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Remarks on some weighted Sobolev inequalities and applications

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INTRODUCTION

Variational problems and the solvability of certain nonlinear equations have a long and rich history beginning with calculus and extending through the calculus of variations. We study "well-connected" pairs of such problems which are related by critical point considerations. We consider elliptic degenerate problems of type

$$-\Delta u(y,z) = \frac{1}{|y|^s} u^{2_*-1}(y,z) \tag{1}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $n \geq 3$, $k \geq 2$, $2_* = \frac{2(n-s)}{n-2}$, $0 < s < 2$. This equation is critical because is invariant with respect to the scalings and z-translations $u_\beta(x) = \beta^{\frac{n-2}{2}} u(\beta y, \beta(z - z_0))$, $\forall \beta > 0$. The lack of compactness of problem (1) is also given by the fact that we are looking for entire solutions, that is solutions for all space. Moreover (1) is the Euler equation associated to a weighted Sobolev inequality, proved in [3]: There exists $S > 0$ such that for $u \in C_0^\infty(\mathbb{R}^k \times \mathbb{R}^{n-k})$

$$\left(\iint_{\mathbb{R}^k\, \mathbb{R}^{n-k}} \frac{|u|^{\frac{2(n-s)}{n-2}}}{|y|^s}\, dy\, dz \right)^{\frac{n-2}{n-s}} \leq S \iint_{\mathbb{R}^k\, \mathbb{R}^{n-k}} |\nabla u|^2\, dy\, dz. \tag{2}$$

for $n \geq 3$, $0 \leq s < 2$ and $2 \leq k \leq n$.
Always in [3],using the method of concentration compactness of P.L.Lions, the authors prove that for $0 < s < 2$, the best constant in (2) is achieved, that is there exists a function u, called extremal, such that verifies the equality. If $s = 0$, (2) corresponds to the classical Sobolev inequality which has been exhaustively studied by Aubin [1] and Talenti [51] who computed exactly the best constant and proved existence of extremal functions, exhibiting them explicitly. If $s = 2$ and $2 < k \leq n$, (2) still holds true (see [3]) thus providing an extension of the classical Hardy, which is known not to possess extremal functions. While for the case $k = n$, an extensive literature was produced concerning more general weighted Sobolev inequalities, only few existence results for variational problems with cylindrical symmetry, $(k < n)$, are known (see [3],[2] and [42]). We recall, for $k = n$, the Caffarelli-Kohn-

Nirenberg inequality ([13]) which establishes: For all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^p |x|^{-bp}\, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2\, dx \tag{3}$$

where $n \geq 3, -\infty < a < \frac{n-2}{2}, a \leq b \leq a+1$, and $p = \frac{2n}{(n-2+2(b-a))}$. Catrina F. and Z.Q.Wang in ([17]) have obtained several results concerning best constants and corresponding extremal functions for (3) in the case $a < 0$, while the case $a = 0$, $0 < b < 1$, was thoroughly investigated by [28] and extremal functions have been computed by Lieb ([34]). Moreover, for $a = 0, bp = s$, we get (2) with $k = n$. For any $a \in \mathbb{R}$, positive solutions of the Euler equation associated to (3), on a properly weighted Sobolev space, turn out to be radially symmetric (see [18] for the case $a \geq 0$ and [17] for $a < 0$) and they can be explicitly computed just solving an ODE.
The main purpose of this thesis is to characterize the extremals in (2) for the case $0 < s < 2$ *and* $2 \leq k < n$, or equivalently, to look for all solutions to (1), but when $k < n$, we note that the extremals cannot be anymore radially symmetric and then they cannot be searched among solutions of an ODE, but of a PDE.

This thesis consists of four chapter which are organized as follows. In chapter 1, we will recall several problems with lack of compactness and some useful tools for these problems, which are employed in the rest of thesis, as the method of Concentration-Compactness of P.L.Lions and the moving plane method of Alexandrov and Serrin. In the second chapter, we give existence and nonexistence results obtained by [3],[2] and [45] for the problem

$$-\Delta u(y,z) = \phi(|y|)\, u^{2_*-1}(y,z). \tag{4}$$

where $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $n \geq 3$, $k \geq 2$, $2_* = \frac{2(n-s)}{n-2}$, $0 < s < 2$. This equation, for $n = 3$ and $\phi(r) = \frac{r^{2\alpha}}{(1+r^2)^{\alpha+\frac{1}{2}}}$, $\alpha > 0$ has been proposed by G.Bertin and L.Ciotti as a model describing the dynamics of elliptic galaxies (see the book [7] and lecture notes [19]) . Here u satisfies the usual Poisson equation $\Delta u = 4\pi G\rho$ relating the gravitational potential u to the density of matter ρ and satisfies the condition $\int_{\mathbb{R}^3} \phi(r) u^{p-1} dx < +\infty$ which guarantees that the given solution carries a finite total mass. The cylindrical symmetry of the problem is derived from the assumptions that the elliptic galaxies are axially symmetric. Several elliptic equations similar to (4) arising from models of globular cluster of stars have been investigated, but we know most of these models are radial. In our case, we suppose that $\phi(r)$ is asymptotic, at 0 or at ∞ (or both) to the function $1/r^s$. Then, (4) admits (1) as limit problem. In third chapter we prove, using moving planes techniques, that all the solutions of (1) are cylindrically symmetric. Thanks to these symmetries, (1) reduces to an elliptic equation in the positive cone in $\mathbb{R}^2$ and in a particular case, when $s = 1$, leads to a complete identification of all the solutions of the equation

$$-\Delta u(y,z) = \frac{u^{\frac{n}{n-2}}}{|y|} \tag{5}$$

In fact, we prove

Theorem 1. *Let u_0 be the function given by*

$$u_0(x) = u_0(y,z) = c_{n,k}\left((1+|y|)^2 + |z|^2\right)^{-\frac{n-2}{2}}$$

where $c_{n,k} = \{(n-2)(k-1)\}^{\frac{n-2}{2}}$. Then u is a solution of (5) if and only if $u(y,z) = \lambda^{\frac{n-2}{2}} u_0(\lambda y, \lambda(z+z_0))$ for some $\lambda > 0$ and $z_0 \in \mathbb{R}^{n-k}$.

As a consequence of the above Theorem, we have

Theorem 2. *The best constant S in the weighted Sobolev inequality (2), with $s = 1$, is given by*

$$S = (n-2)(k-1)\left\{2(\pi)^{\frac{n}{2}} \frac{(k-2)!}{(n+k-3)!} \frac{\Gamma(\frac{n+k-2}{2})}{\Gamma(k/2)}\right\}^{\frac{1}{n-1}}.$$

The identification of the solutions to (5) is based on a mysterious identity which goes back to the work of Jerison and Lee ([31]) on the CR-Yamabe problem. More precisely, it is related to the identification of the extremals for the Sobolev inequality on the Heisenberg group ([32]). Actually, we follow closely the approach by Garofalo-Vassilev ([25]) in the search of entire solutions of Yamabe-type equations on more general groups of Heisenberg type. We also remark that, while symmetry properties hold true for (2), we didn't succeeded in getting an efficient Jerison-Lee type identity in the general case, and this is why a classification of solutions is missing if, in (1), $s \neq 1$. Similar difficulties are encountered in dealing with Grushin type operators $-\Delta_x - (\alpha+1)^2|x|^{2\alpha}\Delta_y, \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^k$ with critical nonlinearity (see [6] for a related sharp Sobolev inequality and identification of extremals in case $\alpha = 1$). As noticed in [41], the Heisenberg sublaplacian is in fact a Grushin operator with $\alpha = 1$, m an even integer and $k = 1$ (the work of Garofalo-Vassiliev actually deals with more general values of m and k) and identification of solutions is available only in case $\alpha = 1$. While it is relatively easy to see that weak solutions to (1) are bounded, further regularity on the subspace $C = \{(y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \; : \; y = 0\}$ is based on more intricate estimates given by following lemma:

Lemma 3. *Let u be a solution of (1). Assume*
$s < 1 + \frac{k}{n}$ *if* $n \geq 4$
$s < \frac{3}{2}$ *if* $n = 3$
Then u is C^∞ in the z variable and $C^{0,\alpha}$, for some $\alpha < 1$, in the y variable.

In fourth chapter, we consider a perturbation problem of type

$$-\Delta u = \frac{u^{2_*-1}}{|y|^s} + \lambda u \tag{6}$$

with $u > 0$, $u \in H_0^1(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain containing the origin. With the help of a Pohozaev type identity, we prove that the equation (6) does not have non-trivial non-negative solutions on starlike bounded domains and the following result holds:

Theorem 4. *Assume $n \geq 4$. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of the problem*

$$\left.\begin{array}{rlll} -\Delta u & = \frac{u^{2_*(1)-1}}{|y|} + \lambda u & \text{in} & \Omega \\ u & > 0 & \text{in} & \Omega \\ u & = 0 & \text{on} & \partial\Omega. \end{array}\right\} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^n$ containing the origin and $2_*(1) = \frac{2(n-1)}{n-2}$.

1. ISOPERIMETRIC INEQUALITIES

In this chapter, we shall give an historical background on isoperimetric inequalities that are examples of scale invariance and hence of lack of compactness.

Definition 1.0.1. We will say an "isoperimetric inequality" to be any inequality which relates two or more geometric or physical quantities associated with the same domain. The inequality must be optimal in the sense that the equality sign holds for the same domain or in the limit as the domain degenerates.

Definition 1.0.2. An extremal function or maximizing f, is a function that gives equality.

Having a precise geometric meaning, these inequalities are often invariant under the action of the conformal group, i.e. translations, rotations, reflections, scaling symmetries, inversions on the unit sphere.

Since ancient times it has been known that among all plane domains of a given area, the circle has the shortest boundary. The extremal property is expressed in the classical isoperimetric inequality

$$L^2 \geq 4\pi\, A \tag{1.1}$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle. Analogously, in $\mathbb{R}^n$, with $n \geq 3$, we get isoperimetric inequality for currents

$$[M(\partial T)]^{\frac{n}{n-1}} \geq 4\pi\, M(T) \tag{1.2}$$

The problem in $\mathbb{R}^n$ is to maximize the volume among all domains whose boundary surfaces has a fixed $(n-1)$-dimensional area. The solution is that the unique extremal is the domain bounded by a sphere. Other examples are analytic inequalities such as Sobolev inequality, Rayleigh quotient, eigenvalues of the Laplacian, Poincaré inequality. Extensive discussions of such inequalities can be found in the book of Pólya and Szegö and in the review by Payne. The most famous inequalities are the following.

Theorem 1.0.1. *(Hardy-Littlewood-Sobolev)*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)\,dx\,dy \right| \leq N_{p,\lambda,n}\, \|f\|_p\, \|g\|_t \tag{1.3}$$

for all $f \in L^p(\mathbb{R}^n), g \in L^t(\mathbb{R}^n), 1 < p\,,\, t < \infty, \frac{1}{p} + \frac{1}{t} + \frac{\lambda}{n} = 2$ *and* $0 < \lambda < n$, *where* $N_{p,\lambda,n}$ *is the best constant. The sharp constant satisfies*

$$N_{p,\lambda,n} \leq \frac{n}{(n-\lambda)} \left(\frac{\omega_{n-1}}{n}\right)^{\lambda/n} \frac{1}{p\,t} \left(\left(\frac{\lambda/n}{1-1/p}\right)^{\lambda/n} + \left(\frac{\lambda/n}{1-1/t}\right)^{\lambda/n} \right).$$

where ω_{n-1} = *surface area of the* $(n-1)$*-dimensional unit sphere.*

Remark 1.0.1. We notice that the constraint $\frac{1}{p} + \frac{1}{t} + \frac{\lambda}{n} = 2$ implies the scale invariance of (1.3).

Remark 1.0.2. Inequality (1.3) was proved by Hardy-Littlewood-Sobolev in the 1930's. The sharp version with the constant, was proved by Lieb in 1983 in [34], where he shows that a maximizing pair f, g exists for (1.3). However, neither this constant nor the optimizers are known when $p \neq t$. This requires the use of two rearrangement inequalities and a new compactness technique for maximizing sequence. For the case of $t = p$ and, as corollary, for the case $t = 2$ or $p = 2$, f and g are explicitly computed. As an application of Lieb' s method, the existence of a maximizing function f on $\mathbb{R}^n$, can be proved for the sharp constant in the Sobolev inequality.

1.1 Sobolev inequality

The following estimates, valid for all functions in certain classes, have become a standard tool in existence and regularity theories for solutions of partial differential equations, in the calculus of variations, in geometric measure theory and in many other branches of analysis.

Definition 1.1.1. A "Sobolev inequality" has come to mean an estimation of lower order derivatives of a function in terms of its higher order derivatives.

Theorem 1.1.1. *(Gagliardo-Nirenberg-Sobolev). Assume* $1 \leq p < n$. *There exists a constant* C *depending only on* p, n, *such that*

$$||u||_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C\,||\nabla u||_{L^p(\mathbb{R}^n)} \tag{1.4}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

The sharp constant $C = C_{p,n}$ and the extremal functions were derived by Talenti [51].

1.2 Hardy inequality

This classical result, has been proved by Garcia and Peral in [24], using the following one dimensional Hardy inequality which holds for all $u \in C_0^\infty(0, \infty)$ and for $p > 1$:

$$\int_0^\infty |u'(t)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \left|\frac{u(t)}{t}\right|^p dt \tag{1.5}$$

Theorem 1.2.1. *Assume $1 < p < n$. Then the following inequality*

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p}\,dx \le C_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p\,dx \tag{1.6}$$

hods for all $u \in D^{1,p}(\mathbb{R}^n)$, where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $||u||_{D^{1,p}(\mathbb{R}^n)} = ||\nabla u||_{L^p(\mathbb{R}^n)}$. Moreover, $C_{n,p} = (p/(n-p))^p$ is the optimal constant.

Remark 1.2.1. The (1.6) inequality is false for $p = n \ge 2$, that is, there does not exist any constant $C > 0$ such that $\int_{\mathbb{R}^n} |\nabla u(x)|^n\,dx \ge C \int_{\mathbb{R}^n} |\frac{u(x)}{x}|^n\,dx$. This can be seen easily by taking a ball $B(0,R)$ and a smooth function $0 \le \phi \le 1$ such that $\phi = 1$ on $B(0,R/2)$ and 0 outside the ball $B(0,R)$, so that $\int_{\mathbb{R}^n} |\frac{\phi(x)}{x}|^n\,dx = +\infty$.

Remark 1.2.2. The constant $C_{n,p} = (\frac{p}{n-p})^p$ is the optimal constant, i.e.

$$C_{n,p} = \sup_{\substack{u \in D^{1,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\frac{u(x)}{x}|^p\,dx}{\int_{\mathbb{R}^n} |\nabla u|^p\,dx}. \tag{1.7}$$

Moreover, for any bounded domain Ω containing 0 with $1 < p < n$, the following inequality

$$\int_{\Omega} \frac{|u|^p}{|x|^p}\,dx \le C_{n,p} \int_{\Omega} |\nabla u|^p\,dx \tag{1.8}$$

holds for any $u \in W_0^{1,p}(\Omega)$, which is the completion of $C_0^\infty(\Omega)$ in the norm $||u||_{1,p,\Omega} := \left(\int_\Omega |u(x)|^p\,dx + \int_\Omega |\nabla u|^p\,dx\right)^{1/p}$ and $C_{n,p}$ is the best constant.

Through these two results, we can say that the embedding of $W_0^{1,p}(\Omega)$ *in* $L^p(\Omega)$, where Ω is a bounded domain containing 0, *and* $1 < p < n$, with respect to the weight $|x|^{-p}$ is continuous. Moreover, the constant $C_{n,p}$ is not achieved, unlike Sobolev inequality. In fact, we can give an explicit minimizing sequence for the best constant. For $\epsilon > 0$, we define $u_\epsilon = |x|^{-\frac{n-p}{p}+\epsilon}$; for any $x \in \mathbb{R}^n, u_\epsilon$ is in $D^{1,p}(\mathbb{R}^n)$ and verifies

$$\left(\frac{n-p}{p}\right)^p = \lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}^n} |\nabla u_\epsilon|^p dx}{\int_{\mathbb{R}^n} \frac{|u_\epsilon(x)|^p}{|x|^p}\,dx}.$$

But the limit function $u(x) := |x|^{-\frac{n-p}{p}}$ does not belong to $D^{1,p}(\mathbb{R}^n)$ and satisfies the equation

$$-\ \mathrm{div}(|\nabla u|^{p-2}\nabla u) = C_{n,p}|u|^{p-1} \quad in\ \mathbb{R}^n$$

in the sense of distribution, i.e.

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} < \nabla u, \nabla\phi > \,dx = C_{n,p} \int_{\mathbb{R}^n} |u|^{p-1}\,\phi\,dx \quad \textit{for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

Remark 1.2.3. Let us consider the nonlinear operator

$$\mathcal{L}_\lambda u \equiv -\Delta_p - \frac{\lambda}{|x|^p}\,|u|^{p-2}u \;\; in \;\; W_0^{1,p}(\Omega).$$

Then

1. If $\lambda \leq C_{n,p}$, $\mathcal{L}_\lambda$ is a positive operator .
2. If $\lambda > C_{n,p}$, $\mathcal{L}_\lambda$ is unbounded from below.

(1) it is obvious from (2.7) inequality. (2) By density argument and optimality of the constant, there exists a function $\phi \in C_0^\infty(\Omega)$ such that $\langle \mathcal{L}_\lambda\, \phi, \phi\rangle < 0$. We can assume that $||\phi||_{L^2} = 1$ and then by defining $u_\mu(x) = \mu^{n/2}\,\phi(\mu x)$ we have $||u_\mu||_{L^2} = 1$ and the homogeneity of the operator allows us to conclude that $\langle\, \mathcal{L}_\lambda\, u_\mu, u_\mu\rangle = \mu^2\,\langle \mathcal{L}_\lambda\, \phi, \phi\rangle\; < 0$.

1.3 Weighted Sobolev inequalities

There are many generalizations of (2.5) and (2.6). We can cite

Theorem 1.3.1. *(Caffarelli-Kohn-Nirenberg) The inequality*

$$(\int_{\mathbb{R}^n} |x|^{-bp}|u|^p dx)^{\frac{2}{p}} \leq C_{a,b} \int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2 dx \tag{1.9}$$

holds for $u \in C_0^\infty(\mathbb{R}^n)$, $\quad n \geq 3, \quad -\infty < a < \frac{n-2}{2}, \quad a \leq b \leq a+1$ *and* $p = \frac{2n}{n-2+2(b-a)}$.

Remark 1.3.1. The classical Sobolev inequality ($a = 0, b = 0$) and the Hardy inequality ($a = 0, b = 1$) are special cases.

Note. In [34], Lieb considered the case $a = 0, 0 < b < 1$. In [18], Chou and Chu considered the case $a \geq 0$ and gave the best constants and explicit minimizers. Moreover Lions in [38] (for $a = 0$) and Wang and Willem in [55] (for $a > 0$), have established the compactness of all minimizing sequences up to dilations. The symmetry of the minimizers has been studied in [34] and [18]. In fact, all the nonnegative solutions in $D_a^{1,2}(\mathbb{R}^n)$ for the corresponding Euler equation are radial solutions and explicitly given (see Aubin, Talenti, Lieb, Chou and Chu). This was established in [18], using a generalization of the moving plane method (e.g., Gidas-Ni-Nirenberg, Caffarelli-Gidas-Spruck, Chen).

Taking in (1.9) $a = 0$ and $-b\,p = \nu$, we get

Theorem 1.3.2. *(Egnell-Maz'ya) The inequality*

$$(\int_{\mathbb{R}^n} |u|^{p+1}|x|^\nu dx)^{\frac{2}{p+1}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 dx \tag{1.10}$$

holds for $u \in C_0^\infty(\mathbb{R}^n)$ *if* $-2 \leq \nu \leq 0$ *and* $p + 1 = \frac{2(n+\nu)}{n-2}$.

1.4 The Riemannian Yamabe problem

Let (M, g) be a compact Riemannian manifold, without boundary, of dimension $m \geq 3$. If $\tilde{g} = \phi^{q-2} g$, with $q = \frac{2m}{m-2}$, is a new metric conformal to g, the scalar curvature $\tilde{K}$ of $\tilde{g}$ is given by

$$\tilde{K} = \phi^{1-q}(a_m \Delta\phi + K\phi), \quad a_m = 4(m-1)/(m-2),$$

in which Δ is the Laplace-Beltrami operator of g and K its scalar curvature (see [1]). Thus the problem of finding a conformal metric with constant scalar curvature $\tilde{K} \equiv \mu$ is equivalent to finding a positive, C^∞ solution ϕ to the Yamabe equation:

$$a_m \Delta\phi + K\phi = \mu\phi^{q-1}. \tag{1.11}$$

This problem has the following variational formulation. We consider the constrained variational problem

$$\mu(M) = \inf\{\int_M (a_m |d\phi|^2 + K\phi^2)\, dV_g \;:\; \int_M |\phi|^q dV_g = 1\}. \tag{1.12}$$

We can compute readily that the Euler-Lagrange equations for (1.12) is the Yamabe equation, provided $\phi \geq 0$. Thus we want to search for the extremals for (1.12). One of the major milestones in the solution of the Yamabe problem was the following theorem, due to H.Yamabe [56], N. Trudinger [52], and T. Aubin [1].

Theorem 1.4.1. *Let (M, g) be a compact Riemannian manifold of dimension $m \geq 3$.*

(a) $\mu(M)$, depends only on the conformal class of g.

(b) $\mu(M) \leq \mu(\mathbb{S}^m)$, in which the sphere S^m has the standard metric.

(c) If $\mu(M) < \mu(\mathbb{S}^m)$, then the infimum in (1.12) is attained by a positive, C^∞ solution to (1.11). Thus the metric $\tilde{g} = \phi^{q-2} g$ has constant scalar curvature $\mu(M)$.

Aubin also proved that $\mu(M) < \mu(\mathbb{S}^m)$ in all cases in which M is not locally conformally flat and $m \geq 6$. After a time, in 1984, R. Schoen [46] has completed the solution of the Yamabe problem by proving that $\mu(M) < \mu(\mathbb{S}^m)$ unless M is the sphere. The proof of part (a) consists of the fundamental observation that problem (1.12) is conformally invariant in the following sense. Under the conformal change of metric $\tilde{g} = t^{q-2} g$, if we let $\tilde{\Delta}$ and $\tilde{K}$ denote the Laplacian and scalar curvature of $\tilde{g}$, then we have

$$(a_m \tilde{\Delta} + \tilde{K})\tilde{\phi} = t^{1-q}(a_m + \Delta + K)\phi, \quad \textit{with } \tilde{\phi} = t^{-1}\phi. \tag{1.13}$$

It follows that the integral in (1.12) is unchanged if we replace g by $\tilde{g}$ and ϕ by $\tilde{\phi}$, thus $\mu(M)$ is conformal invariant. The analysis of (1.12) begins with a thorough understanding of the special case of the sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$. We center $\mathbb{S}^m$ at the origin of $\mathbb{R}^m \times \mathbb{R}$. By stereographic projection

$$\mathbb{R}^m \ni x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right) \in \mathbb{S}^m \subset \mathbb{R}^{m+1},$$

coupled with the transformation law (1.13) converts the variational problem on $\mathbb{S}^m$ to the problem on $\mathbb{R}^m$:

$$\mu(\mathbb{S}^m) = \inf\{a_m \int_{\mathbb{R}^m} |df|^2 \, dx \; : \; \int_{\mathbb{R}^m} |f|^q \, dx = 1\}. \tag{1.14}$$

This is just the problem of finding the best constant and the extremal functions for Sobolev's inequality. Thus, the stereographic projection induces a metric $\tilde{g} = u^{\frac{4}{m-2}} g = \frac{4}{(1+|x|^2)^2} g$ of constant curvature $\mu = m(m-1)$ on $\mathbb{R}^m$, conformal to the Euclidean metric g on $\mathbb{R}^m$, where $u(x) = \frac{[m(m-2)]^{\frac{m-2}{4}}}{[1+|x|^2]^{\frac{m-2}{2}}}$ is an extremal for (2.12). In the last twenty years, several authors have generalized the Yamabe problem to Cauchy-Riemann manifolds, to Heisenberg group and Carnot groups, see Jerison and Lee [31],[32], and Garofalo-Vassilev [25].

1.5 Plateau's Problem

The study of parametric two-dimensional surfaces with prescribed mean curvature, satisfying different kinds of geometrical or topological side conditions, has constituted a very challenging problem and has played a prominent role in the history of the Calculus of Variations. Minimal surfaces are defined as surfaces with zero mean curvature. Surfaces with prescribed constant mean curvature are usually known as "soap films" ($H = 0$) or "soap bubbles" ($H = const.$). Minimal surfaces may also be characterized as surfaces of minimal surface area for given boundary conditions. Finding a minimal surface spanning a given constraint is known at Plateau's problem. In general, there may be one, multiple, or no minimal surfaces spanning a given closed curve in space. The existence of a solution to the general case was independently proved by Douglas ([20]) and Radó ([44]) in the 1930's, although their analysis could not exclude the possibility of singularities. In the 1970's Osserman ([43]) and Gulliver ([29]) showed that a minimizing solution cannot have singularities. For the case of nonconstant prescribed mean curvature, only few existence results of variational type are known, see for example [47], [48]. In the last ten years many variational type results hold true in a perturbative setting, namely, for curvature of the forms $H(u) = H_0 + H_1(u)$ where $H_0 \in \mathbb{R} \setminus \{0\}$ and $H_1 \in C^1(\mathbb{R}^3) \cap L^\infty$, having $||H_1||_\infty$ small. In particular, we can mention Struwe [49], Wang [54], Bethuel and Rey [8], Caldiroli and Musina [15]. In analogy with soap bubbles, we can give the following

Definition 1.5.1. A H-bubble is a regular surface $\mathcal{M} \hookrightarrow \mathbb{R}^3$ parametrized by a conformal map $u : \mathbb{S}^2 \to \mathbb{R}^3$, having mean curvature $H(u)$ at every regular point $u \in \mathcal{M}$.

Up to the composition with the stereographic projection $\mathbb{S}^2 \to \mathbb{R}^2$, every conformal parametrization u of $\mathcal{M}$ is a solution to

$$\begin{cases} \Delta u = 2H(u)\, u_x \wedge u_y \ \ in\ \ \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla u|^2 < +\infty. \end{cases} \tag{1.15}$$

Here $u_x = (\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x})$, $u_y = (\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, \frac{\partial u_3}{\partial y})$, $\Delta u = u_{xx} + u_{yy}, \nabla u = (u_x, u_y)$, and $\wedge$ denotes the exterior product in $\mathbb{R}^3$. The invariance of problem (1.15) with respect to the action of the conformal group of $\mathbb{S}^2 \equiv \mathbb{R}^2 \cup \{\infty\}$ means that the true unknown in this problem is a parametric surface, rather than its parametrization. We note that the problem (1.15) has a natural variational structure, since solutions to (1.15) are the critical points of the functional

$$\mathcal{E}_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \mathcal{V}_H(u),$$

where $\mathcal{V}_H(u) = 2\int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$ and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $divQ = H$. When H is bounded, E_H turns out to be well defined (by continuous extension) and sufficiently regular on some Sobolev space. Roughly, the integral $2\int_{\mathbb{R}^2} Q(u) \cdot u_x \wedge u_y$ has the meaning of the algebraic H-weighted volume of the region enclosed by range u and it is essentially cubic in u. Therefore, the energy functional $\mathcal{E}_H$, which is unbounded from below and from above, actually admits a saddle type geometry. In case of nonzero constant mean curvature $H(u) = H_0$, Brezis and Coron [9] proved that the only nonconstant solutions to (1.15) are spheres of radius $|H_0|^{-1}$ anywhere placed in $\mathbb{R}^3$. In order to make precise the geometrical structure of $\mathcal{E}_H$, we consider the restriction of $\mathcal{E}_H$ to the space of smooth functions $C^1_0(\mathbb{R}^2, \mathbb{R}^3)$ and we introduce the value

$$c_H = \inf_{\substack{u \in C^1_0(\mathbb{R}^2,\mathbb{R}^3) \\ u \neq 0}} \sup_{s>0} \mathcal{E}_H(su)$$

which represents the mountain pass level along radial paths. Caldiroli and Musina, in [15], proved the following existence result.

Theorem 1.5.1. *Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a function $C^1(\mathbb{R}^3)$, satisfying:*

(h_1) $H(u) \to H_\infty$, *for some* $H_\infty \in \mathbb{R}$,

(h_2) $\sup_{u\in\mathbb{R}^3} |\nabla H(u + \xi) \cdot u\, u| < 1$, *for some* $\xi \in \mathbb{R}^3$,

(h_3) $c_H < \frac{4\pi}{3\,H^2_\infty}$.

Then there exists an H-bubble ω such that $\mathcal{E}_H(\omega) = c_H$. Moreover, called $\mathcal{B}_H$ the set of H-bubbles, it holds that $c_H = \inf_{\omega\in\mathcal{B}_H} \mathcal{E}_H(\omega)$.

1.6 Historical references: works by Talenti, Lieb, Lions

Theorem 1.6.1. *(Talenti) Let $n \geq 3$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then, for $1 < p < n$, the following inequality holds*

$$||f||_{L^q(\mathbb{R}^n)} \leq C_{p,n} \, ||\nabla f||_{L^p(\mathbb{R}^n)} \tag{1.16}$$

where $q = \frac{np}{n-p}$ and $C_{p,n} = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{(p-1)/p} \left[\frac{\Gamma(n)\Gamma((n+2)/2)}{\Gamma(n/p)\Gamma((p+pn-n)/p)}\right]^{1/n}$.
The equality sign holds in equation (1.16) iff f is a multiple of the function $(a + b|x|^{p/(p-1)})^{(p-n)/p}$ with $a, b > 0$ and $x \in \mathbb{R}^n$.

Definition 1.6.1. We will denote by
$S = \inf\{\int_{\mathbb{R}^n} |\nabla u|^p dx \,|\, u \in D^{1,p}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^q dx = 1\}$.

We recall some useful notions for one-dimensional variational problems.
Let Ω be an open, simply connected region in $\mathbb{R}^{n+1}$, $(t, x_1, \cdots, x_n) = (t, x)$ a point of Ω and $F = F(t, x, p) \in C^2(\Omega \times \mathbb{R}^n)$ a Lagrangian and let (t_1, a) and (t_2, b) two points in Ω. The space

$$\Gamma := \{\gamma : t \to x(t) \in \Omega \,:\, x \in C^1[t_1, t_2],\, x(t_1) = a, x(t_2) = b\}$$

consists of all continuously differentiable curves which start at (t_1, a) and end at (t_2, b). On Γ is defined the functional

$$I(\gamma) = \int_{t_1}^{t_2} F(t, x(t), \dot{x}(t))\, dt.$$

Definition 1.6.2. We say that $\gamma^* \in \Gamma$ is minimal in Γ if

$$I(\gamma) \geq I(\gamma^*), \quad \forall \gamma \in \Gamma.$$

Proposition 1.6.2. *If γ^* is is a regular minimal in Γ,, that is det$(F_{p_i p_j}) \neq 0$ for $x = x^*, p = \dot{x}^*$, then $x^* \in C^2[t_1, t_2]$ and we have for $j = 1, \cdots, n$*

$$\frac{d}{dt} F_{p_j}(t, x^*, \dot{x}^*) = F_{x_j}(t, x^*, \dot{x}^*) \tag{1.17}$$

These equations are called Euler equations.

Definition 1.6.3. An element $\gamma^* \in \Gamma$ satisfying the Euler equations (1.17) is called an extremal in Γ.

Definition 1.6.4. An extremal field in Ω is a vector field $\dot{x} = \psi(t, x), \psi \in C^1(\Omega)$ which is defined in a wide neighborhood U of an extremal solution and which has the property that every solution $x(t)$ of the differential equation $\dot{x} = \psi(t, x)$ is also a solution of the Euler equations.

Definition 1.6.5. A vector field $\dot{x} = \psi(t,x)$ is called a Mayer field if there is a function $g(t,x)$ which satisfies the fundamental equations

$$g_t = F(t,x,\psi) - \sum_{j=1}^{n} \psi_j F_{p_j}(t,x,\psi),$$
$$g_x = F_{p_j}(t,x,\psi). \tag{1.18}$$

Remark 1.6.1. A vector field is a Mayer field if and only if it is an extremal field which satisfies the compatibility condition

$$D_\psi F_{p_j}(t,x,\psi) = F_{x_j}(t,x,\psi) \tag{1.19}$$

where $D_\psi := \partial_t + \psi\partial_x + (\psi_t + \psi\psi_x)\partial_p$.

Definition 1.6.6. A Hilbert invariant integral is a functional defined for every curve $\gamma : t \mapsto x(t)$ as

$$I(\gamma) = \int_\gamma F dt = \int_\gamma F\, dt - F_p\, \dot{x}\, dt + F_p\, dx.$$

Remark 1.6.2. Especially for the path γ^* of the extremal field $\dot{x} = \psi(t,x)$, we have

$$I(\gamma^*) = \int_{\gamma^*} (F - F_p\psi)\, dt + F_p\, dx.$$

Then, the difference with another curve γ is

$$\begin{aligned} I(\gamma) - I(\gamma^*) &= \int_\gamma F(t,x,\dot{x}) - F(t,x,\psi) - (\dot{x}-\psi)F_p(t,x,\psi)dt \\ &= \int_\gamma E(t,x,\dot{x},\psi)dt \end{aligned}$$

where $E(t,x,\dot{x},p,q) = F(t,x,p) - F(t,x,q) - (p-q)F_p(t,x,q)$ is called the Weierstrass excess function.

Note. The proof of Talenti's paper consists of two steps. In the first step the author proves that if the quotient $J(u) = \frac{||u||_{L^q(\mathbb{R}^n)}}{||\nabla u||_{L^p(\mathbb{R}^n)}}$ attains its supremum value, then, by rearrangement technique, he can take spherically symmetric extremals. Moreover, he shows that the critical radial points of $J(u)$ are functions of the form $\phi(r) = (a + b\, r^{\frac{p}{p-1}})^{1-n/p}$ with a, b positive constants. In the second part, he shows that the found extremals actually give the maximum, using techniques of one dimensional calculus of variations. The original idea is to consider a Lagrange constrained problem, i.e.

$$(*) \begin{cases} \int_0^{+\infty} r^{n-1}|u_1(r)|^q dr = \max \\ where \\ u_2'(r) = r^{n-1}|u_1'(r)|^p \\ and \\ u_2(0) = 0, u_1(+\infty) = 0, u_2(+\infty) = 1 \end{cases}$$

and to find the solution of this. It is easy to see that if u satisfies $J(u) = \max$, getting

$$\begin{cases} u_1(r) = u(r)(\int_0^{+\infty} t^{n-1}|u_1'(t)|^p dt)^{-1/p} \\ u_2(r) = \int_0^r t^{n-1}|u_1'(t)|^p dt \end{cases}$$

the couple (u_1, u_2) is a solution of $(*)$. Conversely, if (u_1, u_2) is a solution of $(*)$, then $u = u_1$ is such that $J(u) = \max$. He proves that the two-parameters family of extremals

$$(**)\begin{cases} \phi_1(r) = a(1 + br)^{1-n/p} \\ \phi_2(r) = \int_0^r t^{n-1}|\phi_1'(t)|^p dt \end{cases}$$

is a Mayer field (see Definition 1.6.5) in the first octant of the three-dimensional space. Then, by Remark 1.6.1, there exists an exact differential dW such that, along any path $]0, \infty[\ni r \to (r, u_1(r), u_2(r))$ which satisfies the constraint $u_2'(r) = r^{n-1}|u_1'(r)|^p$, the integral $\int dW \geq \int_0^\infty r^{n-1}|u_1(r)|^q dr$ and equality holds when the path is an extremal belonging to the Mayer field $(**)$.

As seen in the Remark 2.0.2., Lieb in [34], proved the following theorem.

Theorem 1.6.3. *Assume the hypotheses as in theorem 2.0.1. If* $p = t = \frac{2n}{(n-\lambda)}$, *then*

$$N_{p,\lambda,n} = N_{\lambda,n} = \pi^{\lambda/2}\frac{\Gamma(n/2 - \lambda/2)}{\Gamma(n - \lambda/2)}\{\frac{\Gamma(n/2)}{\Gamma(n)}\}^{-1+\lambda/n}. \qquad (1.20)$$

In this case there is equality in (1.3) if and only if $h \equiv$ (*const.*) f *and* $f(x) = (\gamma^2 + (x - a)^2)^{-(2n-\lambda)/2}, 0 \neq \gamma \in \mathbb{R}$ *and* $a \in \mathbb{R}^n$.

The technique used by Lieb to prove this theorem is called "Competing Symmetries".

Note. The name "Competing Symmetries" alludes to the fact that the symmetrization due to the rearrangement and the conformal symmetry strive together to produce the limiting function L_f.

Lieb studied all the symmetries of (1.3) inequality. Some of them are obvious. Certainly (1.3) is invariant by translations, by the orthogonal group of rotations and reflections of $\mathbb{R}^n$. Another important symmetry is the scaling symmetry. If we replace $f(x), h(x)$ by $\lambda^{n/p} f(\lambda x), \lambda^{q/n} h(\lambda x)$ for $\lambda > 0$, then the integral (1.3) is again invariant. It remains to check that the (1.3) inequality in invariant under the conformal group, that is the group of deformations that preserve angles. We can consider the inversion of the unit sphere, $\mathcal{I} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$, such that $x \mapsto \frac{x}{|x|^2}$.

By stereographic projection, there exists a map $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n$ such that $x \mapsto s = \left(\frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)$ and the inverse, $\mathcal{S}^{-1}$ is given by $x_i =$

$\frac{s_i}{1+s_{n+1}}$ for $i = 1, \cdots, n$. By calculation, writing $s = \mathcal{S}(x)$ and $t = \mathcal{S}(y)$, we have that

$$\sum_{i=1}^{n+1}(s_i - t_i)^2 = |s-t|^2 = \frac{4}{(1+|x|^2)(1+|y|^2)}\,|x-y|^2 \tag{1.21}$$

that is $\mathcal{S}$ is conformal. Moreover, the Euclidean group, together with scaling and inversion generates all conformal transformations. We recall that the conformal group on $\mathbb{R}^n$, which we denote by $\mathcal{C}$, is isomorphic to the Lorentz group, $O(n+1,1)$ which has dimension $\frac{(n+2)(n+1)}{2}$. We can define the action of $\mathcal{S}$ on a function $f \in L^p(\mathbb{R}^n)$ as

$$\mathcal{S}^* f(x) = F(s) = |\mathcal{J}_{\mathcal{S}^{-1}}(s)|^{1/p} f(\mathcal{S}^{-1}(s)) \tag{1.22}$$

and we have that $||F||_{L^p(\mathbb{S}^n)} = ||f||_{L^p(\mathbb{R}^n)}$. Computing the Jacobian of the stereographic projection $\mathcal{J}_{\mathcal{S}}(x)$ and its inverse, we have

$$\mathcal{J}_{\mathcal{S}}(x) = (\frac{2}{1+|x|^2})^n \textit{ and } \mathcal{J}_{\mathcal{S}^{-1}}(x) = (1+s_{n+1})^n. \tag{1.23}$$

With these remarks, we can state the following.

Theorem 1.6.4. *(Conformal invariance of the Hardy-Littlewood-Sobolev inequality) Assume that $p = r$ in (1.3) and that $F \in L^p(\mathbb{S}^n)$ and $f \in L^p(\mathbb{R}^n)$ are related by (1.22). Let H and h another pair related in the same way. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}\, h(y)\, dx\, dy = \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} F(s)|s-t|^{-\lambda}\, H(t)\, ds\, dt \tag{1.24}$$

and $||F||_p = ||f||_p$. Here $|s-t|^2 = \sum_{i=1}^{n+1}(s_i - t_i)^2$ is the euclidean distance of $\mathbb{R}^{n+1}$. Manifestly, this shows the invariance under all isometries of $\mathbb{S}^n$, i.e. invariance under the group $O(n+1)$.

Proof. We can write

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}\, h(y)\, dx\, dy \\ &= \textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\frac{1+|x|^2}{2})^{n/p} f(x)\left(\frac{2}{1+|x|^2}|x-y|^2\frac{2}{1+|y|^2}\right)^{-\lambda/2} \\ &\quad \times (\frac{1+|y|^2}{2})^{n/p}\, h(y) (\frac{2}{1+|x|^2})^n\, dx\, (\frac{2}{1+|y|^2})^n dy \end{aligned} \tag{1.25}$$

using that $2/p + \lambda/n = 2$. Then, (1.24) can be rewritten as

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} F(s)|s-t|^{-\lambda}\, H(t)\, ds\, dt. \tag{1.26}$$

As we have seen in the remark, the HLS inequality is invariant under all isometries of $\mathbb{S}^n$, the translations and the scaling, that generates all conformal group $\mathcal{C}$. □

Consider now the rotation by 90°, that is $D : \mathbb{S}^n \to \mathbb{S}^n$,
$Ds = (s_1, \cdots, s_{n+1}, -s_n)$ which maps the north pole $n = (0, \cdots, 1)$ in the vector $e = (0, \cdots, 1, 0)$. The function $F(D^{-1}s)$ is now rotationally symmetric about the e-axis and about n-axis. Thus, oh one hand, $F(s) = \phi(s_{n+1})$ for some $\phi : \mathbb{S}^n \to \mathbb{R}$ and, on the other hand, $F(D^{-1}s) = \psi(s_{n+1})$ for some $\psi : \mathbb{S}^n \to \mathbb{R}$. Then, $\phi(s_{n+1}) = F(s) = \psi((DS)_{n+1}) = \psi(-s_n)$ for all $s \in \mathbb{S}^n$, which is only possible if F is a constant on $\mathbb{S}^n$ and hence

$$f(x) = C\,(1 + |x|^2)^{-n/p}.$$

It is easy to see that the function on $\mathbb{R}^n$ corresponding to $F(D^{-1}s)$ is given by

$$(D^* f)(x) = |x + a|^{-2n/p}\, f(\frac{2\,x_1}{|x + a|^2}, \cdots, \frac{2\,x_{n-1}}{|x + a|^2}, \frac{1 - |x|^2}{|x + a|^2}) \tag{1.27}$$

where $a = (0, \cdots, 1) \in \mathbb{R}^n$. If F is the function on $\mathbb{S}^n$ corresponding to f via D, we set

$$(DF)(s) = F(D^{-1}s) \tag{1.28}$$

and we denote the symmetric-decreasing rearrangement of f by $(\mathcal{R}f)(x) = f^*(x)$. Recall that $\mathcal{R}$ is norm-preserving, i.e., $||\mathcal{R}f||_p = ||f||_p$ and that if we apply a general conformal transformation (as D^*), to a radial function, (as $\mathcal{R}f$), the result will generally no longer be radial. So, we shall consider the map

$$\mathcal{R}D^* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

The following theorem, proved by Carlen and Loss in [16], utilizing the map $\mathcal{R}D^*$ repeatedly, produce a specific optimizing sequence which strongly converges.

Theorem 1.6.5. *Let $1 < p < \infty$ and let $f \in L^p(\mathbb{R}^n)$ be any nonnegative function. Then the sequence $f^m = (\mathcal{R}D^*)^m f$ converges strongly in $L^p(\mathbb{R}^n)$, as $m \to \infty$, to the function $L_f := ||f||_p\, l(x)$ where*

$$l(x) = (\omega_n)^{-1/p} \left(\frac{2}{1 + |x|^2} \right)^{n/p}. \tag{1.29}$$

Proof of the Theorem 1.6.3

We want to find the sharp constant in HLS inequality when $p = r = \frac{2n}{n-\lambda}$, $0 \le \lambda < n$. We can restrict our attention to the case $h = f$ and $f \ge 0$. We consider

$$N_{\lambda,n} = \sup\{\mathcal{H}(f)\ :\ f \in L^p(\mathbb{R}^n), f \ge 0, f \ne 0\} \tag{1.30}$$

where

$$\mathcal{H}(f) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x - y|^{-\lambda} f(y) dx\, dy \Big/ ||f||_p^2 \tag{1.31}$$

The Theorem 1.6.3 can be shown as a corollary of Theorem 1.6.5. Replace f by $f^j(x) = min(f(x), j\, L_f(x))$ so that f^j converges monotically to $f(x)$ pointwise as $j \to \infty$. If we can show that $\mathcal{H}(f^j) \to \mathcal{H}(f)$ then, by monotone convergence, $\mathcal{H}(f^j) \to \mathcal{H}(f)$ and thus $\mathcal{H}(f) \leq N_{n,\lambda}$. Since $\mathcal{H}(D^* f) = \mathcal{H}(f)$ and $\mathcal{H}(\mathcal{R}f) \geq \mathcal{H}(f)$ by Riesz's rearrangement inequality, we have that $\mathcal{H}(f^m)$ is a nondecreasing sequence where $f^m = (\mathcal{R}D^*)^m f$. Since, by the previous theorem, $f^m \to L_f$ in $L^p(\mathbb{R}^n)$ as $m \to \infty$, we can pass to a subsequence (again denoted by m) and assume that $f^m \to L_f$ pointwise. Since

$$f^m \leq C\,(1+|x|^2)^{-n/p}\, for\, all\, m,$$

we know, by dominated convergence, that as $m \to \infty, \mathcal{H}(f^m) \to \mathcal{H}(f)$ from below. By calculations, we get (1.20). It remains to determine the case of equality. Let f a nonnegative function. The equality in (1.3) can occur only if $l = (const.)f$ and if $\mathcal{H}(f) = N_{\lambda,n}$. Then, by the strict rearrangement inequality, we know that f must be a translate of a symmetric-decreasing function. Moreover, the same is true for $D^* f$ since it is also an optimizer by the conformal invariance of $\mathcal{H}(f)$. Thus, the operation $\mathcal{R}D^*$ acting on f does nothing but translate $D^* f$ to the origin, and hence $\mathcal{R}D^* f$ is nothing but a conformal translation of f. The same is true for the whole sequence $f^m = (\mathcal{R}D^*)^m f$ is a conformal image of f and we can write $f^m = C_m f$, where C_m is a sequence of conformal transformations. Since $f^m \xrightarrow{s.} L_f$, and since the conformal transformations act as isometries on $L^p(\mathbb{R}^n)$, we have that

$$\lim_{m\to\infty} ||f - C_m^{-1} L_f||_p = 0. \tag{1.32}$$

Moreover, the following lemma holds.

Lemma 1.6.6. *Let $C \in \mathcal{C}$ be a conformal transformation and let l be given by (1.29). If C acts on l, there exists $\lambda \neq 0$ and $a \in \mathbb{R}^n$ (depending on C) such that*

$$(Cl)(x) = (\omega_n)^{-1/p}\, \lambda^{n/p} \left(\frac{2}{\lambda^2 + (x-a)^2} \right)^{n/p}$$

Proof. We know that every element in $\mathcal{C}$ is a product of elements of the Euclidean group, scaling and inversions. It is easy to see it for scaling and for Euclidean transformations. It remains to check that the inversion $\mathcal{I}$ maps the function $u(x) = (\omega_n)^{-1/p} \mu^{n/p} \big(\frac{2}{\mu^2+(x-b)^2}\big)^{n/p}$ into a function of the same type. In fact,

$$\mathcal{I}u(x) = (\omega_n)^{-1/p} \mu^{n/p} |x|^{-2n/p} \left(\frac{2}{\mu^2 + (x/|x|^2 - b)^2} \right)^{n/p}$$

$$= (\omega_n)^{-1/p} \left(\frac{\mu}{b^2+\mu^2} \right)^{n/p} \left(\frac{2}{[\mu/(b^2+\mu^2)]^2 + [x - b/(b^2+\mu^2)]^2} \right)^{n/p}.$$

□

By Lemma 1.6.6, we have

$$(C_m^{-1}L_f)(x) = \lambda_m^{n/p}(\omega_n)^{-1/p}\,||f||_p\left(\frac{2}{\lambda_m^2+(x-a_m)^2}\right)^{n/p} \tag{1.33}$$

for sequences $\lambda_m \neq 0$ and $a_m \in \mathbb{R}^n$. Since, by (1.32), $C_m^{-1}\,L_f \xrightarrow{s.} f$, it is plain that $\lambda_m \to \lambda \neq 0$ and $a_m \to a \in \mathbb{R}^n$. Hence

$$f(x) = \lambda^{n/p}\,(\omega_n)^{-1/p}\,||f||_p\left(\frac{2}{\lambda^2+(x-a)^2}\right)^{n/p}$$

and the Theorem 1.6.3 is proved. □

In 1985 P.L.Lions, in [38]-[39], developed his concentration-compactness principle to analyze necessary and sufficient condition for the convergence of minimizing sequence satisfying the given constraint. The (CC) principle is based on these lemmas.

Lemma 1.6.7. *Let $\{\mu_h\}_h$ a sequence of probability measures on $\mathbb{R}^n$ such that $\mu_h \geq 0, \int_{\mathbb{R}^n} d\mu_h = 1$. Along a subsequence, still denoted by $\{\mu_h\}_h$, one of the following three alternatives holds:*

i) (Compactness) There exists a sequence $\{\xi_h\}_h \subset \mathbb{R}^n$ such that, for all $\epsilon > 0$ there exists $R > 0$ with the property that

$$\int_{B_R(\xi_h)} d\mu_h \geq 1-\epsilon \quad \forall h.$$

ii) (Vanishing) For all $R > 0$ $\lim_{h\to\infty}\left(\sup_{\xi_h\in\mathbb{R}^n}\int_{B_R(\xi_h)} d\mu_h\right) = 0.$

iii) (Dichotomy) There exists a number $\lambda \in]0,1[$ such that for all $\epsilon > 0$ there exist $R > 0$, and a sequence $\{\xi_h\}_h \subset \mathbb{R}^n$ with the following property: Given $R' > R$ there are non-negative measures μ_h^1, μ_h^2 such that

$$0 \leq \mu_h^1 + \mu_h^2 \leq \mu_h$$
$$supp\,\{\mu_h^1\} \subset B_R(\xi_h),\ supp\,\{\mu_h^2\} \subset \mathbb{R}^n \setminus B_{R'}(\xi_h),$$
$$\limsup_{h\to\infty}\left(|\lambda - \int_{\mathbb{R}^n} d\mu_h^1| + |(1-\lambda) - \int_{\mathbb{R}^n} d\mu_h^2|\right) \leq \epsilon.$$

Note. For any positive measure $\mu \in L^1(\mathbb{R}^n)$, we define concentration function introduced by P. Lévi as

$$Q(r) = \sup_{x\in\mathbb{R}^n}(\int_{B_r(x)} d\mu)$$

Let Q_h be the concentration functions associated with μ_h. Then, $\{Q_h\}$ is a sequence of non-decreasing, positive bounded functions on $[0,\infty[$ with $\lim_{R\to\infty} Q_h(R) = 1$. Hence, $\{Q_h\}$ is locally bounded in BV *on* $[0,\infty[$ and there exist a subsequence $\{\mu_h\}$ and a bounded, positive, non-decreasing function Q such that $Q_h(R) \to Q(R)$ *for* $h \to \infty$, for almost every $R > 0$. Normalizing Q to be continuous from the left, we have:

$$Q(R) \leq \liminf_{h\to\infty} Q_h(R).$$

Getting

$$\lambda = \lim_{R\to\infty} Q(R)$$

we have $0 \leq \lambda \leq 1$. If $\lambda = 0$, we have "vanishing" case, while if $\lambda = 1$, we can achieve "compactness" case and if $0 < \lambda < 1$, we get "dichotomy " case.

The second is

Lemma 1.6.8. *Let* $1 \leq p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. *Let* μ, ν *bounded non-negative measures on* $\mathbb{R}^n$.

i) $u_m \rightharpoonup u$ *weakly in* $D^{1,p}(\mathbb{R}^n)$.

ii) $\nu_m = |u_h|^q dx \rightharpoonup \nu$, *weakly in the sense of measures in* $\mathbb{R}^n$.

iii) $\mu_m = |\nabla u_m|^p dx \rightharpoonup \mu$, *weakly in the sense of measures in* $\mathbb{R}^n$.

Then,there exist an at most countable set of indices J, *a corresponding set of distinct points* $\{x_j \in \mathbb{R}^n \mid j \in J\}$, *and two sets* $\{\mu_j\}_j, \{\nu_j\}_j$ *of non negative numbers such that*

$$\nu = |u|^q dx + \sum_{j\in J} \nu_j \delta_j, \quad \mu \geq |\nabla u|^p dx + \sum_{j\in J} \mu_j \delta_j \ \text{ and } \ \mu_j \geq S\, \nu_j^{p/q}\, \forall j \in J,$$

where $\delta_j = \delta_{x_j}$ *is the Dirac measure with pole at* $\{x_j\} \in \mathbb{R}^n$.
In particular, $\sum_{j\in J} (\nu_j)^{p/q} < \infty$.

Then, the following result holds

Theorem 1.6.9. *Let* $1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. *Suppose* $\{u_m\}$ *is a minimizing sequence for* S *in* $D^{1,p}(\mathbb{R}^n)$ *with* $||u_m||_{L^q} = 1$. *Then* $\{u_m\}$ *up to translations and dilatations is relatively compact in* $D^{1,p}(\mathbb{R}^n)$.

Proof. Choose $\widetilde{x}_m \in \mathbb{R}^n$, $\widetilde{R}_m > 0$ such that for the rescaled sequence

$$v_m(x) = \widetilde{R}_m^{-n/q} u_m\left(\frac{x - \widetilde{x}_m}{\widetilde{R}_m}\right)$$

there holds

$$Q_m(1) = \sup_{x\in\mathbb{R}^n} \int_{B_1(x)} |v_m|^q dx = \int_{B_1(0)} |v_m|^q dx = \frac{1}{2}. \tag{1.34}$$

Since $p > 1$, we can assume that $v_m \rightharpoonup v$ weakly in $L^q(\mathbb{R}^n)$ and weakly in $D^{1,p}(\mathbb{R}^n)$. Consider the families of measures

$$\mu_m = |\nabla v_m|^p dx$$
$$\nu_m = |v_m|^q dx$$

and apply Lemma (1.6.7) to the sequence $\{\nu_m\}$. Vanishing cannot occur. If we have dichotomy, let $\lambda \in]0,1[$ be as in Lemma, and for $\epsilon > 0$ determine $R > 0$, a sequence $\{x_m\}$ and measures ν_m^1, ν_m^2 such that

$$0 \leq \nu_m^1 + \nu_m^2 \leq \nu_m$$
$$supp\,(\nu_m^1) \subset B_R(x_m),\ supp\,(\nu_m^2) \subset \mathbb{R}^n \setminus B_{2R}(x_m),$$
$$\limsup_{m\to\infty}\left(|\lambda - \int_{\mathbb{R}^n} d\nu_m^1| + |(1-\lambda) - \int_{\mathbb{R}^n} d\nu_m^2|\right) \leq \epsilon.$$

Choosing a sequence $\epsilon_m \to 0$, corresponding $R_m > 0$ and passing to a subsequence (ν_m) if necessary, we can achieve that

$$supp\,(\nu_m^1) \subset B_R(x_m),\ supp\,(\nu_m^2) \subset \mathbb{R}^n \setminus B_{2R}(x_m),$$

and

$$\limsup_{m\to\infty}\left(|\lambda - \int_{\mathbb{R}^n} d\nu_m^1| + |(1-\lambda) - \int_{\mathbb{R}^n} d\nu_m^2|\right) = 0.$$

Moreover, we can suppose $R_m \to \infty$. Choose $\phi \in C_0^\infty(B_2(0))$ such that $\phi \equiv 1$ *in* $B_1(0)$ and let $\phi_m(x) = \phi(\frac{x-x_m}{R_m})$. Decompose $v_m = v_m\phi_m + v_m(1-\phi_m)$. Then $\int_{\mathbb{R}^n} |\nabla v_m|^p dx = \int_{\mathbb{R}^n} |\nabla(v_m\phi_m)|^p dx + \int_{\mathbb{R}^n} |\nabla(v_m(1-\phi_m))|^p dx + \delta_m$ where the error terms δ_m can be estimated from below

$$\delta_m \geq -C \int_{B_{2R_m}(x_m)\setminus B_{R_m}(x_m)} |v_m|^p |\nabla\phi_m|^p dx.$$

using the fact that $0 \leq \phi \leq 1$ *and* $p > 1$. Let A_m denote the annulus $A_m = B_{2R_m}(x_m) \setminus B_{R_m}(x_m)$. Estimating $|\nabla\phi_m| \leq CR_m^{-1}$, we can bound

$$||\,|v_m|\,|\nabla\phi_m|\,||_{L^p(A_m)} \leq CR_m^{-1}||v_m||_{L^p(A_m)}. \tag{1.35}$$

Moreover, by Hölder's inequality

$$\begin{aligned} R_m^{-1}||v_m||_{L^p(A_m)} &\leq R_m^{-1}|A_m|^{\frac{1}{p}-\frac{1}{q}}||v_m||_{L^q(A_m)} \\ &= C||v_m||_{L^q(A_m)} \\ &\leq C\left[\int_{\mathbb{R}^n} d\nu_m - \left(\int_{\mathbb{R}^n} d\nu_m^1 + \int_{\mathbb{R}^n} d\nu_m^2\right)\right]^{\frac{1}{q}} \end{aligned}$$

Hence this term tends to 0 as $m \to \infty$, while $||\nabla v_m||^p_{L^p(A_m)} \le ||v_m||^p_{D^{1,p}(\mathbb{R}^n)}$ remains uniformly bounded and for (1.35), we obtain that $\delta_m \ge o(1)$, where $o(1) \to 0 \;\; as\, m \to \infty$. Now by Sobolev's inequality

$$\begin{aligned} ||v_m||^p_{D^{1,p}(\mathbb{R}^n)} &= ||v_m\phi_m|^p_{D^{1,p}(\mathbb{R}^n)} + ||v_m(1-\phi_m)||_{D^{1,p}(\mathbb{R}^n)} + \delta_m \\ &\ge S\,(||v_m\phi_m||^p_{L^q(\mathbb{R}^n)} + ||v_m(1-\phi_m)||^p_{L^q(\mathbb{R}^n)}) + \delta_m \\ &\ge S\Bigg[\Bigg(\int_{B_{R_m}(x_m)} d\nu_m\Bigg)^{p/q} + \Bigg(\int_{\mathbb{R}^n\setminus B_{2R_m}(x_m)} d\nu_m\Bigg)^{p/q}\Bigg] + \delta_m \\ &\ge S\Bigg[\Bigg(\int_{\mathbb{R}^n} d\nu^1_m\Bigg)^{p/q} + \Bigg(\int_{\mathbb{R}^n} d\nu^2_m\Bigg)^{p/q}\Bigg] + \delta_m \\ &\ge S\left(\lambda^{p/q} + (1-\lambda)^{p/q}\right) + o(1) \end{aligned}$$

But for $0 < \lambda < 1 \;\; and\, p < q$ we have $\lambda^{p/q} + (1-\lambda)^{p/q} > 1$, contradicting the initial assumption that $||v_m||^p_{D^{1,p}(\mathbb{R}^n)} = ||u_m||^p_{D^{1,p}(\mathbb{R}^n)} \to S$. It remains the case $\lambda = 1$. Let x_m be as in the previous lemma and for $\epsilon > 0$ choose $R = R(\epsilon)$ such that $\int_{B_R(x_m)} d\nu_m \ge 1-\epsilon$. If $\epsilon < \frac{1}{2}$, our condition (1.34) implies $B_r(x_m) \cap B_1(0) \neq \emptyset$. Hence the conclusion of Lemma (1.6.7) also holds with $x_m = 0$, replacing $R(\epsilon)$ by $2R(\epsilon)+1$ if necessary. Thus, if $\nu_m \stackrel{w}{\rightharpoonup} \nu$, it follows that $\int_{\mathbb{R}^n} d\nu = 1$. By Lemma (1.6.8), we can assume that

$$\mu_m \rightharpoonup \mu \ge |\nabla v|^p dx + \sum_{j\in J} \mu^j \delta_{x_j}$$
$$\nu_m \stackrel{w}{\rightharpoonup} \nu \ge |v|^q dx + \sum_{j\in J} \nu^j \delta_{x_j}$$

for certain points $x_j \in \mathbb{R}^n, j \in J$ and positive numbers μ^j, ν^j satisfying $S\,(\nu^j)^{p/q} \le \nu^j, \quad$ for all $j \in J$. By Sobolev's inequality then

$$\begin{aligned} S + o(1) = ||v_m||^p_{D^{1,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} d\mu_m \ge ||v||^p_{D^{1,p}(\mathbb{R}^n)} + \sum_{j\in J} \mu^j + o(1) \\ &\ge S\left(||v||^{p/q}_{L^q(\mathbb{R}^n)} + (\sum_{j\in J} \nu^j)^{p/q}\right) + o(1) \end{aligned}$$

where $o(1) \to 0$ for $m \to \infty$.By strict concavity of the map $\lambda \to \lambda^{p/q}$ now the latter will be

$$\begin{aligned} &\ge S\left(||v||^q_{L^q(\mathbb{R}^n)} + (\sum_{j\in J} \nu^j)\right)^{p/q} + o(1) \\ &= S\left(\int_{\mathbb{R}^n} d\nu\right)^{p/q} + o(1) = S + o(1) \end{aligned} \tag{1.36}$$

and equality holds if and only if at most one of the terms $||v||_{L^q}, \nu^j, j \in J$, is different from 0. We note that our normalization (1.34) assures that $\nu^j \leq \frac{1}{2}$ $for \ all\, j \in J$. Hence all ν^j must vanish, $||v||_{L^q} = 1$, $and\, v_m \to v$ strongly in $L^q(\mathbb{R}^n)$. But by Sobolev's inequality $||v||^p_{D^{1,p}(\mathbb{R}^n)} \geq S$ $and\, ||v_m||_{D^{1,p}(\mathbb{R}^n)} \to ||v||_{D^{1,p}(\mathbb{R}^n)}$ $as\, m \to \infty$. It follows that $v_m \to v$ $in\, D^{1,p}(\mathbb{R}^n)$, as desired. □

As a consequence we obtain

Corollary 1.6.10. *For $1 < p < n$ there exists a function $u \in D^{1,p}(\mathbb{R}^n)$ with $||u||_{L^q(\mathbb{R}^n)} = 1$ and $||u||_{D^{1,p}(\mathbb{R}^n)} = S$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and where $S = S(k, p, n)$ is the Sobolev constant.*

Over the past quarter century, one field of intense research activity has been the study of what symmetry properties the solution of a nonlinear elliptic boundary value problem can inherit from the domain on which it is being solved. A classic paper is that of Gidas-Ni-Nirenberg. In [26], they prove symmetry without symmetrization and related properties of positive solutions of second order elliptic equations vanishing on the boundary, using just the maximum principle and moving plane method.

Theorem 1.6.11. *(Gidas-Ni-Nirenberg) In the ball $\Omega : |x| < R$ in $\mathbb{R}^n$, let $u > 0$ be a positive solution in $C^2(\Omega)$ of*

$$\Delta u + f(u) = 0 \ \ with\, u = 0 \ \ on\, |x| = R. \tag{1.37}$$

Here f is of class C^1. Then u is radially symmetric and

$$\frac{\partial u}{\partial r} < 0, \ \ for \ \ 0 < r < R.$$

Theorem 1.6.12. *(Gidas-Ni-Nirenberg) Let $u > 0$ be a C^2 solution of (1.37) in a ring-shaped domain $R' < |x| \leq R$. Then*

$$\frac{\partial u}{\partial r} < 0, \ \ for \ \ 0 < \frac{R' + R}{2} < x < R.$$

The following is a generalization of Theorem 1.6.11 in $\mathbb{R}^n$.

Theorem 1.6.13. *(Gidas-Ni-Nirenberg) Let $v > 0$ be a C^2 solution of an elliptic equation*

$$F(v, |\nabla v|^2, \sum v_j v_k v_{jk}, trA, trA^2, \cdots, trA^n) = 0 \ \ in\, \mathbb{R}^n \tag{1.38}$$

where $A =$ the Hessian matrix $\{v_{ij}\}$, here F is C^1, for $v > 0$, and all values of the other arguments. Assume that near infinity, v and its first derivatives admit the asymptotic expansion (using summation convention):

$$v = \frac{1}{|x|^m}(a_0 + \frac{a_j x_j}{|x|^2} + \frac{a_{jk} x_j x_k}{|x|^4} + o(\frac{1}{|x|^2})) \tag{1.39}$$

$$v_{x_i} = -\frac{m}{|x|^{m+2}} x_i (a_0 + \frac{a_j x_j}{|x|^2}) + \frac{a_i}{|x|^{m+2}} - \frac{2x_i}{|x|^{m+4}} a_j x_j + O(\frac{1}{|x|^{m+3}})$$

for some $m, a_0 > 0$. Then v is rotationally symmetric about some point and $v_r < 0$ for $r > 0$ where r is the radial coordinate about that point.

1.7 Extremals for the Sobolev inequality

Corollary 1.7.1. *Let u be an extremal of the Sobolev inequality, that is a nonnegative solution of corresponding Euler equations*

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = u^{q-1} \tag{1.40}$$

such that

$$u(x) \to 0 \;\; for \;\; |x| \to +\infty \;\; and \;\; \int_{\mathbb{R}^n} |\nabla u|^p dx < +\infty. \tag{1.41}$$

Then, u is of the form $u(x) = (\lambda + |x|^{p/(p-1)})^{1-n/p}$, with λ positive constant.

Proof. By (GNN) theorem, we know that the solution of (1.40), are radial functions. So, we can write the equation (1.40), as

$$(r^{n-1}|u'|^{p-1}\, sign\, u')' + r^{n-1}|u|^{q-1}\, sign\, u = 0 \tag{1.42}$$

Moreover, we can suppose u radial, decreasing function and, we get

$$(-\, r^{n-1}|u'|^{p-1})' + r^{n-1}|u|^{q-1} = 0 \tag{1.43}$$

with the condition (1.41). It is possible to represent in closed form a set of solutions, satisfying the condition (1.41), of the differential equation (1.43). In fact, if $p = 2$, (1.43) is a particular case of Emden-Fowler equations and all its solution can be obtained for quadratures. Let $p = 2, n \geq 3, q = \frac{2n}{n-2}$ and we consider only positive decreasing solutions. Then (1.40) rewrites as

$$(n-1)u' + r\, u'' + r\, u^{\frac{n+2}{n-2}} = 0 \tag{1.44}$$

Setting $t = -\log r$ and $u(t) = r^{(n-2)/2}u(r)$, we find for the new unknown of the equation

$$u' = -\frac{n-2}{2}\, r^{-n/2}\, u(t) \;-\; r^{-n/2}\, u_t\, .$$

$$u'' = \frac{(n-2)n}{4}\, r^{-\frac{n+2}{2}}\, u(t) \;+\; (n-1)\, r^{-n/2-1}\, u_t \;+\; r^{-\frac{n+2}{2}}\, u_{tt}$$

Then, for any $t \in \mathbb{R}$ we have

$$0 = (n-1)[-\frac{n-2}{2} r^{-n/2} u(t) - r^{-n/2} u_t] + r[\frac{(n-2)n}{4} r^{-\frac{n+2}{2}} u(t) + (n-1) r^{-n/2-1} u_t$$
$$+ r^{-\frac{n+2}{2}} u_{tt}] + r[r^{-(n-2)/2} u(t)]^{\frac{n+2}{n-2}}$$
$$= -(n-1)\frac{n-2}{2} r^{-n/2} u(t) - (n-1) r^{-n/2} u_t + \frac{(n-2)n}{4} r^{-n/2} u(t) + (n-1) r^{-n/2} u_t$$
$$+ r^{-\frac{n}{2}} u_{tt} + r^{-n/2} u(t)^{\frac{n+2}{n-2}}$$
$$= \frac{n-2}{2}[-(n-1) + \frac{n}{2}] r^{-n/2} u(t) + r^{-\frac{n}{2}} u_{tt} + r^{-n/2} u(t)^{\frac{n+2}{n-2}}$$
$$= -\frac{(n-2)^2}{4} r^{-n/2} u(t) + r^{-\frac{n}{2}} u_{tt} + r^{-n/2} u(t)^{\frac{n+2}{n-2}}$$
$$= u_{tt} - \frac{(n-2)^2}{4} u(t) + u(t)^{\frac{n+2}{n-2}}. \tag{1.45}$$

The simplest singular solution of (1.45) corresponds to $u \equiv c = (\frac{n-2}{2})^{(n-2)/2}$. Equation (1.45) can be integrated to give

$$(u')^2 = (\frac{n-2}{2})^2 u^2 - \frac{n-2}{n} u^{\frac{2n}{n-2}} + D\,. \tag{1.46}$$

It follows from (1.46) that the behavior of u is determined by the roots of

$$(\frac{n-2}{2})^2 u^2 - \frac{n-2}{n} u^{\frac{2n}{n-2}} + D = 0.$$

By the maximum principle, u cannot vanish for any finite t unless $u \equiv 0$, and this forces D to lie in the interval $0 \geq D \geq -(\frac{2}{n})(\frac{n-2}{n})^n$. The case $D = 0$ corresponds to the regular family of solutions

$$v = \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + r^2}\right)^{(n-2)/2} \qquad \lambda > 0, \tag{1.47}$$

while for all other D there is a periodic translation invariant positive family of solutions $u_D(t)$ of (1.45). The other extreme case, $D = -(\frac{2}{n})(\frac{n-2}{n})^n$ corresponds to the solution $u = c$ or $u = \frac{c}{r^{(n-2)/2}}$. Hence, the only positive solution to (1.44) that are in $H^1(\mathbb{R}_+)$ are of type (1.47). In the general case ($1 < p < n, q = \frac{pn}{n-p}$), it is easy to check that the solutions are $(\lambda + r^{\frac{p}{p-1}})^{1-n/p}$ with $\lambda > 0$. □

1.8 Extremals for the Caffarelli-Kohn-Nirenberg inequalities

An other important result has been obtained by Catrina and Wang in [17] in which they also compute explicitly all radial solution of (1.9) for $a < 0$. Let $D_a^{1,2}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the inner product

$$(u, v) = \int_{\mathbb{R}^n} |x|^{-2a} \nabla u \cdot \nabla w \, dx. \tag{1.48}$$

We define

$$E_{a,b}(u) = \frac{\int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^n} |x|^{-bp} |u|^p \, dx)^{2/p}} \tag{1.49}$$

We want to find $S(a,b) = \inf_{u \in D_a^{1,2}(\mathbb{R}^n) \setminus \{0\}} E_{a,b}(u)$. We recall that the extremal functions of CKN inequalities satisfy the Euler equation

$$-\mathrm{div}(|x|^{-2a} \nabla u) = |x|^{-bp} u^{p-1} \tag{1.50}$$

for $n \geq 3, a < \frac{n-2}{2}, a \leq b \leq a+1$, and $p = \frac{2n}{n-2+2(b-a)}$. Moreover (1.50) is invariant under dilation, i.e. if u is a solution on (1.50), also $u_\lambda(x) = \lambda^{\frac{n-2-2a}{n-2}} u(\lambda x)$ is a solution. Then, Catrina and Wang have proved the following

Theorem 1.8.1. *Up to dilations, all radial solution of (1.50) are explicitly given, i.e.*

$$u(x) = \left(\frac{n(n-2-2a)^2}{n-2(1+a-b)} \right)^{\frac{n-2(1+a-b)}{4(1+a-b)}} \frac{1}{\left(1 + |x|^{\frac{2(n-2-2a)(1+a-b)}{n-2(1+a-b)}}\right)^{\frac{n-2(1+a-b)}{2(1+a-b)}}}$$

The proof is based on a standard thecnique. By GNN theorem, we know that the solution of (1.50), are radial functions. So, we can write the equation (1.50), as

$$(r^{n-1-2a} u')' + r^{n-1-bp} u^{p-1} = 0 \tag{1.51}$$

that is

$$(n-1-2a) r^{n-2-2a} u' + r^{n-1-2a} u'' + r^{n-1-\frac{2nb}{n-2+2(b-a)}} u^{p-1} = 0$$

i.e.

$$(n-1-2a) r^{-2a} u' + r^{1-2a} u'' + r^{1-\frac{2nb}{n-2+2(b-a)}} u^{p-1} = 0 \tag{1.52}$$

Setting $t = -\log r$ and $u(t) = r^{(n-2-2a)/2} u(r)$, we have

$$u' = -\frac{(n-2-2a)}{2} r^{-(n-2a)/2} u(t) - r^{-(n-2a)/2} u_t \, .$$

$$u'' = \frac{(n-2-2a)(n-2a)}{4} r^{-\frac{(n-2a+2)}{2}} u(t) + \frac{(n-2a-2)}{2} r^{-(n-2a+2)/2} u_t$$

$$+ \frac{(n-2a)}{2} r^{-(n-2a+2)/2} u_t + r^{-(n-2a+2)/2} u_{tt} \, .$$

Then

$$
\begin{aligned}
0 &= (n-1-2a)\, r^{-2a}\, [-\frac{(n-2-2a)}{2}\, r^{-a-n/2}\, u(t) \;-\; r^{-a-n/2}\, u_t] \\
&+ r^{1-2a}\, [\frac{(n-2-2a)(n-2a)}{4}\, r^{-\frac{(n-2a+2)}{2}}\, u(t) \\
&+ \frac{(n-2a-2)}{2}\, r^{-(n-2a+2)/2}\, u_t \;+\; \frac{(n-2a)}{2}\, r^{-\frac{(n-2a+2)}{2}}\, u_t \;+\; r^{-\frac{(n-2a+2)}{2}}\, u_{tt}] \\
&+ r^{1-\frac{2nb}{n-2+2(b-a)}}\, [r^{-(n-2-2a)/2}\, u(t)]^{\frac{2n}{n-2+2(b-a)}\,-1} \\
&= -\frac{(n-1-2a)(n-2-2a)}{2}\, r^{-n/2-a}\, u(t) - (n-1-2a)\, r^{-n/2-a}\, u_t \\
&+ \frac{(n-2-2a)(n-2a)}{4}\, r^{1-2a-n/2+a-1}\, u(t) + \, (n-2a-1)\, r^{1-2a-n/2+a-1}\, u_t \\
&+ r^{1-2a-n/2+a-1}\, u_{tt} \;+\; r^{1-\frac{2nb}{n-2+2(b-a)}}\, r^{-\frac{(n-2-a)n}{n-2+2(b-a)}+\frac{n-2-2a}{2}}\, u(t)^{\frac{n+2-2(b-a)}{n-2+2(b-a)}} \\
&= \frac{(n-2-2a)}{2}\left(-(n-1-2a) + \frac{(n-2a)}{2}\right) r^{-a-n/2}\, u(t) \\
&+ [n-2a-1-n+1+2a]\; r^{-a-n/2}\, u_t + r^{-a-n/2}\, u_{tt} \;+\; r^{-n/2-a}\, u(t)^{\frac{n+2-2(b-a)}{n-2+2(b-a)}} \\
&= \frac{(n-2-2a)}{2}\,(-n/2+1+a)\; r^{-a-n/2}\, u(t) + \, r^{-a-n/2}\, u_{tt} \;+\; r^{-a-n/2}\, u(t)^{\frac{n+2-2(b-a)}{n-2+2(b-a)}} \\
&= -\frac{(n-2-2a)^2}{4}\, r^{-a-n/2}\, u(t) \;+\; r^{-a-n/2}\, u_{tt} \;+\; r^{-a-n/2}\, u(t)^{\frac{n+2-2(b-a)}{n-2+2(b-a)}} \\
&= -\frac{(n-2-2a)^2}{4}\, u(t) \;+\; u_{tt} \;+\; u(t)^{\frac{n+2-2(b-a)}{n-2+2(b-a)}}\,. \qquad (1.53)
\end{aligned}
$$

Again, $u(t)$ satisfies

$$
u_{tt} \;-\; \frac{(n-2-2a)^2}{4}\, u \;+\; u^{\frac{n+2-2(b-a)}{n-2+2(b-a)}} = 0
$$

that is a nonlinear autonomous ordinary differential equation of the second order and it has as first integral

$$
(u_t)^2 = \frac{(n-2-2a)^2}{4}\, u^2 \;-\; \frac{(n-2+2(b-a))}{n}\; u^{\frac{2n}{n-2+2(b-a)}} \;+ C\,.
$$

Remark 1.8.1. Let p, γ be real numbers with $p > 2$. If v satisfies the equation

$$
(v_t)^2 = \gamma^2\, v^2 \;-\; \frac{2}{p}\, v^p \qquad (1.54)
$$

then

$$
v(t) = \left(\frac{\gamma^2 p}{2}\right)^{\frac{1}{p-2}} \left(\cosh(\frac{p-2}{2}\,\gamma t)\right)^{\frac{-2}{p-2}}. \qquad (1.55)
$$

Proof. If v satisfies (1.54), then $v_t = \pm\sqrt{\gamma^2\, v^2 - \frac{2}{p}\, v^p} \; = \pm v\, \sqrt{\gamma^2 \;-\; \frac{2}{p}\, v^{p-2}}\,$. Setting $w = \sqrt{\gamma^2 \;-\; \frac{2}{p}\, v^{p-2}}$, we have

$\frac{dw}{dv} = -(\frac{p-2}{p})\,\frac{1}{w\,v}\,v^{p-2} = \frac{1}{v\,w}\,(\frac{p-2}{2})\,(w^2-\gamma^2)$. Thus,
$\pm dt = \frac{dv}{v\,\sqrt{\gamma^2 - \frac{2}{p}\,v^{p-2}}} = (\frac{2}{p-2})\,\frac{dw}{(w^2-\gamma^2)}$. Integrating, we have

$$t = \mp\,(\frac{2}{p-2})\int \frac{dw}{(w^2-\gamma^2)} = \pm\,\frac{2\,\gamma}{(p-2)}(\tanh)^{-1}(\frac{w}{\gamma})$$
$$= \pm\,\frac{2\,\gamma}{(p-2)}(\tanh)^{-1}\left(\frac{\sqrt{\gamma^2 - \frac{2}{p}\,v^{p-2}}}{\gamma}\right).$$

This implies

$$\tanh(\frac{p-2}{2\gamma}\,t) = \pm\,\frac{\sqrt{\gamma^2 - \frac{2}{p}\,v^{p-2}}}{\gamma}$$

that is

$$\tanh^2(\frac{p-2}{2\gamma}\,t) = \frac{\gamma^2 - \frac{2}{p}\,v^{p-2}}{\gamma^2}$$

i.e.

$$\left(\frac{e^{\frac{p-2}{2\gamma}t} - e^{-\frac{p-2}{2\gamma}t}}{e^{\frac{p-2}{2\gamma}t} + e^{-\frac{p-2}{2\gamma}t}}\right)^2 = \left(\frac{1 + e^{\frac{2(p-2)}{\gamma}t} - 2\,e^{\frac{p-2}{\gamma}\,t}}{1 + e^{\frac{2(p-2)}{\gamma}t} + 2\,e^{\frac{p-2}{\gamma}t}}\right) = 1 - \frac{2}{p\,\gamma^2}\,v^{p-2}.$$

Then

$$\left(1 - \frac{4\,e^{\frac{p-2}{\gamma}t}}{(1 + e^{\frac{2(p-2)}{\gamma}t} + 2\,e^{\frac{p-2}{\gamma}t})}\right) = (1 - \frac{2}{p\gamma^2}\,v^{p-2})$$

and

$$-\frac{4\,e^{\frac{p-2}{\gamma}t}}{(1 + e^{\frac{2(p-2)}{\gamma}t} + 2\,e^{\frac{p-2}{\gamma}t})} = -\frac{4\,e^{\frac{p-2}{\gamma}t}}{(1 + e^{\frac{(p-2)}{\gamma}t})^2} = -\frac{2}{p\gamma^2}\,v^{p-2}$$

Finally,

$$v^{p-2} = \frac{p\gamma^2}{2}\,\frac{4\,e^{\frac{p-2}{\gamma}t}}{(1 + e^{\frac{p-2}{\gamma}t})^2} = \frac{p\gamma^2}{2}\,\cosh\left(\frac{(p-2)}{2\gamma}t\right)^{-\frac{2}{p-2}}$$

and we get the thesis. □

As in the previous paragraph, we get that the only positive solution in $H^1(\mathbb{R})$ for the equation (1.53) verify the equation (1.54) and it is of the type (1.55) where $\gamma = \frac{(n-2-2a)}{2}$ and $p = \frac{2n}{n-2+2(b-a)}$. Then, the radial solution in $\mathbb{R}^n$ for (1.50) corresponding to this u is

$$u(x) = \left(\frac{n(n-2-2a)^2}{n-2(1+a-b)}\right)^{\frac{n-2(1+a-b)}{4(1+a-b)}} \frac{1}{\left(1 + |x|^{\frac{2(n-2-2a)(1+a-b)}{n-2(1+a-b)}}\right)^{\frac{n-2(1+a-b)}{2(1+a-b)}}}.$$

2. A WEIGHTED SOBOLEV INEQUALITY

In this chapter we study existence and non-existence of cylindrical solutions for a nonlinear elliptic equation in $\mathbb{R}^3$. The problem is the following:

$$\left.\begin{array}{l} -\Delta u = \phi(r)|u|^{p-2}u \quad in \ \ \mathbb{R}^3 \\ u(x) > 0 \\ \int_{\mathbb{R}^3} \phi(r) u^{p-1} dx < +\infty \end{array}\right\} \tag{2.1}$$

with $p > 1, x = (x_1, x_2, x_3) \in \mathbb{R}^3, r = \sqrt{x_1^2 + x_2^2}$ and $u = u(r, z)$. The model function for ϕ is

$$\phi(r) = \frac{r^{2\alpha}}{(1 + r^2)^{\alpha + \frac{1}{2}}}, \tag{2.2}$$

where $\alpha \geq 0$. This equation has been proposed by G.Bertin and L.Ciotti as a model describing the dynamics of elliptic galaxies (see the book [7] and lecture notes [19]) . Here u satisfies the usual Poisson equation $\Delta u = 4\pi G\rho$ relating the gravitational potential u to the density of matter ρ. The derivation of such a model is the following. A galaxy can be conceived as a "gas of stars" with a distribution function $f(x, v, t)$ so that $\rho = \int_{\mathbb{R}^3} f(x, v, t) dv$, where v is the velocity of the stars. The integration over the velocity space is restricted to a domain defined by the requirement that the distribution function is positive. This condition is formulated in terms of the integrals of motion for the chosen potential. As a consequence of this approach, we are led to an implicit relation between the galaxy density and galaxy potential. In general this relation is nonlinear, moreover it is not known if the Poisson equation admits physically acceptable solutions. Thus, any system for which solutions can be established is of great interest in the applications. The cylindrical symmetry of the problem is derived from the assumptions that the elliptic galaxies are axially symmetric. Moreover, the condition $\int_{\mathbb{R}^3} \phi(r) u^{p-1} dx < +\infty$ guarantees that the given solution carries a finite total mass. Several elliptic equations similar to (2.1) arising from models of globular cluster of stars have been investigated, but we know most of these models are radial. Motivated by the Bertin and Ciotti choice (2.2), M.Badiale and G.Tarantello in [3], have considered the following hypothesis:

$$\phi \in C(\mathbb{R}^+), \, \phi \geq 0, \, \phi(r) = 0 \ \ if \ \ and \ \ only \ \ if \, r = 0, \, r\phi(r) \in L^\infty(\mathbb{R}^+), \tag{2.3}$$

where we set $\mathbb{R}^+ = [0, +\infty[$. We note that the functions in (2.2) satisfy (2.3) for $\alpha > 0$. They proved that the problem (2.1), for $p \in [4, 6]$, can be handled by a variational approach in the Sobolev space $D^{1,2}(\mathbb{R}^3)$ which also guarantees the finite-total-mass condition. This fact rests upon a new Sobolev inequality, which has been derived for any dimension $n \geq 3$ and extends the Caffarelli-Kohn-Nirenberg inequality (see [13]). This inequality can be seen as the "cylindrical" version of the CKN inequality, the most natural formulation of CKN inequality for cylindrical functions.

Theorem 2.0.2. *(Badiale-Tarantello) Let* $n \geq 3$, $2 \leq k \leq n, x = (y, z) \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, 1 < q < n, 0 \leq s \leq q,$ *and* $s < k$. *There exists a positive constant* $C = C(n, q, k, s)$ *such that for all* $u \in D^{1,q}(\mathbb{R}^n)$ *we have*

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{\frac{q(n-s)}{n-q}}}{|y|^s} \right)^{\frac{n-q}{n-s}} \leq C \int_{\mathbb{R}^n} |\nabla u|^q \tag{2.4}$$

More general inequalities of (2.4), in the case of $k = n$, have been considered by Caffarelli-Cohn-Nirenberg ([13]), while Catrina F. and Z.Q.Wang in ([17]) have obtained several results concerning best constants and corresponding extremal functions. When $k = n$, (2.4) was thoroughly investigated in [28] and extremal functions in this case have been computed by Lieb ([34]).

2.1 The case $q = 2$

From now on, we will always consider the inequality (2.4) with $q = 2$, i.e.

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{\frac{2(n-s)}{n-2}}}{|y|^s} \right)^{\frac{n-2}{n-s}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 \tag{2.5}$$

holds for any $u \in D^{1,2}(\mathbb{R}^n)$, $n \geq 3$ and $0 \leq s \leq 2$. Moreover, setting $\sigma p_\sigma = s$, we can write the equation (2.5), as

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{p_\sigma}}{|y|^{\sigma p_\sigma}} \right)^{\frac{2}{p_\sigma}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 \tag{2.6}$$

where $n \geq 3$, $0 \leq \sigma \leq 1$ *and* $p_\sigma = \frac{2n}{n-2+2\sigma}$. We note that the case $\sigma = 0$ corresponds to the classical Sobolev inequality, while, if $\sigma = 1$, (2.6) is the classical Hardy inequality which is known not to have extremal functions. So, our interest, is for the case $0 < \sigma < 1$ *and* $2 \leq k < n$.
In the next section we will introduce fundamental definitions and some relevant results which will be needed in the sequel. We collect below a list of the main notations.

- $D^{1,2}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $||u|| = (\int_{\mathbb{R}^n} |\nabla u|^2 dx)^{1/2}$.

- $L^q(\mathbb{R}^n), L^q_{loc}(\mathbb{R}^n)$ are the usual Lebesgue spaces.
- $2_* = \frac{2(n-s)}{n-2}$ defines the critical exponent for (2.5) inequality.
- We set $\mathbb{R}^+ = [0, +\infty[$ and $\mathbb{R}_+ =]0, +\infty[$.

Remark 2.1.1. For $s > 0$, $2_* = 2_*(s) = \frac{2(n-s)}{n-2} < 2^*$ where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent.

In analogy to the Sobolev's best constant, we introduce the best constant in (2.6) by setting

$$S = \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^2 dx \,|\, u \in D^{1,2}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s} dx = 1\right\} \tag{2.7}$$

Clearly, S depends on $k, s, 2$ and n.

2.2 Existence of extremal functions

Through the Concentration-Compactness principle of P.L.Lions, in [3] it is proved

Theorem 2.2.1. *(q=2) Assume* $2 \le k \le n, x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}, 0 < s < 2$. *Then, the extremal problem (2.7) attains its infimum:* $\exists$ *a function* u *which satisfies*

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx = S, \quad \int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s} dx = 1.$$

Corollary 2.2.2. *As a consequence, we get that the problem*

$$\begin{cases} -\,div(\nabla v) = \dfrac{1}{|y|^s} v^{2_*-1} \\ v(x) > 0 \ \ in\, \mathbb{R}^n \\ v \in D^{1,2}(\mathbb{R}^n) \end{cases} \tag{2.8}$$

admits the non trivial solution $v = S^{-\frac{1}{2_*-1}} u$ *where* u *is an extremal function for (2.7) as given by the last theorem.*

One of the crucial aspects is the following

Remark 2.2.1. For every $\lambda > 0$ and $\xi \in \mathbb{R}^{n-k}$, the problem (2.8) is invariant under the transformation $u \to u_{\lambda,\xi}$ with

$$u_{\lambda,\xi}(y, z) = \lambda^{\frac{n-2}{2}} u(\lambda y, \lambda(z - \xi)) \tag{2.9}$$

In particular, if u minimizes (2.7), so does $u_{\lambda,\xi}$ for every λ *and* $\xi \in \mathbb{R}^{n-k}$.

Definition 2.2.1. A minimizing sequence $\{u_h\}_h \in D^{1,2}(\mathbb{R}^n)$ for (2.7), is characterized by the properties

$$\int_{\mathbb{R}^n} |\nabla u_h|^2 dx \to S, \quad \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{|y|^s} dx = 1. \tag{2.10}$$

For $R > 0$ and $\xi \in \mathbb{R}^k$, we set

$$\Omega_R(\xi) = \{(y,z) \in \mathbb{R}^k \times \in \mathbb{R}^{n-k} \mid \quad |y| + |z-\xi| < R\}.$$

The first lemma of Concentration-Compactness principle is the following

Lemma 2.2.3. *Let $\{u_h\}_h$ satisfying (2.10). Along a subsequence, still denoted by $\{u_h\}_h$, one of the following three alternatives holds:*

i) There exists a sequence $\{\xi_h\}_h \subset \mathbb{R}^{n-k}$ such that, for all $\epsilon > 0$ there exist $h_\epsilon \in \mathbb{N}$ and $R_\epsilon > 0$ such that

$$\int_{\Omega_R(\xi_h)} \frac{|u_h|^{2_*}}{|y|^s} dx > 1 - \epsilon, \quad \forall h \geq h_\epsilon, \forall R \geq R_\epsilon.$$

ii) For all $R > 0$ $\lim_h \left[\sup_{\xi \in \mathbb{R}^{n-k}} \int_{\Omega_R(\xi)} \frac{|u_h|^{2_*}}{|y|^s} dx \right] = 0.$

iii) There exists $\overline{\alpha} \in]0,1[$ such that for all $\epsilon > 0$ there exist $R_\epsilon > 0$, a sequence of positive numbers $R_h \to \infty$ and a sequence $\{\xi_h\}_h \subset \mathbb{R}^{n-k}$ such that

$$\left| \int_{\Omega_{R_\epsilon}(\xi_h)} \frac{|u_h|^{2_*}}{|y|^s} dx - \overline{\alpha} \right| < \epsilon, \left| \int_{\mathbb{R}^n \setminus \Omega_{R_h}(\xi_h)} \frac{|u_h|^{2_*}}{|y|^s} dx - (1 - \overline{\alpha}) \right| > \epsilon \tag{2.11}$$

$$\int_{\Omega_{R_h}(\xi_h) \setminus \Omega_{R_\epsilon}(\xi_h)} \frac{|u_h|^{2_*}}{|y|^s} dx < \epsilon \tag{2.12}$$

The proof of theorem (2.2.1) can be divided in to two more steps.
Step 1. Under the condition (2.10), only alternative i) can occur.

Proof. In fact, by Remark (2.2.1), given a minimizing sequence $\tilde{u}_h$, also $u_h = \lambda_h^{\frac{n-2}{2}} \tilde{u}(\lambda_h y, \lambda_h(z - \xi_h))$ is a minimizing sequence. Then, for a suitable choice of $\lambda_h > 0$ and $\xi_h \in \mathbb{R}^{n-k}$, we can assume

$$\sup_{\xi \in \mathbb{R}^{n-k}} \int_{|z-\xi|<1} \int_{|y|<1} \frac{|u_h|^{2_*}}{|y|^s} dydz = \int_{|z|<1} \int_{|y|<1} \frac{|u_h|^{2_*}}{|y|^s} dydz = \frac{1}{2} \tag{2.13}$$

So, the case ii) in the above lemma, cannot occur. The third case is more delicate and it is proved by contradiction. To show this, the following is necessary

Remark 2.2.2. Let $p > 1$. Then $\exists C = C_p > 0$ such that $\forall a, b \in \mathbb{R}^n$,

$$\big|\,|a+b|^p - |a|^p - |b|^p\,\big| \leq C_p(|a|^{p-1}|b| + |a||b|^{p-1}). \tag{2.14}$$

$$1 - x^p \geq (1-x)^p \;\forall x \in [0,1],\; \forall p > 1. \tag{2.15}$$

Proposition 2.2.4. *Let $\{u_h\}_h$ be a sequence satisfying (2.10),(2.11),(2.12). Let $\psi \in C_0^\infty([0,+\infty[,\mathbb{R})$ be such that $\psi(t) = 1$ for $0 \leq t \leq 1$, $\psi(t) = 0$ for $t \geq 2$, $0 \leq \psi(t) \leq 1$ and $-C \leq \psi'(t) \leq 0, \forall t \geq 0$. For fixed $\epsilon > 0$ small enough, we define*

$$\phi_{h,\epsilon}(y,z) = \psi\left(\frac{1}{R_h - R_\epsilon}(|y| + |z - \xi_h|) + \frac{R_h - 2R_\epsilon}{R_h - R_\epsilon}\right)$$

Under the hypothesis of theorem (2.2.1), we get

$$\int_{\mathbb{R}^n} |\nabla u_h|^2 dx \geq \int_{\mathbb{R}^n} |\nabla \phi_{h,\epsilon} u_h|^2 dx + \int_{\mathbb{R}^n} |\nabla(1 - \phi_{h,\epsilon})u_h|^2 dx - \sigma_\epsilon, \tag{2.16}$$

where σ_ϵ is a constant depending on ϵ only and such that $\sigma_\epsilon \to 0$ as $\epsilon \to 0$.

Proof. To prove (2.16), we will first prove the following claim:

$$\int_{\mathbb{R}^n} |\nabla u_h|^2 dx \geq \int_{\mathbb{R}^n} |\nabla u_h|^2 \phi_{h,\epsilon}^2 dx + \int_{\mathbb{R}^n} (1 - \phi_{h,\epsilon})^2 |\nabla u_h|^2 dx. \tag{2.17}$$

We denote by K_h the support of $\phi_{h,\epsilon}$ and $\Omega_{R_\epsilon(\xi_h)} = \Omega_h$. We notice that $\Omega_h \subset K_h$. Then

$$\int_{\mathbb{R}^n} |\nabla u_h|^2 dx = \int_{\Omega_h} |\nabla u_h|^2 dx + \int_{K_h \setminus \Omega_h} |\nabla u_h|^2 dx =$$

$$\int_{\Omega_h} \phi_{h,\epsilon}^2 |\nabla u_h|^2 dx + \int_{\mathbb{R}^n \setminus K_h} (1 - \phi_{h,\epsilon})^2 |\nabla u_h|^2 dx + \int_{K_h \setminus \Omega_h} |\nabla u_h|^2 dx.$$

But, by Remark (2.2.2),

$$\begin{aligned} \int_{K_h \setminus \Omega_h} |\nabla u_h|^2 dx &= \int_{K_h \setminus \Omega_h} \phi_{h,\epsilon}^2 |\nabla u_h|^2 dx + \int_{K_h \setminus \Omega_h} (1 - \phi_{h,\epsilon}^2) |\nabla u_h|^2 dx \\ &\geq \int_{K_h \setminus \Omega_h} \phi_{h,\epsilon}^2 |\nabla u_h|^2 dx + \int_{K_h \setminus \Omega_h} (1 - \phi_{h,\epsilon})^2 |\nabla u_h|^2 dx \end{aligned}$$

and (2.17) holds. Always by Remark (2.2.2),

$$\int_{\mathbb{R}^n} \phi_{h,\epsilon}^2 |\nabla u_h|^2 dx + \int_{\mathbb{R}^n} (1-\phi_{h,\epsilon})^2 |\nabla u_h|^2 dx$$
$$= \int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon} u_h) - u_h \nabla \phi_{h,\epsilon}|^2 dx + \int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon}) u_h + u_h \nabla\phi_{h,\epsilon}|^2 dx$$
$$\geq \int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon} u_h)|^2 dx + \int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon}) u_h|^2 dx + 2\int_{\mathbb{R}^n} |\nabla\phi_{h,\epsilon}|^2 |u_h|^2 dx$$
$$- C\Big(\int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon})u_h|^{2-1}\, |u_h|\, |\nabla\phi_{h,\epsilon}| dx + \int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon})u_h| |u_h|^{2-1} |\nabla\phi_{h,\epsilon}|^{2-1} dx\Big)$$
$$- C\Big(\int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon}u_h)|^{2-1}\, |u_h|\, |\nabla\phi_{h,\epsilon}| dx + \int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon}u_h)| |u_h|^{2-1} |\nabla\phi_{h,\epsilon}|^{2-1} dx\Big)$$
$$= \int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon} u_h)|^2 dx + \int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon}) u_h|^2 dx + 2\int_{\mathbb{R}^n} |\nabla\phi_{h,\epsilon}|^2 |u_h|^2 dx$$
$$- 2C\Big(\int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon})u_h|\, |u_h|\, |\nabla\phi_{h,\epsilon}|\, dx + \int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon}u_h)|\, |u_h|\, |\nabla\phi_{h,\epsilon}|\, dx\Big).$$

Now, we will prove that the last term vanishes as $\epsilon \to 0$, uniformly in h. In fact, we note that

$$\int_{\mathbb{R}^n} |\nabla(\phi_{h,\epsilon}u_h)|\, |u_h|\, |\nabla\phi_{h,\epsilon}|\, dx$$
$$\leq \int_{\mathbb{R}^n} |u_h|^2 |\nabla\phi_{h,\epsilon}|^2 dx + \int_{\mathbb{R}^n} |\nabla u_h| |u_h| \phi_{h,\epsilon} |\nabla\phi_{h,\epsilon}| dx. \tag{2.18}$$

Estimating each term of (2.18) separately, we get:

$$\int_{\mathbb{R}^n} |u_h|^2 |\nabla\phi_{h,\epsilon}|^2 dx \leq C\Big(\frac{1}{R_h - R_\epsilon}\Big)^2 \int_{A_{h,\epsilon}} |u_h|^2 dx$$

where $A_{h,\epsilon} = \Omega_{R_h}(\xi_h) \setminus \Omega_{R_\epsilon}(\xi_h)$. By the Hölder inequality, we obtain

$$\int_{A_{h,\epsilon}} |u_h|^2 dx = \int_{A_{h,\epsilon}} |u_h|^2 \frac{1}{|y|^{2s/2_*}} |y|^{2s/2_*} dx$$
$$\leq \Big(\int_{A_{h,\epsilon}} |u_h|^{2_*} \frac{1}{|y|^s} dx\Big)^{2/2_*} \Big(\int_{A_{h,\epsilon}} |y|^{\frac{2s}{2_*-2}} dx\Big)^{\frac{2_*-2}{2_*}}$$
$$\epsilon^{2/2_*} \Big(\int_{A_{h,\epsilon}} |y|^{\frac{2s}{2_*-2}} dx\Big)^{\frac{2_*-2}{2_*}} \tag{2.19}$$

We remember that in Lemma (2.2.3), case (iii), we can always choose $R_h \geq 2R_\epsilon$. Setting $\gamma = \frac{2s}{2_*-2}$, we have

$$\int_{A_{h,\epsilon}} |y|^\gamma\, dx$$
$$\leq \int_{|z|<R_h} dz \int_{|y|<R_h} |y|^\gamma dy \leq C{R_h}^{n-k} \int_0^{R_h} \rho^\gamma \rho^{k-1} d\rho \leq C{R_h}^{n+\gamma}.$$

Thus, from (2.19), we get

$$\left(\frac{1}{R_h - R_\epsilon}\right)^2 \int_{A_{h,\epsilon}} |u_h|^2 dx \leq \left(\frac{1}{R_h - R_\epsilon}\right)^2 C\epsilon^{2/2_*} ({R_h}^{n+\gamma})^{\frac{2_*-2}{2_*}}.$$

Since
$(n+\gamma)\frac{2_*-2}{2_*} = 2$ and $\frac{R_h}{R_h-R_\epsilon} = 1 + \frac{R_\epsilon}{R_h-R_\epsilon} \leq 2$, we conclude

$$\int_{\mathbb{R}^n} |\nabla\phi_{h,\epsilon}|^2 |u_h|^2 dx \leq C\epsilon^{2/2_*}, \tag{2.20}$$

with the constant $C > 0$ independent of ϵ and h. The next estimate follows from (2.19) and the fact that, by (2.10), $\int_{\mathbb{R}^n} |\nabla u_h|^2 dx$ is uniformly bounded.

$$\int_{\mathbb{R}^n} \phi_{h,\epsilon} |\nabla\phi_{h,\epsilon}|\, |u_h|\, |\nabla u_h|\, dx = \int_{A_{h,\epsilon}} \phi_{h,\epsilon} |\nabla\phi_{h,\epsilon}|\, |u_h|\, |\nabla u_h|\, dx$$
$$\left(\int_{A_{h,\epsilon}} |\nabla\phi_{h,\epsilon}|^2 |u_h|^2\, dx\right)^{1/2} \left(\int_{A_{h,\epsilon}} \phi_{h,\epsilon}^2\, |\nabla u_h|^2\, dx\right)^{1/2} \leq C(\epsilon^{2/2_*})^{1/2} = C\epsilon^{1/2_*}. \tag{2.21}$$

Besides

$$|\int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon})u_h|\, |u_h|\, |\nabla\phi_{h,\epsilon}|\, dx| = |\int_{\mathbb{R}^n} |\nabla\phi_{h,\epsilon} u_h|\, |u_h|\, |\nabla\phi_{h,\epsilon}|\, dx|$$
$$= |\int_{\mathbb{R}^n} |u_h|^2\, |\nabla\phi_{h,\epsilon}|^2\, dx| \leq \sigma_\epsilon$$

where $\sigma_\epsilon \to 0$ as $\epsilon \to 0$, uniformly with respect to h. Putting these estimates in (2.2.2), for (2.17), we have proved the proposition. □

Now we can proceed by contradiction. For any $\epsilon > 0$, we may extract a subsequence and assume that

$$\int_{\Omega_{R_\epsilon(\xi_h)}} \frac{|u_h|^{2_*}}{|y|^s}\, dx \to \alpha_\epsilon$$

as $h \to \infty$. We note that such a sequence $\{u_h\}_h$ depends also on ϵ. By the assumption (2.11), we have $|\alpha_\epsilon - \overline{\alpha}| \leq \epsilon$, so that $\alpha_\epsilon \to \overline{\alpha}$ as $\epsilon \to 0$. Set $\sigma_{h,\epsilon} = \int_{\mathbb{R}^n} |\nabla u|^2 dx - S$. By (2.10), $\sigma_{h,\epsilon} \to 0$ as $h \to \infty$. But by proposition (3.2.4) and (2.11), we have:

$$S + \sigma_{h,\epsilon} = \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} |\nabla\phi_{h,\epsilon} u_h|^2 dx + \int_{\mathbb{R}^n} |\nabla(1-\phi_{h,\epsilon})u_h|^2 dx - \sigma_\epsilon$$
$$\geq S\left[\left(\int_{\mathbb{R}^n} \frac{|\phi_{h,\epsilon} u_h|^{2_*}}{|y|^s} dx\right)^{2/2_*} + \left(\int_{\mathbb{R}^n} \frac{|(1-\phi_{h,\epsilon})u_h|^{2_*}}{|y|^s} dx\right)^{2/2_*}\right] - \sigma_\epsilon$$
$$\geq S\left[\int_{\Omega_{R_\epsilon(\xi_h)}} \frac{|\phi_{h,\epsilon} u_h|^{2_*}}{|y|^s} dx\right)^{2/2_*} + \left(1 - \int_{\Omega_{R_\epsilon(\xi_h)}} \frac{|\phi_h u_h|^{2_*}}{|y|^s} dx + \sigma_\epsilon\right)^{2/2_*}\right] - \sigma_\epsilon$$

Then, for a fixed $\epsilon > 0$, if we pass to the limit foe $h \to \infty$, we get

$$S \geq S(\alpha_\epsilon^{2/2_*} + (1 - \alpha_\epsilon + \sigma_\epsilon)^{2/2_*}) - \sigma_\epsilon$$

Letting $\epsilon \to 0$, we have:

$$S \geq S(\,\overline{\alpha}^{2/2_*} + (1 - \overline{\alpha})^{2/2_*})$$

which gives the contradiction, as $\overline{\alpha} \in]0, 1[$. In conclusion, we can always find a minimizing sequence for (2.7) such that

$$\forall \epsilon > 0, \exists h_\epsilon \in \mathbb{N}, \exists R_\epsilon > 0 \; \textit{such that} \, \forall h \geq h_\epsilon, \forall R \geq R_\epsilon : \int_{\Omega_R(0)} \frac{|u_h|^{2_*}}{|y|^s} dx > 1 - \epsilon. \tag{2.22}$$

□

Step 2. To prove the second form of Concentration-Compactness principle, i.e.

Lemma 2.2.5. *Let $\{u_h\}_h \subset D^{1,2}(\mathbb{R}^n)$ be a sequence with the property that there exist two Radon measures μ, ν and a function $u \in D^{1,2}(\mathbb{R}^n)$ such that for some $s > 0$,*

i) $u_h \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^n)$.

ii) $\nu_h = |u_h|^{2_*} \frac{1}{|y|^s} dx \rightharpoonup \nu$, weakly in the sense of measures.

iii) $\mu_h = |\nabla u_h|^2 dx \rightharpoonup \mu$, weakly in the sense of measures.

Then, there exist an at most countable set of indices J, a corresponding set of points $\{z_j \in \mathbb{R}^{n-k} \,|\, j \in J\}$, and two sets $\{\mu_j\}_j, \{\nu_j\}_j$ of non negative numbers such that

$$\nu = |u|^{2_*} \frac{1}{|y|^s} dx + \sum_{j \in J} \nu_j \delta_j, \; \mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu_j \delta_j \; \textit{and} \; \mu_j \geq S\, \nu_j^{2/2_*} \, \forall j \in J,$$

where $\delta_j = \delta_{(0,z_j)}$ is the Dirac measure with pole at $(0, z_j) \in \mathbb{R}^n$.

Proof. By the Remark (2.1.1) and by Sobolev's embedding, we can assume that

$$u_h \xrightarrow{s} u \; \textit{in} \, L^{2_*}_{loc}(\mathbb{R}^n) \; \textit{and pointwise almost everywhere.}$$

In particular,

$$\frac{u_h}{|y|^s} \to \frac{u}{|y|^s} \; \textit{pointwise a.e.}$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$. Then we have

$$\int_{\mathbb{R}^n} |\phi|^{2_*} |u_h|^{2_*} \frac{1}{|y|^s} dx \le \frac{1}{S^{2/2_*}} \left(\int_{\mathbb{R}^n} |\nabla \phi \, u_h|^2 dx \right)^{2/2_*} \tag{2.23}$$

$$\int_{\mathbb{R}^n} |\nabla \phi u_h|^2 dx \le \int_{\mathbb{R}^n} |\phi|^2 d\mu + \int_{\mathbb{R}^n} |\nabla \phi|^2 |u|^2 dx + 2C(\int_{\mathbb{R}^n} |\phi|^2 d\mu)^{1/2} (\int_{\mathbb{R}^n} |\nabla \phi|^2 |u|^2 dx)^{1/2} + o(1) \tag{2.24}$$

where $o(1) \to 0$ for $h \to \infty$.
The first follows easily, so we prove (2.24).

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla \phi u_h|^2 dx &= \int_{\mathbb{R}^n} |\phi \nabla u_h + u_h \nabla \phi|^2 \, dx \\
&\le \int_{\mathbb{R}^n} |\phi|^2 \, |\nabla u_h|^2 \, dx + \int_{\mathbb{R}^n} |\nabla \phi|^2 \, |u_h|^2 \, dx \\
&+ 2C \int_{\mathbb{R}^n} |\phi| \, |\nabla \phi| \, |\nabla u_h| \, |u_h| \, dx \\
&\le \int_{\mathbb{R}^n} |\phi|^2 \, |\nabla u_h|^2 \, dx + \int_{\mathbb{R}^n} |\nabla \phi|^2 \, |u_h|^2 \, dx \\
&2C \left(\int_{\mathbb{R}^n} |\phi|^2 \, |\nabla u_h|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla \phi|^2 \, |u_h|^2 \, dx \right)^{\frac{1}{2}} \\
&\le \left(\int_{\mathbb{R}^n} |\phi|^2 \, d\mu \right)^{\frac{1}{2}} + \int_{\mathbb{R}^n} |\nabla \phi|^2 \, |u|^2 \, dx \\
&2C \left(\int_{\mathbb{R}^n} |\phi|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla \phi|^2 \, |u|^2 \, d\mu \right)^{\frac{1}{2}} + o(1)
\end{aligned}$$

as $h \to \infty$, since

$$\int_{\mathbb{R}^n} |\phi|^2 |\nabla u_h|^2 \, dx \to \int_{\mathbb{R}^n} |\phi|^2 \, d\mu \;\; and \;\; u_h \to u \;\; in \;\; L^2_{loc}(\mathbb{R}^n)$$

as $1 < 2 \le 2_*$. Thus the estimate (2.24) holds.
First case. If $u = 0$, from (2.23),(2.24) we have

$$\int_{\mathbb{R}^n} |\phi|^{2_*} d\nu \le \frac{1}{S^{2/2_*}} \left(\int_{\mathbb{R}^n} |\phi|^2 d\mu \right)^{2/2_*} \tag{2.25}$$

By an approximation procedure, we get

$$\nu(E) \le \frac{1}{S^{2/2_*}} \mu(E)^{2/2_*} \tag{2.26}$$

for all bounded borelian sets E. Thus, we can write

$$\nu = \nu_0 + \sum_{j \in J} \nu_j \delta_{x_j}$$

where $\nu_j = \nu(\{x_j\}), \{x_j | j \in J\}$ is the set of atoms of ν and ν_0 is free of atoms. As $\nu(\mathbb{R}^n) < +\infty$, J is a countable set. Moreover ν_0 is absolutely continuous with respect to μ, hence $\nu_0 = f d\mu$ with $f \in L^1(\mathbb{R}^n, d\mu)$ and

$$f(x) = \lim_{r \to 0} \frac{\nu_0(B_r(x))}{\mu(B_r(x))} \ \mu - \ a.e. \tag{2.27}$$

As $\nu_0 \leq \nu$, we obtain that, if x is not atom of μ, then $f(x) = 0$ and $\nu_0 = 0$. In fact, since $\mu(\mathbb{R}^n) < +\infty$, also the set G of atoms for μ is almost countable. Since ν_0 is free of atoms, $\nu_0(G) = 0$. Then, for any borelian set B, we have

$$\nu_0(B) = \nu_0(B \cap G) + \nu_0(B \setminus G) = \nu_0(B \setminus G) = \int_{B \setminus G} f(x) d\mu = 0$$

as $f(x) = 0$ in $B \setminus G$. Hence,

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}$$

and by setting, $\mu_j = \mu(\{x_j\})$, we derive

$$\nu_j \leq \frac{1}{S^{2/2_*}} \mu_j^{2/2_*}$$

that is

$$\mu_j \geq S\, \nu_j^{2_*/2}.$$

Clearly,

$$\mu \geq \sum_j \mu_j \delta_{x_j}$$

and hence the lemma is proved in the case $u = 0$.
Second case: $u \neq 0$.
By result of Brezis and Lieb in [10], we know

$$|u_h|^{2_*} \frac{1}{|y|^s} dx - |u_h - u|^{2_*} \frac{1}{|y|^s} dx - |u|^{2_*} \frac{1}{|y|^s} dx \rightharpoonup 0$$

Applying the same result to the sequence $\{u_h - u\}_h$, we have

$$|u_h - u|^{2_*} \frac{1}{|y|^s} dx \rightharpoonup \tilde{\nu} = \sum_{j \in J} \nu_j \delta_{x_j}.$$

We are going to apply (2.23) and (2.24), for an appropriate choice of test functions. Let $\phi \in C_0^\infty(B_2), 0 \leq \phi \leq 1$ *and* $\phi = 1$ *in* B_1. For $\epsilon > 0$, we define:

$$\phi_\epsilon(x) = \phi(\frac{x - x_j}{\epsilon})$$

for a given $j \in J$. Then, we obtain

$$\int_{\mathbb{R}^n} \phi_\epsilon^{2_*} d\nu \leq \frac{1}{S^{2/2_*}} \left[\int_{\mathbb{R}^n} \phi_\epsilon^2 d\mu + \int_{\mathbb{R}^n} |u|^2 |\nabla \phi_\epsilon|^2 dx + 2C \left(\int_{\mathbb{R}^n} |u|^2 |\nabla \phi_\epsilon|^2 dx \right)^{1/2} \right]^{2/2_*} \tag{2.28}$$

Also

$$\int_{\mathbb{R}^n} |u|^2 |\nabla \phi_\epsilon|^2 dx = \int_{B_{2\epsilon}(x_j)} |u|^2 |\nabla \phi_\epsilon|^2 dx \leq \left(\int_{B_{2\epsilon}(x_j)} |u|^{2_*} \right)^{2/2_*} \left(\int_{B_{2\epsilon}(x_j)} |\nabla \phi_\epsilon|^n dx \right)^{2/n}.$$

But with a change of variables, we get

$$\int_{B_{2\epsilon}(x_j)} |\nabla \phi_\epsilon|^n dx = \int_{B_{2\epsilon}(x_j)} \frac{1}{\epsilon^n} |(\nabla \phi_\epsilon)(\frac{x - x_j}{\epsilon})|^n dx = \int_{B_2} |\nabla \phi_\epsilon|^n dt.$$

Moreover, $\int_{B_{2\epsilon}(x_j)} |u|^{2_*} dx \to 0 \ \ for \ \epsilon \to 0$. Passing to the limit in (2.28), for $\epsilon \to 0$, we have

$$\nu(\{x_j\}) \leq \frac{1}{S^{2/2_*}} \mu(\{x_j\})^{2/2_*}$$

that is

$$\mu_j \geq S \nu_j^{2/2_*}$$

since x_j is also an atom for the measure μ. By weak convergence, $\mu \geq |\nabla u|^2 dx$. Clearly, $\mu \geq \mu_j \delta_{x_j}$. As the measures $|\nabla u|^2$ and δ_{x_j} are orthogonal to each other, we derive

$$\mu \geq |\nabla u|^2 dx + \sum_j \mu_j \delta_{x_j}$$

At this point, we will prove that $x_j = (0, z_j)$ for some $z_j \in \mathbb{R}^{n-k}$. We notice that, since $u_h \to u$ in $L^{2_*}_{loc}(\mathbb{R}^n)$, then for all $\epsilon > 0$, we have

$$\int_{|z|<R} \int_{\epsilon<|y|<R} |u_h - u|^{2_*} \frac{1}{|y|^s} dx \leq \frac{1}{\epsilon^s} \int_{|z|<R} \int_{\epsilon<|y|<R} |u_h - u|^{2_*} dx \to 0. \tag{2.29}$$

Let $A = \{(0, z) \,|\, z \in \mathbb{R}^{n-k}\}$ and let $B \in \mathbb{R}^{n-k}$ be any ball such that $dist(A, B) > 0$. Then, (2.18) implies $\nu(B) = \int_B \frac{|u|^{2_*}}{|y|^s} dx$ and necessarily all atoms of ν must lie in A, that is $x_j \in A$ for all $j \in J$. Finally, the proof of lemma is supplemented. □

Step 3. Proof of Theorem 2.2.1.
We only consider the case $s > 0$. We are going to apply Lemma 2.2.5 to a minimizing sequence for (2.7), with the properties (i),(ii),(iii) of Lemma 2.2.5. We can also assume to satisfy (2.13) and (2.22). For the limiting measure ν notice that, (2.22) implies that, for any $\epsilon > 0$,

$$1 \geq \int_{\mathbb{R}^n} d\nu \geq \int_{\Omega_R(0)} d\nu \geq \limsup_n \int_{\Omega_R(0)} \frac{|u_n|^{2_*}}{|y|^s} dx \geq 1 - \epsilon$$

provided $R \geq R_\epsilon$. Then $\int_{\mathbb{R}^n} d\nu = 1$ and $\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx \leq 1$. By Lemma 2.2.5, we have that any possible atom for ν must lie on the subspace $y = 0$, and by (2.13), we find that $\nu_j \in [0, 1/2]$, for all $j \in J$. Since $1 < 2 < 2_*$, we derive the inequalities

$$\begin{aligned} S \geq \int_{\mathbb{R}^n} d\nu &\geq \int_{\mathbb{R}^n} |\nabla u|^2 dx + \sum_{j\in J} \mu_j \\ &\geq S\Big[\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx\Big)^{2/2_*} + \sum_{j\in J} \nu_j^{2/2_*}\Big] \\ &\geq S\Big[\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx + \sum_{j\in J} \nu_j\Big] \\ &= S\int_{\mathbb{R}^n} d\nu = S. \end{aligned}$$

Thus all inequalities above must reduce to equalities. In particular

$$\Big[\Big(\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx\Big)^{2/2_*} + \sum_{j\in J} \nu_j^{2/2_*}\Big] = \int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx + \sum_{j\in J} \nu_j$$

which for $2/2_* < 1$ implies that all terms above are forced to be equal to either 1 or 0. As their sum equals 1, only one of these terms is equal to 1 and the others must be 0. But since $0 \leq \nu_j \leq 1/2$, we only have the possibility that $\nu_j = 0$ for all j and $\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s}\, dx = 1$. Consequently $S = \int_{\mathbb{R}^n} |\nabla u|^2\, dx$, so u is the desired minimizer.

2.3 A variational approach for the problem (2.1)

Through the Sobolev inequality (2.5), M.Badiale and G.Tarantello in [3] derive a variational principle for the problem (2.1) in the Sobolev space $D^{1,2}(\mathbb{R}^3)$, provided ϕ satisfies (2.2), and $p \in [4, 6]$. Indeed, they prove:

Lemma 2.3.1. *For each $p \in [4, 6]$ there exists a constant C_p such that, for $u \in D^{1,2}(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3} \phi(r)\, |u|^p(x) dx \leq C_p\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{p/2}. \tag{2.30}$$

Proof. In view of (2.2), since $r\phi(r)$ is bounded, we can use (2.5), with $n = 3$, with $k = 2$, and $s = 1$, (so $2_* = 4$) to conclude

$$\int_{\mathbb{R}^3} \phi(r)|u|^4(x)dx = \int_{\mathbb{R}^3} r\phi(r)\frac{|u|^4}{r} dx \leq C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2.$$

Moreover, since $\phi \in L^\infty(\mathbb{R}^+)$, by the Sobolev embedding, we obtain

$$\int_{\mathbb{R}^3} \phi(r)|u|^6(x)dx \leq C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^3.$$

For $4 < p < 6$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \phi(r)|u|^p dx &= \int_{\mathbb{R}^3} \phi(r)^{\frac{p-4}{2}}|u|^{3(p-4)}\phi(r)^{\frac{6-p}{2}}|u|^{2(6-p)}dx \\ &\leq \left(\int_{\mathbb{R}^3} \phi(r)|u|^6\right)^{\frac{p-4}{2}}\left(\int_{\mathbb{R}^3} \phi(r)|u|^4\right)^{\frac{6-p}{2}} \\ &\leq C_1\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{3(p-4)}{2}} C_2\left(\int_{\mathbb{R}^3} \phi(r)|u|^4\right)^{6-p} \\ &= C_p\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{p}{2}}.\end{aligned}$$

□

Moreover, always in [3], the authors got the asymptotic behavior of solutions for the problem (2.32) by a result of Egnell [21].

Lemma 2.3.2. *Assume $p \in [4,6]$, ϕ verifying (2.2), and let $u \in D^{1,2}(\mathbb{R}^3)$, $u \geq 0$ be a weak solution of the equation $-\Delta u = \phi u^{p-1}$. There exists a constant positive C such that*

$$u(x) \leq \frac{C}{|x|} \tag{2.31}$$

as $|x| \to \infty$.

By Lemma 2.3.1 and Lemma 2.3.2, it follows that, for $p \in [4,6]$ the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} \phi(r)|u|^p\, dx$$

is well defined, Fréchet differentiable in $D^{1,2}(\mathbb{R}^3)$, and its critical points satisfy

$$\begin{cases} -\Delta u(x) & = \phi(r)|u|^{p-2}u \ \ in\ \mathbb{R}^3, \\ u(x) & > 0\ \ in\ \mathbb{R}^3 \\ & u \in D^{1,2}(\mathbb{R}^3) \end{cases} \tag{2.32}$$

Thus, every solution of (2.32), satisfies the finite-mass condition and it is solution of (2.1). In fact, by (2.3) and (2.31), it holds

$$\int_{\mathbb{R}^3} \phi(r)u^{p-1}dx \leq C\int_{\mathbb{R}}\int_0^\infty \frac{1}{r}\frac{1}{(1+r^2+z^2)^{\frac{p-1}{2}}}\, r\, dr dz$$
$$= C\int_{\mathbb{R}}\int_0^\infty \frac{1}{(1+r^2+z^2)^{\frac{p-1}{2}}} dr dz < +\infty.$$

for $p \in [4,6]$. So, for $p \in [4,6]$, the problem reduces to search non-negative critical points for I in $D^{1,2}(\mathbb{R}^3)$. Also, the problem (2.32) has the following variational formulation. Any extremal function for the minimization problem

$$\inf \{|\nabla u|^2 dx \mid u \in D^{1,2}(\mathbb{R}^3), \int_{\mathbb{R}^3} \phi(r)|u|^p dx = 1\} \tag{2.33}$$

yields a solution for (2.32). Moreover, to obtain solutions with cylindrical symmetry, it needs to consider the restriction of I over the subspace $D_c^{1,2}(\mathbb{R}^3) = \{u \in D^{1,2}(\mathbb{R}^3)$ *with cylindrical symmetry*$\}$. Let M and M_c be the Nehari manifold

$$M = \{u \in D^{1,2}(\mathbb{R}^3) \,:\, u \neq 0, \int_{\mathbb{R}^3} |\nabla u|^2 = \int_{\mathbb{R}^3} \phi(r)|u|^p\}.$$

$$M_c = \{u \in D_c^{1,2}(\mathbb{R}^3) \,:\, u \neq 0, \int_{\mathbb{R}^3} |\nabla u|^2 = \int_{\mathbb{R}^3} \phi(r)|u|^p\}.$$

It is easy to verify that the functional I is coercive and bounded from below on M, (respectively I_c on M), and

$$I(u) = (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \quad \forall u \in M$$

$$I_c(u) = (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}} \int_0^\infty |\nabla u|^2 \, r \, dr dz \quad \forall u \in M_c.$$

For any $u \in M, u \neq 0$,there exists a unique scalar λ given by

$$\lambda = \left(\frac{\int_{\mathbb{R}^3} |\nabla u|^2}{\int_{\mathbb{R}^3} \phi |u|^p} \right)^{\frac{1}{p-2}}$$

such that $\lambda u \in M$. Therefore if we define the best constant in (2.30)

$$S_\phi = \inf \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 dx \,:\, u \in D^{1,2}(\mathbb{R}^3), \int_{\mathbb{R}^3} \phi(r)|u|^p dx = 1 \right\} \tag{2.34}$$

$$S_{c,\phi} = \inf \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 dx \,:\, u \in D_c^{1,2}(\mathbb{R}^3), \int_{\mathbb{R}^3} \phi(r)|u|^p dx = 1 \right\} \tag{2.35}$$

as an immediate consequence, we have

Lemma 2.3.3. *S is achieved if and only if S_M is achieved (respectively $S_{\phi,c}$ is achieved if and only if S_{M_c} is achieved) and*

$$S_M = \inf_{u \in M} I(u) = \left(\frac{1}{2} - \frac{1}{p} \right) S_\phi^{\frac{p}{p-2}} \tag{2.36}$$

$$S_{M_c} = \inf_{u \in M} I(u) = \left(\frac{1}{2} - \frac{1}{p} \right) S_{\phi,c}^{\frac{p}{p-2}} \tag{2.37}$$

Remark 2.3.1. The particular case is when $p = 4$. In this situation, we obtain the nonexistence for the minimization problems (2.32) and (2.33) also for the function $\phi(r)$ of type (2.2). We note that Bertin and Ciotti model, $\phi(r)$ is asymptotic to $1/r$ when $r \to \infty$, so the problem (2.1) admits the limit problem

$$\begin{cases} -\Delta u(x) & = \frac{1}{r}\, u^3 \;\; in \;\; \mathbb{R}^3, \\ \quad u(x) & > 0 \;\; in \;\; \mathbb{R}^3 \\ & u \in D^{1,2}(\mathbb{R}^3) \end{cases} \tag{2.38}$$

which has a solution, given by the extremals of (2.5) inequality when $n = 3, k = 2, s = 1$.

2.4 Existence and nonexistence results

In [3], it is proved the following theorem:

Theorem 2.4.1. *Let ϕ satisfy (2.3).*
i) If $4 < p < 6$ then I (or I_c) attains its infimum over M (respectively on M_c) at a solution u (respectively, a cylindrically symmetric solution u_c) for (2.1). These solutions are also extremals for the best constant in (2.30) over $D^{1,2}(\mathbb{R}^3)$ (respectively $D_c^{1,2}(\mathbb{R}^3)$)). Furthermore, there exists $p_0 \in (4, 6)$ such that, if $p_0 < p < 6$, then $S_\phi < S_{\phi,c}$; therefore u and u_c define two different solution for (2.1) in this case, and the best constant in (2.30) is attained at a function which is not cylindrically symmetric.
ii) If $p = 6$, then I cannot attain its infimum over M. On the contrary, I_c attains its infimum over M_c at a cylindrically solution for (2.1) which corresponds to an extremal for the best constant in (2.30) over $D_c^{1,2}(\mathbb{R}^3)$.
iii) If $p = 4$ and the function $r \to r\phi(r)$ is assumed increasing and not constant then neither I or I_c can attain their infimum over M and M_c respectively.

M.Badiale and E.Serra in [2] continue the work begun in [3] and obtained several results for a more general problem in $\mathbb{R}^n$, with $n \geq 3$, given by:

$$\begin{cases} -\Delta u(x) & = \phi(|y|)\, u^{2_*-1} \;\; in \;\; \mathbb{R}^n, \\ \quad u(x) & > 0 \;\; in \;\; \mathbb{R}^n \\ & u \in D^{1,2}(\mathbb{R}^n) \end{cases} \tag{2.39}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $k \geq 2$, $0 < s < 2$, $2_* = 2\frac{n-s}{n-2}$. We denote $|y|$ by r and suppose that the function $\phi(r)$ is asymptotic, at 0 or at ∞ (or both) to $1/r^s$. Then, the problem (2.39) admits the problem (2.8) as limit problem, invariant under transformation (2.9). The hypotheses on ϕ are the following. For some $\eta \in (0, 1)$ and $s \in (0, 2)$

$$\begin{gathered} \phi \in C^{0,\eta}_{loc}(\mathbb{R}_+, \mathbb{R}^+), \quad \phi(r) r^s \in L^\infty(\mathbb{R}_+), \\ \textit{and at last one between } \lim_{r\to 0} \phi(r) r^s = 1 \textit{ and } \lim_{r\to\infty} \phi(r) r^s = 1 \textit{ holds.} \end{gathered} \tag{2.40}$$

$$\limsup_{r\to 0,+\infty} \phi(r)r^s \le 1. \tag{2.41}$$

The Theorem 2.0.2 implies that in the space $D^{1,2}(\mathbb{R}^n)$ the integral $\int_{\mathbb{R}^n} \frac{1}{|y|^s}|u|^{2_*}dx$ is finite, so, for $u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}$, we can define

$$J(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{(\int_{\mathbb{R}^n} \frac{1}{|y|^s}|u|^{2_*}dx)^{2/2_*}}, \qquad \textit{and} \qquad J_\phi(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{(\int_{\mathbb{R}^n} \phi(|y|)|u|^{2_*}dx)^{2/2_*}}$$

We can see that J and J_ϕ are C^1 functionals over $D^{1,2}(\mathbb{R}^n)$. We also define

$$S = \inf\{J(u) \,:\, u \in D^{1,2}(\mathbb{R}^n)\backslash\{0\}\} \; \textit{and} \; S_\phi = \inf\{J_\phi(u) \,:\, u \in D^{1,2}(\mathbb{R}^n)\backslash\{0\}\}$$

and we consider the minimization problem

$$\textit{find } u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}, u \ge 0, \;\; \textit{such that} \;\; J_\phi(u) = S_\phi. \tag{2.42}$$

Badiale and Serra in [2] investigate the regularity and positivity of solution for (2.39) in some particular cases and get the following results.

Lemma 2.4.2. *Assume (2.40) and let $u \in D^{1,2}(\mathbb{R}^n), u \ge 0$, be a weak solution of the equation*

$$-\Delta u(x) = \phi(|y|)u^{2_*-1}$$

If $s\,n < 4$, then $u \in C^{0,\eta}_{loc}(\mathbb{R}^n)$ for some $\eta \in (0,1)$. If $s\,n < 2$, then $u \in C^{1,\theta}_{loc}(\mathbb{R}^n)$ for some $\eta \in (0,1)$.

Proof. We first claim that $u \in L^q_{loc}(\mathbb{R}^n)$ for all $q < +\infty$. We write $\phi(|y|)u^{2_*-1} = \phi(|y|)\frac{u^{2_*-1}}{1+u}(1+u)$, and we set

$$a(x) = \phi(|y|)\frac{u^{2_*-1}}{1+u},$$

so that u satisfies

$$-\Delta u(x) = a(x)(1+u).$$

For Lemma B.3 in the book of Struwe [50], it is enough to prove that $a \in L^{n/2}(\mathbb{R}^n)$. We rewrite

$$a(x) = \phi(|y|)|y|^s \frac{u^{2_*-2}}{|y|^s}\frac{u}{1+u}$$

Of course, the quantity $\frac{u}{1+u}$ is bounded (as $u \ge 0$), while $\phi(|y|)|y|^s$ is bounded by (2.40). Hence we have to prove that

$$\frac{u^{2_*-2}}{|y|^s} \in L^{n/2}, \;\; \textit{that is}, \; \frac{u^{(2_*-2)n/2}}{|y|^{ns/2}} \in L^1.$$

To show this, we notice that since $s\,n < 4$, so that $n\,s/2 < 2 < k$. We can use the Theorem 2.0.2, with $n\,s/2$. Indeed, we have

$$(2_* - 2)\frac{n}{2} = \frac{2(n - n\,s/2)}{n-2}.$$

In this way, we have proved the claim. Moreover, if $s\,n < 4$ we can take some $p > \frac{n}{2}$ such that $sp < 2$; since $k \geq 2$, we see that $(\frac{1}{|y|^{sp}})$ is locally integrable. This implies that $\frac{u^{2_*-1}}{|y|^s}$ is locally in $L^q(\mathbb{R}^n)$ for some $p > q > \frac{n}{2}$. By the usual elliptic regularity theory and the Sobolev embedding, it follows that $u \in C^{0,\theta}_{loc}(\mathbb{R}^n)$ for some $\theta \in (0,1)$. Finally, if $s\,n < 2$ we can repeat the same argument using some $p > n$ such that again $s\,p < 2$, to obtain that $u \in C^{1,\theta}_{loc}(\mathbb{R}^n)$ for some $\theta \in (0,1)$. □

Lemma 2.4.3. *Assume (2.40) and let* $u \in D^{1,2}(\mathbb{R}^n), u \geq 0$, *be a weak solution of the equation*

$$-\Delta u(x) = \phi(|y|)u^{2_*-1}$$

If $s\,n < 4$, *then* $u > 0$ *in* $\mathbb{R}^n$.

Proof of Lemma 2.4.3. We define $A = \{x = (y,z) \in \mathbb{R}^n \,:\, y \neq 0\}$. By standard elliptic regularity, $u \in C^{2,\eta}_{loc}(A)$ for some $\eta \in (0,1)$; hence we can apply the classical strong maximum principle to obtain that $u > 0$ in A. Moreover, by Lemma 2.4.2, $u \in C^{0,\theta}_{loc}$, for some $\theta \in (0,1)$. Let $x_0 \in \mathbb{R}^n$be such that $y_0 = 0$ and consider the ball $B = B_1(x_0)$. We define $\tilde{\phi}(r) = \min\{1, \phi(r)\}$ and we remark that $\tilde{\phi}$ is Hölder continuous, because so is ϕ. Let v be the classical solution of the problem

$$\begin{cases} -\Delta v(x) &= \tilde{\phi}(|y|)\,u^{2_*-1} \;\; in \;\; B, \\ \quad v(x) &= 0 \;\; on \;\; \partial B. \end{cases} \tag{2.43}$$

We notice that $\tilde{\phi}(|y|)\,u^{2_*-1}$ is Hölder continuous and now $-\Delta u \geq -\Delta v$ in the weak sense in B, and $u \geq v$ on ∂B, so that, using the maximum principle for weak solutions, we have $u \geq v$ in all B. But v is a classical solution of (2.43), $-\Delta v \geq 0$ and v is not a constant, so by the strong maximum principle we have $v > 0$ in all of B and hence $u(x_0) > 0$. □

The first nonexistence result is a consequence of the following identity of Pohozaev type (see Proposition 2.5 in [2]).

Proposition 2.4.4. *Let* $a : \mathbb{R}^n \to \mathbb{R}$ *be a function such that*

$$a(x)|y|^s \in L^\infty(\mathbb{R}^n), \;\; with \;\; a \in C(A), \tag{2.44}$$

and let $u \in D^{1,2}(\mathbb{R}^n)$ *be a weak solution of*

$$-\Delta u(x) = a(x)|u|^{p-2}u \;\; in \;\; \mathbb{R}^n. \tag{2.45}$$

Assume that $u \in C^{1,\theta}_{loc}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap C^2(A)$ *for some* $\theta \in (0,1)$ *and also that*

$$a|u|^p \in L^1(\mathbb{R}^n) \ \ and \, \nabla a(x) \cdot x|u|^p \in L^1(\mathbb{R}^n). \tag{2.46}$$

Then the following identity holds

$$\int_{\mathbb{R}^n} \left[\left(\frac{n-2}{2} - \frac{n}{p} \right) a(x) - \frac{1}{p} \nabla a(x) \cdot x \right] |u|^p dx = 0 \tag{2.47}$$

Applying (2.47) to problem (2.39), with the suitable limitations on the values of s, n, we get

Corollary 2.4.5. *Assume that* $s\,n < 2$, *that* ϕ *satisfies (2.40) and also that*

$$\phi' \in C(\mathbb{R}_+) \ \ and \ \ \phi'(r) r^{s+1} \in L^\infty(\mathbb{R}_+). \tag{2.48}$$

Define $\psi(r) = \phi(r) r^s$ *and assume that* ψ *is monotone (increasing or decreasing) and not constant. Then the problem (2.39) has no solutions.*

Proof. We apply Proposition 2.4.4 with $p = 2_*$ and $a(x) = \phi(|y|)$. We note that $\nabla a(x) \cdot x = \phi'(|y|)|y|$, so that (2.40), (2.48) and Theorem 2.0.2 imply that the hypotheses of Proposition 2.4.4 are satisfied. So, if we assume that u is a solution of (2.39), by Lemma 2.4.2 we have $u \in C^{1,\theta}_{loc}(\mathbb{R}^n)$ and by Lemma 2.4.3 we obtain that $u > 0$ everywhere. Moreover, computing

$$\begin{aligned}
&\left(\frac{n-2}{2} - \frac{n}{2_*} \right) a(x) - \frac{1}{2_*} \nabla a(x) \cdot x = -\frac{s}{2_*} a(x) - \frac{1}{2_*} \nabla a(x) \cdot x \\
&= -\frac{1}{2_*} (s a(x) + \nabla a(x) \cdot x) = -\frac{1}{2_*} (s\phi(|y|) + \phi'(|y|)|y|) \\
&= -\frac{1}{2_*} \frac{1}{|y|^{s-1}} \left(s|y|^{s-1} \phi(|y|) + \phi'(|y|)|y|^s \right) = -\frac{1}{2_* |y|^{s-1}} \psi'(|y|),
\end{aligned}$$

we see that (2.47) gives

$$0 = \int_{\mathbb{R}^n} \frac{1}{|y|^{s-1}} \, \psi'(|y|) \, |u|^{2_*} \, dx. \tag{2.49}$$

Under the hypothesis that ψ is monotone and not constant we obtain that ψ' is not zero and has constant sign. Therefore (2.47) gives a contradiction and this implies that (2.39) has no solutions. □

A natural question is the following: there does a $\phi \neq \frac{1}{r^s}$ exist for which the infimum in (2.33), when $p = q_*(s) = q_*$, is achieved? The answer was given by K. Sandeep in [45], who discussed the existence and nonexistence of minimizer for the constraint minimization problem

$$S_\phi = \inf_{u \in D^{1,q}(\mathbb{R}^n)} \{ \int_{\mathbb{R}^n} |\nabla u|^q \ : \ \int_{\mathbb{R}^n} \phi(r) |u|^{q_*} dx = 1 \}, \tag{2.50}$$

where $r = |y|$, $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}, 2 \leq k \leq n, 0 \leq s < q, s < k, 1 < q < n$, $q_* = \frac{q(n-s)}{n-q}$. For the sake of simplicity, we will consider only the case $q = 2$. We will denote $I(u)$ the functional given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{2_*} \int_{\mathbb{R}^n} \phi(r)|u|^{2_*}.$$

Notice that its critical points satisfy the problem (2.39). In [45] the following theorem is proved.

Theorem 2.4.6. *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and satisfies* $\lim_{r \to 0} r^s \phi(r) = C_0$ *and* $\lim_{r \to \infty} r^s \phi(r) = C_\infty$, *where* $0 \leq C_0 < \infty$ *and* $0 \leq C_\infty < \infty$, *then*

i)

$$S_\phi \leq \min\{\frac{S}{C_0^{\frac{2}{2_*}}}, \frac{S}{C_\infty^{\frac{2}{2_*}}}\} \tag{2.51}$$

ii) and S_ϕ *is achieved if* S_ϕ *satisfies*

$$S_\phi < \min\{\frac{S}{C_0^{\frac{2}{2_*}}}, \frac{S}{C_\infty^{\frac{2}{2_*}}}\} \tag{2.52}$$

iii) If ϕ *satisfies in addition* $r^s \phi(r) \leq \max\{C_0, C_\infty\}$ *for all* r, *then*

$$S_\phi = \min\{\frac{S}{C_0^{\frac{2}{2_*}}}, \frac{S}{C_\infty^{\frac{2}{2_*}}}\}$$

and S_ϕ *is achieved only when* $\phi(r) r^s \equiv \max\{C_0, C_\infty\}$.

Proof. (i) By Theorem 2.0.2, we know that S is achieved at some $w \in D^{1,2}(\mathbb{R}^n)$. We define, for $\lambda > 0, w_\lambda(x) = \lambda^{\frac{n-2}{2}} w(\lambda x)$. Then, by a change of variables, $J(w_\lambda) = J(w) = S$, while

$$J_\phi(w_\lambda) = \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^n} \phi(\frac{|y|}{\lambda}) \frac{|y|^s}{\lambda^s} \frac{|w|^{2_*}}{|y|^s} \, dx)^{2/2_*}}.$$

Since $\frac{|w|^{2_*}}{|y|^s} \in L^1(\mathbb{R}^n)$, it is easy to check that $J_\phi(w_\lambda) \to \frac{S}{C_\infty^{2/2_*}}$ as $\lambda \to 0$. On the other hand, by definition, we have that $S_\phi \leq J_\phi(w_\lambda)$ for all $\lambda > 0$, therefore for $\lambda \to 0$, we have $S_\phi \leq \frac{S}{C_\infty^{2/2_*}}$. Analogously, letting $\lambda \to +\infty$, we have $S_\phi \leq \frac{S}{C_0^{2/2_*}}$. Thus we get

$$S_\phi \leq \min\{\frac{S}{C_0^{2/2_*}}, \frac{S}{C_\infty^{2/2_*}}\}.$$

□

To prove (ii), we need some preliminaries. We denote the sets $\{(y,z) \in \mathbb{R}^n : r_0 \le |y| \le r_1\}$, $\{(y,z) \in \mathbb{R}^n : |y| \le r_0\,(\ge r_1)\}$ by $\{r_0 \le r \le r_1\}$ and $\{r \le r_0\}\,(\ge r_1)$ respectively. We recall a well-known application of Ekeland's principle to select the best minimizing sequence for a functional over the corresponding Nehari' manifold.

Lemma 2.4.7. *There exists a sequence $\{u_h\}_h \in M$ such that $I(u_h) \to S_M$ and $I^{'}(u_h) \to 0$ where $I^{'}$ denotes the Frechet derivative of I.*

Proceeding in the same lines of Badiale and Tarantello, in [45] the compactness properties of the minimizing sequence are discussed. The first step is to show that the assumption (ii) prevents "vanishing" of the minimizing sequence in the y-direction .

Lemma 2.4.8. *Let $S_\phi < \min\{\frac{S}{C_0^{2/2_*}}, \frac{S}{C_\infty^{2/2_*}}\}$ and $\{u_h\}_h$ be a minimizing sequence as in above Lemma, then there exist r_0, r_1 with $0 < r_0 < \infty$, $0 < r_1 < \infty$ and a constant $K_0 > 0$ such that*

$$\liminf_{n\to\infty} \int_{r_0 \le r \le r_1} \phi(r)|u_h|^{2_*}\, dx \ge K_0. \tag{2.53}$$

Proof of Lemma 2.4.8.
Let C be such that $C = \max\{C_0^{2/2_*}, C_\infty^{2/2_*}\}$, then we have $S_\phi < \frac{S}{C}$ and hence, by Lemma 2.3.3, we have

$$S_M < \left(\frac{1}{2} - \frac{1}{2_*}\right)\left(\frac{S}{C}\right)^{\frac{2_*}{2_*-2}} \tag{2.54}$$

We will prove our lemma by contradiction. For absurd, we suppose that for all $r_0, r_1 > 0$

$$\liminf_{n\to\infty} \int_{r_0 \le r \le r_1} \phi(r)|u_h|^{2_*}\, dx = 0.$$

We claim that this contradict (2.54). Let $\epsilon > 0$. Then, we can find $r_0, r_1 > 0$ such that

$$\Big| \int_{r \le r_0} \left(\phi(r) - \frac{C_0}{r^s}\right)|u_h|^{2_*}\, dx + \int_{r \ge r_1} \left(\phi(r) - \frac{C_\infty}{r^s}\right)|u_h|^{2_*}\, dx \Big| < \epsilon. \tag{2.55}$$

uniformly for all h. By our assumptions, passing to a subsequence if necessary, we can assume that

$$\lim_{h\to\infty} \int_{r_0 \le r \le r_1} \phi(r)|u_h|^{2_*}\, dx = 0. \tag{2.56}$$

Now, by (2.55), (2.56), we get

$$\int_{\mathbb{R}^n} \phi(r)|u_h|^{2_*}\,dx = \int_{r\le r_0} \phi(r)|u_h|^{2_*}\,dx + \int_{r_0<r<r_1} \phi(r)|u_h|^{2_*}\,dx + \int_{r\ge r_1} \phi(r)|u_h|^{2_*}\,dx$$
$$\le C_0 \int_{r\le r_0} \frac{|u_h|^{2_*}}{r^s}\,dx + o(1) + C_\infty \int_{r\ge r_1} \frac{|u_h|^{2_*}}{r^s}\,dx + \epsilon$$
$$\le \max\{C_0, C_\infty\} \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx + \epsilon + o(1) \tag{2.57}$$

where $o(1)$ denotes the terms which go to zero as $h \to \infty$. Since $u_h \in M$, we have

$$1 = \left(\frac{\int_{\mathbb{R}^n} |\nabla u|^2\,dx}{\int_{\mathbb{R}^n} \phi(r)|u_h|^{2_*}\,dx}\right)^{\frac{2}{2_*-2}} \ge \left(\frac{\int_{\mathbb{R}^n} |\nabla u|^2\,dx}{C^{\frac{2_*}{2}} \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx + \epsilon + o(1)}\right)^{\frac{2}{2_*-2}}$$

Therefore

$$S_M = \lim_{h\to\infty} \left(\frac{1}{2} - \frac{1}{2_*}\right) \int_{\mathbb{R}^n} |\nabla u_h|^2\,dx$$
$$\ge \lim_{h\to\infty} \left(\frac{1}{2} - \frac{1}{2_*}\right) \frac{\left(\int_{\mathbb{R}^n} |\nabla u_h|^2\,dx\right)^{\frac{2}{2_*-2}+1}}{\left(C^{\frac{2_*}{2}} \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx + \epsilon + o(1)\right)^{\frac{2}{2_*-2}}}$$
$$= \lim_{h\to\infty} (\frac{1}{2} - \frac{1}{2_*}) \left(\frac{\int_{\mathbb{R}^n} |\nabla u_h|^2\,dx}{C(\int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx)^{\frac{2}{2_*}}}\right)^{\frac{2_*}{2_*-2}} \left(1 + \frac{\epsilon + o(1)}{C^{\frac{2_*}{2}} \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx}\right)^{\frac{-2}{2_*-2}}$$
$$\ge \left(\frac{1}{2} - \frac{1}{2_*}\right) \left(\frac{S}{C}\right)^{\frac{2_*}{2_*-2}} \lim_{h\to\infty} \left(1 + \frac{\epsilon + o(1)}{C^{\frac{2_*}{2}} \int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx}\right)^{\frac{-2}{2_*-2}} \tag{2.58}$$

Since $r^s\phi(r)$ is bounded, we have

$$\int_{\mathbb{R}^n} \frac{|u_h|^{2_*}}{r^s}\,dx \ge K \int_{\mathbb{R}^n} \phi(r)|u_h|^{2_*}\,dx = K \int_{\mathbb{R}^n} |\nabla u_h|^2\,dx \to K\,\frac{22_*}{2_*-2}\,S_M > 0$$

where K is a positive constant. Therefore by taking $h \to \infty$ and $\epsilon \to 0$ in (2.58), we get

$$S_M \ge \left(\frac{1}{2} - \frac{1}{2_*}\right) \left(\frac{S}{C}\right)^{\frac{2_*}{2_*-2}}.$$

This contradicts (2.55) and hence the lemma follows. □

Fix an r_0 and r_1 satisfying (2.55) and let Q denotes

$$Q = \{(y,z) \in \mathbb{R}^n \,:\, r_0 \le r \le r_1, z \in [0,1]^{n-k}\}$$

and

$$Q_j = \{(y,z) \in \mathbb{R}^n \,:\, r_0 \le r \le r_1 \,:\, z \in \xi_j + [0,1]^{n-k}\}$$

where $\xi_j \in \mathbb{N}^{n-k}$. Now by translating the minimizing sequence $\{u_h\}$ in the z direction, if necessary, we can assume that the minimizing sequence guaranteed by Lemma 2.4.9 satisfies in addition

$$\sup_j \int_{Q_j} \phi(r)|u_h|^{2_*}\,dx = \int_Q \phi(r)|u_h|^{2_*}\,dx \tag{2.59}$$

The second step is to extend the last lemma in the z-direction.

Lemma 2.4.9. *Let $\{u_h\}_h$ be a minimizing sequence as in Lemma 2.4.9, and satisfying (2.59) and r_0, r_1 chosen as in Lemma 2.4.8, then*

$$\liminf_{h\to\infty} \int_Q \phi(r)|u_h|^{2_*}\,dx > 0 \tag{2.60}$$

Proof. Let $s_1 > 0$ be such that $s < s_1 < 2$. We define

$$2_*^{'} = 2_*(s_1) = \frac{2(n-s_1)}{n-2}.$$

Then $2 < 2_*^{'} < 2_*$ and by Theorem 2.0.2 there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{|u|^{p_*^{'}}}{r^{s_1}}\,dx \le C\left(\int_{\mathbb{R}^n} |\nabla u|^2\,dx\right)^{\frac{n-s_1}{n-2}} \tag{2.61}$$

holds for $u \in D^{1,2}(\mathbb{R}^n)$. Now by using (2.59) and the boundedness of u_h in $D^{1,2}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left(\int_{Q_j} \phi(r)|u_h|^{2_*}\,dx\right)^{\frac{2_*^{'}}{2_*}} &\le K\left(\int_{Q_j} |u_h|^{2_*}\,dx\right)^{\frac{2_*^{'}}{2_*}} \\
&= K\left(\int_{Q} |u_h(x+\xi_j)|^{2_*}\,dx\right)^{\frac{2_*^{'}}{2_*}} \\
&\le K\left(\left(\int_Q |\nabla u_h(x+\xi_j)|^2\,dx\right)^{1/2} + \left(\int_Q |u_h(x+\xi_j)|^{2_*^{'}})\,dx\right)^{\frac{1}{2_*^{'}}}\right)^{2_*^{'}} \\
&\le K\left(\left(\int_{Q_j} |\nabla u_h(x)|^2\,dx\right)^{\frac{2_*^{'}}{2}} + \int_{Q_j} |u_h(x)|^{2_*^{'}}\,dx\right) \\
&\le K\left(\int_{Q_j} |\nabla u_h(x)|^2\,dx + \left(\int_{Q_j} |u_h(x)|^{2_*^{'}}\,dx\right)\right. \\
&\le K\left(\int_{Q_j} |\nabla u_h(x)|^2\,dx + \left(\int_{Q_j} \frac{|u_h(x)|^{2_*^{'}}}{r^{s_1}}\,dx\right)\right.
\end{aligned} \tag{2.62}$$

where K stands for a positive constant independent of n, k. Now using (2.59), (2.61) and (2.62), we get

$$\begin{aligned}
\int_{r_0\le r\le r_1} \phi(r)|u_h|^{2_*}\,dx &= \sum_{i=1}^{\infty} \int_{Q_j} \phi(r)|u_h|^{2_*}\,dx \\
&= \sum_{i=1}^{\infty} \Big(\int_{Q_j} \phi(r)|u_h|^{2_*}\,dx\Big)^{\frac{2_*'}{2_*}} \Big(\int_{Q_j} \phi(r)|u_h|^{2_*}\,dx\Big)^{1-\frac{2_*'}{2_*}} \\
&\le K \Big(\int_{Q} \phi(r)|u_h|^{2_*}\,dx\Big)^{1-\frac{2_*'}{2_*}} \Big(\int_{\mathbb{R}^n} |\nabla u_h|^2\,dx + \int_{\mathbb{R}^n} \frac{|u_h|^{2_*'}}{r^{s_1}}\,dx\Big) \\
&\le K \Big(\int_{Q} \phi(r)|u_h|^{2_*}\,dx\Big)^{1-\frac{2_*'}{2_*}} \qquad (2.63)
\end{aligned}$$

The conclusion follows immediately by Lemma 2.4.8. □

Now we are ready to prove the part (ii) of the Theorem 2.4.6.

Proof of (ii).

Let S_ϕ satisfy (ii). Then from previous Lemmas, we can find a sequence $\{u_h\}_h \in M$ such that $I(u_h) \to S_M, I'(u_h) \to 0$ and

$$\lim_{h\to\infty} \int_Q \phi(r)|u_h|^{2_*}\,dx > 0.$$

Then $\{u_h\}_h$ is a bounded sequence in $D^{1,2}(\mathbb{R}^n)$ and therefore by passing to a subsequence if necessary we can assume that $\{u_h\}_h$ converges weakly to say $u_0 \in D^{1,2}(\mathbb{R}^n), u_h \to u_0$ in $L^{2_*}_{loc}(\mathbb{R}^n)$ and $u_h \to u_0$ pointwise almost everywhere. Hence

$$\int_Q \phi(r)|u_0|^{2_*}\,dx = \lim_{h\to\infty} \int_Q \phi(r)|u_h|^{2_*}\,dx > 0.$$

So, $u_0 \neq 0$, and since $I'(u_h) \to 0$, we know that u_0 satisfies the equation

$$-\Delta u_0 = \phi(r)|u_0|^{2_*-2}u_0 \, in \, \mathbb{R}^n$$

i.e.

$$\int_{\mathbb{R}^n} |\nabla u_0|^2\,dx = \int_{\mathbb{R}^n} \phi(r)|u_0|^{2_*}\,dx$$

i.e. $u_0 \in M$. Consequently, using the weak lower semicontinuity of the norm in $D^{1,2}(\mathbb{R}^n)$, we get

$$I(u_0) = \Big(\frac{1}{2} - \frac{1}{2_*}\Big) \int_{\mathbb{R}^n} |\nabla u_0|^2\,dx \le \liminf_{h\to\infty} \Big(\frac{1}{2} - \frac{1}{2_*}\Big) \int_{\mathbb{R}^n} |\nabla u_h|^2\,dx = \lim_{h\to\infty} I(u_h) = S_M.$$

Thus, $\lim_{h\to\infty}||u_h||^2 = ||u_0||^2$, that is $\{u_h\}_h$ converges strongly to u_0 in $D^{1,2}(\mathbb{R}^n)$ and $I(u_0) = \inf_M I(u) = S_M$. □

Proof of (iii).

From our assumptions we have that

$$\phi(r) \le \max\{C_0, C_\infty\}\frac{1}{r^s}. \tag{2.64}$$

Thus from definition of S_ϕ and S, we get

$$S_\phi \ge \min\{\frac{S}{C_0^{2/2_*}}, \frac{S}{C_\infty^{2/2_*}}\} \tag{2.65}$$

and hence equality holds in (2.65). It is clear that when $r^s\phi(r) \equiv \max\{C_0, C_\infty\}$ then S_ϕ is achieved. So we assume that the inequality in (2.64) is strict at least at one point. Let us suppose that S_ϕ is achieved for such a ϕ say at $v \in D^{1,2}(\mathbb{R}^n)$. Then we can take $v \ge 0$ and by strong maximum principle of Vazquez [53], $v > 0$. Then

$$\begin{aligned} S_\phi &= \frac{\int_{\mathbb{R}^n}|\nabla v|^2\,dx}{(\int_{\mathbb{R}^n}\phi(r)|v|^{2_*}\,dx)^{2/2_*}} \\ &> \frac{\int_{\mathbb{R}^n}|\nabla v|^2\,dx}{(\max\{C_0, C_\infty\}\int_{\mathbb{R}^n}\frac{|v|^{2_*}}{r^s}\,dx)^{2/2_*}} \ge \min\{\frac{S}{C_0^{2/2_*}}, \frac{S}{C_\infty^{2/2_*}}\} = S_\phi \end{aligned}$$

which is a contradiction. Hence S_ϕ is not achieved. This completes the proof oh Theorem 2.4.6. □

The research of a condition for ϕ which guarantees a solution to the problem (2.39) was studied also by Badiale and Serra in [2] by Concentration-Compactness principle, having analogous results. Here there is a list.

Theorem 2.4.10. *(i) Assume that ϕ satisfies (2.40) and also that*

$$\phi(r)r^s \le 1 \forall r > 0 \;\; and \;\; \phi(r_0)r_0^s < 1 \;\; for \;\; some\, r_0 > 0. \tag{2.66}$$

(ii) Assume that ϕ satisfies (2.40) and (2.41) and $S_\phi < S$.

(iii) Assume that ϕ satisfies (2.40) and

$$\lim_{r\to 0}\phi(r)r^s = \lim_{r\to+\infty} = 1 \;\; with\, \phi(r)r^s \ge 1 \;\; for \;\; all\, r > 0. \tag{2.67}$$

(iv) Assume that $s\,n < 4$ and that in addition to (2.40) and (2.41), ϕ satisfies also $\lim_{r\to+\infty}\phi(r)r^s = 1$ and

$$\exists r_0 > 0, \exists \beta \in (0, k-s) \;\; such \;\; that\, \phi(r)r^s \ge 1 + \frac{1}{r^\beta} \;\; for \;\; all\, r \ge r_0.$$

Then, in the case (i), the problem (2.42) has no positive solutions, while in the cases (ii),(iii),(iv) the problem (2.42) has a solution.

Proof. The proof of (i),(ii) is the same as in Theorem 2.4.6.
iii) Let $\phi \in C^{0,\eta}_{loc}(\mathbb{R}_+, \mathbb{R}^+), \phi(r)r^s \in L^\infty(\mathbb{R}_+)$ and $\lim_{r\to 0}\phi(r)r^s = \lim_{r\to\infty}\phi(r)r^s = 1$, with $\phi(r)r^s \geq 1$ for all $r > 0$. If $\phi(r)r^s \equiv 1$, then the problem reduces to (2.8) and of course $S = S_\phi$. The existence of a solution has been proved in [3]. Otherwise there exists a $r_1 > 0$ such that

$$\phi(r_1)r_1^s > 1. \tag{2.68}$$

Let $w \in D^{1,2}(\mathbb{R}^n), w > 0$ be such that

$$\int_{\mathbb{R}^n} |\nabla w|^2 dx = S, \quad and \quad \int_{\mathbb{R}^n} \frac{1}{|y|^s}|w|^{2_*}dx = 1 \tag{2.69}$$

Hence, by (2.67) and (3.67) we see that

$$\int_{\mathbb{R}^n} \phi(|y|)|w|^{2_*}dx > \int_{\mathbb{R}^n} \frac{1}{|y|^s}|w|^{2_*}dx = 1,$$

which implies that

$$S_\phi \leq \frac{\int_{\mathbb{R}^n} |\nabla w|^2\, dx}{(\int_{\mathbb{R}^n} \phi(|y|)|w|^{2_*}dx)^{2/2_*}} < S$$

The conclusion follows then from Theorem 2.4.6. □

(iv) In order to prove (iv), we need some estimates on the decay of w, the solution of problem (2.8). We recall this following result, due to Egnell (see [21]).

Theorem 2.4.11. *Let $u \in D^{1,2}(\mathbb{R}^n), u \geq 0$ be a weak solution of the equation*

$$-\Delta u = f(x, u),$$

where $0 \leq f(x,u) \leq b(x)u^\sigma$ and

$$1 < \sigma < \frac{n+2}{n-2}, b \in L^\tau(\mathbb{R}^n), \tau = \frac{2n}{n+2-(n-2)\sigma}.$$

Then $\limsup_{|x|\to+\infty} |x|^{n-2}u(x) < +\infty$.

By an application of Theorem 2.4.11, Badiale and Serra in [2] prove this result.

Lemma 2.4.12. *Let w be a solution of problem (2.8) and assume that $s\,n < 4$. There exists $C > 0$ such that*

$$w(x) \leq \frac{C}{|x|^{n-2}} \quad as \quad |x| \to +\infty.$$

Proof. We want to show that the hypotheses of Theorem 2.4.11 are verified. Fix some $\sigma \in (1, \frac{n+2}{n-2})$ very close to 1 and write the equation as $-\Delta w = b(x)w^\sigma$, where $b(x) = \frac{1}{|y|^s} w^{2_*-1-\sigma}$. We want to prove that $b \in L^\tau(\mathbb{R}^n)$, where

$$\tau = \frac{2n}{n+2-\sigma(n-2)}$$

as in Theorem 2.4.11, that is

$$\int_{\mathbb{R}^n} b(x)^\tau \, dx = \int_{\mathbb{R}^n} (\frac{1}{|y|^s} w^{2_*-1-\sigma})^\tau \, dx < +\infty. \tag{2.70}$$

But

$$\int_{\mathbb{R}^n} (\frac{1}{|y|^s} w^{2_*-1-\sigma})^\tau \, dx = \int_{\mathbb{R}^n} w^{(2_*-1-\sigma)\tau} \frac{1}{|y|^{s\tau}} \, dx. \tag{2.71}$$

Applying the (2.5) with $a = s\tau$, to the second member of (2.71), we have:

$$\int_{\mathbb{R}^n} w^{(2_*-1-\sigma)\tau} \frac{1}{|y|^{s\tau}} \, dx = \int_{\mathbb{R}^n} \frac{1}{|y|^a} w^{2_*(a)} \, dx < +\infty.$$

We just have to check that $s\tau < 2$ and the exponents,

$$2_*(s\tau) = \frac{2(n-s\tau)}{n-2} = (2_*-1-\sigma)\tau. \tag{2.72}$$

Now since $s < 4/n$ and τ is as close as we wish to $n/2$, by taking σ sufficiently close to 1, we see that $s\tau < 2$. By computations, we have (2.72). □

Proof of (iv). We want to prove that $S_\phi < S$ and apply Theorem 2.4.6. Let w verify (2.69) and define $w_\lambda(x) = \lambda^{\frac{n-2}{2}} w(\lambda x)$, where $x = (y, z)$. We compute, setting $x' = \lambda x$,

$$\int_{\mathbb{R}^n} \phi(|y|) w_\lambda^{2_*}(x) dx = \int_{\mathbb{R}^n} \phi\left(\frac{|y'|}{|\lambda|}\right) \frac{1}{\lambda^s} w(x')^{2_*} dx' = \int_{\mathbb{R}^n} \phi\left(\frac{|y'|}{|\lambda|}\right) \frac{|y'|^s}{\lambda^s} \frac{1}{|y'|^s} w(x')^{2_*} dx'$$
$$= \int_{\mathbb{R}^n} \frac{1}{|y'|^s} w^{2_*} \, dx' + \int_{\mathbb{R}^n} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} \, dx' = 1 + \int_{\mathbb{R}^n} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} \, dx'.$$

Let us study the last integral. We write it as

$$\int_{\mathbb{R}^n} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} dx'$$
$$= \int_{|y'|<\lambda r_0} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} dx' + \int_{|y'|\geq\lambda r_0} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} dx'$$

and we consider these two integrals separately. We have, by our assumptions,

$$\int_{|y'|\geq\lambda r_0} [\phi\left(|\frac{y'}{\lambda}|\right) \frac{|y'|^s}{\lambda^s} - 1] \frac{1}{|y'|^s} w^{2_*} dx' \geq \int_{|y'|\geq\lambda r_0} \frac{1}{|\frac{y'}{\lambda}|^\beta} \frac{1}{|y'|^s} w^{2_*} dx' = \lambda^\beta \int_{|y'|\geq\lambda r_0} \frac{1}{|y'|^{s+\beta}} w^{2_*} dx'.$$

When $\lambda \geq 1/r_0$ we obtain

$$\lambda^\beta \int_{|y'|\geq \lambda r_0} \frac{1}{|y'|^{s+\beta}} w^{2_*} dx' \geq \lambda^\beta \int_{|y'|\geq 1} \frac{1}{|y'|^{s+\beta}} w^{2_*} dx' = a\,\lambda^\beta,$$

where $a = \int_{|y'|\geq 1} \frac{1}{|y'|^{s+\beta}} w^{2_*} dx'$ is (positive) and finite since

$$\int_{|y'|\geq 1} \frac{1}{|y'|^{s+\beta}} w^{2_*} dx' \leq \int_{|y'|\geq 1} \frac{1}{|y'|^{s}} w^{2_*} dx' \leq \int_{\mathbb{R}^n} \frac{1}{|y'|^{s}} w^{2_*} dx' = 1.$$

Therefore we have proved that

$$\int_{|y'|\geq \lambda r_0} [\phi\Big(|\frac{y'}{\lambda}|\Big)\frac{|y'|^s}{\lambda^s} - 1]\frac{1}{|y'|^s} w^{2_*} dx' \geq a\,\lambda^\beta,$$

with $a > 0$. Concerning the other integral we obtain, writing $x' = (y'z')$,

$$\begin{aligned}&\left|\int_{|y'|\leq \lambda r_0} [\phi\Big(|\frac{y'}{\lambda}|\Big)\frac{|y'|^s}{\lambda^s} - 1]\frac{1}{|y'|^s} w^{2_*} dx'\right| \\ &\leq C\int_{|y'|\leq \lambda r_0, |z'|\leq C_1} \frac{1}{|y'|^s} w^{2_*} dx' + C\int_{|y'|\leq \lambda r_0, |z'|\geq C_1} \frac{1}{|y'|^s} w^{2_*} dx'.\end{aligned}$$

As w is continuous (by Lemma 2.4.3) we firs see that

$$\begin{aligned}&\int_{|y'|\leq \lambda r_0, |z'|\leq C_1} \frac{1}{|y'|^s}\, w^{2_*}\, dx' \leq C\int_{|y'|\leq \lambda r_0, |z'|\leq C_1} \frac{1}{|y'|^s}\, dx' \\ &\leq C\int_0^{\lambda r_0} \frac{1}{\rho^s}\rho^{k-1}\, d\rho = C\,\lambda^{k-s}.\end{aligned}$$

Next, from Lemma 3.3.14 we obtain that for $|x| \to \infty$, $w(x) \leq \frac{C}{|x|^{n-2}}$ that implies $w(x)^{\frac{2(n-s)}{n-2}} \leq \frac{C^{2_*}}{|x|^{2(n-s)}}$. Then

$$\begin{aligned}&\int_{|y'|\leq \lambda r_0, |z'|\geq C_1} \frac{1}{|y'|^s} w^{2_*} dx' \leq C\int_{|y'|\leq \lambda r_0, |z'|\geq C_1} \frac{1}{|y'|^s}\frac{1}{|x'|^{2(n-s)}} dx' \\ &\leq C\int_{|y'|\leq \lambda r_0, |z'|\geq C_1} \frac{1}{|y'|^s}\frac{1}{|z'|^{2(n-s)}} dy'dz' \leq C\lambda^{k-s}\int_{|z'|\geq C_1} \frac{1}{|z'|^{2(n-s)}} dz' \\ &= C\lambda^{k-s}\int_{C_1}^{+\infty} \frac{1}{\rho^{2(n-s)}}\,\rho^{n-k-1}\, d\rho = C\lambda^{k-s}\int_{C_1}^{+\infty} \frac{1}{\rho^{n+k-2s+1}}\, d\rho.\end{aligned}$$

But since $n+k-2s+1 > 1$, because $s < k \leq n$, we have that $\int_{C_1}^{+\infty} \frac{1}{\rho^{n+k-2s+1}}\, d\rho < +\infty$, thus

$$\int_{|y'|\leq \lambda r_0, |z'|\geq C_1} \frac{1}{|y'|^s} w^{2_*} dx' \leq C\lambda^{k-s}.$$

Collecting all above all estimates we finally arrive at

$$\int_{\mathbb{R}^n} \phi(|y|) w_\lambda^{2_*}(x)\, dx \geq 1 + a\,\lambda^\beta - b\lambda^{k-s},$$

with $a, b > 0$. As $\beta < k - s$, when λ is small, (i.e. $\frac{1}{r_0} \leq \lambda < 1$), we obtain

$$\int_{\mathbb{R}^n} \phi(|y|)\, w_\lambda^{2_*}\, dx > 1,$$

which implies that

$$S_\phi \leq J_\phi(w_\lambda) = \frac{\int_{\mathbb{R}^n} |\nabla w|^2\, dx}{(\int_{\mathbb{R}^n} \phi(|y|) w_\lambda^{2_*}\, dx)^{2/2_*}} < S.$$

The conclusion follows easily. □

Summing up, we get existence of solutions for (2.42) when $\phi(r)r^s \geq 1$ and nonexistence when $\phi(r)r^s \leq 1$ and $\phi(r)r^s \neq 1$. By iv), we require $\phi(r)r^s$ to be above 1 just for large values of r.

Remark 2.4.1. We have just seen that the problem (2.32) has no solutions for $n = 3, p = 2_* = 4$. However, for the Theorem 2.4.6 shows that by a small perturbation of Ciotti-Bertin function ϕ, we can obtain a problem which does have a solution. Indeed, we can fix $\gamma \in (2\alpha - 1, 2\alpha)$ and $\epsilon > 0$ and define

$$\phi_\epsilon(r) = \phi(r) + \epsilon \frac{r^\gamma}{(1 + r^2)^{\alpha + \frac{1}{2}}} = \frac{r^{2\alpha} + \epsilon r^\gamma}{(1 + r^2)^{\alpha + \frac{1}{2}}}$$

By computations, we have, for large r's

$$\phi_\epsilon(r) r - 1 \geq \frac{C}{r^{2\alpha - \gamma}}.$$

As $0 < 2\alpha - \gamma < 1 < k - s$, the hypotheses of the Theorem 2.4.6 are satisfied and we obtain a solution for problem (2.32) with ϕ_ϵ replacing ϕ.

3. SYMMETRY PROPERTIES AND IDENTIFICATION OF THE EXTREMALS OF A WEIGHTED SOBOLEV INEQUALITY

In this chapter, we study the symmetry properties of the minimizers of a weighted Sobolev inequality which establishes:
Let $n \geq 3, 2 \leq k \leq n, x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}, 0 \leq s < 2, 2_* = 2\frac{n-s}{n-2}$. There exists a optimal constant $S = S_{n,k,s}$ such that

$$S \left(\int_{\mathbb{R}^n} \frac{|u|^{2_*}}{|y|^s} \right)^{\frac{2}{2_*}} \leq \int_{\mathbb{R}^n} |\nabla u|^2 \qquad \forall\ u \in D^{1,2}(\mathbb{R}^n) \tag{3.1}$$

If $s = 2$, (3.1) still holds true (see [3], Remark 2.3) if $2 < k \leq n$, thus providing an extension of the classical Hardy inequality (which is known not to possess extremal functions). The limiting case $s = 0$ corresponds to the classical Sobolev inequality. It has been exhaustively studied by Aubin [1] and Talenti [51] who computed exactly the best constant

$$S_{0,n} = n(n-2) \left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)\omega_{n-1}}{\Gamma(n+1)} \right)^{\frac{2}{n}}$$

and proved existence of extremal functions, exhibiting them explicitly. In the more general case $0 \leq s < 2$ (but $k = n$) the best constant has been computed in [28] and extremal functions have been identified by Lieb ([34]): they are given, up to dilations and translations by

$$U(x) = \frac{1}{(1 + |x|^{2-s})^{\frac{n-2}{2-s}}}$$

More general inequalities of type (3.1), still in case $k = n$, have been considered by Caffarelli-Kohn-Nirenberg [13]. Also here, positive solutions of the associated Euler equation (on a properly weighted Sobolev space) turn out to be radially symmetric (see [7]) and they can be explicitly computed just solving an ODE (see [17]). They turns out to be of the same form as in the case $a = 0$. When $k < n$, extremals cannot be anymore radially symmetric and then they cannot be searched among solutions of an ODE.
In this chapter, at first, we will recall an important result proved by Mancini and Sandeep in [40] where the cylindrical symmetry of minimizers of (3.1)

has been proved, by symmetrization arguments applied to a related Hardy-Littlewood-Sobolev inequality. In the second part of this chapter, we study entire solutions for the following problem

$$\left.\begin{array}{rcll} -\Delta u &=& \frac{u^{\frac{n+2-2t}{n-2}}}{|y|^t} & \text{in} \quad \mathbb{R}^n \\ u &>& 0 & \\ u &\in& D^{1,2}(\mathbb{R}^n) & \end{array}\right\} \tag{3.2}$$

where $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, n \geq 3, 2 \leq k < n, t \in (0,2)$ and a point $x \in \mathbb{R}^n$ is denoted by $x = (y,z)$. In [3] it was proved that (3.2) is the Euler equation associated to (3.1) inequality. It is known that (3.1) admits extremals (see [3]) and so, the problem (3.2), (with $t = s$), has a solution. We will prove, using moving planes techniques, that all the solutions of the Euler equation associated to (3.1) are cylindrically symmetric. This result is from [22].

3.1 Rearrangement inequalities

In this section, we will give some preliminaries which will be useful later.

Definition 3.1.1. $f : \mathbb{R}^n \to \mathbb{R}$ is said to be vanishing at ∞ if it satisfies $\{|f| > t\}| < \infty$ for all $t > 0$, where $|\cdot|$ denotes the Lebesgue measures.

Definition 3.1.2. Let $A \subset \mathbb{R}^n$ a Borel set of finite measure. Define A^*, the symmetric rearrangement of the set A, as the open ball centered in the origin whose volume is that of A. In formula, $A^* = \{x \,|\, |x| < r\}$ with $(|\mathbb{S}^{n-1}|/n)r^n = |A|$.

Definition 3.1.3. The symmetric-decreasing rearrangement of a characteristic function of a set is, $\chi_A^* = \chi_{A^*}$.

Definition 3.1.4. The symmetric decreasing rearrangement of f is $f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$.

Lemma 3.1.1. *(Riesz's rearrangement inequality) Let f, g, h be three non negative functions on $\mathbb{R}^n$, vanishing at ∞, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy. \tag{3.3}$$

Further, if g is strictly symmetric decreasing, then equality in (3.3) holds if and only if $f(x) = f^*(x-x_0)$ and $h(x) = h^*(x-x_0)$ for some common $x_0 \in \mathbb{R}^n$.

Lemma 3.1.2. *Let f, g, h be three non negative functions on $\mathbb{R}^n$, vanishing at ∞ and $a > 0$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)}{|x|^a}\, g(x-y)\, \frac{h(y)}{|y|^a}\, dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f^*(x)}{|x|^a}\, g^*(x-y)\, \frac{h^*(y)}{|y|^a}\, dxdy. \tag{3.4}$$

Moreover, if $g = g^*$ equality holds in (3.4) if and only if $f = f^*$ and $h = h^*$.

Proof. For $s>0$, we denote by B_s the ball in $\mathbb{R}^n$ with center at zero and radius $(\frac{1}{s})^{\frac{1}{a}}$. With this notation, we have

$$\frac{1}{|x|^a}=\int_0^\infty \chi_{B_s}(x)\,ds.$$

Using this representation, Fubini's theorem and Lemma 3.1.1, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^a}g(x-y)\frac{h(y)}{|y|^a}dxdy &= \int_0^\infty\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f\chi_{B_s})(x)g(x-y)(h\chi_{B_t})(y)dydx\right)dsdt\\ &\le \int_0^\infty\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f\chi_{B_s})^*(x)g^*(x-y)(h\chi_{B_t})^*(y)dydx\right)dsdt.\end{aligned} \tag{3.5}$$

Further, we note that $(f\chi_{B_s})^*\le f^*\chi_{B_s}$, $\forall s>0$. In fact, by the inclusion between level sets, we have

$$\begin{aligned}\{(f\chi_{B_s})^*>t\}=\{f\chi_{B_s}>t\}^*=(\{(f>t\}\cap B_s)^*\subset(\{(f>t\}^*\cap B_s)\\ =\{f^*>t\}\cap B_s=\{f^*\chi_{B_s}>t\}\end{aligned}$$

and hence

$$(f\chi_{B_s})^*(x)=\int_0^\infty \chi_{\{(f\chi_{B_s})^*>t\}}(x)dt\le\int_0^\infty\chi_{\{f^*\chi_{B_s}>t\}}(x)dt=(f^*\chi_{B_s})(x).$$

Then

$$\begin{aligned}&\int_0^\infty\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f\chi_{B_s})^*(x)g^*(x-y)(h\chi_{B_t})^*(y)dydx\right)dsdt\\ &\le\int_0^\infty\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f^*\chi_{B_s})(x)\,g^*(x-y)\,(h^*\chi_{B_t})(y)dydx\right)dsdt\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f^*(x)}{|x|^a}g^*(x-y)\frac{h^*(y)}{|y|^a}dxdy.\end{aligned} \tag{3.6}$$

This proves (3.4). Let g be strictly symmetric decreasing. Assume that equality (3.4) holds for an f and h. We want to show that $f=f^*$ and $h=h^*$.
It follows from (3.3), that if equality (3.4) holds, then for almost $s>0$,

$$\begin{aligned}&\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f\chi_{B_s})^*(x)g^*(x-y)(h\chi_{B_t})^*(y)dydx\right)dt\\ &=\int_0^\infty\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f^*\chi_{B_s})(x)\,g^*(x-y)\,(h^*\chi_{B_t})(y)dydx\right)dt\end{aligned} \tag{3.7}$$

We fix an s_0 for which (3.4) holds. Then

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f\chi_{B_{s_0}})^*(x)g^*(x-y)(h\chi_{B_t})^*(y)dydx\\ &=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f^*\chi_{B_{s_0}})(x)\,g^*(x-y)\,(h^*\chi_{B_t})(y)dydx\end{aligned} \tag{3.8}$$

for almost all $t > 0$. Therefore, by Lemma 3.1.1, we get:
There exists $x_0 \in \mathbb{R}^n$ such that $(f\chi_{B_{s_0}})$ and $(h\chi_{B_t})$ are symmetrically decreasing with respect to x_0. We claim that $x_0 = 0$, unless $f \equiv 0$ or $h \equiv 0$. Suppose $x_0 \neq 0$. Let $t > 0$ such that (3.4) holds for this t and $(\frac{1}{t})^{\frac{1}{a}} < |x_0|$. Then $(h\chi_{B_t})$ is symmetric decreasing with respect to x_0 and $(h\chi_{B_t})$ is zero near $x_0 \Rightarrow h\chi_{B_t} \equiv 0 \Rightarrow h \equiv 0$ in B_t. Now varying t, we get $h \equiv 0$ in the ball $B(0, |x_0|) := \{x : |x| < |x_0|\}$. Again, since $(h\chi_{B_t})$ is symmetrically decreasing with respect to x_0 for almost $t > 0$ and $h \equiv 0$ in the ball center at zero and radius $|x_0|$, we get $h \equiv 0$ in $\mathbb{R}^n$.
This proves that $x_0 = 0$, i.e. $(h\chi_{B_t})$ and $(f\chi_{B_{s_0}})$ are radially decreasing functions for $a.e.\, t \in (0, \infty) \Rightarrow h$ is radially decreasing in $\mathbb{R}^n$ and f is radially decreasing in B_{s_0}. Varying s_0, we get $f = f^*$ and $h = h^*$. This proves lemma. □

We have seen that (3.1) inequality can be written as

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{p_\sigma}}{|y|^{\sigma p_\sigma}}\right)^{\frac{2}{p_\sigma}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 \tag{3.9}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}, k \geq 2, n \geq 3,\ 0 \leq \sigma \leq 1$ *and* $p_\sigma = \frac{2n}{n-2+2\sigma}$. Now, we will remember the important results obtained by Mancini and Sandeep in [40]. We can define

$$S = S(\sigma) = \sup_{\substack{u \in D^{1,2}(\mathbb{R}^n) \\ \int |\nabla u|^2 = 1}} \left(\int_{\mathbb{R}^n} \frac{|u|^{p_\sigma}}{|y|^{\sigma p_\sigma}}\right)^{\frac{2}{p_\sigma}} \tag{3.10}$$

and notice that S is achieved by $u \in D^{1,2}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} \frac{|u|^{p_\sigma - 2} u\varphi}{|y|^{\sigma p_\sigma}} = S^{\frac{p_\sigma}{2}} \int_{\mathbb{R}^n} \nabla u \nabla \varphi \quad \varphi \in D^{1,2}(\mathbb{R}^n). \tag{3.11}$$

Let $G = \frac{1}{(n-2)\omega_n |x|^{n-2}}$ denote the fundamental solutions of $-\Delta$ in $\mathbb{R}^n$. We have

Theorem 3.1.3. *Let* $\sigma \in [0, 1), p = \frac{2n}{n-2+2\sigma}, \frac{1}{q} + \frac{1}{p} = 1$. *Let* $x = (y, z), x' = (y', z')$ *be two different points in* $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. *Then*

$$\sup_{h \in L^q, ||h||_q = 1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{h(x)}{|y|^\sigma} G(x - x') \frac{h(x')}{|y'|^\sigma} \, dx' \, dx = S. \tag{3.12}$$

Furthermore, u *is an extremal for (3.10) if and only if* $f := \frac{|u|^{p-2}u}{|y|^{\sigma p_\sigma}}$ *is an extremal for (3.12).*

Proof. By (3.1), we derive a doubly weighted Hardy-Littlewood-Sobolev inequality:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{h(x)}{|y|^\sigma} G(x-x') \frac{f(x')}{|y'|^\sigma}\, dx'\, dx \leq S||h||_q\, ||f||_q \quad \forall h, f \in L^q. \tag{3.13}$$

By density argument, it is enough to prove (3.13) for positive C_0^∞ functions. For $f \in C_0^\infty$, let $f_\sigma(x) := \frac{f(x)}{|y|^\sigma}$. Then $f_\sigma \in L^{\frac{2n}{n+2}}$ and hence, by the classical Hardy-Littlewood-Sobolev inequality, $G * f_\sigma \in D^{1,2}(\mathbb{R}^n)$ and

$$\int \nabla\phi\nabla(G * f_\sigma) = \int \phi f_\sigma \qquad \forall\phi \in D^{1,2}(\mathbb{R}^n). \tag{3.14}$$

Thus, by the Hölder inequality, we get

$$\int (G * f_\sigma) h_\sigma \leq ||h||_q \left(\int \frac{(G * f_\sigma)^p}{|y|^{\sigma p}}\right)^{1/p} \leq ||h||_q \left(S \int |\nabla(G * f_\sigma)|^2\right)^{1/2}. \tag{3.15}$$

Analogously, we obtain from (3.14)

$$\int |\nabla(G * f_\sigma)|^2 = \int f_\sigma (G * f_\sigma)$$

$$\leq ||f||_q \left(\int \frac{(G * f_\sigma)^p}{|y|^{\sigma p}}\right)^{\frac{1}{p}} \leq ||f||_q\, S^{\frac{1}{2}}\, ||\nabla(G * f_\sigma)||_2$$

and hence

$$||\nabla(G * f_\sigma)||_2 \leq S^{\frac{1}{2}} ||f||_q. \tag{3.16}$$

This, jointly with (3.15) gives (3.13) for $h, f \in C_0^\infty(\mathbb{R}^n)$. A density argument using Fatou's lemma gives the result. Moreover, we note that (3.13) holds true also if we replace q with $\frac{2n}{n+2}$ and h_σ with any $h \in L^{\frac{2n}{n+2}}$. Hence $G * f_\sigma \in L^{\frac{2n}{n+2}}$ for any $f \in L^q$, and in fact, applying (3.14) to $f_j \to f$ in L^q, we see that $G * f_\sigma \in D^{1,2}(\mathbb{R}^n)$ and (3.15) holds for all $f \in L^q$. Now, if we set

$$\widehat{S} = \widehat{S}(\sigma) := \sup_{||f||_q=1} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{f(x)}{|y|^\sigma} G(x-x') \frac{f(x')}{|y'|^\sigma}\, dx\, dx' \leq S \tag{3.17}$$

We want to prove the reverse inequality. Since $\sigma < 1$, (3.9) posses an extremal function u. Since $|u|$ is extremal as well, and hence solves (3.11), it turns out to be positive away from $\{y = 0\}$ by the maximum principle, i.e. u does not change sign , and we can assume $u \geq 0$. After normalization, we have

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|y|^{\sigma p}} = 1, \quad \int_{\mathbb{R}^n} \frac{|u|^{p-1}\varphi}{|y|^{\sigma p}} = S\int_{\mathbb{R}^n} \nabla u \nabla\varphi \quad \varphi \in D^{1,2}(\mathbb{R}^n). \tag{3.18}$$

If $f := \frac{u^{p-1}}{|y|^{\sigma(p-1)}}$, then $\int_{\mathbb{R}^n} f^q = \int_{\mathbb{R}^n} \frac{u^p}{|y|^{\sigma p}} = 1$. In particular, as noted above, we have that $G * f_\sigma \in D^{1,2}(\mathbb{R}^n)$, and hence, using (3.18),

$$S \int_{\mathbb{R}^n} u\, f_\sigma = S \int_{\mathbb{R}^n} \nabla(G * f_\sigma)\nabla u = \int_{\mathbb{R}^n} \frac{|u|^{p-1}}{|y|^{\sigma p}}\, G * f_\sigma.$$

But $f_\sigma = \frac{|u|^{p-1}}{|y|^{\sigma p}}$ and hence $S = \int_{\mathbb{R}^n} f_\sigma G * f_\sigma \leq \widehat{S}$ because $\int_{\mathbb{R}^n} f^q = 1$. This proves that $S = \widehat{S}$ and that $f = \frac{u^{p-1}}{|y|^{\sigma(p-1)}}$ is an extremal function for (3.12). Finally, let f be an extremal function for the weighted Hardy-Littlewood-Sobolev quotient (3.12). Clearly, f cannot change sign, so we can assume $f \geq 0$. We have:

$$||f||_q = 1 \;\; and \;\; \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)}{|y|^\sigma}\, G(x-x')\, \frac{g(x')}{|y'|^\sigma}\, dx\, dx' = S \int_{\mathbb{R}^n} f^{q-1}\, g \quad \forall g \in L^q.$$

Let $u := f^{q-1}\, |y|^\sigma$ so that $\frac{u^{p-1}}{|y|^{\sigma(p-1)}} = f \in L^q$. The Euler-Lagrange equation for f rewrites

$$\int_{\mathbb{R}^n} \frac{g(x)}{|y|^\sigma}\, G(x-x')\, \frac{u^{p-1}}{|y'|^{\sigma p}} = S \int_{\mathbb{R}^n} \frac{u}{|y|^\sigma}\, g \quad \forall g \in L^q.$$

In particular, we find that $Su = G * f_\sigma$ and hence, as remarked above, $u \in D^{1,2}(\mathbb{R}^n)$ and $-S\Delta u = f_\sigma = \frac{u^{p-1}}{|y|^{\sigma p}}$. Thus $S \int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} \frac{u^p}{|y|^{\sigma p}} = \int_{\mathbb{R}^n} f^q = 1$ i.e. u is an extremal function for (HS). □

3.2 Cylindrical symmetry of extremals

In this section, we will show the following result, proved by G.Mancini and K.Sandeep which evidences symmetry properties of the extremals of (3.1) inequality.

Theorem 3.2.1. *Extremals of the Sobolev inequality (3.1) are cylindrically symmetric, i.e., if the supremum in (3.1) is attained at $u \in D^{1,2}(\mathbb{R}^n)$, then*

i) $u(\cdot, z)$ is radially symmetric decreasing function in $\mathbb{R}^k$ for all $z \in \mathbb{R}^{n-k}$.

ii) There exists $z_0 \in \mathbb{R}^{n-k}$ such that $u(y, \cdot + z_0)$ is radially symmetric decreasing function in $\mathbb{R}^{n-k}$ for all $y \in \mathbb{R}^k$, $y \neq 0$.

Proof. In view of the last theorem, it is enough to prove that the extremals of (3.13) have the required symmetry properties.Let $f \in L^q(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f^q = 1 \;\; and \;\; \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|y|^\sigma}\, G(x-x')\, \frac{f(x')}{|y'|^\sigma}\, dx'\, dx = S.$$

Let $f^{*'}$ denote the obtained by taking the k-dim rearrangement of $f(\cdot, z)$ for all $z \in \mathbb{R}^{n-k}$ i.e. $f^{*'}(\cdot, z) = f(\cdot, z)^*$, where $*$ denotes the rearrangement of functions in $\mathbb{R}^k$. Then, from the properties of symmetrization, Lemma 3.1.2 and the fact that f is an extremal for (3.13), we get $\int_{\mathbb{R}^n} (f^{*'})^q = 1$ and

$$\begin{aligned} S &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|y|^\sigma} G(x - x') \frac{f(x')}{|y'|^\sigma} dx' \, dx \\ &= \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} (\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{f^{*'}(y, z)}{|y|^\sigma} G(y - y', z - z') \frac{f^{*'}(y', z')}{|y'|^\sigma}) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^{*'}(x)}{|y|^\sigma} G(x - x') \frac{f^{*'}(x')}{|y'|^\sigma} dx' \, dx \\ &\leq S. \end{aligned}$$

Hence we get equality in all the steps and therefore from Lemma 3.1.2, we obtain $u(\cdot, z)$ is symmetrically decreasing for almost all $z \in \mathbb{R}^{n-k}$. Now let us prove the symmetry in the z variable. Let v be the function obtained by taking the $(n - k)$ dim symmetrization of $f(y, \cdot)$ for a.e. $y \in \mathbb{R}^k$ i.e. $v(y, \cdot) = (f(y, \cdot))^*$, where $*$ denotes the rearrangement in $\mathbb{R}^{n-k}$. Then, from Lemma 3.1.1, we have $\int_{\mathbb{R}^n} v^q = 1$ and

$$\begin{aligned} S &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{1}{|y|^\sigma} \frac{1}{|y'|^\sigma} (\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} f(y, z) G(y - y', z - z') f(y', z')) \\ &\leq \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{1}{|y|^\sigma} \frac{1}{|y'|^\sigma} (\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} v(y, z) G(y - y', z - z') v(y', z')) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{v(x)}{|y|^\sigma} G(x - x') \frac{v(x')}{|y'|^\sigma} \leq S \end{aligned}$$

Then, we have equality in the second step, i.e., for a.e. $y \in \mathbb{R}^k$

$$\begin{aligned} &\int_{\mathbb{R}^k} \frac{1}{|y'|^\sigma} (\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} f(y, z) G(y - y', z - z') f(y', z')) \, dy' \\ &= \int_{\mathbb{R}^k} \frac{1}{|y'|^\sigma} (\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} v(y, z) G(y - y', z - z') v(y', z')) \, dy'. \end{aligned}$$

Fix a $y_0 \in \mathbb{R}^k$. Then, from the above equality, we have

$$\begin{aligned} &\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} f(y_0, z) G(y_0 - y', z - z') f(y', z') dz \, dz' \\ &= \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} v(y_0, z) G(y_0 - y', z - z') v(y', z') dz \, dz' \end{aligned}$$

holds for a.e. $y' \in \mathbb{R}^k$. Since $G(y, \cdot)$ is strictly symmetric decreasing in $\mathbb{R}^{n-k}$, we get from Lemma 3.1.1, that $f(y_0, \cdot)$ and $f(y', \cdot)$ are symmetrically decreasing with respect to a common point in $\mathbb{R}^{n-k}$ for almost all $y' \in \mathbb{R}^k$, i.e. there exists $z_0 \in \mathbb{R}^{n-k}$ such that $z \to f(y, z + z_0)$ is symmetrically decreasing in $\mathbb{R}^{n-k}$. This completes the proof of theorem. □

3.3 Symmetry Properties of solutions

In this section we establish, using the moving planes method (see [26],[27]), the symmetry properties of solutions of the Euler equation associated to (3.1). The principal result in this section, is the following theorem.

Theorem 3.3.1. *If u is a solution of (3.2), then u is cylindrically symmetric, i.e.:*

(i) for any choice of $z \in \mathbb{R}^{n-k}$, $u(\cdot, z)$ is symmetric decreasing in $\mathbb{R}^k$.

(ii) there exists $z_0 \in \mathbb{R}^{n-k}$ such that, for any choice of $y \in \mathbb{R}^k, y \neq 0$, $u(y, \cdot)$ is symmetric decreasing about z_0 in $\mathbb{R}^{n-k}$.

Proof. Let us denote a point $x \in \mathbb{R}^n$ as $x = (y, z) = (y_1, ...y_k, z_1...z_{n-k})$. First we will show that u is symmetric decreasing in the y variable.
For $\lambda > 0$ define

$$\Omega_\lambda = \{(y, z) : y_1 > \lambda\} \tag{3.19}$$

and for $x \in \Omega_\lambda$ denote by x^λ its reflection with respect to the hyper plane $y_1 = \lambda$. i.e, $x^\lambda = (2\lambda - y_1, y_2...y_k, z) = (y^\lambda, z)$. Now let us define

$$u_\lambda(x) := u(x^\lambda) \ , \ x \in \Omega_\lambda \ , \qquad w_\lambda := u_\lambda - u \tag{3.20}$$

Notice that w_λ is smooth away from the subspace $\{(2\lambda, 0...0, z) : z \in \mathbb{R}^{n-k}\}, w_\lambda = 0$ on $\partial\Omega_\lambda$, $w_\lambda \in D^{1,2}(\Omega_\lambda)$. We first claim that $w_\lambda \geq 0$ in Ω_λ for λ large enough. First, since $\lambda > 0 \Rightarrow |y^\lambda| < |y|$ in Ω_λ , w_λ satisfies

$$-\Delta w_\lambda \geq A(x)\frac{w_\lambda}{|y|^t} \tag{3.21}$$

$$\text{where} \qquad 0 \leq A(x) := \frac{u_\lambda^{2_*(t)-1} - u^{2_*(t)-1}}{u_\lambda - u} \leq$$

$$\leq (2_*(t) - 1)\,[\max\{u_\lambda(x), u(x)\}]^{2_*(t)-2} \ , \qquad 2_*(t) - 2 = 2\frac{2-t}{n-2} \tag{3.22}$$

Multiplying (3.21) by w_λ^- and integrating by parts over Ω_λ, we get

$$\int_{\Omega_\lambda} |\nabla w_\lambda^-|^2 \leq \int_{\Omega_\lambda} A(x)\frac{(w_\lambda^-)^2}{|y|^t} \leq \left(\int_{\Omega_\lambda\cap\{w_\lambda<0\}} (A(x))^{\frac{n}{2-t}}\right)^{\frac{2-t}{n}} \left(\int_{\Omega_\lambda} \frac{(w_\lambda^-)^{2_*(s)}}{|y|^s}\right)^{\frac{2}{2_*(s)}}$$

where we have set $s := \frac{nt}{n-2+t} \in (0, 2)$, so that $\frac{2_*(s)}{2} = \frac{2n}{n-2+t}$ is the exponent conjugate to $\frac{n}{2-t}$. An application of (3.1) leads to

$$S\left(\int_{\Omega_\lambda} \frac{(w_\lambda^-)^{2_*(s)}}{|y|^s}\right)^{\frac{2}{2_*(s)}} \leq \left(\int_{\Omega_\lambda\cap\{w_\lambda<0\}} (A(x))^{\frac{n}{2-t}}\right)^{\frac{2-t}{n}} \left(\int_{\Omega_\lambda} \frac{(w_\lambda^-)^{2_*(s)}}{|y|^s}\right)^{\frac{2}{2_*(s)}} \tag{3.23}$$

Since $u_\lambda < u$ on $\{w_\lambda < 0\}$, (3.22) gives

$$\int\limits_{\Omega_\lambda \cap \{w_\lambda<0\}} (A(x))^{\frac{n}{2-t}} \leq C \int\limits_{\Omega_\lambda} u^{\frac{2n}{n-2}} \tag{3.24}$$

and the right hand side goes to zero as λ goes to infinity. Hence it follows from (3.23) that for λ large enough $w_\lambda^- = 0$ and hence $u_\lambda \geq u$. Now, let

$$A = \{\lambda > 0 : u_{\lambda'} \geq u \text{ in } \Omega_{\lambda'} \text{ for all } \lambda' > \lambda\}, \qquad \lambda_0 := \inf A$$

We will show that $\lambda_0 = 0$.
Assume that $\lambda_0 > 0$. Define $w_{\lambda_0} = u_{\lambda_0} - u$. Then $w_{\lambda_0} \geq 0$ and satisfies $-\Delta w_{\lambda_0} \geq 0$ in Ω_{λ_0} and away from the (n-k) dimensional subspace $y = (2\lambda_0, 0...0)$. Hence in this region $w_{\lambda_0} > 0$. Let $\epsilon > 0$. Choose $R > 0$ and $\delta_0 > 0$ such that

$$\int\limits_{|x|\geq R} u^{\frac{2n}{n-2}} < \frac{\epsilon}{2} \tag{3.25}$$

$$\int\limits_{\lambda_0-\delta_0<y_1<\lambda_0+\delta_0} u^{\frac{2n}{n-2}} + \int\limits_{2\lambda_0-\delta_0<y_1<2\lambda_0+\delta_0} u^{\frac{2n}{n-2}} < \frac{\epsilon}{2}. \tag{3.26}$$

Let us consider the set

$$K = \{x = (y, z) : \lambda_0+\delta_0 \leq y_1 \leq 2\lambda_0-\delta_0 \text{ or } y_1 \geq 2\lambda_0+\delta_0\} \cap \{x = (y, z) : |x| \leq R\}.$$

Then K is compact and $w_{\lambda_0} > 0$ in K. Choose $0 < \delta_1 < \delta_0$ such that $u_{\lambda_0-\delta} - u > 0$ in K for all $0 < \delta < \delta_1$. Let $\lambda_1 = \lambda_0 - \delta$ with $0 < \delta < \delta_1$. We claim that when ϵ is small enough $u_{\lambda_1} \geq u$ in Ω_{λ_1}, which contradicts the definition of λ_0 and hence $\lambda_0 = 0$. Now to see this, we define $w_{\lambda_1} = u_{\lambda_1} - u$ and we proceed as in the case of (3.23), to get

$$S\left(\int\limits_{\Omega_{\lambda_1}} \frac{(w_{\lambda_1}^-)^{2_*(s)}}{|y|^s}\right)^{\frac{2}{2_*(s)}} \leq$$

$$\leq \left(\int\limits_{\Omega_{\lambda_1}\cap\{w_{\lambda_1}<0\}} (A(x))^{\frac{n}{2-t}}\right)^{\frac{2-t}{n}} \left(\int\limits_{\Omega_{\lambda_1}} \frac{(w_{\lambda_1}^-)^{2_*(s)}}{|y|^s}\right)^{\frac{2}{2_*(s)}} \tag{3.27}$$

By the choice of λ_1, $w_{\lambda_1} > 0$ in K . This together with (3.25) and (3.26) gives

$$\int\limits_{\Omega_{\lambda_1}\cap\{w_{\lambda_1}<0\}} (A(x))^{\frac{n}{2-t}} \leq C \int\limits_{\Omega_{\lambda_1}\cap\{w_{\lambda_1}<0\}} u^{\frac{2n}{n-2}} < \epsilon. \tag{3.28}$$

(3.27) and (3.28) together implies that for ϵ small enough $w^-_{\lambda_1} = 0$ and this completes the proof. Hence $\lambda_0 = 0$ and consequently

$$u(-y_1, ...y_k, z) \geq u(y_1, ...y_k, z)$$

for all $y_1 > 0$. Doing the same arguments for $v(y, z) = u(-y_1, y_2..., y_k, z)$ leads to $u(-y_1, ...y_k, z) \leq u(y_1, ...y_k, z)$ for $y_1 > 0$ and hence

$$u(-y_1, ...y_k, z) = u(y_1, ...y_k, z) \text{ in } \mathbb{R}^n.$$

Now the symmetry in the y variable follows as one can do the moving plane argument in any direction in the y plane instead of the y_1 direction.

Next we will prove the symmetry in the z direction. Let $\Omega_\lambda, x^\lambda$ and u_λ be as defined in (3.19),(3.20) with z_1 in place of y_1 and this time for all $\lambda \in \mathbb{R}$. Now exactly as in the previous case one gets the existence of a $\lambda_1 > 0$ such that $u_\lambda \geq u$ in Ω_λ for all $\lambda \geq \lambda_1$. The same arguments applied to $v(y, z_1, ...z_{n-k}) = u(y, -z_1, ...z_{n-k})$ yields the existence of $\lambda_2 < 0$ such that $u_\lambda \leq u$ in Ω_λ for all $\lambda \leq \lambda_2$.. Now let $A = \{\lambda \in \mathbb{R} : u_{\lambda'} \geq u \text{ in } \Omega_{\lambda'} \text{ for all } \lambda' > \lambda\}$. Then A is nonempty and bounded below, let $\lambda_0 = \inf A$. We claim that $u_{\lambda_0} = u$ in Ω_{λ_0}.
Let $w_{\lambda_0} = u_{\lambda_0} - u$, then w_{λ_0} is smooth in Ω_{λ_0} except on the subspace $y = 0$, $w_{\lambda_0} \geq 0$ and satisfies

$$-\Delta w_{\lambda_0} = \frac{1}{|y|^t}\left(u_{\lambda_0}^{\frac{n+2-2t}{n-2}} - u^{\frac{n+2-2t}{n-2}}\right) = A(x)\frac{w_{\lambda_0}}{|y|^t} \tag{3.29}$$

where A satisfies the estimate (3.22). Since $\Omega_{\lambda_0} \setminus \{(0, y) : y \in \mathbb{R}^{n-k}\}$ is connected, by strong maximum principle either $w_{\lambda_0} \equiv 0$ or $w_{\lambda_0} > 0$. If the second case happens we can argue as in the previous case (where we showed $\lambda_0 > 0 \Rightarrow \lambda_0$ is not the infimum), to get a contradiction. Hence $w_{\lambda_0} = 0$ and therefore u is symmetric decreasing in the z_1 direction with respect to $z_1^0 = \lambda_0$. Similarly once can show that u is symmetric decreasing in the z_i direction with respect to some z_i^0 for $i = 1, ..., n-k$. Now it is easy to show that $u(y, .)$ is symmetric decreasing with respect to $z_0 = (z_1^0, ..., z_{n-k}^0)$ for all $y \neq 0$. This completes the proof of Theorem 3.3.1. □

3.4 A priori Estimates and regularity of solutions

We want to know the properties of regularity and boundedness of solutions for (3.2). Define the Móbius inversion $\mathcal{I} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ by

$$\mathcal{I}(w) = \frac{w}{|w|^2}$$

and the following Kelvin transform, $u^* : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$u^* = |x|^{2-n}u(\mathcal{I}(x)) = |x|^{2-n}u(\frac{x}{|x|^2})$$

Then, the following result holds.

Remark 3.4.1. If u satisfies the equation (3.2), then also $u^*(x)$ verifies (3.2).
In fact,

$$-\Delta u^*(x) = -|x|^{-n-2}\Delta u(\frac{x}{|x|^2}) = |x|^{-n-2}u(\frac{x}{|x|^2})^{\frac{n+2-2t}{n-2}}\frac{|x|^{2t}}{|y|^t}$$

$$= |x|^{-n-2+2t}u(\frac{x}{|x|^2})^{\frac{n+2-2t}{n-2}}\frac{1}{|y|^t} = \frac{[u(\frac{x}{|x|^2})|x|^{2-n}]^{\frac{n+2-2t}{n-2}}}{|y|^t} = \frac{[u^*(x)]^{\frac{n+2-2t}{n-2}}}{|y|^t}$$

First we prove a lemma which is the key in obtaining estimates on u and its z derivatives.

Lemma 3.4.1. *Let $u \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$-\Delta u = \frac{f(x)u}{|y|^t} + \frac{g(x)}{|y|^t} \tag{3.30}$$

where $t \in (0,2)$, f and g are in $L^p_{loc}(\mathbb{R}^n)$ for some $p > \frac{n}{2-t}$. Then u is locally bounded in $\mathbb{R}^n$.

Proof. We prove the result using the well known Moser iteration scheme (see [30]). To start with let us define the test function to be used.
Fix $R > 0$ and let $R < r_{i+1} < r_i < 2R$, and η be a smooth cut off function satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B(0, r_{i+1})$, $\eta = 0$ outside $B(0, r_i)$ and $|\nabla\eta| \leq C(r_i - r_{i+1})^{-1}$ for some constant C independent of r_{i+1} and r_i.
Let $k = ||g||_{L^p(B(0,2R))}$. For $m > 0$ define $\overline{u} = u^+ + k$ and

$$u_m = \begin{cases} \overline{u} & \text{if} \quad u < m \\ k+m & \text{if} \quad u \geq m \end{cases} \tag{3.31}$$

Now for $\beta \geq 0$ define the test function $v = v_\beta$ as

$$v = \eta^2(u_m^{2\beta}\overline{u} - k^{2\beta+1}).$$

Then $0 \leq v \in H^1_0(B(0, r_i))$ and hence from (3.30)

$$\int_{\mathbb{R}^n} \nabla u.\nabla v = \int_{\mathbb{R}^n} \frac{f(x)uv}{|y|^t} + \int_{\mathbb{R}^n} \frac{g(x)v}{|y|^t} \tag{3.32}$$

Now by direct calculation

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla u.\nabla v &= \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u + 2\beta\int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u \\ &\quad +2\int_{\mathbb{R}^n} \eta(u_m^{2\beta}\overline{u} - k^{2\beta+1})\nabla u.\nabla\eta \end{aligned} \tag{3.33}$$

Observe that in the support of the integrand of the first integral, $\nabla\overline{u} = \nabla u$ and in the second integral $u_m = \overline{u}$, $\nabla u_m = \nabla u$. Also $u_m^{2\beta}\overline{u} - k^{2\beta+1} \leq u_m^{2\beta}\overline{u}$. These facts together with Cauchy Schwartz give

$$\begin{aligned}
\int_{\mathbb{R}^n} \nabla u.\nabla v &\geq 2\beta \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u + \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u - 2\int_{\mathbb{R}^n} \eta(u_m^{2\beta}\overline{u} - k^{2\beta+1})|\nabla u||\nabla\eta| \\
&\geq 2\beta \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u + \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u - 2\int_{\mathbb{R}^n} \eta\, u_m^{2\beta}\overline{u}\,|\nabla u||\nabla\eta| \\
&= 2\beta \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u + \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u - 2\int_{\mathbb{R}^n} u_m^{2\beta}\,(\frac{1}{\sqrt{2}}\,\eta\,|\nabla u|)(\sqrt{2}\,\overline{u}\,|\nabla\eta|) \\
&\geq 2\beta \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u + \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u - \int_{\mathbb{R}^n} u_m^{2\beta}\,(\frac{1}{2}\eta^2\,|\nabla u|^2 + 2\overline{u}^2\,|\nabla\eta|^2) \\
&= 2\beta \int_{\mathbb{R}^n} \eta^2 u_m^{2\beta-1}\overline{u}\nabla u_m.\nabla u + \frac{1}{2}\int_{\mathbb{R}^n} \eta^2 u_m^{2\beta}\nabla\overline{u}.\nabla u - 2\int_{\mathbb{R}^n} u_m^{2\beta}\,\overline{u}^2\,|\nabla\eta|^2 \\
&\quad \frac{1}{2}\int_{\mathbb{R}^n} \eta^2(u_m^{2\beta}|\nabla\overline{u}\,|^2 + 4\beta u_m^{2\beta-2}\overline{u}^2|\nabla u_m|^2) - 2\int_{\mathbb{R}^n} u_m^{2\beta}\,\overline{u}^2\,|\nabla\eta|^2
\end{aligned} \tag{3.34}$$

Setting $w = u_m^{\beta}\overline{u}$, we have

$$\int_{\mathbb{R}^n} |\nabla w|^2\eta \leq \int_{\mathbb{R}^n} |\nabla w|^2 \leq (1+2\beta)\int_{\mathbb{R}^n} 4\beta|u_m|^{2\beta}\,|\nabla u_m|^2 + |u_m|^{2\beta}|\nabla u|^2 \tag{3.35}$$

$$|\nabla w|^2\eta^2 \geq \frac{1}{2}[|\nabla(\eta w)|^2 - 2|w|^2|\nabla\eta|^2] \tag{3.36}$$

and (3.34) rewrites as

$$\int_{\mathbb{R}^n} \nabla u.\nabla v \geq \frac{1}{4(1+2\beta)}\int_{\mathbb{R}^n} |\nabla(\eta w)|^2 - \frac{C}{(r_i - r_{i+1})^2}\int_{B(0,r_i)} w^2 \tag{3.37}$$

The right hand side of (3.32) can be estimated as follows.

$$\begin{aligned}
|\int_{\mathbb{R}^n} \left(\tfrac{f(x)uv}{|y|^t} + \tfrac{g(x)v}{|y|^t}\right)| &\leq \int_{\mathbb{R}^n} \left(|f(x)| + \tfrac{|g(x)|}{k}\right)\tfrac{(\eta w)^2}{|y|^t} \\
&\leq ||\tfrac{(\eta w)^2}{|y|^t}||_{L^q}||f + \tfrac{g}{k}||_{L^p(B(0,2R))}
\end{aligned} \tag{3.38}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > \frac{n}{2-t}$ and hence $q < r := \frac{n}{n-2+t}$. Let $\frac{1}{q} = \theta + \frac{1-\theta}{r}$. The interpolation inequality gives

$$||\frac{(\eta w)^2}{|y|^t}||_{L^q} \leq \epsilon(1-\theta)||\frac{(\eta w)^2}{|y|^t}||_{L^r} + \epsilon^{-\frac{1-\theta}{\theta}}||\frac{(\eta w)^2}{|y|^t}||_{L^1}\ , \quad \forall\epsilon > 0 \tag{3.39}$$

The weighted Sobolev inequality (3.1) with $s = \frac{nt}{n-2+t}$, $\frac{2_*(s)}{2} = \frac{n}{n-2+t} = r$ gives

$$||\frac{(\eta w)^2}{|y|^t}||_{L^r} = (\int \frac{|\eta w|^{2_*(s)}}{|y|^s})^{\frac{2}{2_*(s)}} \leq C\|\nabla(\eta w)\|_{L^2}^2$$

and hence, from (3.39), we have

$$||\frac{(\eta w)^2}{|y|^t}||_{L^q} \leq C\epsilon||\nabla(\eta w)||_{L^2}^2 + C\epsilon^{-\alpha}||\frac{(\eta w)^2}{|y|^t}||_{L^1} \tag{3.40}$$

where C and α are constants depending only on n, q and t.
Note that by the choice of k , $||f + \frac{g}{k}||_{L^p(B(0,2R))} \leq C$, where C depends only on R . Hence by choosing ϵ suitably in (3.40) and from (3.32), (3.37) and (3.38), we get

$$\int_{\mathbb{R}^n} |\nabla(\eta w)|^2 \leq \frac{C(1+\beta)^\alpha}{(r_i - r_{i+1})^2} \int_{B(0,r_i)} \frac{w^2}{|y|^t} \tag{3.41}$$

where C and α depends only on R, p and n.
Combining with the weighted Sobolev inequality (3.1) leads to

$$\left(\int_{B(0,r_{i+1})} \frac{w^{2\frac{n-t}{n-2}}}{|y|^t} \right)^{\frac{n-2}{n-t}} \leq \frac{C(1+\beta)^\alpha}{(r_i - r_{i+1})^2} \int_{B(0,r_i)} \frac{w^2}{|y|^t}. \tag{3.42}$$

Substituting w, using $u_m \leq \overline{u}$ and setting $\gamma = 2\beta + 2$ and $\chi = \frac{n-t}{n-2}$, (3.42) becomes,

$$\left(\int_{B(0,r_{i+1})} \frac{u_m^{\gamma\chi}}{|y|^t} \right)^{\frac{1}{\gamma\chi}} \leq \left(\frac{C(1+\beta)^\alpha}{(r_i - r_{i+1})^2} \right)^{\frac{1}{\gamma}} \left(\int_{B(0,r_i)} \frac{\overline{u}^\gamma}{|y|^t} \right)^{\frac{1}{\gamma}}. \tag{3.43}$$

Passing to the limit as $m \to \infty$ we get

$$\left(\int_{B(0,r_{i+1})} \frac{\overline{u}^{\gamma\chi}}{|y|^t} \right)^{\frac{1}{\gamma\chi}} \leq \left(\frac{C(1+\beta)^\alpha}{(r_i - r_{i+1})^2} \right)^{\frac{1}{\gamma}} \left(\int_{B(0,r_i)} \frac{\overline{u}^\gamma}{|y|^t} \right)^{\frac{1}{\gamma}} \tag{3.44}$$

provided the right hand side is finite.
We prove our lemma by iterating the above relation. For $i = 0, 1, 2, ...,$ define $\gamma_i = 2\chi^i$ and $r_i = R + \frac{R}{2^i}$. Then $\chi\gamma_i = \gamma_{i+1}$, $r_i - r_{i+1} = \frac{R}{2^{i+1}}$ and hence from (3.44), with $\gamma = \gamma_i$, we have

$$\left(\int_{B(0,r_{i+1})} \frac{\overline{u}^{\gamma_{i+1}}}{|y|^t} \right)^{\frac{1}{\gamma_{i+1}}} \leq C^{\frac{i}{\chi^i}} \left(\int_{B(0,r_i)} \frac{\overline{u}^{\gamma_i}}{|y|^t} \right)^{\frac{1}{\gamma_i}}, \quad i = 0, 1, 2.... \tag{3.45}$$

where C is a constant depending only on R and hence by iteration,

$$\left(\int_{B(0,r_{i+1})} \frac{\overline{u}^{\gamma_{i+1}}}{|y|^t}\right)^{\frac{1}{\gamma_{i+1}}} \leq C^{\Sigma \frac{j}{\chi^j}} \left(\int_{B(0,r_0)} \frac{\overline{u}^{\gamma_0}}{|y|^t}\right)^{\frac{1}{\gamma_0}}, \quad i = 0, 1, 2, ... \tag{3.46}$$

Taking the limit as $i \to \infty$, we get

$$\sup_{B(0,R)} \overline{u} \leq C \left(\int_{B(0,2R)} \frac{\overline{u}^2}{|y|^t}\right)^{\frac{1}{2}} \tag{3.47}$$

Hence u^+ is bounded in $B(0, R)$. Applying the same argument to $-u$ instead of u we get the boundedness of u^- and hence u is locally bounded. This proves the lemma. □

Lemma 3.4.2. *Let u be a solution of (3.2). Then*

$$\exists c_2 > c_1 > 0 : \qquad \frac{c_1}{1 + |x|^{n-2}} \leq u(x) \leq \frac{c_2}{1 + |x|^{n-2}}, \qquad \forall x \in \mathbb{R}^n.$$

Proof. To show that u is locally bounded it is enough to show, in view of Lemma 3.4.1, that $f(x) = u^{2_*-2} \in L^p_{loc}(\mathbb{R}^n)$ for some $p > \frac{n}{2-t}$, that is $u^{2\frac{2-t}{n-2}} \in L^p_{loc}(\mathbb{R}^n)$ for some $p > \frac{n}{2-t}$, that is $u \in L^p_{loc}(\mathbb{R}^n)$ for some $p > \frac{2n}{n-2}$. To prove this additional integrability of u let us multiply the equation (3.2) by v the test function used in the proof of Lemma 3.4.1, with $k = 0$ and $\beta = \frac{2-t}{n-2}$. Then from (3.37) we get

$$\int_{\mathbb{R}^n} |\nabla(\eta w)|^2 \leq \frac{C(1+\beta)}{(r_i - r_{i+1})^2} \int_{B(0,r_i)} w^2 + 2(1+2\beta) \int_{\mathbb{R}^n} u^{\frac{2(2-t)}{n-2}} \frac{(\eta w)^2}{|y|^t} \tag{3.48}$$

Now choose $M > 0$ such that $\left(\int_{u \geq M} u^{\frac{2n}{n-2}}\right)^{\frac{2-t}{n}} < \frac{S}{8(1+2\beta)}$, where S is the constant appearing in (3.1) with , instead of t, $s := \frac{nt}{n-2+t}$. Then, from Cauchy Schwartz and weighted Sobolev inequality (3.1)

$$\int_{\mathbb{R}^n} u^{\frac{2(2-t)}{n-2}} \frac{(\eta w)^2}{|y|^t} \leq M^{\frac{2(2-t)}{n-2}} \int_{\mathbb{R}^n} \frac{(\eta w)^2}{|y|^t} + \left(\int_{u \geq M} u^{\frac{2n}{n-2}}\right)^{\frac{2-t}{n}} \left(\int_{\mathbb{R}^n} \left(\frac{(\eta w)^2}{|y|^t}\right)^{\frac{n}{n-2+t}}\right)^{\frac{n-2+t}{n}}$$

$$\leq M^{\frac{2(2-t)}{n-2}} \int_{\mathbb{R}^n} \frac{(\eta w)^2}{|y|^t} + \frac{1}{8(1+2\beta)} \int_{\mathbb{R}^n} |\nabla(\eta w)|^2 \tag{3.49}$$

Substituting (3.49) in (3.48) and using $w \leq u^{\frac{n-t}{n-2}}$ we get

$$\int_{\mathbb{R}^n} |\nabla(\eta w)|^2 \leq C \left(\int_{B(0,r_i)} w^2 + \int_{B(0,r_i)} \frac{(\eta w)^2}{|y|^t} \right) \leq$$

$$\leq C \left(\int_{B(0,r_i)} u^{2\frac{n-t}{n-2}} + \int_{B(0,r_i)} \frac{u^{2\frac{n-t}{n-2}}}{|y|^t} \right) \tag{3.50}$$

where C is a constant depending on β, r_i and r_{i+1}. Now using the Sobolev inequality and then passing to the limit as $m \to \infty$ we get

$$\int_{B(0,R)} u^{(1+\beta)\frac{2n}{n-2}} \leq C \left(\int_{B(0,r_i)} u^{2\frac{n-t}{n-2}} + \int_{B(0,r_i)} \frac{u^{2\frac{n-t}{n-2}}}{|y|^t} \right)^{\frac{2n}{n-2}} \tag{3.51}$$

and the right hand side is finite thanks to (3.1). This shows the required integrability and hence u is locally bounded. Also, since u is radially decreasing in y and z, we get $0 < c_1 \leq u(x) \leq c_2, \quad \forall \ |x| \leq 1$. Finally, the bounds at infinity follow from the fact that if u is a solution of (3.2), by Remark 3.4.1, also $u^*(x) = |x|^{2-n} u(|x|^{-2}x)$ solves (3.2). □

We already know that any solution of (3.2) is $C^{0,\alpha}$ if $t < \frac{2k}{n}$ (compare with Lemma 2.2 in [2]). As a consequence of Lemma 3.4.1 and standard elliptic regularity, we get a more precise result which extends Badiale and Serra' Lemma.

Lemma 3.4.3. *Let u be a solution of (3.2). Assume*
$t < 1 + \frac{k}{n}$ *if* $n \geq 4$
$t < \frac{3}{2}$ *if* $n = 3$
Then u is C^∞ in the z variable and $C^{0,\alpha}$, for some $\alpha < 1$, in the y variable.

Proof. First we show the (local) boundedness of the z derivatives of u. Note that from (3.2) and standard elliptic estimates $u \in W^{2,p}_{loc}(\mathbb{R}^n)$ for all $p < \frac{k}{t}$. Therefore $u_{z_i} \in W^{1,p}_{loc}(\mathbb{R}^n)$ for all $p < \frac{k}{t}$ and it satisfies

$$-\Delta u_{z_i} = \frac{(n+2-2t)}{(n-2)} \frac{u^{\frac{2(2-t)}{n-2}}}{|y|^t} u_{z_i} \tag{3.52}$$

In order to prove u_{z_i} is locally bounded, it is enough to prove, in view of Lemma 3.4.1, that $u_{z_i} \in W^{1,2}_{loc}(\mathbb{R}^n)$.
This clearly happens if $k \geq 4$, or $t < 1$, since $\frac{k}{t} > 2$ in all these cases. So, let $k = 2, 3$ and $t \in [1, 2)$.
Write $\frac{u^{\frac{2(2-t)}{n-2}}}{|y|^t} u_{z_i} = \frac{u^{\frac{2(2-t)}{n-2}}}{|y|^{t-1}} \frac{u_{z_i}}{|y|}$. From Sobolev (see [3]) $u_{z_i} \in W^{1,p}_{loc}(\mathbb{R}^n) \Rightarrow$

$u_{z_i} \in L^p_{loc}(\mathbb{R}^n)$. Now, under the given assumptions, $\frac{2t-1}{k} < 1$, and $n < \frac{k}{t-1}$. Hence one can find $q \in (n, \frac{k}{t-1})$ and $p_1 < \frac{k}{t}$, such that $\frac{1}{r_1} := \frac{1}{p_1} + \frac{1}{q} < 1$. Thus, $\frac{u^{\frac{2(2-t)}{n-2}}}{|y|^t} u_{z_i} \in L^{r_1}_{loc}(\mathbb{R}^n)$. By elliptic regularity and Sobolev embedding, $u_{z_i} \in W^{1,p_2}_{loc}$, $p_2 := \frac{nr_1}{n-r_1} \geq p_1\gamma$ where $\gamma := \frac{nq}{nq+(n-q)} > 1$ by the choice of q. A bootstrap argument gives eventually $u_{z_i} \in W^{1,2}_{loc}(\mathbb{R}^n)$.
Similar arguments proves the local boundedness of $u_{z_i}, u_{z_i z_j}, u_{z_i z_j z_k}, \ldots$..
To prove holder continuity in the y variable, let us define $v(y) = u(y,z)$ for $y \in \mathbb{R}^k$. Then from the local bounds on u and its z derivatives, we get $\Delta v \in L^p_{loc}(\mathbb{R}^k)$ locally uniformly in z, for all $p < \frac{k}{t}$. Hence $v \in W^{1,\frac{kp}{k-p}}_{loc}(\mathbb{R}^k)$ for all $p < \frac{k}{t}$. So, for $t \leq 1$ v is in fact holder continuous of any exponent by Morrey's theorem. More in general, since $t < 1 + \frac{k}{n} \Leftrightarrow \frac{nk}{nt-k} > k$ and hence $\frac{np}{n-p} > k$ for suitable $p < \frac{k}{t}$, we still obtain $v \in C^\alpha_{loc}(\mathbb{R}^k)$ for some $\alpha \in (0,1)$. This proves the lemma. □

3.5 Classification of solutions of a critical weighted operator

In this section we classify all the solutions of

$$\left.\begin{array}{rcll} -\Delta u & = & \frac{u^{\frac{n}{n-2}}}{|y|} & \text{in} \quad \mathbb{R}^n \\ u & > & 0 & \\ u & \in & D^{1,2}(\mathbb{R}^n) & \end{array}\right\} \tag{3.53}$$

where $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $k \geq 2, n \geq 3$ and a point $x \in \mathbb{R}^n$ is denoted as $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.
The problem (3.53) is a particular case, (in which $t = 1$), of the problem (3.2) studied in the previous section. Note that $t = 1$ is the unique integer belonging to the interval $(0,2)$. We have already proved that all solution of the Euler equations associated to (3.1) inequality (see [22]) have cylindrical symmetry. Thanks to these symmetries, (3.53) reduces to an elliptic equation in the positive cone in $\mathbb{R}^2$ which eventually leads to a complete identification of all the solutions of (3.53). The identification of the solutions to (3.53) is based on a mysterious identity which goes back to the work of Jerison and Lee ([31]) on the CR-Yamabe problem. More precisely, it is related to the identification of the extremals for the Sobolev inequality on the Heisenberg group ([32]). Actually, we follow closely the approach by Garofalo-Vassilev ([25]) in the search of entire solutions of Yamabe-type equations on more general groups of Heisenberg type. We also remark that, while symmetry properties hold true for (3.2), we didn't succeeded in getting an efficient Jerison-Lee type identity in the general case, and this is why a classification of solutions is missing if, in (1), $s \neq 1$. Similar difficulties are encountered in dealing with Grushin type operators $-\Delta_x - (\alpha+1)^2|x|^{2\alpha}\Delta_y$, $(x,y) \in \mathbb{R}^m \times \mathbb{R}^k$ with critical nonlinearity (see [6] for a related sharp Sobolev inequality and

identification of extremals in case $\alpha = 1$). As noticed in [41], the Heisenberg sublaplacian is in fact a Grushin operator with $\alpha = 1$, m an even integer and $k = 1$ (the work of Garofalo-Vassiliev actually deals with more general values of m and k) and identification of solutions is available only in case $\alpha = 1$.

As a consequence we obtain the best constant and extremals of the related weighted Sobolev inequality:

$$S\left(\int_{\mathbb{R}^n} \frac{|u|^{2_*(1)}}{|y|}\right)^{\frac{2}{2_*(1)}} \leq \int_{\mathbb{R}^n} |\nabla u|^2 \quad \forall u \in D^{1,2}(\mathbb{R}^n) \tag{3.54}$$

where $2_*(1) = 2\frac{n-1}{n-2}, S = S_{n,k,1}$.

Let u be a solution of (3.53). By Theorem 3.3.1, we know that u can be written, up to a translation in the z variable, as

$$u(y, z) = \theta(|y|, |z|). \tag{3.55}$$

where $\theta(r, s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. By Lemma 3.4.1 and 3.4.2, we have

$$|\theta(r, s)| + |\theta_s(r, s)| + |\theta_{ss}(r, s)| \leq C \text{ for all } |(r, s)| \leq R \tag{3.56}$$

for some constant depending only on R. Define for $r > 0$, and $s > 0$

$$\phi(r, s) = \left(\frac{n-2}{2}\right)^2 \theta^{\frac{-2}{n-2}}(r, s) \tag{3.57}$$

Then from the equation θ satisfies, we get

$$\Delta\phi + \frac{k-1}{r}\phi_r + \frac{n-k-1}{s}\phi_s = \frac{n}{2}\frac{|\nabla\phi|^2}{\phi} + \frac{n-2}{2r} \tag{3.58}$$

Let us also define

$$F = 2\nabla\phi.\nabla\phi_r - \frac{|\nabla\phi|^2}{\phi}\phi_r + \frac{n-2}{2(k-1)}\frac{|\nabla\phi|^2}{\phi} - \frac{n-2}{(k-1)}\phi_{rr} \tag{3.59}$$

$$G = 2\nabla\phi.\nabla\phi_s - \frac{|\nabla\phi|^2}{\phi}\phi_s - \frac{n-2}{(k-1)}\phi_{rs} \tag{3.60}$$

$$h(r, s) = r^{k-1}s^{n-k-1}\phi^{-(n-1)} \tag{3.61}$$

and X be the vector field

$$X = (hF, hG). \tag{3.62}$$

With these definitions we can state our Jerison-Lee identity as

Lemma 3.5.1. *Let ϕ and X be as above, then*

$$divX = h\left[\left(2|\nabla^2\phi|^2 - (\Delta\phi)^2\right) + \frac{n}{n-2}\left(\Delta\phi - \frac{|\nabla\phi|^2}{\phi}\right)^2\right]$$
$$+h\left[\frac{2(k-1)(n-k-1)}{n-2}\left(\frac{\phi_s}{s} - \frac{\phi_r}{r} + \frac{n-2}{2(k-1)r}\right)^2\right]$$

Proof. For convenience we split X as $X = Y + \frac{n-2}{k-1}Z$, where

$$Z := h\left(\frac{1}{2}\frac{|\nabla\phi|^2}{\phi} - \phi_{rr}, -\phi_{rs}\right), \qquad Y := X - \frac{n-2}{k-1}Z$$

Then by direct calculation

$$divY = h\left[\left(2|\nabla^2\phi|^2 - (\Delta\phi)^2\right) + \frac{n}{n-2}\left(\Delta\phi - \frac{|\nabla\phi|^2}{\phi}\right)^2 + 2\nabla\phi.\nabla\psi\right]$$
$$-h\left[\frac{|\nabla\phi|^2}{\phi}\psi - \frac{2}{n-2}\psi^2 + \frac{k-1}{r}F + \frac{n-k-1}{s}G\right] \tag{3.63}$$

where $\psi = \Delta\phi - \frac{n}{2}\frac{|\nabla\phi|^2}{\phi}$. Substituting ψ from (3.58) as $\psi = \frac{n-2}{2r} - \frac{k-1}{r}\phi_r - \frac{n-k-1}{s}\phi_s$, (3.63) becomes

$$divY = h\left\{\left(2|\nabla^2\phi|^2 - (\Delta\phi)^2\right) + \frac{n}{n-2}\left(\Delta\phi - \frac{|\nabla\phi|^2}{\phi}\right)^2\right\} +$$
$$+h\left\{\frac{2(k-1)(n-k-1)}{n-2}\left(\frac{\phi_r}{r} - \frac{\phi_s}{s}\right)^2\right\} -$$
$$-h\left\{\frac{n-2}{2r}\frac{|\nabla\phi|^2}{\phi} + (2k-n)\frac{\phi_r}{r^2} + 2(n-k-1)\frac{\phi_s}{rs} - \frac{n-2}{2r^2}\right\} \tag{3.64}$$

Again by direct calculation

$$\frac{n-2}{k-1}divZ = -\frac{n-2}{r^2}\phi_r + \frac{(n-2)^2}{2(k-1)r^2} + \frac{n-2}{2r}\frac{|\nabla\phi|^2}{\phi} \tag{3.65}$$

Now (3.63) follows by adding (3.63) and (3.65). □

To put at work the Jerison-Lee identity, we need more estimates on our solution written in cylindrical coordinates:

Lemma 3.5.2. *Let u be a solution of (3.53) and θ be as in (3.55). Then there exists a constant $C > 0$ such that*
(i) $\frac{1}{C} \leq \left(1 + (r^2+s^2)^{\frac{n-2}{2}}\right)\theta(r,s) \leq C$
(ii) $\left(1 + (r^2+s^2)^{\frac{n-1}{2}}\right)|\nabla\theta(r,s)| \leq C$
(iii) $\left(1 + (r^2+s^2)^{\frac{n}{2}}\right)|\nabla^2\theta(r,s)| \leq C.$

Proof. Let as above $v(y) = u(y,z)$ for $y \in \mathbb{R}^k$. We know that $v \in C^{\alpha}_{loc}(\mathbb{R}^k)$ for all $\alpha \in (0,1)$. Hence $|y|^{\beta}|\nabla v(y)| \to 0$ as $|y| \to 0$ for all $\beta > 0$. Hence

$$|r^{\beta}\theta_r(r,s)| \to 0 \text{ for all } \beta > 0 \tag{3.66}$$

Now from (3.53), θ satisfies the equation

$$\theta_{rr} + \frac{k-1}{r}\theta_r = -\frac{\theta^{\frac{n}{n-2}}}{r} - f(r,s) \tag{3.67}$$

where $f(r,s)$ is the Laplacian in the z variable, which we know is locally bounded. Multiplying (3.67) by r^{k-1} and integrating using (3.66) leads to

$$\theta_r(r,s) = \frac{-1}{r^{k-1}}\int_0^r t^{k-2}\theta^{\frac{n}{n-2}}(t,s)dt - \frac{1}{r^{k-1}}\int_0^r t^{k-1}f(t,s) \tag{3.68}$$

and the later is bounded. This together with the (3.56) gives

$$|\nabla\theta(r,s)| \leq C \text{ for all } |(r,s)| \leq R \tag{3.69}$$

An integration by parts in (3.68) gives

$$\theta_r = -\frac{\theta^{\frac{n}{n-2}}}{k-1} + \frac{C}{r^{k-1}}\int_0^r t^{k-1}\theta^{\frac{2}{n-2}}\theta_r - \frac{1}{r^{k-1}}\int_0^r t^{k-1}f(t,s) \tag{3.70}$$

and hence

$$\frac{k-1}{r}\theta_r + \frac{\theta^{\frac{n}{n-2}}}{r} = \frac{C}{r^k}\int_0^r t^{k-1}\theta^{\frac{2}{n-2}}\theta_r - \frac{1}{r^k}\int_0^r t^{k-1}f(t,s) \tag{3.71}$$

and the later is locally bounded. Plugging back this information in (3.67) gives the local boundedness of θ_{rr}. The local boundedness of θ_{rs} follows similarly by integrating the o.d.e u_{z_i} satisfies in the variable r. Thus we have

$$||\theta(r,s)||_{C^2} \leq C \text{ for all } |(r,s)| \leq R \tag{3.72}$$

Now to prove the bounds at infinity, let us observe that if u is a solution of (3.53) then $u^*(x) = |x|^{2-n}u(|x|^{-2}x)$ also solves (3.53). Therefore if we define $\sigma(r,s) = (r^2+s^2)^{\frac{2-n}{2}}\theta((r^2+s^2)^{-1}(r,s))$ then σ is locally bounded in C^2. This immediately proves (i) of Lemma 3.5.2 once we note that σ and θ are bounded below by positive constants locally as they are decreasing in both variables. Also by direct computation

$$|\nabla\theta(r,s)| \leq (r^2+s^2)^{\frac{-n}{2}}|\nabla\sigma((r^2+s^2)^{-1}(r,s))| + (n-2)(r^2+s^2)^{\frac{-1}{2}}\theta(r,s) \tag{3.73}$$

This together with (i) of Lemma 3.5.2 proves (ii). Again by direct calculation

$$|\nabla^2\theta(r,s)| \le (r^2+s^2)^{\frac{-(n+2)}{2}}|\nabla^2\sigma((r^2+s^2)^{-1}(r,s))|+$$
$$+C(r^2+s^2)^{\frac{-1}{2}}|\nabla\theta(r,s)|+C(r^2+s^2)^{-1}\theta(r,s) \tag{3.74}$$

This together with (i) and (ii) proves (iii) and hence the lemma. □

Theorem 3.5.3. *Let u_0 be the function given by*

$$u_0(x)=u_0(y,z)=c_{n,k}\left((1+|y|)^2+|z|^2\right)^{-\frac{n-2}{2}}$$

where $c_{n,k}=\{(n-2)(k-1)\}^{\frac{n-2}{2}}$. Then u is a solution of (3.53) if and only if $u(y,z)=\lambda^{\frac{n-2}{2}}u_0(\lambda y,\lambda(z+z_0))$ for some $\lambda>0$ and $z_0\in\mathbb{R}^{n-k}$.

Proof. Let $0<\epsilon<R$ and define

$$\Omega_{\epsilon,R}=\left\{(r,s)\ :\ s>0, r>\epsilon, r^2+s^2<R^2\right\}$$

Integration by parts gives

$$\int\limits_{\Omega_{\epsilon,R}} divX=\int\limits_{\partial\Omega_{\epsilon,R}} X.\nu dH^1$$

where ν is the outward normal to $\partial\Omega_{\epsilon,R}$ and dH^1 is the surface measure on the boundary. Now the boundary integral can be split as

$$\int\limits_{\partial\Omega_{\epsilon,R}} X.\nu dH^1=-\int\limits_{\Gamma_1} hGdH^1-\int\limits_{\Gamma_2} hFdH^1+\int\limits_{\Gamma_3} X.\frac{1}{R}(r,s)dH^1$$

where $\Gamma_1=\partial\Omega_{\epsilon,R}\cap\{s=0\}$, $\Gamma_2=\partial\Omega_{\epsilon,R}\cap\{r=\epsilon\}$ and $\Gamma_3=\partial\Omega_{\epsilon,R}\cap\{r^2+s^2=R^2\}$. Now note that since our original solution u is smooth away from $\{|y|=0\}$, θ_s,θ_{rs} and hence ϕ_s and ϕ_{rs} vanishes on Γ_1. Consequently $G=0$ on Γ_1. Hence

$$\int\limits_{\Omega_{\epsilon,R}} divX=-\epsilon^{k-1}\int\limits_{\Gamma_2} s^{n-k-1}\frac{F(\epsilon,s)}{\phi^{n-1}}+\frac{1}{R}\int\limits_{\Gamma_3}\left\{r^k s^{n-k-1}\frac{F(r,s)}{\phi^{n-1}}+r^{k-1}s^{n-k}\frac{G(r,s)}{\phi^{n-1}}\right\}dH^1$$

It follows from Lemma 3.4.2 that ϕ is locally bounded in C^2 and for $r^2+s^2>1$, $\frac{1}{C}(r^2+s^2)\le\phi(r,s)\le C(r^2+s^2)$, $|\nabla\phi(r,s)|\le C(r^2+s^2)^{\frac{1}{2}}$ and $|\nabla^2\phi(r,s)|\le C$, for some positive constant C. Hence F and G are locally bounded and $|F(r,s)|\le C(r^2+s^2)^{\frac{1}{2}}$ and $|G(r,s)|\le C(r^2+s^2)^{\frac{1}{2}}$ for $r^2+s^2>1$, and for some positive constant C. Using this estimates we get

$$|\int\limits_{\Omega_{\epsilon,R}} divX|\le C\left(\epsilon^{k-1}+R^{-(n-2)}\right).$$

where C is independent of both R and ϵ. Now passing to the limit as $R \to \infty$ and then $\epsilon \to 0$ we get $\int\limits_{\mathbb{R}^+ \times \mathbb{R}^+} divX = 0$. Note from the right hand side of (3.63) that $divX \geq 0$, and hence $divX = 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$ and thus from (3.63), we obtain

$$2|\nabla^2 \phi|^2 - (\Delta\phi)^2 = 0 \tag{3.75}$$

and

$$\Delta\phi - \frac{|\nabla\phi|^2}{\phi} = 0 \tag{3.76}$$

It follows from (3.75) that

$$\phi_{rr} = \phi_{ss} \text{ and } \phi_{rs} = 0$$

This immediately tells us that all third order derivatives of ϕ vanishes and hence ϕ can be written as

$$\phi(r,s) = a_0 + a_1 r + a_2 s + a_3 rs + a_4 r^2 + a_5 s^2$$

Recall that $\phi_s(r,0) = 0$ and this gives,

$$a_2 = a_3 = 0 \tag{3.77}$$

Since $\phi_{rr} = \phi_{ss}$, $a_4 = a_5$. Also from (3.76) and the fact that ϕ is not a constant, we can write

$$a_4 = a_5 = m, \ m > 0. \tag{3.78}$$

Now from (3.63) and (3.76), ϕ satisfies

$$\frac{n-2}{2}\Delta\phi + \frac{n-2}{2r} = (k-1)\frac{\phi_r}{r} + (n-k-1)\frac{\phi_s}{s} \tag{3.79}$$

and this gives

$$a_1 = \frac{n-2}{2(k-1)} \tag{3.80}$$

Finally from (3.76),

$$a_0 = \left(\frac{a_1}{2m}\right)^2 \tag{3.81}$$

Now (3.77), (3.78), (3.80) and (3.81) together gives

$$\phi(r,s) = m\left\{\left(r + \frac{n-2}{4(k-1)m}\right)^2 + s^2\right\} \tag{3.82}$$

Writing $m = \frac{n-2}{4(k-1)}\lambda$, $\lambda > 0$, we get

$$\phi(r,s) = \frac{n-2}{4(k-1)}\lambda\left\{\left(r + \frac{1}{\lambda}\right)^2 + s^2\right\}$$

and thus

$$u(y,z) = \lambda^{\frac{n-2}{2}} u_0(\lambda y, \lambda z),$$

where u_0 is as in Theorem 5.1.1. Conversely by direct calculation one can see that the function $u(y,z) = \lambda^{\frac{n-2}{2}} u_0(\lambda y, \lambda(z+z_0))$, indeed solve (3.53) for any $\lambda > 0$ and $z_0 \in \mathbb{R}^{n-k}$. This completes the proof of Theorem 3.5.3. $\square$

Theorem 3.5.4. *The best constant S in the weighted Sobolev inequality (3.54) is given by*

$$S = (n-2)(k-1)\left\{2(\pi)^{\frac{n}{2}} \frac{(k-2)!}{(n+k-3)!} \frac{\Gamma(\frac{n+k-2}{2})}{\Gamma(k/2)}\right\}^{\frac{1}{n-1}}.$$

Proof. Let u_0 be as in Theorem 3.5.3. We note that

$$\int_{\mathbb{R}^n} |\nabla u_0|^2\, dx = \int_{\mathbb{R}^n} \frac{u_0^{\frac{2(n-1)}{n-2}}}{|y|}\, dx.$$

Then

$$S = \frac{\int_{\mathbb{R}^n} |\nabla u_0|^2\, dx}{\left(\int_{\mathbb{R}^n} \frac{u_0^{2_*}}{|y|}\, dx\right)^{2/2_*}} = \frac{\int_{\mathbb{R}^n} \frac{u_0^{2_*}}{|y|}\, dx}{\left(\int_{\mathbb{R}^n} \frac{u_0^{2_*}}{|y|}\, dx\right)^{2/2_*}} = \left(\int_{\mathbb{R}^n} \frac{u_0^{2_*}}{|y|}\, dx\right)^{1-2/2_*} = \left(\int_{\mathbb{R}^n} \frac{u_0^{2_*}}{|y|}\, dx\right)^{1-\frac{n-2}{n-1}}.$$

Therefore

$$S^{n-1} = \int_{\mathbb{R}^n} |\nabla u_0|^2\, dx = \int_{\mathbb{R}^n} \frac{u_0^{\frac{2(n-1)}{n-2}}}{|y|}\, dx = \int_{\mathbb{R}^n} [c_{n,k}(1+|y|)^2+|z|^2)^{-(n-2)/2}]^{\frac{2(n-1)}{n-2}} \frac{1}{|y|}\, dx$$

$$= c_{n,k}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^n} ((1+|y|)^2+|z|^2))^{-(n-1)} \frac{1}{|y|}\, dx = \{(n-2)(k-1)\}^{n-1} \int_{\mathbb{R}^n} \frac{1}{((1+|y|)^2+|z|^2)^{n-1}} \frac{1}{|y|}\, dx$$

$$= \{(n-2)(k-1)\}^{n-1}\, \omega_{k-1}\omega_{n-k-1} \int_0^\infty \int_0^\infty \frac{r^{k-2}s^{n-k-1}}{((1+r)^2+s^2)^{n-1}}\, dr\, ds$$

where ω_{k-1} and ω_{n-k-1} are the surface measure of the $k-1$ and $n-k-1$ dimensional sphere in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively. Now

$$\int_0^\infty \int_0^\infty \frac{r^{k-2}s^{n-k-1}}{((1+r)^2+s^2)^{n-1}}\, dr\, ds \quad = \quad \int_0^\infty \frac{r^{k-2}dr}{(1+r)^{n+k-2}} \int_0^\infty \frac{s^{n-k-1}ds}{(1+s^2)^{n-1}}$$

Now consider, for $j \in \mathbb{N}, a \in \mathbb{R}^+, a > \frac{j}{2}$ the integral

$$\int_0^\infty \frac{\rho^{j-1} ds}{(1+\rho^2)^a} = \frac{1}{2} B(\frac{j}{2}, a - \frac{j}{2})$$

where $B(x,y)$ is the Beta function. Recalling that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, where Γ indicates the Euler's Gamma function, we conclude that

$$\int_0^\infty \frac{\rho^{j-1} ds}{(1+\rho^2)^a} = \frac{1}{2} \frac{\Gamma(\frac{j}{2})\Gamma(a-\frac{j}{2})}{\Gamma(a)} \tag{3.83}$$

Moreover, if p is an integer, $\Gamma(p) = (p-1)!$

$$\int_0^\infty \frac{s^{n-k-1} ds}{(1+s^2)^{n-1}} = \frac{1}{2} \frac{\Gamma(\frac{n-k}{2})\,\Gamma(n-1-\frac{n-k}{2})}{\Gamma(n-1)} = \frac{1}{2} \frac{\Gamma(\frac{n-k}{2})\,\Gamma(\frac{n+k-2}{2})}{(n-2)!} \tag{3.84}$$

Analogously,

$$\int_0^\infty \frac{r^{k-2} dr}{(1+r)^{n+k-2}} = 2 \int_0^\infty \frac{t^{2k-3}\, dt}{(1+t^2)^{n+k-2}} = \frac{\Gamma(k-1)\Gamma(n+k-2-k+1)}{\Gamma(n+k-2)} = \frac{(k-2)!(n-2)!}{(n+k-3)!} \tag{3.85}$$

Recalling the formula $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$, we get

$$S^{n-1} = \{(n-2)(k-1)\}^{n-1}\, \omega_{k-1}\omega_{n-k-1} \int_0^\infty \int_0^\infty \frac{r^{k-2} s^{n-k-1}}{((1+r)^2+s^2)^{n-1}}\, dr\, ds$$

$$= \{(n-2)(k-1)\}^{n-1} \frac{2\pi^{k/2}}{\Gamma(k/2)} \frac{2\pi^{(n-k)/2}}{\Gamma(\frac{n-k}{2})} \frac{1}{2} \frac{\Gamma(\frac{n-k}{2})\,\Gamma(\frac{n+k-2}{2})}{(n-2)!} \frac{(k-2)!(n-2)!}{(n+k-3)!}$$

$$= \{(n-2)(k-1)\}^{n-1}\, 2\pi^{n/2} \frac{\Gamma(\frac{n+k-2}{2})}{\Gamma(k/2)} \frac{(k-2)!}{(n+k-3)!}$$

□

3.6 Properties of solutions

Remark 3.6.1. We note that the function $U = ((1+|y|)^2 + |z|^2)^{1-n/2}$ satisfies $U = (U)^*$.

In fact, for any $\lambda > 0$, we get:

$$
\begin{aligned}
U(\frac{\lambda x}{|\lambda x|^2}) &= U(\frac{\lambda y}{|\lambda x|^2}, \frac{\lambda z}{|\lambda x|^2}) = U(\frac{y}{\lambda |x|^2}, \frac{z}{\lambda |x|^2}) = \left(\frac{1}{(1+\frac{r}{\lambda(r^2+s^2)})^2 + (\frac{s}{\lambda(r^2+s^2)})^2} \right)^{(n-2)/2} \\
&= \frac{1}{((\frac{\lambda(r^2+s^2)+r}{\lambda(r^2+s^2)})^2 + (\frac{s}{\lambda(r^2+s^2)})^2)^{(n-2)/2}} \\
&= \frac{1}{\frac{1}{[\lambda(r^2+s^2)]^{n-2}}((\lambda(r^2+s^2)+r)^2+s^2)^{(n-2)/2}} \\
&= \frac{[\lambda(r^2+s^2)]^{n-2}}{((\lambda(r^2+s^2)+r)^2+s^2)^{(n-2)/2}} \\
&= \frac{[\lambda(r^2+s^2)]^{n-2}}{(\lambda^2(r^2+s^2)^2+r^2+2\lambda r(r^2+s^2)+s^2)^{(n-2)/2}} \\
&= \frac{[\lambda(r^2+s^2)]^{n-2}}{((r^2+s^2)(\lambda(r^2+s^2)+2\lambda r+1))^{(n-2)/2}} \\
&= \frac{\lambda^{n-2}(r^2+s^2)^{(n-2)/2}}{((1+\lambda r)^2+(\lambda s)^2)^{(n-2)/2}} \\
&= \lambda^{n-2}|x|^{n-2}U(\lambda x)
\end{aligned}
$$

Thus, the function $w(y,z) = \lambda^{\frac{n-2}{2}} U(\lambda y, \lambda z) = \lambda^{\frac{n-2}{2}} (U)^*(\lambda x)$
$= \lambda^{\frac{n-2}{2}} |\lambda x|^{2-n} U(\lambda x |\lambda x|^{-2}) = \lambda^{\frac{2-n}{2}} |x|^{2-n}\, U(\lambda x |\lambda x|^{-2})$.

Remark 3.6.2. If we define

$$V(y,z) = |y|^{2-n}\, U(\frac{y}{|y|^2}, \frac{z}{|y|}), \tag{3.86}$$

then $V = U$.

In fact,

$$
\begin{aligned}
U(\frac{y}{|y|^2}, \frac{z}{|y|}) &= \frac{1}{((1+|\frac{y}{y^2}|)^2 + (|\frac{z}{y}|)^2)^{\frac{n-2}{2}}} = \frac{1}{((\frac{|y|+1}{|y|})^2 + \frac{|z|^2}{|y|^2})^{\frac{n-2}{2}}} \\
&= \frac{|y|^{n-2}}{((1+|y|)^2+|z|^2)^{\frac{n-2}{2}}} = |y|^{n-2}\, U(y,z)\,.
\end{aligned}
$$

We want to compute $\Delta_{y,z} V(y,z)$.
Let $x = (y,z)$ be a point in $\mathbb{R}^n$, $x' = (\frac{y}{|y|^2}, \frac{z}{|y|}) = (y', z')$.
Note that $|y'| = \frac{1}{|y|}$, $y = \frac{y'}{|y'|^2}$, $|z'| = \frac{|z|}{|y|} = |z|\,|y'|$, $z = \frac{z'}{|y'|}$. Then

i) $\nabla_y V = \frac{(2-n)y}{|y|^n}\, U + \frac{1}{|y|^n}\, \nabla_{y'} U - \frac{2\,y}{|y|^{n+2}}\, \nabla_{y'} U \cdot y - \frac{y}{|y|^{n+1}}\, \nabla_{z'} U \cdot z$.

ii) $\nabla_z V = |y|^{2-n}\,\nabla_{z'} U\,\frac{1}{|y|} = \frac{1}{|y|^{n-1}}\nabla_{z'} U.$

iii) $\frac{1}{|y|^2}\nabla_y V\cdot y = \frac{(2-n)}{|y|^{n-2}}\,U\, + \frac{1}{|y|^n}\,\nabla_{y'} U\cdot y - \frac{2}{|y|^n}\,\nabla_{y'} U\cdot y - \frac{1}{|y|^{n-1}}\,\nabla_{z'} U\cdot z$
$= \frac{(2-n)}{|y|^{n-2}}\,U\, - \frac{1}{|y|^n}\,\nabla_{y'} U\cdot y - \frac{1}{|y|^{n-1}}\,\nabla_{z'} U\cdot z.$

iv) $\nabla_z V\cdot\frac{z}{|y|^2} = \frac{1}{|y|^{n+1}}\,\nabla_{z'} U\cdot z.$

We know that
$\Delta_x V = \Delta_x(|y|^{2-n})\,U + |y|^{2-n}\Delta_x U(y',z') + 2\nabla_x(|y|^{2-n})\,\cdot\nabla_x U$. So, we must compute three therms.
First therm: $|y|^{2-n}\Delta_x\,U(\frac{y}{|y|^2},\frac{z}{|y|})$.
Now

$$\nabla_y U(\frac{y}{|y|^2},\frac{z}{|y|}) = \nabla_{y'} U\,\frac{\partial}{\partial y}(\frac{y}{|y|^2}) + \nabla_{z'} U\,\frac{\partial}{\partial y}(\frac{z}{|y|}) \tag{3.87}$$

$$\begin{aligned}&\nabla_y\cdot\left(\nabla_y U(\frac{y}{|y|^2},\frac{z}{|y|})\right)\\ &= \sum_{i,j=1}^{k}\frac{\partial^2 U}{\partial^2 y_i' y_j'}\left(\sum_{\alpha=1}^{k}\frac{\partial y_i'}{\partial x_\alpha}\cdot\frac{\partial y_j'}{\partial x_\alpha}\right)\\ &+\sum_{i=1}^{k}\frac{\partial U}{\partial y_i'}\left(\sum_{\alpha=1}^{k}\frac{\partial^2 y_i'}{\partial x_\alpha}\right)+\sum_{i=k+1}^{n}\frac{\partial U}{\partial z_i'}\left(\sum_{\alpha=1}^{k}\frac{\partial z_i'}{\partial x_\alpha}\right)\end{aligned} \tag{3.88}$$

So,

$$\begin{aligned}\Delta_y U(\frac{y}{|y|^2},\frac{z}{|y|}) &= \sum_{i=1}^{k}\frac{\partial^2 U}{\partial {y_i'}^2}\frac{1}{|y|^4} - \frac{2(k-2)}{|y|^4}\,\nabla_{y'} U\cdot y - \frac{(k-3)}{|y|^3}\,\nabla_{z'} U\cdot z\\ &= \frac{1}{|y|^4}\,\Delta_{y'} U - \frac{2(k-2)}{|y|^4}\,\nabla_{y'} U\cdot y - \frac{(k-3)}{|y|^3}\,\nabla_{z'} U\cdot z\\ &= |y'|^4\,\Delta_{y'} U - 2(k-2)|y'|^2\,\nabla_{y'} U\cdot y' - (k-3)|y'|^2\nabla_{z'} U\cdot z'\end{aligned} \tag{3.89}$$

while

$$\nabla_z U(\frac{y}{|y|^2},\frac{z}{|y|}) = \frac{1}{|y|}\,\nabla_{z'} U \tag{3.90}$$

$$\Delta_z U(\frac{y}{|y|^2},\frac{z}{|y|}) = \frac{1}{|y|^2}\,\Delta_{z'} U = |y'|^2\Delta_{z'} U \tag{3.91}$$

Then

$$\Delta_x U(\frac{y}{|y|^2},\frac{z}{|y|}) = |y'|^4\,\Delta_{y'} U + |y'|^2\Delta_{z'} U - 2(k-2)|y'|^2\,\nabla_{y'} U\cdot y' - (k-3)|y'|^2\nabla_{z'} U\cdot z' \tag{3.92}$$

and

$$
\begin{aligned}
&|y|^{2-n}\,\Delta_{y,z}\,U(\frac{y}{|y|^2},\frac{z}{|y|})\\
&=|y|^{2-n}\,[|y\,'|^4\,\Delta_{y'}U+|y\,'|^2\Delta_{z'}U-2(k-2)|y\,'|^2\,\nabla_{y'}U\cdot y'-(k-3)|y\,'|^2\nabla_{z'}U\cdot z']\\
&=\frac{1}{|y|^{n+2}}\,\Delta_{y'}U+\frac{1}{|y|^n}\,\Delta_{z'}U-\frac{2(k-2)}{|y|^n}\,\nabla_{y'}U\cdot\frac{y}{|y|^2}\;-\frac{(k-3)}{|y|^n}\nabla_{z'}U\cdot\frac{z}{|y|}\,.
\end{aligned}
\tag{3.93}
$$

Second therm: $\Delta_x(|y|^{2-n})\,U$.
But

$$
\Delta_x(|y|^{2-n})=\Big(\Delta_y+\Delta_z\Big)(|y|^{2-n})=\Delta_y(|y|^{2-n})=\frac{(n-2)(n-k)}{|y|^n}
$$

and

$$
\Delta_x(|y|^{2-n})\,U=\,\frac{(n-2)(n-k)}{|y|^n}\,U\,. \tag{3.94}
$$

Third therm: $2\,\left\langle\nabla_x|y|^{2-n},\nabla_xU(y',z')\right\rangle$. Now

$$
\nabla_x|y|^{2-n}=\nabla_y|y|^{2-n}=\frac{(2-n)y}{|y|^n}\,.
$$

We can compute

$$
\begin{aligned}
\nabla_yU(y',z')&=\nabla_yU(\frac{y}{|y|^2},\frac{z}{|y|})\\
&=\frac{1}{|y|^2}\,\nabla U_{y'}-\frac{2\,y}{|y|^4}\,\nabla U_{y'}U\cdot y-\frac{y}{|y|^3}\,\nabla_{z'}U\cdot z
\end{aligned}
\tag{3.95}
$$

Then

$$
\begin{aligned}
&2\,\left\langle\nabla_x|y|^{2-n},\nabla_xU(y',z')\right\rangle\,=2\,\left\langle\nabla_y|y|^{2-n},\nabla_yU(y',z')\right\rangle\,=2\,\left\langle\nabla_x|y|^{2-n},\nabla_xU(y',z')\right\rangle\\
&=2\left\langle\frac{(2-n)\,y}{|y|^n},\frac{1}{|y|^2}\,\nabla_{y'}U-\frac{2\,y}{|y|^4}\,\nabla_{y'}U\cdot y-\frac{y}{|y|^3}\,\nabla_{z'}U\cdot z\right\rangle\\
&=\frac{2\,(2-n)}{|y|^{n+2}}\,y\cdot\nabla_{y'}U-\frac{4(2-n)|y|^2}{|y|^{n+4}}\,y\cdot\nabla_{y'}U-\frac{2\,(2-n)|y|^2}{|y|^{n+3}}\,z\cdot\nabla_{z'}U\\
&=-\frac{2\,(2-n)}{|y|^{n+2}}\,y\cdot\nabla_{y'}U-\frac{2\,(2-n)}{|y|^{n+1}}\,z\cdot\nabla_{z'}U
\end{aligned}
\tag{3.96}
$$

Summing (3.93), (3.94), (3.96), we get

$$\Delta_{y,z}V = \frac{1}{|y|^{n+2}}\,\Delta_{y'}U + \frac{1}{|y|^{n}}\,\Delta_{z'}U - \frac{2(k-2)}{|y|^{n}}\,\nabla_{y'}U\cdot\frac{y}{|y|^2} - \frac{(k-3)}{|y|^{n}}\nabla_{z'}U\cdot\frac{z}{|y|}$$
$$+\frac{(n-2)(n-k)}{|y|^{n}}\,U - \frac{2\,(2-n)}{|y|^{n+2}}\,y\cdot\nabla_{y'}U - \frac{2\,(2-n)}{|y|^{n+1}}\,z\cdot\nabla_{z'}U$$
$$=\frac{(n-2)(n-k)}{|y|^{n}}\,U \;+\frac{1}{|y|^{n+2}}\,\Delta_{y'}U + \frac{1}{|y|^{n}}\,\Delta_{z'}U$$
$$-\frac{2\,(2-n+k-2)}{|y|^{n+2}}\,y\cdot\nabla_{y'}U + \frac{(-4+n-k+3)}{|y|^{n+1}}\,z\cdot\nabla_{z'}U$$
$$=\frac{(n-2)(n-k)}{|y|^{n}}\,U \;+\frac{1}{|y|^{n+2}}\,\Delta_{y'}U + \frac{1}{|y|^{n}}\,\Delta_{z'}U$$
$$+\frac{2\,(n-k)}{|y|^{n+2}}\,y\cdot\nabla_{y'}U + \frac{(n-k-1)}{|y|^{n+1}}\,z\cdot\nabla_{z'} \tag{3.97}$$

But the function V satisfies the properties (1)–(4), and so (3.97) rewrites

$$\Delta_{y,z}V + \frac{2(n-k)}{|y|^2}y\cdot\nabla_y V - \frac{(k-1)}{|y|^2}z\cdot\nabla_z V + \frac{(n-2)(n-k)}{|y|^{n}}\,U$$
$$=\frac{1}{|y|^{n+2}}\,\Delta_{y'}U(y',z') + \frac{1}{|y|^{n}}\,\Delta_{z'}U(y',z') \tag{3.98}$$

that implies

$$\Delta_y V + \frac{1}{|y|^2}\Delta_z V + \frac{2(n-k)}{|y|^2}y\cdot\nabla_y V - \frac{(k-1)}{|y|^2}z\cdot\nabla_z V + (n-2)(n-k)\,\frac{V}{|y|^2}$$
$$=\frac{1}{|y|^{n+2}}\,\Delta_{y'}U(y',z') + \frac{1}{|y|^{n+2}}\,\Delta_{z'}U(y',z') = \frac{1}{|y|^{n+2}}\,\Delta_{y',z'}U(y',z')$$
$$=-\frac{1}{|y|^{n+2}}\frac{U(y',z')^{\frac{n}{n-2}}}{|y'|} = -\,|y|^{-n-2}\,|y|\Big(|y|^{n-2}V(y,z)\Big)^{\frac{n}{n-2}} = -\,\frac{V(y,z)^{\frac{n}{n-2}}}{|y|}\,. \tag{3.99}$$

Finally, we get the following theorem

Theorem 3.6.1. *Let U a solution to the problem (3.53). Let V be defined as (3.86). Then V satisfies the following semilinear equation:*

$$\Delta_y V + \frac{1}{|y|^2}\Delta_z V + \frac{2(n-k)}{|y|^2}y\cdot\nabla_y V - \frac{(k-1)}{|y|^2}z\cdot\nabla_z V + (n-2)(n-k)\,\frac{V}{|y|^2}$$
$$=-\,\frac{V(y,z)^{\frac{n}{n-2}}}{|y|}\,. \tag{3.100}$$

Thus, to any function U, solution to (3.53), we can associate a function $V(y,z) = |y|^{2-n}U(\frac{y}{|y|^2},\frac{z}{|y|})$ which solves the equation

$$\Delta_H V + (n-2)(n-k)\,\frac{V}{|y|^2} = -\,\frac{V(y,z)^{n/(n-2)}}{|y|} \tag{3.101}$$

where $\Delta_H = \Delta_y + \frac{1}{|y|^2}\Delta_z + \frac{2(n-k)}{|y|^2}\, y \cdot \nabla_y - \frac{(k-1)}{|y|^2}\, z \cdot \nabla_z$
is a Grushin operator.

Corollary 3.6.2. *Let U a solution to the problem (3.53). Let W be defined as*

$$W(y,z) = |y|^{2-n}\, U(\frac{y}{|y|^2}, z) \tag{3.102}$$

Then W satisfies the following semilinear equation

$$\Delta_y W + \frac{1}{|y|^4}\, \Delta_z W + \frac{2(n-k)}{|y|^2}\, y \cdot \nabla_y W + (n-2)(n-k)\frac{W}{|y|^2} = -\frac{W^{\frac{n}{n-2}}}{|y|} \tag{3.103}$$

Proof. As before, we must compute

$$\Delta_x(\frac{y}{|y|^2}, z)$$

We have

$$\Delta_y(\frac{y}{|y|^2}, z) = |y'|^4\, \Delta_{y'} U - 2(k-2)|y'|^2\, \nabla_{y'} U \cdot y' \tag{3.104}$$

$$\Delta_z(\frac{y}{|y|^2}, z) = \Delta_{z'} U \tag{3.105}$$

and we get

$$\Delta_{y,z} W = \frac{(n-2)(n-k)}{|y|^n}\, U + \frac{1}{|y|^{n+2}}\, \Delta_{y'} U + \frac{1}{|y|^{n+2}} \Delta_{z'} U + \frac{2(n-k)}{|y|^{n-2}} y \cdot \nabla_{y'} U \tag{3.106}$$

that is

$$\begin{aligned} &\Delta_y W + \frac{1}{|y|^4}\, \Delta_z W + \frac{2(n-k)}{|y|^2} y \cdot \nabla_y W + (n-2)(n-k)\frac{W}{|y|^2} \\ &= \frac{1}{|y|^{n+2}} \Delta_{y'} U + \frac{1}{|y|^{n-2}}\, \Delta_{z'} U = \frac{1}{|y|^{n-2}}\, \Delta_{y',z'} U(y', z') = -\frac{W^{\frac{n}{n-2}}}{|y|} \end{aligned} \tag{3.107}$$

equivalently,

$$\begin{aligned} &|y|\, \Delta_y W + \frac{1}{|y|^3}\, \Delta_z W + \frac{2(n-k)}{|y|} y \cdot \nabla_y W + (n-2)(n-k)\frac{W}{|y|} \\ &= W^{\frac{n}{n-2}} \end{aligned} \tag{3.108}$$

We can rewrite (3.107) as

$$\mathcal{L} W + (n-2)(n-k)\frac{W}{|y|^2} = -\frac{W^{n/(n-2)}}{|y|} \tag{3.109}$$

where $\mathcal{L} = \Delta_y + \frac{1}{|y|^4}\, \Delta_z + \frac{2(n-k)}{|y|^2} y \cdot \nabla_y$ is a Grushin operator. □

The next work in progress is the study of hyperbolic symmetries for the solutions to (3.2).

4. BREZIS NIRENBERG TYPE PROBLEMS

In this chapter we shall give some existence and nonexistence results for the boundary value problem

$$\left.\begin{array}{rlll} -\Delta u & = \frac{|u|^{2_*-2}u}{|y|^s} + \lambda u & \text{in} & \Omega \\ u & > 0 & \text{in} & \Omega \\ u & = 0 & \in & \partial\Omega. \end{array}\right\} \tag{4.1}$$

where $x = (y, z) \in \Omega \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ is a bounded smooth domain containing the origin in its interior and $n \geq 3$, $2 \leq k < n$, $s \in (0, 2)$ and $2_* = \frac{2(n-s)}{n-2}$. The exponent $2_* - 1$ is critical from the viewpoint of weighted Sobolev embedding, that is $H_0^1(\Omega) \hookrightarrow L^{2_*}(\Omega, |y|^{-s}\,dx)$ is continuous but not compact. We can approach this problem by a direct method and attempt to obtain non-trivial solutions of (4.1) as constrained minima of the functional

$$\Phi_\lambda u = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda|u|^2)\,dx - \frac{1}{2_*}\int_\Omega \frac{|u|^{2_*}}{|y|^s}\,dx$$

Equivalently, we may seek to minimize

$$Q_\lambda(u, \Omega) = \int_\Omega (|\nabla u|^2 - \lambda|u|^2)dx, \quad u \neq 0, \quad \int_\Omega \frac{|u|^{2_*}}{|y|^s}dx = 1.$$

We will consider the second method.

4.1 Preliminaries

We recall the weighted Hardy-Sobolev inequality proved by [3].
For $1 < p < n$, $D^{1,p}(\mathbb{R}^n)$ is embedded continuously in $L^{p_*}(\mathbb{R}^n, |y|^{-s})$ and

$$C\left(\int_{\mathbb{R}^n} \frac{|u|^{p_*}}{|y|^s}\,dx\right)^{p/p_*} \leq \int_{\mathbb{R}^n} |\nabla u|^p\,dx \quad \forall u \in D^{1,p}(\mathbb{R}^n) \tag{4.2}$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $p_* = \frac{p(n-s)}{n-p}$ and $D^{1,p}(\mathbb{R}^n)$. If $s = p$, $s < k$, we get the partial Hardy inequality

$$C\int_{\mathbb{R}^n} \frac{|u|^p}{|y|^p}\,dx \leq \int_{\mathbb{R}^n} |\nabla u|^p\,dx \quad \forall u \in D^{1,p}(\mathbb{R}^n) \tag{4.3}$$

If we take $\Omega = B_R = \{(y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \,:\, |y| + |z| < R\}$, a ball of radius $R > 0$, then also (4.2), (4.3) are true for any $u \in W_0^{1,p}(B_R)$. If Ω is a any general bounded domain, we can verify the validity of (4.2) for $W_0^{1,p}(\Omega)$ which is the completion of $C_0^\infty(\Omega)$ in the norm
$||u||_{1,p,\Omega} := \left(\int\limits_\Omega |u(x)|^p dx + \int\limits_\Omega |\nabla u|^p \, dx \right)^{1/p}$.
We can define the best Hardy-Sobolev constant of a domain $\Omega \subset \mathbb{R}^n, n \geq 3$, as

$$S_{s,k,n}(\Omega) = S := \inf\left\{ \int\limits_\Omega |\nabla u|^2 \, dx \,:\, u \in H_0^1(\Omega) \, and \int\limits_\Omega \frac{|u|^{2_*}}{|y|^s} \, dx = 1 \right\} \quad (4.4)$$

where $0 \leq s < 2, 2_* = \frac{2(n-s)}{n-2}$. Moreover, we set

$$S_\lambda := \inf\left\{ ||\nabla u||_2^2 - \lambda ||u||_2^2 \,:\, u \in H_0^1(\Omega) \, and \int\limits_\Omega \frac{|u|^{2_*}}{|y|^s} \, dx = 1 \right\}, \, \lambda \in \mathbb{R}. \quad (4.5)$$

Another relevant parameter is the first eigenvalue of the Laplacian, defined as

$$\lambda_1 = \inf\{ \int\limits_\Omega |\nabla w|^2 dx \,:\, w \in H_0^1(\Omega), \int\limits_\Omega |w|^2 dx = 1 \}.$$

Definition 4.1.1. Let V be a Banach space, $E \in C^1(V), \beta \in \mathbb{R}$. We say that E satisfies condition $(P.S.)_\beta$, if any sequence $\{u_m\}$ in V such that $E(u_m) \to \beta$ while $DE(u_m) \to 0$ as $m \to \infty$ is relatively compact. Such sequences in the sequel for brevity will be referred to as $(P.S.)_\beta$-sequences.

The following lemmas are an immediate consequence of (4.2), (4.3) and Sobolev embeddings. We begin with an analogue of the Rellich-Kondrachov compactness theorem for the space $H_0^1(\Omega)$.

Lemma 4.1.1. *For $1 \leq q < 2_*, 0 \leq s < 2$, the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega, |y|^{-s}dx)$ is compact.*

Proof. Let $\{u_k\}$ be a bounded sequence in $H_0^1(\Omega)$. Since $q < p < \frac{2n}{n-2}$, the Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_k\}$ in $L^q(\rho < |y| < 1)$ for any $\rho > 0$. By taking diagonal sequence one may assume without loss of generality that $\{u_j\}$ converges in $L^q(\rho < |y| < 1)$ for any $\rho > 0$. On the other hand, by the Hölder inequality and the weighted Sobolev inequality,

$$\iint\limits_{\substack{|y|<\delta \\ |z|<\delta}} |y|^{-s} \, |u_j|^q \, dy \, dz \leq \left(\iint\limits_{\substack{|y|<\delta \\ |z|<\delta}} |y|^{-s} \, dx \right)^{1-q/p} \left(\int_\Omega |y|^{-s} \, |u_j|^p \, dy \, dz \right)^{q/p}$$

for some constant C. Therefore, for a given $\epsilon > 0$, we first fix a δ such that

$$\iint\limits_{\substack{|y|<\delta \\ |z|<\delta}} |y|^{-s} |u_j|^q \, dy \, dz \le \frac{\varepsilon}{2}.$$

Then

$$\int_\Omega |y|^{-s} |u_j - u_k|^q \, dy \, dz < \frac{1}{2}\varepsilon + \delta^{-s} \iint\limits_{\substack{|y|<\delta \\ |z|<\delta}} |u_j - u_k|^q \, dy \, dz < \varepsilon$$

for sufficiently large j, k. Hence $\{u_j\}$ is also a Cauchy sequence in $L^q(\Omega, |y|^{-s}\, dx)$. □

4.2 Nonexistence results

In this section we will establish some nonexistence results. The main tools are different variants of the Pohozahev's identity.

Proposition 4.2.1. *There is no solution of (4.1) when $\lambda \ge \lambda_1$.*

Indeed, let ϕ_1 be the eigenfunction of $-\Delta$, with $\phi_1 > 0$ on Ω. Suppose u is a solution of (4.1). We have

$$-\int\limits_\Omega \Delta u \; \phi_1 \, dx = \lambda_1 \int\limits_\Omega u \, \phi_1 \, dx = \int\limits_\Omega \frac{u^{2_*-1}}{|y|^s} \phi_1 \, dx + \lambda \int\limits_\Omega u \, \phi_1 \, dx > \lambda \int\limits_\Omega u \, \phi_1 \, dx.$$

and thus $\lambda < \lambda_1$.

Theorem 4.2.2. *There is no solution of (4.1) when $\lambda \le 0$ and Ω is a smooth starshaped domain.*

Proof. The proof is based on the following "Pohozaev identity".

Proposition 4.2.3. *Let $a : \mathbb{R}^n \to \mathbb{R}$ be such that $a(x)|y|^s \in L^\infty(\mathbb{R}^n)$, with $a \in C(A)$, where $A = \{(y, z) \in \mathbb{R}^n \; : \; y \ne 0\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous with primitive $G(u) = \int\limits_0^u g(v)\, dv$. Let $u \in H^1_0(\Omega)$ be a weak solution of*

$$\begin{cases} -\Delta u & = a(x)|u|^{2_*-2}u + g(u) \quad \text{in} \quad \Omega \\ \quad u & = 0 \qquad\qquad\qquad\qquad\; \text{in} \quad \partial\Omega \end{cases} \tag{4.6}$$

Assume that $u \in C^{1,\theta}(\Omega) \cap L^p(\Omega) \cap C^2(A)$ for some $\theta \in (0, 1)$. Then u satisfies the identity

$$\int\limits_{\partial\Omega} -\frac{1}{2}|\nabla u|^2 x\cdot\nu \, d\sigma = \int\limits_\Omega \frac{n-2}{2} |\nabla u|^2 - n\, G(u) - \frac{1}{2_*}(n\, a(x) + \nabla a(x)\cdot x)|u|^{2_*} \, dx. \tag{4.7}$$

where ν is the outer unit normal of $\partial\Omega$.

Proof. For any $\varepsilon > 0$ we consider the sets

$$V_1 = V_1(\epsilon) = \{x \in \Omega \,:\, y_i > \varepsilon\}, \quad V_2 = V_2(\epsilon) = \{x \in \Omega \,:\, y_i < -\varepsilon\}.$$

for $i = 1, \cdots, n$. For simplicity we will take $i = 1$. By our assumptions, u is a classical solution to (4.6) in $V_1 \cup V_2$, so in each set, we can multiply (4.6) by $x \cdot \nabla u(x)$ and we can integrate. We recall that

$$(\Delta u + g(u))(x \cdot \nabla u)) = \operatorname{div}(\nabla u(x \cdot \nabla u)) - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2}\right) + x \cdot \nabla G(u)$$
$$= \operatorname{div}\left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) + \frac{n-2}{2}|\nabla u|^2 - nG(u)$$

and

$$x \cdot \nabla u\, a(x)|u|^{2_*-2}u = \operatorname{div}\left(\frac{1}{2_*}a(x)|u|^{2_*}x\right) - \frac{n}{2_*}a(x)|u|^{2_*} - \frac{1}{2_*}|u|^{2_*}\nabla a(x) \cdot x$$

The, setting $V = V_1 \cup V_2$, we have

$$\int_{\partial V} \left(-x \cdot \nabla u\frac{\partial u}{\partial \nu} + \frac{1}{2}|\nabla u|^2 x \cdot \nu - G(u)x \cdot \nu\right) d\sigma - \frac{n-2}{2}\int_V |\nabla u|^2\, dx + n\int_V G(u)\, dx$$
$$= \int_{\partial V} \frac{1}{2_*}\, a(x)|u|^{2_*}\, x \cdot \nu\, d\sigma - \frac{n}{2_*}\int_V a(x)|u|^{2_*}\, dx - \frac{1}{2_*}\int_V x \cdot \nabla a\, |u|^{2_*}\, dx$$

that is

$$\int_{\partial V} \left(-x \cdot \nabla u\frac{\partial u}{\partial \nu} + \frac{1}{2}|\nabla u|^2 x \cdot \nu - \frac{1}{2_*}\, a(x)|u|^{2_*}\, x \cdot \nu\, - G(u)x \cdot \nu\right) d\sigma$$
$$= \int_V \left(\frac{n-2}{2}|\nabla u|^2\, - nG(u)\, - \frac{n}{2_*}a(x)|u|^{2_*}\, dx - \frac{1}{2_*}x \cdot \nabla a\, |u|^{2_*}\right) dx \quad (4.8)$$

We pass to the limit in (4.8) as $\varepsilon \to 0$. The limit of the right hand side in (4.8) is

$$\int_\Omega \frac{n-2}{2}|\nabla u|^2\, - n\, G(u) - \frac{1}{2_*}(n\, a(x)\, + \nabla a(x) \cdot x)|u|^{2_*}\, dx$$

For the left hand side, we have

$$\partial\Omega = \{x \in \partial\Omega \,:\, x_1 < \varepsilon\} \cup \{x \in \partial\Omega \,:\, x_1 < -\varepsilon\}$$
$$\cup \{x \in \Omega \,:\, x_1 = \varepsilon\} \cup \{x \in \Omega \,:\, x_1 = -\varepsilon\} = S_1 \cup S_2 \cup S_3 \cup S_4.$$

Then

$$\lim_{\varepsilon\to 0} \int_{S_1 \cup S_2} \left(-x \cdot \nabla u\frac{\partial u}{\partial \nu} + \frac{1}{2}|\nabla u|^2 x \cdot \nu - \frac{1}{2_*}\, a(x)|u|^{2_*}\, x \cdot \nu\, - G(u)x \cdot \nu\right) d\sigma$$

$$= \int_{\partial\Omega} \left(-x \cdot \nabla u \frac{\partial u}{\partial \nu} + \frac{1}{2} |\nabla u|^2 x \cdot \nu - \frac{1}{2_*} \, a(x)|u|^{2_*} \, x \cdot \nu \, - G(u) x \cdot \nu \right) \, d\sigma$$

We have now to see what happens in $S_3 \cup S_4$, as $\varepsilon \to 0$. Notice that in $S_3, \nu = (-1, 0, \cdots, 0)$, while in $S_4, \nu = (1, 0, \cdots, 0)$. Hence we can write

$$\int_{S_3 \cup S_4} \left(-x \cdot \nabla u \frac{\partial u}{\partial \nu} + \frac{1}{2} |\nabla u|^2 x \cdot \nu - \frac{1}{2_*} \, a(x)|u|^{2_*} \, x \cdot \nu \, - G(u) x \cdot \nu \right) \, d\sigma$$
$$= \int_{S_3} \left(x \cdot \nabla u D_1 u - \frac{1}{2} |\nabla u|^2 x_1 + \frac{1}{2_*} \, a(x)|u|^{2_*} \, x_1 + G(u) \, x_1 \right) \, d\sigma$$
$$+ \int_{S_4} \left(-x \cdot \nabla u D_1 u + \frac{1}{2} |\nabla u|^2 x_1 - \frac{1}{2_*} \, a(x)|u|^{2_*} \, x_1 \, - G(u) \, x_1 \right) \, d\sigma$$

But the hypotheses, the function $-x \cdot \nabla u D_1 u + \frac{1}{2} |\nabla u|^2 \, x_1 - \frac{1}{2_*} a(x)|u|^{2_*} x_1 - G(u) x_1$ is continuous everywhere, so that we can conclude that

$$\lim_{\varepsilon \to 0} \int_{S_1 \cup S_2} \left(-x \cdot \nabla u \frac{\partial u}{\partial \nu} + \frac{1}{2} |\nabla u|^2 x \cdot \nu - \frac{1}{2_*} \, a(x)|u|^{2_*} \, x \cdot \nu \, - G(u) x \cdot \nu \right) \, d\sigma = 0.$$

Hence, passing to the limit as $\varepsilon \to 0$, (4.8) implies

$$\int_{\partial\Omega} \left(-x \cdot \nabla u \frac{\partial u}{\partial \nu} + \frac{1}{2} |\nabla u|^2 x \cdot \nu - \frac{1}{2_*} \, a(x)|u|^{2_*} \, x \cdot \nu \, - G(u) x \cdot \nu \right) \, d\sigma$$
$$= \int_{\Omega} \frac{n-2}{2} \, |\nabla u|^2 \, - n \, G(u) - \frac{1}{2_*} (n \, a(x) \, + \nabla a(x) \cdot x) |u|^{2_*} \, dx. \tag{4.9}$$

But, on $\partial\Omega, u = 0$ so that $G(u) = 0, \nabla u = \nabla u \cdot \nu\nu$ and this implies

$$\int_{\partial\Omega} -\frac{1}{2} |\nabla u|^2 x \cdot \nu \, d\sigma = \int_{\Omega} \frac{n-2}{2} \, |\nabla u|^2 \, - n \, G(u) - \frac{1}{2_*} (n \, a(x) \, + \nabla a(x) \cdot x) |u|^{2_*} \, dx. \tag{4.10}$$

□

Proof of Theorem 4.2.2. Let u is a solution to (4.1) and suppose $u \in C^{1,\theta}(\Omega)$. Then, we can apply Proposition (4.2.3) with $a(x) = |y|^{-s}$.

Notice that $\nabla a(x) \cdot x = (-s)|y|^{-s-1}|y|$. Then we have:

$$
\begin{aligned}
&-\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu \, d\sigma \\
&= \frac{n-2}{2}\int_{\Omega} |\nabla u|^2 \, dx - \frac{n}{2}\int_{\Omega} u^2 \, dx - \frac{n(n-2)}{2(n-s)}\int_{\Omega} |y|^{-s}|u|^{2_*} \, dx + \frac{s(n-2)}{2(n-s)}\int_{\Omega} |y|^{-s}|u|^{2_*} \\
&= \frac{n-2}{2}\int_{\Omega} \frac{|u|^{2_*}}{|y|^s} \, dx + \lambda\frac{(n-2)}{2}\int_{\Omega} u^2 \\
&- \frac{n}{2}\int_{\Omega} u^2 \, dx - \frac{n(n-2)}{2(n-s)}\int_{\Omega} |y|^{-s}|u|^{2_*} \, dx + \frac{s(n-2)}{2(n-s)}\int_{\Omega} |y|^{-s}|u|^{2_*} \\
&= \lambda\left(\frac{n-2}{2} - \frac{n}{2}\right)\int_{\Omega} u^2 \, dx + \left(\frac{n-2}{2} - \frac{n(n-2)}{2(n-s)} + \frac{s(n-2)}{2(n-s)}\right)\int_{\Omega} |y|^{-s}|u|^{2_*} \\
&= -\lambda\int_{\Omega} u^2 \, dx
\end{aligned} \tag{4.11}
$$

So,

$$
\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu \, d\sigma = \lambda\int_{\Omega} u^2 \, dx \tag{4.12}
$$

If Ω is starshaped about the origin, we have $(x \cdot \nu) > 0$ a.e. on $\partial\Omega$. When $\lambda < 0$ we deduce from (4.12) that $u \equiv 0$. When $\lambda = 0$ we deduce from (4.12) that $\frac{\partial u}{\partial \nu} = 0$ on Ω and then, by (4.1) we have

$$
0 = -\int_{\Omega} \Delta u \, dx = \int_{\Omega} \frac{u^{2_*-1}}{|y|^s} \, dx
$$

that is $u \equiv 0$. □

Remark 4.2.1. We can note that

i) $S_{k,n,s}$ is independent of Ω and depends only by n, k, s.

ii) The infimum $S = S_{k,n,s}$ is never achieved when Ω is a bounded domain.

i) This follows from the fact that the ratio $\frac{||\nabla u||_{L^2(\Omega)}}{||u||_{L^{2_*}(\Omega,|y|^{-s}dx)}}$ is invariant under the scaling $u_\beta(x) = \beta^{(n-2)/2}u(\beta x) \, \forall \beta > 0$.
ii) Suppose that S were attained by some function $u \in H_0^1(\Omega)$. We may assume that $u \geq 0$ on Ω (otherwise replace u by $|u|$). Fix a ball B containing Ω and set

$$
\tilde{u} = \begin{cases} \tilde{u} & \text{on} \quad \Omega \\ 0 & \text{on} \quad B \setminus \Omega. \end{cases}
$$

Thus S is also achieved on B by $\tilde{u}$ and $\tilde{u}$ satisfies $-\Delta\tilde{u} = \mu\,\frac{\tilde{u}^{2_*-1}}{|y|^s}$ for some constant $\mu > 0$ (since $\int\limits_B \frac{\tilde{u}^{2_*}}{|y|^s}\,dx = 1$ and $\int\limits_B |\nabla u|^2\,dx = \mu = S > 0$) which is impossible by Pohozaev's identity.

4.3 The special case $s = 1$

When $\Omega = \mathbb{R}^n$ and $s = 1$, we know that the infimum for S is achieved by function

$$U(x) = U(y,z) = C_{n,k}\left[(1+|y|)^2 + |z|^2\right]^{-\frac{(n-2)}{2}} \tag{4.13}$$

where $C_{n,k} = \{(n-2)(k-1)\}^{\frac{n-2}{2}}$ or after scaling, and translations in the z-variable, by any of the functions

$$\begin{aligned} U_t(x) &= t^{\frac{n-2}{2}}U(t\,y, t(z+z_0)) = t^{\frac{n-2}{2}}\,C_{n,k}\left[(1+|ty|)^2 + |t(z+z_0)|^2\right]^{-\frac{(n-2)}{2}} \\ &= \left(\frac{(n-2)(k-1)}{t}\right)^{\frac{n-2}{2}} [(\frac{1}{t}+|y|)^2 + |z+z_0|^2]^{-\frac{(n-2)}{2}} \\ &= C_{n,k,t}\,[(\frac{1}{t}+|y|)^2 + |z+z_0|^2]^{-\frac{(n-2)}{2}} \end{aligned} \tag{4.14}$$

for $t > 0$. We want to show the following lemma

Lemma 4.3.1. *We have*

$$S_\lambda < S \;\; for \;\; all \;\; \lambda > 0. \tag{4.15}$$

Proof. Following Aubin's method [1], also used by [11] we define

$$Q_\lambda(u) = \frac{||\nabla u||_2^2 - \lambda||u||_2^2}{||u||^2_{2_*,\,|y|^{-1}}} \tag{4.16}$$

for $u \in H_0^1(\Omega), u \neq 0$. We assume that $0 \in \Omega$ and we set

$$u_\varepsilon(y,z) = \frac{\psi(y,z)}{((\varepsilon+|y|)^2 + |z|^2)^{(n-2)/2}} \tag{4.17}$$

for some $\varepsilon \in (0,1]$ and $\psi(y,z) \in C_0^\infty(\Omega)$ are test functions such that $0 \le \psi \le 1$ and $\psi \equiv 1$ in a neighborhood of 0. The idea is to estimate $Q_\lambda(u_\varepsilon)$ where $\varepsilon = \frac{1}{t} \to 0$.

Step 1. We claim that

$$||\nabla u_\varepsilon||_2^2 = \frac{K_1}{\varepsilon^{n-2}} + O(1) \tag{4.18}$$

where $K_1 = K_1(n,k,2,1)$. From (4.17), we have

$$\nabla_y u_\varepsilon(y,z) = \frac{\nabla_y\psi(y,z)}{[(\varepsilon+|y|)^2 + |z|^2]^{(n-2)/2}} - \frac{(n-2)\,(\varepsilon+|y|)\,y\,\psi(y,z)}{|y|\,[(\varepsilon+|y|)^2 + |z|^2]^{n/2}} \tag{4.19}$$

$$\nabla_z u_\varepsilon(y,z) = \frac{\nabla_z\psi(y,z)}{[(\varepsilon+|y|)^2 + |z|^2]^{(n-2)/2}} - \frac{(n-2)z\,\psi(y,z)}{[(\varepsilon+|y|)^2 + |z|^2]^{n/2}} \tag{4.20}$$

Since $\psi \equiv 1$ near $(y,z)=(0,0)$, writing $\psi^2 = 1+\psi^2-1$, it follows that

$$\int_\Omega |\nabla u_\varepsilon|^2\,dy\,dz = \int_\Omega |\nabla_y u_\varepsilon|^2 + |\nabla_z u_\varepsilon|^2\,dy\,dz$$

$$= \int_\Omega dy\,dz \left|\frac{(n-2)\,(\varepsilon+|y|)\,y\,\psi}{|y|\,[(\varepsilon+|y|)^2+|z|^2]^{n/2}}\right|^2 + \int_\Omega dy\,dz\left|\frac{(n-2)z\,\psi}{[(\varepsilon+|y|)^2+|z|^2]^{n/2}}\right|^2 + O(1)$$

$$= (n-2)^2\int_\Omega dy\,dz\,\psi^2\frac{(\varepsilon+|y|)^2+|z|^2}{[(\varepsilon+|y|)^2+|z|^2]^n} + O(1)$$

$$= (n-2)^2\int_\Omega dy\,dz\,\psi^2\,\frac{1}{[(\varepsilon+|y|)^2+|z|^2]^{n-1}} + O(1)$$

$$= (n-2)^2\int_\Omega dy\,dz\,\frac{1}{[(\varepsilon+|y|)^2+|z|^2]^{n-1}} + (n-2)^2\int_\Omega \frac{\psi^2-1}{[(\varepsilon+|y|)^2+|z|^2]^{n-1}} + O(1)$$

$$= (n-2)^2\int_\Omega dy\,dz\,\frac{1}{\varepsilon^{2(n-1)}[(1+|\frac{y}{\varepsilon}|)^2+|\frac{z}{\varepsilon}|^2]^{n-1}} + O(1)$$

$$= (n-2)^2\int_{\mathbb{R}^n}\frac{dv\,\varepsilon^k\,dw\,\varepsilon^{n-k}}{\varepsilon^{2(n-1)}[(1+|v|)^2+|w|^2]^{n-1}} + O(1)$$

$$= (n-2)^2\,\varepsilon^{-n+2}\int_{\mathbb{R}^n}\frac{dv\,dw}{[(1+|v|)^2+|w|^2]^{n-1}} + O(1)$$

$$= \frac{K_1}{\varepsilon^{n-2}} + O(1), \quad \left(= \frac{K_1}{\varepsilon^{(n-m)/(m-s)}} + O(1)\right)$$

where $K_1 = (n-2)^2\int_{\mathbb{R}^n}\frac{1}{[(1+|y|)^2+|z|^2]^{n-1}}\,dx = \frac{||\nabla U||_2^2}{C_{n,k}^2}$. □

Step 2. We claim that

$$||u_\varepsilon||^2_{2_*,|y|^{-1}} = \frac{K_2}{\varepsilon^{n-2}} + O(1) \tag{4.21}$$

Proof. We have

$$
\begin{aligned}
\int\limits_\Omega \frac{|u_\varepsilon|^{2_*}}{|y|}\,dx &= \int\limits_\Omega \frac{\psi^{2_*}}{|y|\ [(\varepsilon+|y|)^2+|z|^2]^{n-1}}\,dx \\
&= \int\limits_\Omega \frac{1}{|y|\ [(\varepsilon+|y|)^2+|z|^2]^{n-1}}\,dx + \int\limits_\Omega \frac{\psi^{2_*}-1}{|y|\ [(\varepsilon+|y|)^2+|z|^2]^{n-1}}\,dx \\
&= \int\limits_\Omega \frac{1}{|y|\ [(\varepsilon+|y|)^2+|z|^2]^{n-1}}\,dx\ + O(1) \\
&= \int\limits_\Omega \frac{1}{\varepsilon^{2(n-1)}[(1+|\frac{y}{\varepsilon}|)^2+|\frac{z}{\varepsilon}|^2]^{n-1}}\frac{1}{|y|} + O(1) \\
&= \int\limits_{\mathbb{R}^n} \frac{dv\,\varepsilon^k\,dw\,\varepsilon^{n-k}}{\varepsilon^{2(n-1)}[(1+|v|)^2+|w|^2]^{n-1}}\,\frac{1}{|v|\varepsilon}\,dv\,dw + O(1) \\
&= \varepsilon^{-n+1}\int\limits_{\mathbb{R}^n} \frac{dv\,dw}{[(1+|v|)^2+|w|^2]^{n-1}}\,\frac{1}{|v|}\,dv\,dw\ + O(1) \\
&= \varepsilon^{-n+1}\int\limits_{\mathbb{R}^n} |U|^{2_*}|y|^{-1}\,dx + O(1) = \frac{K_2'}{\varepsilon^{n-1}} + O(1),
\end{aligned}
$$

where $K_2' = \int\limits_{\mathbb{R}^n} |U|^{2_*}|y|^{-1}\,dx = (\frac{U_{2_*,|y|^{-1}}}{C_{n,k}})^{2_*}$. Then

$$
\begin{aligned}
||u_\varepsilon||^2_{2_*,|y|^{-1}} &= \left(\int\limits_\Omega \frac{|u_\varepsilon|^{2_*}}{|y|}\,dx\right)^{2/2_*} = \left(\frac{\varepsilon^{-n+1}}{C_{n,k}^{2_*}}\int\limits_{\mathbb{R}^n} |U|^{2_*}|y|^{-1}\,dx\right)^{2/2_*} + O(1) \\
&= \frac{K_2}{\varepsilon^{n-2}} + O(1) \quad (= \frac{K_2}{\varepsilon^{(n-m)/(2-s)}} + O(1))
\end{aligned}
\tag{4.22}
$$

where $K_2 = \frac{1}{(C_{n,k})^2}\left(\int\limits_{\mathbb{R}^n} |U|^{2_*}|y|^{-1}\,dx\right)^{2/2_*}$ and $\frac{K_1}{K_2} = S$. □

Step 3. We claim that

$$||u_\varepsilon||_2^2 = \frac{K_3}{\varepsilon^{n-4}} + O(1) \quad if \quad n > m^2 = 4 \tag{4.23}$$

$$||u_\varepsilon||_2^2 > C|\log\varepsilon| \quad if \;\; n = m^2 = 4 \tag{4.24}$$

Proof. In the first case we have

$$\int_\Omega |u_\varepsilon|^2\,dx = \int_\Omega \frac{\psi^2}{[(\varepsilon+|y|)^2+|z|^2]^{n-2}}\,dx$$

$$= \int_\Omega \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^{n-2}}\,dx + \int_\Omega \frac{\psi^2-1}{[(\varepsilon+|y|)^2+|z|^2]^{n-2}}\,dx$$

$$= \int_\Omega \frac{1}{\varepsilon^{2(n-2)}[(1+|\frac{y}{\varepsilon}|)^2+|\frac{z}{\varepsilon}|^2]^{n-2}}\,dx + O(1)$$

$$= \int_{\mathbb{R}^n} \frac{dv\,\varepsilon^k\,dw\,\varepsilon^{n-k}}{\varepsilon^{2(n-2)}[(1+|v|)^2+|w|^2]^{n-2}} + O(1)$$

$$= \frac{1}{\varepsilon^{n-4}}\int_{\mathbb{R}^n} \frac{1}{[(1+|v|)^2+|w|^2]^{n-2}}\,dv\,dw + O(1) = \frac{K_3}{\varepsilon^{n-4}} \quad \Big(= \frac{K_3}{\varepsilon^{(n-m^2)/(2-s)}}\Big) \tag{4.25}$$

where $K_3 = \frac{1}{(C_{n,k})^2}\int_{\mathbb{R}^n} U^2\,dx$. In the second case we have

$$\int_\Omega |u_\varepsilon|^2\,dx = O(1) + 2(k-1)\int_\Omega \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^2}\,dx = O(1) + 2(k-1)\,I_k(\varepsilon)$$

and there exist $0 < R_1 < R_2$ such that

$$\iint_{\substack{|y|<R_1\\|z|<R_1}} \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^2}\,dx \le I_k(\varepsilon) \le \iint_{\substack{|y|<R_2\\|z|<R_2}} \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^2}\,dx$$

and it is clear that for a fixed $R > 0$ we have

$$\int_\Omega \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^2}\,dx = \int_\Omega \frac{1}{[(\varepsilon+|y|)^2+|z|^2]^2}\,dx$$

$$= \omega_k\,\omega_{4-k}\int_0^R\int_0^R \frac{r^{k-1}s^{4-k-1}}{[(\varepsilon+r)^2+s^2]^2}\,dr\,ds = \omega_k\,\omega_{4-k}\int_0^R\int_0^R \frac{r^{k-1}s^{3-k}}{\varepsilon^4\,\big[(1+\frac{r}{\varepsilon})^2+(\frac{s}{\varepsilon})^2\big]^2}\,dr\,ds$$

$$= \omega_k\,\omega_{4-k}\int_0^{R/\varepsilon}\int_0^{R/\varepsilon} \frac{(v\varepsilon)^{k-1}(w\varepsilon)^{3-k}\varepsilon\,dv\,\varepsilon\,dw}{\varepsilon^4[(1+v)^2+w^2]^2} = \omega_k\,\omega_{4-k}\int_0^{R/\varepsilon}\int_0^{R/\varepsilon} \frac{v^{k-1}w^{3-k}}{[(1+v)^2+w^2)]^2}\,dv\,dw$$

where ω_{k-1} and ω_{n-k-1} are the surface measure of the $k-1$ and $n-k-1$ dimensional sphere in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$. Hence, if $k=2$, we get

$$I_2(\varepsilon) = \int_0^R\int_0^R \frac{vw}{[(\varepsilon+v)^2+w^2)]^2}\,dv\,dw = -\frac{R\log\varepsilon}{2(R+\varepsilon)} + O(1) \tag{4.26}$$

If $k = 3$, we have two sums, J_1 and J_2. But $J_1 = O(1)$ and

$$J_2 \geq \frac{R\arctan(\frac{R}{R+\varepsilon})(-3R^2 - 2R\varepsilon - 2R^2\log\varepsilon - 4R\varepsilon\log\varepsilon - 2\varepsilon^2\log\varepsilon)}{4((R+\varepsilon)^2}$$
$$+ \frac{R\arctan(\frac{R}{R+\varepsilon})(2R^2\log(R+\varepsilon) + 4R\varepsilon\log(R+\varepsilon) + 2\varepsilon^2\log(R+\varepsilon))}{4((R+\varepsilon)^2} + O(1)$$
$$> -\frac{R^3\arctan(\frac{R}{R+\varepsilon})\log\varepsilon}{2(R+\varepsilon)^2} + O(1) = C|\log\varepsilon| + O(1). \tag{4.27}$$

If $k = n = 4$, we have that our function $U(y,z) = U(x) = C(1+|x|)^{n-2}$ is radial and

$\int\limits_{|x|<R} \frac{1}{(\varepsilon+|y|)^4}\,dx = \omega_n \int\limits_0^R \frac{r^3\,dr}{(\varepsilon+r)^4} = O(1) + C|\log\varepsilon|.$ □

Step 4. End of the proof.

Proof. Combining Steps 1-3, we obtain

$$\begin{cases} Q_\lambda(u_\varepsilon) = S - \lambda\dfrac{K_3}{K_2}\varepsilon^2 + O(\varepsilon^{n-2}) & \text{if} \quad n \geq 5, \\ Q_\lambda(u_\varepsilon) < S - \lambda\dfrac{C}{K_2}\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if} \quad n = 4. \end{cases} \tag{4.28}$$

In all cases we deduce that $S_\lambda \leq Q_\lambda(u_\varepsilon) < S$ provided $\varepsilon > 0$ is small enough. □

Lemma 4.3.2. *If* $0 < S_\lambda < S$, *then* S_λ *is achieved.*

Proof. First we prove the existence of a minimizer of S_λ. Choose a minimizing sequence such that $\{u_n\}_n$ such that

$$\int\limits_\Omega \frac{u_n^{2_*}}{|y|}\,dx = 1 \tag{4.29}$$

$$||\nabla u_n||_2^2 - \lambda||u_n||_2^2 = S_\lambda + o(1) \quad as\, n \to \infty. \tag{4.30}$$

Since $\{u_n\}_n$ is also bounded in $H_0^1(\Omega)$ we can assume that the sequence converges weakly in $H_0^1(\Omega)$ and $L^{\frac{2n}{n-2}}(\Omega)$ and that it converges in $L^2(\Omega)$ and pointwise a.e. in Ω with $\int\limits_\Omega \frac{u^{2_*}}{|y|}\,dx \leq 1$. Set $v_n = u_n - u$, so that $v_n \rightharpoonup 0$ weakly in H_0^1 and $v_n \to 0$ a.e. on Ω. By definition of S and (4.29), we have $||\nabla u_n|| \geq S$. From (4.30) it follows that $\lambda||u||_2^2 \geq S - S_\lambda > 0$ and therefore $u \neq 0$. Using (4.30) we obtain

$$||\nabla u||_2^2 + ||\nabla v_n||_2^2 - \lambda||u||_2^2 = S_\lambda + o(1) \tag{4.31}$$

since $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. On the other hand, a lemma of Brezis and Lieb [10] yields

$$1 = \int_\Omega \frac{|u_n|^{2_*}}{|y|}\,dx = \int_\Omega \frac{|u|^{2_*}}{|y|}\,dx + \int_\Omega \frac{|v_n|^{2_*}}{|y|}\,dx + o(1)$$

$$\leq \Big(\int_\Omega \frac{|u|^{2_*}}{|y|}\,dx\Big)^{2/2_*} + \Big(\int_\Omega \frac{|v_n|^{2_*}}{|y|}\,dx\Big)^{2/2_*} + o(1). \tag{4.32}$$

which leads to

$$1 \leq \Big(\int_\Omega \frac{|u|^{2_*}}{|y|}\,dx\Big)^{2/2_*} + \frac{1}{S}||\nabla v_n||_2^2 + o(1). \tag{4.33}$$

We claim that

$$||\nabla u||_2^2 - \lambda ||u||_2^2 \leq S_\lambda ||u||_{2_*,\,|y|^{-1}}^2 \tag{4.34}$$

to conclude the proof of lemma since $u \neq 0$. We consider the case $S_\lambda > 0$ (i.e. $0 < \lambda < \lambda_1$). We deduce from (4.33) that

$$S_\lambda \leq S_\lambda ||u||_{2_*,\,|y|^{-1}}^2 + (\frac{S_\lambda}{S})\,||\nabla v_n||_2^2 + o(1). \tag{4.35}$$

Combining (4.31) and (4.35) we obtain (4.34).

□

Then, our main result is the following:

Theorem 4.3.3. *Assume $n \geq 4$. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of the problem*

$$\left.\begin{array}{rll} -\Delta u & = \frac{u^{2_*(1)-1}}{|y|} + \lambda u & \text{in } \Omega \\ u & > 0 & \text{in } \Omega \\ u & = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{4.36}$$

where Ω is a bounded domain in $\mathbb{R}^n$ containing the origin and $2_*(1) = \frac{2(n-1)}{n-2}$.

Proof of theorem 4.3.3

Let $u \in H_0^1(\Omega)$ be given by Lemma 5.2.2, that is

$$||u||_{2_*,|y|^{-1}} = 1 \quad \textit{and} \quad ||\nabla u||_2^2 - \lambda ||u||_2^2 = S_\lambda.$$

We may assume that $u \geq 0$ on Ω. Since u is a minimizer for S_λ, we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta u - \lambda u = \mu \frac{u^{2_*-1}}{|y|} \quad \textit{on}\,\Omega.$$

In fact, $\mu = S_\lambda$ and $S_\lambda > 0$ since $\lambda < \lambda_1$. It follows that $c\,u$ satisfies (4.36) for some appropriate constant $c > 0$ ($c = S_\lambda^{1/(2_*-2)}$). We note that $u > 0$ on Ω by strong maximum principle of Vazquez [53]. □

BIBLIOGRAPHY

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), no. 4, 573–598.

[2] M. Badiale and E. Serra, *Critical nonlinear elliptic equations with singularities and cylindrical symmetry*, Rev. Mat. Iberoamericana **20** (2004), no. 1, 33–66.

[3] M. Badiale and G. Tarantello, *A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics*, Arch. Ration. Mech. Anal. **163** (2002), no. 4, 259–293.

[4] R. Beals, B. Gaveau, P. Greiner, and Y. Kannai, *Exact fundamental solutions for a class of degenerate elliptic operators*, Comm. Partial Differential Equations **24** (1999), no. 3-4, 719–742.

[5] R. Beals, P. Greiner, and B. Gaveau, *Green's functions for some highly degenerate elliptic operators*, J. Funct. Anal. **165** (1999), no. 2, 407–429.

[6] W. Beckner, *On the Grushin operator and hyperbolic symmetry*, Proc. Amer. Math. Soc. **129** (2001), no. 4, 1233–1246.

[7] G. Bertin, *Dynamics of galaxies*, Cambridge University Press, Cambridge, 2000.

[8] F. Bethuel and O. Rey, *Multiple solutions to the Plateau problem for nonconstant mean curvature*, Duke Math. J. **73** (1994), no. 3, 593–646.

[9] H. Brezis and J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. **37** (1984), no. 2, 149–187.

[10] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.

[11] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[12] L. A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.

[13] L. A. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275.

[14] P. Caldiroli and R. Musina, *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent*, Calc. Var. Partial Differential Equations **8** (1999), no. 4, 365–387.

[15] ______, *Existence of minimal H-bubbles*, Commun. Contemp. Math. **4** (2002), no. 2, 177–209.

[16] E. A. Carlen and M. Loss, *Extremals of functionals with competing symmetries*, J. Funct. Anal. **88** (1990), no. 2, 437–456.

[17] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math. **54** (2001), no. 2, 229–258.

[18] K. S. Chou and C. W. Chu, *On the best constant for a weighted Sobolev-Hardy inequality*, J. London Math. Soc. (2) **48** (1993), no. 1, 137–151.

[19] L. Ciotti, *Dynamical models in astrophysics*, Scuola Normale Superiore, Pisa, 2001.

[20] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), no. 1, 263–321.

[21] H. Egnell, *Asymptotic results for finite energy solutions of semilinear elliptic equations*, J. Differential Equations **98** (1992), no. 1, 34–56.

[22] I. Fabbri, G. Mancini, and Sandeep K., *Classification of solutions of a critical hardy sobolev operator*, Preprint (2005).

[23] V. Felli and M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Differential Equations **191** (2003), no. 1, 121–142.

[24] J. P. García Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations **144** (1998), no. 2, 441–476.

[25] N. Garofalo and D. Vassilev, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type*, Duke Math. J. **106** (2001), no. 3, 411–448.

[26] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[27] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud., vol. 7, Academic Press, New York, 1981, pp. 369–402.

[28] Grosse H. Glaser V., Martin A. and Thirring W., *A family of optimal condiions for absence of bound states in a poential*, University Press, Princeton, 1976.

[29] R.D.II Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. (2) **97** (1973), 275–305.

[30] Q. Han and F. Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, vol. 1, New York University Courant Institute of Mathematical Sciences, New York, 1997.

[31] D. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. **25** (1987), no. 2, 167–197.

[32] ———, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), no. 1, 1–13.

[33] I. Kuzin and S. Pohozaev, *Entire solutions of semilinear elliptic equations*, Progress in Nonlinear Differential Equations and their Applications, 33, Birkhäuser Verlag, Basel, 1997.

[34] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374.

[35] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997.

[36] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145.

[37] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283.

[38] ———, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201.

[39] ———, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **1** (1985), no. 2, 45–121.

[40] G. Mancini and K. Sandeep, *Cylindrical symmetry of extremals of a Hardy-Sobolev inequality*, Ann. Mat. Pura Appl. (4) **183** (2004), no. 2, 165–172.

[41] R. Monti and D. Morbidelli, *On the elliptic equation Kelvin transform for grushin operators and critical semilinear equations*, Preprint (2005).

[42] W. M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$*, its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), no. 4, 493–529.

[43] R. Osserman, *A proof of the regularity everywhere of the classical solution to Plateau's problem*, Ann. of Math. (2) **91** (1970), 550–569.

[44] T. Radó, *An iterative process in the problem of plateau*, Trans. Amer. Math. Soc. **35** (1933), no. 4, 869–887.

[45] K. Sandeep, *On a noncompact minimization problem of Hardy-Sobolev type*, Adv. Nonlinear Stud. **2** (2002), no. 1, 81–91.

[46] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.

[47] K. Steffen, *Isoperimetric inequalities and the problem of Plateau*, Math. Ann. **222** (1976), no. 2, 97–144.

[48] ______, *On the existence of surfaces with prescribed mean curvature and boundary*, Math. Z. **146** (1976), no. 2, 113–135.

[49] M. Struwe, *Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature*, Analysis, et cetera, Academic Press, Boston, MA, 1990, pp. 639–666.

[50] ______, *Variational methods*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 34, Springer-Verlag, Berlin, 2000.

[51] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[52] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[53] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

[54] G. F. Wang, *The Dirichlet problem for the equation of prescribed mean curvature*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 6, 643–655.

[55] Z.-Q. Wang and M. Willem, *Singular minimization problems*, J. Differential Equations **161** (2000), no. 2, 307–320.

[56] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.