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# Equivariant localization methods, orientations and modularity

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# Equivariant localization methods, orientations and modularity

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**Summary/Thesis**

## Introduction

This thesis is made of the papers I wrote together with my Ph.D. colleague Mattia Coloma and our advisor Prof. Domenico Fiorenza.

In this introduction I will briefly summarise the contents and the main results and techniques of each of the papers. After that, the main body of the thesis will be comprised of the latest version of each paper (i.e., the in-press version for the accepted articles and the latest ar$\chi$iv version for the one still under the refereeing process), each followed by appendices to expand on some points that could not find enough space in the articles or to present alternative points of view on some aspects.

The first paper, "An exposition of the topological half of the Grothendieck–Hirzebruch–Riemann–Roch theorem in the fancy language of spectra", uploaded to the ar$\chi$iv on 27/11/2019 and accepted for publication in *Expositiones Mathematicæ* on 17/11/2021, explains at a fundamental level the reasons for the presence of the Todd class in the GHRR theorem, highlighting how natural and inevitable it is.

In the second paper, "A very short note on the (rational) graded Hori map", uploaded to the ar$\chi$iv on 29/03/2020 and accepted for publication in *Communications in Algebra* on 08/11/2021, we show how the so-called graded Hori map of [HM20] emerges at the rational level from the canonical equivalence between left and right gerbes associated with a $T$-duality configuration and, in particular, how it can be recovered as a pull-iso-push transform.

In the third paper, "The (anti-)holomorphic sector in $\mathbb{C}/\Lambda$-equivariant cohomology, and the Witten class", uploaded to the ar$\chi$iv on 28/06/2021 and currently under review by the *Journal of Geometry and Physics* in its revised version after a first positive report, we investigate the Witten genus from the point of view of equivariant localization. We show how the a priori unjustified statement that two formal degree 2 independent variables $u$ and $v$ have ratio $\frac{u}{v} = \tau \in \mathbb{H}$ can be made rigorous via a generalization of the Atiyah and Bott localization theorem in equivariant cohomology, involving a suitably defined (anti-)holomorphic sector in the equivariant cohomology for the action of an elliptic curve $\mathbb{C}/\Lambda$.

## 1. The fancy language of spectra and the topological Grothendieck–Hirzebruch–Riemann–Roch theorem

In the the paper "An exposition of the topological half of the Grothendieck–Hirzebruch–Riemann–Roch theorem in the fancy language of spectra" we give an informal exposition of the theory of pushforwards and orientations in cohomology with emphasis on categorical features of the category of spectra coming into play, and show how the topological content of the GHRR theorem naturally emerges from this.

While we explicitly reference the $\infty$-category $\mathsf{Sp}$ of Spectra in the article, we just barely use its actual $\infty$-categorical structure explicitly. For our scopes we just need to keep in mind that $\mathsf{Sp}$ is topologically enriched and that its homotopy category has hom sets the $\pi_0$ of the spaces of morphisms. We denote by $\mathsf{Sp}(X, Y)$ the space of morphisms between spectra $X$ and $Y$ and by $[X, Y] := \pi_0\mathsf{Sp}(X, Y)$ the hom set in the homotopy category.

Given a space $X$ one obtains a spectrum out of it by first adjoining a basepoint via the functor $(-)_+ : \mathsf{Top} \to \mathsf{Top}_*$ and then building the suspension spectrum $\Sigma^\infty_+ X$ by applying the functor $\Sigma^\infty : \mathsf{Top}_* \to \mathsf{Sp}$. It is actually convenient denote this spectrum just by $X$ by a slight abuse of notation. We systematically do this in the article, the only notable exception being writing $\mathbb{S}$ instead of $\Sigma^\infty_+ *$ for the sphere spectrum. The $\infty$-category $\mathsf{Sp}$ of spectra is stable (in particular it is the stabilization of the category of (nice) topological spaces). As such, it comes with a fundamental family of self-equivalences given by the suspensions/shifts. In view of their topological origin, the shift by 1 in positive degree is called the suspension while in negative degree is called the looping.

Another important categorical feature of $\mathsf{Sp}$ we make a prominent use of is its closed monoidality: we have a tensor product given by the smash product of spectra and it admits a right adjoint $F$ given by the internal hom. The tensor product allows us to define monoids (that is, ring spectra, representing multiplicative cohomology theories) and comonoids (and we remark that all suspension spectra admit a comonoid structure, inherited from their base space). As a consequence, given a space $X$ and a ring spectrum $E$ we have a natural ring structure on $[X, E]$ and a natural map $\pi_0(E) = [\mathbb{S}, E] \to [X, E]$ making $[X, E]$ a $\pi_0(E)$ algebra.

Given a vector bundle $V \to X$ one defines its Thom spectrum $X^V$ to be the suspension spectrum of its Thom space, the latter defined as the homotopy cofibre of the map $V\backslash X \to V$, with $V\backslash X$ given by $V$ minus the image of the zero section of the bundle. Since trivial bundles become suspensions under the Thom space functor, we can extend our definition of Thom spectrum to virtual bundles (formal differences of bundles): if $V = W - Z$, and if $Y + Z$ equals the trivial bundle of rank $n$, we define $X^V = X^{W+Y}[-n]$. This is useful to define the Thom spectrum of the negative tangent bundle and to look at the Pontryagin-Thom collapse map as a map of spectra $\gamma_{PT} : \mathbb{S} \to X^{-TX}$. A theorem of Atiyah states that, if $X$ is a compact smooth manifold, there is an isomorphism of spectra under $\mathbb{S}$ between $\gamma_{PT} : \mathbb{S} \to X^{-TX}$ and $\varphi_X : \mathbb{S} \to DX$, where the Alexander–Spanier dual $DX = F(X, \mathbb{S})$ of $X$ is the internal hom spectrum of maps from $X$ to $\mathbb{S}$ and $\varphi_X$ is the map induced by the terminal morphism $X \to \mathbb{S}$ by duality.

This leads to integration as follows. Given a spectrum $E$ one says that a compact smooth $n$-dimensional manifold $X$ is $E$-orientable if $[X, E]$ and $[DX[n], E]$ are isomorphic as $[X, E]$-modules, and one calls $E$-orientation of $X$ an isomorphism between them. Given an $E$-oriented manifold one then has a natural integration map given by $[X, E] \cong [DX[n], E] \xrightarrow{\varphi^*} [\mathbb{S}[n], E]$.

The definition of orientability can be immediately extended to arbitrary vector bundles as the condition $[X, E] \cong [X^V[-\operatorname{rk} V], E]$. Having introduced this notion of $E$-orientability for arbitrary vector bundles, we consider what we call closed families, a family $\mathcal{F}$ being closed if it is closed under box sums and pullbacks. In a thesis appendix to the article, containing extra material with respect to the article itself, we show how closed families defined this way correspond to families of vector bundles associated with tangential structures and vice versa. A coherent $E$-orientation of a closed family is the datum of an $E$-orientation for every vector bundle in the family, such that our orientation isomorphisms behave well under pullback and products. After we define stable families as closed families containing every trivial bundle and such that given a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ if two of them are in $\mathcal{F}$ so is the third, we show how a coherent orientation of a stable family allows us to extend the definition of $E$-orientation to virtual bundles that are formal differences of bundles in the stable family.

Defining the tangent bundle to a map $f : X \to Y$ as the virtual bundle $T_f := TX - f^*TY$, we say that $f$ is $E$-oriented if $T_f$ is. Given an $E$-oriented map $f$ we describe how to obtain pushforwards in cohomology, by first factoring $f : X \to Y$ as an inclusion $i : X \to P$ composed with a fibration $p : P \to Y$, and then defining the pushforwards along $i$ and $p$. It should be remarked that the pushforward along an $E$-oriented fibration $p$ is quite technical, making use of the theory of parametrized spectra of May and Sigurdsson [MS06] to build the parametrized version $\varphi_{PT} : Y \to P^{-T_p}$ of the Pontryagin–Thom collapse map. The pushforward along the inclusion is on the other hand more standard, as the construction of the Pontryagin–Thom map in this case is straightforward. Once one has the Pontryagin–Thom morphism $\varphi_{PT} : Y \to X^{-T_f}$ one can define the pushforward $f_* : [X, E] \to [Y[\dim X - \dim Y], E]$ as the composition

$$[X, E] \cong [X^{-T_f}[\dim X - \dim Y], E] \xrightarrow{\varphi_{PT}^*} [Y[\dim X - \dim Y], E].$$

We notice that $\varphi_{PT}$ is a morphism of comodules so the map above is a map of $[Y, E]$-modules. In particular, this gives the projection formula

$$f_*((f^*a)x) = af_*(x)$$

for any $a \in [Y, E]$ and $x \in [X, E]$.

In the second part of the article we focus on stably complex vector bundles, that is, on the smallest stable family containing all the vector bundles with structure group $U(n)$, with $n \geqslant 0$, and prove that an orientation of this family for a cohomology theory $E$ amounts to a map $MU \to E$, with $MU$ the spectrum of complex cobordism. We call this map, and the corresponding $E$-orientation for stably complex vector bundles, a *complex orientation* of $E$. For an even 2-periodic ring spectrum $E$, i.e., for a ring spectrum such that $\pi^{\mathsf{Sp}}_{2k+1}E = 0$ for all $k \in \mathbb{Z}$ and satisfying $E \cong E[2]$, this definition is shown to be equivalent to the classical one in terms of the choice of a generator of $E(S^2)$.

We then study the case of two complex orientations $A$ and $B$ for an even 2-periodic spectrum $E$ and prove a first version of the Grothendieck–Hirzebruch–Riemann-Roch theorem stating that, given $f$ a complex oriented map,

$$f_*^A(a) = f_*^B(a \cdot \mathrm{td}_{A,B}(T_f))$$

for any $a \in [X, E]$, with $f_*^A$ and $f_*^B$ the pushforwards induced by the two orientations and $\mathrm{td}_{A,B}(T_f)$ a distinguished invertible element generalizing the classical Todd class of $T_f$. Equivalently, the above identity can be written as

$$f_*^A(a) \cdot \mathrm{td}_{A,B}(TY) = f_*^B(a \cdot \mathrm{td}_{A,B}(TX)).$$

As a slight generalization of this first version of the theorem, we get a second version that more directly generalizes the classical statement. We consider two even 2-periodic spectra $E, F$ each endowed with a fixed complex orientation, and a map of ring spectra $\psi : E \to F$. Pushing forward the orientation on $E$ to $F$ one falls back into the first version of the theorem, where the two involved orientations are now the original orientation on $F$ and the one pushed forward from $E$ via $\psi$. This way we obtain the following:

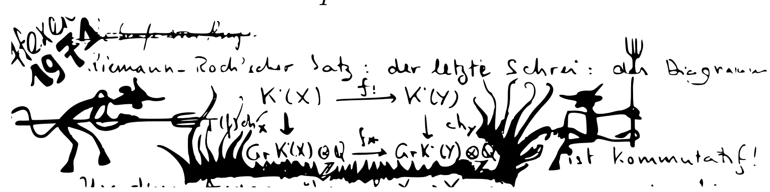
**Proposition.** *Let $\rho_E \colon MU \to E$ and $\rho_F \colon MU \to F$ be complex orientations for the ring spectra $E$ and $F$, respectively, and let $\psi \colon E \to F$ be a morphism of ring spectra. Then, for any stably complex map $f \colon X \to Y$ and any $a \in [X, E]$, the following Grothendieck–Hirzebruch–Riemann–Roch-like identity holds:*

$$\psi_*(f_*^{\rho_E}(a)) = f_*^{\rho_F}(\psi_*(a) \cdot \mathrm{td}_{\psi_*\rho_E, \rho_F}(T_f)).$$

Finally we show how taking $E$ and $F$ to be complex $K$-theory and periodic rational cohomology, respectively, and taking $\psi$ to be the Chern character one recovers the classical statement of the Theorem:

**Theorem.** *Let $f : X \to Y$ be a complex oriented map, and let $K$ and $HP_{\mathbb{Q}}$ denote the complex $K$-theory and the periodic singular cohomology with rational coefficients, respectively. Let $f_*^K$ and $f_*^{HP_{\mathbb{Q}}}$ be the standard pushforwards in along $f$ in $K$-theory and in periodic rational cohomology, respectively. Finally, let $ch : K \to HP_{\mathbb{Q}}$ be the Chern character.*

*Then we have a commutative square:*




*with $td(f)$ the Todd class of the map $f$.*

This is obtained by identifying the general Todd-type class $\mathrm{td}_{\psi_*\rho_E,\rho_F}$ appearing in the Proposition above with the classical Todd class in the special case given by the assumptions in the statement of the Theorem.

## 2. The (rational) graded Hori map in a very short note

In the article "A very short note on the (rational) graded Hori map" we bring to light how the seemingly *ad hoc* construction of Han–Mathai's graded Hori map of [HM20] can be naturally seen as a pull-iso-push transform. This generalizes the classical isomorphism between the twisted cohomologies of two $T$-dual principal bundles known as the Hori map.

We work $\mathbb{K}$-rationally over a fixed characteristic zero field $\mathbb{K}$; doing this allows us to work in the simple world of differential graded commutative algebras (DGCAs) letting us do everything rather explicitly leaving the geometric/topological translation of the various constructions to the interested reader.

We begin by recalling how to extend DGCAs trivialising some cocycle as an homotopy pushout (unsurprisingly, this is dual to the geometric operation of forming the homotopy pullback giving the total space of a gerbe). In particular we consider the DGCA $\mathbb{K}[x_{2L}, x_{2R}]$ with two degree 2 generators and extend it by trivialising the 4-cocycle given by product $x_{2L}x_{2R}$. This extension $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ will be the protagonist of the paper, as it is the DGCA (co)classifying $\mathbb{K}$-rational $T$-duality configurations.

We then consider a map $f$ from $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ to some DGCA $A$ and out of this, by means of a rational "homotopy fibre/cyclification adjunction", we build several other extensions and $\mathbb{K}$-rational gerbes, participating in the following diagram where every square is a homotopy pushout and the isomorphism $\nu$ comes from the automorphism of $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ switching the cocycles:

$$
\begin{tikzcd}
 & \mathcal{G}_{L\{f(x_{2R})\}} \arrow[rr, bend left, "\nu"] & & \mathcal{G}_{R\{f(x_{2L})\}} \arrow[ll, bend left, "\nu^{-1}"] & \\
\mathcal{G}_L \arrow[ur, "\iota_R"] & & A_{LR} \arrow[ul, "i_L"'] \arrow[ur, "i_R"] & & \mathcal{G}_R \arrow[ul, "\iota_L"'] \\
 & A_L \arrow[ul, "i_L"] \arrow[ur, "\iota_R"'] & & A_R \arrow[ul, "\iota_L"] \arrow[ur, "i_R"'] & \\
 & & A \arrow[ul, "\iota_L"] \arrow[ur, "\iota_R"'] & &
\end{tikzcd}
$$

We focus then on the upper part of this diagram and extend every DGCA $B$ appearing into it to the ring $\hat{B} = B[[\xi^{-1}, \xi]$ of bounded above formal Laurent series in a variable $\xi$ of degree 2. Thanks to a fairly general result allowing us to construct a degree-shifting map $\pi : \mathcal{G}_{R\{f(x_{2L})\}} \to \mathcal{G}_R[-1]$, we give a projection map $\hat{\pi} \colon \widehat{\mathcal{G}_{R\{f(x_{2L})\}}} \to \widehat{\mathcal{G}_R}[-1]$ which lets us define the pull-iso-push transform $\mathcal{T}_{L\to R} := \hat{\pi}\hat{\nu}\hat{\iota}_R : \widehat{\mathcal{G}_L} \to \widehat{\mathcal{G}_R}[-1]$. This reproduces/generalizes the graded Hori map of Han-Mathai. Switching the roles of the left and right generators one obtains the map $\mathcal{T}_{R\to L}$.

Once these maps have been defined in the formal setting above, we specialize them first by considering the algebra of meromorphic functions with coefficients in a given DGCA that are holomorphic on $\mathbb{C}\setminus\{0\}$, and then by considering the algebra of index 0 Jacobi forms (again with values in a given DGCA). Doing this we prove that the composition $\mathcal{T}_{L\to R} \circ \mathcal{T}_{R\to L}$ is naturally identified with the operator $-\frac{1}{2\pi i}\frac{\partial}{\partial z}$, thus reproducing Theorem 2.2 from Han-Mathai paper.

In a thesis appendix to the article, containing extra material with respect to the article itself, we give a sketch of the above constructions in the topological setting, emphasizing how the lack of an isomorphism between $K$-theory and even singular cohomology (which only become isomorphic after rationalization) makes everything more involved.

## 3. The Witten class and the (anti-)holomorphic sector in $\mathbb{C}/\Lambda$-equivariant cohomology

For a nice topological space $X$ acted upon by a topological group $G$, the $G$-equivariant (singular) cohomology of $X$ is the (singular) cohomology of the homotopy quotient $X//G$ or, equivalently, of the space $(X \times_G EG)/G$.

Such a quotient always admits a map to $BG \cong *//G$ and thus gives us a map in cohomology $H(BG) = H_G(*) \to H_G(X) = H(X//G)$. In other words, given a $G$-space $X$ its $G$-equivariant cohomology is an algebra over $H_G(*)$. We will be mostly concerned with circle and torus actions, so we recall that $H_{S^1}(*;\mathbb{R}) = H(BS^1;\mathbb{R}) = H(\mathbb{CP}^\infty;\mathbb{R}) = \mathbb{R}[u]$ and $H_{S^1\times S^1}(*;\mathbb{R}) = \mathbb{R}[u,v]$, where $u$ and $v$ are variables of degree 2.

In our paper "The (anti-)holomorphic sector in $\mathbb{C}/\Lambda$-equivariant cohomology, and the Witten class" we recall the how to compute the equivariant cohomology of a smooth manifold with a circle action via the Cartan (bi)complex. To be more precise, rather than an action of the topological 1-torus, we consider the actions of 1-dimensional Euclidean tori $\mathbb{R}/\Lambda$, i.e., of Euclidean circles of length $\lambda$, where $\lambda$ is the positive generator of the lattice $\Lambda \subseteq \mathbb{R}$. We remark that the $\mathbb{R}/\Lambda$-equivariant cohomologies will all be naturally isomorphic making this, at a first sight, quite trivial and uninteresting. This is just a manifestation of the contractibility of the moduli stack of real 1-dimensional lattices that is isomorphic to $\mathbb{R}_{>0}$; the case we are actually interested about, that of flat 2-dimensional tori, will yield more interesting features, again manifesting the nontrivial topology of the moduli stack of 2-dimensional lattices.

For a given smooth manifold $M$ with an $\mathbb{R}/\Lambda$-action we therefore consider the Cartan bicomplex

$$\Omega^\bullet(M,\mathbb{R})^{\mathbb{R}/\Lambda} \otimes_{\mathbb{R}} \mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2]),$$

whose two differentials are de Rham differential and the contraction along the infinitesimal generator of the rotations, and whose total cohomology is the $\mathbb{R}/\Lambda$-equivariant cohomology of $M$. We also provide a version of the Cartan (bi)complex written in terms of the coordinates relative to a choice of a linear basis for the Lie algebra $t_\Lambda$ of $\mathbb{R}/\Lambda$, which is useful for explicit computations.

Next, we turn our attention to equivariant characteristic classes of equivariant vector bundles over $\mathbb{R}/\Lambda$-trivial bases, i.e., such that the $\mathbb{R}/\Lambda$-action on the base of the bundle is trivial. Since an $\mathbb{R}/\Lambda$-equivariant complex line bundle over a $\mathbb{R}/\Lambda$-trivial base can be seen as the datum of a standard complex line bundle together with a complex character, we can define, using the splitting principle, the weight polynomial $wp(E^{\mathrm{eff}})$ of (the effectively acted part of) a complex $\mathbb{R}/\Lambda$-equivariant vector bundle $E$ as the product of the weights of the characters associated with equivariant line bundles it formally splits into. Here the weight $w_\chi$ of a character $\chi : \mathbb{R}/\Lambda \to U(1)$ is defined to be $(2\pi i)^{-1}$ times the Lie algebra morphism $2\pi i w_\chi$ induced by $\chi$.

Once the weight polynomial has been defined, we can define the normalized top Chern class of $E^{\mathrm{eff}}$ as the element, in the $\mathbb{R}/\Lambda$-equivariant cohomology localized at $wp(E^{\mathrm{eff}})$, as $\widehat{c_{top,\mathbb{R}/\Lambda}}(E^{\mathrm{eff}}) = c_{top,\mathbb{R}/\Lambda}(E^{\mathrm{eff}})/wp(E^{\mathrm{eff}})$. Analogously, we define the Euler class and normalized Euler class of (the effectively acted part of) a real $\mathbb{R}/\Lambda$-equivariant vector bundle $V$ and show that

$$\widehat{\mathrm{eul}}_{\mathbb{R}/\Lambda}(V^{\mathrm{eff}}) := \sqrt{\widehat{c_{top,\mathbb{R}/\Lambda}}(V^{\mathrm{eff}} \otimes \mathbb{C})}.$$

After having presented the main constructions in the simplified setting of Euclidean 1-tori, we then turn our attention to the case of 2-dimensional flat tori equipped with a complex structure, or, equivalently, of complex Lie groups $\mathbb{C}/\Lambda$.

Here the analogue of the Cartan bicomplex above admits a tricomplex structure, that is, we have that the $\mathbb{C}/\Lambda$-equivariant cohomology of $M$ with coefficients in $\mathbb{C}$ is the total cohomology of

$$\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_{\mathbb{C}} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{1,0})^\vee[-2]) \otimes_{\mathbb{C}} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{0,1})^\vee[-2]),$$

endowed with the three differentials given by the de Rham differential and by the contractions in the complex directions $\partial/\partial z$ and $\partial/\partial \overline{z}$.

Again, we also give a version with coordinates. Restricting our attention to ${\mathfrak{t}_\Lambda}^{1,0}$ or ${\mathfrak{t}_\Lambda}^{0,1}$ we get the holomorphic and antiholomorphic sector of the Cartan complex. To be precise, we define the antiholomorphic sector of the Cartan complex as the total complex of the bicomplex

$$\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_{\mathbb{C}} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{0,1})^\vee[-2])$$

whose two differentials are the de Rham differential and the contraction in the complex direction $\partial/\partial \overline{z}$. Here too one can give a convenient description using coordinates. We remark that one has a natural restriction map from the Cartan tricomplex to the holomorphic and antiholomorphic sectors.

Unsurprisingly, everything we did in the 1-dimensional case admits an analogue in 2 dimensions, in particular we define weights, the weight polynomial, Chern and Euler classes and their normalized versions. The new, interesting feature is that we can define these in the antiholomorphic sector by simply taking the restriction map.

A well known theorem of Atiyah and Bott (see [AB84]) states that if $M$ is a smooth, orientable manifold with an $S^1$-action and $X$ is the submanifold of fixed points, then there is an isomorphism of localized modules

$$H_{S^1}(M)_{(u)} \xrightarrow{i^*/\mathrm{eul}(\nu)} H_{S^1}(X)_{(u)}$$

with $\mathrm{eul}(\nu)$ the Euler class of the normal bundle relative to the inclusion $i : X \hookrightarrow M$. Here one is localizing at the generator $u$ of $H_{S^1}(*;\mathbb{R}) = \mathbb{R}[u]$.

In particular, by functoriality of the pushforwards, one obtains from this that

$$\int_M \omega = \int_X \frac{i^*(\omega)}{\mathrm{eul}(\nu)}$$

for any $S^1$-equivariant form $\omega$ on $M$. By formally using the localization formula, in the seminal paper [Ati85] on localization techniques for $S^1$-actions on loop spaces, Atiyah obtains the $\hat{A}$ genus of a spin manifold $X$ via an extremely neat construction. Atiyah considers the (free) loop space $\mathcal{L}X$ of smooth unbased maps from $S^1$ to $X$; this space admits a smooth, infinite dimensional manifold structure and an action of $S^1$ given by rotations of loops. One may then want to compute the (equivariant) volume of such manifold. By formally applying the localization formula with $M = \mathcal{L}X$ (and so pretending to forget that the localization formula was given only for finite dimensional manifolds) one can simply define integrals on $\mathcal{L}X$ to be given by the localization formula. In particular one has this way that the equivariant volume of $\mathcal{L}X$ is given by

$$\int_{\mathcal{L}X} \mathrm{vol} = \int_X \frac{i^*\mathrm{vol}}{\mathrm{eul}(\nu)}.$$

By using $\zeta$-regularization techniques, Atiyah proved that, for a suitably chosen "symplectic" volume form on $\mathcal{L}X$, this actually yields the $\hat{A}$ genus of $X$. It is interesting to notice that from Atiyah's computation one can extract a derivation of the $\hat{A}$ genus not involving the ill-defined infinite-dimensional volume form: one finds

$$\hat{A}(X) = \int_X \frac{1}{\widehat{\mathrm{eul}^\zeta}(\nu)},$$

where $\widehat{\mathrm{eul}^\zeta}(\nu)$ is the $\zeta$-regularized normalized Euler class of the normal bundle to the inclusion of $X$ in $\mathcal{L}X$ as the submanifold of fixed points.

In our article we prove a version of the localization formula in the antiholomorphic sector. More precisely, we show that if an equivariantly closed form $\omega$ in the antiholomorphic sector is in the image of the restriction map from the closed forms in the full Cartan complex, then

$$\int_M \omega = \int_{\mathrm{Fix}(M)} \frac{\iota^*\omega}{\mathrm{eul}^{\bar{\partial}}_{\mathbb{C}/\Lambda}(\nu)}.$$

Having proved this antiholomorphic localization theorem, we want to apply it to the mapping space $M = \mathrm{Maps}(\mathbb{C}/\Lambda, X)$. To do this we have to work formally since again the localization formula is actually true only for finite dimensional manifolds. In particular the normal bundle of $X = \mathrm{Fix}(M) \hookrightarrow M$ has infinite rank and its Euler class is thus not well defined a priori. The trick is to proceed with a Fourier decomposition and to notice that the (complexified) normal bundle is isomorphic to the complexified tangent bundle of $X$ tensored with the direct sum of all nontrivial 1-dimensional representations of $\mathbb{C}/\Lambda$. This in turn implies that we can write our normalized Chern and Euler classes in the antiholomorphic sector as infinite products that can be computed via a $\zeta$-regularization procedure.

We then obtain that, when $X$ is a rational string manifold of even dimension $d$ then

$$\frac{1}{\widehat{\mathrm{eul}_{\mathbb{C}/\Lambda}^{\bar{\partial};\zeta}}(\nu)} = \sum_{j=0}^{\infty} \mathrm{Wit}_{\mathbb{R};j}(X)\overline{\xi}_\lambda^{-j},$$

where $\mathrm{Wit}_{\mathbb{R};j}(X)$ is the degree $2j$ homogeneous component of the real Witten class $\mathrm{Wit}_{\mathbb{R}}(X)$ of $X$, i.e., of the nonhomogeneous cohomology class defined by

$$\mathrm{Wit}_{\mathbb{R}}(X) = \prod_{j=1}^{d/2} \frac{\sqrt{z}}{\sigma_\Lambda(\sqrt{z})}\bigg|_{z=\beta_j(TX)},$$

where the $\beta_j(TX)$ are the Pontryagin roots of $TX$ and $\sigma_\Lambda$ is the Weierstraß $\sigma$-function of the lattice $\Lambda$.

In a thesis appendix to the article, containing extra material with respect to the article itself, we provide a general framework in which to view the construction in our paper. In particular we suggest that it may be interesting to associate to a manifold $\Sigma$ and a subgroup $G$ of its diffeomorphism group, the integral of the normalized ($\zeta$-regularized) Euler class of the $G$-equivariant normal bundle relative to the inclusion $\iota : X \hookrightarrow \mathrm{Maps}(\Sigma, X)$. We give a minimal example, with $\Sigma = S^0$ and $G = \mathbb{Z}_2$ and prove that, working in singular cohomology with coefficients in $\mathbb{F}_2$, the aforementioned integral is always 0.

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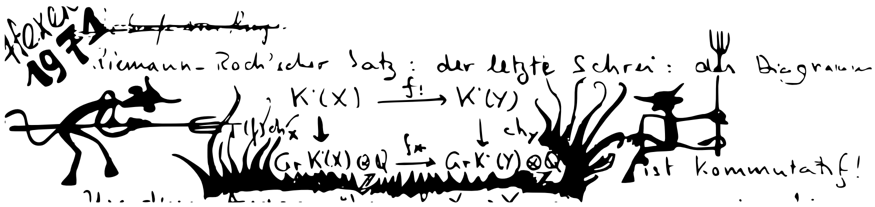
**An exposition of the topological half of the Grothendieck–Hirzebruch–Riemann–Roch theorem in the fancy language of spectra**

Abstract. We give an informal exposition of pushforwards and orientations in generalized cohomology theories in the language of spectra. The whole note can be seen as an attempt at convincing the reader that Todd classes in Grothendieck–Hirzebruch–Riemann–Roch type formulas are not Devil's appearances but rather that things just go in the most natural possible way.




## 1. Introduction

After tensoring by $\mathbb{Q}$, the Chern character induces a natural isomorphism of rings

$$\mathrm{ch}\colon K^0(X)\otimes\mathbb{Q} \to \bigoplus_{i\in\mathbb{Z}} H^{2i}(X;\mathbb{Q})$$

from the complex $K$-theory of a (nice) topological space $X$ to its even singular cohomology with rational coefficients. Moreover, both complex $K$-theory and singular cohomology have natural complex orientations. Simplifying a bit, this means that if $X$ is a compact complex manifold, one has pushforward maps

$$\pi^{KU}_{X,*}\colon K^0(X)\otimes\mathbb{Q} \to K^{-2\dim_{\mathbb{C}} X}(*)\otimes\mathbb{Q} = \mathbb{Q}$$

and

$$\pi^{H\mathbb{Q}}_{X,*} = \int_X\colon \bigoplus_{i\in\mathbb{Z}} H^{2i}(X;\mathbb{Q}) \to \bigoplus_{i\in\mathbb{Z}} H^{2i-2\dim_{\mathbb{C}} X}(*;\mathbb{Q}) = \mathbb{Q}$$

Since everything is very natural here, one would expect that in the best of possible worlds the diagram

$$\begin{tikzcd} K^0(X)\otimes\mathbb{Q} \arrow[rr, "\mathrm{ch}", "\sim"'] \arrow[dr, "\pi^{KU}_{X,*}"'] & & \bigoplus_{i\in\mathbb{Z}} H^{2i}(X;\mathbb{Q}) \arrow[dl, "\int_X"] \\ & \mathbb{Q} & \end{tikzcd}$$

would commute, i.e., to have an integral formula of the form

$$\pi^{KU}_{X,*}([V]) = \int_X \mathrm{ch}(V)$$

for any complex vector bundle $V$ on $X$. Unfortunately, this formula is notoriously not correct: it becomes so only after the introduction of a suitable multiplicative correction factor, the Todd class of $X$, given by the cohomology class

$$\mathrm{td}(X) = 1 + \frac{c_1(X)}{2} + \frac{c_1(X)^2 + c_2(X)}{12} + \frac{c_1(X)c_2(X)}{24} + \cdots$$

This rather formidable expression is obtained from the characteristic power series

$$\frac{u}{1-e^{-u}} = 1 + \frac{u}{2} + \frac{u^2}{12} - \frac{u^4}{720} + \cdots$$

by applying the splitting principle to the holomorphic tangent bundle $TX$ of $X$, i.e., if $TX$ splits as a direct sum of complex line bundles $L_i$, then

$$\mathrm{td}(X) = \prod_{i=1}^{\dim_{\mathbb{C}} X} \left.\frac{u}{1-e^{-u}}\right|_{u=c_1(L_i)}.$$

On first sight, both the presence of the factor $\mathrm{td}(X)$ in the corrected formula

$$\pi_{X,*}^{KU}([V]) = \int_X \mathrm{ch}(V)\,\mathrm{td}(X)$$

and its expression in terms of Chern classes may appear rather mysterious. The main goal of this paper is to try to convince the reader that there is actually no mystery here, and that on the contrary the specific correction factor $\mathrm{td}(X)$ is precisely what one should have expected from the very beginning. We will try to achieve this within an informal exposition of the theory of pushforwards and orientations in cohomology, with an emphasis on categorical features of the category of spectra coming into play.

We make no claim of originality. Everything we write can be found elsewhere, and plenty of references will be given throughout the text. We owe most of our gratitude for the inspiration to Ando, Blumberg and Gepner and their paper [ABG10], to Panin and Smirnov for their [PS02] and [Pan02], and of course to Quillen [Qui71]. We have also taken from Lurie's [Lur09a] and [Lur10], from Ando, Blumberg Gepner, Hopkins and Rezk's [ABGHR08] and [AHR10], and from May's classic [May06].

We wish to reassure the reader that, although at the very beginning we will be mentioning that the category of spectra is a symmetric monoidal stable $\infty$-category in the sense Lurie's Higher Algebra [Lur12], no advanced knowledge of $\infty$-category theory is actually required to read this note as we will treat the higher categorical aspects of spectra only very intuitively and we will be mostly working in the homotopy category of spectra, which is an ordinary category (i.e., a 1-category).

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## 2. The stable $\infty$-category of spectra

We will be working in the stable $\infty$-category $\mathsf{Sp}$ of spectra. We reassure the reader possibly unfamiliar both with spectra and/or with $\infty$-categories (stable or not) that a previous knowledge of these may be useful but not necessary at all. A good motto to keep in mind is the following informal analogy: spectra are to spaces as real numbers are to rational numbers. By this we do not only mean that spectra are certain sequences of (nice) topological spaces just like real numbers are (equivalence classes of) certain sequences of rational numbers, but also –and this is the main content of the motto– that one actually works with spectra by knowing the categorical features of the category they form and its relations to the category of topological spaces, more than with their actual definition entailing these features. In terms of the analogy, this is to say that $\mathbb{R}$ can be defined as a suitable quotient of the set of Cauchy sequences of rational numbers, but what one generally uses when working with real numbers is not this definition but the properties it implies: e.g., that $\mathbb{R}$ is a complete ordered field. This is actually a complete definition of $\mathbb{R}$: a complete ordered field, if it exists, is unique and contains $\mathbb{Q}$. From this point of view, the whole business of equivalence classes of Cauchy sequences can be seen as a proof of the existence of a field with the completeness and ordering properties. Turning back to spectra, we are saying that the only point we are asking the reader to trust us on is that there exists a category $\mathsf{Sp}$ having the properties we will attribute to it.

Actually, $\mathsf{Sp}$ will not be an ordinary category, but an $(\infty, 1)$-category. We address the interested reader to [Lur09b] for a comprehensive treatment, and here content us with saying that for two given objects $X$ and $Y$ in $\mathsf{Sp}$, we do not have just a *set* of morphisms, but a whole *space* of morphisms $\mathsf{Sp}(X, Y)$ between $X$ and $Y$. This allows us to say not only that two morphisms are possibly equal, but that they are possibly homotopic (i.e., they lie in the same connected component of $\mathsf{Sp}(X, Y)$), or to talk of homotopies between homotopies between morphisms, and so on. Also, commutative diagrams of spectra will not be strictly commutative, but always commutative up to some given homotopy, which is part of the data defining a commutative diagram in an $\infty$-category.

*Remark* 2.1. Calling the objects of $\mathsf{Sp}$ "0-morphisms", the elements in $\mathsf{Sp}(X, Y)$ "1-morphisms, the homotopies between 1-morphisms "2-morphisms" and so on, one sees that in $\mathsf{Sp}$ one has $k$-morphisms for every $k \geqslant 0$. This motivates the terminology $\infty$-category. Moreover, since every homotopy is invertible up to a homotopy between homotopies, one sees that for $k > 1$ every $k$-morphism in $\mathsf{Sp}$ is invertible up to $(k + 1)$-morphisms. One indicates that 1 is the threshold for invertibility by saying that the $\infty$-category $\mathsf{Sp}$ is an $(\infty, 1)$-category.

Taking connected components of the hom-spaces one gets the homotopy category of spectra, usually denoted by $h\mathsf{Sp}$. As a matter of notation, hom-sets in the category $h\mathsf{Sp}$ will be denoted by $[-, -]$ rather than by $h\mathsf{Sp}(-, -)$, that is one writes

$$[X, Y] := \pi_0 \mathsf{Sp}(X, Y),$$

for any $X, Y$ in $\mathsf{Sp}$. For a morphism $f\colon X \to Y$ in $\mathsf{Sp}$, we will write

$$f^*\colon \mathsf{Sp}(Y, Z) \to \mathsf{Sp}(X, Z)$$

and

$$f_*\colon \mathsf{Sp}(T, X) \to \mathsf{Sp}(T, Y)$$

for the continuous maps between hom-spaces induced by precompostion and postcomposition with $f$, respectively. We will use the same symbols to denote the induced maps between hom-sets in $h\mathsf{Sp}$.

*Remark* 2.2. Even by the few informal lines above, one should deduce that $\infty$-categories are a nice context for doing homotopical constructions. In classical category theory, such a context is provided by the notion of a model structure on a category. Starting with a model structure on an ordinary category is indeed one of the most powerful ways to produce a rigorously defined $(\infty, 1)$-category. In these cases one says that the $(\infty, 1)$-category is *presented* by the model structure. For instance, a rigorous definition of the $(\infty, 1)$-category of spectra we are talking about is as the $(\infty, 1)$-category presented by the standard model structure on the (ordinary) category of orthogonal spectra [MMSS01].

Other two $(\infty, 1)$-categories we will meet in this note are the $\infty$-category $\mathsf{Top}$ of (nice) topological spaces, presented by the classical (or Quillen) model structure, and whose homotopy category $h\mathsf{Top}$ is the usual homotopy category of (nice) topological spaces, and its pointed version $\mathsf{Top}_*$. The $\infty$-categories $\mathsf{Top}$, $\mathsf{Top}_*$ and $\mathsf{Sp}$ are related by the functors (or, better, $\infty$-functors)

$$\mathsf{Top} \xrightarrow{(-)_+} \mathsf{Top}_* \ ,$$

and

$$\mathsf{Top}_* \xrightarrow{\Sigma^\infty} \mathsf{Sp} \ ,$$

where $(-)_+ \colon \mathsf{Top} \to \mathsf{Top}_*$ is the functor adjoining a free basepoint, i.e., $X_+ := X \sqcup *$, and $\Sigma^\infty \colon \mathsf{Top}_* \to \mathsf{Sp}$ is the infinite suspension, mapping a pointed space $Y$ to the spectrum given by sequence of pointed spaces $\Sigma^\infty Y$ recursively defined by $(\Sigma^\infty Y)_0 = Y$ and $(\Sigma^\infty Y)_{n+1} = \Sigma((\Sigma^\infty Y)_n)$ for any $n > 0$. The morphisms $\Sigma((\Sigma^\infty Y)_n) \to (\Sigma^\infty Y)_{n+1}$ giving $\Sigma^\infty Y$ the structure of a (sequential) spectrum are the identities.

We will always look at (nice) topological spaces and pointed (nice) topological spaces as spectra via the sequence of functors

$$\mathsf{Top} \xrightarrow{(-)_+} \mathsf{Top}_* \xrightarrow{\Sigma^\infty} \mathsf{Sp} \ ,$$

that is, given a space $X$, we will denote by the same symbol $X$ its stabilization, i.e., the spectrum $\Sigma^\infty_+ X$, and similarly for pointed spaces. As a notable exception to this rule, when $X$ is the space $*$ consisting of a single point, we will use the classical notation $\mathbb{S}$ to denote the spectrum $\Sigma^\infty_+ *$.

It should be remarked that stabilization is not an embedding: by stabilising one loses all the unstable information about a space $X$. Nevertheless, as we will only be concerned with stable aspects, no confusion should arise in this note, and we will find it an extremely convenient notation to denote by the same symbol both a space and its stabilization.

*Remark* 2.3. By definition, the spectrum $\mathbb{S}$ has $\mathbb{S}_0 = *_+ = * \sqcup * = S^0$ and so, inductively, $\mathbb{S}_n = S^n$ for any $n \geqslant 0$. It is called the *sphere spectrum*.

*Remark* 2.4. In an $(\infty, 1)$-category, all universal properties are given "up to homotopy" (with the homotopies being part of the universal property). For instance, in an ordinary category $\mathsf{C}$ the property of an object $\varnothing$ of being initial is the fact that the hom-set $\mathsf{C}(\varnothing, X)$ is a singleton for every object $X$ in $\mathsf{C}$. In the $(\infty, 1)$-categorical setting this is translated into requiring that the hom-space $\mathsf{C}(\varnothing, X)$ is "a singleton up to homotopy", i.e., into the requirement that $\mathsf{C}(\varnothing, X)$ is contractible. This means that there is not a unique initial morphism, but that the initial morphism is unique up to homotopies that are in turn unique up to higher homotopies, and so on. With this in mind, one continues to talk of "the" initial morphism $\varnothing \to X$. Similarly, the universal property of the pullback, declined into an $(\infty, 1)$-categorical setting, becomes the fact that the commutative diagram

$$\begin{tikzcd} \mathsf{C}(T, X \times_Z Y) \arrow[r] \arrow[d] & \mathsf{C}(T, X) \arrow[d] \\ \mathsf{C}(T, Y) \arrow[r] & \mathsf{C}(T, Z) \end{tikzcd}$$

is a *homotopy pullback* of topological spaces for every $T$ in $\mathsf{C}$. We refer the reader to [Doe98] for an introduction to homotopical pullbacks and pushouts in the category of (nice) topological spaces.

By saying that the $\infty$-category of spectra is stable one means it satisfies two very simple axioms (see [Lur12] for a comprehensive account):

(1) it has a zero object $\mathbf{0}$, i.e., an object that is at the same time initial and terminal;
(2) it has pullbacks and pushouts, and every pullback is a pushout, and vice versa.

*Remark* 2.5. Since pullback and pushout diagrams coincide in a stable $(\infty, 1)$-category, one sometimes calls them "pullout" diagrams. Pullout diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ \mathbf{0} \arrow[r] & Z \end{tikzcd}$$

are called fibre/cofibre sequences. In this case, one says that $X$ is the fibre of $g$ and $Z$ is the cofibre of $g$.

*Remark* 2.6. One uses a special notation to denote the fibre of the initial morphism $\mathbf{0} \to X$ and the cofibre of the terminal morphism $X \to \mathbf{0}$. Namely one denotes them by the symbols $X[-1]$ and $X[1]$, respectively, so that one has the following defining pullout diagrams for these:

$$\begin{tikzcd} X[-1] \arrow[r] \arrow[d] & \mathbf{0} \arrow[d] \\ \mathbf{0} \arrow[r] & X \end{tikzcd} \quad ; \quad \begin{tikzcd} X \arrow[r] \arrow[d] & \mathbf{0} \arrow[d] \\ \mathbf{0} \arrow[r] & X[1] \end{tikzcd} .$$

Comparison of the two pullout diagrams above immediately shows that in a stable $(\infty, 1)$-category $\mathsf{C}$ the shifts

$$[1],\ [-1] \colon \mathsf{C} \to \mathsf{C}$$

are autoequivalences of $\mathsf{C}$ inverse to each other. One calls $[1]$ the *shift* functor and, for any $n \in \mathbb{Z}$ one writes $[n]$ for $[1]^n$.

*Remark* 2.7. The reader used to the language of triangulated categories will have found this sketchy description of stable $\infty$-categories echoing something familiar. And indeed the homotopy category $h\mathsf{C}$ of a stable $(\infty, 1)$-category carries a natural structure of a triangulated category, where distinguished triangles are the image in $h\mathsf{C}$ of fibre/cofibre sequences in $\mathsf{C}$ and where the shift functor is induced by the shift functor of $\mathsf{C}$. It is a nice exercise to show that the two simple axioms of stable $(\infty, 1)$-category imply the somehow less transparent "octahedral axiom" of triangulated categories.

*Remark* 2.8. The shift functor in the $(\infty, 1)$-category of spectra is induced by the suspension functor on pointed topological spaces. By this reason it is commonly denoted as $X \mapsto \Sigma X$ in algebraic topology textbooks. Here we prefer denoting it by $X \mapsto X[1]$ to stress that it is the shift functor. Since $[1]$ and $[-1]$ are inverse autoequivalences of $\mathsf{C}$, one has a natural homotopy equivalence $\mathsf{Sp}(X[1], Y) \cong \mathsf{Sp}(X, Y[-1])$ for any $X, Y$ in $\mathsf{Sp}$. The identification of $[1]$ with the suspension functor then identifies the negative shift $Y \mapsto Y[-1]$ with the loop space functor.

*Remark* 2.9. As in every stable $\infty$-category, the hom-sets $[X, Y]$ in the homotopy category $h\mathsf{Sp}$ have a natural structure of abelian groups. Namely, since $\mathbf{0}$ is the zero object, we have natural homotopy equivalences $\mathsf{Sp}(X, \mathbf{0}) \cong *$ and so, by definition of the negative shift functor, the hom-space $\mathsf{Sp}(X, Y[-1])$ is defined by the homotopy pullback

$$\begin{tikzcd} \mathsf{Sp}(X, Y[-1]) \arrow[r] \arrow[d] & \mathbf{0} \arrow[d] \\ \mathbf{0} \arrow[r] & \mathsf{Sp}(X, Y) \end{tikzcd} .$$

This gives a natural identification of $\mathsf{Sp}(X, Y[-1])$ with the loop space $\Omega\mathsf{Sp}(X, Y)$ based at the zero morphism $X \to Y$, i.e., at the morphism $X \to \mathbf{0} \to Y$. Iterating this, one gets an identification $\mathsf{Sp}(X, Y[-n]) \cong \Omega^n\mathsf{Sp}(X, Y)$ for every $n \geqslant 1$. In particular, by using $Y \cong Y[2][-2]$, this gives the the natural identification

$$[X, Y] = \pi_0\mathsf{Sp}(X, Y) = \pi_0\Omega^2\mathsf{Sp}(X, Y[2]) = \pi_2\mathsf{Sp}(X, Y[2]).$$

**Definition 2.10.** Let $X$ be a spectrum. For any $n \in \mathbb{Z}$, one writes

$$\pi_n^{\mathsf{Sp}}(X) := [\mathbb{S}[n], X]$$

and calls the abelian group $\pi_n^{\mathsf{Sp}}(X)$ the $n$-th homotopy group of $X$.

*Remark* 2.11. Notice that if $X$ is a pointed space seen as a spectrum, then for $n \geqslant 0$, the homotopy group $\pi_n^{\mathsf{Sp}}(X)$ is *not* the $n$-th homotopy group of $X$: it is the $n$-th *stable* homotopy group of $X$.

Another remarkable feature of the $\infty$-category of spectra is that it is symmetric monoidal with the smash product as tensor product and the sphere spectrum $\mathbb{S}$ as unit object. Following [Lur10] we will denote the smash product of spectra with the symbol $\otimes$ rather than with the usual $\wedge$ to stress it is the monoidal product. Similarly, we will use $\oplus$ instead of $\vee$ to denote the coproduct of spectra. Also $\mathsf{Top}$ and $\mathsf{Top}_*$ are symmetric monoidal, with tensor product given by the Cartesian product and by the smash product, respectively. Moreover, both $(-)_+ \colon \mathsf{Top} \to \mathsf{Top}_*$ and $\Sigma^\infty \colon \mathsf{Top}_* \to \mathsf{Sp}$ are monoidal functors [MMSS01]. This fact has an immediate important consequence: spaces are special objects within spectra with respect to the monoidal structure. Namely, every object in $\mathsf{Top}$ is a coassociative cocommutative comonoid via the diagonal morphism $\Delta \colon X \to X \times X$ and the terminal morphism $X \to *$. Since $\Sigma^\infty_+$ is symmetric monoidal we have that a space $X$, seen as a spectrum, comes with natural distinguished morphisms

$$\Delta \colon X \to X \otimes X$$
$$\epsilon \colon X \to \mathbb{S}$$

making it a coassociative cocommutative comonoid in $\mathsf{Sp}$.

*Remark* 2.12. Tensor product of spectra is compatible with the shift functor: one has natural isomorphisms

$$X \otimes (Y[1]) \cong (X \otimes Y)[1] \cong (X[1]) \otimes Y.$$

These identify the shift functor with the tensor product with the object $\mathbb{S}[1]$.

2.1. **Periodic ring spectra.**

**Definition 2.13.** Algebra objects (or monoids) in $\mathsf{Sp}$, i.e., spectra $E$ endowed with morphisms

$$m \colon E \otimes E \to E$$
$$e \colon \mathbb{S} \to E$$

satisfying the usual unit and associativity conditions will be called *ring spectra*. If, moreover, the multiplication is commutative (up to coherent homotopies) they will be called *commutative ring spectra* or $E_\infty$-ring spectra.

*Remark* 2.14. Since shift commutes with tensor product, if $E$ is a ring spectrum then $E[k]$ is an $E$-module for any $k \in \mathbb{Z}$.

As in every monoidal category, if $X$ is a comonoid and $E$ is a monoid, then $[X, E]$ is a monoid with multiplication given by the composition

$$[X, E] \otimes [X, E] \xrightarrow{\otimes} [X \otimes X, E \otimes E] \xrightarrow{(\Delta^*, m_*)} [X, E]$$

As this multiplication is compatible with the abelian group structure on $[X, E]$, when $X$ is a comonoid and $E$ is a monoid the hom-set $[X, E]$ has a canonical ring structure, which is commutative if both $E$ is commutative and $X$ is cocommutative. So, in particular, if $X$ is a space and $E$ is an $E_\infty$-ring spectrum, then $[X, E]$ is a commutative ring.

*Remark* 2.15. Thanks to the compatibility of the shift functor with the tensor product, if $X$ is a space then the direct sum

$$\bigoplus_{n \in \mathbb{Z}} [X, E[n]]$$

has a natural structure of graded commutative ring. This is called the graded $E$-cohomology ring of $X$.

*Remark* 2.16. If $\psi\colon E \to F$ is a morphism of $E_\infty$-ring spectra, then

$$\psi_*\colon [X, E] \to [X, F]$$

is a homomorphism of rings, for any comonoid $X$. Dually, if $\varphi\colon X \to Y$ is a morphism of comonoids, then

$$\varphi^*\colon [Y, E] \to [X, E]$$

is a homomorphism of rings, for any $E_\infty$-ring spectrum $E$. In particular, since any continuous map of (nice) topological spaces is a comonoid map with respect to the comultiplication given by the diagonal embedding, any continuous map between spaces induces a pullback ring homomorphism $\varphi^*\colon [Y, E] \to [X, E]$. As a special case of this, by taking the terminal morphism we see that for any space $X$ the ring $[X, E]$ comes with a natural ring homomorphism

$$\pi_0^{\mathsf{Sp}}(E) = [\mathbb{S}, E] \to [X, E],$$

making $[X, E]$ a $\pi_0^{\mathsf{Sp}}(E)$-algebra. One says that $\pi_0^{\mathsf{Sp}}(E)$ is the *ring of coefficients* for the multiplicative cohomology theory defined by the $E_\infty$-ring spectrum $E$.

*Example* 2.17. Let $A$ be an abelian ring. If $E = HA$, the Eilenberg–Mac Lane $E_\infty$-ring spectrum defining singular cohomology with coefficients in $A$, one has $\pi_0^{\mathsf{Sp}}(HA) = A$, so that the coefficients of singular cohomology are indeed the coefficients in the sense of Remark 2.16.

*Remark* 2.18. Since shift commutes with tensor product, if $X$ is a comonoid in $\mathsf{Sp}$ and $Y$ is a comodule over $X$, i.e., we have a morphism

$$\rho\colon Y \to X \otimes Y$$

making the obvious diagrams commute, then for any $n \in \mathbb{Z}$ also $Y[n]$ is a $X$-comodule. In particular, for any $n$ the hom-set $[Y[n], E]$ is an $[X, E]$-module via the $[X, E]$-action

$$[X, E] \otimes [Y[n], E] \xrightarrow{\otimes} [X \otimes Y[n], E \otimes E] \xrightarrow{(\rho[n]^*, m_*)} [Y[n], E].$$

A particular instance of this construction is obtained by taking $X = \mathbb{S}$ and $\rho\colon Y \xrightarrow{\sim} \mathbb{S} \otimes Y$ the natural isomorphism, for any spectrum $Y$. This way, one sees that $[Y[n], E]$ is a $[\mathbb{S}, E]$-module, i.e., a $\pi_0^{\mathsf{Sp}}(E)$-module, for any $n \in \mathbb{Z}$. In particular, if $Y$ is a space, this module structure coincides with the one induced by the ring homomorphism $\pi_0^{\mathsf{Sp}}(E) \to [Y, E]$ described in Remark 2.16.

*Remark* 2.19. If $X$ and $Y$ are two (nice) topological spaces and $E$ is an $E_\infty$-ring spectrum, then the commutative diagram of spaces

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

induces the commutative diagram of commutative rings

$$\begin{tikzcd} {[X \otimes Y, E]} & {[X, E]} \arrow[l] \\ {[Y, E]} \arrow[u] & {[\mathbb{S}, E]} \arrow[l] \arrow[u] \end{tikzcd}$$

and so, by the universal property of the tensor product of commutative rings, a morphism of rings

$$\otimes_E\colon [X, E] \otimes_{\pi_0^{\mathsf{Sp}}(E)} [Y, E] \to [X \otimes Y, E].$$

One sees that $\otimes_E$ is induced by the external multiplication

$$[X, E] \otimes [Y, E] \xrightarrow{\otimes} [X \otimes Y, E \otimes E] \xrightarrow{m_*} [X \otimes Y, E],$$

by factoring the latter through $[X, E] \otimes [Y, E] \to [X, E] \otimes_{\pi_0^{\mathsf{Sp}}(E)} [Y, E]$.

**Definition 2.20.** We will say that the $E_\infty$-ring spectrum $E$ is *periodic of period $k$* or simply *$k$-periodic* if we are given an isomorphism of $E$-modules $\varphi : E \to E[-k]$. The composition

$$\beta_E\colon \mathbb{S} \to E \xrightarrow{\varphi} E[-k]$$

is called the *Bott element* of the $k$-periodic $E_\infty$-ring spectrum $(E, \varphi)$.

*Remark* 2.21. At the level of homotopy classes, the Bott element $\beta_E$ is an element in $\pi_k^{\mathsf{Sp}}(E)$.

*Remark* 2.22. Since $\varphi$ is an isomorphism, the Bott element is invertible: there exists an element $\beta_E^{-1}\colon \mathbb{S} \to E[k]$ with $\beta_E\beta_E^{-1} = \beta_E^{-1}\beta_E = e\colon \mathbb{S} \to E$. For any spectrum $X$, the multiplication by the Bott element induces a natural isomorphism of abelian groups $[X, E] \xrightarrow{\beta_E\cdot} [X[k], E]$. In particular, the choice of an element $\xi \in [X[-k], E]$ is equivalent to the choice of an element $\beta_E\xi \in [X, E]$ via the periodicity isomorphisms. If $X$ is a space, by the universal property of polynomial rings the latter are in bijection with ring homomorphisms $\pi_0^{\mathsf{Sp}}(E)[u] \to [X, E]$ extending the canonical homomorphism $\epsilon^*\colon [\mathbb{S}, E] \to [X, E]$, via the association $u \mapsto \beta_E\xi$.

*Remark* 2.23. If $(E, \varphi)$ is a $k$-periodic $E_\infty$-ring spectrum, then the isomorphism $\varphi$ is the multiplication by the Bott element, i.e., $\varphi$ is the composition

$$E \cong E \otimes \mathbb{S} \xrightarrow{\mathrm{id}\otimes\beta} E \otimes E[k] \cong (E \otimes E)[-k] \xrightarrow{m[-k]} E[-k].$$

Therefore, one can equivalently define a $k$-periodic $E_\infty$-ring spectrum as an $E_\infty$-ring spectrum endowed with a morphism $\beta_E\colon \mathbb{S}[k] \to E$ such that multiplication by $\beta_E$ induces an isomorphism of $E$-modules $E \to E[-k]$.

**Definition 2.24.** An $E_\infty$-ring spectrum $E$ is called *even* 2*-periodic* if it is 2-periodic and $\pi_{2k+1}^{\mathsf{Sp}}(E) = 0, \forall k \in \mathbb{Z}$.

*Example* 2.25. The first and most natural example of an even 2-periodic cohomology theory is complex $K$-theory $KU$. The Bott element $\beta_{KU} \in \pi_2^{\mathsf{Sp}}(KU) = K^0(S^2)$ can be identified with (the class of the) virtual line bundle $\mathbf{1}_{\mathbb{C}} - L^{-1}$ in the $K$-theory of $S^2$, where $L = \mathcal{O}_{\mathbb{P}^1\mathbb{C}}(1)$ is the universal complex line bundle over the complex projective line $\mathbb{P}^1\mathbb{C} \cong S^2$ and $\mathbf{1}_{\mathbb{C}}$ is the trivial rank 1 complex vector bundle $S^2 \times \mathbb{C} \to S^2$.

Another even 2-periodic cohomology theory we will come back to later is even 2-periodic rational singular cohomology $HP_{\mathrm{ev}}\mathbb{Q} := \bigoplus_{i\in\mathbb{Z}} H\mathbb{Q}[2i]$. Here, as usual, $H\mathbb{Q}$ denotes the Eilenberg-Mac Lane spectrum of the abelian group $(\mathbb{Q}, +)$, so that $[X, H\mathbb{Q}[2i]] = H^{2i}(X; \mathbb{Q})$ for any (nice) topological space $X$. Its homotopy groups are given by

$$\pi_i^{\mathsf{Sp}}(H\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 0 \\ 0 \text{ otherwise.} \end{cases}$$

The Bott element $\beta_{HP_{\mathrm{ev}}\mathbb{Q}}$ of $HP_{\mathrm{ev}}\mathbb{Q}$ can be identified with $1 \in \pi_0^{\mathsf{Sp}}(H\mathbb{Q})$ via

$$\pi_2^{\mathsf{Sp}}(HP_{\mathrm{ev}}\mathbb{Q}) = \prod_{i\in\mathbb{Z}} \pi_{2-2i}^{\mathsf{Sp}}(H\mathbb{Q})\beta^i = \pi_0^{\mathsf{Sp}}(H\mathbb{Q})\beta = \mathbb{Q}\beta,$$

where $\beta$ is a degree $-2$ formal variable used to keep track of the degree shiftings. In other words, the Bott element $\beta_{HP_{\mathrm{ev}}\mathbb{Q}}$ of $HP_{\mathrm{ev}}\mathbb{Q}$ is naturally identified with the formal variable $\beta$ used in forming the even 2-periodic rational singular cohomology of a space $X$:

$$HP_{\mathrm{ev}}^k(X;\mathbb{Q}) = \bigoplus_{i\in\mathbb{Z}} H^{k+2i}(X;\mathbb{Q})\beta^i.$$

Equivalently, the Bott element $\beta_{HP_{\mathrm{ev}}\mathbb{Q}}$ is the fundamental class of $S^2$ via

$$\pi_2^{\mathsf{Sp}}(HP_{\mathrm{ev}}\mathbb{Q}) = \prod_{i\in\mathbb{Z}} \pi_2^{\mathsf{Sp}}(H\mathbb{Q}[2i]) = \pi_2^{\mathsf{Sp}}(H\mathbb{Q}[2]) = \mathbb{Q}\beta.$$

Notice that at the level of $\pi_2^{\mathsf{Sp}}$'s, the Chern character $\mathrm{ch}\colon KU \to HP_{\mathrm{ev}}\mathbb{Q}$ gives $\pi_2^{\mathsf{Sp}}(\mathrm{ch})\colon \pi_2^{\mathsf{Sp}}(KU) \to \pi_2^{\mathsf{Sp}}(HP_{\mathrm{ev}}\mathbb{Q})$ mapping $\mathbf{1}_{\mathbb{C}} - L^{-1}$ to $1 - e^{\beta c_1(L^{-1})} = \beta c_1(L)$. As the first Chern class of the universal line bundle on $\mathbb{P}^1\mathbb{C}$ represents the fundamental class of $\mathbb{P}^1\mathbb{C}$ in singular cohomology (with $\mathbb{Z}$- and so also) with $\mathbb{Q}$-coefficients, we see that the Chern character maps the Bott element of $KU$ to the Bott element of $HP_{\mathrm{ev}}\mathbb{Q}$.

2.2. **Vector bundles and Thom spectra.** We will also be prominently considering the category $\mathsf{VectBun}_{\mathbb{R}}$ of real vector bundles over (nice) topological spaces, whose objects are vector bundles $V \to X$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\tilde{f}$ is fibrewise linear. The category $\mathsf{VectBun}_{\mathbb{R}}$ is symmetric monoidal with monoidal product given by $(V \to X) \otimes (W \to Y) = (V \boxplus W \to (X \times Y))$, where $(V \boxplus W)_{(x,y)} = V_x \oplus W_y$, and unit object the zero vector bundle over the point.

*Notation* 2.26. With $K$-theory in mind, in what follows we will usually denote the direct sum of vector spaces and of vector bundles by $+$ instead than by $\oplus$.

**Definition 2.27.** The *Thom space* $\mathrm{Th}(V \to X)$ of a real vector bundle $V \to X$ is defined as the homotopy pushout

$$\begin{tikzcd} V\backslash X \arrow[r] \arrow[d] & V \arrow[d] \\ * \arrow[r, "f"] & \mathrm{Th}(V \to X) \end{tikzcd}$$

where $V\backslash X$ denotes the total space of the vector bundle $V$ minus the copy of $X$ inside it given by the zero section.

Notice that, by its very definition, $\mathrm{Th}(V \to X)$ is a pointed space. The basepoint of $\mathrm{Th}(V \to X)$ is customary denoted by $\infty$: it is the "point at infinity" of the total space of the vector bundle $V \to X$ in the "vertical directions".

*Remark* 2.28. A more concrete description of the Thom space $\mathrm{Th}(V \to X)$ is obtained as follows. In order to compute the homotopy pushout defining it, one replaces the inclusion $V\backslash X \hookrightarrow V$ with the homotopy equivalent cofibration $V\backslash B(V) \hookrightarrow V$, where $B(V)$ is the open unit disk bundle of $V$ for some chosen Riemannian metric. By retracting on the closed unit disk bundle, this is in turn equivalent to the cofibration $S(V) \hookrightarrow D(V)$, where $S(V)$ denotes the unit sphere bundle and $D(V)$ the closed unit disk bundle of $V$, respectively. The Thom space of $V$ is then realized as the quotient space $D(V)/S(V)$, with base point the equivalence class $S(V)$. One sees from this description that when the base space $X$ of the vector bundle $V \to X$ is compact, the Thom space $\mathrm{Th}(V \to X)$ reduces to the one-point compactification of the total space $V$, pointed at the point at infinity.

*Remark* 2.29. The Thom space construction,

$$\mathrm{Th}\colon \mathsf{VectBun}_{\mathbb{R}} \to \mathsf{Top}_*$$

is a symmetric monoidal functor: one has a natural isomorphism

$$\mathrm{Th}(V \boxplus W \to (X \times Y)) \cong \mathrm{Th}(V \to X) \wedge \mathrm{Th}(W \to Y).$$

The above remark immediately leads to the following

**Definition 2.30.** The *Thom spectrum* functor $(V \to X) \mapsto X^V$ is the symmetric monoidal functor $\mathsf{VectBun}_{\mathbb{R}} \to \mathsf{Sp}$ given by the composition

$$\mathsf{VectBun}_{\mathbb{R}} \xrightarrow{\mathrm{Th}} \mathsf{Top}_* \xrightarrow{\Sigma^\infty} \mathsf{Sp} \ .$$

*Example* 2.31. If $V$ is the rank zero vector bundle $\mathbf{0} = X \times \mathbb{R}^0 \to X$, then $V\backslash X = \varnothing$ and so $\mathrm{Th}(\mathbf{0} \to X) = X_+$. This implies $X^{\mathbf{0}} = X$, where as usual on the right hand side we are writing $X$ for the suspension spectrum $\Sigma^\infty_+ X$. Another simple example is the following. As $\mathrm{Th}(\mathbb{R} \to *) = (S^1, \infty) = \Sigma_+(*)$, we have $*^{\mathbb{R}} = \mathbb{S}[1]$.

*Remark* 2.32. In the categorical spirit of this note, the actual definition of the Thom spectrum functor is less important than its properties. What the reader should keep in mind is that to a real vector bundle $V \to X$ is associated a spectrum $X^V$ in such a way that $(X \times Y)^{V \boxplus W} \cong X^V \otimes Y^W$, and that $*^{\mathbb{R}} = \mathbb{S}[1]$. For instance, from these two properties one derives

$$X^{V+\mathbf{1}} = (X \times \{*\})^{V \boxplus \mathbf{1}} = X^V \otimes \mathbb{S}[1] = X^V[1],$$

where we have written $\mathbf{1}$ for the trivial rank 1 vector bundle $\mathbf{1} := X \times \mathbb{R} \to X$. More generally, writing $\mathbf{n} := X \times \mathbb{R}^n \to X$ for the trivial rank $n$ bundle, one has $X^{V+\mathbf{n}} = X^V[n]$ for any nonnegative $n$.

The other categorical property of Thom spectra that we will need is that Thom spectra are comodules over their bases: for every (nice) topological space $X$ and every vector bundle $V \to X$ the Thom spectrum $X^V$ has a canonical $X$-comodule structure

$$\Delta_V \colon X^V \to X \otimes X^V.$$

This is simply obtained by applying the Thom spectrum functor to the morphism in $\mathsf{VectBun}_{\mathbb{R}}$ given by the commutative diagram

$$\begin{tikzcd} V \arrow[r, "{(0,\mathrm{id}_V)}"] \arrow[d] & \mathbf{0} \boxplus V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd},$$

where $\Delta \colon X \to X \times X$ is the diagonal embedding. This construction is natural: if $(f, \tilde{f}) \colon (V \to X) \to (W \to Y)$ is a morphism in $\mathsf{VectBun}_{\mathbb{R}}$,then we have a commutative diagram

$$\begin{tikzcd} X^V \arrow[r, "\Delta_V"] \arrow[d, "f^V"] & X \otimes X^V \arrow[d, "f \otimes f^V"] \\ Y^W \arrow[r, "\Delta_V"] & Y \otimes Y^W \end{tikzcd}$$

In other words, the construction of Thom spectra gives a functor

$$\mathsf{VectBun}_{\mathbb{R}} \to \mathsf{Comod}(\mathsf{Sp})$$
$$(V \to X) \mapsto (X, X^V),$$

where for any monoidal category $\mathsf{C}$ we denote by $\mathsf{Comod}(\mathsf{C})$ the category whose objects are pairs $(A, M)$, where $A$ is a coalgebra object in $\mathsf{C}$ and $M$ is an $A$-comodule, and morphisms from $(A, M)$ to $(B, N)$ are commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\hat{f}"] & A \otimes M \arrow[d, "f \otimes \hat{f}"] \\ N \arrow[r, "\Delta_N"] & B \otimes N \end{tikzcd}$$

where $f \colon A \to B$ is a morphism of coalgebras.

This gives a natural $[X, E]$-module structure on the $E$-cohomology of $X^V$.

*Remark* 2.33. If $f \colon X \to Y$ is a morphism of (nice) topological spaces and $V$ is a vector bundle on $Y$ we have a naturally induced morphism

$$(f, f^V) \colon (X, X^{f^*V}) \to (Y, Y^V)$$

in $\mathsf{Comod}(\mathsf{Sp})$. In particular, given two vector bundles $V$ and $W$ over a space $X$, we have a natural morphism

$$(\Delta, \Delta^{V \boxplus W})\colon (X, X^{V+W}) \to (X \otimes X, X^V \otimes X^W)$$

as a consequence of the pullback isomorphism $V + W = \Delta^*(V \boxplus W)$, where $\Delta\colon X \to X \times X$ is the diagonal embedding.

*Remark* 2.34. The category $\mathsf{VectBun}_{\mathbb{R}}$ has a positive shift given by tensoring with the trivial rank 1 line bundle $\mathbb{R} \to *$. As we have seen in Remark 2.32, the Thom spectrum functor changes the positive shift on $\mathsf{VectBun}_{\mathbb{R}}$ into the shift functor on $\mathsf{Sp}$. As a consequence, we have a well defined notion of Thom spectrum of a virtual bundle $V = V_0 - V_1$ on $X$, given by $X^V = X^{V_0+\tilde{V}_1}[-n]$, where $\tilde{V}_1$ is a vector bundle on $X$ such that $V_1 + \tilde{V}_1$ is the trivial rank $n$ vector bundle, for some $n$. As a shifted comodule is again a comodule, we see that $X^V$ is naturally an $X$-comodule for any virtual vector bundle $V$ on $X$.

*Example* 2.35. Let $X$ be a compact smooth manifold, and let $i\colon X \hookrightarrow \mathbb{R}^n$ be an embedding. If $\nu$ denotes the normal bundle to $X$ in $\mathbb{R}^n$, then $TX + \nu$ is the trivial rank $n$ vector bundle over $X$, so that $X^{-TX} = X^{\nu}[-n]$. By considering $\mathbb{R}^n$ inside its one-point compactification $S^n$, one can look at $i$ as an embedding $i\colon X \hookrightarrow S^n$. A tubular neighborhood $U$ of $X$ in $S^n$ is homeomorphic to the open unit disk bundle $B(\nu)$ of the normal bundle $\nu$. Mapping all points of $S^n$ outside $U$ to the point at infinity in the Thom space of $\nu$ while keeping the points of $U$ fixed defines a map of pointed spaces $S^n \to \mathrm{Th}(\nu)$. Applying $\Sigma^\infty$, this defines a map of spectra $\mathbb{S}[n] \to X^{\nu}$ and so, equivalently, a map of spectra

$$\gamma_{\mathrm{PT}}\colon \mathbb{S} \to X^{-TX}.$$

The latter is, up to homotopy, independent of all the choices involved in its definition. It is called the *Pontryagin–Thom collapse map.*

## 3. $E$-orientations and Integration

A second main feature of the category of spectra we will use is that it is monoidally closed, i.e., for any object $X$ the functors

$$- \otimes X\colon \mathsf{Sp} \to \mathsf{Sp}$$
$$X \otimes -\colon \mathsf{Sp} \to \mathsf{Sp}$$

have right adjoints. We will write

$$F(X, -)\colon \mathsf{Sp} \to \mathsf{Sp}$$

for the right adjoint of the right multiplication functor $-\otimes X$, i.e., we have natural isomorphisms

$$\mathsf{Sp}(Y\otimes X,Z)\cong \mathsf{Sp}(Y,F(X,Z))$$

for any spectra $X,Y,Z$.

**Definition 3.1.** The *Alexander–Spanier dual* $DX$ of a spectrum $X$ is

$$DX:=F(X,\mathbb{S}),$$

i.e., it is the spectrum characterized by

$$\mathsf{Sp}(Y,DX)\cong \mathsf{Sp}(Y\otimes X,\mathbb{S}),$$

naturally, for any spectrum $Y$.

It is immediate from the definition that $D$ is a contravariant functor $D\colon \mathsf{Sp}\to \mathsf{Sp}^{\mathrm{op}}$. Moreover, as $\mathbb{S}$ is the unit object, we have a natural isomorphism $D\mathbb{S}\cong\mathbb{S}$. In particular, for any space $X$ we have a distinguished morphism

$$\varphi\colon \mathbb{S}\to DX,$$

which is the image under the duality functor $D$ of the distinguished morphism $X\to\mathbb{S}$.

We have already noticed (nice) topological spaces are special among spectra. A somehow surprising fact is that, even from the spectral point of view, compact smooth manifolds are special among (nice) topological spaces. We have the following

**Theorem 3.2** (Atiyah)**.** *[Ati61] Let $X$ be a compact smooth manifold. Then there is an isomorphism of spectra under $\mathbb{S}$*

$$\begin{tikzcd} & \mathbb{S} \arrow[dl, "\gamma_{\mathrm{PT}}"'] \arrow[dr, "\varphi"] & \\ X^{-TX} \arrow[rr, "\sim"] & & DX \end{tikzcd}$$

*where $TX$ is the tangent bundle of $X$, $X^{-TX}$ is the* Thom spectrum *of the virtual bundle $-TX$, and $\gamma_{\mathrm{PT}}\colon\mathbb{S}\to X^{-TX}$ is the Pontryagin–Thom collapse map from Example 2.35.*

From Theorem 3.2, we get

**Corollary 3.3.** *Let $X$ be a compact smooth manifold. Then $DX$ carries a natural $X$-comodule structure. In particular, if $E$ is a ring spectrum, then $[DX,E]$ carries a natural $[X,E]$-module structure.*

We can now give the main definition of this Section:

**Definition 3.4.** Let $E$ be a commutative ring spectrum. A compact $n$-dimensional smooth manifold $X$ is *$E$-orientable* if $[DX[n], E]$ is an $[X, E]$-module of rank 1. An *$E$-orientation* of $X$ is an isomorphism of $[X, E]$-modules $[X, E] \to [DX[n], E]$. An *$E$-oriented* compact $n$-dimensional smooth manifold $X$ is a pair $(X, \sigma)$ where $X$ is a compact $n$-dimensional smooth manifold and $\sigma : [X, E] \to [DX[n], E]$ is an isomorphism of $[X, E]$-modules.

Notice that, if $(X, \sigma)$ is an $E$-oriented compact $n$-dimensional smooth manifold, then the datum of $\sigma$ is equivalent to the datum of a generator $\tau = \tau_{(X,\sigma)}$ of $[DX[n], E]$ as an $[X, E]$-module. The element $\tau$ will be called the *Thom* class of the $E$-orientation of $X$.

**Definition 3.5.** For $(X, \sigma)$ an $E$-oriented compact $n$-dimensional smooth manifold we have a naturally defined *integration map*

$$\int_{(X,\sigma)}^{E} : [X, E] \to [\mathbb{S}[n], E]$$

given by the composition

$$[X, E] \xrightarrow{\ \sigma\ } [DX[n], E] \xrightarrow{\ \varphi^*\ } [\mathbb{S}[n], E]$$

*Remark* 3.6. When $E$ is the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ representing integral cohomology, the datum of an $E$-orientation of a compact smooth manifold $X$ is equivalent to the datum of an orientation of $X$ in the sense of differential geometry, and the integration map defined above is naturally identified with integration in singular cohomology.

## 4. $E$-oriented Vector Bundles

So far we have been considering $E$-orientations of compact smooth manifolds, defined in terms of Thom spectra of opposite tangent bundles. It will be convenient to generalize this construction to arbitrary real virtual vector bundles (recall from Remark 2.34 that we have a well defined notion of Thom spectrum $X^V$ for any real virtual vector bundle $V$ over a (nice) topological space $X$).

**Definition 4.1.** Let $V$ be a rank $r$ virtual real vector bundle over $X$. $V$ is is said to be *$E$-orientable* if $[X^V[-r], E]$ is an $[X, E]$-module of rank 1. An *$E$-orientation* of $V$ is the choice of an isomorphism $\sigma_V \colon [X, E] \xrightarrow{\sim} [X^V[-r], E]$.

*Remark* 4.2. Notice that with this definition each trivial real vector bundle $\mathbf{n} = X \times \mathbb{R}^n \to X$ has a canonical $E$-orientation. Namely, $X^{\mathbf{n}} = X[n]$, and so the shift isomorphism provides a distinguished isomorphism $[X, E] \to [X^{\mathbf{n}}[-n], E]$.

**Definition 4.3.** The datum of an $E$-orientation $\sigma_V$ of a rank $r$ virtual vector bundle $V$ is equivalent to the datum of a generator $\tau_V$ of $[X^V[-r], E]$ as an $[X, E]$-module; namely, the image of the unit $1 \in [X, E]$ via the isomorphism $\sigma_V$. The element

$$\tau_V \colon X^V[-r] \to E$$

will be called the *Thom element* of the $E$-oriented bundle $V$.

*Remark* 4.4. Since orientations are isomorphisms of $[X, E]$-modules from $[X, E]$, they are a torsor for the group of $[X, E]$-module automorphisms of $[X, E]$, i.e. for the group $GL_1[X, E]$ of units of $[X, E]$. In other words, if $\sigma_V, \tilde{\sigma}_V$ are two $E$-orientations on an $E$-orientable rank $r$ (virtual) vector bundle $V$ over $X$, there exists a unique element $m_V$ in $GL_1[X, E] \subseteq [X, E]$ making the following diagram

$$\begin{tikzcd}
{[X,E]} \arrow[rr, "m_V \cdot -"] \arrow[dr, "\sigma_V"'] & & {[X,E]} \arrow[dl, "\tilde{\sigma}_V"] \\
& {[X^V[-r], E],} &
\end{tikzcd}$$

commute, where $\cdot$ is the product in $[X, E]$. Equivalently, we have $\tau_V = m_V \cdot \tilde{\tau}_V$.

*Remark* 4.5. The functor $X \mapsto GL_1[X, E]$ is representable: there exists a spectrum $GL_1 E$ with a natural isomorphism

$$GL_1[X, E] \cong [X, GL_1 E].$$

Therefore a multiplier $m_V$ can be equivalently seen as a morphism $m_V \colon X \to GL_1 E$.

*Remark* 4.6. A noteworthy property of $E$-orientations is that they satisfy the 2-out-of-3 property: if a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of vector bundles on a topological space is given, then $E$-orientations on two of the vector spaces in the sequence canonically determine an $E$-orientation of the third one, see [May06].

The main idea now is to study not only an $E$-orientation of a single vector bundle, but to look at systems of coherent $E$-orientations of a family of vector bundles.

**Definition 4.7.** A family of vector bundles $\mathcal{F}$ is said to be *closed* if it closed under the operations of pullback and box sum. This means that:

(1) if the vector bundles $V \to X$ and $W \to Y$ are in the family $\mathcal{F}$, then the box sum $V \boxplus W \to X \otimes Y$ is in $\mathcal{F}$,

(2) for any map $f : X \to Y$ and every vector bundle $V \to Y$ in the family $\mathcal{F}$, then the pullback bundle $f^*V \to X$ is in $\mathcal{F}$.

*Remark* 4.8. Notice that, if $V, W$ are vector bundles over $X$, then via the isomorphism $\Delta^*(V \boxplus W) \cong V + W$ from Remark 2.33 any closed family is automatically closed under the operation of direct sum of vector bundles.

*Example* 4.9. Examples of closed families are the family of all (finite rank) real vector bundles and the family of all (finite rank) trivial real vector bundles. Other, more interesting examples are given by the family of all real oriented vector bundles (i.e. real vector bundles $V$ with a reduction of the structure group to $SO(\operatorname{rk} V)$, where $\operatorname{rk} V$ denotes the rank of $V$ as a real vector bundle), spin bundles (i.e, with reduction of the structure group to $Spin(\operatorname{rk} V)$), and complex vector bundles (i.e., even rank real vector bundles $V$ with a reduction of the structure group to $U(\frac{1}{2} \operatorname{rk} V)$).

**Definition 4.10.** A *coherent system of $E$-orientations* on a closed family of vector bundles $\mathcal{F}$ (or an $E$-orientation of $\mathcal{F}$) is the datum of an $E$-orientation $\sigma_V : [X, E] \xrightarrow{\sim} [X^V[-\operatorname{rk} V], E]$, for each $V \in \mathcal{F}$, satisfying the following coherence conditions:

(1) Given $V, W \in \mathcal{F}$ and $f : X \to Y$ the following diagrams commute.

$$\begin{tikzcd}
[X,E] \otimes [Y,E] \arrow[r, "\sigma_V \otimes \sigma_W"] \arrow[d] & {[X^V[-\operatorname{rk} V], E] \otimes [Y^W[-\operatorname{rk} W], E]} \arrow[d] \\
{[X \otimes Y, E]} \arrow[r, "\sigma_{V \boxplus W}"] & {[X^V \otimes Y^W[-\operatorname{rk} V - \operatorname{rk} W], E]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[Y,E]} \arrow[r, "\sigma_V"] \arrow[d, "f^*"'] & {[Y^V[-\operatorname{rk} V], E]} \arrow[d, "(f^V)^*"] \\
{[X,E]} \arrow[r, "\sigma_{f^*V}"'] & {[X^{f^*V}[-\operatorname{rk} f^*V], E]}
\end{tikzcd}$$

(2) if a trivial vector bundle $\mathbf{n}$ is in $\mathcal{F}$, then $\sigma_{\mathbf{n}}$ is the canonical orientation of $\mathbf{n}$.

The next definition is motivated by the following two facts we noticed in Remarks 4.2 and 4.6: every trivial bundle is canonically oriented, and $E$-orientations satisfy a 2-out-of-3 property.

**Definition 4.11.** A closed family of vector bundles $\mathcal{F}$ is said to be *stable* if it contains the family of trivial bundles and satisfies the 2-out-of-3 property, i.e., if for a short exact sequence of vector bundles

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

two of the $V_i$'s are in $\mathcal{F}$ then also the third is in $\mathcal{F}$.

*Example* 4.12. Let us consider the family of complex vector bundles. This is a closed family but not a stable family: it contains only "half" of the trivial bundles (those with even rank as real vector bundles). To obtain a stable family out of the closed family of complex bundles we need to stabilize it, i.e. consider vector bundles $V$ such that $V + \mathbf{k}$ is complex for some $k \geqslant 0$. Such bundles are called stably complex. The family of stably complex vector bundles is a stable family and it is the smallest stable family that contains the family of complex bundles. More generally, for any closed family of vector bundles we can consider its stabilization.

*Remark* 4.13. An $E$-orientation of a stable family $\mathcal{F}$ is defined as an $E$-orientation of $\mathcal{F}$ as a closed family. It is immediate to see that the datum of an $E$-orientation of a closed family is equivalent to the datum of an $E$-orientation on its stabilization.

*Remark* 4.14. Equivalently, conditions (1–2) above can be expressed in terms of Thom elements as

(1) $\tau_{V \boxplus W} = \tau_V \otimes_E \tau_W$;
(2) $\tau_{f^*V} = (f^V)^* \tau_V$;
(3) $\tau_{\mathbf{n}} = 1 \in [X, E] = [X^{\mathbf{n}}[-n], E]$.

*Remark* 4.15. Let $\{\sigma_V\}_{V \in \mathcal{F}}$ be a an $E$-orientation of a stable family $\mathcal{F}$. Then it makes sense to talk about the $E$-orientation $\sigma_V$ of a virtual bundle $V = V_0 - V_1$ where both $V_0, V_1 \in \mathcal{F}$. Namely, by the 2 out of 3 property, any complement $\tilde{V}_1$ of $V_1$ is in $\mathcal{F}$. Writing $V = V_0 + \tilde{V}_1 - \mathbf{n}$, we have natural isomorphisms of shifted Thom spectra

$$X^V[-\operatorname{rk} V] \cong X^{V_0 + \tilde{V}_1 - \mathbf{n}}[-\operatorname{rk} V] \cong$$

$$\cong X^{V_0 + \tilde{V}_1}[-\operatorname{rk} V - n] \cong X^{V_0 + \tilde{V}_1}[-\operatorname{rk}(V_0 + \tilde{V}_1)].$$

Since $V_0 + \tilde{V}_1 \in \mathcal{F}$, we have an $E$-orientation $\sigma_{V_0 + \tilde{V}_1}$, which we can take as the $E$-orientation $\sigma_V$ of $V$. One checks that $\sigma_V$ is well defined, i.e., it is independent of the choice of the complement $\tilde{V}_1$. This way the notion of $E$-orientation of a stable family of vector bundles extends to stable families of virtual vector bundles.

*Remark* 4.16. Let $V, W$ be two (virtual) vector bundles over the manifold $X$ in the $E$-oriented family $\mathcal{F}$. Then the pullback isomorphism $V + W = \Delta^*(V \boxplus W)$, and conditions (1–2) in Remark 4.14 give

$$\tau_{V+W} = (\Delta^{V \boxplus W})^*(\tau_V \otimes_E \tau_W) =: \tau_V \cdot \tau_W.$$

The normalisation condition (3) then gives $\tau_V \cdot \tau_{-V} = 1$.

*Remark* 4.17. It follows from conditions (1–2) in Definition 4.10 that for any (virtual) vector bundle $V$ in the $E$-oriented stable family $\mathcal{F}$ on $X$, the multiplication by the Thom class $\tau_V \in [X^V[-\operatorname{rk} V], E]$ gives natural morphisms of $[X, E]$-modules

$$[X^W, E] \xrightarrow{\tau_V \cdot} [X^{V+W}[-\operatorname{rk} V], E].$$

As $\tau_V \cdot \tau_{-V} = 1$, these are actually isomorphisms. In particular, by taking $W = \mathbf{n}$, we see that multiplication by the Thom class $\tau_V$ is an isomorphism of $[X, E]$-modules

$$[X[n], E] \xrightarrow{\tau_V \cdot} [X^V[n - \operatorname{rk} V], E],$$

for any $n \in \mathbb{Z}$.

So far we have defined compatible systems of $E$-orientations of a stable family of vector bundles $\mathcal{F}$. Again, rather than study a single system, it is interesting to study how two compatible systems interact. As in Remark 4.4, two systems of $E$-orientations $\{\sigma_V, s_V\}_{V \in \mathcal{F}}$ define a set of Thom classes $\{\tau_V, t_V\}_{V \in \mathcal{F}}$ as well as a set of multipliers $\{m_V\}_{V \in \mathcal{F}}$, uniquely defined by the property $\tau_V = m_V t_V$. Now we want to state some properties satisfied by the set multipliers $\{m_V\}_{V \in \mathcal{F}}$.

**Proposition 4.18.** *Let $\mathcal{F}$ be a stable family of vector bundles, and $\sigma$ and $s$ two $E$-orientations of $\mathcal{F}$, with associated Thom classes $\tau$ and $t$, and system of multipliers $m$. Then for any two (virtual) vector bundles in the family $\mathcal{F}$ over a space $X$, we have $m_{V+W} = m_V m_W$, and for any trivial vector bundle $\mathbf{n}$ on $X$ we have $m_{\mathbf{n}} = 1$. In particular, $m_{V+\mathbf{n}} = m_V$ for any $V$ and $\mathbf{n}$.*

*Proof.* As the $X$-comodule structure on $X^{V+W}$ and the $X \otimes X$-comodule structure on $X^V \otimes X^W$ are compatible (Remark 2.33), for any $\lambda, \mu \in [X, E]$, we have $(\lambda\mu)(t_V \cdot t_W) = \lambda t_V \cdot \mu t_W$. Therefore, by Remark 4.16 we have that

$$\begin{aligned} m_{V+W} t_{V+W} &= \tau_{V+W} \\ &= \tau_V \cdot \tau_W \\ &= m_W t_V \cdot m_W t_W \\ &= m_V m_W t_V \cdot t_W \\ &= m_V m_W t_{V+W}. \end{aligned}$$

The conclusion follows from uniqueness of multipliers. The statement for trivial bundles is immediate from the normalisation condition $\tau_{\mathbf{n}} = t_{\mathbf{n}} = 1$. $\square$

**Corollary 4.19.** *For the multipliers associated to a (virtual) vector bundle $V \in \mathcal{F}$ and to its opposite $-V$, we have $m_V m_{-V} = 1$.*

*Remark* 4.20. By Remark 4.5, the collection of multipliers can be seen as a collection of morphisms $m_V : X \to GL_1E$, for any (virtual) vector bundle $V$ over $X$, and the compatibility of multipliers with pullbacks amounts to the commutativity of the diagrams

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "m_{f^*V}"'] & & X \arrow[dl, "m_V"] \\ & GL_1E, & \end{tikzcd}$$

for any map $f : Y \to X$.

## 5. Pushforwards in $E$-cohomology

We can now introduce the notion of an $E$-oriented map between compact smooth manifolds and define a pushforward in $E$-cohomology along an $E$-oriented map. This will generalize the construction of the integration map in Section 3, which will be recovered as the pushforward along the terminal morphism $X \to *$ for an $E$-oriented manifold $X$.

For a smooth map $f : X \to Y$ we write $T_f$ for the virtual bundle over $X$ defined by

$$T_f := TX - f^*TY.$$

**Definition 5.1.** An $E$-orientation of a smooth map $f : X \to Y$ is an $E$-orientation of the virtual bundle $-T_f$.

*Remark* 5.2. Notice that an $E$-orientation of a manifold is a special case of $E$-orientation for a morphism: $X$ is $E$-oriented precisely when $t : X \to *$ is.

*Remark* 5.3. The 2-out-of-3 property of $E$-orientation of (virtual) vector bundles implies that also $E$-orientations of maps have a 2-out-of-3 property: if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are smooth maps and two out of $\{f, g, g \circ f\}$ are $E$-oriented then so is the third. In particular, if $f : X \to Y$ is a smooth map between $E$-oriented manifolds then $f$ is canonically $E$-oriented. See [May06].

We now describe how to define a pushforward map

$$f_* \colon [X, E] \to [Y[\dim X - \dim Y], E]$$

for an $E$-oriented map $f \colon X \to Y$ which is a smooth fibration with typical fibre a smooth compact manifold $F$ of dimension $d = \dim X - \dim Y$. In this situation we can think of $f$ as a parametrized family over $Y$ of (nice) topological spaces, i.e., as an $\infty$-functor $f \colon Y \to \mathsf{Top}$, where on the left we are writing $Y$ for its $\infty$-Poincaré groupoid (i.e., for the $\infty$-groupoid with objects the points of $Y$, 1-morphisms the paths between points, 2-morphisms the homotopies between paths, etc), and on the right $\mathsf{Top}$ is the $\infty$-category of (nice) topological spaces, with homotopies between continuous maps, homotopies between homotopies, etc. as higher morphisms. Namely, the definition of (Serre) fibration is precisely a way of encoding this idea. Now, we are always looking at topological spaces as spectra via $\Sigma^\infty_+$, so we look at $f$ as a family of spectra parametrized by $Y$,

$$Y \xrightarrow{f} \mathsf{Top} \xrightarrow{\Sigma^\infty_+} \mathsf{Sp}.$$

We refer the interested reader to [ABG11] or [MS06] for a detailed and rigorous treatment of the category $\mathsf{Sp}_Y$ of all such *parametrized spectra* with parameter space $Y$; for the aim of this note the intuitive definition sketched above will be sufficient.

The $\infty$-category $\mathsf{Sp}_Y$ inherits from $\mathsf{Sp}$ the pointwise monoidal structure with unit object the pointwise unit $\mathbb{S}_Y$ given by the constant family with fibre $\mathbb{S}$ over $Y$, and so pointwise duals. In particular we will have an Alexander–Spanier dual $D_Y(f)$ coming with a distinguished morphism

$$\varphi_f \colon \mathbb{S}_Y \to D_Y(f)$$

in $\mathsf{Sp}_Y$. By putting everything together (formally, this means taking the $\infty$-colimit of the natural transformation $\varphi_f$ of $\infty$-functors $Y \to \mathsf{Sp}$), we get a map

$$Y \to \mathrm{colim}_Y D_Y(f).$$

If we denote by $F_y$ the fibre of $f \colon X \to Y$ over $y \in Y$, then Atiyah duality pointwise identifies $D_Y(f)_y = D(F_y)$ with $F_y^{-T_f}$ so the one above is a map

$$Y \to \mathrm{colim}_Y F_y^{-T_f}.$$

Maybe not surprisingly, the colimit on the right hand side is the Thom spectrum $X^{-T_f}$ (see [ABG11] for a rigorous proof). Therefore, at least when $f \colon X \to Y$ is a smooth fibration with compact fibres, we have a natural Pontryagin–Thom morphism

$$\varphi_{PT} \colon Y \to X^{-T_f}.$$

In the best of possible worlds, this would be true for any smooth map $f$, not necessarily a fibration with compact fibres. And indeed it is. This can be shown by factoring the map $f$ as the composition of an embedding and a smooth fibration with compact fibres and noticing that for an embedding $\iota : X \hookrightarrow Y$ one has the geometrically defined Pontryagin–Thom collapse map $Y \to X^{\nu_\iota} = X^{-T_\iota}$. However, to our knowledge, the general Pontryagin–Thom morphism $Y \to X^{-T_f}$ for a general $f$ has not yet been given a transparent interpretation in terms of the monoidal closedness of the category of spectra. See, however [ABG11, Remark 4.17] .

The morphism $\varphi_{PT}\colon Y \to X^{-T_f}$ is a morphism of $Y$-comodules, where the $Y$-comodule structure on $X^{-T_f}$ is induced by its $X$-comodule structure via $f$, i.e., the diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi_{PT}"] \arrow[d, "\Delta_Y"'] & X^{-T_f} \arrow[d, "(f\otimes\mathrm{id})\circ\Delta_{T_f}"] \\ Y \otimes Y \arrow[r, "\mathrm{id}\otimes\varphi_{PT}"'] & Y \otimes X^{-T_f} \end{tikzcd}$$

commutes.

Therefore, if $f$ is $E$-oriented we can define a *pushforward map*

$$f_*\colon [X, E] \to [Y[\dim X - \dim Y], E]$$

as the composition

$$[X, E] \xrightarrow{\sigma_{-T_f}} [X^{-T_f}[\dim X - \dim Y], E] \xrightarrow{\varphi_{PT}^*} [Y[\dim X - \dim Y], E].$$

When $f$ is the terminal morphism, the pushforward map $f_*$ is $E$-integration over $X$.

*Remark* 5.4. Since $\varphi_{PT}$ is a morphism of $Y$-comodules,

$$\varphi_{PT}^*\colon [X^{-T_f}[\dim X - \dim Y], E] \to [Y[\dim X - \dim Y], E]$$

is a morphism of $[Y, E]$-modules. Since $[X, E] \xrightarrow{\sigma_{-T_f}} [X^{-T_f}[\dim X - \dim Y], E]$ is an isomorphism of $[X, E]$-modules by definition of $E$-orientation, it will also be an isomorphism of $[Y, E]$-modules where we look at every $[X, E]$ module as an $[Y, E]$-module via the ring homomorphism $f^*\colon [Y, E] \to [X, E]$. Summing up, $f_*$ is a morphism of $[Y, E]$-modules, where we look at $[X, E]$ as an $[Y, E]$-module via $f^*$. This is the *projection formula*

$$f_*((f^* a) \cdot x) = a \cdot f_*(x)$$

for any $a \in [Y, E]$ and $x \in [X, E]$.

*Remark* 5.5. As we noticed, if $X$ and $Y$ are $E$-oriented, then $f\colon X \to Y$ gets a canonical $E$-orientation by the 2-out-of-3 property. If moreover $X$ and $Y$ are compact, multiplications by the Thom classes of $TY$ and of $f^*TY$ intertwine the morphisms induced in $E$-cohomology by $\varphi_{PT}$ and the dual of $f$, i.e., we have a commutative diagram

$$\begin{tikzcd}
{[DX[\dim X], E]} \arrow[rr, "(Df)^*"] \arrow[d, "\wr"'] & & {[DY[\dim X], E]} \arrow[d, "\wr"] \\
{[X^{-TX}[\dim X], E]} \arrow[d, "f^*\tau_{TY}"', "\wr"] & & {[Y^{-TY}[\dim X], E]} \arrow[d, "\tau_{TY}", "\wr"'] \\
{[X^{-T_f}[\dim X - \dim Y], E]} \arrow[rr, "\varphi_{PT}^*"'] & & {[Y[\dim X - \dim Y], E]}
\end{tikzcd}$$

of $[Y, E]$-modules. So we see that under these assumptions the pushforward $f_*$ can be written as the composition

$$[X, E] \xrightarrow{\sim} [DX[\dim X], E] \xrightarrow{(Df)^*} [DY[\dim X], E] \xrightarrow{\sim} [Y[\dim X - \dim Y], E].$$

*Remark* 5.6. Since they are pullbacks along dualized morphisms, pushforwards are covariantly functorial: if we are given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $E$-oriented maps, then we have $(g \circ f)_* = g_* \circ f_*$. This is immediately seen via Remark 5.5 in case $X, Y$ and $Z$ are compact $E$-oriented manifolds, and by a similar argument in the general case of $E$-oriented maps. See [Dye69] for details.

## 6. The spectrum $MU$ and complex orientations

Let us consider complex vector bundles. As every rank $k$ complex vector bundle over a manifold $X$ is a pullback of the tautological vector bundle $V_k \to BU(k)$, the naturality of orientations with respect to pullbacks tells us that, in order to $E$-orient all complex vector bundles, we only need to orient all the tautological bundles $V_k \to BU(k)$. Moreover, from

$$X^{V+\mathbf{1}_{\mathbb{C}}} = X^V[2],$$

where $\mathbf{1}_{\mathbb{C}} := \mathbb{C} \times X \to X$ is the trivial rank 1 complex vector bundle over $X$, and from $j^*V_{k+1} = V_k + \mathbf{1}_{\mathbb{C}}$, where $j\colon BU(k) \to BU(k+1)$ is the canonical embedding, we see that a coherent system of $E$-orientations on all of the $V_k$'s is equivalent to the datum of a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & MU(k-1) \arrow[r] \arrow[dr, "\tau_{k-1}"'] & MU(k) \arrow[r] \arrow[d, "\tau_k"] & MU(k+1) \arrow[r] \arrow[dl, "\tau_{k+1}"] & \cdots \\
& & E & &
\end{tikzcd}$$

of maps of spectra, where we have written $MU(k)$ for $BU(k)^{V_k}[-2k]$. By the universal property of the limit, this is equivalent to a single map of spectra

$$\rho\colon MU \to E,$$

where by definition

$$MU = \varinjlim MU(k)$$

This spectrum $MU$ may informally be thought of as the infinite desuspension of the Thom spectrum of the infinite dimensional tautological bundle over $BU$.

So far we have not used compatibility of orientations with formation of direct sums, the latter operation giving the $\infty$-abelian group structure on $BU$. Requiring this is equivalent to requiring that $\rho\colon MU \to E$ is a morphism of homotopy commutative ring spectra. The morphism $\rho$ described above has by construction the rather special property that all of its components $\rho_k = \tau_k\colon MU(k) \to E$ are generators of $[MU(k), E]$, which are free rank 1 $[BU(k), E]$-modules. Quite remarkably, this property is actually not special at all: *every* morphism of homotopy commutative ring spectra $\rho\colon MU \to E$ has this property. Namely, one has $MU(0) = \mathbb{S}$ and the morphism $MU(0) \to MU$ is the unit of $MU$. As a ring morphism preserves the unit, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{S} = MU(0) \arrow[r] \arrow[dr] & MU(1) \arrow[r] \arrow[d, "\rho_1"] & \cdots \arrow[r] & MU \arrow[dll, "\rho"] \\
& E & &
\end{tikzcd}$$

Now we have the following

**Lemma 6.1.** *The zero section $\iota\colon BU(1) \to BU(1)^{V_1}$ is a homotopy equivalence.*

*Proof.* Let $L_{1;n} = \mathcal{O}_{\mathbb{P}^n\mathbb{C}}(1)$ be the universal line bundle over $\mathbb{P}^n\mathbb{C}$. There is a natural isomorphism $(\mathbb{P}^n\mathbb{C})^{L_{1;n}} \cong \mathbb{P}^{n+1}\mathbb{C}$ such that the zero section $\mathbb{P}^n\mathbb{C} \to (\mathbb{P}^n\mathbb{C})^{V_{1;n}} \cong \mathbb{P}^{n+1}\mathbb{C}$ is identified with the standard inclusion $\mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^{n+1}\mathbb{C}$ as the hyperplane at infinity. Namely, let $p$ be a point in $\mathbb{P}^{n+1}\mathbb{C}\backslash\mathbb{P}^n\mathbb{C}$. Then the collection of projective lines through $p$, with the point $p$ removed, is a holomorphic line bundle over $\mathbb{P}^n\mathbb{C}$ which is immediately seen to be isomorphic to $\mathcal{O}_{\mathbb{P}^n\mathbb{C}}(1)$. This realises the above mentioned identification. Taking the limit over $n$ we get an isomorphism $BU(1) \cong BU(1)^L$, where $L = \mathcal{O}(1)$ is the universal line bundle over $BU(1) \cong \mathbb{P}^\infty$. The universal line bundle $L$ is obtained from the tautological line bundle $V_1$ by pullback along the equivalence $\mathrm{inv}\colon BU(1) \to BU(1)$ mapping each line bundle to its inverse induced by the group automorphism of $U(1)$ mapping $z$ to $z^{-1}$. By Remark 2.33 we therefore have a commutative diagram

$$\begin{tikzcd}
BU(1) \arrow[r, "\iota"] \arrow[d, "\mathrm{inv}"] & BU(1)^L \arrow[d, "\mathrm{inv}^{V_1}"] \\
BU(1) \arrow[r, "\iota"] & BU(1)^{V_1}
\end{tikzcd}$$

where three of the arrows are homotopy equivalences, and so also the fourth is. ☐

By the argument in Lemma 6.1 we see that $\rho_1$ provides an extension $\varepsilon$ of the unit of $E$ to $\mathbb{P}^\infty[-2]$:

$$\begin{tikzcd}
 & \mathbb{P}^\infty[-2] \arrow[d, "\mathrm{inv}", "\wr"'] \arrow[ddd, bend left, dashed, "\varepsilon"] \\
\mathbb{P}^1\mathbb{C}[-2] \arrow[ur, "i_1"] \arrow[d, equal] & \mathbb{P}^\infty[-2] \arrow[d, "\iota", "\wr"'] \\
\mathbb{S} \arrow[r] \arrow[dr] & MU(1) \arrow[d, "\rho_1"] \\
 & E
\end{tikzcd},$$

where $i_1 \colon \mathbb{P}^1\mathbb{C} \to \mathbb{P}^\infty$ is the standard inclusion. As a consequence of the fact that $\mathbb{P}^\infty$ has a CW-complex structure with only even dimensional cells, one can show that the datum of such an extension $\varepsilon$ gives an isomorphism of $[\mathbb{P}^\infty, E]$-modules $[\mathbb{P}^\infty, E] \cong [\mathbb{P}^\infty[-2], E] \cong [MU(1), E]$ and that $\varepsilon$ as an element in $[\mathbb{P}^\infty[-2], E]$ is a generator of this $[\mathbb{P}^\infty, E]$-module. This is a generalized version of the usual isomorphism $H^2(\mathbb{P}^\infty; \mathbb{Z}) \cong H^0(\mathbb{P}^\infty, \mathbb{Z})$ in singular cohomology, see [Lur10, Lecture 4] for details. Therefore, $\rho_1$ is an $E$-orientation of the tautological line bundle $V_1 \to BU(1)$. To conclude, we need to show that also the $\rho_k$'s with $k \geqslant 2$ are $E$-orientations of the tautological vector bundles $V_k \to BU(k)$. This is a corollary of the splitting principle for complex bundles in $E$-cohomology, for $E$ an $E_\infty$-ring spectrum. Namely, the pullback of the tautological bundle $V_k \to BU(k)$ along the morphism of topological spaces $j_k \colon BU(1)^k \to BU(k)$ splits as $j_k^* V_k = \boxplus_{i=1}^k V_1$ and this induces identifications

$$j_k^* \colon [BU(k), E] \xrightarrow{\sim} [BU(1)^{\otimes k}, E]^{\mathrm{Sym}_k}$$
$$(j_k^{V_k})^* \colon [MU(k), E] \xrightarrow{\sim} [MU(1)^{\otimes k}, E]^{\mathrm{Sym}_k}$$

where $(-)^{\mathrm{Sym}_k}$ denotes the $\mathrm{Sym}_k$-invariants. As the $j_k$'s and the actions of the symmetric groups are compatible with the canonical embeddings $BU(k) \to BU(k+1)$, this identifies $\rho_k$ with the symmetric element $(\rho_1)^{\otimes k}$. The latter is manifestly the datum of an isomorphism $[BU(1)^{\otimes k}, E]^{\mathrm{Sym}_k} \cong [MU(1)^{\otimes k}, E]^{\mathrm{Sym}_k}$

of $[BU(1)^{\otimes k}, E]^{\mathrm{Sym}_k}$-modules, so the $\rho_k$'s are $E$-orientations. Again, see [Lur10, Lectures 4 & 6] for details.

Summing up, by the above reasoning we have proven the following

**Proposition 6.2.** *A system of compatible $E$-orientations on complex (and so also on stably complex) vector bundles is equivalent to the datum of a single morphism of homotopy commutative ring spectra $\rho\colon MU \to E$.*

In view of Proposition 6.2 it is natural to give the following definition.

**Definition 6.3.** Let $E$ be an $E_\infty$-ring spectrum. A morphism of homotopy commutative ring spectra $\rho\colon MU \to E$ is called a *complex orientation* of the $E_\infty$-ring spectrum $E$.

*Remark* 6.4. In particular, the identity morphisms $\mathrm{id}_{MU}\colon MU \to MU$ defines the canonical complex orientation of $MU$.

*Remark* 6.5. As both $MU$ and $E$ are $E_\infty$-ring spectra, one may be tempted to think the morphism $MU \to E$ defined by a system of compatible $E$-orientations on complex vector bundles is actually a morphism of $E_\infty$-ring spectra, but it is actually not necessarily so. Namely, differently from the case of commutative rings inside all rings, the $\infty$-category of $E_\infty$-ring spectra is not a full $\infty$-subcategory of the $\infty$-category of ring spectra. This is so because the enhancement of a ring spectra morphism between two $E_\infty$-rings to an $E_\infty$-ring morphism is structure and not property: it is the additional datum of all the coherent homotopies involved in the definition of a morphism of $E_\infty$-ring spectra. By the same argument for commutative rings inside all rings, a morphism of ring spectra between $E_\infty$-rings is automatically a morphism of homotopy commutative ring spectra, but an enhancement to a full morphism of $E_\infty$-rings may not exist. See [HL18] for a detailed discussion. Clearly, if $MU \to E$ is a morphism of $E_\infty$-ring spectra then it is in particular a morphism of homotopy commutative ring spectra, and so a complex orientation of $E$.

*Remark* 6.6. The above discussion shows that any lift of $1 \in [\mathbb{S}, E]$ to an element $\varepsilon \in [\mathbb{P}^\infty[-2], E]$ under the pullback morphism $[\mathbb{P}^\infty[-2], E] \to [\mathbb{P}^1\mathbb{C}[-2], E] = [\mathbb{S}, E]$ defines a morphism $\rho_1\colon MU(1) \to E$ that uniquely extends to a morphism of $E_\infty$-ring spectra $\rho\colon MU \to E$. Therefore, (homotopy classes of) complex orientations of $E$ bijectively correspond to these lifts. Moreover, one sees by construction that $\varepsilon$ is identified with the pullback of the Thom class $\tau_L \in [BU(1)^L[-2], E]$ along the zero section $\iota\colon BU(1) \to BU(1)^L$

*Remark* 6.7. Although we are not going to make use of this fact, it is worth mentioning that the spectrum $MU$ has an interesting geometric characterization:

it is the *complex cobordism* spectrum. For $X$ a finite dimensional smooth manifold the hom-set $[X[n], MU]$ is the set of complex cobordism classes of proper complex oriented maps $f : Z \to X$ with $\dim f := \dim Z - \dim X = n$ [Qui71]. In particular, when $X$ is a point we get the complex cobordism ring

$$\Omega^U := \bigoplus_{n\geqslant 0}[\mathbb{S}[n], MU] = \bigoplus_{n\geqslant 0}\pi_n^{\mathsf{Sp}}MU.$$

The spectrum $MU$ is $(-1)$-connected, i.e., its homotopy groups vanish in negative degree or, equivalently, $MU \cong MU_{\geqslant 0}$. This gives $MU_{\leqslant 0} \cong (MU_{\geqslant 0})_{\leqslant 0}$ $\cong H\pi_0^{\mathsf{Sp}}(MU)$, and so the fibre sequence associated to the 0-truncation of $MU$ is

$$MU_{>0} \to MU \to MU_{\leqslant 0} \cong H\pi_0^{\mathsf{Sp}}(MU) = H\mathbb{Z}.$$

As the 0-truncation morphism for a $E_\infty$-ring spectrum is an $E_\infty$-ring map, we read from the above sequence an $E_\infty$-ring map

$$MU \to H\mathbb{Z},$$

and so a canonical complex orientation for $\mathbb{Z}$-valued singular cohomology. Equivalently, this tells us that the family of all complex vector bundles has a natural theory of Thom classes in $\mathbb{Z}$-valued singular cohomology. We notice that, by the functoriality of $H- : \mathsf{CommRings} \to \mathsf{E}_\infty\text{-}\mathsf{RingSp}$ and since $\mathbb{Z}$ is initial in the category of commutative rings, for every commutative ring $A$ there is a distinguished morphism of $E_\infty$-ring spectra $H\mathbb{Z} \to HA$. As a consequence, singular cohomology with coefficients in $A$ has a canonical complex orientation for every commutative ring $A$.

*Remark* 6.8. If $\psi\colon E \to F$ is an homomorphism of homotopy commutative ring spectra, then, a complex orientation $\rho\colon MU \to E$ of $E$ can be pushed forward to a complex orientation $\psi_*\rho$ of $F$. In terms of Thom classes of (virtual) stably complex vector bundles the orientation $\psi_*\rho$ is simply defined by

$$\tau_V^{\psi_*\rho} = \psi_*(\tau_V^\rho).$$

*Remark* 6.9. Assume we are given two complex orientations

$$\rho_1, \rho_2\colon MU \to E$$

of a multiplicative cohomology theory $E$. We have seen in Remarks 4.4 and 4.5 that the collection of multipliers between $\rho_1$ and $\rho_2$ is equivalent to the datum of a compatible family of morphism $m_V\colon X \to GL_1E$, indexed by vector bundles $V \to X$. Due to compatibility with pullbacks, we can restrict to universal bundles: the datum of the whole collection of multipliers $\{m_V\}$ is equivalent to the datum of the multipliers

$$m_k \colon BU(k) \to GL_1E,$$

and so to the datum of a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & BU(k-1) \arrow[r] \arrow[dr, "m_{k-1}"'] & BU(k) \arrow[r] \arrow[d, "m_k"] & BU(k+1) \arrow[r] \arrow[dl, "m_{k+1}"] & \cdots \\ & & GL_1E & & \end{tikzcd}$$

As $BU = \varinjlim BU(k)$, this is in turn equivalent to the datum of a single morphism

$$m \colon BU \to GL_1E.$$

Notice that, since $GL_1[X, E]$ is the multiplicative subgroup of the commutative ring $[X, E]$, the group $GL_1[X, E]$ is an abelian group and so the spectrum $GL_1E$ is an $\infty$-abelian group. As the direct sum of vector bundles is the group operation on $BU$, the equation $m_{V+W} = m_V \cdot m_W$ implies that $m \colon BU \to GL_1E$ is a morphism of homotopy abelian $\infty$-groups.

Group homomorphisms into an abelian group are themselves an abelian group (more concretely: the bundlewise product of two compatible systems of multipliers is again a compatible system of multipliers), and as we have already noticed multiplying a compatible system of orientations with a compatible system of multipliers one gets a multiplicative system of multipliers. In other words, the space of complex orientations of $E$, i.e., the space of homotopy commutative ring spectra morphisms $MU \to E$ is a torsor over the group $\mathrm{Hom}_{\mathrm{Grp}}(BU, GL_1E)$.

*Remark* 6.10. The analysis that we made to establish the equivalence between compatible systems of $E$-orientations on complex vector bundles and homotopy commutative ring spectra maps $MU \to E$ can be done in a completely analogous way to establish an equivalence between compatible systems of $E$-orientations on real (resp. oriented real) vector bundles and homotopy commutative ring spectra maps $MO \to E$ (resp. $MSO \to E$). By Thom's theorem (see [Koc96, Theorem 1.5.10]), both $MO$ and $MSO$ are cobordism spectra. More precisely, $MO$ is the real cobordism spectrum while $MSO$ is the real oriented cobordism spectrum and the respective cobordism rings are obtained as

$$\Omega^O \cong \bigoplus_{n \geqslant 0} [\mathbb{S}[n], MO] = \bigoplus_{n \geqslant 0} \pi_n^{\mathsf{Sp}} MO$$

and

$$\Omega^{SO} \cong \bigoplus_{n \geqslant 0} [\mathbb{S}[n], MSO] = \bigoplus_{n \geqslant 0} \pi_n^{\mathsf{Sp}} MSO$$

Both $MO$ and $MSO$ are $(-1)$-connected spectra, so the fibre sequences associated to their 0-truncations are

$$MO_{>0} \to MO \to MO_{\leqslant 0} \cong H\pi_0^{\mathsf{Sp}}(MO) = H\mathbb{Z}/2$$

$$MSO_{>0} \to MSO \to MSO_{\leqslant 0} \cong H\pi_0^{\mathsf{Sp}}(MSO) = H\mathbb{Z}.$$

and we get two natural $E_\infty$-ring maps

$$MO \to H\mathbb{Z}/2$$

$$MSO \to H\mathbb{Z}.$$

This tells us that the family of all real vector bundles has a canonical orientation/Thom classes in $\mathbb{Z}/2$-valued singular cohomology and the family of all oriented real vector bundles has a canonical orientation/Thom classes in $\mathbb{Z}$-valued singular cohomology.

## 7. Euler classes of $E$-oriented vector bundles

Assume an $E$-orientation of a stable family $\mathcal{F}$ is given. Thom spectra of actual (i.e., nonvirtual) vector bundles come equipped with natural *zero section* maps $\iota_V \colon X \to X^V$ that are morphisms of $X$-comodules, i.e., the diagrams

$$\begin{tikzcd}
X \arrow[r, "\iota_V"] \arrow[d, "\Delta"'] & X^V \arrow[d, "\Delta^V"] \\
X \otimes X \arrow[r, "\mathrm{id}\otimes\iota_V"'] & X \otimes X^V
\end{tikzcd}$$

commute. So, for any $V$ in $\mathcal{F}$ we have pullbacks

$$\iota_V^* \colon [X^V[-\operatorname{rk} V], E] \to [X[-\operatorname{rk} V], E]$$

which are morphisms of $[X, E]$-modules.

**Definition 7.1.** The element $e_V := \iota^*\tau_V$ of $[X[-\operatorname{rk} V], E]$ is called the *Euler class* of $V$.

*Remark* 7.2. From the commutativity of

$$\begin{tikzcd}
X \arrow[r, "\iota_{V+W}"] \arrow[d, "\Delta"'] & X^{V+W} \arrow[d, "\Delta^{V\boxplus W}"] \\
X \otimes X \arrow[r, "\iota_V\otimes\iota_W"'] & X^V \otimes X^W
\end{tikzcd}$$

one gets the multiplicativity of Euler classes,

$$e_{V+W} = e_V \cdot e_W,$$

where $\cdot$ is the product in $[X, E]$. For the trivial bundle $\mathbf{1}$, the zero section $\iota_{\mathbf{1}}\colon X \to X^{\mathbf{1}} = X[1]$ is the inclusion of $X$ into its suspension, so it is homotopically trivial. From this and the multiplicativity of Euler classes it follows that if $V$ has a never zero section, so that $V = V_0 + \mathbf{1}$ then, $e_V = 0$.

The above general discussion applies in particular to complex orientations of $E$, i.e., to a compatible system of $E$-orientation of complex vector bundles. By Remark 6.6, the datum of such an orientation is equivalent to the the datum of a Thom class $\tau_L$ such that, under the isomorphism of $[\mathbb{P}^\infty\mathbb{C}, E]$-modules $[MU(1), E] \cong [\mathbb{P}^\infty\mathbb{C}[-2], E]$ induced by the pullback along the zero section $\iota\colon BU(1) \to BU(1)^L$, it lifts the unit element $1 \in [\mathbb{S}, E]$. By definition of the Euler class, we therefore have that the Euler class $e_L \in [\mathbb{P}^\infty[-2]\mathbb{C}, E]$ is a generator of $[\mathbb{P}^\infty\mathbb{C}[-2], E]$ as a $[\mathbb{P}^\infty\mathbb{C}, E]$-module, with $i_1^* e_L = 1$, where $i_1\colon \mathbb{P}^1\mathbb{C} \to \mathbb{P}^\infty\mathbb{C}$ is the inclusion. For a fixed complex orientation on $E$, we will write $e^E$ for the Euler class of the universal line bundle $L = \mathcal{O}(1)$, i.e, we will write $e^E = e_L$.

*Example* 7.3. It follows from the proof of Lemma 6.1 and by the description of $MU$ given in Remark 6.7 that the Euler class $e^{MU}$ of the canonical complex orientation of $MU$ is the hyperplane inclusion $\mathbb{P}^{\infty-1}\mathbb{C} \hookrightarrow \mathbb{P}^\infty$ seen as a complex oriented proper map of dimension $-2$ to $\mathbb{P}^\infty$, i.e., more precisely, that for any $n \geqslant 1$ the pullback $i_n^* e^{MU}$ along the standard inclusion $i_n\colon \mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^\infty$ is the complex oriented proper map of dimension $-2$ to $\mathbb{P}^n\mathbb{C}$ given by the hyperplane inclusion $\mathbb{P}^{n-1}\mathbb{C} \hookrightarrow \mathbb{P}^n\mathbb{C}$.

Let now $E$ be an even 2-periodic $E_\infty$-ring spectrum with Bott element $\beta_E$. One of the most important features of even 2-periodic spectra, and, in general, of complex orientable spectra, lies in the cohomology of (complex) projective spaces. Since the $E$-cohomology of $\mathbb{P}^\infty$ is defined to be the limit of the $E$-cohomologies of the $\mathbb{P}^n\mathbb{C}$'s over the (pullback of the) inclusions $i_n : \mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^{n+1}\mathbb{C}$, one gets that the choice of an element $\beta_E\xi$ in $[\mathbb{P}^\infty, E]$ is equivalent to the datum of a compatible sequence $\{\beta_E\xi_n \in [\mathbb{P}^n\mathbb{C}, E]\}_{n\in\mathbb{N}}$, where $\xi_n$ is the pullback of $\xi$ along the inclusion $i : \mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^\infty$ and hence equivalent to a commutative diagram of rings

$$\begin{tikzcd} & & {[\mathbb{S}, E][u]} \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots \arrow[r] & {[\mathbb{P}^{n+1}\mathbb{C}, E]} \arrow[r, "i_n^*"'] & {[\mathbb{P}^n\mathbb{C}, E]} \arrow[r, "i_{n-1}^*"'] & {[\mathbb{P}^{n-1}\mathbb{C}, E]} \arrow[r] & \cdots \end{tikzcd}$$

where the $n$-th arrow maps $u$ to $\beta\xi_n$.

**Proposition 7.4.** *If $e^E \in [\mathbb{P}^\infty[-2], E]$ is the Euler class of a complex orientation of $E$, then the above diagram induces a sequence of compatible isomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & {[\mathbb{S}, E][u]/(u^{n+2})} \arrow[r] \arrow[d, "\cong"'] & {[\mathbb{S}, E][u]/(u^{n+1})} \arrow[r] \arrow[d, "\cong"'] & {[\mathbb{S}, E][u]/(u^{n})} \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & {[\mathbb{P}^{n+1}\mathbb{C}, E]} \arrow[r, "i_n^*"] & {[\mathbb{P}^{n}\mathbb{C}, E]} \arrow[r, "i_{n-1}^*"] & {[\mathbb{P}^{n-1}\mathbb{C}, E]} \arrow[r] & \cdots
\end{tikzcd}$$

The proof of this proposition is just a consequence of the collapsing at the second page of the (cohomological) Atiyah–Hirzebruch spectral sequence for $\mathbb{P}^n\mathbb{C}$; a more detailed account of this argument can be found in [Ada74] or [Koc96]. If we take the limit of the above diagram we get a commutative diagram

$$\begin{tikzcd}
{[\mathbb{S}, E][[u]]} \arrow[rr, "\cong", "u \mapsto \beta_E e^E"'] \arrow[dr, "u \mapsto \beta_E 1"'] & & {[\mathbb{P}^\infty, E]} \arrow[dl, "i_1^*"] \\
& {[\mathbb{P}^1\mathbb{C}, E]} &
\end{tikzcd}$$

where the top horizontal arrow is an isomorphism, and in the left diagonal arrow 1 denotes the unit of $[\mathbb{S}, E]$ seen as an element in $[\mathbb{P}^1\mathbb{C}[-2], E]$ via the isomorphism $\mathbb{S} \cong \mathbb{P}^1\mathbb{C}[-2]$. Conversely, the existence of such a commutative diagram is equivalent to the existence of a complex orientation of $E$.

**Corollary 7.5.** *Every even 2-periodic ring spectrum is complex orientable.*

*Proof.* Let $E$ be an even 2-periodic ring spectrum. By the above discussion, to prove that $E$ is complex orientable we have to show that the distinguished ring morphism $[\mathbb{S}, E][u] \to [\mathbb{P}^1\mathbb{C}, E]$ sending $u \mapsto \beta_E 1$ admits a lift through $i_1^*$. The obstructions to such a lift are the obstructions to extending $\beta_E 1 \colon \mathbb{P}^1\mathbb{C} \to E$ to a map $\beta e^E \colon \mathbb{P}^\infty \to E$, through the sequence of skeleta inclusions

$$\underbrace{\mathbb{P}^1\mathbb{C}}_{\text{2-skeleton}} = \underbrace{\mathbb{P}^1\mathbb{C}}_{\text{3-skeleton}} \hookrightarrow \underbrace{\mathbb{P}^2\mathbb{C}}_{\text{4-skeleton}} = \underbrace{\mathbb{P}^2\mathbb{C}}_{\text{5-skeleton}} \hookrightarrow \cdots$$

so they lie in the singular cohomologies of $\mathbb{P}^\infty$ with coefficients in the abelian groups $\pi_{2n+1}^{\mathsf{Sp}}(E) = [\mathbb{S}[2n+1], E]$. For an even cohomology theory these are all zero, so all the obstructions vanish. $\square$

## 8. The GHRR Theorem, Case I: One Spectrum, Two Orientations

Let $\rho_A$ and $\rho_B$ be two complex orientations for an even 2-periodic $E_\infty$-ring spectrum $E$, and for every stably complex virtual vector bundle $V$ over $X$, denote by $\sigma_V^A, \sigma_V^B$ the corresponding isomorphisms of $[X, E]$-modules $\sigma_V^A, \sigma_V^B$ : $[X, E] \xrightarrow{\cong} [X^V[-\operatorname{rk} V], E]$. If $f \colon X \to Y$ is a stably complex map, i.e., it is a smooth map such that $T_f$ is a stably complex virtual vector bundle over $X$, then we have two $E$-orientations on $-T_f$ and a commutative diagram

$$\begin{tikzcd}
{[X,E]} \arrow[rr, "\cdot m_{-T_f}"] \arrow[dr, "\sigma_f^A"'] \arrow[ddr, bend right, "f_*^A"'] & & {[X,E]} \arrow[dl, "\sigma_f^B"] \arrow[ddl, bend left, "f_*^B"] \\
& {[X^{-T_f}[\dim X - \dim Y], E]} \arrow[d, "\varphi_{PT}^*"] & \\
& {[Y[\dim X - \dim Y], E]} &
\end{tikzcd}$$

In other words, we have

$$f_*^A(a) = f_*^B(a \cdot m_{-Tf})$$

for any $a \in [X, E]$. We want to determine the multiplier $m_{-T_f} = f^*(m_{TY}) \cdot m_{TX}^{-1}$. To do so, let $V$ be a stably complex vector bundle over $X$. As $m_{V+\mathbf{n}} = m_V$ for any $\mathbf{n}$, we may assume that $V$ is a complex vector bundle. Also, as the splitting principle works for every even 2-periodic $E_\infty$-ring spectrum $E$ as recalled above, it is not restrictive to assume that $V$ splits as a direct sum of complex line bundles $L_i$ classified by maps $\lambda_i\colon X \to \mathbb{P}^\infty$. We then have

$$m_V = m_{L_1 + \cdots L_k} = \prod_{i=1}^{k} m_{L_i} = \prod_{i=1}^{k} m_{\lambda_i^* L} = \prod_{i=1}^{k} \lambda_i^* m_L,$$

where $L = \mathcal{O}(1)$ is the universal line bundle on $\mathbb{P}^\infty$. So we just need to compute a single multiplier, namely $m_L$. This is defined by the equation $\tau_L^A = m_L \cdot \tau_L^B$. As the pullback along the zero section $\iota\colon \mathbb{P}^\infty \to (\mathbb{P}^\infty)^L$ is an isomorphism of $[\mathbb{P}^\infty, E]$-modules, this is equivalent to the equation $e^A = m_L \cdot e^B$, where $e^A$ and $e^B$ are the Euler classes of the $E$-orientations $\rho_A$ and $\rho_B$, respectively. By the conclusion of Section 7, the Euler classes $e^A$ and $e^B$ uniquely determine a commutative diagram

$$\begin{tikzcd}
{[\mathbb{S}, E][[u]]} \arrow[rr, "u \mapsto \varphi(u)"] \arrow[dr, "u \mapsto \beta_E e^A"'] & & {[\mathbb{S}, E][[u]]} \arrow[dl, "u \mapsto \beta_E e^B"] \\
& {[\mathbb{P}^\infty\mathbb{C}, E]} &
\end{tikzcd}$$

where every arrow is a ring isomorphism, and so

$$m_L = \left.\frac{\varphi(u)}{u}\right|_{u=\beta_E e^B}.$$

Therefore, for any complex vector bundle $V$ on $X$ we have

$$m_V = \prod_i \left.\frac{\varphi(u)}{u}\right|_{u=\beta_E \lambda_i^* e^B}.$$

**Definition 8.1.** It is customary to call $m_{-V}$ the *Todd class* of $V$ relative to the two orientations $\rho_A$ and $\rho_B$ and to denote it $\mathrm{td}_{A,B}(V)$, i.e.,

$$\mathrm{td}_{A,B}(V) = \prod_i \frac{u}{\varphi(u)}\bigg|_{u=\beta_E\lambda_i^* e^B}.$$

By introducing the Todd function

$$\mathrm{td}_{A,B}(u) = \frac{u}{\varphi(u)},$$

the above takes the form

$$\mathrm{td}_{A,B}(V) = \prod_i \mathrm{td}_{A,B}(u)\bigg|_{u=\beta_E\lambda_i^* e^B}.$$

*Remark* 8.2. Notice that, since the restrictions of $\beta_E e^A$ and $\beta_E e^B$ to $\mathbb{P}^1\mathbb{C}$ coincide (they are both equal to $\beta_E 1$), one has $\varphi(u) = u + o(u)$, hence

$$\mathrm{td}_{A,B}(u) = 1 + o(1).$$

With this notation we have

$$m_{-T_f} = \mathrm{td}_{A,B}(T_f)$$

and the product formula for the Todd class from Definition 8.1 tells how to express this as a product of characteristic classes for $TX$ and $TY$. Summing up, we have proven the following

**Theorem 8.3** (GHRR for a pair of complex orientations)**.** *Let $\rho_A$ and $\rho_B$ be two complex orientations for an even 2-periodic $E_\infty$-ring spectrum $E$, and let $f\colon X \to Y$ be a stably complex map. Then the two pushforward maps $f_*^A, f_*^B\colon [X, E] \to [Y[\dim X - \dim Y], E]$ are related by the Grothendieck–Hirzebruch–Riemann–Roch formula*

$$f_*^A(a) = f_*^B(a \cdot \mathrm{td}_{A,B}(T_f)),$$

*for any $a \in [X, E]$.*

*Remark* 8.4. By multiplicativity and functoriality of the multipliers and by the projection formula, the above identity can be written as

$$f_*^A(a) \cdot \mathrm{td}_{AB}(TY) = f_*^B(a \cdot \mathrm{td}_{AB}(TX)).$$

## 9. The GHRR Theorem, Case II: Two Oriented Spectra, One Morphism

Let now $\rho\colon MU \to E$ be a complex orientation of $E$ and let $\psi\colon E \to F$ be a morphism of homotopy commutative ring spectra. Then we have the following.

**Lemma 9.1.** *For any stably complex map $f\colon X \to Y$, the diagram*

$$\begin{array}{ccc} [X,E] & \xrightarrow{\quad\psi_*\quad} & [X,F] \\ {\scriptstyle f_*^{\rho}}\downarrow & & \downarrow{\scriptstyle f_*^{\psi_*\rho}} \\ [Y[\dim X-\dim Y],E] & \xrightarrow{\psi_*} & [Y[\dim X-\dim Y],F] \end{array}$$

*commutes.*

*Proof.* The diagram

$$\begin{array}{ccc} [X,E] & \xrightarrow{\quad\psi_*\quad} & [X,F] \\ {\scriptstyle \sigma_{-T_f}}\downarrow & & \downarrow{\scriptstyle \psi_*\sigma_{-T_f}} \\ [X^{-T_f}[\dim X-\dim Y],E] & \xrightarrow{\psi_*} & [X^{-T_f}[\dim X-\dim Y],F] \end{array}$$

commutes, as $\psi_*$ maps the element 1 in $[X,E]$ to the element 1 in $[X,F]$ and $\tau^{\psi_*\rho}_{-T_f} = \psi(\tau^{\rho}_{-T_f})$ by definition of $\psi_*\rho$. The diagram

$$\begin{array}{ccc} [X^{-T_f}[\dim X-\dim Y],E] & \xrightarrow{\psi_*} & [X^{-T_f}[\dim X-\dim Y],F \\ {\scriptstyle \varphi_{PT}^*}\downarrow & & \downarrow{\scriptstyle \varphi_{PT}^*} \\ [Y[\dim X-\dim Y],E] & \xrightarrow{\quad\psi_*\quad} & [Y[\dim X-\dim Y],F] \end{array}$$

trivially commutes as post-compositions and pre-compositions of morphisms commute. $\square$

We also have

**Proposition 9.2.** *Let $\rho_E\colon MU \to E$ and $\rho_F\colon MU \to F$ be complex orientations for the $E_\infty$-ring spectra $E$ and $F$, respectively, and let $\psi\colon E \to F$ be a morphism of ring spectra. Then, for any stably complex map $f\colon X \to Y$ and any $a \in [X,E]$, the following Grothendieck–Hirzebruch–Riemann–Roch like identity holds:*

$$\psi_*(f_*^{\rho_E}(a)) = f_*^{\rho_F}(\psi_*(a)\cdot \mathrm{td}_{\psi_*\rho_E,\rho_F}(T_f))$$

*Proof.* By Lemma 9.1, we have

$$\psi_*(f_*^{\rho_E}(a)) = f_*^{\psi_*\rho_E}(\psi_*(a)).$$

The conclusion then follows from the second-last equation in Section 8. $\square$

The statement of Proposition 9.2 is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} [X,E] & \xrightarrow{\quad\mathrm{td}_{\psi_*\rho_E,\rho_F}(T_f)\cdot\psi_*\quad} & [X,F] \\ {\scriptstyle f_*^{\rho_E}}\downarrow & & \downarrow{\scriptstyle f_*^{\rho_F}} \\ [Y[\dim X-\dim Y],E] & \xrightarrow{\psi_*} & [Y[\dim X-\dim Y],F] \end{array}$$

*Remark* 9.3. As a particular case, one can consider $E = MU$ and $\rho_E$ to be the identity morphism of $MU$. Taking $F = HP_{\mathrm{ev}}\mathbb{Q}$ and $\rho_F \colon MU \to HP_{\mathrm{ev}}\mathbb{Q}$ the standard complex orientation $\rho_H$ of $HP_{\mathrm{ev}}\mathbb{Q}$, one sees from Proposition 9.2 that for any complex orientation $\psi \colon MU \to HP_{\mathrm{ev}}\mathbb{Q}$ and any complex manifold $X$ of complex dimension $n$ one has a commutative diagram

$$\begin{tikzcd}
{[X, MU]} \arrow[r, "\mathrm{td}_{\psi,\rho_H}(TX)\cdot\psi_*"] \arrow[d, "\pi_*^{\mathrm{id}_{MU}}"'] & {[X, HP_{\mathrm{ev}}\mathbb{Q}]} \arrow[d, "\pi_*^{\rho_H}"] \\
\Omega^U_{2n} \arrow[r, "\psi_*"] & \mathbb{Q}\beta^n,
\end{tikzcd}$$

where $\Omega^U_{2n}$ is the $2n$-dimensional complex cobordism group. The image of the unit element $1 \in [X, MU]$ via $\pi_*^{\mathrm{id}_{MU}}$ is the complex cobordism class of $X$, see [Qui71]. The pushforward map in even periodic rational singular cohomology induced by its standard complex orientation is the periodic version of the usual pushforward map in rational singular cohomology. In particular, if $X$ is a compact complex manifold, the pushforward map

$$\int_X^{HP_{\mathrm{ev}}\mathbb{Q}} \colon \bigoplus_{i\in\mathbb{Z}} H^{2i}(X;\mathbb{Q})\beta^i \to \mathbb{Q}\beta^{\dim_{\mathbb{C}} X}$$

along the terminal morphism $\pi_X \colon X \to *$ is

$$\int_X^{HP_{\mathrm{ev}}\mathbb{Q}} = \beta^{\dim_{\mathbb{C}} X} \int_X,$$

where $\int_X$ is the usual integral in singular cohomology.. Finally, the morphism of abelian groups $\psi_* \colon \Omega^U_{2n} \to \mathbb{Q}\beta^n$ is the degree $-2n$ component of the Hirzebruch $\psi_*$-genus, i.e., of the graded rings homomorphism

$$\psi_* \colon \bigoplus_{n\in\mathbb{Z}} \Omega^U_{2n} \to \mathbb{Q}[\beta, \beta^{-1}].$$

Therefore, as a particular case of Proposition 9.2 one finds Hirzebruch's genus formula:

$$\psi_*([X]) = \beta^{\dim_{\mathbb{C}} X} \int_X \mathrm{td}_{\psi,\rho_H}(TX).$$

## 10. Specializing to the Classical Statement

The archetype of the formula in Proposition 9.2 is obviously the classical Grothendieck–Hirzebruch–Riemann–Roch formula. Denote by $KU$ the spectrum representing complex $K$-theory and by $HP_{\mathrm{ev}}\mathbb{Q}$ the spectrum representing even periodic rational singular cohomology. Both spectra are multiplicative and even 2-periodic. With their standard complex orientations, their shifted Euler classes are given by $\beta_K e^{\rho_K} = \mathbf{1}_{\mathbb{C}} - L^{-1} \in [\mathbb{P}^\infty, KU]$ and by $\beta_H e^{\rho_H} = \beta c_1(L) \in$

$[\mathbb{P}^\infty, HP_{\mathrm{ev}}\mathbb{Q}]$, where $c_1$ is the first Chern class in singular cohomology and $\beta$ is a formal degree $-2$ variable; see, e.g. [LM07, Example 1.1.5] for the convention on the orientation of complex $K$-theory.

The Chern character $\mathrm{ch}\colon KU \to HP_{\mathrm{ev}}\mathbb{Q}$ provides a multiplicative map of even 2-periodic cohomology theories (i.e. $\mathrm{ch}(\beta_K) = \beta_H$, see Example 2.25) from complex $K$-theory to periodic rational singular cohomology. It can be seen as the composition

$$\begin{tikzcd} KU \arrow[r, "(-)_{\mathbb{Q}}"'] \arrow[rr, bend left, "\mathrm{ch}"] & KU_{\mathbb{Q}} \arrow[r, "\Phi", "\cong"'] & HP_{\mathrm{ev}}\mathbb{Q}, \end{tikzcd}$$

where $(-)_{\mathbb{Q}}$ is the rationalization map and $\Phi\colon KU_{\mathbb{Q}} \cong HP_{\mathrm{ev}}\mathbb{Q}$ is the equivalence given by the splitting of rational spectra in sums of Eilenberg–Mac Lane spectra, normalised so as to map the Bott element of $KU_{\mathbb{Q}}$ to the Bott element of $HP_{\mathrm{ev}}\mathbb{Q}$. By the general splitting for rational spectra we have

$$KU_{\mathbb{Q}} \cong \oplus_{i\in\mathbb{Z}} H\pi_i^{\mathsf{Sp}}(KU_{\mathbb{Q}})[i] \cong \oplus_{i\in\mathbb{Z}} H(\pi_i^{\mathsf{Sp}} KU)\otimes\mathbb{Q}[i] \cong \oplus_{j\in\mathbb{Z}} H\mathbb{Q}[2j] = HP_{\mathrm{ev}}\mathbb{Q}.$$

Moreover, naturality of these equivalences with respect to the monoidal structure of spectra implies that $KU_{\mathbb{Q}} \cong HP_{\mathrm{ev}}\mathbb{Q}$ is an equivalence if $E_\infty$-ring spectra. A good reference for this result in a cohomological flavour can be found in [Hil71]. Now, to get an explicit expression for the equivalence $\Phi\colon KU_{\mathbb{Q}} \cong HP_{\mathrm{ev}}\mathbb{Q}$ induced by rationalization, recall that $KU$ is generated by the class of the universal line bundle $L$, so we only need to determine $\Phi(L)$. This is an element in $[\mathbb{P}^\infty, HP_{\mathrm{ev}}\mathbb{Q}]$, so by the results in Section 7, there exists a unique formal power series

$$f(u) = \sum_{k=0}^{\infty} f_k u^k$$

with coefficients in $\mathbb{Q} = [\mathbb{S}, HP_{\mathrm{ev}}\mathbb{Q}]$ such that

$$\Phi(L) = f(\beta c_1(L)).$$

By naturality with respect to pullbacks, and since $\Phi$ is a ring homomorphism, we have

$$1 = \Phi(\mathbf{1}_{\mathbb{C}}) = f_0,$$

so we can write

$$f(u) = e^{g(u)}$$

for a unique formal power series

$$g(u) = \sum_{k=0}^{\infty} g_k u^k$$

with $g_0 = 0$. Again by naturality with respect to pullbacks and since the tensor product of vector bundles induces the product in $K$-theory, we have, for any $n \in \mathbb{Z}$,

$$e^{g(n\,u)}\Big|_{\beta c_1(L)} = e^{g(u)}\Big|_{n\beta c_1(L)} = \Phi(L^{\otimes n}) = \Phi(L)^n = e^{n\,g(u)}\Big|_{\beta c_1(L)}.$$

As evaluating at $\beta c_1(L)$ is an isomorphism $\mathbb{Q}[[u]] \xrightarrow{\sim} HP_{\mathrm{ev}}\mathbb{Q}(\mathbb{P}^\infty)$, this gives

$$g(n\,u) = n\;g(u), \qquad \text{for any } n \in \mathbb{Z},$$

so $g(u)$ is a linear function: $g(u) = g_1\,u$ for some $g_1 \in \mathbb{Q}$. Imposing that $\Phi$ preserves Bott elements we find

$$\beta c_1(L|_{\mathbb{P}^1\mathbb{C}}) = 1 - e^{-g_1\beta c_1(L|_{\mathbb{P}^1\mathbb{C}})} = g_1\beta c_1(L|_{\mathbb{P}^1\mathbb{C}}),$$

and so $g_1 = 1$, i.e., $f(u) = e^u$ and $\Phi = \mathrm{ch}$. As we are interested in

$$\mathrm{td}_{\mathrm{ch}_*\rho_K,\rho_H}(u),$$

we have to identify the map $\varphi$ determined by the commutative diagram

$$\begin{tikzcd}
{[\mathbb{S}, HP_{\mathrm{ev}}\mathbb{Q}][[u]]} \arrow[rr, "u\mapsto\varphi(u)"] \arrow[dr, "u\mapsto\mathrm{ch}(\mathbf{1}_{\mathbb{C}}-L^{-1})"'] & & {[\mathbb{S}, HP_{\mathrm{ev}}\mathbb{Q}][[u]]} \arrow[dl, "u\mapsto\beta c_1(L)"] \\
& {[\mathbb{P}^\infty\mathbb{C}, HP_{\mathrm{ev}}\mathbb{Q}]} &
\end{tikzcd}$$

As $\mathrm{ch}(\mathbf{1}_{\mathbb{C}} - L^{-1}) = 1 - e^{-\beta c_1(L)}$, the map $\varphi$ is

$$\varphi(u) = 1 - e^{-u}.$$

Therefore the Todd function associated with the Chern character and the standard orientations of complex $K$-theory and even periodic rational cohomology is

$$\mathrm{td}_{\mathrm{ch}_*\rho_K,\rho_H}(u) = \frac{u}{1-e^{-u}}.$$

## 11. Why "the topological half"?

Readers so patient to have read until here may be wondering what is the non-topological half of the Grothendieck–Hirzebruch–Riemann–Roch theorem we were hinting at in the title. To explain this, consider a compact complex manifold $X$ together with a *holomorphic* vector bundle $V$ over it. Then the Hirzebruch–Riemann–Roch theorem can be stated as the identity

$$\chi(X;V) = \int_X \mathrm{ch}(V)\,\mathrm{td}(TX),$$

where on the left we have the holomorphic Euler characteristic of $X$ with coefficients in the holomorphic bundle $V$ (or, equivalently, in its sheaf $\mathcal{V}$ of holomorphic sections). This Euler characteristic is the pushforward in the analytical $K$-theory of $X$ (or equivalently in the bounded derived category of coherent sheaves on $X$) of the element $[V]$ in $KU^{\mathrm{an}}(X)$ to an element in $KU^{\mathrm{an}}(*) = \mathbb{Z}$ along the terminal morphism $\pi_X : X \to *$. That is, we have

$$\chi(X; V) = \pi_{X,*}^{KU,\mathrm{an}}([V]).$$

By the discussion in the previous sections, the statement of the Hirzebruch–Riemann–Roch theorem can be rewritten as

$$\overbrace{\pi_{X,*}^{KU,\mathrm{an}}([V]) = \pi_{X,*}^{KU}([V])}^{\text{analytical/algebro-geometrical part}} \underbrace{{}= \int_X \mathrm{ch}(V)\,\mathrm{td}(TX)}_{\text{topological part}},$$

where the right part of the identity is what we have discussed in this note, while the left part of the identity, i.e., the identification between the push-forward in analytic (or algebro-geometric) $K$-theory and the pushforward in topological $K$-theory is a deep result in the holomorphic (or algebro-geometric) setting, unattainable by purely topological methods. Analogous considerations apply to the more general case of the pushforward along a proper holomorphic map between holomorphic manifolds considered in the Grothendieck–Hirzebruch–Riemann–Roch theorem.

As a conclusion, let us recall how to determine the Todd function $u/(1-e^{-u})$ just by assuming the identity $\pi_{X,*}^{KU,\mathrm{an}}([V]) = \pi_{X,*}^{KU}([V])$, where $V$ is a holomorphic vector bundle $V$ over a compact complex manifold $X$, holds for some complex orientation $\rho\colon MU \to KU$ of the topological complex $K$-theory. Under these assumptions, by the discussion in the previous sections we will have in particular the identities

$$\chi(\mathbb{P}^n\mathbb{C}; \mathcal{O}_{\mathbb{P}^n\mathbb{C}}) = \int_{\mathbb{P}^n\mathbb{C}} \mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(T\mathbb{P}^n\mathbb{C}) = \beta^{-n} \int_{\mathbb{P}^n\mathbb{C}}^{HP_{\mathrm{ev}}\mathbb{Q}} \mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(T\mathbb{P}^n\mathbb{C})$$

for a suitable formal power series $\mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(u)$. By the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n\mathbb{C}} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n\mathbb{C}}(1) \to T\mathbb{P}^n\mathbb{C} \to 0$$

we get

$$\mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(T\mathbb{P}^n\mathbb{C}) = \mathrm{td}_{\mathrm{ch}_*\rho_K,\rho_H}(\beta c_1(\mathcal{O}_{\mathbb{P}^n\mathbb{C}}(1)))^{n+1}$$

so that, writing

$$(\mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(u))^{n+1} = \sum_{k=0}^{\infty} a_{n+1,k} u^k$$

we obtain

$$\chi(\mathbb{P}^n\mathbb{C}; \mathcal{O}_{\mathbb{P}^n\mathbb{C}}) = a_{n+1,n}.$$

On the other hand

$$\chi(\mathbb{P}^n\mathbb{C}; \mathcal{O}_{\mathbb{P}^n\mathbb{C}}) = \sum_{q=0}^{n} h^{0,q}(\mathbb{P}^n\mathbb{C}) = 1.$$

Therefore the formal series $\mathrm{td}_{\mathrm{ch}_*\rho,\rho_H}(u)$ must satisfy $a_{n+1,n} = 1$ for every $n$, i.e., it must have the property that the coefficient of $u^n$ in $(\mathrm{td}_\rho(u))^{n+1}$ equals 1 for any $n$. A classical computation using the Lagrange inversion formula shows that there exists a unique formal power series with this property: the power series expansion of

$$\mathrm{td}_{\mathrm{ch}_*\rho_K,\rho_H}(u) = \frac{u}{1-e^{-u}}.$$

As the Chern character is an isomorphism from rationalized complex $K$-theory to even 2-periodic rational singular cohomology, this shows that the only possible complex orientation of topological $K$-theory for which one can have the complex analytic/algebro-geometric half of the Grothendieck–Hirzebruch–Riemann–Roch theorem is $\rho_K$, i.e., the one with (shifted) Euler class $\beta_{KU}e^{KU} = \mathbf{1}_\mathbb{C} - L^{-1}$, thus motivating this apparently less natural choice with respect to $\beta_{KU}e^{KU} = L - \mathbf{1}_\mathbb{C}$. Clearly, as far as one is not concerned with the complex analytic/algebro-geometric half of the theorem, this second orientation, with corresponding Todd function $u/(e^u - 1)$, is an equally valid choice, and it is actually quite a common choice for defining a complex orientation of topological $K$-theory in algebraic topology.

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**Appendices**

## 1. Orthogonal Spectra

In an effort to keep the exposition as friendly as possible, in the article we avoided a detailed description of $\mathsf{Sp}$ as a stable $\infty$-category, barely mentioning it as presented by the standard model structure on the category of orthogonal spectra. Also in this appendix we are not going to give details on the the $\infty$-category structure $\mathsf{Sp}$ but at least we recall the basics of the model structure on the category of orthogonal spectra. Standard references include [MMSS01] and [MMSS98].

**Definition 1.1.** An *orthogonal spectrum* $E$ is given by a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pointed spaces along with, for all $n \in \mathbb{N}$,

(1) an action of the orthogonal group $O(n)$ on $E_n$,
(2) a map $\sigma_n : E_n \wedge S^1 \to E_{n+1}$.

For all $n \geqslant 0, k \geqslant 1$ we require the map

$$\sigma^k : E_n \wedge S^k \xrightarrow{\sigma_n \wedge S^{k-1}} E_{n+1} \wedge S^{k-1} \to \cdots \to E_{n+k}$$

to be $O(n) \times O(k)$ equivariant, with $O(k)$ acting on $S^k$ the standard way and $O(n) \times O(k)$ acting on $E_{n+k}$ via the inclusion $O(n) \times O(k) \hookrightarrow O(n+k)$.

A morphism $f : E \to F$ is a sequence of $O(n)$-equivariant maps $f_n : E_n \to F_n$ commuting with the structure maps $\sigma_n$:

$$\begin{tikzcd} E_n \wedge S^1 \arrow[r, "\sigma_n"] \arrow[d, "f_n \wedge S^1"] & E_{n+1} \arrow[d, "f_{n+1}"] \\ F_n \wedge S^1 \arrow[r, "\sigma_n"] & F_{n+1}. \end{tikzcd}$$

We denote the category of orthogonal spectra by $\mathsf{OrthSpectra}$

**Definition 1.2.** Given an orthogonal spectrum $E$ consider the maps

$$\iota_{n,k} : \pi_{n+k} X_n \xrightarrow{- \wedge S^1} \pi_{n+k+1} E_n \wedge S^1 \xrightarrow{(\sigma_{n+1})_*} \pi_{n+k+1} E_{n+1}.$$

We define the *stable homotopy groups* to be $\pi_k E := \operatorname{colim}_n \pi_{n+k} E_n$ along the $\iota_{n,k}$.

**Definition 1.3.** Let $f : E \to F$ be a map of orthogonal spectra. We say that $f$ is a weak homotopy equivalence if it induces isomorphisms on all the stable homotopy groups

$$\pi_\bullet f : \pi_\bullet E \to \pi_\bullet F.$$

**Definition 1.4.** Let $f : E \to F$ be a map of orthogonal spectra, then $f$ is a *$q$-fibration* if it is a level-wise Serre fibration. It is an *acyclic fibration* if it is both a $q$-fibration and a weak homotopy equivalence. We say that $f : E \to F$ is a *$q$-cofibration* if it satisfies the LLP with respect to the acyclic $q$-fibrations.

**Theorem 1.5.** *The category* OrthSpectra *admits a model structure with fibrations and cofibrations those of Definition 1.4 and weak equivalences the weak homotopy equivalences. This model structure presents the $\infty$-category* Sp *of spectra.*

An important feature of OrthSpectra is that it admits a closed monoidal structure with the tensor product given by the smash defined as the following coequalizer:

$$\begin{tikzcd}
\bigvee_{p+1+q=n} O(n)_+ \underset{O(p)\times 1\times O(q)}{\wedge} X_p \wedge S^1 \wedge Y_q \arrow[d, shift right, "\sigma_p^X \wedge Y_q"'] \arrow[d, shift left, "X_p \wedge \sigma'^Y_q"] \\
\bigvee_{p+q=n} O(n)_+ \underset{O(p)\times O(q)}{\wedge} X_p \wedge Y_q \arrow[d] \\
(X \wedge Y)_n
\end{tikzcd}$$

with $\sigma'^Y_q : S^1 \wedge Y_q \to Y_{q+1}$ obtained by composing the braiding with $\sigma^Y_q$.

The monoidal unit is given by the sphere spectrum $\mathbb{S}$ with $S^n$ at level $n$ and with the obvious $O(n)$-action.

## 2. Closed Families, Tangential Structures and Orientations

We now describe how to give orientations to those families of vector bundles coming from tangential structures, also known as $(B, f)$-structures. The simplest, but incomplete, definition of a tangential structure is as follows.

Consider a sequence of spaces $B_n, B_0 = *$ with maps $i_{m,n} : B_m \to B_n$, a sequence of Serre fibrations $f_n : B_n \to BO(n)$ such that the diagrams

$$\begin{tikzcd}
B_m \arrow[r, "i_{m,n}"] \arrow[d, "f_m"] & B_n \arrow[d, "f_n"] \\
BO(m) \arrow[r] & BO(n)
\end{tikzcd}$$

commute.

We also ask for maps $\mu_{m,n} : B_n \times B_m \to B_{m+n}$ such that the diagrams

$$\begin{tikzcd}
B_n \times B_m \arrow[r, "\mu_{m,n}"] \arrow[d, "f_m \times f_n"] & B_{n+m} \arrow[d, "f_{n+m}"] \\
BO(n) \times BO(m) \arrow[r] & BO(n+m)
\end{tikzcd}$$

commute.

We denote by $V_n \to B_n$ the rank $n$ real vector bundle on $B_n$ classified by the map $f_n \colon B_n \to BO(n)$. In other words, $V_n$ is the pullback along $f_n$ of the tautological vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n$ over $BO(n)$.

*Example* 2.1. The Whitehead tower of $O(n)$, with $n \geqslant 3$, gives us plenty of nice and useful tangential structures. We recall that we have

$$* \to \cdots \to String(n) \to Spin(n) \to SO(n) \to O(n).$$

Here, starting with $O(n)$ one *kills* its $\pi_0$ taking the connected component of the identity to obtain $SO(n)$. Then one can kill its $\pi_1$ taking the universal cover to obtain the group $Spin(n)$. The second homotopy group $\pi_2 Spin(n)$ is already trivial; killing the third homotopy group $\pi_3 Spin(n) \cong \mathbb{Z}$ gives the String group $String(n)$. One continues this way ending with the (weakly homotopically) trivial group $*$ after killing all of the homotopy groups of $O(n)$. Let us denote $O(n)\langle k \rangle$ the (homotopy) group appearing at the $k$-th stage of the tower. The classifying spaces $BO(n)\langle k \rangle$ provide a natural example of a tangential structure.

Consider now a rank $n$ vector bundle $V \to X$, classified by a map $v : X \to BO(n)$. By this we mean that it is isomorphic to the pullback along $v$ of the tautological vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n$ over $BO(n)$. Let $B = \{B_n, f_n, \mu_{nm}\}$ be a tangential structure; we say that $V$ admits a $B$-structure if there exists a lift of $v$ along $f_n$. In other words we ask for the existence of the dashed maps in the following diagram

$$\begin{tikzcd}
V \arrow[r, dashed] \arrow[d] & V_n \arrow[r] \arrow[d] & EO(n) \times_{O(n)} \mathbb{R}^n \arrow[d] \\
X \arrow[r, dashed, "\tilde{v}"] \arrow[rr, bend right, "v"'] & B_n \arrow[r, "f_n"] & BO(n).
\end{tikzcd}$$

Notice that if they do exist then $V$ is automatically the pullback of $V_n$ since both the right square and the big square are pullbacks. Given a $(B, f)$-structure $B$, we will denote the family of vector bundles with $B$-structure by $\mathcal{F}_B$. This way one recovers the definition of orientation on a real vector bundle (in the classical sense), or of a Spin and String structure on a real vector bundle as an $BSO(n)$-structure, a $BSpin(n)$-structure and $BString(n)$-structure in the sense of tangential structures, respectively. The family given by the limit of the Whitehead tower, i.e., by the trivial tangential structure $B_n = *$ gives real vector bundles with a framing.

As we said, this definition of tangential structure is not complete, in particular complex vector bundles are not related to any $(B, f)$-structure for the simple reason that they only have even rank. By weakening the notion of $(B, f)$-structure asking $B_n$ and $f_n$ to be indexed only by some multiple of a natural number $k$ we get the notion of $S^k$-$(B, f)$-structure. For simplicity we will refer to these as $(B, f)$-structures as well. One can then define the complex tangential structure as that induced by the canonical maps $BU(n) \to BO(2n)$. The real vector bundles admitting such a structure are, obviously, the complex vector bundles.

Our reason to define these tangential structures is that they yield closed families of vector bundles (and vice versa, as shown later). As such we want to describe how to use them in order to endow such families with coherent orientations with respect to a given cohomology theory $E$.

To begin with, recall that coherent $E$-orientations behave well under pullbacks. In particular we just need to orient the universal vector bundles $V_n \to B_n$ to orient every other vector bundle in $\mathcal{F}_B$. From the pullback diagram

$$\begin{tikzcd}
(EO(n) \times_{O(n)} \mathbb{R}^n) \oplus \mathbb{R} \arrow[r] \arrow[d] & EO(n+1) \times_{O(n+1)} \mathbb{R}^{n+1} \arrow[d] \\
BO(n) \arrow[r] & BO(n+1)
\end{tikzcd}$$

one deduces that the pullback of $V_{n+k}$ along $i_{n,n+k}$ is $V_n \oplus \mathbb{R}^k$. A coherent system of $E$-orientations on the $V_n$'s then amounts to a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & MB_{n-k} \arrow[r] \arrow[dr, "\tau_{n-k}"'] & MB_n \arrow[r] \arrow[d, "\tau_n"] & MB_{n+k} \arrow[r] \arrow[dl, "\tau_{n+k}"] & \cdots \\
& & E & &
\end{tikzcd}$$

with $B_n^{V_n}[-n] = MB_n$. The datum of this diagram is obviously equivalent to a single map of spectra

$$\rho : MB \to E$$

with $MB := \lim MB_n$ called the *$B$-cobordism spectrum.* Moreover, asking for compatibility with the formation of direct sums amounts to asking for $\rho$ to be a map of homotopy commutative ring spectra.

Notice that, by construction, the spectrum $MB$ is connective, i.e., has trivial homotopy groups in negative degree. We recall that given a spectrum $E$ we always have a truncation fibre sequence of the form

$$E_{>k} \to E \to E_{\leqslant k},$$

with $E_{>k}$ (respectively $E_{\leqslant k}$) a spectrum having all homotopy groups of index $> k$ isomorphic to those of $E$ and the others trivial (respectively, the homotopy groups of index $\leqslant k$ isomorphic to those of $E$ and the others trivial). In particular we have a map

$$MB \to MB_{\leqslant 0} \cong H\pi_0 MB$$

giving a canonical orientation for singular cohomology with $\pi_0 MB$ coefficients to bundles with $B$-structure. Here the equivalence $MB_{\leqslant 0} \cong H\pi_0 MB$ is an immediate consequence of the connectiveness of $MB$: since $MB \cong MB_{\geqslant 0}$, we have $MB_{\leqslant 0} \cong (MB_{\leqslant 0})\geqslant 0$ and the latter is (essentially by definition) $H\pi_0 MB$.

*Example* 2.2. By considering complex tangential structures one obtains this way the complex cobordism spectrum $MU$. By considering orthogonal structures, so actually imposing no additional structure on the real vector bundles in the family, one obtains the orthogonal cobordism spectrum $MO$. At the other extreme of the Whitehead tower of $O(n)$, the cobordism spectrum associated with framed vector bundles is the sphere spectrum $\mathbb{S}$.

Let us now explain how any closed family $\mathcal{F}$ comes from a suitable $(B, f)$-structure whose associated bundles are again those in $\mathcal{F}$, thus yielding an equivalence between these two concepts.

Starting with a closed family $\mathcal{F}$ one considers for any $n \in \mathbb{N}$ the subcategory $\mathcal{F}_n$ of $\mathsf{VectBun}_{\mathbb{R}}$ whose objects are rank $n$ vector bundles in $\mathcal{F}$ and whose morphisms are pullback squares. By a colimit procedure analogue to that used to produce classifying spaces for topological groups one sees that $\mathcal{F}$ has a terminal object $E_n \to B_n$. The forgetful morphisms/inclusion functors $\mathcal{F}_n \to \mathsf{VectBun}_{\mathbb{R}}$ induce the maps $B_n \to BO(n)$. The maps $i_{m,n}$ are obtained from the universal property of the pullback by considering the diagrams

$$\begin{tikzcd}
B_m \arrow[r] \arrow[rr, bend left, "i_{m,n}"] \arrow[dr] & \iota^* B_n \arrow[r] \arrow[d] & B_n \arrow[d] \\
& BO(m) \arrow[r, "\iota"] & BO(n).
\end{tikzcd}$$

Finally the morphisms $\mu_{m,n}$ are obtained from the functors $\mathcal{F}_m \times \mathcal{F}_n \to \mathcal{F}_{n+m}$ induced by the box sum axiom for the closed family.

*Remark* 2.3. Using the equivalence between $(B, f)$-structures and closed families one can define the stabilization of a $(B, f)$-structure as the $(\hat{B}, \hat{f})$ structure induced by the stabilization $\widehat{\mathcal{F}_B}$ of the closed family $\mathcal{F}_B$. For instance, this way one defines stably complex structures as tangential structures.

As hinted to in the article, the GHRR theorem that we state there for complex oriented cohomology theories has a natural generalization to cohomology theories oriented by more general cobordism spectra. By the results of the previous section in this appendix, this general setting is that of cohomology theories oriented by tangential structures. Let therefore, in the notation of the previous section, $(B, f)$ be a (stable) tangential structure and let $MB$ be the corresponding cobordism spectrum. For a ring spectrum $E$, we call $B$-orientation of $E$ a morphism of homotopy ring spectra $\rho\colon MB \to E$.

Let now $\rho_A, \rho_B : MB \to E$ be two $B$-orientations. The collection of multipliers for these two $B$-orientations is equivalent to the datum of a compatible family of morphisms $m_V : X \to GL_1E$ indexed by vector bundles $V \to X$ in the closed family corresponding to the tangential structure $(B, f)$. By compatibility with the pullbacks one only needs to consider the multipliers for the universal bundles $V_n \to B_n$. That is, the collection of all the $\{m_V\}_{V\in\mathcal{F}_B}$ is equivalent to the datum of a sequence of multipliers

$$m_k : B_k \to GL_1E$$

making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & B_{k-1} \arrow[r] \arrow[dr, "m_{k-1}"'] & B_k \arrow[r] \arrow[d, "m_k"] & B_{k+1} \arrow[r] \arrow[dl, "m_{k+1}"] & \cdots \\
& & GL_1E & &
\end{tikzcd}$$

commute. This is in turn the datum of a single map $B_\infty := \operatorname{colim} B_k \to GL_1E$ of homotopy abelian $\infty$-groups. Here the group structure on $B_\infty$ coming from the structure maps $\mu_{m,n} : B_n \times B_m \to B_{m+n}$ of the tangential structure, and the fact that the multipliers give rise to a map of homotopy abelian groups is the compatibility of multipliers with box sums of vector bundles. This in particular tells us that the space of $E$-orientations for $B$-bundles, that is, of ring spectra maps $MB \to E$ is a torsor over the group $\mathrm{Hom}_{\mathsf{Grp}}(B_\infty, GL_1E)$.

Now everything works just as in the case of complex orientations described in the article. Let $B$ be a tangential structure, $E$ an $E_\infty$-ring spectrum, $\rho_A, \rho_B : MB \to E$ be two $B$-orientations of $E$, $f : X \to Y$ be a $B$-oriented map, and $\sigma_f^A, \sigma_f^B : [X, E] \cong [X^{-T_f}[\dim_X - \dim Y], E]$ be the isomorphisms corresponding to the two orientations $\rho_A$ and $\rho_B$, respectively. Then we have the following commutative diagram giving a Grothendieck–Hirzebruch–Riemann–Roch theorem for $B$-oriented cohomology theories:

$$\begin{tikzcd}
{[X,E]} \arrow[rr, "\cdot m_{-T_f}"] \arrow[dr, "\sigma_f^A"'] \arrow[ddr, bend right, "f_*^A"'] & & {[X,E]} \arrow[dl, "\sigma_f^B"] \arrow[ddl, bend left, "f_*^B"] \\
& {[X^{-T_f}[\dim X - \dim Y], E]} \arrow[d, "\varphi_{PT}^*"] & \\
& {[Y[\dim X - \dim Y], E]} &
\end{tikzcd}$$

As for complex orientations, also for $B$-orientations one can use pushforward of orientations to get a version of the GHRR theorem closer to the classical statement. If $\psi : E \to F$ is a map of ring spectra, $\rho_E : MB \to E, \rho_F : MB \to F$ are two $B$-orientations, and $f : X \to Y$ is a $B$-oriented map. Then one has the commutative diagram

$$\begin{tikzcd}
{[X,E]} \arrow[r, "\psi_*"] \arrow[d, "f_*^{\rho_E}"'] & {[X,F]} \arrow[d, "f_*^{\psi_*\rho_E}"] \\
{[Y[\dim X - \dim Y], E]} \arrow[r, "\psi_*"] & {[Y[\dim X - \dim Y], F]}
\end{tikzcd}$$

yielding two orientations on $F$, namely $\rho_F$ and $\psi_*(\rho_E)$. This brings us back to the situation considered above and so, denoting by $\{m_V\}_{V\in\mathcal{F}_B}$ the multipliers relative to these two orientations, we have the commutative diagram

$$\begin{tikzcd}
{[X,E]} \arrow[r, "f_*^{\rho_E}"] \arrow[d, "m_{-T_f}\cdot\psi_*"'] & {[Y[\dim X - \dim Y], E]} \arrow[d, "\psi_*"] \\
{[X,F]} \arrow[r, "f_*^{\rho_F}"] & {[Y[\dim X - \dim Y], F],}
\end{tikzcd}$$

or, equivalently, the GHRR identity

$$\psi_* \left(f_*^{\rho_E}(a)\right) = f_*^{\rho_F} \left(\psi_*(a) \cdot m_{-T_f}\right).$$

For more information about tangential structures we refer the reader to [Koc96].

## 3. Localizations of spectra and the Chern-Dold character

In the article the main example of a morphism of ring spectra we considered was the Chern character $ch\colon KU \to HP_{\mathrm{ev}}\mathbb{Q}$. We will now explain how this is actually a particular instance of a general construction of Chern-Dold characters.

Let $A$ be a spectrum. One says that a spectrum $Y$ is $A$-acyclic if $Y \otimes A \cong 0$. A map of spectra $f : X \to Z$ is called $A$-equivalence if $f \otimes A : X \otimes A \to Z \otimes A$ is an equivalence. Finally, one says that a spectrum $X$ is $A$-local if the only morphism $Y \to X$ from an $A$-acyclic spectrum $Y$ to $X$ is the zero morphism (up to homotopy, i.e., if $\mathsf{Sp}(Y, X)$ is contractible). For every spectrum $X$ there exists (and it is unique up to equivalence) an $A$-local spectrum $L_A X$ together with an $A$-equivalence $X \to L_A X$. The spectrum $L_A$ is called the $A$-localization of $X$. The construction of $A$-localizations is functorial, so it gives an endofunctor $L_A$ of $\mathsf{Sp}$.

The localization functor $L_A$ preserves finite direct sums but it does not necessarily preserve arbitrary direct sums. When this happens, it has a particularly simple description; namely, if $L_A$ preserves arbitrary direct sums then one has a natural isomorphisms

$$L_A X \cong X \otimes L_A \mathbb{S}.$$

In other words, if $L_A$ preserves arbitrary direct sums then $A$-localization is the tensor product with the $A$-localization of the sphere spectrum. Since the tensor product in $\mathsf{Sp}$ is given by the smash product of spectra, one calls such a localization functor a *smashing localization*

If $A = HG$, with $G = \mathbb{Z}/p\mathbb{Z}$, $p$ a prime, or $G = \mathbb{Q}$ then $A$-localization (called $G$-localization in this case) is smashing and so is given by the tensor product with $L_{HG}\mathbb{S}$. The spectrum $L_{HG}\mathbb{S}$ is called the Moore spectrum of $G$ and it is usually denoted by the symbol $SG$. It can be characterised as the unique connective spectrum with $\pi_0 SG = G$ and $\pi_{>0}(SG \otimes H\mathbb{Z}) = 0$.

When $G = \mathbb{Q}$ one has an equivalence $S\mathbb{Q} \cong H\mathbb{Q}$, and so $\mathbb{Q}$-localization is just the tensor product with $H\mathbb{Q}$. If now $E$ is the ring spectrum associated with a given multiplicative cohomology theory, we can consider the $\mathbb{Q}$-localization of $E$ to get a map $\lambda_{E,\mathbb{Q}} \colon E \to L_{H\mathbb{Q}} E = E \otimes H\mathbb{Q}$. Now, one uses that a tensor product with an Eilenberg-MacLane spectrum splits as a direct sum of Eilenberg-MacLane spectra:

$$E \otimes H\mathbb{Q} \cong \bigoplus_{i \in \mathbb{Z}} H(\pi_i^{\mathsf{Sp}} E) \otimes \mathbb{Q}[i]$$

to write the localization map as

$$chd_E \colon E \to \bigoplus_{i \in \mathbb{Z}} H(\pi_i^{\mathsf{Sp}} E) \otimes \mathbb{Q}[i].$$

Written this way, the localization map is called the Chern-Dold character. Compatibility of localization with tensor product implies that the Chern-Dold character is a map of homotopy ring spectra, where the ring spectrum structure on the right hand side is induced by the natural maps

$$\pi_i^{\mathsf{Sp}} E \times \pi_j^{\mathsf{Sp}} E = [\mathbb{S}, E[-i]] \times [\mathbb{S}, E[-j]] \to [\mathbb{S}, E[-i-j]] = \pi_{i+j}^{\mathsf{Sp}}(E).$$

As a consequence, for every (nice) topological space $X$ we have an induced ring homomorphism, called again the Chern-Dold character,

$$chd_E\colon E^0(X) \xrightarrow{\lambda_{E,\mathbb{Q};*}} \bigoplus_{i\in\mathbb{Z}} H^i(X; \pi_i^{\mathsf{Sp}} E \otimes \mathbb{Q}),$$

and generalizing the Chern character for complex $K$-theory

$$ch\colon K^0(X) \to H^{\mathrm{even}}(X;\mathbb{Q}).$$

Notice that from the point of view of the cohomology theories represented by these spectra we have

$$chd : E^\bullet(Y) \xrightarrow{L_{H\mathbb{Q}*}} L_{H\mathbb{Q}} E^\bullet(Y) \cong H^\bullet(X, \pi_\bullet(E) \otimes \mathbb{Q}).$$

For details see [Hil71].

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## A very short note on the (rational) graded Hori map

Abstract. The graded Hori map has been recently introduced by Han-Mathai in the context of T-duality as a $\mathbb{Z}$-graded transform whose homogeneous components are the Hori-Fourier transforms in twisted cohomology associated with integral multiples of a basic pair of T-dual closed 3-forms. We show how in the rational homotopy theory approximation of T-duality, such a map is naturally realized as a pull-iso-push transform, where the isomorphism part corresponds to the canonical equivalence between the left and the right gerbes associated with a T-duality configuration.

### 1. Foreword

The graded Hori map has been recently introduced in [HM20], by assembling together the $\mathbb{Z}$-family of Hori maps associated with a certain $\mathbb{Z}$-family of T-duality configuration data naturally associated to a single T-duality configuration. This may at first sight appear as a rather ad hoc construction. The aim of this note is to show how, on the contrary, the graded Hori map as a whole naturally emerges from the geometry associated with a T-duality configuration. One only needs to look at a step higher with respect to the T-dual bundles: the graded Hori map is a manifestation of a canonical equivalence between the left and the right gerbes associated with a T-duality configuration. More precisely, we show that, in the rational homotopy theory approximation of T-duality, such a map is naturally realized as a pull-iso-push transform, where the isomorphism part corresponds to the left gerbe/right gerbe canonical equivalence.

We will construct this pull-iso-push transform using only purely algebraic constructions related to the category $\mathsf{DGCA}$ of differential graded commutative algebras (DGCAs) over a characteristic zero field $\mathbb{K}$, which can be assumed to be the field $\mathbb{Q}$ of rational numbers. In particular, we will heavily use the language of extensions of DGCAs associated with DGCA cocycles. The reader familiar with rational homotopy theory will immediately recognize every step in the construction we are going to present as a translation of phenomena appearing in the rational homotopy theory approximation of T-duality. We point the unfamiliar reader to [FSS18b] for an introduction very close to the spirit of this note. We also borrow from [FSS18a] the rational homotopy theory description of the equivalence of the gerbes associated to a T-duality configuration. See [BS05] for the topological origin of this equivalence. Here we choose to present the construction in purely algebraic terms, leaving to the reader the job of connecting to rational homotopy theory.

The note is organized as follows. First we review topological $T$-duality in rational homotopy theory, in particular, in Section 2 we recall a few basic constructions on extensions of DGCAs and define the DGCA (co)classifying rational $T$-duality configurations, and in Section 3 we recall the definition of the two isomorphic rational gerbes associated with a rational $T$-duality configuration, whose isomorphism will be the "iso" part in the "pull-iso-push" transform.

After this review, in Section 4 we define the graded Hori map $\mathcal{T}_{L\to R}$ associated with these data and extend it to Laurent series. In Section 5 we show how, when the base field is the field $\mathbb{C}$ of complex numbers, this allows one to describe the graded Hori map as an operator on rings of meromorphic functions with a single pole at the origin taking values in a DGCA $A_0$ endowed with a rational T-duality configuration. It turns out that in this translation the graded Hori map becomes the antidiagonal matrix

$$\begin{pmatrix} 0 & 1 \\ -q\frac{d}{dq} & 0 \end{pmatrix}$$

where $q$ is the complex coordinate on $\mathbb{C}$. Finally, in Section 6 we show how one can further extend coefficients to $A_0$-valued index 0 Jacobi forms in the two variables $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, by means of their $q$-expansion, where $q = e^{2\pi i z}$. This way we recover the original definition of the graded Hori map by Han-Mathai, as well as its main properties. In particular, one identifies the graded Hori map on Jacobi forms with the antidiagonal matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{2\pi i}\frac{\partial}{\partial z} & 0 \end{pmatrix},$$

and therefore the composition of two graded Hori transforms as the operator $-\frac{1}{2\pi i}\frac{\partial}{\partial z}$ on the ring of $A_0$-valued index 0 Jacobi-forms [HM20, Theorem 2.2].

## 2. Cocycles and extensions of DGCAs

We start with a (non-negatively graded) differential graded commutative algebra $(A,d)$ over the field $\mathbb{K}$ and with a 2-cocycle, i.e., a closed homogeneous element of degree 2, $t_2 \in A$. We can extend our base DGCA $A$ in such a way to trivialize the 2-cocycle $t_2$ by adding a formal generator $e_1$ of degree 1 and declaring our extension to be

$$(A,d) \xhookrightarrow{\ \iota\ } A_{\{t_2\}} := (A[e_1], de_1 = t_2),$$

where the differential of $A_{\{t_2\}}$ coincides with the differential $d$ on the subalgebra $A$.

The choice of a 2-cocycle for the DGCA $A$ is the same datum as a DGCA map from the polynomial DGCA $(\mathbb{K}[x_2], 0)$ to $A$, where $(\mathbb{K}[x_2], 0)$ is the polynomial algebra over $\mathbb{K}$ on a single degree 2 generator $x_2$, endowed with the trivial differential. This in turn means regarding $A$ as an object under $(\mathbb{K}[x_2], 0)$, a point of view that will be useful later.

More generally, the datum of a DGCA map from $(\mathbb{K}[x_n], 0)$ to $A$, where now $x_n$ is a degree $n$ variable, is the same as that of an $n$-cocycle in $A$ and, again, given such a cocycle $t_n \in A$ it is possible to extend $A$ to trivialize $t_n$ by

$$(A, d) \xhookrightarrow{\iota} A_{\{t_n\}} := (A[e_{n-1}], de_{n-1} := t_n).$$

The construction of $A_{\{t_n\}}$ out of the pair $(A, t_n)$ is universal: $A_{\{t_n\}}$ together with the embedding of the sub-DGCA $A$ is the homotopy cofiber of $t_n$, i.e., the homotopy pushout of the diagram:

$$\begin{tikzcd} \mathbb{K}[x_n] \arrow[r, "\psi_{t_n}"] \arrow[d] & (A, d) \\ 0 & \end{tikzcd}$$

of DGCAs, where $\psi_{t_n}$ is the unique DGCA morphism with $\psi(x_n) = t_n$, in the projective model structure on non-negatively graded DGCAs, see, e.g., [BG76]. Indeed, in order to compute a model for this cofiber one has to replace the vertical map by a cofibration followed by a weak equivalence, and the easiest way of doing this is to consider

$$\mathbb{K}[x_n] \hookrightarrow (\mathbb{K}[x_n, e_{n-1}], de_{n-1} = x_n) \cong 0,$$

and then compute the ordinary pushout of the diagram

$$\begin{tikzcd} \mathbb{K}[x_n] \arrow[r, "\psi_{t_n}"] \arrow[d] & (A, d) \\ (\mathbb{K}[x_n, e_{n-1}], de_{n-1} = x_n) & \end{tikzcd}$$

to obtain

$$\begin{tikzcd} \mathbb{K}[x_n] \arrow[r, "\psi_{t_n}"] \arrow[d] & (A, d) \arrow[d] \\ (\mathbb{K}[x_n, e_{n-1}], de_{n-1} = x_n) \arrow[r] & (P, d_P), \end{tikzcd}$$

with

$$\begin{aligned} (P, d_P) &= (A[e_{n-1}], d_P a = da \text{ for } a \in A, d_P e_{n-1} = \psi_{t_n}(x_n)) \\ &= (A[e_{n-1}], de_{n-1} = t_n) = A_{\{t_n\}}. \end{aligned}$$

Universality implies in particular that the construction $(A, t_n) \rightsquigarrow A_{\{t_n\}}$ is natural, a fact that can also be easily checked directly: if $f\colon (A, t_n) \to (B, s_n)$ is a morphism of DGCAs endowed with $n$-cocycles, i.e., if $f$ is a morphism of DGCAs, $f\colon A \to B$, such that $f(t_n) = s_n$, then we get a morphism of DGCAs $\hat{f}\colon A_{\{t_n\}} \to B_{\{s_n\}}$ by setting $\hat{f}(a) = f(a)$ for any $a \in A$ and $\hat{f}(e_{n-1;A}) = e_{n-1;B}$. This is manifestly compatible with compositions of morphisms of DGCAs endowed with $n$-cocycles.

*Remark* 2.1. If $n$ is even, every degree $k$ element $a_k$ in $A_{\{t_n\}}$ can be uniquely written as $a_k = \alpha_k + e_{n-1}\beta_{k-n+1}$, for some degree $k$ element $\alpha_k$ and some degree $k-n+1$ element $\beta_{k-n+1}$ in $A$. The map

$$\begin{aligned} \pi\colon A_{\{t_n\}} &\to A[-n+1] \\ \alpha_k + e_{n-1}\beta_{k-n+1} &\mapsto \beta_{k-n+1} \end{aligned}$$

is a map of chain complexes. Namely, we have

$$\begin{aligned} d_{[-n+1]}(\pi(a_k) &= d_{[-n+1]}(\pi(\alpha_k + e_{n-1}\beta_{k-n+1})) \\ &= d_{[-n+1]}\beta_{k-n+1} \\ &= (-1)^{(n-1)}d\beta_{k-n+1} \end{aligned}$$

and

$$\begin{aligned} \pi(da_k) &= \pi(d(\alpha_k + e_{n-1}\beta_{k-n+1})) \\ &= \pi(d\alpha_k + t_n\beta_{k-n+1} + (-1)^{n-1}d\beta_{k-n+1}) \\ &= (-1)^{n-1}d\beta_{k-n+1}. \end{aligned}$$

Of course, $\pi$ is not a map of DGCAs (the shifted complex $A[-n+1]$ does not even have a natural DGCA structure). But it is a map of right DG-$A$-modules: if $\gamma_l$ is a degree $l$ element in $A$, then

$$\pi(a_k\gamma_l) = \pi((\alpha_k + t_n\beta_{k-n})\gamma_l) = \pi((\alpha_k\gamma_l) + t_n(\beta_{k-n}\gamma_l) = \beta_{k-n}\gamma_l = \pi(a_k)\gamma_l.$$

As a side remark, by thinking of $\iota\colon A \to A_{\{t_n\}}$ as a pullback $p^*$ and of $\pi\colon A_{\{t_n\}} \to A[-n+1]$ as the pushforward $p_*$, the above identity is the projection formula:

$$p_*(a_k p^*(\gamma_l)) = p_*(a_k)\gamma_l.$$

Finally, the map of right DG-$A$-modules $\pi\colon A_{\{t_n\}} \to A[-n+1]$ has an evident section

$$e_{n-1} \cdot -\colon A[-n+1] \to A_{\{t_n\}}$$

given by the left multiplication by $e_{n-1}$.

An example of the construction $(A, t_n) \rightsquigarrow A_{\{t_n\}}$ we will be interested in is the following. Consider the polynomial algebra

$$\mathbb{K}[x_{2L}, x_{2R}] \cong \mathbb{K}[x_{2L}] \otimes \mathbb{K}[x_{2R}]$$

on two degree 2 generators $x_{2L}$ and $x_{2R}$, endowed with the trivial differential.[1] Then the element $x_{2L}x_{2R}$ is a 4-cocycle and so defines a DGCA map

$$\begin{aligned} \mathbb{K}[t_4] &\to \mathbb{K}[x_{2L}, x_{2R}] \\ t_4 &\mapsto x_{2L}x_{2R} \end{aligned}$$

The associated extension is the DGCA

$$\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} = (\mathbb{K}[x_{2L}, x_{2R}, y_3], dx_{2L} = dx_{2R} = 0, dy_3 = x_{2L}x_{2R})$$

Notice that $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ carries two distinguished 2-cocycles $x_{2L}$ and $x_{2R}$ and that $\sigma\colon x_{2L} \leftrightarrow x_{2R}$ is a DGCA automorphism of $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ exchanging the two cocycles. We denote by $p_L, p_R\colon \mathbb{K}[x_2] \to \mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ the two maps corresponding to the cocycles $x_{2L}, x_{2R}$, respectively.

## 3. Two equivalent rational gerbes

In order to get the DGCA construction corresponding to the rational homotopy description of the pull-iso-push transform between gerbes associated with a T-duality configuration, we consider a DGCA $A$ together with a map $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \xrightarrow{f} A$. As we noticed above, the source of $f$ has two distinct 2-cocycles corresponding to maps $p_L, p_R : \mathbb{K}[x_2] \to \mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ sending the generator $x_2$ in $x_{2L}$ and in $x_{2R}$, respectively. Composing with the map $f$ we therefore get maps $f_L, f_R : \mathbb{K}[x_2] \to A$, corresponding to two distinct 2-cocycles in $A$, and we end up with following commutative diagram of DGCAs:

$$\begin{tikzcd}
& & \mathbb{K}[x_2] \arrow[dll, bend right, "f_L"'] \arrow[drr, bend left, "f_R"] \arrow[ddl, "p_L"'] \arrow[ddr, "p_R"] & & \\
A & & & & A \\
& \mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \arrow[ul, bend left, "f"] \arrow[rr, leftrightarrow, "\sigma"] & & \mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

The previous diagram shows that the map $f$ can be read in two different ways as a map in the undercategory ${}^{\mathbb{K}[x_2]/}\mathsf{DGCA}$ of DGCAs endowed with a distinguished 2-cocycle, i.e., with a distinguished morphism from $\mathbb{K}[x_2]$. In

[1] Here and below, all tensor products are over $\mathbb{K}$.

particular, we have that $f$ is both a map between $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}$ and $A$ decorated with their left 2-cocycles

$$(\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_L) \xrightarrow{f} (A, f_L),$$

and with their right 2-cocycles

$$(\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_R) \xrightarrow{f} (A, f_R).$$

This will be crucial in order to define the equivalence between the algebraic structures corresponding to the left and right gerbes of topological T-duality. We begin with the following, which is a particular case of the "hofib/cyc adjunction" of [FSS18a; FSS18b], and whose proof in this specific case we give for the sake of completeness.

**Proposition 3.1.** *Let $(A, t_2)$ be a DGCA with a distinguished 2-cocycle $t_2$. Then the association*

$$\mathrm{Hom}_{\mathbb{K}[x_2]/\mathsf{DGCA}}\left((\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_L), (A, \psi_{t_2})\right) \to \mathrm{Hom}_{\mathsf{DGCA}}\left(\mathbb{K}[x_3], A_{\{t_2\}}\right)$$

$$\varphi \longmapsto \tilde{\varphi}$$

*where $\tilde{\varphi}$ is defined by*

$$\tilde{\varphi} \colon x_3 \mapsto \varphi(y_3) - e_1\varphi(x_{2R}),$$

*is a natural bijection. Clearly, everything identically works if we exchange $p_L$ with $p_R$ and $x_{2R}$ with $x_{2L}$.*

*Proof.* We begin by showing that $\tilde{\varphi}(x_3)$ is a 3-cocycle. If

$$\varphi \colon (\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_L) \to (A, \psi_{t_2})$$

is a map in the undercategory ${}^{\mathbb{K}[x_2]/}\mathsf{DGCA}$ , then

$$\varphi(x_{2L}) = (\varphi \circ p_L)(x_2) = \psi_{t_2}(x_2) = t_2.$$

Therefore,

$$\begin{aligned} d(\tilde{\varphi}(x_3)) &= d(\varphi(y_3) - e_1\varphi(x_{2R})) = \\ &= \varphi(x_{2L})\varphi(x_{2R}) - t_2\varphi(x_{2R}) + e_1\varphi(dx_{2R}) = \\ &= 0. \end{aligned}$$

This shows that the map $\varphi \mapsto \tilde{\varphi}$ actually takes values in $\mathrm{Hom}_{\mathsf{DGCA}}(\mathbb{K}[x_3], A_{\{t_2\}})$. Now we define a map in the opposite direction. For a DGCA morphism

$\psi\colon \mathbb{K}[x_3] \to A_{\{t_2\}}$, let $t_3$ be the 3-cocycle $t_3 = \psi(x_3)$ in $A_{\{t_2\}}$. The 3-cocycle $t_3$ can be uniquely written as $t_3 = a_3 - e_1 b_2$ with $a_3, b_2 \in A$. The association

$$y_3 \mapsto a_3, \quad x_{2R} \mapsto b_2, \quad x_{2L} \mapsto t_2$$

defines a map $\tilde{\psi}\colon (\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_L) \to (A, t_2)$ in ${}^{\mathbb{K}[x_2]/}\mathsf{DGCA}$. It is immediate to check that $\tilde{\tilde{\varphi}} = \varphi$ and $\tilde{\tilde{\psi}} = \psi$, so the two maps are inverse each other. $\square$

Now, let us go back to our DGCA $A$ endowed with a DGCA morphism $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \xrightarrow{f} A$. To avoid confusion, let us denote by $e_{1L}$ and $e_{1R}$ the additional degree 1 generators in the extensions $A_L := A_{f(x_{2L})}$ and $A_R := A_{f(x_{2R})}$ of $A$, respectively. By the above proposition, and looking at $f$ both as a morphism from $(\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_L)$ to $(A, f_L)$ and as a morphism from $(\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}}, p_R)$ to $(A, f_R)$, we end up with distinguished 3-cocycles

$$f(y_3) - e_{1L} f(x_{2R}) \in A_L, \quad f(y_3) - e_{1R} f(x_{2L}) \in A_R$$

and again, we can define extensions of $A_L$ and $A_R$ by trivializing the above 3-cocycles. We define the *left rational gerbe* $\mathcal{G}_L$ and the *right rational gerbe* $\mathcal{G}_R$ of the rational $T$-configuration $f$ as the DGCAs

$$\begin{aligned} \mathcal{G}_L &:= A_{L\{f(y_3) - e_{1L} f(x_{2R})\}} \\ \mathcal{G}_R &:= A_{R\{f(y_3) - e_{1R} f(x_{2L})\}}. \end{aligned}$$

Again, to avoid confusion, we denote by $\xi_{2L}$ and $\xi_{2R}$ the additional degree 2 generators of $\mathcal{G}_L$ and $\mathcal{G}_R$ as extensions of $A_L$ and of $A_R$, respectively. Both $\mathcal{G}_L$ and $\mathcal{G}_R$ are extensions of $A$ (since both $A_L$ and $A_R$ were extensions), and this tower of extensions of $A$ can be depicted in the diagram

$$\begin{tikzcd}
\mathcal{G}_L & & & & \mathcal{G}_R \\
& A_L \arrow[ul, "i_L"'] & & A_R \arrow[ur, "i_L"'] & \\
& & A \arrow[ul, "\iota_L"] \arrow[ur, "\iota_R"'] & &
\end{tikzcd}$$

We can add to this diagram the DGCA $A_{LR} := A_L \otimes_A A_R$, i.e., the DGCA $(A[e_{1L}, e_{1R}], de_{1L} = f(x_{2L}), de_{1R} = f(x_{2R}))$, obtaining the diagram

$$\begin{tikzcd}
\mathcal{G}_L & & A_{LR} & & \mathcal{G}_R \\
& A_L \arrow[ul, "i_L"] \arrow[ur, "\iota_R"'] & & A_R \arrow[ul, "\iota_L"] \arrow[ur, "i_L"'] & \\
& & A \arrow[ul, "\iota_L"] \arrow[ur, "\iota_R"'] & &
\end{tikzcd}$$

where the central square commutes. As a matter of notation, in the above diagram we are writing $\iota_L$ (resp. $\iota_R$) wherever the extension is made by means of the 1-form $e_{1L}$ (resp. $e_{1R}$) and $i_L$ (resp. $i_R$) whenever the extension is made by means of the 2-form $\xi_{2L}$ (resp. $\xi_{2R}$).

We can extend $\mathcal{G}_L$ and $\mathcal{G}_R$ by computing the obvious (homotopy) pushouts to get the further extensions

$$\begin{tikzcd}
& \mathcal{G}_{L\{f(x_{2R})\}} & & \mathcal{G}_{R\{f(x_{2L})\}} & \\
\mathcal{G}_L \arrow[ur, "\iota_R"'] & & A_{LR} \arrow[ul, "i_L"] \arrow[ur, "i_R"'] & & \mathcal{G}_R \arrow[ul, "\iota_L"] \\
& A_L \arrow[ul, "i_L"] \arrow[ur, "\iota_R"'] & & A_R \arrow[ul, "\iota_L"] \arrow[ur, "i_R"'] & \\
& & A \arrow[ul, "\iota_L"] \arrow[ur, "\iota_R"'] & &
\end{tikzcd}$$

Explicitly,

$$\mathcal{G}_{L\{f(x_{2R})\}} = \left( A[e_{1L}, e_{1R}, \xi_{2L}], \begin{cases} de_{1L} = f(x_{2L}) \\ de_{1R} = f(x_{2R}) \\ d\xi_{2L} = f(y_3) - e_{1L} f(x_{2R}) \end{cases} \right)$$

and

$$\mathcal{G}_{R\{f(x_{2L})\}} = \left( A[e_{1L}, e_{1R}, \xi_{2R}], \begin{cases} de_{1L} = f(x_{2L}) \\ de_{1R} = f(x_{2R}) \\ d\xi_{2R} = f(y_3) - e_{1R} f(x_{2L}) \end{cases} \right).$$

We can now make explicit the iso part of our pull-iso-push transform.

**Proposition 3.2.** *The DGCAs $\mathcal{G}_{L\{f(x_{2R})\}}$ and $\mathcal{G}_{R\{f(x_{2L})\}}$ are isomorphic via an isomorphism*

$$\mathcal{G}_{L\{f(x_{2R})\}} \xrightarrow{\nu} \mathcal{G}_{R\{f(x_{2L})\}}$$

*that is the identity on $A_{LR}$ and acts as*

$$\xi_{2L} \mapsto \xi_{2R} + e_{1L}e_{1R},$$

*on the degree two generator. The inverse isomorphism is, clearly, $\nu^{-1}\colon \xi_{2R} \mapsto \xi_{2L} - e_{1L}e_{1R}$.*

*Proof.* The map $\nu$ is a map of graded commutative algebras, and it is of course a bijection since an explicit inverse is given by the map of graded commutative algebras $\nu^{-1}$ which is the identity on $A_{LR}$ and sending $\xi_{2R}$ to $\xi_{2L} - e_{1L}e_{1R}$. To see that $\nu$ is a map of DGCAs we need to show that it is a map of chain complexes. This can be checked on the generators of the polynomial algebra $\mathcal{G}_{L\{f(x_{2R})\}}$, so we only need to compute $d\nu(\xi_{2L})$. We have

$$\begin{aligned}
d\nu(\xi_{2L}) &= d(\xi_{2R} + e_{1L}e_{1R}) = \\
&= f(y_3) - e_{1R}f(x_{2L}) + f(x_{2L})e_{1R} - e_{1L}f(x_{2R}) = \\
&= f(y_3) - e_{1L}f(x_{2R}) = \\
&= \nu(f(y_3) - e_{1L}f(x_{2R})) = \\
&= \nu(d\xi_{2L}),
\end{aligned}$$

where we used that $f(y_3) - e_{1L}f(x_{2R}) \in A_{LR}$ and $\nu$ is the identity on $A_{LR}$. $\square$

The isomorphisms $\nu$ and $\nu^{-1}$ complete our previous diagram to the commutative diagram

$$\begin{tikzcd}
 & \mathcal{G}_{L\{f(x_{2R})\}} \arrow[rr, bend left, "\nu"] & & \mathcal{G}_{R\{f(x_{2L})\}} \arrow[ll, bend left, "\nu^{-1}"] & \\
\mathcal{G}_L \arrow[ur, "\iota_R"] & & A_{LR} \arrow[ul, "i_L"'] \arrow[ur, "i_R"] & & \mathcal{G}_R \arrow[ul, "\iota_L"'] \\
 & A_L \arrow[ul, "i_L"] \arrow[ur, "\iota_R"] & & A_R \arrow[ul, "\iota_L"'] \arrow[ur, "i_R"'] & \\
 & & A \arrow[ul, "\iota_L"] \arrow[ur, "\iota_R"'] & &
\end{tikzcd}$$

## 4. The graded Hori map from rational equivalences of gerbes

All the maps and the DGCAs appearing in the upper part of our diagram

$$\begin{tikzcd}
 & \mathcal{G}_{L\{f(x_{2R})\}} \arrow[r, "\nu"] & \mathcal{G}_{R\{f(x_{2L})\}} & \\
\mathcal{G}_L \arrow[ur, "\iota_R"] & & & \mathcal{G}_R \arrow[ul, "\iota_L"']
\end{tikzcd}$$

can be extended to the rings of (bounded above) formal Laurent series in the degree 2 generators. For instance, as a graded commutative algebra the DGCA $\mathcal{G}_L$ is the polynomial algebra $A_L[\xi_L]$ over $A_L$ and so embeds as a subalgebra into the ring of Laurent series

$$\widehat{\mathcal{G}_L} := A_L[[\xi_{2L}^{-1}]][\xi_{2L}] =: A_L[[\xi_{2L}^{-1}, \xi_{2L}].$$

The ring $\hat{\mathcal{G}}_L$ has moreover a natural DGCA structure, by setting

$$d\xi_{2L}^{-1} = -\xi_{2L}^{-2}\left(f(y_3) - e_{1L}f(x_{2R})\right),$$

making $\mathcal{G}_L \hookrightarrow \hat{\mathcal{G}}_L$ an inclusion of DGCAs. One similarly extends the other DGCAs $\mathcal{G}_R, \mathcal{G}_{L\{f(x_{2R})\}}$ and $\mathcal{G}_{R\{f(x_{2L})\}}$ appearing in the above diagram.

The maps $\iota_R, \iota_L$ obviously extend to the rings of Laurent series. We denote by $\hat{\iota}_L$ and $\hat{\iota}_R$ these extensions. We notice that $\nu$ extends too, we only need to be careful in defining the extension $\hat{\nu}$. As $e_{1L}e_{1R}$ is nilpotent, this is done by using the formal power series inverse for $1-\eta$, i.e., by declaring that the action of $\hat{\nu}$ on $\xi_{2L}^{-1}$ is given by

$$\hat{\nu}(\xi_{2L}^{-1}) = (\xi_{2R} + e_{1L}e_{1R})^{-1} = \sum_{i \geqslant 0}(-1)^i(e_{1L}e_{1R})^i\xi_{2R}^{-i-1} = \xi_{2R}^{-1} - e_{1L}e_{1R}\xi_{2R}^{-2},$$

where we used that $(e_{1L}e_{1R})^2 = 0$. One easily checks that $\hat{\nu}$ is indeed a DGCA morphism: it is compatible with the relation $\xi_{2L}^{-1}\xi_{2L} = 1$ as

$$\hat{\nu}(\xi_{2L}^{-1})\hat{\nu}(\xi_{2L}) = (\xi_{2R}^{-1} - e_{1L}e_{1R}\xi_{2R}^{-2})(\xi_{2R} + e_{1L}e_{1R}) = 1$$

and with the differential as

$$\begin{aligned}
\hat{\nu}(d\xi_{2L}^{-1}) &= \hat{\nu}(-\xi_{2L}^{-2}\left(f(y_3) - e_{1L}f(x_{2R})\right)) \\
&= -(\xi_{2R}^{-1} - e_{1L}e_{1R}\xi_{2R}^{-2})^2(f(y_3) - e_{1L}f(x_{2R})) \\
&= -(\xi_{2R}^{-2} - 2e_{1L}e_{1R}\xi_{2R}^{-3})(f(y_3) - e_{1L}f(x_{2R})) \\
&= -\xi_{2R}^{-2}f(y_3) + 2e_{1L}e_{1R}\xi_{2R}^{-3}f(y_3) + \xi_{2R}^{-2}e_{1L}f(x_{2R})
\end{aligned}$$

and

$$\begin{aligned}
&d\hat{\nu}(\xi_{2L}^{-1}) = d(\xi_{2R}^{-1} - e_{1L}e_{1R}\xi_{2R}^{-2}) \\
&= -\xi_{2R}^{-2}\left(f(y_3) - e_{1R}f(x_{2L})\right) - (de_{1L}e_{1R})\xi_{2R}^{-2} + 2e_{1L}e_{1R}\xi_{2R}^{-3}\left(f(y_3) - e_{1R}f(x_{2L})\right) \\
&= -\xi_{2R}^{-2}\left(f(y_3) - e_{1R}f(x_{2L})\right) - (f(x_{2L})e_{1R} - e_{1L}f(x_{2R}))\xi_{2R}^{-2} + 2e_{1L}e_{1R}\xi_{2R}^{-3}f(y_3) \\
&= -\xi_{2R}^{-2}f(y_3) + e_{1L}f(x_{2R})\xi_{2R}^{-2} + 2e_{1L}e_{1R}\xi_{2R}^{-3}f(y_3)
\end{aligned}$$

As $f(x_{2L})$ is an even cocycle, by Remark 2.1 we have a projection

$$\pi\colon \mathcal{G}_{R\{f(x_{2L})\}} \to \mathcal{G}_R[-1]$$

mapping $\alpha_k + e_{1L}\beta_{k-1}$ to $\beta_{k-1}$, which is a morphism of right DG-$\mathcal{G}_R$-modules. Also the projection $\pi\colon : \mathcal{G}_{R\{f(x_{2L})\}} \to \mathcal{G}_R[-1]$ naturally extends to formal Laurent series modules to a map

$$\hat{\pi}\colon \widehat{\mathcal{G}_{R\{f(x_{2L})\}}} \to \widehat{\mathcal{G}_R}[-1]$$

and so it is possible to build a pull-iso-push transform $\mathcal{T}_{L\to R}$ as the composition

$$\begin{tikzcd}
& \widehat{\mathcal{G}_{L\{f(x_{2R})\}}} \arrow[r, "\hat{\nu}"] & \widehat{\mathcal{G}_{R\{f(x_{2L})\}}} \arrow[dr, "\hat{\pi}"] & \\
\widehat{\mathcal{G}_L} \arrow[ur, "\hat{\iota}_R"'] \arrow[rrr, "\mathcal{T}_{L\to R}"] & & & \widehat{\mathcal{G}_R}[-1].
\end{tikzcd}$$

The transform

$$\mathcal{T}_{L\to R}\colon \widehat{\mathcal{G}_L} \to \widehat{\mathcal{G}_R}[-1]$$

associated to the initial rational $T$-duality configuration $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \xrightarrow{f} A$ is seen to coincide with the graded Hori map introduced by Han and Mathai in [HM20]. Namely, the action of $\hat{\nu}$ on a generic degree $k$ element

$$\omega_k = \sum_{n\in\mathbb{Z}} (\alpha_{2n+k} + e_{1L}\beta_{2n+k-1} + e_{1R}\gamma_{2n+k-1} + e_{1L}e_{1R}\delta_{2n+k-2})\xi_{2L}^{-n}$$

in $\widehat{\mathcal{G}_{L\{f(x_{2R})\}}}$ is given by

$$\begin{aligned}
&\hat{\nu}(\omega_k) = \sum_{n\in\mathbb{Z}} (\alpha_{2n+k} + e_{1L}\beta_{2n+k-1} + e_{1R}\gamma_{2n+k-1} + e_{1L}e_{1R}\delta_{2n+k-2})\hat{\nu}(\xi_{2L}^{-n}) = \\
&= \sum_{n\in\mathbb{Z}} (\alpha_{2n+k} + e_{1L}\beta_{2n+k-1} + e_{1R}\gamma_{2n+k-1} + e_{1L}e_{1R}\delta_{2n+k-2})(\xi_{2R}^{-n} - ne_{1L}e_{1R}\xi_{2R}^{-n-1}) = \\
&= \sum_{n\in\mathbb{Z}} (\alpha_{2n+k} + e_{1L}\beta_{2n+k-1} + e_{1R}\gamma_{2n+k-1} + e_{1L}e_{1R}\delta_{2n+k-2})\xi_{2R}^{-n} - ne_{1L}e_{1R}\alpha_{2n+k}\xi_{2R}^{-n-1} = \\
&= \sum_{n\in\mathbb{Z}} (\alpha_{2n+k} + e_{1L}\beta_{2n+k-1} + e_{1R}\gamma_{2n+k-1} + e_{1L}e_{1R}(\delta_{2n+k-2} - (n-1)a_{2n+k-2}))\xi_{2R}^{-n},
\end{aligned}$$

hence the action of $\hat{\nu}$ on the coefficients of a generic degree $k$ Laurent series in $\widehat{\mathcal{G}_{L\{f(x_{2R})\}}}$ is given by

$$\begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ \gamma_{2n+k-1} \\ \delta_{2n+k-2} \end{pmatrix} \overset{\hat{\nu}}{\longmapsto} \begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ \gamma_{2n+k-1} \\ \delta_{2n+k-2} - (n-1)\alpha_{2n+k-2} \end{pmatrix}$$

The inclusion $\widehat{\mathcal{G}_L} \xrightarrow{\hat{\iota_R}} \widehat{\mathcal{G}_{L\{f(x_{2R})\}}}$ and the projection $\hat{\pi}\colon \widehat{\mathcal{G}_{R\{f(x_{2L})\}}} \to \widehat{\mathcal{G}_R}[-1]$ can be displayed in a similar way

$$\begin{pmatrix} \alpha_{2n+k} \\ b_{2n+k-1} \end{pmatrix} \overset{\hat{\iota_R}}{\mapsto} \begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ 0 \\ 0 \end{pmatrix} \qquad ; \qquad \begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ \gamma_{2n+k-1} \\ \delta_{2n+k-2} \end{pmatrix} \overset{\hat{\pi}}{\mapsto} \begin{pmatrix} \beta_{2n+k-1} \\ \delta_{2n+k-2} \end{pmatrix}$$

The left-to-right transform $\mathcal{T}_{L\to R}$ therefore act on the coefficients of a generic degree $k$ element $\sum_{n\in\mathbb{Z}}(\alpha_{2n+k} + e_{1L}\beta_{2n+k-1})\xi_{2L}^{-n} \in \widehat{\mathcal{G}_L}$ as

$$\begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \end{pmatrix} \overset{\hat{\iota_R}}{\mapsto} \begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ 0 \\ 0 \end{pmatrix} \overset{\hat{\nu}}{\mapsto} \begin{pmatrix} \alpha_{2n+k} \\ \beta_{2n+k-1} \\ 0 \\ -(n-1)\alpha_{2n+k-2} \end{pmatrix} \overset{\hat{\pi}}{\mapsto} \begin{pmatrix} \beta_{2n+k-1} \\ -(n-1)\alpha_{2n+k-2} \end{pmatrix}$$

i.e., it acts on the degree $k$ element $\sum_{n\in\mathbb{Z}}(\alpha_{2n+k} + e_{1L}b_{2n+k-1})\xi_{2L}^{-n} \in \widehat{\mathcal{G}_L}$ as

$$\begin{aligned} \sum_{n\in\mathbb{Z}}(\alpha_{2n+k} + e_{1L}\beta_{2n+k-1})\xi_{2L}^{-n} \mapsto & \sum_{n\in\mathbb{Z}} \left(\beta_{2n+k-1} - (n-1)e_{1R}\alpha_{2n+k-2}\right)\xi_{2R}^{-n} \\ &= \sum_{n\in\mathbb{Z}} \beta_{2n+k-1}\xi_{2R}^{-n} + e_{1R}\sum_{n\in\mathbb{Z}} -n\alpha_{2n+k}\xi_{2R}^{-n-1} \end{aligned}$$

The above expressions can be conveniently packaged by introducing, for every sequence $\{\eta_{2n+k}\}_{n\in\mathbb{Z}}$ of elements of $A$ with $\deg(\eta_{2n+k}) = 2n+k$, the Laurent series in a degree 2 variable $\xi$

$$\eta_{(k)}(\xi) = \sum_{n\in\mathbb{Z}} \eta_{2n+k}\xi^{-n}.$$

We have manifest isomorphisms of graded vector spaces

$$\widehat{\mathcal{G}_{L\{f(x_{2R})\}}} \xleftarrow{\sim} \begin{pmatrix} A[[\xi^{-1},\xi] \\ \oplus \\ A[[\xi^{-1},\xi][-1] \\ \oplus \\ A[[\xi^{-1},\xi][-1] \\ \oplus \\ A[[\xi^{-1},\xi][-2] \end{pmatrix} \xrightarrow{\sim} \widehat{\mathcal{G}_{R\{f(x_{2L})\}}}$$

and

$$\widehat{\mathcal{G}_L} \xleftarrow{\sim} \begin{pmatrix} A[[\xi^{-1},\xi] \\ \oplus \\ A[[\xi^{-1},\xi][-1] \end{pmatrix} \xrightarrow{\sim} \widehat{\mathcal{G}_R}.$$

In terms of these isomorphisms, the maps $\hat{\nu}$, $\hat{\iota}_R$ and $\hat{\pi}$ are represented by the following matrices:

$$\hat{\nu} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{d}{d\xi} & 0 & 0 & 1 \end{pmatrix}; \qquad \hat{\iota}_R \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \qquad \hat{\pi} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that the graded left-to-right Hori transform $\mathcal{T}_{L\to R}$ is represented in matrix form as:

$$\mathcal{T} \mapsto \begin{pmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0 \end{pmatrix}.$$

One similarly defines the right-to-left Hori transform $\mathcal{T}_{R\to L}$. As

$$\begin{pmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0 \end{pmatrix} = \begin{pmatrix} \frac{d}{d\xi} & 0 \\ 0 & \frac{d}{d\xi} \end{pmatrix}$$

one sees that

$$\mathcal{T}_{R\to L} \circ \mathcal{T}_{L\to R} = \frac{d}{d\xi_{2L}} : \widehat{\mathcal{G}_L} \to \widehat{\mathcal{G}_L}[-2]$$

and

$$\mathcal{T}_{L\to R} \circ \mathcal{T}_{R\to L} = \frac{d}{d\xi_{2R}} : \widehat{\mathcal{G}_R} \to \widehat{\mathcal{G}_R}[-2].$$

## 5. Hori transforms of meromorphic functions

Before extending the ring of coefficients to the ring of Jacobi forms we start with a one variable intermediate step. We will need an extra degree 2 variable in order to keep the following computations within the context of graded maps. So we assume that our base DGCA $A$ is of the form

$$A := A_0[u^{-1}, u]]$$

where $u$ is a degree 2 variable and $A_0$ is a DGCA endowed with a rational T-duality configuration $\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \xrightarrow{f} A_0$. Notice the a T-duality configuration on $A_0$ induces a T-duality configuration

$$\mathbb{K}[x_{2L}, x_{2R}]_{\{x_{2L}x_{2R}\}} \xrightarrow{f} A_0 \hookrightarrow A$$

on $A$ simply by composing $f$ with the inclusion $A_0 \hookrightarrow A$. All the extension and gerbes below are computed with respect to this T-duality configuration on $A$.

For instance, the extended left gerbe $\widehat{\mathcal{G}_L}$ will be

$$\widehat{\mathcal{G}_L} = \overbrace{A_0}^{\text{starting DGCA}} \underbrace{[u^{-1}, u]]}_{\substack{\text{additional}\\ \text{variable}}} \overbrace{[e_{1L}]}^{\text{trivialize } f(x_{2L})} \underbrace{[[\xi_{2L}^{-1}, \xi_{2L}]}_{\substack{\text{trivialize } f(y_3)-\\ e_{1L}f(x_{2R}) \text{ and ex-}\\ \text{tend to Laurent}\\ \text{series}}}.$$

Assume now the base field $\mathbb{K}$ to be the field $\mathbb{C}$ of complex numbers and let $\mathcal{M}_0$ be the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{C}$ that are holomorphic on the punctured plane $\mathbb{C}\backslash\{0\}$, i.e. meromorphic functions that admit at most a polar singularity in the origin. By looking at the algebra $\mathcal{M}_0$ as a DGCA concentrated in degree zero, we can then consider the DGCA $\mathcal{M}_0(A_0) := \mathcal{M}_0 \otimes A_0$, that we will call the DGCA of meromorphic functions with values in $A_0$ and with at most polar singularities in the origin. A degree $k$ element in $\mathcal{M}_0(A_0)$ has a Laurent series expansion around the origin of the form

$$f(q) = \sum_n f_{n;k} q^n$$

where the $f_{n;k}$ are degree $k$ elements in $A_0$, with $f_{n;k} = 0$ for $n \ll 0$. For any $i \in \mathbb{Z}$ we have an isomorphism $\mu_{2i}$ of graded vector spaces

$$\begin{aligned} \mathcal{M}_0(A_0) &\xrightarrow{\mu_{2i}} A[[\xi^{-1}, \xi][2i] \\ f(q) &\mapsto \xi^i f(u\xi^{-1}) \end{aligned}$$

Notice that there exists a commutative diagram

$$\begin{tikzcd} \mathcal{M}_0(A_0) \arrow[r, "-q\frac{d}{dq}"] \arrow[d, "\mu_0"'] & \mathcal{M}_0(A_0) \arrow[d, "\mu_{-2}"] \\ A[[\xi^{-1}, \xi] \arrow[r, "\frac{d}{d\xi}"] & A[[\xi^{-1}, \xi][-2] \end{tikzcd}$$

As remarked at the end of the previous section, the natural isomorphism of graded vector spaces of $\widehat{\mathcal{G}_L}$ and $\widehat{\mathcal{G}_R}$ with $A[[\xi^{-1}, \xi] \oplus A[[\xi^{-1}, \xi][-1]$ identifies the graded Hori map $\mathcal{T}_{L\to R}$ with the antidiagonal matrix $\begin{pmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0 \end{pmatrix}$, i.e., we have a commutative diagram

$$\begin{tikzcd} A[[\xi^{-1}, \xi] \oplus A[[\xi^{-1}, \xi][-1] \arrow[r, "\begin{pmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0 \end{pmatrix}"] \arrow[d, "\wr"] & A[[\xi^{-1}, \xi][-1] \oplus A[[\xi^{-1}, \xi][-2] \arrow[d, "\wr"] \\ \widehat{\mathcal{G}_L} \arrow[r, "\mathcal{T}_{L\to R}"] & \widehat{\mathcal{G}_R}[-1] \end{tikzcd}$$

Therefore, we see that the graded Hori map $\mathcal{T}_{L\to R}$ participates into a commutative diagram of graded vector spaces

$$\begin{tikzcd}
\mathcal{M}_0(A_0)\oplus\mathcal{M}_0(A_0)[-1] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \\ -q\frac{d}{dq} & 0\end{smallmatrix}\right)"] \arrow[d, "\mu_0\oplus\mu_0[-1]"', "\wr"] & \mathcal{M}_0(A_0)[-1]\oplus\mathcal{M}_0(A_0) \arrow[d, "\mu_0[-1]\oplus\mu_{-2}", "\wr"'] \\
A[[\xi^{-1},\xi]\oplus A[[\xi^{-1},\xi][-1] \arrow[r, "\left(\begin{smallmatrix} 0 & 1 \\ \frac{d}{d\xi} & 0\end{smallmatrix}\right)"] \arrow[d, "\wr"] & A[[\xi^{-1},\xi][-1]\oplus A[[\xi^{-1},\xi][-2] \arrow[d, "\wr"] \\
\widehat{\mathcal{G}_L} \arrow[r, "\mathcal{T}_{L\to R}"] & \widehat{\mathcal{G}_R}[-1]
\end{tikzcd}$$

The same happens for the graded Hori map $\mathcal{T}_{R\to L}$, so that the composition $\mathcal{T}_{R\to L}\circ\mathcal{T}_{L\to R}$ is identified with the endomorphism

$$\begin{pmatrix} -q\frac{d}{dq} & 0 \\ 0 & -q\frac{d}{dq}\end{pmatrix}$$

of $\mathcal{M}_0(A_0)\oplus\mathcal{M}_0(A_0)[-1]$, and similarly for $\mathcal{T}_{L\to R}\circ\mathcal{T}_{R\to L}$.

## 6. Extending coefficients to the ring of Jacobi forms

In this concluding section we extend the ring of coefficients for our extended gerbes to the graded ring of Jacobi forms of index 0. We address the reader to the classic [EZ85] for a complete and detailed account of the general theory of Jacobi forms of arbitrary index, and here we content us in briefly recalling the definition of a (meromorphic) Jacobi form of index 0.

**Definition 6.1.** A *(meromorphic) Jacobi form of weight $s$ and index* 0 is a function

$$\mathbb{C}\times\mathbb{H}\xrightarrow{J}\mathbb{C}$$

which is meromorphic in the variable $z$ and holomorphic in the variable $\tau$, such that $J$

- is modular in $\tau$ i.e. $J(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d})=(c\tau+d)^sJ(z,\tau)$ for any $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$;
- is elliptic in $z$ i.e. $J(z+\lambda\tau+\mu,\tau)=J(z,\tau)$ for any $(\lambda,\mu)$ in $\mathbb{Z}^2$;
- has a polar behaviour for $z\to+i\infty$.

We notice two important features of Jacobi forms. First, by applying the operator $\partial/\partial z$ to both sides of the modularity and of the ellipticity equations, one sees that if $J(z,\tau)$ is a Jacobi form of weight $s$ and index 0 then $\frac{\partial}{\partial z}J(z,\tau)$ is a Jacobi form of weight $s+1$ and index 0.

Secondly, from the ellipticity condition for the pair $(0,1) \in \mathbb{Z}^2$ one sees that every Jacobi form is periodic in $z$ of period 1, hence it has a series expansion in the variable $q = e^{2\pi i z}$ of the form

$$J(z,\tau) = \sum_{n=n_0}^{\infty} \alpha_n(\tau) q^n,$$

for some $n_0 \in \mathbb{Z}$, where the fact that this Laurent series is bounded below is a consequence of the polar behaviour of $J$ for $z \to +i\infty$.

As the weight $s$ ranges over the integers, Jacobi form of index 0 form a graded ring

$$\mathcal{J}_0 = \bigoplus_{s\in\mathbb{Z}} \mathcal{J}_0(s),$$

(with degree given by the weight). The fact that $\frac{\partial}{\partial z}$ maps weight $s$ index 0 Jacobi forms to weight $s+1$ index 0 Jacobi forms then can be expressed by saying that $\frac{\partial}{\partial z}$ is a degree 1 derivation of the graded ring $\mathcal{J}_0$. Moreover, from the identity

$$-q\frac{\partial}{\partial q} = -\frac{1}{2\pi i}\frac{\partial}{\partial z}$$

we see that the ring of $q$-expansions of index 0 Jacobi forms (a subring of the ring of bounded below Laurent series in the variable $q$ with coefficients in the ring of holomorphic function on $\mathbb{H}$) is closed under the action of the operator $-q\dfrac{\partial}{\partial q}$.

We can now verbatim repeat the construction of Section 5. For a DGCA $B$ over $\mathbb{C}$, let us write $B(\tau)$ for the DGCA

$$B(\tau) := B \otimes \mathcal{H}ol(\mathbb{H})$$

where the ring $\mathcal{H}ol(\mathbb{H})$ of holomorphic function in the variable $\tau \in \mathbb{H}$ is seen as a DGCA concentrated in degree zero. Also, let us write

$$\mathcal{J}_0(A_0) = \bigoplus_{s\in\mathbb{Z}} \mathcal{J}_0^{(s)}(A_0)$$

for the bigraded ring $\mathcal{J}_0(A_0) := \mathcal{J}_0 \otimes A_0$ of index 0 Jacobi forms with values in a DGCA $A_0$. Then the commutative diagram at the end of Section 5 induces the commutative diagram

$$\begin{tikzcd}
\mathcal{J}_0^{(s_1)}(A_0)\oplus\mathcal{J}_0^{(s_2)}(A_0)[-1] \arrow[r, "\begin{pmatrix} 0 & 1\\ -\frac{1}{2\pi i}\frac{\partial}{\partial z} & 0\end{pmatrix}"] \arrow[d, "q\text{-expansion}"'] & \mathcal{J}_0^{(s_2)}(A_0)[-1]\oplus\mathcal{J}_0^{(s_1+1)}(A_0) \arrow[d, "q\text{-expansion}"] \\
\mathcal{M}_0(A_0(\tau))\oplus\mathcal{M}_0(A_0(\tau))[-1] \arrow[r, "\begin{pmatrix} 0 & 1\\ -q\frac{\partial}{\partial q} & 0\end{pmatrix}"] \arrow[d, "\mu_0\oplus\mu_0[-1]"', "\wr"] & \mathcal{M}_0(A_0(\tau))[-1]\oplus\mathcal{M}_0(A_0(\tau)) \arrow[d, "\mu_0[-1]\oplus\mu_{-2}", "\wr"'] \\
A(\tau)[[\xi^{-1},\xi]\oplus A(\tau)[[\xi^{-1},\xi][-1] \arrow[r, "\begin{pmatrix} 0 & 1\\ \frac{d}{d\xi} & 0\end{pmatrix}"] \arrow[d, "\wr"] & A[[\xi^{-1},\xi][-1]\oplus A[[\xi^{-1},\xi][-2] \arrow[d, "\wr"'] \\
\widehat{\mathcal{G}_L}(\tau) \arrow[r, "\mathcal{T}_{L\to R}"] & \widehat{\mathcal{G}_R}(\tau)[-1]
\end{tikzcd}$$

That is, the graded Hori transform $\mathcal{T}_{L\to R}$ induces the morphism

$$\begin{pmatrix} 0 & 1\\ -\frac{1}{2\pi i}\frac{\partial}{\partial z} & 0\end{pmatrix} : \mathcal{J}_0^{(s_1)}(A_0)\oplus\mathcal{J}_0^{(s_2)}(A_0)[-1]\to\mathcal{J}_0^{(s_2)}(A_0)[-1]\oplus\mathcal{J}_0^{(s_1+1)}(A_0)$$

at the level of index 0 $A_0$-valued Jacobi forms, for any weights $s_1, s_2$ in $\mathbb{Z}$. The same holds for the graded Hori transform $\mathcal{T}_{R\to L}$, so that the composition $\mathcal{T}_{L\to R}\circ\mathcal{T}_{R\to L}$ acts as

$$\begin{pmatrix} -\frac{1}{2\pi i}\frac{\partial}{\partial z} & 0\\ 0 & -\frac{1}{2\pi i}\frac{\partial}{\partial z}\end{pmatrix} : \mathcal{J}_0^{(s_1)}(A_0)\oplus\mathcal{J}_0^{(s_2)}(A_0)[-1]\to\mathcal{J}_0^{(s_1+1)}(A_0)\oplus\mathcal{J}_0^{(s_2+1)}(A_0)[-1]$$

and similarly for $\mathcal{T}_{L\to R}\circ\mathcal{T}_{R\to L}$. This reproduces [HM20, Theorem 2.2].

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**Appendices**

## 1. Twisted Cohomology

The idea behind twisted cohomology is actually really simple. One starts by observing that, in view of Brown's representability theorem if one follows a classical approach, or by definition if one directly starts with spectra, the degree $n$ $E$-cohomology of a space $X$ is the space of maps from $X$ to a space $E_n$ (or rather, the set of homotopy classes of these maps). Now, maps $X \to E_n$ can be equivalently seen as the sections of the trivial bundle $X \times E_n \to X$. This immediately suggests to think of sections of a nonnecessarily trivial bundle $E_n \to P_n \to X$ as "twisted maps" from $X$ to $E_n$, and their homotopy classes as degree $n$ twisted $E$-cohomology classes of $X$. A further step consists in looking at the collection of all the spaces $P_n$ as a parametrized spectrum $P$ with parameter space $X$, and so define the twisted cohomologies of a space $X$ as the spaces of sections of parametrized spectra $P \in \mathsf{Sp}_X$.

Of course, one has to take some care in defining what is meant by a section of a parametrized spectrum, as we do not have an actual projection map $P \to X$ from our parametrized spectrum $P$ to our space $X$. Also, in order to show that this definition of twisted cohomology is not just a generalization for the sake of generality, we need a recipe to build interesting examples.

Let $F$ a spectrum. Since $F$ is an object in an $\infty$-category its automorphisms form a group object in $\infty$-groupoids/(nice)topological spaces. In particular we have a classifying space $BAut(F)$ (or classifying $\infty$-groupoid $\mathsf{BAut}(F)$) and we can envision the action of $Aut(F)$ on $F$ as the datum of an $\infty$-functor $\mathsf{BAut}(F) \to \mathsf{Sp}$ sending the unique object of the $\infty$-groupoid $\mathsf{BAut}(F)$ to $F$, with an obvious definition on the morphisms. This map is the spectral analogue to the universal fibre bundle with fibre $F$. Given a space $X$, one calls *twist* for the $F$-cohomology of $X$ a map of spaces $\chi \colon X \to BAut(F)$, or equivalently a map of $\infty$-groupoids $\Pi_\infty X \to \mathsf{BAut}(F)$, were $\Pi_\infty X$ denotes the Poincaré $\infty$-groupoid of $X$. Composing $\chi$ with the $\infty$-functor $\mathsf{BAut}(F) \to \mathsf{Sp}$ encoding the action of $Aut(F)$ on $F$ gives a $X$-parametrized spectrum where over each point of $X$ one has a copy of the spectrum $F$. The stable homotopy groups of the spaces of sections of this $X$-parametrized spectrum are called the $\chi$-twisted $F$-cohomologies of $X$. By an abuse of notation (as actually this map does not exist) reminiscent of the Grothendieck construction, one denotes the $\infty$-functor $\mathsf{BAut}(F) \to \mathsf{Sp}$ as $p \colon F//Aut(F) \to *//Aut(F)$. In this notation, the composite functor $\Pi_\infty X \to \mathsf{BAut}(F)$ corresponds to the pullback

$$\begin{array}{ccc} E & \longrightarrow & F//Aut(F) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow{\chi} & *//Aut(F), \end{array}$$

and the $\chi$-twisted $F$ cohomologies of $X$ are the stable homotopy groups of the spaces of sections of $\pi$.

Given a ring $R$ it is typically more interesting to study bundles with fibre $R$ where the fibres are not bare sets but maintain an $R$-module structure. Things are the same in the spectral case and the definition of twisted cohomology with coefficients in a ring spectrum just follows this point of view. Indeed, consider a ring spectrum $R$ and recall the existence of the space $GL_1(R)$ mentioned in the article on the GHRR theorem. Taking a space $X$ and a twist $\chi : X \to BGL_1(R)$ we can do everything as we did above to define the $\chi$-twisted $R$ cohomology of $X$, but looking at the $GL_1(R)$-action on $R$ not just as at an $\infty$-functor $\mathsf{B}GL_1(R) \to \mathsf{Sp}$ but as at an $\infty$-functor $\mathsf{B}GL_1(R) \to \mathsf{RMod}$. The $\chi$-twisted cohomology groups of $X$ with coefficients in the ring spectrum $R$ are then defined as the homotopy groups of the spaces of sections of the $X$-parametrized $R$-module

$$X \xrightarrow{\chi} BGL_1(R) \to \mathsf{RMod}.$$

For more information about twisted cohomology we refer the reader to [MS06] and [ABG10].

## 2. Towards a topological graded Hori map

We now specialize the general construction of twisted cohomology sketched above to the specific case of twisted $K$-theory. In this case it is known, see, e.g., [MS06], that $GL_1(KU) \cong \mathbb{Z}/2 \times BU_\otimes$, where the factor $\mathbb{Z}/2$ acts on complex vector bundles by taking the conjugate bundle and $BU_\otimes$ acts by tensoring with a vector bundle. These actions naturally extend to virtual complex vector bundles and so to $KU$. Moreover, one has $BU_\otimes \cong K(\mathbb{Z}, 2) \times BSU_\otimes$, so that $GL_1(KU) \cong \mathbb{Z}/2 \times K(\mathbb{Z}, 2) \times BSU_\otimes$. It is customary to restrict the attention to the middle factor and thus to consider twists $\chi : Y \to K(\mathbb{Z}, 3) = BK(\mathbb{Z}, 2) \to BGL_1(KU)$. In other words, the classical twists of complex $K$-theory are elements in the third singular cohomology group with integer coefficients.

This fact suggests the idea that the right object to study in the topological version of T-duality is not singular cohomology twisted by a 3-cocycle, but twisted complex $K$-theory. The first part of the construction presented in our article is then easily lifted form the rational approximation to a genuinely topological form as follows.

First, one defines the space BTfold as the homotopy pullback:

$$\begin{tikzcd} \mathrm{BTfold} \arrow[r] \arrow[d] & * \arrow[d] \\ K(\mathbb{Z},2)\times K(\mathbb{Z},2) \arrow[r] & K(\mathbb{Z},4). \end{tikzcd}$$

and calls *T-duality configuration* on a space $X$ a map $f : X \to \mathrm{BTfold}$. The composition

$$X \xrightarrow{f} \mathrm{BTfold} \to K(\mathbb{Z},2)\times K(\mathbb{Z},2)$$

gives us two maps $l, r : X \to K(\mathbb{Z},2)$. We denote the homotopy fibres of $l$ and $r$ by $P_l$ and $P_r$ respectively. These are topological $S^1$-fibrations over $X$. Also in the topological setting we have a homotopy fibre - cyclification adjunction and moreover one can interpret the results in [BS05] as the topological equivalence

$$\mathcal{L}\,\mathrm{BTfold}//S^1 \cong B^2U(1).$$

By definition of the map $l$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "l"'] & & \mathrm{BTfold} \arrow[dl, "p_l"] \\ & K(\mathbb{Z},2) & \end{tikzcd}$$

and so we can view $f$ as a map in the overcategory of topological spaces over $K(\mathbb{Z},2)$. By the homotopy fibre - cyclification adjunction, we will have a corresponding adjoint map $f_l\colon P_l \to \mathcal{L}\,\mathrm{BTfold}//S^1$, and so, by the Bunke-Schick result, as a map

$$f_l\colon P_l \to B^2U(1) \cong K(\mathbb{Z},3).$$

The same is true on the right side, giving a map $f_r : P_r \to K(\mathbb{Z},3)$.

We can then form the homotopy commutative diagram

$$\begin{tikzcd} & & P \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_r"] & & \\ & P_l \arrow[dl, "f_l"'] \arrow[dr] & & P_r \arrow[dl] \arrow[dr, "f_r"] & \\ K(\mathbb{Z},3) & & X & & K(\mathbb{Z},3), \end{tikzcd}$$

where $P = P_l \times_X P_r$ is a topological $S^1$-fibration both over $P_l$ and over $P_r$. The two maps $f_l\colon P_l \to K(\mathbb{Z},3)$ and $f_r\colon P_r \to K(\mathbb{Z},3)$ serve as twists for the complex K-theory of $P_l$ and of $P_r$ respectively. The composite maps $f_l \circ \pi_l$ and $f_r \circ \pi_r$ serve as twists for the complex K-theory of $P$, and the defining property of a T-fold configuration amounts to the datum of an isomorphism of twisted K-theory groups[2]

$$K_{f_l \circ \pi_l}(P) \cong K_{f_r \circ \pi_r}(P).$$

Then one has a pull-iso-push transform

$$K_{f_l}(P_l) \xrightarrow{\pi_l^*} K_{f_l \circ \pi_l}(P) \xrightarrow{\sim} K_{f_r \circ \pi_r}(P) \xrightarrow{\pi_{r;*}} K_{f_r}(P_r),$$

giving the topological Hori map in complex K-theory. The possibility of defining a graded version of this map at the rational level and its interplay with the theory of Jacobi forms strongly suggests that the topological Hori map in complex K-theory should be the $q \to 0$ limit of a topological Hori map in elliptic cohomology, involving a suitable pull-iso-push transform in twisted elliptic cohomology. This fits nicely in the geometric framework in which Han and Mathai derive the graded Hori map of [HM20] and will hopefully be investigated in detail in forthcoming research.

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## The (anti-)holomorphic sector in $\mathbb{C}/\Lambda$-equivariant cohomology, and the Witten class

Abstract. Atiyah's classical work on circular symmetry and stationary phase shows how the $\hat{A}$-genus is obtained by formally applying the equivariant cohomology localization formula to the loop space of a simply connected spin manifold. The same technique, applied to a suitable "antiholomorphic sector" in the $\mathbb{C}/\Lambda$-equivariant cohomology of the conformal double loop space $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$ of a rationally string manifold $X$ produces the Witten genus of $X$. This can be seen as an equivariant localization counterpart to Berwick-Evans supersymmetric localization derivation of the Witten genus.

*Se vogliamo che tutto rimanga come è, bisogna che tutto cambi.*

### 1. Introduction

In the classic work [Ati85], Atiyah shows how to recover the $\hat{A}$-class of a compact smooth spin manifold $X$ via a formal infinite dimensional version of the Duistermaat-Heckman formula applied to the smooth loop space $\mathrm{Maps}(\mathbb{T}, X)$ of maps from a circle to $X$. Such a formula is a particular case of the well known localization formula for torus equivariant cohomology, extensively treated in [AB84]. The appearance of the $\hat{A}$-class in such an infinite dimensional version of localization techniques in torus equivariant cohomology was pointed out by Atiyah as "a brilliant observation of the physicist E. Witten" and suggests that, reasoning as in [Ati85], the Witten class $\mathrm{Wit}(X)$ [Wit87; Wit88], should emerge from a localization formula for the torus equivariant cohomology of the double loop space $\mathrm{Maps}(\mathbb{T}^2, X)$ of maps from a 2-dimensional torus to $X$. This is indeed the case, as long as one makes an a priori unjustified assumption: that the generators $u, v$ of the $\mathbb{T}^2$-equivariant cohomology of a point over $\mathbb{C}$,

$$H^*_{\mathbb{T}^2}(\mathrm{pt}; \mathbb{C}) \cong \mathbb{C}[u, v],$$

are not independent but rather satisfy a $\mathbb{C}$-linear dependence condition of the form

$$v = \tau u$$

where $\tau$ is a point in the complex upper half plane $\mathbb{H}$, see [Lu08]. Although the hypothesis of $\mathbb{C}$-linear dependence of the polynomial variables $u, v$ may appear somewhat "ad hoc" to make the computations work out, yet it suggests that if instead of looking at a topological torus $\mathbb{T}^2$ we consider a complex torus $\mathbb{C}/\Lambda$ then there should exist a version of the localization theorem for torus equivariant cohomology, where only a holomorphic variable $\xi$ (or its conjugate $\overline{\xi}$) appears, instead of the two real variables $u, v$. In this paper we show that such a *holomorphic* (resp. *antiholomorphic*) *sector* of the $\mathbb{C}/\Lambda$-equivariant cohomology can indeed be defined and that an (anti-)holomorphic localization formula holds. Going back to what inspired it, in the final part of the paper we show how the Witten class of a compact smooth manifold emerges from the antiholomorphic localization formula for the $\mathbb{C}/\Lambda$-equivariant cohomology of the double loop space $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$. More precisely, the idea is to formally apply the finite-dimensional antiholomorphic localization formula obtained in the first part of the paper to the inclusion of $X$ in $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$ as the submanifold of constant maps (that are the fixed points for the $\mathbb{C}/\Lambda$-translation action on $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$. It turns out, however, that the infinite products that would naively define the equivariant Euler class for the normal bundle $\nu_\Lambda$ of $X$ in $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$ do not converge, so a suitable $\zeta$-regularization is needed in order to make sense of these infinite products. Once this is done, one obtains that if $X$ is a compact rational string manifold, i.e., if $X$ is a compact spin manifold with torsion first Pontryagin class, then the inverse of the normalized Euler class of $\nu_\Lambda$ defines a modular form with values in the complex cohomology of $X$, which turns out to be the Witten class of $X$. In particular, the integral over $X$ of the inverse of the normalized Euler class of $\nu_\Lambda$ is the Witten genus of $X$. From the point of view of $\mathbb{C}/\Lambda$-equivariant cohomology, the geometric condition that $X$ needs to be a rationally string manifold will emerge as the condition ensuring that the $\zeta$-regularization procedure involved in the infinite rank localization formula is independent of the choice of arguments for the nonzero elements in the lattice $\Lambda \subset \mathbb{C}$. This condition will also imply that the expected modular properties of the inverse normalized Euler class are not disrupted by the $\zeta$-regularization.

Equivariant cohomology and the Atiyah–Bott localization formula admit an elegant rephrasing in terms of supergeometry, see, e.g., [PZ17]. Reversing this point of view, every differential geometric construction obtained through supersymmetric localization techniques in quantum field theory should in principle admit a derivation internal to the setting of equivariant cohomology. In this sense, the results of this paper can be seen as an equivariant localization counterpart to Berwick-Evans supersymmetric localization derivation of the Witten genus [BE13; BE19], with the Weierstraß $\zeta$-regularization of equivariant Euler classes playing the role of the $\zeta$-regularization of infinite dimensional determinants in supersymmetric quantum field theory.

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## 2. 1d Euclidean tori equivariant cohomology

2.1. **The Euclidean Cartan complex for circle actions.** As a half-way step towards two dimensional real tori endowed with a complex structure $\mathbb{C}/\Lambda$, we start by recalling a few basic constructions in the equivariant cohomology for 1-dimensional torus actions, formulating them for 1-dimensional Euclidean tori $\mathbb{R}/\Lambda$ rather than for the topological 1-dimensional torus $\mathbb{T}$. Here $\Lambda \subset \mathbb{R}$ is a lattice in $\mathbb{R}$, i.e. an additive subgroup of $\mathbb{R}$ isomorphic to $\mathbb{Z}$.

The quotient $\mathbb{R}/\Lambda$ can be thought of as a circle of length $\ell$, with $\ell$ the minimum strictly positive element of $\Lambda$. It has a natural structure of real Lie group; we will denote its Lie algebra by $\mathfrak{t}_\Lambda$. Next we consider a compact smooth manifold $M$ with a smooth action of $\mathbb{R}/\Lambda$, denote by $\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}/\Lambda}$ the $\mathbb{R}/\Lambda$-invariant part of the de Rham algebra of $M$, and endow

$$\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}/\Lambda} \otimes_\mathbb{R} \mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$$

with a bigrading where the component of bidegree $(k,l)$ is $\Omega^{k-l}(M;\mathbb{R})^{\mathbb{R}/\Lambda} \otimes_\mathbb{R} \mathrm{Sym}^l(\mathfrak{t}_\Lambda{}^\vee[-2])$. This bigraded vector space comes equipped with a structure of bicomplex where the differential of degree $(1,0)$ is the de Rham differential (acting trivially on $\mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$) and the differential of degree $(0,1)$ is the operator $e_\Lambda{}^\vee[-2]\iota_{v_{e_\Lambda}}$, where $(e_\Lambda, e_\Lambda{}^\vee)$ is a pair consisting of a linear generator of $\mathfrak{t}_\Lambda$ and of its dual element in $\mathfrak{t}_\Lambda^\vee$, and $v_{e_\Lambda}$ is the vector field on $M$ corresponding to $e_\Lambda$ via the differential of the action. The operator $\iota$ is the contraction operator. It is immediate to see that $e_\Lambda{}^\vee[-2]\iota_{v_{e_\Lambda}}$ is independent of the choice of the generator $e_\Lambda$.

**Definition 2.1.** The Cartan complex of $\mathbb{R}/\Lambda \circlearrowright M$ is the total complex of the bicomplex

$$(\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}/\Lambda} \otimes_\mathbb{R} \mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2]); d_{\mathrm{dR}}, e_\Lambda{}^\vee[-2]\iota_{v_{e_\Lambda}}).$$

The total differential in the Cartan complex is denoted by $d_{\mathbb{R}/\Lambda}$ and is called the equivariant differential. Elements in the Cartan complex that are $d_{\mathbb{R}/\Lambda}$-closed are called *equivariantly closed* forms.

*Remark* 2.2. The importance of the Cartan complex resides in the fact its cohomology is the real $\mathbb{R}/\Lambda$-equivariant cohomology $H^\bullet_{\mathbb{R}/\Lambda}(M;\mathbb{R})$ of $M$. So it provides a differential geometric tool to compute this cohomology. It is the generalization to the equivariant setting of the de Rham complex computing real singular cohomology.

*Remark* 2.3. Evaluation at $0 \in \mathfrak{t}_\Lambda^\vee[-2]$ is a morphism of complexes from the Cartan complex to the de Rham complex $(\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}/\Lambda}, d_{\mathrm{dR}})$ of $\mathbb{R}/\Lambda$-invariant forms. One says that an element $\widetilde{\omega}$ in the Cartan complex is an extension of an invariant form $\omega$ if $\widetilde{\omega}|_0 = \omega$.

*Remark* 2.4. The quotient map $\mathbb{R} \to \mathbb{R}/\Lambda$ gives a distinguished Lie algebra isomorphism $\mathrm{Lie}(\mathbb{R}) \xrightarrow{\sim} \mathfrak{t}_\Lambda$. By means of this isomorphism, the Cartan bicomplex is isomorphic to

$$(\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}/\Lambda}[u]; d_{\mathrm{dR}}, u\iota_{v_{d/dx}})$$

where $d/dx$ is the standard basis vector in $\mathrm{Lie}(\mathbb{R})$ and $u$ is a degree 2 formal variable corresponding to the dual 1-form $dx$ placed in degree 2. Notice that with respect to the bigrading, the variable $u$ has bidegree $(1,1)$.

2.2. **$\mathbb{R}/\Lambda$-equivariant characteristic classes.** Equivariant vector bundles over an $\mathbb{R}/\Lambda$-manifold come with a natural notion of equivariant characteristic classes. When the action on the manifold is trivial[3], equivariant characteristic classes admit a simple combinatorial/representation theoretic description that we recall below.

*Remark* 2.5. A typical situation where one meets equivariant vector bundles on a $\mathbb{R}/\Lambda$-trivial base is by considering equivariant vector bundles on the $\mathbb{R}/\Lambda$-fixed point locus $\mathrm{Fix}(M)$ in an $\mathbb{R}/\Lambda$-manifold $M$. Notice that, since $\mathbb{R}/\Lambda$ is a compact Lie group, its action on $M$ is automatically proper and so $\mathrm{Fix}(M)$ is a smooth submanifold of $M$. Equivariant vector bundles on $\mathrm{Fix}(M)$ one considers need not be restrictions of equivariant vector bundles on $M$. A classical example is the normal bundle $\nu$ for the inclusion $\mathrm{Fix}(M) \hookrightarrow M$.

For ease of exposition, we will tacitly assume $\mathrm{Fix}(M)$ to be connected: in the more general situation of a possibly nonconnected fixed point locus all the constructions we recall in this section are to be repeated for each of the connected components of $\mathrm{Fix}(M)$.

*Remark* 2.6. For a $\mathbb{R}/\Lambda$-trivial manifold one has $M = \mathrm{Fix}(M)$, so it is actually not restrictive to work with submanifolds of the form $\mathrm{Fix}(M)$ when one is interested into equivariant vector bundles over $\mathbb{R}/\Lambda$-trivial base manifolds.

*Remark* 2.7. As the $\mathbb{R}/\Lambda$-action is trivial on $\mathrm{Fix}(M)$, the associated Cartan bicomplex is

$$(\Omega^\bullet(\mathrm{Fix}(M);\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{Sym}({\mathfrak{t}_\Lambda}^\vee[-2]); d_{\mathrm{dR}}, 0)$$

and so the $\mathbb{R}/\Lambda$-equivariant cohomology of $\mathrm{Fix}(M)$ is

$$H^\bullet_{\mathbb{R}/\Lambda}(\mathrm{Fix}(M);\mathbb{R}) = H^\bullet(\mathrm{Fix}(M);\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{Sym}({\mathfrak{t}_\Lambda}^\vee[-2]).$$

[3]This does not imply that the action is trivial on the bundle.

The key to the combinatorial description of equivariant complex vector bundles over $\mathbb{R}/\Lambda$-trivial base manifolds is the following statement, which is an immediate consequence of the regularity of the decomposition into isotypic components of smooth families of complex representations of compact Lie groups. The statement is well known, see, e.g., [Sin01, Proposition 4.6] where however it is given without a proof. For completeness, we provide a proof for the particular case we are interested in.

**Lemma 2.8.** *An $\mathbb{R}/\Lambda$-equivariant complex line bundle on $\mathrm{Fix}(M)$ is equivalently the datum of a pair $(L, \chi)$, where $L$ is a complex line bundle on $\mathrm{Fix}(M)$ and $\chi\colon \mathbb{R}/\Lambda \to U(1)$ is a character of $\mathbb{R}/\Lambda$.*

*Proof.* Let us denote by $L_p$ the fiber of $L$ on the point $p \in M$. The datum of an $\mathbb{R}/\Lambda$-equivariant complex line bundle $L$ on $\mathrm{Fix}(M)$ is the datum of a collection of group homomorphisms $\mathbb{R}/\Lambda \to \mathrm{Aut}_{\mathbb{C}}(L_p)$, smoothly depending on $p \in \mathrm{Fix}(M)$. Since $L_p$ is 1-dimensional, one has a canonical isomorphism $\mathrm{Aut}_{\mathbb{C}}(L_p) \cong \mathbb{C}^*$, so our datum is the datum of a smooth family of Lie group homomorphisms $\chi_p\colon \mathbb{R}/\Lambda \to \mathbb{C}^*$. Since $\mathbb{R}/\Lambda$ is compact, these have to factor through $U(1)$ and so they form a smooth family of characters $\chi_p\colon \mathbb{R}/\Lambda \to U(1)$. Since $U(1)$-valued characters of $\mathbb{R}/\Lambda$ are uniquely defined by their topological degree as smooth maps $\mathbb{R}/\Lambda \to U(1)$ and the topological degree is a homotopy invariant, we have that $\chi_p$ is constant on connected components of $\mathrm{Fix}(M)$. Therefore, if $\mathrm{Fix}(M)$ is connected, as we are assuming, we are reduced with the datum of a single $U(1)$-valued character $\chi$ of $\mathbb{R}/\Lambda$. $\square$

By the above Lemma, in what follows we will write an $\mathbb{R}/\Lambda$-equivariant complex line bundle over $\mathrm{Fix}(M)$ as a pair $(L, \chi)$.

**Definition 2.9.** Let $\chi\colon \mathbb{R}/\Lambda \to U(1)$ be a character. The *weight* of $\chi$ is the linear map

$$w_\chi\colon \mathfrak{t}_\Lambda \to \mathbb{R}$$

defined as follows: $2\pi i w_\chi$ is the Lie algebra homomorphism $2\pi i w_\chi\colon \mathfrak{t}_\Lambda \to 2\pi i\mathbb{R} = \mathrm{Lie}(U(1))$ associated with the Lie group homomorphism $\chi$, i.e., $2\pi i w_\chi$ is the linear map making the diagram

$$\begin{tikzcd}
\mathfrak{t}_\Lambda \cong \mathbb{R} \arrow[r, "2\pi i w_\chi"] \arrow[d, "\mathrm{proj}"'] & 2\pi i\mathbb{R} \arrow[d, "\exp(-)"] \\
\mathbb{R}/\Lambda \cong \mathbb{R}/\lambda\mathbb{Z} \arrow[r, "\chi"'] & U(1)
\end{tikzcd}$$

commute.

*Remark* 2.10. Notice that, by definition, $w_\chi$ is an element of $\mathfrak{t}_\Lambda{}^\vee$, and so $w_\chi[2] \in \mathfrak{t}_\Lambda{}^\vee[-2] \subseteq \mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$.

**Definition 2.11.** Let $(L, \chi)$ be an $\mathbb{R}/\Lambda$-equivariant complex line bundle over $\mathrm{Fix}(M)$. The equivariant first Chern class of $(L, \chi)$ is the element of $H^\bullet(\mathrm{Fix}(M); \mathbb{R}) \otimes_\mathbb{R} \mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$ given by

$$c_{1,\mathbb{R}/\Lambda}(L, \chi) := c_1(L) + w_\chi[-2].$$

*Remark* 2.12. It is convenient to give a more explicit description of $c_{1,\mathbb{R}/\Lambda}(L, \chi)$ in terms of the isomorphism

$$H^\bullet_{\mathbb{R}/\Lambda}(\mathrm{Fix}(M); \mathbb{R}) \cong H^\bullet(\mathrm{Fix}(M); \mathbb{R})[u]$$

induced by the Lie algebra isomorphism $\mathrm{Lie}(\mathbb{R}) \xrightarrow{\sim} \mathfrak{t}_\Lambda$. In order to do so, recall that characters of $\mathbb{R}/\Lambda$ are indexed by the dual lattice $\Lambda^\vee$ of $\Lambda$ and that via the standard inner product in $\mathbb{R}$ this is identified with $\Lambda$: every character of $\mathbb{R}/\Lambda$ is of the form

$$\chi(x) = \rho_\lambda(x) := \exp(2\pi i \lambda \mathrm{vol}(\mathbb{R}/\Lambda)^{-2} x),$$

for some $\lambda \in \Lambda$. The associated weight $w_\lambda$ is then $w_\lambda = \lambda \mathrm{vol}(\mathbb{R}/\Lambda)^{-2} dx$ so that $w_\lambda[-2] = \lambda \mathrm{vol}(\mathbb{R}/\Lambda)^{-2} u$. The equivariant first Chern class of $(L, \rho_\lambda)$ is then written as $c_{1,\mathbb{R}/\Lambda}(L, \rho_\lambda) = c_1(L) + \lambda \mathrm{vol}(\mathbb{R}/\Lambda)^{-2} u$. Introducing the rescaled formal variable $u_\Lambda := \mathrm{vol}(\mathbb{R}/\Lambda)^{-2} u$, of the same bidegree as $u$, this is written

$$c_{1,\mathbb{R}/\Lambda}(L, \rho_\lambda) = c_1(L) + \lambda u_\Lambda.$$

For a $\mathbb{R}/\Lambda$-equivariant complex vector bundle $E$ on $\mathrm{Fix}(M)$ one defines the equivariant Chern classes of $E$ by the equivariant splitting principle. Namely, first one decomposes $E$ as the direct sum of its isotypic components,

$$E = \bigoplus_{\chi \in \Lambda^\vee} E_\chi;$$

next, one define the equivariant Chern roots of each component $E_\chi$ via the splitting principle:

$$\{\alpha_{i,\mathbb{R}/\Lambda}(E_\chi)\}_{i=1,\dots \mathrm{rk} E_\chi} = \{\alpha_i(E_\chi) + w_\chi[-2]\}_{i=1,\dots \mathrm{rk} E_\chi},$$

where the $\alpha_i(E_\chi)$'s are the Chern roots of $E_\chi$. Finally one defines the total $\mathbb{R}/\Lambda$-equivariant Chern class of $E$ by means of these equivariant Chern roots.

**Definition 2.13.** In the same notation as above, the total $\mathbb{R}/\Lambda$-equivariant Chern class of $E$ is

$$c_{\mathbb{R}/\Lambda}(E) := \prod_{\chi \in \Lambda^\vee} c_{\mathbb{R}/\Lambda}(E_\chi),$$

with

$$c_{\mathbb{R}/\Lambda}(E_\chi) := \prod_{i=1}^{\mathrm{rk}E_\chi}(1+\alpha_{i,\mathbb{R}/\Lambda}(E_\chi)).$$

In particular, the top $\mathbb{R}/\Lambda$-equivariant Chern class of $E$ is

$$c_{\mathrm{top},\mathbb{R}/\Lambda}(E) = \prod_{\chi\in\Lambda^\vee}\prod_{i=1}^{\mathrm{rk}E_\chi}(\alpha_i(E_\chi)+w_\chi[-2]).$$

*Remark* 2.14. In terms of the formal variable $u_\Lambda$ and the identification between $\Lambda^\vee$ and $\Lambda$, these read

$$c_{\mathbb{R}/\Lambda}(E) = \prod_{\lambda\in\Lambda}\prod_{i=1}^{\mathrm{rk}E_{\rho_\lambda}}(1+\alpha_{i,\mathbb{R}/\Lambda}(E_{\rho_\lambda})+\lambda u_\Lambda)$$

and

$$c_{\mathrm{top},\mathbb{R}/\Lambda}(E) = \prod_{\lambda\in\Lambda}\prod_{i=1}^{\mathrm{rk}E_{\rho_\lambda}}(\alpha_i(E_{\rho_\lambda})+\lambda u_\Lambda).$$

It is convenient to isolate the contribution from the isotypic component of the trivial character $\mathbf{0}\in\Lambda^\vee$, corresponding to the zero weight. We write

$$E = E_{\mathbf{0}}\oplus E^{\mathrm{eff}} = E_{\mathbf{0}}\oplus\bigoplus_{\chi\in\Lambda^\vee\setminus\{\mathbf{0}\}}E_\chi = E_{\mathbf{0}}\oplus\bigoplus_{\lambda\in\Lambda\setminus\{0\}}E_{\rho_\lambda},$$

and call $E^{\mathrm{eff}}$ the *effectively acted* bundle. By multiplicativity of the total Chern class and of the top Chern class one finds

$$c_{\mathbb{R}/\Lambda}(E) = c(E_{\mathbf{0}})c_{\mathbb{R}/\Lambda}(E^{\mathrm{eff}}); \qquad c_{\mathrm{top},\mathbb{R}/\Lambda}(E) = c_{\mathrm{top}}(E_{\mathbf{0}})c_{\mathrm{top},\mathbb{R}/\Lambda}(E^{\mathrm{eff}}).$$

**Definition 2.15.** The *weight polynomial* of $E^{\mathrm{eff}}$ is the element in $\mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$ given by

$$wp(E^{\mathrm{eff}}) := \prod_{\chi\in\Lambda^\vee\setminus\{\mathbf{0}\}}(w_\chi[-2])^{\mathrm{rk}E_\chi} = \prod_{\chi\in\Lambda^\vee\setminus\{\mathbf{0}\}}w_\chi^{\mathrm{rk}E_\chi}[-2\mathrm{rk}E].$$

*Remark* 2.16. By construction, the weight polynomial $wp(E^{\mathrm{eff}})$ is a nonzero element in $\mathrm{Sym}(\mathfrak{t}_\Lambda{}^\vee[-2])$.

By localizing the $\mathbb{R}/\Lambda$-equivariant cohomology of $\mathrm{Fix}(M)$ at $wp(E^{\mathrm{eff}})$, i.e., by formally inverting $wp(E^{\mathrm{eff}})$ one can rewrite the top $\mathbb{R}/\Lambda$-equivariant Chern class of $E^{\mathrm{eff}}$ as

$$c_{\mathrm{top},\mathbb{R}/\Lambda}(E^{\mathrm{eff}}) = wp(E^{\mathrm{eff}})\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}}), \tag{2.1}$$

where

$$(2.2) \qquad \widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}}) := \prod_{\chi\in\Lambda^\vee\setminus\{\mathbf{0}\}} \prod_{i=1}^{\mathrm{rk}E_\chi} \left(1 + \frac{\alpha_i(E_\chi)}{w_\chi[-2]}\right).$$

**Definition 2.17.** The degree zero element $\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}})$ in the localization $H^\bullet_{\mathbb{R}/\Lambda}(\mathrm{Fix}(M);\mathbb{R})_{(wp(E^{\mathrm{eff}}))}$ is called the *normalized top Chern class* of $E^{\mathrm{eff}}$.

*Remark* 2.18. Notice that $\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}})$ is an invertible element in the localization $H^\bullet_{\mathbb{R}/\Lambda}(\mathrm{Fix}(M);\mathbb{R})_{(wp(E^{\mathrm{eff}}))}$.

*Remark* 2.19. Equivalently, in terms of the variable $u_\Lambda$ one writes

$$c_{\mathrm{top},\mathbb{R}/\Lambda}(E^{\mathrm{eff}}) = u_\Lambda^{\mathrm{rk}E^{\mathrm{eff}}} \left(\prod_{\lambda\in\Lambda\setminus\{0\}} \lambda^{\mathrm{rk}E_{\rho_\lambda}}\right) \underbrace{\prod_{\lambda\in\Lambda\setminus\{0\}} \prod_{i=1}^{\mathrm{rk}E_{\rho_\lambda}} \left(1 + \frac{\alpha_i(E_{\rho_n})u_\Lambda^{-1}}{\lambda}\right)}_{\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}})}.$$

2.3. **Equivariant Euler classes of real vector bundles.** Since equivariant vector bundles come naturally with a notion of equivariant characteristic classes, real oriented equivariant vector bundles come with a natural notion of equivariant Euler class. And again, if the equivariant vector bundle has a trivial base space, the combinatorics behind the computation of an equivariant Euler class is purely representation theoretic.

Real irreducible representations of $\mathbb{R}/\Lambda$ are indexed by the quotient set $\Lambda^\vee/\pm$. The unique fixed point $\mathbf{0}$ corresponds to the trivial representation, which is the unique 1-dimensional real representation of $\mathbb{R}/\Lambda$; the equivalence class $[\chi]$ of the complex character $\chi$ corresponds to the irreducible real 2-dimensional representation $\chi_\mathbb{R}$. As $\chi^{-1} \cong \overline{\chi}$, we see that $(\chi^{-1})_\mathbb{R}$ and $\chi_\mathbb{R}$ are isomorphic as real representations. In terms of the distinguished isomorphism of $\Lambda^\vee \cong \Lambda$ induced by the inner product, the involution on $\Lambda^\vee$ reads $\lambda \leftrightarrow -\lambda$ and the above isomorphism of complex characters is $\rho_{-\lambda} \cong \overline{\rho_\lambda}$. In particular, we see that every nontrivial irreducible real representation of $\mathbb{R}/\Lambda$ factors through a complex character via the standard inclusion $U(1) \cong SO(2) \hookrightarrow O(2)$:

$$\begin{tikzcd} \mathbb{R}/\Lambda \arrow[r, "\chi"'] \arrow[rr, bend left, "\varphi"] & U(1) \arrow[r] & O(2) \end{tikzcd}$$

As a consequence, if we decompose an $\mathbb{R}/\Lambda$-equivariant real vector bundle $V$ over $\mathrm{Fix}(M)$ as

$$V = V_{[\mathbf{0}]} \oplus V^{\mathrm{eff}} = V_{[\mathbf{0}]} \oplus \bigoplus_{[\chi]\in\Lambda^\vee\setminus\{\mathbf{0}\}/\pm} V_{[\chi]},$$

we see that the effective component $V^{\mathrm{eff}}$ can always be endowed (non canonically) with a complex structure. In particular $V^{\mathrm{eff}}$ is always an even rank orientable vector bundle.

*Remark* 2.20. By choosing an orientation for $V^{\mathrm{eff}}$ one has a well defined equivariant Euler class for it, and a change in the choice of the orientation corresponds to a sign change in the equivariant Euler class.

The above remark leads to the following doubling trick. The two possible equivariant Euler classes for $V^{\mathrm{eff}}$, corresponding to the two possible orientations, are precisely the two solutions of the equation

$$[\omega]^2 = (-1)^{\frac{\mathrm{rk}\, V^{\mathrm{eff}}}{2}} c_{\mathrm{top},\mathbb{R}/\Lambda}(V^{\mathrm{eff}} \otimes \mathbb{C}) \tag{2.3}$$

with $[\omega]$ of degree $\frac{1}{2}\,\mathrm{rk}_{\mathbb{R}}\, V^{\mathrm{eff}}$. The choice of one solution then determines an orientation of $V^{\mathrm{eff}}$ whose corresponding equivariant Euler class is the chosen solution.

*Remark* 2.21. Since characteristic classes with real or complex coefficients can be computed via Chern-Weil theory, equation (2.3) has a simple origin in linear algebra: if $F_\nabla \in \Omega^2(M, \mathfrak{so}(2k))$ is the curvature 2-form for a Riemannian connection $\nabla$ on an even rank orientable vector bundle $V$ on a smooth manifold $M$, then the top Chern class of $V \otimes \mathbb{C}$ has a closed form representative given by the determinant $\det(\frac{1}{2\pi i}F_\nabla) = (-1)^k \det(\frac{1}{2\pi}F_\nabla)$, while the Euler class of $V$ has a closed form representative given by the Pfaffian $\mathrm{Pf}(\frac{1}{2\pi}F_\nabla)$, and for any skew-symmetric matrix $A$ in $\mathfrak{so}(2k)$ one has $\mathrm{Pf}(A)^2 = \det(A)$.

**Definition 2.22.** Let a choice of arguments for the elements $\lambda \in \Lambda\backslash\{0\}$ be fixed. The equivariant Euler class $\mathrm{eul}_{\mathbb{R}/\Lambda}(V^{\mathrm{eff}})$ defined by this choice is the distinguished solution of equation (2.3), given by

$$\mathrm{eul}_{\mathbb{R}/\Lambda}(V^{\mathrm{eff}}) := (iu_\Lambda)^{\frac{\mathrm{rk}\, V^{\mathrm{eff}}}{2}} \left( \prod_{\lambda \in \Lambda\backslash\{0\}} \lambda^{\frac{\mathrm{rk}(V^{\mathrm{eff}} \otimes \mathbb{C})_{\rho_\lambda}}{2}} \right) \sqrt{\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(V^{\mathrm{eff}} \otimes \mathbb{C})}, \tag{2.4}$$

where the determination of the square root is such that $\sqrt{1+t} = 1 + t/2 + \cdots$. The distinguished orientation on $V^{\mathrm{eff}}$ defined by the given choice of arguments is the one that is coherent with this choice of equivariant Euler class.

*Remark* 2.23. Since we are assuming $V$ is a finite rank vector bundle, only finitely many ranks $\mathrm{rk}(V^{\mathrm{eff}} \otimes \mathbb{C})_{\rho_\lambda}$ are nonzero. So the product in (2.4) is actually a finite product and one actually only needs to choose arguments for the finitely many $\lambda$'s in $\Lambda\backslash 0$ such that $\mathrm{rk}(V^{\mathrm{eff}} \otimes \mathbb{C})_{\rho_\lambda}$ is nonzero.

**Definition 2.24.** The $(\mathbb{R}/\Lambda)$-equivariant cohomology class

$$\widehat{\mathrm{eul}_{\mathbb{R}/\Lambda}}(V^{\mathrm{eff}}) := \sqrt{\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(V^{\mathrm{eff}} \otimes \mathbb{C})}$$

is called the normalized equivariant Euler class of $V^{\mathrm{eff}}$.

*Remark* 2.25. The normalized equivariant Euler class $\widehat{\mathrm{eul}_{\mathbb{R}/\Lambda}}(V^{\mathrm{eff}})$ is independent of any choice of arguments, and so is canonically associated with the real equivariant vector bundle $V$.

*Remark* 2.26. If the $\mathbb{R}/\Lambda$-equivariant vector bundle $V$ is oriented, one endows $V_{[\mathbf{0}]}$ with the orientation compatible with those of $V$ and $V^{\mathrm{eff}}$. By this procedure, applied to the tangent bundle of an oriented $\mathbb{R}/\Lambda$-manifold $M$, one gets a canonical orientation for $\mathrm{Fix}(M)$ once a choice of arguments for the nonzero elements in the lattice $\Lambda$ has been fixed.

## 3. The (anti-)holomorphic sector for a complex torus action

With this short reminder of equivariant cohomology for 1d Euclidean tori actions, we have set up the stage to describe the Cartan complex and equivariant cohomology classes for the action of 2d flat tori equipped with a complex structure.

By definition, these tori are given by the quotients $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by two dimensional lattices $\Lambda \subset \mathbb{C}$, so they are the natural generalization of the Euclidean 1d tori $\mathbb{R}/\Lambda$ considered in the previous section. The quotients $\mathbb{C}/\Lambda$ have a natural structure of Lie groups and, as in the 1d case, we will denote their Lie algebra by $\mathfrak{t}_\Lambda$. Moreover $\mathbb{C}/\Lambda$, carries a holomorphic structure compatible with the group addition, so that complex tori are an example of holomorphic Lie groups. This gives the Lie algebra $\mathfrak{t}_\Lambda$ a complex Lie algebra structure that will allow us to give the complexified Cartan complex of a $\mathbb{C}/\Lambda$-action a holomorphic kick. The following statement is immediate.

**Lemma 3.1.** *Let $M$ be a compact smooth manifold $M$ equipped with a smooth action by $\mathbb{C}/\Lambda \circlearrowright M$. The complex structure on $\mathfrak{t}_\Lambda$ gives a natural splitting $\mathfrak{t}_\Lambda \otimes_{\mathbb{R}} \mathbb{C} \cong {\mathfrak{t}_\Lambda}^{1,0} \oplus {\mathfrak{t}_\Lambda}^{0,1}$ inducing a decomposition*

$$\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_{\mathbb{C}} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{1,0})^\vee[-2]) \otimes_{\mathbb{C}} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{0,1})^\vee[-2]),$$

*of the complexified Cartan complex*

$$\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_{\mathbb{C}} \mathrm{Sym}((\mathfrak{t}_\Lambda \otimes_{\mathbb{R}} \mathbb{C})^\vee[-2])$$

*computing the equivariant cohomology of $\mathbb{C}/\Lambda \circlearrowright M$ with coefficients in $\mathbb{C}$. This realizes the complexified Cartan complex as the total complex of a tricomplex with $\Omega^{k-p-q}(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_{\mathbb{C}} \mathrm{Sym}^p(\mathfrak{t}_\Lambda^{1,0\,\vee}[-2]) \otimes_{\mathbb{C}} \mathrm{Sym}^q(\mathfrak{t}_\Lambda^{0,1\,\vee}[-2])$ in tridegree $(k,p,q)$.*

*The differential of degree* $(1,0,0)$ *in this tricomplex is the de Rham differential; the differential of degree* $(0,1,0)$ *is the operator* ${e_\Lambda}^\vee[-2]\iota_{v_{e_\Lambda}}$*, where* $(e_\Lambda, {e_\Lambda}^\vee)$ *is a pair consisting of a* $\mathbb{C}$*-linear generator of* ${\mathfrak{t}_\Lambda}^{1,0}$ *and of its dual element in* $({\mathfrak{t}_\Lambda}^{1,0})^\vee$*, and* $v_{e_\Lambda}$ *is the complex vector field on* $M$ *corresponding to* $e_\Lambda$ *via the differential of the action; the differential of degree* $(0,0,1)$ *is the operator* ${\overline{e}_\Lambda}^\vee[-2]\iota_{v_{\overline{e}_\Lambda}}$*, where* $(\overline{e}_\Lambda, {\overline{e}_\Lambda}^\vee)$ *is a pair consisting of a* $\mathbb{C}$*-linear generator of* ${\mathfrak{t}_\Lambda}^{0,1}$ *and of its dual element in* $({\mathfrak{t}_\Lambda}^{0,1})^\vee$*.*

*Remark* 3.2. The isomorphism $\mathrm{Lie}(\mathbb{C}) \xrightarrow{\sim} \mathfrak{t}_\Lambda$ induced by the projection $\mathbb{C} \to \mathbb{C}/\Lambda$ induces natural $\mathbb{C}$-linear generators for $\mathfrak{t}_\Lambda^{1,0}$ and $\mathfrak{t}_\Lambda^{0,1}$, given by the images of the complex invariant vector fields $\partial/\partial z$ and $\partial/\partial\overline{z}$, respectively. Denoting by $\xi$ and $\overline{\xi}$ the dual invariant 1-forms $dz$ ad $d\overline{z}$ placed in degree 2, the complexified Cartan tricomplex is written

$$(\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\xi,\overline{\xi}]; d_{\mathrm{dR}}, \xi\iota_{v_{\partial/\partial z}}, \overline{\xi}\iota_{v_{\partial/\partial\overline{z}}}).$$

With respect to the given trigrading, the variables $\xi$ and $\overline{\xi}$ have tridegree $(1,1,0)$ and $(1,0,1)$, respectively.

By restricting the Cartan tricomplex only to the antiholomorphic (resp. holomorphic) part, i.e. by taking only ${\mathfrak{t}_\Lambda}^{0,1}$ (resp. ${\mathfrak{t}_\Lambda}^{1,0}$) instead of $\mathfrak{t}_\Lambda \otimes_\mathbb{R} \mathbb{C}$, and taking the associated total complex, we end up with the definition of the *antiholomorphic* (resp. *holomorphic*) *sector* of the Cartan complex over $\mathbb{C}$.

**Definition 3.3.** In the same assumptions as in Lemma 3.1, the antiholomorphic sector of the complexified Cartan complex is the total complex associated with the bicomplex

$$(\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_\mathbb{C} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{0,1})^\vee[-2]); d_{\mathrm{dR}}, {\overline{e}_\Lambda}^\vee[-2]\iota_{v_{\overline{e}_\Lambda}}).$$

Its total differential will be denoted by $\overline{\partial}_{\mathbb{C}/\Lambda}$ and its cohomology by the symbol $H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M;\mathbb{C})$. By changing ${\mathfrak{t}_\Lambda}^{0,1}$ into ${\mathfrak{t}_\Lambda}^{1,0}$ one obtains the definition of the holomorphic sector.

*Remark* 3.4. In terms of the distinguished basis $\{\partial/\partial z, \partial/\partial\overline{z}\}$ of $\mathrm{Lie}(\mathbb{C}) \otimes \mathbb{C}$, the antiholomorphic sector of the Cartan complex over $\mathbb{C}$ is the total complex associated to the bicomplex

$$(\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\overline{\xi}]; d_{\mathrm{dR}}, \overline{\xi}\iota_{v_{\partial/\partial\overline{z}}}).$$

3.1. **$\mathbb{C}/\Lambda$-equivariant Chern classes.** Exactly as in the $\mathbb{R}/\Lambda$ case, $\mathbb{C}/\Lambda$-equivariant complex line bundles over $\mathrm{Fix}(M)$ are equivalently pairs $(L,\chi)$ consisting of a complex line bundle $L$ over $\mathrm{Fix}(M)$ and a character $\chi : \mathbb{C}/\Lambda \to$

$U(1)$, and the first equivariant Chern class of $(L,\chi)$ in the $\mathbb{C}/\Lambda$-equivariant Cartan complex is

$$c_{1,\mathbb{C}/\Lambda}(L,\chi) = c_1(L) + w_\chi[-2],$$

where $w_\chi$ is the weight of $\chi$, i.e., the $\mathbb{R}$-linear map defined by the commutative diagram

$$\begin{tikzcd} \mathfrak{t}_\Lambda \cong \mathbb{C} \arrow[r, "2\pi i w_\chi"] \arrow[d, "\mathrm{proj}"'] & 2\pi i\mathbb{R} \arrow[d, "\exp(-)"] \\ \mathbb{C}/\Lambda \arrow[r, "\chi"'] & U(1) \end{tikzcd}.$$

Chern classes of higher rank $\mathbb{C}/\Lambda$-equivariant complex vector bundles are defined exactly as in the $\mathbb{R}/\Lambda$ setting: one first decomposes the bundle as the direct sum of its isotypic components, and then formally splits each of these a direct sum of line bundles. This way one defines the equivariant Euler classes $\mathrm{eul}_{\mathbb{C}/\Lambda}(V^{\mathrm{eff}})$ and the normalized equivariant Euler classes $\widehat{\mathrm{eul}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})$ by generalizing Definitions 2.22 and 2.24.

*Remark* 3.5. By means of the standard Hermitian pairing on $\mathbb{C}$, the dual lattice $\Lambda^\vee$ of characters of $\mathbb{C}/\Lambda$ is identified with $\Lambda$: every character of $\mathbb{C}/\Lambda$ is of the form

$$\rho_\lambda(z) = \exp\left(\pi\frac{\lambda\bar{z} - \bar{\lambda}z}{\mathrm{vol}(\mathbb{C}/\Lambda)}\right),$$

for some $\lambda \in \Lambda$. The corresponding weight is

$$w_\lambda = \frac{\lambda d\bar{z} - \bar{\lambda}dz}{2i\mathrm{vol}(\mathbb{C}/\Lambda)}.$$

The first equivariant Chern class of $(L,\rho_\lambda)$ is given by

$$c_{1,\mathbb{C}/\Lambda}(L,\rho_\lambda) = c_1(L) + \lambda\bar{\xi}_\Lambda - \bar{\lambda}\xi_\Lambda,$$

where

$$\xi_\Lambda = \frac{\xi}{2i\mathrm{vol}(\mathbb{C}/\Lambda)}; \qquad \bar{\xi}_\Lambda = \frac{\bar{\xi}}{2i\mathrm{vol}(\mathbb{C}/\Lambda)}.$$

We will be particularly interested in the antiholomorphic part of the $\mathbb{C}/\Lambda$-equivariant Chern classes, i.e. the classes in the antiholomorphic sector obtained by evaluating the holomorphic parameter $\xi$ at 0. By the splitting principle, these are determined by the antiholomorphic parts of the equivariant first Chern classes,

$$c^{\bar{\partial}}_{1,\mathbb{C}/\Lambda}(L,\rho_\lambda) = c_{1,\mathbb{C}/\Lambda}(L,\chi)|_{\xi=0} = c_1(L) + \lambda\bar{\xi}_\Lambda. \tag{3.1}$$

From this, one has the following immediate generalization of (2.1, 2.2):

$$c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}(E^{\mathrm{eff}}) = \underbrace{\overline{\xi}_\Lambda^{\,\mathrm{rk}\,E^{\mathrm{eff}}}\left(\prod_{\lambda\in\Lambda\setminus\{0\}}\lambda^{\mathrm{rk}E_{\rho_\lambda}}\right)}_{wp^{\overline{\partial}}(E^{\mathrm{eff}})}\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E^{\mathrm{eff}}), \tag{3.2}$$

where

$$\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E^{\mathrm{eff}}) := \prod_{\lambda\in\Lambda\setminus\{0\}}\prod_{i=1}^{\mathrm{rk}E_{\rho_\lambda}}\left(1+\frac{\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}}{\lambda}\right).$$

*Remark* 3.6. The polynomial

$$wp^{\overline{\partial}}(E^{\mathrm{eff}}) = \overline{\xi}_\Lambda^{\,\mathrm{rk}\,E^{\mathrm{eff}}}\left(\prod_{\lambda\in\Lambda\setminus\{0\}}\lambda^{\mathrm{rk}E_{\rho_\lambda}}\right)$$

in the variable $\overline{\xi}_\Lambda$ is the weight polynomial of $E^{\mathrm{eff}}$ (or, more precisely, its complexification) evaluated at $\xi = 0$. One calls it the antiholomorphic weight polynomial. By construction, it is a nonzero element in $H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M);\mathbb{C})$.

**Definition 3.7.** The degree zero element $\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E^{\mathrm{eff}})$ in the localization $H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M);\mathbb{C})_{(wp^{\overline{\partial}}(E^{\mathrm{eff}}))}$ is called the *normalized antiholomorphic top Chern class* of $E^{\mathrm{eff}}$.

*Remark* 3.8. There is no particular reason to prefer the antiholomorphic sector over the holomorphic sector if not this: when $\Lambda = \Lambda_\tau$ is the lattice $\mathbb{Z}\oplus\mathbb{Z}\tau$, the association

$$\tau \longmapsto c^{\overline{\partial}}_{1,\mathbb{C}/\Lambda_\tau}(L,\rho_{m+n\tau}) = c_1(L) + (m+n\tau)\,\overline{\xi}_{\Lambda_\tau}$$

is holomorphic in terms of the modular parameter $\tau$ rather than in terms of the conjugate parameter $\overline{\tau}$.

By analogy with the construction in Section 2.3, for real $\mathbb{C}/\Lambda$-equivariant bundles we have a notion of (normalized) equivariant Euler classes in the antiholomorphic sector for their effectively acted parts.

**Definition 3.9.** Let $V$ be a real $\mathbb{C}/\Lambda$-equivariant bundle on $\mathrm{Fix}(M)$ and let $V^{\mathrm{eff}}$ be its effectively acted subbundle. For a fixed choice of the arguments for the elements $\lambda\in\Lambda\setminus\{0\}$, the equivariant Euler class of $V^{\mathrm{eff}}$ in the antiholomorphic sector is the element in $H^\bullet_{\mathbb{C}/\lambda;\overline{\partial}}(\mathrm{Fix}(M);\mathbb{C})$ defined by

$$\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(V^{\mathrm{eff}}) = (i\overline{\xi}_\Lambda)^{\frac{\mathrm{rk}\,V^{\mathrm{eff}}}{2}}\left(\prod_{\lambda\in\Lambda\setminus\{0\}}\lambda^{\frac{\mathrm{rk}(V^{\mathrm{eff}}\otimes\mathbb{C}))_{\rho_\lambda}}{2}}\right)\underbrace{\sqrt{\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{R}/\Lambda}}(V^{\mathrm{eff}}\otimes\mathbb{C})}}_{\widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})}.$$

The invertible degree zero element $\widehat{\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})$ in the localization of $H^\bullet_{\mathbb{C}/\lambda;\bar\partial}(\mathrm{Fix}(M);\mathbb{C})$ at $wp^{\bar\partial}(V^{\mathrm{eff}}\otimes_{\mathbb{R}}\mathbb{C})$ is called the normalized equivariant Euler class of $V^{\mathrm{eff}}$ in the antiholomorphic sector.

*Remark* 3.10. The normalized Euler class $\widehat{\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})$ in the antiholomorphic sector is independent of the choice of arguments for the elements $\lambda$'s.

*Remark* 3.11. Since the real vector bundle $V^{\mathrm{eff}}$ carries a complex structure the nonzero Chern roots of its complexification $V^{\mathrm{eff}}\otimes\mathbb{C}$ come in opposite pairs. From

$$\left(1+\frac{\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}}{\lambda}\right)\left(1-\frac{\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}}{\lambda}\right)=1-\frac{\alpha_i(E_{\rho_\lambda})^2\overline{\xi}_\Lambda^{-2}}{\lambda^2}$$

we see that only even powers of $\overline{\xi}_\Lambda^{-1}$ appear in the expansion of $\widehat{\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})$ as a polynomial in the variable $\overline{\xi}_\Lambda^{-1}$.

The following statement is immediate from the definitions. As it will be used several times in what follows, we make it stand out as a Lemma.

**Lemma 3.12.** *In the same notation as in Definition 3.9, the following identities hold:*

$$\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}(V^{\mathrm{eff}})=\mathrm{eul}_{\mathbb{C}/\Lambda}(V^{\mathrm{eff}})\big|_{\xi=0}$$

*and*

$$\widehat{\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})=\widehat{\mathrm{eul}_{\mathbb{C}/\Lambda}}(V^{\mathrm{eff}})\big|_{\xi=0}.$$

## 4. The antiholomorphic localization theorem

Localization techniques are a very common and powerful tool in equivariant cohomology. We will briefly recall the main theorem, the Atiyah-Bott localization theorem for a $d$-dimensional torus actions [AB84] declined in its Euclidean version, i.e., for flat tori of the form $\mathbb{R}^d/\Lambda$, and then show how for complex tori $\mathbb{C}/\Lambda$ the result continues to hold even when we restrict our attention to the antiholomorphic sector.

4.1. **The localization formula for a Euclidean torus actions.** Let $\mathbb{R}^d/\Lambda$ be a $d$-dimensional Euclidean torus, with Lie algebra $\mathfrak{t}_\Lambda$, and let $M$ be a smooth compact connected oriented finite dimensional manifold endowed with a smooth $\mathbb{R}^d/\Lambda$-action. Assume $\mathrm{Fix}(M)$ is a nonempty smooth submanifold of $M$, and denote by $\nu$ the normal bundle to the inclusion $\iota\colon \mathrm{Fix}(M)\hookrightarrow M$. The $\mathbb{R}^d/\Lambda$-action on $\nu$ is completely effective, i.e., $\nu_{\{\mathbf{0}\}}=0$ and so, by the same argument used above in the case $d=1$, the real bundle $\nu$ carries a complex structure. In particular, it is of even rank and orientable. Once an orientation is fixed, one has a well defined equivariant Euler class for $\nu$, that can be written as

$$\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu) = wp(\nu)\widehat{\mathrm{eul}_{\mathbb{R}^d/\Lambda}}(\nu)$$

with $wp(\nu)$ a degree $2\,\mathrm{rk}\,\nu$ element in $\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])$, called the weight polynomial, and $\widehat{\mathrm{eul}_{\mathbb{R}^d/\Lambda}}(\nu)$ a degree zero invertible element in the $\mathbb{R}^d/\Lambda$-equivariant cohomology of $\mathrm{Fix}(M)$ localized at $wp(\nu)$, of the form $1 + \cdots$. One orients $\mathrm{Fix}(M)$ in such a way that its orientation is compatible with those on $M$ and on $\nu$. Having fixed this notation, the Atiyah-Bott localization theorem reads as follows.

**Theorem 4.1** (Localization isomorphism)**.** *After localization at the weight polynomial $wp(\nu)$, the equivariant cohomologies of $M$ and* $\mathrm{Fix}(M)$ *become isomorphic* $\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}$*-modules. An explicit isomorphism is given by:*

$$H^\bullet_{\mathbb{R}^d/\Lambda}(M,\mathbb{R})_{(wp(\nu))} \xrightarrow{\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)^{-1}\cdot\iota^*} H^\bullet_{\mathbb{R}^d/\Lambda}(\mathrm{Fix}(M),\mathbb{R})_{(wp(\nu))}[-\mathrm{rk}\nu].$$

*The inverse isomorphism is given by the equivariant pushforward $\iota_*$.*

*Remark* 4.2. The localization isomorphism is induced by a morphism between the Cartan complexes. To realize such a morphism one only needs to choose closed forms representatives in $\Omega^\bullet(\mathrm{Fix}(M);\mathbb{C})^{\mathbb{R}^d/\Lambda}$ for the Chern classes of the normal bundle $\nu$, endowed with a chosen complex structure. Such a choice determines a representative for $\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)^{-1}$ in $\Omega^\bullet(\mathrm{Fix}(M);\mathbb{R})^{\mathbb{R}^d/\Lambda}\otimes_{\mathbb{R}}\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}$, which we will denote by the same symbol $\mathrm{eul}_{\mathbb{R}/\Lambda}(\nu)^{-1}$, and one has a morphism of differential graded $\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}$-modules

$$\begin{array}{c}\Omega^\bullet(M,\mathbb{R})^{\mathbb{R}^d/\Lambda}\otimes_{\mathbb{R}}\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}\\ \Big\downarrow{\scriptstyle \mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)^{-1}\cdot\iota^*}\\ \Omega^\bullet(\mathrm{Fix}(M),\mathbb{R})^{\mathbb{R}^d/\Lambda}\otimes_{\mathbb{R}}\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}[-\mathrm{rk}\nu].\end{array}$$

The Atiyah-Bott theorem then says that this morphism is a quasi-isomorphism.

The fact that the inverse of $\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)^{-1}\cdot\iota^*$ is the equivariant pushforward $\iota_*$ has the following important consequence.

**Corollary 4.3** (Localization formula)**.** *Let $\tilde{\omega}\in(\Omega^\bullet(M;\mathbb{R})^{\mathbb{R}^d/\Lambda})_{wp(\nu)}$ be an equivariantly closed form in the localization of the Cartan complex of $M$. Then*

$$\int_M\tilde{\omega} = \int_{\mathrm{Fix}(M)}\frac{\iota^*\tilde{\omega}}{\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)}. \tag{4.1}$$

Corollary 4.3 is often used in the following version, to compute integrals of invariant forms on $M$.

**Corollary 4.4.** *Let $\omega \in \Omega^{\dim M}(M;\mathbb{R})^{\mathbb{R}^d/\Lambda}$ be an invariant top degree form on $M$. Assume one has an equivariantly closed extension $\tilde{\omega} \in (\Omega^\bullet(M,\mathbb{R})^{\mathbb{R}^d/\Lambda} \otimes \mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2]))^{\dim M}$ of $\omega$. Then*

$$\int_M \omega = \int_{\mathrm{Fix}(M)} \frac{\iota^*\tilde{\omega}}{\mathrm{eul}_{\mathbb{R}^d/\Lambda}(\nu)}. \tag{4.2}$$

*Remark* 4.5. In the particular setting of Corollary 4.4, the localization formula (4.2) tells us that the term on its right hand side, which is a priori an element in the $\mathbb{R}$-algebra $\mathrm{Sym}(\mathfrak{t}_\Lambda^\vee[-2])_{(wp(\nu))}$, is actually a constant, i.e., an element of $\mathbb{R}$. Also notice that despite the right-hand side in (4.2) appears on first sight to depend on the choice of an orientation of $\nu$ it actually does not depend on it, as the orientation of $\mathrm{Fix}(M)$ is not fixed a priori but is determined by that of $\nu$ in such a way that they are jointly compatible with the orientation of $M$.

*Remark* 4.6. When $d = 1$, one can use (2.4) to write the localization formula (4.2) as

$$\int_M \omega = (iu_\Lambda)^{-\frac{\mathrm{rk}\,\nu}{2}} \left(\prod_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-\frac{\mathrm{rk}\nu_{\rho_\lambda}}{2}}\right) \int_{\mathrm{Fix}(M)} \frac{\iota^*\widetilde{\omega}}{\widehat{\mathrm{eul}_{\mathrm{top},\mathbb{R}/\Lambda}}(\nu)}. \tag{4.3}$$

The right hand side of (4.3) a priori depends on the choice of the arguments for the elements $\lambda \in \Lambda\setminus\{0\}$, and it is actually independent of it due to the same argument as in remark 4.5.

4.2. **The antiholomorphic localization theorem.** Let us now consider complex tori $\mathbb{C}/\Lambda$. In this situation, Theorem 4.1 becomes the following.

**Theorem 4.7** (Localization Isomorphism in the Antiholomorphic Sector)**.** *After localization at the antiholomorphic weight polynomial $wp^{\bar\partial}(\nu)$, the antiholomorphic sectors of equivariant cohomologies of $M$ and $\mathrm{Fix}(M)$ become isomorphic $\mathbb{C}[\bar\xi]_{(wp^{\bar\partial}(\nu))}$-modules. An explicit isomorphism is given by:*

$$H^\bullet_{\mathbb{C}/\Lambda;\bar\partial}(M,\mathbb{C})_{(wp^{\bar\partial}(\nu))} \xrightarrow{\mathrm{eul}^{\bar\partial}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*} H^\bullet_{\mathbb{C}/\Lambda;\bar\partial}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\bar\partial}(\nu))}[-\mathrm{rk}\nu].$$

*The inverse isomorphism is given by the restriction of the equivariant pushforward $\iota_*$ to the antiholomorphic sector.*

*Proof.* In terms of the distinguished variables $\xi$ and $\bar\xi$ introduced in Remark 3.2, the localization quasi-isomorphism is written as the quasi isomorphism of differential graded $\mathbb{C}[\xi,\bar\xi]_{(wp(\nu))}$-modules

$$\Omega^\bullet(M,\mathbb{C})^{\mathbb{C}/\Lambda}[\xi,\bar\xi]_{(wp(\nu))} \xrightarrow{\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*} \Omega^\bullet(\mathrm{Fix}(M),\mathbb{C})^{\mathbb{C}/\Lambda}[\xi,\bar\xi]_{(wp(\nu))}[-\mathrm{rk}\nu].$$

Evaluation at $\xi = 0$ induces a surjective homomorphism

$$\mathbb{C}[\xi, \overline{\xi}]_{(wp(\nu))} \xrightarrow{|_{\xi=0}} \mathbb{C}[\overline{\xi}]_{(wp^{\overline{\partial}}(\nu))}.$$

From this we get the morphism of short exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\xi\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\xi,\overline{\xi}]_{(wp(\nu))} \arrow[r, "\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*"] \arrow[d] & \xi\Omega^\bullet(\mathrm{Fix}(M);\mathbb{C})[\xi,\overline{\xi}]_{(wp(\nu))}[-\mathrm{rk}\nu] \arrow[d] \\
\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\xi,\overline{\xi}]_{(wp(\nu))} \arrow[r, "\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*"] \arrow[d, "|_{\xi=0}"'] & \Omega^\bullet(\mathrm{Fix}(M);\mathbb{C})[\xi,\overline{\xi}]_{(wp(\nu))}[-\mathrm{rk}\nu] \arrow[d, "|_{\xi=0}"] \\
\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\overline{\xi}]_{(wp^{\overline{\partial}}(\nu))} \arrow[r, "\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*"] \arrow[d] & \Omega^\bullet(\mathrm{Fix}(M);\mathbb{C})[\overline{\xi}]_{(wp^{\overline{\partial}}(\nu))}[-\mathrm{rk}\nu] \arrow[d] \\
0 & 0,
\end{tikzcd}$$

where the commutativity of the bottom square follows from Lemma 3.12.

Since the first two horizontal arrows are quasi-isomorphisms by the Atiyah-Bott localization theorem, so is the third one. This proves the first part of the statement. Since the differential on the Cartan complexes for the fixed point loci reduces to the de Rham differential acting trivially on the variables $\xi, \overline{\xi}$, the induced linear map

$$H^\bullet_{\mathbb{C}/\Lambda}(\mathrm{Fix}(M),\mathbb{C})_{(wp(\nu))} \xrightarrow{|_{\xi=0}} H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}$$

is just the evaluation at $\xi = 0$. Writing it as

$$H^\bullet(\mathrm{Fix}(M),\mathbb{C})[\xi,\overline{\xi}]_{(wp(\nu))} \xrightarrow{|_{\xi=0}} H^\bullet(\mathrm{Fix}(M),\mathbb{C})[\overline{\xi}]_{(wp^{\overline{\partial}}(\nu))}$$

one sees it is manifestly surjective. By choosing a linear section $\sigma$ to this map one defines a morphism

$$\iota_*^\sigma \colon H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}[-\mathrm{rk}\nu] \to H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M,\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}$$

as the composition

$$\begin{tikzcd}
H^\bullet_{\mathbb{C}/\Lambda}(\mathrm{Fix}(M),\mathbb{C})_{(wp(\nu))}[-\mathrm{rk}\nu] \arrow[r, "\iota_*"] & H^\bullet_{\mathbb{C}/\Lambda}(M,\mathbb{C})_{(wp(\nu))} \arrow[d, "|_{\xi=0}"] \\
H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}[-\mathrm{rk}\nu] \arrow[u, "\sigma"] & H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M,\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}
\end{tikzcd}.$$

The linear map $\iota_*^\sigma$ is an inverse to the isomorphism

$$H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M,\mathbb{C})_{(wp^{\overline{\partial}}(\nu))} \xrightarrow[\sim]{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*} H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}[-\mathrm{rk}\nu].$$

Namely, we have in cohomology

$$\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*\iota_*^\sigma = \left(\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*\iota_*\sigma\right)\big|_{\xi=0} = \sigma\big|_{\xi=0} = \mathrm{id}.$$

By uniqueness of the inverse, this in particular shows that $\iota_*^\sigma$ is actually independent of the choice of the section $\sigma$ and so we can unambiguously write $\iota_*$ for it. By construction, the morphism

$$\iota_*\colon H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M),\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}[-\mathrm{rk}\nu] \to H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M,\mathbb{C})_{(wp^{\overline{\partial}}(\nu))}$$

serves as the pushforward map between the antiholomorphic sectors. □

**Corollary 4.8** (Localization Formula in the Antiholomorphic Sector)**.** *Let $\omega_{\overline{\xi}}$ be a $\overline{\partial}_{\mathbb{C}/\Lambda}$-closed form of degree $\dim M$ in $\Omega^\bullet(M;\mathbb{C})^{\mathbb{C}/\Lambda}[\overline{\xi}]$. If $\omega_{\overline{\xi}}$ admits a degree $\dim M$ $d_{\mathbb{C}/\Lambda}$-closed extension to $\Omega^\bullet(M;\mathbb{C})[\xi,\overline{\xi}]$ then*

$$\int_M \omega_{\overline{\xi}} = (i\overline{\xi}_\Lambda)^{-\frac{\mathrm{rk}\,\nu}{2}}\left(\prod_{\lambda\in\Lambda\setminus\{0\}}\lambda^{-\frac{\mathrm{rk}(\nu\otimes\mathbb{C})_{\rho_\lambda}}{2}}\right)\int_{\mathrm{Fix}(M)}\frac{\iota^*\omega_{\overline{\xi}}}{\widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(\nu)}}.$$

*Proof.* Let $\tilde{\omega}(\xi,\overline{\xi})$ be a degree $\dim M$ $d_{\mathbb{C}/\Lambda}$-closed extension of $\omega_{\overline{\xi}}$ to $\Omega^\bullet(M;\mathbb{C})[\xi,\overline{\xi}]$. By the localization formula (Corollary 4.4) we have

$$\int_M \omega_{\overline{\xi}} = \int_{\mathrm{Fix}(M)}\frac{\iota^*\widetilde{\omega}(\xi,\overline{\xi})}{\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)}.$$

Since the left hand side is independent of $\xi$, so is the right hand side. Therefore we can write

$$\int_M \omega_{\overline{\xi}} = \left(\int_{\mathrm{Fix}(M)}\frac{\iota^*\widetilde{\omega}(\xi,\overline{\xi})}{\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)}\right)\Bigg|_{\xi=0}.$$

The rational expression $\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)^{-1}\cdot\iota^*\widetilde{\omega}(\xi,\overline{\xi})$ is defined at $\xi=0$ and evaluation at $\xi=0$ commutes with equivariant integration (which in the Cartan model is just componentwise integration of the differential form parts). So, by Lemma 3.12, we find

$$\int_M \omega_{\overline{\xi}} = \int_{\mathrm{Fix}(M)}\frac{\iota^*\widetilde{\omega}(\xi,\overline{\xi})}{\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)}\Bigg|_{\xi=0} = \int_{\mathrm{Fix}(M)}\frac{\iota^*\omega_{\overline{\xi}}}{\mathrm{eul}_{\mathbb{C}/\Lambda}(\nu)|_{\xi=0}} = \int_{\mathrm{Fix}(M)}\frac{\iota^*\omega_{\overline{\xi}}}{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}(\nu)}.$$

□

*Example* 4.9. Let $M=S^2$ with its standard metric induced by the canonical embedding $S^2\hookrightarrow\mathbb{R}^3$, and let $\omega$ be its volume form. Let us make $U(1)\cong SO(2)$ act on $S^2$ by rotations around the vertical axis, i.e., via the embedding

$SO(2) \hookrightarrow SO(3)$ given by $A \mapsto \mathrm{diag}(A, 1)$. For any nonzero $\lambda \in \Lambda \subset \mathbb{C}$, use the character $\rho_\lambda\colon \mathbb{C}/\Lambda \to U(1)$ to define a $\mathbb{C}/\Lambda$-action on $S^2$. Using stereographic coordinates on $S^2$ and polar coordinates on $\mathbb{R}^2$, one sees that

$$\omega_{\overline{\xi}} = \omega - \frac{4\pi}{1+\rho^2}\lambda\overline{\xi}_\Lambda$$

is a degree 2 $\overline{\partial}_{\mathbb{C}/\Lambda}$-closed form and that

$$\omega(\xi, \overline{\xi}) = \omega - \frac{4\pi}{1+\rho^2}(\lambda\overline{\xi}_\Lambda - \overline{\lambda}\xi_\Lambda)$$

is a degree 2 $d_{\mathbb{C}/\Lambda}$-closed extension of $\omega_{\overline{\xi}}$. The $\mathbb{C}/\Lambda$-action on $S^2$ has exactly two fixed points, the North pole corresponding to $\rho = \infty$ and the South pole corresponding to $\rho = 0$. Since the manifold of fixed points is 0-dimensional, the normalized equivariant Euler class in the antiholomorphic sector reduces to the constant 1. Choosing the arguments of $\lambda$ and $-\lambda$ in such a way that $\arg(-\lambda) = \arg(\lambda) - \pi$, the induced orientation on the manifold of fixed points gives positive orientation to the North pole and negative orientation to the South pole. From Corollary 4.8 we then find

$$\int_{S^2} \omega = \int_{S^2} \omega_{\overline{\xi}} = (i\overline{\xi}_\Lambda)^{-1}\lambda^{-1/2}(-\lambda)^{-1/2}\int_{\mathrm{Fix}(S^2)}\left(-\frac{4\pi}{1+\rho^2}\right)\lambda\overline{\xi}_\Lambda = 4\pi.$$

## 5. Conformal families of $\mathbb{C}/\Lambda$-manifolds and Modularity

So far we have been considering a single $\mathbb{C}/\Lambda$-manifold $M$, for a fixed lattice $\Lambda$. Interesting phenomena happen if we let both the lattice an the manifold vary. More precisely, we will be interested into a smooth family $M_\Lambda$ of $\mathbb{C}/\Lambda$-manifolds, with $\Lambda$ ranging over all oriented lattices in $\mathbb{C}$.

*Remark* 5.1. By saying that the family is smooth we are implicitly saying that the set of all oriented lattices in $\mathbb{C}$ has a natural structure of a smooth manifold. It is indeed so: any lattice $\Lambda \subseteq \mathbb{C}$ admits a basis given by an ordered pair $(\omega_1, \omega_2) \in \mathbb{C}^2$ with $\Im(\overline{\omega}_1\omega_2) > 0$. Denoting this open subset of $\mathbb{C}^2$ by $\mathcal{U}$, one has that

$$\mathrm{Lattices}^+(\mathbb{C}) = \mathcal{U}/SL(2;\mathbb{Z}).$$

Since the $SL(2;\mathbb{Z})$-action on $\mathcal{U}$ is free and properly discontinuous, one sees that $\mathrm{Lattices}^+(\mathbb{C})$ is naturally a smooth manifold. The intuitive notion of a smooth family of manifolds parametrized by lattices can then be formalized as the datum of a smooth and proper submersion $\mathcal{M} \to \mathrm{Lattices}^+(\mathbb{C})$. Similarly, one formalizes the notion of a smooth family of $\mathbb{C}/\Lambda$-manifolds by working in the category of smooth group actions over the base manifold $\mathrm{Lattices}^+(\mathbb{C})$. Sice writing definitions and constructions over a base makes the exposition more obscure on first reading without really adding a mathematical content, we will content ourselves in giving "fibrewise definitions", leaving to the interested reader the straightforward but a bit tedious task of writing them in terms of global objects over a base.

The multiplicative group $\mathbb{C}^*$ of nonzero complex numbers smoothly acts on the set of lattices by homotheties. The multiplication by a nonzero complex number $z$ induces a complex Lie group isomorphism

$$m_{a,\Lambda}\colon \mathbb{C}/\Lambda \xrightarrow{\sim} \mathbb{C}/a\Lambda$$

for any $\Lambda$ in $\mathrm{Lattices}^+(\mathbb{C})$, and these isomorphisms satisfy

$$m_{1,\Lambda} = \mathrm{id}_{\mathbb{C}/\Lambda}; \qquad m_{a_1a_2,\Lambda} = m_{a_1,a_2\Lambda} \circ m_{a_2,\Lambda} = m_{a_2,a_1\Lambda} \circ m_{a_1,\Lambda}$$

We say that the family $\{M_\Lambda\}$ of $\mathbb{C}/\Lambda$-manifolds is a conformal family if it is compatible with this $\mathbb{C}^*$-action. More precisely, we give the following.

**Definition 5.2.** An *conformal family* of $\mathbb{C}/\Lambda$-manifolds is a smooth family $\{M_\Lambda\}$ of $\mathbb{C}/\Lambda$-manifolds such that, for any $a \in \mathbb{C}^*$ and any oriented lattice $\Lambda$ one is given diffeomorphisms

$$\varphi_{a,\Lambda}\colon M_\Lambda \xrightarrow{\sim} M_{\mathbb{C}/a\Lambda}$$

such that

$$\varphi_{1,\Lambda} = \mathrm{id}_{M_\Lambda}; \qquad \varphi_{a_1a_2,\Lambda} = \varphi_{a_1,a_2\Lambda} \circ \varphi_{a_2\Lambda} = \varphi_{a_2,a_1\Lambda} \circ \varphi_{a_1,\Lambda}$$

and the diagram

$$\begin{tikzcd} \mathbb{C}/\Lambda \times M_\Lambda \arrow[r] \arrow[d, "(m_{a,\Lambda},\varphi_{a,\Lambda})"'] & M_\Lambda \arrow[d, "\varphi_{a,\Lambda}"] \\ \mathbb{C}/a\Lambda \times M_{a\Lambda} \arrow[r] & M_{a\Lambda} \end{tikzcd} \tag{5.1}$$

commutes.

*Remark* 5.3. In the same spirit of Remark 5.1, one can express the notion of a conformal family in terms of smooth fiber bundles over the moduli stack

$$\mathcal{M}_{1,1}(\mathbb{C}) = \mathrm{Lattices}^+(\mathbb{C})/\!/\mathbb{C}^* = \mathbb{H}/\!/SL(2,\mathbb{Z})$$

of elliptic curves over $\mathbb{C}$. Notice that neither the $\mathbb{C}^*$-action on $\mathrm{Lattices}^+(\mathbb{C})$ nor the $SL(2,\mathbb{Z})$ on the upper complex half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \,:\, \Im(\tau) > 0\}$ are free due the fact that the multiplication by $-1$ acts trivially. In terms of elliptic curves this corresponds to the standard involution realizing them as ramified double covers of $\mathbb{P}^1\mathbb{C}$. Additionally, there are points with larger stabilizers, corresponding to elliptic curves with complex multiplication.

*Remark* 5.4. It is immediate from Definition 5.2 that the diffeomorphisms $\varphi_{a,\Lambda}$ induce, by restriction to the fixed points loci, diffeomorphisms

$$\varphi_{a,\Lambda}\colon \mathrm{Fix}(M_\Lambda) \xrightarrow{\sim} \mathrm{Fix}(M_{a\Lambda}).$$

Thanks to the compatibilities between the morphisms $(\varphi_{a,\Lambda}, m_{a,\Lambda})$ and the elliptic curve actions in a conformal family, one sees that the pullback morphisms $\varphi^*_{a,\Lambda}\colon \Omega^\bullet(M_{a\Lambda};\mathbb{C}) \to \Omega^\bullet(M_\Lambda;\mathbb{C})$ induce, by restriction to invariant forms, pullback morphisms

$$\varphi^*_{a,\Lambda}\colon \Omega^\bullet(M_{a\Lambda};\mathbb{C})^{\mathbb{C}/a\Lambda} \to \Omega^\bullet(M_\Lambda;\mathbb{C})^{\mathbb{C}/\Lambda}.$$

The complex Lie group isomorphism $m_{a,\Lambda}$ induce, by passing to Lie algebras, complex linear isomorphisms of abelian Lie algebras $dm_{a,\Lambda}\colon \mathfrak{t}_\Lambda \to \mathfrak{t}_{a\Lambda}$. Under the isomorphism $\mathrm{Lie}(\mathbb{C}) \xrightarrow{\sim} \mathfrak{t}_\Lambda$ induced by the projection $\mathbb{C} \to \mathbb{C}/\Lambda$, these linear isomorphisms are just multiplications by the complex number $a$. Complexifying and dualizing we obtain the $\mathbb{C}$-linear automorphism of $\mathrm{Lie}(\mathbb{C})^\vee \otimes \mathbb{C}$ that acts on the distinguished basis $(dz, d\overline{z})$ as $dz \mapsto a\,dz$ and $d\overline{z} \mapsto \overline{a}\,d\overline{z}$. Therefore,in terms of the distinguished basis $(\xi, \overline{\xi})$ of $(\mathfrak{t}_\Lambda \otimes_\mathbb{R} \mathbb{C})^\vee[-2]$ consisting of $dz$ and $d\overline{z}$ placed in degree 2, the $\mathbb{C}$-linear isomorphism

$$\varphi^*_{\mathfrak{t};a,\Lambda}\colon \mathfrak{t}^\vee_{a\Lambda} \otimes \mathbb{C} \xrightarrow{\sim} \mathfrak{t}^\vee_\Lambda \otimes \mathbb{C}$$

induced by $m_a$ is given by

$$\varphi^*_{\mathfrak{t};a,\Lambda}\colon \xi \mapsto a\xi; \qquad \varphi^*_{\mathfrak{t};a,\Lambda}\colon \overline{\xi} \mapsto \overline{a\xi}$$

*Remark* 5.5. From

$$\mathrm{vol}(\mathbb{C}/a\Lambda) = a\overline{a}\,\mathrm{vol}(\mathbb{C}/\Lambda),$$

one sees that that in terms of the distinguished basis $(\xi_{a\Lambda}, \overline{\xi}_{a\Lambda})$ and $(\xi_\Lambda, \overline{\xi}_\Lambda)$ the isomorphism $\varphi^*_{\mathfrak{t};a,\Lambda}$ reads

$$\varphi^*_{\mathfrak{t};a,\Lambda}\colon \xi_{a\Lambda} \mapsto \overline{a}^{-1}\xi_\Lambda; \qquad \varphi^*_{\mathfrak{t};a,\Lambda}\colon \overline{\xi}_{a\Lambda} \mapsto a^{-1}\overline{\xi}_\Lambda.$$

**Lemma 5.6.** *The data of a conformal family of $\mathbb{C}/\Lambda$-manifolds induce isomorphisms of complexified Cartan tricomplexes*

$$\begin{aligned}\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathfrak{t};a,\Lambda}\colon \Omega^\bullet(M_{a\Lambda};\mathbb{C})^{\mathbb{C}/a\Lambda} &\otimes_\mathbb{C} \mathrm{Sym}(({\mathfrak{t}_{a\Lambda}}^{1,0})^\vee[-2]) \otimes_\mathbb{C} \mathrm{Sym}(({\mathfrak{t}_{a\Lambda}}^{0,1})^\vee[-2])\\ &\xrightarrow{\sim} \Omega^\bullet(M_\Lambda;\mathbb{C})^{\mathbb{C}/\Lambda} \otimes_\mathbb{C} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{1,0})^\vee[-2]) \otimes_\mathbb{C} \mathrm{Sym}(({\mathfrak{t}_\Lambda}^{0,1})^\vee[-2]).\end{aligned}$$

*Proof.* Since the three differentials are trivial on the generators coming from $\mathfrak{t}^\vee_{a\Lambda}$, we only need to check compatibility with differentials on $\mathbb{C}/a\Lambda$-invariant differential forms on $M_{a\Lambda}$. That is, for an element $\omega \in \Omega^\bullet(M_{a\Lambda};\mathbb{C})^{\mathbb{C}/a\Lambda}$ we have to check the three identities

$$\begin{aligned}
d_{\mathrm{dR}}(\varphi^*_{a,\Lambda}\omega) &= \varphi^*_{a,\Lambda}(d_{\mathrm{dR}}\omega);\\
\xi\iota_{v^{\Lambda}_{\partial/\partial z}}(\varphi^*_{a,\Lambda}\omega) &= \varphi^*_{\mathfrak{t};a,\Lambda}(\xi)\varphi^*_{a,\Lambda}(\iota_{v^{a\Lambda}_{\partial/\partial z}}\omega)\\
\overline{\xi}\iota_{v^{\Lambda}_{\partial/\partial \overline{z}}}(\varphi^*_{a,\Lambda}\omega) &= \varphi^*_{\mathfrak{t};a,\Lambda}(\overline{\xi})\varphi^*_{a,\Lambda}(\iota_{v^{a\Lambda}_{\partial/\partial \overline{z}}}\omega),
\end{aligned}$$

where $v^{\Lambda}_{\partial/\partial z}$ and $v^{a\Lambda}_{\partial/\partial z}$ are the complex vector fields on $M_\Lambda$ and $M_{a\Lambda}$ corresponding to $\partial/\partial z$ via the differentials of the actions of $\mathbb{C}/\Lambda$ and $C/a\Lambda$, respectively, and similarly for $v^{\Lambda}_{\partial/\partial \overline{z}}$ and $v^{a\Lambda}_{\partial/\partial \overline{z}}$. The first identity is obvious. Using $\varphi^*_{\mathfrak{t};a,\lambda}(\xi) = a\xi$, the second identity reduces to $\iota_{v^{\Lambda}_{\partial/\partial z}}(\varphi^*_{a,\Lambda}\omega) = a\varphi^*_{a,\Lambda}(\iota_{v^{a\Lambda}_{\partial/\partial z}}\omega)$. By definition of the pullback of differential forms, $\iota_{v^{\Lambda}_{\partial/\partial z}}(\varphi^*_{a,\Lambda}\omega) = \varphi^*_{a,\Lambda}(\iota_{d\varphi_{a,\Lambda}(v^{\Lambda}_{\partial/\partial z})}\omega)$, and so we are reduced to proving the identity $d\varphi_{a,\Lambda}(v^{\Lambda}_{\partial/\partial z}) = v^{a\Lambda}_{a\partial/\partial z}$. Since $a\partial/\partial z = dm_{a,\Lambda}\partial/\partial z$, the identity we have to prove is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{t}_\Lambda \arrow[r, "v^\Lambda"] \arrow[d, "dm_{a,\Lambda}"'] & \mathrm{VectorFields}(M_\Lambda) \arrow[d, "d\varphi_{a,\Lambda}"] \\
\mathfrak{t}_{a\Lambda} \arrow[r, "v^{a\Lambda}"] & \mathrm{VectorFields}(M_{a\Lambda}),
\end{tikzcd}$$

which is immediate from (5.1). The proof of the third identity is identical. $\square$

**Corollary 5.7.** *The data of a conformal family of $\mathbb{C}/\Lambda$-manifolds induce isomorphisms between the antiholomorphic sector of the complexified Cartan complex of $M_{a\Lambda}$ and that of $M_\Lambda$, for any oriented lattice $\Lambda$ and every $a \in \mathbb{C}^*$.*

**Corollary 5.8.** *In a conformal family, the $\mathbb{C}/\Lambda$-equivariant cohomology of $M_\Lambda$ and the $\mathbb{C}/a\Lambda$-equivariant cohomology of $M_{a\Lambda}$ are canonically isomorphic. The same holds for their (anti-)holomorphic sectors.*

*Remark* 5.9. In global terms, Corollary 5.8 amounts to saying that $H^\bullet_{\mathbb{C}/\Lambda}(M_\Lambda;\mathbb{C})$ and $H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(M_\Lambda;\mathbb{C})$ define complex vector bundles over the moduli stack $\mathcal{M}_{1,1}(\mathbb{C})$.

5.1. **Equivariant vector bundles over conformal families.** Given a family $\{M_\Lambda\}$ of $\mathbb{C}/\Lambda$-manifolds, we can consider a family of $\mathbb{C}/\Lambda$-equivariant vector bundles $E_\Lambda$ over the fixed loci $\mathrm{Fix}(M_\Lambda)$. Again, regularity of the family $\{E_\Lambda\}$ can be expressed in terms of a single equivariant vector bundle $\mathcal{E}$ over $\mathrm{Fix}(\mathcal{M})$, where $\mathcal{M} \to \mathrm{Lattices}^+(\mathbb{C})$ is the smooth and proper submersion from Remark 5.1, but here too we will content us with fibrewise definitions. When $\{M_\Lambda\}$ is a conformal family, it is natural to consider vector bundles $E_\Lambda$ that form a conformal family, too. This leads to the following.

**Definition 5.10.** A conformal family of $\mathbb{C}/\Lambda$-equivariant vector bundles $E_\Lambda$ over the fixed loci of a conformal family of $\mathbb{C}/\Lambda$-manifolds is a smooth family of equivariant vector bundles equipped with isomorphisms of vector bundles

$$\psi_{a,\Lambda}\colon E_\Lambda \xrightarrow{\sim} \varphi_{a,\Lambda}^* E_{a\Lambda}$$

making the diagrams

$$\begin{tikzcd} \mathbb{C}/\Lambda \times E_\Lambda \arrow[r] \arrow[d, "(m_{a,\Lambda},\psi_{a,\Lambda})"'] & E_\Lambda \arrow[d, "\psi_{a,\Lambda}"] \\ \mathbb{C}/a\Lambda \times \varphi_{a,\Lambda}^* E_{a\Lambda} \arrow[r] & \varphi_{a,\Lambda}^* E_{a\Lambda} \end{tikzcd}$$

commute for any oriented lattice $\Lambda$ and any $a \in \mathbb{C}^*$, such that

$$\psi_{a_1a_2,\Lambda} = \varphi_{a_1,\Lambda}^*(\psi_{a_2,a_1\Lambda}) \circ \psi_{a_1,\Lambda} \tag{5.2}$$

for any $a_1, a_2$.

*Example* 5.11. The restrictions to the fixed loci of the tangent bundles $TM_\Lambda$ for a conformal family $\{M_\Lambda\}$ are a conformal family of $\mathbb{C}/\Lambda$-equivariant vector bundles, with isomorphisms $\psi_{a,\Lambda}$ given by the differentials of the diffeomorphisms $\varphi_{a,\Lambda}$:

$$d\varphi_{a,\Lambda}\colon TM_\Lambda \xrightarrow{\sim} \varphi_{a,\Lambda}^* TM_{a\Lambda}.$$

Equation (5.2) in this case is

$$d\varphi_{a_1a_2,\Lambda} = \varphi_{a_1,\Lambda}^*(d\varphi_{a_2,a_1\Lambda}) \circ d\varphi_{a_1,\Lambda}$$

and so it is satisfied due to the chain rule for differentials, since $\varphi_{a_1a_2,\Lambda} = \varphi_{a_2,a_1\Lambda} \circ \varphi_{a_1,\Lambda}$. The tangent bundles $T\mathrm{Fix}(M_\Lambda)$ to the fixed loci form a conformal family of subbundles of $\{TM_\Lambda\big|_{\mathrm{Fix}(M_\Lambda)}\}$, and so the normal bundles to the fixed loci

$$\nu_\Lambda = \frac{TM_\Lambda\big|_{\mathrm{Fix}(M_\Lambda)}}{T\mathrm{Fix}(M_\Lambda)}$$

are a conformal family.

**Lemma 5.12.** *Let $\{(L_\Lambda, \chi_\Lambda)\}$ be a conformal family of $\mathbb{C}/\Lambda$-equivariant complex line bundles on* $\mathrm{Fix}(M_\Lambda)$. *Then*

$$(\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{ta},\Lambda}^*)(c_{1,\mathbb{C}/a\Lambda}(L_{a\Lambda}, \chi_{a\Lambda})) = c_{1,\mathbb{C}/\Lambda}(L_\Lambda, \chi_\Lambda)$$

*and*

$$(\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{ta},\Lambda}^*)(c^{\overline{\partial}}_{1,\mathbb{C}/a\Lambda}(L_{a\Lambda}, \chi_{a\Lambda})) = c^{\overline{\partial}}_{1,\mathbb{C}/\Lambda}(L_\Lambda, \chi_\Lambda).$$

*Proof.* In the notation of Remark 3.5, the character $\chi_\Lambda$ will be of the form $\rho_\lambda$ for some $\lambda \in \Lambda$. By definition of conformal family, the diagram

$$\begin{tikzcd}
\mathbb{C}/\Lambda \times L_\Lambda \arrow[r, "\chi_\Lambda \cdot"] \arrow[d, "(m_{a,\Lambda}, \psi_{a,\Lambda})"'] & L_\Lambda \arrow[d, "\psi_{a,\Lambda}"] \\
\mathbb{C}/a\Lambda \times \varphi_{a\Lambda}^* L_{a\Lambda} \arrow[r, "\chi_{a\Lambda} \cdot"] & \varphi_{a\Lambda}^* L_{a\Lambda}
\end{tikzcd}$$

commutes, and so $\chi_\Lambda(z) = \chi_{a\Lambda}(az)$ for any $z \in \mathbb{C}$. This means that $\chi_{a\Lambda} = \rho_{a\lambda}$. Now we compute

$$c_{1,\mathbb{C}/a\Lambda}(L_{a\Lambda}, \chi_{a\Lambda}) = c_{1,\mathbb{C}/a\Lambda}(L_{a\Lambda}, \rho_{a\lambda}) = c_1(L_{a\Lambda}) + a\lambda\overline{\xi}_{a\Lambda} - \overline{a\lambda}\xi_{a\Lambda},$$

and so by Remark 5.5 we have

$$\begin{aligned}
(\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{t}a,\Lambda}^*)(c_{1,\mathbb{C}/a\Lambda}(L_{a\Lambda}, \chi_{a\Lambda})) &= (\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{t}a,\Lambda}^*)(c_1(L_{a\Lambda}) + a\lambda\overline{\xi}_{a\Lambda} - \overline{a\lambda}\xi_{a\Lambda}) \\
&= c_1(\varphi_{a,\Lambda}^* L_{a\Lambda}) + \lambda\overline{\xi}_\Lambda - \overline{\lambda}\xi_\Lambda \\
&= c_1(L_\Lambda) + \lambda\overline{\xi}_\Lambda - \overline{\lambda}\xi_\Lambda \\
&= c_{1,\mathbb{C}/\Lambda}(L_\Lambda, \chi_\Lambda)).
\end{aligned}$$

The proof for the antiholomorphic part is analogous. □

By the splitting principle we therefore get the following.

**Proposition 5.13.** *Let* $\{E_\Lambda\}$ *be a conformal family of* $\mathbb{C}/\Lambda$*-equivariant complex vector bundles over* $\mathrm{Fix}(M_\Lambda)$*. Then we have*

$$(\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{t};a,\Lambda}^*)(c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/a\Lambda}(E^{\mathrm{eff}}_{a\Lambda})) = c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}(E^{\mathrm{eff}}_\Lambda)$$

*and*

$$(\varphi_{a,\Lambda}^* \otimes \varphi_{\mathrm{t};a,\Lambda}^*)(\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/a\Lambda}}(E^{\mathrm{eff}}_{a\Lambda})) = \widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E^{\mathrm{eff}}_\Lambda).$$

*Proof.* The first identity is immediate from Lemma 5.12 and the splitting principle. To prove the second identity, we write

$$c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/a\Lambda}(E^{\mathrm{eff}}_{a\Lambda}) = \overline{\xi}_{a\Lambda}^{\;\mathrm{rk}\, E^{\mathrm{eff}}_{a\Lambda}} \left( \prod_{a\lambda \in a\Lambda \setminus \{0\}} (a\lambda)^{\mathrm{rk} E_{a\Lambda;\rho_{a\lambda}}} \right) \widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E^{\mathrm{eff}}_{a\Lambda}),$$

notice that $\mathrm{rk}E_{a\Lambda;\rho_{a\lambda}} = \mathrm{rk}E_{\Lambda;\rho_\lambda}$for every $\lambda \in \Lambda$, and use the first identity to get

$$\begin{aligned}
&\overline{\xi}_\Lambda^{\;\mathrm{rk}\, E_\Lambda^{\mathrm{eff}}} \left( \prod_{\lambda\in\Lambda\setminus\{0\}} \lambda^{\mathrm{rk}E_{\Lambda;\rho_\lambda}} \right) \widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E_\Lambda^{\mathrm{eff}}) = c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}(E_\Lambda^{\mathrm{eff}}) \\
&\quad = (\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathrm{t};a,\Lambda})(\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/a\Lambda}(E_{a\Lambda}^{\mathrm{eff}}}) ) \\
&\quad = (\varphi^*_{\mathrm{t};a,\Lambda}\overline{\xi}_{a\Lambda})^{\mathrm{rk}\, E_\Lambda^{\mathrm{eff}}} \left( \prod_{\lambda\in\Lambda\setminus\{0\}} (a\lambda)^{\mathrm{rk}E_{\Lambda;\rho_\lambda}} \right) (\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathrm{t};a,\Lambda})\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E_{a\Lambda}^{\mathrm{eff}}) \\
&\quad = \overline{\xi}_\Lambda^{\mathrm{rk}\, E_\Lambda^{\mathrm{eff}}} \left( \prod_{\lambda\in\Lambda\setminus\{0\}} \lambda^{\mathrm{rk}E_{\Lambda;\rho_\lambda}} \right) (\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathrm{t};a,\Lambda})\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E_{a\Lambda}^{\mathrm{eff}}),
\end{aligned}$$

and the conclusion follows. $\square$

**Corollary 5.14.** *Let* $\{V_\Lambda\}$ *be a conformal family of* $\mathbb{C}/\Lambda$*-equivariant real vector bundles on* $\mathrm{Fix}(M_\Lambda)$*. Then we have*

$$(\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathrm{t};a,\Lambda})(\widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/a\Lambda}}(V_{a\Lambda}^{\mathrm{eff}})) = \widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}}(V_\Lambda^{\mathrm{eff}})$$

Specializing this to the case considered in Example 5.11 we find the following.

**Corollary 5.15.** *Let* $\{M_\Lambda\}$ *be a conformal family of* $\mathbb{C}/\Lambda$*-manifolds, and let* $\nu_\Lambda$ *be the normal bundle for* $\mathrm{Fix}(M_\Lambda) \hookrightarrow M_\Lambda$*. Then we have*

$$(\varphi^*_{a,\Lambda} \otimes \varphi^*_{\mathrm{t};a,\Lambda})(\widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/a\Lambda}}(\nu_{a\Lambda})) = \widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}}(\nu_\Lambda)$$

*for any oriented lattice* $\Lambda$ *and any* $a \in \mathbb{C}^*$*.*

5.2. **Trivializations of the fixed points bundle and modular forms.** Assume now we have a trivialization of the family of fixed points submanifolds of a conformal family $\{M_\Lambda\}$. In terms of smooth bundles over the moduli stack $\mathcal{M}_{1,1}$, this is a trivialization of the smooth fiber bundle $\mathrm{Fix}(\mathcal{M}) \to \mathcal{M}_{1,1}$. In terms of fibrewise definitions, this is the following.

**Definition 5.16.** Let $\{M_\Lambda\}$ be a conformal family. A trivialization of the conformal family $\{\mathrm{Fix}(M_\Lambda)\}$ is the datum of a smooth manifold $X$ and of a collection of diffeomorphisms

$$j_\Lambda \colon X \xrightarrow{\sim} \mathrm{Fix}(M_\Lambda)$$

such that the diagrams

$$
\begin{tikzcd}
 & \mathrm{Fix}(M_\Lambda) \arrow[dd, "\varphi_{a,\Lambda}"] \\
X \arrow[ur, "j_\Lambda"] \arrow[dr, "j_{a\Lambda}"'] & \\
 & \mathrm{Fix}(M_{a\Lambda})
\end{tikzcd}
\tag{5.3}
$$

commute for any oriented lattice $\Lambda$ and any $a \in \mathbb{C}^*$.

A trivialization of the fixed points bundle produces a trivialization of the $\mathbb{C}/\Lambda$-equivariant cohomologies of the fixed loci and of their antiholomorphic sectors. More precisely, we have the following lemma, whose proof is immediate from Remark 5.5 and Definition 5.16.

**Lemma 5.17.** *Let* $\xi_X$ *and* $\overline{\xi}_X$ *be two variables in degree 2, and let* $j^*_{\mathrm{t};X}\colon \mathbb{C}[\xi_\Lambda, \overline{\xi}_\Lambda] \xrightarrow{\sim} \mathbb{C}[\xi_X, \overline{\xi}_X]$ *be the ring isomorphism induced by*

$$j^*_{\mathrm{t};\Lambda}\colon \xi_\Lambda \mapsto \xi_X; \quad j^*_{\mathrm{t};\Lambda}\colon \overline{\xi}_\Lambda \mapsto \overline{\xi}_X.$$

*For any* $a \in \mathbb{C}^*$, *let* $\mu_a\colon \mathbb{C}[\xi_X, \overline{\xi}_X] \xrightarrow{\sim} \mathbb{C}[\xi_X, \overline{\xi}_X]$ *be the ring isomorphism induced by*

$$\mu_a\colon \xi_X \mapsto \overline{a}^{-1}\xi_X; \quad \mu_a\colon \overline{\xi}_X \mapsto a^{-1}\overline{\xi}_X.$$

*Then*

$$H^\bullet_{\mathbb{C}/\Lambda}(\mathrm{Fix}(M_\Lambda), \mathbb{C}) \cong H^\bullet(\mathrm{Fix}(M_\Lambda), \mathbb{C})[\xi_\Lambda, \overline{\xi}_\Lambda] \xrightarrow{j^*_\Lambda \otimes j^*_{\mathrm{t};\Lambda}} H^\bullet(X, \mathbb{C})[\xi_X, \overline{\xi}_X]$$

*is an isomorphism of graded rings, and all the diagrams*

$$
\begin{tikzcd}
H^\bullet_{\mathbb{C}/\Lambda}(\mathrm{Fix}(M_{a\Lambda}), \mathbb{C}) \arrow[r, "j^*_{a\Lambda}\otimes j^*_{\mathrm{t};a\Lambda}"] \arrow[d, "\varphi^*_{a,\Lambda}\otimes\varphi^*_{\mathrm{t};a,\Lambda}"'] & H^\bullet(X, \mathbb{C})[\xi_X \overline{\xi}_X] \arrow[d, "\mathrm{id}\otimes\mu_a"] \\
H^\bullet_{\mathbb{C}/\Lambda}(\mathrm{Fix}(M_\Lambda), \mathbb{C}) \arrow[r, "j^*_\Lambda\otimes j^*_{\mathrm{t};\Lambda}"'] & H^\bullet(X, \mathbb{C})[\xi_X, \overline{\xi}_X]
\end{tikzcd}
$$

*commute. The same hold for the antiholomorphic sectors:*

$$H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M_\Lambda), \mathbb{C}) \cong H^\bullet(\mathrm{Fix}(M_\Lambda), \mathbb{C})[\overline{\xi}_\Lambda] \xrightarrow{j^*_\Lambda \otimes j^*_{\mathrm{t};\Lambda}} H^\bullet(X, \mathbb{C})[\overline{\xi}_X]$$

*is an isomorphism of graded rings, and all the diagrams*

$$
\begin{tikzcd}
H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M_{a\Lambda}), \mathbb{C}) \arrow[r, "j^*_{a\Lambda}\otimes j^*_{\mathrm{t};a\Lambda}"] \arrow[d, "\varphi^*_{a,\Lambda}\otimes\varphi^*_{\mathrm{t};a,\Lambda}"'] & H^\bullet(X, \mathbb{C})[\overline{\xi}_X] \arrow[d, "\mathrm{id}\otimes\mu_a"] \\
H^\bullet_{\mathbb{C}/\Lambda;\overline{\partial}}(\mathrm{Fix}(M_\Lambda), \mathbb{C}) \arrow[r, "j^*_\Lambda\otimes j^*_{\mathrm{t};\Lambda}"'] & H^\bullet(X, \mathbb{C})[\overline{\xi}_X]
\end{tikzcd}
$$

*commute.*

The second statement in Lemma 5.17 clearly continues to hold after we localize $H^\bullet(\mathrm{Fix}(M_\Lambda),\mathbb{C})[\overline{\xi}_\Lambda]$ at $\overline{\xi}_\Lambda$ and $H^\bullet(X,\mathbb{C})[\overline{\xi}_X]$ at $\overline{\xi}_X$. Recalling Remark 3.11, we can now give the main definition of this section.

**Definition 5.18.** Let $\{M_\Lambda\}$ be a conformal family of $\mathbb{C}/\Lambda$-manifolds with a given trivialization $(X,\{j_\Lambda\})$ of its fixed points, and let $\mathcal{E}=\{E_\Lambda\}$ and $\mathcal{V}=\{V_\Lambda\}$ be a conformal family of $\mathbb{C}/\Lambda$-equivariant complex and real vector bundles over $\mathrm{Fix}(M_\Lambda)$, respectively. The *total complex Witten class* of $\mathcal{E}$ is the function

$$\mathrm{Wit}_{\mathcal{E}}\colon \mathrm{Lattices}^+(\mathbb{C})\to H^\bullet(X,\mathbb{C})[\overline{\xi}_X^{-1}]$$

given by

$$\mathrm{Wit}_{\mathcal{E}}(\Lambda):=(j_\Lambda^*\otimes j_{\mathrm{t};\Lambda}^*)\left(\frac{1}{\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(E_\Lambda^{\mathrm{eff}})}\right).$$

The coefficient

$$\mathrm{Wit}_{\mathcal{E};k}\colon \mathrm{Lattices}^+(\mathbb{C})\to H^{2k}(X,\mathbb{C})$$

defined by the expansion

$$\mathrm{Wit}_{\mathcal{E}}(\Lambda)=\sum_{k=0}^{\infty}\mathrm{Wit}_{\mathcal{E};k}(\Lambda)\xi_X^{-k}$$

will be called the $k$-th complex Witten class of $\mathcal{E}$. The *total real Witten class* of $\mathcal{V}$ is the function

$$\mathrm{Wit}_{\mathbb{R};\mathcal{V}}\colon \mathrm{Lattices}^+(\mathbb{C})\to H^\bullet(X,\mathbb{C})[\overline{\xi}_X^{-1}]$$

given by

$$\mathrm{Wit}_{\mathbb{R};\mathcal{V}}(\Lambda):=(j_\Lambda^*\otimes j_{\mathrm{t};\Lambda}^*)\left(\frac{1}{\widehat{\mathrm{eul}^{\overline{\partial}}_{\mathbb{C}/\Lambda}}(V_\Lambda^{\mathrm{eff}})}\right).$$

The coefficient

$$\mathrm{Wit}_{\mathcal{V};k}\colon \mathrm{Lattices}^+(\mathbb{C})\to H^{4k}(X,\mathbb{C})$$

defined by the expansion

$$\mathrm{Wit}_{\mathbb{R};\mathcal{V};k}(\Lambda)=\sum_{k=0}^{\infty}\mathrm{Wit}_{\mathbb{R};\mathcal{V};k}(\Lambda)\xi_X^{-2k}$$

will be called the $k$-th real Witten class of $\mathcal{V}$.

**Proposition 5.19.** *The $k$-th complex Witten class* $\mathrm{Wit}_{\mathcal{E};k}$ *is a modular form of weight $k$. The $k$-th real Witten class* $\mathrm{Wit}_{\mathbb{R};\mathcal{V};k}$ *is a modular form of weight $2k$.*

*Proof.* We give a proof for the complex Witten classes first. We have to show that for any $a \in \mathbb{C}^*$ and any oriented lattice $\Lambda$ we have

$$\mathrm{Wit}_{\mathcal{E};k}(a\Lambda) = a^k \mathrm{Wit}_{\mathcal{E},k}(\Lambda)$$

and that, denoting by $\Lambda_\tau$ the lattice $\Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$ for a complex number $\tau$ in the upper complex half-plane $\mathbb{H}$ one has that

$$\begin{aligned} \mathbb{H} &\to H^{2k}(X, \mathbb{C})[\overline{\xi}_X^{-1}] \\ \tau &\mapsto \mathrm{Wit}_{\mathcal{E};k}(\Lambda_\tau) \end{aligned}$$

is a holomorphic function of $\tau$. The first identity follows from Lemma 5.17 and Proposition 5.13. Indeed, we have

$$\begin{aligned}
\mathrm{Wit}_{\mathcal{E}}(a\Lambda) &= (j_{a\Lambda}^* \otimes j_{\mathrm{t};a\Lambda}^*) \left( \frac{1}{\widehat{c_{\mathrm{top},\mathbb{C}/a\Lambda}^{\overline{\partial}}(E_{a\Lambda}^{\mathrm{eff}})}} \right) \\
&= (\mathrm{id} \otimes \mu_{a^{-1}}) \circ (j_{\Lambda}^* \otimes j_{\mathrm{t};\Lambda}^*) \circ (\varphi_{a\Lambda}^* \otimes \varphi_{\mathrm{t};a\Lambda}^*) \left( \frac{1}{\widehat{c_{\mathrm{top},\mathbb{C}/a\Lambda}^{\overline{\partial}}(E_{a\Lambda}^{\mathrm{eff}})}} \right) \\
&= (\mathrm{id} \otimes \mu_{a^{-1}}) \circ (j_{\Lambda}^* \otimes j_{\mathrm{t};\Lambda}^*) \left( \frac{1}{\widehat{c_{\mathrm{top},\mathbb{C}/\Lambda}^{\overline{\partial}}(E_{\Lambda}^{\mathrm{eff}})}} \right) \\
&= (\mathrm{id} \otimes \mu_{a^{-1}}) \mathrm{Wit}_{\mathcal{E}}(\Lambda).
\end{aligned}$$

Expanding this identity gives

$$\sum_{k=0}^{\infty} \mathrm{Wit}_{\mathcal{E};k}(a\Lambda)\xi_X^{-k} = \sum_{k=0}^{\infty} a^k \mathrm{Wit}_{\mathcal{E};k}(\Lambda)\xi_X^{-k}.$$

Holomorphicity of $\tau \mapsto \mathrm{Wit}_{\mathcal{E};k}(\Lambda_\tau)$ is immediate from Remark 3.8. The proof for the real Witten classes is identical, by using Corollary 5.14. ☐

**Corollary 5.20.** *In the same assumptions as in Definition 5.18, if $X$ is a manifold of even dimension $d$, then*

$$\int_X \mathrm{Wit}_{\mathcal{E};d/2}$$

*is a complex valued modular form of weight $d/2$. If $X$ is a manifold of dimension $d$ with $d \equiv 0 \mod 4$, then*

$$\int_X \mathrm{Wit}_{\mathbb{R};\mathcal{V};d/4}$$

*is a complex valued modular form of weight $d/2$.*

We conclude this section with a definition.

**Definition 5.21.** Let $\mathcal{M} = \{M_\Lambda\}$ be a conformal family of $\mathbb{C}/\Lambda$-manifolds with a given trivialization $(X, \{j_\Lambda\})$ of its fixed points. The real an complex Witten classes of $\mathcal{M}$, denoted by $\mathrm{Wit}_{\mathbb{R};\mathcal{M};k}$ and $\mathrm{Wit}_{\mathcal{M};k}$, are defined as the real Witten classes of the normal bundles $\{\nu_\Lambda\}$ and the complex Witten classes of the complexified normal bundles $\{\nu_\Lambda \otimes \mathbb{C}\}$, That is,

$$\mathrm{Wit}_{\mathbb{R};\mathcal{M};k} := \mathrm{Wit}_{\mathbb{R};\{\nu_\Lambda\};k}; \qquad \mathrm{Wit}_{\mathcal{M};k} := \mathrm{Wit}_{\{\nu_\Lambda\otimes\mathbb{C}\};k}.$$

## 6. The Witten class of double loop spaces

Let now $X$ be a smooth $d$-dimensional manifold. A paradigmatic example of a conformal family of $\mathbb{C}/\Lambda$-manifolds is given by

$$M_\Lambda := \mathrm{Maps}(\mathbb{C}/\Lambda, X),$$

the spaces of smooth maps from $\mathbb{C}/\Lambda$ to $X$, with their standard Fréchet infinite-dimensional smooth manifold structures, and with $\mathbb{C}/\Lambda$-actions given by translation: $z \star \gamma \colon p \mapsto \gamma(p+z)$. The isomorphisms

$$\varphi_{a,\Lambda} \colon \mathrm{Maps}(\mathbb{C}/\Lambda, X) \xrightarrow{\sim} \mathrm{Maps}(\mathbb{C}/a\Lambda, X)$$

are given by the pullbacks along $m_{a^{-1},a\Lambda} \colon \mathbb{C}/a\Lambda \to \mathbb{C}/\Lambda$. The commutativity of (5.1) is then the trivial identity $a^{-1}(p+az) = a^{-1}p + z$. The submanifold of fixed points for this action consists of the submanifold of constant loops, so that we have a canonical trivialization $j_\Lambda \colon X \xrightarrow{\sim} \mathrm{Fix}(M_\Lambda)$ mapping a point $x \in X$ to the constant map $\gamma_x \colon \mathbb{C}/\Lambda \to X$ with constant value $x$. The commutativity of (5.3) is trivial. We are thus in the situation considered in Section 5.2 and so, by Proposition 5.19, Corollary 5.20 and Definition 5.21, modular forms are naturally associated with the conformal family $\{\mathrm{Maps}(\mathbb{C}/\Lambda, X)\}$ and so, ultimately, to $X$.

Things are however not so straightforward. Indeed, as the normal bundle to $X$ in $\mathrm{Maps}(\mathbb{C}/\Lambda, X)$ has infinite rank, we will need to make sense of the now infinite products defining normalized equivariant top Chern classes and Euler classes. The idea here is to write

$$\left(1 + \frac{\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}}{\lambda}\right) = \left(1 + \frac{z}{\lambda}\right)\Big|_{z=\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}}$$

where we think of the degree zero variable $z$ as of a complex variable, to compute the product of the factors $1 + z/\lambda$ so to obtain an entire function $\Phi(z)$ and then to compute $\Phi(\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1})$ by putting $z = \alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}$ in the Taylor expansion of $\Phi$ at $z = 0$. Notice that this last operation makes sense without any convergence issue as, by the finite dimensionality of $X$, the degree zero element $\alpha_i(E_{\rho_\lambda})\overline{\xi}_\Lambda^{-1}$ is nilpotent. Yet, there is no guarantee that the infinite product of the factors $1 + z/\lambda$ will converge, and actually it does not. So one has to suitably regularize it in order to get a convergent product. A convenient way of doing so is by the technique of Weierstraß $\zeta$-regularization that we recall below. Our problems are not over, yet: $\zeta$-regularization may disrupt the expected modularity of Witten classes, so we will have to check in the end whether this is preserved. It will turn out that a topological constraint on $X$ has to be imposed in order to maintain modularity: $X$ has to be a rational string manifold.

With these premises, we can now determine the Witten classes of the conformal family $\{M_\Lambda\} = \{\mathrm{Maps}(\mathbb{C}/\Lambda, X)\}$. We will assume $X$ to be 2-connected so that $M_\Lambda$ is connected. One can weaken this assumption by requiring $X$ to be only connected, and taking $M_\Lambda$ to be the space $\mathrm{Maps}_0(\mathbb{C}/\Lambda, X)$ of homotopically trivial maps from $\mathbb{C}/\Lambda$ to $X$. Or one can even remove any connectedness assumption on $X$ by working separately on each of the connected components of $\mathrm{Maps}_0(\mathbb{C}/\Lambda, X)$ (which bijectively correspond to the connected components of $X$).

As the smooth structure on $M_\Lambda$ is the standard Fréchet one, the tangent space at the point $\gamma \in \mathrm{Maps}(\mathbb{C}/\Lambda, X)$ is the space $H^0(\mathbb{C}/\Lambda; \gamma^*TX)$ of smooth sections of the pullback of the tangent bundle of $X$ via $\gamma$. In particular, for any $x \in X$ we have

$$T_{\gamma_x} M_\Lambda = C^\infty(\mathbb{C}/\Lambda; T_x X) = C^\infty(\mathbb{C}/\Lambda; \mathbb{R}) \otimes_{\mathbb{R}} T_x X,$$

and so the restriction of the complexified tangent bundle of $M_\Lambda$ to $X = \mathrm{Fix}(M_\Lambda)$ is

$$TM \otimes \mathbb{C}\big|_X = C^\infty(\mathbb{C}/\Lambda; \mathbb{C}) \otimes_{\mathbb{C}} (TX \otimes_{\mathbb{R}} \mathbb{C}).$$

*Remark* 6.1. In writing $X = \mathrm{Fix}(M_\Lambda)$ we have identified $X$ with $\mathrm{Fix}(M_\Lambda)$ via $j_\Lambda$. We will keep this identification fixed in all that follows, so that we will see $X$ as a submanifold of $M_\Lambda$ and consequently reduce ${}_\Lambda$ to the identity of $X$.

Since Fourier polynomials are dense in the Fréchet topology of $C^\infty(\mathbb{C}/\Lambda; \mathbb{C})$, in the same vein of [Ati85], we consider the Fourier topological direct sum decomposition

$$TM_\Lambda \otimes \mathbb{C}\big|_X = \left( \bigoplus_{\lambda \in \Lambda} \mathbb{C}_{(\lambda)} \right) \otimes_{\mathbb{C}} (TX \otimes_{\mathbb{R}} \mathbb{C}),$$

where $\mathbb{C}_{(\lambda)}$ is the 1-dimensional representation of $\mathbb{C}/\Lambda$ with character $\rho_\lambda$. This immediately implies

$$\nu_\Lambda \otimes \mathbb{C} = \left( \bigoplus_{\lambda \in \Lambda \setminus \{0\}} \mathbb{C}_{(\lambda)} \right) \otimes_{\mathbb{C}} (TX \otimes_{\mathbb{R}} \mathbb{C}),$$

where $\nu_\Lambda$ denotes the normal bundle for the inclusion $X \hookrightarrow \mathrm{Maps}(\mathbb{C}/\Lambda, X)$.

By formally applying formula (3.2) to this infinite rank situation, we obtain

$$\begin{aligned}
\widehat{c^{\overline{\partial}}_{\mathrm{top},\mathbb{C}/\Lambda}}(\nu_\Lambda \otimes \mathbb{C}) &= \prod_{\lambda\in\Lambda\setminus\{0\}} \prod_{i=1}^{d} \left(1 + \frac{\alpha_i(X)\overline{\xi}_\Lambda^{-1}}{\lambda}\right) \\
&= \prod_{i=1}^{d} \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 + \frac{z}{\lambda}\right)\bigg|_{z=\alpha_i(X)\overline{\xi}_\Lambda^{-1}}, \qquad (6.1)
\end{aligned}$$

where $\alpha_1(X), \ldots, \alpha_d(X)$ are the Chern roots of the complexified tangent bundle $TX \otimes_{\mathbb{R}} \mathbb{C}$ of $X$. As anticipated, to compute (and actually give a meaning to) the infinite product

$$\prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 + \frac{z}{\lambda}\right)$$

one uses Weierstraß $\zeta$-regularization. For the reader's convenience we recall the basics of the procedure here. A detailed treatment can be found, e.g., in [Rud87, Chapter 15]. For any $r \geqslant 0$, let

$$P_r(z) := \sum_{j=1}^{r} \frac{z^j}{j} = \begin{cases} 0 & \text{if } r = 0 \\ z + \frac{z^2}{2} + \cdots + \frac{z^r}{r} & \text{if } r > 0. \end{cases}$$

If $\{\kappa_n\}$ is a sequence of nonzero complex numbers with $|\kappa_n| \to +\infty$ for $n \to +\infty$ and $\{p_n\}$ is a sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} \left(\frac{R}{|\kappa_n|}\right)^{1+p_n} < +\infty$$

for every $R > 0$, then the infinite product

$$\mathrm{Wei}_{\vec{\kappa},\vec{p}}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\kappa_n}\right) e^{P_{p_n}(z/\kappa_n)}$$

converges and defines an entire function which has a zero at each point $\kappa_n$ and no other zeroes. More precisely, if $\kappa$ occurs with multiplicity $m$ in the sequence $\{\kappa_n\}$ then $\mathrm{Wei}_{\vec{\kappa},\vec{p}}(z)$ has a zero of order $m$ at $z = \kappa$. Moreover, the infinite product defining $\mathrm{Wei}_{\vec{\kappa},\vec{p}}(z)$ is unchanged under simultaneous renumbering of $\{\kappa_n\}$ and $\{p_n\}$. If $\Lambda \subseteq \mathbb{C}$ is a lattice, then the series

$$\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{|\lambda|^s} \qquad (6.2)$$

converges for $\Re(s) > 2$. This implies that for any $r \geqslant 2$ the series

$$\sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{R}{|\lambda|}\right)^{1+r} = R^{r+1} \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{|\lambda|^{r+1}}$$

converges, and so, by choosing $p_n$ to be the constant sequence $p_n \equiv r$ one sees that the infinite product

$$\prod_{\lambda\in\Lambda\backslash\{0\}} \left(1+\frac{z}{\lambda}\right) e^{P_r(-z/\lambda)}$$

defines an entire function of $z$. Here we used the renumbering invariance to write the product as a product over $\Lambda\backslash\{0\}$. Now one makes a choice of arguments for the elements $\lambda \in \Lambda\backslash\{0\}$ in such a way that the $\zeta$-function $\zeta_{\Lambda\backslash\{0\}}$, defined by analytic extension of the holomorphic function

$$\zeta_{\Lambda\backslash\{0\}}(s) = \sum_{\lambda\in\Lambda\backslash\{0\}} \frac{1}{\lambda^s}; \qquad \Re(s) > 2$$

is defined at $s = 1$ and at $s = 2$. By convergence of (6.2) for $\Re(s) > 2$, the $\zeta$-function $\zeta_{\Lambda\backslash\{0\}}$ will then be defined at every positive integer, and one can formally write

$$\begin{aligned}
\prod_{\lambda\in\Lambda\backslash\{0\}} \left(1+\frac{z}{\lambda}\right) &= \prod_{\lambda\in\Lambda\backslash\{0\}} e^{-P_r(-z/\lambda)} \left(1+\frac{z}{\lambda}\right) e^{P_r(-z/\lambda)} \\
&= \left(\prod_{\lambda\in\Lambda\backslash\{0\}} e^{-P_r(-z/\lambda)}\right) \prod_{\lambda\in\Lambda\backslash\{0\}} \left(1+\frac{z}{\lambda}\right) e^{P_r(-z/\lambda)} \\
&= e^{-\left(-\zeta_{\Lambda\backslash\{0\}}(1)z+\zeta_{\Lambda\backslash\{0\}}(2)\frac{z^2}{2}+\cdots+\zeta_{\Lambda\backslash\{0\}}(r)\frac{(-z)^r}{r}\right)} \prod_{\lambda\in\Lambda\backslash\{0\}} \left(1+\frac{z}{\lambda}\right) e^{P_r(-z/\lambda)},
\end{aligned}$$

where in the last step one has replaced the possibly divergent sums $\sum_{\lambda\in\Lambda\backslash\{0\}} \frac{1}{\lambda^j}$, for $j = 1, \ldots, r$ with their $\zeta$-regularizations. Thanks to absolute convergence, the terms $e^{(-z/\lambda)^k}$ freely move in and out from the product over $\Lambda\backslash\{0\}$ for $k > 2$. Therefore, the last term in the above chain of formal identities is independent of $r$ as soon as $r \geqslant 2$ and we arrive at the following.

**Definition 6.2.** Let a choice of arguments for the elements $\lambda \in \Lambda\backslash\{0\}$ in such a way that $\zeta_{\Lambda\backslash\{0\}}$ is defined at $r = 1$ and at $r = 2$ be fixed. The Weierstraß $\zeta$-regularized product of the factors $(1 + z/\lambda)$ with $\lambda$ ranging in $\Lambda\backslash\{0\}$ is

$${\prod_{\lambda\in\Lambda\backslash\{0\}}}^{\zeta} \left(1+\frac{z}{\lambda}\right) := e^{\zeta_{\Lambda\backslash\{0\}}(1)z-\zeta_{\Lambda\backslash\{0\}}(2)\frac{z^2}{2}} \prod_{\lambda\in\Lambda\backslash\{0\}} \left(1+\frac{z}{\lambda}\right) e^{-\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}.$$

*Remark* 6.3. Choices of arguments for the elements $\lambda \in \Lambda\backslash\{0\}$ such that $\zeta_{\Lambda\backslash\{0\}}(1)$ and $\zeta_{\Lambda\backslash\{0\}}(2)$ are defined do actually exist and moreover there are quite natural choices with this property, see Remark 6.10 below.

We can now turn (6.1) into a formal definition.

**Definition 6.4.** Let $\nu_\Lambda$ be the normal bundle for the inclusion $X \hookrightarrow \mathrm{Maps}(\mathbb{C}/\Lambda, X)$. The Weierstraß $\zeta$-regularized equivariant top Chern class of $\nu_\Lambda \otimes \mathbb{C}$ in the antiholomorphic sector is defined as

$$\widehat{c^{\overline{\partial};\zeta}_{\mathrm{top},\mathbb{C}/\Lambda}}(\nu_\Lambda \otimes \mathbb{C}) := \prod_{i=1}^{d} \prod_{\lambda\in\Lambda\setminus\{0\}}^{\zeta} \left(1+\frac{z}{\lambda}\right)\Big|_{z=\alpha_i(X)\overline{\xi}_\Lambda^{-1}}$$

By analogy with Definition 5.18 and Definition 5.21 we then give the following.

**Definition 6.5.** In the same assumptions as in Definition 6.2, the *$\zeta$-regularized total complex Witten class* of $\{\mathrm{Maps}(\mathbb{C}/\Lambda, X)\}$ is the function

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}} \colon \mathrm{Lattices}^{+}(\mathbb{C}) \to H^{\bullet}(X,\mathbb{C})[\overline{\xi}_X^{-1}]$$

given by

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) := (j_\Lambda^* \otimes j_{\mathrm{t};\Lambda}^*)\left(\frac{1}{\widehat{c^{\overline{\partial};\zeta}_{\mathrm{top},\mathbb{C}/\Lambda}}(\nu_\Lambda \otimes \mathbb{C})}\right).$$

The coefficient

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k} \colon \mathrm{Lattices}^{+}(\mathbb{C}) \to H^{2k}(X,\mathbb{C})$$

defined by the expansion

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = \sum_{k=0}^{\infty} \mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k}(\Lambda)\xi_X^{-k}$$

will be called the $k$-th $\zeta$-regularized complex Witten class of $\{\mathrm{Maps}(\mathbb{C}/\Lambda, X)\}$.

**Lemma 6.6.** *In the same assumptions as in Definition 6.2 one has*

$$\prod_{\lambda\in\Lambda\setminus\{0\}}^{\zeta} \left(1+\frac{z}{\lambda}\right) = e^{\zeta_{\Lambda\setminus\{0\}}(1)z - \zeta_{\Lambda\setminus\{0\}}(2)\frac{z^2}{2}} \frac{\sigma_\Lambda(z)}{z},$$

*where $\sigma_\Lambda(z)$ is the Weierstraß $\sigma$-function of the lattice $\Lambda$.*

*Proof.* By definition of the Weierstraß $\sigma$-function, see, e.g., [WW15, Section 20.42], one has

$$\sigma_\Lambda(z) = z \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1-\frac{z}{\lambda}\right) e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}. \tag{6.3}$$

The statement then follows by changing $\lambda$ in $-\lambda$ in the above product and by comparing with Definition 6.2. □

**Corollary 6.7.** *In the same assumptions as in Definition 6.2 one has*

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = e^{\zeta_{\Lambda\setminus\{0\}}(2)p_1(TX)\bar{\xi}_X^{-2}} \prod_{i=1}^{d} \frac{z}{\sigma_\Lambda(z)}\bigg|_{z=\alpha_i(X)\bar{\xi}_X^{-1}}, \tag{6.4}$$

*where $p_1(TX)$ denotes the first Pontryagin class of $TX$ seen as an element in $H^4(X;\mathbb{C})$.*

*Proof.* From Definitions 6.2 and 6.4 and from Lemma 6.6, recalling Remark 6.1 and that $j^*_{\mathrm{t};\Lambda}(\overline{xi}_\Lambda) = \bar{\xi}_X$, one has

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = \prod_{i=1}^{d} \left( e^{-\zeta_{\Lambda\setminus\{0\}}(1)z+\zeta_{\Lambda\setminus\{0\}}(2)\frac{z^2}{2}} \frac{z}{\sigma_\Lambda(z)} \right)\bigg|_{z=\alpha_i(X)\bar{\xi}_X^{-1}}. \tag{6.5}$$

One then rewrites the right hand side of (6.5) as

$$e^{-\zeta_{\Lambda\setminus\{0\}}(1)c_1(TX\otimes\mathbb{C})\bar{\xi}_\Lambda^{-1}+\zeta_{\Lambda\setminus\{0\}}(2)\left(\frac{1}{2}c_1(TX\otimes\mathbb{C})^2-c_2(TX\otimes\mathbb{C})\right)\bar{\xi}_\Lambda^{-1}} \prod_{i=1}^{d} \frac{z}{\sigma_\Lambda(z)}\bigg|_{z=\alpha_i(X)\bar{\xi}_X^{-1}},$$

recalls that the odd Chern classes of the complexification of a real vector bundle vanish, and uses the relation $c_2(TX \otimes_{\mathbb{R}} \mathbb{C}) = -p_1(TX)$ to conclude. □

It is important to stress that the $\zeta$-function $\zeta_{\Lambda\setminus\{0\}}$ and its value at 2 depend on the choice of arguments for the elements $\lambda$ in $\Lambda\setminus\{0\}$. One removes this dependence by requiring that $p_1(X)$ is zero in $H^4(X;\mathbb{Q})$, i.e., by requiring that $X$ is a rational string manifold. With this assumption, formula (6.4) reduces to

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = \prod_{i=1}^{d} \frac{z}{\sigma_\Lambda(z)}\bigg|_{z=\alpha_i(X)\bar{\xi}_\Lambda^{-1}}, \tag{6.6}$$

where now the right hand side is a canonically defined equivariant cohomology class in the antiholomorphic sector.

The entire function $z/\sigma_\Lambda(z)$ is the characteristic power series for the complex Witten genus [AHR10]. Therefore, summing up, we have obtained the following.

**Proposition 6.8.** *Let $X$ be a $d$-dimensional rational string manifold.*

*Then $\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}$ is the Witten class of the manifold $X$. In particular, if $d$ is even then*

$$\int_X \mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};d/2}$$

*is the complex Witten genus of $X$.*

*Remark* 6.9. Proposition 6.8 in particular tells us that if $X$ is a rational string manifold, then the $\zeta$-regularized complex Witten classes

$$\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k} \colon \mathrm{Lattices}^+(\mathbb{C}) \to H^{2k}(X,\mathbb{C})$$

are modular forms of weight $k$. This can be directly seen from (6.6) by the modular properties of the function $(\Lambda, z) \mapsto z/\sigma_\Lambda(x)$, which are in turn immediate from the product formula (6.3).

*Remark* 6.10. When $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\Im(\tau) > 0$, with the standard choice of arguments $-\pi \leqslant \arg(\lambda) < \pi$, one gets

$$\zeta_{\Lambda\setminus\{0\}}(2) = -4\pi i \frac{\eta'(\tau)}{\eta(\tau)} = G_2(\tau)$$

where $\eta$ is the Dedekind $\eta$-function and $G_2$ is the quasi-modular Eisenstein series

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{n\in\mathbb{Z}\setminus 0} \sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^2},$$

see [Apo12, Chapter 3, Ex.1] and [QHS93, Example 13]. This explains the exponential prefactor

$$e^{-G_2(\tau)p_1(X)}$$

appearing in the expression of the Witten class for a non rationally string manifold for lattices of the standard form $\mathbb{Z} \oplus \mathbb{Z}\tau$. More generally, once an oriented basis $(\omega_1, \omega_2)$ for the lattice $\Lambda$ is chosen, one can write $\Lambda = \omega_1^{-1}(\mathbb{Z}\oplus\tau\mathbb{Z})$ with $\tau = \omega_2/\omega_1$, choose an argument for $\omega_1$ in $[-\pi, \pi)$ and choose the arguments of the elements $\lambda \in \Lambda\setminus\{0\}$ so that

$$-\pi + \arg(\omega_1) \leqslant \arg(\lambda) < \pi + \arg(\omega_1).$$

With this choice one has $\zeta_{\Lambda\setminus\{0\}}(2) = \omega_1^{-2} G_2(\tau)$.

As an immediate corollary, we get the analogous of Proposition 6.8 for the real Witten class. We first need an couple of obvious definitions.

**Definition 6.11.** Let $\nu_\Lambda$ be the normal bundle for the inclusion $X \hookrightarrow \mathrm{Maps}(\mathbb{C}/\Lambda, X)$. The Weierstraß $\zeta$-regularized equivariant Euler class of $\nu_\Lambda$ in the antiholomorphic sector is defined as

$$\mathrm{eul}^{\bar\partial;\zeta}_{\mathbb{C}/\Lambda}(\nu_\Lambda) := \sqrt{\widehat{c^{\bar\partial;\zeta}_{\mathrm{top},\mathbb{C}/\Lambda}}(\nu_\Lambda \otimes \mathbb{C})},$$

where the determination of the square root is such that $\sqrt{1+\cdots} = 1 + \cdots$.

**Definition 6.12.** In the same assumptions as in Definition 6.2, the *$\zeta$-regularized total real Witten class* of $\{\mathrm{Maps}(\mathbb{C}/\Lambda, X)\}$ is the function

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}} \colon \mathrm{Lattices}^+(\mathbb{C}) \to H^\bullet(X,\mathbb{C})[\bar\xi_X^{-1}]$$

given by

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) := (j^*_\Lambda \otimes j^*_{\mathrm{t};\Lambda})\left(\frac{1}{\widehat{\mathrm{eul}^{\bar\partial;\zeta}_{\mathbb{C}/\Lambda}}(\nu_\Lambda)}\right).$$

The coefficient

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k}\colon \mathrm{Lattices}^+(\mathbb{C}) \to H^{4k}(X,\mathbb{C})$$

defined by the expansion

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = \sum_{k=0}^{\infty} \mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k}(\Lambda)\xi_X^{-2k}$$

will be called the $k$-th $\zeta$-regularized real Witten class of $\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}$.

From equation (6.6) and by the usual $\sqrt{z} \leftrightarrow z$ rule for passing from passing from Pontryagin classes of a real vector bundle to Chern classes of its complexification in characteristic power series for genera (see, e.g., [Hir78, Section 1.3]), we have that if $X$ is a rational string manifold of even dimension $d$, then

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}(\Lambda) = \prod_{i=1}^{d/2} \frac{\sqrt{z}}{\sigma_\Lambda(z)}\bigg|_{z=\beta_i(X)\xi_X^{-2}}, \tag{6.7}$$

where the $\beta_j(X)$ are the Pontryagin roots of $TX$. From this, one has the real Witten classes analogue of Proposition 6.8.

**Proposition 6.13.** *Let $X$ be a rational string manifold of even dimension $d$. Then $\mathrm{Wit}^{\zeta}_{\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\}}$ is the Witten class of the manifold $X$. In particular, if $d \equiv 0 \mod 4$ then*

$$\int_X \mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};d/2}$$

*is the real Witten genus of $X$.*

Clearly, we also have the analogue of Remark 6.9

*Remark* 6.14. If $X$ is a rational string manifold of even dimension $d$, then the $\zeta$-regularized real Witten classes

$$\mathrm{Wit}^{\zeta}_{\mathbb{R};\{\mathrm{Maps}(\mathbb{C}/\Lambda,X)\};k}\colon \mathrm{Lattices}^+(\mathbb{C}) \to H^{4k}(X,\mathbb{C})$$

are modular forms of weight $2k$.

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**Appendices**

## 1. Generalising to other mapping spaces, Transgressed Bundles, Todd, $\hat{A}$, and a toy example

In the paper we considered the normal bundle relative to the inclusion $X \hookrightarrow L^2X = \mathrm{Maps}(\mathbb{T}^2, X)$. This was, of course, due to us wanting to use the localization theorem for torus actions. The careful reader may have noticed how here the torus $\mathbb{T}^2$ plays two roles: it is the source space for the space of maps $\mathrm{Maps}(\mathbb{T}^2, X)$ and it is the group acting on this space with $X$ as manifold of fixed points.

An obvious generalization to consider, especially in the context of TFTs, consists in allowing arbitrary closed compact manifolds as source manifolds for the space of maps, and arbitrary subgroups of the diffeomorphism groups of these manifolds as groups of symmetries: given a manifold $\Sigma$ acted smoothly upon by a group $D$, the mapping space $\mathrm{Maps}(\Sigma, X)$ naturally inherits a $D$-action. Moreover, if the $D$-action on $\Sigma$ is transitive, then the fixed point locus for the $D$-action on $\mathrm{Maps}(\Sigma, X)$ is the submanifold of constant maps and so it is canonically identified with $X$.

When $\mathrm{Maps}(\Sigma, X)$ is given its standard Fréchet manifold structure, the tangent bundle to $\mathrm{Maps}(\Sigma, X)$ can be described in terms of a pull-push procedure. Consider the span

$$\mathrm{Maps}(\Sigma, X) \xleftarrow{\pi} \mathrm{Maps}(\Sigma, X) \times \Sigma \xrightarrow{ev} X$$

with $\pi$ the projection on the first factor and $ev$ sending a pair $(\gamma, s)$ to $\gamma(s)$. Then $T(\mathrm{Maps}(\Sigma, X)) = \pi_* ev^* TX$, where the fibre of $\pi_* V$ over a point $\gamma$ is given by the space of sections of $V$ over the fibre $\Sigma \cong \pi^{-1}(\gamma)$. In other words, the tangent space over a map $\gamma : \Sigma \to X$ is given by the space $\Gamma(\Sigma; \gamma^* TX)$. When $\gamma$ is an embedding, this can be thought of as the space of sections of $TX$ over the image of $\gamma$.

With this in mind, it is clear how the restriction of the tangent bundle of $\mathrm{Maps}(\Sigma, X)$ to $X \hookrightarrow \mathrm{Maps}(\Sigma, X)$ identified with the submanifold of constant maps is given, over a point $x$, by $C^\infty(\Sigma, TX_x)$. Thus the normal bundle of $\iota : X \hookrightarrow \mathrm{Maps}(\Sigma, X)$ will be given (over $x$) by $C^\infty(\Sigma, TX_x)/TX_x$.

We generalize this process with the following definition:

**Definition 1.1.** Given two manifolds $\Sigma, X$, and a vector bundle $V \to X$, we define the transgressed bundle $\tau(V)$ to be $\iota^* \pi_* ev^*(V)$ with $\pi$ and $ev$ as above. We also define $\psi(V) := \tau(V)/V$.

If $\Sigma$ admits a $D$-action, then both $\tau(V)$ and $\psi(V)$ will be $D$-equivariant bundles over $X$.

If $V \to X$ is a real or complex vector bundle on $X$, characteristic classes of $V$ appearing in various classical genus formulas can be expressed as inverse equivariant normalized top Chern or Euler classes of $\psi(V)$ for a suitable choice of the manifold $\Sigma$ and of the group $D$. For instance, the result of Atiyah recovering $\hat{A}(X)$ from circular symmetry [Ati85] can be immediately generalized to an arbitrary real vector bundle $V \to X$ obtaining $\hat{A}(V)$ as the inverse of the equivariant normalized Euler class of $\psi(V)$ for $\Sigma = \mathbb{T}^1$ and $D = \mathbb{T}^1$ acting on itself by translations. We show this below. In doing this we will also see how the Todd class of a complex vector bundle $F \to X$ is recovered as the inverse of the equivariant normalized top Chern class of $\psi(V)$, again with $(\Sigma, D) = (\mathbb{T}^1, \mathbb{T}^1)$.

To begin with we recall from our article that, given a lattice $\Lambda$ in $\mathbb{R}$ and given a $\mathbb{R}/\Lambda$-equivariant complex bundle $E$ on a $\mathbb{R}/\Lambda$-trivial manifold, its normalized top Chern class is given by

$$\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(E^{\mathrm{eff}}) = \prod_{\lambda\in\Lambda\setminus\{0\}} \prod_{i=1}^{\mathrm{rk} E_{\rho_\lambda}} \left(1 + \frac{\alpha_i(E_{\rho_n}) u_\Lambda^{-1}}{\lambda}\right) =$$

$$= \prod_{\lambda\in\Lambda\setminus\{0\}} \prod_{i=1}^{\mathrm{rk} E_{\rho_\lambda}} \left(1 + \frac{z}{\lambda}\right)\bigg|_{z=\alpha_i(E_{\rho_n})u_\Lambda^{-1}}.$$

Unfortunately this product is only convergent for bundles whose effective part is finite dimensional. This is obviously true for finite dimensional bundles and obviously false for transgressed bundles. We can fix this non-convergence issue for a transgressed bundle $E = \tau(F)$ exactly as we did in the paper for the $\mathbb{C}/\Lambda$-equivariant bundles. The effectively acted part of $\tau(F)$ is $\psi(F)$ and the 1d version of the 2d computation done in the article gives( for details on the $\zeta$-regularizations used in this computation see [QHS93])

$$\prod_{\lambda\in\Lambda\setminus\{0\}}^{\zeta} \left(1 + \frac{z}{\lambda}\right) = e^{-\operatorname{vol}(\mathbb{R}/\Lambda)^{-1} i\pi z} \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 + \frac{z}{\lambda}\right) e^{-z/\lambda}.$$

Recalling that

$$\prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 + \frac{z}{\lambda}\right) e^{-z/\lambda} = \frac{\sin\left(\pi z \operatorname{vol}(\mathbb{R}/\Lambda)^{-1}\right)}{\pi z \operatorname{vol}(\mathbb{R}/\Lambda)^{-1}}.$$

and setting $\tilde{u_\Lambda} = \frac{u}{2\pi i \operatorname{vol}(\mathbb{R}/\Lambda)}$, one then obtains

$$\widehat{c^{\zeta}_{\mathrm{top},\mathbb{R}/\Lambda}}(\psi(F)) = \prod_{i=1}^{\mathrm{rk}\, F} e^{-i\pi z} \frac{\sin(\pi z)}{\pi z}\bigg|_{z=\alpha_i(F)(\operatorname{vol}(\mathbb{R}/\Lambda)u_\Lambda)^{-1}} =$$

$$= \prod_{i=1}^{\mathrm{rk}\, F} \frac{1 - e^{-z}}{z}\bigg|_{z=\alpha_i(F)\tilde{u_\Lambda}^{-1}},$$

i.e.,

$$\widehat{c^{\zeta}_{\mathrm{top},\mathbb{R}/\Lambda}}(\psi(F)) = \mathrm{td}(F, \tilde{u_\Lambda})^{-1},$$

with $\mathrm{td}(F, \tilde{u_\Lambda})$ the homogeneous Todd polynomial of $F$ in the variable $\tilde{u}_\Lambda$. From the point of view of equivariant characteristic classes, this is nicely rewritten as

$$\mathrm{td}(F, \tilde{u_\Lambda}) = \frac{1}{\widehat{c^{\zeta}_{\mathrm{top},\mathbb{R}/\Lambda}}(\psi(F))}.$$

Next, we recall that

$$\frac{1-e^{-z}}{z} = e^{-z/2}\frac{\sinh z/2}{z/2}.$$

This implies that, if $V \to X$ is a real vector bundle and $F = V \otimes \mathbb{C}$ is its complexification, then

$$\widehat{c^{\zeta}_{\mathrm{top},\mathbb{R}/\Lambda}}(\psi(V\otimes\mathbb{C})) = \mathrm{td}(V\otimes\mathbb{C}, \tilde{u_\Lambda})^{-1} = \hat{A}(V, \tilde{u_\Lambda})^{-2}.$$

In other words, recalling that

$$\widehat{\mathrm{eul}_{\mathbb{R}/\Lambda}}(W^{\mathrm{eff}}) := \sqrt{\widehat{c_{\mathrm{top},\mathbb{R}/\Lambda}}(W^{\mathrm{eff}}\otimes\mathbb{C})},$$

where the determination of the square root is such that $\sqrt{1+\cdots} = 1+\cdots$, we have

$$\hat{A}(V, \tilde{u_\Lambda}) = \frac{1}{\widehat{\mathrm{eul}^{\zeta}_{\mathbb{R}/\Lambda}}(\psi(V))}.$$

*Remark* 1.2. In the article on the Witten genus, its modular properties have been derived through conformal invariance. On the other hand, these modular properties have a topological origin. This is seen by noticing that instead of considering just $\mathbb{T}^2$-translations on $\mathrm{Maps}(\mathbb{T}^2, \Sigma)$, one could have considered the action of the whole group $\mathrm{Diff}^+(\mathbb{T}^2)$ of oriented diffeomorphisms of $\mathbb{T}^2$. The modular group $SL(2;\mathbb{Z})$ action on $\mathbb{T}^2$-equivariant cohomology of the fixed locus $X \hookrightarrow \mathrm{Maps}(\mathbb{T}^2, \Sigma)$ then shows up by noticing that $\mathrm{Diff}^+(\mathbb{T}^2)$ retracts on the subgroup $D = SL(2;\mathbb{Z}) \ltimes \mathbb{T}^2$. From the point of view of oriented lattices $\Lambda \subseteq \mathbb{C}$, this $SL(2;\mathbb{Z})$-action is reflected in the fact that the moduli stack of oriented lattices in $\mathbb{C}$ is equivalent to $BSL(2;\mathbb{Z})$. A detailed topological derivation of the Witten genus via the $(SL(2;\mathbb{Z}) \ltimes \mathbb{T}^2)$-equivariant cohomology of $\mathrm{Maps}(\mathbb{T}^2, \Sigma)$ will be hopefully carried out in forthcoming research.

1.1. **A toy example with Stiefel-Whitney classes.** As a conclusion, we investigate a toy example, based on the simplest nontrivial space of maps: the one obtained by taking $\Sigma = S^0$, endowed with the transitive action of the symmetric group on 2 elements. Equivalently this is the translation action of the group $S^0 = \{1, -1\}$ on itself. For a manifold $X$, we have $\mathrm{Maps}(S^0, X) = X \times X$, and the induced symmetric group action is the permutation action. The fixed points are the diagonal of $X \times X$, and so again a copy of $X$. For a real vector bundle $V \to X$, the transgressed bundle $\tau(V)$ is isomorphic to $V \oplus V$ in this case, $S^0$ acts on this direct sum by acting trivially on one copy of $V$ and via the sign representation on the other copy. The bundle $\psi(V)$ is therefore a copy of $V$ as a vector bundle, with $S^0$ acting through the sign representation.

In order to handle localization formulas in $S^0$-equivariant cohomology we will need to work with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. As a consequence, the relevant characteristic classes will be the equivariant Stiefel-Whitney classes. The $S^0$-equivariant cohomology ring of the point with $\mathbb{F}_2$ coefficients is

$$H^\bullet_{S^0}(*; \mathbb{F}_2) = H^\bullet(BS^0; \mathbb{F}_2) = H^\bullet(\mathbb{P}^\infty\mathbb{R}; \mathbb{F}_2) = \mathbb{F}_2[x],$$

with $x$ in degree 1. The equivariant Stiefel-Whitney classes of $\psi(V)$ will therefore be elements in $H^\bullet_{S^0}(X, \mathbb{F}_2) = H^\bullet(X, \mathbb{F}_2)[x]$.

For an equivariant real line bundle over $X = \mathrm{Fix}(X \times X)$ whose character is the sign representation of $S^0$, one has that the first equivariant Stiefel-Whitney class is

$$w_{1,S^0}(L) = w_1(L) + x,$$

and so the normalized first equivariant Stiefel-Whitney class of $L$ is

$$\widehat{w}_{1,S^0}(L) = 1 + \frac{w_1(L)}{x}.$$

*Remark* 1.3. Notice that it is possible to localize at $x$ precisely because we are working with $\mathbb{F}_2$ coefficients, so that the fact that $x$ has degree 1 does not make it a zero divisor: the equation $x^2 = -x^2$ is trivially satisfied and does not impose a nontrivial constraint on $x$; in particular the equivalent equation $2x^2 = 0$ is trivial and does not make up a pair of zero divisors $(2x, x)$ as it would be with coefficients in a field of characteristic different from 2.

By applying the splitting principle to $\psi(V)$, one sees that the top normalized Stiefel-Whitney class of $\psi(V)$ is

$$\widehat{w}_{\mathrm{top},S^0}(\psi(V)) = \prod_{i=1}^{\mathrm{rk}V} 1 + \frac{w_1(L_i)}{x},$$

where the $w_1(L_i)$ are the Stiefel-Whitney roots of $V$. We therefore find

$$\begin{aligned}\frac{1}{\widehat{w}_{\mathrm{top},S^0}(\psi(V))} &= \prod_{i=1}^{\mathrm{rk}\,V} \frac{1}{1+z}\Big|_{z=w_1(L_i)x^{-1}} = \prod_{i=1}^{\mathrm{rk}\,V}\sum_{k=0}^{\infty} z^k\Big|_{z=w_1(L_i)x^{-1}} = \\ &= 1+\sum_{i=1}^{\infty}\overline{w}_k(V)x^{-k}\end{aligned}$$

where the classes $\overline{w}_k(V)$ are recursively defined by $\overline{w}_1(V) = w_1(V)$ and

$$\overline{w}_k(V) = \overline{w}_{k-1}(V)w_1(V) + \cdots + \overline{w}_1(V)w_{k-1}(V) + w_k(V).$$

Expanding this relation one finds

$$\begin{aligned}
\overline{w}_1 &= w_1\\
\overline{w}_2 &= w_1^2+w_2\\
\overline{w}_3 &= w_1^3+w_3\\
\overline{w}_4 &= w_1^4+w_1^2w_2+w_2^2+w_4\\
\overline{w}_5 &= w_1^5+w_1w_2^2+w_1^2w_3+w_5\\
\overline{w}_6 &= w_1^6+w_1^4w_2+w_2^3+w_3^2+w_1^2w_4+w_6\\
\overline{w}_7 &= w_1^7+w_1^4w_3+w_2^2w_3+w_1w_3^2+w_1^2w_5+w_7\\
\overline{w}_8 &= w_1^8+w_1^6w_2+w_1^4w_2^2+w_2^4+w_2w_3^2+w_1^4w_4+w_2^2w_4+w_4^2+w_1^2w_6+w_8\\
\overline{w}_9 &= w_1^9+w_1^5w_2^2+w_1w_2^4+w_1^6w_3+w_3^3+w_1w_4^2+w_1^4w_5+w_2^2w_5+w_1^2w_7+w_9,
\end{aligned}$$

and so on.

By comparison with what happens for $\mathrm{Maps}(\mathbb{T}^1, X)$, where one recovers the $\hat{A}$-class and the $\hat{A}$-genus of $X$, one would expect that $\overline{w}_n(TX)$ is an interesting class of an $n$-dimensional manifold $X$, and that $\int_X \overline{w}_n(TX)$ is an interesting invariant of $X$ taking values in $\mathbb{F}_2$. Unfortunately, it is not so. By the multiplicativity of Stiefel–Whitney classes, one can equivalently characterize $\overline{w}(V)$ as the total Stiefel–Whitney class $w(W)$ of a vector bundle $W$ such that $V \oplus W = \mathbb{R}^d \times X$. Specializing to the case of $V = TX$ and recalling that by the Whitney embedding theorem every smooth $n$-manifold $X$ can be smoothly embedded into $\mathbb{R}^{2n}$, we have $\overline{w}_n(TX) = w_n(\nu_{X/\mathbb{R}^{2n}})$, where $\nu_{X/\mathbb{R}^{2n}}$ denotes the normal bundle for a fixed embedding $X \hookrightarrow \mathbb{R}^{2n}$. Now, it is a (nontrivial) fact (see [MS74, Corollary 11.4]) that $w_n(\nu_{X/\mathbb{R}^{2n}}) = 0$. Therefore, in particular

$$\int_X \frac{1}{\widehat{w}_{\mathrm{top},S^0}(\psi(TX))} = 0$$

for any $X$.

Even though this may appear a trivial result, this is far from the truth: the invariant is trivial, but its triviality is a nontrivial result. Indeed, for an arbitrary vector bundle $V \to X$ on an $n$-dimensional manifold $X$ there is no reason for $\overline{w}_n(V)$ to be 0. A simple counterexample is the case of the tautological bundle $\xi \to \mathbb{P}^n\mathbb{R}$. In this case one has $\overline{w}_n(\xi) = w_1(\xi)^n$, and so

$$\int_{\mathbb{P}^n\mathbb{R}} \frac{1}{\widehat{w}_{\mathrm{top},S^0}(\psi(\xi))} = \int_{\mathbb{P}^n} w_1(\xi)^n x^{-n} = x^{-n}$$

is a nontrivial element in $H^\bullet(BS^0, \mathbb{F}_2)_{(x^n)}$.

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