^{\mathrm{ps}}_{d,g} \cong \widetilde{H}_d /\!\!/ \mathrm{GL}_{d-g+1}$ *if* $2(2g-2) < d \leq \frac{7}{2}(2g-2)$.

(4) *We have the following commutative diagrams*

$$\begin{array}{ccc} \overline{\mathcal{J}}_{d,g} & \longrightarrow & \overline{J}_{d,g} \\ {\scriptstyle \Psi^{\mathrm{s}}}\downarrow & & \downarrow{\scriptstyle \Phi^{\mathrm{s}}} \\ \overline{\mathcal{M}}_g & \longrightarrow & \overline{M}_g \end{array} \qquad \begin{array}{ccc} \overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} & \longrightarrow & \overline{J}^{\mathrm{wp}}_{d,g} \\ {\scriptstyle \Psi^{\mathrm{wp}}}\downarrow & & \downarrow{\scriptstyle \Phi^{\mathrm{wp}}} \\ \overline{\mathcal{M}}^{\mathrm{wp}}_g & \longrightarrow & \overline{M}^{\mathrm{p}}_g \end{array} \qquad \begin{array}{ccc} \overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} & \longrightarrow & \overline{J}^{\mathrm{ps}}_{d,g} \\ {\scriptstyle \Psi^{\mathrm{ps}}}\downarrow & & \downarrow{\scriptstyle \Phi^{\mathrm{ps}}} \\ \overline{\mathcal{M}}^{\mathrm{p}}_g & \longrightarrow & \overline{M}^{\mathrm{p}}_g \end{array}$$

*where* $\Psi^{\mathrm{s}}$ *(resp.* $\Psi^{\mathrm{wp}}$, $\Psi^{\mathrm{ps}}$*) is universally closed and surjective and* $\Phi^{\mathrm{s}}$ *(resp.* $\Phi^{\mathrm{wp}}$*, resp.* $\Phi^{\mathrm{ps}}$*) is projective and surjective. Moreover:*

*(i) The morphisms* $\Phi^{\mathrm{s}} : \overline{J}_{d,g} \to \overline{M}_g$ *and* $\Phi^{\mathrm{ps}} : \overline{J}^{\mathrm{ps}}_{d,g} \to \overline{M}^{\mathrm{p}}_g$ *have equidimensional fibers of dimension g; moreover, if* $\mathrm{char}(k) = 0$, $\Phi^{\mathrm{s}}$ *and* $\Phi^{\mathrm{ps}}$ *are flat over the smooth locus of* $\overline{M}_g$ *and* $\overline{M}^{\mathrm{p}}_g$*, respectively.*

*(ii) The fiber of the morphism* $\Phi^{\mathrm{wp}} : \overline{J}^{\mathrm{wp}}_{d,g} \to \overline{M}^{\mathrm{p}}_g$ *over a p-stable curve* $X \in \overline{M}^{\mathrm{p}}_g$ *has dimension equal to the sum of g with the number of cusps of X.*

(5) *Let* $\overline{\mathcal{J}}^{\star}_{d,g}$ *be equal to either* $\overline{\mathcal{J}}_{d,g}$ *or* $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$ *or* $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$*. Denote by* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *the rigidification of* $\overline{\mathcal{J}}^{\star}_{d,g}$ *by* $\mathbb{G}_m$ *and by* $\widehat{\Psi}^{\star} : \overline{\mathcal{J}}^{\star}_{d,g} \to \overline{\mathcal{M}}^{\star}_g$ *the associated morphism, where* $\overline{\mathcal{M}}^{\star}_g$ *is equal to either* $\overline{\mathcal{M}}_g$ *or* $\overline{\mathcal{M}}^{\mathrm{wp}}_g$ *or* $\overline{\mathcal{M}}^{\mathrm{p}}_g$*. Then the following conditions are equivalent:*

*(i)* $\gcd(d+1-g, 2g-2) = 1$;

*(ii) The stack* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *is a DM-stack;*

*(iii) The stack* $\overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m$ *is proper;*

*(iv) The morphism* $\widehat{\Psi}^{\star} : \overline{\mathcal{J}}^{\star}_{d,g} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}^{\star}_g$ *is representable.*

(6) *If* $\mathrm{char}(k) = 0$*, then it holds*

*(i)* $(\Phi^{\mathrm{st}})^{-1}(X) \cong \overline{\mathrm{Jac}_{\mathrm{d}}}(X)/\mathrm{Aut}(X)$ *for any* $X \in \overline{M}_g$,

*(ii)* $(\Phi^{\mathrm{ps}})^{-1}(X) \cong \overline{\mathrm{Jac}_{\mathrm{d}}}(X)/\mathrm{Aut}(X)$ *for any* $X \in \overline{M}^{\mathrm{p}}_g$,

*where* $\overline{\mathrm{Jac}_{\mathrm{d}}}(X)$ *is the moduli space of of rank-*1*, torsion-free sheaves on X of degree d that are slope-semistable with respect to* $\omega_X$ *(and it is called the canonical compactified Jacobian of X in degree d).*

The stack (resp. variety) $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{J}_{d,g}$) was introduced by Caporaso in [Cap94] and [Cap05] and is therefore called the *Caporaso's compactified universal Jacobian stack* (resp. *variety*). The properties of $\overline{\mathcal{J}}_{d,g}$ and $\overline{J}_{d,g}$ stated in the above theorem were indeed already known (also for $g = 2$), by the work of Caporaso [Cap94], [Cap05] and of the third author [Mel09].

In Section §16.4, we provide also an alternative description of the stack $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$, resp. $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$) via certain rank-1, torsion-free sheaves on stable (resp. wp-stable, resp. p-stable) that are semistable with respect to the canonical line bundle (see Theorem 16.19).

1.4. **Open problems.** This work leaves unsolved some natural problems for further investigation, that we briefly discuss here.

As we observed above, Theorem E does not hold for $d = 2(2g-2)$. The first problem is thus the following.

**Problem A.**

*(i) Describe the (semi-,poly-)stable points of* $\mathrm{Hilb}_d$ *and* $\mathrm{Chow}_d$ *in the case* $d = 2(2g-2)$.

*(ii) Describe the (semi-,poly-)stable points of* $\mathrm{Hilb}_d$ *and* $\mathrm{Chow}_d$ *in the case* $d = 2(2g-2) - \epsilon$ *(for small* $\epsilon$*).*

*(iii) What is the next critical value of* $\frac{d}{2g-2} < 2$ *at which the GIT quotients change?*

As an output of the GIT analysis proposed in Problem A, one experts to find new compactifications of the universal Jacobian over the Hassett-Hyeon [HH13] moduli spaces $\overline{M}_g^h$ and $\overline{M}_g^c$ of c-semistable and h-semistable curves, respectively.

In order to understand the relation between the three compactifications $\overline{\mathcal{J}}_{d,g}$, $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ and $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ of the universal Jacobian stack $\mathcal{J}_{d,g}$, the following problem seems natural.

**Problem B.** *Describe the birational maps fitting into the following commutative diagram*

$$\begin{array}{ccccc}
\overline{\mathcal{J}}_{d,g} & \dashrightarrow & \overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} & \dashleftarrow & \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \\
\Big\downarrow{\scriptstyle \Psi^{\mathrm{s}}} & & \Big\downarrow{\scriptstyle \Psi^{\mathrm{wp}}} & & \Big\downarrow{\scriptstyle \Psi^{\mathrm{ps}}} \\
\overline{\mathcal{M}}_g & \hookrightarrow & \overline{\mathcal{M}}_g^{\mathrm{wp}} & \hookleftarrow & \overline{\mathcal{M}}_g^{\mathrm{p}}
\end{array}$$

More generally, one would like to set up a Hassett-Keel program for the Caporaso's compactified universal Jacobian stack $\overline{\mathcal{J}}_{d,g}$ and give an interpretation of the alternative compactifications $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$ and $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ of $\mathcal{J}_{d,g}$ as the first two steps in this program. Moreover it would be interesting to study how the new settled Hassett-Keel program for $\overline{\mathcal{J}}_{d,g}$ relates with the classical Hassett-Keel program for $\overline{\mathcal{M}}_g$.

1.5. **Outline of the paper.** We now give a detailed outline of the paper.

In Section 2, we discuss the singular curves that will appear throughout the paper: stable, wp-stable and p-stable curves together with their associated stacks in §2.1; quasi-stable, quasi-wp-stable and quasi-p-stable curves in §2.2. Moreover, we introduce two operations on families of curves: the p-stable reduction that contracts elliptic tails of wp-stable curves to cusps (see Proposition 2.6) and the wp-stable reduction that contracts exceptional components of quasi-wp-stable curve to either nodes or cusps (see Proposition 2.11).

In Section 3, we first collect in §3.1 several combinatorial results on balanced multidegrees and on the degree class group of Gorenstein curves; then, we introduce and study in §3.2 stably and strictly balanced multidegrees on quasi-wp-stable curves.

In Section 4, we collect all the general results from GIT that we will need in this work. In §4.1 we set up our GIT problem for $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$. In §4.2 we recall the Hilbert-Mumford numerical criterion for $m$-th Hilbert and Chow (semi)stability. Then we recall several classical results that will be used in our GIT analysis: basins of attraction (§4.3); flat limits and Gröbner basis (§4.4); the parabolic subgroup associated to a one-parameter subgroup (§4.5). We end this section by recalling in §4.6 two classical results due to Mumford and Gieseker: the Chow (or Hilbert) stability of smooth curves of genus $g$ embedded by line bundles of degree $d \geq 2g+1$; and the Potential stability Theorem giving necessary conditions for a point of $\mathrm{Hilb}_d$ or of $\mathrm{Chow}_d$ to be semistable, provided that $d > 4(2g-2)$.

In Section 5, we prove the Potential pseudo-stability Theorem 5.1 which gives necessary conditions for a point of $\mathrm{Hilb}_d$ or of $\mathrm{Chow}_d$ to be semistable, provided that $d > 2(2g-2)$.

In Section 6, we compute the stabilizer subgroup of a point of $\mathrm{Hilb}_d$, under the assumption that $d > 2(2g-2)$.

In Section 7, we investigate the isotrivial specializations that arise when one of the extremes of the basic inequalities is achieved.

In Section 8, we give a criterion for the (semi-, poly-)stability of a point of $\mathrm{Hilb}_d$ or $\mathrm{Chow}_d$ whose underlying curve has a tail.

In Section 9, we deal with the Hilbert or Chow semistability of curves having an elliptic tail (special or not) or having a tacnode with a line. We prove that special elliptic tails become Chow unstable for $d < 4(2g-2)$ (see Theorem 9.1), ordinary elliptic tails become Chow unstable for $d < \frac{7}{2}(2g-2)$ (see Theorem 9.6), tacnodes with a line are Chow unstable for $d > \frac{7}{2}(2g-2)$ (see Theorem 9.3). Moreover, we examine the basins of attraction of the curves in Figure 1 and 2 (see Theorem 9.2 and 9.8).

In Section 10, we introduce a stratification of the Chow semistable locus by fixing the isomorphism class of a curve and the multidegree of the line bundle that embeds it. We study the closure of the strata in §10.1 and we prove a completeness result for these strata in §10.2.

In Section 11 we characterize (semi, poly)-stable points in $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$ if either $4(2g-2) < d$ or $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ and $g \geq 3$, thus proving Theorems A, D and E.

In Section 12, we study the stability of elliptic tails in the range $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$.

In Section 13, we characterize (semi, poly)-stable points in $\mathrm{Hilb}_d$ and $\mathrm{Chow}_d$ in the range $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$, thus proving Theorems B and C.

In Section 14, we study the geometric properties of the Hilbert and Chow GIT quotients and of their modular map towards the moduli space of p-stable curves.

In Section 15, we determine when the Hilbert or Chow semistable locus admits extra-components made entirely of non-connected curves.

In Section 16, we first recall in §16.1 the properties of the Caporaso's compactified universal Jacobian stack $\overline{\mathcal{J}}_{d,g}$ over the moduli stack of stable curves and of its moduli space $\overline{J}_{d,g}$. Then, in §16.2, we define and study the two new compactifications $\overline{\mathcal{J}}_{d,g}^{\rm wp}$ and $\overline{\mathcal{J}}_{d,g}^{\rm ps}$ of the universal Jacobian stack $\mathcal{J}_{d,g}$ over the moduli stack of wp-stable curves and p-stable curves, respectively. In §16.3, we prove that $\overline{\mathcal{J}}_{d,g}^{\rm wp}$ and $\overline{\mathcal{J}}_{d,g}^{\rm ps}$ admit projective moduli spaces $\overline{J}_{d,g}^{\rm wp}$ and $\overline{J}_{d,g}^{\rm ps}$, respectively, and we study their properties. Finally, in §16.4, we provide an alternative description of the stack $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\rm wp}$, resp. $\overline{\mathcal{J}}_{d,g}^{\rm ps}$) and of its moduli space via certain rank-1, torsion-free sheaves on stable (resp. wp-stable, resp. p-stable) curves that are semistable with respect to the canonical line bundle (see Theorem 16.19).

The Appendix 17 contains some positivity results for balanced line bundles on Gorenstein curves which are used throughout the paper and that we find interesting in their own.

Some of the results of this paper (more precisely, Theorems A and E and Theorem F for $\overline{\mathcal{J}}_{d,g}^{\rm ps}$) were originally obtained by the first, third and fourth author and then announced in [BMV12]. However, the GIT analysis in the range $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ was left as an open question (see [BMV12, Question A]). The second author solved this open problem by proving Theorems B, C, D and Theorem F for $\overline{\mathcal{J}}_{d,g}^{\rm wp}$ and then became a coauthor of this work. Moreover, the presence of extra-components in the Hilbert or Chow semistable locus made of non-connected curves was also left as an open question in loc. cit. (see [BMV12, Question C]); this was also solved by the second author and resulted in Section 15 of the present work.

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**Conventions.**

**1.1.** *$k$ will denote an algebraically closed field (of arbitrary characteristic). All **schemes** are $k$-schemes, and all morphisms are implicitly assumed to respect the $k$-structure.*

**1.2.** *A **curve** is a complete, reduced and separated scheme (over $k$) of pure dimension 1 (not necessarily connected). The **genus** $g(X)$ of a curve $X$ is $g(X) := h^1(X, \mathcal{O}_X)$. The set of **singular points** of a curve $X$ is denoted by $X_{\mathrm{sing}}$.*

**1.3.** *A **subcurve** $Z$ of a curve $X$ is a closed $k$-scheme $Z \subseteq X$ that is reduced and of pure dimension 1. We say that a subcurve $Z \subseteq X$ is proper if $Z \neq \emptyset, X$.*

*Given two subcurves $Z$ and $W$ of $X$ without common irreducible components, we denote by $Z \cap W$ the 0-dimensional subscheme of $X$ that is obtained as the scheme-theoretic intersection of $Z$ and $W$ and we denote by $|Z \cap W|$ its length.*

*Given a subcurve $Z \subseteq X$, we denote by $Z^c := \overline{X \setminus Z}$ the **complementary subcurve** of $Z$ and we set $k_Z = k_{Z^c} := |Z \cap Z^c|$.*

**1.4.** *Let $X$ be a curve. A point $p$ of $X$ is said to be*

- *a **node** if $\widehat{\mathcal{O}_{X,p}} \cong k[[x,y]]/(y^2 - x^2)$, where $\widehat{\mathcal{O}_{X,p}}$ is the completion of the local ring $\mathcal{O}_{X,p}$ of $X$ at $p$;*
- *a **cusp** if $\widehat{\mathcal{O}_{X,p}} \cong k[[x,y]]/(y^2 - x^3)$;*
- *a **tacnode** if $\widehat{\mathcal{O}_{X,p}} \cong k[[x,y]]/(y^2 - x^4)$.*

*A **tacnode with a line** of a curve $X$ is a tacnode $p$ of $X$ at which two irreducible components $D_1$ and $D_2$ of $X$ meet with a simple tangency so that $D_1 \cong \mathbb{P}^1$ and $k_{D_1} = 2$ (or equivalently $p$ is the set-theoretical intersection of $D_1$ and $D_1^c$).*

**1.5.** *An **elliptic tail** of a curve $X$ is a connected subcurve $F$ of genus 1 meeting the rest of the curve in one point; i.e. a connected subcurve $F \subseteq X$ such that $g(F) = 1$ and $k_F = |F \cap F^c| = 1$. Moreover we say that $F$ is*

- ***nodal** if $F$ is an irreducible rational curve with one node;*
- ***cuspidal** if $F$ is an irreducible rational curve with one cusp;*
- ***reducible nodal** if $F$ consists of two smooth rational subcurves meeting in two nodes;*
- ***tacnodal** if $F$ consists of two smooth rational subcurves meeting in a tacnode.*

*Moreover we define the **elliptic locus**, which we denote by $X_{\mathrm{ell}}$, as the union of all the elliptic tails of $X$.*

**1.6.** *A curve $X$ is called **Gorenstein** if its dualizing sheaf $\omega_X$ is a line bundle.*

**1.7.** *A **family of curves** is a proper, flat morphism $X \to T$ whose geometric fibers are curves. Given a class $\mathcal{C}$ of curves, a family of curves of $\mathcal{C}$ is a family of curves $X \to T$ whose geometric fibers belong to the class $\mathcal{C}$. For example: if $\mathcal{C}$ is the class of nodal curves of genus $g$, then a family of nodal curves of genus $g$ is a family of curves whose geometric fibers are nodal curves of genus $g$.*

## 2. Singular curves

The aim of this section is to collect the definitions and basic properties of the curves that we will deal with throughout the paper.

### 2.1. **Stable, p-stable and wp-stable curves.**

We begin by recalling the definition of stable curves ([DM69]), pseudo-stable curves ([Sch91]) and weakly-pseudo-stable curves ([HM10, Pag. 8]) of genus $g \geq 2$.

**Definition 2.1.** A connected curve $X$ of arithmetic genus $g \geq 2$ is

- (i) *stable* if
  - (a) $X$ has only nodes as singularities;
  - (b) the canonical sheaf $\omega_X$ is ample.
- (ii) *p-stable* (or pseudo-stable) if
  - (a) $X$ has only nodes and cusps as singularities;
  - (b) $X$ does not have elliptic tails, i.e. $X_{\mathrm{ell}} = \emptyset$;
  - (c) the canonical sheaf $\omega_X$ is ample.
- (iii) *wp-stable* (or weakly-pseudo-stable) if
  - (a) $X$ has only nodes and cusps as singularities;
  - (b) the canonical sheaf $\omega_X$ is ample.

Note that, in each of the three cases, $\omega_X$ is ample if and only if each connected subcurve $Z$ of $X$ of genus zero is such that $k_Z = |Z \cap Z^c| \geq 3$.

**Remark 2.2.** Note that stable curves and p-stable curves are wp-stable. More precisely:

- (i) stable curves are exactly those wp-stable curves without cusps.
- (ii) p-stable curves are exactly those wp-stable curves without elliptic tails.

We will work throughout the paper with the following stacks.

**Definition 2.3.** Let $g \geq 2$. We denote by $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}_g^{\mathrm{p}}$, resp. $\overline{\mathcal{M}}_g^{\mathrm{wp}}$) the stack parametrizing families of stable (resp. p-stable, resp. wp-stable) curves of genus $g$.

The properties of the above stacks can be summarized in the following

**Theorem 2.4.** *Let $g \geq 2$.*

- *(i) $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ is a smooth, irreducible algebraic stack of dimension $3g-3$, containing $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\mathrm{p}}$ as open substacks.*
- *(ii) $\overline{\mathcal{M}}_g$ is a proper stack; $\overline{\mathcal{M}}_g^{\mathrm{p}}$ is a proper stack if $g \geq 3$ and a weakly proper stack if $g = 2$; $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ is a weakly proper stack.*
- *(iii) $\overline{\mathcal{M}}_g$ admits a coarse moduli space $\overline{M}_g$; $\overline{\mathcal{M}}_g^{\mathrm{p}}$ admits a coarse moduli space $\overline{M}_g^{\mathrm{p}}$ for $g \geq 3$ and an adequate moduli space $\overline{M}_g^{\mathrm{p}}$ for $g = 2$. $\overline{M}_g^{\mathrm{p}}$ is also an adequate moduli space for $\overline{\mathcal{M}}_g^{\mathrm{wp}}$.*

  *Moreover, $\overline{M}_g$ and $\overline{M}_g^{\mathrm{p}}$ are irreducible projective varieties of dimension $3g-3$.*

*Proof.* Part (i): $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ is an algebraic stack since it is an open substack of the stack of all genus $g$ curves, which is well known to be algebraic (see e.g. [Hal]). By [Ser06, Prop. 2.4.8], an obstruction space for the deformation functor $\mathrm{Def}_X$ of a wp-stable curve $X$ is the vector space $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ which is zero according to [DM69, Lemma 1.3] since $X$ is a reduced curve with locally complete intersection singularities. This implies that $\mathrm{Def}_X$ is formally smooth, hence that $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ is smooth at $X$. Moreover, from [Ser06, Thm. 2.4.1] and [Ser06, Cor. 3.1.13], it follows that a reduced curve with locally complete intersection singularities can always be smoothened; therefore the open substack $\mathcal{M}_g \subset \overline{\mathcal{M}}_g^{\mathrm{wp}}$ of smooth curves is dense. Since $\mathcal{M}_g$ is irreducible of dimension $3g-3$ (see [DM69]), we deduce that $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ is irreducible of dimension $3g-3$ as well. Clearly, $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\mathrm{p}}$ are open substacks of $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ because the condition of having no cusps or no elliptic tails is an open condition.

Part (ii): for any $m \geq 2$, denote by $\mathrm{Chow}^{ss}_{m,\mathrm{can}}$ the locally closed sub-locus of the Chow scheme of 1-cycles of degree $m(2g-2)$ in $\mathbb{P}^N$ (where $N := m(2g-2)-g$) consisting of curves which are embedded by the $m$-pluricanonical map and semistable (see Section 4.1 for more details). It is known that: $\mathrm{Chow}^{ss}_{m,\mathrm{can}}$ consists of stable curves if $m \geq 5$ (see [Mum77]); $\mathrm{Chow}^{ss}_{4,\mathrm{can}}$ consists of wp-stable curves (see [HM10]); $\mathrm{Chow}^{ss}_{3,\mathrm{can}}$ consists of p-stable curves (see [Sch91] for $g \geq 3$ and [HL07] for $g = 2$). Now, a standard argument (see [Edi00, Thm. 3.2] and [ACG11, Chap. XII, Thm. 5.6]) yields the following isomorphisms of stacks:

$$\begin{aligned}
&\overline{\mathcal{M}}_g \cong [\mathrm{Chow}^{ss}_{m,\mathrm{can}}/\mathrm{PGL}_{N+1}] \text{ for any } m \geq 5,\\
&\overline{\mathcal{M}}_g^{\mathrm{wp}} \cong [\mathrm{Chow}^{ss}_{4,\mathrm{can}}/\mathrm{PGL}_{N+1}],\\
&\overline{\mathcal{M}}_g^{\mathrm{p}} \cong [\mathrm{Chow}^{ss}_{3,\mathrm{can}}/\mathrm{PGL}_{N+1}].
\end{aligned} \tag{2.1}$$

In particular, it follows that all the above stacks are weakly proper (see [ASvdW, Section 2]). Moreover, it is well known that there are no strictly semistable points in $\mathrm{Chow}^{ss}_{m,\mathrm{can}}$ for $m \geq 5$ (see [Mum77]) and in $\mathrm{Chow}^{ss}_{3,\mathrm{can}}$ for $g \geq 3$ (see [Gie82]). This yields that $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\mathrm{p}}$ for $g \geq 3$ are proper stacks (see [ASvdW, Section 2]).

Part (iii): define the GIT quotients

$$\begin{aligned}
&\overline{M}_g := \mathrm{Chow}^{ss}_{5,\mathrm{can}}/\!\!/\mathrm{PGL}_{N+1},\\
&\overline{M}_g^{\mathrm{p}} := \mathrm{Chow}^{ss}_{3,\mathrm{can}}/\!\!/\mathrm{PGL}_{N+1}.
\end{aligned} \tag{2.2}$$

By combining (2.1), (2.2) and what said above on the strictly semistable points, it follows that $\overline{M}_g$ is a coarse moduli for $\overline{\mathcal{M}}_g$ and $\overline{M}_g^{\mathrm{p}}$ is a coarse (resp. adequate) moduli space of $\overline{\mathcal{M}}_g^{\mathrm{p}}$ for $g \geq 3$ (resp. $g = 2$), see [Alp2]. It was proved in [HM10] that

$$\overline{M}_g^{\mathrm{p}} \cong \mathrm{Chow}^{ss}_{4,\mathrm{can}}/\!\!/\mathrm{PGL}_{N+1},$$

which – combined with (2.1) – implies that $\overline{M}_g^{\mathrm{p}}$ is an adequate moduli space for $\overline{\mathcal{M}}_g^{\mathrm{wp}}$.

The fact that $\overline{M}_g$ and $\overline{M}_g^{\mathrm{p}}$ are irreducible projective varieties of dimension $3g-3$ is well-known (see [DM69] and [HH09]).

□

Note that our stacks $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_g^{\mathrm{p}}$ and $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ correspond to the stacks $\overline{M}_g(A_2^-)$, $\overline{M}_g(A_2^+)$ and $\overline{M}_g(A_2)$ in [ASvdW], respectively.

**Remark 2.5.**

(i) The stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ of wp-stable curves is not proper since in $\mathrm{Chow}_{4,\mathrm{can}}^{ss}$ there are strictly semistable points. Indeed, Hyeon-Morrison proved in [HM10] that the unique orbit specializations occurring in $\mathrm{Chow}_{4,\mathrm{can}}^{ss}$ (for $g \geq 3$) are the ones depicted in figure 3 below:

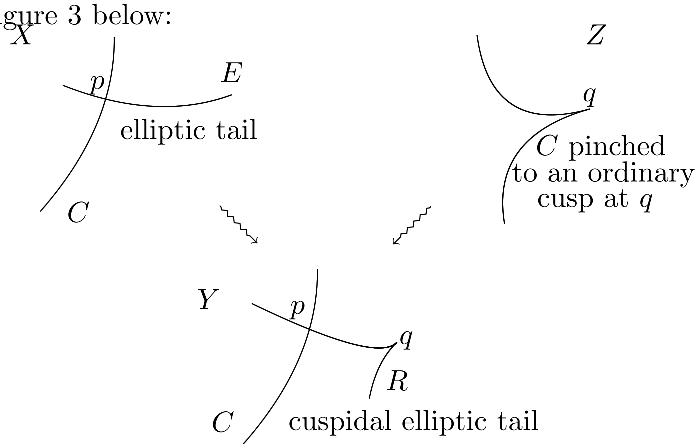



FIGURE 3. Orbit specializations in $\mathrm{Chow}_{4,\mathrm{can}}^{ss}$, i.e. isotrivial specializations in $\overline{\mathcal{M}}_g^{\mathrm{wp}}$.

The above orbit specializations correspond to isotrivial specializations in the stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ (see [ASvdW]). Therefore, the closed points of $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ are the wp-stable curves $X$ such that every elliptic tail of $X$ is cuspidal and every cusp of $X$ is contained in an elliptic tail.

(ii) If $\mathrm{char}(k) = 0$ then the adequate moduli spaces appearing in the above Theorem 2.4 are indeed good moduli spaces (see [Alp2, Prop. 5.1.4]).

Given a wp-stable curve $Y$, it is possible to obtain a p-stable curve, called its p-stable reduction and denoted by ps(Y), by contracting the elliptic tails of $Y$ to cusps. The p-stable reduction works even for families.

**Proposition 2.6.** *Let $v : \mathcal{Y} \to S$ be a family of wp-stable curves of genus $g \geq 2$. There exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\psi} & \mathrm{ps}(\mathcal{Y}) \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle \mathrm{ps}(v)} & \\ & S & \end{array}$$

*where* $\mathrm{ps}(v) : \mathrm{ps}(\mathcal{Y}) \to \mathrm{S}$ *is a family of p-stable curves of genus $g$, called the* p-stable reduction *of $v : \mathcal{Y} \to S$. For every geometric point $s \in S$, the morphism $\psi_s : \mathcal{Y}_s \to$* $\mathrm{ps}(\mathcal{Y})_{\mathrm{s}}$ *contracts the elliptic tails of $\mathcal{Y}_s$ to cusps of* $\mathrm{ps}(\mathcal{Y})_{\mathrm{s}}$. *Moreover, the formation of the p-stable reduction commutes with base change.*

*This defines a morphism of stacks* $\mathrm{ps}: \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{wp}} \to \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{p}}$.

*Proof.* If $v: \mathcal{Y} \to S$ is a family of stable curves, the statement was proved by Hassett-Hyeon in [HH09, Sec. 3] under the assumption that $g \geq 3$ and then extended to $g = 2$ with a similar argument by Heyon-Lee in [HL07, Sec. 4]. In what follows, we will show how to adapt the argument of loc. cit. in order to work out in our case.

First of all, if $S = k$, then the statement follows from Proposition 3.1 in [HH09], which asserts that given a stable curve $C$, there is a replacement morphism $\xi_C : C \to \mathcal{T}(C)$, where $\mathcal{T}(C)$ is a pseudo-stable curve of genus $g$, which is an isomorphism away from the loci of elliptic tails and that replaces elliptic tails with cusps. The argumentation is local on the nodes connecting each genus-one subcurve meeting the rest of the curve in a single node. Since in a wp-stable curve all elliptic tails are connected to the rest of the curve via a single node, the same argumentation works also in our case with no further modifications.

Now, we have to prove the statement over an arbitrary base $S$. The approach of Hassett-Hyeon is to consider a faithfully flat atlas $V \to \overline{\mathcal{M}}_g$ and define the p-stable reduction for the family of stable curves over $V$ induced by the above morphism. The case of a family over an arbitrary base will follow by base-change from $V \to \overline{\mathcal{M}}_g$ to $S$.

In our case, we consider a faithful atlas $\rho_\pi : U \to \overline{\mathcal{M}}_g^{\mathrm{wp}}$ of the stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ of wp-stable curves and we let $\pi : Z \to U$ be the associated (universal) family of wp-stable curves. The idea is now to consider an invertible sheaf $L$ on $Z$, which will be a twisted version of the relative dualizing sheaf of $\pi$, such that $L$ is very ample away from the locus of elliptic tails, and instead has relative degree 0 over all elliptic tails. Then use $L$ to define an $S$-morphism from $Z$ to a family of p-stable curves which coincides with the previous one over all geometric fibers of $\pi$.

To be precise, denote by $\delta_1 \subset \overline{\mathcal{M}}_{g,1}^{\mathrm{wp}}$ the boundary divisor of elliptic tails on the universal stack $\overline{\mathcal{M}}_{g,1}^{\mathrm{wp}}$ over $\overline{\mathcal{M}}_g^{\mathrm{wp}}$. An argument similar to the proof of Theorem 2.4(i) shows that $\overline{\mathcal{M}}_{g,1}^{\mathrm{wp}}$ is smooth; hence $\delta_1$ is a Cartier divisor. Let $\mu_\pi : Z \to \overline{\mathcal{M}}_{g,1}^{\mathrm{wp}}$ be the classifying morphism corresponding to the family $\pi : Z \to U$ and set $L := \omega_\pi(\mu_\pi^* \delta_1)$. The whole point is now to prove that $\pi_*(L^n)$ is locally free and that $L^n$ is relatively globally generated for $n > 0$ and that the associated morphism factors through

$$Z \xrightarrow{\xi_Z} \mathcal{T}(Z) \hookrightarrow \mathbb{P}(\pi_* L^n)$$

where $\mathcal{T}(Z)$ is a family of p-stable curves and $\xi_Z$ coincides with the replacement morphism $\xi_C$ for all geometric fibers $C$ of $\pi$. By browsing carefully through Hassett-Hyeon's argumentation, we easily conclude that everything holds also in our case.

□

**Remark 2.7.** From the above Proposition, we get the existence of a morphism of stacks

$$\mathrm{ps}: \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{wp}} \to \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{p}}, \tag{2.3}$$

which, passing to the adequate moduli spaces, induces the morphism $T : \overline{M}_g \to \overline{M}_g^{\mathrm{p}}$ studied by Hassett-Hyeon in [HH09] for $g \geq 3$ and by Hyeon-Lee in [HL07] for $g = 2$. Indeed, it is proved in loc. cit. that $T$ is the contraction of the divisor $\Delta_1 \subset \overline{M}_g$ of curves having an elliptic tail.

2.2. **Quasi-wp-stable curves and wp-stable reduction.** The most general class of singular curves that we will meet throughout this work is the one given in the following:

**Definition 2.8.**

(i) A connected curve $X$ is said to be *pre-wp-stable* if the only singularities of $X$ are nodes, cusps or tacnodes with a line.
(ii) A connected curve $X$ is said to be *pre-p-stable* if it is pre-wp-stable and it does not have elliptic tails.
(iii) A connected curve $X$ is said to be *pre-stable* if the only singularities of $X$ are nodes.

Note that wp-stable (resp. p-stable, resp. stable) curves are pre-wp-stable (resp. pre-p-stable, resp. pre-stable) curves. Moreover, if $p \in X$ is a tacnode with a line lying in $D_1 \cong \mathbb{P}^1$ and $D_2$ as in 1.4, then $(\omega_X)_{|D_1} = \mathcal{O}_{D_1}$, hence $\omega_X$ is not ample. From this, we get easily that

**Remark 2.9.** $X$ is wp-stable (resp. p-stable, resp. stable) if and only if $X$ is pre-wp-stable (resp. pre-p-stable, resp. pre-stable) and $\omega_X$ is ample.

The pre-wp-stable curves that we will meet in this paper, even when non wp-stable, will satisfy a very strong condition on connected subcurves where the restriction of the canonical line bundle is not ample, i.e., on connected subcurves of genus zero that meet the complementary subcurve in less than three points. This justifies the following

**Definition 2.10.** A pre-wp-stable curve $X$ is said to be

(i) *quasi-wp-stable* if every connected subcurve $E \subset X$ such that $g_E = 0$ and $k_E \leq 2$ satisfies $E \cong \mathbb{P}^1$ and $k_E = 2$ (and therefore it meets the complementary subcurve $E^c$ either in two distinct nodal points of $X$ or in one tacnode of $X$).
(ii) *quasi-p-stable* if it is quasi-wp-stable and pre-p-stable.
(iii) *quasi-stable* if it is quasi-wp-stable and pre-stable.

The irreducible components $E$ such $E \cong \mathbb{P}^1$ and $k_E = 2$ are called *exceptional* and the subcurve of $X$ given by the union of the exceptional components is denoted by $X_{\mathrm{exc}}$. The complementary subcurve $X_{\mathrm{exc}}^c = \overline{X \setminus X_{\mathrm{exc}}}$ is called the *non-exceptional* subcurve and is denoted by $\widetilde{X}$.

Equivalently, a quasi-wp-stable curve is a pre-wp-stable $X$ such that $\omega_X$ is nef (i.e. it has non-negative degree on every subcurve of $X$) and such that all the connected subcurves $E \subseteq X$ such that $\deg_E \omega_X = 0$ (which are called exceptional subcurves) are irreducible. Note that the term quasi-stable curve was introduced in [Cap94, Sec. 3.3].

We summarize the different types of curves that we have introduced so far in Table 1.

| SINGULARITIES | $\omega_X$ NEF + IRREDUCIBLE EXCEPTIONAL SUB-CURVES | $\omega_X$ AMPLE |
|---|---|---|
| **pre-wp-stable** = nodes, cusps, tacnodes with a line | **quasi-wp-stable** | **wp-stable** |
| **pre-p-stable** = pre-wp-stable without elliptic tails | **quasi-p-stable** | **p-stable** |
| **pre-stable** = nodes | **quasi-stable** | **stable** |

TABLE 1. Singular curves

Given a quasi-wp-stable curve $Y$, it is possible to contract all the exceptional components in order to obtain a wp-stable curve, which is called the wp-stable reduction of $Y$ and is denoted by wps(Y). This construction indeed works for families.

**Proposition 2.11.** *Let $S$ be a scheme and $u : \mathcal{X} \to S$ a family of quasi-wp-stable curves. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & \mathrm{wps}(\mathcal{X}) \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle \mathrm{wps(u)}} & \\ & S & \end{array}$$

*where* $\mathrm{wps(u)} : \mathrm{wps}(\mathcal{X}) \to \mathrm{S}$ *is a family of wp-stable curves, called the* wp-stable reduction *of $u$.*

*For every geometric point $s \in S$, the morphism $\phi_s : \mathcal{X}_s \to \mathrm{wps}(\mathcal{X})_\mathrm{s}$ contracts the exceptional components $E$ of $\mathcal{X}_s$ so that*

*(1) If $E \cap E^c$ consists of two distinct nodal points of $X$, then $E$ is contracted to a node;*

*(2) If $E \cap E^c$ consists of one tacnode of $X$, then $E$ is contracted to a cusp.*

*The formation of the wp-stable reduction commutes with base change. Furthermore, if $u$ is a family of quasi-p-stable (resp. quasi-stable) curves then* $\mathrm{wps(u)}$ *is a family of p-stable (resp. stable) curves.*

*Proof.* We will follow the same ideas as in the proof of [Knu83, Prop. 2.1] and of [Mel11, Prop. 6.6]. Consider the relative dualizing sheaf $\omega_u := \omega_{\mathcal{X}/S}$ of the family $u : \mathcal{X} \to S$. It is a line bundle because the geometric fibers of $u$ are Gorenstein curves by our assumption. From Corollary 17.7 in the Appendix we get that for all $i \geq 2$, the restriction of $\omega_u^i$ to a geometric fiber $\mathcal{X}_s$ is non-special, globally generated and, if

$i \geq 3$, normally generated. Then, we can apply [Knu83, Cor. 1.5] to get the following properties for $\omega_u$:

(a) $R^1u_*(\omega_u^i) = (0)$ for all $i \geq 2$;
(b) $u_*(\omega_u^i)$ is $S$-flat for all $i \geq 2$;
(c) for any morphism $T \to S$ there are canonical isomorphisms

$$u_*(\omega_u^i) \otimes_{\mathcal{O}_S} \mathcal{O}_T \to (u \times 1)_*(\omega_u^i \otimes_{\mathcal{O}_S} \mathcal{O}_T)$$

for any $i \geq 2$;
(d) the canonical map $u^*u_*(\omega_u^i) \to \omega_u^i$ is surjective for all $i \geq 3$;
(e) the natural maps $(u_*\omega_u^3)^i \otimes u_*\omega_u^3 \to (u_*\omega_u^3)^{i+1}$ are surjective for $i \geq 1$.

Define now

$$\mathcal{S}_i := u_*(\omega_u^i), \text{ for all } i \geq 0.$$

By (a) and (b) above, we know that $\mathcal{S}_i$ is locally free and flat on $S$ for $i \geq 2$. Consider

$$\mathbb{P}(\mathcal{S}_3) \to S.$$

Since, by (d) above, the natural map

$$u^*u_*(\omega_u^3) \to \omega_u^3$$

is surjective, we get that there is a natural $S$-morphism

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \mathbb{P}(\mathcal{S}_3) \\ & {\scriptstyle u}\searrow \quad \swarrow & \\ & S & \end{array}$$

Denote by $\mathcal{Y}$ the image of $\mathcal{X}$ via $q$ and by $\phi$ the (surjective) $S$-morphism from $\mathcal{X}$ to $\mathcal{Y}$. By (e) above, we get that

$$\mathcal{Y} = \mathcal{P}roj(\oplus_{i\geq 0}\mathcal{S}_i).$$

So, $\phi : \mathcal{Y} \to S$ is flat since the $\mathcal{S}_i$'s are flat for $i \geq 2$. To conclude that $\mathcal{Y} \to S$ is a family of wp-stable curves note that the restriction of $\omega_u^3$ to the geometric fibers of $u$ has positive degree in all irreducible components except the exceptional ones, where it has degree 0. Indeed, it is easy to see (see for example [Cat82, Rmk. 1.20]) that, on each geometric fiber $\mathcal{X}_s$, $\phi$ contracts an exceptional component $E \subseteq \mathcal{X}_s$ to a node if $E$ meets the complementary curve in two distinct nodal points and to a cusp if $E$ meets the complementary subcurve in one tacnode. Moreover, $\Phi$ is an isomorphism outside the non exceptional locus. We conclude that $\mathcal{Y} \to S$ is a family of wp-stable curves, so we set $\mathrm{wps}(\mathcal{X}) := \mathcal{Y}$ and $\mathrm{wps(u)} := \mathcal{Y} \to \mathrm{S}$.

Property (c) above implies that forming wps is compatible with base-change.

The last assertion is clear from the above geometric description of the contraction $\phi_s : \mathcal{X}_s \to \mathrm{wps}(\mathcal{X})_\mathrm{s}$ on each geometric point of $u$.

□

**Remark 2.12.** If $u : \mathcal{X} \to S$ is a family of quasi-stable curves then the wp-stable reduction $\mathrm{wps(u)} : \mathrm{wps}(\mathcal{X}) \to \mathrm{S}$ coincides with the usual stable reduction $\mathrm{s(u)}$ of $u$ (see e.g. [Knu83]).

The wp-stable reduction allows us to give a more explicit description of the quasi-wp-stable curves.

**Corollary 2.13.** *A curve $X$ is quasi-wp-stable (resp. quasi-p-stable, resp. quasi-stable) if and only if it can be obtained from a wp-stable (resp. p-stable, resp. stable) curve $Y$ via an iteration of the following construction:*

*(i) Normalize $Y$ at a node $p$ and insert a $\mathbb{P}^1$ meeting the rest of the curve in the two branches of the node.*

*(ii) Normalize $Y$ at a cusp and insert a $\mathbb{P}^1$ tangent to the rest of the curve at the branch of the cusp.*

*In this case,* $Y = \mathrm{wps(X)}$*. In particular, given a wp-stable (resp. p-stable, resp. stable) curve $Y$ there exists only a finite number of quasi-wp-stable (resp. quasi-p-stable, resp. quasi-stable) curves $X$ such that* $\mathrm{wps(X)} = \mathrm{Y}$*, which we call* quasi-wp-stable *(resp.* quasi-p-stable*, resp.* quasi-stable*)* models *of $Y$.*

Note that the above operation (ii) cannot occur for quasi-stable curves. With a slight abuse of terminology, we call the above operation (i) (resp. (ii)) the **blow-up** of a node (resp. of a cusp).

*Proof.* We will prove the Corollary only for quasi-wp-stable curves. The remaining cases are dealt with in the same way.

Let $X$ be a quasi-wp-stable curve and set $Y := \mathrm{wps(X)}$. By Proposition 2.11, the wp-stablization $\phi : X \to Y = \mathrm{wps(X)}$ contracts each exceptional component $E$ of $X$ to a node or a cusp according to whether $E \cap E^c$ consists of two distinct points or one point with multiplicity two. Therefore, $X$ is obtained from $Y$ by a sequence of the two operations (i) and (ii).

Conversely, if $X$ is obtained from a wp-stable curve $Y$ by a sequence of operations (i) and (ii), then clearly $X$ is quasi-wp-stable and $Y = \mathrm{wps(X)}$.

The last assertion is now clear. $\square$

We end this section with an extension of the p-stable reduction of Proposition 2.6 to families of quasi-wp-stable curves.

**Definition 2.14.** Let $S$ be a scheme and $u : \mathcal{X} \to S$ be a family of quasi-wp-stable curves of genus $g \geq 3$. Then there exists a commutative digram

$$\begin{array}{ccccc} \phi : \mathcal{X} & \xrightarrow{\phi} & \mathrm{wps}(\mathcal{X}) & \xrightarrow{\psi} & \mathrm{ps}(\mathrm{wps}(\mathcal{X})) := \mathrm{ps}(\mathcal{X}) \\ & {\scriptstyle u} \searrow & \downarrow {\scriptstyle \mathrm{wps}(u)} & \swarrow {\scriptstyle \mathrm{ps}(u) := \mathrm{ps}(\mathrm{wps}(u))} & \\ & & S & & \end{array}$$

where the family wps(u) is the wp-stable reduction of the family $u$ (see Proposition 2.11) and the family ps(wps(u)) is the p-stable reduction of the family wps(u) (see Proposition 2.6).

We set ps(u) := ps(wps(u)) and we call it the *p-stable reduction* of $u$.

## 3. Combinatorial results

The aim of this section is to collect all the combinatorial results that will be used in the sequel.

3.1. **Balanced multidegree and the degree class group.** Let us first recall some combinatorial definitions and results from [Cap94, Sec. 4]. Although the results in loc. cit. are stated for nodal curves, a close inspection of the proofs reveals that the same results are true – more in general – for Gorenstein curves.

Fix a connected Gorenstein curve $X$ of genus $g \geq 2$ and we denote by $C_1, \ldots, C_\gamma$ the irreducible components of $X$. A *multidegree* on $X$ is an ordered $\gamma$-tuple of integers

$$\underline{d} = (\underline{d}_{C_1}, \ldots, \underline{d}_{C_\gamma}) \in \mathbb{Z}^\gamma.$$

We denote by $|\underline{d}| = \sum_{i=1}^{\gamma} \underline{d}_{C_i}$ the total degree of $\underline{d}$. Given a subcurve $Z \subseteq X$, we set $\underline{d}_Z := \sum_{C_i \subseteq Z} \underline{d}_{C_i}$. The term multidegree comes from the fact that every line bundle $L$ on $X$ has a multidegree $\underline{\deg} L$ given by $\underline{\deg} L := (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L)$ whose total degree $|\underline{\deg} L|$ is the degree $\deg L$ of $L$.

We now introduce an inequality condition on the multidegree of a line bundle which will play a key role in the sequel.

**Definition 3.1.** Let $\underline{d}$ be a multidegree of total degree $|\underline{d}| = d$. We say that $\underline{d}$ is *balanced* if it satisfies the inequality (called *basic inequality*)

$$\left| \underline{d}_Z - \frac{d}{2g-2} \deg_Z \omega_X \right| \leq \frac{k_Z}{2}, \tag{3.1}$$

for every subcurve $Z \subseteq X$.

We denote by $\widetilde{B}_X^d$ the set of all balanced multidegrees on $X$ of total degree $d$.

Following [Cap94, Sec. 4.1], we now define an equivalence relation on the set of multidegrees on $X$. For every irreducible component $C_i$ of $X$, consider the multidegree $\underline{C_i} = ((\underline{C_i})_1, \ldots, (\underline{C_i})_\gamma)$ of total degree 0 defined by

$$(\underline{C_i})_j = \begin{cases} |C_i \cap C_j| & \text{if } i \neq j, \\ -\sum_{k \neq i} |C_i \cap C_k| & \text{if } i = j. \end{cases}$$

More generally, for any subcurve $Z \subseteq X$, we set $\underline{Z} := \sum_{C_i \subseteq Z} \underline{C_i}$.

Denote by $\Lambda_X \subseteq \mathbb{Z}^\gamma$ the subgroup of $\mathbb{Z}^\gamma$ generated by the multidegrees $\underline{C_i}$ for $i = 1, \ldots, \gamma$. It is easy to see that $\sum_i \underline{C_i} = 0$ and this is the only relation among the multidegrees $\underline{C_i}$. Therefore, $\Lambda_X$ is a free abelian group of rank $\gamma - 1$.

**Definition 3.2.** Two multidegrees $\underline{d}$ and $\underline{d}'$ are said to be *equivalent*, and we write $\underline{d} \equiv \underline{d}'$, if $\underline{d} - \underline{d}' \in \Lambda_X$. In particular, if $\underline{d} \equiv \underline{d}'$ then $|\underline{d}| = |\underline{d}'|$.

For every $d \in \mathbb{Z}$, we denote by $\Delta_X^d$ the set of equivalence classes of multidegrees of total degree $d = |\underline{d}|$. Clearly, $\Delta_X^0$ is a finite group under component-wise addition of multidegrees (called the *degree class group* of $X$) and each $\Delta_X^d$ is a torsor under $\Delta_X^0$.

The following two facts will be used in the sequel. The first result says that every element in $\Delta_X^d$ has a balanced representative. The second result investigates the relation between balanced multidegrees that have the same class in $\Delta_X^d$.

**Fact 3.3** (Caporaso). *For every multidegree $\underline{d}$ on $X$ of total degree $d = |\underline{d}|$, there exists $\underline{d}' \in \widetilde{B}_X^d$ such that $\underline{d} \equiv \underline{d}'$.*

For a proof see [Cap94, Prop. 4.1]. Note that in loc. cit. the above result is only stated for a nodal curve $X$ and $d = 0$. Nonetheless, a closer inspection at the proof shows that it extends verbatim to our case. See also [MV12, Prop. 2.8] for a refinement of the above result.

**Fact 3.4** (Caporaso). *Let $\underline{d}, \underline{d}' \in \widetilde{B}_X^d$. Then $\underline{d} \equiv \underline{d}'$ if and only if there exist subcurves $Z_1 \subseteq \ldots \subseteq Z_m$ of $X$ such that*

$$\begin{cases} \underline{d}_{Z_i} = \dfrac{d}{2g-2}\deg_{Z_i}\omega_X + \dfrac{k_{Z_i}}{2} \text{ for } 1 \leq i \leq m, \\ \underline{d}' = \underline{d} + \displaystyle\sum_{i=1}^{m} \underline{Z_i}. \end{cases}$$

*Moreover, the subcurves $Z_i$ can be chosen so that $Z_i^c \cap Z_j = \emptyset$ for $i > j$.*

For a proof see [Cap94, p. 620 and p. 625]. In loc. cit., the result is stated for DM-semistable curves but the proof extends verbatim to our case.

3.2. **Stably and strictly balanced multidegrees on quasi-wp-stable curves.** We now specialize to the case where $X$ is a quasi-wp-stable curve of genus $g \geq 2$ (see Definition 2.10) [3].

Given a balanced multidegree $\underline{d}$ on $X$, the basic inequality (3.1) gives that $\underline{d}_E = -1, 0, 1$ for every exceptional component $E \subset X$. The multidegrees such that $\underline{d}_E = 1$ on each exceptional component $E \subset X$ will play a special role in the sequel; hence they deserve a special name.

**Definition 3.5.** We say that a multidegree $\underline{d}$ on $X$ is *properly balanced* if $\underline{d}$ is balanced and $\underline{d}_E = 1$ for every exceptional component $E$ of $X$.

We denote by $B_X^d$ the set of all properly balanced multidegrees on $X$ of total degree $d$.

[3] Actually, the reader can easily check that all the results of this subsection are valid more in general if $X$ is a G-quasistable curve of genus $g \geq 2$ in the sense of Definition 17.1.

The aim of this subsection is to investigate the behavior of properly balanced multidegrees on a quasi-wp-stable curve $X$, which attain the equality in the basic inequality (3.1) relative to some subcurve $Z \subseteq X$. Let us denote the two extremes of the basic inequality relative to $Z$ by

$$\begin{cases} m_Z := \dfrac{d}{2g-2}\deg_Z\omega_X - \dfrac{k_Z}{2}, \\ M_Z := \dfrac{d}{2g-2}\deg_Z\omega_X + \dfrac{k_Z}{2}, \end{cases} \tag{3.2}$$

Note that $m_Z = M_{Z^c}$ and $M_Z = m_{Z^c}$. The definitions below will be important in what follows.

**Definition 3.6.** A properly balanced multidegree $\underline{d}$ on $X$ is said to be

(i) *strictly balanced* if any proper subcurve $Z \subset X$ such that $\underline{d}_Z = M_Z$ satisfies $Z \cap Z^c \subset X_{\text{exc}}$.

(ii) *stably balanced* if any proper subcurve $Z \subset X$ such that $\underline{d}_Z = M_Z$ satisfies $Z \subseteq X_{\text{exc}}$.

When $X$ is a quasi-stable curve, the above Definition 3.6(i) coincides with the definition of extremal in [Cap94, Sec. 5.2], while Definition 3.6(ii) coincides with the definition of G-stable in [Cap94, Sec. 6.2]. Here we adopt the terminology of [BFV12, Def. 2.4].

**Definition 3.7.** We will say that a line bundle $L$ on $X$ is balanced if and only if its multidegree $\underline{\deg} L$ is balanced, and similarly for properly balanced, strictly balanced, stably balanced.

**Remark 3.8.** In order to check that a multidegree $\underline{d}$ on $X$ is balanced (resp. strictly balanced, resp. stably balanced), it is enough to check the conditions of Definitions 3.1 and 3.6 only for the subcurves $Z \subset X$ such that $Z$ and $Z^c$ are *connected*. This follows easily from the following facts. If $Z$ is a subcurve of $X$ and we denote by $\{Z_1, \ldots, Z_c\}$ the connected components of $Z$, then the following hold:

(i) The upper (resp. lower) inequality in (3.1) is achieved for $Z$ if and only if the upper (resp. lower) inequality in (3.1) is achieved for every $Z_i$. This follows from the (easily checked) additivity relations

$$\begin{cases} \deg_Z L = \sum_i \deg_{Z_i} L, \\ \deg_Z \omega_X = \sum_i \deg_{Z_i}\omega_X, \\ k_Z = \sum_i k_{Z_i}. \end{cases}$$

(ii) $Z \cap Z^c \subseteq X_{\text{exc}}$ if and only if $Z_i \cap Z_i^c \subseteq X_{\text{exc}}$ for every $i$. Similarly, $Z \subseteq X_{\text{exc}}$ if and only if $Z_i \subseteq X_{\text{exc}}$ for every $Z_i$.

(iii) If $Z^c$ is connected, then $Z_i^c = \cup_{j\neq i} Z_j \cup Z^c$ is connected for every $Z_i$.

The next result explains the relation between stably balanced and strictly balanced line bundles.

**Lemma 3.9.** *A multidegree $\underline{d}$ on a quasi-wp-stable curve $X$ of genus $g \geq 2$ is stably balanced if and only if $\underline{d}$ is strictly balanced and $\widetilde{X} = \overline{X \setminus X_{\text{exc}}}$ is connected.*

*Proof.* The proof of [BFV12, Lemma 2.7] extends verbatim from quasi-stable curves to quasi-wp-stable curves. □

The importance of strictly balanced multidegrees is that they are unique in their equivalence class in $\Delta_X^d$, at least among the properly balanced multidegrees.

**Lemma 3.10.** *Let $\underline{d}, \underline{d}' \in B_X^d$ be two properly balanced multidegrees of total degree $d$ on a quasi-wp-stable curve $X$ of genus $g \geq 2$. If $\underline{d} \equiv \underline{d}'$ and $\underline{d}$ is strictly balanced, then $\underline{d} = \underline{d}'$.*

*Proof.* According to Fact 3.4, there exist subcurves $Z_1 \subseteq \ldots \subseteq Z_m$ of $X$ such that

$$\underline{d}' = \underline{d} + \sum_{i=1}^{m} \underline{Z_i}, \tag{3.3}$$

$$\underline{d}_{Z_i} = \frac{d}{2g-2}\deg_{Z_i}\omega_X + \frac{k_{Z_i}}{2} \text{ for } 1 \leq i \leq m, \tag{3.4}$$

$$Z_i^c \cap Z_j = \emptyset \text{ for } i > j. \tag{3.5}$$

Assume, by contradiction, that $\underline{d} \not\equiv \underline{d}'$; hence, using (3.3), we can assume that $Z := Z_1$ is a proper subcurve of $X$. From (3.4) and the fact that $\underline{d}$ is strictly balanced, we deduce that $Z \cap Z^c \subset X_{\text{exc}}$. Therefore, there exists an exceptional component $E \subseteq X_{\text{exc}}$ such that one of the following four possibilities occurs:

$$\begin{aligned}
&\text{Case (I): } E \subseteq Z \text{ and } |E \cap Z^c| = 1,\\
&\text{Case (II): } E \subseteq Z \text{ and } |E \cap Z^c| = 2,\\
&\text{Case (III): } E \subseteq Z^c \text{ and } |E \cap Z| = 1,\\
&\text{Case (IV): } E \subseteq Z^c \text{ and } |E \cap Z| = 2.
\end{aligned}$$

Note that in Cases (II) or (IV), the intersection of $E$ with $Z$ or $Z^c$ consists either of two distinct points or of one point of multiplicity two.

<u>Claim</u>: Cases (III) and (IV) cannot occur.

By contradiction, assume first that case (III) occurs. Consider the subcurve $Z \cup E$ of $X$. We have clearly that

$$\begin{cases}
\underline{d}_{Z\cup E} = \underline{d}_Z + 1,\\
\deg_{Z\cup E}\omega_X = \deg_Z\omega_X,\\
k_{Z\cup E} = k_Z.
\end{cases}$$

Therefore, using (3.4), we have that

$$\underline{d}_{Z\cup E} = \underline{d}_Z + 1 = \frac{d}{2g-2}\deg_Z\omega_X + \frac{k_Z}{2} + 1 = \frac{d}{2g-2}\deg_{Z\cup E}\omega_X + \frac{k_{Z\cup E}}{2} + 1,$$

which contradicts the basic inequality (3.1) for $\underline{d}$ with respect to the subcurve $Z\cup E \subsetneq X$.

Assume now that case (IV) occurs. For the subcurve $Z\cup E \subsetneq X$, we have that

$$\begin{cases} \underline{d}_{Z\cup E} = \underline{d}_Z + 1, \\ \deg_{Z\cup E}\omega_X = \deg_Z\omega_X, \\ k_{Z\cup E} = k_Z - 2. \end{cases}$$

Therefore, using (3.4), it follows that

$$\underline{d}_{Z\cup E} = \underline{d}_Z + 1 = \frac{d}{2g-2}\deg_Z\omega_X + \frac{k_Z}{2} + 1 = \frac{d}{2g-2}\deg_{Z\cup E}\omega_X + \frac{k_{Z\cup E}}{2} + 2,$$

which contradicts the basic inequality (3.1) for $\underline{d}$ with respect to the subcurve $Z\cup E \subsetneq X$. The claim is now proved.

Therefore, only cases (I) or (II) can occur. Note that

$$\underline{Z}_E = -|E\cap Z^c| = \begin{cases} -1 & \text{if case (I) occurs,} \\ -2 & \text{if case (II) occurs.} \end{cases} \tag{3.6}$$

Note also that, in any case, we must have that $E \subseteq Z = Z_1$. Using (3.5), we get that $E\cap Z_i^c = \emptyset$ for any $i>1$, which implies that

$$\underline{Z_i}_E = 0 \ \text{ for any } i>1. \tag{3.7}$$

We now evaluate (3.3) at the subcurve $E$: using that $\underline{d}_E = 1$ because $\underline{d}$ is strictly balanced and equations (3.6) and (3.7), we conclude that

$$\underline{d}'_E = \begin{cases} 0 & \text{if case (I) occurs,} \\ -1 & \text{if case (II) occurs.} \end{cases}$$

In both cases, this contradicts the assumption that $\underline{d}'$ is properly balanced. □

We conclude this subsection with the following Lemma, that will be used several times in what follows.

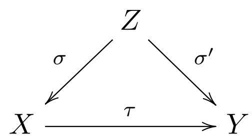
**Lemma 3.11.** *Let $X$, $Y$ and $Z$ be quasi-wp-stable curves of genus $g\geq 2$. Let $\sigma : Z\to X$ and $\sigma' : Z \to Y$ be two surjective maps given by blowing down some of the exceptional components of $Z$. Let $\underline{d}$ (resp. $\underline{d}'$) be a properly balanced multidegree on $X$ (resp. on $Y$). Denote by $\widetilde{\underline{d}}$ the pull-back of $\underline{d}$ on $Z$ via $\sigma$, i.e., the multidegree on $Z$ given on a subcurve $W\subseteq Z$ by*

$$\widetilde{\underline{d}}_W = \begin{cases} \underline{d}_{\sigma(W)} & \text{if } \sigma(W) \text{ is a subcurve of } X, \\ 0 & \text{if } W \text{ is contracted by } \sigma \text{ to a point.} \end{cases}$$

*In a similar way, we define the pull-back $\widetilde{\underline{d}'}$ of $\underline{d}'$ on $Z$ via $\sigma'$. The following is true:*

*(i)* $\widetilde{\underline{d}}$ *and* $\widetilde{\underline{d}'}$ *are balanced multidegrees.*

*(ii) If* $\underline{d}$ *is strictly balanced and* $\widetilde{\underline{d}} \equiv \widetilde{\underline{d}'}$ *then there exists a map* $\tau : X \to Y$ *such that the following diagram commutes*




*Proof.* Part (i): let us prove that $\widetilde{\underline{d}}$ is balanced; the proof for $\widetilde{\underline{d}'}$ being analogous. Consider a connected subcurve $W \subseteq Z$ and let us show that $\widetilde{\underline{d}}$ satisfies the basic inequality (3.1) with respect to the subcurve $W \subseteq Z$. If $W$ is contracted by $\sigma$ to a point, then $W$ must be an exceptional component of $Z$. In this case, we have that $\widetilde{\underline{d}}_Z = 0$, $k_W = 2$ and $\deg_W(\omega_Z) = 0$ so that (3.1) is satisfied. If $\sigma(W)$ is a subcurve of $X$, then $\widetilde{\underline{d}}_W = \underline{d}_{\sigma(W)}$ and, since $\sigma$ contracts only exceptional components of $Z$, it is easy to see that $\deg_W(\omega_Z) = \deg_{\sigma(W)}(\omega_X)$ and that $|W \cap W^c| = |\sigma(W) \cap \sigma(W)^c|$. Therefore, in this case, the basic inequality for $\widetilde{\underline{d}}$ with respect to $W$ follows from the basic inequality for $\underline{d}$ with respect to $\sigma(W)$.

Part (ii): start by noticing that if every exceptional component $E \subset Z$ which is contracted by $\sigma$ is also contracted by $\sigma'$ then $\sigma'$ factors through $\sigma$, so the map $\tau$ exists. Let us now prove that in order for the map $\tau$ to exist, it is also necessary that every exceptional component $E \subset Z$ which is contracted by $\sigma$ is also contracted by $\sigma'$. By contradiction, assume that $\tau$ exists and that there exists an exceptional component $E \subset Z$ which is contracted by $\sigma$ but not by $\sigma'$. Then we have that

$$\begin{cases} \widetilde{\underline{d}}_E = 0, \\ \widetilde{\underline{d}'}_E = \underline{d}'_{\sigma(E)} = 1, \end{cases} \tag{3.8}$$

where in the last equation we have used that $\sigma(E')$ is an exceptional component of $Y$ and that $\underline{d}'$ is properly balanced.

Since $\widetilde{\underline{d}}$ is equivalent to $\widetilde{\underline{d}'}$ by assumption, Fact 3.4 implies that we can find subcurves $W_1 \subset \ldots \subset W_m \subseteq Z$ such that

$$\widetilde{\underline{d}} = \widetilde{\underline{d}'} + \sum_{i=1}^{m} \underline{W_i}, \tag{3.9}$$

$$\widetilde{\underline{d}'}_{W_i} = \frac{d}{2g-2}\deg_{W_i}\omega_Z + \frac{k_{W_i}}{2} \text{ for } 1 \leq i \leq m, \tag{3.10}$$

$$W_i^c \cap W_j = \emptyset \text{ for } i > j. \tag{3.11}$$

From (3.8) and (3.9), we get that

$$\sum_{i=1}^{m} \underline{W_i}_E = -1. \tag{3.12}$$

Denote by $C_1$ and $C_2$ the irreducible components of $Y$ that intersect $E$, with the convention that $C_1 = C_2$ if there is only one such irreducible component of $Y$ that meets $E$ in two distinct points or in one point with multiplicity 2. It follows directly from the definition of $\underline{W}$ (see Section 3) that for any subcurve $W \subseteq Z$ with complementary subcurve $W^c$ we have that

$$\underline{W}_E = \begin{cases} 2 \text{ if } E \subseteq W^c \text{ and } C_1 \cup C_2 \subseteq W, \\ 1 \text{ if } E \subseteq W^c \text{ and exactly one among } C_1 \text{ and } C_2 \text{ is a subcurve of } W, \\ 0 \text{ if } E \cup C_1 \cup C_2 \subseteq W^c \text{ or } E \cup C_1 \cup C_2 \subseteq W, \\ -1 \text{ if } E \subseteq W \text{ and exactly one among } C_1 \text{ and } C_2 \text{ is a subcurve of } W, \\ -2 \text{ if } E \subseteq W \text{ and } C_1 \cup C_2 \subseteq W^c. \end{cases}$$

Using this formula, together with (3.12) and (3.11), it is easy to see that $C_1$ must be different from $C_2$ and that, up to exchanging $C_1$ with $C_2$, there exists an integer $1 \leq q \leq m$ such that

$$\begin{cases} E \cup C_1 \cup C_2 \subseteq W_i^c & \text{if } i < q, \\ E \cup C_1 \subset W_q \text{ and } C_2 \subseteq W_q^c, \\ E \cup C_1 \cup C_2 \subseteq W_i & \text{if } i > q. \end{cases} \tag{3.13}$$

Let us now compute $\widetilde{\underline{d}}_{W_q}$. From (3.11), we get that

$$\underline{W_i}_{W_q} = \begin{cases} -k_{W_q} & \text{if } i = q, \\ 0 & \text{if } i \neq q. \end{cases}$$

Combining this with (3.9) and (3.10), we get that

$$\widetilde{\underline{d}}_{W_q} = \widetilde{\underline{d}'}_{W_q} - k_{W_q} = \frac{d}{2g-2}\deg_{W_q}\omega_Z - \frac{k_{W_q}}{2}. \tag{3.14}$$

Consider now the subcurve $\sigma(W_q)$ of $X$. By (3.14), we have that

$$\underline{d}_{\sigma(W_q)} = \widetilde{\underline{d}}_{W_q} = \frac{d}{2g-2}\deg_{W_q}\omega_Z - \frac{k_{W_q}}{2} = \frac{d}{2g-2}\deg_{\sigma(W_q)}\omega_X - \frac{k_{\sigma(W_q)}}{2},$$

and by (3.13) we have that

$$\sigma(W_q) \cap \sigma(W_q)^c \not\subseteq X_{\text{exc}}.$$

This contradicts the fact that $\underline{d}$ is strictly balanced.

□

## 4. Preliminaries on GIT

4.1. **Hilbert and Chow schemes of curves.** Fix, throughout this paper, two integers $d$ and $g \geq 2$ and write $d := v(2g-2) = 2v(g-1)$ for some (uniquely determined) rational number $v$. Set $r + 1 := d - g + 1 = (2v-1)(g-1)$.

Let $\text{Hilb}_{d,g}$ (or $\text{Hilb}_d$ when $g$ is clear from the context) be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^r = \mathbb{P}(V)$ having Hilbert polynomial $P(m) := md+1-g$, i.e., subschemes of $\mathbb{P}^r$ of dimension 1, degree $d$ and arithmetic genus $g$. An element $[X \subset \mathbb{P}^r]$ of $\mathrm{Hilb}_d$ is thus a 1-dimensional scheme $X$ of arithmetic genus $g$ together with an embedding $X \xhookrightarrow{i} \mathbb{P}^r$ of degree $d$. We let $\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^r}(1) \in \mathrm{Pic^d(X)}$. The group $\mathrm{GL}(V) \cong \mathrm{GL}_{r+1}$ (hence its subgroup $\mathrm{SL}(V) \cong \mathrm{SL}_{r+1}$) acts on $\mathrm{Hilb}_d$ via its natural action on $\mathbb{P}^r = \mathbb{P}(V)$. Given an element $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$, we will denote by $\mathrm{Orb}([X \subset \mathbb{P}^r])$ its *orbit* with respect to the above action of $\mathrm{GL}(V)$ (or equivalently of $\mathrm{SL}(V)$).

It is well-known (see [MS11, Lemma 2.1]) that for any $m \geq M := \binom{d}{2} + 1 - g$ and any $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ it holds that:

- $\mathcal{O}_X(m)$ has no higher cohomology;
- The natural map
$$\mathrm{Sym}^m V^\vee \to \Gamma(\mathcal{O}_X(m)) = H^0(X, \mathcal{O}_X(m))$$
is surjective.

Under these hypotheses, the $m$-th Hilbert point of $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is defined to be

$$[X \subset \mathbb{P}^r]_m := \left[\mathrm{Sym}^m V^\vee \twoheadrightarrow \Gamma(\mathcal{O}_X(m))\right] \in \mathrm{Gr}(P(m), \mathrm{Sym}^m V^\vee) \hookrightarrow \mathbb{P}\left(\bigwedge^{P(m)} \mathrm{Sym}^m V^\vee\right),$$

where $\mathrm{Gr}(P(m), \mathrm{Sym}^m V^\vee)$ is the Grassmannian variety parametrizing $P(m)$-dimensional quotients of $\mathrm{Sym}^m V^\vee$, which lies naturally in $\mathbb{P}\left(\bigwedge^{P(m)} \mathrm{Sym}^m V^\vee\right)$ via the Plücker embedding.
For any $m \geq M$, we get a closed $\mathrm{SL}(V)$-equivariant embedding (see [Mum66, Lect. 15]):

$$\begin{array}{rcl} j_m : \mathrm{Hilb}_d & \hookrightarrow & \mathrm{Gr}(P(m), \mathrm{Sym}^m V^\vee) \hookrightarrow \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V^\vee) := \mathbb{P} \\ {[X \subset \mathbb{P}^r]} & \mapsto & [X \subset \mathbb{P}^r]_m. \end{array}$$

Therefore, for any $m \geq M$, we get an ample $\mathrm{SL}(V)$-linearized line bundle $\Lambda_m := j_m^* \mathcal{O}_{\mathbb{P}}(1)$ and we denote by

$$\mathrm{Hilb}_d^{s,m} \subseteq \mathrm{Hilb}_d^{ss,m} \subseteq \mathrm{Hilb}_d$$

the locus of points that are stable and semistable with respect to $\Lambda_m$, respectively. If $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{s,m}$ (resp. $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss,m}$), we say that $[X \subset \mathbb{P}^r]$ is *$m$-Hilbert stable* (resp. *semistable*).

The ample cone of $\mathrm{Hilb}_d$ admits a finite decomposition into locally-closed cells, such that the stable and the semistable locus are constant for linearizations taken from a given cell [DH98, Theorem 0.2.3(i)]. In particular, $\mathrm{Hilb}_d^{s,m}$ and $\mathrm{Hilb}_d^{ss,m}$ are constant for $m \gg 0$. We set

$$\begin{cases} \mathrm{Hilb}_d^{s} := \mathrm{Hilb}_d^{s,m} \text{ for } m \gg 0, \\ \mathrm{Hilb}_d^{ss} := \mathrm{Hilb}_d^{ss,m} \text{ for } m \gg 0. \end{cases}$$

If $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^s$ (resp. $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$, $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss} \setminus \mathrm{Hilb}_d^s$), we say that $[X \subset \mathbb{P}^r]$ is *Hilbert stable* (resp. *semistable*, *strictly semistable*). If $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ is such that the SL($V$)-orbit Orb($[X \subset \mathbb{P}^r]$) of $[X \subset \mathbb{P}^r]$ is closed inside $\mathrm{Hilb}_d^{ss}$ then we say that $[X \subset \mathbb{P}^r]$ is *Hilbert polystable.*

Let $\mathrm{Chow}_d \stackrel{j}{\hookrightarrow} \mathbb{P}(\otimes^2 \mathrm{Sym}^d V^\vee) := \mathbb{P}'$ the Chow scheme parametrizing 1-cycles of $\mathbb{P}^r$ of degree $d$ together with its natural SL($V$)-equivariant embedding $j$ into the projective space $\mathbb{P}(\otimes^2 \mathrm{Sym}^d V^\vee)$ (see [Mum66, Lect. 16]). Therefore, we have an ample SL($V$)-linearized line bundle $\Lambda := j^* \mathcal{O}_{\mathbb{P}'}(1)$ and we denote by

$$\mathrm{Chow}_d^s \subseteq \mathrm{Chow}_d^{ss} \subseteq \mathrm{Chow}_d$$

the locus of points of $\mathrm{Chow}_d$ that are, respectively, stable and semistable with respect to $\Lambda$.

There is an SL($V$)-equivariant cycle map (see [MFK94, §5.4]):

$$\begin{array}{rcl} \mathrm{Ch}: \mathrm{Hilb}_d & \to & \mathrm{Chow}_d \\ {[X \subset \mathbb{P}^r]} & \mapsto & \mathrm{Ch}([X \subset \mathbb{P}^r]). \end{array}$$

We say that $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is *Chow stable* (resp. *semistable*, *strictly semistable*) if $\mathrm{Ch}([X \subset \mathbb{P}^r]) \in \mathrm{Chow}_d^s$ (resp. $\mathrm{Chow}_d^{ss}$, $\mathrm{Chow}_d^{ss} \setminus \mathrm{Chow}_d^s$). We say that $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is *Chow polystable* if $\mathrm{Ch}([X \subset \mathbb{P}^r]) \in \mathrm{Chow}_d^{ss}$ and its SL($V$)-orbit is closed inside $\mathrm{Chow}_d^{ss}$. Clearly, this is equivalent to asking that $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ and that the SL($V$)-orbit Orb($[X \subset \mathbb{P}^r]$) of $[X \subset \mathbb{P}^r]$ is closed inside $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$.

The relation between asympotically Hilbert (semi)stability and Chow (semi)stability is given by the following (see [HH13, Prop. 3.13])

**Fact 4.1.** *There are inclusions*

$$\mathrm{Ch}^{-1}(\mathrm{Chow}_d^s) \subseteq \mathrm{Hilb}_d^s \subseteq \mathrm{Hilb}_d^{ss} \subseteq \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}).$$

*In particular, there is a natural morphism of GIT-quotients*

$$\mathrm{Hilb}_d^{ss}/SL(V) \to \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})/SL(V).$$

Note also that in general there is no obvious relation between Hilbert and Chow polystability.

4.2. **Hilbert-Mumford numerical criterion for $m$-Hilbert and Chow (semi)stability.** Let us now recall the Hilbert-Mumford numerical criterion for the $m$-Hilbert (semi)stability and Chow (semi)stability of a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$, following [Gie82, Sec. 0.B] and [Mum77, Sec. 2] (see also [HM98, Chap. 4.B]). Although the criterion in its original form involves one-parameter subgroups (in short 1ps) of SL($V$), it is technically convenient to work with 1ps of GL($V$) (see [Gie82, pp. 9-10] for an explanation on how to pass from 1ps of SL($V$) to 1ps of GL($V$), and conversely).

Let $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ be a 1ps and let $x_0, \ldots, x_r$ be coordinates of $V$ that diagonalize the action of $\rho$, so that for $i = 0, \ldots, r$ we have

$$\rho(t) \cdot x_i = t^{w_i} x_i \text{ with } w_i \in \mathbb{Z}.$$

The total weight of $\rho$ is by definition

$$w(\rho) := \sum_{i=0}^{r} w_i.$$

Given a monomial $B = x_0^{\beta_0} \dots x_r^{\beta_r}$, we define the weight of $B$ with respect to $\rho$ to be

$$w_\rho(B) = \sum_{i=0}^{r} \beta_i w_i.$$

For any $m \geq M$ as in Section 4.1 and any 1ps $\rho$ of $\mathrm{GL}(V)$, we introduce the following function

$$W_{X,\rho}(m) := \min \left\{ \sum_{i=1}^{P(m)} w_\rho(B_i) \right\}, \tag{4.1}$$

where the minimum runs over all the collections of $P(m)$ monomials $\{B_1, \dots, B_{P(m)}\} \subset \mathrm{Sym}^m V^\vee$ which restrict to a basis of $H^0(X, \mathcal{O}_X(m))$. It is easy to check that $W_{X,\rho}(m)$ coincide with the filtered Hilbert function of [HH13, Def. 3.15]. In the sequel, we will often write $W_\rho(m)$ instead of $W_{X,\rho}(m)$ when there is no danger of confusion.

The Hilbert-Mumford numerical criterion for $m$-th Hilbert (semi)stability translates into the following (see [Gie82, p. 10] and also [HM98, Prop. 4.23]).

**Fact 4.2** (**Numerical criterion for $m$-Hilbert (semi)stability).** *Let $m \geq M$ as before. A point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is $m$-Hilbert stable (resp. semistable) if and only if for every 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ of total weight $w(\rho)$ we have that*

$$\mu([X \subset \mathbb{P}^r]_m, \rho) := \frac{w(\rho)}{r+1} m P(m) - W_{X,\rho}(m) > 0$$

*(resp. $\geq$).*

Indeed, the function $\mu([X \subset \mathbb{P}^r]_m, \rho)$ introduced above coincides with the Hilbert-Mumford index of $[X \subset \mathbb{P}^r]_m \in \mathbb{P}\left(\bigwedge^{P(m)} \mathrm{Sym}^m V^\vee\right)$ relative to the 1ps $\rho$ (see [MFK94, 2.1]).

The function $W_{X,\rho}(m)$ also allows one to state the numerical criterion for Chow (semi)stability. According to [Mum77, Prop. 2.11] (see also [HH13, Prop. 3.16]), the function $W_{X,\rho}(m)$ is an integer valued polynomial of degree 2 for $m \gg 0$. We define $e_{X,\rho}$ (or $e_\rho$ when there is no danger of confusion) to be the normalized leading coefficient of $W_{X,\rho}(m)$, i.e.,

$$\left| W_{X,\rho}(m) - e_{X,\rho} \frac{m^2}{2} \right| < Cm, \tag{4.2}$$

for $m \gg 0$ and for some constant $C$.

The Hilbert-Mumford numerical criterion for Chow (semi)stability translates into the following (see [Mum77, Thm. 2.9]).

**Fact 4.3** (**Numerical criterion for Chow (semi)stability).** *A point* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *is Chow stable (resp. semistable) if and only if for every 1ps* $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ *of total weight* $w(\rho)$ *we have that*

$$e_{X,\rho} < 2d \cdot \frac{w(\rho)}{r+1}$$

*(resp.* $\leq$*).*

**Remark 4.4.** Observe that $2d \cdot \frac{w(\rho)}{r+1}$ is the normalized leading coefficient of the polynomial $\frac{w(\rho)}{r+1} mP(m) = \frac{w(\rho)}{r+1} m(dm+1-g)$. Therefore, combining Fact 4.3 and Fact 4.2 for $m \gg 0$, one gets a proof of Fact 4.1.

The following definition is very natural.

**Definition 4.5.** Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and $\rho$ be a one-parameter subgroup of $\mathrm{GL}_{r+1}$. We say that

(i) $[X \subset \mathbb{P}^r]$ is *Hilbert semistable* (resp. *Chow semistable*) *with respect to* $\rho$ if

$$W_{X,\rho}(m) \leq \frac{w(\rho)}{r+1} mP(m) \quad \text{for } m \gg 0 \quad \left(\text{resp. } e_{X,\rho} \leq \frac{2d}{r+1} w(\rho)\right);$$

Moreover, we say that $[X \subset \mathbb{P}^r]$ is *Hilbert strictly semistable* (resp. *Chow strictly semistable*) *with respect to* $\rho$ if

$$W_{X,\rho}(m) = \frac{w(\rho)}{r+1} mP(m) \quad \text{for } m \gg 0 \quad \left(\text{resp. } e_{X,\rho} = \frac{2d}{r+1} w(\rho)\right);$$

(ii) $[X \subset \mathbb{P}^r]$ is *Hilbert stable* (resp. *Chow stable*) *with respect to* $\rho$ if

$$W_{X,\rho}(m) < \frac{w(\rho)}{r+1} mP(m) \quad \text{for } m \gg 0 \quad \left(\text{resp. } e_{X,\rho} < \frac{2d}{r+1} w(\rho)\right);$$

(iii) $[X \subset \mathbb{P}^r]$ is *Hilbert polystable* (resp. *Chow polystable*) *with respect to* $\rho$ if one of the following conditions is satisfied:

(a) $[X \subset \mathbb{P}^r]$ is Hilbert stable (resp. Chow stable) with respect to $\rho$;

(b) $[X \subset \mathbb{P}^r]$ is Hilbert strictly semistable (resp. Chow strictly semistable) with respect to $\rho$ and

$$\lim_{t \to 0} \rho(t)[X \subset \mathbb{P}^r] \in \mathrm{Orb}([X \subset \mathbb{P}^r]).$$

**Remark 4.6.** Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and $\rho$ be a one-parameter subgroup of $\mathrm{GL}_{r+1}$. Applying Definition 4.5, Fact 4.2 and Fact 4.3 we have that $[X \subset \mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable) if and only if $[X \subset \mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable) with respect to any one-parameter subgroup of $\mathrm{GL}_{r+1}$. The same holds for the Chow semistability (resp. polystability, stability).

Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$. If $C$ is a subscheme of $X$ of aritmetic genus $g_C$, we can consider the new point $[C \subset \mathbb{P}^r] \in \mathrm{Hilb}_{\deg \mathcal{O}_C(1), g_C}$ and also $W_{C,\rho}(m)$ and $e_{C,\rho}$ with respect to a one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$. The next result says that we can estimate or compute $e_{X,\rho}$ in terms of the weights of the subschemes of $X$.

**Proposition 4.7.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and let $\rho$ be a one-parameter subgroup of* $\mathrm{GL}_{r+1}$.

*(i) If $Y$ is a subscheme of $X$ and the weights of $\rho$ are non-negative, then $W_{X,\rho}(m) \geq W_{Y,\rho}(m)$ (in particular $e_{X,\rho} \geq e_{Y,\rho}$).*

*(ii) If $X$ is reduced (possibly non connected), has pure dimension 1 and $\{X_i\}_{i=1,\dots,n}$ is a collection of subcurves of $X$ such that*

$$(X_i)^c = \bigcup_{k \neq i} X_k$$

*for each $i = 1, \dots, n$, then*

$$e_{X,\rho} = \sum_{i=1}^{n} e_{X_i,\rho}.$$

*Proof.* Let us prove (i). Denote by $P_X$ and $P_Y$ the Hilbert polynomials of $X$ and $Y$, respectively, and consider a monomials basis $\{B_1, \dots, B_{P_X(m)}\}$ of $H^0(X, \mathcal{O}_X(m))$ such that

$$W_{X,\rho}(m) = \sum_{i=1}^{P_X(m)} w_\rho(B_i).$$

Since the restriction map $H^0(X, \mathcal{O}_X(m)) \longrightarrow H^0(Y, \mathcal{O}_Y(m))$ is onto for $m \gg 0$, up to reordering the monomials, we can assume that $\{B_1, \dots, B_{P_Y(m)}\}$ is a monomial basis of $H^0(Y, \mathcal{O}_Y(m))$. Hence

$$W_{Y,\rho}(m) \leq \sum_{i=1}^{P_Y(m)} w_\rho(B_i) \leq \sum_{i=1}^{P_X(m)} w_\rho(B_i) = W_{X,\rho}(m)$$

and (i) is proved.

Now we will prove (ii). We can assume that $n = 2$. Let $x_1, \dots, x_{r+1}$ be the coordinates of $V$ that diagonalize $\rho$ and denote by $w_1, \dots, w_{r+1} \in \mathbb{Z}$ the weights of $\rho$. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_{X_1 \cap X_2} \longrightarrow 0 \tag{4.3}$$

and the other ones obtained by tensoring (4.3) by $\mathcal{O}_X(m)$ with $m \in \mathbb{Z}$. For $m \gg 0$ we get the exact sequence

$$0 \to H^0(\mathcal{O}_X(m)) \overset{(|_{X_1}, |_{X_2})}{\longrightarrow} H^0(\mathcal{O}_{X_1}(m)) \oplus H^0(\mathcal{O}_{X_2}(m)) \longrightarrow H^0(\mathcal{O}_{X_1 \cap X_2}(m)) \to 0.$$

Since $X_1 \cap X_2$ is a 0-dimensional of length $k := k_{X_1} = k_{X_2}$, we have $h^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}(m)) = k$ for each $m \in \mathbb{Z}$. Denote by $P(m), P_1(m), P_2(m)$ the Hilbert polynomials of $X$, $X_1$, and $X_2$ respectively (satisfying $P_1(m) + P_2(m) = P(m) + k$ by the last exact sequence) and let $\{B_1, \dots, B_{P(m)}\}$ be a monomial basis of $H^0(X, \mathcal{O}_X(m))$ such that

$$W_{X,\rho}(m) = \sum_{i=1}^{P(m)} w_\rho(B_i).$$

Now, consider the linear independent vectors

$$C_1 = (B_{1|X_1}, B_{1|X_2}), \dots, C_{P(m)} = (B_{P(m)|X_1}, B_{P(m)|X_2}) \in H^0(\mathcal{O}_{X_1}(m)) \oplus H^0(\mathcal{O}_{X_2}(m))$$

Adding other vectors $C_{P(m)+j} = (B_{1j}, B_{2j})$ for $j = 1, \dots, k$, we can complete the linear independent set $\{C_1, \dots, C_{P(m)}\}$ to a basis of $H^0(X_1, \mathcal{O}_{X_1}(m)) \oplus H^0(X_2, \mathcal{O}_{X_2}(m))$. Now, it is easy to check that, up to reordering the vectors, $\pi_1(C_1), \dots, \pi_1(C_{P_1(m)})$ are linear independent in $H^0(X_1, \mathcal{O}_{X_1}(m))$ and $\pi_2(C_{P_1(m)+1}), \dots, \pi_2(C_{P(m)+k})$ are linear independent in $H^0(X_2, \mathcal{O}_{X_2}(m))$, where we denote by $\pi_i$ the projection of $H^0(X_1, \mathcal{O}_{X_1}(m)) \oplus H^0(X_2, \mathcal{O}_{X_2}(m))$ onto the $i$-th factor. This implies that, up to reordering the vectors again, there exists $k_1 \in \mathbb{Z}$ with $k_1 \leq k$ such that $B_{1|X_1}, \dots, B_{P_1(m)-k_1|X_1}$ are linear independent in $H^0(X_1, \mathcal{O}_{X_1}(m))$ and $B_{P_1(m)-k_1+1|X_2}, \dots, B_{P(m)|X_2}$ are linear independent in $H^0(X_2, \mathcal{O}_{X_2}(m))$. Finally, setting $k_2 := k - k_1$, we can consider other monomials $B'_1, \dots, B'_{k_1}, B''_1, \dots, B''_{k_2}$ so that

$$\{B_1, \dots, B_{P_1(m)-k_1}, B'_1, \dots, B'_{k_1}\} \quad \text{is a monomial basis for } H^0(X_1, \mathcal{O}_{X_1}(m)),$$

$$\{B_{P_1(m)-k_1+1}, \dots, B_{P(m)}, B''_1, \dots, B''_{k_2}\} \quad \text{is a monomial basis for } H^0(X_2, \mathcal{O}_{X_2}(m)).$$

Denoting by $\widetilde{w} = \max_i\{w_i\}$, we have

$$\begin{aligned} W_{X,\rho}(m) &= \sum_{i=1}^{P(m)} w_\rho(B_i) = \sum_{i=1}^{P_1(m)-k_1} w_\rho(B_i) + \sum_{i=P_1(m)-k_1+1}^{P(m)} w_\rho(B_i) \\ &\geq W_{X_1,\rho}(m) - \sum_{i=1}^{k_1} w_\rho(B'_i) + W_{X_2,\rho}(m) - \sum_{i=1}^{k_2} w_\rho(B''_i) \\ &\geq W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - k\widetilde{w}\, m = \left(\frac{e_{X_1,\rho} + e_{X_2,\rho}}{2}\right) m^2 + O(m), \end{aligned}$$

which implies that

$$e_{X_\rho} \geq e_{X_1,\rho} + e_{X_2,\rho}.$$

Now, we will prove the reverse inequality. Let $F$ be a homogeneous polynomial of degree $h \geq 1$ vanishing identically on $X_1$ and regular on $X_2$. Let $\{B_1, \dots, B_{P_1(m)}\}$ be a monomial basis of $H^0(X_1, \mathcal{O}_{X_1}(m))$ and $\{B'_1, \dots, B'_{P_2(m-h)}\}$ a monomial basis of $H^0(X_2, \mathcal{O}_{X_2}(m-h))$ such that

$$W_{X_1,\rho}(m) = \sum_{i=1}^{P_1(m)} w_\rho(B_i) \qquad \text{and} \qquad W_{X_2,\rho}(m-h) = \sum_{i=1}^{P_2(m-h)} w_\rho(B'_i).$$

It is easy to check that $B_1, \dots, B_{P_1(m)}, FB'_1, \dots, FB'_{P_2(m-h)}$ are linearly independent in $H^0(X, \mathcal{O}_X(m))$, so that, setting $d_2 = \deg X_2$, we have

$$\begin{aligned} \dim \left\langle B_1, \dots, B_{P_1(m)}, FB'_1, \dots, FB'_{P_2(m-h)} \right\rangle &= P_1(m) + P_2(m-h) \\ &= P_1(m) + P_2(m) - d_2 h \\ &= P(m) + k - d_2 h \leq P(m). \qquad (4.4) \end{aligned}$$

Adding possibly other monomials $B''_1, \dots, B''_{d_2h-k}$, we get a basis of $H^0(X, \mathcal{O}_X(m))$. Actually we would like to work with a monomial basis in order to apply the Hilbert-Mumford numerical criterion (Fact 4.3), so suppose that $F = M_1 + \dots + M_p$, where $M_1, \dots, M_p$ are monomials of degree $h$. It is an easy exercise to prove that for $j = 1, \dots, P_2(m-h)$ we can choose monomials $M_{i_j}$ such that

$$B_1, \dots, B_{P_1(m)}, M_{i_1}B'_1, \dots, M_{i_{P_2(m-h)}}B'_{P_2(m-h)}, B''_1, \dots, B''_{d_2h-l}$$

are linearly independent. For each $m \gg 0$ we get

$$\begin{aligned} W_{X,\rho}(m) &\leq \sum_{j=1}^{P_1(m)} w_\rho(B_j) + \sum_{j=1}^{P_2(m-h)} w_\rho(M_{i_j}B'_j) + \sum_{j=1}^{d_2h-k} w_\rho(B''_j) \\ &\leq W_{X_1,\rho}(m) + W_{X_2,\rho}(m-h) + h\widetilde{w}P_2(m-h) + (d_2h-k)\widetilde{w}m \\ &= \left(\frac{e_{X_1,\rho} + e_{X_2,\rho}}{2}\right)m^2 + O(m). \end{aligned}$$

This implies that

$$e_{X_\rho} \leq e_{X_1,\rho} + e_{X_2,\rho}$$

and we are done. □

**Remark 4.8.** Proposition 4.7(ii) improves the estimate of [HM98, Chap. 4, Ex. 4.49], which however holds even for non-reduced 1-dimensional complete subschemes of $\mathbb{P}^r$.

Proposition 4.7(ii) holds only for the Chow weight. We will see later a class of examples with $n = 2$ (see Lemma 8.1) that in general do not satisfy the equality $W_{X,\rho}(m) = W_{X_1,\rho}(m) + W_{X_2,\rho}(m)$.

4.3. **Basins of attraction.** Basins of attraction represent a useful tool in the study of the orbits which are identified in a GIT quotient. We review the basic definitions, following the presentation in [HH13, Sec. 4].

**Definition 4.9.** Let $[X_0 \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ a 1ps of $\mathrm{GL}_{r+1}$ that stabilizes $[X_0 \subset \mathbb{P}^r]$. The *$\rho$-basin of attraction* of $[X_0 \subset \mathbb{P}^r]$ is the subset

$$A_\rho([X_0 \subset \mathbb{P}^r]) := \{[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d \ : \ \lim_{t\to 0} \rho(t) \cdot [X \subset \mathbb{P}^r] = [X_0 \subset \mathbb{P}^r]\}.$$

Clearly, if $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ then $[X_0 \subset \mathbb{P}^r]$ belongs to the closure of the $\mathrm{SL}_{r+1}$-orbit $O([X \subset \mathbb{P}^r])$ of $[X \subset \mathbb{P}^r]$. Therefore, if $[X_0 \subset \mathbb{P}^r]$ is Hilbert semistable (resp. Chow semistable) then every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is Hilbert semistable (resp. Chow semistable) and is identified with $[X_0 \subset \mathbb{P}^r]$ in the GIT quotient $\mathrm{Hilb}_d^{ss}/SL_{r+1}$ (resp. $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})/SL_{r+1}$).

The following well-known properties of the basins of attraction (see e.g. [HH13, p. 24-25]) will be used in the sequel.

**Fact 4.10.** *Same notation as in Definition 4.9 and let $m \geq M$ as in Section 4.1.*

*(i) If $\mu([X_0 \subset \mathbb{P}^r]_m, \rho) < 0$ (resp. $e_{X_0,\rho} > 2d \cdot \frac{w(\rho)}{r+1}$) then every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is not $m$-Hilbert semistable (resp. not Chow semistable).*

*(ii) If $\mu([X_0 \subset \mathbb{P}^r]_m, \rho) = 0$ (resp. $e_{X_0,\rho} = 2d \cdot \frac{w(\rho)}{r+1}$) then $[X_0 \subset \mathbb{P}^r]$ is m-Hilbert semistable (resp. Chow semistable) if and only if every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is m-Hilbert semistable (resp. Chow semistable).*

4.4. **Flat limits and Gröbner bases.** A useful technique for computing the limit $\lim_{t\to 0} \rho(t)[X \subset \mathbb{P}^r]$ is based on the theory of Gröbner bases (see [HeHi11] for the general theory and [HHL07] and for its applications to GIT). Let $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ be a 1ps and let $\{x_1, \ldots, x_{r+1}\}$ be coordinates of $V$ that diagonalize the action of $\rho$, so that for $i = 1, \ldots, r+1$ we have

$$\rho(t) \cdot x_i = t^{w_i} x_i \text{ for some } w_i \in \mathbb{Z}.$$

If $a = (a_1, \ldots, a_{r+1}) \in \mathbb{N}^{r+1}$, we define the monomial

$$x^a := x_1^{a_1} x_2^{a_2} \ldots x_{r+1}^{a_{r+1}} \in S := k[x_1, \ldots, x_{r+1}].$$

Let us define the following order $\prec_\rho$ (called the *$\rho$-weighted graded order*) on the set of monomials of $S$. If $x^a$ and $x^b$ are monomials, we say that $x^a \prec_\rho x^b$ if

(1) $\deg x^a < \deg x^b$ or
(2) $\deg x^a = \deg x^b$ and $w_\rho(x^a) < w_\rho(x^b)$.

It is easy to notice that the order $\prec_\rho$ is not total, in general. In order to have a monomial order $\prec$ that refines $\prec_\rho$, it suffices to fix a lexicographical order $<$ on the set of monomials of $S$, for example the one induced by declaring that $x_1 < x_2 < \ldots < x_{r+1}$, and to say that $x^a \prec x^b$ if

(1) $x^a \prec_\rho x^b$ or
(2) $\deg x^a = \deg x^b$, $w_\rho(x^a) = w_\rho(x_b)$ and $x^a < x^b$.

We call the above monomial order $\prec$ a *$\rho$-weighted lexicographic order*. Moreover, if $f = \sum c_a x^a \in S$ and $I$ is an ideal of $S$, we denote by

(1) $\mathrm{in}_{\prec_\rho}(f)$ the sum of the terms of $f$ of maximal order with respect to $\prec_\rho$;
(2) $\mathrm{in}_{\prec_\rho}(I) = \langle \mathrm{in}_{\prec_\rho}(f) \,|\, f \in I\rangle$;
(3) $\mathrm{in}_\prec(f)$ the monomial (hence without coefficient) of maximal order with respect to $\prec$;
(4) $c_\prec(f)$ the coefficient of $\mathrm{in}_\prec(f)$ in $f$;
(5) $\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f) \,|\, f \in I\rangle$;
(6) $w(f) = \max\{w_\rho(x^a) \,|\, c_a \neq 0\}$ and $\widetilde{f}(x_1, \ldots, x_{r+1}, t) = t^{w(f)} f(t^{-w_1}x_1, \ldots, t^{-w_{r+1}}x_{r+1})$;
(7) $\widetilde{I} = \langle \widetilde{f}, f \,|\, f \in I\rangle \subset S[t]$.

Now, we recall the definition of Gröbner basis with respect to a monomial order (see [HeHi11, Definition 2.1.5]).

**Definition 4.11.** Let $I$ be an ideal of $S$ and $\prec$ a monomial order. A system of generators $\{f_1, \ldots, f_n\}$ of $I$ is said to be a **Gröbner basis** for $I$ with respect to $\prec$ if $\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f_1), \ldots, \mathrm{in}_\prec(f_n)\rangle$.

In the sequel, we will use some facts about Gröbner bases. First of all, we recall a famous criterion to determine if a system of generators of an ideal is a Gröbner basis or not (cf. [HeHi11, Theorem 2.3.2]). Let $f_1, f_2 \in S$ be two homogeneous polynomial and define

$$S(f_1,f_2) = \frac{\text{l.c.m.}(\text{in}_\prec(f_1),\text{in}_\prec(f_2))}{c_\prec(f_1)\,\text{in}_\prec(f_1)}\,f_1 - \frac{\text{l.c.m.}(\text{in}_\prec(f_1),\text{in}_\prec(f_2))}{c_\prec(f_2)\,\text{in}_\prec(f_2)}\,f_2.$$

where $\text{l.c.m.}(\text{in}_\prec(f_1),\text{in}_\prec(f_2))$ is the least common multiple of $\text{in}_\prec(f_1)$ and $\text{in}_\prec(f_2)$.

**Fact 4.12.** (Buchberger's criterion) Let $I = \langle f_1, \ldots, f_n\rangle$ be an ideal in $S$ and $\prec$ a monomial order. The system of generators $\{f_1, \ldots, f_n\}$ is a Gröbner basis with respect to $\prec$ if and only if

$$\text{in}_\prec(S(f_i,f_j)) \in \langle \text{in}_\prec(f_1), \ldots, \text{in}_\prec(f_n)\rangle$$

for each $i, j \in \{1, \ldots, n\}$.

Now, we recall a basic fact about the relation between Gröbner bases and flat limits (see [HHL07, Theorem 3] or for more details [HeHi11, Sec. 3.2]).

**Fact 4.13.** *If $I \subset S$ is an ideal, then the $k[t]$-algebra $S[t]/\widetilde{I}$ is free as a $k[t]$-algebra. Moreover, the following hold:*

$$S[t]/\widetilde{I} \otimes_{k[t]} k[t,t^{-1}] \cong (S/I)[t,t^{-1}] \quad \text{and} \quad S[t]/\widetilde{I} \otimes_{k[t]} k[t]/(t) \cong S/\text{in}_{\prec_\rho}(I). \tag{4.5}$$

We obtain a useful corollary.

**Corollary 4.14.** *Let $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ and let $\rho$ be a one-parameter subgroup of $\text{GL}_{r+1}$. Denote by $I$ the homogeneous ideal of $X$. Then $[V(\text{in}_{\prec_\rho}(I)) \subset \mathbb{P}^r] = \lim_{t\to 0}\rho(t)[X \subset \mathbb{P}^r]$.*

*Proof.* By Fact 4.13 we have a family of curves $\mathcal{X} \to \mathbb{A}^1$ whose central fiber is $V(\text{in}_{\prec_\rho}(I)) \subset \mathbb{P}^r$. This yields a map $\beta : \mathbb{A}^1 \to \text{Hilb}_d$ which coincides away from $0 \in \mathbb{A}^1$ with the map $\alpha : \mathbb{A}^1 \to \text{Hilb}_d$ induced by $\rho$. Since $\text{Hilb}_d$ is projective, the maps $\alpha$ and $\beta$ coincides everywhere and we are done. □

Finally, the following fact allows us to compute explicitly the ideal $\text{in}_{\prec_\rho}(I)$ (see [HHL07, Theorem 3] or for more details [HeHi11, Sec. 3.2]).

**Fact 4.15.** *Let $\{f_1, \ldots, f_n\}$ be a Gröbner basis for $I$ with respect to a $\rho$-weighted lexicographical order $\prec$ that refines $\prec_\rho$. Then*

*(i)* $\widetilde{f_1}, \ldots, \widetilde{f_n}$ *generate* $\widetilde{I}$*;*
*(ii)* $\text{in}_{\prec_\rho}(f_1), \ldots, \text{in}_{\prec_\rho}(f_n)$ *generate* $\text{in}_{\prec_\rho}(I)$*.*

4.5. **The parabolic group.** Here we recall a classical result due to J. Tits (see for more details [Dol03, Sec. 9.5] or [MFK94, Chap. 2, Sec. 2]) which is very useful to study the semistable locus of the action of a reductive group $G$ on an algebraic variety. Let $X \subset \mathbb{P}(V)$ be a projective variety and $G$ a reductive group that acts on $X$ via a linear representation in $V$. For the sake of simplicity we assume that

$G = \mathrm{GL}(W)$ for some vector space $W$. By the Hilbert-Mumford criterion, $x \in X$ is semistable if and only if for every one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}(W)$ we have that $\mu(x, \rho) \geq 0$. We know that every one-parameter subgroup is diagonalized by some basis of $V$. A priori, it does not suffice to check the condition of the Hilbert-Mumford criterion for all one-parameter subgroups, which are diagonalized by a fixed basis of $V$: this represents the main difficulty in characterizing the semistable locus. Tit's result allows one to identify the one-parameter subgroups which give the "worst" weights so that the research of a destabilizing one-parameter subgroup is less intricate.

**Definition 4.16.** We define the **parabolic group** with respect to a one-parameter subgroup $\rho$ by setting

$$P(\rho) = \left\{ g \in \mathrm{GL}(W) \,|\, \text{ there exists } \lim_{t \to 0} \rho(t) g \rho(t)^{-1} \right\} \subset \mathrm{GL}(W).$$

**Fact 4.17.** *The group $P(\rho)$ is a parabolic subgroup of $\mathrm{GL}(W)$, i.e. it contains a Borel subgroup. Moreover, if $x \in X$, then $\mu(x, \rho) = \mu(x, A^{-1}\rho A)$ for each $A \in P(\rho)$.*

For a proof see [Dol03, Lemma 9.2, Lemma 9.3] or [MFK94, Def. 2.3/Prop. 2.6]. It is not difficult to show that when we consider the action of $\mathrm{GL}_{r+1}$ on $\mathrm{Hilb}_d$, if the weights of the 1ps $\rho$ with respect to a diagonalizing basis $\{x_1, \ldots, x_{r+1}\}$ of $V$ satisfy the inequalities $w_1 \geq \ldots \geq w_{r+1}$, then $P(\rho)$ contains the group of the upper triangular matrices with respect to the coordinates $\{x_1, \ldots, x_{r+1}\}$. This fact has a useful consequence.

**Corollary 4.18.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and let $Y := (y_1, \ldots, y_{r+1})^t$ be an arbitrary basis of $V$.*

*(i) Let $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ be a 1ps diagonalized by the basis coordinates $X = (x_1, \ldots, x_{r+1})^t$ with weights $w_1, \ldots, w_{r+1}$, respectively. Then there exist a lower unitriangular matrix $A = (a_{ij})$ and a one-parameter subgroup $\rho' : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ diagonalized by the new coordinates $(z_1, \ldots, z_{r+1})^t =: Z = AY$ such that*

$$\rho'(t) z_i = t^{w_{\sigma(i)}} z_i \quad \text{for some } \sigma \in S_{r+1} \quad \text{and} \quad W_{X,\rho}(m) = W_{X,\rho'}(m) \ \text{ for } m \gg 0.$$

*(ii) $[X \subset \mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable) if and only if it is Hilbert semistable (resp. polystable, stable) with respect to all the one-parameter subgroups which are diagonalized by $Z = AY$ for every lower unitriangular matrix $A$. The same holds for the Chow semistability (resp. polystability, stability).*

*Proof.* In order to prove (i), it suffices to assume that $w_1 \geq \ldots \geq w_{r+1}$ and that

$$y_1 = x_1, \ldots,\ y_{l-1} = x_{l-1},\ y_l = y = \sum_{i=1}^{r+1} \lambda_i x_i,\ y_{l+1} = x_{l+1}, \ldots, y_{r+1} = x_{r+1},$$

where $\lambda_1, \ldots, \lambda_{r+1} \in k$. Now, we define the following basis of coordinates:

$$z_i = x_i \text{ if } i \neq l \text{ and } z_l = y - \sum_{i=1}^{l-1} \lambda_i x_i = \sum_{i=l}^{r+1} \lambda_i x_i.$$

Let $A$ and $B$ be the matrices such that $Z = AY = BX$. By construction, $A$ is lower unitriangular and $B$ is upper unitriangular, hence $B \in P(\rho)$ by Fact 4.17 and

$$W_{X,\rho}(m) = W_{X,B^{-1}\rho B}(m) \tag{4.6}$$

for $m \gg 0$. Now, if we define $\rho' = B^{-1}\rho B$, then $\rho'$ is diagonalized by the coordinates $Z$.

It remains to prove (ii). The "only if" implication follows from Remark 4.6. In order to prove the "if" direction, consider a 1ps $\rho$ of $\mathrm{GL}_{r+1}$ diagonalized by a basis $X = (x_1, \ldots, x_{r+1})$. Using (i), we can find a lower unitriangular matrix $A$ such that the 1ps $\rho' := A^{-1}\rho A$ is diagonalized by the basis $Z = AY$ and is such that

$$w(\rho') = w(\rho) \quad \text{and} \quad W_{X,\rho'}(m) = W_{X;\rho}(m) \quad \text{for} \quad \gg 0. \tag{4.7}$$

The equalities (4.7) imply (ii) for the (semi)stability. Now, let us prove (ii) for the polystability. Since $A \in P(\rho)$, there exists $\lim_{t\to 0}(\rho(t)A\rho(t)^{-1})$: call it $B$. We have

$$\begin{aligned}
\lim_{t\to 0} \rho'(t)[X \subset \mathbb{P}^r] &= \lim_{t\to 0}(A^{-1}\rho(t)A[X \subset \mathbb{P}^r]) = \lim_{t\to 0} A^{-1}(\rho(t)A\rho(t)^{-1})(\rho(t)[X \subset \mathbb{P}^r]) \\
&= A^{-1} \cdot \lim_{t\to 0}(\rho(t)A\rho(t)^{-1}) \cdot \lim_{t\to 0} \rho(t)[X \subset \mathbb{P}^r] \\
&= A^{-1}B \cdot \lim_{t\to 0} \rho(t)[X \subset \mathbb{P}^r]. \qquad (4.8)
\end{aligned}$$

Combining (4.7) and (4.8), we see that $[X \subset \mathbb{P}^r]$ is Hilbert (resp. Chow) polystable with respect to $\rho'$ if and only if it Hilbert (resp. Chow) polystable with respect to $\rho$; combined with Remark 4.6, this concludes the proof of (ii). ☐

4.6. **Stability of smooth curves and Potential stability.** Here we recall two basic results due to Mumford and Gieseker: the stability of smooth curves of high degree and the (so-called) potential stability theorem.

**Fact 4.19.** *If $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is connected and smooth and $d \geq 2g+1$ then $[X \subset \mathbb{P}^r]$ is Chow stable.*

For a proof, see [Mum77, Thm. 4.15]. In [Gie82, Thm. 1.0.0], a weaker form of the above Fact is proved: if $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is connected and smooth and $d \geq 10(2g-2)$ then $[X \subset \mathbb{P}^r]$ is Hilbert stable. See also [HM98, Chap. 4.B] and [Mor10, Sec. 2.4] for an overview of the proof.

**Fact 4.20** (Potential stability)**.** *If $d > 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ (with $X$ possibly non connected) then:*

*(i) $X$ is reduced of pure dimension one and has at most nodes as singularities. In particular, $X$ is a pre-stable curve whenever it is connected.*
*(ii) $X \subset \mathbb{P}^r$ is non-degenerate, linearly normal (i.e., $X$ is embedded by the complete linear system $|\mathcal{O}_X(1)|$) and $\mathcal{O}_X(1)$ is non-special (i.e., $H^1(X, \mathcal{O}_X(1)) = 0$).*
*(iii) The line bundle $\mathcal{O}_X(1)$ on $X$ is balanced (see Definition 3.7).*

*Proof.* For the connected case, see [Mum77, Prop. 4.5]. In [Gie82, Thm. 1.0.1, Prop. 1.0.11], the same conclusions are shown to hold under the stronger hypothesis that $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ and $d \geq 10(2g-2)$. See also [HM98, Chap. 4.C] and [Mor10, Sec. 3.2] for an overview of the proof. If $X$ is not connected, the argument is analogous to Theorem 5.1 below. □

**Remark 4.21.** The hypothesis that $d > 4(2g-2)$ in Fact 4.20 is sharp: in [HM10] it is proved that all the 4-canonical p-stable curves (which in particular can have cusps) belong to $\mathrm{Hilb}^s_{4(2g-2)}$.

## 5. Potential pseudo-stability theorem

The aim of this section is to generalize the Potential stability theorem (see Fact 4.20) for smaller values of $d$. The main result is the following theorem, which we call Potential pseudo-stability Theorem for the relations with the pseudo-stable curves (see Definition 2.1(ii)).

**Theorem 5.1.** (Potential pseudo-stability theorem) *If* $d > 2(2g-2)$ *and* $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ *(with* $X$ *possibly not connected), then*

- *(i)* $X$ *is a pre-wp-stable curve, i.e. it is reduced and its singularities are at most nodes, cusps and tacnodes with a line.*
- *(ii)* $X \subset \mathbb{P}^r$ *is non-degenerate, linearly normal (i.e.,* $X$ *is embedded by the complete linear system* $|\mathcal{O}_X(1)|$*) and* $\mathcal{O}_X(1)$ *is non-special (i.e.,* $H^1(X, \mathcal{O}_X(1)) = 0$*);*
- *(iii) The line bundle* $\mathcal{O}_X(1)$ *on* $X$ *is balanced (see Definition 3.7).*

*Proof.* To prove the claim, we adapt various results in [Mum77], [Gie82], [Sch91] and [HH13, Sec. 7]. Let us indicate the different steps of the proof. Suppose that $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ (with $X$ possibly non connected). We will denote by $X' \subset X$ the union of the connected components of $X$ of dimension 1.

- $X'$ is non-degenerate: the same proof of [Gie82, Prop. 1.0.2] shows that if $X'$ is degenerate, then there exists a 1ps $\rho$ with positive weights such that
$$e_{X',\rho} > \frac{2d}{r+1} w(\rho),$$
without any assumption on $d$. By Proposition 4.7(i), we have $e_{X,\rho} \geq e_{X',\rho}$, so that $[X \subset \mathbb{P}^r]$ is Chow unstable.
- $X' = Y \sqcup Z$, where $Y$ is generically reduced and $Z$ is a disjoint union of $\mathbb{P}^1$'s of multiplicity 2.

  If $X'$ is connected, this follows, using Proposition 4.7(i), from [Sch91, Lemma 2.4], which works under the assumption that $d > 2(2g-2)$, or from [HH13, Lemma 7.4], which works under the assumption that $d > \frac{3}{2}(2g-2)$. Notice that if $X'$ is connected then $X' = Y$.

  Now suppose that $X'$ is not connected. We use some ideas from [Sch91, Lemma 2.4]. Let $C$ be a connected component of $X'$ such that is not generically reduced. There are two cases:

(1) $C$ is reducible;
(2) $C$ is irreducible.

Suppose that case (1) occurs. Pick an irreducible component $D$ of $C$ having multiplicity greater than 1. Following the first part of the proof of [Sch91, Lemma 2.4], we get that there exists a 1ps $\rho$ with $w(\rho) = 1$ such that $e_{C,\rho} \geq 3$. By Proposition 4.7(i), we have

$$e_{X,\rho} \geq e_{C,\rho} \geq 3 > \frac{2d}{r+1} \quad \text{if} \quad \frac{d}{r+1} < \frac{3}{2} \quad \left( \iff d > \frac{3}{2}(2g-2) \right);$$

hence, under our assumption on $d$, $[X \subset \mathbb{P}^r]$ is Chow unstable.

It remains to analyze case (2). If $\deg \mathcal{O}_X(1)_{|C_{\text{red}}} \geq 2$, then the second part of the proof of [Sch91, Lemma 2.4] gives a 1ps $\rho$ with positive weights and $w(\rho) = 3$ such that $e_{C,\rho} \geq 8$, so that by Proposition 4.7(i) we have

$$e_{X,\rho} \geq e_{C,\rho} \geq 8 > \frac{6d}{r+1} \quad \text{if} \quad \frac{d}{r+1} < \frac{4}{3} \quad \left( \iff d > 2(2g-2) \right);$$

hence, $[X \subset \mathbb{P}^r]$ is Chow unstable. If $\deg \mathcal{O}_X(1)_{|C_{\text{red}}} = 1$ (i. e. $C_{\text{red}}$ is a line), the last proof does not work. Suppose that the multiplicity of $C$ is $n$, with $n \geq 3$. Let $\{x_1 \ldots, x_{r+1}\}$ be a system of coordinates in $\mathbb{P}^r$ such that

$$C_{\text{red}} = \bigcap_{i=3}^{r+1} \{x_i = 0\}$$

and consider a 1ps $\rho$ diagonalized by $\{x_1 \ldots, x_{r+1}\}$ with weights $w_1 = w_2 = 1$ and $w_3 = \ldots = w_{r+1} = 0$. We obtain that

$$e_{X,\rho} \geq n\, e_{C_{\text{red}},\rho} \geq 6 > \frac{2d}{r+1} w(\rho) = \frac{4d}{r+1} \quad \text{if} \quad \frac{d}{r+1} < \frac{3}{2} \quad \left( \iff d > \frac{3}{2}(2g-2) \right)$$

and again $[X \subset \mathbb{P}^r]$ is Chow unstable. We deduce that if $C$ is a non-reduced irreducible component of $X'$, then $C$ is a $\mathbb{P}^1$ with multiplicity two.

- $Y$ does not have triple points: using Proposition 4.7(i), same proof as that of [Mum77, Prop. 3.1, p. 69] (see also [Sch91, Lemma 2.1]) or [Gie82, Prop. 1.0.4], both of which are easily seen, by direct inspection, to work under the assumption that $d > \frac{3}{2}(2g-2)$.
- $Y$ does not have non-ordinary cusps: using Proposition 4.7(i), same proof of [Sch91, Lemma 2.3] which works for $d > 2(2g-2)$ or [HH13, Lemma 7.2] which works for $d > \frac{25}{14}(2g-2)$.
- $Y$ does not have higher order tacnodes or tacnodes in which one of the two branches does not belong to a line: using Proposition 4.7(i), same proof of [Sch91, Lemma 2.2], which works for $d > 2(2g-2)$.
- $H^1(X_{\text{red}}, \mathcal{O}_{X_{\text{red}}}(1)) = 0$. Since it is obvious that

$$H^1(X_{\text{red}}, \mathcal{O}_{X_{\text{red}}}(1)) = H^1(X'_{\text{red}}, \mathcal{O}_{X'_{\text{red}}}(1))$$

and $H^1(Z_{\text{red}}, \mathcal{O}_{Z_{\text{red}}}(1)) = 0$, it suffices to prove that $H^1(Y_{\text{red}}, \mathcal{O}_{Y_{\text{red}}}(1)) = 0$. We use the Clifford's theorem for reduced curves with nodes, cusps and tacnodes of [HH13, Thm. 7.7] (generalizing the Clifford's theorem of Gieseker-Morrison for nodal curves in [Gie82, Thm. 0.2.3]). From the proof of [HH13, Thm. 7.7], we deduce the following

**Fact 5.2** (**Clifford's theorem**). *Let $X$ be a reduced connected curve with nodes, cusps and tacnodes and let $L$ be a line bundle on $X$ generated by global sections. Assume that $H^1(X,L)\neq 0$ and consider a non-zero section $s\in H^0(X,\omega_X\otimes L^{-1})\cong H^1(X,L)^\vee$. Let $C$ be the subcurve of $X$ which is the union of all the irreducible components of $X$ where $s$ is not identically zero. Then*

$$h^0(C,L_{|C})\leq \frac{\deg_C L}{2}+1. \tag{5.1}$$

We will follow the same argument used by Gieseker in the Claim of [Gie82, Prop. 1.0.8] with some modifications. Suppose, by contradiction, that

$$H^1(Y_{\mathrm{red}},\mathcal{O}_{Y_{\mathrm{red}}}(1))\neq 0:$$

there exists a connected component $W\subset Y_{\mathrm{red}}$ such that $H^1(W,\mathcal{O}_W(1))\neq 0$. Choose a non-zero section

$$0\neq s\in H^0(W,\omega_W\otimes\mathcal{O}_W(-1))\cong H^1(W,\mathcal{O}_W(1))^\vee.$$

Let $C$ be the subcurve of $W$ which is the union of all the irreducible components of $W$ where $s$ is not identically zero. Fact 5.2 implies that

$$h^0(C,\mathcal{O}_C(1))\leq\frac{\deg_C\mathcal{O}(1)}{2}+1.$$

We notice that $s$ vanishes at the points of intersection of $C$ with the complementary subcurve $C^c$. It is easy to check that the proof of [Gie82, Prop. 1.0.7] works without any assumption on $d$ and even if $k_C=0$. We obtain

$$2\deg_C\mathcal{O}(1)\leq\frac{2d}{r+1}h^0(C,\mathcal{O}_C(1))\leq\frac{2d}{r+1}\left(\frac{\deg_C\mathcal{O}(1)}{2}+1\right).$$

Using our assumption $d>2(2g-2)$, which is equivalent to the inequality $\frac{d}{r+1}<\frac{4}{3}$, we get that

$$2\deg_C\mathcal{O}(1)<\frac{4}{3}(\deg_C\mathcal{O}(1)+2)\quad\Longleftrightarrow\quad\deg_C\mathcal{O}(1)<4\quad\Longleftrightarrow\quad\deg_C\mathcal{O}(1)=1,\ 2\text{ or }3.$$

Firstly suppose that $\deg_C\mathcal{O}_C(1)=1$ or $2$. If $C$ is irreducible, then $C\cong\mathbb{P}^1$ and we get a contradiction since $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))=0$ and $H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))=0$. If $C$ is reducible, then $\deg_C\mathcal{O}(1)=2$ and we can write $C=C_1\cup C_2$ where $C_1\cong C_2\cong\mathbb{P}^1$, $\deg_{C_1}\mathcal{O}(1)=\deg_{C_2}\mathcal{O}(1)=1$ (i.e. $C_1$ and $C_2$ are lines) and $|C_1\cap C_2|=1$. This gives the exact sequence

$$0\longrightarrow\mathcal{O}_{C_1}\oplus\mathcal{O}_{C_2}\longrightarrow\mathcal{O}_C(1)\longrightarrow\mathcal{O}_{C_1\cap C_2}\longrightarrow 0.$$

From the exact sequence of cohomology we get that $H^1(C,\mathcal{O}_C(1))=0$ and again we have a contradiction.

Now suppose that $\deg_C \mathcal{O}_C(1) = 3$. If $C$ is irreducible, then either $C \cong \mathbb{P}^1$ or $C$ is an elliptic curve (smooth, nodal or cuspidal) in $\langle C \rangle \cong \mathbb{P}^2$, so we obtain that $H^1(C, \mathcal{O}_C(1)) = 0$, which is absurd. Finally assume that $C$ is reducible. If $C$ has 2 irreducible components $C_1$ and $C_2$, then $C_1 \cong C_2 \cong \mathbb{P}^1$ and, up to reordering, we can assume that $\deg_{C_1} \mathcal{O}(1) = 1$ (i.e. $C_1$ is a line) and $\deg_{C_2} \mathcal{O}(1) = 2$ (i.e. $C_2$ is a conic). There are two cases: either $|C_1 \cap C_2| = 2$ (which happens if and only if $C_1$ and $C_2$ lie in the same plane) or $|C_1 \cap C_2| = 1$. In the former case, we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{O}_{C_1 \cap C_2} \longrightarrow 0.$$

Again using the exact sequence of cohomology, we obtain that $H^1(C, \mathcal{O}_C(1)) = 0$, absurd. The latter case is dealt with similarly and it is left to the reader. If $C$ has 3 irreducible components $C_1$, $C_2$ and $C_3$, then $C_1 \cong C_2 \cong C_3 \cong \mathbb{P}^1$ and $\deg_{C_1} \mathcal{O}(1) = \deg_{C_2} \mathcal{O}(1) = \deg_{C_3} \mathcal{O}(1) = 1$ (i.e $C_1$, $C_2$ and $C_3$ are lines). There are two cases: either each of the $C_i$'s intersects all the others (which happens if and only if the $C_i$'s lie on the same plane) or the $C_i$'s form a chain. In the former case, there is the exact sequence

$$0 \longrightarrow \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_2}(-1) \oplus \mathcal{O}_{C_3}(-1) \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{O}_p \oplus \mathcal{O}_q \oplus \mathcal{O}_r \longrightarrow 0$$

from which we obtain again $H^1(C, \mathcal{O}_C(1)) = 0$, absurd. The latter case is similar and left to the reader.

- $X'$ is generically reduced, i.e. $Z = \emptyset$. The proof uses some ideas from [Gie82, Prop. 1.0.2]. Suppose, by contradiction, that $Z \neq \emptyset$ and let $E \subset Z$ be a connected component of $Z$. By hypothesis $E$ is a double line. Setting $I = I(E)$, consider a primary decomposition

$$I = J_1 \cap \ldots \cap J_k.$$

where $J_1$ is $I(E_{\text{red}})$-primary. We notice that $J_1$ is uniquely determined by [Mat89, Thm. 6.8(iii)] and there exists a system of coordinates $\{x_1 \ldots, x_{r+1}\}$ in $\mathbb{P}^r$ such that

$$J_1 \subset \langle x_3, \ldots, x_r, x_{r+1}^2 \rangle := J.$$

Denote by $E_0$ the subscheme of $E$ defined by $J$ and set $W := E^c$, $n := h^0(Z, \mathcal{O}_Z)$ and $m := h^0(Y, \mathcal{O}_Y)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{W_{\text{red}}} \longrightarrow \mathcal{O}(1)_{|W_{\text{red}}} \longrightarrow \mathcal{O}_D \longrightarrow 0, \tag{5.2}$$

where $D$ is a divisor associated to $\mathcal{O}(1)_{|W_{\text{red}}}$ with support on the smooth locus of $W_{\text{red}}$. Observing that $g(E) \leq g(E_0) = 0$, it is easy to check that

$$h^0(\mathcal{O}_{W_{\text{red}}}) = m + n - 1, \ h^1(\mathcal{O}_{W_{\text{red}}}) \geq g + 1 + m + n - 2 \ \text{ and } h^0(\mathcal{O}_D) = d - 1 - n,$$

so that from the exact sequence of cohomology associated to (5.2) we obtain that

$$\begin{aligned} h^0(W_{\mathrm{red}}, \mathcal{O}(1)_{|W_{\mathrm{red}}}) &= h^0(W_{\mathrm{red}}, \mathcal{O}_{W_{\mathrm{red}}}) + h^0(W_{\mathrm{red}}, \mathcal{O}_D) - h^1(W_{\mathrm{red}}, \mathcal{O}_{W_{\mathrm{red}}}) \\ &= d-g-n-1 < d-g+1 = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)). \end{aligned}$$

This implies that the restriction map

$$\pi : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \longrightarrow H^0(W_{\mathrm{red}}, \mathcal{O}_{\mathbb{P}^r}(1)_{|W_{\mathrm{red}}})$$

has kernel $K \neq 0$. Since $[X \subset \mathbb{P}^r]$ is Chow semistable, $X$ is non-degenerate in $\mathbb{P}^r$, hence there exists a non-zero section $s \in K$ which is regular on $E$. Let $\{x_1 \dots, x_{r+1}\}$ be a system of coordinates such that $x_1 = s$ and

$$E_{\mathrm{red}} = \bigcap_{i=3}^{r+1} \{x_i = 0\}.$$

Consider a 1ps $\rho$ diagonalized by $\{x_1 \dots, x_{r+1}\}$ with weights $w_1 = 0$ and $w_2 = \dots = w_{r+1} = 1$. It is not difficult to check that $e_{E_0,\rho} = 2$ and $e_{W,\rho} = 2(d-2)$. By [HM98, Chap. 4, Ex. 4.49] and Proposition 4.7(i) we get that

$$e_{X,\rho} \geq e_{E_0} + e_{W,\rho} = 2d-2 > 2d - \frac{2d}{r+1} = \frac{2d}{r+1} r \quad \text{if} \quad \frac{d}{r+1} > 1 \quad (\Longleftrightarrow g \geq 2),$$

hence $[X \subset \mathbb{P}^r]$ is Chow unstable.

- $X'$ is reduced and 5.1(*ii*) and 5.1(*iii*) hold: it follows from the proofs of [Gie82, Prop. 1.0.7-1.0.12], which work also for disconnected curves.
- $X$ has pure dimension 1: suppose by contradiction that there exists an irreducible component of dimension 0. This implies that, denoting by $g(X')$ the arithmetic genus of $X'$, $g(X') > g$. By Riemann-Roch and the vanishing of $H^1(X', \mathcal{O}_{X'}(1))$, we get

$$h^0(X', \mathcal{O}_{X'}(1)) = d - g(X') + 1 < d - g + 1 = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)),$$

hence $X'$ is degenerate, absurd.

□

**Remark 5.3.** The hypothesis that $d > 2(2g-2)$ in the above Theorem (5.1) is sharp: in [HH13, Thm. 2.14] it is proved that all the 2-canonical h-stable curves in the sense of [HH13, Def. 2.5, Def. 2.6] (which in particular can have arbitrary tacnodes and not only tacnodes with a line) belong to $\mathrm{Hilb}^s_{2(2g-2)}$.

5.1. **Balanced line bundles and quasi-wp-stable curves.** The aim of this subsection is to study the following

**Question 5.4.** *Given a pre-wp-stale curve $X$, what kind of restrictions does the existence of an* ample *balanced line bundle $L$ impose on $X$?*

The following result gives an answer to the above question.

**Proposition 5.5.** *Let $X$ be a pre-wp-stable curve of genus $g \geq 2$. If there exists an ample balanced line bundle $L$ on $X$ of degree $d \geq g-1$, then $X$ is quasi-wp-stable and $L$ is properly balanced.*

*Proof.* Let $Z$ be a connected rational subcurve of $X$ (equivalently $Z$ is a chain of $\mathbb{P}^1$'s) such that $k_Z \leq 2$. Clearly, $k_Z \geq 1$ since $X$ is connected and $Z \neq X$ because $g \geq 2$.

If $k_Z = 1$ then $\deg_Z(\omega_X) = -1$ and the basic inequality (3.1) together with the hypothesis that $d \geq g-1$ gives that

$$\deg_Z(L) \leq \frac{d}{2g-2}\deg_Z(\omega_X) + \frac{k_Z}{2} = -\frac{d}{2g-2} + \frac{1}{2} \leq 0.$$

This contradicts the fact that $L$ is ample.

If $k_Z = 2$ then $\deg_Z(\omega_X) = 0$ and the basic inequality (3.1) gives that

$$\deg_Z(L) \leq \frac{d}{2g-2}\deg_Z(\omega_X) + \frac{k_Z}{2} = 1.$$

Since $L$ is ample, it has positive degree on each irreducible component of $Z$; therefore $Z$ must be irreducible which implies that $Z \cong \mathbb{P}^1$ and $\deg_Z L = 1$. □

Combining the previous Proposition 5.5 with the potential stability Theorem (see Fact 4.20) and the Potential pseudo-stability Theorem 5.1, we get the following

**Corollary 5.6.**

*(i) If $d > 2(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ with $X$ connected then $X$ is a quasi-wp-stable curve and $\mathcal{O}_X(1)$ is properly balanced.*

*(ii) If $d > 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ with $X$ connected then $X$ is a quasi-stable curve and $\mathcal{O}_X(1)$ is properly balanced.*

Note that, by Proposition 17.3(ii) of the Appendix, we have the following Remark, which can be seen as a partial converse to Proposition 5.5.

**Remark 5.7.** A balanced line bundle of degree $d > \frac{3}{2}(2g-2)$ on a quasi-wp-stable curve $X$ is properly balanced if and only if it is ample. Therefore, for $d > \frac{3}{2}(2g-2)$, the set $B_X^d$ is the set of all the multidegrees of ample balanced line bundles on $X$.

## 6. Stabilizer subgroups

Let $[X \subset \mathbb{P}^r]$ be a Chow semistable point of $\mathrm{Hilb}_d$ with $X$ connected and $d > 2(2g-2)$. Note that $X$ is a quasi-wp-stable curve by Corollary 5.6(i), $L := \mathcal{O}_X(1)$ is balanced and $X$ is non-degenerate and linearly normal in $\mathbb{P}^r$ by the Potential pseudo-stability Theorem 5.1.

The aim of this section is to describe the stabilizer subgroup of an element $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ as above. We denote by $\mathrm{Stab}_{\mathrm{GL}_{r+1}}([X \subset \mathbb{P}^r])$ the stabilizer subgroup of $[X \subset \mathbb{P}^r]$ in $\mathrm{GL}_{r+1}$, i.e. the subgroup of $\mathrm{GL}_{r+1}$ fixing $[X \subset \mathbb{P}^r]$. Similarly, $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is the stabilizer subgroup of $[X \subset \mathbb{P}^r]$ in $\mathrm{PGL}_{r+1}$. Clearly, $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r]) = \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X \subset \mathbb{P}^r])/\mathbb{G}_m$, where $\mathbb{G}_m$ denotes the diagonal subgroup of $\mathrm{GL}_{r+1}$ which clearly belongs to $\mathrm{Stab}_{\mathrm{GL}_{r+1}}([X \subset \mathbb{P}^r])$.

It turns out that the stabilizer subgroup of $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is related to the automorphism group of the pair $(X, \mathcal{O}_X(1))$, which is defined as follows.

Given a variety $X$ and a line bundle $L$ on $X$, an automorphism of $(X, L)$ is given by a pair $(\sigma, \psi)$ such that $\sigma \in \mathrm{Aut}(X)$ and $\psi$ is an isomorphism between the line bundles $L$ and $\sigma^*(L)$. The group of automorphisms of $(X, L)$ is naturally an algebraic group denoted by $\mathrm{Aut}(X, L)$. We get a natural forgetful homomorphism

$$\begin{aligned} F : \mathrm{Aut}(X, L) &\to \mathrm{Aut}(X) \\ (\sigma, \psi) &\mapsto \sigma, \end{aligned} \tag{6.1}$$

whose kernel is the multiplicative group $\mathbb{G}_m$, acting as fiberwise multiplication on $L$, and whose image is the subgroup of $\mathrm{Aut}(X)$ consisting of automorphisms $\sigma$ such that $\sigma^*(L) \cong L$. The quotient $\mathrm{Aut}(X, L)/\mathbb{G}_m$ is denoted by $\overline{\mathrm{Aut}(X, L)}$ and is called the *reduced* automorphism group of $(X, L)$.

The relation between the stabilizer subgroup of an embedded variety $X \subset \mathbb{P}^r$ and the automorphism group of the pair $(X, \mathcal{O}_X(1))$ is provided by the following result.

**Lemma 6.1.** *Given a projective embedded variety $X \subset \mathbb{P}^r$ which is non-degenerate and linearly normal, there are isomorphisms of algebraic groups*

$$\begin{cases} \mathrm{Aut}(X, \mathcal{O}_X(1)) \cong \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X \subset \mathbb{P}^r]), \\ \overline{\mathrm{Aut}(X, \mathcal{O}_X(1))} \cong \mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r]). \end{cases}$$

*Proof.* This result is certainly well-known to experts. However, since we could not find a suitable reference, we sketch a proof for the reader's convenience.

Observe first that the natural restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(X, \mathcal{O}_X(1))$ is an isomorphism because by assumption the embedding $X \subset \mathbb{P}^r$ is non-degenerate and linearly normal. Therefore, we identify the above two vector spaces and we denote them by $V$. Note that $\mathbb{P}^r = \mathbb{P}(V^\vee)$ and that the standard coordinates on $\mathbb{P}^r$ induce a basis of $V$, which we call the standard basis of $V$.

Let us now define a homomorphism

$$\eta : \mathrm{Aut}(X, \mathcal{O}_X(1)) \to \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X \subset \mathbb{P}^r]) \subseteq \mathrm{GL}_{r+1} = \mathrm{GL}(V^\vee). \tag{6.2}$$

Given $(\sigma, \psi) \in \mathrm{Aut}(X, \mathcal{O}_X(1))$, where $\sigma \in \mathrm{Aut}(X)$ and $\psi$ is an isomorphism between $\mathcal{O}_X(1)$ and $\sigma^*\mathcal{O}_X(1)$, we define $\eta((\sigma, \psi)) \in \mathrm{GL}(V^\vee)$ as the composition

$$\eta((\sigma, \psi)) : V^\vee = H^0(X, \mathcal{O}_X(1))^\vee \xrightarrow[\cong]{\widehat{\psi^{-1}}} H^0(X, \sigma^*\mathcal{O}_X(1))^\vee \xrightarrow[\cong]{\widehat{\sigma^*}} H^0(X, \mathcal{O}_X(1))^\vee = V^\vee,$$

where $\widehat{\psi^{-1}}$ is the dual of the isomorphism induced by $\psi^{-1}$ and $\widehat{\sigma^*}$ is the dual of the isomorphism induced by $\sigma^*$. Let us denote by $\phi_{|\mathcal{O}_X(1)|}$ (resp. $\phi_{|\sigma^*\mathcal{O}_X(1)|}$) the embedding of $X$ in $\mathbb{P}^r$ given by the complete linear series $|\mathcal{O}_X(1)|$ (resp. by $|\sigma^*\mathcal{O}_X(1)|$) with respect to the basis of $H^0(X, \mathcal{O}_X(1))$ (resp. $H^0(X, \sigma^*\mathcal{O}_X(1))$) induced by the standard basis of $V$ via the above isomorphisms. By construction, the following diagram commutes:

$$
\begin{array}{ccc}
X & \xhookrightarrow{\phi_{|\mathcal{O}_X(1)|}} & \mathbb{P}(H^0(X,\mathcal{O}_X(1))^\vee) \\
 & \searrow^{\phi_{|\sigma^*\mathcal{O}_X(1)|}} & \downarrow \widehat{\psi^{-1}} \\
\downarrow \sigma & & \mathbb{P}(H^0(X,\sigma^*\mathcal{O}_X(1))^\vee) \\
 & & \downarrow \widehat{\sigma^*} \\
X & \xhookrightarrow{\phi_{|\mathcal{O}_X(1)|}} & \mathbb{P}(H^0(X,\mathcal{O}_X(1))^\vee).
\end{array}
\tag{6.3}
$$

Thus we get that $\eta((\sigma,\psi))$ belongs to $\mathrm{Stab}_{\mathrm{GL}_{r+1}}([X\subset\mathbb{P}^r])\subseteq \mathrm{GL}(V^\vee)$ and $\eta$ is well-defined.

Conversely, we define a homomorphism

$$
\tau : \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X\subset\mathbb{P}^r]) \to \mathrm{Aut}(X,L) \tag{6.4}
$$

as follows. An element $g\in \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X\subset\mathbb{P}^r])\subseteq \mathrm{GL}_{r+1}=\mathrm{GL}(V^\vee)$ will send $X$ isomorphically onto itself, and thus induces an automorphism $\sigma\in\mathrm{Aut}(X)$. Consider now the isomorphism

$$
\widetilde{\psi} : V = H^0(X,\mathcal{O}_X(1)) \xrightarrow[\cong]{\widehat{g^{-1}}} V = H^0(X,\mathcal{O}_X(1)) \xrightarrow[\cong]{\sigma^*} H^0(X,\sigma^*\mathcal{O}_X(1)),
$$

where $\widehat{g^{-1}}$ is the dual of $g^{-1}$ and $\sigma^*$ is the isomorphism induced by $\sigma$. The isomorphism $\widetilde{\psi}$ induces an isomorphism $\psi$ between $\mathcal{O}_X(1)$ and $\sigma^*\mathcal{O}_X(1)$ making the following diagram commutative

$$
\begin{array}{ccc}
H^0(X,\mathcal{O}_X(1))\otimes\mathcal{O}_X & \twoheadrightarrow & \mathcal{O}_X(1) \\
\widetilde{\psi}\downarrow & & \downarrow\psi \\
H^0(X,\sigma^*\mathcal{O}_X(1))\otimes\mathcal{O}_X & \twoheadrightarrow & \sigma^*\mathcal{O}_X(1).
\end{array}
$$

We define $\tau(g):=(\sigma,\psi)\in\mathrm{Aut}(X,\mathcal{O}_X(1))$.

We leave to the reader the task of checking that the homomorphisms $\eta$ and $\tau$ are induced by morphisms of algebraic groups and that they are one the inverse of the other.

The map $\eta$ sends the subgroup $\mathbb{G}_m\subseteq \mathrm{Aut}(X,\mathcal{O}_X(1))$ of scalar multiplications on $\mathcal{O}_X(1)$ into the diagonal subgroup $\mathbb{G}_m\subset\mathrm{GL}_{r+1}$ and therefore it induces an isomorphism $\overline{\mathrm{Aut}(X,\mathcal{O}_X(1))}\cong \mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X\subset\mathbb{P}^r])$. □

In Theorem 6.4 below, we describe the connected component $\mathrm{Aut}(X,L)^0$ of $\mathrm{Aut}(X,L)$ containing the identity for the pairs we will be interested in. By Definition 2.10, recall that for a quasi-wp-stable curve $X$ we denote by $X_{\mathrm{exc}}\subset X$ the subcurve of $X$ consisting of the union of the exceptional components $E$ of $X$, i.e., the subcurves $E\subset X$ such that $E\cong\mathbb{P}^1$ and $k_E=2$. We denote by $\widetilde{X}:=X_{\mathrm{exc}}^c$ the complementary subcurve of $X_{\mathrm{exc}}$ and by $\gamma(\widetilde{X})$ the number of connected components of $\widetilde{X}$. Certain elliptic tails of $X$ will play a special role in what follows.

**Definition 6.2.** Let $F$ be an irreducible elliptic tail of $X$ (i.e., an irreducible subcurve of $X$ such that $g_F = 1$ and $k_F = 1$) and let $p$ denote the intersection point between $F$ and the complementary subcurve $F^c$. Given an ample line bundle $L$ on $X$, we can write $L_{|F} = \mathcal{O}_F((d_F - 1)p + q)$, where $d_F = \deg_F L$ denotes the degree of $L$ on $F$, for a uniquely determined smooth point $q$ of $F$. We say that $F$ is *special* with respect to $L$ if $q = p$ and *non-special* otherwise. We denote by $\epsilon(X, L)$ the number of cuspidal elliptic tails of $X$ that are special with respect to $L$.

**Remark 6.3.** If $F$ is a reducible elliptic tail of $X$ (for example reducible nodal or tacnodal), $F$ cannot be special. Indeed, using the same notation as in Definition 6.2, if $L_{|F} = \mathcal{O}_F(d_F p)$, there exists an irreducible component $E \subset F$ such that $\deg L_{|E} = 0$, hence $L$ is not ample.

Before stating Theorem 6.4, we introduce another notation: we denote by $\tau(X)$ the number of tacnodal elliptic tails of $X$.

**Theorem 6.4.** *Let $X$ be either a quasi-stable curve of genus $g \geq 2$ or a quasi-wp-stable curve of genus $g \geq 3$ and let $L$ be a properly balanced line bundle of degree $d \in \mathbb{Z}$ on $X$. Then the connected component $\mathrm{Aut}(X, L)^0$ of $\mathrm{Aut}(X, L)$ containing the identity is isomorphic to $\mathbb{G}_m^{\gamma(\widetilde{X})+\epsilon(X,L)+\tau(X)}$.*

*Proof.* Consider the wp-stable reduction $X \to \mathrm{wps(X)}$ of $X$ (see Proposition 2.11). Note that since $\mathrm{wps(X)} = \mathcal{P}\mathrm{roj}(\oplus_{\mathrm{i}\geq 0} \mathrm{H}^0(\mathrm{X}, \omega_{\mathrm{X}}^{\mathrm{i}}))$, an automorphism of $X$ naturally induces an automorphism of $\mathrm{wps(X)}$, so by composing the homomorphism $F$ (see (6.1)) with the homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut(wps(X))}$ induced by the wp-stable reduction, we get a homomorphism

$$G : \mathrm{Aut}(X, L) \longrightarrow \mathrm{Aut(wps(X))}. \tag{6.5}$$

We will determine the connected component $\mathrm{Ker}(G)^0$ of the kernel of $G$ and the connected component $\mathrm{Im}(G)^0$ of the image of $G$ in the two claims below.

CLAIM 1: $\mathrm{Ker}(G) = \mathrm{Ker}(G)^0 = \mathbb{G}_m^{\gamma(\widetilde{X})}$.

Recall from Proposition 2.11 that the wp-stable reduction $X \to \mathrm{wps(X)}$ is the contraction of every exceptional component $E \cong \mathbb{P}^1$ of $X$ to a node or a cusp if $E \cap E^c$ consists of two nodes or one tacnode, respectively. We can factor the wp-stable reduction of $X$ as

$$X \to Y \to \mathrm{wps(X)},$$

where $c : X \to Y$ is obtained by contracting all the exceptional components $E$ of $X$ such that $E \cap E^c$ consists of two nodes and $Y \to \mathrm{wps(X)}$ is obtained by contracting all the exceptional components $E$ of $Y$ such that $E \cap E^c$ consists of a tacnode. Now, since an automorphism of $X$ must send exceptional components of $X$ meeting the rest of $X$ in two distinct points to exceptional components of the same type, we can factor the map $G$ of (6.5) as

$$G : \mathrm{Aut}(X, L) \xrightarrow{G_1} \mathrm{Aut}(Y) \xrightarrow{G_2} \mathrm{Aut(wps(X))}.$$

This gives an exact sequence

$$0 \to \mathrm{Ker}(G_1) \to \mathrm{Ker}(G) \xrightarrow{G_{1|\mathrm{Ker}(G)}} \mathrm{Ker}(G_2). \tag{6.6}$$

The same proof of [BFV12, Lemma 2.13] applied to the contraction map $X \to Y$ gives that

$$\mathrm{Ker}(G_1) = \mathbb{G}_m^{\gamma(\widetilde{X})}. \tag{6.7}$$

Using (6.6) and (6.7), Claim 1 follows if we prove that

$$\mathrm{Im}(G_1) \cap \mathrm{Ker}(G_2) = \{\mathrm{id}\}. \tag{6.8}$$

In order to prove (6.8), we need first to describe explicitly $\mathrm{Ker}(G_2)$. Recall that, by construction, all the exceptional components $E \cong \mathbb{P}^1$ of $Y$ are such that $E \cap E^c$ consists of a tacnode of $Y$ and all of them are contracted to a cusp of wps(X) by the map $Y \to$ wps(X). Therefore, $\mathrm{Ker}(G_2)$ consists of all the automorphisms $\gamma \in \mathrm{Aut}(Y)$ such that $\gamma$ restricts to the identity on $\overline{Y \setminus \cup E}$ where the union runs over all the exceptional subcurves $E$ of $Y$. Consider one of these exceptional components $E \subset Y$ and let $\{p\} = E \cap E^c$. Since $p$ is a tacnode of $Y$, there is an isomorphism (see [HH13, Sec. 6.2])

$$i : T_pE \xrightarrow{\cong} T_pE^c,$$

where $T_pE$ is the tangent space of $E$ at $p$ and similarly for $T_pE^c$. Any $\gamma \in \mathrm{Aut}(Y)$ preserves the isomorphism $i$. If moreover $\gamma \in \mathrm{Ker}(G_2) \subseteq \mathrm{Aut}(Y)$ then $\gamma$ acts trivially on the irreducible component of $E^c$ containing $p$, hence it acts trivially also on $T_pE^c$. Therefore, the restriction of $\gamma \in \mathrm{Ker}(G_2)$ to $E$ will be an element $\phi \in \mathrm{Aut}(E)$ that fixes $p$ and induces the identity on $T_pE$. Fix the identification $(E, p) \cong (\mathbb{P}^1, 0)$ and consider the transformations in $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ of the form

$$\phi_\lambda(z) = \frac{z}{\lambda z + 1} \qquad \psi_\mu(z) = \mu z \tag{6.9}$$

for $\lambda \in k$ and $\mu \in k^*$. All the elements that fix $p$ and induce the identity on $T_pE$ form a subgroup of $\mathrm{Aut}(E)$, which is isomorphic to the additive subgroup $\mathbb{G}_a$ of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ given by all the transformations $\phi_\lambda$ (for $\lambda \in k$). Conversely, every such $\phi$ extends to an automorphism of $\mathrm{Aut}(Y)$, which is the identity on $E^c$ and therefore lies on $\mathrm{Ker}(G_2)$. From this discussion, we deduce that

$$\mathrm{Ker}(G_2) = \prod_E \mathbb{G}_a, \tag{6.10}$$

where the product runs over all the exceptional components $E$ of $Y$.

We can now prove (6.8). Take an element $(\sigma, \psi) \in \mathrm{Aut}(X, L)$ such that $G_1(\sigma, \psi) \in \mathrm{Ker}(G_2)$. Consider an exceptional component $E$ of $Y$; let $\{p\} = E \cap E^c$ and let $C$ be the irreducible component of $E^c$ containing $p$. By (6.10) and the discussion preceding it, we get that $G_1(\sigma, \psi)_{|E} = \phi_\lambda$ for some $\lambda \in k$ (as in (6.9)) and $G_1(\sigma, \psi)_{|C} = \mathrm{id}_C$. By construction, the map $c : X \to Y$ is an isomorphism in a neighborhood of $E \subset Y$. Therefore, by abuse of notation, we identify $E$ with its inverse image via $c$, similarly for $p$, and we call $C'$ the irreducible component of $X$ such that $\{p\} = E \cap C'$. From the above properties of $G_1(\sigma, \psi)$, we deduce that $\sigma_{|E} = \phi_\lambda$ and $\sigma_{|C} = \mathrm{id}_C$. Consider now $\widehat{X} \cong E \coprod E^c$ the partial normalization of $X$ at $p$ and let $\nu : \widehat{X} \to X$ be the natural map. We have an exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Pic(X)} \xrightarrow{\nu^*} \mathrm{Pic}(\widehat{\mathrm{X}}) = \mathrm{Pic(E)} \times \mathrm{Pic(E^c)} \to 0.$$

By looking at the gluing data defining line bundles on $X$, it is easy to check that the above automorphism $\sigma \in \mathrm{Aut}(X)$ acts as the identity on $\mathrm{Pic}(\widehat{\mathrm{X}})$ and that it acts on $\mathbb{G}_a$ by sending $\mu$ in $\mu + \lambda$. Since, by assumption, there exists an isomorphism $\psi$ between $\sigma^*(L)$ and $L$, we must have that $\lambda = 0$, or in other words that $\sigma_{|E} = \phi_0 = \mathrm{id}_E$. Since this is true for all the exceptional components $E$ of $Y$, from (6.10) we get that $G_1(\sigma, \psi) = \mathrm{id}$ and (6.8) is now proved.

CLAIM 2: $\mathrm{Im}(G)^0 = \mathbb{G}_m^{\epsilon(X,L)+\tau(X)}$.

Note that if $X$ is quasi-stable of genus $g \geq 2$ then wps(X) is stable of genus $g \geq 2$ and that if $X$ is quasi-p-stable of genus $g \geq 3$ then wps(X) is p-stable of genus $g \geq 3$. In both cases, we have that Aut(wps(X)) is a finite group (see [DM69] for stable curves and [Sch91, Proof of Lemma 5.3] for p-stable curves); hence $\mathrm{Im}(G)^0 = \{\mathrm{id}\}$ and Claim 2 is proved.

In the general case, consider the p-stable reduction wps(X) $\to$ ps(wps(X)) := ps(X) of wps(X) (see Definition 2.14) and the induced map

$$H : \mathrm{Aut(wps(X))} \to \mathrm{Aut(ps(X))}.$$

As recalled before, Aut(ps(X)) is a finite group if $g \geq 3$; hence we get that

$$\mathrm{Aut(wps(X))}^0 = \mathrm{Ker(H)}^0. \tag{6.11}$$

The p-stable reduction wps(X) $\to$ ps(X) contracts all the elliptic tails of wps(X) to cusps of ps(X). This easily implies that

$$\mathrm{Ker}(H)^0 = \prod_F \mathrm{Aut}(F, p)^0, \tag{6.12}$$

where the product is over all the elliptic tails $F$ of wps(X), $\{p\} = F \cap F^c$ and $\mathrm{Aut}(F, p)^0$ is the connected component of the automorphism group of the pointed curve $(F, p)$. There are 3 possibilities for the elliptic tails of the quasi-wp-stable wps(X) according to Figure 1 below.

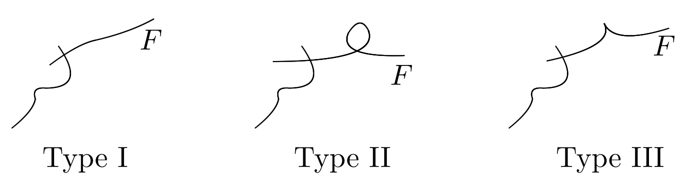



FIGURE 4. All the possible elliptic tails of a wp-stable curve.

We claim that, for an elliptic tail $F$ of wps(X), the following holds

$$\mathrm{Aut}(F,p)^0=\begin{cases}\{\mathrm{id}\} & \text{if } F \text{ is smooth or nodal (type I or II)},\\ \mathbb{G}_m & \text{if } F \text{ is cuspidal (type III)}.\end{cases} \tag{6.13}$$

If $F$ is of type I, this follows from the well-known fact that a 1-pointed smooth curve of genus 1 has only finitely many automorphisms. If $F$ is of type II (resp. of type III), this follows from the identification of $\mathrm{Aut}(F,p)^0$ with the subgroup of automorphism of $F^\nu \cong \mathbb{P}^1$ fixing three points (resp. two points), namely the inverse image of $p$ and of the singular locus of $F$ via the normalization map $\nu: F^\nu \to F$.

Now, suppose that $F$ is an elliptic tail of type III. Obviously $F$ is the image of an elliptic tail $F'$ of X via the wp-stable reduction. Since the wp-stable reduction contracts the exceptional subcurves of $X$, $F'$ can be chosen is such a way that $F'$ is cuspidal irreducible or tacnodal with two irreducible components. Using (6.11), (6.12) and (6.13), Claim 2 follows if we prove that $\mathrm{Aut}(F,p)^0=\mathbb{G}_m\subset \mathrm{Im}(G)$ if and only if one of the following cases is satisfied:

(i) $F'$ is cuspidal and special with respect to $L$,
(ii) $F'$ is tacnodal.

If $F'$ is cuspidal, we can identify $F$ with $F'$ and clearly $\mathrm{Aut}(F,p)^0\subset \mathrm{Im}(G)$ if and only if $L_{|F}\in \mathrm{Pic}^{\mathrm{d_F}}(\mathrm{F})$ is fixed by $\mathbb{G}_m$. Consider the $\mathbb{G}_m$-equivariant isomorphism $\rho: F_{\mathrm{sm}}\xrightarrow{\cong}\mathrm{Pic}^{\mathrm{d_F}}(\mathrm{F})$ which maps $r$ to $\mathcal{O}_F((d_F-1)p+r)$. The unique $\mathbb{G}_m$-fixed point is the point $p$, which is sent to $\mathcal{O}_F(d_Fp)$ by $\rho$. Therefore, $L_{|F}$ is fixed by $\mathrm{Aut}(F,p)^0=\mathbb{G}_m$ if and only if $L_{|F}=\mathcal{O}_F(d_Fp)$, or in other words when $F$ is special with respect to $L$.

Now, suppose that $F'$ is tacnodal, i.e. $F'$ is the union of two smooth rational subcurve $E_1$ and $E_2$ meeting in a tacnode. Let $\{p\}=F'\cap(F')^c$ and let $q$ be the tacnode; assume that $p\in E_2$. Consider

$$\widehat{X}=(F')^\nu\coprod(F')^c=E_1\coprod E_2\coprod(F')^c$$

the partial normalization of $X$, $\nu:\widehat{X}\to X$ the natural map and $\{q_1,q_2\}$ the inverse image of $q$ via $\nu$, where we assume that $q_1\in E_1$ and $q_2\in E_2$. The following holds:

$$\mathrm{Aut}((F')^\nu,q_1,q_2,\nu^{-1}(p))^0=\mathrm{Aut}(E_1,q_1)^o\times\mathrm{Aut}(E_2,q_2,\nu^{-1}(p))^0\cong(\mathbb{G}_m\ltimes\mathbb{G}_a)\times\mathbb{G}_m.$$

Indeed, if we fix the identifications $(E_1,q_1)\cong(\mathbb{P}^1,0)$ and $(E_2,q_2,\nu^{-1}(p))\cong(\mathbb{P}^1,0,\infty)$, we can consider the transformations of the form (6.9) and it is well-known that

(1) $\mathrm{Aut}(\mathbb{P}^1,0)^0$ is generated by the automorphisms $\phi_\lambda$, $\psi_\mu\in\mathrm{PGL}_2$ for $\lambda\in k$ and $\mu\in k^*$,
(2) $\mathrm{Aut}(\mathbb{P}^1,0,\infty)^0$ is generated by $\psi_\mu\in\mathrm{PGL}_2$ for $\mu\in k^*$.

As explained in the proof of Claim 1, every $\gamma\in\mathrm{Aut}(X)$ preserves the isomorphism $i: T_qE_1\xrightarrow{\cong}T_qE_2$, so that there is an identification of $\mathrm{Aut}(F',p)^0$ with the subgroup of $\mathrm{Aut}(E_1,q_1)\times\mathrm{Aut}(E_2,q_2,\nu^{-1}(p))^0$ corresponding to the elements $(\psi_{\mu_1},\phi_\lambda,\psi_{\mu_2})$ such that $\mu_1 = \mu_2$. Hence

$$\mathrm{Aut}(F', p)^0 \cong \mathbb{G}_m \ltimes \mathbb{G}_a.$$

Now, the wp-stable reduction $F' \to F$ induces a surjective map $\mathrm{Aut}(F', p)^0 \to \mathrm{Aut}(F, p)^0$ and $(\psi_\mu, \phi_\lambda, \psi_\mu) \in \mathrm{Aut}(F', p)^0$ is mapped to the identity if and only if its restriction to $E_2$ is the identity, i.e. if and only if $\mu = 1$. We obtain an exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Aut}(F', p)^0 \cong \mathbb{G}_m \ltimes \mathbb{G}_a \to \mathrm{Aut}(F, p)^0 \cong \mathbb{G}_m \to 0,$$

which allows one to identify $\mathrm{Aut}(F, p)^0$ with the subgroup of $\mathrm{Aut}(F', p)$ consisting of all the elements of the form $(\psi_\mu, \mathrm{id}, \psi_\mu)$. If $L \in \mathrm{Pic(X)}$, for any such $\gamma = (\psi_\mu, \mathrm{id}, \psi_\mu) \in \mathrm{Aut}(F', p)^0 \subset \mathrm{Aut}(X)$ we have that $\gamma^* L \cong L$ since, given the exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Pic(X)} \xrightarrow{\nu^*} \mathrm{Pic}(\widehat{\mathrm{X}}) = \mathrm{Pic(E_1)} \times \mathrm{Pic(E_2)} \times \mathrm{Pic((F')^c)} \to 0,$$

the automorphism $\gamma$ acts as the identity both on $\mathrm{Pic}(\widehat{\mathrm{X}}) \cong \mathbb{Z}^2 \times \mathrm{Pic((F')^c)}$ and on the gluing data $\mathbb{G}_a$. Hence $\mathrm{Aut}(F, p)^0 \subset \mathrm{Im}(G)$ and the claim 2 is completely proven.

□

## 7. Behaviour at the extremes of the basic inequality

Recall from Corollary 5.6(i) that if $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is Chow semistable with $X$ connected and $d > 2(2g-2)$, then $X$ is quasi-wp-stable and $\mathcal{O}_X(1)$ is properly balanced.

The aim of this section is to investigate the properties of the Chow semistable points $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ such that $\mathcal{O}_X(1)$ is stably balanced or strictly balanced (see Definition 3.7).

Our fist result is the following

**Theorem 7.1.** *If $d > 2(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^s \subseteq \mathrm{Hilb}_d$ with $X$ connected, then $\mathcal{O}_X(1)$ is stably balanced.*

*Proof.* The proof uses some ideas from [Gie82, Prop. 1.0.7] and [Cap94, Lemma 3.1].

Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^s \subseteq \mathrm{Hilb}_d$ with $X$ connected and assume that $d > 2(2g-2)$. By the Potential pseudo-stability Theorem 5.1 and Corollary 5.6(i), we get that $X$ is a quasi-wp-stable curve and $L := \mathcal{O}_X(1)$ is properly balanced and non-special.

By contradiction, suppose that $\mathcal{O}_X(1)$ is not stably balanced. Then, by Definition 3.6 and Remark 3.8, we can find a connected subcurve $Y$ with connected complementary subcurve $Y^c$ such that

$$(7.1)\quad \begin{cases} Y^c \not\subset X_{\mathrm{exc}} \text{ or equivalently } g_{Y^c} = 0 \Longrightarrow k_{Y^c} = k_Y \geq 3, \\ \deg_{Y^c} L = M_Y = \dfrac{d}{2g-2}\deg_{Y^c}\omega_X + \dfrac{k_{Y^c}}{2} = \dfrac{d}{2g-2}(2g_{Y^c} - 2 + k_{Y^c}) + \dfrac{k_{Y^c}}{2}, \\ \deg_Y L = m_Y = \dfrac{d}{2g-2}\deg_Y \omega_X - \dfrac{k_Y}{2} = \dfrac{d}{2g-2}(2g_Y - 2 + k_Y) - \dfrac{k_Y}{2}. \end{cases}$$

In order to produce the desired contradiction, we will use the numerical criterion for Hilbert stability (see Fact 4.2). Let $V := H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = H^0(X, \mathcal{O}_X(1))$ and consider the vector subspace

$$U := \operatorname{Ker}\left\{H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(Y, L_{|Y})\right\} \subseteq V.$$

Set $N+1 := \dim U$. Choose a basis $\{x_0, \ldots, x_N, \ldots, x_r\}$ of $V$ relative to the filtration $U \subseteq V$, i.e., $x_i \in U$ if and only if $0 \le i \le N$. Define a 1ps $\rho$ of $\mathrm{GL}_{r+1}$ by

$$\rho(t)\cdot x_i = \begin{cases} x_i & \text{if } 0 \le i \le N, \\ t x_i & \text{if } N+1 \le i \le r. \end{cases}$$

We will estimate the two polynomials appearing in Fact 4.2 for the 1ps $\rho$.

First of all, the total weight $w(\rho)$ of $\rho$ satisfies $w(\rho) = r - N = \dim V - \dim U \le h^0(Y, L_{|Y})$. Since $L$ is non-special and $H^0(X, L) \twoheadrightarrow H^0(Y, L_{|Y})$ because $X$ is a curve, we get that $h^0(Y, L_{|Y}) = \deg_Y L + 1 - g_Y$. Therefore, we conclude that

$$\text{(7.2)}\quad \frac{w(\rho)}{r+1} m P(m) \le \frac{h^0(Y, L_{|Y})}{r+1} m(dm+1-g) = \frac{\deg_Y L + 1 - g_Y}{d+1-g}\left[dm^2 + (1-g)m\right].$$

In order to compute the polynomial $W_\rho(m)$ for $m \gg 0$, consider the filtration of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$:

$$0 \subseteq U^m \subseteq U^{m-1}V \subseteq \ldots \subseteq U^{m-i}V^i \subseteq \ldots \subseteq V^m = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)),$$

where $U^{m-i}V^i$ is the subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ generated by the monomials containing at least $(m-i)$-terms among the variables $\{x_0, \ldots, x_N\}$. Note that for a monomial $B$ of degree $m$, it holds that

$$\text{(7.3)}\qquad B \in U^{m-i}V^i \setminus U^{m-i+1}V^{i-1} \iff w_\rho(B) = i.$$

Via the surjective restriction map $\mu_m : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \twoheadrightarrow H^0(X, L^m)$, the above filtration on $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ induces a filtration

$$0 \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^i \subseteq \ldots \subseteq F^m = H^0(X, L^m),$$

where $F^i := \mu_m(U^{m-i}V^i)$. Using (7.3), we get that

$$\text{(7.4)}\qquad W_\rho(m) = \sum_{i=1}^{m} i\left[\dim(F^i) - \dim(F^{i-1})\right] = m\dim(F^m) - \sum_{i=1}^{m-1}\dim(F^i) =$$

$$= m(dm+1-g) - \sum_{i=0}^{m-1}\dim(F^i).$$

It remains to estimate $\dim F^i$ for $0 \le i \le m-1$. To that aim, consider the partial normalization $\tau : \hat{X} \to X$ of $X$ at the nodes laying on $Y \cap Y^c$. Observe that $\hat{X}$ is the disjoint union of $Y$ and $Y^c$. We denote by $\widetilde{D}$ the inverse image of $Y \cap Y^c$ via $\tau$. Since $Y \cap Y^c$ consists of $k_Y$ nodes of $X$, $\widetilde{D}$ is the disjoint union of $D_Y$ and $D_{Y^c}$, where $D_Y$ consists of $k_Y$ smooth points on $Y$ and $D_{Y^c}$ consists of $k_Y$ smooth points on $Y^c$. Consider now the injective pull-back morphism

$$\tau^* : H^0(X, L^m) \hookrightarrow H^0(\hat{X}, \tau^* L^m) = H^0(Y, L^m_{|Y}) \oplus H^0(Y^c, L^m_{|Y^c}),$$

which clearly coincides with the restriction maps to $Y$ and $Y^c$.

Note that if $B$ is a monomial belonging to $U^{m-i}V^i \subseteq H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ for some $i \leq m-1$, then $B$ contains at least $m-i \geq 1$ variables among the $x_j$'s such that $x_j \in U$; hence the order of vanishing of $B$ along the subcurve $Y$ is at least equal to $m-i$ . This implies that any $s \in F^i \subseteq H^0(X, L^m)$ with $i \leq m-1$ vanishes identically on $Y$ and vanishes on the points of $D_{Y^c}$ with order at least $(m-i)$. We deduce that

$$\tau^*(F^i) \subseteq H^0(Y^c, L^m_{|Y^c}((i-m)D_{Y^c})) \text{ for } 0 \leq i \leq m-1. \tag{7.5}$$

<u>CLAIM:</u> $H^1(Y^c, L^m_{|Y^c}((i-m)D_{Y^c})) = 0$ for $0 \leq i \leq m-1$ and $m \gg 0$.

Let us prove the claim. Clearly, if the claim is true for $i=0$ then it is true for every $i>0$; so we can assume that $i=0$. According to Fact 17.4(i) of the Appendix, it is enough to prove that for any connected subcurve $Z \subseteq Y^c$, we have that

$$\deg_Z(L^m_{|Z}(-mD_Z)) > 2g_Z - 2 \text{ for } m \gg 0, \tag{7.6}$$

where $D_Z := D_{Y^c} \cap Z$. Indeed, (7.6) is equivalent to

$$\deg_Z L \geq |D_Z| \text{ with strict inequality if } g_Z \geq 1. \tag{7.7}$$

Observe that, since each point of $D_Z$ is the intersection of $Z$ with $Y = \overline{X \setminus Y^c}$ and $Z \cap \overline{Y^c \setminus Z} \neq \emptyset$ unless $Z = Y^c$ because $Y^c$ is connected, the following holds:

$$|D_Z| \leq k_Z \text{ with equality if and only if } Z = Y^c, \tag{7.8}$$

where $k_Z$ is, as usual, the length of the schematic intersection of $Z$ with the complementary subcurve $\overline{X \setminus Z}$ in $X$. In order to prove (7.7), we consider different cases.

If $g_Z \geq 1$ then using the basic inequality (3.1) for $L$ relative to the subcurve $Z$ and the assumption $d > 2(2g-2)$, we compute

$$\deg_Z L \geq \frac{d}{2g-2}\deg_Z \omega_X - \frac{k_Z}{2} > 2(2g_Z - 2 + k_Z) - \frac{k_Z}{2} \geq \frac{3k_Z}{2} \geq \frac{3|D_Z|}{2} \geq |D_Z|,$$

which shows that (7.7) holds in this case.

If $g_Z = 0$ and $Z = Y^c$ then, using that $\deg_{Y^c} L = M_{Y^c}$ and $k_{Y^c} \geq 3$ by (7.1), we get

$$\deg_{Y^c} L = M_{Y^c} = \frac{d}{2g-2}(2g_{Y^c} - 2 + k_{Y^c}) + \frac{k_{Y^c}}{2} > 2(k_{Y^c} - 2) + \frac{k_{Y^c}}{2} > k_{Y^c} = |D_{Y^c}|,$$

which shows that (7.7) holds also in this case.

It remains to consider the case $g_Z = 0$ and $Z \subsetneq Y^c$. If $k_Z \leq 2$ then, since $X$ is quasi-wp-stable and $Z$ is connected, we must have that $Z$ is an exceptional component of $X$, i.e., $Z \cong \mathbb{P}^1$ and $k_Z = 2$. By Proposition 5.5 it follows that $\deg_Z L = 1$. Since $|D_Z| \leq 1$ by (7.8), we deduce that (7.7) is satisfied also in this case. Finally, assume that $k_Z \geq 3$. Consider the subcurve $W := Z^c \cap Y^c \subset Y^c$. It is easy to check that

$$k_{Y^c} - k_W = |Z \cap Y| - |W \cap Z| = |D_Z| - (k_Z - |D_Z|) = 2|D_Z| - k_Z. \tag{7.9}$$

Using the basic inequality of $L$ with respect to $W$ together with (7.1), (7.9) and $k_Z \geq 3$, we get

$$\deg_Z L = \deg_{Y^c} L - \deg_W L \geq \frac{d}{2g-2}\deg_{Y^c}\omega_X + \frac{k_{Y^c}}{2} - \frac{d}{2g-2}\deg_W \omega_X - \frac{k_W}{2} =$$

$$= \frac{d}{2g-2}\deg_Z\omega_X + |D_Z| - \frac{k_Z}{2} > 2(k_Z-2) + |D_Z| - \frac{k_Z}{2} > |D_Z|.$$

The claim is now proved.

Using the claim above, we get from (7.5) that

(7.10)

$$\dim F^i = \dim \tau^*(F^i) \leq m\deg_{Y^c}L + (i-m)k_Y + 1 - g_{Y^c} \text{ for } 0 \leq i \leq m-1 \text{ and } m \gg 0.$$

Combining (7.10) and (7.4), we get that

$$W_\rho(m) \geq m(dm+1-g) - \sum_{i=0}^{m-1} [m\deg_{Y^c}L + (i-m)k_Y + 1 - g_{Y^c}] =$$

$$= m(dm+1-g) - m\left[m\deg_{Y^c}L - mk_Y + 1 - g_{Y^c}\right] - k_Y\frac{m(m-1)}{2} =$$

$$= m^2\left[\deg_Y L + \frac{k_Y}{2}\right] + m\left[1 - g_Y - \frac{k_Y}{2}\right], \tag{7.11}$$

where in the last equality we have used $d = \deg L = \deg_Y L + \deg_{Y^c} L$ and $g = g_Y + g_{Y^c} + k_Y - 1$.

Using that $\deg_Y L = m_Y$ by (7.1), we easily check that

$$\deg_Y L + \frac{k_Y}{2} = d\,\frac{\deg_Y L + 1 - g_Y}{d+1-g} \tag{7.12}$$

and

$$1 - g_Y - \frac{k_Y}{2} = (1-g)\,\frac{\deg_Y L + 1 - g_Y}{d+1-g}. \tag{7.13}$$

By combining (7.2), (7.11), (7.12), (7.13), we get for $m \gg 0$:

$$W_\rho(m) \geq m^2\left[\deg_Y L + \frac{k_Y}{2}\right] + m\left[1 - g_Y - \frac{k_Y}{2}\right] = \tag{7.14}$$

$$= \frac{\deg_Y L + 1 - g_Y}{d+1-g}\left[dm^2 + (1-g)m\right] \geq \frac{w(\rho)}{r+1} mP(m),$$

which contradicts the numerical criterion for Hilbert stability (see Fact 4.2).

□

7.1. **Closure of orbits.** Given a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$, denote by $\mathrm{Orb}([X \subset \mathbb{P}^r])$ the orbit of $[X \subset \mathbb{P}^r]$ under the action of $\mathrm{SL}(V) = \mathrm{SL}_{r+1}$. Clearly, $\mathrm{Orb}([X \subset \mathbb{P}^r])$ depends only on $X$ and on the line bundle $L := \mathcal{O}_X(1)$ and not on the chosen embedding $X \subset \mathbb{P}^r$.

The aim of this subsection is to investigate the following

**Question 7.2.** *Given two points $[X \subset \mathbb{P}^r]$, $[X' \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ with $X$ and $X'$ connected, when does it hold that*

$$[X' \subset \mathbb{P}^r] \in \overline{\mathrm{Orb}([X \subset \mathbb{P}^r])}?$$

We start by introducing an order relation on the set of pairs $(X, L)$ where $X$ is a quasi-wp-stable curve and $L$ is a properly balanced line bundle on $X$ of degree $d$.

**Definition 7.3.** Let $(X', L')$ and $(X, L)$ be two pairs consisting of a quasi-wp-stable curve together with a properly balanced line bundle of degree $d$ on it.

(i) We say that $(X', L')$ is an *elementary isotrivial specialization* of $(X, L)$, and we write $(X, L) \stackrel{\text{el}}{\rightsquigarrow} (X', L')$, if there exists a proper connected subcurve $Z \subset X'$ with $\deg_Z L' = m_Z$, $Z^c$ connected and $Z \cap Z^c \subseteq X'_{\text{exc}}$ such that $(X, L)$ is obtained from $(X', L')$ by smoothing some nodes of $Z \cap Z^c$, i.e., there exists a smooth pointed curve $(B, b_0)$ and a flat projective morphism $\mathcal{X} \to B$ together with a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{L})_{b_0} \cong (X', L')$ and $(\mathcal{X}, \mathcal{L})_b \cong (X, L)$ for every $b_0 \neq b \in B$.

(ii) We say that $(X', L')$ is an *isotrivial specialization* of $(X, L)$, and we write $(X, L) \rightsquigarrow (X', L')$ if $(X', L')$ is obtained from $(X, L)$ via a sequence of elementary isotrivial specializations.

There is a close relation between the existence of isotrivial specializations and strictly balanced line bundles, as explained in the following

**Lemma 7.4.** *Notations as in Definition 7.3.*

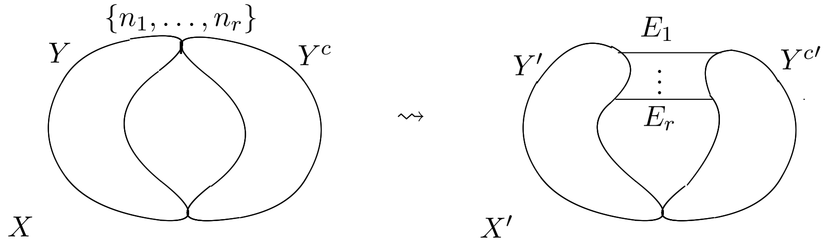
*(i) If $(X, L) \rightsquigarrow (X', L')$ then $L$ is not strictly balanced.*

*(ii) If $L$ is not strictly balanced then there exists an isotrivial specialization $(X, L) \rightsquigarrow (X', L')$ such that $L'$ is strictly balanced.*

*Proof.* Part (i): clearly, it is enough to consider the case where $(X, L) \stackrel{\text{el}}{\rightsquigarrow} (X', L')$ is an elementary isotrivial specialization as in Definition 7.3(i). For $Z \subseteq X'$ as in Definition 7.3(i), decompose $Z^c$ as the union of all the exceptional components $\{E_i\}_{i=1,\cdots,k_Z}$ of $X'$ that meet $Z$ and a subcurve $W$. By applying Remark 3.8(i) to the subcurve $E_1 \cup \cdots \cup E_{k_Z}$, where the basic inequality achieves its maximal value, it is easy to see that $\deg_W L' = m_W$. Let now $\widetilde{W}$ be the subcurve of $X$ given by the union of the irreducible components of $X$ that specialize to an irreducible component of $W \subset X'$. Since $(X, L)$ is obtained from $(X', L')$ by smoothing some nodes which belong to $Z \cap \cup_i E_i$ and therefore are not in $W$, we clearly have that $\widetilde{W} \cong W$, $k_{\widetilde{W}} = k_W$ and $L_{\widetilde{W}} \cong L'_W$. Hence $\deg_{\widetilde{W}} L' = m_{\widetilde{W}}$ and, since $\widetilde{W} \cap \widetilde{W}^c \not\subset X_{\text{exc}}$, we conclude that $L$ is not strictly balanced.

Part (ii): if $L$ is not strictly balanced, we can find a subcurve $Y \subset X$ such that $\deg_Y L = M_Y$ and $Y \cap Y^c \subsetneq X_{\text{exc}}$. Using that $\deg_Y L = M_Y$, or equivalently that $\deg_{Y^c} L = m_{Y^c}$, it is easy to check that if $n \in Y \cap Y^c \cap X_{\text{exc}}$ then there exists a unique exceptional component $E$ of $X$ such that $n \in E \subset Y$.

Let us denote by $\{n_1, \ldots, n_r\}$ the points belonging to $Y \cap Y^c \setminus X_{\text{exc}}$. Let $X'$ be the blow-up of $X$ at $\{n_1, \ldots, n_r\}$ and let $E_Y := E_1 \cup \cdots \cup E_r$ be the new exceptional components of $X'$. Given a subcurve $Z \subseteq X$ denote by $Z'$ the strict transform of $Z$ via the blow-up morphism and define $k_{Z'}^Y := |Z' \cap E_Y \cap Y|$.




Define a multidegree $\underline{d}$ on $X'$ such that $\underline{d}_{E_i} = 1$, for $i = 1, \ldots, r$ and, given an irreducible component $C$ of $X$,

$$\underline{d}_{C'} = \deg_C L - k^Y_{C'}.$$

From [Cap94, Important Remark 5.1.1] we know that there is a flat and proper family $\mathcal{X} \to B$ over a pointed curve $(B, b_0)$ and a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $(\mathcal{X}_b, \mathcal{L}_{|\mathcal{X}_b}) \cong (X, L)$ for $b \neq b_0$ and $(\mathcal{X}_{b_0}, \mathcal{L}_{|\mathcal{X}_{b_0}}) \cong (X', L')$ where $X'$ is the blow-up of $X$ at $\{n_1, \ldots, n_r\}$ and $\underline{\deg} L' = \underline{d}$.

Let us check that $L'$ is properly balanced. It is clear that the degree of $L'$ is equal to one on all the exceptional components of $X'$. Let $W \subseteq X'$ and let us check that $L'$ satisfies the basic inequality (3.1). Start by assuming that $W = Z'$ for some $Z \subseteq Y$. Then we have that

(7.15)

$$\deg_{Z'} L' = \deg_Z L - k^Y_{Z'} = \deg_Y L - \deg_{\overline{Y \setminus Z}} L - k^Y_{Z'} = M_Y - \deg_{\overline{Y \setminus Z}} L - k^Y_{Z'} \geq M_Y - M_{\overline{Y \setminus Z}} - k^Y_{Z'}$$

$$= M_Z - |Z \cap \overline{Y \setminus Z}| - k^Y_{Z'} = M_Z - k_Z + |Z' \cap Y^{c'}| = m_Z + |Z' \cap Y^{c'}|.$$

Suppose now that $W = Z'_{Y^c} \cup Z'_Y \cup E_W$ where $Z_{Y^c} \subseteq Y^c$, $Z_Y \subseteq Y$ and $E_W \subseteq E_Y$. Then, $\deg_W L' = \deg_{Z'_{Y^c}} L' + \deg_{Z'_Y} L' + |E_W|$ and, by (7.15), it follows that

$$\deg_W L' = \deg_{Z_{Y^c}} L + m_Z + |Z'_Y \cap Y^{c'}| + |E_W| \geq$$

$$\frac{d\omega_W}{2g-2} - \frac{k_{Z_{Y^c}}}{2} - \frac{k_{Z_Y}}{2} + |Z'_Y \cap Y^{c'}| + |E_W| = m_W + |E_W| - |E_W \cap Z'_Y \cap Z'_{Y^c}| \geq m_W$$

Analogously, we can show that $\deg_W L' \leq M_W$, so we conclude that $L'$ is properly balanced.

Now, if $L'$ is strictly balanced we are done. If not, we repeat the same procedure and after a finite number of steps we will find the desired pair $(X'', L'')$ with $L''$ strictly balanced.

□

We can now give a partial answer to Question 7.2.

**Theorem 7.5.** *Let* $[X \subset \mathbb{P}^r], [X' \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *and assume that* $X$ *and* $X'$ *are quasi-wp-stable curves and* $\mathcal{O}_X(1)$ *and* $\mathcal{O}_{X'}(1)$ *are properly balanced and non-special. Suppose that* $(X, \mathcal{O}_X(1)) \rightsquigarrow (X', \mathcal{O}_{X'}(1))$. *Then*

*(i)* $[X' \subset \mathbb{P}^r] \in \overline{\mathrm{Orb}([X \subset \mathbb{P}^r])}$.

*(ii)* $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}^{ss}_d)$ *(resp.* $\mathrm{Hilb}^{ss}_d$*) if and only if* $[X' \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}^{ss}_d)$ *(resp.* $\mathrm{Hilb}^{ss}_d$*).*

*Proof.* It is enough, in view of Fact 4.10, to find a 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ that stabilizes $[X' \subset \mathbb{P}^r]$ and such that $\mu([X' \subset \mathbb{P}^r]_m, \rho) \leq 0$ for $m \gg 0$ and $[X \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r])$.

We can clearly assume that $(X, \mathcal{O}_X(1)) \overset{\mathrm{el}}{\rightsquigarrow} (X', \mathcal{O}_{X'}(1))$. Using the notation of Definition 7.3(i), this means that there exists a connected subcurve $Z \subset X'$ with $Z^c$ connected and $Z \cap Z^c \subset X'_{\mathrm{exc}}$ and $\deg_Z L' = m_Z$ such that $(X, \mathcal{O}_X(1))$ is obtained from $(X', \mathcal{O}_{X'}(1))$ by smoothing some of the nodes of $Z \cap Z^c$. Moreover, we can decompose the connected complementary subcurve $Z^c$ as

$$Z^c = \bigcup_{1 \leq i \leq k_Z} E_i \cup W,$$

where the $E_i$'s are the exceptional subcurves of $X'$ that meet the subcurve $Z$ and $W := \overline{Z^c \setminus \cup_i E_i}$ is clearly connected as well. Since $\deg_{E_i} L' = 1$, it follows from Remark 3.8 applied to the subcurve $E_1 \cup \cdots \cup E_{k_Z}$ that $\deg_W L' = m_W$.

The required 1ps $\rho$ of $\mathrm{GL}_{r+1}$ is similar to the 1ps considered in the proof of Theorem 7.1. More precisely, consider the restriction map

$$\mathrm{res} : H^0(X', \mathcal{O}_{X'}(1)) \longrightarrow H^0(Z, \mathcal{O}_Z(1)) \oplus H^0(W, \mathcal{O}_W(1)).$$

The map res is injective since the complementary subcurve of $Z \cup W$ is made of the exceptional components $E_i \cong \mathbb{P}^1$, each of which meets both $Z$ and $W$ in one point. Moreover, since $\mathcal{O}_{X'}(1)$ is non-special by assumption, which implies that also $\mathcal{O}_Z(1)$ and $\mathcal{O}_W(1)$ are non-special, we have that

$$\dim H^0(Z, \mathcal{O}_Z(1)) + \dim H^0(W, \mathcal{O}_W(1)) = \deg_Z \mathcal{O}_{X'}(1) - g_Z + 1 + \deg_W \mathcal{O}_{X'}(1) - g_W + 1 =$$

$$= m_Z - g_Z + 1 + m_W - g_W + 1 = d - g + 1 = \dim H^0(X', \mathcal{O}_{X'}(1)),$$

where we have used that $m_Z + m_W = d - k_Z$ and $g = g_W + g_Z + k_Z - 1$. This implies that res is an isomorphism. Define now the 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ so that

$$\begin{cases} \rho(t)_{|H^0(W, \mathcal{O}_W(1))} = t \cdot \mathrm{Id}, \\ \rho(t)_{|H^0(Z, \mathcal{O}_Z(1))} = \mathrm{Id}. \end{cases}$$

Let us check that the above 1ps $\rho$ satisfies all the desired properties.

<u>CLAIM 1</u>: $\mu([X' \subset \mathbb{P}^r]_m, \rho) \leq 0$ for $m \gg 0$.

This is proved exactly as in Theorem 7.1: see (7.14) and the equation for $\mu([X \subset \mathbb{P}^r]_m, \rho)$ given in Fact (4.2).

<u>CLAIM 2</u>: $\rho$ stabilizes $[X' \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$.

Using Lemma 6.1, it is enough to check that

$$\mathrm{Im}\rho \subseteq \mathrm{Aut}(X', \mathcal{O}_{X'}(1)) \cong \mathrm{Stab}_{\mathrm{GL}_{r+1}}([X' \subset \mathbb{P}^r]) \subseteq \mathrm{GL}_{r+1}.$$

Since the non exceptional subcurve $\widetilde{X'} \subset X'$ is contained in $Z \coprod W$, it follows from the proof of Theorem 6.4 that $\mathrm{Aut}(X', \mathcal{O}_{X'}(1))$ contains a subgroup $H$ isomorphic to $\mathbb{G}_m^2$ and such that $(\lambda, \mu) \in H \cong \mathbb{G}_m^2$ acts via multiplication by $\lambda$ on $H^0(W, \mathcal{O}_W(1))$ and by $\mu$ on $H^0(Z, \mathcal{O}_Z(1))$. By construction, it follows that $\mathrm{Im}\rho \subseteq H$ and we are done.

<u>CLAIM 3</u>: $[X \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r])$.

Recall that, by assumption, $(X, \mathcal{O}_X(1))$ is obtained from $(X', \mathcal{O}_{X'}(1))$ by smoothing some of the nodes of $Z \cap Z^c = \cup_i (Z \cap E_i)$. Denote by $n_i$ the node given by the intersection of $Z$ with $E_i$ and by $\mathrm{Def}_{(X',n_i)}$ the functor of infinitesimal deformations of the complete local ring $\widehat{\mathcal{O}}_{X',n_i}$ (see [Ser06, Sec. 2.4]). According to [Ser06, Cor. 3.1.2, Exa. 3.1.4(a)], if we write $\widehat{\mathcal{O}}_{X',n_i} = k[[u_i, v_i]]/(u_i v_i)$, then $\mathrm{Def}_{(X',n_i)}$ has a semiuniversal ring equal to $k[[a_i]]$ with universal family given by $k[[u_i, v_i, a_i]]/(u_i v_i - a_i)$.

Consider now the local Hilbert functor $H_{X'}^{\mathbb{P}^r}$ parametrizing infinitesimal deformations of $X'$ in $\mathbb{P}^r$ (see [Ser06, Sec. 3.2.1]). Clearly, $H_{X'}^{\mathbb{P}^r}$ is pro-represented by the complete local ring of $\mathrm{Hilb}_d$ at $[X \subset \mathbb{P}^r]$. Since $X'$ is a curve with locally complete intersection singularities and $\mathcal{O}_{X'}(1)$ is non-special, from [Kol96, I.6.10] we get that the natural morphism of functors

$$H_{X'}^{\mathbb{P}^r} \longrightarrow \mathrm{Def}_{X'} \tag{7.16}$$

is formally smooth, where $\mathrm{Def}_{X'}$ is the functor of infinitesimal deformations of $X'$. It follows easily from [Ser06, Thm. 2.4.1], that also the natural morphism of functors

$$\mathrm{Def}_{X'} \longrightarrow \prod_i \mathrm{Def}_{(X',n_i)} \tag{7.17}$$

is formally smooth. Moreover, since $\rho$ stabilizes $[X' \subset \mathbb{P}^r]$ by Claim 2, the above morphisms (7.16) and (7.17) are equivariant under the natural action of $\rho$ on each functor. Therefore, in order to prove that $[X \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r])$, it is enough to prove that $\rho$ acts on each $k[[a_i]]$ with positive weight (compare also with the proof of [HM10, Lemma 4] and of [HH13, Cor. 7.9]).

Fix a node $n_i = E_i \cap Z$ for some $1 \leq i \leq k_Z$. We can choose coordinates $\{x_1, \ldots, x_{r+1}\}$ of $V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = H^0(X', \mathcal{O}_{X'}(1))$ so that $x_i$ is the unique coordinate which does not vanish at $n_i$, the exceptional component $E_i$ is given by the linear span $\langle x_i, x_{i+1} \rangle$ and the tangent $T_{Z,n_i}$ of $Z$ at $n_i$ is given by the linear span $\langle x_{i-1}, x_i \rangle$. Then the completion of the local ring $\mathcal{O}_{X',n_i}$ is equal to $k[[u_i, v_i]]/(u_i v_i)$ where $u_i = x_{i-1}/x_i$ and $v_i = x_{i+1}/x_i$. Since $T_{Z,n_i}$ is contained in the linear span $\langle Z \rangle$ of $Z$ and $\rho(t)_{|H^0(W, \mathcal{O}_W(1))} = \mathrm{Id}$ by construction, we have that $\rho(t) \cdot x_i = x_i$ and $\rho(t) \cdot x_{i-1} = x_{i-1}$; hence $\rho(t) \cdot u_i = u_i$. On the other hand, the point $q_i$ defined by $x_k = 0$ for every $k \neq i+1$ is clearly the node given by the intersection of $E_i$ with $W$. Since $\rho(t)_{|H^0(W, \mathcal{O}_W(1))} = t \cdot \mathrm{Id}$ by construction, we have that $\rho(t) \cdot x_{i+1} = t x_{i+1}$; hence $\rho(t) \cdot v_i = t v_i$. Since the equation of the universal family over $k[[a_i]]$ is given by $u_i v_i - a_i = 0$ and $\rho$ acts on this universal family, we deduce that $\rho(t) \cdot a_i = t a_i$, which concludes our proof.

□

From the above theorem, we deduce now the following

**Corollary 7.6.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $X$ connected and $d > 2(2g-2)$. If $[X \subset \mathbb{P}^r]$ is Chow polystable or Hilbert polystable then $\mathcal{O}_X(1)$ is strictly balanced.*

*Proof.* Let us prove the statement for the Chow polystability; the Hilbert polystability being analogous.

Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ for $d > 2(2g-2)$ with $X$ connected and assume that $[X \subset \mathbb{P}^r]$ is Chow-polystable. Recall that $X$ is quasi-wp-stable by Corollary 5.6(i) and $\mathcal{O}_X(1)$ is properly balanced by Theorem 5.1 and Proposition 5.5. By Lemma 7.4, we can find a pair $(X', L')$ consisting of a quasi-wp-stable curve $X'$ and a strictly balanced line bundle $L'$ on $X'$ such that $(X, \mathcal{O}_X(1)) \rightsquigarrow (X', L')$. Note that $L'$ is ample by Remark 5.7; moreover $X'$ does not have elliptic tails if $d < 5/2(2g-2)$ because otherwise, by the basic inequality (3.1), $L'$ would have degree at most 2 on each elliptic tail, hence it would not be very ample. Therefore, we can apply Theorem 17.5 which allows us to conclude that $L'$ is non-special and very ample; we get a point $[X' \stackrel{|L'|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$. The above Theorem 7.5 gives that $[X' \subset \mathbb{P}^r] \in \overline{\mathrm{Orb}([X \subset \mathbb{P}^r])}$ and $[X' \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$. Since $[X \subset \mathbb{P}^r]$ is Chow polystable, we must have that $[X' \subset \mathbb{P}^r] \in \mathrm{Orb}([X \subset \mathbb{P}^r])$; hence $X' = X$ and $\mathcal{O}_X(1) = \mathcal{O}_{X'}(1) = L'$ is strictly balanced.

□

## 8. A criterion of stability for tails

In this section we would like to state a criterion of stability for tails based on the Hilbert-Mumford criterion and the parabolic group. Let $[X \hookrightarrow \mathbb{P}^r] \in \mathrm{Hilb_d}$ with $d > 2(2g-2)$, where $X$ is the union of two curves $X_1$ and $X_2$ (of degrees $d_1, d_2$ and genus $g_1, g_2$) that intersect each other trasversally in a single point $p$. By the Potential pseudo-stability Theorem 5.1(ii), we can assume that $h^1(X, \mathcal{O}_X(1)) = 0$, which implies that $h^0(X_i, \mathcal{O}_{X_i}(1)) = d_i + 1 - g_i =: r_i + 1$. Hence, denoting by $\langle X_1 \rangle$ and $\langle X_2 \rangle$ respectively the linear spans of $X_1$ and $X_2$, we can find a system of coordinates $\{x_1, \dots, x_{r+1}\}$ such that

$$\langle X_1 \rangle = \bigcap_{i=r_1+2}^{r+1} \{x_i = 0\} \quad \textit{and} \quad \langle X_2 \rangle = \bigcap_{i=1}^{r_1} \{x_i = 0\}. \tag{8.1}$$

Using this type of coordinates to find destabilizing one-parameter subgroups is very convenient because we can study the two subcurves separately, as the results below show.

Let $\rho$ be a 1ps of $\mathrm{GL}_{r+1}$. By Proposition 4.7, we know that $e_{X,\rho} = e_{X_1,\rho} + e_{X_2,\rho}$, but in general we cannot say something similar for the Hilbert weight $W_{X,\rho}(m)$. If $\rho$ is diagonalized by coordinates of type (8.1), we can do it.

**Lemma 8.1.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ as above and let $\rho$ be a 1ps of $\mathrm{GL}_{r+1}$ diagonalized by coordinates of type* (8.1)*. Then*

$$W_{X,\rho}(m) = W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - w_{r_1+1} m. \tag{8.2}$$

*Proof.* Let $m$ be a positive integer and consider a monomial basis $\{B_1, \dots, B_{P_1(m)}\}$ of $H^0(X_1, \mathcal{O}_{X_1}(m))$. Since the point $p = [x_1 = 0, \dots, x_{r_1} = 0, x_{r_1+1} = 1, x_{r_1+2} =$

$0, \dots, x_{r+1} = 0]$ belongs to $X_1$, there exists a monomial (for example $B_{P_1(m)}$) such that $B_{P_1(m)} = x_{r_1+1}^m$. The same holds for each monomial basis $\{B'_1, \dots, B'_{P_2(m)}\}$ of $H^0(X_2, \mathcal{O}_{X_2}(m))$ (for example $B'_{P_2(m)} = x_{r_1+1}^m$). By the exact sequence

$$0 \to H^0(\mathcal{O}_X(m)) \overset{(|_{X_1}, |_{X_2})}{\longrightarrow} H^0(\mathcal{O}_{X_1}(m)) \oplus H^0(\mathcal{O}_{X_2}(m)) \longrightarrow H^0(\mathcal{O}_{X_1 \cap X_2}(m)) \to 0,$$

we obtain that $\{B_1, \dots, B_{P_1(m)}, B'_1, \dots, B'_{P_2(m)-1}\}$ is a monomial basis of $H^0(X, \mathcal{O}_X(m))$. Therefore, if we choose the monomial basis $\{B_1, \dots, B_{P_1(m)}\}$ and $\{B'_1, \dots, B'_{P_2(m)}\}$ so that

$$W_{X_1,\rho}(m) = \sum_{i=1}^{P_1(m)} w_\rho(B_i) \quad \text{and} \quad W_{X_2,\rho}(m) = \sum_{i=1}^{P_2(m)} w_\rho(B'_i),$$

then we get

$$\begin{aligned} W_{X,\rho}(m) &\leq \sum_{i=1}^{P_1(m)} w_\rho(B_i) + \sum_{i=1}^{P_2(m)-1} w_\rho(B'_i) \\ &= \sum_{i=1}^{P_1(m)} w_\rho(B_i) + \sum_{i=1}^{P_2(m)} w_\rho(B'_i) - w_\rho(B'_{P_2(m)}) \\ &= W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - w_{r_1+1} m. \end{aligned}$$

Now, we will prove the reverse inequality. Choose a monomial basis $\{B_1, \dots, B_{P(m)}\}$ of $H^0(X, \mathcal{O}_X(m))$ such that

$$W_{X,\rho}(m) = \sum_{i=1}^{P(m)} w_\rho(B_i).$$

The same argument used to prove the inequality $\geq$ of Proposition 4.7 shows that for each monomial basis $\{B_1, \dots, B_{P(m)}\}$ of $H^0(X, \mathcal{O}_X(m))$, we can reorder the monomials so that

(1) $\{B_1, \dots, B_{P_1(m)}\}$ is a monomial basis of $H^0(X_1, \mathcal{O}_{X_1}(m))$,
(2) $\{B_{P_1(m)}, \dots, B_{P(m)}\}$ is a monomial basis of $H^0(X_2, \mathcal{O}_{X_2}(m))$,
(3) $B_{P_1(m)} = x_{r_1+1}^m$.

We obtain

$$\begin{aligned} W_{X,\rho}(m) &= \sum_{i=1}^{P(m)} w_\rho(B_i) = \sum_{i=1}^{P_1(m)} w_\rho(B_i) + \sum_{i=P_1(m)}^{P(m)} w_\rho(B_i) - w_\rho(B_{P_1(m)}) \\ &\geq W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - w_{r_1+1} m \end{aligned}$$

and we are done. □

Let $I$, $I_1$ and $I_2$ be the ideals of $X$, $X_1$ and $X_2$, respectively. If $\rho$ is diagonalized by coordinates of type (8.1), we can compute easily the flat limit

$$\lim_{t \to 0} \rho(t)[X \subset \mathbb{P}^r]$$

by computing the flat limits of $X_1$ and $X_2$ separately.

**Lemma 8.2.** *Let $X = X_1 \cup X_2 \subset \mathbb{P}^r$ be a connected (possibly not reduced) curve and let $\{x_1, \dots, x_{r+1}\}$ coordinates such that*

$$\{x_i \,|\, r_1 + 2 \leq i \leq r+1\} \subset I_1, \quad \{x_i \,|\, 1 \leq i \leq r_1\} \subset I_2 \quad \text{and} \quad I_1 + I_2 = \langle x_i \,|\, i \neq r_1 + 1\rangle.$$

*Let $\rho$ be a 1ps of $\mathrm{GL}_{r+1}$ diagonalized by $\{x_1, \dots, x_{r+1}\}$ and denote by $\prec$ a $\rho$-weighted lexicographical order in $k[x_1, \dots, x_{r+1}]$ that refines $\prec_\rho$.*

*(i) If $\{f_1, \dots, f_n\} \subset k[x_1, \dots, x_{r_1+1}]$ is a system of generators for $I_1 \cap k[x_1, \dots, x_{r_1+1}]$ and $\{g_1, \dots, g_m\} \subset k[x_{r_1+1}, \dots, x_{r+1}]$ is a system of generators for $I_2 \cap k[x_{r_1+1}, \dots, x_{r+1}]$, then*

$$I = \langle f_1, \dots, f_n, g_1, \dots, g_m, x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1\rangle.$$

*(ii) Moreover, if $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_m\}$ are Gröbner bases with respect to $\prec$, then*

*(a) $\{f_1, \dots, f_n, x_{r_1+2}, \dots, x_{r+1}\}$ and $\{x_1, \dots, x_{r_1}, g_1, \dots, g_m\}$ are Gröbner bases respectively for $I_1$ and $I_2$;*

*(b) $\{f_1, \dots, f_n, g_1, \dots, g_m, x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1\}$ is a Gröbner basis for $I$.*

*(iii) We have that*

$$\mathrm{in}_\prec(I) = \mathrm{in}_\prec(I_1) \cap \mathrm{in}_\prec(I_2) \quad \text{and} \quad \mathrm{in}_{\prec_\rho}(I) = \mathrm{in}_{\prec_\rho}(I_1) \cap \mathrm{in}_{\prec_\rho}(I_2).$$

*Proof.* Let us first prove part (i). Consider $f \in I = I_1 \cap I_2$. Since $f \in I_2$, there exist $p_1, \dots, p_{r_1} \in k[x_1, \dots, x_{r+1}]$ and $q_1, \dots, q_m \in k[x_{r_1+1}, \dots, x_{r+1}]$ such that

$$f = \sum_{i=1}^{r_1} x_i p_i + \sum_{k=1}^{m} q_k g_k.$$

Let $\widetilde{p}_i \in k[x_1, \dots, x_{r_1+1}]$ for $i = 1, \dots, r_1$ such that each monomial of $p_i - \widetilde{p}_i$ contains one among the coordinates $x_{r_1+2}, \dots, x_{r+1}$. Analogously, let $\widetilde{q}_k \in k[x_1, \dots, x_{r_1+1}]$ for $k = 1, \dots, m$ such that each monomial of $q_k - \widetilde{q}_k$ contains one among the coordinates $x_1, \dots, x_{r_1}$. In this way for $i = 1, \dots, r_1$ and $j = r_1+2, \dots, r+1$ there exist polynomials $l_{ij}$ which satisfy

$$f = \sum_{i=1}^{r_1} x_i \widetilde{p}_i + \sum_{i=1}^{r_1} \sum_{j=r_1+2}^{r+1} x_i x_j l_{ij} + \sum_{k=1}^{m} \widetilde{q}_k g_k.$$

Since in each term of the above summation the monomial $x_{r_1+1}^a$ does not appear, we get that

$$\sum_{i=1}^{r_1} \sum_{j=r_1+2}^{r+1} x_i x_j l_{ij} + \sum_{k=1}^{m} \widetilde{q}_k g_k \in I_1.$$

Moreover, since $f \in I_1$ by assumption, we get also that

$$\sum_{i=1}^{r_1} x_i \widetilde{p}_i \in I_1,$$

hence there exist $h_1, \dots, h_n \in k[x_1, \dots, x_{r_1+1}]$ such that

$$\sum_{i=1}^{r_1} x_i \widetilde{p_i} = \sum_{i=1}^{n} h_i f_i.$$

Substituting into the above expression of $f$, we get

$$f = \sum_{i=1}^{n} h_i f_i + \sum_{i=1}^{r_1} \sum_{j=r_1+2}^{r+1} x_i x_j l_{ij} + \sum_{k=1}^{m} \widetilde{q}_k g_k, \tag{8.3}$$

which shows that $\{f_1, \dots, f_n, g_1, \dots, g_m, x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1\}$ is a system of generators for $I$, q.e.d.

Now, suppose that $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_m\}$ are Gröbner bases with respect to $\prec$ and let us prove part (ii). The assertion (a) follows easily from the Buchberger's criterion (see Fact 4.12). In order to prove the assertion (b), consider an element $f \in I_1 \cap I_2$ and write it as in (8.3). By definition the three polynomials

$$F := \sum_{i=1}^{n} h_i f_i, \quad G := \sum_{i=1}^{r_1} \sum_{j=r_1+2}^{r+1} x_i x_j l_{ij} \quad \text{and} \quad H := \sum_{k=1}^{m} \widetilde{q}_k g_k$$

have no common similar monomials, so that

$$\mathrm{in}_\prec(f) = \mathrm{in}_\prec(\mathrm{in}_\prec(F) + \mathrm{in}_\prec(G) + \mathrm{in}_\prec(H)).$$

Obviously $\mathrm{in}_\prec(G) \in \langle x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1 \rangle$. We know that $\{f_1, \dots, f_n\}$ is a Gröbner basis, hence $\mathrm{in}_\prec(F) \in \langle \mathrm{in}_\prec(f_1), \dots, \mathrm{in}_\prec(f_n) \rangle$. Similarly, $\mathrm{in}_\prec(H) \in \langle \mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_n) \rangle$, hence

$$\{f_1, \dots, f_n, g_1, \dots, g_m, x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1\}$$

is a Gröbner basis for $I$, q.e.d.

Let us now prove part (iii). According to (ii)(a), the ideals $\mathrm{in}_\prec(I_1)$ and $\mathrm{in}_\prec(I_2)$ satisfy the hypothesis of (i) with respect to the generators $\{\mathrm{in}_{\prec_\rho}(f_1), \dots, \mathrm{in}_{\prec_\rho}(f_n)\}$ of $\mathrm{in}_\prec(I_1) \cap k[x_1, \dots, x_{r_1+1}]$ and $\{\mathrm{in}_{\prec_\rho}(g_1), \dots, \mathrm{in}_{\prec_\rho}(g_m)\}$ of $\mathrm{in}_\prec(I_2) \cap k[x_{r_1+1}, \dots, x_{r+1}]$. Therefore, part (i) gives that

$$\{\mathrm{in}_{\prec_\rho}(f_1), \dots, \mathrm{in}_{\prec_\rho}(f_n), \mathrm{in}_{\prec_\rho}(g_1), \dots, \mathrm{in}_{\prec_\rho}(g_m), x_i x_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1 + 2 \leq j \leq r+1\}$$

is a system of generators of $\mathrm{in}_\prec(I_1) \cap \mathrm{in}_\prec(I_2)$. However, the above elements generate also $\mathrm{in}_\prec(I)$ by (ii)(b), and the first assertion of part (iii) follows. The second assertion follows in a similar way once we apply Fact 4.15 and (ii) to get

$$\begin{cases} \mathrm{in}_{\prec_\rho}(I_1) = \langle \mathrm{in}_{\prec_\rho}(f_1), \dots, \mathrm{in}_{\prec_\rho}(f_n), x_{r_1+2}, \dots, x_{r+1} \rangle, \\ \mathrm{in}_{\prec_\rho}(I_2) = \langle \mathrm{in}_{\prec_\rho}(g_1), \dots, \mathrm{in}_{\prec_\rho}(g_m), x_1, \dots, x_{r_1} \rangle, \\ \mathrm{in}_{\prec_\rho}(I) = \langle \mathrm{in}_{\prec_\rho}(f_k), \mathrm{in}_{\prec_\rho}(g_l), x_i x_j \,|\, 1 \leq i \leq r_1,\ r_1 + 2 \leq j \leq r+1,\ 1 \leq k \leq n,\ 1 \leq l \leq m \rangle. \end{cases}$$

□

The criterion of stability for tails we are going to explain states that coordinates of type (8.1) diagonalize the one-parameter subgroups that give the "worst" weights.

**Proposition 8.3. (Criterion of stability for tails.)** *Let* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb_d}$ *as above. The following conditions are equivalent:*

*(1)* $[X \subset \mathbb{P}^r]$ *is Hilbert semistable (resp. polystable, stable);*
*(2)* $[X \subset \mathbb{P}^r]$ *is Hilbert semistable (resp. polystable, stable) with respect to any one-parameter subgroup* $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ *diagonalized by coordinates of type (8.1);*
*(3)* $[X \subset \mathbb{P}^r]$ *is Hilbert semistable (resp. polystable, stable) with respect to any one-parameter subgroup* $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ *diagonalized by coordinates of type (8.1) with weights* $w_1, \ldots, w_{r+1}$ *such that*

$$w_1 = w_2 = \ldots = w_{r_1+1} = 0 \quad or \quad w_{r_1+1} = w_{r_1+2} = \ldots = w_{r+1} = 0.$$

*The same holds for the Chow semistability (resp. polystability, stability).*

*Proof.* The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are clear for each type of stability.

Let us now prove the implication (2) $\Longrightarrow$ (1). Let $X = (x_1, \ldots, x_{r+1})^t$ be a basis of coordinates of type (8.1). By Corollary 4.18 applied to $(x_1, \ldots, x_{r_1}, x_{r_1+2}, \ldots, x_{r+1}, x_{r_1+1})$, it is enough to consider a 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ that is diagonalized by the coordinates

$$(z_1, \ldots, z_{r+1})^t = Z = AX \tag{8.4}$$

where
(8.5)

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r_1,1} & a_{r_1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_{r_1+1,1} & a_{r_1+1,2} & \cdots & a_{r_1+1,r_1} & 1 & a_{r_1+1,r_1+2} & a_{r_1+1,r_1+3} & \cdots & a_{r_1+1,r+1} \\ a_{r_1+2,1} & a_{r_1+2,2} & \cdots & a_{r_1+2,r_1} & 0 & 1 & 0 & \cdots & 0 \\ a_{r_1+3,1} & a_{r_1+3,2} & \cdots & a_{r_1+3,r_1} & 0 & a_{r_1+3,r_1+2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \ddots & \vdots \\ a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,r_1} & 0 & a_{r+1,r_1+2} & a_{r+1,r_1+3} & \cdots & 1 \end{pmatrix}.$$

Define the new matrix $A' = (a'_{ij})$ as follows

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } i \leq r_1 + 1 \text{ or } j \geq r_1 + 1 \\ 0 & \text{if } i \geq r_1 + 2 \text{ and } j \leq r_1 \end{cases}$$

so that

(8.6)

$$A' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r_1,1} & a_{r_1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_{r_1+1,1} & a_{r_1+1,2} & \cdots & a_{r_1+1,r_1} & 1 & a_{r_1+1,r_1+2} & a_{r_1+1,r_1+3} & \cdots & a_{r_1+1,r+1} \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_{r_1+3,r_1+2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & a_{r+1,r_1+2} & a_{r+1,r_1+3} & \cdots & 1 \end{pmatrix}.$$

Now, set $(z'_1, \ldots, z'_{r+1})^t =: Z' = A'X$; the coordinates $Z'$ are of type (8.1). Consider the one-parameter subgroup $\rho'$ diagonalized by the coordinates $Z'$ with the same weights of $\rho$ (in particular $w(\rho) = w(\rho')$). Since $z'_i = z_i$ for $i = 1, \ldots, r_1 + 1$, if $\{B_1(Z'), \ldots, B_{P_1(m)}(Z')\}$ is a monomial basis of $H^0(\mathcal{O}_{X_1}(m))$, then $\{B_1(Z), \ldots, B_{P_1(m)}(Z)\}$ is again a monomial basis of $H^0(\mathcal{O}_{X_1}(m))$, hence

$$W_{X_1,\rho}(m) \le W_{X_1,\rho'}(m) \quad \text{and} \quad e_{X_1,\rho} \le e_{X_1,\rho'}. \tag{8.7}$$

Similarly, the set of monomial bases of the subcurve $X_2$ with respect to $Z$ and the one with respect to $Z'$ are the same, so that

$$W_{X_2,\rho}(m) = W_{X_2,\rho'}(m) \quad \text{and} \quad e_{X_2,\rho} = e_{X_2,\rho'}. \tag{8.8}$$

Suppose that $[X \subset \mathbb{P}^r]$ is Chow semistable (resp. stable) with respect to $\rho'$, i.e.

$$e_{X,\rho'} \le \frac{2d}{r+1} w(\rho') \quad \text{(resp. } <\text{)}.$$

Combining the formulas (8.7) and (8.8) with Proposition 4.7, we get

$$e_{X,\rho} = e_{X_1,\rho} + e_{X_2,\rho} \le e_{X_1,\rho'} + e_{X_2,\rho'} = e_{X,\rho'} \le \frac{2d}{r+1} w(\rho') = \frac{2d}{r+1} w(\rho) \quad \text{(resp. } <\text{)}$$

and the implication (2) $\Longrightarrow$ (1) for Chow semistability (resp. stability) follows. We notice that this last step does not work for the Hilbert semistability (resp. stability) because in general

$$W_{X,\rho}(m) \ne W_{X_1,\rho}(m) + W_{X_2,\rho}(m),$$

as Lemma 8.1 shows. But the argument used to prove the part $\le$ of 8.2 can be applied to $\rho$, so that

$$W_{X,\rho}(m) \le W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - w_{r+1}m, \tag{8.9}$$

By (8.7), (8.8), (8.9) and Lemma 8.1 we obtain

$$\begin{aligned} W_{X,\rho}(m) &\le W_{X_1,\rho}(m) + W_{X_2,\rho}(m) - w_{r_1+1}m \\ &\le W_{X_1,\rho'}(m) + W_{X_2,\rho'}(m) - w_{r_1+1}m = W_{X,\rho'}(m). \end{aligned}$$

and the implication (2) $\implies$ (1) for the Hilbert semistability (resp. stability) follows.

Now, we will prove the implication (2) $\implies$ (1) for the Chow polystability (for the Hilbert polystability the argument is analogous using Lemma 8.1 instead of Proposition 4.7). By what proved above, we get that $[X \subset \mathbb{P}^r]$ is Chow semistable. By Corollary 4.18, it is enough to prove that $[X \subset \mathbb{P}^r]$ is Chow polystable with respect a 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ that is diagonalized by coordinates $Z$ as in (8.4). We can assume that

$$e_{X,\rho} = \frac{2d}{r+1}\, w(\rho), \tag{8.10}$$

because, if $e_{X,\rho} > \frac{2d}{r+1}\, w(\rho)$ then there is nothing to prove. As above consider the one-parameter subgroup $\rho'$ with the same weights of $\rho$ and diagonalized by the coordinates $Z' = A'X$, which are of type (8.1). Denoting by $B = (b_{ij})$ the matrix $A(A')^{-1}$, we have that

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ b_{r_1+2,1} & b_{r_1+2,2} & \cdots & b_{r_1+2,r_1} & 0 & 1 & 0 & \cdots & 0 \\ b_{r_1+3,1} & b_{r_1+3,2} & \cdots & b_{r_1+3,r_1} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & \ddots & \vdots \\ b_{r+1,1} & b_{r+1,2} & \cdots & b_{r+1,r_1} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and $Z = BZ'$.

<u>CLAIM</u>: If $r_1 + 2 \leq j \leq r+1$, $1 \leq i \leq r_1$ and $b_{ji} \neq 0$ then $w_j \geq w_i$.

Suppose by contradiction that $w_j < w_i$. Define a one-parameter subgroup $\widetilde{\rho}$ diagonalized by the new coordinates $Y = (y_1, \ldots, y_{r+1})^t$ where

$$y_k = \begin{cases} z'_k & \text{if } k \neq i, \\ \sum_{l=1}^{r_1} b_{jl} z'_l & \text{if } k = i \end{cases} \tag{8.11}$$

with weights

$$\widetilde{w}_k = \begin{cases} w_k & \text{if } k \neq i, \\ w_j & \text{if } k = i. \end{cases} \tag{8.12}$$

Notice that $Y$ is a set of coordinates of type (8.1) and $w(\widetilde{\rho}) < w(\rho)$. Let $B_1, \ldots, B_{P_1(m)}$ be a monomial basis of $H^0(X_1, \mathcal{O}_{X_1}(m))$ with respect to $Y$ such that

$$e_{X_1,\widetilde{\rho}} = \text{n.l.c.}\Big(\sum_{l=1}^{P_1(m)} w_\rho(B_l)\Big),$$

where n.l.c denotes the normalized leading coefficient. The set of coordinates $Y$ is of type (8.1), hence $B_l \in k[y_1, \ldots, y_{r_1+1}]$ for $l = 1, \ldots, P_1(m)$. Notice that $y_{i|X_1} = z_{j|X_1}$ and $y_k = z_k$ for $k = 1, \ldots, i-1, i+1, \ldots, r_1+1$. Therefore, if

$$\{B_1(y_1, \ldots, y_{r_1+1}), \ldots, B_{P_1(m)}(y_1, \ldots, y_{r_1+1})\}$$

is a monomial basis of $H^0(X_1, \mathcal{O}_{X_1}(m))$, then the same holds for

$$\{B_1(z_1, \ldots, z_{i-1}, z_j, z_{i+1} \ldots, z_{r_1+1}), \ldots, B_{P_1(m)}(z_1, \ldots, z_{i-1}, z_j, z_{i+1} \ldots, z_{r_1+1})\};$$

hence

$$e_{X_1,\rho} \leq \text{n.l.c.}\left(\sum_{l=1}^{P_1(m)} w_\rho(B_l)\right) = e_{X_1,\widetilde{\rho}}. \tag{8.13}$$

Moreover, since $y_{l|X_2} = z_{l|X_2}$ for $r_1+1 \leq l \leq r+1$, the monomial bases of $H^0(\mathcal{O}_{X_2}(m))$ with respect to $Y$ and $Z$ are the same, hence

$$e_{X_1,\rho} = e_{X_1,\widetilde{\rho}}. \tag{8.14}$$

Since we already know that $[X \subset \mathbb{P}^r]$ is Chow semistable, Fact 4.3 gives that

$$e_{X,\widetilde{\rho}} \leq \frac{2d}{r+1}\, w(\widetilde{\rho}). \tag{8.15}$$

Combining (8.13), (8.14), (8.15) and Proposition 4.7, we get

$$e_{X,\rho} = e_{X_1,\rho} + e_{X_2,\rho} \leq e_{X_1,\widetilde{\rho}} + e_{X_2,\widetilde{\rho}} = e_{X,\widetilde{\rho}} \leq \frac{2d}{r+1}\, w(\widetilde{\rho}) < \frac{2d}{r+1}\, w(\rho),$$

which contradicts (8.10) and the Claim is proved.

Consider the weighted graded orders $\prec_\rho$, $\prec_{\rho'}$ and two weighted graded lexicographical orders $\prec$ and $\prec'$ that refine respectively $\prec_\rho$ and $\prec_{\rho'}$ and are induced by the lexicographical orders $z_1 < z_2 < \ldots < z_{r+1}$ and $z'_1 < z'_2 < \ldots < z'_{r+1}$. Denote by $I$, $I_1$ and $I_2$ the ideals of $X$, $X_1$ and $X_2$ respectively. Let $f_1, \ldots, f_n \in k[z'_1, \ldots, z'_{r_1+1}]$ and $g_1, \ldots, g_m \in k[z'_{r_1+1}, \ldots, z'_{r+1}]$ such that $\{f_1, \ldots, f_n, z'_{r_1+2}, \ldots, z'_{r+1}\}$ and $\{z'_1, \ldots, z'_{r_1}, g_1, \ldots, g_m\}$ are Gröbner bases respectively of $I_1$ and $I_2$ with respect to $\prec'$. Fact 4.15 implies that

$$\begin{cases} I'_1 := \text{in}_{\prec_{\rho'}}(I_1) = \langle \text{in}_{\prec_{\rho'}}(f_1), \ldots, \text{in}_{\prec_{\rho'}}(f_n), z'_{r_1+2}, \ldots, z'_{r+1}\rangle, \\ I'_2 := \text{in}_{\prec_{\rho'}}(I_2) = \langle z'_1, \ldots, z'_{r_1}, \text{in}_{\prec_{\rho'}}(g_1), \ldots, \text{in}_{\prec_{\rho'}}(g_m)\rangle. \end{cases}$$

Applying Lemma 8.2, we obtain that

$$\{\text{in}_{\prec_{\rho'}}(f_1), \ldots, \text{in}_{\prec_{\rho'}}(f_n), \text{in}_{\prec_{\rho'}}(g_1), \ldots, \text{in}_{\prec_{\rho'}}(g_m), z'_i z'_j \,|\, 1 \leq i \leq r_1 \text{ and } r_1+2 \leq j \leq r+1\}$$

is a system of generators of $I' := \text{in}_{\prec_\rho}(I)$. Denoting by $X' := V(I') \subset \mathbb{P}^r$ and applying Fact 4.13, we get

$$[X' \subset \mathbb{P}^r] = \lim_{t \to 0} \rho'(t)[X \subset \mathbb{P}^r].$$

Now, define

$$[X'' \subset \mathbb{P}^r] = \lim_{t \to 0} \rho(t)[X \subset \mathbb{P}^r]$$

and consider the matrix $B' = (b'_{ji})$ defined as follows:

$$b'_{ji} = \begin{cases} b_{ji} & \text{if } 1 \le j \le r_1+1 \text{ or } r_1+1 \le i \le r+1, \\ b_{ji} & \text{if } r_1+2 \le j \le r+1,\ 1 \le i \le r_1 \text{ and } w_j = w_i, \\ 0 & \text{if } r_1+2 \le j \le r+1,\ 1 \le i \le r_1 \text{ and } w_j > w_i. \end{cases}$$

By the above CLAIM, we have

$$\mathrm{in}_{\prec_\rho}\left(z_k - \sum_{i=1}^{r_1} b_{ki} z_i\right) = z_k - \sum_{i=1}^{r_1} b'_{ki} z_i$$

for $k = r_1+2, \ldots, r+1$. Since $z'_i = z_i$ for $i = 1, \ldots, r_1+1$, we get that $f_i(Z) = f_i(B^{-1}Z)$, hence, by Buchberger's criterion (see Fact 4.12), the system of generators

$$\left\{ f_1(B^{-1}Z), \ldots, f_n(B^{-1}Z), z_{r_1+2} - \sum_{k=1}^{r_1} b_{r_1+2,k} z_k, \ldots, z_{r+1} - \sum_{k=1}^{r_1} b_{r+1,k} z_k \right\}$$

is a Gröbner basis of $I_1$ with respect to $\prec$, so that

$$\mathrm{in}_\prec(I_1) = \langle \mathrm{in}_\prec(f_1(B^{-1}Z)), \ldots, \mathrm{in}_\prec(f_n(B^{-1}Z)), z_{r_1+2} \ldots, z_{r+1} \rangle.$$

Now, consider $I_2$. For each $j = 1, \ldots, m$ there exists $h_j \in k[z_1, \ldots, z_{r+1}]$ such that each of its monomials contains one among the coordinates $z_1, \ldots, z_{r_1}$ and such that it holds

$$g_j(B^{-1}Z) = g_j(Z) + h_j(Z); \tag{8.16}$$

hence

$$\langle z_1, \ldots, z_{r_1}, g_1(B^{-1}Z), \ldots, g_m(B^{-1}Z) \rangle = \langle z_1, \ldots, z_{r_1}, g_1(Z), \ldots, g_m(Z) \rangle.$$

Applying $\mathrm{in}_\prec$ to (8.16) we obtain $\mathrm{in}_\prec(g_j(B^{-1}Z)) = \mathrm{in}_\prec(g_j(Z))$; hence

$$\begin{aligned} &\mathrm{in}_\prec(\langle z_1, \ldots, z_{r_1}, g_1(B^{-1}Z), \ldots, g_m(B^{-1}Z) \rangle) = \mathrm{in}_\prec(\langle z_1, \ldots, z_{r_1}, g_1(Z), \ldots, g_m(Z) \rangle) \\ &= \langle z_1, \ldots, z_{r_1}, \mathrm{in}_\prec(g_1(Z)), \ldots, \mathrm{in}_\prec(g_m(Z)) \rangle \\ &= \langle z_1, \ldots, z_{r_1}, \mathrm{in}_\prec(g_1(B^{-1}Z)), \ldots, \mathrm{in}_\prec(g_m(B^{-1}Z)) \rangle. \end{aligned}$$

By definition $\{z_1, \ldots, z_{r_1}, g_1(B^{-1}Z), \ldots, g_m(B^{-1}Z)\}$ is a Gröbner basis of $I_2$ with respect to $\prec$. We notice that $\mathrm{in}_\prec(I) \subset \mathrm{in}_\prec(I_1) \cap \mathrm{in}_\prec(I_2)$. Applying Lemma 8.2 to the ideals $\mathrm{in}_\prec(I_1)$ and $\mathrm{in}_\prec(I_2)$ we deduce that

$$\{\mathrm{in}_\prec(f_1(B^{-1}Z)), \ldots, \mathrm{in}_\prec(f_n(B^{-1}Z)), \mathrm{in}_\prec(g_1(B^{-1}Z)), \ldots, \mathrm{in}_\prec(g_m(B^{-1}Z)), z_i z_j\}$$

generate $\mathrm{in}_\prec(I_1) \cap \mathrm{in}_\prec(I_2)$ for $1 \le i \le r_1, r_1+2 \le j \le r+1$; hence

$$\left\{ f_1(B^{-1}Z), \ldots, f_n(B^{-1}Z), g_1(B^{-1}Z), \ldots, g_m(B^{-1}Z), z_i \left( z_j - \sum_{k=1}^{r_1} b_{jk} z_k \right) \right\}$$

for $1 \le i \le r_1, r_1+2 \le j \le r+1$ is a Gröbner basis for $I$ with respect to $\prec$. By Fact 4.15, we obtain that

$$\left\{ \mathrm{in}_{\prec_\rho}(f_h(B^{-1}Z)), \mathrm{in}_{\prec_\rho}(g_l(B^{-1}Z)), z_i \left( z_j - \sum_{k=1}^{r_1} b'_{jk} z_k \right) \right\}$$

generate $\mathrm{in}_{\prec_\rho}(I)$ for $1 \le i \le r_1$, $r_1+2 \le j \le r+1$, $1 \le h \le n$ and $1 \le l \le m$. Let $M_i$ be the monomials in $Z'$ such that

$$g_j = \sum M_i$$

and the sum is not redundant. Denoting by $\widetilde{w} = \max_i\{w_{\rho'}(M_i)\}$ we have that

$$\begin{aligned} \mathrm{in}_{\prec_\rho}(g_j(B^{-1}Z)) &= \mathrm{in}_{\prec_\rho}\left(\sum M_i(B^{-1}Z)\right) \\ &= \sum_{i \,|\, w_{\rho'}(M_i)=\widetilde{w}} \mathrm{in}_{\prec_\rho}(M_i(B^{-1}Z)) \\ &= \sum_{i \,|\, w_{\rho'}(M_i)=\widetilde{w}} M_i((B')^{-1}Z)) \\ &= \mathrm{in}_{\prec_{\rho'}}(g_j)((B')^{-1}Z) \end{aligned} \tag{8.17}$$

Moreover, as we said before, $B$ and $B'$ do not change the first $r_1+1$ coordinates, hence

$$\mathrm{in}_{\prec_{\rho'}}(f)((B')^{-1}Z) = \mathrm{in}_{\prec_\rho}(f((B)^{-1}Z)). \tag{8.18}$$

Combining (8.17) and (8.18), we deduce that $[X'' \subset \mathbb{P}^r] \in \mathrm{Orb}([X' \subset \mathbb{P}^r])$. By our hypothesis, $[X \subset \mathbb{P}^r]$ is Chow polystable with respect to $\rho'$; thus there exists $C \in \mathrm{GL}_{r+1}$ such that

$$[X' \subset \mathbb{P}^r] = C[X \subset \mathbb{P}^r] \tag{8.19}$$

We deduce that $[X'' \subset \mathbb{P}^r] \in \mathrm{Orb}([X \subset \mathbb{P}^r])$ and we are done.

Let us finally prove the implication (3) $\Longrightarrow$ (2). Consider $\rho'$ as above (which we will rename $\rho$). Up to translating the weights, we can assume that $w_{r_1+1} = 0$. Define $\rho_1$ and $\rho_2$ with weights respectively $w_1^1, \dots, w_{r+1}^1$ and $w_1^2, \dots, w_{r+1}^2$ so that

$$w_i^1 = \begin{cases} w_i & \text{if } i \le r_1 \\ 0 & \text{if } i \ge r_1+1 \end{cases} \quad \text{and} \quad w_i^2 = \begin{cases} 0 & \text{if } i \le r_1 \\ w_i & \text{if } i \ge r_1+1 \end{cases} \tag{8.20}$$

so that $w_i^1 + w_i^2 = w_i$ for all $i$ and $w(\rho_1) + w(\rho_2) = w(\rho)$. Now, notice that

$$W_{X_1,\rho}(m) = W_{X,\rho_1}(m), \quad W_{X_2,\rho}(m) = W_{X,\rho_2}(m), \quad e_{X_1,\rho} = e_{X,\rho_1} \quad \text{and} \quad e_{X_2,\rho} = e_{X,\rho_2}. \tag{8.21}$$

If $[X \subset \mathbb{P}^r]$ is Chow semistable (resp. stable) with respect to $\rho_1$ and $\rho_2$, i.e.

$$e_{X,\rho_1} \le \frac{2d}{r+1} w(\rho_1) \quad (\text{resp. } <) \quad \text{and} \quad e_{X,\rho_2} \le \frac{2d}{r+1} w(\rho_2) \quad (\text{resp. } <)$$

then, applying Proposition 4.7 and (8.21), we get that

$$e_{X,\rho} = e_{X_1,\rho} + e_{X_2,\rho} = e_{X,\rho_1} + e_{X,\rho_2} \le \frac{2d}{r+1} w(\rho_1) + \frac{2d}{r+1} w(\rho_2) = \frac{2d}{r+1} w(\rho) \quad (\text{resp. } <),$$

or, in other words, that $[X \subset \mathbb{P}^r]$ is Chow semistable (resp. stable) with respect to $\rho$. The same argument goes through for the Hilbert semistability (resp. stability) replacing Proposition 4.7 with Lemma 8.1.

It remains to prove the implication (3) $\Longrightarrow$ (2) for the polystability. Suppose that there exist matrices $A_1, A_2 \in \mathrm{GL}_{r+1}$ such that

$$\lim_{t\to 0}\rho_1(t)[X\subset \mathbb{P}^r] = A_1[X\subset \mathbb{P}^r] \quad \text{and} \quad \lim_{t\to 0}\rho_2(t)[X\subset \mathbb{P}^r] = A_2[X\subset \mathbb{P}^r]$$

By Lemma 8.2

$$\lim_{t\to 0}\rho(t)[X\subset \mathbb{P}^r] = \lim_{t\to 0}\rho_2(t)\Big(\lim_{t\to 0}\rho_1(t)[X\subset \mathbb{P}^r]\Big).$$

Moreover, each point of $X_2$ is fixed by $A_1$, hence $\rho_2(t)A_1 = A_1\rho_2(t)$. We deduce that

$$\begin{aligned}\lim_{t\to 0}\rho(t)[X\subset \mathbb{P}^r] &= \lim_{t\to 0}\rho_2(t)\Big(\lim_{t\to 0}\rho_1(t)[X\subset \mathbb{P}^r]\Big) = \lim_{t\to 0}\rho_2(t)(A_1[X\subset \mathbb{P}^r])\\ &= A_1\lim_{t\to 0}\rho_2(t)[X\subset \mathbb{P}^r] = A_1A_2[X\subset \mathbb{P}^r]\end{aligned}$$

and we are done. ☐

The proof of this criterion suggests us an important remark. First we need a definition.

**Definition 8.4.** Let $X$ be a quasi wp-stable curve such that $X = X_1 \cup X_2$, $k_{X_1} = 1$, and denote by $p$ the intersection point of $X_1$ and $X_2$. Let $(X_1', q)$ be a pointed curve, where $q$ is a smooth point, and define the new curve $X'$ by gluing $X_1'$ and $X_2$ so that $p$ is identified with $q$ and represents a separating node of the new curve. We say that $X'$ **is obtained from** $X$ **by replacing** $X_1$ **with** $(X_1', q)$. Since $\mathrm{Pic}(X) = \mathrm{Pic}(X_1)\times \mathrm{Pic}(X_2)$, it makes sense to introduce another definition. Let $(X, L)$ and $(X_1', q, L_1')$ be a couple and a triple where $X$, $X'$ and $q$ are as above, $L \in \mathrm{Pic}(X)$ and $L_1' \in \mathrm{Pic}(X_1')$. Consider the new curve $X'$ as above and the line bundle $L'$ defined as follows:

$$L' := (L_1, L_{|X_2}) \in \mathrm{Pic}(X_1') \times \mathrm{Pic}(X_2) = \mathrm{Pic}(X').$$

We say that the couple $(X', L')$ **is obtained from** $(X, L)$ **by replacing** $X_1$ **with** $(X_1', q, L_1')$. If $L$ is very ample we will identify $(X, L)$ and $[X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^{\dim |L|}]$.

**Remark 8.5.** Let $X$, $\rho_1$ and $\rho_2$ be as in the proof of the implication (3) $\Longrightarrow$ (2) in Proposition 8.3 and denote by $L := \mathcal{O}_X(1)$ (we keep the same notation). Suppose that the system of coordinates $\{z_1, \dots, z_{r+1}\}$ is of type (8.1) and diagonalizes $\rho_1$ and $\rho_2$. By formulas (8.21), $W_{X,\rho_1}(m)$ (hence $e_{X,\rho_1}$) depends only on the curve $X_1$ and the embedding $L_1 := L_{|X_1}$ in $\cap_{i=r_1+2}^{r+1}\{x_i = 0\}$. In other words, if we replace $X_2$ with another curve $X_2'$ so that the embedding $L_1$ in $\cap_{i=r_1+2}^{r+1}\{x_i = 0\}$ is the same, then $W_{X,\rho_1}(m)$ does not change.

We can use this remark in order to prove a useful result.

**Corollary 8.6.** *Let* $[X_1 \subset \mathbb{P}^{r_1}] \in \mathrm{Hilb}_{d_1,g_1}$, $[X_2 \subset \mathbb{P}^{r_2}] \in \mathrm{Hilb}_{d_2,g_2}$ *and* $[X_3 \subset \mathbb{P}^{r_3}] \in \mathrm{Hilb}_{d_3,g_3}$ *such that* $X = C_i \cup D_i$ *and* $k_{C_i} = 1$ *for* $i = 1, 2, 3$. *Denote by*

$$\{p_i\} = C_i \cap D_i \quad , \quad L_i = \mathcal{O}_{X_i}(1) \quad \textit{and} \quad \nu_i = \frac{d_i}{2g_i - 2}$$

*for $i = 1, 2, 3$. Suppose that $\nu_1 = \nu_2 = \nu_3$ and $(X_3, L_3)$ is obtained from $(X_1, L_1)$ by replacing $D_1$ with $(D_2, p_2, L_{2|D_2})$. If $[X_1 \subset \mathbb{P}^{r_1}]$ and $[X_2 \subset \mathbb{P}^{r_2}]$ are Chow semistable (resp. polystable, stable) then $[X_3 \subset \mathbb{P}^{r_3}]$ is Chow semistable (resp. polystable, stable). The same holds for the Hilbert semistability (resp. polystability, stability).*

*Proof.* We will first prove the statement for the Chow (semi-, poly-) stability; the case of Hilbert (semi-, poly-) stability is completely analogous. Denoting by $s_i = h^0(C_i, L_{i|C_i}) - 1$ for $i = 1, 2, 3$, we have that $s_1 = s_3$. By Proposition 8.3, it suffices to consider one-parameter subgroups $\rho_1$ and $\rho_2$ diagonalized by coordinates $(x_1, \ldots, x_{r_3+1})$ of $\mathbb{P}^{r_3}$ such that

$$\langle C_3 \subset \mathbb{P}^{r_3} \rangle = \bigcap_{i=s_3+2}^{r_3+1} \{x_i = 0\} \quad \text{and} \quad \langle D_3 \subset \mathbb{P}^{r_3} \rangle = \bigcap_{i=1}^{s_3} \{x_i = 0\}$$

with weights

$$\rho_1(t) \cdot x_i = \begin{cases} t^{w_i} x_i & \text{if } i \leq s_3 \\ x_i & \text{if } i \geq s_3 + 1 \end{cases} \quad \text{and} \quad \rho_2(t) \cdot x_i = \begin{cases} x_i & \text{if } i \leq s_3 + 1 \\ t^{w_i} x_i & \text{if } i \geq s_3 + 2 \end{cases}$$

By hypothesis $C_1 = C_3$ and $L_{|C_1} = L_{|C_3}$, hence we can find coordinates $(x'_1, \ldots, x'_{r_1+1})$ in $\mathbb{P}^{r_1}$ such that for each $i = 1, \ldots, s_1 + 1$

$$x'_{i|C_1} = x_{i|C_3}$$

where we identify $\langle C_1 \subset \mathbb{P}^{r_1} \rangle$ with $\langle C_3 \subset \mathbb{P}^{r_3} \rangle$. If $\rho'_1$ is a one-parameter subgroup diagonalized by $(x'_1, \ldots, x'_{r_1+1})$ with weights

$$w'_i = \begin{cases} w_i & \text{if } i \leq r_1 \\ 0 & \text{if } i \geq r_1 + 1 \end{cases}$$

then $w(\rho) = w(\rho')$ and

$$e_{X_1, \rho'_1} = e_{C_1, \rho'_1} = e_{C_3, \rho_1} = e_{X_3, \rho_1},$$

since the embeddings $C_1 \hookrightarrow \langle C_1 \subset \mathbb{P}^{r_1} \rangle$ and $C_3 \hookrightarrow \langle C_3 \subset \mathbb{P}^{r_3} \rangle$ are the same (see Remark 8.5). Notice that the equalities $\nu_1 = \nu_2 = \nu_3$ are equivalent to

$$\frac{d_1}{r_1 + 1} = \frac{d_2}{r_2 + 1} = \frac{d_3}{r_3 + 1}.$$

Since $[X_1 \subset \mathbb{P}^{r_1}]$ is Chow semistable (resp. stable) by assumption, the Hilbert-Mumford numerical criterion (Fact 4.3) gives that

$$e_{X_1, \rho'_1} \leq \frac{2d_1}{r_1 + 1} w(\rho'_1) \quad \text{(resp. } <\text{)}.$$

Combining this inequality with the previous relations, we obtain

$$e_{X_3, \rho_1} = e_{X_1, \rho'_1} \leq \frac{2d_1}{r_1 + 1} w(\rho'_1) = \frac{2d_3}{r_3 + 1} w(\rho_1) \quad \text{(resp. } <\text{)},$$

i.e. that $[X_3 \subset \mathbb{P}^{r_3}]$ is Chow semistable (resp. stable) with respect to $\rho_1$. Analogously, using that $[X_2 \subset \mathbb{P}^{r_2}]$ is Chow semistable (resp. stable), we obtain that $[X_3 \subset \mathbb{P}^{r_3}]$ is Chow semistable (resp. stable) with respect to $\rho_2$.

It remains to prove the polystability. Suppose that $[X_3 \subset \mathbb{P}^{r_3}]$ is strictly Chow semistable, so that there exists a one-parameter subgroup, for example $\rho_1$ as above, such that

$$e_{X_3,\rho_1} = \frac{2d_3}{r_3+1}\, w(\rho_1)$$

and consider the one-parameter subgroup $\rho_1'$ as above. Let $I_1$ and $I_3$ be the ideals of $C_1$ and $C_3$ respectively in $\mathbb{P}^{r_1}$ and $\mathbb{P}^{r_3}$. It is easy to check that $I_1 \cap k[x_1,\dots,x_{s_1+1}] = I_3 \cap k[x_1,\dots,x_{s_3+1}]$, hence $\mathrm{in}_{\prec_{\rho_1'}}(I_1) = \mathrm{in}_{\prec_{\rho_1}}(I_3)$. Since $[X_1 \subset \mathbb{P}^{r_1}]$ is Chow polystable by hypothesis, there exists $A = (a_{ij}) \in \mathrm{GL}_{r_1+1}$ such that

$$\lim_{t\to 0} \rho_1'(t)[X_1 \subset \mathbb{P}^{r_1}] = A[X_1 \subset \mathbb{P}^{r_1}].$$

Define the matrix $A' = (a'_{ij}) \in \mathrm{GL}_{r_3+1}$ as follows:

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i \le s_1 \text{ and } 1 \le j \le s_1, \\ 1 & \text{if } (s_1+1 \le i \le r_3+1 \text{ or } s_1+1 \le j \le r_3+1) \text{ and } i = j, \\ 0 & \text{if } (s_1+1 \le i \le r_3+1 \text{ or } s_1+1 \le j \le r_3+1) \text{ and } i \neq j. \end{cases}$$

Now, we notice that $\rho_1$ fixes each point of $D_3 \subset \mathbb{P}^{r_3}$. Moreover, the actions of $\rho_1$ and $\rho_1'$ on $C_3 \subset \mathbb{P}^{r_3}$ coincide, hence by Lemma 8.2 and Corollary 4.14 we get

$$\lim_{t\to 0} \rho_1(t)[X_3 \subset \mathbb{P}^{r_3}] = A'[X_3 \subset \mathbb{P}^{r_1}].$$

This implies that $[X_3 \subset \mathbb{P}^{r_3}]$ is Chow polystable with respect to $\rho_1$. Analogously, using that $[X_2 \subset \mathbb{P}^{r_2}]$ is Chow polystable, we obtain that $[X_3 \subset \mathbb{P}^{r_3}]$ is Chow polystable with respect to $\rho_2$, and we are done.

□

## 9. Elliptic tails and Tacnodes with a line

According to Corollary 5.6(i), if $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ is Chow semistable with $X$ connected and $2(2g-2) < d \le 4(2g-2)$, then $X$ is quasi-wp-stable. The aim of this section is to investigate whether $X$ can have tacnodes with a line or elliptic tails.

We begin this section by recalling the relation between the presence of cusps and the presence of special elliptic tails (in the sense of Definition 6.2). We refer to [HM10].

**Theorem 9.1.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $X$ connected and $2(2g-2) < d$. Assume that one of the following two conditions is satisfied*

*(i) $d < 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$;*

*(ii) $d = 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$.*

*Then $X$ does not have special elliptic tails.*

*Proof.* According to the hypothesis on $[X \subset \mathbb{P}^r]$, we get that $X$ is quasi-wp-stable by Corollary 5.6(ii) and that $L := \mathcal{O}_X(1)$ is very ample, non-special and balanced of degree $d$ by the Potential pseudo-stability Theorem 5.1.

Suppose that $X$ has a special elliptic tail, i.e., $X = F \cup C$ where $F \subseteq X$ is an irreducible subcurve of arithmetic genus 1, $C \subseteq X$ is a connected subcurve of arithmetic genus $g-1$ and $p := F \cap C$ is a nodal point of $X$ which is a smooth point of both $F$ and $C$, and $L_{|F} = \mathcal{O}_F(\nu p)$ for some $\nu \in \mathbb{N}$. We want to show, by contradiction, that $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ (resp. $\mathrm{Hilb}_d^{ss}$) if (i) (resp. (ii)) holds. Since $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ and $\mathrm{Hilb}_d^{ss}$ are open in $\mathrm{Hilb}_d$, we can assume that $F$ is a smooth elliptic tail.

Note that the basic inequality (3.1) applied to $F$ gives that $\nu := \deg L_{|F} \leq 4$ if (i) holds and $\nu = 4$ if (ii) holds.

Consider the linear spans $V_F := \langle F \rangle$ of $F$ and $V_C := \langle C \rangle$ of $C$ on $\mathbb{P}^r = \mathbb{P}(V)$. It follows from Riemann-Roch theorem, using that $L$ (hence $L_{|C}$ and $L_{|F}$) is non-special, that $V_F$ has dimension $\nu-1$ and $V_C$ has dimension $d-\nu-(g-1) = r-\nu+1$. Therefore, we can choose a system of coordinates $\{x_1, \ldots, x_{r+1}\}$ of type (8.1), i.e. such that

$$V_F = \bigcap_{i=\nu+1}^{r+1} \{x_i = 0\} \quad \text{and} \quad V_C = \bigcap_{i=1}^{\nu-1} \{x_i = 0\}. \tag{9.1}$$

Hence $p$ is the point where all the $x_i$'s vanish except $x_\nu$. For $1 \leq i \leq \nu$, we will identify $x_i$ with the section of $H^0(F, L_{|F})$ it determines and we will denote by $\mathrm{ord}_p(x_i)$ the order of vanishing of $x_i$ at $p$. By Riemann-Roch theorem applied to the line bundles $L_{|F}(-ip) = \mathcal{O}_F((\nu-i)p)$ for $i = 0, \ldots, \nu$, we may choose the first $\nu$ coordinates $\{x_1, \ldots, x_\nu\}$ of $V$ so that

$$\mathrm{ord}_p(x_i) = \begin{cases} \nu & \text{if } i = 1, \\ \nu - i & \text{if } 2 \leq i \leq \nu. \end{cases} \tag{9.2}$$

Consider the one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ which, in the above coordinates, has the diagonal form $\rho(t) \cdot x_i = t^{w_i} x_i$ for $i = 1, \ldots, r+1$, with weights $w_i$ such that

$$\begin{cases} w_1 = w_\rho(x_1) = 0 \\ w_i = w_\rho(x_i) = i & \text{for } 2 \leq i \leq \nu, \\ w_j = w_\rho(x_j) = \nu & \text{for } j \geq \nu + 1. \end{cases} \tag{9.3}$$

The proof of [HM10, Lemma 1] extends verbatim to our case and gives that

$$W_{X,\rho}(m) = m^2 \left[ d\nu - \frac{\nu^2}{2} \right] + m \left[ \frac{3\nu}{2} - g\nu \right] - 1 \text{ for any } m \geq 2.$$

In particular, the normalized leading coefficient of $W_{X,\rho}$ is equal to

$$e_{X,\rho} = 2d\nu - \nu^2. \tag{9.4}$$

From (9.3), it is easy to compute that the total weight of $\rho$ is equal to

$$w(\rho) = \sum_{i=2}^{\nu} i + \nu(r+1-\nu) = \nu(r+1) + \frac{-\nu^2 + \nu - 2}{2}. \tag{9.5}$$

If (i) holds, i.e. if $v := \frac{d}{2g-2} < 4$, combining (9.4), (9.5) and the fact that $r = d - g$, we get that

$$2d\frac{w(\rho)}{r+1} = 2d\nu + \frac{2v}{2v-1}(-\nu^2+\nu-2) < 2d\nu + \frac{8}{7}(-\nu^2+\nu-2) = 2d\nu - \nu^2 - \frac{(\nu-4)^2}{7} \leq e_{X,\rho}.$$

This implies that $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ by Fact 4.3.

On the other hand, if (ii) holds, i.e. if $v := \frac{d}{2g-2} = 4$ (hence $\nu = 4$), then

$$W_{X,\rho}(m) = (4d-8)m^2+(6-4g)m-1 > (4d-8)m^2+(5-4g)m = \frac{w(\rho)}{r+1}mP(m)\text{for } m \gg 0.$$

This implies that $[X \subset \mathbb{P}^r] \notin \mathrm{Hilb}_d^{ss}$ by Fact 4.2.

□

As a corollary, if $d = 4(2g-2)$ and $X \subset \mathbb{P}^r$ admits a special elliptic tail, then $[X \subset \mathbb{P}^r]$ is strictly Chow semistable with respect to the 1ps $\rho$ as in (9.3). It would be interesting to find the equations of $F$ in its linear span $\langle F \rangle$ in order to determine the flat limit

$$[X_0 \subset \mathbb{P}^r] = \lim_{t \to 0} \rho(t)[X \subset \mathbb{P}^r]$$

using Corollary 4.14. This is not very difficult to do (we leave it to the reader as an exercise) and one obtains that $X_0$ is given by the union of $C$ and a special cuspidal elliptic tail, which we denote by $F_0$. Here we do not use this fact, we consider directly $[X_0 \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ where $X_0 = F_0 \cup C$ and $F_0$ is cuspidal and special. Using the same system of coordinates $\{x_1, \ldots, x_{r+1}\}$ in $\mathbb{P}^r$ as in Theorem 9.1, we can parameterize $F_0$ by

$$[s,t] \in \mathbb{P}^1 \mapsto [s^4, s^2t^2, st^3, t^4, 0, \ldots, 0],$$

so that $F$ is special since $\mathrm{ord}_p(x_1) = 4$, the cusp $q$ is the point $[1,0,0,\ldots,0]$ and $\rho$ stabilizes $[X_0 \subset \mathbb{P}^r]$. We are ready to show explictly the relation between the cusps and the special elliptic tails we outlined at the beginning of this section by studying $A_\rho([X_0 \subset \mathbb{P}^r])$ and $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$ (see [HM10, Lemma 4]).

**Theorem 9.2.** *Let $[X_0 \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ as above and let $\rho$ as in* (9.3)*. Then $A_\rho([X_0 \subset \mathbb{P}^r])$ (resp. $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$) contains smoothings of the cusp (resp. separating node), but not smoothings of the separating node (resp. cusp).*

*Proof.* We use the same tecniques used to prove CLAIM 3 in the proof of Theorem 7.5. We have that the tangent space $T_{X,q}$ is given by $\langle x_2, x_3 \rangle$, so that the completion of the local ring $\mathcal{O}_{X,q}$ is given by $k[[u,v]]/(u^2 - v^3)$ where $u = x_3/x_1$ and $v = x_2/x_1$. Since $\rho(t) \cdot x_1 = x_1$, $\rho(t) \cdot x_2 = t^2 x_2$ and $\rho(t) \cdot x_3 = t^3 x_3$, we obtain that $\rho(t) \cdot u = t^3 u$ and $\rho(t) \cdot v = t^2 v$. We recall that $\mathrm{Def}_{(X,q)}$ has a semiuniversal ring equal to $k[[a,b]]$ with universal family $k[[u,v,a,b]]/(u^2 - v^3 - av - b)$. This implies that $\rho(t) \cdot a = t^4 a$ and $\rho(t) \cdot b = t^6 b$, so that $A_\rho([X_0 \subset \mathbb{P}^r])$ contains smoothings of $q$. If we consider the action of $\rho$ on the universal family of the separating node $p$, we obtain non positive weights, hence $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$ does not contain smoothings of $p$. The converse result holds if we consider $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$. □

The original result of this section is that there is a similar relation between tacnodes with a line and non-special elliptic tails. Concerning the presence of tacnodes with a line, we recall the following result.

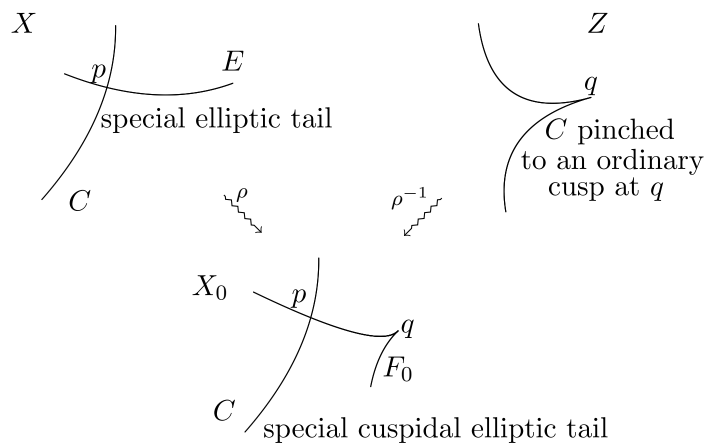



FIGURE 5. The basin of attraction of a curve $X_0$ with a special cuspidal elliptic tail $F_0$.

**Theorem 9.3.** *If* $\frac{7}{2}(2g-2) < d$ *and* $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ *with* $X$ *connected, then* $X$ *does not have tacnodes with a line.*

*Proof.* This follows from [Gie82, Prop. 1.0.6, Case 2]; however, for the reader's convenience and also because we will need it later, we give a sketch of the proof.

Using the hypothesis on $[X \subset \mathbb{P}^r]$, we get that $X$ is quasi-wp-stable by Corollary 5.6(ii) and $L := \mathcal{O}_X(1)$ is very ample, non-special and balanced of degree $d$ by the Potential pseudo-stability Theorem 5.1. Suppose that $X$ has a tacnode with a line, i.e. we can write $X = Y \cup E$ with $E \cong \mathbb{P}^1$, $\{p\} = E \cap Y$ is a tacnode of $X$ and $\deg L_{|E} = 1$. We want to show, by contradiction, that $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ if $\frac{7}{2}(2g-2) < d$.

Since $E$ and $Y$ are tangent in $p \in E \cap Y$, we can choose coordinates $\{x_1, \ldots, x_{r+1}\}$ of $H^0(X, L)$ so that

$$\begin{cases} \mathrm{ord}_p(x_{i|E}) \geq 2 \text{ and } \mathrm{ord}_p(x_{i|Y}) \geq 2 & \text{for any } 1 \leq i \leq r-1, \\ \mathrm{ord}_p(x_{r|E}) = \mathrm{ord}_p(x_{r|Y}) = 1, \\ \mathrm{ord}_p(x_{r+1|E}) = \mathrm{ord}_p(x_{r+1|Y}) = 0, \end{cases}$$

where $x_{i|E}$ (resp. $x_{i|Y}$) denotes the image of $x_i \in H^0(X, L)$ via the restriction map $H^0(X, L) \to H^0(E, L_{|E})$ (resp. $H^0(X, L) \to H^0(Y, L_{|Y})$), and $\mathrm{ord}_p$ denotes the order of vanishing of a section at the point $p$ (considered as a smooth point of $E$ and of $Y$).

Consider now the one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ which, in the above coordinates, has the diagonal form $\rho(t) \cdot x_i = t^{w_i} x_i$ for $i = 1, \ldots, r+1$, with weights $w_i$ such that

$$\begin{cases} w_i = w_\rho(x_i) = 0 & \text{for } 1 \leq i \leq r-1, \\ w_r = w_\rho(x_r) = 1, \\ w_{r+1} = w_\rho(x_{r+1}) = 2. \end{cases} \tag{9.6}$$

Clearly, the total weight of $\rho$ is equal to $w(\rho) = 3$. The proof of [Gie82, Prop. 1.0.6, Case 2] gives that

$$e_{X,\rho} \geq 7. \tag{9.7}$$

Therefore, if $\frac{7}{2} < v := \frac{d}{2g-2}$ then we have that

$$2w(\rho)\frac{d}{r+1} = 6\frac{2v}{2v-1} < 6 \cdot \frac{7}{6} \leq e_{X,\rho}.$$

which implies that $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ by Fact 4.3. $\square$

Combining Corollary 5.6(i) with Theorem 9.3 and Theorem 9.1, we get the following

**Corollary 9.4.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $X$ connected and $2(2g-2) < d$. Assume that one of the following two conditions is satisfied*

*(i) $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$;*
*(ii) $d = 4(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$.*

*Then $X$ is a quasi-wp-stable curve without tacnodes nor special elliptic tails.*

We turn now our attention to the stability of generic elliptic tails (hence also non-special ones).

**Remark 9.5.** Suppose that $[X \subset \mathbb{P}^r]$ has an elliptic tail, i.e. we can write $X = Y \cup F$ where $F \subseteq X$ is a connected subcurve of arithmetic genus 1, $Y \subseteq X$ is a connected subcurve of arithmetic genus $g-1$ and $F \cap Y = \{p\}$ where $p$ is a nodal point of $X$ which is smooth in $F$ and $Y$. Our goal is to determine under which hypothesis $[X \subset \mathbb{P}^r]$ is Hilbert or Chow semistable. Using Corollary 5.6(i) and the Potential pseudo-stability Theorem 5.1, we can assume that $X$ is quasi-wp-stable and $L := \mathcal{O}_X(1)$ is very ample, non-special and balanced of degree $d$.

Let $\nu := \deg L_{|F}$. Since $L$ (and hence $L_{|F}$) is very ample by construction, we must have $\nu \geq 3$. On the other hand, by applying the basic inequality (3.1) to the subcurve $F \subseteq X$ we get

$$\left|\nu - \frac{d}{2g-2}\right| \leq \frac{1}{2},$$

so that

$$\begin{cases} \nu \leq 3 & \text{if } d < \frac{7}{2}(2g-2), \\ \nu = 3,4 & \text{if } d = \frac{7}{2}(2g-2), \\ \nu \geq 4 & \text{if } d > \frac{7}{2}(2g-2). \end{cases}$$

If $\nu = 4$ then there exists an isotrivial specialization $(X, L) \rightsquigarrow (X', L')$ (in the sense of Definition 7.3), where $X' = Y \cup E \cup F$ is obtained from $X$ by blowing up the node $p$ (i.e. inserting an exceptional component $E \cong \mathbb{P}^1$ meeting $Y$ and $F$ in one point) and $L'$ is a properly balanced line bundle on $X'$ such that $\deg L'_{|Y} = \deg L_{|Y} = d-4$, $\deg L'_{|E} = 1$ and $\deg L'_{|F} = 3$. Using Theorem 17.5 from the Appendix, it is easy to see that $L'$ is non-special and very ample; therefore there exists $[X' \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ such that $\mathcal{O}_{X'}(1) = L'$. Thus the basic inequality (3.1) and Theorem 7.5 imply

(1) $[X' \subset \mathbb{P}^r] \in \overline{\mathrm{Orb}([X \subset \mathbb{P}^r])}$;
(2) if $d = \frac{7}{2}(2g-2)$ then $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ (resp. $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$) if and only if $[X' \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ (resp. $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$).

**Theorem 9.6.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $X$ connected and assume that one of the following conditions is satisfied*

*(i) $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$;*
*(ii) $d = \frac{7}{2}(2g-2)$ and $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$.*

*Then $X$ does not have elliptic tails, i.e. $X_{\mathrm{ell}} = \emptyset$.*

*Proof.* Denote by $L := \mathcal{O}_X(1)$. We want to show, by contradiction, that $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ (resp. $[X \subset \mathbb{P}^r] \notin \mathrm{Hilb}_d^{ss}$) if (i) (resp. (ii)) holds. Note that since $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ and $\mathrm{Hilb}_d^{ss}$ are open in $\mathrm{Hilb}_d$, we can assume that $F$ is a generic connected curve of arithmetic genus one, and in particular that it is a smooth elliptic curve. Moreover, by Remark 9.5 we can assume that $\deg L_{|F} = 3$, so that we can write

$$L_{|F} = \mathcal{O}_F(2p+q) \tag{9.8}$$

for some (uniquely determined) $q \in F$. By our generic assumption on $F$, we can assume that $q \neq p$.

Consider now the linear spans $V_F := \langle F \rangle$ of $F$ and $V_Y := \langle Y \rangle$ of $Y$ on $\mathbb{P}^r = \mathbb{P}(V)$. It follows from Riemann-Roch theorem, using that $L$ (hence $L_{|Y}$ and $L_{|F}$) is non-special, that $V_F$ has dimension 2 and $V_Y$ has dimension $d-3-(g-1) = r-2$. Therefore, we can choose coordinates $\{x_1, \ldots, x_{r+1}\}$ of $V$ such that

$$V_F = \bigcap_{i=4}^{r+1} \{x_i = 0\} \quad , \quad V_Y = \bigcap_{i=1}^{2} \{x_i = 0\}$$

and $p$ is the point where all the $x_i$'s vanish except $x_3$. For $1 \le i \le 3$, we will identify $x_i$ with the section of $H^0(F, L_{|F})$ it determines and we will denote by $\mathrm{ord}_p(x_i)$ the order of vanishing of $x_i$ at $p$. By Riemann-Roch theorem applied to the line bundles $L_{|F}(-ip)$ for $i = 0, \ldots, 3$ and using (9.8) with $q \neq p$, we may choose the first three coordinates $\{x_1, \ldots, x_3\}$ of $V$ so that

$$\mathrm{ord}_p(x_i) = 3 - i \text{ for } 1 \le i \le 3. \tag{9.9}$$

Consider the one-parameter subgroup $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ which, in the above coordinates, has the diagonal form $\rho(t) \cdot x_i = t^{w_i} x_i$ for $i = 1, \ldots, r+1$, with weights $w_i$ such that

$$\begin{cases} w_1 = w_\rho(x_1) = -2, \\ w_2 = w_\rho(x_2) = -1, \\ w_j = w_\rho(x_j) = 0 & \text{for } j \ge 3. \end{cases} \tag{9.10}$$

The total weight of $\rho$ is equal to

$$w(\rho) = -2 - 1 = -3. \tag{9.11}$$

We want now to compute the polynomial $W_{X,\rho}(m)$. By Lemma 8.1

$$W_{X,\rho}(m) = W_{F,\rho}(m) + W_{Y,\rho}(m)$$

since $w_3 = 0$. Moreover, each coordinate of $V_Y = \{x_1 = x_2 = 0\}$ has weight 0, hence $W_{Y,\rho}(m) = 0$ and

$$W_{X,\rho}(m) = W_{F,\rho}(m) \tag{9.12}$$

In order to compute the polynomial $W_{F,\rho}(m)$, consider the embedding of $F$ as a cubic curve in $\mathbb{P}^2 = \mathbb{P}(H^0(F, L_{|F})^\vee)$ given by the complete linear system $|L_{|F}|$ . Let $f \in k[x_1, x_2, x_3]_3$ be a homogenous polynomial of degree 3 defining $F$. The conditions (9.9) on the order at $p$ of the coordinates $\{x_1, x_2, x_3\}$ translate directly into conditions on the polynomial $f$. More specifically, the point $p$ has coordinates $(0, 0, 1)$ and $p \in F$ if and only if the coefficient of $x_3^3$ in $f$ is equal to zero. The condition that $\mathrm{ord}_p x_1 \geq 2$ says that the tangent space of $F$ at $p$ must have equation $\{x_1 = 0\}$ which translates into the fact that the coefficient of $x_3^2 x_2$ in $f$ is zero while the coefficient of $x_3^2 x_1$ is not zero. Finally, we have that $\mathrm{ord}_p(x_1) = 2$ (i.e. $p$ is not a flex point of $F$) if and only if the coefficient of $x_2^2 x_3$ in $f$ is not zero. Summing up, every polynomial $f$ such that the coordinates $\{x_1, x_2, x_3\}$ satisfy (9.9) is of the form

(9.13)

$$f = a_{300}x_1^3+a_{210}x_1^2x_2+a_{201}x_1^2x_3+a_{120}x_1x_2^2+a_{102}x_1x_3^2+a_{111}x_1x_2x_3+a_{030}x_2^3+a_{021}x_2^2x_3,$$

where $a_{102} \neq 0$ and $a_{021} \neq 0$.

Because of the choice (9.10) of the one-parameter subgroup $\rho$, it is easy to see that the monomial $x_2^2 x_3$ has the maximal $\rho$-weight among all the monomials appearing in the above equation (9.13) of $f$. Moreover, the same monomial appears with non-zero coefficient in $f$. Therefore, a collection of $3m$ monomials that compute the polynomial $W_{F,\rho}(m)$ is represented by the monomials which are not divisible by $x_2^2 x_3$, namely

$$\left\{\{x_1^{m-k}x_3^k\}_{0\leq k\leq m}, \{x_2 x_1^{m-1-h}x_3^h\}_{0\leq h\leq m-1}, \{x_2^2 x_1^{m-2-j}x_2^j\}_{0\leq j\leq m-2}\right\}.$$

We get

$$W_{F,\rho}(m) = \sum_{k=0}^{m}[w_1(m-k)+kw_3]+\sum_{h=0}^{m-1}[w_2+(m-1-h)w_1+hw_3]+\sum_{j=0}^{m-2}[(j+2)w_2+(m-2-j)w_1] =$$

$$= \left[\frac{3}{2}w_1 + \frac{1}{2}w_2 + w_3\right] m^2 + \left[-\frac{3}{2}w_1 + \frac{3}{2}w_2\right] m + [w_1 - w_2] = -\frac{7}{2}m^2 + \frac{3}{2}m - 1. \tag{9.14}$$

Combining (9.14) with (9.12), we get

$$W_{X,\rho}(m) = -\frac{7}{2}m^2 + \frac{3}{2}m - 1. \tag{9.15}$$

In particular, the normalized leading coefficient of $W_{X,\rho}(m)$ is equal to

$$e_{X,\rho} = -7. \tag{9.16}$$

Let us first assume that condition (i) holds, and in particular that $v := \frac{d}{2g-2} < \frac{7}{2}$. The right hand side of the numerical criterion for Chow (semi)stability (see Fact 4.3) can be bounded above as follows:

$$(9.17) \qquad 2d\,\frac{w(\rho)}{r+1} = -\frac{6d}{r+1} = -6\,\frac{d}{d-g+1} = -6\,\frac{2v}{2v-1} < -7.$$

From (9.16) and (9.17), we deduce that the chosen 1ps $\rho$ satisfies

$$e_{X,\rho} > 2d\,\frac{w(\rho)}{r+1}.$$

In other words, $\rho$ violates the numerical criterion for Chow semistability of $[X \subset \mathbb{P}^r]$ (see Fact 4.3); hence $[X \subset \mathbb{P}^r] \notin \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ which is the desired contradiction.

Finally, let us assume that condition (ii) holds, namely $v := \frac{d}{2g-2} = \frac{7}{2}$. One of the two polynomials appearing in the numerical criterion for Hilbert (semi)stability (see Fact 4.2) is equal to

(9.18)

$$\frac{w(\rho)}{r+1}\,mP(m) = -\frac{3}{r+1}\,m(md+1-g) = -\frac{3d}{r+1}\,m^2 + \frac{3}{r+1}\,(g-1)m = \frac{7}{2}\,m^2 + \frac{1}{2}\,m.$$

From (9.14) and (9.18), it follows that

$$\frac{w(\rho)}{r+1}\,mP(m) - W_{X,\rho}(m) < 0 \text{ for } m \gg 0,$$

which implies that $[X \subset \mathbb{P}^r] \notin \mathrm{Hilb}_d^{ss}$ by Fact 4.2. □

Combining the previous Theorem 9.6 with Corollary 5.6(i) and Definition 2.10(ii), we get the following

**Corollary 9.7.** *Let* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *with* $X$ *connected and assume that one of the following two conditions is satisfied*

*(i)* $2(2g-2) < d < \frac{7}{2}(2g-2)$ *and* $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$*;*

*(ii)* $d = \frac{7}{2}(2g-2)$ *and* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$.

*Then* $X$ *is a quasi-p-stable curve.*

The Chow case for $d = \frac{7}{2}(2g-2)$ is very interesting. In the proof of the last theorem we said that if $[X \subset \mathbb{P}^r]$ admits an elliptic tail $F$ that satisfies (9.8) (we use the same notation of the last proof) and $\rho : \mathbb{G}_m \to \mathrm{GL}_{r+1}$ is the one-parameter subgroup defined above with weights (9.10), then

$$e_{X,\rho} = \frac{7}{3}\,w(\rho)$$

Thus $[X \subset \mathbb{P}^r]$ is Chow strictly semistable with respect to $\rho$. In the next theorem, we determine

$$[X_0 \subset \mathbb{P}^r] := \lim_{t \to 0} \rho(t) \cdot [X \subset \mathbb{P}^r]$$

and we study $A_\rho([X_0 \subset \mathbb{P}^r])$ and $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$.

**Theorem 9.8.** *Let $X$ and $\rho$ be as above. Then $X_0 = F_0 \cup Y$, where $F_0$ is a tacnodal elliptic tail (in particular $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$). Moreover, $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$ contains smoothings of the separating node, but not smoothings of the tacnode.*

*Proof.* Denote by $I$ the homogeneous ideal defined by $X \subset \mathbb{P}^r$. In the proof of Theorem 9.6 we said that $F \subset V_F$ satisfies the equation

$$f = a_{300}x_1^3 + a_{210}x_1^2x_2 + a_{201}x_1^2x_3 + a_{120}x_1x_2^2 + a_{102}x_1x_3^2 + a_{111}x_1x_2x_3 + a_{030}x_2^3 + a_{021}x_2^2x_3 = 0,$$

where $a_{102} \neq 0$ and $a_{021} \neq 0$. Let us consider the $\rho$-weighted graded order $\mathrm{in}_{\prec_\rho}$ as in §4.4 .We obtain that $g := \mathrm{in}_{\prec_\rho}(f) = a_{102}x_1x_3^2 + a_{021}x_2^2x_3$, which is the equation of an elliptic tacnodal curve $F_0$ in $V_F \cong \mathbb{P}^2$. Denote by $I_F$ and $I_Y$ respectively the ideals of $F$ and $C$, respectively. Suppose that $h_1, \dots, h_n \in k[x_3, x_4 \dots, x_{r+1}]$ is a Gröbner basis for $I(Y) \cap k[x_3, \dots, x_{r+1}]$: by Lemma 8.2 we deduce that

$$\{f, h_1, \dots, h_n, x_ix_j \,|\, 1 \leq i \leq 2 \text{ and } 4 \leq j \leq r+1\}$$

is a Gröbner basis for $I = I_Y \cap I_F$. Applying the definition we get

$$\mathrm{in}_{\prec_\rho}(I) = \langle g, h_1, \dots, h_n, x_ix_j \,|\, 1 \leq i \leq 2 \text{ and } 4 \leq j \leq r+1\rangle,$$

which is exactly the ideal defining the variety $X_0 = F_0 \cup Y \subset \mathbb{P}^r$. Now, it is enough to apply Corollary 4.14.

In order to show the last part of the theorem, we use the same tecniques used to prove CLAIM 3 in the proof of Theorem 7.5. Consider the separating node $\{p\} = \{[0, 0, 1, 0, \dots, 0]\} = V_{F_0} \cap V_Y$. We can assume that the tangent space $T_{X,p}$ is given by $\langle x_2, x_4\rangle$, hence the completion of the local ring $\mathcal{O}_{X,p}$ is given by $k[[u, v]]/(uv)$ where $u = x_2/x_3$ and $v = x_4/x_3$. Since $\rho(t)^{-1} \cdot x_2 = tx_2$ and $\rho(t)^{-1} \cdot x_4 = x_4$, we get $\rho(t)^{-1} \cdot u = tu$ and $\rho(t)^{-1} \cdot v = v$. We recall that $\mathrm{Def}_{(X,p)}$ has a semiuniversal ring equal to $k[[a]]$ with universal family $k[[u, v, a]]/(uv - a)$. This implies that $\rho(t)^{-1} \cdot a = ta$ and $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$ contains smoothings of $p$. If we consider the action of $\rho$ on the universal family of the tacnode we obtain non positive weights, hence $A_{\rho^{-1}}([X_0 \subset \mathbb{P}^r])$ does not contain smoothings of the tacnode. □

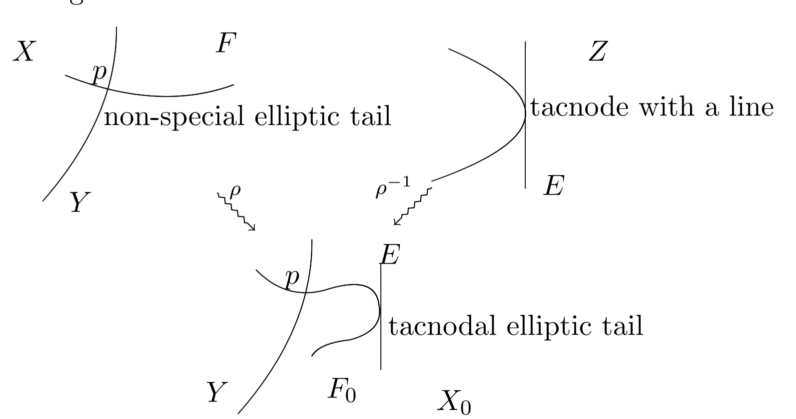



FIGURE 6. The basin of attraction of a curve $X_0$ with a tacnodal elliptic tail $F_0$.

## 10. A stratification of the semistable locus

Consider the following sublocus of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$:

$$\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o := \{[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d \,:\, X \text{ is connected}\}. \tag{10.1}$$

Note that if $d > 2(2g-2)$ then the condition of being connected is both closed and open in $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$: it is closed because of its natural interpretation as a topological condition; it is open because if $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ then $X$ is a reduced curve by the Potential pseudo-stability Theorem 5.1 and therefore $X$ is connected if and only if $h^0(X, \mathcal{O}_X) = 1$, which is an open condition by upper-semicontinuity. Therefore, $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ is both open and closed in $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$; or, in other words, it is a disjoint union of connected components of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$.

Inspired by [Cap94, Sec. 5], we introduce in this section an $\mathrm{SL}_{r+1}$-invariant stratification of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ and we establish some properties of it.

Recall that if $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ and $d > 2(2g-2)$ then $X$ is quasi-wp-stable and $\mathcal{O}_X(1)$ is properly balanced by Corollary 5.6(i). Recall also that $B_X^d$ denotes the set of multidegrees of properly balanced line bundles on $X$ of total degree $d$ (see Definition 3.5).

Following [Cap94, Sec. 5.1], consider, for any quasi-wp-stable curve $X$ of genus $g$ and any $\underline{d} \in B_X^d$, the (locally closed) stratum of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$:

$$M_X^{\underline{d}} := \{[Y \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o \,:\, \exists\, \phi : X \xrightarrow{\cong} Y \text{ such that } \underline{\deg}\, \phi^*\mathcal{O}_Y(1) = \underline{d}\}. \tag{10.2}$$

Note in particular, that the isomorphism $\phi$ between the abstract curve $X$ and the embedded curve $Y$ is not specified. However, with a slight abuse of notation, we will often represent points of $M_X^{\underline{d}}$ by $[X \subset \mathbb{P}^r]$.

Each stratum $M_X^{\underline{d}}$ is $\mathrm{SL}_{r+1}$-invariant since $\mathrm{SL}_{r+1}$ acts on $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ by changing the embedding of $X$ inside $\mathbb{P}^r$ and thus it preserves $X$ and the multidegree $\underline{d}$. Note that $M_X^{\underline{d}}$ may be empty for certain pairs $(X, \underline{d})$ as above.

10.1. **Specializations of strata.** The aim of this subsection is to describe all pairs $(X', \underline{d}')$ with $X'$ quasi-wp-stable of genus $g$ and $\underline{d}' \in B_{X'}^d$ such that $M_{X'}^{\underline{d}'} \subseteq \overline{M_X^{\underline{d}}}$.

Generalizing the refinement relation of [Cap94, Sec. 5.2], we define an order relation on the sets of pairs $(X, \underline{d})$ where $X$ is a quasi-wp-stable curve of genus $g$ and $\underline{d} \in B_X^d$.

**Definition 10.1.** Let $(X', d')$ and $(X'', d'')$ be such that $X'$ and $X''$ are two quasi-wp-stable curves of genus $g$ and $\underline{d}' \in B_{X'}^d$, $\underline{d}'' \in B_{X''}^d$. We say that $(X'', \underline{d}'') \preceq (X', \underline{d}')$ if $(X'', d'')$ can be obtained from $(X', d')$ via a sequence of elementary operations as depicted in Figures 7, 8, 9, 10, 11 and 12 below.

**Remark 10.2.** Given two quasi-wp-stable curves $X'$ and $X''$ (not necessarily endowed with any multidegree), we can also say that $X'' \preceq X'$ if $X''$ can be obtained from $X'$ via a sequence of the elementary operations depicted in Figures 7, 8, 11, 9, 10 and 12, ignoring the degrees.

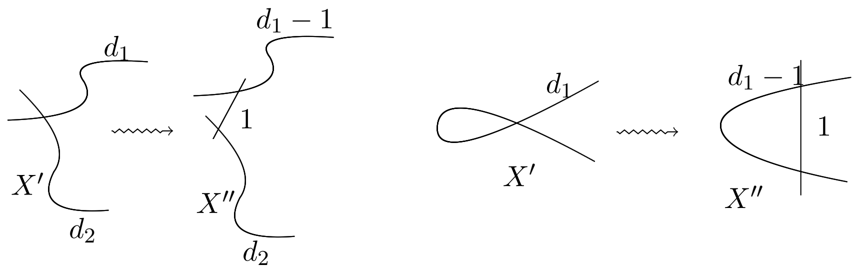



Figure 7. Blow-up of a node: external and internal cases.

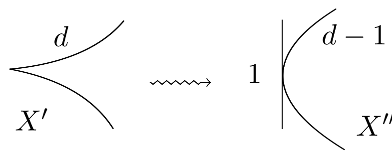



Figure 8. Blow-up of a cusp.

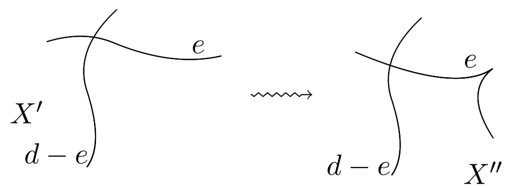



Figure 9. Replacing an elliptic tail by a cuspidal elliptic tail.

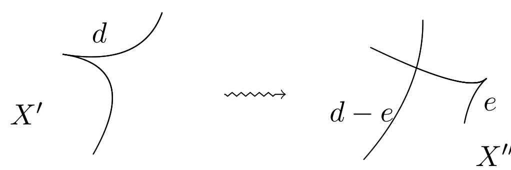



Figure 10. Replacing a cuspidal singularity by a cuspidal elliptic tail.

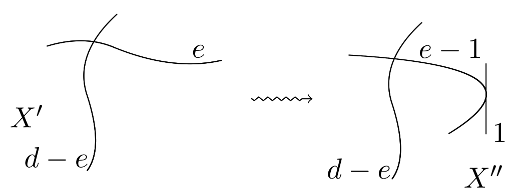



Figure 11. Replacing an elliptic tail by a tacnodal elliptic tail.

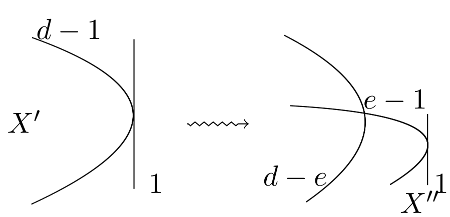



Figure 12. Replacing a tacnode with a line by a tacnodal elliptic tail.

From the above description it is easy to see that there is a relation between the isotrivial specialization introduced in Definition 7.3 and the order relation $\preceq$. More precisely, we have the following

**Remark 10.3.** Let $(X', L')$ and $(X'', L'')$ be two pairs consisting of a quasi-wp-stable curve of genus $g$ and a properly balanced line bundle of degree $d$. If $(X', L') \rightsquigarrow (X'', L'')$ then $(X'', \underline{\deg} L'') \preceq (X', \underline{\deg} L')$.

The following elementary property of the order relation $\preceq$ will be used in what follows.

**Lemma 10.4.** *Notations as in Definition 10.1. If $X'' \preceq X'$ and $\underline{d}'' \in B^d_{X''}$ then there exists $\underline{d}' \in B^d_{X'}$ such that $(X'', \underline{d}'') \preceq (X', \underline{d}')$.*

*Proof.* Start by assuming that $X''$ is obtained from $X'$ by blowing up an external node $n$, as in the picture on the left of Figure 7.

Denote by $\{C'_1, \ldots, C'_\gamma\}$ the irreducible components of $X'$, by $\{C''_1, \ldots, C''_\gamma\}$ their proper transforms in $X''$ and by $E$ the exceptional component that is contracted to the node $n$ by the map $\sigma : X'' \to X'$. Assume that $C'_1$ and $C'_2$ are the two irreducible components of $X'$ that contain the node $n$. Define a multidegree $\underline{d}'$ on $X'$ in the following way:

$$\underline{d}'_{C'_i} := \begin{cases} \underline{d}''_{C''_i} & \text{for } i \neq 1, \\ \underline{d}''_{C''_1} + 1 & \text{for } i = 1. \end{cases}$$

It is clear that $|\underline{d}'| = d$, so we must check that $\underline{d}'$ satisfies the basic inequality (3.1). Given a subcurve $Z'$ of $X'$, we denote by $Z''$ the subcurve of $X''$ that is the proper transform of $Z'$ under the blow-up map $X'' \to X'$. Define $W_{Z'}$ to be the subcurve of $X''$ such that $W_{Z'} = Z''$ if $C'_1 \subsetneq Z'$ and $W_{Z'} = Z'' \cup E$ if $C'_1 \subseteq Z'$. Then it is easy to see that

$$\begin{cases} d'_{Z'} = d''_{W_{Z''}}, \\ g_{Z'} = g_{W_{Z''}}, \\ k_{Z'} = k_{W_{Z''}}. \end{cases}$$

Hence the basic inequality (3.1) for $\underline{d}'$ relative to the subcurve $Z'$ is the same as the basic inequality for $\underline{d}''$ relative to the subcurve $W_{Z''}$. We conclude that if $d'' \in B^d_{X''}$ then $d' \in B^d_{X'}$.

The remaining cases are similar (and easier) and are therefore left to the reader. $\square$

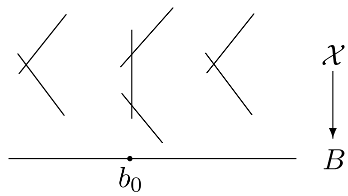
We will now prove that the above order relation $\preceq$ determines the inclusion relations among the closures of the strata $M^{\underline{d}}_X \subset \mathrm{Ch}^{-1}(\mathrm{Chow}^{ss}_d)^o$ of (10.2). The following result is a generalization of [Cap94, Prop. 5.1].

**Proposition 10.5.** *Assume that $d > 2(2g-2)$ and moreover that $g \geq 3$ if $d \leq 4(2g-2)$. Let $X'$ and $X''$ be two quasi-wp-stable curves of genus $g$ and let $\underline{d}' \in B^d_{X'}$ and $\underline{d}'' \in B^d_{X''}$. Assume that $M^{\underline{d}''}_{X''} \neq \emptyset$. Then*

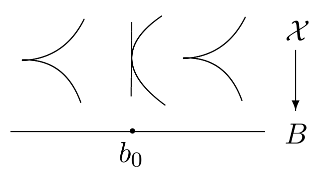
$$M^{\underline{d}''}_{X''} \subseteq \overline{M^{\underline{d}'}_{X'}} \Longleftrightarrow (X'', \underline{d}'') \preceq (X', \underline{d}').$$

*Proof.* $\Longleftarrow$ We will start by showing that if $X'' \preceq X'$ then there is a family $u : \mathcal{X} \to B$ over a smooth curve $B$ whose geometric fiber $\mathcal{X}_b$ over a point $b \in B$ is such that $\mathcal{X}_b \cong X'$ for all $b \neq b_0$ and $\mathcal{X}_{b_0} \cong X''$.

Start by assuming that $X''$ is obtained from $X'$ by blowing up a node, say $n$. Let $B$ be a smooth curve and consider the trivial family $X' \times B$ over $B$. By blowing up the surface $X \times B$ at the node $n$ belonging to the fiber over a point $b_0 \in B$, we get a family $u : \mathcal{X} \to B$ whose geometric fiber $\mathcal{X}_b$ over a point $b \in B$ is such that $\mathcal{X}_b \cong X'$ for all $b \neq b_0$ and $\mathcal{X}_{b_0} \cong X''$ as in the figure below (where we have depicted an external node, but the case of an internal node is completely similar).




In the case when $X''$ is obtained from $X'$ by blowing up a cusp we proceed in the same way as in the previous case: we consider the trivial family $X' \times B$ over $B$ and by blowing up the surface $X \times B$ on the cusp $p$ belonging to the fiber over a point $b_0 \in B$, we get a family $u : \mathcal{X} \to B$ whose geometric fiber $\mathcal{X}_b$ over a point $b \in B$ is such that $\mathcal{X}_b \cong X'$ for all $b \neq b_0$ and $\mathcal{X}_{b_0} \cong X''$ as in the figure below.




The cases when $X''$ is a cuspidal elliptic tail as in Figures 9 and 10 are direct consequences of Remark 2.5(i). The cases depicted in Figures 11 and 12 can then be obtained from these as follows. Let $u : \mathcal{X} \to B$ be a family such that, for $b \neq b_0$, $\mathcal{X}_b \cong X'$ is a curve with an elliptic tail and such that $\mathcal{X}_{b_0}$ is a curve with a cuspidal elliptic tail as in Figure 9. By blowing up the surface $\mathcal{X}$ at the cusp $p$ of the central fiber $\mathcal{X}_{b_0}$, we get a new family $u' : \mathcal{X}' \to B$ such that $\mathcal{X}'_b \cong X'$ is a curve with an elliptic tail as before and such that $\mathcal{X}'_{b_0}$ has a tacnodal elliptic tail as in Figure 11. Finally, in order to deal with the situation depicted in Figure 12, we consider an isotrivial family $u : \mathcal{X} \to B$ where for $b \neq b_0$, $\mathcal{X}_b$ is a curve with a cuspidal singularity and such that $\mathcal{X}_{b_0}$ has a cuspidal tail, as in Figure 10. The locus in $\mathcal{X}$ corresponding to the cusp in each fiber $\mathcal{X}_b$ of $u$ over $B$ is a Weil divisor on the surface $\mathcal{X}$; by blowing up this divisor, we get a new family $u' : \mathcal{X}' \to B$ such that, for $b \neq b_0$, $\mathcal{X}'_b \cong X'$ is a curve having a tacnode with a line while $\mathcal{X}'_{b_0}$ has a tacnodal elliptic tail, as in Figure 12.

Consider now the relative Picard scheme $\pi : \mathrm{Pic}_{\mathcal{X}/\mathrm{B}} \to \mathrm{B}$ of the family $u : \mathcal{X} \to B$, which exists by a well-known result of Mumford (see [BLR90, Sec. 8.2, Thm. 2]). Since

$H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) = 0$ for any $b \in B$ because $\mathcal{X}_b$ is a curve, we get that $\pi : \mathrm{Pic}_{\mathcal{X}/\mathrm{B}} \to \mathrm{B}$ is smooth by [BLR90, Sec. 8.4, Prop. 2].

Let now $[X'' \subset \mathbb{P}^r = \mathbb{P}(V)] \in M_{X''}^{\underline{d}''}$ and set $L'' = \mathcal{O}_{X''}(1) \in \mathrm{Pic}^{\underline{d}''}(\mathrm{X}'')$. Note that the embedding $X'' \subset \mathbb{P}^r$ defines an isomorphism $\phi : H^0(X'', L'') \stackrel{\cong}{\to} V$.

We can view $L''$ as a geometric point of $(\mathrm{Pic}_{\mathcal{X}/\mathrm{B}})_{\mathrm{b}_0} \cong \mathrm{Pic}(\mathrm{X}'')$. Since the morphism $\pi : \mathrm{Pic}_{\mathcal{X}/\mathrm{B}} \to \mathrm{B}$ is smooth, up to shrinking $B$ (i.e., replacing it with an étale open neighborhood of $b_0$), we can find a section $\sigma$ of $\pi$ such that $\sigma(b_0) = L''$. Moreover, by definition of the order relation $\preceq$ (see Figures 7, 8, 9, 10, 11 and 12 above), it is clear that we can choose the section $\sigma$ so that $\sigma(b)$ is a line bundle of multidegree $\underline{d}'$ on $\mathcal{X}_b \cong X'$ for every $b \neq b_0$.

Up to shrinking $B$ again, we can assume that the section $\sigma$ corresponds to a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $\mathcal{L}_{|\mathcal{X}_{b_0}} \cong L''$ and $\mathcal{L}_{|\mathcal{X}_b}$ has multidegree $\underline{d}'$ for $b \neq b_0$. Since $L''$ is very ample and non-special and these conditions are open, up to shrinking $B$ once more, we can assume that $\mathcal{L}$ is relatively very ample and we can fix an isomorphism $\Phi : u_*\mathcal{L} \stackrel{\cong}{\to} \mathcal{O}_B \otimes V$ of sheaves on $B$ such that $\Phi_{|b_0} = \phi$. Via the isomorphism $\Phi$, the relatively very ample line bundle $\mathcal{L}$ defines an embedding

$$\begin{array}{ccc} \mathcal{X} & \stackrel{i}{\hookrightarrow} & \mathbb{P}(\mathcal{O}_B \otimes V) = \mathbb{P}^r_B \\ & {\scriptstyle u}\searrow \quad \swarrow & \\ & B & \end{array}$$

whose restriction over $b_0 \in B$ is the embedding $X'' \subset \mathbb{P}^r$. The family $u : \mathcal{X} \to B$ together with the embedding $i$ defines a morphism $f : B \to \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ such that $f(b_0) = [X'' \subset \mathbb{P}^r] \in M_{X''}^{\underline{d}''}$ and $f(b) \in M_{X'}^{\underline{d}'}$ for every $b \neq b_0$, so we conclude that $M_{X''}^{\underline{d}''} \subseteq \overline{M_{X'}^{\underline{d}'}}$.

$\Longrightarrow$ Suppose now that $M_{X''}^{\underline{d}''} \subseteq \overline{M_{X'}^{\underline{d}'}}$. Then we can find a smooth curve $B$ and a morphism $f : B \to \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ such that $f(b_0) \in M_{X''}^{\underline{d}''}$ for some $b_0 \in B$ and $f(b) \in M_{X'}^{\underline{d}'}$ for every $b_0 \neq b \in B$. By pulling back the universal family above $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ along the morphism $f$, we get a family

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & B \times \mathbb{P}^r \\ \downarrow {\scriptstyle u} & & \\ B & & \end{array}$$

such that $\mathcal{X}_{b_0} = X''$ and $\mathcal{X}_b = X'$ for every $b \neq b_0$. In particular, $u$ yields an isotrivial specialization of $X'$ into $X''$. Let $\overline{\mathcal{X}} \to B$ be the wp-stabilization of $u$; for $b \neq b_0$, $\overline{X}' := \overline{\mathcal{X}}_b$ is the wp-stabilization of $X'$ while $\overline{X}'' := \overline{\mathcal{X}}_{b_0}$ is the wp-stabilization of $X''$. According to Remark 2.5(i), $\overline{X}'$ and $\overline{X}''$ may differ by replacing elliptic tails by cuspidal elliptic tails or by replacing cuspidal singularities by cuspidal elliptic tails as in Figures 9 and 10, so $\overline{X}'' \preceq \overline{X}'$. Then, as $\mathcal{X}$ is a family of quasi-wp-stable curves, it is obtained from $\overline{\mathcal{X}}$ in two steps: first by blowing up the surface $\overline{\mathcal{X}}$ on the locus of some nodal or cuspidal singularities along all the fibers of $\overline{\mathcal{X}}$ giving rise to a new family $\widetilde{\mathcal{X}}$, and then by further blowing up $\widetilde{\mathcal{X}}$ on nodal or cuspidal singularities of the fiber over $b_0$. Denote $\widetilde{X}' := \widetilde{\mathcal{X}}_b$ for $b \neq b_0$ and $\widetilde{X}'' := \widetilde{\mathcal{X}}_{b_0}$. Then it is easy to see that $\widetilde{X}'' \preceq \widetilde{X}'$: the only situation that needs some care is when blowing up $\overline{\mathcal{X}}$ on a cuspidal singularity in the case when $\overline{X}''$ is obtained from $\overline{X}'$ by replacing a cuspidal singularity by a cuspidal elliptic tail as in Figure 10. But in this case, we easily see that we get a situation as described on Figure 12, so $\widetilde{X}'' \preceq \widetilde{X}'$. Finally, by blowing up $\widetilde{\mathcal{X}}$ on nodes or cusps of $\widetilde{\mathcal{X}}_{b_0}$ we get the situations described in Figures 7, 8 and 11, so $X'' \preceq X'$.

Consider now the line bundles $L_{b_0} := \mathcal{O}_{\mathcal{X}}(1)_{|\mathcal{X}_{b_0}} \in \mathrm{Pic}^{\underline{\mathrm{d}}''}(\mathrm{X}'')$ and $L_b := \mathcal{O}_{\mathcal{X}}(1)_{|\mathcal{X}_b} \in \mathrm{Pic}^{\underline{\mathrm{d}}'}(\mathrm{X}')$ for any $b_0 \neq b \in B$. Let $Y' \subseteq X'$ be a subcurve of $X'$. Consider the subcurve $Y'' \subseteq X'' = \mathcal{X}_{b_0}$ given by the union of all the irreducible components $C_i$ of $X''$ for which there exists a section $s$ of $u : \mathcal{X} \to B$ such that $s(b_0) \in C_i$ and $s(b) \in Y' \subseteq X' = \mathcal{X}_{b'}$ for every $b \neq b_0$. By construction, we get that $\underline{d}'_{Y'} = \deg_{Y'} L_b = \deg_{Y''} L_{b_0} = \underline{d}''_{Y''}$. According to Definition 10.1, this yields that $(X'', \underline{d}'') \preceq (X', \underline{d}')$.

□

10.2. **A completeness result.** Given any quasi-wp-stable curve $X$ of genus $g$ and a multidegree $\underline{d} \in B_X^d$, consider the subgroup of the automorphism group $\mathrm{Aut}(X)$ of $X$ given by

$$\mathrm{Aut}^{\underline{d}}(X) = \{\phi \in \mathrm{Aut}(X) \ : \ \phi^* L \in \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X}) \ \text{ for any } \mathrm{L} \in \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})\}. \tag{10.3}$$

Note that given a point $[Y \overset{i}{\hookrightarrow} \mathbb{P}^r]$ belonging to the stratum $M_X^{\underline{d}}$ as in (10.2), the line bundle $\phi^* \mathcal{O}_Y(1) \in \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$ is only well-defined up to the action of $\mathrm{Aut}^{\underline{d}}(X)$. We denote by $[\mathcal{O}_X(1)]$ the class of this line bundle in the quotient $\mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})/\mathrm{Aut}^{\underline{\mathrm{d}}}(\mathrm{X})$. Therefore, we have a well-defined (set-theoretic) map

$$\begin{aligned} p : M_X^{\underline{d}} &\to \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})/\mathrm{Aut}^{\underline{\mathrm{d}}}(\mathrm{X}) \\ [X \subset \mathbb{P}^r] &\mapsto [\mathcal{O}_X(1)]. \end{aligned} \tag{10.4}$$

Note that the fibers of the map $p$ are exactly the $\mathrm{SL}_{r+1}$-orbits on $M_X^{\underline{d}}$. The image of $p$ can be nicely described using the following useful result about the relation between the automorphism group of $X$ and the stability of $[X \subset \mathbb{P}^r]$.

**Lemma 10.6.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ where $X$ is non-degenerate and linearly normal in $\mathbb{P}^r$. Set $L = \mathcal{O}_X(1)$. If $\phi \in \mathrm{Aut}(X)$, then $[X \subset \mathbb{P}^r]$ is Chow semistable (resp. stable) if and only if $[X \overset{|\phi^* L|}{\hookrightarrow} \mathbb{P}^r]$ is Chow semistable (resp. stable). The same holds for the Hilbert (semi)stability.*

*Proof.* For $m \gg 0$ we have the commutative diagram

$$\begin{array}{ccc} S^m H^0(X, L) & \xrightarrow{\phi^*} & S^m H^0(X, \phi^* L) \\ \downarrow & & \downarrow \\ H^0(X, L^m) & \xrightarrow{\phi^*} & H^0(X, \phi^* L^m), \end{array}$$

which allows us to identify the monomial bases of $H^0(X, L^m)$ and of $H^0(X, \phi^* L^m)$. More precisely, if we fix a system of coordinates $\{x_1, \ldots, x_{r+1}\}$ and $\{B_1, \ldots, B_{P(m)}\}$ is a monomial basis of $H^0(X, L^m)$ with respect to $\{x_1, \ldots, x_{r+1}\}$, then $\{\phi^*(B_1), \ldots, \phi^*(B_{P(m)})\}$ is a monomial basis of $H^0(X, \phi^* L^m)$ with respect to the system of coordinates

$$\{\phi^*(x_1), \ldots, \phi^*(x_{r+1})\}.$$

Now, let $\rho$ be a 1ps diagonalized by $\{x_1, \ldots, x_{r+1}\}$ with weights $w_1, \ldots, w_{r+1}$ and define another 1ps $\rho^*$ diagonalized by $\{\phi^*(x_1), \ldots, \phi^*(x_{r+1})\}$ with the same ordered weights. By the identification of monomial bases, we have that $W_{X,\rho}(m) = W_{X,\rho^*}(m)$ for $m \gg 0$[4]. Now, it suffices to apply the Hilbert-Mumford criterion (Fact 4.2) and our statement is proved. □

**Corollary 10.7.** *Let $X$ be a quasi-wp-stable curve and $\underline{d} \in B_X^d$. Let $L \in \mathrm{Pic}^{\underline{d}}(X)$ and assume that $L$ is very ample and non-special. Consider the point $[X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$, which is well-defined up to the action of $\mathrm{SL}_{r+1}$. Then*

$$[L] \in \mathrm{Im}(p) \Leftrightarrow [X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o.$$

The aim of this subsection is to prove the following completeness result, which generalizes [Cap94, Prop. 5.2].

**Proposition 10.8.** *Let $X$ be a quasi-wp-stable curve and $\underline{d} \in B_X^d$. Assume that one of the following conditions is satisfied:*

*(i)* $d > 4(2g-2)$;

*(ii)* $2(2g-2) < d < \frac{7}{2}(2g-2)$ *and* $g \geq 3$;

*(iii)* *$X$ is quasi-p-stable,* $\frac{7}{2}(2g-2) < d < 4(2g-2)$ *and* $g \geq 3$.

*Then either $M_X^{\underline{d}} = \emptyset$ or $p : M_X^{\underline{d}} \to \mathrm{Pic}^{\underline{d}}(X)/\mathrm{Aut}^{\underline{d}}(X)$ is surjective.*

*Proof.* Note that the statement of the Proposition is equivalent to the fact that either $\theta^{-1}(\mathrm{Im}(p)) = \emptyset$ or $\theta^{-1}(\mathrm{Im}(p)) = \mathrm{Pic}^{\underline{d}}(X)$, where $\theta : \mathrm{Pic}^{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}}(X)/\mathrm{Aut}^{\underline{d}}(X)$ is the projection to the quotient. We first make the following two reductions.

Reduction 1: We can assume that if $d < \frac{5}{2}(2g-2)$ then $X$ does not contain elliptic tails; in this case every $L \in \mathrm{Pic}^{\underline{d}}(X)$ is non-special and very ample.

Indeed, according to Theorem 17.5(i), $L \in \mathrm{Pic}^{\underline{d}}(X)$ is non-special since $X$ is quasi-wp-stable, hence $G$-semistable, and $\deg L = d > 2(2g-2) > 2g-2$ (recall that $g \geq 2$). Now, if $d < \frac{5}{2}(2g-2)$ and $X$ contains some elliptic tail $F$, then from the basic inequality it follows easily that $\underline{d}_F = 2$. But no line bundle of degree 2 on a curve of genus 1 is very ample, hence no line bundle of multidegree $\underline{d}$ on $X$ can be very ample. Otherwise, since any $L \in \mathrm{Pic}^{\underline{d}}(X)$ is ample by Remark 5.7, it follows from Theorem 17.5(iii) that $L$ is very ample, q.e.d.

Reduction 2: We can assume that $\underline{d}$ is strictly balanced.

[4] Here there is an abuse of notation: $W_{X,\rho}(m)$ is referred to $[X \subset \mathbb{P}^r]$, while $W_{X,\rho^*}(m)$ is referred to $[X \stackrel{|\phi^* L|}{\hookrightarrow} \mathbb{P}^r]$.

Indeed, suppose the proposition is true for all strictly balanced line bundles on quasi-wp-stable curves and let us show that it is true for our multidegree $\underline{d}$ on $X$, assuming that $\underline{d}$ is not strictly balanced. Let $L \in \mathrm{Pic}^{\underline{d}}(X)$. Since $\underline{d}$ is not strictly balanced, by Lemma 7.4(ii) there exists an isotrivial specialization $(X, L) \rightsquigarrow (X', L')$ such that $\underline{d}' := \deg L'$ is a strictly balanced multidegree on $X'$. Moreover, from the proof of the cited Lemma, it follows easily that the curve $X'$ and the multidegree $\underline{d}'$ depend only on $X$ and $\underline{d}$ and not on $L \in \mathrm{Pic}^{\underline{d}}(X)$. Note that, since $X'$ is obtained from $X$ by blowing up some nodes of $X$, then $X$ has some elliptic tails if and only if $X'$ has some elliptic tails. Therefore, according to Reduction 1, $L$ and $L'$ are non-special and very ample. Up to the choice of a basis of $H^0(X, L)$ and of $H^0(X', L')$, we get two points of $\mathrm{Hilb}_d$, namely $[X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r]$ and $[X' \stackrel{|L'|}{\hookrightarrow} \mathbb{P}^r]$. These two points are indeed well-defined only up to the action of the group $\mathrm{SL}_{r+1}$. From Corollary 10.7, we get that $[L] \in \mathrm{Im}(M_X^{\underline{d}} \stackrel{p}{\to} \mathrm{Pic}^{\underline{d}}(X)/\mathrm{Aut}^{\underline{d}}(X))$ if and only if $[X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$, and similarly that $[L'] \in \mathrm{Im}(M_{X'}^{\underline{d}'} \stackrel{p'}{\to} \mathrm{Pic}^{\underline{d}'}(X')/\mathrm{Aut}^{\underline{d}'}(X'))$ if and only if $[X' \stackrel{|L'|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$. Therefore, Theorem 7.5(ii) gives that $[L] \in \mathrm{Im}(p)$ if and only if $[L'] \in \mathrm{Im}(p')$. In other words, we have defined a set-theoretic map

$$\begin{aligned} \Upsilon : \mathrm{Pic}^{\underline{d}}(X) &\to \mathrm{Pic}^{\underline{d}'}(X') \\ L &\mapsto L' \end{aligned}$$

such that $\Upsilon^{-1}(\theta'^{-1}(\mathrm{Im}(p'))) = \theta^{-1}(\mathrm{Im}(p))$., where $\theta : \mathrm{Pic}^{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}}/\mathrm{Aut}^{\underline{d}}(X)$ and $\theta' : \mathrm{Pic}^{\underline{d}'}(X') \to \mathrm{Pic}^{\underline{d}'}/\mathrm{Aut}^{\underline{d}'}(X')$ are the projection maps. The proposition for $\underline{d}'$ is equivalent to the fact that either $\theta'^{-1}(\mathrm{Im}(p')) = \emptyset$ or $\theta'^{-1}(\mathrm{Im}(p')) = \mathrm{Pic}^{\underline{d}'}(X')$. Using the above map $\Upsilon$, it is easy to see that the above properties hold also for $\underline{d}$, q.e.d.

We now prove the proposition for a pair $(X, \underline{d})$ satisfying the properties of Reduction 1 and Reduction 2 and one of the conditions (i), (ii) and (iii). Assume that $M_X^{\underline{d}} \neq \emptyset$, for otherwise there is nothing to prove. Let us first prove the following

CLAIM: $\theta^{-1}(\mathrm{Im}(p)) \subset \mathrm{Pic}^{\underline{d}}(X)$ is open and dense.

Consider a Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^{\underline{d}}(X)$, i.e., a line bundle $\mathcal{P}$ such that $\mathcal{P}_{|X \times \{L\}} \cong L$ for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ (see [Kle05, Ex. 4.3]). By Reduction 1, it follows that $\mathcal{P}$ is relatively very ample with respect to the projection $\pi_2 : X \times \mathrm{Pic}^{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}}(X)$ and that $(\pi_2)_*(\mathcal{P})$ is locally free of rank equal to $r + 1 = d - g$ . We can therefore find a Zariski open cover $\{U_i\}_{i \in I}$ of $\mathrm{Pic}^{\underline{d}}(X)$ such that $(\pi_2)_*(\mathcal{P})_{|U_i} \cong \mathcal{O}_{U_i}^{\oplus(r+1)}$ and the line bundle $\mathcal{P}$ induces an embedding

$$\begin{array}{ccc} X \times U_i & \stackrel{\eta_i}{\hookrightarrow} & \mathbb{P}(\mathcal{O}_{U_i}^{r+1}) = \mathbb{P}^r_{U_i} \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow & \\ & U_i & \end{array}$$

The above embedding corresponds to a map $f_i : U_i \to \mathrm{Hilb}_d$ and, using Corollary 10.7, we get that

$$\theta^{-1}(\mathrm{Im}(p)) = \bigcup_i f_i^{-1}(\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o).$$

Since $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ is open inside $\mathrm{Chow}^{-1}(\mathrm{Chow}_d^{ss})$ (by the discussion at the beginning of Section 10) and $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ is open in $\mathrm{Hilb}_d$ (because any GIT-semistability condition is open), it follows that $f_i^{-1}(\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o)$ is open inside $U_i$; hence

$$\theta^{-1}(\mathrm{Im}(p)) \subseteq \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$$

is open as well. Moreover, since $\mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$ is irreducible and $M_X^{\underline{d}} \neq \emptyset$, we get that $\theta^{-1}(\mathrm{Im}(p)) \subseteq \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$ is also dense, q.e.d.

In order to finish the proof, it remains to show that $\theta^{-1}(\mathrm{Im}(p)) \subseteq \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$ is closed. Since $\theta^{-1}(\mathrm{Im}(p))$ is open by the CLAIM, it is enough to prove that $\theta^{-1}(\mathrm{Im}(p))$ is closed under specializations (see [Har77, Ex. II.3.18(c)]), i.e., if $B \subseteq \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})$ is a smooth curve such that $B \setminus \{b_0\} \subseteq \theta^{-1}(\mathrm{Im}(p))$ then $b_0 \in \theta^{-1}(\mathrm{Im}(p))$. The same construction as in the proof of the Claim gives, up to shrinking $B$ around $b_0$, a map $f : B \to \mathrm{Hilb}_d$ such that $f(B \setminus \{b_0\}) \subset \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o \subseteq \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$. We denote by $\mathcal{L}$ the relatively ample line bundle on $\pi_1 : \mathcal{X} := X \times B \to B$ which gives the embedding into $\mathbb{P}^r_B$.

We can now apply a fundamental result in GIT, called *polystable replacement property* (see e.g. [HH13, Thm. 4.5]), which implies that, up to replacing $B$ with a finite cover ramified over $b_0$, we can find two maps $g : B \to \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ and $h : B \setminus \{b_0\} \to \mathrm{SL}_{r+1}$ such that

(10.5) $$f(b) = h(b) \cdot g(b) \text{ for every } b_0 \neq b \in B,$$

(10.6) $$g(b_0) \text{ is Chow polystable.}$$

We denote by $\pi_2 : \mathcal{Y} \to B$ the pull-back of the universal family over $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ via the map $g$ and by $\mathcal{M}$ the line bundle on $\mathcal{Y}$ which is the pull-back of the universal line bundle via $g$. Property (10.5) implies that $X \cong \mathcal{Y}_b$ and $\underline{\deg}\mathcal{M}_{|\mathcal{Y}_b} = \underline{d}$ for every $b_0 \neq b \in B$. Moreover, if we set $Y := \mathcal{Y}_{b_0}$, $M := \mathcal{M}_{|Y_0}$ and $\underline{d}' := \underline{\deg}M$, then Proposition 10.5 implies that $(Y, \underline{d}') \preceq (X, \underline{d})$. Observe also that (10.6) together with Corollary 7.6 imply that $M$ is strictly balanced.

Assume now that (i) holds. By Corollary 5.6(ii), $X$ and $Y$ are quasi-stable curves. Therefore, $(Y, \underline{d}')$ is obtained from $(X, \underline{d})$ via a sequence of blowing up of nodes, as depicted in Figure 7. In particular, there exists a surjective map $\sigma : Y \to X$ that contracts the new exceptional components produced by blowing up some of the nodes of $X$. Hence there exists a map $\Sigma : \mathcal{Y} \to \mathcal{X}$ over $B$ which is an isomorphism away from the fiber over $b_0$ and whose restriction over $b_0 \in B$ is the contraction map $\sigma : Y \to X$. Consider the line bundle $\widetilde{\mathcal{L}} := \Sigma^*(\mathcal{L})$ on $\mathcal{Y}$ and set $\widetilde{L} := \widetilde{\mathcal{L}}_{|\mathcal{Y}_{b_0}} = \sigma^*(L)$ and $\underline{\widetilde{d}} = \underline{\deg}(\widetilde{L})$. Property (10.5) implies that, up to shrinking $B$ around $b_0$, $\widetilde{\mathcal{L}}$ and $\mathcal{M}$ are isomorphic away from the central fiber $\mathcal{Y}_{b_0} = Y$; hence, by Lemma 10.9, we can find a Cartier divisor $T$ on $\mathcal{Y}$ supported on the central fiber $\mathcal{Y}_{b_0} = Y$ such that

$$\widetilde{\mathcal{L}} = \mathcal{M} \otimes \mathcal{O}_{\mathcal{Y}}(T). \tag{10.7}$$

This implies that the multidegrees $\underline{d}'$ and $\widetilde{\underline{d}}$ on $Y$ are equivalent in the sense of Definition 3.2. Since $\underline{d}$ is strictly balanced by Reduction 1, we can now apply Lemma 3.11 (with $Z = Y$ and $\sigma' = \mathrm{id}$) in order to conclude that $X = Y$ or, equivalently, $\mathcal{X} = \mathcal{Y}$. Since we have already observed that $(Y, \underline{d}') \preceq (X, \underline{d})$, we must have that $\underline{d} = \underline{d}'$. Combining this with (10.7), we get that $L := \mathcal{L}_{\mathcal{X}_{b_0}} = \mathcal{M}_{\mathcal{X}_{b_0}} = M$. We deduce that $b_0 = L = M \in \theta^{-1}(\mathrm{Im}(p))$ by combining Corollary 10.7 with (10.6), q.e.d.

Next, assume that (ii) holds. By Corollary 9.7, $X$ and $Y$ are quasi-p-stable. Therefore, $(Y, \underline{d}')$ is obtained from $(X, \underline{d})$ via a sequence of blowing up of nodes and cusps, as depicted in Figures 7 and 8. Thus we get again a map $\Sigma : \mathcal{Y} \to \mathcal{X}$ over $B$ with the same properties as in case (i) and the argument is completely analogous to the previous one.

Finally, assume that (iii) holds. In this case, the curve $X$ does not contain elliptic tails by assumption, whereas the curve $Y$ might contain some elliptic tails. Denote by $M'$ the line bundle on $Y$ such that $M'_{|Y_{\mathrm{ell}}^c} = M_{|Y_{\mathrm{ell}}^c}$ and $M'_{|F}$ is special for each elliptic tail $F$ of $Y$. As in the proof of ($\Leftarrow$) in Proposition 10.5, up to shrinking $B$ around $b_0$, we can find a relatively very ample line bundle $\mathcal{M}'$ on $\mathcal{Y}$ so that $\mathcal{M}'_{|Y} = M'$ and $\mathcal{M}'_{|\mathcal{Y}_b}$ has the same multidegree as $\mathcal{M}_{|\mathcal{Y}_b}$ for each $b \neq b_0$. Let $F_1, \ldots, F_n$ be the elliptic tails that compose the elliptic locus $Y_{\mathrm{ell}}$, denote by $p_i$ the intersection point of $F_i$ with $(F_i)^c$ and define the line bundle $\mathcal{N}$ as follows:

$$\mathcal{N} = \mathcal{M}' \otimes \mathcal{O}_{\mathcal{Y}}(4F_1 + \ldots + 4F_n).$$

Since $\mathcal{O}_{\mathcal{Y}}(4F_1 + \ldots + 4F_n)_{|F_i} = \mathcal{O}_{F_i}(-4p_i)$ and $\mathcal{M}'_{|F_i} = \mathcal{O}_{F_i}(4p_i)$, we deduce that $N := \mathcal{N}_{|Y}$ is trivial on each $F_i$. Therefore, the line bundle $\mathcal{N}$ is relatively globally generated and the induced map

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\phi_{|\mathcal{N}|}} & \mathbb{P}^r_B \\ & {\scriptstyle \pi_2}\searrow \quad \swarrow & \\ & B & \end{array}$$

embeds $\pi_2^{-1}(B \setminus \{b_0\})$ in $\mathbb{P}^r_B$ and contracts exactly $Y_{\mathrm{ell}} \subset Y$ over $b_0$. Denote by $\mathcal{Z}$ the image of $\mathcal{Y}$ in $\mathbb{P}^r_B$ via $\phi_{|\mathcal{N}|}$ and $\pi_3 : \mathcal{Z} \to B$ the restriction of $\mathbb{P}^r_B \to B$ to $\mathcal{Z} \subset \mathbb{P}^r_B$. Setting $Z = \mathcal{Z}_{b_0}$ and $\underline{d}'' := \underline{\deg}\, \mathcal{O}_{\mathbb{P}^r_B}(1)_{|Z}$, Proposition 10.5 implies that $(Z, \underline{d}'') \preceq (X, \underline{d})$. Since $Z$ does not contain elliptic tails, $(Z, \underline{d}'')$ is obtained from $(X, \underline{d})$ via a sequence of blowing up of nodes and cusps, as depicted in Figures 7 and 8. Hence, as in part (i), there exists a map $\Sigma : \mathcal{Z} \to \mathcal{X}$ which is an isomorphism away from the central fiber and whose restriction to the central fiber is the contraction of the exceptional components of $Z$ produced by blowing up some nodes and cusps of $X$. Summing up, we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\phi_{|\mathcal{N}|}} & \mathcal{Z} & \xrightarrow{\Sigma} & \mathcal{X}, \\ & \searrow^{\pi_2} & \downarrow^{\pi_3} & \swarrow^{\pi_1} & \\ & & B & & \end{array}$$

Composing $\phi_{|\mathcal{N}|}$ with $\Sigma$, we obtain a map $\Sigma' : \mathcal{Y} \to \mathcal{X}$ over $B$, whose restriction over $b_0$ contracts $Y_{\rm ell}$ and possibly some exceptional components of $Y$. As above, consider the line bundle $\widetilde{\mathcal{L}} := (\Sigma')^*\mathcal{L}$ and let $T$ be the Cartier divisor on $\mathcal{Y}$, supported on $\mathcal{Y}_{b_0} = Y$, such that

$$\widetilde{\mathcal{L}} = \mathcal{M} \otimes \mathcal{O}_{\mathcal{Y}}(T). \tag{10.8}$$

If we prove that $Y_{\rm ell} = \emptyset$, we are done since $\Sigma' : \mathcal{Y} \to \mathcal{X}$ contracts only exceptional components as in the case (i) and (ii). Suppose, by contradiction, that $Y$ admits an elliptic tail, which we denote by $F$. Set $C = F^c$ and denote by $p$ the intersection point of $F$ with $F^c$. Note that $\mathcal{O}_{\mathcal{Y}}(T)_{|F}$ is equal to the line bundle associated to some multiple of $p$, i.e. $F$ is special with respect to $\mathcal{O}_{\mathcal{Y}}(T)_{|F}$ (see Definition 6.2). On the other hand, since $F$ is contracted by $\Sigma'$ and $\widetilde{\mathcal{L}}$ is the pull-back of $\mathcal{L}$ via $\Sigma'$, we have that $\widetilde{\mathcal{L}}_{|F} = \mathcal{O}_F$. Therefore, using (10.8), we deduce that $F$ is also special with respect to $M$. This is absurd because the point $[Y \stackrel{|M|}{\hookrightarrow} \mathbb{P}^r]$ is Chow semistable and cannot have special elliptic tails by Theorem 9.1.

□

The following well-known Lemma (see e.g. the proof of [Ray70, Prop. 6.1.3]) was used in the above proof of Proposition 10.8.

**Lemma 10.9.** *Let $B$ be a smooth curve and let $f : \mathcal{X} \to B$ be a flat and proper morphism. Fix a point $b_0 \in B$ and set $B^* = B \setminus \{b_0\}$. Let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on $\mathcal{X}$ such that $\mathcal{L}_{|f^{-1}(B^*)} = \mathcal{M}_{|f^{-1}(B^*)}$. Then*

$$\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_{\mathcal{X}}(D),$$

*where $D$ is a Cartier divisor on $\mathcal{X}$ supported on $f^{-1}(b_0)$.*

**Remark 10.10.** Since in the proof of Proposition 10.8 we applied the *polystable replacement property*, a stronger result holds: if $[X \subset \mathbb{P}^r]$ is Hilbert (resp. Chow) semistable, $\mathcal{O}_X(1)$ is strictly balanced and one of the conditions of Proposition 10.8 is satisfied, then $[X \subset \mathbb{P}^r]$ is Hilbert (resp. Chow) polystable. This result can be seen as a partial converse to Corollary 7.6.

**Remark 10.11.**

(i) The above Proposition 10.8 is false in the case $\frac{7}{2}(2g-2) \leq d < 4(2g-2)$ if the curve $X$ is not assumed quasi-p-stable (see Remark 11.4 and also Theorem 11.5(2) and Theorem 13.2).

(ii) The careful reader will have noticed that we do not say anything if $d = \frac{7}{2}(2g-2)$ or $4(2g-2)$. Actually Proposition 10.8 can be extended to the cases $d = 4(2g-2)$ (for $X$ quasi-wp-stable) and $d = \frac{7}{2}(2g-2)$ (for $X$ quasi-p-stable). In our presentation, we will only use the extension to $d = 4(2g-2)$ but we are not yet ready to prove it. Its proof requires the analysis of stability of elliptic tails and will be dealt with later (see Proposition 12.4).

The following result is an immediate consequence of Proposition 10.8.

**Corollary 10.12.** *Let* $[X \stackrel{i}{\hookrightarrow} \mathbb{P}^r], [X \stackrel{i'}{\hookrightarrow} \mathbb{P}^r]$ *points in* $\mathrm{Hilb}_d$. *Assume that one of the conditions of Proposition 10.8 is satisfied and* $\underline{\deg}\, i^*\mathcal{O}_{\mathbb{P}^r}(1) = \underline{\deg}\, i'^*\mathcal{O}_{\mathbb{P}^r}(1)$. *Then* $[X \stackrel{i}{\hookrightarrow} \mathbb{P}^r]$ *belongs to* $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ *(resp.* $\mathrm{Hilb}_d^{ss}$*) if and only if* $[X \stackrel{i'}{\hookrightarrow} \mathbb{P}^r]$ *belongs to* $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ *(resp.* $\mathrm{Hilb}_d^{ss}$*).*

*Proof.* Let us first prove the statement for the Chow semistability. Assume that $[X \stackrel{i}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$. This is equivalent to saying that $[X \stackrel{i}{\hookrightarrow} \mathbb{P}^r] \in M_X^{\underline{d}}$ where $\underline{d} := \underline{\deg}\, i^*\mathcal{O}_{\mathbb{P}^r}(1) = \underline{\deg}\, i'^*\mathcal{O}_{\mathbb{P}^r}(1)$. In particular, $M_X^{\underline{d}} \neq \emptyset$; hence, from Proposition 10.8 and Corollary 10.7, we deduce that $[X \stackrel{i'}{\hookrightarrow} \mathbb{P}^r] \in M_X^{\underline{d}}$, or in other words $[X' \stackrel{i'}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$, q.e.d.

The proof for the Hilbert semistability is similar: we can define a stratification of

$$\mathrm{Hilb}_d^{ss,o} := \{[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss} \ : \ X \text{ is connected}\},$$

whose strata are given by

$$\widetilde{M}_X^{\underline{d}} = \{[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss,o} \ : \ \underline{\deg}\, \mathcal{O}_X(1) = \underline{d}\} \subseteq M_X^{\underline{d}}.$$

It is clear that Propositions 10.5 and 10.8 remain valid if we substitute $M_X^{\underline{d}}$ with $\widetilde{M}_X^{\underline{d}}$. Therefore, the above proof for the Chow semistability extends verbatim to the Hilbert semistability.

□

## 11. Semistable, polystable and stable points (part I)

The aim of this section is to describe the points of $\mathrm{Hilb}_d$ that are Hilbert or Chow semistable, polystable and stable for

$$2(2g-2) < d \leq \frac{7}{2}(2g-2) \quad \text{and} \quad d > 4(2g-2).$$

The range $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$ will be studied later.

Let us begin with the semistable points.

**Theorem 11.1.** *Consider a point* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *and assume that* $X$ *is connected.*

*(1) If* $d > 4(2g-2)$ *then the following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^r]$ *is Hilbert semistable;*

*(ii)* $[X \subset \mathbb{P}^r]$ *is Chow semistable;*

*(iii)* *$X$ is quasi-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*

*(iv)* *$X$ is quasi-stable and $\mathcal{O}_X(1)$ is properly balanced;*

*(v)* *$X$ is quasi-stable and $\mathcal{O}_X(1)$ is balanced.*

*(2)* *If $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$ then the following conditions are equivalent:*

*(i)* *$[X \subset \mathbb{P}^r]$ is Hilbert semistable;*

*(ii)* *$[X \subset \mathbb{P}^r]$ is Chow semistable;*

*(iii)* *$X$ is quasi-p-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*

*(iv)* *$X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is properly balanced;*

*(v)* *$X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is balanced.*

*Proof.* Let us first prove part (1).

(1i) $\Longrightarrow$ (1ii) follows from Fact 4.1.

(1ii) $\Longrightarrow$ (1iii) follows from the potential stability theorem (see Fact 4.20) and Corollary 5.6(ii).

(1iii) $\Longrightarrow$ (1iv) is clear.

(1iv) $\Longleftrightarrow$ (1v) follows from Remark 5.7, using that $\mathcal{O}_X(1)$ is ample.

(1iv) $\Longrightarrow$ (1i) First of all, we make the following

Reduction: We can assume that $\mathcal{O}_X(1)$ is strictly balanced.

Indeed, by Lemma 7.4(ii), there exists an isotrivial specialization $(X, \mathcal{O}_X(1)) \rightsquigarrow (X', L')$ such that $X'$ is quasi-stable and $L'$ is a strictly balanced line bundle on $X'$ of total degree $d$. According to Theorem 17.5 and using that $d > 4(2g-2)$, we conclude that $L'$ is very ample and non-special. Therefore, by choosing a basis of $H^0(X', L')$, we get a point $[X' \stackrel{|L'|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$. According to Theorem 7.5, $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ if and only if $[X' \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$. Therefore, up to replacing $X$ with $X'$, we can assume that $\mathcal{O}_X(1)$ is strictly balanced, q.e.d.

Now, since $X$ is quasi-stable, we can find a smooth curve $B \stackrel{f}{\hookrightarrow} \mathrm{Hilb}_d$ and a point $b_0 \in B$ such that, if we denote by $\mathbb{P}^r \times B \stackrel{i}{\hookleftarrow} \mathcal{X} \stackrel{\pi}{\to} B$ the pull-back via $f$ of the universal family over $\mathrm{Hilb}_d$ and we set $\mathcal{L} := i^*(\mathcal{O}_{\mathbb{P}^r}(1) \boxtimes \mathcal{O}_B)$, then $[\mathcal{X} \stackrel{i}{\hookrightarrow} \mathbb{P}^r \times B]_{b_0} = [X \subset \mathbb{P}^r]$ and $\mathcal{X}_{|\pi^{-1}(b)}$ is a connected smooth curve for every $b \in B \setminus \{b_0\}$. Note that, by construction, $\pi$ is a family of quasi-stable curves of genus $g$. As in the proof of Proposition 10.8, we can now apply the *semistable replacement property*, which implies that, up to replacing $B$ with a finite cover ramified over $b_0$, we can find two maps $g : B \to \mathrm{Hilb}_d$ and $h : B \setminus \{b_0\} \to \mathrm{SL}_{r+1}$ such that

$$f(b) = h(b) \cdot g(b) \text{ for every } b_0 \neq b \in B, \tag{11.1}$$

$$g(b_0) \text{ is Hilbert polystable.} \tag{11.2}$$

We denote by $\mathbb{P}^r \times B \stackrel{i'}{\hookleftarrow} \mathcal{Y} \stackrel{\pi'}{\to} B$ the pull-back via $g$ of the universal family over $\mathrm{Hilb}_d$ and we set $\mathcal{M} := (i')^*(\mathcal{O}_{\mathbb{P}^r}(1) \boxtimes \mathcal{O}_B)$. Property (11.1) implies that, up to shrinking again $B$ around $b_0$, we have that

$$(\mathcal{X}, \mathcal{L})_{|\pi^{-1}(B\setminus\{b_0\})} \cong (\mathcal{Y}, \mathcal{M})_{|(\pi')^{-1}(B\setminus\{b_0\})}. \tag{11.3}$$

Note that this fact together with (11.2) and the potential stability Theorem (Fact 4.20) implies that $\pi'$ is also a family of quasi-stable curves of genus $g$.

Consider now the stable reductions $\mathrm{s}(\pi) : \mathrm{s}(\mathcal{X}) \to \mathrm{B}$ of $\pi : \mathcal{X} \to B$ and $\mathrm{s}(\pi') : \mathrm{s}(\mathcal{Y}) \to \mathrm{B}$ of $\pi' : \mathcal{Y} \to B$ (see Remark 2.12). From (11.3), it follows that $\mathrm{s}(\pi)$ and $\mathrm{s}(\pi')$ are two families of stable curves, which are isomorphic away from the fibers over $b_0$. Since the stack $\overline{\mathcal{M}}_g$ of stable curves is separated, we conclude that

$$\begin{array}{ccc} \mathrm{s}(\mathcal{X}) & \xrightarrow{\cong} & \mathrm{s}(\mathcal{Y}) \\ & {\scriptstyle \mathrm{s}(\pi)} \searrow \quad \swarrow {\scriptstyle \mathrm{s}(\pi')} & \\ & B & \end{array} \tag{11.4}$$

Therefore, $\pi$ and $\pi'$ are two families of quasi-stable curves with the same stable reduction (from now on, we identify $\mathrm{s}(\mathcal{X}) \stackrel{\mathrm{s}(\pi)}{\longrightarrow} \mathrm{B}$ and $\mathrm{s}(\mathcal{Y}) \stackrel{\mathrm{s}(\pi')}{\longrightarrow} \mathrm{B}$ via the above isomorphism). If we blow-up all the nodes of the fiber over $b_0$ of the stable reduction $\mathrm{s}(\pi) = \mathrm{s}(\pi')$, we get a new family of quasi-stable curves $\widetilde{\pi} : \mathcal{Z} \to B$ with the same stable reduction as that of $\pi$ and of $\pi'$, which moreover dominates $\pi$ and $\pi'$, i.e., such that there exists a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & {\scriptstyle \Sigma} \swarrow & & \searrow {\scriptstyle \Sigma'} & \\ \mathcal{X} & & \downarrow {\scriptstyle \widetilde{\pi}} & & \mathcal{Y} \\ & {\scriptstyle \pi} \searrow & & \swarrow {\scriptstyle \pi'} & \\ & & B & & \end{array} \tag{11.5}$$

where the morphisms $\Sigma$ and $\Sigma'$ induce an isomorphism of the corresponding stable reductions. Equivalently, the maps $\Sigma$ and $\Sigma'$ are obtained by blowing down some of the exceptional components of the fiber of $\mathcal{Z}$ over $b_0$. If we set $\widetilde{\mathcal{L}} := \Sigma^*(\mathcal{L})$ and $\widetilde{\mathcal{M}} := (\Sigma')^*(\mathcal{M})$, then (11.3) gives that

$$\widetilde{\mathcal{L}}_{\widetilde{\pi}^{-1}(B\setminus b_0)} \cong \widetilde{\mathcal{M}}_{\widetilde{\pi}^{-1}(B\setminus b_0)}.$$

Lemma 10.9 now gives that there exists a Cartier divisor $D$ on $\mathcal{Z}$ supported on $\widetilde{\pi}^{-1}(b_0)$ such that

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{M}} \otimes \mathcal{O}_{\mathcal{Z}}(D). \tag{11.6}$$

We now set $(X, L) := (\mathcal{X}, \mathcal{L})_{b_0}$ and $\underline{d} := \underline{\deg} L$, $(Y, M) := (\mathcal{Y}, \mathcal{M})_{b_0}$ and $\underline{d}' := \underline{\deg} M$, $Z := \mathcal{Z}_{b_0}$, $\widetilde{L} := \widetilde{\mathcal{L}}_{b_0}$ and $\widetilde{\underline{d}} := \deg \widetilde{L}$, $\widetilde{M} := \widetilde{\mathcal{M}}_{b_0}$ and $\widetilde{\underline{d}'} := \deg \widetilde{M}$. Equation (11.6) gives that $\widetilde{\underline{d}}$ and $\widetilde{\underline{d}'}$ are equivalent on $Z$. Moreover, $\underline{d}$ is strictly balanced by the above

Reduction and $\underline{d}$' is strictly balanced by the assumption (11.2) together with Corollary 7.6. Therefore, we can apply Lemma 3.11 twice to conclude that $\mathcal{X} = \mathcal{Y}$. Now, the relation (11.3) together with the Lemma 10.9 imply that there exists a Cartier divisor $D'$ on $\mathcal{X} = \mathcal{Y}$ supported on $\pi^{-1}(b_0)$ such that

$$\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_{\mathcal{X}}(D'). \tag{11.7}$$

In particular, we get that $\underline{d}$ is equivalent to $\underline{d}'$. Since $\underline{d}$ and $\underline{d}'$ are strictly balanced, Lemma 3.10 implies that $\underline{d} = \underline{d}'$. Since $[\mathcal{Y} \stackrel{i'}{\hookrightarrow} \mathbb{P}^r \times B]_{b_0} = [Y \hookrightarrow \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$ by assumption (11.2), Corollary 10.12 gives that $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss}$, q.e.d.

The proof of part (2) is similar: it is enough to replace quasi-stable curves by quasi-p-stable curves (using Corollaty 9.7), to replace the stable reduction by the p-stable reduction and using the fact that the stack $\overline{\mathcal{M}}_g^{\mathrm{p}}$ of p-stable curves of genus $g \geq 3$ is separated.

□

From the above Theorem 11.1, we can deduce a description of the Hilbert and Chow polystable and stable points of $\mathrm{Hilb}_d$.

**Corollary 11.2.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ and assume that $X$ is connected.*

*(1) If $d > 4(2g-2)$ then the following conditions are equivalent:*
- *(i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;*
- *(ii) $[X \subset \mathbb{P}^r]$ is Chow polystable;*
- *(iii) $X$ is quasi-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;*
- *(iv) $X$ is quasi-stable and $\mathcal{O}_X(1)$ is strictly balanced.*

*(2) If $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$ then the following conditions are equivalent:*
- *(i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;*
- *(ii) $[X \subset \mathbb{P}^r]$ is Chow polystable;*
- *(iii) $X$ is quasi-p-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;*
- *(iv) $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is strictly balanced.*

*Proof.* Let us prove part (1).

(1i) $\Longleftrightarrow$ (1ii): from Theorem 11.1(1) we get that the Hilbert semistable locus inside $\mathrm{Hilb}_d$ is equal to the Chow semistable locus. Since a point of $\mathrm{Hilb}_d$ is Hilbert (resp. Chow) polystable if and only if it is Hilbert (resp. Chow) semistable and its orbit is closed inside the Hilbert (resp. Chow) semistable locus, we conclude that also the locus of Hilbert polystable points is equal to the locus of Chow polystable points.

(1ii) $\Longrightarrow$ (1iii) follows from the potential stability theorem (see Fact 4.20), Corollary 5.6(ii) and Corollary 7.6.

(1iii) $\Longrightarrow$ (1iv) is obvious.

(1iv) $\Longrightarrow$ (1i) follows from Theorem 11.1 and Remark 10.10. □

**Corollary 11.3.** *Consider a point* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *and assume that* $X$ *is connected.*

*(1) If* $d > 4(2g-2)$ *then the following conditions are equivalent:*
   *(i)* $[X \subset \mathbb{P}^r]$ *is Hilbert stable;*
   *(ii)* $[X \subset \mathbb{P}^r]$ *is Chow stable;*
   *(iii)* $X$ *is quasi-stable, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special;*
   *(iv)* $X$ *is quasi-stable and* $\mathcal{O}_X(1)$ *is stably balanced.*

*(2) If* $2(2g-2) < d < \frac{7}{2}(2g-2)$ *and* $g \geq 3$ *then the following conditions are equivalent:*
   *(i)* $[X \subset \mathbb{P}^r]$ *is Hilbert stable;*
   *(ii)* $[X \subset \mathbb{P}^r]$ *is Chow stable;*
   *(iii)* $X$ *is quasi-p-stable, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special;*
   *(iv)* $X$ *is quasi-p-stable and* $\mathcal{O}_X(1)$ *is stably balanced.*

*Proof.* Let us prove part (1).

(1ii) $\Longrightarrow$ (1i) follows from Fact 4.1.

(1i) $\Longrightarrow$ (1iii) follows from the potential stability theorem (see Fact 4.20) and Theorem 7.1.

(1iii) $\Longrightarrow$ (1iv) is obvious.

(1iv) $\Longrightarrow$ (1ii): from Corollary 11.2(1), we get that $[X \subset \mathbb{P}^r]$ is Chow polystable. Lemma 3.9 gives that $\widetilde{X} := \overline{X \setminus X_{\mathrm{exc}}}$ is connected; hence, combining Lemma 6.1 and Theorem 6.4, we deduce that $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is a finite group. This implies that $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^s)$ since a point of $\mathrm{Hilb}_d$ is Hilbert (resp. Chow) stable if and only if it is Hilbert (resp. Chow) polystable and it has finite stabilizers with respect to the action of $\mathrm{PGL}_{r+1}$.

The proof of part (2) is similar, using the Potential pseudo-stability Theorem 5.1 and Corollary 11.2(2). □

The characterization of the GIT semistable locus for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ is a bit more intricate and requires other arguments. We can understand this by the following remark.

**Remark 11.4.** Let $X = C \cup E$ be a curve of genus $g \geq 3$ whose only singularity is a tacnode with the line $E$ and let us fix a balanced line bundle $L$ of degree $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$. Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $\mathcal{O}_X(1) = L$ and let us try going over again the argument of the proof of Theorem 11.1(1). Using the same notation, since $X$ is quasi-p-stable, we can find a polarized family $(\mathcal{X} \to B, \mathcal{L})$ over a smooth curve $B \stackrel{f}{\hookrightarrow} \mathrm{Hilb}_d$ such that $[\mathcal{X} \stackrel{i}{\hookrightarrow} \mathbb{P}^r \times B]_{b_0} = [X \subset \mathbb{P}^r]$ and $\mathcal{X}_{|\pi^{-1}(b)}$ is a connected smooth curve for every $b \in B \setminus \{b_0\}$. We apply the *polystable replacement property* and we obtain a new polarized family $(\mathcal{Y} \to B, \mathcal{M})$. Consider now the p-stable reductions

$\mathrm{ps}(\pi) : \mathrm{ps}(\mathcal{X}) \to \mathrm{B}$ and $\mathrm{ps}(\pi) : \mathrm{ps}(\mathcal{Y}) \to \mathrm{B}$. Up to shrinking $B$ around $b_0$ we have

$$(\mathcal{X}, \mathcal{L})_{|\pi^{-1}(B\setminus\{b_0\})} \cong (\mathcal{Y}, \mathcal{M})_{|(\pi')^{-1}(B\setminus\{b_0\})} \tag{11.8}$$

so that $\mathrm{ps}(\pi) : \mathrm{ps}(\mathcal{X}) \to \mathrm{B}$ and $\mathrm{ps}(\pi) : \mathrm{ps}(\mathcal{Y}) \to \mathrm{B}$ are isomorphic away from the fibers over $b_0$, hence isomorphic everywhere since the stack $\overline{\mathcal{M}}_g^{\mathrm{p}}$ of p-stable curves is separated for $g \geq 3$. In particular $\mathrm{ps(Y)} \cong \mathrm{ps(X)}$. There are three cases:

(1) $Y \cong X$;
(2) $Y \cong \mathrm{wps(X)}$, or in other words, $Y$ is irreducible with only a cusp and no nodes;
(3) $Y$ admits an elliptic tail.

We claim that (3) occurs. Indeed, case (1) is absurd because $[Y \overset{|M|}{\hookrightarrow} \mathbb{P}^r]$ is Chow polystable by construction but the tacnodes with a line are not Chow polystable for $d = \frac{7}{2}(2g-2)$ by Theorem 9.8 and they are Chow unstable for $d > \frac{7}{2}(2g-2)$ by Theorem 9.3. Suppose by contradiction that (2) occurs. We have a map

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathrm{wps}} & \mathcal{Y} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & B & \end{array} \tag{11.9}$$

Denote by $\widetilde{\mathcal{L}}$ the pull-back of $\mathcal{M}$ via wps, $L' = \widetilde{\mathcal{L}}_{|b_0}$, $\underline{d} = \underline{\deg} L = (\deg_C L, \deg_E L)$ and $\underline{d}' = \underline{\deg} L'$. By Lemma 10.9, there exists a Cartier divisor $T$ on $\mathcal{X}$ such that $\widetilde{\mathcal{L}} = \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(T)$. This implies that

$$(d-1, 1) = \underline{d} \equiv \underline{d}' = (d, 0),$$

which is absurd since $|C \cap E| = 2$. We conclude that in $\mathrm{Hilb}_d$ there are examples of Chow semistable points that admit elliptic tails. This fact is the origin of some new difficulties in the range $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$. So far our techniques worked well since the stacks $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_g^{\mathrm{p}}$ (for $g \geq 3$) are separated, but for $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ they are not enough to determine the semistable locus of $\mathrm{Hilb}_d$ because we have to work with the stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ of wp-stable curves, which is not separated. Notice also that, if we could use the same techniques in the range $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ successfully, we would prove, for instance, the completeness result of Proposition 10.8 for every quasi-wp-stable curve, which is false since special elliptic curves are Chow unstable by Theorem 9.1.

To conclude this section we study the extremal case $d = \frac{7}{2}(2g-2)$, a very interesting case, because the semistable locus with respect to Hilbert stability and Chow stability are different.

**Theorem 11.5.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $d = \frac{7}{2}(2g-2)$ and $g \geq 3$ and assume that $X$ is connected.*

*(1) The following conditions are equivalent:*

*(i) $[X \subset \mathbb{P}^r]$ is Hilbert semistable;*

*(ii) $X$ is quasi-p-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*

*(iii) $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is properly balanced;*

*(iv) $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is balanced.*

*(2) The following conditions are equivalent:*

*(i) $[X \subset \mathbb{P}^r]$ is Chow semistable;*

*(ii) $X$ is quasi-wp-stable without special elliptic tails, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*

*(iii) $X$ is quasi-wp-stable without special elliptic tails and $\mathcal{O}_X(1)$ is properly balanced;*

*(iv) $X$ is quasi-wp-stable without special elliptic tails and $\mathcal{O}_X(1)$ is balanced.*

*Proof.* The proof of (1) is analogous to Theorem 11.1(2) since for $d = \frac{7}{2}(2g-2)$ the elliptic tails are Hilbert unstable by Theorem 9.6. Let us prove (2).

(2i) $\Longrightarrow$ (2ii) follows from the Potential pseudo-stability Theorem 5.1 and Theorem 9.1.

(2ii) $\Longrightarrow$ (2iii) is clear.

(2iii) $\Longrightarrow$ (2iv) is obvious.

(2iv) $\Longrightarrow$ (2i). By Theorem 9.8 and Theorem 7.5 we can assume that

(a) each elliptic tail $F$ of degree 4 contains an elliptic tail $F'$ of degree 3 as a subcurve;

(b) each elliptic tail $F$ of degree 3 is tacnodal and $F^c$ consists of the union of subcurves $C$ and $E \cong \mathbb{P}^1$, where $E$ meets $C$ and $F$ in one point;

Let $n$ be the number of elliptic tails of degree 3. We prove our statement by induction on $n$. If $n = 0$, then $[X \subset \mathbb{P}^r]$ is Chow semistable by (1). Suppose that $n > 0$. Consider an elliptic tail $F$ of degree 3 and the 1ps $\rho$ as in (9.10) which, as observed before Theorem 9.8, satisfies $e_{X,\rho} = 2d\dfrac{w(\rho)}{r+1} = \dfrac{7}{3}w(\rho)$. By Theorem 9.8 there exists $[Y \subset \mathbb{P}^r] \in A_{\rho^{-1}}([X \subset \mathbb{P}^r])$ that satisfies (2iv) and contains $n-1$ elliptic tail. By induction $[Y \subset \mathbb{P}^r]$ is Chow semistable and Fact 4.10 implies that also $[X \subset \mathbb{P}^r]$ is Chow semistable. $\square$

**Corollary 11.6.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $d = \frac{7}{2}(2g-2)$ and $g \geq 3$ and assume that $X$ is connected.*

*(1) The following conditions are equivalent:*

*(i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;*

*(ii) $X$ is quasi-p-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;*

*(iii) $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is strictly balanced.*

*(2) The following conditions are equivalent:*

*(i) $[X \subset \mathbb{P}^r]$ is Chow polystable;*

*(ii)* *$X$ is quasi-wp-stable, each elliptic tail of $X$ is tacnodal, each tacnode is contained in an elliptic tail, $X$ is non-degenerate and linearly normal in $\mathbb{P}^r$, $\mathcal{O}_X(1)$ is strictly balanced and non-special;*

*(iii)* *$X$ is quasi-wp-stable, each elliptic tail is tacnodal, each tacnode is contained in an elliptic tail and $\mathcal{O}_X(1)$ is strictly balanced.*

*Proof.* Since for $d = \frac{7}{2}(2g-2)$ the elliptic tails are Hilbert unstable by Theorem 9.6, the argument of Corollary 11.2(2) goes through for (1). Let us prove (2).

(2i) $\Longrightarrow$ (2ii) is implied by Theorem 9.8.

(2ii) $\Longrightarrow$ (2iii) is clear.

(2iii) $\Longrightarrow$ (2i). Let $X$ and $L := \mathcal{O}_X(1)$ be as in (2iii). By Theorem 9.8 and Theorem 7.5 we have to work under the assumptions (a) and (b) of the proof of Theorem 9.6. Let $n$ be the number of elliptic tails of degree 3. We prove our statement by induction over $n$. (For a sketch of the proof strategy, see Construction 13.1.) If $n = 0$, $[X \subset \mathbb{P}^r]$ is Hilbert polystable by (1), hence also Chow polystable. Suppose now that $n > 0$. Consider an elliptic tail $F$ of degree 3 and denote by $C_1 = F^c$ and $\{p\} = F \cap C_1$. Let $C_2$ be a smooth curve of genus $g$, $q$ a point of $C_2$ and $L'_{C_2} \in \mathrm{Pic}^{d+3}(C_2)$. Denote by $(X', L')$ the couple consisting of a curve $X'$ of genus $g'$ and a line bundle $L'$ on $X'$ obtained from $(X, L)$ by replacing $F$ with $(C_2, q, L'_{C_2})$. The line bundle $L'$ has degree $d' = 2d$ and is very ample, hence we can consider the point $[X' \subset \mathbb{P}^{r'}] \in \mathrm{Hilb}_{d',g'}$ with $\mathcal{O}_{X'}(1) = L'$. We notice that

$$\nu' := \frac{d'}{2g'-2} = \frac{d}{2g-2} =: \nu. \tag{11.10}$$

Now, we claim that $L'$ is strictly balanced. As we noticed in Remark 3.8, it suffices to check the basic inequality (3.1) for each connected subcurves such that its complementary is connected. Let $D \subset X'$ a connected subcurve. If $D = C_2$ then obviously the basic inequality (3.1) is satisfied. Otherwise, up to replacing $D$ with $D^c$, we can assume that $D$ does not contain $C_2$ as a subcurve. This implies that $D$ can be seen as a subcurve of $X$. Since $\deg L_{|D} = \deg L'_{|D}$ and $|D \cap \overline{X \setminus D}| = |D \cap \overline{X' \setminus D}|$, the basic inequality (3.1) is satisfied by (11.10). Now, the point $[X' \subset \mathbb{P}^{r'}]$ admits $n-1$ elliptic tails, hence it is Chow polystable. Consider now $[Y \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ such that $Y = F \cup E \cup C$, where $C$ is smooth, $E \cong \mathbb{P}^1$, $E$ meets $F$ and $C$ in one point and $\mathcal{O}_Y(1)$ is balanced. By Theorem 11.5 this point is Chow semistable. Let $[Y' \subset \mathbb{P}^r] \in \overline{\mathrm{Orb}([Y \subset P^r])} \cap \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$. Denoting by $\underline{d}$ and $\underline{d}'$ the multidegrees of $\mathcal{O}_Y(1)$ and $\mathcal{O}_{Y'}(1)$ respectively, by Proposition 10.5 we get that $(Y', \underline{d}') \preceq (Y, \underline{d})$, so that $Y \cong Y'$ and $\dim(\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([Y \subset \mathbb{P}^r])) = \dim(\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([Y \subset \mathbb{P}^r]))$. This implies that $[Y \subset \mathbb{P}^r]$ is Chow polystable. Since (11.10) holds and $(X, L)$ can be obtained again from $(X', L')$ by replacing $C_2$ with $(F, p, L|_F)$, $[X \subset \mathbb{P}^r]$ is Chow polystable by Corollary 8.6 and we are done. ☐

**Corollary 11.7.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $d = \frac{7}{2}(2g-2)$ and $g \geq 3$ and assume that $X$ is connected.*

*(1) The following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^r]$ *is Hilbert stable;*

*(ii)* $X$ *is quasi-p-stable, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special;*

*(iii)* $X$ *is quasi-p-stable and* $\mathcal{O}_X(1)$ *is stably balanced.*

*(2) The following conditions are equivalent:*

*(i)* $[X \subset \mathbb{P}^r]$ *is Chow stable;*

*(ii)* $X$ *is quasi-p-stable without tacnodes, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special;*

*(iii)* $X$ *is quasi-p-stable without tacnodes and* $\mathcal{O}_X(1)$ *is stably balanced.*

*Proof.* Since for $d = \frac{7}{2}(2g-2)$ the elliptic tails are Hilbert unstable by Theorem 9.6, the argument of Corollary 11.3(2) goes through for (1). Let us prove (2).

(2i) $\Longrightarrow$ (2ii) follows from Corollary 11.6 and Theorem 9.6.

(2ii) $\Longrightarrow$ (2iii) is clear.

(2iii) $\Longrightarrow$ (2i): from Corollary 11.6(2), we get that $[X \subset \mathbb{P}^r]$ is Chow polystable. Since $\mathcal{O}_X(1)$ is stably balanced, Lemma 3.9 gives that $\widetilde{X} := \overline{X \setminus X_{\text{exc}}}$ is connected; hence, combining Lemma 6.1 and Theorem 6.4, we deduce that $\text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is a finite group. This implies that $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^s)$ since a point of $\text{Hilb}_d$ is Hilbert (resp. Chow) stable if and only if it is Hilbert (resp. Chow) polystable and it has finite stabilizers with respect to the action of $\text{PGL}_{r+1}$. □

## 12. Stability of elliptic tails

In this section we will use the criterion of stability for tails (Proposition 8.3) in order to study the stability of elliptic curves for $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$. We notice that in this range - by the basic inequality (3.1) - it suffices to consider the elliptic curves of degree 4. In particular if $F$ is an elliptic curve of $[X \subset \mathbb{P}^r]$, then $r_1 := h^0(F, \mathcal{O}_X(1)|_F) - 1 = 3$.

**Lemma 12.1.** *Let* $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$ *and let* $[X \subset \mathbb{P}^r] \in \text{Hilb}_\text{d}$ *such that* $X = F \cup C$ *where* $F$ *is an elliptic tail (smooth, nodal, cuspidal or reducible nodal). Denote by* $\{p\} = F \cap C$ *and*

$$\mathcal{O}_X(1) = (\mathcal{O}_X(1)|_F, L_2 := \mathcal{O}_X(1)|_C) \in \text{Pic}^4(F) \times \text{Pic}^{d-4}(C).$$

*Let* $(F', q)$ *be a pointed elliptic curve and denote by* $X'$ *be the curve obtained from* $X$ *by replacing* $F$ *with* $(F', q)$*. Then*

*(1) If* $[X \subset \mathbb{P}^r]$ *is Hilbert semistable (resp. stable) then* $[X' \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$ *is Hilbert semistable (resp. stable) for each properly balanced line bundle*

$$L \in (\text{Pic}^4(F') \setminus \{\mathcal{O}_{F'}(4p)\}) \times \{L_2\}$$

*(2) If* $[X \subset \mathbb{P}^r]$ *is Chow semistable (resp. stable) then*

*(i) If* $\frac{7}{2}(2g-2) < d < 4(2g-2)$ *then* $[X' \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r]$ *is Chow semistable (resp. stable) for each properly balanced line bundle*

$$L \in (\mathrm{Pic}^4(F') \setminus \{\mathcal{O}_{F'}(4p)\}) \times \{L_2\}$$

*(ii) If* $d = 4(2g-2)$ *then* $[X' \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r]$ *is Chow semistable for each properly balanced line bundle*

$$L \in \mathrm{Pic}^4(F') \times \{L_2\}$$

*(resp. Chow stable if* $F'$ *is not cuspidal and* $L \in (\mathrm{Pic}^4(F') \setminus \{\mathcal{O}_{F'}(4p)\}) \times \{L_2\}$*).*

*Proof.* Consider $X'$ and a properly balanced line bundle $L = (L_1, L_2) \in \mathrm{Pic}^4(F') \times \mathrm{Pic}^{d-4}(C)$. By Theorem 17.5(iiia) the line bundle $L$ is very ample and non-special, hence we can consider the point $[X \stackrel{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$. Let $\rho_1$ and $\rho_2$ be two one-parameter subgroups diagonalized by a system of coordinates $\{x_1, \ldots, x_{r+1}\}$ of type (8.1), i.e. such that

$$\langle F' \rangle = \bigcap_{i=5}^{r+1} \{x_i = 0\} \quad \text{and} \quad \langle C \rangle = \bigcap_{i=1}^{3} \{x_i = 0\},$$

and having weights

$$\rho_1(t) \cdot x_i = \begin{cases} t^{w_i} x_i & \text{if } i \leq 3, \\ x_i & \text{if } i \geq 4, \end{cases} \quad \text{and} \quad \rho_2(t) \cdot x_i = \begin{cases} x_i & \text{if } i \leq 4, \\ t^{w_i} x_i & \text{if } i \geq 5. \end{cases} \tag{12.1}$$

By Proposition 8.3, it suffices to prove that $[X' \subset \mathbb{P}^r]$ is Chow or Hilbert (semi-)stable with respect to any such $\rho_1$ and $\rho_2$.

By Remark 8.5 and the Hilbert-Mumford criterion (see Fact 4.2 and Fact 4.3), if $[X \subset \mathbb{P}^r]$ is Hilbert semistable (resp. Chow semistable) we have

$$W_{X',\rho_2}(m) = W_{X,\rho_2}(m) \leq \frac{w(\rho_2)}{r+1} mP(m) \quad \left(\text{resp. } e_{X',\rho_2} = e_{X,\rho_2} \leq \frac{2d}{r+1} w(\rho_2)\right),$$

while if $[X \subset \mathbb{P}^r]$ is Hilbert stable (resp. Chow stable) then

$$W_{X',\rho_2}(m) = W_{X,\rho_2}(m) < \frac{w(\rho_2)}{r+1} mP(m) \quad \left(\text{resp. } e_{X',\rho_2} = e_{X,\rho_2} < \frac{2d}{r+1} w(\rho_2)\right).$$

This proves the Hilbert or Chow (semi-)stability of $[X' \subset \mathbb{P}^r]$ with respect to $\rho_2$.

The Hilbert or Chow (semi-)stability of $[X' \subset \mathbb{P}^r]$ with respect to $\rho_1$ will follow from the next lemma, that completes our proof. □

**Lemma 12.2.** *Let* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb_d}$ *such that* $X = F \cup C$ *where* $F$ *is an elliptic tail (smooth, nodal, cuspidal or reducible nodal) and the line bundle*

$$L := \mathcal{O}_X(1) = (L_1 := L_{|F}, L_2 := L_{|C}) \in \mathrm{Pic}^4(F) \times \mathrm{Pic}^{d-4}(C)$$

*is properly balanced. Let* $\rho_1$ *be a one-parameter subgroup as in (12.1). Then* *(i) if* $\frac{7}{2}(2g-2) < d < 4(2g-2)$ *and* $L_1 \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$ *then*

$$e_{F,\rho_1} < \frac{2d}{r+1}\, w(\rho_1)$$

*(ii) if* $d = 4(2g-2)$ *then*

$$e_{F,\rho_1} \le \frac{2d}{r+1}\, w(\rho_1) = \frac{16}{7}\, w(\rho_1).$$

*Moreover, if* $L_1 \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$ *then*

$$e_{F,\rho_1} < \frac{16}{7}\, w(\rho_1) \qquad \textit{if } F \textit{ is not cuspidal,} \tag{12.2}$$

$$W_{F,\rho_1}(m) < \frac{w(\rho_1)}{7}\, mP(m) \ \textit{ for } m \gg 0 \qquad \textit{if } F \textit{ is cuspidal.}$$

*Proof.* Since $e_{F,\rho_1}$ does not depend on $C$, we can prove these two claims by considering $F$ as an elliptic tail of polarized curves whose semistability is known.

Firstly assume that $F$ is smooth, nodal or reducible nodal. Let $C$ be a smooth curve of genus 2 and consider the new curve $X = F \cup C$ with $\{p\} = F \cap C$ embedded in $\mathbb{P}^{11}$ via a properly balanced line bundle $M = (M_1, M_2)$ (indeed $M$ is very ample and non-special by Theorem 17.5(iiia)) with $M_1 \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$, $\deg M_{|F} = 4$ and $\deg M_{|C} = 10$. Since the curve $X$ is quasi-wp-stable, $\frac{d}{2g-2} = \frac{7}{2}$ and $M$ is properly balanced, by Theorem 11.5 we know that $[X \subset \mathbb{P}^{11}]$ with $M = \mathcal{O}_X(1)$ is Chow semistable; hence

$$e_{F,\rho_1} \le \frac{2d}{r+1} w(\rho_1) = \frac{7}{3}\, w(\rho_1) \tag{12.3}$$

by the Hilbert-Mumford numerical criterion (Fact 4.3). In the same way we can consider another properly balanced line bundle $M'$ such that $\deg M'_{|F} = 4$ and $\deg M'_{|C} = 13$. Since the curve $X$ is quasi-stable, $4 < \frac{d}{2g-2} = \frac{17}{4} < 4,5$ and $M'$ is stably balanced, by Corollary 11.3(1) we know that $[X \subset \mathbb{P}^{14}]$ with $M' = \mathcal{O}_X(1)$ is Chow stable; hence

$$e_{F,\rho_1} < \frac{2d}{r+1} w(\rho_1) = \frac{34}{15}\, w(\rho_1). \tag{12.4}$$

Now, consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ that satisfies the hypothesis of our lemma. Assume that $\frac{7}{2}(2g-2) < d \le 4(2g-2)$ and $L_1 \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$. Since $\frac{17}{15} < \frac{8}{7} \le \frac{d}{r+1} < \frac{7}{6}$, combining (12.3) and (12.4) we deduce that

$$\text{if } w(\rho_1) \ge 0, \text{ then } e_{F,\rho_1} < \frac{34}{15}\, w(\rho_1) \le \frac{2d}{r+1}\, w(\rho_1),$$

and

$$\text{if } w(\rho_1) < 0, \text{ then } e_{F,\rho_1} \le \frac{7}{3}\, w(\rho_1) < \frac{2d}{r+1}\, w(\rho_1),$$

so that (i) and (ii) are proved for smooth, nodal and reducible nodal elliptic tails under the hypothesis that $L_1 \ne \mathcal{O}_F(4p)$.

Let $X = F \cup C$ be a curve as above, with $F$ an irreducible elliptic tail (smooth, nodal, or cuspidal). By [HM10, Proposition 6], we know that $[X \subset \mathbb{P}^{13}]$ with $\mathcal{O}_X(1) = \omega_X^{\otimes 4}$ is strictly Chow semistable. Hence if $d = 4(2g-2)$ and $L_1 = \mathcal{O}_F(4p) = (\omega_X^{\otimes 4})_{|F}$ we get

$$e_{F,\rho_1} \leq \frac{2d}{r+1}\, w(\rho_1) = \frac{16}{7}\, w(\rho_1),$$

and the first part of (ii) is proved.

It remains to prove (i) and (ii) for the cuspidal case. Suppose that $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $F$ is cuspidal. In order to prove (i), it suffices to exhibit a non-special line bundle $L_1$ for which the inequality (12.2) is satisfied. Indeed, $\mathrm{Aut}(F,p)$ acts transitively on $\mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$ and we can apply Lemma 10.6. Consider the Chow semistable point $[Y \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ obtained in Remark 11.4 and denote by $F$ its elliptic tail. Since the semistability is an open condition, up to smoothing arbitrarily $Y$, we can assume that $F$ is smooth. Now, let $B \subseteq \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{Y})$ be a smooth curve such that

$$B \setminus \{b_0\} \subseteq (\mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}) \times \{\mathcal{O}_C(1)\} \quad \text{and} \quad b_0 = \{\mathcal{O}_F(4p)\} \times \{\mathcal{O}_C(1)\}.$$

Consider the trivial family $\mathcal{Y} = Y \times B \to B$ and denote by $\mathcal{L}$ the Poincaré bundle $\mathcal{P}$ on $Y \times \mathrm{Pic}^{\underline{d}}(Y)$ restricted to $\mathcal{Y}$. As in the proof of Proposition 10.8, up to shrinking $B$ around $b_0$, we obtain an embedding $\mathcal{Y} \hookrightarrow \mathbb{P}^r_B$, which yields a map $f : B \to \mathrm{Hilb}_d$ such that $f(B \setminus \{b_0\}) \subset \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$. Now, apply the *polystable replacement property.* Up to replacing $B$ with a finite cover ramified over $b_0$, we get a polarized family $(\mathcal{Z} \to B, \mathcal{M})$ such that, denoting $Z := \mathcal{Z}_{b_0}$ and $M := \mathcal{M}_{|Z}$, the point $[Z \subset \mathbb{P}^r]$ with $M = \mathcal{O}_Z(1)$ is Chow polystable. Denote by $F'$ the elliptic tail of $Z$. Since $\mathcal{Z}$ is an isotrivial family of curves over $B$, either $Z \cong Y$ or $F'$ is cuspidal. If $Z \cong Y$, then $\mathcal{O}_Z(1)_{|F} = \mathcal{O}_F(4p)$, which is absurd by Theorem 9.1, hence the second case occurs. Since $F' \subset Z$ is not special, $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([Z \subset \mathbb{P}^r])$ is finite by Theorem 6.4 and Lemma 6.1, hence $[Z \subset \mathbb{P}^r]$ is Chow stable. This proves the inequality

$$e_{F,\rho_1} < \frac{2d}{r+1}\, w(\rho_1)$$

if $F$ is cuspidal. The last inequality of (ii) can be proved in the same way applying the *polystable replacement property* for the Hilbert stability. □

**Corollary 12.3.** *Let $X = F \cup C$ be a connected curve where $F$ is an elliptic tail (smooth, nodal, cuspidal or reducible nodal) and $C$ is smooth. Denote by $p$ the intersection point of $F$ with $C$ and consider a properly balanced line bundle $L \in \mathrm{Pic}^d(X)$ with $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$. Then there exists $M \in \mathrm{Pic}^{d-4}(C)$ such that*

*(1) if $\frac{7}{2}(2g-2) < d < 4(2g-2)$, $L_{|C} = M$ and $L_{|F} \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$, then $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$ is Chow stable;*

*(2) if $d = 4(2g-2)$, $L_{|C} = M$, $L_{|F} \in \mathrm{Pic}^4(F) \setminus \{\mathcal{O}_F(4p)\}$ and $F$ is cuspidal (resp. not cuspidal), then $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$ is Hilbert (resp. Chow) stable; moreover if $\mathcal{O}_X(1)_{|F} = \mathcal{O}_F(4p)$ and $F$ is cuspidal, then $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$ is Chow polystable.*

*Proof.* For (1) and the first statement of (2), it suffices to consider the curve $Y$ obtained in Remark 11.4 applying the *polystable replacement property* to a quasi-wp-stable curve $X' = C \cup E$ where $C$ and $E \cong \mathbb{P}^1$ meet together in a tacnode (in this case $p \in C$) and apply Lemma 12.1. For the last statement of (2), we notice that $X$ is a closed point in the stack $\overline{\mathcal{M}}_g^{\mathrm{wp}}$ by Remark 2.5(i). Hence if $\rho$ is a one-parameter subgroup such that

$$e_{X,\rho} = \frac{16}{7}\, w(\rho)$$

then, setting $[X_0 \subset \mathbb{P}^r] = \lim_{t\to 0} \rho(t)[X \subset \mathbb{P}^r]$, we have that $X \cong X_0$ and

$$\dim \mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r]) \geq \dim \mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X_0 \subset \mathbb{P}^r])$$

by Theorem 6.4 and Lemma 6.1. This implies that $[X_0 \subset \mathbb{P}^r] \in \mathrm{Orb}([X \subset \mathbb{P}^r])$ and we are done. ☐

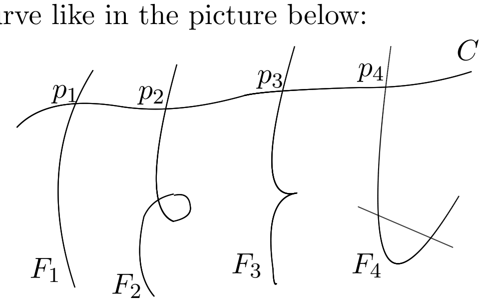
Now, we are ready to extend the completeness result of Proposition 10.8 to the case $d = 4(2g-2)$.

**Proposition 12.4.** *Let $X$ be a quasi-wp-stable curve and $\underline{d} \in B_X^d$. Assume that $d = 4(2g-2)$. Then either $M_X^{\underline{d}} = \emptyset$ or the map $p : M_X^{\underline{d}} \to \mathrm{Pic}^{\underline{d}}(X)/\mathrm{Aut}(X)$ is surjective.*

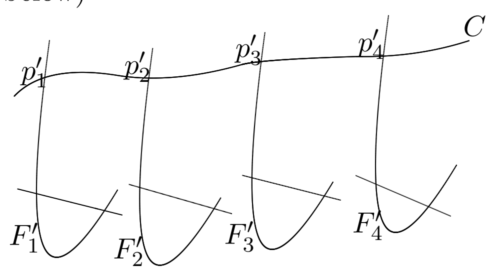
*Proof.* Assume that $M_X^{\underline{d}} \neq \emptyset$, for otherwise there is nothing to prove. According to Corollary 10.7, the surjectivity of $p$ is equivalent to the fact that $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$ is Chow semistable for every $L \in \mathrm{Pic}^{\underline{d}}(X)$. With this aim, let $E = \{F_1, \ldots, F_k\}$ be the set of the elliptic tails of $X$, set $C = X_{\mathrm{ell}}^c$ and denote by $p_i$ the intersection point of $F_i$ with $(F_i)^c$ for each $i = 1, \ldots, k$. By standard arguments of basins of attraction we can assume that:

(1) the multidegree $\underline{d}$ is strictly balanced (same proof as that of <u>Reduction 2</u> in Proposition 10.8);
(2) each elliptic tails $F$ with $\mathcal{O}_X(1)|_F = \mathcal{O}_F(4p)$ is cuspidal (by Theorem 9.2 and Fact 4.10);
(3) each cusp is contained in an elliptic curve (same proof as that of (2)).

In this way we have a curve like in the picture below:




Let $F = E_1 \cup E_2$ be a reducible nodal elliptic curve where $E_1$ and $E_2$ are two smooth rational curves. Consider a smooth point $p \in E_1 \subset F$ and a line bundle $M \in \mathrm{Pic}^4(F)$ such that $\underline{\deg}\, M = (\deg M_{|E_1}, \deg M_{|E_2}) = (3,1)$. By Lemma 12.1, if we replace each elliptic tail $F_i$ with a pointed polarized curve $(F'_i, p'_i, M_i) \cong (F, p, M)$, we obtain a new curve $X'$ (see the picture below)




and a multidegree $\underline{d}'$ for which there exists a properly balanced line bundles $L' \in \mathrm{Pic}^{\underline{d}'}(X')$ such that $[X' \stackrel{|L'|}{\hookrightarrow} \mathbb{P}^r]$ is Chow semistable. We notice that $X'$ is quasi-stable and each Chow semistable isotrivial specialization is again a quasi-stable curve, so that by the proof of Proposition 10.8 (case $d > 4(2g-2)$) we obtain that our statement is true for $X'$ and $\underline{d}'$. In order to complete the proof, it is enough to replace again each $F'_i$ with $(F_i, p_i, \mathcal{O}_X(1)_{|F_i})$ and to apply Lemma 12.1. $\square$

## 13. Semistable, polystable and stable points (part II)

The aim of this section is to describe the points of $\mathrm{Hilb}_d$ that are Hilbert or Chow semistable, polystable and stable for

$$\frac{7}{2}(2g-2) < d \leq 4(2g-2) \quad \text{and} \quad g \geq 3.$$

The GIT analysis in this range is based on a nice numerical trick which uses the following

**Construction 13.1.** Given a quasi-wp-stable curve $[X \subset \mathbb{P}^{d-g}] \in \mathrm{Hilb}_d$ which admits a non-special elliptic tail $F$, we define a new polarized curve $X'$ by replacing the polarized subcurve $F$ with a polarized smooth curve $Y$ of genus $g$ and degree $d - d_F$ in such a way $Y$ and $X \setminus F$ intersect again in one node. If we denote by $d'$ and $g'$ respectively the degree of the new line bundle $L'$ and the genus of $X'$, one can consider the Hilbert point $[X' \subset \mathbb{P}^{d'-g'}] \in \mathrm{Hilb}_{d'}$. We easily check that

$$\nu' := \frac{d'}{2g'-2} = \frac{2d}{2(2g-1)-2} = \frac{d}{2g-2} =: \nu.$$

Moreover we claim that

$$\mathcal{O}_X(1) \text{ is balanced } \Longleftrightarrow \mathcal{O}_{X'}(1) \text{ is balanced.}$$

Let us prove the implication $\Longrightarrow$. As we noticed in Remark 3.8, it suffices to check the basic inequality (3.1) for each connected subcurve such that its complementary is connected. Let $D \subset X'$ a connected subcurve. If $D = Y$ then obviously the basic inequality (3.1) is satisfied. Otherwise, up to replacing $D$ with $D^c$, we can assume that $D$ does not contain $Y$ as a subcurve. This implies that $D$ can be seen as a subcurve of $X$. Since $\nu' = \nu$, $\deg L_{|D} = \deg L'_{|D}$ and $|D \cap \overline{X \setminus D}| = |D \cap \overline{X' \setminus D}|$, the basic inequality (3.1) is satisfied. The proof of the reverse implication $\Longleftarrow$ is analogous.

We notice that from $X$ to $X'$ the number of non-special elliptic tails decreases by 1. Applying the results about elliptic tails of the previous section and Corollary 8.6, one prove that

$$[X' \subset \mathbb{P}^{d'-g'}] \text{ is semistable} \quad \Longrightarrow \quad [X \subset \mathbb{P}^{d-g}] \text{ is semistable},$$

so that the GIT analysis can be completed by an induction argument on the number of non-special elliptic tails of $X$. (For $d = 4(2g-2)$the induction argument will be on the number of all elliptic tails of $X$.) Applying arguments based on specializations of strata (Proposition 10.5) and results of completeness (Proposition 10.8 and Proposition 12.4), one can prove the basis of the induction as well. Notice that we have already used this construction in the proof of Corollary 11.6.

Let us begin with the case $\frac{7}{2}(2g-2) < d < 4(2g-2)$.

**Theorem 13.2.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$ and assume that $X$ is connected.*

- *(i) $[X \subset \mathbb{P}^r]$ is Hilbert semistable;*
- *(ii) $[X \subset \mathbb{P}^r]$ is Chow semistable;*
- *(iii) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*
- *(iv) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails and $\mathcal{O}_X(1)$ is properly balanced;*
- *(v) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails and $\mathcal{O}_X(1)$ is balanced.*

*Proof.* The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are clear.

(ii) $\Rightarrow$ (iii) follows from Corollary 5.6(i) and Corollary 9.4.

(v) $\Rightarrow$ (i). The proof is based on Construction 13.1. Let $X$ and $L := \mathcal{O}_X(1)$ be as in (v) and let $n$ be the number of elliptic tails of $X$. We will prove our statement by induction over $n$.

Assume first that each cusp of $X$ is contained in an elliptic tail of $X$. If $n = 0$ then $X$ is quasi-stable without elliptic tail and the same argument used to prove Theorem 11.1 (case $d > 4(2g-2)$) goes through. Suppose that $n > 0$. Consider an elliptic tail $F$ (which is non-special by assumption) and denote by $C_1 = F^c$ and $\{p\} = F \cap C_1$. Let $C_2$ be a smooth curve of genus $g$, $q$ a point of $C_2$ and $L'_{C_2} \in \mathrm{Pic}^{d+4}(C_2)$. Denote by $(X', L')$ the couple consisting of a curve $X'$ of genus $g'$ and a line bundle $L'$ on $X'$ obtained from $(X, L)$ by replacing $F$ with $(C_2, q, L'_{C_2})$. The line bundle $L'$ is ample of degree $d' = 2d$, moreover we have

$$\nu' := \frac{d'}{2g'-2} = \frac{2d}{2(2g-1)-2} = \frac{d}{2g-2} =: \nu. \tag{13.1}$$

By the same argument used in the proof of Corollary 11.6 and Construction 13.1, $L'$ is properly balanced, therefore $L'$ is non-special and very ample by Theorem 17.5; hence we can consider the point $[X' \subset \mathbb{P}^{r'}] \in \mathrm{Hilb}_{d',g'}$ with $\mathcal{O}_{X'}(1) = L'$. Now, $X'$ contains $n-1$ elliptic tails and $L'$ is balanced, hence by induction $[X' \subset \mathbb{P}^{r'}]$ is Hilbert semistable. By Corollary 12.3(1), there exists a Hilbert semistable point $[Y \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ such that $Y$ admits the elliptic tail $F$ with $\mathcal{O}_Y(1)_{|F} = L_{|F}$. Since (13.1) holds and $(X, L)$ can be obtained again from $(X', L')$ by replacing $C_2$ with $(F, p, L_{|F})$, $[X \subset \mathbb{P}^r]$ is Hilbert semistable by Corollary 8.6.

Consider now the general case, where $X$ can have cusps not contained in an elliptic tail. As before, we prove our statement by induction over $n$. If $n = 0$ then $X$ is quasi-p-stable and, by Corollary 10.12, it is enough to prove that for each balanced multidegree $\underline{d}$ there exists a line bundle $L$ of multidegree $\underline{d}$ such that $[X \subset \mathbb{P}^r]$ is Hilbert semistable with $\mathcal{O}_X(1) = L$. By Proposition 10.5 the curve $X$ specializes isotrivially to a curve $X'$ such that each cusp in contained in an elliptic tail. Let $F_1, \ldots, F_m$ be the elliptic tails of $X'$ and denote by $p_i$ the intersection point of $F_i$ with $F_i^c$. Replacing each cuspidal elliptic tail $F_i$ with a pointed reducible nodal one $(F_i', q_i)$ we obtain a quasi-stable curve $X''$, which is Hilbert semistable for each balanced polarization $L''$ by the argument above. If we replace again each reducible nodal elliptic tail $F_i'$ with the pointed polarized curve $(F_i, p_i, L_{|F_i})$, by Lemma 12.1 $[X' \subset \mathbb{P}^r]$ is Hilbert semistable. The semistability is an open condition, so that the theorem is true for a generic element of $\mathrm{Pic}^{\underline{d}}(X)$. If $n > 0$ we apply the same argument based on replacement of elliptic tails used above and the proof is complete. □

**Corollary 13.3.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$ and assume that $X$ is connected. The following conditions are equivalent:*

*(i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;*

*(ii) $[X \subset \mathbb{P}^r]$ is Chow polystable;*

*(iii) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;*

*(iv) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails and $\mathcal{O}_X(1)$ is strictly balanced.*

*Proof.* The implications (i) ⇒ (ii) ⇒ (iii) are proved in the same way as in Corollary 11.2.

(iii) ⇒ (iv) is obvious.

(iv) ⇒ (i): the proof of this implication is based on Construction 13.1 and is very similar to the one used to prove Corollary 11.6(2). Denote by $L = \mathcal{O}_X(1)$ and $n$ the number of elliptic tails of $X$. We will prove our corollary by induction over $n$. If $n = 0$, then $X$ is quasi-p-stable and the same argument used to prove Proposition 10.8 shows that $[X \subset \mathbb{P}^r]$ is Hilbert polystable. Suppose that $n > 0$. Consider an elliptic tail $F$ and denote by $C_1 = F^c$ and $\{p\} = F \cap C_1$. As in the proof of Corollary 11.6, let $C_2$ be a smooth curve of genus $g$, $q$ a point of $C_2$ and $L'_{C_2} \in \mathrm{Pic}^{d+4}(C_2)$. Denote by $(X', L')$

the couple consisting of a curve $X'$ of genus $g'$ and an ample line bundle $L'$ of degree $d' = 2d$ on $X'$ obtained from $(X, L)$ by replacing $F$ with $(C_2, q, L'_{C_2})$. By construction we have that

$$\nu' := \frac{d'}{2g'-2} = \frac{2d}{2(2g-1)-2} = \frac{d}{2g-2} =: \nu. \tag{13.2}$$

By the same argument used in the proof of Corollary 11.6 and Construction 13.1, the line bundle is strictly balanced. Therefore, the line bundle $L'$ is very ample and non-special by Theorem 17.5, and we can consider the point $[X' \subset \mathbb{P}^{r'}] \in \mathrm{Hilb}_{d',g'}$, with $\mathcal{O}_{X'}(1) = L'$. Now, $[X' \subset \mathbb{P}^{r'}]$ admits $n-1$ elliptic tails, hence it is Hilbert polystable by induction. By Corollary 12.3, there exists a Hilbert stable (hence polystable) point $[Y \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ such that $Y$ admits the elliptic tail $F$ with $\mathcal{O}_Y(1)|_F = L|_F$. Since 13.2 holds and $(X, L)$ can be obtained from $(X', L')$ by replacing $C_2$ with $(F, p, L|_F)$, we get that $[X \subset \mathbb{P}^r]$ is Hilbert polystable by Corollary 8.6. □

**Corollary 13.4.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $\frac{7}{2}(2g-2) < d < 4(2g-2)$ and $g \geq 3$ and assume that $X$ is connected. The following conditions are equivalent:*

- *(i) $[X \subset \mathbb{P}^r]$ is Hilbert stable;*
- *(ii) $[X \subset \mathbb{P}^r]$ is Chow stable;*
- *(iii) $X$ is quasi-wp-stable without tacnodes and special elliptic tails, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is stably balanced and non-special;*
- *(iv) $X$ is quasi-wp-stable without tacnodes and special elliptic tails and $\mathcal{O}_X(1)$ is stably balanced.*

*Proof.* The implications (ii) ⇒ (i) and (iii) ⇒ (iv) are clear.

(i) ⇒ (iii) follows from Theorem 13.2 and Theorem 7.1.

(iv) ⇒ (ii). By Corollary 13.3, $[X \subset \mathbb{P}^r]$ is Chow polystable; hence it suffices to prove that $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is a finite group. Since the line bundle $\mathcal{O}_X(1)$ is stably balanced, Lemma 3.9 gives that $\widetilde{X} := \overline{X \setminus X_{\mathrm{exc}}}$ is connected; hence, combining Lemma 6.1 and Theorem 6.4, we deduce that $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is a finite group □

To conclude this section we study the extremal case $d = 4(2g-2)$, where the Chow semistable locus differs from the Hilbert semistable locus.

**Theorem 13.5.** *Consider a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $d = 4(2g-2)$ and $g \geq 3$ and assume that $X$ is connected.*

*(1) The following conditions are equivalent:*
  - *(i) $[X \subset \mathbb{P}^r]$ is Hilbert semistable;*
  - *(ii) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is properly balanced and non-special;*
  - *(iii) $X$ is quasi-wp-stable without tacnodes nor special elliptic tails and $\mathcal{O}_X(1)$ is properly balanced;*

(iv) $X$ *is quasi-wp-stable without tacnodes nor special elliptic tails and* $\mathcal{O}_X(1)$ *is balanced.*

(2) *The following conditions are equivalent:*

(i) $[X \subset \mathbb{P}^r]$ *is Chow semistable;*

(ii) $X$ *is quasi-wp-stable without tacnodes, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is properly balanced and non-special.*

(iii) $X$ *is quasi-wp-stable without tacnodes and* $\mathcal{O}_X(1)$ *is properly balanced;*

(iv) $X$ *is quasi-wp-stable without tacnodes and* $\mathcal{O}_X(1)$ *is balanced.*

*Proof.* The proof of (1) is the same as the proof of Theorem 13.2, using the fact that Corollary 9.4 does hold true also in the present case. Let us prove (2).

(2i) $\Rightarrow$ (2ii) follows from Theorem 5.1, Corollary 5.6 and Theorem 9.3.

(2ii) $\Rightarrow$ (2iii) $\Rightarrow$ (2iv) are clear.

(2iv) $\Rightarrow$ (2i) is proved with the same argument used to prove the implication (v)$\Rightarrow$(i): the only difference is that we do not assume that the elliptic tails are non-special and we use Corollary 12.3(2) instead of Corollary 12.3(1). ☐

**Corollary 13.6.** *Consider a point* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *with* $d = 4(2g-2)$ *and* $g \geq 3$ *and assume that* $X$ *is connected. The following conditions are equivalent:*

(1) *The following conditions are equivalent:*

(i) $[X \subset \mathbb{P}^r]$ *is Hilbert polystable;*

(ii) $X$ *is quasi-wp-stable without tacnodes and special elliptic tails, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is strictly balanced and non-special;*

(iii) $X$ *is quasi-wp-stable without tacnodes and special elliptic tails and* $\mathcal{O}_X(1)$ *is strictly balanced.*

(2) *The following conditions are equivalent:*

(i) $[X \subset \mathbb{P}^r]$ *is Chow polystable;*

(ii) $X$ *is quasi-wp-stable without tacnodes, each special elliptic tail of* $X$ *is cuspidal, each cuspidal elliptic tail of* $X$ *is special, each cusp of* $X$ *is contained in an elliptic tail,* $X$ *is non-degenerate and linearly normal in* $\mathbb{P}^r$*,* $\mathcal{O}_X(1)$ *is strictly balanced and non-special;*

(iii) $X$ *is quasi-wp-stable without tacnodes, each special elliptic tail of* $X$ *is cuspidal, each cuspidal elliptic tail of* $X$ *is special, each cusp of* $X$ *is contained in an elliptic tail and* $\mathcal{O}_X(1)$ *is strictly balanced.*

*Proof.* The same argument of Corollary 13.3 proves (1). Let us prove (2).

(2i) $\Rightarrow$ (2ii) follows from Theorem 13.5(2), Theorem 5.1 and Theorem 9.2.

(2ii) $\Rightarrow$ (2iii) is clear.

(2iii) $\Rightarrow$ (2i). We use Construction 13.1 again, as in the proof of Corollary 13.3. Let $n$ be the number of elliptic tails of $X$. If $n = 0$, then $X$ is quasi-stable and the proof of Corollary 11.2 goes through. Suppose that $n > 0$ and consider the point $[X' \subset \mathbb{P}^{r'}] \in \mathrm{Hilb}_{d',g'}$ obtained from $(X, L)$ by replacing an elliptic tail $F$ with a smooth curve $C_2$ of genus $g$ and degree $d+4$. The line bundle $L' := \mathcal{O}_{X'}(1)$ is strictly balanced on $X'$ and the point $[X' \subset \mathbb{P}^{r'}]$ is Chow polystable since $X$ admits $n-1$ elliptic tails. By Corollary 12.3(2), there exists a Chow polystable point $[Y \subset \mathbb{P}^r]$ such that $Y$ admits $F$ as an elliptic tail and $\mathcal{O}_Y(1)_{|F} = \mathcal{O}_X(1)_{|F}$. Finally, applying Corollary 8.6 to the points $[X' \subset \mathbb{P}^{r'}]$, $[Y \subset \mathbb{P}^r]$ and $[X \subset \mathbb{P}^r]$, we deduce that $[X \subset \mathbb{P}^r]$ is Chow polystable. □

**Corollary 13.7.** *Consider a point* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *with* $d = 4(2g-2)$ *and* $g \geq 3$ *and assume that* $X$ *is connected.*

*(1) The following conditions are equivalent:*
- *(i)* $[X \subset \mathbb{P}^r]$ *is Hilbert stable;*
- *(ii)* $X$ *is quasi-wp-stable without tacnodes and special elliptic tails, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special;*
- *(iii)* $X$ *is quasi-wp-stable without tacnodes and special elliptic tails and* $\mathcal{O}_X(1)$ *is stably balanced.*

*(2) The following conditions are equivalent:*
- *(i)* $[X \subset \mathbb{P}^r]$ *is Chow stable;*
- *(ii)* $X$ *is quasi-stable without special elliptic tails, non-degenerate and linearly normal in* $\mathbb{P}^r$ *and* $\mathcal{O}_X(1)$ *is stably balanced and non-special.*
- *(iii)* $X$ *is quasi-stable without special elliptic tails and* $\mathcal{O}_X(1)$ *is stably balanced.*

*Proof.* The same argument of Corollary 13.4 proves (1).

Let us now prove (2). Note that $[X \subset \mathbb{P}^r]$ is Chow stable if and only if it is Chow polystable and its stabilizer $\mathrm{Stab}_{\mathrm{PGL}_{r+1}}([X \subset \mathbb{P}^r])$ is a finite group. Lemma 6.1 and Theorem 6.4 give that a Chow polystable point $[X \subset \mathbb{P}^r]$ as in Corollary 13.6(2) has finite stabilizer subgroup if and only if

- $X$ does not have special cuspidal elliptical tails;
- $\widetilde{X} := \overline{X \setminus X_{\mathrm{exc}}}$ is connected.

The first condition is equivalent to the fact that $X$ does not have cusps (hence it is quasi-stable) nor special elliptic tails. The second condition is equivalent to the fact that $\mathcal{O}_X(1)$ is stably balanced by Lemma 3.9. Part (2) follows now from this fact together with Corollary 13.6(2).

□

## 14. Geometric properties of the GIT quotient

For any $d > 2(2g-2)$, consider the open and closed subscheme $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ of the Chow-semistable locus $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss}) \subset \mathrm{Hilb}_d$ consisting of connected curves, see (10.1). From now on, in order to shorten the notation, we set

$$H_d := \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o \subset \mathrm{Hilb}_d \tag{14.1}$$

and we call $H_d$ the *main component* of the Chow-semistable locus. Similarly, the locus

$$\widetilde{H}_d := \mathrm{Hilb}_d^{ss,o} := \{[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d^{ss} \,:\, X \text{ is connected}\} \tag{14.2}$$

is an open and closed subscheme of $\mathrm{Hilb}_d^{ss}$, that we call the *main component* of the Hilbert semi-stable locus. Note that $\widetilde{H}_d$ is an open subset of $H_d$ by Fact 4.1 and that $\widetilde{H}_d = H_d$ if and only if $d \notin \{\frac{7}{2}(2g-2), 4(2g-2)\}$ by Theorems 11.1, 11.5, 13.2 and 13.5. The name "main component" is justified by the fact that $H_d$ (resp. $\widetilde{H}_d$) is an irreducible component of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ (resp. $\mathrm{Hilb}_d^{ss}$), as we will prove in Corollary 14.7, together with the fact that for some values of $d$ and $g$ there could exist other irreducible components of $\mathrm{Chow}_d^{ss}$ (resp. $\mathrm{Hilb}_d^{ss}$) made of non-connected curves (see Section 15).

Since $H_d$ is clearly an $\mathrm{SL}_{r+1}$-invariant closed and open subscheme of $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$, GIT tells us that there exists a projective scheme

$$\overline{Q}_{d,g}^c := H_d /\!\!/ \mathrm{SL}_{r+1} \tag{14.3}$$

which is a good categorical quotient of $H_d$ by $\mathrm{SL}_{r+1}$ (see e.g. [Dol03, Sec. 6.1]). Similarly, there exists a projective scheme

$$\overline{Q}_{d,g}^h := \widetilde{H}_d /\!\!/ \mathrm{SL}_{r+1} \tag{14.4}$$

which is a good categorical quotient of $\widetilde{H}_d$ by $\mathrm{SL}_{r+1}$. Moreover, since $\widetilde{H}_d \subseteq H_d$, there exists a projective morphism

$$\Xi : \overline{Q}_{d,g}^h = \widetilde{H}_d /\!\!/ \mathrm{SL}_{r+1} \to H_d /\!\!/ \mathrm{SL}_{r+1} = \overline{Q}_{d,g}^c. \tag{14.5}$$

If $d \notin \{\frac{7}{2}(2g-2), 4(2g-2)\}$ then $\widetilde{H}_d = H_d$ (as observed above), which implies that $\Xi$ is an isomorphism. We will therefore set

$$\overline{Q}_{d,g} := \overline{Q}_{d,g}^h = \overline{Q}_{d,g}^c \ \text{ if } d \notin \left\{\frac{7}{2}(2g-2), 4(2g-2)\right\}. \tag{14.6}$$

Indeed, we will prove that $\Xi$ is an isomorphism if $d = \frac{7}{2}(2g-2)$ (see Proposition 14.5(i)) while it is not an isomorphism if $d = 4(2g-2)$ (see Proposition 14.6(i)).

**Remark 14.1.** By the well-known properties of GIT quotients (see [Dol03, Cor. 6.1]), it follows that the closed points of $\overline{Q}_{d,g}^h = \widetilde{H}_d /\!\!/ \mathrm{SL}_{r+1}$ (resp. $\overline{Q}_{d,g}^c = H_d /\!\!/ \mathrm{SL}_{r+1}$) correspond bijectively to orbits of Hilbert polystable points $[X \subset \mathbb{P}^r]$ in $\widetilde{H}_d$ (resp. Chow polystable points in $H_d$). Moreover, note that the orbit of a point $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ only determines the curve $X$ and the line bundle $\mathcal{O}_X(1)$ up to automorphisms of $X$ (compare with the discussion at the beginning of §10.2).

We now focus on the geometric properties of $\overline{Q}_{d,g}^h$ and $\overline{Q}_{d,g}^c$. We begin with the following result, which says that the singularities of $\overline{Q}_{d,g}^h$ and $\overline{Q}_{d,g}^c$ are not too bad.

**Proposition 14.2.** *Assume that $d > 2(2g-2)$ and, moreover, that $g \geq 3$ if $d \leq 4(2g-2)$. Then:*

*(i) $H_d$ (resp. $\widetilde{H}_d$) is non-singular of pure dimension $r(r+2)+4g-3$.*
*(ii) $\overline{Q}_{d,g}^c$ (resp. $\overline{Q}_{d,g}^h$) is reduced and normal of pure dimension $4g-3$. Moreover, if $\mathrm{char}(k) = 0$, then $\overline{Q}_{d,g}^c$ (resp. $\overline{Q}_{d,g}^h$) has rational singularities, hence it is Cohen-Macauly.*

*Proof.* Part (i): the fact that $H_d$ (resp. $\widetilde{H}_d$) is non-singular of pure dimension $r(r+2)+4g-3$ is proved exactly as in [Cap94, Lemma 2.2], whose proof uses only the fact that if $X \in H_d$ (resp. $\widetilde{H}_d$) then $X$ is reduced curve with locally complete intersection singularities and embedded by a non-special linear system; these conditions are satisfied by the Potential pseudo-stability Theorem 5.1. See also [HH13, Cor. 6.3] for another proof.

Part (ii): $\overline{Q}_{d,g}^c$ is reduced and normal since $H_d$ is such (see e.g. [Dol03, Prop. 3.1]). The dimension of $\overline{Q}_{d,g}^c$ is $4g-3$ since $H_d$ has dimension $r(r+2)+4g-3$, $\mathrm{SL}_{r+1}$ has dimension $r(r+2)$ and the action of $\mathrm{SL}_{r+1}$ has generically finite stabilizers. If $\mathrm{char}(k)=0$ then $\overline{Q}_{d,g}^c$ has rational singularities by [Bou87], using that $H_d$ is smooth. This implies that $\overline{Q}_{d,g}^c$ is Cohen-Macauly since, in characteristic zero, a variety having rational singularities is Cohen-Macauly (see [KoM98, Lemma 5.12]). Alternatively, the fact that $\overline{Q}_{d,g}^c$ is Cohen-Macauly follows from [HR74], using the fact that $H_d$ is smooth. The same argument works for $\overline{Q}_{d,g}^h$. □

We mention that, if $\mathrm{char}(k)=0$, $d>4(2g-2)$ and $g\geq 4$, then $\overline{Q}_{d,g}$ is known to have canonical singularities (see [BFV12] in the case where $\gcd(d+1-g,2g-2)=1$ and [CMKV2] in the general case). This result has been used in loc. cit. to compute the Kodaira dimension and the Iitaka fibration of $\overline{Q}_{d,g}$.

The GIT quotient $\overline{Q}_{d,g}^c$ admits a modular morphism to the moduli space $\overline{M}_g^{\mathrm{p}}$ of p-stable curves.

**Theorem 14.3.** *Assume that $d>2(2g-2)$ and, moreover, that $g\geq 3$ if $d\leq 4(2g-2)$. Then:*

*(i) There exists a surjective natural map $\Phi^{\mathrm{ps}}:\overline{Q}_{d,g}^c\to\overline{M}_g^{\mathrm{p}}$.*

*(ii) If $d>4(2g-2)$ then the above map $\Phi^{\mathrm{ps}}$ factors as*

$$\Phi^{\mathrm{ps}}:\overline{Q}_{d,g}^c\xrightarrow{\Phi^{\mathrm{s}}}\overline{M}_g\xrightarrow{T}\overline{M}_g^{\mathrm{p}},$$

*where $T$ is the map of Remark 2.7.*

*(iii) We have that*

$$(\Phi^{\mathrm{ps}})^{-1}(M_g^0)\cong J_{d,g}^0,$$

*where $M_g^0$ is the open subset of $M_g$ parametrizing curves without non-trivial automorphisms and $J_{d,g}^0$ is the degree $d$ universal Jacobian over $M_g^0$. In particular, $(\Phi^{\mathrm{ps}})^{-1}(C)\cong\mathrm{Pic^d(C)}$ for every geometric point $C\in M_g^0\subset\overline{M}_g^{\mathrm{p}}$.*

*If $d>4(2g-2)$ then the same conclusions hold for the morphism $\Phi^{\mathrm{s}}$.*

*Proof.* The proof is an adaptation of the ideas from [Cap94, Sec. 2].

Part (i): consider the restriction to $H_d$ of the universal family over $\mathrm{Hilb}_d$ and denote it by

$$\begin{array}{ccc}\mathcal{C}_d & \hookrightarrow & H_d\times\mathbb{P}^r\\ {\scriptstyle u_d}\downarrow & & \\ H_d & & \end{array}$$

The morphism $u_d$ is flat, proper and its geometric fibers are quasi-wp-stable curves by Corollary 9.7(ii). Consider the p-stable reduction of $u_d$ (see Definition 2.14):

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathrm{ps}(\mathcal{C}_\mathrm{d}) \\ & {\scriptstyle u_d}\searrow \quad \swarrow{\scriptstyle \mathrm{ps}(\mathrm{u_d})} & \\ & H_d & \end{array}$$

The morphism $\mathrm{ps}(\mathrm{u_d})$ is flat, proper and its geometric fibers are p-stable curves of genus $g$. Therefore, by the modular properties of $\overline{M}_g^{\mathrm{p}}$, the family $\mathrm{ps}(\mathrm{u_d})$ induces a modular map $\phi^{\mathrm{ps}} : H_d \to \overline{M}_g^{\mathrm{p}}$. Since the group $\mathrm{SL}_{r+1}$ acts on the family $\mathcal{C}_d$ by only changing the embedding of the fibers of $u_d$ into $\mathbb{P}^r$, the map $\phi^{\mathrm{ps}}$ is $\mathrm{SL}_{r+1}$-invariant and therefore it factors via a map $\Phi^{\mathrm{ps}} : \overline{Q}_{d,g}^c \to \overline{M}_g^{\mathrm{p}}$.

Let us show that $\Phi^{\mathrm{ps}}$ is surjective. Let $C$ be any connected smooth curve over $k$ of genus $g \geq 2$ and $L$ be any line bundle on $C$ of degree $d > 2(2g-2)$. Note that $d = \deg L \geq 2g+1$ since $g \geq 2$. Hence $L$ is very ample and non-special and therefore it embeds $C$ in $\mathbb{P}^r = \mathbb{P}^{d-g}$. By Fact 4.19, the corresponding point $[C \overset{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$ belongs to $H_d$ and clearly it is mapped to $C \in M_g \subset \overline{M}_g^{\mathrm{p}}$ by $\Phi^{\mathrm{ps}}$. We conclude that the image of $\Phi^{\mathrm{ps}}$ contains the open dense subset $M_g \subset \overline{M}_g^{\mathrm{p}}$. Moreover, $\Phi^{\mathrm{ps}}$ is projective since $\overline{Q}_{d,g}^c$ is projective. Therefore, being projective and dominant, $\Phi^{\mathrm{ps}}$ has to be surjective. This finishes the proof of part (i).

Consider now Part (ii). If $d > 4(2g-2)$, then the potential stability Theorem (see Fact 4.20) says that the geometric fibers of the morphism $u_d$ are quasi-stable curves. From Definition 2.14 and Proposition 2.11, it follows that the p-stable reduction $\mathrm{ps}(\mathrm{u_d})$ of $u_d$ factors through the wp-stable reduction $\mathrm{wps}(\mathrm{u_d})$ of $u_d$ and that the latter one is a family of stable curves. This implies that the map $\Phi^{\mathrm{ps}} : \overline{Q}_{d,g}^c \to \overline{M}_g^{\mathrm{p}}$ factors via a map $\Phi^{\mathrm{s}} : \overline{Q}_{d,g}^c \to \overline{M}_g$ followed by the contraction map $T : \overline{M}_g \to \overline{M}_g^{\mathrm{p}}$.

Part (iii): the proof of [Cap94, Thm. 2.1(2)] extends verbatim to our case.

□

We now determine the dimension of the fibers of the morphisms $\Phi^{\mathrm{s}}$ and $\Phi^{\mathrm{ps}}$, starting from the cases $d \notin \{\frac{7}{2}(2g-2), 4(2g-2)\}$.

**Proposition 14.4.**

*(i) Assume that $d > 4(2g-2)$. The morphism $\Phi^{\mathrm{s}} : \overline{Q}_{d,g} \to \overline{M}_g$ has equidimensional fibers of dimension $g$ and, if $\mathrm{char}(k) = 0$, $\Phi^{\mathrm{s}}$ is flat over the smooth locus of $\overline{M}_g$.*

*(ii) Assume that $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$. The morphism $\Phi^{\mathrm{ps}} : \overline{Q}_{d,g} \to \overline{M}_g^{\mathrm{p}}$ has equidimensional fibers of dimension $g$ and, if $\mathrm{char}(k) = 0$, $\Phi^{\mathrm{ps}}$ is flat over the smooth locus of $\overline{M}_g^{\mathrm{p}}$.*

*(iii) Assume that $\frac{7}{2}(2g-2) < d$, $d \neq 4(2g-2)$ and $g \geq 3$. The fiber of $\Phi^{\mathrm{ps}} : \overline{Q}_{d,g} \to \overline{M}_g^{\mathrm{p}}$ over a p-stable curve $X$ has dimension equal to the sum of $g$ and the number of cusps of $X$.*

*Proof.* The flatness assertions in (i) and (ii) follow from the equidimensionality of the fibers and the fact that $\overline{Q}_{d,g}$ is Cohen-Macauly if $\mathrm{char}(k) = 0$ (see Theorem 14.3(ii)) by using the following well-know flatness's criterion.

Fact (see [Mat89, Cor. of Thm 23.1, p. 179]): Let $f : X \to Y$ be a dominant morphism between irreducible varieties. If $X$ is Cohen-Macauly, $Y$ is smooth and $f$ has equidimensional fibers of the same dimension, then $f$ is flat.

Let us now prove the statements about the dimension of the fibers.

Assume first that $d > 4(2g-2)$. By Corollary 5.6(ii), the fiber of the morphism

$$\phi^{\mathrm{s}} : H_d \to \overline{Q}_{d,g} \xrightarrow{\Phi^{\mathrm{s}}} \overline{M}_g,$$

over a stable curve $X \in \overline{M}_g$, is equal to

$$(\phi^{\mathrm{s}})^{-1}(X) = \bigcup_{\substack{\mathrm{s(X')=X} \\ \underline{d}' \in B^d_{X'}}} M^{\underline{d}'}_{X'}$$

where the union runs over the *quasi-stable* curves $X'$ whose stable reduction $\mathrm{s(X')} = \mathrm{wps(X')}$ is equal to $X$ and $\underline{d}' \in B^d_{X'}$. Since every such $X'$ is obtained from $X$ by blowing up some of the nodes of $X$, we have that $X' \preceq X$ (see Remark 10.2). Therefore, Lemma 10.4 implies that, for every pair $(X', \underline{d}')$ appearing in the above decomposition, there exists $\underline{d} \in B^d_X$ such that $(X', \underline{d}') \preceq (X, \underline{d})$. This implies that

$$(\phi^{\mathrm{s}})^{-1}(X) = \overline{\bigcup_{\underline{d} \in B^d_X} M^{\underline{d}}_X} \cap H_d.$$

We deduce that the fiber $(\Phi^{\mathrm{s}})^{-1}(X)$ contains an open dense subset isomorphic to

$$\left( \bigcup_{\underline{d} \in B^d_X} M^{\underline{d}}_X \right) / SL_{r+1} = \bigcup_{\underline{d} \in B^d_X} M^{\underline{d}}_X / \mathrm{SL}_{r+1}.$$

For any $\underline{d} \in B^d_X$ the map $p : M^{\underline{d}}_X \to \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})/\mathrm{Aut}^{\underline{\mathrm{d}}}(\mathrm{X})$ of (10.4) is surjective by Theorem 11.1(1) and its fibers are exactly the $\mathrm{SL}_{r+1}$-orbits on $M^{\underline{d}}_X$. Therefore

$$\dim M^{\underline{d}}_X / SL_{r+1} = \dim \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})/\mathrm{Aut}^{\underline{\mathrm{d}}}(\mathrm{X}) = g,$$

where we used that $\mathrm{Aut}^{\underline{d}}(X) \subseteq \mathrm{Aut}(X)$ is a finite group because $X$ is a stable curve. We conclude that $(\Phi^{\mathrm{s}})^{-1}(X)$ is of pure dimension $g$, i.e. part (i) is proved.

Assume now that $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$. By Corollary 9.7, the fiber of the morphism

$$\phi^{\mathrm{ps}} : H_d \to \overline{Q}_{d,g} \xrightarrow{\Phi^{\mathrm{ps}}} \overline{M}^{\mathrm{p}}_g,$$

over a p-stable curve $X \in \overline{M}^{\mathrm{p}}_g$, is given by

$$(\phi^{\mathrm{ps}})^{-1}(X) = \bigcup_{\substack{\mathrm{wps(X')=X} \\ \underline{d}' \in B^d_{X'}}} M^{\underline{d}'}_{X'},$$

where the union is over the possible *quasi-p-stable* curves $X'$ whose wp-stable reduction wps(X') (which coincides with the p-stable reduction ps(X')) is equal to $X$ and $\underline{d}' \in B^d_{X'}$. Since every such $X'$ is obtained from $X$ by blowing up some nodes or cusps of $X$, we have that $X' \preceq X$ (see Remark 10.2). Therefore, Lemma 10.4 implies that, for every pair $(X', \underline{d}')$ appearing in the above decomposition, there exists $\underline{d} \in B^d_X$ such that $(X', \underline{d}') \preceq (X, \underline{d})$. This implies that

$$(\phi^{\rm ps})^{-1}(X) = \overline{\bigcup_{\underline{d}\in B^d_X} M^{\underline{d}}_X} \cap H_d.$$

We now conclude the proof of part (ii) arguing as before (using Theorem 11.1(2)).

Assume finally that $\frac{7}{2}(2g-2) < d$, $d \neq 4(2g-2)$ and $g \geq 3$. By Corollary 5.6(i), the fiber of the morphism

$$\phi^{\rm ps}: H_d \to \overline{Q}_{d,g} \stackrel{\Phi^{\rm ps}}{\to} \overline{M}^{\rm p}_g,$$

over a p-stable curve $X \in \overline{M}^{\rm p}_g$, is given by

$$(\phi^{\rm ps})^{-1}(X) = \bigcup_{\substack{\rm ps(X')=X\\ \underline{d}'\in B^d_{X'}}} M^{\underline{d}'}_{X'}, \tag{14.7}$$

where the union is over the possible *quasi-wp-stable* curves $X'$ whose p-stable reduction ps(X') is equal to $X$ and $\underline{d}' \in B^d_{X'}$. Every such a curve $X'$ is obtained from $X$ by blowing up some of the nodes or cusps of $X$ and by replacing some of the cusps of $X$ by elliptic tails.

We want now to rewrite (14.7) in a more convenient way. With this in mind, let us introduce some notation. Let $\{c_1, \cdots, c_l\}$ be the cusps of $X$. For any subset $\emptyset \subseteq S \subseteq [l] := \{1, \cdots, l\}$, consider the family of wp-stable curves $\eta^S : \mathcal{X}^S \to V^S := (\overline{M}_{1,1})^S$ such the fiber of $\eta^S$ over a point $(F_i, p_i)_{i\in S} \in (\overline{M}_{1,1})^S$ is the wp-stable curve obtained from $X$ by replacing the cusp $c_i$ with the 1-pointed stable elliptic tail $(F_i, p_i)$ for every $i \in S$. Note that $\eta^S : \mathcal{X}^S \to V^S$ is a family of wp-stable curves whose p-stabilization is the trivial family $X \times V^S$. For a point $t \in V^S$, set $\mathcal{X}^S_t := (\eta^S)^{-1}(t)$. We can canonically identify the properly balanced multidegrees of total degree $d$ on $\mathcal{X}^S_t$ as $t$ varies in $V^S$; we therefore set ${}^S B^d := B^d_{\mathcal{X}^S_t}$ for any $t \in V_S$. Moreover, for any given $\underline{d} \in {}^S B^d$, we consider the locally closed subset of $H_d$ given by

$${}^S M^d = \bigcup_{t\in V^S} M^{\underline{d}}_{\mathcal{X}^S_t} \subset H_d.$$

Now, from Definition 10.1, it follows that, among the quasi-wp-stable curves appearing in (14.7), the maximal curves with respect to the order relation $\preceq$ (see Remark 10.2) are the ones of type $\mathcal{X}^S_t := (\eta^S)^{-1}(t)$ for some $t \in V^S$ with $\emptyset \subseteq S \subseteq [l]$. Using this and Lemma 10.4, we can rewrite (14.7) as

$$(\phi^{\rm ps})^{-1}(X) = \overline{\bigcup_{\substack{\emptyset\subseteq S\subseteq[l]\\ \underline{d}\in {}^S B^d}} {}^S M^{\underline{d}}} \cap H_d,$$

from which it follows that

$$\bigcup_{\substack{\emptyset\subseteq S\subseteq[l]\\ \underline{d}\in {}^S B^d}} {}^S M^{\underline{d}}/SL_{r+1} \text{ is open and dense in } (\Phi^{\rm ps})^{-1}(X). \tag{14.8}$$

Using the map $p$ of (10.4) and the fact that $\mathrm{Aut}(\mathcal{X}_t^S)$ is a finite group since $\mathcal{X}_t^S$ is wp-stable, we get for any $\underline{d}\in {}^S B^d$ and any $t\in V^S$:

$$\dim M^{\underline{d}}_{\mathcal{X}_t^S}/SL_{r+1} \leq \dim \mathrm{Pic}^{\underline{d}}(\mathcal{X}_t^S)/\mathrm{Aut}^{\underline{d}}(\mathcal{X}_t^S) = g. \tag{14.9}$$

We deduce that

$$\dim {}^S M^{\underline{d}}/SL_{r+1} \leq \dim V^S + g = |S| + g, \tag{14.10}$$

which using (14.8) implies that $\dim(\Phi^{\rm ps})^{-1}(X)\leq g+l$.

Consider now the special case where $S=[l]$. In this case, the curves $\mathcal{X}_t^{[l]}$ are stable for any $t\in V^{(l)}$ and, for a generic $L_t\in \mathrm{Pic}^{\underline{d}}(\mathcal{X}_t^{[l]})$, any element of the form $[\mathcal{X}_t^{[l]} \stackrel{|L_t|}{\hookrightarrow} \mathbb{P}^r]$ is Chow (or equivalently Hilbert) semistable by Theorems 11.1(1) and 13.2. Therefore, for $S=[l]$ equality does hold in (14.9) and (14.10) and we deduce that $\dim(\Phi^{\rm ps})^{-1}(X)=g+l$. □

Now, we study the dimension of the fibers of the morphisms $\Xi:\overline{Q}^h_{d,g}\to \overline{Q}^c_{d,g}$ and $\Phi^{\rm ps}:\overline{Q}^c_{d,g}\to \overline{M}^{\rm p}_g$, as well as of their composition, in the two special cases $d\notin \{\frac{7}{2}(2g-2), 4(2g-2)\}$.

**Proposition 14.5.** *Assume that $d=\frac{7}{2}(2g-2)$ and that $g\geq 3$.*

- *(i) The morphism $\Xi:\overline{Q}^h_{d,g}\to \overline{Q}^c_{d,g}$ is an isomorphism.*
- *(ii) The morphisms $\Phi^{\rm ps}:\overline{Q}^c_{d,g}\to \overline{M}^{\rm p}_g$ and $\Phi^{\rm ps}\circ\Xi:\overline{Q}^h_{d,g}\to \overline{M}^{\rm p}_g$ have equidimensional fibers of dimension $g$ and, if $\mathrm{char}(k)=0$, then $\Phi^{\rm ps}$ and $\Phi^{\rm ps}\circ\Xi$ are flat over the smooth locus of $\overline{M}^{\rm p}_g$.*

*Proof.* In order to prove part (i), by applying the Zariski's main theorem in the form [EGAIII1, (4.4.9)], it is enough to check that $\overline{Q}^c_{d,g}$ is reduced and normal and that $\Xi$ is birational and injective.

The fact that $\overline{Q}^c_{d,g}$ is reduced and normal follows from Proposition 14.2.

Consider now the open and dense $\mathrm{SL}_{r+1}$-invariant subset $\mathrm{Hilb}^s_d\cap \widetilde{H}_d\subseteq \widetilde{H}_d$. GIT tells us that there exists a good geometric quotient $(\mathrm{Hilb}^s_d\cap\widetilde{H}_d)/\mathrm{SL}_{r+1}$ (in the sense of [Dol03, Sec. 6.1]) which is an open subset of $\widetilde{H}_d/\!\!/\mathrm{SL}_{r+1}=\overline{Q}^h_{d,g}$. Moreover, since $\mathrm{Ch}^{-1}(\mathrm{Chow}^s_d)\cap H_d$ is an open and dense $\mathrm{SL}_{r+1}$-invariant subset of $\mathrm{Hilb}^s_d\cap\widetilde{H}_d$ by Fact 4.1, the properties of the good geometric quotients (see [Dol03, Sec. 6.1]) ensure that there exists a good geometric quotient $(\mathrm{Ch}^{-1}(\mathrm{Chow}^s_d)\cap H_d)/\mathrm{SL}_{r+1}$ which is an open and dense subset of $(\mathrm{Hilb}^s_d\cap\widetilde{H}_d)/\mathrm{SL}_{r+1}$. Clearly, $\Xi$ is an isomorphism over $(\mathrm{Ch}^{-1}(\mathrm{Chow}^s_d)\cap H_d)/\mathrm{SL}_{r+1}$, which shows that $\Xi$ is birational.

Finally, let us show that $\Xi$ is injective, which will conclude the proof. Consider a point of $\overline{Q}^h_{d,g}$ represented by the orbit of an Hilbert polystable point $[X\subset\mathbb{P}^r]\in\widetilde{H}_d$

(see Remark 14.1). According to Corollary 11.6(1), this is equivalent to the fact that $X$ is quasi-p-stable and that $\mathcal{O}_X(1)$ is strictly balanced. From Theorem 9.8 and Corollary 11.6(2) it follows that $\Xi([X \subset \mathbb{P}^r])$ is represented by the orbit of any Chow polystable point $[Y \subset \mathbb{P}^r]$ of $H_d$ such that:

- Let $\{q_1, \ldots, q_n\}$ be the tacnodes with a line of $X$; denote by $E_i$ the line contained in $X$ and passing through $q_i$ (for any $i = 1, \ldots, n$) and let $\widehat{X}$ be the complement of the lines $E_i$ in $X$. Then $Y$ is obtained from $\widehat{X}$ by gluing at each point $q_i$ an elliptic tail $F_i = F_i^1 \cup F_i^2 \cup F_i^3$, where $F_i^j \cong \mathbb{P}^1$ (for each $j = 1, 2, 3$), $F_i^1$ is joined nodally to $\widehat{X}$ in $q_i$ and to $F_i^2$ while $F_i^2$ and $F_i^3$ meets in a tacnode. Note that $F_i^2 \cup F_i^3$ is a tacnodal elliptic tail for each $i$.
- $\mathcal{O}_Y(1)$ is a strictly balanced line bundle on $Y$ such that $\mathcal{O}_Y(1)_{|F_i^1} = \mathcal{O}_{F_i^1}(1)$, $\mathcal{O}_Y(1)_{|F_i^2} = \mathcal{O}_{F_i^2}(2)$, $\mathcal{O}_Y(1)_{|F_i^3} = \mathcal{O}_{F_i^3}(1)$ and $\mathcal{O}_Y(1)_{|\widehat{X}} = \mathcal{O}_X(1)_{|\widehat{X}}$.

Note that two line bundles $\mathcal{O}_Y(1)$ as above differ by an automorphism of $Y$ (as it follows from the proof of Theorem 6.4), so that the orbit of $[Y \subset \mathbb{P}^r]$ is well-defined (see Remark 14.1).

From this explicit description it follows that the curve $X$ and the restriction $\mathcal{O}_X(1)_{|\widetilde{X}}$ are uniquely determined by the orbit of the Chow-polystable point $[Y \subset \mathbb{P}^r] \in H_d$. Since the line bundle $\mathcal{O}_X(1)$ is uniquely determined by its restriction $\mathcal{O}_X(1)_{|\widetilde{X}}$ up to automorphisms of $X$, we can recover the orbit of $[X \subset \mathbb{P}^r]$ from the orbit of $[Y \subset \mathbb{P}^r]$ (see Remark 14.1), which shows the injectivity of $\Xi$, q.e.d.

Part (ii): using part (i), it is enough to prove the result for the morphism $\Phi^{\rm ps} \circ \Xi : \overline{Q}^h_{d,g} \to \overline{M}^{\rm p}_g$. The proof of the statement for $\Phi^{\rm ps} \circ \Xi$ is exactly the same as the proof of Proposition 14.4(ii) replacing Theorem 11.1(2) by Theorem 11.5(1). ☐

**Proposition 14.6.** *Assume that $d = 4(2g-2)$ and that $g \geq 3$.*

*(i) The fiber of the morphism $\Xi : \overline{Q}^h_{d,g} \to \overline{Q}^c_{d,g}$ over the orbit of a Chow polystable point $[X \subset \mathbb{P}^r] \in H_d$ is equal to the number of cuspidal elliptic tails of $X$ that are special with respect to $\mathcal{O}_X(1)$.*

*(ii) The fiber of the morphism $\Phi^{\rm ps} : \overline{Q}^c_{d,g} \to \overline{M}^{\rm p}_g$ (resp. $\Phi^{\rm ps} \circ \Xi : \overline{Q}^h_{d,g} \to \overline{M}^{\rm p}_g$) over a p-stable curve $X$ has dimension equal to the sum of $g$ and the number of cusps of $X$.*

*Proof.* Part (i): consider the point of $\overline{Q}^c_{d,g}$ represented by the orbit of the Chow polystable point $[X \subset \mathbb{P}^r] \in H_d$ (see Remark 14.1). By Corollary 13.6(2), $X$ is quasi-wp-stable without tacnodes, $\mathcal{O}_X(1)$ is strictly balanced and all the cusps of $X$ are contained in special cuspidal elliptic tails of $X$ which, moreover, are the unique special elliptic tails or cuspidal elliptic tails of $X$.

If $X$ does not have special cuspidal elliptic tails (hence it does not have special elliptic tails at all), then $[X \subset \mathbb{P}^r]$ is also Hilbert polystable by Corollary 13.6(1) and its orbit represents the unique point of $\Xi^{-1}([X \subset \mathbb{P}^r])$ and we are done.

In the general case, let $\{F_1, \ldots, F_n\}$ be the special cuspidal elliptic tails of $X$. Set $q_i := F_i \cap \widehat{X}$ and note that $\deg \mathcal{O}_X(1)_{|F_i} = 4$ by the basic inequality (3.1). By Corollary

13.6(1) and Theorem 9.2, any Hilbert polystable point $[Y \subset \mathbb{P}^r] \in \widetilde{H}_d$ such that $\Xi([Y \subset \mathbb{P}^r]) = [X \subset \mathbb{P}^r]$ is of the following form (for some $\emptyset \subseteq S \subseteq [n] = \{1, \ldots, n\}$):

Type $S$: $Y = Y_S$ is obtained from $X$ by contracting to a cuspidal point $q_i'$ all the tails $F_i$ such that $i \in S$; in particular there is a natural morphism $\nu_S : \widehat{X}_S := (\cup_{i\in S} F_i)^c \to Y$ which is the partial normalization of $Y$ at the cusps $q_i'$ (with $i \in S$). Moreover, the line bundle $\mathcal{O}_Y(1)$ is such that $\nu_S^* \mathcal{O}_Y(1) = \mathcal{O}_X(1)_{|\widehat{X}_S}(4 \cdot \sum_{i\in S} q_i)$ and each of the cuspidal elliptic tails $F_i \subset Y$ with $i \notin S$ is non-special with respect to $\mathcal{O}_Y(1)$. Set $\underline{d}_S$ equal to the strictly balanced multidegree of such a line bundle $\mathcal{O}_Y(1)$.

From Definition 10.1 (and in particular Figure 10), it follows that if $\emptyset \subseteq T \subseteq S \subseteq [n]$ then $(Y_T, \underline{d}_T) \preceq (Y_S, \underline{d}_S)$ which then implies that $M_{Y_T}^{\underline{d}_T} \subseteq \overline{M_{Y_S}^{\underline{d}_S}}$ by Proposition 10.5. In other words, inside the fiber $\Xi^{-1}([X \subset \mathbb{P}^r])$, the points of Type $S = [n]$ are dense.

Observe now that for points $[Y \subset \mathbb{P}^r] \in \widetilde{H}_d$ of Type $S = [n]$, the line bundle $\mathcal{O}_Y(1)$ is specified up to the choice of the gluing data for each of the cusps $q_i'$. Since each of the cusps give a one-dimensional space of gluing conditions for $\mathcal{O}_Y(1)$, points of Type $S = [n]$ form an irreducible $n$-dimensional family sitting in the fiber $\Xi^{-1}([X \subset \mathbb{P}^r])$. This shows that the dimension of $\Xi^{-1}([X \subset \mathbb{P}^r])$ is equal to $n$, which was the number of special cuspidal elliptic tails of $X$, q.e.d.

Part (ii): the same proof of Proposition 14.4(iii) works in this case by replacing Theorems 11.1(1) and 13.2 with Theorem 13.5.

□

Using the above Proposition, we can prove the irreducibility of $\overline{Q}_{d,g}^c$ and $\overline{Q}_{d,g}^h$ (and hence of $H_d$ and $\widetilde{H}_d$).

**Corollary 14.7.** *Assume that $d > 2(2g-2)$ and, moreover, that $g \geq 3$ if $d \leq 4(2g-2)$. Then $\overline{Q}_{d,g}^c$ and $\overline{Q}_{d,g}^h$ are irreducible. In particular, $H_d$ and $\widetilde{H}_d$ are also irreducible.*

*Proof.* Let us first prove the irreducibility of $\overline{Q}_{d,g}^c$.

In the case $d \leq 4(2g-2)$ (and $g \geq 3$), look at the surjective morphism $\Phi^{\mathrm{ps}} : \overline{Q}_{d,g}^c \to \overline{M}_g^{\mathrm{p}}$. Since $\overline{M}_g^{\mathrm{p}}$ is irreducible by Theorem 2.4(iii) and the generic fiber of $\Phi^{\mathrm{ps}}$ is irreducible by Theorem 14.3(iii), we get that there exists a unique irreducible component of $\overline{Q}_{d,g}^c$ that dominates $\overline{M}_g^{\mathrm{p}}$. Assume, by contradiction, that there is another irreducible component of $\overline{Q}_{d,g}^c$, call it $Z$, that does not dominate $\overline{M}_g^{\mathrm{p}}$. Let $W := \Phi^{\mathrm{ps}}(Z) \subsetneq \overline{M}_g^{\mathrm{p}}$ and denote by $l \geq 0$ the number of cusps of the generic point $X \in W$. Since each cusp will increase the codimension of $W$ in $\overline{M}_g^{\mathrm{p}}$ by two, we get that

$$\dim W \leq \min\{3g-4, 3g-3-2l\}. \tag{14.11}$$

Propositions 14.4(ii), 14.4(iii), 14.5(ii), 14.6(ii) imply that the generic fiber of the map $Z \twoheadrightarrow W$ has dimension less than or equal to $g + l$. Using this and (14.11), we get

$$\dim Z \leq \min\{4g-4+l, 4g-3-l\} < 4g-3. \tag{14.12}$$

This however contradicts the fact that $\overline{Q}_{d,g}$ is of pure dimension equal to $4g-3$ by Proposition 14.2(ii), q.e.d.

The case $d > 4(2g-2)$ is dealt with in a similar (and easier) way by considering the map $\Phi^s : \overline{Q}_{d,g} \to \overline{M}_g$ and using Proposition 14.4(i).

From the irreducibility of $\overline{Q}_{d,g}^c$ it follows that: $H_d$ is connected (hence irreducible because of its smoothness, see Proposition 14.2(i)) because $\overline{Q}_{d,g}^c$ is the good categorial quotient of $H_d$ by the connected algebraic group $\mathrm{SL}_{r+1}$; $\widetilde{H}_d$ is irreducible because it is an open subset of $H_d$; $\overline{Q}_{d,g}^h$ is irreducible because it is the good categorical quotient of $\widetilde{H}_d$ by $\mathrm{SL}_{r+1}$.

□

## 15. Extra components of the GIT quotient

Until now we considered the action of $\mathrm{GL}_{r+1}$ over $\mathrm{Hilb}_d$ and we restricted our attention to $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})^o$ and $\mathrm{Hilb}_d^{ss,o}$. It is very natural to ask if there are Chow and Hilbert semistable points $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ with $X$ not connected. In this section we would like to answer to this question.

As a corollary of the Potential pseudo-stability Theorem 5.1, we have the following result.

**Corollary 15.1.** *Let* $[X \subset \mathbb{P}^r] \in \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ *(resp.* $\in \mathrm{Hilb}_d^{ss}$*) where* $X = X_1 \cup \ldots \cup X_n$ *and each* $X_i$ *is a connected component of* $X$*. Suppose that* $d > 2(2g-2)$*, set* $d_i := \deg \mathcal{O}_X(1)_{|X_i}$*,* $r_i := \dim\langle X_i\rangle$ *(where* $\langle X_i\rangle$ *is the linear span of* $X_i$*) and denote by* $g_i$ *the genus of* $X_i$*. Then*

*(1)* $h^0(X_i, \mathcal{O}_{X_i}(1)) = d_i - g_i + 1 = r_i + 1$, $h^1(X_i, \mathcal{O}_{X_i}(1)) = 0$ *and*
$$h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = h^0(X, \mathcal{O}_X(1)) = \sum_{i=1}^n h^0(X_i, \mathcal{O}_{X_i}(1)).$$
*In particular,* $\langle X_i\rangle \cap \langle X_j\rangle = \emptyset$ *for every* $i \neq j$*.*

*(2) For each* $i$
$$\frac{d_i}{2g_i-2} = \frac{d}{2g-2} \quad \left( i.\ e.\quad \frac{d_i}{r_i+1} = \frac{d}{r+1}\right).$$
*In particular, if* $n \geq 2$*, then* $\gcd(d, g-1) \neq 1$*.*

*(3) For each* $i$*,* $[X_i \subset \langle X_i\rangle] \in \mathrm{Hilb}_{d_i,g_i}$ *is Chow (resp. Hilbert) semistable.*

*(4) If* $n \geq 2$*,* $[X \subset \mathbb{P}^r]$ *is Chow (resp. Hilbert) strictly semistable.*

*Proof.* (1) follows easily from Theorem 5.1(ii). (2) is an easy consequence of the basic inequality applied to $\mathcal{O}_X(1)$, which holds by Theorem 5.1(iii). Indeed, if $X_i$ is a connected component of $X$, we have $k_{X_i} = 0$, hence
$$d_i = \frac{d}{2g-2}(2g_i - 2)$$
and we are done. If $X$ is not connected and, by contradiction, $\gcd(d, g-1) = 1$, the ratio $\dfrac{d}{g-1}$ is reduced, hence for each connected component $X_i \subset X$, we have $d_i = d$, absurd. Let us prove (3). Consider a 1ps $\rho : \mathbb{G}_m \to \mathrm{GL}_{r_1+1}$ diagonalized by a system

of coordinates $\{x_1,\ldots,x_{r_1+1}\}$ in $\langle X_1\rangle$ and denote by $w_1,\ldots,w_{r_1+1}$ the weights of $\rho$. Let $\{y_1,\ldots,y_{r+1}\}$ be a system of coordinates in $\mathbb{P}^r$ such that $y_{i|X_1}=x_{i|X_1}$ and

$$\langle X_1\rangle=\bigcap_{i=r_1+2}^{r+1}\{y_i=0\}\quad\text{and}\quad\langle X_1^c\rangle=\bigcap_{i=1}^{r_1+1}\{y_i=0\}.$$

Now consider a 1ps $\rho':\mathbb{G}_m\to\mathrm{GL}_{r+1}$ diagonalized by $\{y_1,\ldots,y_{r+1}\}$ with weights $w_1',\ldots,w_{r+1}'$ such that

$$w_i'=\begin{cases} w_i & \text{if } 1\le i\le r_1+1\\ 0 & \text{if } i\ge r_1+2.\end{cases}$$

By Proposition 8.3 we get

$$e_{X_1,\rho}=e_{X\rho'}\le\frac{2d}{r+1}\,w(\rho')=\frac{2d_1}{r_1+1}\,w(\rho),$$

so that $[X_1\subset\langle X_1\rangle]\in\mathrm{Hilb}_{d_1,g_1}$ is Chow semistable (the Hilbert semistability is proved in the same way). In order to prove (4), it suffices to consider $\rho$ and $\rho'$ as above with $w_i=1$ for $i=1,\ldots,r_1+1$. We get

$$e_{X,\rho'}=e_{X_1,\rho}=2d_1=\frac{2d_1}{r_1+1}\,(r_1+1)=\frac{2d}{r+1}\,w(\rho')$$

and we are done. $\square$

Now we would like to say that each point $[X\subset\mathbb{P}^r]\in\mathrm{Hilb}_d$ which satisfies (1), (2) and (3) of Corollary 15.1 is Chow (resp. Hilbert) semistable.

Suppose that $d>2(2g-2)$ and let $[X\hookrightarrow\mathbb{P}^r]\in\mathrm{Hilb_d}$ where $X$ is the disjoint union of two curves (possibly non connected) $X_1$ and $X_2$ (of degrees $d_1,d_2$ and genus $g_1,g_2$ respectively). Under the hypothesis that $h^1(X,\mathcal{O}_X(1))=0$, we have $h^0(X,\mathcal{O}_X(1))=h^0(X_1,\mathcal{O}_{X_1}(1))+h^0(X_2,\mathcal{O}_{X_2}(1))$, hence there exists a system of coordinates $\{x_1,\ldots,x_{r+1}\}$ such that

$$\langle X_1\rangle=\bigcap_{i=r_1+2}^{r+1}\{x_i=0\}\quad\text{and}\quad\langle X_2\rangle=\bigcap_{i=1}^{r_1+1}\{x_i=0\}.\tag{15.1}$$

We have the following criterion (very similar to Proposition 8.3).

**Proposition 15.2. (Criterion of stability for non-connected curves.)** *Let $[X\subset\mathbb{P}^r]\in\mathrm{Hilb_d}$ as above. The following conditions are equivalent:*

*(1) $[X\subset\mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable);*

*(2) $[X\subset\mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable) with respect to any one-parameter subgroup $\rho:\mathbb{G}_m\to\mathrm{GL}_{r+1}$ diagonalized by coordinates of type (15.1);*

*(3) $[X\subset\mathbb{P}^r]$ is Hilbert semistable (resp. polystable, stable) with respect to any one-parameter subgroup $\rho:\mathbb{G}_m\to\mathrm{GL}_{r+1}$ diagonalized by coordinates of type (15.1) with weights $w_1,\ldots,w_{r+1}$ such that*

$$w_1=w_2=\ldots=w_{r_1+1}=0\quad\textit{or}\quad w_{r_1+2}=w_{r_1+3}=\ldots=w_{r+1}=0.$$

*The same holds for the Chow semistability (resp. polystability, stability).*

*Proof.* It is analogous to the proof of Proposition 8.3. $\square$

**Corollary 15.3.** *Let $[X \subset \mathbb{P}^r] \in \mathrm{Hilb_d}$ where $X = X_1 \cup \ldots \cup X_n$ and each $X_i$ is a connected component of $X$. Set $d_i := \deg \mathcal{O}_X(1)_{|X_i}$ and denote by $g_i$ the genus of $X_i$. If $[X \subset \mathbb{P}^r] \in \mathrm{Hilb_d}$ satisfies (1) and (2) of Corollary 15.1, then the following conditions are equivalent:*

*(1) $[X \subset \mathbb{P}^r]$ is Hilbert semistable (resp. polystable);*

*(2) $[X_i \subset \langle X_i \rangle] \in \mathrm{Hilb}_{d_i,g_i}$ is Hilbert semistable (resp. polystable).*

*The same holds for the Chow semistability (resp. polystability).*

Thus the semistable locus and the polystable locus of $\mathrm{Hilb}_d$ for $d > 2g-2$ are completely determined by applying the previous corollary and the results of Section 11 and Section 13 about the stability of connected curves.

We are now able to determine the connected components of $\mathrm{Hilb}_d^{ss}$ and $\mathrm{Chow}_d^{ss}$ for $d > 2(2g-2)$. Set

$$d' := \frac{d}{\gcd(d, g-1)} \quad \text{and} \quad g' := \frac{g-1}{\gcd(d, g-1)} + 1.$$

Let $\mathcal{H}$ be a connected component of $\mathrm{Hilb}_d^{ss}$ and consider $[X \subset \mathbb{P}^r] \in \mathcal{H}$. Using the same notations as in Corollary 15.1, suppose that $d_1 \geq d_2 \geq \ldots \geq d_n$. We get a well-defined (integral) partition $\left(\frac{d_1}{d'}, \ldots, \frac{d_n}{d'}\right)$ of $\gcd(d, g-1)$. Define the function

$$\begin{aligned} \phi : \{\text{connected components of } \mathrm{Hilb}_d^{ss}\} &\longrightarrow \{\text{partitions of } \gcd(d, g-1)\} \\ \mathcal{H} &\longmapsto \left(\frac{d_1}{d'}, \ldots, \frac{d_n}{d'}\right). \end{aligned}$$

Conversely, let $(k_1, \ldots, k_n)$ be a partition of $\gcd(d, g-1)$. For each $i = 1, \ldots, n$ consider a smooth curve $X_i$ of genus $g_i = g'k_i + 1$ and a line bundle $L_i$ on $X_i$ of degree $d_i = d'k_i$. Define the curve $X = \bigsqcup_{i=1}^{n} X_i$ and consider the line bundle $L$ on $X$ such that $L_{|X_i} = L_i$. Using the assumption that $d > 2(2g-2)$, it is easy to see that $d_i \geq 2g_i + 1$, so that $L_i$ is very ample, $[X \overset{|L_i|}{\hookrightarrow} \mathbb{P}^{d_i - g_i}] \in \mathrm{Hilb}_{d_i,g_i}$ is Hilbert stable (notice that $g_i \geq 2$ for every $i$) and $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r] \in \mathrm{Hilb}_d$ is Hilbert semistable by Corollary 15.3. Let $\mathcal{K}$ the connected component which contains $[X \overset{|L|}{\hookrightarrow} \mathbb{P}^r]$. Now define the function

$$\begin{aligned} \psi : \{\text{partitions of } \gcd(d, g-1)\} &\longrightarrow \{\text{connected components of } \mathrm{Hilb}_d^{ss}\} \\ (k_1, \ldots, k_n) &\longmapsto \mathcal{K}. \end{aligned}$$

It is easy to check that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. Summing up, we obtain that

$$\mathrm{Hilb}_d^{ss} = \bigsqcup_{\pi \text{ part. of } \gcd(d,g-1)} \psi(\pi).$$

The same arguments works for $\mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})$ giving a bijection

$$\phi' : \{\text{connected components of } \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})\} \longrightarrow \{\text{partitions of } \gcd(d, g-1)\}.$$

We have proved the following

**Theorem 15.4.** *There is a commutative diagram*

$$\begin{array}{ccc}
\{\text{connected components of } \mathrm{Hilb}_d^{ss}\} & \xrightarrow{\ \phi\ } & \{\text{partitions of } \gcd(d, g-1)\} \\
\Big\downarrow \eta & \nearrow_{\phi'} & \\
\{\text{connected components of } \mathrm{Ch}^{-1}(\mathrm{Chow}_d^{ss})\} & &
\end{array}$$

*where all the maps are one-to-one correspondences and $\eta$ is induced by the inclusion* $\mathrm{Hilb}_d^{ss} \subseteq \mathrm{Ch}^{-1}(\mathrm{Hilb}_d^{ss})$.

## 16. Compactifications of the universal Jacobian

Fix integers $d$ and $g \geq 2$. Consider the stack $\mathcal{J}_{d,g}$, called the *universal Jacobian stack* of genus $g$ and degree $d$, whose section over a scheme $S$ is the groupoid of families of smooth curves of genus $g$ over $S$ together with a line bundle of relative degree $d$. We denote by $J_{d,g}$ its coarse moduli space, and we call it the *universal Jacobian variety* (or simply the universal Jacobian) of degree $d$ and genus $g$[5].

### 16.1. Caporaso's compactification.

From the work of Caporaso ([Cap94]), it is possible to obtain a modular compactification of the universal Jacobian stack and of the universal Jacobian variety. Denote by $\overline{\mathcal{J}}_{d,g}$ the category fibered in groupoids over the category of schemes whose section over a scheme $S$ is the groupoid of families of quasi-stable curves over $S$ of genus $g$ endowed with a line bundle whose restriction to each geometric fiber is a properly balanced line bundle of degree $d$. We summarize the main properties of $\overline{\mathcal{J}}_{d,g}$ into the following

**Fact 16.1.** *Let $g \geq 2$ and $d \in \mathbb{Z}$.*

1. *$\overline{\mathcal{J}}_{d,g}$ is a smooth, irreducible, universally closed Artin stack of finite type over $k$, having dimension $4g-4$ and containing $\mathcal{J}_{d,g}$ as an open substack.*
2. *$\overline{\mathcal{J}}_{d,g}$ admits an adequate moduli space $\overline{J}_{d,g}$ (in the sense of* [Alp2]*), which is a normal irreducible projective variety of dimension $4g-3$ containing $J_{d,g}$ as an open subvariety.*

[5] In [Cap94], this variety is called the universal Picard variety and it is denoted by $P_{d,g}$. We prefer to use the name universal Jacobian, and therefore the symbol $J_{d,g}$, because the word Jacobian variety is used only for curves while the word Picard variety is used also for varieties of higher dimensions and therefore it is more ambiguous. Accordingly, we will denote the Caporaso's compactified universal Jacobian by $\overline{J}_{d,g}$ instead of $\overline{P}_{d,g}$ as in [Cap94] (see Fact 16.1).

(3) *There exists a commutative digram*

$$\begin{array}{ccc} \overline{\mathcal{J}}_{d,g} & \longrightarrow & \overline{J}_{d,g} \\ {\scriptstyle \Psi^{\rm s}}\downarrow & & \downarrow{\scriptstyle \Phi^{\rm s}} \\ \overline{\mathcal{M}}_g & \longrightarrow & \overline{M}_g \end{array}$$

*where* $\Psi^{\rm s}$ *is universally closed and surjective and* $\Phi^{\rm s}$ *is projective, surjective with equidimensional fibers of dimension* $g$.

(4) *If* $\mathrm{char}(k) = 0$, *then for any* $X \in \overline{M}_g$ *we have that*

$$(\Phi^{\rm s})^{-1}(X) \cong \overline{\mathrm{Jac_d}}(X)/\mathrm{Aut}(X),$$

*where* $\overline{\mathrm{Jac_d}}(X)$ *is the canonical compactified Jacobian of* $X$ *in degree* $d$, *parametrizing rank-*1*, torsion-free sheaves on* $X$ *that are slope-semistable with respect to* $\omega_X$ *(see Remark 16.13*(ii)*).*

(5) *If* $4(2g-2) < d$ *then we have that*

$$\begin{cases} \overline{\mathcal{J}}_{d,g} \cong [H_d/GL(r+1)], \\ \overline{J}_{d,g} \cong H_d /\!\!/ GL(r+1) = \overline{Q}_{d,g}, \end{cases}$$

*where* $H_d \subset \mathrm{Hilb}_d$ *is the open subset consisting of points* $[X \subset \mathbb{P}^r] \in \mathrm{Hilb}_d$ *such that* $X$ *is connected and* $[X \subset \mathbb{P}^r]$ *is Chow semistable (or equivalently, Hilbert semistable).*

Parts (1), (2), (3) follow by combining the work of Caporaso ([Cap94], [Cap05]) and of Melo ([Mel09]). Part (5) follows as well from the previous quoted papers if $d \geq 10(2g-2)$ and working with Hilbert semistability. The extension to $d > 4(2g-2)$ and to the Chow semistability follows straightforwardly from our Theorem 11.1(1). Part (4) was observed by Alexeev in [Ale04, Sec. 1.8] (see also [CMKV, Sec. 2.9] for a related discussion and in particular for a discussion about the need for the assumption $\mathrm{char}(k) = 0$).

We call $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{J}_{d,g}$) the *Caporaso's compactified universal Jacobian stack* (resp. *Caporaso's compactified universal Jacobian variety*) of genus $g$ and degree $d$.

16.2. **Two new compactifications of the universal Jacobian stack $\mathcal{J}_{d,g}$.** The aim of this subsection is to define and study two new compactifications of the universal Jacobian stack $\mathcal{J}_{d,g}$, one over the stack $\overline{\mathcal{M}}_g^{\rm p}$ of p-stable curves of genus $g$ and the other over the stack $\overline{\mathcal{M}}_g^{\rm wp}$ of wp-stable curves of genus $g$.

**Definition 16.2.** Fix two integers $d$ and $g \geq 3$.

(i) Let $\overline{\mathcal{J}}_{d,g}^{\rm ps}$ be the category fibered in groupoids over the category of $k$-schemes whose sections over a $k$-scheme $S$ are pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f$ is a family of quasi-p-stable curves of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree

$d$ that is properly balanced on the geometric fibers of $f$. Arrows between such pairs are given by cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

together with a specified isomorphism $\mathcal{L} \xrightarrow{\cong} h^*\mathcal{L}'$ of line bundles over $\mathcal{X}$.

(ii) Let $\overline{\mathcal{J}}^{\rm wp}_{d,g}$ be the category fibered in groupoids over the category of $k$-schemes whose sections over a $k$-scheme $S$ are pairs $(f : \mathcal{X} \to S, \mathcal{L})$ where $f$ is a family of quasi-wp-stable curves of genus $g$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ of relative degree $d$ that is properly balanced on the geometric fibers of $f$ and such that the geometric fibers of $f$ do not contain tacnodes with a line nor special elliptic tails relative to $\mathcal{L}$. Arrows between such pairs are given as in (i) above.

The aim of this subsection is to prove that $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ and $\overline{\mathcal{J}}^{\rm wp}_{d,g}$ are algebraic stacks and to study their properties. Let us first show that $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ and $\overline{\mathcal{J}}^{\rm wp}_{d,g}$ are periodic in $d$ with period $2g-2$.

**Lemma 16.3.** *For any integer $n$, there are natural isomorphisms*

$$\overline{\mathcal{J}}^{\rm ps}_{d,g} \cong \overline{\mathcal{J}}^{\rm ps}_{d+n(2g-2),g} \quad \textit{and} \quad \overline{\mathcal{J}}^{\rm wp}_{d,g} \cong \overline{\mathcal{J}}^{\rm wp}_{d+n(2g-2),g},$$

*of categories fibered in groupoids.*

*Proof.* Note that a line bundle $L$ on a quasi-wp-stable curve $X$ is properly balanced if and only if $L\otimes\omega_X^n$ is properly balanced; moreover an elliptic tail $F$ of $X$ is special with respect to $L$ if and only if $F$ is special with respect to $L \otimes \omega_X^n$. The required isomorphisms will then consist in associating to any section $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}^{\rm ps}_{d,g}(S)$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}(S)$) the section $(f : \mathcal{X} \to S, \mathcal{L} \otimes \omega_f^n) \in \overline{\mathcal{J}}^{\rm ps}_{d+n(2g-2),g}(S)$ (resp. $\overline{\mathcal{J}}^{\rm ps}_{d+n(2g-2),g}(S)$), where by $\omega_f$ we denote the relative dualizing sheaf of the morphism $f$. ☐

Moreover, the stacks $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ and $\overline{\mathcal{J}}^{\rm wp}_{d,g}$ are invariant by changing the sign of degree.

**Lemma 16.4.** *There are natural isomorphisms*

$$\overline{\mathcal{J}}^{\rm ps}_{d,g} \cong \overline{\mathcal{J}}^{\rm ps}_{-d,g} \quad \textit{and} \quad \overline{\mathcal{J}}^{\rm wp}_{d,g} \cong \overline{\mathcal{J}}^{\rm wp}_{-d,g},$$

*of categories fibered in groupoids.*

The proof of this Lemma will be given later (after Theorem 16.19), when an alternative description of $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ and $\overline{\mathcal{J}}^{\rm wp}_{d,g}$ will be available.

We will now show that if $2(2g-2) < d \le \frac{7}{2}(2g-2)$ (resp. $\frac{7}{2}(2g-2) < d \le 4(2g-2)$) then $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}$) is isomorphic to the quotient stack $[\widetilde{H}_d/{\rm GL}_{r+1}]$, where

$$\widetilde{H}_d := {\rm Hilb}_d^{ss,o} := \{[X \subset \mathbb{P}^r] \in {\rm Hilb}_d^{ss} \ : \ X \text{ is connected}\} \tag{16.1}$$

is the *main component* of the Hilbert semi-stable locus and the action of $\mathrm{GL}_{r+1}$ on $\widetilde{H}_d$ is induced by the natural action of $\mathrm{GL}_{r+1}$ on $\mathbb{P}^r$. Note that, according to Fact 4.1, $\widetilde{H}_d$ is contained in the main component $H_d$ of the Chow-semistable locus defined in (14.1); moreover, if $d > 2(2g-2)$ then $\widetilde{H}_d = H_d$ if and only if $d \neq \frac{7}{2}(2g-2)$ and $d \neq 4(2g-2)$ (see Theorems 11.1, 11.5, 13.2, 13.5).

Recall that, given a scheme $S$, $[\widetilde{H}_d/\mathrm{GL}_{r+1}](S)$ consists of $\mathrm{GL}_{r+1}$-principal bundles $\phi : E \to S$ with a $\mathrm{GL}_{r+1}$-equivariant morphism $\psi : E \to \widetilde{H}_d$. Morphisms are given by pullback diagrams which are compatible with the morphism to $\widetilde{H}_d$.

**Theorem 16.5.** *Let $g \geq 3$.*

*(i) If $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ then $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ is isomorphic to the quotient stack $[\widetilde{H}_d/\mathrm{GL}_{r+1}]$.*

*(ii) If $\frac{7}{2}(2g-2) \leq d \leq 4(2g-2)$ then $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$ is isomorphic to the quotient stack $[\widetilde{H}_d/\mathrm{GL}_{r+1}]$.*

*Proof.* To shorten the notation, we set $G := \mathrm{GL}_{r+1}$.

Let us first prove (i). We must show that, for every $k$-scheme $S$, the groupoids $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(S)$ and $[\widetilde{H}_d/G](S)$ are equivalent. Our proof goes along the lines of the proof of [Mel09, Thm. 3.1], so we will explain here the main steps and refer to loc. cit. for further details.

Given $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(S)$, we must produce a principal $G$-bundle $E$ on $S$ and a $G$-equivariant morphism $\psi : E \to \widetilde{H}_d$. Notice that since $d > 2(2g-2)$, Theorem 17.5(i) implies that $H^1(\mathcal{X}_s, \mathcal{L}_{|\mathcal{X}_s}) = 0$ for any geometric fiber $\mathcal{X}_s$ of $f$, so $f_*(\mathcal{L})$ is locally free of rank $r+1 = d-g+1$. We can then consider its frame bundle $E$, which is a principal $G$-bundle: call it $E$. To find the $G$-equivariant morphism to $\widetilde{H}_d$, consider the family $\mathcal{X}_E := \mathcal{X} \times_S E$ of quasi-p-stable curves together with the pullback of $\mathcal{L}$ to $\mathcal{X}_E$, call it $\mathcal{L}_E$, whose restriction to the geometric fibers is properly balanced.

By definition of frame bundle, $f_{E*}(\mathcal{L}_E)$ is isomorphic to $\mathbb{A}^{r+1}_k \times_k E$. Moreover, the line bundle $\mathcal{L}_E$ is relatively ample by Remark 5.7; hence it is relatively very ample by Theorem 17.5(iii). Therefore, $\mathcal{L}_E$ gives an embedding over $E$ of $\mathcal{X}_E$ in $\mathbb{P}^r \times E$. By the universal property of the Hilbert scheme $\mathrm{Hilb}_d$, this family determines a map $\psi : E \to \mathrm{Hilb}_d$ whose image is contained in $\widetilde{H}_d$ by Theorems 11.1(2) and 11.5(1). It follows immediately from the construction that $\psi$ is a $G$-equivariant map.

$$\begin{array}{ccccc}
\mathcal{X} & \longleftarrow & \mathcal{X}_E := \mathcal{X} \times_S E & & \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_E} & & \\
S & \longleftarrow & E & \xrightarrow{\ \psi\ } & \mathrm{Hilb}_d
\end{array}$$

Let us check that isomorphisms in $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(S)$ lead canonically to isomorphisms in $[\widetilde{H}_d/G](S)$. Consider an isomorphism between two pairs $(f : \mathcal{X} \to S, \mathcal{L})$ and $(f' : \mathcal{X}' \to S, \mathcal{L}')$ , i.e., an isomorphism $h : \mathcal{X} \to \mathcal{X}'$ over $S$ and an isomorphism of line bundles $\mathcal{L} \overset{\cong}{\to} h^*\mathcal{L}'$. Since $f'h = f$, we get a unique isomorphism between the vector bundles $f_*(\mathcal{L})$ and $f'_*(\mathcal{L}')$. As taking the frame bundle gives an equivalence between the category of vector bundles of rank $r+1$ over $S$ and the category of principal $G$-bundles over $S$, the isomorphism $f_*(\mathcal{L}) \xrightarrow{\cong} f'_*(\mathcal{L}')$ leads to a unique isomorphism between their frame bundles, call them $E$ and $E'$ respectively. It is clear that this isomorphism is compatible with the $G$-equivariant morphisms $\psi : E \to \widetilde{H}_d$ and $\psi' : E' \to \widetilde{H}_d$.

Conversely, given a section $(\phi : E \to S, \psi : E \to \widetilde{H}_d)$ of $[\widetilde{H}_d/G]$ over a $k$-scheme $S$, let us construct a family of quasi-$p$-stable curves of genus $g$ over $S$ and a line bundle whose restriction to the geometric fibers is properly balanced of degree $d$.

Let $\mathcal{C}_d$ be the restriction to $\widetilde{H}_d$ of the universal family on $\mathrm{Hilb}_d$. By Theorem 11.1(2), the pullback of $\mathcal{C}_d$ by $\psi$ gives a family $\mathcal{C}_E$ on $E$ of quasi-$p$-stable curves of genus $g$ and a line bundle $\mathcal{L}_E$ on $\mathcal{C}_E$ whose restriction to the geometric fibers is properly balanced. As $\psi$ is $G$-invariant and $\phi$ is a $G$-bundle, the family $\mathcal{C}_E$ descends to a family $\mathcal{C}_S$ over $S$, where $\mathcal{C}_S = \mathcal{C}_E/G$. In fact, since $\mathcal{C}_E$ is flat over $E$ and $E$ is faithfully flat over $S$, $\mathcal{C}_S$ is flat over $S$ too.

Now, since $G = \mathrm{GL}_{r+1}$, the action of $G$ on $\mathcal{C}_d$ is naturally linearized. Therefore, the action of $G$ on $E$ can also be linearized to an action on $\mathcal{L}_E$, yielding descent data for $\mathcal{L}_E$. Since $\mathcal{L}_E$ is relatively very ample and $\phi$ is a principal $G$-bundle, a standard descent argument shows that $\mathcal{L}_E$ descends to a relatively very ample line bundle on $\mathcal{C}_S$, call it $\mathcal{L}_S$, whose restriction to the geometric fibers of $\mathcal{C}_S \to S$ is properly balanced by construction.

It is straightforward to check that an isomorphism on $[\widetilde{H}_d/G](S)$ leads to an unique isomorphism in $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(S)$.

We leave to the reader the task of checking that the two functors between the groupoids $[\widetilde{H}_d/G](S)$ and $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(S)$ that we have constructed are one the inverse of the other, which concludes the proof of part (i).

The proof of part (ii) proceeds along the same lines using Theorems 13.2 and 13.5(1). □

From Theorem 16.5 and Lemmas 16.3 and 16.4, we deduce the following consequences for $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$.

**Theorem 16.6.** *Let $g \geq 3$ and $d$ any integer.*

*(i) $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ is a smooth and irreducible universally closed Artin stack of finite type over $k$ and of dimension $4g-4$, endowed with a universally closed morphism $\Psi^{\mathrm{ps}}$ onto the moduli stack of p-stable curves $\overline{\mathcal{M}}^{\mathrm{p}}_g$.*

*(ii) $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$ is a smooth and irreducible universally closed Artin stack of finite type over $k$ and of dimension $4g-4$, endowed with a universally closed morphism $\Psi^{\mathrm{wp}}$ onto the moduli stack of wp-stable curves $\overline{\mathcal{M}}^{\mathrm{wp}}_g$.*

*Proof.* Let us first prove part (i). Using Lemma 16.3, we can assume that $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ and hence that $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} \cong [\widetilde{H}_d/\mathrm{GL}_{r+1}]$ by Theorem 16.5(i). The fact that $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ is a universally closed Artin stack of finite type over $k$ follows from Theorem 16.5 and general properties of stacks coming from GIT problems. $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ is smooth and irreducible since $\widetilde{H}_d \subseteq H_d$ is smooth by Theorem 14.3(i) and irreducible by Proposition 14.4. Using again Theorem 14.3(i), we can compute the dimension of $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ as follows:

$$\dim \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} = \dim \widetilde{H}_d - \dim \mathrm{GL}_{r+1} = r(r+2) + 4g - 3 - (r+1)^2 = 4g - 4.$$

Now, given $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}(S)$, we get an element of $\overline{\mathcal{M}}_g^{\mathrm{ps}}(S)$ by forgetting $\mathcal{L}$ and by considering the p-stable reduction ps(f) : ps($\mathcal{X}$) $\to$ S of $f$ (see Definition 2.14). This defines a morphism of stacks $\Psi^{\mathrm{ps}} : \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \to \overline{\mathcal{M}}_g^{\mathrm{p}}$, which is universally closed since $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ is so.

Let us now prove part (ii). Using Lemmas 16.3 and 16.4, we can assume that $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$ and hence that $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} \cong [\widetilde{H}_d/\mathrm{GL}_{r+1}]$ by Theorem 16.5(ii). Now, the proof proceeds as in part (i). Note that the morphism $\Psi^{\mathrm{wp}} : \overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} \to \overline{\mathcal{M}}_g^{\mathrm{wp}}$ send $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}(S)$ into the wp-stable reduction wps(f) : wps($\mathcal{X}$) $\to$ S of $f$ (see Proposition 2.11).

□

Note that $\mathbb{G}_m$ acts on $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$) by scalar multiplication on the line bundles and leaving the curves fixed. So, $\mathbb{G}_m$ is contained in the stabilizers of any section of $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$). This implies that $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$) are never DM (= Deligne-Mumford) stacks. However, we can quotient out $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$) by the action of $\mathbb{G}_m$ using the rigidification procedure defined by Abramovich, Corti and Vistoli in [ACV01]: denote the rigidified stack by $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} /\!\!/ \mathbb{G}_m$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} /\!\!/ \mathbb{G}_m$).

From the modular description of $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$) it follows that the stack $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} /\!\!/ \mathbb{G}_m$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} /\!\!/ \mathbb{G}_m$) is the stackification of the prestack whose sections over a scheme $S$ are given by pairs $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}}$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}}$) and whose arrows between two such pairs are given by a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & \Box & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

together with an isomorphism $\mathcal{L} \xrightarrow{\cong} h^*\mathcal{L}' \otimes f^*M$, for some $M \in$ Pic(S). We refer to [Mel09, Sec. 4] for more details.

From Theorem 16.5 it follows that $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} /\!\!/ \mathbb{G}_m$ (resp. $\overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} /\!\!/ \mathbb{G}_m$) is isomorphic to the quotient stack $[\widetilde{H}_d/\mathrm{PGL}_{r+1}]$ if $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ (resp. if $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$). Note that, using Theorem 16.6, we get

$$\begin{cases} \dim \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} /\!\!/ \mathbb{G}_m = \dim \overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} + 1 = 4g - 3, \\ \dim \overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} /\!\!/ \mathbb{G}_m = \dim \overline{\mathcal{J}}_{d,g}^{\mathrm{wp}} + 1 = 4g - 3. \end{cases}$$

Moreover, the morphisms $\Psi^{\rm ps}: \overline{\mathcal{J}}_{d,g}^{\rm ps} \to \overline{\mathcal{M}}_g^{\rm p}$ and $\Psi^{\rm wp}: \overline{\mathcal{J}}_{d,g}^{\rm wp} \to \overline{\mathcal{M}}_g^{\rm wp}$ of Theorem 16.6 factor as

$$\text{(16.2)} \qquad \begin{cases} \Psi^{\rm ps}: \overline{\mathcal{J}}_{d,g}^{\rm ps} \to \overline{\mathcal{J}}_{d,g}^{\rm ps} /\!\!/ \mathbb{G}_m \xrightarrow{\widehat{\Psi}^{\rm ps}} \overline{\mathcal{M}}_g^{\rm p}, \\ \Psi^{\rm wp}: \overline{\mathcal{J}}_{d,g}^{\rm wp} \to \overline{\mathcal{J}}_{d,g}^{\rm wp} /\!\!/ \mathbb{G}_m \xrightarrow{\widehat{\Psi}^{\rm wp}} \overline{\mathcal{M}}_g^{\rm wp}, \end{cases}$$

We can now determine when the stacks $\overline{\mathcal{J}}_{d,g}^{\rm ps} /\!\!/ \mathbb{G}_m$ and $\overline{\mathcal{J}}_{d,g}^{\rm wp} /\!\!/ \mathbb{G}_m$ are DM-stacks.

**Proposition 16.7.** *Let $g \geq 3$ and $d$ any integer.*

*(1) The following conditions are equivalent:*
  - *(i)* $\gcd(d+1-g, 2g-2) = 1$;
  - *(ii) For any $d' \equiv \pm d \mod 2g-2$ with $2(2g-2) < d' \leq \frac{7}{2}(2g-2)$, the GIT quotient $\widetilde{H}_{d'}/\mathrm{PGL}_{r+1}$ is geometric, i.e., there are no strictly semistable points;*
  - *(iii) The stack $\overline{\mathcal{J}}_{d,g}^{\rm ps} /\!\!/ \mathbb{G}_m$ is a DM-stack;*
  - *(iv) The stack $\overline{\mathcal{J}}_{d,g}^{\rm ps} /\!\!/ \mathbb{G}_m$ is proper;*
  - *(v) The morphism $\widehat{\Psi}^{\rm ps}: \overline{\mathcal{J}}_{d,g}^{\rm ps} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}_g^{\rm p}$ is representable.*

*(2) The following conditions are equivalent:*
  - *(i)* $\gcd(d+1-g, 2g-2) = 1$;
  - *(ii) For any $d' \equiv \pm d \mod 2g-2$ with $\frac{7}{2}(2g-2) < d' \leq 4(2g-2)$, the GIT quotient $\widetilde{H}_{d'}/\mathrm{PGL}_{r+1}$ is geometric, i.e., there are no strictly semistable points;*
  - *(iii) The stack $\overline{\mathcal{J}}_{d,g}^{\rm wp} /\!\!/ \mathbb{G}_m$ is a DM-stack;*
  - *(iv) The stack $\overline{\mathcal{J}}_{d,g}^{\rm wp} /\!\!/ \mathbb{G}_m$ is proper;*
  - *(v) The morphism $\widehat{\Psi}^{\rm wp}: \overline{\mathcal{J}}_{d,g}^{\rm wp} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}_g^{\rm wp}$ is representable.*

*Proof.* Let us first prove part (1).

(1i) $\Longleftrightarrow$ (1ii): the GIT quotient $\widetilde{H}_{d'}/\mathrm{PGL}_{r+1}$ is geometric if and only if every Hilbert polystable point is also Hilbert stable. From Corollaries 11.2(2), 11.3(2), 11.6(1) and 11.7(1), this happens if and only if, given a quasi-p-stable curve $X$ of genus $g$ and a line bundle $L$ on $X$ of degree $d'$, $L$ is stably balanced whenever it is strictly balanced. Recalling Definition 3.6, it is easy to see that this occurs if and only if, given a quasi-p-stable curve $X$ of genus $g$, any proper connected subcurve $Y \subset X$ such that

$$m_Y = \frac{d'}{2g-2}\deg_Y \omega_X - \frac{k_Y}{2} \in \mathbb{Z},$$

is either an exceptional component or the complementary subcurve of an exceptional component. Now, the combinatorial proof of [Cap94, Lemma 6.3] shows that this happens precisely when $\gcd(d'+1-g, 2g-2) = 1$. We conclude since $\gcd(d+1-g, 2g-2) = \gcd(d'+1-g, 2g-2)$ for any $d \equiv \pm d' \mod 2g-2$.

For the remainder of the proof, using Lemma 16.3, we can (and will) assume that $2(2g-2) < d \leq \frac{7}{2}(2g-2)$.

Let us now show that the conditions (1ii), (1iii) and (1v) are equivalent. From Theorem 6.4 and its proof, we get that for any quasi-p-stable curve $X$ of genus $g \geq 3$ and any properly balanced line bundle $L$ on $X$ we have an exact sequence

$$0 \to \mathbb{G}_m^{\gamma(\widetilde{X})-1} \to \overline{\mathrm{Aut}(X,L)} \to \mathrm{Aut}(\mathrm{ps(X)}), \tag{16.3}$$

where $\gamma(\widetilde{X})$ denotes, as usual, the connected components of the non-exceptional subcurve $\widetilde{X}$ of $X$. Note that $\overline{\mathrm{Aut}(X,L)}$ is the automorphism group of $(X,L) \in (\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \,/\!\!/\, \mathbb{G}_m)(k)$ by the definition of the $\mathbb{G}_m$-rigidification.

We claim that each of the conditions (1ii), (1iii) and (1v) is equivalent to the condition

(*)
$\gamma(\widetilde{X}) = 1$ for any $[X \subset \mathbb{P}^r] \in \widetilde{H}_d$ or, equivalently, for any $(X,L) \in (\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \,/\!\!/\mathbb{G}_m)(k)$.

Indeed:

- Condition (1ii) is equivalent to (*) by Lemma 3.9.
- Condition (1iii) implies (*) because the geometric points of a DM-stack have a finite automorphism group scheme. Conversely, if (*) holds then $\overline{\mathrm{Aut}(X,L)} \subset \mathrm{Aut}(\mathrm{ps(X)})$, which is a finite and reduced group scheme since $\overline{\mathcal{M}}_g^{\mathrm{ps}}$ is a DM-stack if $g \geq 3$. Therefore, also $\overline{\mathrm{Aut}(X,L)}$ is a finite and reduced group scheme, which implies that $\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \,/\!\!/\mathbb{G}_m$ is a DM-stack.
- Condition (1v) is equivalent to the injectivity of the map $\overline{\mathrm{Aut}(X,L)} \to \mathrm{Aut}(\mathrm{ps(X)})$ for any $(X,L) \in (\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \,/\!\!/\mathbb{G}_m)(k)$. This is equivalent to condition (*) by the exact sequence (16.3).

(1ii) $\Longrightarrow$ (1iv): this follows from the well-known fact that the quotient stack associated to a geometric projective GIT quotient is a proper stack.

(1iv) $\Longrightarrow$ (1ii): the automorphism group schemes of the geometric points of a proper stack are complete group schemes. From (16.3), this is only possible if $\gamma(\widetilde{X}) = 1$ for any $(X,L) \in (\overline{\mathcal{J}}_{d,g}^{\mathrm{ps}} \,/\!\!/\mathbb{G}_m)(k)$, or equivalently if condition (*) is satisfied. This implies that (1ii) holds by what proved above.

Let us now prove part (2).

(2i) $\Longleftrightarrow$ (2ii): the proof is similar to the proof of the equivalence (1i) $\Longleftrightarrow$ (1ii), using Corollaries 13.3, 13.4, 13.6(1), 13.7(1).

For the remainder of the proof, using Lemmas 16.3 and 16.4, we can (and will) assume that $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$.

Note that for any quasi-wp-stable curve $X$ of genus $g \geq 3$ and any properly balanced line bundle $L$ on $X$ such that $X$ does not have tachnodes nor special elliptic tails with respect to $L$, Theorem 6.4 and its proof provides an exact sequence

$$0 \to \mathbb{G}_m^{\gamma(\widetilde{X})-1} \to \overline{\mathrm{Aut}(X,L)} \to \mathrm{Aut}(\mathrm{wps(X)}). \tag{16.4}$$

Now, the equivalences (2ii) $\Longleftrightarrow$ (2iii) $\Longleftrightarrow$ (2iv) $\Longleftrightarrow$ (2v) are proved as in part (1) using (16.4) instead of (16.3).

□

**Remark 16.8.** Notice that even if the existence of strictly semistable points in $\widetilde{H}_d$ for $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ (resp. $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$) prevents $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$) to be separated when $\gcd(d+1-g, 2g-2) \neq 1$, the fact that $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$ can be realized as a GIT quotients imply that their non-separatedness is, in some sense, quite mild. Indeed, according to the recent work of Alper, Smyth and van der Wick in [ASvdW], we have that the stacks $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$ are weakly separated, which roughly means that sections of $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ (resp. of $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$) over a punctured disc have unique completions that are closed in $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$); see [ASvdW, Definition 2.1] for the precise statement. Since both $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m$ are also universally closed, then according to loc. cit. we get that they are weakly proper. A similar argument implies that the morphisms $\widehat{\Psi}^{\mathrm{ps}} : \overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}^{\mathrm{p}}_g$ and $\widehat{\Psi}^{\mathrm{wp}} : \overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} /\!\!/ \mathbb{G}_m \to \overline{\mathcal{M}}^{\mathrm{wp}}_g$ are weakly proper.

16.3. **Existence of moduli spaces for $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$.** The aim of this subsection is to define (adequate or good) moduli spaces for the stacks $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ and $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$.

We start by observing that, since from Theorem 16.5 above we have that, for $2(2g-2) < d \leq \frac{7}{2}(2g-2)$ (resp. $\frac{7}{2}(2g-2) < d \leq 4(2g-2)$), the stack $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$) is isomorphic to the quotient stack $[\widetilde{H}_d/\mathrm{GL}_{r+1}]$, there are natural morphisms

$$\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g} \to \overline{Q}^h_{d,g} := \widetilde{H}_d /\!\!/ \mathrm{GL}_{r+1} \text{ for any } 2(2g-2) < d \leq \frac{7}{2}(2g-2), \tag{16.5}$$

$$\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g} \to \overline{Q}^h_{d,g} := \widetilde{H}_d /\!\!/ \mathrm{GL}_{r+1} \text{ for any } \frac{7}{2}(2g-2) < d \leq 4(2g-2). \tag{16.6}$$

From the work of Alper (see [Alp] and [Alp2]), we deduce that the morphism (16.5) (resp. (16.6)) realizes $\overline{Q}^h_{d,g}$ as the *adequate* moduli space of $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$) and even as its *good* moduli space if the characteristic of our base field $k$ is equal to zero or bigger than the order of the automorphism group of every p-stable (rep. wp-stable) curve of genus $g$ (because in this case, all the stabilizers are linearly reductive subgroups of $\mathrm{GL}_{r+1}$, as it follows from Lemma 6.1 and the proof of Theorem 6.4). We do not recall here the definition of an adequate or a good moduli space (we refer to [Alp] and [Alp2] for details). We limit ourselves to point out some consequences of the fact that (16.5) and (16.6) is an adequate moduli space, namely:

- The morphisms (16.5) and (16.6) are surjective and universally closed (see [Alp2, Thm. 5.3.1]);
- The morphism (16.5) (resp. (16.6)) is universal for morphisms from $\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}$) to locally separated algebraic spaces (see [Alp2, Thm. 7.2.1]);
- For any algebraically closed field $k'$ containing $k$, the morphisms (16.5) and (16.6) induce bijections

$$\overline{\mathcal{J}}^{\mathrm{ps}}_{d,g}(k')_{/\sim} \xrightarrow{\cong} \overline{Q}^h_{d,g}(k') \text{ if } 2(2g-2) < d \leq \frac{7}{2}(2g-2),$$

$$\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}(k')_{/\sim} \xrightarrow{\cong} \overline{Q}^h_{d,g}(k') \text{ if } \frac{7}{2}(2g-2) < d \leq 4(2g-2),$$

where we say that two points $x_1, x_2 \in \overline{\mathcal{J}}^{\rm ps}_{d,g}(k')$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}(k')$) are equivalent, and we write $x_1 \sim x_2$, if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $\overline{\mathcal{J}}^{\rm ps}_{d,g} \times_k k'$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g} \times_k k'$); see [Alp2, Thm. 5.3.1].

Moreover, if the GIT-quotient is geometric, which occurs if and only if $\gcd(d-g+1, 2g-2) = 1$ by Proposition 16.7, then it follows from the work of Keel-Mori (see [KeM97]) that actually $\overline{Q}^h_{d,g}$ is the *coarse* moduli space for $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}$), which means that the morphism (16.5) (resp. (16.6)) is universal for morphisms of $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}$) into algebraic spaces and moreover that (16.5) (resp. (16.6)) induces bijections

$$\overline{\mathcal{J}}^{\rm ps}_{d,g}(k') \xrightarrow{\cong} \overline{Q}^h_{d,g}(k') \text{ (resp. } \overline{\mathcal{J}}^{\rm wp}_{d,g}(k') \xrightarrow{\cong} \overline{Q}^h_{d,g}(k'))$$

for any algebraically close field $k'$ containing $k$.

It follows from the above universal properties of the morphism (16.5) that if $2(2g-2) < d, d' \leq \frac{7}{2}(2g-1)$ are such that $\overline{\mathcal{J}}^{\rm ps}_{d,g} \cong \overline{\mathcal{J}}^{\rm ps}_{d',g}$ then $\overline{Q}^h_{d,g} \cong \overline{Q}^h_{d',g}$. Similarly, if $\frac{7}{2}(2g-2) < d, d' \leq 4(2g-1)$ are such that $\overline{\mathcal{J}}^{\rm wp}_{d,g} \cong \overline{\mathcal{J}}^{\rm wp}_{d',g}$ then $\overline{Q}^h_{d,g} \cong \overline{Q}^h_{d',g}$. Using this fact together with Lemmas 16.3 and 16.4, the following definition is well-posed.

**Definition 16.9.** Fix $d \in \mathbb{Z}$ and $g \geq 3$.

(i) Set $\overline{J}^{\rm ps}_{d,g} := \overline{Q}^h_{d',g} = \widetilde{H}_{d'} /\!\!/ \mathrm{GL}_{r+1}$ for any $d' \equiv \pm d \mod 2g-2$ such that $2(2g-2) < d' \leq \frac{7}{2}(2g-2)$.

(ii) Set $\overline{J}^{\rm wp}_{d,g} := \overline{Q}^h_{d',g} = \widetilde{H}_{d'} /\!\!/ \mathrm{GL}_{r+1}$ for any $d' \equiv \pm d \mod 2g-2$ such that $\frac{7}{2}(2g-2) < d' \leq 4(2g-2)$.

Note that for any $d \in \mathbb{Z}$, we have natural morphisms

$$\overline{\mathcal{J}}^{\rm ps}_{d,g} \to \overline{J}^{\rm ps}_{d,g} \text{ and } \overline{\mathcal{J}}^{\rm wp}_{d,g} \to \overline{J}^{\rm wp}_{d,g} \tag{16.7}$$

which are adequate moduli spaces in general and coarse moduli spaces if (and only if) $\gcd(d-g+1, 2g-2) = 1$.

The projective varieties $\overline{J}^{\rm ps}_{d,g}$ and $\overline{J}^{\rm wp}_{d,g}$ are two compactifications of the universal Jacobian variety $J_{d,g}$. We collect some of their properties in the following theorem.

**Theorem 16.10.** *Let $g \geq 3$ and $d \in \mathbb{Z}$.*

*(1) The variety $\overline{J}^{\rm ps}_{d,g}$ satisfies the following properties:*

*(i) $\overline{J}^{\rm ps}_{d,g}$ is a normal integral projective variety of dimension $4g-3$ containing $J_{d,g}$ as a dense open subset. Moreover, if $\mathrm{char}(k) = 0$, then $\overline{J}^{\rm ps}_{d,g}$ has rational singularities, hence it is Cohen-Macauly.*

*(ii) There exists a surjective map $\Phi^{\rm ps} : \overline{J}^{\rm ps}_{d,g} \to \overline{M}^{\rm p}_g$ whose geometric fibers are equidimensional of dimension $g$. Moreover, if $\mathrm{char}(k) = 0$, then $\Phi^{\rm ps}$ is flat over the smooth locus of $\overline{M}^{\rm p}_g$.*

*(iii) The $k$-points of $\overline{J}^{\rm ps}_{d,g}$ are in natural bijection with isomorphism classes of pairs $(X, L)$ where $X$ is a quasi-p-stable curve of genus $g$ and $L$ is a strictly balanced line bundle of degree $d$ on $X$.*

*(2) The variety $\overline{J}^{\rm wp}_{d,g}$ satisfies the following properties:*

(i) $\overline{J}^{\rm wp}_{d,g}$ *is a normal irreducible projective variety of dimension* $4g-3$ *containing* $J_{d,g}$ *as a dense open subset. Moreover, if* $\mathrm{char}(k)=0$*, then* $\overline{J}^{\rm wp}_{d,g}$ *has rational singularities, hence it is Cohen-Macauly.*

(ii) *There exists a surjective map* $\Phi^{\rm wp}:\overline{J}^{\rm wp}_{d,g}\to\overline{M}^{\rm p}_g$ *whose geometric fiber over a p-stable curve* $X$ *has dimension equal to the sum of* $g$ *and the number of cusps of* $X$*.*

(iii) *The k-points of* $\overline{J}^{\rm wp}_{d,g}$ *are in natural bijection with isomorphism classes of pairs* $(X,L)$ *where* $X$ *is a quasi-wp-stable curve of genus* $g$ *without tacnodes and* $L$ *is a strictly balanced line bundle of degree* $d$ *on* $X$ *such that* $X$ *does not have special elliptic tails with respect to* $L$*.*

*Proof.* Let us first prove (1). Clearly, the above properties are preserved by the isomorphisms of Lemmas 16.3 and 16.4. Therefore, we can assume that $2(2g-2)<d\leq\frac{7}{2}(2g-2)$ so that $\overline{J}^{\rm ps}_{d,g}=\overline{Q}^{h}_{d,g}=\widetilde{H}_d/\mathrm{GL}_{r+1}$ by Definition 16.9.

Part (1i) follows by combining Proposition 14.2 and Corollary 14.7.

Part (1ii) follows from Theorem 14.3, Propositions 14.4(ii) and 14.5(ii).

Part (1iii) follows from Remark 14.1 together with Corollaries 11.2(2) and 11.6(1).

Let us first prove (2). Clearly, the above properties are preserved by the isomorphisms of Lemmas 16.3 and 16.4. Therefore, we can assume that $\frac{7}{2}(2g-2)<d\leq4(2g-2)$ so that $\overline{J}^{\rm wp}_{d,g}=\overline{Q}^{h}_{d,g}=\widetilde{H}_d/\mathrm{GL}_{r+1}$ by Definition 16.9.

Part (2i) follows by combining Proposition 14.2 and Corollary 14.7.

Part (2ii) follows from Theorem 14.3, Propositions 14.4(iii) and 14.6(ii).

Part (2iii) follows from Remark 14.1 together with Corollaries 13.3 and 13.6(1).

□

16.4. **An alternative description of $\overline{\mathcal{J}}_{d,g}$, $\overline{\mathcal{J}}^{\rm ps}_{d,g}$ and $\overline{\mathcal{J}}^{\rm wp}_{d,g}$.** The aim of this subsection is to provide an alternative description of the stack $\overline{\mathcal{J}}_{d,g}$ (resp. $\overline{\mathcal{J}}^{\rm ps}_{d,g}$, resp. $\overline{\mathcal{J}}^{\rm wp}_{d,g}$) in terms of certain torsion-free rank-1 sheaves on stable (resp. p-stable, resp. wp-stable) curves rather than line bundles on quasi-stable (resp. quasi-p-stable, resp. quasi-wp-stable) curves. Indeed, the results of this subsection are inspired by the work of Pandharipande in [Pan96, Sec. 10], where he reinterprets Caporaso's compactified universal Jacobian variety $\overline{J}_{d,g}$ as the moduli space of slope-semistable torsion-free, rank-1 sheaves of degree $d$ on stable curves of genus $g$.

Let us first introduce the sheaves we will be working with.

**Definition 16.11.** Let $X$ be a (reduced) curve and let $I$ be a coherent sheaf on $X$.

(i) We say that $I$ is *torsion-free* if the support of $I$ is equal to $X$ and $I$ does not have non-zero subsheaves whose support has dimension zero.
(ii) We say that $I$ is of *rank-1* if $I$ is invertible on a dense open subset of $X$.

Observe that a torsion-free rank-1 can be non locally-free only at the singular points of $X$. Clearly, every line bundle on $X$ is a torsion-free, rank-1 sheaf on $X$.

For each subcurve $Y$ of $X$, let $I_Y$ be the restriction $I_{|Y}$ of $I$ to $Y$ modulo torsion. If $I$ is a torsion-free (resp. rank-1) sheaf on $X$, so is $I_Y$ on $Y$. We let $\deg_Y(I)$ denote the degree of $I_Y$, that is, $\deg_Y(I) := \chi(I_Y) - \chi(\mathcal{O}_Y)$.

**Definition 16.12.** Let $X$ be a Gorenstein curve of arithmetic genus $g \geq 2$ and $I$ a rank-1 torsion-free sheaf of degree $d$ on $X$. We say that $I$ is *$\omega_X$-semistable* if, for every proper subcurve $Z$ of $X$, we have that

$$\deg_Z(I) \geq d\frac{\deg_Z(\omega_X)}{2g-2} - \frac{k_Z}{2} \tag{16.8}$$

where $k_Z$ denotes, as usual, the length of the scheme-theoretic intersection $Z \cap Z^c$ of $X$.

**Remark 16.13.** Let $X$ be a Gorenstein curve such that $\omega_X$ is ample.

(i) A torsion-free rank-1 sheaf $I$ on $X$ is $\omega_X$-semistable in the sense of Definition 16.12 if and only if it is slope-semistable with respect to the polarization $\omega_X$: the proof of this fact for stable curves in [CMKV, Sec. 2.9] extends to the general case.

(ii) Consider the controvariant functor

$$\overline{\mathcal{J}}_{d,X} : \mathrm{SCH} \to \mathrm{SET} \tag{16.9}$$

which associates to a scheme $T$ the set of $T$-flat coherent sheaves on $X \times T$ which are rank-1 torsion-free sheaves and $\omega_X$-semistable on the geometric fibers $X \times \{t\}$ of the second projection morphism $X \times T \to T$. The functor $\overline{\mathcal{J}}_{d,X}$ is co-represented by a projective variety $\overline{\mathrm{Jac}_\mathrm{d}}(X)$, called the *canonical compactified Jacobian* of $X$ in degree $d$; see [CMKV, Section 2] for a detailed discussion on the different constructions of the compactified Jacobians available in the literature.

**Remark 16.14.** Assume that $X$ is a Gorenstein curve such that all its singular points lying on more than one irreducible component are nodes (e.g. $X$ is a wp-stable curve). Then a torsion-free, rank-1 sheaf $I$ is $\omega_X$-semistable if and only if, for any subcurve $Y \subseteq X$, we have that

$$d\frac{\deg_Y(\omega_X)}{2g-2} - \frac{k_Y}{2} \leq \deg_Y(I) \leq d\frac{\deg_Y(\omega_X)}{2g-2} + \frac{k_Y}{2} - |Y \cap Y^c \cap \mathrm{Sing}(I)|, \tag{16.10}$$

where $\mathrm{Sing}(I)$ denotes the set of singular points of $X$ where $I$ is not locally free.

Indeed, under the above assumptions on $X$, we have the exact sequence

$$0 \to I_{Y^c}(-[Y \cap Y^c \setminus \mathrm{Sing}(I)]) \to I \to I_Y \to 0. \tag{16.11}$$

From (16.11), by using that $\deg(I) := \chi(I) - \chi(\mathcal{O}_X)$ by definition (and the analogous formulas for $I_Y$ and $I_{Y^c}$), the additivity of the Euler characteristic and the formula $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(\mathcal{O}_{Y^c}) - |Y \cap Y^c|$, we get

$$\deg(I) = \deg_Y(I) + \deg_{Y^c}(I) + |Y \cap Y^c \cap \mathrm{Sing}(I)|. \tag{16.12}$$

By substituting (16.12) into the inequality (16.8) for $Y^c$, we get the right inequality in (16.10), q.e.d.

Torsion-free, rank-1 sheaves on a wp-stable curve $X$ can be described via certain line bundles on quasi-wp-stable models of $X$.

**Lemma 16.15.** *Let $X$ be a wp-stable curve. For any set $S \subset X_{\rm sing}$, denote by $\widehat{X}_S$ the quasi-wp-stable curve obtained from $X$ by blowing-up the nodes and cusps of $X$ belonging to $S$ and set $\phi^S : \widehat{X}_S \to X$ equal to the wp-stable reduction (as in Proposition 2.11).*

*(1) Let $L$ be a line bundle on $\widehat{X}_S$ such that for every exceptional component $E$ of $\widehat{X}_S$ we have that $\deg_E L \in \{-1,0,1\}$. Then*

*(i) $R^1\phi^S_*(L) = 0$ and $\phi^S_*(L)$ is a torsion-free rank-1 sheaf on $X$ such that $\deg \phi^S_*(L) = \deg L$.*

*(ii) $\phi^S_*(L)$ is $\omega_X$-semistable if and only if $L$ is balanced.*

*(2) Let $I$ be a torsion-free rank-1 sheaf on $X$ and denote by $\mathrm{Sing}(I) \subseteq X_{\rm sing}$ the set of points of $X$ where $I$ is not locally free. Then there exists a a line bundle $L$ on $\widehat{X}_{\mathrm{Sing}(I)}$ such that*

- *$\deg_E L = 1$ for all exceptional subcurves $E$ of $\widehat{X}_{\mathrm{Sing}(I)}$;*
- *$I = \phi^{\mathrm{Sing}(I)}_*(L)$.*

*Moreover, the restriction of $L$ to the non-exceptional subcurve of $\widehat{X}_{\mathrm{Sing}(I)}$ (see Definition 2.10) is unique.*

*Proof.* Let us first prove (1). In order to simplify the notation, set $Y := \widehat{X}_S$ and $\phi := \phi^S$. As in Definition 2.10, write $Y = Y_{\rm exc} \cup \widetilde{Y}$, where $Y_{\rm exc}$ is given by the union of all the exceptional subcurves of $Y$ and $\widetilde{Y} = Y_{\rm exc}^c$ is the non-exceptional subcurve of $Y$. Let $D_{\rm exc} := Y_{\rm exc} \cap \widetilde{Y}$, which we can view as a Cartier divisor on both $Y_{\rm exc}$ and $\widetilde{Y}$. The restrictions of $L$ to $\widetilde{Y}$ and to $Y_{\rm exc}$ give rise to the following two exact sequences of sheaves:

$$\begin{cases} 0 \to L_{|Y_{\rm exc}}(-D_{\rm exc}) \to L \to L_{|\widetilde{Y}} \to 0, \\ 0 \to L_{|\widetilde{Y}}(-D_{\rm exc}) \to L \to L_{|Y_{\rm exc}} \to 0. \end{cases} \tag{16.13}$$

By taking the push-forward of (16.13) via $\phi$, we get the two exact sequences of vector spaces

$$\begin{cases} 0 \to \phi_*(L_{|Y_{\rm exc}}(-D_{\rm exc})) \to \phi_*(L) \to \phi_*(L_{|\widetilde{Y}}), \\ R^1\phi_*(L_{|\widetilde{Y}}(-D_{\rm exc})) \to R^1\phi_*(L) \to R^1\phi_*(L_{|Y_{\rm exc}}) \to 0. \end{cases} \tag{16.14}$$

Since the restriction of $\phi$ to $\widetilde{Y}$ is a finite birational morphism onto $X$, the sheaf $\phi_*(L_{|\widetilde{Y}})$ is torsion-free and of rank 1 on $X$ and $R^1\phi_*(L_{|\widetilde{Y}}(-D_{\rm exc})) = 0$. On the other hand, the sheaves $\phi_*(L_{|Y_{\rm exc}}(-D_{\rm exc}))$ and $R^1\phi_*(L_{|Y_{\rm exc}})$ are torsion sheaves supported on $\phi(Y_{\rm exc})$. For every exceptional component $E \cong \mathbb{P}^1$ of $Y$, we have that $\deg_E L_{|Y_{\rm exc}} \geq -1$ and

$\deg_E\left(L_{|Y_{\rm exc}}(-D_{\rm exc})\right) = \deg_E L - \deg_E \mathcal{O}_E(-D_{\rm exc}) \le 1-2=-1$, which implies that

$$\begin{cases} \phi_*(L_{|Y_{\rm exc}}(-D_{\rm exc}))_{\phi(E)} = H^0(E, L_{|Y_{\rm exc}}(-D_{\rm exc})) = 0, \\ R^1\phi_*(L_{|Y_{\rm exc}})_{\phi(E)} = H^1(E, L_{|Y_{\rm exc}}) = 0. \end{cases}$$

Therefore, using (16.14), we deduce that $\phi_*(L) \subseteq \phi_*(L_{|\widetilde{Y}})$ is torsion-free and of rank 1 on $X$ and $R^1\phi_*(L)=0$. Moreover, we have that $\chi(L) = \chi(\phi_*(L)) - \chi(R^1\phi_*(L)) = \chi(\phi_*(L))$, which, together with the fact that $Y$ and $X$ have the same arithmetic genus, implies that $\deg L = \deg \phi_*(L)$. Part (1i) is now proved.

Let us now prove part (1ii). Assume first that $L$ is properly balanced. Let $Z$ be a subcurve of $X$ and let $\widehat{Z}$ be the subcurve of $Y$ obtained from the subcurve $\phi^{-1}(Z)$ by removing the exceptional subcurves $E \subset \phi^{-1}(Z)$ such that $E \cap \phi^{-1}(Z)^c \neq \emptyset$ and $\deg_E L = 1$. From the definition of $\widehat{W}$, it is easy to check that

$$\begin{cases} k_{\widehat{Z}} = k_Z, \\ p_a(\widehat{Z}) = p_a(Z). \end{cases} \tag{16.15}$$

<u>CLAIM</u>: $\deg_{\widehat{Z}}(L) = \deg_Z(\phi_*(L))$.

Indeed, first of all, by the projection formula, we get

$$\phi_*(L_{|\phi^{-1}(Z)}) = \phi_*(L\otimes \mathcal{O}_{\phi^{-1}(Z)}) = \phi_*(L\otimes \phi^*(\mathcal{O}_Z)) = \phi_*(L)\otimes \mathcal{O}_Z = \phi_*(L)_{|Z}. \tag{16.16}$$

Let $\mathcal{E}$ be the union of the exceptional subcurves of $Y$ contained in $\phi^{-1}(Z)\cap\phi^{-1}(Z^c)$ and set $\mathring{Z}$ to be equal to the complement of $\mathcal{E}$ inside $\phi^{-1}(Z)$. The morphism $\phi:\mathring{Z}\to Z$ is the blow-up of $Z$ at the singular points $S\setminus(Z\cap Z^c)$. Therefore, by what proved in (1i), we get that

$$\begin{cases} \phi_*(L_{|\mathring{Z}}) \text{ is a torsion-free, rank-1 sheaf on } Z, \\ R^1\phi_*(L_{|\mathring{Z}}) = 0. \end{cases} \tag{16.17}$$

We have the following two exact sequences of sheaves on $\phi^{-1}(Z)$ :

$$\begin{cases} 0\to L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}) \to L_{|\phi^{-1}(Z)} \to L_{|\mathring{Z}} \to 0, \\ 0\to L_{|\mathring{Z}}(-\mathcal{E}\cap\mathring{Z}) \to L_{|\phi^{-1}(Z)} \to L_{|\mathcal{E}} \to 0. \end{cases}$$

By taking the push-forward via $\phi$ and using (16.17) and the analogous vanishing $R^1\phi_*(L_{|\mathring{Z}}(-\mathcal{E}\cap\mathring{Z}))=0$, we get the following two exact sequence of sheaves

$$\begin{cases} 0\to \phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z})) \to \phi_*(L_{|\phi^{-1}(Z)}) \to \phi_*(L_{|\mathring{Z}}), \\ \qquad 0\to R^1\phi_*(L_{|\phi^{-1}(Z)}) \to R^1\phi_*(L_{|\mathcal{E}}) \to 0. \end{cases} \tag{16.18}$$

The sheaves $\phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}))$ and $R^1\phi_*(L_{|\mathcal{E}})$ are torsion sheaves supported at $\phi(\mathcal{E}^1)$ and for any $\mathbb{P}^1\cong E\subseteq\mathcal{E}$ we get

$$\begin{cases} \phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}))_{\phi(E)} = H^0(E, L_{|E}(-E\cap\mathring{Z})) = \begin{cases} k & \text{if } \deg_E L = 1, \\ 0 & \text{if } \deg_E L = -1, 0, \end{cases} \\ R^1\phi_*(L_{|\mathcal{E}^1})_{\phi(E)} = H^1(E, L_{|E}) = 0, \end{cases} \tag{16.19}$$

since $\deg_E L = -1, 0, 1$ and $E$ intersects $\mathring{Z}$ in (exactly) one point. The first equation in (16.18) together with (16.17) imply that $\phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}))$ is the biggest torsion subsheaf of $\phi_*(L_{|\phi^{-1}(Z)})$. Taking into account (16.15), we get that

$$\phi_*(L)_Z = \phi_*(L_{|\phi^{-1}(Z)})/\phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z})). \tag{16.20}$$

In order to compute the degree of $\phi_*(L)_Z$, notice first of all that from the first equation in (16.19) it follows that $\phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}))$ is a torsion sheaf of length equal to the number of exceptional components $E \subseteq \mathcal{E}$ such that $\deg_E L = 1$, which is also equal to $\deg_{\phi^{-1}(Z)}(L) - \deg_{\widehat{Z}}(L)$. Moreover, from the second equations in (16.19) and in (16.18) it follows that $R^1\phi_*(L_{|\mathcal{E}}) = R^1\phi_*(L_{|\phi^{-1}(Z)}) = 0$ which implies that $\chi(L_{|\phi^{-1}(Z)}) = \chi(\phi_*(L_{|\phi^{-1}(Z)}))$. Now, we can compute the degree of $\phi_*(L)_Z$ using (16.20):

$$\deg_Z(\phi_*(L)) = \deg\phi_*(L)_Z = \chi(\phi_*(L)_Z) - \chi(\mathcal{O}_Z) = \chi(\phi_*(L_{|\phi^{-1}(Z)})) - \chi(\phi_*(L_{|\mathcal{E}}(-\mathcal{E}\cap\mathring{Z}))) - \chi(\mathcal{O}_Z) =$$

$$= \chi(L_{|\phi^{-1}(Z)}) - \deg_{\phi^{-1}(Z)}(L) + \deg_{\widehat{Z}}(L) - \chi(\mathcal{O}_{\phi^{-1}(Z)}) = \deg_{\widehat{Z}}(L),$$

q.e.d.

Using the above CLAIM and (16.15), the basic inequality (3.1) for $L$ and the subcurve $\widehat{Z} \subseteq Y$ translates into the inequality (16.8) for $\phi_*(L)$ and the subcurve $Z \subseteq X$; hence $\phi_*(L)$ is $\omega_X$-semistable.

Assume next that $\phi_*(L)$ is $\omega_X$-semistable. Let $W$ be a connected subcurve of $Y$. We want to compare the degree of $L$ on $W$ with its degree on the subcurve $\widehat{\phi(W)} \subseteq \phi^{-1}(\phi(W))$ defined above. With this aim, set

- $\mathcal{E}_W^0$ to be the collection of the exceptional subcurves contained in $W$ but not in $\widehat{\phi(W)}$ (or equivalently, contained in $W$, intersecting $\phi^{-1}(\phi(W))^c$ and having degree 1 with respect to $L$);
- $\mathcal{E}_W^1$ to be the collection of the exceptional subcurves contained in $\widehat{\phi(W)} \cap \phi^{-1}(\phi(W)^c)$ but not in $W$.
- $\mathcal{E}_W^2$ to be the collection of the exceptional subcurves contained in $\widehat{\phi(W)} \setminus \phi^{-1}(\phi(W)^c)$ but not in $W$.

Moreover, set $e_W^i$ to be equal to the cardinality of $\mathcal{E}_W^i$ (for $i = 0, 1, 2$). By construction, we have that

$$\widehat{\phi(W)} \coprod \left[\bigcup_{E\in\mathcal{E}_W^0} E\right] = W \coprod \left[\bigcup_{E\in\mathcal{E}_W^1} E\right] \coprod \left[\bigcup_{E\in\mathcal{E}_W^2} E\right]. \tag{16.21}$$

Moreover, the degree of $L$ on the exceptional components belonging to $\mathcal{E}_W^i$ can assume the following values:

$$\deg_E L = \begin{cases} 1 & \text{if } E \in \mathcal{E}_W^0, \\ -1, 0 & \text{if } E \in \mathcal{E}_W^1, \\ -1, 0, 1 & \text{if } E \in \mathcal{E}_W^2. \end{cases} \tag{16.22}$$

Using (16.21) and (16.22), together with the above CLAIM, we get that

$$\deg_{\phi(W)}(\phi_*(L)) = \deg_{\widehat{\phi(W)}} L \leq \deg_{\widehat{\phi(W)}} L + e_W^0 \leq \deg_W L + e_W^2. \tag{16.23}$$

Moreover, by the definition of $\mathcal{E}_W^i$ together with (16.15), it is easily checked that

$$\begin{cases} k_W = k_{\widehat{\phi(W)}} + 2e_W^2 = k_{\phi(W)} + 2e_W^2, \\ p_a(W) = p_a(\widehat{\phi(W)}) - e_W^2 = p_a(\phi(W)) - e_W^2. \end{cases} \tag{16.24}$$

By applying the inequality (16.8) to the sheaf $\phi_*(L)$ and the subcurve $\phi(W)$ and using (16.23) and (16.24), we get

$$\begin{aligned} \deg_W(L) \geq \deg_{\phi(W)}(\phi_*(L)) - e_W^2 \geq & d\,\frac{2[p_a(\phi(W)) - 2] + k_{\phi(W)}}{2g-2} - \frac{k_{\phi(W)}}{2} - e_W^2 = \\ & = d\,\frac{2[p_a(W) - 2] + k_W}{2g-2} - \frac{k_W}{2}, \end{aligned}$$

which shows that $L$ satisfies the basic inequality (3.1) with respect to the subcurve $W \subseteq Y$; hence $L$ is balanced.

Let us now prove (2). In order to simplify the notation, set $Y := \widehat{X}_{\mathrm{Sing}(I)}$ and $\phi := \phi^{\mathrm{Sing}(I)}$. The restriction of $\phi$ to the non-exceptional subcurve $i : \widetilde{Y} \hookrightarrow Y$, which we denote by $\nu : \widetilde{Y} \to X$, coincides the normalization of $X$ at the points of $\mathrm{Sing}(I)$. In other words, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \overset{i}{\hookrightarrow} & Y \\ & \underset{\nu}{\searrow} & \downarrow{\scriptstyle \phi} \\ & & X \end{array} \tag{16.25}$$

<u>CLAIM 1:</u> There is a unique line bundle $M$ on $\widetilde{Y}$ such that $\nu_*(M) = I$.

This is certainly well-known (see [Kas] and the references therein), so we only give a sketch of the proof. Consider the sheaf $\mathcal{E}nd(I)$ of endomorphisms of $I$. Scalar multiplication induces a natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{E}nd(I)$ and this inclusion makes $\mathcal{E}nd(I)$ into a sheaf of finite commutative $\mathcal{O}_X$-algebras. Moreover, there exists a unique rank-1 torsion-free sheaf $J$ on $\mathrm{Spec}(\mathcal{E}nd(I))$ with the property that $f_*(J) = I$, where $f : \mathrm{Spec}(\mathcal{E}nd(I)) \to X$ is the natural map (see [Kas, Lemma 3.7]). The claim now follows from the following two facts

$$\mathcal{E}nd(I) = \nu_*(\mathcal{O}_{\widetilde{Y}}) \text{ and } J \text{ is a line bundle.} \tag{*}$$

Indeed, if (*) is true then $\widetilde{Y} = \mathrm{Spec}(\mathcal{E}nd(I))$ and we can take $M = J$. Property (*) is a local property, i.e. it is enough to prove that for any $p \in X$ with $\nu^{-1}(p) = \{q_1, \cdots, q_r\} \subset \widetilde{Y}$, we have that

$$\begin{cases} \mathrm{End}(\mathrm{I_p}) \cong \oplus_{\mathrm{i}} \mathcal{O}_{\widetilde{\mathrm{Y}},\mathrm{q_i}} \text{ as } \mathcal{O}_{\mathrm{X,p}} - \text{modules}, \\ I_p \text{ is a free module over } \mathrm{End}(\mathrm{I_p}). \end{cases} \tag{**}$$

If $p \notin \mathrm{Sing}(I)$ then (**) is clear: $\nu$ is an isomorphism above $p$ and $I_p = \mathcal{O}_{X,p}$ is a free module over $\mathrm{End}(\mathrm{I_p}) = \mathcal{O}_{\mathrm{X,p}}$. If $p \in \mathrm{Sing}(I)$ (hence $p$ is a node or a cusp of $X$), then it is well-known (see e.g. [Kas, Prop. 5.7]) that $I_p$ is isomorphic to the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_{X,p}$, $\mathrm{End}(\mathfrak{m}_\mathrm{p})$ is isomorphic to the normalization $\widetilde{\mathcal{O}_{X,p}}$ of $\mathcal{O}_{X,p}$ and $\mathfrak{m}_p$ is a free module over $\widetilde{\mathcal{O}_{X,p}}$. Property (**) is proved also in this case, q.e.d.

Let now $E := E_1 \cup \cdots \cup E_n$ be the union of the exceptional subcurves of $Y$. Then we can find a line bundle $L$ on $Y$ such that $L_{|\widetilde{Y}} = M$ and $\deg_{E_i} L = 1$, $i = 1, \ldots, n$. The proof of (2) is now implied by CLAIM 1 together with the following

CLAIM 2: The natural restriction morphism

$$\mathrm{res} : \phi_* L \to \nu_*(L_{|\widetilde{Y}}) = \nu_*(M)$$

is an isomorphism of sheaves on $X$.

We must show that for every open subset $U \subseteq X$, the restriction map

$$\mathrm{res} : L(\phi^{-1}(U)) \to L_{|\widetilde{Y}}(i^{-1}\phi^{-1}(U)) = L_{|\widetilde{Y}}(\nu^{-1}(U))$$

is an isomorphism of $\mathcal{O}_X(U)$-modules. Suppose for simplicity that $U$ contains a unique point $p \in \mathrm{Sing}(I)$ and let $E_0$ be its pre-image under $\phi$. Then every section $s \in L(\phi^{-1}(U))$ can be seen as a couple $(\mathrm{res}(s), s_{|E_0})$ plus a compatibility condition. In the case when $p$ is a node, this condition just says that the value of $s_{|E_0}$ in $i^{-1}(\phi^{-1}(p)) = \nu^{-1}(p)$ must coincide with the values of $\mathrm{res}(s)$ on those points. In the case when $p$ is a cusp, the condition says that the value of $s_{|E_0}$ on the pre-image $\nu^{-1}(p)$ of the cusp must coincide with the value of $\mathrm{res}(s)$ on that point and the same for their derivatives at that point. We conclude using the fact that a section $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is determined either by its value at two distinct points of $\mathbb{P}^1$ or by its value at one point together with its derivative at that point. The general case, where $U$ contains several points of $\mathrm{Sing}(I)$, is dealt with similarly.

□

Given a wp-stable curve $X$ and a torsion-free rank-1 sheaf $I$ on $X$, an automorphism of $(X, I)$ is given by a pair $(\sigma, \psi)$ such that $\sigma \in \mathrm{Aut}(X)$ and $\psi$ is an isomorphism between the sheaves $I$ and $\sigma^*(I)$. The group of automorphisms of $(X, I)$ has a natural structure of an algebraic group, which we denote by $\mathrm{Aut}(X, I)$. We have a natural exact sequence of algebraic groups

$$\begin{aligned} 0 \to \mathrm{Aut}(I) \to \mathrm{Aut}(X, I) \to & \ \mathrm{Aut}(X)^I \to 0 \\ (\sigma, \psi) \mapsto & \ \sigma \end{aligned} \tag{16.26}$$

where $\mathrm{Aut}(I)$ is the group of automorphisms of the sheaf $I$ and $\mathrm{Aut}(X)^I$ is the subgroup of $\mathrm{Aut}(X)$ consisting of all the elements $\sigma \in \mathrm{Aut}(X)$ such that $\sigma^*(I) \cong I$.

If we write the sheaf $I$ on $X$ as the pushforward of a line bundle $L$ on the quasi-wp-stable model $\widehat{X}_{\mathrm{Sing}(I)}$ of $X$ as in Lemma 16.15(2), then $\mathrm{Aut}(X, I)$ can be expressed in terms of the automorphism group $\mathrm{Aut}(\widehat{X}_{\mathrm{Sing}(I)}, L)$, which was studied in Section 6.

**Lemma 16.16.** *Notation as in Lemma 16.15*(2). *There is a natural isomorphism of algebraic groups*

$$\mathrm{Aut}(\widehat{X}_{\mathrm{Sing}(I)}, L) \xrightarrow{\cong} \mathrm{Aut}(X, \phi_*^{\mathrm{Sing}(I)}(L)).$$

*Proof.* To simplify the notation, set $Y := \widehat{X}_{\mathrm{Sing}(I)}$ and $\phi := \phi^{\mathrm{Sing}(I)}$. From the proof of Theorem 6.4, it follows that there is an exact sequence

$$0 \to \mathbb{G}_m^{\gamma(\widetilde{Y})} \to \mathrm{Aut}(Y, L) \xrightarrow{G} \mathrm{Aut}(X), \tag{16.27}$$

where $\gamma(\widetilde{Y})$ is the number of connected components of the non-exceptional subcurve $\widetilde{Y} \subseteq Y$.

The homomorphism $G$ is obtained by composing the forgetful homomorphism $F : \mathrm{Aut}(Y, L) \to \mathrm{Aut}(Y)$ of (6.1) with the homomorphism $\widetilde{\phi} : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)$ induced by the wp-stabilization $\phi : Y \to X$. The image of the homomorphism $F$ is equal to the subgroup $\mathrm{Aut}(Y)^L \subseteq \mathrm{Aut}(Y)$ consisting of all the elements $\sigma \in \mathrm{Aut}(Y)$ such that $\sigma^*(L) \cong L$. Moreover, the homomorphism $\widetilde{\phi}$ induces a surjection $\widetilde{\phi} : \mathrm{Aut}(Y)^L \twoheadrightarrow \mathrm{Aut}(X)^{\phi_*(L)}$. Therefore, we conclude that

$$\mathrm{Im}(G) = \mathrm{Aut}(X)^{\phi_*(L)}. \tag{16.28}$$

From the proof of CLAIM 1 in Lemma 16.15(2), it follows that $\mathcal{E}nd(I) \cong \nu_*(\mathcal{O}_{\widetilde{Y}})$, where $\nu : \widetilde{Y} \to X$ is the restriction of $\phi$ to $\widetilde{Y}$. This implies that

$$\mathrm{Aut}(I) \cong \mathbb{G}_m^{\gamma(\widetilde{Y})}. \tag{16.29}$$

The natural homomorphism $\mathrm{Aut}(Y, L) \xrightarrow{\cong} \mathrm{Aut}(X, \phi_*(L))$ induces a morphism between the exact sequence (16.27) and the exact sequence (16.26) for the pair $(X, \phi_*(L))$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m^{\gamma(\widetilde{Y})} & \longrightarrow & \mathrm{Aut}(Y, L) & \xrightarrow{G} & \mathrm{Aut}(X)^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Aut}(\phi_*(L)) & \longrightarrow & \mathrm{Aut}(X, \phi_*(L)) & \longrightarrow & \mathrm{Aut}(X)^I & \longrightarrow & 0 \end{array} \tag{16.30}$$

where we used (16.28) for the exactness of the first sequence at $\mathrm{Aut}(X)^I$. The right vertical homomorphism of diagram (16.30) is the identity, while the left vertical one is an isomorphism by (16.29); hence also the middle vertical homomorphism is an isomorphism. □

We will need one last definition, namely the concept of a special elliptic tail with respect to a torsion-free, rank-1 sheaf, generalizing Definition 6.2.

**Definition 16.17.** Let $X$ be a quasi-wp-stable curve and let $I$ be a torsion-free, rank-1 sheaf on $X$. Let $F$ be an irreducible elliptic tail of $X$ and let $p$ denote the intersection point between $F$ and the complementary subcurve $F^c$. Denote, as usual, by $I_F$ the restriction of $I$ to $F$ modulo the torsion subsheaf. We say that $F$ is *special* with respect to $I$ if $I_F$ is a line bundle and $I_F = \mathcal{O}_F(d_F \cdot p)$, where $d_F := \deg(I_F)$. Otherwise, we say that $F$ is *non-special* with respect to $I$.

We can now introduce three new categories fibered in groupoids over the category of schemes parametrizing certain torsion-free, rank-1 sheaves on stable (resp. p-stable, resp. wp-stable) curves.

**Definition 16.18.** Fix two integers $d$ and $g \geq 2$.

(i) Let $\overline{\mathcal{S}}_{d,g}$ be the category fibered in groupoids over the category of $k$-schemes whose sections over a $k$-scheme $S$ are pairs $(f : \mathcal{X} \to S, \mathcal{I})$ where $f$ is a family of stable curves of genus $g$ and $\mathcal{I}$ is a coherent sheaf on $\mathcal{X}$, flat over $S$, such that its restriction $\mathcal{X}_s$ to every geometric fiber $\mathcal{X}_s := f^{-1}(s)$ of $f$ is a torsion-free, rank-1, $\omega_{\mathcal{X}_s}$-semistable sheaf of degree $d$. Arrows between such pairs are given by cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

together with a specified isomorphism $\mathcal{I} \xrightarrow{\cong} h^*\mathcal{I}'$ of coherent sheaves over $\mathcal{X}$.

(ii) Let $\overline{\mathcal{S}}^{\text{ps}}_{d,g}$ be the category fibered in groupoids over the category of $k$-schemes whose sections over a $k$-scheme $S$ are pairs $(f : \mathcal{X} \to S, \mathcal{I})$ where $f$ is a family of p-curves of genus $g$ and $\mathcal{I}$ is a coherent sheaf on $\mathcal{X}$, flat over $S$, such that its restriction $\mathcal{I}_s$ to every geometric fiber $\mathcal{X}_s := f^{-1}(s)$ of $f$ is a torsion-free, rank-1, $\omega_{\mathcal{X}_s}$-semistable sheaf of degree $d$. Arrows between such pairs are given as in (i) above.

(iii) Let $\overline{\mathcal{S}}^{\text{wp}}_{d,g}$ be the category fibered in groupoids over the category of $k$-schemes whose sections over a $k$-scheme $S$ are pairs $(f : \mathcal{X} \to S, \mathcal{I})$ where $f$ is a family of quasi-wp-stable curves of genus $g$ and $\mathcal{I}$ is a coherent sheaf on $\mathcal{X}$, flat over $S$, such that its restriction $\mathcal{I}_s$ to ever geometric fiber $\mathcal{X}_s := f^{-1}(\mathcal{X}_s)$ of $f$ is torsion-free, rank-1, $\omega_{\mathcal{X}_s}$-semistable with the property that $\mathcal{I}_s$ is locally free at the cusps of $\mathcal{X}_s$ and each elliptic tail of $\mathcal{X}_s$ is non-special with respect to $\mathcal{I}_s$. Arrows between such pairs are given as in (i) above.

We can now prove that the stacks $\overline{\mathcal{S}}_{d,g}$, $\overline{\mathcal{S}}^{\text{ps}}_{d,g}$ and $\overline{\mathcal{S}}^{\text{wp}}_{d,g}$ are isomorphic to, respectively, the stacks $\overline{\mathcal{J}}_{d,g}$, $\overline{\mathcal{J}}^{\text{ps}}_{d,g}$ and $\overline{\mathcal{J}}^{\text{wp}}_{d,g}$; thus, they provide an alternative modular description of them.

**Theorem 16.19.** *Fix two integers $d$ and $g \geq 2$. There are isomorphisms of Artin stacks*

$$\begin{cases} \overline{\mathcal{J}}_{d,g} \xrightarrow{\cong} \overline{\mathcal{S}}_{d,g} \\ \overline{\mathcal{J}}^{\text{ps}}_{d,g} \xrightarrow{\cong} \overline{\mathcal{S}}^{\text{ps}}_{d,g} \\ \overline{\mathcal{J}}^{\text{wp}}_{d,g} \xrightarrow{\cong} \overline{\mathcal{S}}^{\text{wp}}_{d,g} \end{cases}$$

*obtained by sending $(f : \mathcal{X} \to S, \mathcal{L}) \in \overline{\mathcal{J}}_{d,g}(S)$ (resp. $\overline{\mathcal{J}}^{\text{ps}}_{d,g}(S)$, resp. $\overline{\mathcal{J}}^{\text{wp}}_{d,g}(S)$) into $(\mathrm{wps(f) : wps}(\mathcal{X}) \to \mathrm{S}, \phi_*(\mathcal{L})) \in \overline{\mathcal{S}}_{\mathrm{d,g}}(\mathrm{S})$ (resp. $\overline{\mathcal{S}}^{\text{ps}}_{d,g}(S)$, resp. $\overline{\mathcal{S}}^{\text{wp}}_{d,g}(S)$), where $\phi : \mathcal{X} \to \mathrm{wps}(\mathcal{X})$ is the morphism between the family $f : \mathcal{X} \to S$ and its wp-stable reduction $\mathrm{wps(f) : wps}(\mathcal{X}) \to \mathrm{S}$ (as in Proposition 2.11).*

*Proof.* First of all, the fact that the above defined morphisms of stacks $\Phi^\star : \overline{\mathcal{J}}^\star_{d,g} \longrightarrow \overline{\mathcal{S}}^\star_{d,g}$ (for $\star = \emptyset, \text{ps}, \text{wp}$) are well-defined is proved similarly to [Pan96, Sec. 10]: the flatness over $S$ of the coherent sheaf $\phi_*(\mathcal{L})$ follows from the fact that $R^1\phi_*(\mathcal{L}_s) = 0$ for every $s \in S$ (by Lemma 16.15(1)) using the flatness criterion of [Pan96, Lemma 10.5.2]; the fact that the geometric fibers $\phi_*(\mathcal{L})_s$ of $\phi_*(\mathcal{L})$ are torsion-free, rank-1 of degree $d$ and $\omega_{\mathrm{wps}(\mathcal{X})_s}$-semistable follows from Lemma 16.15(1); moreover the extra-properties of the geometric fibers of $\phi_*(\mathcal{L})$ for elements of $\overline{\mathcal{S}}^{\mathrm{wp}}_{d,g}(S)$ as in Definition 16.18(iii) follows the analogous extra-properties of the geometric fibers of $\mathcal{L}$ for elements of $\overline{\mathcal{J}}^{\mathrm{wp}}_{d,g}(S)$ as in Definition 16.2(ii).

Now, we have to check that the given natural transformations of categories fibered in groupoids are equivalences for every scheme $S$: for simplicity we will verify it for $S = \mathrm{Spec}k$; the general case is dealt with similarly and it will be left to the reader. The essential surjectivity of the given natural transformation of categories follows from Lemma 16.15(1); the fully faithfulness follows from Lemma 16.16.

□

Using Theorem 16.19, we can now prove Lemma 16.4.

*Proof of Lemma 16.4.* For every scheme $S$ (which we can assume to be of finite type over $k$) and for every $\star = \emptyset, \mathrm{ps}, \mathrm{wp}$, consider the natural transformation of functors

$$\begin{aligned} \Lambda^\star_S : \overline{\mathcal{S}}^\star_{d,g}(S) &\longrightarrow \overline{\mathcal{S}}^\star_{-d,g}(S) \\ (f:\mathcal{X}\to S, \mathcal{I}) &\mapsto (f:\mathcal{X}\to S, \mathcal{I}^\vee := \mathcal{R}Hom(\mathcal{I}, \mathbb{D}_{\mathcal{X}}\otimes\omega_f^{-1})), \end{aligned} \tag{16.31}$$

where $\mathbb{D}_{\mathcal{X}}$ is the dualizing complex of $\mathcal{X}$ and $\omega_f$ is the relative dualizing sheaf of $f$ (which is a line bundle because the fibers of $f$ are Gorenstein curves).

Let us check that $\Lambda^\star_S$ is well-defined and an equivalence of groupoids. The coherent sheaf $\mathcal{I}$ is flat over $S$ by assumption and its fibers are Cohen-Macaulay sheaves (because a torsion-free sheaf on a curve is automatically Cohen-Macaulay). Therefore, standard results for families of Cohen-Macaulay sheaves (see e.g. [Ari13, Lemma 2.1]) show that the coherent sheaf $\mathcal{I}^\vee$ is flat over $S$ and that

$$(\mathcal{I}^\vee)_{|f^{-1}(s)} = \mathcal{R}Hom(\mathcal{I}_{|f^{-1}(s)}, (\mathbb{D}_{\mathcal{X}}\otimes\omega_f^{-1})_{|f^{-1}(s)}) = \mathcal{H}om(\mathcal{I}_{|f^{-1}(s)}, \mathcal{O}_{f^{-1}(s)}) = (\mathcal{I}_{|f^{-1}(s)})^\vee.$$

Using this and Lemma 16.20 below, we get that $\Lambda^\star_S$ is well-defined. Similarly, we have that $(\mathcal{I}^\vee)^\vee = \mathcal{I}$ which implies that $\Lambda^\star_S\circ\Lambda^\star_S = \mathrm{id}$, hence $\Lambda^\star_S$ is an an equivalence of groupoids.

Therefore, we get that

$$\overline{\mathcal{S}}^\star_{d,g} \cong \overline{\mathcal{S}}^\star_{-d,g},$$

which, together with Theorem 16.19, concludes the proof of the Lemma.

□

**Lemma 16.20.** *Let $X$ be a Gorenstein curve and let $I$ be a rank-1, torsion-free sheaf on $X$ of degree $d$. Then it holds:*

*(i) The dual $I^\vee := \mathcal{H}om(I, \mathcal{O}_X)$ of $I$ is a rank-1, torsion-free sheaf on $X$ of degree $-d$.*
*(ii) $I$ is reflexive, i.e. $(I^\vee)^\vee = I$.*

*(iii) If moreover $X$ is wp-stable, then $I$ is $\omega_X$-semistable if and only if $I^\vee$ is $\omega_X$-semistable.*

*Proof.* Part (ii) follows from [Har94, Prop. 1.6].

Let us now prove (i). Rank-1, torsion-free (or equivalently reflexive by (ii)) sheaves of degree $d$ on $X$ are in bijection with generalized divisors of degree $d$ on $X$ up to linear equivalence, see [Har94, Prop. 2.8]. Taking the dual of such a sheaf correspond to taking the dual of the corresponding linear equivalence class of generalized divisors by [Har94, Prop. 2.8(d)]. Therefore, $I^\vee$ is a rank-1, reflexive (hence torsion-free) sheaf of degree $-d$ on $X$.

Part (iii): by (ii), it is enough to prove the only if part. So assume that $I$ is $\omega_X$-semistable and let us show that $I^\vee$ is $\omega_X$-semistable. From Lemma 16.15(2), it follows that, setting $\widehat{X} := \widetilde{X}_{\mathrm{Sing}(I)}$ and $\phi := \phi^{\mathrm{Sing}(I)}$, there exists a properly balanced bundle $L$ on $\widehat{X}$ such that $\phi_*(L) = I$. The line bundle $L^{-1}$ is also balanced (although not necessarily properly balanced!) since, given a proper subcurve $Z \subseteq X$, we have that

$$\left|\underline{d}_Z - \frac{d}{2g-2}\deg_Z\omega_X\right| \leq \frac{k_Z}{2} \Leftrightarrow \left|-\underline{d}_Z - \frac{-d}{2g-2}\deg_Z\omega_X\right| \leq \frac{k_Z}{2}.$$

Moreover, by the definitions of $\mathcal{H}om(-,-)$ and $\phi_*$ and using that $\phi_*(L) = I$ and $\phi_*(\mathcal{O}_{\widehat{X}}) = \mathcal{O}_X$, we have that

$$\phi_*(L^{-1}) = \phi_*\mathcal{H}om(L, \mathcal{O}_{\widehat{X}}) = \mathcal{H}om(\phi_*(L), \phi_*(\mathcal{O}_{\widehat{X}})) = \mathcal{H}om(I, \mathcal{O}_X) = I^\vee.$$

We conclude that $I^\vee$ is $\omega_X$-semistable by Lemma 16.15(1). □

From Theorem 16.19, using Fact 16.1, Theorem 16.6 and what discussed in §16.3, we deduce the following corollary.

**Corollary 16.21.** *Let $d \in \mathbb{Z}$ and $g \geq 2$ (resp. $g \geq 3$, resp. $g \geq 3$).*

*(i) $\overline{\mathcal{S}}_{d,g}$ (resp. $\overline{\mathcal{S}}^{\mathrm{ps}}_{d,g}$, resp. $\overline{\mathcal{S}}^{\mathrm{wp}}_{d,g}$) is a smooth and irreducible universally closed Artin stack of finite type over $k$ and of dimension $4g-4$, endowed with a universally closed morphism $\Psi^{\mathrm{s}}$ (resp. $\Psi^{\mathrm{ps}}$, resp. $\Psi^{\mathrm{wp}}$) onto the moduli stack of stable (resp. p-stable, resp. wp-stable) curves $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}^{\mathrm{p}}_g$, resp. $\overline{\mathcal{M}}^{\mathrm{wp}}_g$).*

*(ii) The projective variety $\overline{J}_{d,g}$ (resp. $\overline{J}^{\mathrm{ps}}_{d,g}$, resp. $\overline{J}^{\mathrm{wp}}_{d,g}$) is an adequate moduli space (and even a good moduli space if $\mathrm{char}(k) = 0$) for $\overline{\mathcal{S}}_{d,g}$ (resp. $\overline{\mathcal{S}}^{\mathrm{ps}}_{d,g}$, resp. $\overline{\mathcal{S}}^{\mathrm{wp}}_{d,g}$).*

Another corollary of Theorem 16.19 is a modular description of the fibers of the map $\Phi^{\mathrm{ps}} : \overline{J}^{\mathrm{ps}}_{d,g} \to \overline{M}^{\mathrm{ps}}_g$ in terms of canonical compactified Jacobians (see Remark 16.13(ii)), extending the description of the fibers of the map $\Phi^{\mathrm{s}} : \overline{J}_{d,g} \to \overline{M}_g$ given in Fact 16.1(4).

**Corollary 16.22.** *Let $g \geq 3$ and $d \in \mathbb{Z}$. Assume that $\mathrm{char}(k) = 0$. Then the fiber $(\Phi^{\mathrm{ps}})^{-1}(X)$ of the morphism $\Phi^{\mathrm{ps}} : \overline{J}^{\mathrm{ps}}_{d,g} \to \overline{M}^{\mathrm{ps}}_g$ over $X \in \overline{M}^{\mathrm{p}}_g$ is isomorphic to $\overline{\mathrm{Jac}_{\mathrm{d}}}(X)/\mathrm{Aut}(X)$.*

*Proof.* The proof is the same as the proof of the analogous result for the morphism $\Phi^{\mathrm{s}} : \overline{J}_{d,g} \to \overline{M}_g$ (see Fact 16.1(4)) using that $\overline{J}_{d,g}$ is a good moduli space for $\overline{\mathcal{S}}_{d,g}$ by Corollary 16.21(ii); see e.g. [CMKV, Proof of Fact 2.6(3)] for more details. □

It would be interesting to know if the above Corollary 16.22 is true regardless of the characteristic of the base field $k$. This would follow if one could prove that $\overline{J}_{d,g}^{\rm ps}$ is a good moduli scheme for the stack $\overline{\mathcal{J}}_{d,g}^{\rm ps}$ (or equivalently for the stack $\overline{\mathcal{S}}_{d,g}^{\rm ps}$) also in positive characteristic.

## 17. Appendix: Positivity properties of balanced line bundles

The aim of this Appendix is to investigate positivity properties of balanced line bundles of sufficiently high degree on *reduced* Gorenstein curves. The results obtained here are applied in this paper only for quasi-wp-stable curves. However we decided to present these results in the Gorenstein case for two reasons: firstly, we think that these results are interesting in their own (in particular we will generalize several results of [Cap10] and [Mel11, Sec. 5] in the case of nodal curves); secondly, our proof extends without any modifications to the Gorenstein case.

So, throughout this Appendix, we let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$ and $L$ be a balanced line bundle on $X$ of degree $d$, i.e., a line bundle $L$ of degree $d$ satisfying the basic inequality

$$\left|\deg_Z L - \frac{d}{2g-2}\deg_Z \omega_X\right| \leq \frac{k_Z}{2}, \tag{17.1}$$

for any (connected) subcurve $Z \subseteq X$, where $k_Z$ is as usual the length of the scheme-theoretic intersection of $Z$ with the complementary subcurve $Z^c := \overline{X \setminus Z}$ and $\omega_X$ is the dualizing invertible (since $X$ is Gorenstein) sheaf.

The following definitions are natural generalizations to the Gorenstein case of the familiar concepts for nodal curves.

**Definition 17.1.** Let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$. We say that

(i) $X$ is *G-semistable*[6] if $\omega_X$ is nef, i.e. $\deg_Z \omega_X \geq 0$ for any (connected) subcurve $Z$. The connected subcurves $Z$ such that $\deg_Z \omega_X = 0$ are called *exceptional* subcurves.
(ii) $X$ is *G-quasistable* if $X$ is G-semistable and every exceptional subcurve $Z$ is isomorphic to $\mathbb{P}^1$.
(iii) $X$ is *G-stable* if $\omega_X$ is ample, i.e. $\deg_Z \omega_X > 0$ for any (connected) subcurve $Z$.

Note that G-semistable (resp. G-stable) curves are called semi-canonically positive (resp. canonically positive) in [Cat82, Def. 0.1]. The terminology G-stable was introduced in [CCE08, Def. 2.2]. We refer to [Cat82, Sec. 1] for more details on G-stable and G-semistable curves.

Observe also that quasi-wp-stable, quasi-p-stable and quasi-stable curves are G-quasistable; similarly wp-stable, p-stable and stable curves are G-stable.

[6] The letter G stands for Gorenstein to suggest that these notions are the natural generalizations of the usual notions from nodal to Gorenstein curves.

**Remark 17.2.** Given a subcurve [7] $i : Z \subseteq X$ with complementary subcurve $Z^c$, consider the exact sequence

$$0 \to \omega_X \otimes I_{Z^c} \to \omega_X \to (\omega_X)_{|Z^c} \to 0,$$

where $I_{Z^c}$ is the ideal sheaf of $Z^c$ in $X$. By the definition of the dualizing sheaf $\omega_Z$ of $Z$, it is easy to check that $i_*(\omega_Z) = \omega_X \otimes I_{Z^c}$ which, by restricting to $Z$, gives

$$\omega_Z = (\omega_X \otimes I_{Z^c})_{|Z} = (\omega_X)_{|Z} \otimes I_{Z\cap Z^c/Z},$$

where $I_{Z\cap Z^c/Z}$ is the ideal sheaf of the scheme theoretic intersection $Z \cap Z^c$ seen as a subscheme of $Z$. By taking degrees, we get the adjunction formula

$$\deg_Z \omega_X = 2g_Z - 2 + k_Z. \tag{17.2}$$

Using the above adjunction formula and recalling that $g_Z \geq 0$ if $Z$ is connected, it is easy to see that:

(i) $X$ is $G$-semistable if and only if for any connected subcurve $Z$ such that $g_Z = 0$ we have that $k_Z \geq 2$.
(ii) $X$ is $G$-stable if and only if for any connected subcurve $Z$ such that $g_Z = 0$ we have that $k_Z \geq 3$.

Our first result says when a balanced line bundle of sufficiently high degree is nef or ample.

**Proposition 17.3.** *Let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$ and let $L$ be a balanced line bundle on $X$ of degree $d$. The following is true:*

*(i) If $d > \frac{1}{2}(2g-2) = g-1$ then $L$ is nef if and only if $X$ is $G$-semistable and for every exceptional subcurve $Z$ it holds that $\deg_Z L = 0$ or $1$.*
*(ii) If $d > \frac{3}{2}(2g-2) = 3(g-1)$ then $L$ is ample if and only if $X$ is $G$-quasistable and for every exceptional subcurve $Z$ it holds that $\deg_Z L = 1$.*

*Proof.* Let us first prove part (i). Let $Z \subseteq X$ be a connected subcurve of $X$. If $Z = X$ then $\deg_Z L = \deg L = d > (g-1) > 0$ by assumption. So we can assume that $Z \subsetneq X$. Notice that, since $X$ is connected, this implies that $k_Z \geq 1$.

If $\deg_Z \omega_X = 2g_Z - 2 + k_Z > 0$ then, using the basic inequality (17.1) and the assumption $d > \frac{1}{2}(2g-2)$, we get

$$\deg_Z L \geq d \cdot \frac{2g_Z - 2 + k_Z}{2g-2} - \frac{k_Z}{2} > \frac{2g_Z - 2 + k_Z}{2} - \frac{k_Z}{2} \geq \begin{cases} 0 & \text{if } g_Z \geq 1, \\ -1 & \text{if } g_Z = 0, \end{cases}$$

hence $\deg_Z L \geq 0$. If $g_Z = 0$ and $k_Z = 1$ then, using the basic inequality and the assumption on $d$, we get that

$$\deg_Z L \leq \frac{d}{2g-2}(-1) + \frac{1}{2} < 0.$$

[7]Note that a subcurve of Gorenstein curve need not to be Gorenstein. For example, the curve $X$ given by the union of 4 generic lines through the origin in $\mathbb{A}^3_k$ is Gorenstein, but each subcurve of $X$ given by the union of three lines is not Gorenstein.

Therefore, if $L$ is nef then $X$ must be G-semistable. Finally, if $Z$ is any exceptional subcurve of $X$, then the basic inequality gives

$$|\deg_Z L| \le 1, \tag{17.3}$$

from which we deduce that if $L$ is nef then $\deg_Z L = 0$ or 1. Conversely, it is also clear that if $X$ is G-semistable and $\deg_Z L = 0$ or 1 for every exceptional subcurve $Z$ of $X$ then $L$ is nef.

Let us now prove part (ii). Let $Z \subseteq X$ be a connected subcurve of $X$. If $Z = X$ then $\deg_Z L = \deg L = d > 3(g-1) > 0$ by assumption. So we can assume that $Z \subsetneq X$. Notice that, since $X$ is connected, this implies that $k_Z \ge 1$.

If $\deg_Z \omega_X = 2g_Z - 2 + k_Z > 0$ then, using the basic inequality (17.1) and the inequality $d > \frac{3}{2}(2g-2)$, we get

$$\deg_Z L \ge d\cdot\frac{2g_Z - 2 + k_Z}{2g-2} - \frac{k_Z}{2} > \frac{3(2g_Z - 2 + k_Z)}{2} - \frac{k_Z}{2} \ge \begin{cases} k_Z \ge 1 \text{ if } g_Z \ge 1, \\ \dfrac{2k_Z - 6}{2} \ge 0 \text{ if } g_Z = 0 \text{ and } k_Z \ge 3, \end{cases}$$

hence $\deg_Z L > 0$. From part (i) and equation (17.3), we get that if $L$ is ample then $X$ is G-semistable and for every exceptional subcurve $Z$ we have that $\deg_Z L = 1$. Note that every exceptional subcurve $Z$ of $X$ is a chain of $\mathbb{P}^1$. Assume that this chain has length $l \ge 2$ and denote by $W_i$ (for $i = 1, \ldots, l$) the irreducible components of $Z$. Then each of the $W_i$'s is an exceptional subcurve of $X$. Therefore, the same inequality as before gives that if $L$ is ample then $\deg_{W_i} L = 1$. This is a contradiction since $1 = \deg_Z L = \sum_i \deg_{W_i} L = l > 1$. Hence $Z \cong \mathbb{P}^1$ and $X$ is G-quasistable. Conversely, it is clear that if $X$ is G-semistable and $\deg_Z L = 1$ for every exceptional subcurve $Z$ of $X$ then $L$ is ample.

□

We next investigate when a balanced line bundle on a reduced Gorenstein curve is non-special, globally generated, very ample or normally generated. To this aim, we will use the following criteria, due to Catanese-Franciosi [CF96], Catanese-Franciosi-Hulek-Reid [CFHR99] and Franciosi-Tenni [FT14] (see also [Fra04] and [Fra07]) which generalize the classical criteria for smooth curves.

**Fact 17.4.** *(*[CF96]*,* [CFHR99]*,* [FT14]*) Let L be a line bundle on a reduced Gorenstein curve $X$. Then the following holds:*

- *(i) If $\deg_Z L > 2g_Z - 2$ for all (connected) subcurves $Z$ of $X$, then L is non-special, i.e., $H^1(X, L) = 0$.*
- *(ii) If $\deg_Z L > 2g_Z - 1$ for all (connected) subcurves $Z$ of $X$, then L is globally generated;*
- *(iii) If $\deg_Z L > 2g_Z$ for all (connected) subcurves $Z$ of $X$, then L is very ample.*
- *(iv) If $\deg_Z L > 2g_Z$ for all (connected) subcurves $Z$ of $X$, then L is normally generated, i.e. the multiplication maps*

$$\rho_k : H^0(X, L)^{\otimes k} \to H^0(X, L^k)$$

*are surjective for every $k \geq 2$.*

Recall that if $Z$ is a subcurve that is a disjoint union of two subcurves $Z_1$ and $Z_2$ then $g_Z = g_{Z_1} + g_{Z_2} - 1$. From this, it is easily checked that if the numerical assumptions of (i), (ii), (iii) and (iv) are satisfied for all connected subcurves $Z$ then they are satisfied for all subcurves $Z$. With this in mind, part (i) follows from [CF96, Lemma 2.1]. Note that in loc. cit. this result is only stated for a curve $C$ embedded in a smooth surface; however, a closer inspection of the proof reveals that the same result is true for any Gorenstein curve $C$. Parts (ii) and (iii) follow from [CFHR99, Thm. 1.1]. Part (iv) follows from [FT14, Thm. 4.2], which generalizes the previous results of Franciosi (see [Fra04, Thm. B] and [Fra07, Thm. 1]) for reduced curves with locally planar singularities.

Using the above criteria, we can now investigate when balanced line bundles are non-special, globally generated, very ample or normally generated.

**Theorem 17.5.** *Let $L$ be a balanced line bundle of degree $d$ on a connected reduced Gorenstein curve $X$ of genus $g \geq 2$. Then the following properties hold:*

*(i) If $X$ is G-semistable and $d > 2g-2$ then $L$ is non-special.*

*(ii) Assume that $L$ is nef. If $d > \frac{3}{2}(2g-2) = 3(g-1)$ then $L$ is globally generated.*

*(iii) Assume that $L$ is ample. Then:*

*(a) If $d > \frac{5}{2}(2g-2) = 5(g-1)$ then $L$ is very ample and normally generated.*

*(b) If $d > \max\{\frac{3}{2}(2g-2) = 3(g-1), 2g\}$ and $X$ does not have elliptic tails (i.e., connected subcurves $Z$ such that $g_Z = 1$ and $k_Z = 1$) then $L$ is very ample and normally generated.*

*Proof.* In order to prove part (i), we apply Fact 17.4(i). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then $\deg_Z L = d > 2g-2$ by assumption. Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). Since $X$ is G-semistable, we have that $\deg_Z(\omega_X) = 2g_Z - 2 + k_Z \geq 0$. If $\deg_Z(\omega_X) > 0$ then the basic inequality (17.1) together with the hypothesis on $d$ gives that

$$\deg_Z L \geq \frac{d}{2g-2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} > 2g_Z - 2 + \frac{k_Z}{2} > 2g_Z - 2.$$

If $\deg_Z(\omega_X) = 0$ (which happens if and only if $Z$ is exceptional, i.e., $g_Z = 0$ and $k_Z = 2$) then the basic inequality gives that

$$\deg_Z L \geq \frac{d}{2g-2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} = -1 > -2 = 2g_Z - 2.$$

In order to prove part (ii), we apply Fact 17.4(ii). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then we have that $\deg_Z L = d > 3(g-1) \geq 2g-1$ by the assumption on $d$. Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). If $g_Z = 0$ then $\deg_Z L > -1 = 2g_Z - 1$ since $L$ is nef. Therefore, we can assume that $g_Z \geq 1$. By applying the basic inequality (17.1) and using our assumption on $d$, we get that

$$\deg_Z L \geq \frac{d}{2g-2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} > \frac{3}{2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} = 3(g_Z - 1) + k_Z \geq 2g_Z - 1.$$

In order to prove parts (iiia) and (iiib), we apply Facts 17.4(iii) and 17.4(iv). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then, in each of the cases (iiia) and (iiib), we have that $\deg_Z L = d > 2g$ by the assumption on $d$ (note that $5(g-1) > 2g$ since $g \geq 2$). Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). If $g_Z = 0$ then $\deg_Z L > 0 = 2g_Z$ since $L$ is ample. Therefore, we can assume that $g_Z \geq 1$.

In the first case (iiia), by applying the basic inequality (17.1) and the numerical assumption on $d$, we get that

$$\deg_Z L \geq \frac{d}{2g-2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} > \frac{5}{2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} = 5(g_Z - 1) + 2k_Z \geq 2g_Z.$$

In the second case (iiib), from the basic inequality (17.1) and the numerical assumption on $d$, we get that

$$\deg_Z L \geq \frac{d}{2g-2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} > \frac{3}{2}(2g_Z - 2 + k_Z) - \frac{k_Z}{2} = 3(g_Z - 1) + k_Z \geq 2g_Z,$$

where in the last inequality we used that $g_Z, k_Z \geq 1$ and $(g_Z, k_Z) \neq (1,1)$ because $X$ does not contain elliptic tails.

□

**Remark 17.6.** Theorem 17.5(i) recovers [Cap10, Thm. 2.3(i)] in the case of nodal curves. Theorem 17.5(ii) combined with Proposition 17.3(i) recovers and improves [Cap10, Thm. 2.3(iii)] in the case of nodal curves. Theorem 17.5(iii) improves [Mel11, Cor. 5.11] in the case of nodal curves. See also [Bal09], where the author gives some criteria for the global generation and very ampleness of balanced line bundles on quasi-stable curves.

The previous results can be applied to study the positivity properties of powers of the canonical line bundle on a reduced Gorenstein curve, which is clearly a balanced line bundle.

**Corollary 17.7.** *Let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$. Then the following holds:*

*(i) If $X$ is G-semistable then $\omega_X^i$ is non-special and globally generated for all $i \geq 2$;*
*(ii) If $X$ is G-stable then $\omega_X^i$ is very ample for all $i \geq 3$;*
*(iii) If $X$ is G-quasistable then $\omega_X^i$ is normally generated for all $i \geq 3$.*

*Proof.* Part (i) follows from Theorem 17.5(i) and Theorem 17.5(ii).

Part (ii) follows from Theorem 17.5(iiia).

Let us now prove part (iii). If $X$ is G-stable, then this follows from Theorem 17.5(iiia). In the general case, since $\omega_X^i$ is globally generated by part (i), it defines a morphism

$$q : X \to \mathbb{P} := \mathbb{P}(H^0(X, \omega_X^i)^\vee),$$

whose image we denote by $Y := q(X)$. Since $X$ is $G$-quasistable, the degree of $\omega_X^i$ on a connected subcurve $Z$ of $X$ is zero if and only if $Z = E$ is an exceptional subcurve, i.e., if $E \cong \mathbb{P}^1$ and $k_E = 2$. The map $q$ will contract such an exceptional subcurve $E$

to a node if $E$ meets the complementary subcurve $E^c$ in two distinct points and to a cusp if $E$ meets $E^c$ in one point with multiplicity two. Moreover, using Fact 17.4(iii), it is easy to check that $\omega_X$ is very ample on $X \setminus \cup E$, where the union runs over all exceptional subcurves $E$ of $X$. We deduce that $Y$ is G-stable. By what proved above, $\omega_Y^i$ is normally generated. Clearly, $q^*\omega_Y^i = \omega_X^i$ and moreover, since $q$ has connected fibers, we have that $q_*\mathcal{O}_X = \mathcal{O}_Y$. This implies that $H^0(X, (\omega_X^i)^k) = H^0(Y, (\omega_Y^i)^k)$ from which we deduce that $\omega_X^i$ is normally generated. □

**Remark 17.8.** Part (i) of the above Corollary 17.7 recovers [Cat82, Thm. A and p. 68], while part (ii) recovers [Cat82, Thm B]. Part (iii) was proved for nodal curves in [Mel11, Cor. 5.9].

A closer inspection of the proof reveals that parts (ii) and (iii) continue to hold for $\omega_X^2$ if, moreover, $g \geq 3$ and $X$ does not have elliptic tails (see also [Cat82, Thm. C] and [Fra04, Thm. C]).

Let us end this Appendix by mentioning that it is possible to generalize the above results in order to prove that a balanced line bundle of sufficiently high degree is $k$-very ample in the sense of Beltrametti-Francia-Sommese ([BFS89]). Recall first the definition of $k$-very ampleness.

**Definition 17.9.** Let $L$ be a line bundle on $X$ and let $k \geq 0$ be a integer. We say that $L$ is *$k$-very ample* if for any 0-dimensional subscheme $S \subset X$ of length at most $k+1$ we have that the natural restriction map

$$H^0(X, L) \to H^0(S, L_{|S})$$

is surjective. In particular 0-very ample is equivalent to being globally generated and 1-very ample is equivalent to being very ample.

The proof of the following Theorem is very similar to the proof of the Theorem 17.5 above, using again [CFHR99, Thm. 1.1], and therefore we omit it.

**Theorem 17.10.** *Let $k \geq 2$ and assume that $X$ is G-stable. Then:*

*(i) If $d > \frac{2k+3}{2}(2g-2) = (2k+3)(g-1)$ then $L$ is $k$-very ample.*

*(ii) If $d > \frac{2k+1}{2}(2g-2) = (2k+1)(g-1)$ and $X$ does not have elliptic tails then $L$ is $k$-very ample.*

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