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DYNAMICS OF NULL HYPERSURFACES IN GENERAL RELATIVITY AND APPLICATIONS TO GRAVITATIONAL RADIATION,CONSERVED CHARGES AND QUANTUM GRAVITY

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DYNAMICS OF NULL HYPERSURFACES IN GENERAL RELATIVITY AND APPLICATIONS TO GRAVITATIONAL RADIATION, CONSERVED CHARGES AND QUANTUM GRAVITY

DYNAMIQUE DES HYPERSURFACES DE TYPE LUMIERE ET APPLICATION AU RAYONNEMENT GRAVITATIONNEL,

AUX CHARGES CONSERVEES ET A LA GRAVITATION QUANTIQUE

Soutenue le 28/04/2020 devant le jury composé de:

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Résumé

Le théorème de Noether est l'un des plus beaux piliers de la mécanique classique et de la théorie des champs. Il a permis de démêler une relation entre les symétries et les lois de conservation, et a trouvé des applications dans tous les domaines de la physique. Parmi ses applications, le cas de la relativité générale est probablement l'un des plus subtils. La seule symétrie de la relativité générale est l'invariance sous les transformations de coordonnées, ou difféomorphismes. Mais cela ressemble plus à une symétrie de jauge locale, et comme pour les symétries de jauge locale, une application directe du théorème dit qu'il n'y a pas de charges conservées non triviales. Une analyse plus approfondie montre que si l'on prend correctement en compte les conditions aux limites, il y a des charges non triviales, mais ce ne sont pas des intégrales sur une hypersurface de Cauchy, comme dans les applications aux théories de terrain sur l'espace-temps plat, mais plutôt des intégrales de surface sur les limites bidimensionnelles d'une hypersurface de Cauchy. De telles charges de surface ont joué un rôle clé dans la compréhension de la relativité générale depuis l'analyse Hamiltonienne de l'ADM (Arnowitt-Deser-Misner) et plus tard sur l'article fondateur de Regge et Teitelboim. Dans la recherche actuelle, ces charges de surface jouent un rôle important dans les applications phénoménologiques : par exemple, les quantités mesurées par LIGO et la Vierge, comme la masse et le moment angulaire des trous noirs en coalescence emportés par les ondes gravitationnelles, sont comprises comme des charges de surface. Elles jouent également un rôle dans les développements théoriques : elles décrivent la première loi de la mécanique des trous noirs, entrent dans la description de l'entropie des trous noirs et ont été utilisées pour explorer les résolutions du paradoxe de l'information sur les trous noirs. Il existe actuellement un domaine d'intérêt actif autour des charges, et diverses questions ouvertes sont sur la table, de l'inclusion des multipôles gravitationnels à la compréhension de leur quantification correcte. Il y a un deuxième aspect subtil de la relativité générale que j'ai abordé dans cette thèse. Une conséquence de l'invariance du difféomorphisme de la théorie est la présence de contraintes de première classe, comme la loi de Gauss dans les théories de jauge. Ces contraintes de première classe limitent le choix des conditions initiales admissibles pour le problème de Cauchy d'une manière non linéaire et très non triviale. C'est un problème qui apparaît très clairement dans la relativité numérique, où il faut mettre en $\frac{1}{2}$uvre les contraintes avec soin et s'assurer que les approximations utilisées par la grille numérique n'introduisent pas de trop fortes violations. De bonnes conditions initiales sont connues pour des solutions très simples, et une solution générale des contraintes est inconnue. Ce fait a une conséquence importante également pour les approches de la gravité quantique.

Dans la gravité quantique en boucle par exemple, il y a de l'espace, et des opérateurs géométriques avec des spectres discrets et des propriétés de non-commutativité. Cette image se tient au niveau cinématique, c'est-à-dire avant l'imposition de la version quantique des contraintes du difféorphisme Hamiltonien, et il n'est pas prouvé que la même géométrie quantique décrirait également l'espace physique de Hilbert de la théorie, défini sur-coquille sur les contraintes. Une perspective différente du problème peut être obtenue si l'on passe d'un problème de valeur initiale de Cauchy sur une hypersurface semblable à un espace à un problème de valeur initiale caractéristique sur une hypersurface nulle. Dans ce cas, on sait depuis les travaux de Sachs dans les années 60 que l'on peut identifier des données sans contraintes, sous la forme du cisaillement de la congruence géodésique nulle de l'hypersurface. La question est de savoir si ces données sans contrainte peuvent être interprétées en termes de variables de connexion, puis d'appliquer les techniques de gravité quantique en boucle.

Resume

Noether's theorem is one of the most beautiful pillars of classical mechanics and field theory. It unravelled a relation between symmetries and conservation laws, and found applications in all domains of physics. Among its applications, the case of general relativity is probably one of the most subtle ones. The only symmetry of general relativity is the invariance under coordinate transformations, or diffeomorphisms. But this is more like a local gauge symmetry, and like for local gauge symmetries, a direct application of the theorem says that there are no non-trivial conserved charges. A more careful analysis shows that if one correctly takes into account the boundary conditions, there are non-trivial charges, but these are not integrals over a Cauchy hypersurface, like in applications to field theories on flat spacetime, but rather surface integrals over the two-dimensional boundaries of a Cauchy hypersurface. Such surface charges have played a key role in the understanding of general relativity since the ADM (Arnowitt-Deser-Misner) Hamiltonian analysis and later on the seminal paper by Regge and Teitelboim. In current research, these surface charges play an important role in phenomenological applications: for instance the quantities measured by LIGO and Virgo, like the mass and angular momentum of coalescing black holes carried away by gravitational waves, are understood as surface charges. They also play a role in theoretical developments: they describe the first law of black hole mechanics, enter the description of black hole entropy and have been used to explore resolutions of the black hole information paradox. There is currently an active area of interest around the charges, and various open questions are on the table, from the inclusion of gravitational multipoles to understanding their correct quantization. There is a second subtle aspect of general relativity that I addressed in this thesis. A consequence of the diffeomorphism invariance of the theory is the presence of first class constraints, like the Gauss law in gauge theories. This first class constraints limit the choice of admissible initial conditions for the Cauchy problem in a non-linear, highly non-trivial way. This is a problem that shows up very clearly in numerical relativity, where one has to carefully implement the constraints and make sure that the approximations used by the numerical grid don't introduce too strong violations. Good initial conditions are known for very simple solutions, and a general solutions of the constraints is unknown. This fact has an important consequence also for approaches to quantum gravity. In loop quantum gravity for example, there of space, and geometric operators with discrete spectra and non-commutativity properties. This picture holds at the kinematical level, namely prior to the imposition of the quantum version of the Hamiltonian diffeorphism constraints, and it is not proved that the same quantum geometry would also describe

the physical Hilbert space of the theory, defined on-shell on the constraints. A different perspective to the problem can be gained if one switches attention from a Cauchy initial value problem on a space-like hypersurface to a characteristic initial value problem on a null hypersurface. In this case, it is known since the work of Sachs in the sixties that one can identify constraint-free data, in the form of the shear of the null geodesics congruence of the hypersurface. The question is whether these constraint-free data can be given an interpretation in terms of connection variables, and then the loop quantum gravity techniques be applied.

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Chapter 1

Introduction

One of the most important theoretical frameworks at the ground of quantum field theory consists of the determination of suitable symmetry invariance of the associated Lagrangian density. A fundamental tool of this analysis is the Noether theorem, which allows one to associate such invariance of the field's equations with a conserved quantity known as the Noether charge. A well-known example of how this theorem applies is the derivation of the energy-momentum tensor for a general field in a flat spacetime.
For this purpose, it must be regarded of significant interest the application of this formalism to the description of the gravitational field, which requires to implement this field concept on curved space-time geometry and in presence of a non-Riemannian affine connection. The only symmetry of general relativity is the invariance under coordinate transformations, or diffeomorphisms. But this is more like a local gauge symmetry, and like for local symmetries, a direct application of the theorem says that there are no non-trivial conserved charges as consequence of the fact that they are not global quantities, as they are not associated to a global symmetry. Nevertheless, a more careful analysis shows that if one correctly takes into account the boundary conditions, there are non-trivial charges, but these are not integrals over a Cauchy hypersurface, like in applications to field theories on flat spacetime, but rather surface integrals over the two-dimensional boundaries of a Cauchy hypersurface.
An intuitive idea of what is behind this different application can be given looking at concepts of time and energy in Special Relativity (SR) and General Relativity (GR). In SR, time and energy are relative to the observer but uniquely defined: once we fixed the coordinate system, we have a global inertial reference frame in which we can define the energy of the matter field and particles and we can transform these quantities from inertial frame to an accelerating one. On the contrary, for GR a unique definition of time and a universal notion of energy as a conserved quantity lack as consequence of diffeomorphisms invariance. The time of an inertial observer can flow at a very different speed respect to the time of an inertial one and the diffeomorphisms invariance prevents the construction of a meaningful notion of local energy-momentum tensor for the gravitational field, since the latter can locally be made to vanish going to an inertial frame.
As a consequence, global conserved charges can be defined only for particular solutions of Einstein's Equations in which the spacetime allows one or more global symmetries or for Asymptotically-Flat Spacetime where the Minkowski background at the boundary

allows the identification of a canonical notion of energy. In this context, the conserved quantities result to be 2-d integrals, usually called "surface charges", and are derived using an extension of the Noether theorem known as "second Noether Theorem". In fact, from the standard Noether theorem, we have that for local gauge symmetries it exists only one equivalence class of conserved currents and it corresponds to the trivial one. However, these same trivial currents can generate a conservation law for 2-d charges or fluxes in finite regions: if the theory has gauge parameters which generate global symmetries, we are still able to define associated surface charges that are the same for each observer and whose conservation law is given from the vanishing of the 3-d Noether current.
Such surface charges have played a key role in the understanding of general relativity since the ADM (Arnowitt-Deser-Misner) [Arnowitt et al., 2008] Hamiltonian analysis and later on the seminal paper by Regge and Teitelboim [Regge and Teitelboim, 1974]. In current research, these surface charges play an important role in phenomenological applications: for instance the quantities measured by LIGO and Virgo, like the mass and angular momentum of coalescing black holes carried away by gravitational waves, are understood as surface charges. They also play a role in theoretical developments: they describe the first law of black hole mechanics, enter the description of black hole entropy and have been used to explore resolutions of the black hole information paradox. There is currently an active area of interest around the derivation of these charges, and various open questions are on the table, from the inclusion of gravitational multipoles to understanding their correct quantization.
The study of conserved quantities also plays a fundamental role in scenario of information-loss paradox resolution. The debate around the information paradox is crucially related to the definition of the Bekenstein-Hawking entropy and suggested an analogy between a BH and a thermodynamical system. A well known result of this interpretation is the no-hair theorem [Carter, 1971, Hawking, 1972], stating that any stationary axisymmetric BH can be described, no matter how it has been formed, by means of only three parameters: the mass M, the angular momentum J and the charge Q. This is reminiscent of a description of a system with only macroscopic thermodynamical variable such as energy, temperature and pressure and it connected on Jacobson interpretation of Einsteins equations as emergent from description of more fundamental degrees of freedom [Jacobson, 1995].

There is a second main aspect of GR that we deal in this dissertation. A consequence of the diffeomorphism invariance of the theory is the presence of first class constraints, like the Gauss law in gauge theories. This first class constraints limit the choice of admissible initial conditions for the Cauchy problem in a non-linear, highly non-trivial way. This is a problem that shows up very clearly in numerical relativity, where one has to carefully implement the constraints and make sure that the approximations used by the numerical grid don't introduce too strong violations. Good initial conditions are known for very simple solutions, and a general solutions of the constraints is unknown. This fact has an important consequence also for approaches to quantum gravity. In loop quantum gravity for example, there exists a compelling kinematical picture of quantum spacetime, where the smooth manifold of general relativity is replaced by a collection of quanta of space, and geometric operators with discrete spectra and non-commutativity properties. This picture holds at the kinematical level, namely prior to the imposition of

the quantum version of the Hamiltonian diffeomorphism constraints, and it is not proved that the same quantum geometry would also describe the physical Hilbert space of the theory, defined on-shell on the constraints. A different perspective to the problem can be gained if one switches attention from a Cauchy initial value problem on a space-like hypersurface to a characteristic initial value problem on a null hypersurface. In this case, it is known since the work of Sachs [Sachs, 1962a] that one can identify constraint-free data, in the form of the shear of the null geodesics congruence of the hypersurface. The question is whether these constraint-free data can be given an interpretation in terms of connection variables, and then the loop quantum gravity techniques be applied.
The topic of this thesis concerns the study of these two aspects, constraint-free data in the bulk of a null hypersurface, and surface charges and their conservation or balance laws, using tetrad variables instead of metric variables. This is motivated both by the possibility that these different variables can shed light on the open questions in the classical theory, but also by the potential applications to loop quantum gravity. My thesis only dealt with classical aspects. The structure of my thesis is as follows. I will first provide a basic introduction to the use of differential forms and tetrad variables in general relativity, in Chapter 2. I will then present the original results of my thesis, split in three Chapters based on the papers I published during my first two years of PhD work.

Chapter three

In the first chapter of the original part of this dissertation we present a work [De Lorenzo et al., 2018] related to Jacobson's idea that the continuum structure of spacetime could emerge as the thermodynamical equilibrium description of more fundamental quantum degrees of freedom [Jacobson, 1995].
The analysis that we will report is based on the paper [De Lorenzo et al., 2018] in which we restate the thermodynamical formulation of gravity in the first order formalism, i.e. using tetrads and spin connections as independent variables. Our analysis is related to Jacobson's idea that the continuum structure of spacetime could emerge as the thermodynamical equilibrium description of more fundamental quantum degrees of freedom [Jacobson, 1995] and was motivated to generalised this results in presence of torsion. As we will show in the first introductive chapter, we can interpret torsion as a some extra degrees of freedom that describes the spacetime structure and are "visible" only in presence of a matter field that coupled with the gravitational field. In this interpretation torsion should enter in every definition related to spacetime, as parallel transport, curvature of spacetime, killing vector field etc. This means that if we want to give a thermodynamical interpretation to Einstein-Cartan equations we need to reformulate Jacobson's idea starting from the physical and mathematical set-up (bifurcating horizon) and conserved quantity laws.
We divided our analysis in two main parts. In the first one we study the case of the identification of a conserved energy-momentum tensor for Einstein-Cartan theory coupled with a matter field. We show that this is possible only using the Noether identities of the matter Lagrangian and the torsion field equations.
In the second part we use the conserved energy tensor to show that Jacobson's thermodynamical derivation of the Einstein equations follows as in the metric theory, namely from the equilibrium Clausius relation and the fact that a Killing horizon is metric-geodetic. In the same part we review the laws of black hole mechanics and analyse their dependence on torsion. We didn't specify any matter Lagrangian but we took a general

one so our derivation works for an arbitrary torsion field.
Such a first order formulation finds its motivation in the possibility of extending the thermodynamical picture of gravity towards advanced formulation required by a proper approach to quantum gravity. In particular the set of dynamical equations derived trough this analysis as to be regarded as the starting point for an implementation of the so called Ashetkar-Barbero-Immirzi variables at the ground of loop quantum gravity, once an Hamiltonian formulation is concerned.

Chapter fourth

The previous analysis pointed out that the study of conserved quantities in the first order formalism requires some more attentions respect to the usual derivation in the second order formalism trough the Noether theorem. We treat this general problem in the third chapter, based on the paper [De Paoli and Speziale, 2018], where we extend the analysis on conserved quantities to the presence of extra gauge symmetries which act on tetrad indices and we use the covariant phase space formalism. The power of this method hang on the fact that it allows one to define surface charges for diffeomorphisms as the canonical generators in the covariant phase space and relies on the symplectic potential, which summarizes all the symplectic properties of the covariant phase space. In particular the symplectic two-form, obtained varying such potential, provides the geometrical properties of the covariant phase space.
The analysis that we carried on is based on the identification of a symplectic potential for general relativity in tetrad and connection variables that is fully gauge-invariant, using the freedom to add surface terms. In fact the symplectic structure arising from the Einstein-Cartan action is not fully gauge invariant and it leads to surface charges associated with the internal Lorentz transformations. These charges are not present in the metric analysis and are an evidence of the non gauge invariance of this structure. Moreover we expect that when torsion vanishes the covariant phase space morphology it's equivalent to the metric one and it's not the case as these internal charges are not even defined for the Einstein Hilbert action where this internal gauge symmetry doesn't apply. To find a fully gauge-invariant symplectic potential, it is enough to use the fact that the symplectic potential is defined from an action principle only up to the addition of an exact form. In particular, it reproduces the Komar form when the variation is a Lie derivative, and the geometric expression in terms of extrinsic curvature and 2d corner data for a general variation.The additional surface term vanishes at spatial infinity for asymptotically flat spacetimes, thus the usual Poincaré charges are obtained. We prove that the first law of black hole mechanics follows from the Noether identity associated with the covariant Lie derivative, and that it is independent of the ambiguities in the symplectic potential provided one takes into account the presence of non-trivial Lorentz charges that these ambiguities can introduce.
Among the motivations for these results we mention the study of boundary degrees of freedom, in particular the presence of the Holst term in Palatini action has been shown to lead to an interesting description in terms of a conformal field theory on the boundary [Freidel et al., 2017a, Wieland, 2017a, Geiller, 2018]) and it would be intriguing to see if and how that description is affected by our results.

Chapter fifth

Finally we devoted our study to formulate the Cauchy problem for GR and the as-

sociated constraint structure in the case of a null foliation, still retaining tetrad and spin connection as basic variables. This analysis is based on the paper [De Paoli and Speziale, 2017], in which we discuss the Hamiltonian dynamics of general relativity with real connection variables on a null foliation, and use the Newman-Penrose formalism to shed light on the geometric meaning of the various constraints. An explicit interpretation of the constraint structure for a null foliation is provided clarifying how the diffeomorphisms and internal gauge symmetries emerge in this scenario.
The use of a null-foliation permits a straightforward identification of the real physical degrees of freedom which enter the constraint expressions. We identify the equivalent of Sachs'constraint-free initial data as projections of connection components related to null rotations, i.e. the translational part of the ISO(2) group stabilising the internal null direction soldered to the hypersurface. A pair of second-class constraints reduces these connection components to the shear of a null geodesic congruence, thus establishing equivalence with the second-order formalism, which we show in details at the level of symplectic potentials. A special feature of the first-order formulation is that Sachs' propagating equations for the shear, away from the initial hypersurface, are turned into tertiary constraints; their role is to preserve the relation between connection and shear under retarded time evolution. The conversion of wave-like propagating equations into constraints is possible thanks to an algebraic Bianchi identity; the same one that allows one to describe the radiative data at future null infinity in terms of a shear of a (non-geodesic) asymptotic null vector field in the physical spacetime. Finally, we compute the modification to the spin coefficients and the null congruence in the presence of torsion. This analysis is of impact for the perspective of a quantization procedure of the gravitational field viewed in a null foliation. In this respect has particular value the identification of the physical degrees of freedom within the constraints morphology, since it would facilitate the extension of this picture in a full quantum sector described by optimized variables.

Chapter 2

Einstein-Cartan Action and Local Gauge Invariance

In this chapter, we discuss both the basic formalism that we used in the rest of the dissertation both the background that we consider necessary to fully understand the results explained in the following chapters and put them into context.
In the first part, we introduce the tetrad and connection formalism and the related action. This formalism is an approach to general relativity that describes the gravitational field trough four independent vectors defined in a local reference frame instead of the metric tensor components. It was developed mainly in the study of a quantum theory of gravity, in which tetrad variables are fundamental if one wants to couple the gravitational field with the fermionic matter, but it also has the power to a deeper analysis of GR symmetries ad degrees of freedom. The simplest action that one can build in this formulation is Palatini action, or Einstein-Cartan action if we consider to be at the first order. We will give a brief overview of it in this chapter together with an analysis of the constraints system,
In the second part, we introduce the torsion tensor and we described its basic properties. The torsion contribution arises in the first order formalism if the covariant derivative is not metric compatible, i.e. is not Levi-Civita, or if the connection is not tetrad compatible. In vacuum, on-shell of the fields equation, the torsion tensor identically vanishes everywhere so its presence could be explained only by the interaction of the gravitational field with some sort of matter field. We show what are the basic sources that can generate torsion, i.e. the matter terms action that could be added to the Einstein-Cartan action and what are the physical interpretations of its contribution.
In the last part of this general introduction we recall how to recover the Noether Theorem in the case of scalar field in flat spacetime and how to derive a conserved charges if a global symmetry is present. If we want to apply the classical Noether Theorem to GR (and in general for any gauge theory) to derive conserved quantities some accuracies should be taken. Moreover in presence of tetrad variables we introduce four extra local symmetries in the description of the gravitational field which actually play a fundamental role in the correct derivation of conserved charges. We will discuss these aspects of tetrad gravity in the two followings chapter and we left here the analysis of Noether charges for the gravitational field in metric variables.

2.1 Tetrad and connection variables

Let us now introduce the formalism of the tetrads and connections variables. These represent a useful way to express the gravitational theory in terms of gauge invariant variables.

The tetrad is a collection of four linearly independent one-forms $e^I_\mu(x)$, $I = 0,1,2,3$ that provides a local isomorphism between a general reference frame and an inertial one, characterized by the flat metric η_{IJ}. The relation between these two reference frames is conceptually based on the equivalence principle and is mathematically expressed by the following equation

$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ}, \tag{2.1}$$

which reppresents the formal definition of the tetrad variables this is why the tetrads are sometimes described as the square root of the metric.

The new indices $I = 0,1,2,3$ of the tetrad are called internal indices as they act in a flat spacetime-metric manifold and they and come together with an additional invariance under a local gauge group G which corresponds to the the Lorentz group $SO(3,1)$. This can be easily understood recalling that a local inertial frame is defined up to a Lorentz transformation and as consequence he definition (2.1) is invariant under the following transformation

$$e^I_\mu(x) \longrightarrow \tilde{e}^I_\mu(x) = \Lambda^I_J(x) e^J_\mu(x) \tag{2.2}$$

which act on the internal index and we will sometimes refer to as "internal gauge" symmetry.

We can geometrically interpret the tetrads as the linear map that, for each point p $\in M$ sends the tangent space $T_p(\mathscr{M})$ in the flat space. Then, given a vector v in p, the tetrads are the matrices $e^\mu_I(x)$ that transform the components v^μ (i.e., contravariant vectors) into new components v^I

$$v^\mu \, e^I_\mu = v^I \,. \tag{2.3}$$

which transforms under the Lorentz group.Then, any tensor can be decomposed using internal or spacetime coordinates related by the tetrads and their inverse i.e.,

$$T^{I_1 \dots I_q}_{J_1 \dots J_p} = e^{I_1}_{\mu_1} \dots e^{I_q}_{\mu_q} e^{\nu_1}_{J_1} \dots e^{\nu_p}_{J_p} T^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_p} \,. \tag{2.4}$$

Related to the local gauge symmetry there is a connection ω^{IJ}_μ, that is a 1-form with values in the Lorentz algebra, which we can use to define covariant differentiation of the fibres in the Lorentz tangent bundle. For example:

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega^I_{\mu J}(x) v^J(x). \tag{2.5}$$

This is the analogue of the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma_\mu$ for vectors in $T(\mathscr{M})$. Using these two connections we can also define a covariant derivative for objects with both types of indices as the tetrads:

$$\mathscr{D}_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu - \Gamma^\rho_{\nu\mu} e^I_\rho. \tag{2.6}$$

Using this definition we can require to the fibre bundle space to have a connection structure ω_μ that is tetrad-compatible

$$\mathscr{D}_\mu e^I_\nu \equiv 0\,. \tag{2.7}$$

Under this condition ω_μ^{IJ} is the unique connection that we can write as a function of the tetrads and we call it spin connection. This is the analogue as for the Levi-Civita connection $\Gamma(g)$, which is defined requiring that the covariant derivative is metric-compatible $\nabla_\mu g_{\nu\rho} = 0$. From (2.7) we can derive the relation between the components of ω_μ^{IJ} and the coefficients $\Gamma^\rho_{\mu\nu}$ of the Levi-Civita connection,

$$\partial_{(\mu} e^I_{\nu)} + \omega^I_{(\mu J} e^J_{\nu)} = \Gamma^\rho_{(\nu\mu)} e^I_\rho, \qquad \partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu J} e^J_{\nu]} = \Gamma^\rho_{[\nu\mu]} e^I_\rho \equiv 0, \tag{2.8}$$

where we separed the spacetime indices into their symmetric and antisymmetric combinations. If we used the fact that the Levi-Civita connection $\Gamma(g)$ has no antisymmetric part we immediately obtain the following relation

$$\omega^I_{\mu J} = e^I_\nu \nabla_\mu e^\nu_J. \tag{2.9}$$

A useful way to rewrite the previous equations is with the exterior calculus of forms. Moving from the coordinate basis to the differential form notation we write $e^I = e^I_\mu \, dx^\mu$ and $\omega^{IJ} = \omega^{IJ}_\mu \, dx^\mu$. We denote with d the exterior derivative, with d_ω the covariant exterior derivative and with $\wedge$ the exterior product or wedge product [1].
In this notation equation (2.6) can be rewritten as

$$\mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J = \left(\partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0, \tag{2.10}$$

and it's known as Cartan's first structure equation. Given the connection, we define the usually related tensor that characterized the manifold structure: its curvature

$$F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}, \tag{2.11}$$

whose components are

$$F^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^I{}_{K\mu} \omega^{KJ}_\nu - \omega^J{}_{K\mu} \omega^{KI}_\nu. \tag{2.12}$$

If we substitute in ω the tetrad compatible connection we find the relation between $F^{IJ}_{\mu\nu}$ and the usual Riemann tensor $R_{\mu\nu\rho\sigma}(g)$ which is constructed out of the metric can be expressed in term of the tetrad using (2.1),

$$F^{IJ}_{\mu\nu}\left(\omega\left(e\right)\right) \equiv e^{I\rho} e^{J\sigma} R_{\mu\nu\rho\sigma}(g). \tag{2.13}$$

The relation is known as Cartan second structure equation. It shows that GR is a gauge theory whose local gauge group is the Lorentz group, and the Riemann tensor can be viewed as the field-strength of the spin connection.

2.1.1 The action in terms of tetrads

A theory of gravitation is formulated by using the metric tensor as a dynamical variable. We assume that its dynamics is regulated by a variational principle based on an action

[1] We address the reader at the end of this chapter for a brief introduction on the exterior calculus of forms

functional S, which is invariant under diffeomorphisms. In its original formulation the dynamics of the metric tensor is encoded in the Einstein-Hilbert action,

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\int \sqrt{-g}R[g_{\mu\nu}]. \tag{2.14}$$

The latter expression can be rewritten as a functional of the tetrad,

$$S_{EH}(e^I_\mu) = \frac{1}{2}\varepsilon_{IJKL}\int e^I \wedge e^J \wedge F^{KL}(\omega(e)). \tag{2.15}$$

implying that now the tetrad is the dynamical variables and the variational principle should be taken varying respect to them. As these are sixteen a priori independent variables they bring with them and extra symmetry in the internal index. Moreover, on top of the invariance under diffeomorphism, this reformulation of the theory possesses an additional gauge symmetry under local Lorentz transformations.
A fact which plays an important role in the following is that we can lift the connection to be an *independent variable*, and consider the new action

$$S(e^I_\mu, \omega^{IJ}_\mu) = \frac{1}{2}\varepsilon_{IJKL}\int e^I \wedge e^J \wedge F^{KL}(\omega). \tag{2.16}$$

Although it depends on extra fields, this action remarkably gives the same equations of motion as the Einstein-Hilbert one (2.15). This happens because the extra field equations coming from varying the action with respect to ω do not add anything new: they simply impose the form (2.9) of the spin connection, and general relativity is thus recovered. As it gives the same field equations, (2.15) can be used as the action of general relativity. Notice that only first derivatives appears, thus it provides a first order formulation of general relativity. Furthermore, the action is polynomial in the fields, a desiderable property for quantization. On the other hand, there are two non-trivial aspects to take into account. First the equivalence with GR holds only if the tetrad is non-degenerate, i.e. invertible. On the other hand, (2.15) is also defined for degenerate tetrads, since inverse tetrads never appear. Compare the situation with the Einstein-Hilbert action, where the inverse metric appears explicitly. Hence, the use of (2.15) leads naturally to an extension of general relativity where a sector with degenerate tetrads, and thus degenerate metrics, exists. Second if we insist on the connection being an independent variable, there exists a second term that we can add to the Lagrangian that is compatible with all the symmetries and has mass dimension 4:

$$\delta_{IJKL}e^I \wedge e^J \wedge F^{KL}(\omega), \tag{2.17}$$

where $\delta_{IJKL} \equiv \delta_{I[K}\delta_{L]J}$. This term is not present in the ordinary second order metric, since when (2.9) holds,

$$\delta_{IJKL}e^I \wedge e^J \wedge F^{KL}(\omega(e)) = \varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(e) \equiv 0. \tag{2.18}$$

Adding this second term to (2.15) with a coupling constant $1/\gamma$ leads to the so-called Holst action [Holst, 1996]

$$S(e,\omega) = \left(\frac{1}{2}\varepsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)\int e^I \wedge e^J \wedge F^{KL}(\omega). \tag{2.19}$$

Assuming non-degenerate tetrads, this action leads to the same field equations of general relativity,

$$\omega_\mu^{IJ} = e_\nu^I \nabla_\mu e^{J\nu}, \qquad G_{\mu\nu}(e) = 0. \tag{2.20}$$

This result is *completely independent* of the value of γ, which is thus a parameter irrelevant in classical vacuum general relativity. It will however turn out to play a key role in the quantum theory, where it is known as the Immirzi parameter.

2.1.2 Hamiltonian analysis of tetrad formulation

For the Hamiltonian formulation we proceed as before, assuming a $3+1$ splitting of the space-time ($\mathscr{M} \cong \mathbb{R} \times \Sigma$) and coordinates (t,x). We introduce the lapse function and the shift vector (N, N^a) and the ADM decomposition of the metric. It is easy to see that a tetrad for the ADM metric is given by

$$e_0^I = e_\mu^I \tau^\mu = N n^I + N^a e_a^I, \qquad \delta_{ij} e_a^i e_b^j = g_{ab}, \; i = 1,2,3. \tag{2.21}$$

The "triad" e_a^i is the spatial part of the tetrad. As before, we want to identify canonically conjugated variables and perform the Legendre transform, but we now have two new features which complicate the analysis. The first one is the tetrad formulation, which in particular has introduced a new symmetry in the action: the invariance under local Lorentz transformations. As a consequence, we expect more constraints to appear, corresponding to the generators of the new local symmetry. The second one is the use of the tetrad and the connection as independent fields. Therefore, the conjugate variables are now functions of both e_a^I and ω_a^{IJ} (and their time derivatives), as opposed to be functions of the metric g_{ab} only. The consequence of these novelties is a much more complicated structure than in the metric case. In particular, the constraint algebra is *second class*. However, there is a particular choice of variables which simplifies the analysis, making it possible to implement a part of the constraint and reducing the remaining ones to first class again. These are the famous Ashtekar variables, which we now introduce.[2] To simplify the discussion, it is customary to work in the "time gauge" $e_\mu^I n^\mu = \delta_0^I$, where

$$e_\mu^0 = (N,0) \longrightarrow e_0^I = \left(N, N^a e_a^i\right). \tag{2.22}$$

The crucial change of variables is the following: we define the *densitized triad*

$$E_i^a = e e_i^a = \frac{1}{2} \varepsilon_{ijk} \varepsilon^{abc} e_b^j e_c^k, \tag{2.23}$$

and the *Ashtekar-Barbero connection*

$$A_a^i = \gamma \omega_a^{0i} + \frac{1}{2} \varepsilon^i_{\ jk} \omega_a^{jk}. \tag{2.24}$$

These variables turns out to be conjugated. In fact, we can rewrite the action (2.19) in terms of the new variables as [Barros e Sa, 2001a, Thiemann, 2001]

$$S(A,E,N,N^a) = \frac{1}{\gamma} \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left[\dot{A}_a^i E_i^a - A_0^i D_a E_i^a - NH - N^a H_a\right], \tag{2.25}$$

[2] On the general analysis with the second class constraints see [Barros e Sa, 2001a].

where

$$G_j \equiv D_a E^a_i = \partial_a E^a_j + \varepsilon_{jk\ell} A^j_a E^{a\ell}, \tag{2.26}$$

$$H_a = \frac{1}{\gamma} F^j_{ab} E^b_j - \frac{1+\gamma^2}{\gamma} K^i_a G_i, \tag{2.27}$$

$$H = \left[F^j_{ab} - \left(\gamma^2 + 1\right) \varepsilon_{jmn} K^m_a K^n_b \right] \frac{\varepsilon_{jk\ell} E^a_k E^b_\ell}{\det E} + \frac{1+\gamma^2}{\gamma} G^i \partial_a \frac{E^a_i}{\det E}. \tag{2.28}$$

The resulting action is similar to Einstein-Hilbert action, with (A,E) as canonically conjugated variables. Lapse and shift are still Lagrange multipliers, and consistently we still refer to $H(A,E)$ and $H_a(A,E)$ as the Hamiltonian and space-diffeomorphism constraints.
The algebra is still first class. The new formulation in terms of tetrads has introduced the extra constraint (2.26). The reader familiar with gauge theories will recognize it as the Gauss constraint. Just as the H^μ constraints generate diffeomorphisms, the Gauss constraint generates gauge transformations. It is in fact easy to check that E^b_j and A^i_a transform respectively as an SU(2) vector and as an SU(2) connection under this transformation.

2.2 The role of Torsion field

In this section we want to introduce the basic notion of torsion. This will play an important role in our analysis in the first order formalism where we require that the connection Γ, or ω^{IJ}, is not metric, or tetrad, dependent. In this case the connection is consider as some extra fields, we will discuss in the following the different interpretation as spacetime degrees of freedom. To treat this problem what is usually done in litterature is to split the connection tensor in two parts, one that is still dependent on the metric and corresponds to the usual Christoffel symbol plus a part that is independent. This splitting allows to immediately see the contributions given by the presence of torsion respect to classical gravity theory in the second order formalism with a Levi Civita connection.

2.2.1 The torsion tensor

To understand better how torsion arise we first go back to metric variables and to the construction of the covariant derivative in GR. The partial derivative of a scalar field is a covariant vector and in the formalism of form is given by a one-form. However, the partial derivative of any other tensor field does not form a tensor. But, one can add to the partial derivative some additional term such that the sum is a tensor. The sum of partial derivative and this additional term is called covariant derivative. The covariant derivative is a tensor if and only if the affine connection $\Gamma^\rho_{\mu\nu}$ transforms in a special non-tensor way.
The rule for constructing the covariant derivatives of other tensors immediately follows from requiring that the covariant derivatives of controvariant vectors transforms as a tensor with an index up and an index down and that the covariant derivatives of a covariant vector transforms as a tensor with both indices down. This can be generalise

to the covariant derivative of any tensor with indices up and down. However from this requirements the definition of connection $\Gamma^{\alpha}_{\beta\gamma}$ contains some ambiguity. Indeed the covariant derivative remains a tensor if one adds to $\Gamma^{\alpha}_{\beta\gamma}$ *any* tensor $C^{\alpha}_{\beta\gamma}$ such that

$$\Gamma^{\alpha}_{\beta\gamma} \to \Gamma^{\alpha}_{\beta\gamma} + C^{\alpha}_{\beta\gamma}. \tag{2.29}$$

A special choice of connection which is used in General Relativity for the definition of the covariant derivative is based on two requirement: (i) that is symmetric in the lower indices $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$ and ii) that the covarinat derivative of the metric tensor identically vanishes $\nabla_{\alpha} g_{\mu\nu} = 0$. The former condition is also know as metricity condition. If these conditions are satisfied there is a unique solution for $\Gamma^{\alpha}{}_{\beta\gamma}$:

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}(g) = \frac{1}{2} g^{\alpha\lambda} \left(\partial_{\beta} g_{\lambda\gamma} + \partial_{\gamma} g_{\lambda\beta} - \partial_{\lambda} g_{\beta\gamma} \right) . \tag{2.30}$$

that corresponds to the Christoffel symbol, which can then be interpreted as a particular case of the affine connection and we will refer to it as $\Gamma^{\alpha}_{\beta\gamma}(g)$ to underline its dependence respect to the metric tensor. It's a very important object, because it depends on the metric only and it is the simplest one among all possible affine connections. Other connections can be considered as (2.30) plus some additional tensor as one can prove that the difference between any two connections is a tensor.
When the space-time is flat, the metric and the Levi-Civita connection depend just on the choice of the coordinates, and one can choose them such that $\Gamma(g)^{\alpha}_{\beta\gamma}$ vanishes everywhere. On the contrary, if we consider,

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}(g) + C_{\beta}{}^{\alpha}{}_{\gamma}, \tag{2.31}$$

than the tensor $C^{\alpha}_{\beta\gamma}$ (and, consequently, the whole connection $\Gamma^{\alpha}_{\beta\gamma}$) can not be eliminated by a choice of the coordinates. Even if one takes the flat metric, the covariant derivative based on $\Gamma^{\alpha}_{\beta\gamma}$ does not reduce to the coordinate transform of partial derivative. Thus, the introduction of an affine connection different from Christoffel symbol means that the geometry is not completely described by the metric, but has another, absolutely independent characteristic tensor $C^{\alpha}_{\beta\gamma}$. This ambiguity in the definition of $\Gamma^{\alpha}_{\beta\gamma}$ is very important, for it enables one to introduce gauge fields different from gravity, in the next section we will see how to derive these extra fields from an action and what are the necessary conditions to have it.

2.2.2 Basic properties and notation

Untill now we have introduced the independent term of connection but we still didn't give a proper definition of the torsion tensor. Using the violation of condition (i), that represents the antisymmetry property of the connection, we define the torsion tensor $T^{\alpha}_{\beta\gamma}$ as:

$$T^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta} = 2\Gamma^{\alpha}_{[\beta\gamma]} = -2C_{[\beta\gamma]}{}^{\alpha}. \tag{2.32}$$

From this definition and the relation between the full connection and the Levi-Civita one, (2.31), we derive the following relation

$$C_{\beta\gamma}{}^{\alpha} = \frac{1}{2} \left(T_{\beta\gamma}{}^{\alpha} - T^{\alpha}{}_{\beta\gamma} + T_{\gamma\beta}{}^{\alpha} \right) \tag{2.33}$$

where $C^{\alpha}_{b\gamma}$ is called the contorsion tensor. Here and in the following we use a comma between indices to bundle up those with special symmetry properties.
The indices are raised and lowered by means of the metric. It is worthwhile noticing that the contorsion is antisymmetric in the first two indices: $C_{\beta\gamma}{}^{\alpha} = -C_{\gamma\beta}{}^{\alpha}$, while torsion itself is antisymmetric in the last two indices $T^{\alpha}{}_{\beta\gamma} = -T^{\alpha}{}_{\gamma\beta}$.
In a spacetime with torsion also the curvature definition are influence from this extra field. For example, the commutator of covariant derivatives now depends both on the torsion and on the curvature tensor. We start first from the commutator acting on the scalar field ϕ,

$$\left[\nabla_{\alpha}, \nabla_{\beta}\right] \phi = C^{\gamma}{}_{\alpha\beta}\, \partial_{\gamma}\phi\,, \tag{2.34}$$

which doesn't vanish anymore due to the antisymmetry property of the indices. In the case of a vector, a brief algebraic calculation give the following expression

$$\left[\nabla_{\alpha}, \nabla_{\beta}\right] v^{\gamma} = T^{\delta}{}_{\alpha\beta}\, \nabla_{\delta}\, v^{\gamma} + R^{\gamma}{}_{\delta\alpha\beta}\, v^{\delta}\,, \tag{2.35}$$

where $R^{\gamma}{}_{\delta\alpha\beta}$ is the curvature tensor in the space with torsion:

$$R^{\gamma}_{\delta\alpha\beta} = \partial_{\alpha}\Gamma^{\gamma}_{\delta\beta} - \partial_{\beta}\Gamma^{\gamma}_{\delta\alpha} + + \Gamma^{\gamma}_{\lambda\alpha}\Gamma^{\lambda}_{\delta\beta} - \Gamma^{\lambda}_{\lambda\beta}\Gamma^{\lambda}_{\delta\alpha}\,.$$

Using (2.34), (2.35) and that the product $P^{\lambda}B_{\lambda}$ is a scalar, one can easily derive the commutator of covariant derivatives acting on a 1-form B_{λ} and then calculate such a commutator acting on any tensor. In all cases the commutator is the linear combination of curvature (2.36) and torsion. The curvature (2.36) can be easily expressed through the Riemann tensor (curvature tensor depending only on the metric), covariant derivative ∇_{α} (torsionless) and contorsion as

$$R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}(g) + 2\imath\nabla_{[\rho}C_{\sigma],\mu\nu} + C_{\rho,\mu\lambda}C_{\sigma,}{}^{\lambda}{}_{\nu} - C_{\sigma,\mu\lambda}C_{\rho,}{}^{\lambda}{}_{\nu} \tag{2.36}$$

where we indicate the usual Riemann tensor defined only in terms of the metric connection as $R_{\mu\nu\rho\sigma}(g)$ Similar formulas can be written for the Ricci tensor and for the scalar curvature with torsion:

$$R_{\mu\nu} = R_{\mu\nu}(g) + 2\imath\nabla_{[\nu}C_{\rho],\mu}{}^{\rho} + 2C_{[\nu,\mu\lambda}C_{\rho],}{}^{\lambda\rho} \tag{2.37}$$

$$R = R(g) + 2\overset{e}{\nabla}_{[\nu}C_{\rho],}{}^{\nu\rho} + 2C_{[\mu,}{}^{\mu\lambda}C_{\rho],\lambda}{}^{\rho} \tag{2.38}$$

It proves useful to divide torsion into three irreducible components: i) the trace vector $T_{\beta} = T^{\alpha}_{\cdot\beta\alpha}$, ii) the axial vector $S^{\nu} = \varepsilon^{\alpha\beta\mu\nu}T_{\alpha\beta\mu}$ and iii) the tensor $q^{\alpha}_{\cdot\beta\gamma}$, which satisfies two conditions $q^{\alpha}_{\cdot\beta\alpha} = 0$ and $\varepsilon^{\alpha\beta\mu\nu}q_{\alpha\beta\mu} = 0$.
Then, the torsion field can be expressed through these new fields as

$$T_{\alpha\beta\mu} = \frac{1}{3}\left(T_{\beta}\, g_{\alpha\mu} - T_{\mu}\, g_{\alpha\beta}\right) - \frac{1}{6}\varepsilon_{\alpha\beta\mu\nu}\, S^{\nu} + q_{\alpha\beta\mu}\,. \tag{2.39}$$

Using the above formulas, it is not difficult to express the curvatures (2.36), through these irreducible components. We shall write only the expression for scalar curvature, which will be useful in what follows

$$R = R(g) - 2\nabla_{\alpha}T^{\alpha} - \frac{4}{3}T_{\alpha}T^{\alpha} + \frac{1}{2}q_{\alpha\beta\gamma}q^{\alpha\beta\gamma} + \frac{1}{24}S^{\alpha}S_{\alpha}\,. \tag{2.40}$$

2.3 Noether theorem and conserved charges

To conclude this recapitulation chapter we want to give a basic review on how deriving conserved charges for a general quantum field theory. As we already discussed the fundamental tool to derive conserved charges is the Noether theorem which states that *every differentiable symmetry of the action of a physical system has a corresponding conservation law*. The main idea under this theorem is related to the concept of (global) symmetry: that a system is invariant under a certain transformation means that the physical quantities describing this system don't change. As an example we can consider the translation symmetry, for which case the invariance of the system is associated with the conservation of the energy-momentum tensor of the matter field.
While considering the gravitational field, the Noether theorem doesn't have a straight application since the global symmetry is now a local property in the sense of the diffeomorphism invariance. In fact to recover a proper concept of energy-momentum is necessary to consider the role of the gravitational field as spacetime metric and the affine connection. A proper way to address this subtle question relies on addressing the concept of isolated system. Such a concept can be mathematically expressed by suitable asymptotic boundary conditions, i.e. flatness conditions at spatial and null infinity. This approach has been developed in literature and the conserved charges are derived for some specific solution of spacetime *ADM charges, Ashtekar:2000hw, Ashtekar:2008jw, Corichi:2016eoe*. Here we will not face the full derivation of this method, but we discuss the main problems of a direct application of the Noether Theorem giving a physical intuition behind them.

2.3.1 Noether theorem in QFT in Minkowsky spacetime

In field theory, we can derive conserved quantities associated with a Lagrangian symmetry through the Noether theorem. In particular, for theories defined on flat space-times, the principle of special relativity must be valid and the equations of motion for two observers in two inertial systems must be the same. This translates into in the request that the Lagrangian of the system is invariant under Poincaré transformations and allows us to define conserved quantities for each field defined on a flat space-time. If we consider the subgroup of translation and use the Noether theorem we can define a stress-energy tensor $T^{\mu\nu}$ that satisfies a conservation law.
We consider as an example the Klein Gordon scalar field $\phi(x^\alpha)$, described by the Lagrangian:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.41}$$

and consider an infinitesimal translation ε^μ:

$$x'^\mu = x^\mu + \varepsilon^\mu, \qquad \phi'(x^\mu) = \phi(x^\mu). \tag{2.42}$$

where ε^μ is a global parameter. The Noether theorem tells us that every continuous symmetry that leaves the Lagrangian unchanged is associated with a four-current

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi + \mathscr{L}\,\delta x^\mu, \tag{2.43}$$

which satisfies the conservation equation,

$$\partial_\mu J^\mu = 0, \tag{2.44}$$

and whose integrated temporal component on a spatial surface Σ is preserved in time[3]

$$\int_V d^3x \, \frac{\partial J^0}{\partial t} = -\int_V d^3x \, \partial_i J^i = \int_\Sigma d\sigma \, n \cdot J = 0. \tag{2.45}$$

We defined the conserved quantity as the integral of the zero componente,

$$Q = \int_\Sigma d^3x \, J^0, \tag{2.46}$$

and one can prove that is independent of the chosen surface Σ. Let us now apply these considerations to the Lagrangian (2.41), which is independent of the position of the fields in space-time. To derive the quantities associated with invariance under translations we replace in the (2.43) the variation of the fields:

$$\delta_T \phi = \delta\phi + (\partial_\mu \phi)\varepsilon^\mu = 0, \tag{2.47}$$

$$\delta\phi = -\partial_\mu \phi \varepsilon^\mu. \tag{2.48}$$

where in the first variation the term δ_T represents the total transformation of the fields,

$$\delta_t \phi = \phi(x) - \phi'(x') = \phi(x) - \phi'(x) + \phi'(x) - \phi'(x') = \delta\phi + (\partial_\mu \phi)\delta x^\mu. \tag{2.49}$$

The resulting current is a rank two tensor given by

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \partial^\mu \phi + g^{\mu\nu} \mathscr{L}. \tag{2.50}$$

with

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.51}$$

and we can prove that is the energy-stress tensor associated with the Klein Gordon field. Consider now the four conserved charges (we have a charge for each symmetry, so in in this case we have four charges, one for temporal translation and three for spatial translations),

$$T^{0\mu} = \frac{\partial \mathscr{L}}{\partial(\dot\phi)} \partial^\mu \phi + g^{0\mu} \mathscr{L}. \tag{2.52}$$

The component 00 corresponds to the Hamiltonian density,

$$T^{00} = \frac{\partial \mathscr{L}}{\partial(\dot\phi)} \dot\phi - \mathscr{L} = \pi\dot\phi - \mathscr{L}, \tag{2.53}$$

that integrated gives us the Hamiltonian of the system

$$H = \int d^3x \, T^{00} = E. \tag{2.54}$$

[3]using appropriate fall-off condition for the fields at spatial infinity.

While the integral of the spatial components T^{0j} represents the spatial impulse

$$P^i = \int d^3x \, \frac{\partial L}{\partial \dot{\phi}} \partial^i \phi. \tag{2.55}$$

Let us now move to a curved space-time and then replace the tensor $\eta^{\mu\nu}$ of the flat-spacetime with the metric tensor $g^{\mu\nu}$, and the ordinary derivative ∂_μ with the covariant derivative ∇_μ. The Lagrangian (2.41) will depend explicitly on the gravitational field $g^{\mu\nu}$. In this case we know that the transformations naturally associated with curved spacetime are the diffeomorphisms, under which the dynamics remains unchanged. Also in this case it is possible to obtain an energy-stress tensor for the Klein Gordon field which satisfies the equation,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.56}$$

However, in this case, it is not possible to associate a conserved charge since the former equation no longer represents a global conservation law[4]. The Noether theorem stated above is in fact valid only for global transformations which represent an isometry for the metric. When we move fro a minkowskian space-time to a curved one, the system is no longer invariant under Poincarè transformations but under diffeomorphisms, and the field transforms like a tensor

$$g^{\alpha'\beta'} = \Lambda^{\alpha'}_{\mu} \Lambda^{\beta'}_{\nu} g^{\mu\nu}, \tag{2.57}$$

$$\Lambda^{\alpha'}_{\mu} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}}. \tag{2.58}$$

$$\tag{2.59}$$

The prove that (2.56) does not represent a conservation law is quite immediate it is based on the fact that the we have substitute the partial derivative with the covariant one to consider the curvature of spacetime,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\lambda\mu} T^{\nu\lambda} + \Gamma^{\mu}_{\nu\lambda} T^{\mu\lambda} \tag{2.60}$$

The first can be rewrite as

$$\Gamma^{\mu}_{\lambda\mu} = \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g}, \tag{2.61}$$

from which follow that the Gauss Theorem is not enough anymore to recover a conserved quantities,

$$\partial_\mu (\sqrt{-g} T^{\mu\nu}) = -\sqrt{-g}\, \Gamma^{\mu}_{\nu\lambda} T^{\mu\lambda}. \tag{2.62}$$

From a conceptual point of view this result is not surprising, in fact in the case in which a gravitational field is present, we expect that next to the energy of the matter also that of the gravitational field will appear. However we are not able to define a notion of energy density for the gravitational field.
There are some cases in which it is possible to define conserved quantities, for example

[4]Note that the equation (2.56) still defines a local conservation law, and therefore it is possible to associate an energy density with matter, but this is well defined only locally, where we can always consider putting ourselves in a minkowskian reference system.

in the Schwarzchild metric for a static black hole we identify the parameter M with the energy of the black hole and we know that this quantity is conserved globally. This happens whenever space-time has a Killing vector ξ^{α}, that is, such that $\nabla_{\beta}, \xi_{\alpha} = 0$, which generates an isometry along the geodesics to which it is tangent. In other words, a Killing vector identifies directions of the time space along which the metric remains unchanged: $\mathscr{L}_{\xi} g_{\alpha\beta} = 0$. For example, in a static space-time it is possible to define a temporal killing vector whose geodesics, given by the coordinate lines t, the metric does not change. The quantity conserved in this case corresponds precisely to energy. While in the case of a spherical symmetry time space we are able to define a spatial killing vector and the conserved charge corresponding to the angular momentum. Suppose we have a space-time in which there is a killing vector ξ^{α}, this allows us to write a conservation law for the vector $T^{\alpha\beta}\xi^{\beta}$, given by:

$$\nabla_{\alpha}(T^{\alpha\beta}\xi^{\beta}) = (\nabla_{\alpha}T^{\alpha\beta})\xi^{\beta} + T^{\alpha\beta}\nabla_{\alpha}\xi^{\beta} = 0. \tag{2.63}$$

In this case we can take advantage of the following property, valid only for vectors:

$$\sqrt{-g}\,\nabla_{\alpha}(T^{\alpha\beta}\xi_{\beta}) = \partial_{\alpha}(\sqrt{-g}T^{\alpha\beta}\xi_{\beta}) \tag{2.64}$$

which allows us to use the Gauss theorem. Separating the spatial part from the temporal part we obtain:

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\int_{\Sigma} d^3x\,\sqrt{-g}T^{0\beta}\xi_{\beta} = -\int_{\Sigma} d^3x\,\partial_a(\sqrt{-g}T^{a\beta}\xi_{\beta}) = -\int_{\partial\Sigma} d\sigma\,\sqrt{-g}T^{a\beta}\xi_{\beta}n_a = 0 \tag{2.65}$$

where n_a is the vector normal to the edge surface and also in this case we have assumed the annulment of the fields to infinity. We have thus obtained that it is possible to obtain globally conserved quantities even in general relativity only if space-time has at least one Killing vector. It can be shown that the maximum of independent Killing vectors can be ten. In this case, space-time is flat and the 10 conserved quantities correspond precisely to those obtained from invariance under Poincarè transformations.

At this point we pose the problem of seeing if it is possible to give some notion of energy for the gravitational field. The invariance under diffeomorphisms tells us that it is not possible to have local observables, so what we would expect to find in any case is a global quantity. Let's start from the equation (), as already underlined because of the right term this does not represent a conservation law. What we try to define then is a tensor $t^{\mu\nu}$ that has added to the tensor energy-impulse of the matter satisfying the law of conservation:

$$\partial_{\mu}(\sqrt{-g}(T^{\mu\nu} + t^{\mu\nu})) = 0. \tag{2.66}$$

Furthermore, in order for it to be a well-defined quantity it must only be a function of the metric This problem was studied by Landau Lipschitz, and the tensor form that satisfies the properties listed above was found to be:

$$t^{\mu\nu}_{LL} = -\frac{c^4}{8\pi G}g^{\mu\nu} + -\frac{c^4}{16\pi G(-g)}((-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}))_{,\alpha\beta} \tag{2.67}$$

where it is seen that the impulse energy tensor is only a function of the metric. Also you can easily see that the equation () is satisfied, in fact the term with Einstein's tensor

$G^{\mu\nu}$ is going to just cancel $T^{\mu\nu}$, while using the commutativity of ordinary derivatives and the fact that the term in brackets is anti-symmetrical in β and μ it is possible to set the second term to zero. However, $t_{LL}^{\mu\nu}$ is not a tensor but a pseudotensitor, in fact, although the first term with $G^{\mu\nu}$ is a tensor, the term containing the second derivatives of the metric does not transform like a tensor. It can also be rewritten according to Christoffell's symbols, which highlight its pseudo-sensorial character.
A definition of the impulse energy tensor has also tried to give in the linearized case, where in the second-order expansion of the Einstein equations it is possible to interpret one of the terms of self-interaction as the analogue of the impulse energy tensor. Consider Einstein's second-order equations in the vacuum [5]

$$G^{(1)}_{\alpha\beta}[\gamma^{(2)}_{\lambda\delta}] + G^{(2)}_{\alpha\beta}[\gamma_{\lambda\delta}] = 0, \tag{2.68}$$

where $\gamma_{\lambda\delta}$ and $\gamma^{(2)}_{\lambda\delta}$ are the perturbative contributions to the first and second order metrics:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta} + \gamma^{(2)}_{\alpha\beta}. \tag{2.69}$$

Einstein's equations are not linear, a consequence of the fact that the gravitational field interacts with itself. In this perspective we can interpret the Einstein tensor in the second order

$$G^{(2)}_{\alpha\beta} = R^{(2)}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R^{(2)} \tag{2.70}$$

as the tensor energy impulse that generates the contribution to the first order, and rewrite the previous equation in the form:

$$G^{(1)}_{\alpha\beta}[\gamma^{(2)}_{\lambda\delta}] = 8\pi t_{\alpha\beta}, \tag{2.71}$$

with

$$t_{\alpha\beta} = -\frac{1}{8\pi} G^{(2)}_{\alpha\beta}[\gamma_{\lambda\delta}]. \tag{2.72}$$

From the properties of $G_{\mu\nu}$ we know that $t_{\alpha\beta}$ is symmetric and is preserved, $\partial^{\alpha} t_{\alpha\beta} = 0$, so can to be effectively interpreted as the tensor energy impulse of the gravitational field in the linearized development in the second order. However it can be seen that it is not unique, that is, the equation (2.72) remains unchanged if we add to $t_{\alpha\beta}$ a tensor of the form $\partial^{\lambda}\partial^{\delta}U_{\alpha\lambda\beta\delta}$ constructed out from the metric tensor, quadratic in it and with the following symmetries $U_{\alpha\lambda\beta\delta} = U_{[\alpha\lambda]\beta\delta} = U_{\alpha\lambda[\beta\delta]} = U_{\beta\delta\alpha\lambda}$. In fact, as you can see by comparing $t_{\alpha\beta}$ with the Landau-Lifshitz tensor $t^{\mu\nu}LL$, these two differ by a term that has the same properties stated above .
It is also not a gauge invariant, meaning if we replace $\gamma_{\alpha\beta}$ with $\gamma_{\alpha\beta} + 2\delta_{(\alpha}\xi_{\beta)}$ does not remain unchanged [6] These difficulties reflect the fact that it is not possible to give a local definition of energy for the gravitational field. The problem can be overcome by considering non-local observables as the flow at infinity. We see that although it is not possible to give a local definition of energy it is possible to define the total energy

[5]First-order we have $G^{(1)}_{\alpha\beta}[\gamma_{\lambda\delta}] = 0$.

[6]From the linearized case to the first order we know instead that $g_{\alpha\beta}$ is not completely, but we can make gauge transformations under which Einstein's equations remain unchanged.

for an isolated system, that is, in one this space-time is asymptotically flat. In this case requesting that for $r \to \infty$ the components of $\gamma_{\alpha\beta}$ and its derivatives go to zero as $\gamma_{\alpha\beta} = O(1/r)$ and $\partial_r = O(1/r^2)$, we show that the flow of $\gamma_{\alpha\beta}$:

$$E = \int_{\Sigma} t_{00} d^3x \tag{2.73}$$

it is well defined, that is the integral is convergent, and furthermore it is invariant under gauge transformation ξ^{α} which preserve the asymptotic conditions.
There are different types of infinity. We can consider the spatial infinity or the zero infinite. We will discuss later what properties the metric must have for a space-time to be asymptotically flat. In both cases, by imposing fall-off conditions for the metric it is possible to recover a definition of total mass. In fact, we expect that in a space-time asymptotically flat, since the metric at infinity tends towards that of Minkowsky, we are able to find some isometries and consequently asymptotically conserved charges. Let us first analyze the spatial case studied by ADM and then the null one proposed by Bondi, which has particular importance in the study of energy carried by gravitational waves. The two definitions of mass coincide only for a stationary spacetime, ie in the absence of gravitational waves. In the case where gravitational radiation is present, it is shown that the mass of Bondi is decreasing. It is also possible to make a comparison between the mass of Bondi in the second order and that defined above for the flow of gravitational energy and to see that they coincide.

Chapter 3

Spacetime Thermodynamics with Contorsion

In this chapter we present a work based on the paper [De Lorenzo et al., 2018] in collaboration with Simone Speziale and Tommaso De Lorenzo. Our analysis is related to Jacobson's idea that the continuum structure of spacetime could emerge as the thermodynamical equilibrium description of more fundamental quantum degrees of freedom [Jacobson, 1995] and was motivated to generalised this results in presence of torsion.
The chapter is organised in the following way: in the first section we gave a brief introduction of the physical background. In particular our results build on the discussion of this paper [Dey et al., 2017], albeit with a critique of some of their methods and results, and on the original paper of Jacobson [Jacobson, 1995] where he first proposed a thermodynamical interpretation of Einstein equation. In section two we give a brief review of the Einstein-Cartan Action in tetrad variables and with exterior calculus of forms. Then, before giving the original results, we review the Noether identities and the conservation laws in presence of torsion. In the last section we present a derivation of the laws of black hole thermodynamics for our specific case and we address the reader to [Wald, 1984] for a more general discussion. Finally we give in the last section of the chapter a list of the conventions that we used and some equations in tetrad formalism with the exterior calculus.

3.1 Background and motivations

In a famous paper [Jacobson, 1995], Ted Jacobson proposed that Einstein equations could have a thermodynamical origin, compatible with the thermodynamical interpretation of the laws of black hole thermodynamics [Bardeen et al., 1973]. His argument, based on a geometric interpretation of Clausius relation, has been later extended to include non-equilibrium terms and higher derivative gravity theories [Eling et al., 2006, Chirco and Liberati, 2010, Guedens et al., 2012], and more recently to spacetimes with non-propagating torsion, namely Einstein-Cartan first-order gravity [Dey et al., 2017]. The main difficulty of extending Jacobson's idea to Einstein-Cartan gravity is that there are two sets of independent field equations to be derived: the torsion equations as well as the proper Einstein equations. It was showed in [Dey et al., 2017] that it is possible to derive the latter set for a special type of torsion, and by identifying the torsional terms as a non-equilibrium contribution to Clausius relation. The torsion equations are not derived, and whether they can have also a thermodynamical origin is left as an open question. In the following analysis we also do not provide a derivation of the torsional equations, but we show that if they hold, the tetrad Einstein equations can be derived without the need of non-equilibrium terms nor restrictions on torsion. The technical result that allows us to achieve this is the identification of the conserved energy-momentum tensor. The last point is crucial: in Einstein-Cartan theory, there is no conserved energy-momentum tensor that appears as source of the field equations. Nonetheless, if one restricts to invertible tetrads (and this appears necessary to connect with the metric theory and the familiar notions used in Jacobson's argument), the connection can always be written as a Levi-Civita one plus a contorsion tensor. Using this well-known decomposition, the tetrad Einstein equations can be written as the Levi-Civita Einstein tensor on the left hand side, and a torsion dependent effective energy-momentum tensor T^{eff} on the right hand side. By taking the Levi-Civita covariant derivate of both sides, the left one vanishes due to Bianchi's identities. This in turn implies the vanishing of the right hand side, allowing to identify the conserved energy-momentum tensor also in the presence of torsion.

For the thermodynamical argument, on the other hand, one needs to identify a conserved energy-momentum tensor without using the field equations, since these are to be derived. In order to achieve this result we will show that the conservation of T^{eff} in the Einstein-Cartan theory can be derived *without* using the tetrad field equations. The proof is simple although rather lengthy, and best done using differential forms. It follows from the Noether identities of the theory, and requires the matter and torsion field equations to be satisfied. Our second result is to use this conserved energy-momentum tensor and the contorsion description to show that the tetrad Einstein equations can be derived from the Clausius relation with the same assumptions and hypothesis of the metric case [Jacobson, 1995], without the need of the non-equilibrium terms and the restrictions on torsion used in [Dey et al., 2017]. This is possible because the starting point of Jacobson's argument, a Killing horizon associated with a locally boosted observer, is a notion which is insensitive to the presence of torsion. In particular, the generators of the Killing horizons follow the Levi-Civita geodesic equation. This turns out to suffice to recover the tetrad Einstein equations from the equilibrium Clausius relation, since the torsion terms are identified by the effective energy-momentum tensor. A further advantage of our derivation is that it includes also the Immirzi term in the Einstein-Cartan theory. To complete our discussion, we also look at the laws of black hole mechanics in the presence of torsion. The zeroth law is unaffected, and it can be proven exactly as in the metric case, provided that the energy conditions are imposed on T^{eff}. The first law on the other hand depends on torsion. We consider here the 'physical process' version of the first law [Wald, 1995], which is closely related to Jacobson's argument run backwards. Using the same contorsion decomposition as before, the formal expression of the first law is unchanged, but the quantities appearing depend on torsion through the effective energy-momentum tensor. The second law has a more marginal dependence, in the sense that torsion simply enters the inequalities on the energy conditions required.

3.2 Analysis of Einstein-Cartan field equations and the role of matter

Let us begin by briefly reviewing the field equations of Einstein-Cartan theory and the contorsion decomposition. We refer the reader to [Hehl et al., 1995] for more details, and to the Appendix 3.7 for definitions and notation. We consider the following first-order action,

$$S_{\rm EC}(e,\omega)=\frac{1}{16\pi}\int P_{IJKL}\Big(e^I\wedge e^J\wedge F^{KL}(\omega)-\frac{\Lambda}{6}e^I\wedge e^J\wedge e^K\wedge e^L\Big), \tag{3.1}$$

where

$$P_{IJKL}:=\frac{1}{2\gamma}(\eta_{IK}\eta_{JL}-\eta_{IL}\eta_{JK})+\frac{1}{2}\varepsilon_{IJKL}, \tag{3.2}$$

and γ is the Immirzi parameter. We restrict attention to invertible, right-handed tetrads. The action is then equivalent to first-order general relativity [1]

$$S_{\rm EP}(g,\Gamma)=\frac{1}{16\pi}\int[\sqrt{-g}(g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}(\Gamma)-2\Lambda)+\frac{1}{\gamma}\tilde{\varepsilon}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]d^4x, \tag{3.3}$$

with initially independent metric and connections, which are related to the fields of (4.1) by the familiar formulas

$$g_{\mu\nu}=e^I_\mu e^J_\nu\eta_{IJ}, \qquad \Gamma^\rho_{\mu\nu}=e^\rho_I D_\mu e^I_\nu:=e^\rho_I(\partial_\mu e^I_\nu+\omega^{IJ}_\mu e_{J\nu}). \tag{3.4}$$

We collectively denote the matter fields as ψ, and consider a general matter Lagrangian $L_m(e,\omega,\psi):=\mathscr{L}_m(e,\omega,\psi)d^4x$. Varying the matter action we have

$$\delta S_m=\int\delta L_m=\int\Big(2\tau^\mu{}_I\delta e^I_\mu+\sigma^\mu{}_{IJ}\delta\omega^{IJ}_\mu+E_m\delta\psi\Big)\,e\,d^4x, \tag{3.5}$$

where E_m denotes the matter field equations, and we defined the source terms

$$\tau^\mu{}_I:=\frac{1}{2e}\frac{\delta\mathscr{L}_m}{\delta e^I_\mu}=-\frac{1}{2e}\frac{\delta\mathscr{L}_m}{\delta e^\nu_J}e^\nu_I e^\mu_J=:\tau^J{}_\nu e^\nu_I e^\mu_J, \qquad \sigma^\mu{}_{IJ}=\frac{1}{e}\frac{\delta\mathscr{L}_m}{\delta\omega^{IJ}_\mu}. \tag{3.6}$$

The sign choice in the definition of τ is not universal in the literature. We picked it this way in analogy with the metric energy-momentum tensor $T^\Gamma_{\mu\nu}$,

$$T^\Gamma_{\mu\nu}:=-\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_m(g,\Gamma)}{\delta g^{\mu\nu}}=-\frac{1}{e}\frac{\delta\mathscr{L}_m(e,\omega)}{\delta e^{I(\mu}}e^I_{\nu)}=2\tau^I{}_{(\mu}e_{\nu)I}, \tag{3.7}$$

which coincides with the one of general relativity in the absence of torsion.
The field equations obtaining varying (4.1) and the matter action are

$$G^\mu{}_I(e,\omega)+\Lambda e^\mu_I+\frac{1}{2\gamma}\varepsilon^{\mu\nu\rho\sigma}e^\alpha_I R_{\alpha\nu\rho\sigma}(e,\omega)=16\pi\,\tau^\mu{}_I, \tag{3.8a}$$

$$P_{IJKL}\varepsilon^{\mu\nu\rho\sigma}e^K_\nu T^L_{\rho\sigma}=-16\pi\,\sigma^\mu_{IJ}. \tag{3.8b}$$

[1] Sometimes called Einstein-Palatini general relativity because proving its equivalence to general relativity uses the Palatini identity.

Here

$$G^{\mu}{}_{I}(e,\omega) := \frac{1}{4}\varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma}e^{J}_{\nu}F^{KL}_{\rho\sigma}(\omega) = G^{\mu\nu}(e,\omega)e_{\nu I} \tag{3.9}$$

is the first-order Einstein tensor, the Riemann tensor and curvature are related by $R_{\mu\nu\rho\sigma}(e,\omega) = e_{\mu I}e_{\nu J}F^{IJ}_{\rho\sigma}(\omega)$, and $T^{I} := d_{\omega}e^{I}$ is the torsion. The first set (3.8a) contains the ten Einstein equations, plus six redundant equations: although $G_{\mu\nu}(e,\omega)$ is not symmetric a priori, it is easy to show that the Noether identity associated with invariance of the action under internal Lorentz transformations (see (3.31a) below) implies that the equations for $G^{\mu}{}_{[I}e_{J]\mu}$ are automatically satisfied. The relevant content of (3.8a) is therefore just its symmetric part, which in turn gives the Einstein's equations

$$G_{\mu\nu}(e,\omega) + \Lambda g_{\mu\nu} + \frac{1}{2\gamma}\varepsilon_{(\mu}{}^{\lambda\rho\sigma}R_{\nu)\lambda\rho\sigma}(e,\omega) = 8\pi T^{\Gamma}_{\mu\nu}, \tag{3.10}$$

or equivalently as functions of (g,Γ) via (3.4).
In the following, we will refer to (3.8a) or (3.10) as Einstein's equations (in the presence of torsion), to be distinguished from the torsion Einstein-Cartan equations (3.8b), or torsion equations for short. It is often convenient to write the field equations using the language of differential forms, as we did in the action (4.1). To that end, we use the Hodge dual $\star$ mapping p-forms to $(4-p)$-forms (see Appendix 3.7 for conventions). This allows us to define the Einstein 3-from

$$\star G_{I}(\omega) := -\frac{1}{2}\varepsilon_{IJKL}e^{J}\wedge F^{KL}(\omega), \tag{3.11}$$

where the opposite sign with respect to (3.9) is a consequence of Lorentzian signature, and equivalently the dual source forms $\star\tau_{I}$ and $\star\sigma_{IJ}$. The field equations (3.8) then read

$$\star G_{I}(\omega) + \Lambda\star e_{I} - \frac{1}{\gamma}e^{J}\wedge F_{IJ}(\omega) = 16\pi\star\tau_{I}, \tag{3.12a}$$

$$P_{IJKL}e^{K}\wedge T^{L} = 8\pi\star\sigma_{IJ}. \tag{3.12b}$$

3.2.1 Properties of the contorsion field

Although connections form an affine space with no preferred origin, the presence of an invertible tetrad suggests a natural origin: the Levi-Civita connection $\omega^{IJ}_{\mu}(e)$ associated with the tetrad. We can then always decompose an arbitrary connection into Levi-Civita plus a contorsion tensor C^{IJ}_{μ} as

$$\omega^{IJ}_{\mu} = \omega^{IJ}_{\mu}(e) + C^{IJ}_{\mu}. \tag{3.13}$$

Torsion and curvature are related to the contorsion as follows:

$$T^{I} = C^{IJ}\wedge e_{J}, \tag{3.14}$$

$$F^{JK}(\omega) = F^{JK}(e) + d_{\omega(e)}C^{JK} + C^{JM}\wedge C_{M}{}^{K} = F^{JK}(e) + d_{\omega}C^{JK} - C^{JM}\wedge C_{M}{}^{K}, \tag{3.15}$$

where $d_{\omega(e)}$ is the exterior derivative with respect to the Levi-Civita connection. Plugging this decomposition into the field equations we find

$$\star G_I(e) + \Lambda \star e_I = 16\pi \star \tau_I + P_{IJKL}(d_{\omega(e)} C^{JK} + C^{JM} \wedge C_M{}^K), \tag{3.16}$$

$$P_{IJKL} e^K \wedge C^{LM} \wedge e_M = 8\pi \star \sigma_{IJ}. \tag{3.17}$$

The fact that the field equations for the Einstein-Cartan theory can be recasted as in (3.16) is the source of an old debate in the literature about the role of torsion [Hehl and Weinberg, 2007]: if we forget about the notion of affine parallel transport defined by ω^{IJ}, and use simply the one defined by $\omega^{IJ}(e)$ in the sector of invertible tetrads, then the theory is indistinguishable from ordinary metric theory with some non-minimal matter coupling. The non-minimality is captured by the effective energy-momentum tensor sourcing (3.16), i.e.

$$\star \tau_I^{\text{eff}} := \star \tau_I + \frac{1}{16\pi} P_{IJKL}(d_{\omega(e)} C^{JK} + C^{JM} \wedge C_M{}^K). \tag{3.18}$$

While we take no stand in the debate, we will heavily use this fact in the thermodynamic discussion below. Before getting there, we need to review in the next Section the relation between the conservation of the energy-momentum tensor and the Bianchi identities. For convenience of the reader, we report the relation between torsion and contorsion in tensor language,

$$T^\rho{}_{\mu\nu} := e^\rho_I T^I{}_{\mu\nu} = -2C_{[\mu,\nu]}{}^\rho = 2\Gamma^\rho_{[\mu\nu]}, \tag{3.19}$$

$$C_{\mu,\nu\rho} = \frac{1}{2} T_{\mu,\nu\rho} - T_{[\nu,\rho]\mu}, \qquad C_{(\mu,\nu)\rho} = T_{(\mu,\nu)\rho}. \tag{3.20}$$

The Einstein equations (3.8a) read

$$G_{\mu\nu}(e) + \Lambda g_{\mu\nu} = 8\pi T^{\text{eff}}_{\mu\nu}, \tag{3.21}$$

$$T^{\text{eff}}_{\mu\nu} = 2\tau^I{}_{(\mu} e_{\nu)I} + \frac{1}{16\pi}\left(6 g_{\alpha(\mu} \delta^{\alpha\rho\sigma}_{\nu)\gamma\delta} - \frac{2}{\gamma} g_{\gamma(\mu} \varepsilon_{\nu)\delta}{}^{\rho\sigma}\right)\left(1\nabla_\rho C_{\sigma,}{}^{\gamma\delta} + C_{\rho,}{}^{\gamma\lambda} C_{\sigma,\lambda}{}^\delta\right). \tag{3.22}$$

We refrained from expanding the completely antisymmetric $\delta^{\alpha\rho\sigma}_{\nu\gamma\delta}$ since no useful simplification occurs. Notice that for a given contorsion we have a 1-parameter family of conserved energy momentum tensors, labeled by the Immirzi parameter. Finally, using the torsion equations, T^{eff} can be seen to be linear in the source tensor of the Einstein equations, and contain derivative and quadratic terms in the source tensor of the torsion equations.

3.3 Noether identities and conservation laws in presence of torsion

The gravity action (4.1) is invariant under internal Lorentz transformations

$$\delta_\lambda e^I = \lambda^I{}_J e^J, \qquad \delta_\lambda \omega^{IJ} = -d_\omega \lambda^{IJ}, \tag{3.23}$$

as well as diffeomorphisms,[2]

$$\delta_\xi e^I = \pounds_\xi e^I = de^I \lrcorner \xi + d(e^I \lrcorner \xi) = d_\omega e^I \lrcorner \xi + d_\omega(e^I \lrcorner \xi) - (\omega^I{}_J \lrcorner \xi) e^J, \tag{3.24a}$$

$$\delta_\xi \omega^{IJ} = \pounds_\xi \omega^{IJ} = d\omega^{IJ} \lrcorner \xi + d(\omega^{IJ} \lrcorner \xi) = F^{IJ} \lrcorner \xi + d_\omega(\omega^{IJ} \lrcorner \xi). \tag{3.24b}$$

Specializing the variation of the action (4.1) to (3.23) and (4.21) respectively, and integrating by parts, one obtains the following Noether identities,[3]

$$P_{IJKL} e^K \wedge F^{LM} \wedge e_M = P_{IJKL} e^K \wedge d_\omega T^L, \tag{3.26a}$$

$$d_\omega (P_{IJKL} e^J \wedge F^{KL}) = P_{IJKL} T^J \wedge F^{KL}. \tag{3.26b}$$

These are nothing but contracted forms of the Bianchi identities $d_\omega F^{IJ} = 0$, $d_\omega T^I = F^{IJ} \wedge e_J$. Using the field equations (3.12) in (3.26) one finds additional relations for the matter sources,

$$d_\omega \star \sigma_{IJ} = 2 \star \tau_{[I} \wedge e_{J]}, \tag{3.27a}$$

$$d_\omega \star \tau_I = \frac{1}{2} F^{JK} \lrcorner e_I \wedge \star \sigma_{JK} + T^J \lrcorner e_I \wedge \star \tau_J. \tag{3.27b}$$

These matter Noether identities can also be derived without reference to the field equations (3.12): they follow from invariance of the matter action (3.5) under (3.23) and (4.21), on-shell of the matter field equations. See [Hehl and McCrea, 1986, Hehl et al., 1995, Barnich et al., 2016] for more details.

Recall now that, in the metric formalism, invariance of the matter Lagrangian under diffeomorphisms guarantees the conservation of the energy-momentum tensor,

$$\delta_\xi L_m = d(L_m \lrcorner \xi) \quad \Rightarrow \quad \nabla_\mu T^{\mu\nu} = 0, \tag{3.28}$$

on-shell of the matter field equations. In the first-order formalism with tetrads, the energy-momentum tensor does not appear immediately in the field equations: the closest object we have is the source τ of the Einstein's equations (3.8a). This quantity is however not conserved, as we can see from (3.27b), whose right-hand side does not vanish on-shell. Nevertheless, although τ is not conserved, it is easy to identity an effective energy-momentum tensor which is conserved, thanks to the contorsion decomposition (3.16). If we take the Levi-Civita exterior derivative $d_{\omega(e)}$ on both sides of (3.16), the left-hand side vanishes identically. This in turns implies the vanishing of the right-hand side, which gives a local conservation law

$$d_{\omega(e)} \tau_I^{\text{eff}} = 0 \tag{3.29}$$

valid *also in the presence of torsion*. Equivalently in terms of tensors, the object with vanishing (Levi-Civita) divergence is $T^{\text{eff}}_{\mu\nu}$ as defined in (3.22), and it provides

[2]Note that the Lie derivatives (4.21) are not gauge-covariant objects. It is often convenient to consider the linear combination of transformations $L_\xi = \pounds_\xi + \delta_{\omega \lrcorner \xi}$ which is covariant.

[3]To obtain (3.26a), we used the identity (A.6) below. For the reader's convenience, we report the identities also in the more common γ-less case,

$$\star G_{[I} \wedge e_{J]} = -\frac{1}{2} \varepsilon_{IJKL} e^K \wedge d_\omega T^L, \qquad d_\omega \star G_I = -\frac{1}{2} \varepsilon_{IJKL} T^J \wedge F^{KL}. \tag{3.25}$$

the conserved energy-momentum tensor of the theory.[4] This simple observation is well-known in the literature, see [Hehl et al., 1976, Hehl, 1976, Böhmer and Hehl, 2018] (where it is referred to as 'combined energy-momentum tensor'), and can be taken to provide the basis of energy conservation in Einstein-Cartan theory.

For later purposes, we are interested in whether it is possible to derive the conservation law (3.29) *without* using the Einstein's equations. This is a bit of a strange question if one starts from an action principle, but it is crucial to Jacobson's thermodynamical argument, where this is not the case. We could not find the answer to this question in the literature, which turns out to be affirmative. The result is the following: **Proposition 1:** *The matter Noether identities* (3.27) *on-shell of the matter and torsion field equations imply the conservation law for the effective energy-momentum tensor* (3.29). The proof is a somewhat lengthy exercise in algebraic identities, and we leave it to Appendix 2. We also looked for a stronger result, namely whether (3.29) also holds without imposing the torsion equation, but we did not succeed. The proof in the Appendix (A.2) shows explicitly the step in which we use the torsion field equation. To give an idea of what happens, using the contorsion decomposition (3.27) can be combined to give

$$d_{\omega(e)}\Big(\star\tau_I+\frac{1}{2}C^{JK}\lrcorner e_I\star\sigma_{JK}\Big)=\frac{1}{2}\Big(F^{JK}(\omega)\lrcorner e_I+\pounds_{e_I}C^{JK}\Big)\wedge\star\sigma_{JK} \tag{3.30}$$

(an expression for the Noether identities which appears for instance in [Hehl et al., 2013]), and using the torsion field equation the right-hand side reduces to $d_{\omega(e)}$ of a 3-form. In tensorial language, the Noether identities for a generic gauge and diff-invariant Lagrangian density $\mathscr{L}$ read (see e.g. [Barnich et al., 2016])

$$D_\mu\frac{\delta\mathscr{L}}{\delta\omega^{IJ}_\mu}+\frac{\delta\mathscr{L}}{\delta e^{[I}_\mu}e^\mu_{J]}=0, \tag{3.31a}$$

$$\frac{\delta\mathscr{L}}{\delta\omega^{IJ}_\mu}F^{IJ}_{\nu\mu}(\omega)+\frac{\delta\mathscr{L}}{\delta e^I_\mu}T^I_{\nu\mu}-e^I_\nu D_\mu\frac{\delta\mathscr{L}}{\delta e^I_\mu}=0, \tag{3.31b}$$

on-shell of the matter field equations. For the Lagrangian density in (4.1), these give respectively contractions of the algebraic and differential Bianchi identities,

$$2R_{[\mu\nu]}=-\nabla_\rho T^\rho{}_{\mu\nu}-2\nabla_{[\mu}T^\rho{}_{\nu]\rho}+T^\rho{}_{\rho\sigma}T^\sigma{}_{\mu\nu}, \tag{3.32}$$

$$\nabla_\nu G^\nu{}_\mu=T^\rho{}_{\mu\sigma}R^\sigma{}_\rho-\frac{1}{2}T^\nu{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}, \tag{3.33}$$

from the γ-less terms, and

$$\varepsilon^{\alpha\nu\rho\sigma}R_{\mu\nu\rho\sigma}=\varepsilon^{\alpha\nu\rho\sigma}(\nabla_\nu T_{\mu,\rho\sigma}+T_{\mu,\lambda\nu}T^\lambda{}_{\rho\sigma}), \tag{3.34}$$

$$\varepsilon^{\alpha\beta\rho\sigma}\nabla_\beta R_{\mu\nu\rho\sigma}=\varepsilon^{\alpha\beta\rho\sigma}T^\lambda{}_{\beta\rho}R_{\mu\nu\sigma\lambda} \tag{3.35}$$

[4]An alternative 'conservation law' using the full connection would be of little practical meaning, because it would not lead to hypersurface quantities independent of the choice of space-like slice. Another way to identify this conserved object is to solve the torsion equation – which in the case of Einstein-Cartan is simply algebraic since torsion does not propagate, and plug the solution back into the action. Varying the resulting matter action with respect to the tetrad then immediately gives the effective energy-momentum tensor (3.22).

for the part in $1/\gamma$. As for the matter action,

$$D_\mu(e\sigma^\mu_{IJ}) = -2e\tau^\mu{}_{[I}e_{J]\mu}, \tag{3.36}$$

$$D_\mu(e\tau^\mu{}_I) = ee^\mu_I\left(\frac{1}{2}F^{JK}_{\mu\nu}\sigma^\nu_{JK} + T^J_{\mu\nu}\tau^\nu{}_J\right). \tag{3.37}$$

3.4 Thermodynamics formulation of Einstein equations

We now come to the main motivation for our analysis: show that Proposition 1 allows us to run Jacobson's argument with the usual equilibrium assumptions. To better appreciate our point, let us briefly recall the key steps of the metric case, referring the reader to [Jacobson, 1995] for more details.

3.4.1 Set-up: definition of killing vector field and bifurcating horizon

Consider an arbitrary metric $g_{\mu\nu}$ on a manifold, a point P and a neighbourhood sufficiently small for spacetime to be approximately flat. Denote by ξ^μ the future-pointing (approximate) Killing vector generating a Rindler horizon $\mathscr{H}$ within the approximately flat region, with bifurcating surface $\mathscr{B}$ through the point P. This is by construction hypersurface orthogonal, null at the horizon but not outside, and vanishing at $\mathscr{B}$:

$$\xi^2 \stackrel{\mathscr{H}}{=} 0, \qquad \partial_\mu\xi^2 =: -2\kappa\xi_\mu, \qquad \xi^\mu \stackrel{\mathscr{B}}{=} 0. \tag{3.38}$$

Since it is Killing, it is also geodesic,

$$\xi^\nu{}_1\nabla_\nu\xi^\mu = -\frac{1}{2}\partial_\mu\xi^2 = \kappa\xi^\mu. \tag{3.39}$$

The inaffinity κ can be proven to be constant on the horizon, and it is usually referred to as the horizon surface gravity. For a Rindler horizon, constancy of κ follows immediately from the vanishing of the Riemann tensor.[5] It is useful to introduce an affine parameter λ along the null geodesics, with origin at the point P. It can be easily shown that

$$\xi^\mu = -\lambda\,\kappa l^\mu, \qquad l^\mu\partial_\mu = \partial_\lambda. \tag{3.40}$$

Given this geometric set-up, the first step of Jacobson's argument is to associate to the Rindler horizon its Unruh temperature:

$$(i) \qquad T = \frac{\kappa}{2\pi}, \qquad \kappa = \text{constant}. \tag{3.41}$$

[5]For a stationary black hole horizon, this is the content of the zeroth law of black hole mechanics. This was proved using the Einstein's equations and the dominant energy condition [Bardeen et al., 1973], although in principle one could just require the analogue of the dominant energy condition directly on the Ricci tensor, as done in the generalization to isolated horizons [Ashtekar et al., 2000b].

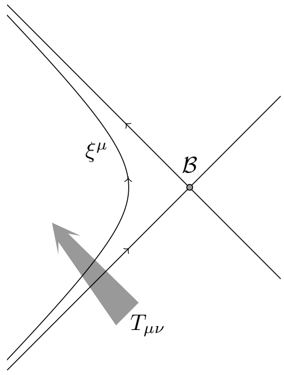


Figure 3.1. *The set-up thermodynamical derivation of Einstein's equation as proposed in [Jacobson, 1995]. Local flatness allows to consider approximate Rindler observers ξ^μ around any point P of a given spacetime. The associate Rindler horizon has bifurcate surface $\mathscr{B}$ passing through P. The system is perturbed by a small flux of matter crossing the past horizon and entering the left wedge. For the derivation to be valid, an infinite family of ξ^μ is actually considered, one per each direction.*

Next, three assumptions are made: first, that there is an energy flux through the horizon in the near past of P, see Fig. 5.1, given by a *conserved* energy-momentum tensor $T_{\mu\nu}$:

$$(ii) \qquad \Delta U := \int_{\mathscr{H}} T_{\mu\nu}\xi^\mu l^\nu d\lambda d^2S = -\kappa \int_{\mathscr{H}} T_{\mu\nu}l^\mu l^\nu \lambda d\lambda d^2S, \qquad \nabla_\mu T^\mu{}_\nu = 0, \tag{3.42}$$

where we used (3.40) and the constancy of κ. This energy flux will be interpreted thermodynamically as a heat flux, $\Delta U = \Delta Q$. Second assumption, that there is a notion of entropy variation associated to the horizon, which is (universally, i.e. independently of the matter state) proportional to the area variation:

$$(iii) \qquad \Delta S = \eta\,\Delta A = \eta \int_{\mathscr{H}} \theta d\lambda d^2S, \tag{3.43}$$

where θ is the expansion of horizon. This is controlled by the Raychadhuri equation for l^μ,

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}l^\mu l^\nu. \tag{3.44}$$

The final, technical assumption made in [Jacobson, 1995] is that at P one can take $\theta = \sigma_{\mu\nu} = 0$, and approximate the solution of the Raychadhuri equation simply by $\theta = -\lambda R_{\mu\nu}l^\mu l^\nu + O(\lambda^2)$.[6] Using this approximation,

$$\Delta S = -\eta \int_{\mathscr{H}} \lambda\, R_{\mu\nu}l^\mu l^\nu d\lambda d^2S. \tag{3.45}$$

[6] Vanishing of the initial expansion and shear are taken to be the equilibrium conditions necessary for the upcoming application of Clausius relation. We find on the other hand the last approximation quite strong in that it implies constant curvature, at least along the horizon's generators. See Appendix 3.5.1 for a discussion of this approximation, and an alternative derivation which uses perturbation theory in the metric fluctuations.

Finally, we observe that using $(i-iii)$ and the approximation (3.45), the Clausius first law of thermodynamics $\Delta Q = T\Delta S$ implies

$$\int_{\mathscr{H}} \left(\frac{2\pi}{\eta} T_{\mu\nu} - R_{\mu\nu}\right) l^\mu l^\nu \lambda d\lambda d^2S = 0. \tag{3.46}$$

Since this is valid for an arbitrary direction of the Killing boost and at any point, we can remove the integral. The Einstein equations (with an undetermined cosmological constant) then follow by imposing the conservation law $\nabla^\mu T_{\mu\nu} = 0$. The Newton constant is identified determined by $G = 1/(4\eta)$.

3.4.2 The effect of torsion on the set-up

In the Einstein-Cartan theory (4.1) the connection is a priori affine, and torsion can be present, affecting the geodesic and Raychaudhuri equations. One may then think that the argument above should be substantially revisited. As we now show, this is actually not the case. The first observation we make is that the starting point of Jacobson's argument, a Killing horizon, is a purely metric notion:

$$\begin{aligned} 0 = \pounds_\xi g_{\mu\nu} &= \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha \\ &= 2\imath\nabla_{(\mu}\xi_{\nu)} = \nabla_{(\mu}\xi_{\nu)} + T_{(\mu}{}^{\rho}{}_{\nu)}\xi_\rho . \end{aligned} \tag{3.47}$$

Hence by definition, it does not depend on torsion, in spite of the apparent presence of the latter in the last expression above. The constancy of κ on the approximate Rindler horizon also follows like in the metric case from the vanishing of the metric Riemann tensor. Being Killing and null, ξ^μ is automatically geodetic with respect to the Levi-Civita connection (which we recall is always well-defined and at disposal since we are only interested in the sector of Einstein-Cartan theory with invertible tetrads), so (3.39) still holds. Hence, we can run most of the argument as in the metric case. Step (i) is unchanged. For step (ii), we follow [Jacobson, 1995] and define the energy flux as the integral of the *conserved* energy-momentum tensor. Proposition 1 identifies this object uniquely as $T^{\text{eff}}_{\mu\nu}$ defined in (3.22), with its torsional dependence. Step (iii) is also unchanged: since the generators of the Killing horizon follow the Levi-Civita geodesics (3.39), the change of the expansion of the generators is governed by the Raychaudhuri equation (3.44) with the metric Ricci tensor $R_{\mu\nu}(e)$ appearing on the right-hand side. Imposing again the equilibrium Clausius relation $\Delta Q = T\Delta S$ with these $(i-iii)$, and using the same approximation (3.45), we arrive exactly at

$$\int_{\mathscr{H}} \left(\frac{2\pi}{\eta} T^{\text{eff}}_{\mu\nu} - R_{\mu\nu}(e)\right) l^\mu l^\nu \lambda d\lambda d^2S. \tag{3.48}$$

We conclude that the torsion-full Einstein equations, in the form (3.21), can be derived à la Jacobson from the equilibrium Clausius relation. No need to consider a torsion-full Raychaudhuri equation, non-equilibrium terms and restrictions on torsion, as argued in [Dey et al., 2017] and reviewed in the next Section. It suffices to use the result of Proposition 1 to identify the correct energy-momentum tensor.
There is however an important caveat to our procedure: we are assuming the torsion equations to hold, since we used them to prove Proposition 1. This may look unsatisfactory, since it is currently not known whether these equations can be derived from

a thermodynamical description. Our logic is that if such a description of the torsion equations exists, then it is consistent to assume that they hold when deriving the Einstein equations. This said, it is also possible that Proposition 1 holds off-shell of the torsion equations, so that these are not needed to derive the Einstein equations. Nonetheless, one would still need to be able to derive the torsion equations from thermodynamics for the whole framework to make sense. Assuming them to hold seems thus to us coherent if a complete thermodynamical framework exists. In any case, the main problem if one does not want to use the conserved energy-momentum tensor is the ambiguity that one faces in defining it, see e.g. [Hehl, 1976]. The prescription used by the authors of [Dey et al., 2017] for instance, is to take what would be the source of the Einstein equations, namely the derivative of the matter Lagrangian with respect to the tetrad (or to the metric, equivalently up to a symmetrization). Notice that this can be tricky in the presence of torsion, because one can work with either the first-order action $S(g,\Gamma)$ or the second-order action $S(g,C)$. The field equations are completely equivalent since the two actions are related by a (non-linear) field redefinition, however for the sources one has

$$T^{\Gamma}_{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta L_m(g,\Gamma)}{\delta g^{\mu\nu}}, \tag{3.49}$$

$$T^{C}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta L_m(g,C)}{\delta g^{\mu\nu}} = T^{\Gamma}_{\mu\nu} + \frac{2}{\sqrt{-g}}\frac{\delta L_m(g,\Gamma)}{\delta \Gamma^{\alpha}_{\beta\gamma}}\frac{\delta \Gamma^{\alpha}_{\beta\gamma}}{\delta g^{\mu\nu}}. \tag{3.50}$$

Both coincide with the general relativity energy-momentum tensor when torsion vanishes, but differ in the presence of torsion. This type of ambiguity reminds us that using a conserved energy-momentum tensor, when available, is always the best choice. We now show how this ambiguity in turn affects the non-equilibrium approach to the derivation of the Einstein equations.

3.4.3 Non-equilibrium terms in the thermodynamical relations and their interpretation

A more general setting including a non-vanishing shear has been considered in [Eling et al., 2006, Chirco and Liberati, 2010]. In this case the presence of additional terms on the right-most side of (3.43) is incompatible with the equilibrium Clausius relation. Hence to run Jacobson's argument one must assume that there are non-equilibrium terms,

$$\Delta Q = T\Delta S + \Delta S_{\text{non-equi}}. \tag{3.51}$$

The interpretation of the shear-squared terms as non-equilibrium is justified a priori from the horizon tidal heating effect [Chirco and Liberati, 2010]. It should be noticed however that the same shear-squared terms enter both the $T\Delta S$ and the $\Delta S_{\text{non-equi}}$ contributions, since one is still assuming (3.43), that the entropy variation is proportional to the area variation. This feature seems to us unusual from a thermodynamical perspective.

In any case, we now discuss the application of the non-equilibrium approach to deriving the Einstein equations, which is more problematic. We start as before from the observation that a Killing horizon is metric-geodetic, and use the same approximations leading to the integrated metric Raychaudhuri equation (3.45), but this time allowing a non-zero shear in (3.44). Then from (3.51) we obtain

$$\frac{2\pi}{\eta}\int_{\mathscr{H}} T^{??}_{\mu\nu} l^\mu l^\nu \lambda d\lambda d^2S = \int_{\mathscr{H}} \Big(R_{\mu\nu}(e) l^\mu l^\nu + \sigma_{\mu\nu}\sigma^{\mu\nu}\Big)\lambda d\lambda d^2S + \Delta S_{\text{non-equi}}. \quad (3.52)$$

The delicate point now is how to define the heat flux, namely what $T^{??}_{\mu\nu}$ needs to be used on the left-hand side of the above equation. Clearly, the identification of the non-equilibrium terms that will be needed to obtain the Einstein equations (3.21) depends on how we define the energy-momentum tensor. If, as in the previous Section, the conserved one is used, the only non-equilibrium term comes from the shear, which can then be argued for as in the metric theory following [Eling et al., 2006, Chirco and Liberati, 2010]. This shows how the derivation of the Einstein equations from the conserved energy-momentum tensor and metric Raychaudhuri equation can be easily extended to the presence of shear.

If we chose instead to define the heat flux via a source tensor, we would need additional non-equilibrium terms in order to fully reproduce the Einstein equations (3.21). The crucial point is whether they can be justified a priori as in the example of the tidal heating, else the construction is artificial. The authors of [Dey et al., 2017] argue that this is possible, if (a) we choose $T^{??}_{\mu\nu} = T^C_{\mu\nu}$ for the heat flux, and (b) we define the non-equilibrium terms as those arising from the torsion-full Raychaudhuri equation that include torsion-full derivatives of l^μ. There are three problems that we can see with this construction. First, a Killing vector is metric-geodesic, but in general not geodesic with respect to the torsion-full connection, since from (3.13) we see that

$$\xi^\nu \nabla_\nu \xi_\mu = \kappa\xi_\mu - C_{\nu,\mu\rho}\,\xi^\nu\xi^\rho = \kappa\xi_\mu - T_{\nu,\mu\rho}\,\xi^\nu\xi^\rho. \quad (3.53)$$

For this reason, the authors of [Dey et al., 2017] restrict torsion to satisfy

$$C_{\nu,\mu\rho}\xi^\nu\xi^\rho = 0. \quad (3.54)$$

This restriction implies that metric and torsion-full geodesics coincides, and one can use the geodesic torsion-full Raychaudhuri equation on the Killing horizon. But since the metric and the torsion-full geodesic expansions also coincide,[7] it follows that the torsion-full Raychaudhuri equation is *identical* to the metric one. Therefore, it is unclear what one gains from this approach, except for a restriction on torsion that in the equilibrium approach presented in the previous Section is not necessary.[8]

Second, the identification of the non-equilibrium contributions as torsion-full covariant derivatives of l^μ is questionable: we are not aware of any proof that in a

[7]In the presence of torsion, the displacement of a vector q^μ Lie dragged along ξ^μ is given by

$$\xi^\nu\nabla_\nu q^\mu = B_{\mu\nu}q^\nu, \qquad B_{\mu\nu} := \nabla_\nu\xi_\mu + T_{\mu,\lambda\nu}\xi^\lambda = \imath\nabla_\nu\xi_\mu + C_{\rho,\mu\nu}\xi^\rho,$$

hence introducing the usual projector $\perp^{\mu\nu}$ on a 2d space-like surface orthogonal to ξ^μ, we have $\theta := \perp^{\mu\nu} B_{\mu\nu} = \imath\theta$. For the reader interested in more details on geodesics with torsion, see e.g. [Luz and Vitagliano, 2017, **?**].

[8]Since in order to recover the Einstein equations we will need to consider arbitrary boost Killing vectors, see discussion below (3.46), the restriction on torsion (3.54) should hold for any ξ^μ. This implies a strong restriction on torsion, that can be satisfied for instance if it is completely antisymmetric. A priori it could be possible to consider a relaxation of (3.54), allowing for a right-hand side proportional to ξ_μ rather than vanishing, since this would only mismatch the inaffinity of metric and torsion-full geodesics. However we don't know whether the derivation of [Dey et al., 2017] can be extended to this case.

spacetime with torsion it is the torsion-full shear that gives the tidal heating. Furthermore, the condition of vanishing initial expansion implies that at the point P we have $\nabla_\mu l^\mu = \perp^{\mu\nu} T_{\mu,\nu\rho} l^\rho$, making some 'non-equilibrium terms' indistinguishable from terms without derivatives, as the authors of [Dey et al., 2017] acknowledge in a footnote.

Third, there is the ambiguity associated with picking a non-conserved $T^{??}_{\mu\nu}$, as discussed before. Had we chosen the alternative source T^Γ, which is also more natural from the perspective of a metric-connection action, the same identification of non-equilibrium contributions would not work, as it would miss the terms with covariant derivatives of the contorsion in (3.21).

Summarizing, although the non-equilibrium approach has the advantage of allowing to relax the assumption of an initial non-vanishing shear [Eling et al., 2006, Chirco and Liberati, 2010], it is in our opinion ambiguous when applied to gravity with torsion.

3.5 Black hole thermodynamics in presence of torsion

As mentioned in the introduction, Jacobson's derivation is inspired by the laws of black hole thermodynamics. Having shown that the derivation works also in the presence of torsion, at least as far as recovering the Einstein equations, the next question we considered is what happens to the these laws.

We have recalled earlier that the surface gravity of the Rindler horizon is constant simply because the Riemann tensor vanishes. For a general horizon, constancy of the surface gravity is the zeroth law, and its proof uses the Einstein equations and the dominant energy conditions. In the presence of torsion, we can follow the proof with the equations (3.21), and the only modification is that the dominant energy condition will be a restriction on the effective energy-momentum tensor.

More interesting is the modification that occurs to the first law. To see this, let us consider the 'physical process' version of the proof [Wald, 1995], in which an initially stationary black hole is perturbed by some matter falling inside the horizon. For our generalization, we suppose that the in-falling matter has spin and sources torsion, and that the metric and connection satisfy the Einstein-Cartan field equations.

As in the metric case, we assume that all matter falls into the black hole, and that the latter is not destroyed by the process, but settles down to a new stationary configuration [Wald, 1995, Gao and Wald, 2001]. These assumptions are motivated by the no-hair theorem and the cosmic censorship conjecture, which keep their value also in a theory with non-propagating torsion. For example, it is known that a compact ball of static or slowly spinning torsion-full Weyssenhoff fluid[9] admits a solution which satisfies the junction conditions with an external Schwarzschild or slowly rotating Kerr [Prasanna, 1975, Arkuszewski et al., 1974].

Following [Wald, 1995], we use the linearized Einstein equation to study the effect on the horizon geometry caused by the in-falling matter at first order in perturbation theory,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \qquad C_{\rho,\mu\nu} = c_{\rho,\mu\nu}. \tag{3.55}$$

Being null and hypersurface orthogonal, the affine horizon generators are metric geodetic, and their expansion is governed by the Raychaudhuri equation (3.44). The

[9] This is a single component of torsion (the trace part) generated by the gradient of a scalar [Griffiths and Jogia, 1982].

background generators l^μ are proportional to the Killing generators ξ^μ satisfying $l^\mu = -(\lambda\kappa)^{-1}\xi^\mu$, with constant κ by the zeroth law. They have vanishing shear and expansion, giving therefore at first order

$$\frac{d}{d\lambda}\delta\theta = -\delta R_{\mu\nu}(h) l^\mu l^\nu. \tag{3.56}$$

Integrating along the horizon $\mathscr{H}$ from the bifurcation surface $\mathscr{B}$ to a cut S_∞ at future null infinity, we have for the total area variation

$$\Delta A = \int_{\mathscr{H}} \delta\theta \, d\lambda d^2S = \int_{\mathscr{H}} \delta R_{\mu\nu}(h) l^\mu l^\nu \, \lambda d\lambda d^2S, \tag{3.57}$$

where we integrated by parts and used that $\lambda|_{\mathscr{B}} = 0$ since $\xi^\mu|_{\mathscr{B}} = 0$, and that $\theta|_{S_\infty} = 0$ by the late time settling down assumption.
In the standard particular case of torsion-less matter with conserved energy-momentum tensor $T_{\mu\nu}$, we have from the linearized Einstein equations

$$\int_{\mathscr{H}} \delta R_{\mu\nu}(h) l^\mu l^\nu \, \lambda d\lambda d^2S = 8\pi \int_{\mathscr{H}} \delta T_{\mu\nu}(h) l^\mu l^\nu \, \lambda d\lambda d^2S. \tag{3.58}$$

At this order, we can substitute $l^\mu = -(\lambda\kappa)^{-1}\xi^\mu$ in the right-hand side integrand

$$-\frac{8\pi}{\kappa}\int_{\mathscr{H}} \delta T_{\mu\nu}(h)\xi^\mu l^\nu \, \lambda d\lambda d^2S = \frac{8\pi}{\kappa}\int_{\mathscr{H}} \delta T_{\mu\nu}(h)\xi^\mu \, dH^\nu = \frac{8\pi}{\kappa}(\Delta M - \Omega_H \Delta J), \tag{3.59}$$

where in the first equality we used that fact the future-pointing volume form on $\mathscr{H}$ is $dH_\mu = -l_\mu d\lambda d^2S$, and in the second the explicit expression $\xi^\mu = \partial_t^\mu + \Omega_H \partial_\phi^\mu$ as well as the definitions of ΔM and ΔJ used in [Wald, 1995]. We conclude that the linearized Einstein equations imply the first law of perturbations around a stationary black hole,[10]

$$\Delta M = \frac{\kappa}{8\pi}\Delta A + \Omega_H \Delta J. \tag{3.60}$$

For torsion-generating matter, we can follow exactly the same procedure, the only difference being that we use the Einstein-Cartan equations (3.21) with the conserved effective energy-momentum tensor on the right-hand side. The first law follows as before but with new mass and angular momentum variations

$$\Delta M - \Omega_H \Delta J = \int_{\mathscr{H}} \delta T^{\text{eff}}_{\mu\nu}(h)\xi^\mu \, dH^\nu \tag{3.61}$$

[10] To make contact between this 'physical process' version of the first law, and the one in terms of ADM (Arnowitt-Deser-Misner) charges, recall that since we are assuming all matter to be falling in the black hole, the integral along the horizon equals the integral on a space-like hypersurface Σ extending from $\mathscr{B}$ to a 2-sphere S_∞ at spatial infinity i^0. Using again the Einstein equations and the explicit form of the conserved Noether current (see [Iyer and Wald, 1994], here κ is the Komar charge and Θ the Einstein-Hilbert symplectic potential) we find

$$\int_{\mathscr{H}} \delta T_{\mu\nu}(h)\xi^\mu dH^\nu = \int_\Sigma \delta T_{\mu\nu}(h)\xi^\mu d\Sigma^\nu = \int_{\mathscr{S}_\infty} (k_\xi - \Theta \lrcorner \xi) - \int_{\mathscr{B}} k_\xi = \Delta M_{ADM} - \Omega_H \Delta J_{ADM},$$

where the final result follows from a standard calculation with $\xi^\mu = \partial_t^\mu + \Omega_H \partial_\phi^\mu$. See [De Paoli and Speziale, 2018] for a derivation of the first law with covariant Hamiltonian methods for Einstein-Cartan theory.

determined by the torsion-dependent $T^{\text{eff}}_{\mu\nu}$. This is consistent with the results of [Arkuszewski et al., 1974] mentioned above, where the mass of the external Schwarzschild has a torsion contribution from an effective energy density profile of the static Weyssenhoff fluid compatible with the formula above.
Following the same approach of treating the effect of torsion as an effective energy-momentum tensor, we can conclude that also the second law of black hole mechanics is still valid, provided the required restrictions on the energy-momentum tensor of matter [Bardeen et al., 1973] are applied to the effective tensor (3.22).
As for the more elusive third law, a discussion would require a prior understanding of extremal black holes in the presence of torsion, we didn't face this analysis in our paper and we postpone this study to a future work.

3.5.1 Jacobson's thermodynamic argument: two possible derivations

We want to introduce here a brief discussion on some details of Jacobson's thermodynamic argument and consider a different derivation that is motivated by the (backwards) similitude with the physical process proof of the first law of black hole thermodynamics. Let us first review the physical set-up and its thermodynamical interpretation. With reference to Fig. 5.1, we see that from the perspective of the boosted observer the energy flux is coming out of its 'white hole horizon', or as the authors of [Guedens et al., 2012] put it, 'one has to think of the heat as going into a reservoir which is behind the horizon'. We suppose this must be the reason why (3.42) is defined with a minus signs with respect to the outgoing energy flux (the future-pointing integration is $dH_\mu = -l_\mu d\lambda d^2S$, as we used in Section 3.5). An alternative set-up was presented in [Guedens et al., 2012], see Fig. 3.2, placing the energy flux in the future of the bifurcation surface, so to have the boosted observer seeing it falling into its Rindler horizon. Spacetime is initially flat, in particular $\theta = \sigma_{\mu\nu} = 0$ at the bifurcation surface.

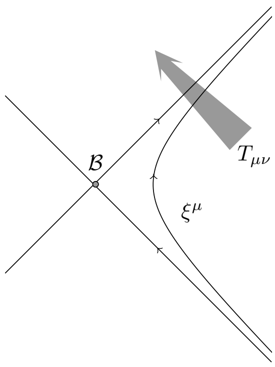


Figure 3.2. *The set-up thermodynamical derivation of Einstein's equation as proposed in [Guedens et al., 2012]. Local flatness allows to consider approximate Rindler observers ξ^μ around any point P of a given spacetime. The associate Rindler horizon has bifurcate surface $\mathscr{B}$ passing through P. The system is perturbed by a small flux of matter crossing the future horizon and leaving the right wedge. An infinite family of ξ^μ is actually considered, one per each direction.*

With the same approximations used in (3.45) (i.e. constant curvature at first order in the affine parameter from $\mathscr{B}$), one can again derive the Einstein equation from the Clausius relation. The physical interpretation of the Clausius law is the same: there is a negative energy flux which corresponds to a reduction in entropy, and assuming an entropy universally proportional to the area this translates into the focusing of geodesics. But now the initial heat reservoir is within the domain of causality of the boosted observer, which is an appreciable feature to have.

Within this '11 set-up, the analogy between the argument and the first law is manifest, and it suggests an alternative procedure, with the advantage of relaxing the constant curvature approximation, at the price of an additional assumption. Consider the same set-up of Fig. 3.2, but let us assume this time that spacetime is initially arbitrary, and that long enough after the flux has crossed and perturbed the horizon, the latter 'settles down' to Rindler again. This is an assumption, which in the case of the first law is motivated by the no-hair theorem; it has no corresponding backing-up in the case of a Rindler horizon that we know of, but we observe that the same assumption is used to derive the results of [**?**, **?**]. We can then treat the Raychaudhuri equation not at first order in l, which implies a constant curvature, but at first order in the metric perturbations, with small but otherwise arbitrary curvature along the horizon. Thanks to the assumption of Rindler behavior at later times we can obtain the area variation integrating by parts as in (3.57) in the main text, without needing to know the explicit solution to the Raychaudhuri equation.

Then, using the same steps $(i-iii)$ (with a small energy-momentum tensor $\delta T_{\mu\nu}$), but replacing (3.45) with (3.57), we can again derive the Einstein equations. This alternative derivation has the nice feature, to our taste, of not requiring constant curvature and energy-momentum tensors, with the consequence of making all $d\lambda$ integrations really not significative. However it can hardly be considered a more solid derivation, as we initially hoped, because of the ad hoc 'Rindler stationarity' assumption at late times. This could be removed if we reverse the boundary conditions, and required that spacetime is initially Rindler, namely at $\mathscr{B}$, and can be arbitrary at later times. However the derivation does not work unfortunately, unless curvature is constant again, which is what allows the authors of [Guedens et al., 2012] to reverse boundary conditions with respect to [Jacobson, 1995].

3.6 Concluding remarks

Prompted by the analysis of [Dey et al., 2017], we looked at one aspect of conservation laws in Einstein-Cartan theory. In the sector of invertible tetrads, where one can choose to split the connection into the Levi-Civita one plus a contorsion tensor, it is immediate to identify a the conservation law of the energy-momentum tensor T^{eff} for matter from the Einstein equations. We showed in our analysis that T^{eff} can be derived *without* using the Einstein equations, starting instead from the Noether identities associated with the gauge and diffeomorphism invariance of the matter Lagrangian, and relating them through the torsion equations.
Thanks to this result, we were able to reproduce Jacobson's thermodynamical argument [Jacobson, 1995], and derive the Einstein equations from the equilibrium Clausius relation. Our derivation is much simpler than the one proposed in [Dey et al., 2017], and

does not require non-equilibrium terms nor any restriction on torsion. On the other hand, like in [Dey et al., 2017], we are only able to derive the tetrad Einstein equations from a thermodynamical argument, and not the torsion equations as well. This remains an open question in order to truly extend Jacobson's argument to theories with independent metric and connection.

For our construction, we used first the equilibrium set-up of [Jacobson, 1995], in particular the initial shear vanishes. Non-equilibrium terms have been advocated in order to relax this assumption [Eling et al., 2006, Chirco and Liberati, 2010, Guedens et al., 2012], and the same can be done in the presence of torsion: we showed that one can treat the shear alone as non-equilibrium feature, and still derive the torsion-full Einstein equations with all the torsional dependence coming from the equilibrium part. On the other hand, non-equilibrium terms could become crucial if one were able to go beyond Einstein-Cartan theory, and apply a thermodynamical reasoning to derive the field equations of modified theories of tetrad and connection with higher order terms, which typically include propagating torsion (and associated ghosts, see e.g. [Tseytlin, 1982]). It could be interesting if the dissipation present in this case would be associated with dissipation of energy to the torsional degrees of freedom. From this perspective it could be intriguing to consider existing condensed matter models in which dissipating lattice defects introduce torsion [Kröner, 2017].

In the next chapter we gave up the thermodynamical set-up for gravity to focus on the general problem of definition of conserved charges in the first order formalism. As we outlined in the introductive chapter, the Noether theorem applies to gauge theories with no global symmetry (e.g. with no killing vector field fo GR) in a form that it's known in litterature as the *second* Noether theorem [Blau, 2011] and a direct application of the theorem states that there are no non-trivial conserved charges. Switching to tetrad variables we add an extra symmetry to the theroy, the internal gauge symmetry, and we want to analyse if it brings new peculiarities in the derivation of conserved charges for gravity. Even this symmetry doesn't bring physical conserved quantities, as we expected, we showed in the next chapter that it leads to some subtelties that one should take care of to have a fair correspondence between the two formulation of GR in tetrad and metric variables.

3.7 Conventions

We take $\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma}$ as the completely antisymmetric spacetime density with $\underset{\sim}{\varepsilon}_{0123} = 1$, and $\tilde{\varepsilon}^{\mu\nu\rho\sigma}\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma} = -4!$. It is related to the volume 4-form by

$$\varepsilon := \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}, \qquad \varepsilon_{\mu\nu\rho\sigma} := \sqrt{-g}\,\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma}. \tag{C.3.1}$$

We define the Hodge dual in components as

$$(\star\omega^{(p)})_{\mu_1..\mu_{4-p}} := \frac{1}{p!}\omega^{(p)\alpha_1..\alpha_p}\varepsilon_{\alpha_1..\alpha_p\mu_1..\mu_{4-p}}. \tag{C.3.2}$$

For the internal Levi-Civita density ε_{IJKL} we refrain from adding the tilde. We use the same convention, $\varepsilon_{0123} = 1$, so the tetrad determinant is

$$e = -\frac{1}{4!}\varepsilon_{IJKL}\tilde{\varepsilon}^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu e^K_\rho e^L_\sigma, \tag{C.3.3}$$

and we take $e > 0$ for a right-handed tetrad.
Curvature and torsion are defined by

$$F^{IJ}(\omega) = d\omega^{IJ} + \omega^{IK} \wedge \omega_K{}^J, \qquad T^I(e,\omega) = d_\omega e^I, \tag{C.3.4}$$

where d_ω is the covariant exterior derivative, whose components we denote by D_μ, to distinguish them from the spacetime covariant derivative ∇_μ with affine connection $\Gamma^\rho_{\mu\nu}$. The relation between the connections on the fiber and on the tangent space is given by

$$D_\mu e^I_\nu = \Gamma^\rho_{\mu\nu} e^I_\rho, \qquad \omega^{IJ}_\mu = e^I_\nu \nabla_\mu e^{\nu J} \tag{C.3.5}$$

for ω and Γ general affine connections, plus the metricity condition $D_\mu \eta^{IJ} = 0$. The compatibility of the internal covariant derivative and the tetrad means that $D_\mu f^I = e^I_\nu \nabla_\mu f^\nu$ and so on.
The commutators of the covariant derivatives satisfy:

$$[D_\mu, D_\nu] f^I = F^I{}_{J\mu\nu}(\omega) f^J, \tag{C.3.6}$$

$$[D_\mu, D_\nu] f = -T^\rho{}_{\mu\nu}(e,\omega)\partial_\rho f, \tag{C.3.7}$$

$$[\nabla_\mu, \nabla_\nu] f^\rho = R^\rho{}_{\sigma\mu\nu}(\Gamma) f^\sigma - T^\sigma{}_{\mu\nu}\nabla_\sigma f^\rho, \tag{C.3.8}$$

where

$$R_{\rho\sigma\mu\nu}(\Gamma) = e_{I\rho} e_{J\sigma} F^{IJ}_{\mu\nu}(\omega) \qquad T^\rho{}_{\mu\nu}(\Gamma) = e^\rho_I T^I{}_{\mu\nu}(\omega). \tag{C.3.9}$$

Finally, torsion and contorsion are related by

$$T^\rho{}_{\mu\nu} := e^\rho_I T^I{}_{\mu\nu}(e,C) = -2C_{[\mu,\nu]}{}^\rho = 2\Gamma^\rho_{[\mu\nu]} \quad \Leftrightarrow \quad C_{\mu,\nu\rho} = \frac{1}{2}T_{\mu,\nu\rho} - T_{[\nu,\rho]\mu}. \tag{C.3.10}$$

Both torsion and contorsion have spinorial decomposition $(\frac{3}{2},\frac{1}{2}) \oplus (\frac{1}{2},\frac{3}{2}) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (\frac{1}{2},\frac{1}{2})$, which corresponds to three irreducible components under Lorentz transformations (since the latter include parity). They can be defined as follows [Hehl et al., 1976],

$$C^{\mu,\nu\rho} = \bar{C}^{\mu,\nu\rho} + \frac{2}{3} g^{\mu[\rho}\check{C}^{\nu]} + \varepsilon^{\mu\nu\rho\sigma}\hat{C}_\sigma, \tag{C.3.11}$$

$$g_{\mu\nu}\bar{C}^{\mu,\nu\rho} = 0 = \varepsilon_{\mu\nu\rho\sigma}\bar{C}^{\mu,\nu\rho}, \qquad \check{C}^\mu := C_{\nu,}{}^{\mu\nu}, \qquad \hat{C}_\sigma := \frac{1}{6}\varepsilon_{\sigma\mu\nu\rho}C^{\mu,\nu\rho}. \tag{C.3.12}$$

As in the rest of the dissertation we use metric signature with mostly plus, and natural units $G = c = \hbar = 1$.

Chapter 4

Derivation of the symplectic potential in tetrad formalism for GR

In this chapter, we continue the discussion on conserved charges for gauge theories using a different approach, based on the covariant phase space formalism. The covariant phase space method is based on the construction of the symplectic potential and the derivation of the boundary terms arising from the variation principles. To derive the conserved quantities in this formalism, instead of starting from a Lagrangian and using the Noether theorem, we start from the action and use the boundary terms[1]. In this case symmetries are studied as properties of the phase space and the associated conserved charges correspond to the integral of the hamiltonian generators of these symmetries.
The analysis on conserved charges for tetrad gravity was carried on in the paper [De Paoli and Speziale, 2018], we report here our results and some features of the derivation of the symplectic potential. We use tetrad variables throughout all the discussion and the exterior calculus which has the merit of compactifying calculations and making them more readable. In particular this formalism avoids confusion in simultaneous employment of internal indices and spacetime indices. All formulas are expressed in the first order formalism, meaning the connection is an independent variable before going on-shell. This has the advantage of a better comparison with the connection description like Ashtekar's phase space at future null infinity and at spacetime infinity. Another property is that our first order formulas can also be applied to extensions of GR, as spacetime with torsion that we already introduced int he previous chapter.
Even if a priopri the use of tetrad variables and the Einstein-Cartan action is just a ridefinition of the theory, it turned out to be non-trivial as one subtlety arises in the study of the symplectic structure: the phase space derived from the Einstein-Cartan action is not gauge-invariant and not equivalent to the usual metric one even in the absence of torsion. As a consequence both the definitions of conserved charges and conservations laws are different respect to the usual metric one. To face these problems we analysed the derivation of a symplectic potential, and we showed that it is possible to define a symplectic structure such that the associated conserved charges match the metric ones as soon as one restores the gauge invariance of the symplectic potential.
The chapter is divided in the following way: in the first part we derive the symplectic

[1] In GR the famous boundary term is the Gibbons-Hawking boundary term which consists of the trace of the extrinsic curvature K.

structure that arises from Einstein-Cartan action. It consists of the symplectic potential Θ, defined as the integral on a Cauchy tridimensional hypersurface of the boundary terms, and from its variation in the fields space, Ω, which plays the role of the metric of the phase-space. In the second part we compared the tetrad symplectic form with the metric one for some particular fields variations, and we proved how to solve the gauge invariance of the first one. We report in the appendix of this chapter A.2 the notation that we used for the exterior calculus formalism, and other conventions.

4.1 Introduction and motivations

Covariant phase space methods [Ashtekar et al., 1991, Crnkovic and Witten, 1986, Lee and Wald, 1990, Wald and Zoupas, 2000] provide powerful tools for the study of symmetries and conservation laws in gauge theories and gravity. These methods have been successfully applied to tetrad gravity, recovering the metric Poincarè charges at spatial infinity, the first law of black hole mechanics and its generalization to isolated horizons [Ashtekar et al., 2000b, Ashtekar et al., 2008, Corichi et al., 2014, Corichi et al., 2016]. Notwithstanding these positive results, the symplectic potential most commonly used has two unappealing features that we wish to improve upon, and which motivate this discussion. The first issue consists of the not fully gauge-invariance of the symplectic potential: the associated pre-symplectic form has degenerate gauge directions inside the Cauchy hypersurface, but not on its boundary, unless this is taken at infinity and with appropriate fall-off conditions. This means that the covariant phase space gives in general non-trivial surface charges for internal Lorentz transformations. Since when torsion vanishes we would like to recover the same physics as in the metric theory, such charges appear unphysical to us.
The second and related issue is that, again when torsion vanishes, the symplectic potential taken from the Einstein-Hilbert action is not equivalent to the one taken from Einstein-Hilbert action. This difference shows up for instance if we look at a variation given by a Lie derivative: the familiar Komar term which appears in the metric case is not present. As a consequence, also the Noether charge is different, which led the authors of [Jacobson and Mohd, 2015, Prabhu, 2017] to point out a potential problem with the derivation of the first law from the Noether identity, and to propose that in tetrad gravity the Noether charge for diffeomorphisms should be associated not to a Lie derivative, but to a modified derivative involving an internal gauge transformation which depends non-linearly on the tetrad. We will see that solving the first issue automatically solves the second.
To find a fully gauge-invariant symplectic potential, it is enough to use the fact that the symplectic potential is defined from an action principle only up to the addition of an exact form. Our first result is to identify an appropriate exact form that makes the pre-symplectic form completely gauge-invariant, thus free of internal Lorentz charges in the absence of torsion. Our second result is to show that the gauge-invariant potential gives exactly the Komar term when the variation coincides with the Lie derivative, thus recovering the expected Noether charge. Finally we prove equivalence to the Einstein-Hilbert symplectic potential for a general variation, in the absence of torsion, by reproducing the geometric formula of [Burnett and Wald, 1990, Lehner et al., 2016] in terms of extrinsic geometry and 2d corner terms. A support for the proposed gauge-

invariant symplectic potential comes also from the fact that it turns out to match the boundary term found in [Bodendorfer et al., 2014] using Hamiltonian method and with the requirement of finding a canonical transformation from the tetrad to the ADM phase space in the presence of 2d corner terms.
The importance of working with a gauge-invariant potential for generic gauge theories has been discussed in details in [Barnich and Compere, 2008], and our construction shows how this can be done for tetrad gravity.
Having established these results, we look at physical applications, in particular to asymptotic charges and to the first law of black hole thermodynamics. Since the modification we propose changes the symplectic form and the phase space structure, it is not guaranteed that the results in the literature still apply: the exact form affects the Hamiltonian charges of the theory. For asymptotic Poincaré charges, it is easy to see that the result of [Ashtekar et al., 2008] is preserved, since with those asymptotic fall-off conditions the additional exact form vanishes at spatial infinity.
For the first law, the situation is more interesting. First of all, having recovered the equivalence with the Einstein-Hilbert symplectic potential, we can immediately show that using our gauge-invariant symplectic potential the first law follows from the Noether identity associated with the covariant Lie derivatives, coherently with the metric case. However, we show that the first law follows also from the non-gauge-invariant potential and the same Lie derivative, provided that one takes into account the non-trivial internal Lorentz charge. The latter has the effect of changing the Hamiltonian Killing flow, because tetrads and connections are preserved by a Killing Lie derivative only up to an internal transformation, and not identically.
Recovering the first law from the non-gauge invariant potential and the Lie derivative is in fact not new: it was already proven in [Ashtekar et al., 2000b] using directly the Hamiltonian generators, not expressing them in terms of the Noether charge and thus without puzzling over that mismatch. The presence of a non-zero Hamiltonian diffeomorphism generator was indeed observed in [Ashtekar et al., 2000b], and referred to as the horizon energy. Our construction clarifies that this *horizon energy* is the internal Lorentz charge produced by using a non-gauge covariant potential.
Therefore, we deal with a situation similar to the metric case, albeit slightly subtler. In the metric case, the first law is invariant under the cohomology ambiguity in the symplectic potential, because the contribution of the exact form to the symplectic form vanishes. In the tetrad formalism this is not the case, because the Killing Lie flow vanishes only up to internal transformations. Nonetheless, the first law is still invariant, provided one takes into account the non-trivial internal Lorentz charges that can be present changing the symplectic potential by an exact form.

4.2 Formulation of a gauge-invariant symplectic potential

We consider the following first order action for Einstein-Cartan gravity (for a review, see [Hehl et al., 1995])

$$S_{\rm EC}(e,\omega) = P_{IJKL}\int_M e^I\wedge e^J\wedge F^{KL}(\omega) - \frac{\Lambda}{6}e^I\wedge e^J\wedge e^K\wedge e^L, \qquad (4.1)$$

where we have taken units $16\pi G = 1$, and

$$P_{IJKL} := \frac{1}{2\gamma}(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK}) + \frac{1}{2}\varepsilon_{IJKL}. \tag{4.2}$$

The action is to be supplemented by appropriate boundary integrals I_{3d} and I_{2d} depending on the boundary conditions chosen, see e.g. [Obukhov, 1987, Bodendorfer and Neiman, 2013, Corichi et al., 2016] for 3d boundaries without corners, and [Jubb et al., 2017] in the presence of corners. The coupling constant γ is referred to as Barbero-Immirzi parameter in most literature,[2] and the associated Lagrangian density corresponds to the additional dimension-two term $\tilde{\varepsilon}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(\Gamma)$ that one can write in the first order formalism.[3] The variation of the action gives the field equations and a boundary term,

$$\int_M d\Big(P_{IJKL}e^I \wedge e^J \wedge \delta\omega^{KL}\Big), \tag{4.3}$$

which will be the centre of attention of this analysis, for the role it plays in the covariant phase space formalism.

Let us denote by $d\theta_{\rm EC}(\delta)$ the integrand.The theory defined by (4.1) differs a priori from general relativity: it is only defined for orientable manifolds, and odd under orientation inversion, instead of even; it allows for degenerate tetrads hence degenerate metrics; it allows for non-vanishing spacetime torsion $T^I := d_\omega e^I$, if matter couples to the affine connection ω^{IJ}. In the following, we restrict attention to an invertible, right-handed tetrad. Then when torsion vanishes $\omega^{IJ} = \imath\omega^{IJ}$ is the Levi-Civita spin connection, (4.1) is equivalent to the Einstein-Hilbert action $S_{\rm EH}$ and thus the theory to general relativity. This equivalence extends to the boundary term:

$$d\theta_{\rm EC}\big|_{\omega=\imath\omega} = d\theta_{\rm EH}, \tag{4.4}$$

as can be easily seen for instance from

$$e^\mu_I e^\nu_J 2\imath D_{[\mu}\delta\imath\omega^{IJ}_{\nu]} = e^\mu_I e^\nu_J \delta F^{IJ}_{\mu\nu}(e) = \delta R - \delta(e^\mu_I e^\nu_J)\delta F^{IJ}_{\mu\nu}(e) = \delta R - 2R^I{}_\mu \delta e^\mu_I = g^{\mu\nu}\delta R_{\mu\nu}. \tag{4.5}$$

The equivalence (4.4) implies also that the 3-forms are equal up to an exact form,

$$\theta_{\rm EH} = \theta_{\rm EC}|_{\omega=\imath\omega} + d\alpha. \tag{4.6}$$

The question we address here is to find an α for which the equality above holds. It is motivated by the covariant phase space formalism, which uses the boundary term to define Noether and Hamiltonian charges of the theory. Let us briefly review the basic points of this formalism, referring the reader to e.g. [Wald and Zoupas, 2000] for details. Suppose that the boundary ∂M of M (which can be the whole spacetime or just a region of interest) admits a canonical split with the identification of a Cauchy hypersurface Σ. Then the boundary term $d\theta(\delta)$ obtained from the variation of a Lagrangian 4-form

[2] Because of the role it plays in the canonical transformation to real Ashtekar-Barbero variables, see e.g. [Thiemann, 2001].

[3] For the interested reader, this parameter has an interesting renormalization flow [Daum and Reuter, 2010, Benedetti and Speziale, 2011a], with an on-shell logarithmic divergence induced by the simultaneous presence of fermions and Λ [Benedetti and Speziale, 2011b].

L can be used to provide a symplectic potential on the space of solutions to the field equations, by taking its integral on Σ:

$$\Theta(\delta) := \int_\Sigma \theta(\delta). \tag{4.7}$$

This defines a one-form in field space, and its exterior derivative is the pre-symplectic two-form

$$\Omega(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) - \Theta([\delta_1, \delta_2]). \tag{4.8}$$

Using $\delta L \approx d\theta(\delta)$, where here and in the following $\approx$ means on-shell of the field equations, one sees that Ω is independent of the choice of hypersurface Σ if the background fields as well as the variations δ_1 and δ_2 satisfy the field equations.

The symplectic structure so defined is not unique. First, the explicit form of the potential depends also on the boundary terms I_{3d} and I_{2d} in the action principle. These however do not affect the pre-symplectic form since the symplectic potential is changed by a total variation, therefore the covariant phase space structure is independent of them. There is nonetheless a certain freedom, since the symplectic potential is defined by the Lagrangian L only up to an exact form, that is the Lagrangian identifies an equivalence class

$$L \longrightarrow \{\theta(\delta) = \theta(\delta) + d\alpha(\delta)\}, \tag{4.9}$$

where α is an arbitrary 2-form in spacetime and 1-form in field space. This cohomology freedom does affect the pre-symplectic form, and it is important to test that physical predictions are independent of it. This freedom plays an important role below.
The simplest set-up for this formalism is when $\partial M = \Sigma_1 \cup \Sigma_2$ joined at a 2d space-like surface, in which case the canonical splitting is obvious. A more general configuration is a topological cylinder, $\partial M = \Sigma_1 \cup \Sigma_2 \cup \mathscr{T}$, with the time-like hypersurface $\mathscr{T}$ connecting the 2d space-like boundary $\partial\Sigma_1$ to $\partial\Sigma_2$. To introduce a canonical split in this case we typically require that $\Theta(\delta)$ vanishes on $\mathscr{T}$.[4] This is a restriction on the admissible solutions if $\mathscr{T}$ is in the spacetime bulk, but can become negligible if the boundary is pushed to infinity, and it is the fall-off conditions on the fields that guarantee the vanishing of $\Theta(\delta)$ on $\mathscr{T}_\infty$. This set-up is relevant for instance in the study of asymptotic charges at spatial infinity with $\Lambda = 0$. The appropriate fall-off conditions for (4.1) where given in [Ashtekar et al., 2008]. We will come back to this point below in Section 4.4.
The power of this formalism for diff-invariant theories is that it allows one to define quasi-local Hamiltonian charges for diffeomorphisms as the canonical generators in the covariant phase space.[5] They are given by the pre-symplectic form with one variation being a Lie derivative $\delta_\xi = \pounds_\xi$,

$$\delta H_\xi[\Sigma] := \Omega(\delta, \delta_\xi) = \int_\Sigma \delta\theta(\delta_\xi) - \delta_\xi \theta(\delta). \tag{4.10}$$

[4] This is because data can generically both inflow and outflow off a time-like boundary, making a canonical split impossible without restricting the phase space. Another useful set-up is when the time-like boundary is replaced by a null hypersurface $\mathscr{N}$. In that case we can have a canonical split with non-zero contribution from $\mathscr{N}$, since it is a one-way only membrane, see e.g. [Ashtekar et al., 2000b, Wieland, 2017b, Ashtekar and Wieland, 2018].

[5] Let us remind the reader less familiar with this formalism that for a general diffeomorphisms, these quasi-local charges are not interesting observables, because their value depends on the shape of the boundary of Σ. It is only when ξ^μ is a Killing or asymptotic Killing vector that the charges are truly useful.

Here we assumed that $[\delta_\xi, \delta] = 0$,[6] and the δ is there to remind us that the quantity on the RHS is not always a total variation. Only when it is, the generator integrates to a proper Hamiltonian charge $H_\xi[\Sigma]$.[7] The integrability condition is $\int_\Sigma \omega(\delta_1, \delta_2) \lrcorner \xi = 0$ [Wald and Zoupas, 2000], where ω is the integrand of Ω, and a sufficient condition familiar from the ADM energy calculations is the existence of a functional B such that $\theta(\delta) \lrcorner \xi = \delta B \lrcorner \xi$.

The origin of this latter condition becomes clear if we recall the relation between the Hamiltonian charges and the Noether charges, which do not coincide for diff-invariant theories. The conserved Noether current is given by (see e.g. [Iyer and Wald, 1994])

$$j(\delta_\xi) := \theta(\delta_\xi) - L \lrcorner \xi, \tag{4.11}$$

since this is the object that is closed on-shell: Using $\delta_\xi L = d(L \lrcorner \xi)$, it is immediate to see that $dj(\delta_\xi) \approx 0$. Furthermore, it is also possible to show that $j(\delta_\xi) \approx dq(\xi)$ for some 2-form $q(\xi)$ [Iyer and Wald, 1994]. It follows that the Noether charge, defined as the integral of the current,[8] is a boundary term:

$$Q_\xi[\partial\Sigma] := \int_\Sigma j(\delta_\xi) \approx \int_{\partial\Sigma} q(\xi). \tag{4.12}$$

To find the relation between the Hamiltonian and Noether charges one takes the variation of (4.11), and replaces it in the definition (4.10) together with the Lie derivative variation $\delta_\xi \theta(\delta) = \pounds_\xi \theta(\delta) = d\theta \lrcorner \xi + d(\theta \lrcorner \xi)$. This gives

$$\delta H_\xi[\Sigma] \approx \delta Q_\xi[\partial\Sigma] - \int_{\partial\Sigma} \theta(\delta) \lrcorner \xi. \tag{4.13}$$

This shows (i) that the Hamiltonian as well as the Noether charge are surface charges, but in general differ by a term $\theta(\delta) \lrcorner \xi$; and (ii) that if $\theta(\delta) \lrcorner \xi = \delta B \lrcorner \xi$, then $\delta H_\xi[\Sigma] = \delta H_\xi[\Sigma]$ is a total variation and thus integrable. In spite of their close relation, the Hamiltonian and Noether charges have an important difference: the former changes only under the ambiguity (4.9) in defining the symplectic potential, whereas the Noether current $j(\delta_\xi)$ and charge $Q_{\partial\Sigma}(\xi)$ are changed also by adding boundary terms I to the action, which makes them less universal objects than the Hamiltonian charges.[9]

To make this quick review more concrete, let us recall that for the Einstein-Hilbert Lagrangian $L_{\text{EH}} = (R - 2\Lambda)\epsilon$ (without boundary terms, for simplicity), we have

$$\theta_{\text{EH}}(\delta) = 2g^{\rho[\sigma} \delta \Gamma^{\mu]}_{\rho\sigma} d\Sigma_\mu, \tag{4.14}$$

with $d\Sigma_\mu$ the oriented volume element. Specializing to a diffeomorphism,

$$\theta_{\text{EH}}(\pounds_\xi) = d\kappa(\xi) + \star(2E \lrcorner \xi) + L_{\text{EH}} \lrcorner \xi, \tag{4.15}$$

[6] This is a customary assumption [Wald and Zoupas, 2000], although it can be argued [Ashtekar et al., 1991] that it is rather a *definition* of what we mean by the perturbation of a diffeomorphed solution.

[7] Since a typical case study is when Σ has two boundaries, $H_\xi[\Sigma]$ is also referred to as a flux, leaving the name charge for the surface integrals whose difference makes up $H_\xi[\Sigma]$, see below.

[8] Noether charges for gravity can also be derived without using covariant phase space methods, see e.g. [Barnich et al., 2000]. For a derivation of Noether charges for first order tetrad gravity with these methods, see [Barnich et al., 2016].

[9] There is also a third ambiguity in the definition of the Noether charge itself, since one can always add an exact 2-form to it. This ambiguity will play no role in the following.

where κ is the Komar form, in components

$$\kappa_{\mu\nu}(\xi) := -\varepsilon_{\mu\nu\rho\sigma}\nabla^{\rho}\xi^{\sigma}, \tag{4.16}$$

$E \lrcorner \xi := (G_{\mu\nu} + \Lambda g_{\mu\nu})\xi^{\mu}dx^{\nu}$ contains the field equations, and $\star$ is the Hodge dual on spacetime forms (see the section at the end of this chapter for conventions). It follows that the Noether charge associated with diffeomorphisms by (4.14) is the Komar form,

$$j(\delta_{\xi}) = \theta_{\text{EH}}(\delta_{\xi}) - L_{\text{EH}} \lrcorner \xi \approx d\kappa(\xi). \tag{4.17}$$

It also enters the Hamiltonian charge,

$$\delta H_{\xi}[\Sigma] := \Omega_{\text{EH}}(\delta, \delta_{\xi}) = \int_{\Sigma} \delta\theta_{\text{EH}}(\delta_{\xi}) - \delta_{\xi}\theta_{\text{EH}}(\delta) \tag{4.18a}$$

$$= \int_{\partial\Sigma} \delta\kappa(\xi) - \theta_{\text{EH}}(\delta) \lrcorner \xi. \tag{4.18b}$$

This equation is the starting point to prove the first law of black hole mechanics.

Coming back to the tetrad action (4.1), we see that it defines the symplectic potential[10]

$$\Theta_{\text{EC}}(\delta) := \int_{\Sigma} \theta_{\text{EC}}(\delta) := \int_{\Sigma} P_{IJKL} e^{I} \wedge e^{J} \wedge \delta\omega^{KL}. \tag{4.19}$$

This turns out not to be equivalent to (4.14) when torsion vanishes, hence a non-zero α is required in (4.6). The difference shows up prominently when one evaluates the symplectic potential for a diffeomorphism variation δ_{ξ}. In the metric case with the Einstein-Hilbert Lagrangian L_{EH}, we have (4.15) with the Komar form. When using tetrads as fundamental variables, we have the additional gauge freedom of performing internal Lorentz transformations. The action (4.1) is thus invariant under $\text{SO}(3,1)$ gauge transformations

$$\delta_{\lambda} e^{I} = \lambda^{I}{}_{J} e^{J}, \qquad \delta_{\lambda}\omega^{IJ} = -d_{\omega}\lambda^{IJ} \tag{4.20}$$

as well as the usual diffeomorphisms,

$$\pounds_{\xi} e^{I} = de^{I} \lrcorner \xi + d(e^{I} \lrcorner \xi) = d_{\omega}e^{I} \lrcorner \xi + d_{\omega}(e^{I} \lrcorner \xi) - (\omega^{I}{}_{J} \lrcorner \xi)e^{J} \tag{4.21a}$$

$$\pounds_{\xi}\omega^{IJ} = d\omega^{IJ} \lrcorner \xi + d(\omega^{IJ} \lrcorner \xi) = F^{IJ} \lrcorner \xi + d_{\omega}(\omega^{IJ} \lrcorner \xi), \tag{4.21b}$$

as well as combinations of the two. In particular, we can consider the gauge-covariant diffeomorphisms

$$L_{\xi} e^{I} = d_{\omega}e^{I} \lrcorner \xi + d_{\omega}(e^{I} \lrcorner \xi) \tag{4.22a}$$

$$L_{\xi}\omega^{IJ} = d_{\omega}\omega^{IJ} \lrcorner \xi + d_{\omega}(\omega^{IJ} \lrcorner \xi) - d(\omega^{IJ} \lrcorner \xi) = F^{IJ} \lrcorner \xi, \tag{4.22b}$$

which are defined adding a gauge transformation to the Lie derivative,

$$L_{\xi} := \pounds_{\xi} + \delta_{\omega \lrcorner \xi}. \tag{4.23}$$

[10] Another common choice is the opposite polarization, obtained adding the extrinsic geometry boundary term to the action. All considerations in this discussion apply also to this alternative choice, although some explicit formulae are different.

These are gauge-covariant, unlike (4.21), and $[L_\xi, \delta_\lambda] = \delta_{d_\omega \lambda \lrcorner \xi}$ is a gauge transformation.[11]

Taking variations given by these Lie derivatives, the potential (4.19) gives

$$\theta_{\rm EC}(\pounds_\xi) = P_{IJKL}\Big[e^I \wedge e^J \wedge F^{KL} \lrcorner \xi + 2e^I \wedge T^J\, \omega^{KL} \lrcorner \xi\Big] + d(P_{IJKL} e^I \wedge e^J\, \omega^{KL} \lrcorner \xi), \tag{4.24a}$$

$$\theta_{\rm EC}(L_\xi) = P_{IJKL} e^I \wedge e^J \wedge F^{KL} \lrcorner \xi. \tag{4.24b}$$

Using

$$\begin{aligned} P_{IJKL} e^I \wedge e^J \wedge F^{KL} \lrcorner \xi &= \frac{1}{3!}\Big(-\frac{1}{\gamma}\varepsilon^{\alpha\beta\gamma\mu} F_{\alpha\beta\gamma\lambda}\xi^\lambda + 2eF^\mu{}_\lambda \xi^\lambda\Big)\varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &\overset{\omega = 1\omega}{=} \big((\star 2E(e) + L_{\rm EH})\lrcorner \xi\big), \end{aligned} \tag{4.25}$$

we see that both options differ from (4.15), even when torsion vanishes. The associated torsionless Noether current is

$$j(\pounds_\xi)|_{\omega = 1\omega} = \star(2E(e) \lrcorner \xi) + d(P_{IJKL} e^I \wedge e^J\, 1\omega^{KL} \lrcorner \xi), \tag{4.26}$$

which is exact on-shell as expected, but lacks the Komar term (4.16), as also the current associated to L_ξ would. Hence (4.19) does not reproduce the Noether charge of the metric theory with neither $\pounds_\xi$ nor L_ξ. This does not affect the evaluation of the asymptotic Poincaré charges, see below in Section 4.4, but it was argued in [Jacobson and Mohd, 2015] to spoil the first law of black hole mechanics. The solution there proposed was to associate the diffeomorphism Noether charge not to the original Lie derivative, but to the following mixing of diffeomorphisms and gauge transformations,

$$K^{(e)}_\xi e^I := L_\xi e^I + \Big(e^{\nu[I} \pounds_\xi e^{J]}_\nu\Big)\, e_J. \tag{4.27}$$

This indeed produces the Komar term (as shown in [Jacobson and Mohd, 2015, Prabhu, 2017], or by direct evaluation of (4.19) with $\delta\omega^{IJ} = K^{(e)}_\xi 1\omega^{IJ}$), and the same proposal has been followed for instance in [Montesinos et al., 2017, Frodden and Hidalgo, 2018]. However, this is not the origin of the alleged problem with the first law, which as we show below in Section 4.5 can be derived also from the Noether identity with the covariant Lie derivative. The key point is that the symplectic potential (4.19) does not define a gauge-invariant symplectic structure. To see this, we look at the pre-symplectic form derived from (4.19). Using the shorthand notation $\Sigma^{IJ} := e^I \wedge e^J$ and the commutativity $[\delta_\lambda, \delta] = 0$ of gauge transformations and variations of the fundamental fields, we have

$$\begin{aligned} \Omega_{\rm EC}(\delta, \delta_\lambda) = \delta\Theta_{\rm EC}(\delta_\lambda) - \delta_\lambda \Theta_{\rm EC}(\delta) = -P_{IJKL}\int_\Sigma [\lambda, \Sigma]^{IJ} \wedge \delta\omega^{KL} + \delta\Sigma^{IJ} \wedge d_\omega \lambda^{KL} = \\ = P_{IJKL}\int_\Sigma \delta(d_\omega \Sigma^{IJ})\lambda^{KL} - P_{IJKL}\int_{\partial\Sigma} \delta\Sigma^{IJ}\lambda^{KL}, \end{aligned} \tag{4.28}$$

[11] The reader familiar with the Hamiltonian analysis of (4.1) will recognise these two covariances as those associated respectively to the generators

$$\mathscr{D}_a := \mathscr{C}_a - \omega^{IJ}_a \mathscr{G}_{IJ}, \qquad \mathscr{C}_a := -2\tilde{P}^b_{IJ} F^{IJ}_{ab}.$$

see e.g. [Thiemann, 2001].

where we used

$$\delta\Sigma^{IJ}\wedge d_\omega\lambda^{KL} = d(\delta\Sigma^{IJ}\lambda^{KL}) - (d_\omega\delta\Sigma^{IJ})\lambda^{KL} = d(\delta\Sigma^{IJ}\lambda^{KL}) - \Big(\delta(d_\omega\Sigma^{IJ}) + [\delta\omega,\Sigma]^{IJ}\Big)\lambda^{KL}, \tag{4.29}$$

and

$$P_{IJKL}\Big([\lambda,\Sigma]^{IJ}\wedge\delta\omega^{KL} - [\delta\omega,\Sigma]^{IJ}\lambda^{KL}\Big) = 0 \tag{4.30}$$

which follows from the Jacobi identity.

On-shell of the field equations, the first term in (4.28) vanishes (or in the presence of torsion it would cancel against the source term coming from the matter contribution to the symplectic potential), and we are left with a surface term, which gives the non-vanishing Lorentz charge

$$\delta H_\lambda = \Omega_{\text{EC}}(\delta,\delta_\lambda) = -P_{IJKL}\int_{\partial\Sigma}\delta\Sigma^{IJ}\lambda^{KL}. \tag{4.31}$$

This means that the symplectic structure induced by Ω_{EC} has degenerate gauge directions in the bulk of Σ, but not on its boundary. Again this fact is well-known in the literature, see e.g. [Corichi et al., 2016, Frodden and Hidalgo, 2018]. While in a gauge theory this is a rather natural fact with a physical meaning, we find it unpalatable in this gravitational context because it would assign charges that are not there in the metric theory, making the covariant phase spaces inequivalent even in the absence of torsion. A fully gauge-invariant symplectic structure can be easily obtained using the ambiguity (4.9) in the definition of the symplectic potential. We find that the required exact form is

$$\int_{\partial\Sigma}\alpha(\delta) := \int_{\partial\Sigma}\frac{1}{\gamma}e^I\wedge\delta e_I + \star e^I\wedge\delta e_I = -P_{IJKL}\int_{\partial\Sigma}e^I\wedge e^J\, e^{\rho K}\delta e^L_\rho. \tag{4.32}$$

In fact, a simple calculation shows that

$$\int_{\partial\Sigma}\delta\alpha(\delta_\lambda) - \delta_\lambda\alpha(\delta) = P_{IJKL}\int_{\partial\Sigma}\delta(e^I\wedge e^J)\lambda^{KL}, \tag{4.33}$$

which cancels the surface term in (4.28). The corrected potential

$$\boxed{\Theta'_{\text{EC}}(\delta) := \Theta_{\text{EC}}(\delta) + \int_{\partial\Sigma}d\alpha(\delta) = P_{IJKL}\int_\Sigma e^I\wedge e^J\wedge\delta\omega^{KL} + \int_{\partial\Sigma}\frac{1}{\gamma}e^I\wedge\delta e_I + \star e^I\wedge\delta e_I} \tag{4.34}$$

is thus gauge-invariant. Notice also that it satisfies $\Theta'_{\text{EC}}(\delta_\lambda) = 0$ for vanishing torsion.[12]

As it turns out, the very same exact form allows us also to recover precisely the Komar expression from a Lie derivative variation. To see this, let us first notice the following identity

$$e^{\nu[I}L_\xi e^{J]}_\nu = D^{[I}\xi^{J]} + T^{[I}_{\mu\nu}e^{J]\mu}\xi^\nu, \tag{4.35}$$

[12]For a gauge transformation

$$\Theta_{\text{EC}}(\delta_\lambda) = \int_\Sigma P_{IJKL}d_\omega(e^I\wedge e^J)\lambda^{KL} - \int_{\partial\Sigma}P_{IJKL}e^I\wedge e^J\lambda^{KL}$$

is a pure boundary term when torsion vanishes, cancelled by the addition of (4.32) since $e^{\rho K}\delta_\lambda e^L_\rho = -\lambda^{KL}$. One could also use this argument to deduce the boundary subtraction term (4.32).

where $D_I\xi_J = e^\mu_I e^\nu_J \nabla_\mu \xi_\nu$ is the covariant derivative corresponding to d_ω. This implies that

$$\alpha(L_\xi)\big|_{\omega^{IJ}=1\omega^{IJ}} = -P_{IJKL} e^I \wedge e^J D^K \xi^L = \kappa(\xi) - \frac{1}{2\gamma} \star\kappa(\xi). \tag{4.36}$$

The last piece is the trivial Komar charge $\star\kappa_{\mu\nu}(\xi) = 2\imath\nabla_{[\mu}\xi_{\nu]}$, similar to the trivial charge associated with the topological Lagrangian $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ in YM theory. This is an exact form; it does not contribute to the boundary integral, and we disregard it in the following.[13] Putting (4.36) together with (4.24b) we find

$$\theta'_{\rm EC}(L_\xi) = \theta_{\rm EC}(L_\xi) + d\alpha(L_\xi) \overset{\omega=1\omega}{=} P_{IJKL} e^I \wedge e^J \wedge F^{KL} \lrcorner \xi + d\kappa(\xi) \equiv \theta_{\rm EH}(\pounds_\xi), \tag{4.37}$$

where in the last step we used (4.25). The gauge-invariant symplectic potential (4.34) reproduces precisely the metric result in the absence of torsion.[14]

For these reasons, it seems to us that (4.34) provides a better symplectic potential for the EC theory than the simple boundary term alone: it satisfies our desiderata

$$\Theta'_{\rm EC}(L_\xi)\big|_{\omega^{IJ}=1\omega^{IJ}} = \Theta_{\rm EH}(\pounds_\xi) \tag{4.38}$$

and

$$\Omega'_{\rm EC}[\delta_\lambda, \delta] = 0. \tag{4.39}$$

As further support for the use of (4.34), we remark that it matches the boundary term derived in the Hamiltonian analysis of [Bodendorfer et al., 2014], starting from the requirement of having a canonical transformation of connection variables to the ADM phase space in the presence of corners. Here we derived it from the requirement of full gauge-invariance of the pre-symplectic structure in the covariant phase space.[15]

4.3 General variations approach and equivalence to the metric casa

Properties (4.38) and (4.39) were the ones we cared the most for. However, it is only a few more steps to prove that the equivalence extends to arbitrary variations. In this Section we prove that

$$\Theta'_{\rm EC}(\delta)\big|_{\omega^{IJ}=1\omega^{IJ}} = \Theta_{\rm EH}(\delta), \tag{4.40}$$

namely that $\alpha(\delta)$ defined in (4.32) satisfies (4.6).
In tensor language, a geometric expression for the generic variation was given in [Burnett and Wald, 1990], and more recently rederived in [Lehner et al., 2016]. Using the notation from the latter paper cited, one has

$$n_\mu \theta^\mu_{\rm EH}(\delta) = 2 n_\mu g^{\rho[\sigma} \delta\Gamma^{\mu]}_{\rho\sigma} = -2\delta K + K_{ab}\delta q^{ab} - s\nabla_a \delta A^a, \tag{4.41}$$

[13] It would be however non-trivial in the presence of torsion.

[14] We point out that this equality holds also with the non-gauge-covariant derivative, since $\theta'_{\rm EC}(\pounds_\xi) = \theta'_{\rm EC}(L_\xi) + 2P_{IJKL}e^I \wedge T^J \omega^{KL} \lrcorner \xi$.

[15] When the paper [De Paoli and Speziale, 2018] appeared on the archives, Matthias Blau showed us some unpublished notes where he had also constructed the same gauge-invariant potential and proved the property (4.38) [Blau, 2018].

where the notation is as follows: n_μ is the unit normal to Σ, with signature $s := n^2 = \pm 1$ and projector $q_{\mu\nu} := g_{\mu\nu} - s n_\mu n_\nu$; $K_{\mu\nu} := q^\rho_\mu q^\rho_\nu \nabla_\rho n_\sigma$ is the extrinsic curvature of the hypersurface. The authors of [Lehner et al., 2016] pick coordinates $y^a(x^\mu)$, $a = 1,2,3$ to parametrize Σ, and $t^\mu_a := \partial x^\mu / y^a$ define tangent vectors and the induced metric $q_{ab} = q_{\mu\nu} t^\mu_a t^\nu_b$ with determinant q. Finally, $t^a_\mu := q^{ab} g_{\mu\nu} t^\nu_b$ are the inverse tangent vectors, ∇_a the induced Levi-Civita covariant derivative and $\delta A^a := -s t^a_\mu \delta n^\mu$ captures the variation of the normal-tangential components of $\delta g^{\mu\nu}$. For our purposes, it is convenient to rewrite this formula in a covariant way, without using tangent vectors and hypersurface tensors. To that end, we denote by $\hat{r}^\mu$ the unit normal to the space-like boundary $\partial\Sigma$ within $T^*\Sigma$: it satisfies $\hat{r}^2 = -s$ and $\hat{r}_\mu n^\mu = 0$ (and in the case when it is time-like we take it future oriented). Since $\delta t^a_\mu = \delta A^a n_\mu$, the second term in (4.41) can be rewritten immediately in covariant form,

$$K_{ab}\delta q^{ab} = K_{\mu\nu}\delta g^{\mu\nu} = -2K^\mu_I \delta e^I_\mu. \tag{4.42}$$

As for the boundary term we have

$$-s\int_\Sigma D_a \delta A^a d\Sigma = -s\int_{\partial\Sigma} \hat{r}_a t^a_\mu \delta n^\mu dS = -s\int_{\partial\Sigma} \hat{r}_\mu q^\mu_\nu \delta n^\nu dS = -s\int_{\partial\Sigma} \hat{r}_\mu \delta n^\mu dS, \tag{4.43}$$

where $d\Sigma := \sqrt{-sq}d^3y$ and dS are the induced volume elements on Σ and $\partial\Sigma$. Hence,[16]

$$\begin{aligned} s\Theta_{\text{EH}}(\delta) &= \int_\Sigma n_\mu \theta^\mu_{\text{EH}}(\delta) d\Sigma \\ &= \int_\Sigma \left[-2\delta(K\sqrt{-sq}) + (K_{\mu\nu} - Kq_{\mu\nu})\sqrt{-sq}\delta q^{\mu\nu} \right] d^3y - s\int_{\partial\Sigma} \hat{r}_\mu \delta n^\mu dS. \end{aligned} \tag{4.44}$$

We will take advantage of this formula to establish the equivalence (4.40), by proving that
$\theta'_{\text{EC}}(\delta) = \theta_{\text{EC}}(\delta) + d\alpha(\delta)$ equals the RHS of (4.44) for vanishing torsion and right-handed tetrads.
First of all, we need an identity which allows us to rewrite the symplectic potential with the hypersurface unit normal n_μ explicitly appearing:

$$\frac{1}{2}\varepsilon_{IJKL}\int_\Sigma e^I \wedge e^J \wedge \delta\omega^{KL} = -2\int_\Sigma e\,\delta\omega_{I,}{}^{IJ} n_J = -s\varepsilon_{IJKL}\int_\Sigma e^I \wedge e^J \wedge \delta\omega^L{}_M n^K n^M. \tag{4.45}$$

To see this, we use the tetrad identity (C.4.6) before and after using $n^K n^M = s(\eta^{KM} - q^{KM})$, getting

$$\begin{aligned} \varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \delta\omega^L_{\rho M} n_\sigma n^K n^M &= 2se\,\delta\omega_{I,}{}^{IJ} n_J \\ &= -s\varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu (\delta\omega^{KL}_\rho + \delta\omega^L_{\rho M} q^{KM}) n_\sigma \\ &= -s\varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \delta\omega^{KL}_\rho n_\sigma + 2se\,\delta\omega_{K,LM} q^{KM} n^L. \end{aligned} \tag{4.46}$$

Since in the last term we can replace q^{KM} with η^{KM} we obtain

$$\varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \delta\omega^{KL}_\rho n_\sigma = 4e\,\delta\omega_{I,}{}^{JI} n_J, \tag{4.47}$$

[16] It is by the way in this covariant form that the equation is presented in [Burnett and Wald, 1990].

from which (4.45) follows. Another needful identity concerns the $1/\gamma$ piece of $\Theta_{\rm EC}$: we have

$$\int_\Sigma e_I \wedge e_J \wedge \delta\omega^{IJ} = \int_\Sigma \Big(T^I \wedge e_J\, (e_I^\rho \delta e^J_\rho) - e_I \wedge \delta T^I \Big) - \int_{\partial\Sigma} e^I \wedge \delta e_I, \tag{4.48}$$

which can be shown using $\omega_\mu^{IJ} = e^{\lambda I}\nabla_\mu e^J_\lambda$ and integrating by parts.
Next, we consider the following boundary term [Obukhov, 1987, Bodendorfer and Neiman, 2013, Wieland, 2013],

$$I_\Sigma := 2\int_\Sigma P_{IJKL} e^I \wedge e^J \wedge n^K d_\omega n^L = 2\int_\Sigma e^\mu_I D_\mu n^I d\Sigma =: 2\int_\Sigma \overset{\circ}{K} d\Sigma \tag{4.49}$$

which represents an 'affine' version $\overset{\circ}{K}$ of the extrinsic curvature – in the sense of being defined without referring to the Levi-Civita connection –, and which reduces to the extrinsic curvature K if there is no torsion. The equality in the middle follows using (C.4.6) and $n_I D_\mu n^I = 0$. Notice also that the term proportional to $1/\gamma$ vanishes identically. We then compute its variation, which gives

$$\begin{aligned}\delta I_\Sigma &= \int_\Sigma \varepsilon_{IJKL} \left[2\delta e^I \wedge e^J \wedge n^K d_\omega n^L + e^I \wedge e^J \wedge (\delta n^K d_\omega n^L + n^K d_\omega \delta n^L + \delta\omega^L{}_M n^K n^M) \right] \\ &= \int_\Sigma \varepsilon_{IJKL} \Big[2\delta e^I \wedge e^J \wedge n^K d_\omega n^L + e^I \wedge e^J \wedge \Big(2\delta n^K d_\omega n^L - \frac{s}{2}\delta\omega^{KL} \Big) + 2e^I \wedge T^J n^K \delta n^L \Big] \\ &\qquad + d(\varepsilon_{IJKL} e^I \wedge e^J n^K \delta n^L)\end{aligned} \tag{4.50}$$

where we used (4.45). The second term vanishes since n^I is unit norm, and isolating the symplectic potential (4.19) in (4.50) we find

$$\begin{aligned}s\Theta_{\rm EC}(\delta) = \varepsilon_{IJKL} \int_\Sigma & -\delta\big(e^I \wedge e^J \wedge n^K d_\omega n^L \big) + 2\delta e^I \wedge e^J \wedge n^K d_\omega n^L + 2e^I \wedge T^J n^K \delta n^L \\ & + \varepsilon_{IJKL} \int_{\partial\Sigma} e^I \wedge e^J n^K \delta n^L + \frac{s}{\gamma}\int_\Sigma e_I \wedge e_J \wedge \delta\omega^{IJ}.\end{aligned} \tag{4.51}$$

We now compare this expression for $\omega = \imath\omega$ and $T = 0$ with (4.44). The first term in (4.51) gives immediately the first term in (4.44), thanks to (4.49). The matching of the second terms in (4.51) and (4.44) is also easily established:

$$\begin{aligned}2\varepsilon_{IJKL} \int_\Sigma \delta e^I \wedge e^J \wedge n^K d_\omega n^L &= -2\int_\Sigma (q^\nu_I \nabla_\nu n^\mu - e^\mu_I \nabla_\rho n^\rho)\delta e^I_\mu d\Sigma \\ &= -2\int_\Sigma (\overset{\circ}{K}_I{}^\mu - \overset{\circ}{K} e^\mu_I)\delta e^I_\mu d\Sigma,\end{aligned} \tag{4.52}$$

which coincides with the second term in (4.44) when torsion vanishes. It remains to look at the boundary term of (4.50), which in tensor form gives

$$\begin{aligned}\varepsilon_{IJKL} \int_{\partial\Sigma} e^I \wedge e^J n^K \delta n^L &= -4\int_{\partial\Sigma} n_{[K}\hat{r}_{L]} n^K \delta n^L dS \\ &= -2s\int_{\partial\Sigma} \hat{r}_L \delta n^L dS = -2s\int_{\partial\Sigma} (\hat{r}_\mu \delta n^\mu + \hat{r}_L n^\mu \delta e^L_\mu) dS.\end{aligned} \tag{4.53}$$

As expected, this surface term alone fails to reproduce the surface term in (4.44). This is fixed by the correcting term (4.32), which gives

$$s\,d\alpha(\delta) = -sP_{IJKL}\int_{\partial\Sigma} e^I\wedge e^J e^{\rho K}\delta e^L_\rho = s\int_{\partial\Sigma}(n^\mu \hat{r}_I\delta e^I_\mu - \hat{r}^\mu n_I \delta e^I_\mu)dS + \frac{s}{\gamma}\int_{\partial\Sigma} e^I\wedge\delta e_I \tag{4.54}$$

The piece in $1/\gamma$ cancels the last term of the second row of (4.51) when torsion vanishes, see (4.48). Adding up (4.53) and the γ-less part of (4.54), and using $\delta(e^I_\mu n^\mu \hat{r}_I)=0$ we obtain

$$-s(2\hat{r}_\mu\delta n^\mu + \hat{r}_I n^\mu \delta e^I_\mu + \hat{r}^\mu n_I\delta e^I_\mu) = -s(2\hat{r}_\mu\delta n^\mu - \hat{r}_\mu\delta n^\mu - n^I\delta\hat{r}_I - n_\mu\delta\hat{r}^\mu - \hat{r}^I\delta n_I) = -s\hat{r}_\mu\delta n^\mu, \tag{4.55}$$

where the final equality follows from $\hat{r}^I\delta n_I = -n_I\delta\hat{r}^I$ which cancels the third with the fifth term, and $n_\mu\delta\hat{r}^\mu = -\hat{r}^\mu\delta n_\mu = (s/2)\hat{r}^\mu n_\mu n_\rho n_\sigma \delta g^{\rho\sigma} = 0$ which cancels the fourth. We have thus proved (4.40).

4.4 Analysis of the Poincaré charges at spatial infinity

Since the modification we propose changes the pre-symplectic form, we should check that it does not spoil established results, such as the recovery of Poincaré charges at spatial infinity with $\Lambda = 0$. It was proved in [Ashtekar et al., 2008] that the original symplectic potential (4.19) vanishes on $\mathscr{T}_\infty$, a necessary condition for the canonical split without reducing the phase space, and that it leads to the correct Poincaré charges as in the metric formalism. Furthermore, the authors showed that the non-gauge-invariance of (4.19) vanishes in the limit to i^0. This already signals that our modification will vanish in that limit, hence preserving those results. Let us show this explicitly, using the boundary and fall-off conditions of [Ashtekar et al., 2008].[17]

One chooses a reference flat metric $\eta_{\mu\nu}$ for the asymptotic behaviour, with hyperbolic slicing given by $\rho^2 := \eta_{\mu\nu}x^\mu x^\nu$ and three angles collectively denoted by Φ. Then the fall-off conditions appropriate to Poincaré symmetries are given for the tetrad by

$$e^I_\mu = {}^0e^I_\mu(\Phi) + \frac{{}^1e^I_\mu(\Phi)}{\rho} + O(\rho^{-2}), \tag{4.56}$$

with

$${}^0e^I_\mu(\Phi) = \delta^I_\mu, \qquad {}^1e^I_\mu(\Phi) = \sigma(\Phi)(2\rho_\mu\rho^I - {}^oe^I_\mu), \tag{4.57}$$

$\sigma(\Phi)$ a reflection-symmetric arbitrary scalar function and $\rho_\mu := \partial_\mu\rho$.

We then have at leading order

$$\int_{\partial\Sigma}\delta_1\alpha(\delta_2) - \delta_2\alpha(\delta_1)$$
$$= -P_{IJKL}\int_{\partial\Sigma}\Big[\Big(2\,{}^0e^{[I}_\mu(\delta_1\,{}^1e^{J]}_\nu)\,{}^0e^{\rho K} - {}^0e^{[I}_\mu\,{}^0e^{J]}_\nu(\delta_1\,{}^1e^{\rho K})\Big)(\delta_2\,{}^1e^L_\rho) - (\delta_1\leftrightarrow\delta_2)\Big]\frac{1}{\rho^2}dS^{\mu\nu} \tag{4.58}$$

[17] These we recall are slightly stronger than strictly necessary, as they are chosen also to eliminate the logarithm and supertranslation freedoms from the asymptotic symmetry group. It would be of course interesting to study relaxations admitting supertranslations, see e.g. [Henneaux and Troessaert, 2018], as motivated by [Hawking et al., 2016].

which vanishes exactly using (4.57) and the antisymmetry in KL. The exact form we added gives no leading contribution to the pre-symplectic form in the limit to i^0, and the recovery of the Poincaré charges established in [Ashtekar et al., 2008] is left unaffected.

4.5 Study of first law of thermodynamics with the simpletic formalism

We now show that our symplectic potential permits to derive the first law of black hole mechanics from the Noether charge associated with a Lie derivative, just like in the metric case [Iyer and Wald, 1994]. For the application of the formalism to derive the first law of black hole mechanics, we take a stationary and axisymmetric background solution, $\Lambda = 0$, and choose Σ to be a Cauchy hypersurface with two boundaries, one at the bifurcation surface $\mathscr{B}$ and one at spatial infinity S_∞. We take ξ^μ to be the Killing vector that generates the horizon. Consider first the metric case. Since ξ^μ is Killing, all variations δ_ξ vanish and by linearity also the Hamiltonian charge,

$$\delta H_\xi = \Omega_{\text{EH}}(\delta, \delta_\xi) = 0. \tag{4.59}$$

Recalling the expression (4.18b) in terms of the Noether charge, we find a conservation law between surface charges at the bifurcating surface and at spatial infinity,

$$\int_{\mathscr{B}} \delta\kappa(\xi) = \int_{S_\infty} \delta\kappa(\xi) - \theta_{\text{EH}}(\delta) \lrcorner \xi, \tag{4.60}$$

where we used that fact that $\xi^\mu|_{\mathscr{B}} = 0$. If the perturbations are asymptotically flat and solution of the linearized field equations (but otherwise general), this equation evaluates to the first law of black hole mechanics (see e.g. [Iyer and Wald, 1994])[18]

$$2k\delta A = \delta M - \Omega_H \delta J. \tag{4.61}$$

Crucially, this first law is invariant under $\theta \mapsto \theta + d\alpha$, since the contribution of this ambiguity to (4.59) always vanishes:

$$d\big(\delta\alpha(\delta_\xi) - \delta_\xi \alpha(\delta)\big) = 0. \tag{4.62}$$

To see this, use the fact that $\alpha(\delta)$ depends linearly on the variations and that $\delta_\xi = 0$ on the background fields. The quantity in square brackets then gives $\alpha(\delta\delta_\xi) - \alpha(\delta_\xi\delta) = 0$ since $[\delta_\xi, \delta] = 0$.
If the same state of affairs held in the tetrad formalism, we would agree with the argument given in [Jacobson and Mohd, 2015, Prabhu, 2017]: neither options presented in (4.24) give the Noether charge of the metric theory, and since the first law should be invariant under redefinitions of the symplectic potential, we are left with the only possibility of looking for a new transformation to which the first law should be associated.

[18] Notice that in this situation the (trivial) Hamiltonian charge is integrable, so both sides of (4.60) are total variations: this is manifest for the LHS, for the RHS it follows from the standard ADM energy result plus the fact that ∂_ϕ is tangent to S_∞. This property is on the other hand not manifest in the final expression (4.61) of the first law, where it is guaranteed by identities relating the variations of the various quantities appearing.

The problem we see with this argument is the assumption that (4.59) still holds in the tetrad theory, namely the requirement that for a Killing vector, $\pounds_\xi e^I = 0$. This is not necessary, and can lead to inconsistencies; it is enough to require that

$$L_\xi e^I = \lambda_\xi{}^I{}_J e^J, \tag{4.63}$$

since this automatically preserves the metric. Contracting with the inverse metric, we get an expression for the gauge transformation:

$$\lambda_\xi{}^{IJ} = -e^{\rho I} L_\xi e^J_\rho = -D^{[I}\xi^{J]}, \tag{4.64}$$

where we used (4.35) in the absence of torsion.[19] Notice that it does not vanish on a bifurcating surface where $\xi^\mu = 0$. This immediately means that

$$\delta H_\xi = \Omega(\delta, L_\xi) = \Omega(\delta, \delta_{\lambda_\xi}), \tag{4.65}$$

namely the Killing diffeomorphism generator for a general potential is an internal Lorentz charge.

Using the gauge-invariant symplectic potential Θ'_{EC}, the Lorentz charge is zero, see (4.39), and thus from (4.65) the vanishing $\delta H_\xi = 0$ of the diffeomorphism generator associated with a Killing vector is preserved. The Noether charge contains the exact Komar form, see (4.38) (there is a priori another ambiguity in the cohomology of κ_ξ, but this is irrelevant for the first law since the boundary of a boundary is zero), and the symplectic potential reduces to the one of the Einstein-Hilbert action, see (4.40). Hence (4.65) gives back precisely the conservation law (4.60), and the first law follows as usual. We conclude that our gauge-invariant potential associates naturally the first law to the invariance of the action under (covariant) Lie derivatives.

One may wonder whether the invariance of the first law under the ambiguity (4.9) is lost. This is not the case. In fact, we now show that the first law can also be derived from the non-gauge-invariant potential (4.19) and the same Lie derivative, without need for the non-linear object (4.27) of [Jacobson and Mohd, 2015] or the automorphism construction of [Prabhu, 2017], *provided* one takes into account the presence of a non-zero Lorentz charge. Starting from the non-gauge-invariant potential (4.19) and using (4.63), the Hamiltonian generator (4.65) for a Killing vector does not vanish anymore but coincides with the Lorentz generator.[20] This evaluates to

$$\Omega_{\text{EC}}(\delta, L_\xi) = \Omega_{\text{EC}}(\delta, \delta_{\lambda_\xi}) = -\int_{\partial\Sigma} P_{IJKL}\,\delta(e^I \wedge e^J)\lambda_\xi{}^{KL}, \tag{4.66}$$

where we used (4.28), which is valid also for a field-dependent gauge parameter like λ_ξ since the contribution from its variation is cancelled by the commutator term

[19] The reader may worry whether the invariance up to a gauge transformation of the tetrad under an isometry is consistent with the transformation of ω^{IJ}, namely whether

$$L_\xi \omega^{IJ} = F^{IJ} \lrcorner\, \xi \stackrel{?}{=} -d_\omega \lambda_\xi^{IJ} = d_\omega D^{[I}\xi^{J]}.$$

The equality is indeed satisfied as it is nothing but the familiar Killing identity $R_{\sigma\mu\nu\rho}\xi^\sigma = \nabla_\mu \nabla_\nu \xi_\rho$ expressed in the tetrad formalism.

[20] A fact that can also be taken as motivation to prefer the gauge-invariant potential.

$\Theta_{\text{EC}}([\delta, \delta_{\lambda_\xi}])$ which is in this case not vanishing.
To evaluate the hand side, we could compute the Noether current associated with L_ξ, but we can also use (4.23) and the bilinearity of the symplectic form to derive

$$\Omega_{\text{EC}}(\delta, L_\xi) = \Omega_{\text{EC}}(\delta, \pounds_\xi) + \Omega_{\text{EC}}(\delta, \delta_{\omega \lrcorner \xi}). \tag{4.67}$$

The first piece gives

$$\begin{aligned}\Omega_{\text{EC}}(\delta, \pounds_\xi) &= \int_{\partial\Sigma} \delta j(\pounds_\xi) - \theta_{\text{EC}}(\delta) \lrcorner \xi = \int_{\partial\Sigma} P_{IJKL} \delta(e^I \wedge e^J\, \omega^{KL} \lrcorner \xi) - \theta_{\text{EH}}(\delta) \lrcorner \xi + d\alpha(\delta) \lrcorner \xi \\ &= \int_{\partial\Sigma} P_{IJKL} \Big[e^I \wedge e^J\, \delta\lambda_\xi^{KL} + \delta(e^I \wedge e^J)\, \omega^{KL} \lrcorner \xi \Big] - \theta_{\text{EH}}(\delta) \lrcorner \xi, \end{aligned} \tag{4.68}$$

where in the second equality we used (4.26) on-shell, and the equivalence $(\theta_{\text{EC}} + d\alpha)|_{\omega^{IJ} = 1\omega^{IJ}} = \theta_{\text{EH}}$ previously established; in the last step we used

$$\int_{\partial\Sigma} d\alpha(\delta) \lrcorner \xi = \int_{\partial\Sigma} \pounds_\xi \alpha(\delta) = \int_{\partial\Sigma} P_{IJKL} \Big[e^I \wedge e^J\, \delta\lambda_\xi^{KL} - e^I \wedge e^J\, \delta\omega^{KL} \lrcorner \xi \Big]. \tag{4.69}$$

This can be proved by explicit calculation using (4.32) and (4.64), but also observing that

$$\pounds_\xi \alpha(\delta) = L_\xi \alpha(\delta) = \delta_{\lambda_\xi} \alpha(\delta) + \alpha([\delta, \delta_{\lambda_\xi} - \delta_{\omega \lrcorner \xi}]) = \alpha([\delta, \delta_{\lambda_\xi} - \delta_{\omega \lrcorner \xi}]), \tag{4.70}$$

which follows using $\delta_\lambda \alpha(\delta) = 0$ for a gauge transformation and (4.64) for the background fields ϕ:

$$L_\xi \alpha(\phi, \delta\phi) = \alpha(L_\xi \phi, \delta\phi) + \alpha(\phi, L_\xi \delta\phi) = \delta_{\lambda_\xi} \alpha(\phi, \delta\phi) - \alpha(\phi, [\delta_{\lambda_\xi}, \delta]\phi) + \alpha(\phi, [L_\xi, \delta]\phi) \tag{4.71}$$

and $[L_\xi, \delta] = [\delta_{\omega \lrcorner \xi}, \delta]$. The second piece in (4.67) is again a Lorentz charge,

$$\Omega_{\text{EC}}(\delta, \delta_{\lambda_\xi}) = -\int_{\partial\Sigma} P_{IJKL} \delta(e^I \wedge e^J)\, \omega^{KL} \lrcorner \xi, \tag{4.72}$$

and cancels the second term in (4.68). We can now equate (4.66) to (4.67) with the above manipulation, and derive

$$-\int_{\partial\Sigma} P_{IJKL} \delta(e^I \wedge e^J) \lambda_\xi{}^{KL} = \int_{\partial\Sigma} P_{IJKL} e^I \wedge e^J\, \delta\lambda_\xi^{KL} - \theta_{\text{EH}}(\delta) \lrcorner \xi. \tag{4.73}$$

Finally, notice that

$$\int_{\partial\Sigma} P_{IJKL} \delta(e^I \wedge e^J\, \lambda_\xi^{KL}) = -\int_{\partial\Sigma} P_{IJKL} \delta(e^I \wedge e^J D^K \xi^L) = \int_{\partial\Sigma} \delta\kappa_\xi, \tag{4.74}$$

hence (4.73) gives the same identity (4.60) as the metric and gauge-invariant symplectic potential calculations, from which the first law follows. The Lorentz charge is thus crucial to recover the Komar form and the first law using the non-gauge-invariant symplectic potential and the ordinary Lie derivative.
We conclude that also with the original potential (4.19) the first law follows from the Noether identity and Lie derivatives. This is consistent with the findings of [Ashtekar et al., 2000b], where the first law for stationary black holes and more in general for

isolated horizons was recovered from the second equality in (4.18a), without going through the Noether current expression (4.18b). In [Ashtekar et al., 2000b] the internal Lorentz symmetry at the isolated horizon was fixed, and the non-vanishing of (4.65) indeed noticed, and referred to as horizon energy. Our results show that this is nothing but the Lorentz charge.

The bottom line of the derivation of the first law (4.73) with the non-gauge-invariant potential and the Lie derivative is that the Komar charge, absent from the symplectic potential, pops up through the Lorentz charge giving the diffeomorphism Hamiltonian generator. This simple reshuffling restoring the first law extends to any symplectic potential in the equivalence class (4.9). Therefore, there still is a perfect invariance of the first law under the cohomology ambiguity in the symplectic potential, albeit in a subtler way than in the metric case. The subtlety is that adding an exact form to the symplectic potential can introduce surface Lorentz charges, which in turn provide non-zero charges also for the Hamiltonian generators of Killing isometries. These have to be taken into account if one wants to recover the first law from the covariant Lie derivative alone.

Let us also compare our results with those of [Jacobson and Mohd, 2015, Prabhu, 2017]. There it was acknowledged that the symplectic potential is not gauge-invariant, and it was shown that one can still work with it and define Hamiltonian diffeomorphism charges vanishing for Killing vectors, provided these diffeomorphism are not associated with Lie derivatives alone, standard or covariant, but with automorphisms of the tetrad. This construction uses the non-linear object (4.27) whose action depends on the tetrad also when acting on other fields, and whose extension in the case of an affine connection with torsion is unclear to us. Our findings show that there is a simpler alternative: keep the covariant Lie derivative and switch to a gauge-invariant potential, or keep the non-gauge-invariant potential but take into account the Lorentz charges and the non-vanishing of (4.65). This said about our alternative, we remark that the motivations of [Jacobson and Mohd, 2015, Prabhu, 2017] include topological issues and smoothness of fields; we have not looked at these aspects, so we are not in a position to assess how they would change our results.

4.6 Conclusions

In this analysis we have proposed a gauge-invariant symplectic potential for tetrad general relativity, implementing what discussed for generic gauge theories in [Barnich and Compere, 2008]. See also [Donnelly and Freidel, 2016, Gomes and Riello, 2018] for additional discussions on the importance of gauge-invariance of the phase space. Our construction uses the freedom to add exact spacetime forms, namely corner terms, to the symplectic potential. A gauge-invariant symplectic potential cannot be directly read off from the action, but additional input is required in the choice of the right corner term, which in turns determines the covariant phase space and resulting Hamiltonian fluxes/surface charges.

The gauge-invariant potential eliminates what we see as spurious internal Lorentz charges produced by the symplectic potentials used so far in the literature. It does not change the Poincaré charges at spatial infinity, since the gauge-breaking terms vanish in that limit. It plays a key role on the other hand in deriving the first law of black hole mechanics from the Noether identity associated with the Lie derivative and a vanishing Killing Hamiltonian flux, like in the metric theory.

We also pointed out that the derivation of the first law from the covariant Lie derivative is in the end invariant under the cohomology ambiguity in the symplectic potential, and thus independent of having chosen a gauge-invariant one: it suffices to take into account the non-trivial Lorentz charges that can be present. The technical statement is that the invariance of the first law under the ambiguity $\theta \mapsto \theta + d\alpha$, which is guaranteed in the metric theory by the fact that for a Killing field we have $\Omega(\delta, \delta_\xi) = 0$, it is now correspondingly guaranteed by the $\Omega(\delta, L_\xi) = \Omega(\delta, \delta_{\lambda_\xi})$. Therefore, the first law is recovered with a *non-vanishing* Killing Hamiltonian flux if one uses the original potential, while a vanishing Killing Hamiltonian flux if one uses the gauge-invariant symplectic potential.

Our gauge-invariant symplectic potential turns out to be exactly equivalent to the Einstein-Hilbert one when torsion vanishes and for arbitrary variations. This was not granted a priori since they could have differed by gauge-invariant exact 2-forms, e.g. terms written directly as variation of the metric. The proof was based on some identities for differential geometry with tetrads that allows us to recover variations of extrinsic curvature and 2d corner terms.

For simplicity, we have neglected boundary terms in the action, and the explicit contribution of matter fields. Boundary terms and topological terms can be added following the previous treatments [Corichi et al., 2016, Jubb et al., 2017]. The matter contribution is worth exploring: we have settled the issue of the equivalence of the symplectic potential when torsion vanishes, this can now be used to study the contribution of torsion to the charges.

Among the applications of our results we mention the study of boundary degrees of freedom, in particular the $1/\gamma$ term in (4.34) has been shown to lead to an interesting description in terms of a conformal field theory on the boundary [Freidel et al., 2017a, Wieland, 2017a, Geiller, 2018]), and it would be interesting to see if and how that description is affected by our results. A related issue concerns calculations of entanglement entropy based on the action (4.1), see e.g. [Ashtekar and Krishnan, 2004, Bodendorfer, 2014]. Finally, approaches to quantization suggest to endow the covariant phase space methods within the Batalin-Fradkin-Vilkovisky framework, which for the non-gauge-invariant potential (4.19) has been discussed in [Cattaneo and Schiavina, 2017].

Throughout our analysis we restricted attention to non-null hypersurfaces. A symplectic potential for tetrad gravity giving vanishing internal Lorentz charges can also be obtained when the 2d corner hinges between a space-like and a null hypersurface [Wieland, 2017b, Ashtekar and Wieland, 2018]. Quasi-local charges and conservation laws are even more interesting when one considers null hypersurfaces (see e.g. [Ashtekar and Streubel, 1981, Wald and Zoupas, 2000, Reisenberger, 2013, Hawking et al., 2016, Wieland, 2017b, Hopfmüller and Freidel, 2018]) and it is natural to ask how our results extend to that case. In the following chapter we explored the Hamiltonian structure of Einstein-Cartan gravity on null hypersurfaces preparing tools for a future study of a symplectic potential for a null foliations.

4.7 Conventions

We define the spacetime Levi-Civita density $\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma}$ as the completely antisymmetric object with $\underset{\sim}{\varepsilon}_{0123} = 1$, and $\tilde{\varepsilon}^{\mu\nu\rho\sigma}\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma} = -4!$. We denote the spacetime volume form as

$$\varepsilon := \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \qquad \varepsilon_{\mu\nu\rho\sigma} := \sqrt{-g}\,\underset{\sim}{\varepsilon}_{\mu\nu\rho\sigma}. \tag{C.4.1}$$

The Hodge dual $\star : \Lambda^p \mapsto \Lambda^{n-p}$ is defined in components as

$$(\star\omega^{(p)})_{\mu_1..\mu_{n-p}} := \frac{1}{p!}\omega^{(p)\alpha_1..\alpha_p}\varepsilon_{\alpha_1..\alpha_p\mu_1..\mu_{n-p}}. \tag{C.4.2}$$

For (non-null) hypersurfaces, we use the following conventions: if the Cartesian equation of Σ is $\varphi(x) = 0$, the unit normal is

$$n_\mu := \frac{s}{\sqrt{g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi}}\partial_\mu\varphi, \qquad s := n^2 = \pm 1, \tag{C.4.3}$$

and the induced volume form

$$\varepsilon^\Sigma := \varepsilon \lrcorner n, \qquad \varepsilon^\Sigma_{\mu\nu\rho} := n^\sigma\varepsilon_{\sigma\mu\nu\rho}, \qquad d\Sigma_\mu = sn_\mu d\Sigma. \tag{C.4.4}$$

On a space-like surface S within Σ, with unit normal $\hat{r}_\mu$, we have $\hat{r}^2 = -s$ and the induced area form

$$\varepsilon^S := \varepsilon^\Sigma \lrcorner \hat{r}, \qquad \varepsilon^S_{\mu\nu} := n^\rho \hat{r}^\sigma \varepsilon_{\mu\nu\rho\sigma}. \tag{C.4.5}$$

For the internal Levi-Civita density ε_{IJKL} we refrain from adding the tilde. We keep the same convention, $\varepsilon_{0123} = 1$, hence the tetrad determinant is

$$e = -\frac{1}{4!}\varepsilon_{IJKL}\tilde{\varepsilon}^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma, \qquad 4ee^{[\mu}_I e^{\nu]}_J = -\varepsilon_{IJKL}\tilde{\varepsilon}^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma, \tag{C.4.6}$$

and we take $e > 0$ for a right-handed tetrad.

Chapter 5

Initial constraint free data in real connection variables

In this last chapter, we extend our analysis of tetrad gravity on a null foliation of the spacetime. This work reported is based on pubblished paper [De Paoli and Speziale, 2017] in which we use the Hamiltonian formalism on a null foliation and the natural action boundary term to identify the constraint free initial data. At different level respect to the previous chapter analysis, where the symplectic potential was derived without considering any vanishing variations of the field at boundary, here we fixed suitable degrees of freedom on null hypersurface. As results of this procedure we are able to identify the physical phase space of the tetrad formulation determining the explicit form of the constraints in term of such physical variables. This identification of the physical degrees of freedom is propaedeutic to the covariant formulation of the phase space, discussed in the previous chapter, for a null boundary.

5.1 Introduction

Null foliations play an important role in general relativity. Among their special features, they admit a gauge-fixing for which the Einstein's equations can be integrated hierarchically as well as constraint-free initial data can be identified, like shown by Sachs [Sachs, 1962a]. Precisely the null foliations provide a framework for the description of gravitational radiation from isolated systems and of conserved charges, starting from the seminal work of Sachs, Bondi, van der Burg and Metzner (henceforth BMS), Newman and Penrose (NP), Geroch and Ashtekar [Bondi, 1960, Sachs, 1962b, Bondi et al., 1962, Newman and Penrose, 1962, Penrose, 1980, Penrose, 1963, Penrose, 1965, Geroch, 1977, Ashtekar, 1981, Ashtekar and Streubel, 1981, Ashtekar, 2014] (see also [Mädler and Winicour, 2016, Wald and Zoupas, 2000, Barnich and Troessaert, 2011] and reference therein). These classic results are based on the Einstein-Hilbert action and the spacetime metric as fundamental framework, and they provide a clear geometric picture of the physical degrees of freedom of General Relativity at the non-linear level. In this chapter we wish to understand some of these results using as the previous chapters a first-order action principle with real connection variables. In particular, we will identify the equivalent of Sachs' free data in terms of some connection components (which will be related to the translational part of the ISO(2) group stabilising the internal null direction soldered to the hypersurface), and highlight some properties of their Hamiltonian dynamics.

We have several reasons to investigate this problem as follows. First of all we know from the work of Ashtekar that the radiative physical degrees of freedom at future null infinity are best described in terms of connections [Ashtekar, 1981, Ashtekar and Streubel, 1981].[1] We then wish to provide a connection description of the physical degrees of freedom in the spacetime bulk, in the sense of constraint-free initial data for the first-order action. Secondly, the connection description later led Ashtekar to the famous reformulation of the action principle of general relativity [Ashtekar, 1986], which is at the root of loop quantum gravity. This approach to quantising general relativity suggests the use of connections as fundamental fields, instead of the metric. There exists a canonical quantisation scheme that leads to the well-known prediction of quantum discreteness of space [Rovelli and Smolin, 1995]. This result uses space-like foliations, and the dynamical restriction to the quanta of space imposed by the Hamiltonian constraint are still not explicitly known. Quantising with analogue connection methods the constraint-free data on null foliations would allow us to study the quantum structure of the physical degrees of freedom directly.[2]

As a preliminary result in this direction, it was shown in [Speziale and Zhang, 2014] that at the kinematical level, discretisations of the 2d space-like metric have quantum area operators with a discrete spectrum given by the helicity quantum numbers. A stronger more recent result appeared in [Wieland, 2017a], based on covariant phase space methods and a spinorial boundary term, which confirms the discrete area spectrum without a discretisation. The aim of our analysis is to extend these results within a Hamiltonian dynamical framework.

The Hamiltonian dynamics of general relativity with real connection variables on a null foliation appeared in [Alexandrov and Speziale, 2015], presenting bit intricated structures, like the conversion of what Sachs called the propagating Einstein's equations into (tertiary) constraints. In this study, we present three results. First, we use the Newman-Penrose formalism to clarify the geometric meaning of the various constraints present in the Hamiltonian structure studied in [Alexandrov and Speziale, 2015]. Second, we identify the connection equivalent of Sachs' free data as the 'shear-like' components of an affine[3] null congruence; we show how they reduce to the shear of a null geodesic congruence in the absence of torsion, and how they are modified in the presence of torsion; we use the Bondi gauge to derive their Dirac brackets, and show the equivalence with the metric formalism at the level of symplectic potentials. Third, we explain the origin and the meaning of the tertiary constraints, and we point out that the algebraic Bianchi identity responsible for the conversion of the propagating equations into constraints is the same one that allows the interpretation of the radiative data at future null infinity $\mathscr{I}^+$.

[1] Another class of null hypersurfaces for which the connection description plays an important role is the one of isolated horizons [Ashtekar and Krishnan, 2004, Ashtekar et al., 2000a, Ghosh and Perez, 2011].

[2] For recent work towards the same goal but in metric variables, see [Fuchs and Reisenberger, 2017].

[3] In the sense of being given by an affine connection, a priori non-Levi-Civita, not of being affinely parametrised.

The identification of the dynamical part of the connection with null rotations[4] (related on-shell to the shear) is a striking difference with respect to the case of a space-like foliation, because these components form a group, albeit a non-compact one, unlike the dynamical components of the space-like formalism which are boosts. We have thus two senses in which a null foliation gives a simpler algebra:the first-class part of the constraint algebra is a genuine Lie algebra (thanks to the fact that the Hamiltonian is second class), and the connection physical degrees of freedom form a group.
This chapter is organised as follows. We first review the background material on the Hamiltonian structure on null foliations that we didn't presenti in the review chapter (2). In the second section, we discuss the use of Bondi coordinates and identification of constraint-free initial data and their symplectic potential in metric variables. In Section 3, we restated this analysis in terms of connection variables. In Section 4, we map the non-adapted tetrad used in the Hamiltonian analysis to a doubly-null tetrad, we identify the constraint-free data and study the effect of the constraints on an affine null congruence. We describe the modifications induced by torsion in the case of minimally and non-minimally coupled fermions, as well as in the presence of a completely general torsion. We rederive the conversion of the propagating equations into constraints using the Newman-Penrose formalism, and single out the algebraic Bianchi identity responsible for it. In Section 5 we specialise to Bondi coordinates, we discuss the Dirac bracket for the constraint-free data and the equivalence of the symplectic potential with that one in metric variables. We finally highlight that the same algebraic Bianchi identity relevant to the understanding of the tertiary constraints plays an interesting role for radiative data at $\mathscr{I}^+$. In Section 6 concluding remarks follow. We also provide an extensive Appendix containing technical material. This includes the detailed relation of our tetrad foliation to the $2+2$ foliation used in the literature, of the metric coefficients we use to those of Sachs and of Newman and Penrose, the explicit expression of all NP spin coefficients in the first-order variables, and some details on the mixing between internal boost gauge-fixing and lapse fixing via radial diffeomorphisms.
For the purposes of this analys, we will mostly restrict attention to local considerations on a single null hypersurface. Moreover we neglect boundary conditions and surface terms. These carry of course very important physics, as discussed in the previous chapter, but we will address the study of null boundaries as future perspective.

5.2 Hamiltonian formulation and the problem initial data

Before presenting the first-order connection formulation, let us review some basic facts of the metric formulation, that will be useful in the following: the details of the Bondi coordinate gauge-fixing, and the description of constraint-free data and their associated symplectic potential.

The typical set-up is a $2+2$ foliation with a doubly-null initial slice, see Fig. 5.1. Sachs' constraint-free data for a local evolution can then be identified with the conformal class of the two-dimensional induced metric along the initial slice, or alternatively its shear, plus corner data at the 2d space-like intersection. With some additional

[4]here we denote by null rotations the generators of rotations on the null hypersurface

regularity assumptions, one can also use a $3+1$ foliation by null cones radiated by a time-like world-line. See [Rendall, 1990, Friedrich, 1986, Frittelli et al., 1995, Choquet-Bruhat et al., 2011, Chrusciel, 2014] for the formal analysis of solutions and existence theorems. Both evolution schemes are typically local because of the development of caustics, however for situations with sufficiently weak gravitational radiation like those of [Christodoulou and Klainerman, 1993], null cones can foliate all of spacetime. A case of special interest is the study of radiating isolated gravitational systems in asymptotically flat spacetimes. In the asymptotic $2+2$ problem, one puts the second null hypersurface at future null infinity $\mathscr{I}^+$, and the foliation describes null hypersurfaces (or null cones) attached to $\mathscr{I}^+$. In this case the assignment of initial data is subtler (see e.g. [Friedrich, 1981]), because of the compactification involved in the definition of $\mathscr{I}$. In particular, $\mathscr{I}^+$ is shear-free by construction. Nonetheless, the data are still described by an asymptotic shear, *transverse* to $\mathscr{I}^+$ [Penrose, 1963, Newman and Penrose, 1962, Ashtekar, 1981], and Ashtekar's result was to show that these degrees of freedom and the phase space they describe are better thought of in terms of connections living on $\mathscr{I}^+$, a construction which is useful for the understanding of conserved charges. Notice that one can not take $\mathscr{I}^+$ itself as null cone of a $3+1$ foliation, because of the 'hole' at i^+ where tails and bound states escape null infinity (see e.g. [Geroch, 1978]), nor $\mathscr{I}^-$ for the same reason. We will mostly focuse on local properties of null hypersurfaces, and not discuss the non-trivial features associated for instance with boundary data at corners, residual diffeomorphisms, caustics and cone-vertex regularity, for which we refer the reader to literature cited above and below.

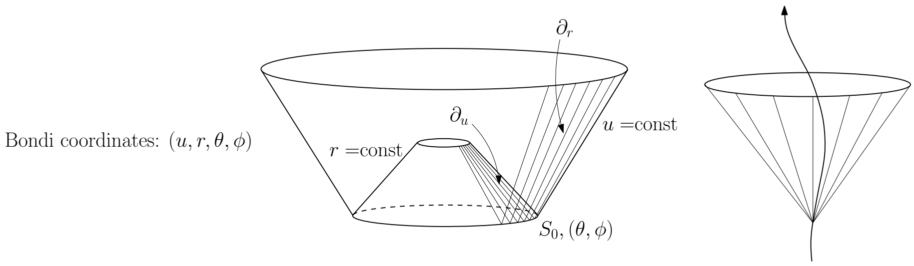


Figure 5.1. Left: *Set-up of the characteristic* $2+2$ *initial-value problem. Two null hypersurfaces intersect on a space-like 2d surface* S_0. *When the two null hypersurfaces are intersecting light cones, as in the picture,* S_0 *has topology of a sphere. The (partial) Bondi gauge is such that* (θ,ϕ) *are constant along* ∂_r, *and* ∂_r *is null for all values of* u. *On the other hand,* ∂_u *is null at at most one value of* r, *unless the spacetime has special isometries.* Right: *Further requiring suitable regularity conditions one can consider also a local* $3+1$ *foliation of light-cones generated by a time-like world-line.*

5.2.1 Bondi gauge and Sachs constraint-free initial data

The Bondi coordinate gauge is specified as follows: we take spherical coordinates in a local patch of spacetime, $x^\mu = (u,r,\theta,\phi)$, with the level sets of u to provide a foliation into null hypersurfaces Σ. du is thus a null 1-form, implying the gauge-fixing condition $g^{00}=0$, and the associated future-pointing null vector $l^\mu = -g^{\mu\nu}\partial_\mu u$ is tangent to the null geodesics of Σ. The second gauge condition is to require the angular coordinates $x^A = (\theta,\phi)$, $A=2,3$, to be preserved along r, i.e. $l^\mu\partial_\mu x^A = 0$. This implies $g^{0A}=0$ and makes r a parameter along the null geodesics: the level sets of r thus provide a

$2+1$ foliation orthogonal to the null geodesics. At this point, the metric and its inverse can be conveniently parametrized as follows,

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\beta}\frac{V}{r} + \gamma_{AB}U^A U^B & -e^{2\beta} & -\gamma_{AB}U^B \\ & 0 & 0 \\ & & \gamma_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ & e^{-2\beta}\frac{V}{r} & -e^{-2\beta}U^A \\ & & \gamma^{AB} \end{pmatrix}, \tag{5.1}$$

in terms of seven functions $(\beta, V, U^A, \gamma_{AB})$. Being the coordinates adapted to the $2+2$ foliation defined by u and r, $g_{AB} \equiv \gamma_{AB}$ is the metric induced on the 2d space-like surfaces, and we denote its determinant γ and its inverse γ^{AB}. The gauge-fixing has the property that $g^{AB} = \gamma^{AB}$, so it is analogue to the shift-free (partial) gauge $N^a = 0$ for a space-like foliation. There still remains one coordinate freedom, for which two different choices are customary in the literature: we can require as in [Sachs, 1962b, Bondi et al., 1962] the radial coordinate to be an areal parameter R (called 'luminosity distance' by Sachs), namely fix $\sqrt{\gamma} = R^2 f(\theta,\phi)$; or we can follow the Newman-Penrose (NP) literature [Newman and Penrose, 1962, Penrose, 1980] and require $g^{01} = -1$, with no restrictions on γ, which makes r an affine parameter for the congruence generated by l^μ. The relation between the two choices is given by $\partial r/\partial R = e^{2\beta}$. As we will review below, $e^{2\beta}$ plays the role of the lapse function in the canonical theory, and these two choices correspond to two different gauge-fixings of the radial diffeomorphism constraint. Accordingly, we will denote from now on $e^{2\beta} = N > 0$. In the following, we will often keep this last gauge fixing unspecified, for our results to be easily adapted to both choices. We will then refer to the partial gauge-fixing $g^{00} = 0 = g^{0A}$ as partial Bondi gauge.[5]

To set up the characteristic $2+2$ initial-value problem, one chooses initial data on two null hypersurfaces intersecting on a space-like 2d surface S_0, see Fig. 5.1. Working with a null foliation, any fixed value of u identifies the first null hypersurface. On the other hand, with r affine or areal at most one r =constant hypersurface will also be null, for a generic spacetime. Its location can be fixed with a measure-zero gauge-fixing $g^{11}|_{r_0} = 0$. Then, as shown originally in [Sachs, 1962a] (see also [d'Inverno and Smallwood, 1980, Torre, 1986, Mädler and Winicour, 2016]), constraint-free initial data for general relativity can be identified with the conformal class of 2d space-like metrics γ_{AB}, of which we take the uni-modular representative $\check{\gamma}_{AB} := \gamma^{-1/2}\gamma_{AB}$; supplemented by boundary data at the corner S_0 between the two initial slices.[6] Up to the measure-zero corner data, the two independent components of $\check{\gamma}_{AB}$ are the two physical degrees of freedom of general relativity on a null hypersurface. In the associated hierarchical integration scheme, the Hamiltonian constraint can be solved as a radial linear equation for V, and one can identify the propagating equations for the constraint-free data as (the

[5]A third option to complete the partial Bondi gauge is to take dr null, so to have also $g^{11} = 0$. This choice, used in the original Sachs paper [Sachs, 1962a], is not adapted to the asymptotic problem, and will be not considered in the following.

[6]Explicitly, Sachs' also fixes the residual hypersurface gauge, and provides the corner data

$$(\gamma, \quad \partial_u\gamma, \quad \partial_r\gamma, \quad \partial_r U^A)|_{S_0}.$$

They provide the area of S_0, the initial expansion of the null geodesic congruences along the two hypersurfaces, and the non-integrability of the two null directions: $U_{A,1}$ gives in these coordinates the Lie bracket among the two normal vectors ∂_u and ∂_r at S_0.

traceless part of) the projection of the Einstein's equations on the space-like surface. These give the evolution of $\breve{\gamma}_{AB}$ away from the initial slice. The price to pay for the identification of constraint-free data is that the dynamical spacetime can be reconstructed only locally in a neighbourhood of the characteristic surface (neighbourhood that may well be smaller then the maximal Cauchy development, see e.g. [Rendall, 1990]), as caustics develop and stop the validity of the coordinate patch. See e.g. [Frittelli et al., 1995, d'Inverno, 2005, Reisenberger, 2013, Mädler and Winicour, 2016] for various discussions on this.

The geometric interpretation of the constraint-free data is most commonly given in terms of the shear of null geodesic congruences, which is directly determined by the induced 2d metric. To see this, let us consider the normal 1-form $l_\mu = -\partial_\mu u$. Since it is null, it is automatically geodesic and twist-free; and since the level sets of u provide a null foliation, it is affinely parametrised. The associated congruence tensor coincides then with the Lie derivative of the induced metric, which in partial Bondi gauge is proportional to the radial derivative,

$$\nabla_A l_B = \frac{1}{2}\pounds_l \gamma_{AB} = \frac{1}{2N}\partial_r \gamma_{AB}. \tag{5.2}$$

This surface tensor can be familiarly decomposed into shear σ_{AB} and expansion θ as the trace-less and trace parts,

$$\frac{1}{2}\pounds_l \gamma_{AB} = \frac{\sqrt{\gamma}}{2}\pounds_l \breve{\gamma}_{AB} + \frac{1}{2}\gamma_{AB}\pounds_l \ln\sqrt{\gamma} = \sigma_{AB} + \frac{1}{2}\gamma_{AB}\theta. \tag{5.3}$$

Hence, the shear of the null congruence carries the same information of the conformal 2d metric, up to zero modes lost in the derivative and which are part of the corner data. The fact that (the bulk of the) constraint-free data can be described in terms of shear will allow us to easily identify them in the first-order formalism, where ∇_μ is an affine connection.

Here we used the Bondi gauge in order to identify the tangent vector field to the null geodesic congruence with a coordinate vector, thus simplifying Lie derivatives. A 2d space-like metric in Σ, its Lie derivative defining a shear, and associated Sachs' propagating equations, can be identified without this gauge-fixing: it suffices to use a $2+2$ decomposition, either in terms of two scalar fields defining a $2+2$ foliation (one being u), or in terms of a null dyad (one element being l_μ), as we will review below. The role of the gauge-fixing is nonetheless crucial to specify the explicit integration scheme of the constraints and the other field equations. Hence, it is possible to talk about physical degrees of freedom in a completely covariant way, as often done in the literature, although only once the gauge is completely fixed one can truly identify constraint-free initial data.

5.2.2 Hamiltonian structure in the metric formalism

The fact that the constraint-free data can be either described by the metric or the shear, its null-radial derivative, captures a well-known property of field theories on the light cone: the momentum conjugated to the fields does not depend on velocities, but on the null radial derivative of the field. Consider for instance a scalar field in Minkowski spacetime. Defining $x^\pm := t \pm r$, and choosing x^+ as 'time' for the canonical analysis, the conjugate momentum is

$$\pi(x^-, x^A) := \frac{\delta \mathscr{L}}{\delta \dot{\phi}} = \partial_- \phi(x^-, x^A), \tag{5.4}$$

where $A = 2, 3$ are the transverse coordinates. The independence of the momentum from the velocities gives rise to a primary constraint $\Phi := \pi - \partial_- \phi$, which is second class with itself, up to zero modes, see e.g. [Alexandrov and Speziale, 2015]. In the following, we will refer to this constraint as light-cone condition. This fact, which is just a direct consequence of the fact that the normal vector to a null hypersurface is tangent to it, means that the momentum is not an independent variable, and can then be eliminated from the phase space. The physical phase space has thus ∞^1 dimensions per degree of freedom, instead of ∞^2 as in the space-like formulation, and the fields satisfy Dirac brackets defined by a suitable regularisation of ∂_-^{-1}. Since we are not interested in this paper in the subtle infrared issues and boundary conditions, let us content ourselves to describe the symplectic structure of the theory looking at the symplectic potential. To that end, one can use the covariant phase space method (see e.g. [Ashtekar et al., 1991]), and read the symplectic potential from the variation of the action in presence of a null boundary. Consider for simplicity a free scalar field, and a null boundary given by a single light-cone Σ ruled by x^-. Then the variation of the action gives the following boundary contribution,

$$\Theta = \int_\Sigma \partial_- \phi \, \delta \phi. \tag{5.5}$$

This symplectic potential shows that the conjugate momentum to ϕ satisfies the light-cone condition (5.4), and announces the presence of ∂_-^{-1} in the Dirac bracket among the ϕ's.

The same structure arises in gauge theories (see e.g. [Grange et al., 1998]) and linearised general relativity around Minkowski [Scherk and Schwarz, 1975, Evens et al., 1987]: the physical phase space has ∞^1 dimensions for each physical degree of freedom (a transverse mode in these examples), and the conjugate momentum is given by the null radial derivative of the mode itself. Remarkably, it is also true in full, non-linear general relativity, with the momentum given by the shear, again a null radial derivative of the physical degrees of freedom as shown in (5.2). The Hamiltonian analysis of general relativity on a null hypersurface has been performed in [Torre, 1986] using the $2+2$ formalism of [d'Inverno and Smallwood, 1980]. Starting with a covariant kinematical phase space of canonical variables $(g_{\mu\nu}, \Pi^{\mu\nu} := \delta \mathscr{L} / \delta \partial_u g_{\mu\nu})$, one finds 6 first class and 6 second class constraints, for a resulting 2-dimensional physical phase space, as expected. The six first class constraints split in 3 hypersurface diffeomorphism generators plus three primary constraints imposing the vanishing of the conjugate momenta to the chosen shift vectors. The six second class are: the null hypersurface condition $g^{00} = 0$, which in turns gauge-fixes the Hamiltonian constraint and makes it second class;[7] two light-cone conditions, the non-linear version of (5.4); the vanishing of the momentum conjugated to the lapse N, and the vanishing of $\partial_u g^{00}$.[8]

[7]Up to zero modes: Measure-zero 'parallel' time diffeomorphisms are still allowed. For instance, these contain the BMS super-translations [Bondi et al., 1962] for asymptotically flat spacetimes.

[8]This last constraint may look puzzling. The problem is that imposing g^{00} strongly in the action

The analysis of [Torre, 1986] is general and does not require the Bondi gauge: we introduce a $2+2$ foliation by two closed 1-forms, $n^\alpha = d\phi^\alpha$ locally, with $\alpha = 0,1$, normals to a pair of hypersurfaces. Instead of lapse and shift, we have two shift vectors and a 'lapse matrix' $N_{\alpha\beta}$, with inverse $N^{\alpha\beta} := n^\alpha_\mu n^{\beta\mu}$, and dual basis $n^\mu_\alpha := g^{\mu\nu} N_{\alpha\beta} n^\beta_\nu$. The only gauge-fixing is to take a null foliation defined say by the level sets of ϕ^0, so that $N^{00} = 0 = N_{11}$, and the lapse (i.e. the Lagrange multiplier of the Hamiltonian constraint) turns out to be the off-diagonal component, $N_{01} = -N$.[9] The induced space-like metric on the 2-dimensional surface orthogonal to both normals is then $\gamma_{\mu\nu} = g_{\mu\nu} - N_{\alpha\beta} n^\alpha_\mu n^\beta_\nu$. In this formalism, we can identify covariantly the two physical degrees of freedom with $\breve{\gamma}_{\mu\nu}$; their propagating equations as the two components of the Einstein equations obtained from the trace-less projection onto the 2d surface; and their Hamiltonian counterpart as the multiplier equations arising from the stabilisation of the two light-cone conditions.

If we adapt the null coordinate, $\phi^0 = u$, we have $n^\mu_1 = N_{01} n^{0\mu} = N l^\mu$. Unlike l^μ, n^μ_1 has non-vanishing affinity, given by $k_{(n_1)} = \pounds_{n_1} \ln N$, and its shear and expansion are N times those of l^μ. The partial Bondi gauge corresponds to putting to zero one of the two shift vectors, and only in this gauge the coordinate vector ∂_{ϕ^1} is tangent to the null geodesics on Σ. As discussed above, the gauge-fixing is convenient for many reasons, principally to provide the explicit integration scheme of the Einstein's equations, in particular solving the constraints. Another advantage is that due to the presence of complicated second class constraints, it is difficult to write the explicit Dirac bracket for the physical phase space. Gauge-fixing gets rid of gauge quantities and simplifies this problem. It becomes for instance straightforward to write the symplectic potential purely in terms of physical data. For our purposes, we specialise here the analysis of [Torre, 1986] to the partial Bondi gauge, adapting coordinates so that $\phi^0 = u$ and requiring $g^{0A} = 0$, but keeping r unfixed as to see explicitly the role of lapse and $\sqrt{\gamma}$. This partial gauge-fixing eliminates various gauge fields from the phase space, and one can isolate the induced 2d metric γ_{AB} and its conjugate momentum density, which turns out to be

$$\hat{\Pi}_{AB} := \sqrt{\gamma}\Pi^{AB} = \frac{\delta\mathscr{L}}{\delta\dot{\gamma}_{AB}} = \frac{\sqrt{\gamma}}{2}(\gamma_{AB}\gamma_{CD} - \gamma_{AC}\gamma_{BD})\pounds_{n_1}\gamma^{CD} - \sqrt{\gamma}\gamma_{AB}(\pounds_{n_1}\ln N + \frac{1}{2N}\pounds_{n_0}N^{00}), \tag{5.6}$$

in terms of the dual basis (n_0, n_1) defined above. Taking the trace-less and trace parts, it is immediate to identify them as the shear and expansion of the null-geodesic congruence

would lead to a variational principle missing one of the Einstein's equations. To avoid this 'missing equation', the Hamiltonian in [Torre, 1986] is first constructed with arbitrary g^{00}, and $g^{00} = 0$ is later imposed as initial-value constraint on the phase space. The additional constraint $\partial_u g^{00} = 0$ then simply arises as a secondary constraint preserving the first one under evolution. As explained in [Alexandrov and Speziale, 2015], an advantage of working with a first order formalism is that one does not need this somewhat artificial construction: we can impose the gauge-fixing condition strongly in the action and still have a complete well-defined variational principle, thanks to the appearance in the action of the variable canonically conjugated to g^{00}. Furthermore, the on-shell value of the Lagrange multiplier for $g^{00} = 0$, which is fixed by hand in [Torre, 1986], comes up dynamically as a multiplier equation.

[9] The sign we use in this definition is opposite to the one of [Torre, 1986], to match with our earlier choice $N > 0$.

of n_1,

$$\Pi_{AB} - \frac{1}{2}\gamma_{AB}\Pi = \frac{\sqrt{\gamma}}{2}\pounds_{n_1}\check{\gamma}_{AB} = \sigma_{(n_1)AB}, \tag{5.7}$$

$$\Pi := \gamma^{AB}\Pi_{AB} = -\theta_{(n_1)} - 2k_{(n_1)} - \frac{1}{N}\pounds_{n_0}N^{00}. \tag{5.8}$$

The first equation above is precisely the light-cone condition (5.4) for non-linear gravity: the two physical momenta are the null radial derivatives of the two physical modes of the metric, namely, the shear of n_1. The second equation shows that the trace of the momentum does not carry any additional information, although this may require a few words: first, the expansion can be determined from the dynamical fields (up to boundary values) using the Raychaudhuri equation; the lapse can always be fixed to 1 with a radial diffeomorphism as mentioned above, thus removing the non-affinity term;[10] finally, the last term vanishes using the equations of motion.

In this partial Bondi gauge, the symplectic potential computed in [Torre, 1986] reads[11]

$$\Theta = \int_\Sigma d^3x\, \hat{\Pi}^{AB}\delta\gamma_{AB} = -\int_\Sigma d^3x \left[\sigma_{(n_1)AB}\delta\hat{\gamma}^{AB} + (\theta_{(n_1)} + 2k_{(n_1)})\delta\sqrt{\gamma}\right], \tag{5.9}$$

where we used $\delta\gamma_{AB} = -\gamma_{AC}\gamma_{BD}\delta\gamma^{BD}$ and defined the densitised inverse metric $\hat{\gamma}^{AB} := \sqrt{\gamma}\gamma^{AB}$. Notice also that the shear term can be rewritten using $-\sigma_{AB}\delta\hat{\gamma}^{AB} = \sqrt{\gamma}\sigma^{AB}\delta\gamma_{AB}$. The non-affinity term vanishes if we fix a constant lapse, and using the explicit metric form of shear and expansion, the symplectic potential takes the form

$$\Theta = -\int_\Sigma d^3x \left[\frac{\sqrt{\gamma}}{2}\pounds_{n_1}\check{\gamma}_{AB}\delta\hat{\gamma}^{AB} + \pounds_{n_1}\ln\sqrt{\gamma}\,\delta\sqrt{\gamma}\right]. \tag{5.10}$$

The first term has precisely the form (5.5) for the 2 physical degrees of freedom, which is the main point we wanted to make. The second term is just a corner contribution thanks to the Bondi gauge. In this paper we are interested in bulk degrees of freedom, hence we neglect corner terms in the symplectic potential.

This symplectic potential for the shear, here adapted from [Torre, 1986] to the Bondi gauge, can also be derived with covariant phase space methods (see e.g. [Ashtekar et al., 1991]), without referring to a special coordinate system but only to the field equations. It plays a crucial role in the study of BMS charges at null infinity (see e.g. [Ashtekar, 2014, Wald and Zoupas, 2000, Barnich and Troessaert, 2011]), which has recently received much attention for its possible relation to the information black hole paradox argued for in [Hawking et al., 2016]. For a careful treatment of caustics, corner data and residual diffeos, see [Reisenberger, 2008, Reisenberger, 2013], as well as [Duch et al., 2017] in a related context. For a more general expression of Θ without

[10] Canonically, the fact that changing r can be used to fix $N = 1$ follows from the fact that lapse transforms under radial diffeos like the radial component of a tangent vector. The alternative gauge-fixing, r areal coordinate with lapse free, turns the non-affinity term into a corner contribution to the symplectic potential, see e.g. [Reisenberger, 2013]. As mentioned above, we do not discuss corner terms in the present paper.

[11] As usual, deriving the symplectic potential requires an integration by part. Although [Torre, 1986] does not give the associated boundary term, this is known to be $2\int_\Sigma(\theta + k)\sqrt{\gamma}$, see e.g. [Parattu et al., 2016]. Note the different factors of 2 between the boundary term and the symplectic potential.

a full foliation and a discussion of corner terms without any coordinate gauge fixing, and its relevance to capture the full information about the charges, see [Hopfmüller and Freidel, 2016]. See also [Parattu et al., 2016, Lehner et al., 2016, ?, Wieland, 2017b] for additional discussions on corner terms.

5.3 Canonical structure in real connection variables

5.3.1 Spacetime Foliation and the tetrad gauge-fixing

In this section we briefly review the canonical structure of general relativity in connection variables on a null hypersurface [Alexandrov and Speziale, 2015]. In units $16\pi G = 1$, we work with the Einstein-Cartan action

$$S[e,\omega] = \frac{1}{2}\int_{\mathscr{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) - \frac{\Lambda}{6} e^K \wedge e^L\right), \tag{5.11}$$

where e^I is the tetrad 1-form, and $F^{IJ}(\omega) = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$ the curvature of the spin connection ω^{IJ}. As in the ADM formalism, we fix a $3+1$ foliation with adapted coordinates $x^\mu = (t, x^a)$, and hypersurfaces Σ described by the level sets of t. We parametrise the tetrad as follows [Barros e Sa, 2001b, Alexandrov and Vassilevich, 1998, Alexandrov, 2000],

$$e^0 = \hat{N} dt + \chi_i E^i_a dx^a, \qquad e^i = N^a E^i_a dt + E^i_a dx^a. \tag{5.12}$$

The hypersurface normal is then the soldering of the internal direction $x^I_+ := (1, \chi^i)$:

$$n^\Sigma_\mu := e^I_\mu x_{+I} = (-N, 0, 0, 0), \qquad N = \hat{N} - N^a E^i_a \chi_i. \tag{5.13}$$

For space-like Σ, the usual tetrad adapted to the ADM coordinates is recovered for vanishing χ^i, which makes e^0 parallel to the hypersurface normal. Using a non-adapted tetrad may appear as an unnecessary complication, but has the advantage that allows one to control the nature of the foliation. The metric induced by (5.12) on Σ is

$$q_{ab} := e^I_a e^J_b \eta_{IJ} = \mathscr{X}_{ij} E^i_a E^j_b, \qquad \mathscr{X}_{ij} := \delta_{ij} - \chi_i \chi_j, \qquad \det q_{ab} = E^2(1-\chi^2), \tag{5.14}$$

where $\chi^2 := \chi_i \chi^i$. It is respectively space-like for $\chi^2 < 1$, null for $\chi^2 = 1$, and time-like for $\chi^2 > 1$. In other words, we control with χ^i the signature of the hypersurface normal, while e^0 is always time-like.

We are interested here in the case of a foliation by null hypersurfaces. Notice that even though the induced hypersurface metric is degenerate, we can still assume an invertible triad, with inverse denoted by E^a_i. This means that we can use the triad determinant, $E := \det E^i_a \neq 0$, to define tensor densities. We denote such densities with a tilde respectively above or below the tensor, e.g. $\tilde{E}^a_i := E E^a_i$ for density weight 1 and $\underset{\sim}{E}^i_a := E^{-1} E^i_a$ for density weight -1. The triad invertibility is an advantage of the tetrad formalism for null foliations, and it allows us to write the null direction of the induced metric on Σ as $(E^a_i \chi^i)\partial_a$. Further, although the induced metric q_{ab} is not invertible, we can raise and lower its indices with the triad. We define the projector $q^a{}_b := E^{ai} E^j_b \mathscr{X}_{ij}$, which projects hypersurface vectors on 2d space-like spaces orthogonal to the null direction $E^a_i\chi^i$; and $q^{ab} := E^a_i E^b_j \mathscr{X}^{ij}$, which satisfies $q^{ab}q_{bc} = q^a{}_c$.

On the other hand, $\hat{N}$ and N^a should not be immediately identified with the lapse and shift functions, defined as the Lagrange multipliers of the diffeomorphism constraints. The true lapse can be identified from (5.13) or by computing the tetrad determinant, which turns out to be $e = NE$. As for the shift vector, there is no canonical choice on a null foliation, corresponding to the fact that there is no canonical Hamiltonian.[12] Following the canonical analysis of [Alexandrov and Speziale, 2015], to be recalled below, we keep N^a as the shift vector. In terms of the lapse N, the metric associated with the tetrad (5.12) reads

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N^b q_{ab} - 2NN^a E^i_a \chi_i & q_{bc}N^c - NE^i_b\chi_i \\ q_{ac}N^c - NE^i_a\chi_i & q_{ab} \end{pmatrix}, \tag{5.15}$$

with inverse

$$g^{\mu\nu} = \frac{1}{N}\begin{pmatrix} 0 & -E^b_i\chi^i \\ -E^a_i\chi^i & NE^a_i E^b_i + (N^a E^b_i + N^b E^a_i)\chi_i \end{pmatrix}. \tag{5.16}$$

The coordinate t being adapted to the null foliation, $g_{ab} \equiv q_{ab}$ is the degenerate induced metric on Σ. We can also write the projector on the 2d space-like spaces in a covariant form, using the null dyad provided by the internal null vectors $x^I_\pm = (\pm 1, \chi^i)$ soldered by the tetrad,

$$x^I_\pm := (\pm 1, \chi^i), \qquad x_{\pm\mu} = e^I_\mu x_{\pm I} = \begin{cases} (-N, 0) \\ (N + 2N^a E_a\chi, 2E_a\chi) \end{cases}, \qquad x_{+\mu}x^\mu_- = 2. \tag{5.17}$$

We then have

$$\perp^\mu{}_\nu := \delta^\mu{}_\nu - \frac{1}{2}x^\mu_+ x_{-\nu} - \frac{1}{2}x^\mu_- x_{+\nu} = \begin{pmatrix} 0 & 0 \\ q^a{}_b N^b & q^a{}_b \end{pmatrix}, \tag{5.18}$$

and

$$\gamma_{\mu\nu} := g_{\mu\nu} - x_{+(\mu}x_{-\nu)} = \begin{pmatrix} q_{ab}N^a N^b & q_{bc}N^c \\ q_{ab}N^b & q_{ab} \end{pmatrix} \tag{5.19}$$

is the induced metric in covariant form. For later purposes, let us identify here the propagating Einstein's equations, which are given by the components

$$(\perp \mathbb{G}^{\mathrm{T}})^{ab} := \Big(\perp^a{}_{(\rho} \perp^b{}_{\sigma)} - \tfrac{1}{2}\perp^{ab}\perp_{\rho\sigma}\Big)\mathbb{G}^{\rho\sigma} = \Pi^{ab}_{cd}\Big(\mathbb{G}^{cd} + N^d\mathbb{G}^{c0} + N^c\mathbb{G}^{0d} + N^c N^d \mathbb{G}^{00}\Big). \tag{5.20}$$

Here

$$\Pi^{ab}_{cd} := q^a_{(c}q^b_{d)} - \frac{1}{2}q^{ab}q_{cd} \tag{5.21}$$

is the traceless part of the projector on S for symmetric hypersurface tensors, and we used the notation $\mathbb{G}^\mu_I := G^\mu_I + \Lambda e^\mu_I = 0$ where G^μ_I is the Einstein tensor in tetrad indices. The explicit form of (5.20) is given in [Alexandrov and Speziale, 2015], and it will not be needed here.

[12] In the sense that it is not possible to express the Hamiltonian constraint purely in terms of hypersurface data, see for instance [Alexandrov and Speziale, 2015] and [Torre, 1986].

An advantage of the tetrad formulation is that we can perform the canonical analysis with the $3+1$ null foliation [Alexandrov and Speziale, 2015], without the need of introducing a further $2+2$ foliation like in the metric case. Nonetheless, it is instructive to review how the two formalisms compare in the absence of torsion. Our coordinates are adapted to the $3+1$ foliation by null hypersurfaces with normal 1-form dt, and to match notations with the literature, we rename from now on $t = u$; however the 2d space-like spaces defined by (5.18) are in general not integrable, hence they do not foliate spacetime. Nonetheless, we can choose a $2+2$ foliation and adapt our tetrad to it. For the sake of simplicity let us choose the foliation given by the normals

$$n^0 = du, \qquad n^1 = dr, \tag{5.22}$$

so that our coordinates $x^a = (r, x^A)$ are already adapted, and the induced 2d metric is $\gamma_{AB} \equiv g_{AB} = q_{AB}$. To adapt the null dyad $x_{\pm\mu}$ to this foliation we use the translational part of the ISO(2) group stabilising x^I_+ to remove the components $x_{-A} = E^i_A \chi_i = 0$. This gauge transformation makes the tangent vectors to $\{S\}$ integrable. The same can be done in the Newman-Penrose formalism, see Appendix A.7 for details and a general discussion. Comparing then the metric coefficients of (5.1) and (5.16) we see that the lapse functions used in the metric and connection formulations differ by a factor $E^r_i \chi^i$. This can be always set to one with an internal boost along x^I_+, as explained in the next Section. Hence, using this boost and the translational part of the stabiliser we can always reach the internal 'radial gauge'

$$E^i_a \chi_i = (1,0,0) \quad \Leftrightarrow \quad E^r_i \chi^i = 1, \ \mathscr{X}^{ij} E^r_j = 0, \tag{5.23}$$

where the equivalence follows from the invertibility of the triad. In this internal gauge N coincides with the lapse of the metric formalism, given by $-1/g^{01}$ in adapted coordinates, $E = \sqrt{\gamma}$ and $\sqrt{-g} = NE = N\sqrt{\gamma}$, and the induced metrics coincide, $g_{\mu\nu} - x_{+(\mu} x_{-\nu)} = g_{\mu\nu} - N_{\alpha\beta} n^\alpha_\mu n^\beta_\nu$. Proofs and more details on the relation between the χ-tetrad and the 2+2 formalism are reported in Appendix A.8.

5.3.2 Constraint structure

On a null hypersurface, each degree of freedom is characterised by a single dimension in phase space, as recalled above. This means that the constraint structure associated to the gravitational action should lead to a phase space of dimensions $2 \times \infty^3$ on Σ (plus eventual zero modes and corner data, not discussed here). We now review from [Alexandrov and Speziale, 2015] how this counting comes about, as the result has some peculiar aspects that we wish to analyse in this paper.

From (5.11), we see that the canonical momentum conjugated to ω^{IJ}_a is $\tilde{P}^a_{IJ} := (1/2)\varepsilon^{abc}\varepsilon_{IJKL} e^K_b e^L_c$, namely, it is simple as a bi-vector in the internal indices. This results in a set of (primary) simplicity constraints, which fixing an internal null direction, can be written in linear form as $\Phi^a_I := \varepsilon_{IJ}{}^{KL} \tilde{P}^a_{KL} x^J_+ = 0$. Two different canonical analysis were presented in [Alexandrov and Speziale, 2015]. The first is manifestly covariant, with only $\chi^2 = 1$ as a gauge-fixing condition. The second gauge-fixes instead all three components, that is $\chi^i = \hat{\chi}^i$ for a fixed $\hat{\chi}^i$ with $\hat{\chi}^2 = 1$. Since in this paper we are interested in the identification of constraint-free data that arises through a complete gauge-fixing, we recall only the details of the second analysis, and refer the reader interested in the covariant expressions to [Alexandrov and Speziale, 2015].

Working with a gauge-fixed internal direction, we can solve explicitly the primary simplicity constraints in terms of $\widetilde{P}^a_{0i} = \widetilde{E}^a_i, \widetilde{P}^a_{ij} = 2\widetilde{E}^a_{[i}\chi_{j]}$. The kinetic term of the action is then diagonalised by the same change of connection variables as in the space-like case [Alexandrov and Vassilevich, 1998],

$$\omega^{0i}_a = \eta^i_a - \omega^{ij}_a \chi_j, \qquad \omega^{ij}_a = \varepsilon^{ijk}\left(\tilde{r}_{kl} + \frac{1}{2}\varepsilon_{klm}\tilde{\omega}^m\right) \underset{\sim}{E}{}^l_a, \tag{5.24}$$

with $\tilde{r}_{ij}$ symmetric. After this change of variables and an integration by parts, the action reads[13]

$$S = \int dt \int_\Sigma 2(\widetilde{E}^a_i \partial_t \eta^i_a + \pi^{ij}\partial_t \tilde{r}_{ij} + \chi_i \partial_t \tilde{\omega}^i) + \lambda_{ij}\Phi^{ij} + \mu_i \varphi^i + n^{IJ}\mathscr{G}_{IJ} + N^a \mathscr{D}_a + \underset{\sim}{N}\mathscr{H}, \tag{5.25}$$

where

$$\mathscr{G}_{IJ} := D_a \widetilde{P}^a_{IJ}, \qquad \mathscr{D}_a := -\widetilde{P}^b_{IJ} F^{IJ}_{ab} + \omega^{IJ}_a \mathscr{G}_{IJ}, \qquad \mathscr{H} := \widetilde{E}^a_i \widetilde{E}^b_j F^{ij}_{ab} - 2\Lambda E^2, \tag{5.26}$$

are the gauge and diffeomorphism constraints, written in covariant form for practical reasons. Notice that as in the space-like case, the generator of spatial diffeomorphism includes internal gauge transformations (and accordingly, we have $n^{IJ} = \omega^{IJ}_0 - N^a \omega^{IJ}_a$). Next, the constraint

$$\Phi^{ij} = \pi^{ij} \tag{5.27}$$

imposes the vanishing of the momentum conjugated to r^{ij}, and is the left-over of the primary simplicity constraints in this non-covariant analysis. Finally, the constraint

$$\varphi^i = \chi^i - \hat{\chi}^i \tag{5.28}$$

gauge-fixes the internal vector. In particular, the projection $(\chi_i + \hat{\chi}^i)\varphi^i$ gives the null-foliation condition $\chi^2 = 1$, namely $g^{00} = 0$, and its stabilisation plays an important role in recovering all of Einstein's equations.[14]

The phase space of the theory has initially 36 dimensions, with Poisson brackets

$$\begin{aligned} &\{\eta^i_a(x), \widetilde{E}^b_j(x')\} = \frac{1}{2}\delta^i_j \delta^b_a \delta^{(3)}(x,x'), \\ &\{\tilde{r}^{ij}(x), \pi_{kl}(x')\} = \frac{1}{2}\delta^{ij}_{(kl)}\delta^{(3)}(x,x'), \qquad \{\tilde{\omega}^i(x), \chi_j(x')\} = \frac{1}{2}\delta^i_j \delta^{(3)}(x,x'). \end{aligned} \tag{5.29}$$

The explicit form of the constraints is considerably more compact and elegant than in the metric case [Torre, 1986], a fact familiar from the use of Ashtekar variables in other foliations. On the other hand, many of the constraints are second class. The reader familiar with the Hamiltonian analysis in the space-like case will recall that the

[13] In [Alexandrov and Speziale, 2015] we rescaled the action by a factor 1/2, to avoid a number of factors of 2 when computing Poisson brackets. Here we restore the conventional units. Accordingly, the parametrization of $\widetilde{P}^a_{IJ}$ in terms of $\widetilde{E}^a_i$, as well as the explicit expressions for the constraints presented below in (5.26), are twice those of [Alexandrov and Speziale, 2015].

[14] This plays the role of the $\partial_u g^{00} = 0$ condition of [Torre, 1986], and the advantage of the first-order formalism is that it can be imposed prior to computing the Hamiltonian.

stabilisation of the primary simplicity constraints leads to six secondary constraints which are second class with the primary. The secondary constraints thus obtained, together with the six Gauss constraints, recover half of the torsion-less conditions; the remaining half goes in Hamiltonian equations of motion. In the null case the situation becomes more subtle: there are again six secondary constraints, given by

$$\Psi^{ij} = -\varepsilon^{(ikl}\widetilde{E}^a_k \underset{\sim}{E}^{j)}_b \partial_a \widetilde{E}^b_l + \varepsilon^{(ikl}\widetilde{E}^a_k \chi_l \eta^{j)}_a - \mathscr{M}^{ij,kl} r_{kl}, \tag{5.30}$$

where

$$\mathscr{M}^{ij,kl} = \varepsilon^{(ikm}\varepsilon^{j)ln}\mathscr{X}_{mn}. \tag{5.31}$$

These have the same geometric interpretation of being six of the torsion-less conditions. However, only four of them are now automatically preserved. This is a consequence of the fact that (5.31) has a two-dimensional kernel: $\Pi^{ij}_{kl}\mathscr{M}^{kl,mn} \equiv 0$, where Π^{ij}_{kl} is the internal version of the symmetric-traceless projector (5.21) obtained via the triad. Then, stabilisation of the two secondary constraints

$$\hat{\Psi}^{ij} = \Pi^{ij}_{kl}\Psi^{kl}, \tag{5.32}$$

requires two additional, tertiary constraints

$$\Upsilon^{ab} := \frac{1}{2}\Pi^{ab}_{cd} E^{(c}_i \varepsilon^{d)ef}\left(F^{0i}_{ef} - \chi_j F^{ij}_{ef}\right) = 0. \tag{5.33}$$

As pointed out in [Alexandrov and Speziale, 2015], the two constraints (5.32) are the light-cone conditions imposing the proportionality of physical momenta to the hypersurface derivatives in the null direction: As we will show below, they reproduce precisely the metric relation (5.7) between momenta and shear. What is peculiar to the formalism, is that this condition is not automatically preserved under the evolution, but requires the additional constraints (5.33). These additional constraints are not torsion-less conditions; they will be discussed in details in Section 5.4.4 below.

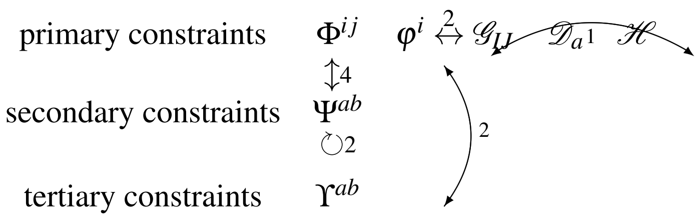
Concerning the nature of the constraints and the dimension of the reduced phase space, we have the following situation. The hypersurface diffeos $\mathscr{D}_a$ are first class, but not the Hamiltonian $\mathscr{H}$, which forms a second class pair with $\chi_i \varphi^i$. The other two components $\mathscr{X}_{ij}\varphi^j$ gauge-fix two of the six Gauss constraints, those that would change the internal direction. The other four Gauss constraints remain first class. This is different from the canonical analysis on a space-like or time-like hypersurface, where fixing the internal direction gives a 3-dimensional isometry group. Here instead we have a 4-dimensional isometry group, given by the little group ISO(2) of the internal direction given by χ^i, plus boosts along χ^i. The fact that the isometry group on a null hypersurface is one dimension larger than for other foliations is of course a well-known property, that led Dirac himself to suggest the use of null foliations as preferred ones. In the context of first-order general relativity with complex self-dual variables, it has for instance been pointed out in [Goldberg et al., 1992, d'Inverno et al., 2006].

However, there is a subtle way in which this extra isometry is realised in our context, because the action of internal boosts along χ^i mixes with that of radial diffeomorphisms. Let us spend a few words explaining it. Notice that right from the start we fixed to unity the 0-th component of the internal null direction x^I_+. This choice, implicit in the parametrization (5.12) of the tetrad, deprives us of the possibility of changing χ^i with an internal boost along χ^i, since in the absence of a variable x^0_+ this would not preserve the light-likeness of the internal direction. Nonetheless, the explicit calculation of the constraint structure shows that $K_\chi := \mathscr{G}_{0i}\chi^i$ is still a first class constraint: simply, its action is not to change χ^i, which it leaves invariant, but rather to rescale the lapse function. Using the transformation properties for Lagrange multipliers (see e.g. [Henneaux and Teitelboim, 1992]), we find for the smeared constraint the transformation

$$K_\chi(\lambda) \triangleright N = e^\lambda N. \tag{5.34}$$

In other words, the lapse function is in our formalism soldered to the extent of the internal null direction, see (5.17), and this is the reason why it transforms under internal radial boosts. As already discussed at the end of Sec. 5.3.1, our lapse coincides with the lapse of the metric formalism only if we fix the radial boosts to have $E^r_i\chi^i = 1$. Hence, there is in our formalism a partial mixing of the action of internal boosts along χ^i and radial diffeomorphisms.

To complete the review of the constraints structure, it remains to discuss the simplicity constraints. They are all second class, but in different ways: $\hat{\Psi}^{ij}$ among themselves, just like those encoding the light-cone conditions (5.4), the remaining four Ψ^{ij} are second class with four of the primary Φ^{ij}, and the remaining two Φ^{ij} are second class with the two tertiary constraints. The overall canonical structure established in [Alexandrov and Speziale, 2015] leads to the following diagram, where the arrows indicate which constraints are mutually second class:



We have 7 first class constraints (forming a proper Lie algebra), and 20 second class constraints, for a $2 \times \infty^3$-dimensional physical phase space, as expected for the use of a null hypersurface. Among those, the pair Hamiltonian-null hypersurface condition.

5.4 Geometric interpretation of the constraint solution

5.4.1 Newman-Penrose tetrad

To elucidate the geometric content of the canonical structure in the first order formalism, it is convenient to use the Newman-Penrose (NP) formalism. To that end, we want to map our tetrad (5.12) to a doubly-null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, where

$$l_\mu n^\mu = -1 = -m_\mu \bar{m}^\mu, \qquad g_{\mu\nu} = -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)}. \tag{5.35}$$

We have already partially done so, when we introduced the soldered internal null vectors $x^\mu_\pm = e^\mu_I x^I_\pm$, $x^I_\pm = (\pm 1, \chi^i)$, which provide the first pair. For the second pair, we have to choose a spatial dyad for the induced metric (5.19), that is $\gamma_{\mu\nu} = 2m_{(\mu}\bar{m}_{\nu)}$; we can do so taking m^μ to be a complex linear combination of the two orthogonal tetrad directions $\mathscr{X}^{ij}e^\mu_j$, normalised by $m_\mu \bar{m}^\mu = 1$. The set

$$(x^\mu_+, -x^\mu_-, m^\mu, \bar{m}^\mu) \tag{5.36}$$

so defined is an NP tetrad. Notice that $x_{+\mu} = -N\partial_\mu u$, so the first vector chosen is normal to the null hypersurface. The minus sign in front of the second vector is to follow the conventions to have all vectors future-pointing.

Before adopting the traditional notation with l^μ and n^μ for the first two vectors, let us briefly discuss the frame freedom. Using the nomenclature of [Chandrasekhar, 1985], we have rotations of class *I* leaving l^μ unchanged, of class *II* leaving n^μ unchanged, and of class *III* rescaling l^μ and n^μ and rotating m^μ:

$$l^\mu \mapsto A^{-1} l^\mu, \qquad n^\mu \mapsto A n^\mu, \qquad m^\mu \mapsto e^{i\theta} m^\mu. \tag{5.37}$$

Conforming with standard literature on null hypersurfaces, we want the first null co-vector to be normal to the null hypersurface and future pointing, that is $l_\mu \propto -\partial_\mu u$. Concerning its 'normalisation', a reasonable choice is to take it proportional to the lapse function, like in the space-like Arnowitt-Deser-Misner (ADM) canonical analysis: $l^{\mathrm{ADM}}_\mu = -N\partial_\mu u$. This analogy with ADM is confirmed by Torre's analysis, which as we recalled above, identifies in $n_{1\mu} \equiv l^{\mathrm{ADM}}_\mu$ the normal relevant to the Hamiltonian structure, namely whose shear gives the conjugate momentum in the action. However, most of the literature on null hypersurfaces uses a gradient normal, $l_\mu = -\partial_\mu u$, and we'll conform to that, by taking

$$l^\mu = \frac{1}{N} x^\mu_+, \qquad n^\mu = -\frac{N}{2} x^\mu_-. \tag{5.38}$$

This rescaling of $x^\mu_\pm$ means paying off a large number of N factors in the spin coefficients, see the explicit expressions reported in Appendix A.4. In any case, the relation between the two choices is a class *III* transformation, and all NP quantities are related by simple and already tabulated transformations that can be found in [Chandrasekhar, 1985], some of which are reported in Appendix A.4.[15]

We fix from now on the following internal direction,

$$\chi^i = (1,0,0), \tag{5.39}$$

and introduce the notation $v^\pm \equiv v_\pm := \frac{1}{\sqrt{2}}(v^2 \pm i v^3)$ for the internal indices $M = 2,3$ orthogonal to it. This choice is done only for the convenience of writing explicitly the tetrad components of m^μ and $\bar{m}^\mu$ when needed, and we will keep referring to χ^i in the formulas as to make them immediately adaptable to other equivalent choices. Summarising, our NP tetrad and co-tetrad, and their expressions in terms of the metric

[15] The rescaling also means that while all Lorentz transformations of (5.36) are generated canonically via $\mathscr{G}_{IJ}$, this is not the case for (l,n) defined via (5.38): we disconnect the canonical action of the radial boost K_χ, which leaves them invariant instead of generating the class *III* rescaling. We see then again that $l^{\mathrm{ADM}}_\mu = x_{+\mu}$ is a more canonical choice of null tetrad adapted to the foliation.

coefficients (5.12), are

$$l^\mu = \frac{1}{N}(e_0^\mu + e_1^\mu) = (0, \frac{1}{N}E_i^a\chi^i), \tag{5.40a}$$

$$n^\mu = \frac{N}{2}(e_0^\mu - e_1^\mu) = (1, -N^a - \tfrac{1}{2}NE_i^a\chi^i), \tag{5.40b}$$

$$m^\mu = \frac{1}{\sqrt{2}}(e_2^\mu - ie_3^\mu) = (0, E_-^a), \tag{5.40c}$$

and

$$l_\mu = \frac{1}{N}(-e_\mu^0 + e_\mu^1) = (-1, 0), \tag{5.41a}$$

$$n_\mu = -\frac{N}{2}(e_\mu^0 + e_\mu^1) = -\Big(\frac{N}{2}(N + 2N^aE_a^i\chi_i), NE_a^i\chi_i\Big), \tag{5.41b}$$

$$m_\mu = \frac{1}{\sqrt{2}}(e_\mu^2 - ie_\mu^3) = (N^aE_a^-, E_a^-). \tag{5.41c}$$

The NP tetrad thus constructed is adapted to a null foliation like the one used in most literature [Newman and Unti, 1962, Newman and Tod, 1981, Adamo et al., 2009]. A detailed comparison and discussion of the special cases corresponding to a tetrad further adapted to a $2+2$ foliation or to the Bondi gauge can be found in Appendix A.7 and A.8.

Associated with the NP tetrad are the spin coefficients, namely 12 complex scalars projections of the connection ω_μ^{IJ}, e.g. (minus) the complex shear $\sigma := -m^\mu m^\nu \nabla_\nu l_\mu$.[16] If the connection is Levi-Civita, these are functions of the metric. In the first order formalism on the other hand, the connection is an independent variable, and the NP spin coefficients will be functions of the metric and of the connection components. To distinguish the two situations, we will keep the original NP notation, e.g. σ, for the Levi-Civita coefficients, and add an apex $\circ$ for the spin coefficients with an affine off-shell connection, e.g. $\overset{\circ}{\sigma}$. On-shell of the torsion-less condition, $\omega^{IJ} = \omega^{IJ}(e)$ and $\overset{\circ}{\sigma} = \sigma$. Explicit expressions for all the spin coefficients are in Appendix A.4, and we will report in the main text only those relevant for the discussion.

5.4.2 The affine null congruence

Since the normal vector l^μ is null, it would be automatically geodesic with respect to the spacetime Levi-Civita connection. Furthermore it would have vanishing non-affinity since it is the unit normal to a null foliation. With an off-shell, affine connection ω_a^{IJ} on the other hand, these familiar properties do not hold. Using Newman-Penrose notation with an apex $\circ$ for the spin coefficients of the affine off-shell connection, what we have is

$$l^\nu \nabla_\nu l^\mu = \overset{\circ}{\varepsilon}\, l^\mu - \overset{\circ}{\kappa}\bar{m}^\mu + \text{cc}, \tag{5.42}$$

with 'non-affinity' and 'non-geodesicity' that are given respectively by

$$k_{(l)} := \overset{\circ}{\varepsilon} + \text{cc} = -\frac{1}{N}E_i^a\chi^i(\eta_a^i\chi_i - \partial_a \ln N), \qquad \overset{\circ}{\kappa} = -\frac{1}{N^2}E_i^a\chi^i\eta_a^-. \tag{5.43}$$

[16] The reader familiar with the NP formalism will notice an opposite sign in this definition. This is a consequence of the fact that we work with mostly plus signature.

For the same reason, the congruence $\nabla_\mu l_\nu$ is not twist-free, even though l_μ is the gradient of a scalar, nor defined intrinsically on S: it also carries components away from it. Nonetheless, we can still take its projection $\perp^\rho{}_\mu \perp^\sigma{}_\nu \nabla_\rho l_\sigma$, and decompose it into irreducible components: we will refer to the traceless-symmetric $\overset{\circ}{\sigma}_{\mu\nu}$, trace $\overset{\circ}{\theta}$ and antisymmetric parts $\overset{\circ}{\omega}_{\mu\nu}$ as 'connection shear', 'connection expansion', and 'connection twist'. The components away from the hypersurface Σ, which are all proportional to the shift vector N^a, are not directly relevant for us and we leave them to Appendix A.5. Using the definition $\nabla_\mu e^I_\nu = -\omega^{IJ}_\mu e_{\nu J}$ and the decomposition (5.24), we have for the hypersurface components

$$\nabla_a l_b = \frac{1}{N}\mathscr{X}_{ij}\eta^i_a E^j_b, \tag{5.44}$$

and

$$\overset{\circ}{\sigma}_{(l)\,ab} := \frac{1}{N} q_a{}^c q_b{}^d \mathscr{X}_{ij}\eta^i_{(c}E^j_{d)} - \frac{1}{2}q_{ab}\overset{\circ}{\theta}_{(l)}, \qquad \overset{\circ}{\theta}_{(l)} := \frac{1}{N}\mathscr{X}_{ij}\eta^i_a E^a_j, \qquad \overset{\circ}{\omega}_{(l)\,ab} := \frac{1}{N} q_a{}^c q_b{}^d \mathscr{X}_{ij}\eta^i_{[c}E^j_{d]}. \tag{5.45}$$

In NP notation, shear, twist and expansion are described by the following two complex scalars,

$$\overset{\circ}{\sigma} := -m^\mu m^\nu \nabla_\nu l_\mu = -m^\mu m^\nu \overset{\circ}{\sigma}_{(l)\,\mu\nu} = -\frac{1}{N}E^a_- \eta^-_a, \tag{5.46}$$

$$\overset{\circ}{\rho} := -m^\mu \bar{m}^\nu \nabla_\nu l_\mu = -\frac{1}{2}\overset{\circ}{\theta}_{(l)} - m^\mu \bar{m}^\nu \overset{\circ}{\omega}_{(l)\,\mu\nu} = -\frac{1}{N}E^a_+ \eta^-_a, \tag{5.47}$$

where the real and imaginary parts of $\overset{\circ}{\rho}$ carry respectively the connection expansion and twist. It is also convenient to introduce the complex shear $\overset{\circ}{\sigma}_{(l)} := m^\mu m^\nu \overset{\circ}{\sigma}_{(l)\,\mu\nu} = -\overset{\circ}{\sigma}$. This comes up awkwardly opposite in sign to the NP spin coefficient, but the minus sign is an unavoidable consequence of the fact that we work with mostly plus signature, the opposite to NP.

The connection shear so computed allows us to identify Sachs' constraint-free initial data for first-order general relativity in terms of real connection variables: in the absence of torsion, $\overset{\circ}{\sigma} = \sigma$ and we can follow the same hierarchical integration scheme. From the connection perspective, the relevant piece of information is thus $E^a_- \eta^-_a$; namely the contraction with the triad of η^-_a, which is the translation part of the ISO(2) stabilising the null direction x^I_+. Notice that both connection term and triad term have the same internal helicity: loosely speaking, it is this coherence that allows to reproduce the spin-2 behaviour in metric language.

Notice that at the level of Poisson brackets, the shear components commute: trivially in $\{\overset{\circ}{\sigma}_{(l)}, \overset{\circ}{\sigma}_{(l)}\} = 0$, but also when the conjugate appears, since[17]

$$\{\overset{\circ}{\sigma}_{(l)}, \bar{\overset{\circ}{\sigma}}_{(l)}\} = \frac{2i}{NE}\mathrm{Im}(\overset{\circ}{\rho}), \tag{5.48}$$

which vanishes on-shell of the Gauss law, as we show in the next Section. This is to be expected, since it is only at the level of the Dirac bracket that the shear components

[17]Using the brackets (5.29), and notice that $\{\eta^i_a, \tilde{E}^b_j/(NE)\} = 1/(2NE)(\delta^b_a\delta^j_i - E^a_i E^j_b/2)$.

do not commute with themselves, that is when the light-cone constraints are used. We will show below in Section 5.5.1 that the Dirac bracket reproduces exactly the metric structure of (5.10).

In terms of the covariant connection, the shear, twist and expansion are described as follows,

$$\overset{\circ}{\sigma}_{(l)} = e^\nu_I e^\rho_J \, m^\mu m_\nu l_\rho \, \omega^{IJ}_\mu, \qquad \overset{\circ}{\rho} = e^\nu_I e^\rho_J \, \bar{m}^\mu m_\nu l_\rho \, \omega^{JI}_\mu. \tag{5.49}$$

Using these covariant expressions, it is easy to see how the congruence is affected by the presence of torsion, writing $\omega^{IJ}_\mu = \omega^{IJ}_\mu(e) + C^{IJ}_\mu$ where C^{IJ}_μ is the contorsion tensor. For instance, consider the case of fermions with a non-minimal coupling [Alexandrov, 2008]

$$S_\psi = -\frac{i}{4}\int e\bar{\psi}e^\mu_I \gamma^I(a - ib\gamma^5)D_\mu(\omega)\psi + \text{cc}, \qquad a,b \in \mathbb{C}, \qquad \text{Re}(a) \equiv 1. \tag{5.50}$$

(The minimal coupling would be $a = 1$, $b = 0$). Solving Cartan's equation, one gets (restoring for a moment Newton's constant G)

$$C^{IJ}_\mu = 2\pi e^K_\mu G\left[\frac{1}{2}\varepsilon^{IJ}{}_{KL}\Big(A^L - \text{Im}(b)V^L\Big) - \delta^{[I}_K\Big(\text{Re}(b)A^{J]} + \text{Im}(a)V^{J]}\Big)\right], \tag{5.51}$$

where $V^I = \bar{\psi}\gamma^I\psi$ and $A^I = \bar{\psi}\gamma^I\gamma^5\psi$ are the vectorial and axial currents. Plugging this decomposition into (5.49) we find

$$\overset{\circ}{\sigma} = \sigma, \qquad \overset{\circ}{\rho} = \rho - \pi G\Big[in_\mu\Big(A^\mu - \text{Im}(b)V^\mu\Big) - l_\mu\Big(\text{Re}(b)A^\mu + \text{Im}(a)V^\mu\Big)\Big]. \tag{5.52}$$

The connection shear recovers its usual metric expression, whereas twist is introduced proportional to the axial current; for non-minimal coupling, the twist depends also on the vectorial current, and furthermore the expansion is modified, picking up an extra term proportional to the time-like component of the vectorial and axial currents. More in general, for an arbitrary contorsion decomposed into its three irreducible components $(\mathbf{3/2},\mathbf{1/2}) \oplus (\mathbf{1/2},\mathbf{3/2}) \oplus (\mathbf{1/2},\mathbf{1/2}) \oplus (\mathbf{1/2},\mathbf{1/2})$,

$$C^{\mu,\nu\rho} = \bar{C}^{\mu,\nu\rho} + \frac{2}{3}g^{\mu[\rho}\check{C}^{\nu]} + \frac{1}{e}\varepsilon^{\mu\nu\rho\sigma}\hat{C}_\sigma, \tag{5.53}$$

we have

$$\overset{\circ}{\sigma} = \sigma - m_\mu m_\nu l_\rho \bar{C}^{\mu,\nu\rho}, \qquad \overset{\circ}{\rho} = \rho - \bar{m}_\mu m_\nu l_\rho \bar{C}^{\mu,\nu\rho} + \frac{1}{3}l_\mu \check{C}^\mu - in_\mu \hat{C}^\mu, \tag{5.54}$$

as well as

$$\overset{\circ}{\kappa} = \kappa - l_\mu m_\nu l_\rho \bar{C}^{\mu,\nu\rho}, \qquad \overset{\circ}{k}_{(l)} = k_{(l)} - l_\mu n_\nu l_\rho \bar{C}^{\mu,\nu\rho} - \frac{1}{3}l_\mu \check{C}^\mu \tag{5.55}$$

for the non-geodesicity and inaffinity.

It is now instructive to see how the various quantities introduced above, and associated with an affine geodesic, are put on-shell by the constraints present in the Hamiltonian formulation of the theory, and thus (in the absence of torsion) take their values as in the more familiar metric formalism. As we show in details in the next subsection, the congruence is made geodesic by the Gauss law, which also puts on-shell the connection twist and expansion; the non-affinity vanishes as a consequence of the equation of motion stabilising $\chi^2 = 1$, namely the condition of null foliation; and finally, the connection shear is put on-shell by the two secondary simplicity constraints (5.32).[18]

5.4.3 Torsionless nature of the affine null congruence

In this subsection we use the affine congruence defined above to study the geometric meaning of the various constraints present in the theory, in particular those responsible for the metricity of the congruence. Let us begin with the Gauss constraint $\mathscr{G}$ in (5.26). First, we decompose it into rotations $L_i := \frac{1}{2}\varepsilon_{ijk}\mathscr{G}^{jk}$ and boosts $K_i := \mathscr{G}_{0i}$. Then, we consider the projections along χ^i, and perpendicular to it, defined by $v^i_\perp := \varepsilon^{ijk}\chi_j v_k$ (notice that $v^i_\perp = -iv^-$). These various components have the following explicit forms (see Appendix A.6),

$$L_\chi := \frac{1}{2}\varepsilon_{ijk}\chi^i\mathscr{G}^{jk} \overset{\varphi}{\approx} \varepsilon_{ijk}\chi^i\widetilde{E}^{aj}\eta^k_a, \tag{5.56a}$$

$$L^i_\perp := \mathscr{G}^{ij}\chi_j \overset{\varphi}{\approx} \partial_a\widetilde{E}^{ai} + \widetilde{E}^{ai}\eta_a\chi - \eta^i_a\widetilde{E}^a_j\chi^j - \tilde{\omega}^i + \chi^i(\tilde{\omega}_j\chi^j - \partial_a\widetilde{E}^a_j\chi^j), \tag{5.56b}$$

$$K_\chi := \chi^i\mathscr{G}_{0i} \overset{\varphi}{\approx} \partial_a\widetilde{E}^a_i\chi^i - \mathscr{X}_{ij}\widetilde{E}^{ai}\eta^j_a, \tag{5.56c}$$

$$K^i_\perp := \varepsilon^{ijk}\chi_j\mathscr{G}_{0k} \overset{\varphi}{\approx} \varepsilon^{ijk}\chi_j(\partial_a\widetilde{E}^a_k + \widetilde{E}^a_k\,\eta^i_a\chi_i - \tilde{\omega}_k), \tag{5.56d}$$

where $\overset{\varphi}{\approx}$ means on-shell of the φ constraint only, namely assuming χ^i constant.

The two second class constraints are the linear combinations $\hat{T}^i_\perp := L^i_\perp - \varepsilon^i{}_{jk}\chi^j K^k_\perp$, whose action would change the internal direction χ^i. On the other hand, $T^i_\perp := L^i_\perp + \varepsilon^i{}_{jk}\chi^j K^k_\perp$ and L_χ belong to the ISO(2) subgroup stabilising x^I_+ and are first class, together with K_χ.[19] Using the explicit expressions for the spin coefficients (see Appendix A.4), we immediately identify the x^I_+-stabilisers with

$$L_\chi = 2EN\mathrm{Im}(\mathring{\rho}), \qquad L^-_\perp + \varepsilon^{-jk}\chi_j K^k_\perp = EN^2\mathring{\kappa}. \tag{5.57}$$

These first class constraints are thus responsible for the congruence being geodesic and twist-free. For the remaining first class constraint, we have

$$K_\chi = E(E^a_i\chi^i\partial_a\ln|E| - N\mathring{\theta}_{(l)} + \partial_a E^a_i\chi^i). \tag{5.58}$$

Recalling that $\sqrt{-g} = NE$ and the explicit form of l^μ from (5.41), we see that this constraint puts the expansion on-shell:

$$\mathring{\theta}_{(l)} \approx \theta_{(l)} = l^\mu\partial_\mu\ln\sqrt{-g} + \partial_\mu l^\mu. \tag{5.59}$$

[18] Notice that here we are defining the congruence in the presence of torsion using a displacement vector η^μ such that $B_{\mu\nu}\eta^\nu := l^\nu\nabla_\nu\eta^\mu = \eta^\nu\nabla_\nu l_\mu$. This is suggested as to keep the geometric interpretation of the spin coefficients $\mathring{\sigma}$ and $\mathring{\rho}$, however it means that the displacement vector is not Lie dragged: $\pounds_l\eta^\mu = l^\nu\nabla_\nu\eta^\mu - \eta^\nu\nabla_\nu l^\mu + 2C^\mu{}_{\sigma\nu}\eta^\sigma l^\nu = 2C^\mu{}_{\sigma\nu}\eta^\sigma l^\nu$. In spite of the fact that in the presence of torsion differential parallelograms do not close, it is natural to still require the Lie dragging of η^μ (see e.g. [Luz and Vitagliano, 2017]). With this definition of the congruence, shear and expansion are never modified by torsion, but only the twist. The NP spin coefficients $\mathring{\sigma}$ and $\mathring{\rho}$ lose their geometric interpretation.

[19] Covariantly, the stabilisers can be written as $T^I := 1/2\varepsilon^{IJKL}x_{+J}J_{KL}$ and $\hat{T}^I := -1/2\varepsilon^{IJKL}x_{-J}J_{KL}$. With $\chi^i = (1,0,0)$ and $M = 2,3$, we have $v^M_\perp = (-v^3, v^2)$; the second class constraints read $\hat{T}^M = (L^2 + K^3, L^3 - K^2)$, whereas the first class ones are $T^M = (L^2 - K^3, L^3 + K^2)$.

Let us remark the central role played by the radial boost: as a constraint, it is responsible for the metricity of the expansion; as a symmetry generator, it rescales lapse as discussed in (5.34).[20] Finally, the second class constraints fix the two components of $\tilde{\omega}^i$ orthogonal to χ^i, and have no direct implication for the affine congruence.

Let us now come to the non-affinity: even on-shell of the Gauss law, the now-geodesic congruence still carries non-affinity $k_{(l)}$, in spite of l_μ being the gradient of a scalar. This is because the Gauss law only captures half of the torsion-less conditions. Where is then the equation setting $k_{(l)} = 0$? It must come from the Hamiltonian equation of motion that gives the stability of $\varphi^i \chi_i$, namely the equation capturing the fact that the level sets of u provide a null foliation. Indeed, this stability condition was identified in [Alexandrov and Speziale, 2015] as the multiplier equation expressing lapse in terms of canonical variables,[21] which reads

$$E^a_i \chi^i (\partial_a \ln N - \eta^i_a \chi_i) = 0. \tag{5.60}$$

Comparing this expression with the first of (5.43), we see that it implies the vanishing of the non-affinity.

It remains to put on shell the connection shear. To that end, we look at the light-cone conditions (5.32). With our gauge-fixing $\chi^i = (1,0,0)$ the two components of $\hat{\Psi}^{ij}$ are $\hat{\Psi}^{23}$ are $\hat{\Psi}^{22} - \hat{\Psi}^{33}$. We combine them into a single complex equation, which gives

$$-\frac{1}{2}\Big(\Psi^{23} + \frac{i}{2}(\Psi^{22} - \Psi^{33})\Big) = \widetilde{N}\,\overset{\circ}{\sigma}_{(l)} - E^a_- E^-_b\, \partial_a \widetilde{E}^b_1 + E^a_1 E^-_b\, \partial_a \widetilde{E}^b_- = 0, \tag{5.61}$$

from which it follows that

$$\overset{\circ}{\sigma}_{(l)} = \frac{1}{N} E^-_b \left(E^a_- \partial_a E^b_1 - E^a_1 \partial_a E^b_-\right) = l^\mu m^\nu (\partial_\mu m_\nu - \partial_\nu m_\mu) = \frac{1}{2} m^\mu m^\mu \pounds_l \gamma_{\mu\nu} \equiv \sigma_{(l)}, \tag{5.62}$$

where in the second equality we used the explicit expressions (5.41) for the NP tetrad. Hence, the two secondary simplicity constraints corresponding to the light-cone conditions make the connection shear metric. Comparing this result with the analysis in metric variables of [Torre, 1986], we expect the connection shear to be the conjugate momentum to the conformal metric. This expectation is indeed borne out, as we will show below in Section 5.5.1.

Summarising, the congruence generated by l^μ is made geodesic by three first-class Gauss constraints. The fourth first-class one gives the relation between the connection expansion and the metric expansion. All these conditions are automatically preserved under evolution in u, since there are no secondary constraints arising from the stabilisation of the Gauss law. As for the connection shear, its relation to the metric shear is realised by the light-cone secondary simplicity constraints, and they are not automatically preserved. Tertiary constraints are required, to whose analysis we turn next.

[20] As well as transforming the connection component determining lapse, $\{K_\chi(\lambda), (\eta_a \chi)(E^a \chi)\} = \partial_r \lambda$. At the level of covariant field equations, the radial boost constraint corresponds to the equation $\varepsilon^{abc} \varepsilon_{ijk} e^j_c \chi^k D_a e^i_b = 0$.

[21] Recall that on a null hypersurface, the Hamiltonian constraint is second class, therefore its Lagrange multiplier satisfies an equation of motion, which fixes it up to zero modes. Concerning the zero modes, these are the left-over diffeomorphisms that on $\mathscr{I}$ become the supertranslations of the BMS transformations.

5.4.4 Tertiary constraints as the propagating equations

Let us now discuss the tertiary constraints (5.33), whose presence is something quite unfamiliar within general relativity, and which is due to the combined use of a first-order formalism and a null foliation: each feature taken individually introduces a secondary layer of constraints in the Hamiltonian structure. Perhaps even more surprising is which of the field equations are described by these constraints: the propagating Einstein's equations, namely the dynamical equations describing the evolution (in retarded time u) of the shear away from the null hypersurface. In fact, it was shown in [Alexandrov and Speziale, 2015] that

$$\Upsilon^{ab} = -\frac{1}{2N}\Pi^{ab}_{cd}\left[4g\,\varepsilon_{efh}\,g^{0e}g^{cf}(\perp \mathbb{G}^{\mathrm{T}})^{dh} + E^c_i\,(\mathbb{B}^{di} + N^d\mathbb{B}^{0i})\right], \tag{5.63}$$

where in the first term we recognise the propagating Einstein's equations, and

$$\mathbb{B}^{\mu I} := \varepsilon^{\mu\nu\rho\sigma}e_{\nu J}F^{IJ}_{\rho\sigma}(\omega) \equiv 0 \tag{5.64}$$

denotes the algebraic Bianchi identities. This means that in the first-order formalism, the only time derivative present in the propagating equations (5.20) can be completely encoded in algebraic Bianchi equations.

The equivalence (5.63) may appear geometrically obscure, and it is furthermore not completely trivial to derive as a tensorial equation. On the other hand, it becomes transparent using the Newman-Penrose formalism, as we now show. To that end, let us first identify the propagating equations in the Newman-Penrose formalism. A straightforward calculation of the propagating equations gives

$$m^\mu m^\nu \perp^{\rho\sigma}_{\mu\nu} G_{\rho\sigma}(\omega,e) = m^\mu m^\nu G_{\mu\nu}(\omega,e) = m^\mu m^\nu R_{\mu\nu}(\omega,e) = -R_{lmnm}(\omega,e) - R_{nmlm}(\omega,e) \approx 2R_{lmmn}(e),$$

where in the last equality $\approx$ means on-shell of the torsion-less condition.[22] Next, let us look at the tertiary constraints in its form (5.33), and project it in the same way on S:

$$m_a m_b \Upsilon^{ab} = \frac{1}{2} m_a m_b E^{(a}_i\varepsilon^{b)ef}\left(F^{0i}_{ef} - \chi_j F^{ij}_{ef}\right). \tag{5.65}$$

First, we have that

$$F^{0i}_{ef}(\omega,e) - \chi_j F^{ij}_{ef}(\omega,e) = -\frac{2}{N}n^\mu e^{i\nu}R_{\mu\nu ef}(\omega,e). \tag{5.66}$$

Then, to obtain the hypersurface Levi-Civita symbol, we observe that n^μ is the only vector with a u-component, therefore we can write[23]

$$\varepsilon^{def} = -e6l^{[d}m^e\bar{m}^{f]}. \tag{5.67}$$

Finally, using the fact that $m_a E^a_i e^{i\nu} = m^\nu$, we have

$$\frac{1}{E}m_a m_b \Upsilon^{ab} = -\frac{1}{e}n^\mu m^\nu m_d \varepsilon^{def}R_{\mu\nu ef}(e,\omega) = 2n^\mu m^\nu l^\rho m^\sigma R_{\mu\nu\rho\sigma}(e,\omega) = 2R_{nmlm}(e,\omega), \tag{5.68}$$

[22] These equations are not be confused with Sachs' optical equations R_{lmlm} and $R_{lml\bar{m}}$, which relate Weyl and Ricci to the variation of shear and twist *along* the null hypersurface, not away from it.

[23] With conventions $\varepsilon^{0123} = 1$, $e = -1/4!\varepsilon_{IJKL}\varepsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma$.

which coincides with (minus) the propagating equations on-shell of the torsion-less condition,

$$m_a m_b \Upsilon^{ab} \approx -E m^\mu m^\nu G_{\mu\nu}(e). \tag{5.69}$$

It is also instructive to see the explicit role played by the algebraic Bianchi identity. For vanishing torsion and NP gauge,[24] the propagating equation reads

$$\Delta\sigma - \delta\tau + \bar{\lambda}\rho + (\mu + \bar{\gamma} - 3\gamma)\sigma + 2\beta\tau + \Phi_{02} = 0, \tag{5.70}$$

where we can further set $\Phi_{02} = 0$ since we are interested in the vacuum equations. Here $\Delta := n^\mu \nabla_\mu$ and $\delta := m^\mu \nabla_\mu$ is conventional NP notation, see Appendix A.4. For an expression of this equation in metric language, see e.g. [Mädler and Winicour, 2016]. The point is that if the connection is initially independent from the metric, this is a PDE with a single time derivative in the term $\Delta\sigma$; but this term can be eliminated using an algebraic Bianchi identity, or 'eliminant relation' in the terminology of [Chandrasekhar, 1985]. Using equation (g) on page 48 of [Chandrasekhar, 1985], which in NP gauge reads

$$D\lambda + \Delta\bar{\sigma} - \bar{\delta}(\alpha + \bar{\beta}) = \bar{\sigma}(3\bar{\gamma} - \gamma + \mu - \bar{\mu}) - 2\bar{\beta}(\alpha + \bar{\beta}), \tag{5.71}$$

we can replace $\Delta\sigma$ with $\delta(\bar{\alpha} + \beta) - D\bar{\lambda}$ plus squares of spin coefficients. In metric variables, this would indeed be a trivial manipulation, since the time derivative is now simply shifted from $\Delta\sigma = -m^\mu m^\nu \partial_u \partial_r \gamma_{\mu\nu} + \dots$ to $D\bar{\lambda} = -m^\mu m^\nu \partial_r \partial_u \gamma_{\mu\nu} + \dots$. But used in the first order formalism with an independent connection (where now (5.71) holds with all $\overset{\circ}{\sigma}$ quantities and it is derived from (5.64)), relates non-trivially the propagating equations to the tertiary constraint.

Finally, concerning the geometric interpretation of this constraint, recall from Section 5.3.2 that it is there to stabilise the light-cone conditions: hence, Einstein's propagating equations can be seen as the condition that a metric-compatible connection shear on the initial null slice, remains metric at later retarded times.[25]

5.5 Implementation of the Bondi gauge

The discussion in the last two Sections has been completely general: apart from the condition of having a null foliation, we have not specified further the coordinate system. We now specialise to Bondi coordinates, presenting the simplified formulas that one obtains in this case. We will then use this gauge to prove the equivalence of the symplectic potentials of the first-order and metric formalisms, which in particular identifies the connection shear with the momentum conjugated to the conformal 2d metric; and to discuss a property of radiative data at $\mathscr{I}^+$.

To that end, we completely fix the internal gauge, adapting the doubly-null tetrad to a $2+2$ foliation. For the interested reader, the Bondi gauge for our tetrad without the complete internal gauge-fixing is described in Appendix A.8.1. We take $\chi^i = (1,0,0)$

[24] Namely $\rho = \bar{\rho}$, $\kappa = \varepsilon = \pi = 0$, $\tau = \bar{\alpha} + \beta$. See Appendix A.7.2 for details.

[25] This can be compared with the metric formalism of [Torre, 1986], where the propagating Einstein's equations also arise from the stabilisation of the light-cone shear-metric conditions, but as multiplier equations, not as constraints.

as in (5.39), and use the first-class generators K_χ and T_M to fix $E^r_i = (1,0,0)$. This internal 'radial gauge' adapts the tetrad to the $2+1$ foliation of constant-r slices:

$$\chi^i = (1,0,0), \quad E^r_i = (1,0,0) \quad \Rightarrow \quad E^1_a = (1,0,0), \quad E = \sqrt{\gamma}, \quad m^\mu = (0,0,E^A_-). \tag{5.72}$$

The determinant of the triad now coincides with that of the induced metric γ_{AB} (hence triad and metric densities now conveniently coincide). This fixes five of the internal transformations, leaving us with the SO(2) freedom of rotations in the 2d plane of mappings $m^\mu \mapsto e^{i\delta} m^\mu$. We will not use this freedom in the following, and if desired can be fixed for instance requiring the triad to be lower-triangular. Now we impose the coordinate gauge-fixing. On top of the null foliation condition $g^{00} = 0$, the Bondi gauge conditions are $g^{0A} = 0$, plus a condition on r, typically either the areal choice $\sqrt{\gamma} = r^2 f(\theta, \phi)$, or the affine choice $g^{01} = -1$. We take here the affine Bondi gauge, and report the details on Sachs areal gauge in Appendix A.8.2. From the parametrisation (5.16), we can read these conditions in terms of our tetrad variables:

$$g^{0a} = -\frac{1}{N} E^a_i \chi^i = (-1,0,0). \tag{5.73}$$

Using the internal gauge-fixing (5.72), $E^r_i \chi^i = E^r_1 = 1$, hence (5.73) implies $E^A_1 = 0$ and $N = 1$, as in the metric formalism. The metric (5.15) and its inverse reduce to the following form,

$$g_{\mu\nu} = \begin{pmatrix} 2U + \gamma_{AB} N^A N^B & -1 & \gamma_{AB} N^B \\ & 0 & 0 \\ & & \gamma_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ & -2U & N^A \\ & & \gamma^{AB} \end{pmatrix}, \tag{5.74}$$

where we redefined $2U := -1 - 2N^r$ for convenience. The triad and its inverse are

$$E^i_a = \begin{pmatrix} 1 & 0 \\ -E^M_A E^A_1 & E^M_A \end{pmatrix}, \qquad E^a_i = \begin{pmatrix} 1 & 0 \\ E^A_1 & E^A_M \end{pmatrix}, \tag{5.75}$$

where as before we use $M = 2,3$ for the internal hypersurface coordinates orthogonal to χ^i, and E^A_M is the inverse of the dyad E^M_A.

The structure of the null congruence of l_μ reduces to:

$$k_{(l)} = -\eta^1_r, \qquad \mathring{\kappa} = \eta^-_r, \tag{5.76}$$

$$\mathring{\sigma}_{(l)AB} = \eta^M_{(A} E_{B)M}, \qquad \mathring{\theta}_{(l)} = \eta^M_A E^A_M, \qquad \mathring{\omega}_{(l)AB} = \eta^M_{[A} E_{B]M}. \tag{5.77}$$

The lapse equation (5.60) simplifies to

$$\eta^1_r = 0, \tag{5.78}$$

so this connection component is set to zero by working with a constant lapse. The vanishing of the twist imposed by L_χ (in absence of torsion) now reads

$$\eta_{[AB]} := \eta^M_{[A} \widetilde{E}^M_{B]} = 0. \tag{5.79}$$

This equation is the null-hypersurface analogue of the familiar symmetry of the extrinsic curvature in the spatial hypersurface case, there analogously imposed by part of the Gauss constraint: $K_{[ab]} := K^i_{[a}E^i_{b]} = 0$. The radial boost K_χ simplifies to

$$K_\chi = \sqrt{\gamma}\Big(\overset{\circ}{\theta}_{(l)} - \partial_r \ln\sqrt{\gamma}\Big), \tag{5.80}$$

and its solutions give the affine Bondi-gauge formula for the expansion, $\overset{\circ}{\theta}_{(l)} = \theta_{(l)} := \partial_r \ln\sqrt{\gamma}$. The solution of the light-cone secondary simplicity constraints (5.61) now gives

$$\overset{\circ}{\sigma}_{(l)} = E^-_A \partial_r E^A_-, \tag{5.81}$$

namely the expression for the shear in affine Bondi gauge, written here in terms of the dyad E^A_M.

5.5.1 Equivalence of symplectic potentials

We now show the equivalence between the symplectic potential in connection variables (which we can read from the $p\delta q$ part of (5.25)) and the one in metric variables (5.9), thereby identifying the canonical momentum to the conformal 2d metric in the connection language. It will turn out to be the connection shear of the canonical normal $n^1_\mu = Nl_\mu$, as to be expected from the on-shell equivalence of the first and second order pure gravity action principles. As in the usual space-like canonical analysis, the equivalence of symplectic potentials will require the Gauss law. We begin by eliminating χ^i and π^{ij} from the phase space, completely fixing the internal gauge and using the primary simplicity constraints, and consider then only the first term of (5.25) for the symplectic potential. Since our main focus are the bulk physical data, we will neglect boundary contributions to the symplectic potential, and show the equivalence in the partial Bondi gauge $g^{0A} = 0$. The reason not to fix completely the Bondi gauge is to keep both lapse and an arbitrary $\sqrt{\gamma}$, to show a more general equivalence holding regardless of the choice of coordinate r. Hence, we want to show that

$$\Theta = \int_\Sigma 2\tilde{E}^a_i \delta\eta^i_a \overset{(g^{0A}=0)}{\approx} \int_\Sigma \sqrt{\gamma}\Pi^{AB}\delta\gamma_{AB}, \tag{5.82}$$

with Π_{AB} given by (5.6).

The partial Bondi gauge is $E^A_i\chi^i = E^A_1 = 0$, which implies $\eta^M_r = 0$ on shell of the Gauss law, see (5.76). This eliminates two monomials from the integrand, and we are left with the following two terms:

$$\Theta = \int_\Sigma 2\tilde{E}^a_i \delta\eta^i_a \approx \int_\Sigma 2(\tilde{E}^A_M \delta\eta^M_A + \sqrt{\gamma}\delta\eta^1_r). \tag{5.83}$$

Accordingly, here and in the following we will restrict attention to variations preserving the gauge and the Gauss constraint surface. Let us look at the right-hand side of (5.82). We expect from the metric formalism that the conjugate momentum is build from the congruence of $n^\mu_1 = Nl^\mu$. Its shear and expansion are just N times those of l^μ, which we can read from (5.45); its non-affinity is $k_{(n_1)} = \partial_r \ln N = \eta^1_r$ using the lapse equation

(5.60) in partial Bondi gauge. Accordingly, we consider the following ansatz for the momentum,

$$\Pi_{AB} := \eta_A^M E_{BM} - \gamma_{AB}(E_M^A \eta_A^M + \eta_r^1), \tag{5.84}$$

whose decomposition gives

$$\Pi_{AB} - \frac{1}{2}\gamma_{AB}\Pi = \eta^M_{(A}E_{B)M} - \frac{1}{2}\gamma_{AB}\overset{\circ}{\theta}_{(n_1)} \equiv \overset{\circ}{\sigma}_{(n_1)AB}, \qquad \Pi = -\overset{\circ}{\theta}_{(n_1)} - 2\eta_r^1, \tag{5.85}$$

where we used $\eta_r^M = 0 = \eta^M_{[A}E_{B]M}$ from the Gauss law. This momentum reduces to the one in the metric formalism (5.6) by construction, and we now show it satisfies (5.82). To that end, we first observe that $E = \sqrt{\gamma}$ is now a 2×2 determinant. This means that $\det E_A^M = \det \widetilde{E}_M^A$, and the inverse induced metric has the following expression in terms of canonical variables,

$$\gamma^{AB} = \frac{\widetilde{E}_M^A \widetilde{E}^{BM}}{(\det \widetilde{E}_M^A)^2}. \tag{5.86}$$

A simple calculation then gives

$$\begin{aligned}\Pi^{AB}\delta\gamma_{AB} &= -\Pi_{AB}\delta\gamma^{AB} = -2\left[\Pi_{(AB)}E^{BM} - \Pi E_A^M\right]\delta\widetilde{E}_M^A \\ &= -2\left[\eta^M_{(A}E_{B)M}E^{BN}\delta\widetilde{E}_N^A + \eta_r^1 E_A^M \delta\widetilde{E}_M^A\right].\end{aligned} \tag{5.87}$$

where we used $\delta\det\widetilde{E} = E_A^M\delta\widetilde{E}_M^A$. Next, we use again $\eta^M_{[A}\widetilde{E}_{B]M} = 0$ from the Gauss law, so the first symmetrised term above gives twice the same contribution. Using the fact that $\delta\sqrt{\gamma} = NE_A^M\delta\widetilde{E}_M^A$, we finally get

$$\Pi^{AB}\delta\gamma_{AB} \approx -2\left[\eta_A^M\delta\widetilde{E}_M^A + \eta_r^1\delta\sqrt{\gamma}\right], \tag{5.88}$$

and (5.82) follows up to boundary terms. We have thus verified that in the first order formalism the (traceless part of the) conjugate momentum to the induced metric is the connection shear of $n_1 = Nl_\mu$.

We also remark the presence of a term proportional to the 2d area. As in the metric formalism, this is a measure-zero degree of freedom, that can be pushed to a corner contribution and describes one of Sachs' corner data. A similar corner term appears in the spinorial construction of [Wieland, 2017a], where it is shown to admit a quantisation compatible with that of the loop quantum gravity area operator. See also [Freidel and Perez, 2015] for related results on 2d discreteness.

This result provides an answer to one of the open questions of [Alexandrov and Speziale, 2015], namely that of identifying the Dirac brackets for the reduced phase space variables. We did so looking at the symplectic potential as in covariant phase space methods, and completely fixing the gauge: this introduced additional second class constraints that could be easily solved, e.g (5.78). Whether it is possible to write covariant Dirac brackets without a complete gauge-fixing remains an open and difficult question, because of the non-trivial field equations satisfied by the second class Lagrange multipliers.

It is interesting to compare the situation with the space-like case, where the dynamical part of the connection is also contained in components of η_a^i, except now χ^i belongs to a time-like 4-vector (and we can always set $\chi^i = 0$, a choice often referred to as 'time gauge', since $e^0 \propto dt$). These dynamical components describe boosts and therefore do not form a group. An SU(2) group structure can be obtained via a canonical transformation, to either complex self-dual variables, as in the original formulation [Ashtekar, 1986], or to the auxiliary Ashtekar-Barbero real SU(2) connection (see e.g. [Thiemann, 2001]): the transformation requires adding the Immirzi term to the action, and the price to pay is either additional reality conditions, or use of an auxiliary object instead of a proper spacetime connection. Using a null foliation appears to improve the situation: the three internal components of η^i_a can be naively[26] associated with the radial boost K_χ and the two 'translations' $T^i_\perp$, or null rotations, related to the ISO(2) group stabilising the null direction of the hypersurface. But as we have seen above only the translation components η^M_a enter the bulk physical degrees of freedom, which are described by the connection shear. The component $\eta^i_a \chi_i$ is on a different footing: it enters the spin coefficients α, β, γ and ε (see Appendix A.4), and is treated in a way similar to the expansion θ, in that it is fully determined from initial data on a corner. We plan to develop these ideas in future research, in particular investigating the relation with a loop quantum gravity quantization based on the translation components of the connection, representing bulk physical degrees of freedom.

To complete the comparison between null and space-like foliations, in the latter case the canonical momentum conjugated to the induced metric is build from the triad projection K^i_a of the extrinsic curvature (see e.g. [Thiemann, 2001] for details). For a null foliation, the canonical momentum conjugated to the induced metric is related to the shear of the null congruence. The comparison is summarised by the following table:[27]

foliation	**space-like**	**null**
relevant internal group	SU(2)	ISO(2)
momentum conjugated to metric	$\Pi_{ab} = K^i_a E_{bi} - q_{ab} K^i_c E^c_i$	$\Pi_{ab} = \mathscr{X}_{ij}(\eta^i_a E^j_b - q_{ab} \eta^i_c E^{cj})$

To help the comparison in the table above, we have used the fact that in our formalism we can define the raised-indices hypersurface metric q^{ab}, and use it to prescribe an extension Π_{ab} of (5.84) on the whole hypersurface.[28]

5.5.2 Radiative data at future null infinity

As a final consideration, we would like to come back to the geometric interpretation of the tertiary constraints, and point out that the very same algebraic Bianchi identity that links them to the propagating equations, also plays an interesting role in the interpretation of the radiative data at $\mathscr{I}^+$.

To that end, we consider in this subsection the case of an asymptotically flat spacetime, and the u =constant null foliation attached to future null infinity $\mathscr{I}^+$. In this

[26]To make the argument precise, we should embed the dynamical components into a covariant connection whose non-dynamical parts are put to zero by linear combinations of constraints, see e.g. [Alexandrov and Livine, 2003] for an analogue treatment in the space-like case.

[27]For the reader interested in the time-like case, see [Alexandrov and Kadar, 2005].

[28]The equivalence (5.82) can then be written with $\Pi^{ab}\delta q_{ab}$ on the right-hand side, and trivially holds because the extra pieces now present are put to zero by the constraints and/or gauge conditions.

setting, we can compare our metric (5.74) and doubly-null tetrad to those of Newman-Unti [Newman and Unti, 1962, Newman and Tod, 1981, Adamo et al., 2009] mostly used in the literature, and use the asymptotic fall-off conditions for the spin coefficients there computed.[29] We refer the interested reader to Appendix A.8.1 for the details, and report here only the most relevant results. In particular,

$$\sigma = \frac{\sigma^0}{r^2} + O(r^{-4}), \tag{5.89}$$

and the asymptotic shear $-\sigma^0(u,\theta,\phi)$ fully characterises the radiative data at $\mathscr{I}^+$ [Penrose, 1963, Newman and Penrose, 1962, Ashtekar, 1981]. Ashtekar's result [Ashtekar, 1981] (see also [Ashtekar, 2014] for a recent review) is that the data can be described in terms of a connection D_μ defined intrinsically on $\mathscr{I}^+$, related to the shear by $\sigma^0_{\mu\nu} = D_\mu l_\nu - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\rho\sigma}D_\rho l_\sigma$. This description has led to a deeper understanding of the physics of future null infinity, showing among other things that the phase space at $\mathscr{I}^+$ is an affine space (there is no super-translational invariant classical vacuum). The connection description at $\mathscr{I}^+$ inspired and is exactly analogous to the local spacetime connection description studied in this paper.

From the perspective of the $2+2$ characteristic initial-value formulation (with backward evolution – or we should rather say final-value formulation), this means that one can think of $\mathscr{I}^+$ as one of the two null hypersurfaces, but the relevant datum there is not the shear along it (which vanishes!), but the transverse asymptotic shear $-\sigma^0(u,\theta,\phi)$ at varying u, see Fig. 5.2. However, we now show that thanks to the Bianchi identity (5.71), this datum can also be identified as shear of a vector field in the physical spacetime.

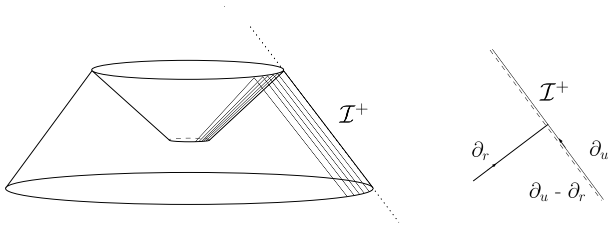


Figure 5.2. *Characteristic initial-value problem at $\mathscr{I}^+$. One prescribes data on a chosen u_0 hypersurface of the foliation attached to future null infinity, plus the asymptotic transverse shear $-s^0(u,\theta,\phi)$. Thanks to the algebraic Bianchi identity (5.71), this can also be understood as prescribing a certain shear for the non-geodetic asymptotic null vector $\partial_u - \partial_r$ in the physical spacetime.*

To that end, consider the second null vector of the tetrad, n^μ. It is null everywhere but non-geodesic, with

$$n^\nu \nabla_\nu n^\mu = -\gamma n^\mu + \nu m^\mu + \text{cc}. \tag{5.90}$$

In the asymptotic expansion,

$$n^\mu \partial_\mu \overset{r\mapsto\infty}{\longrightarrow} \partial_u - \partial_r \tag{5.91}$$

[29] In using these results, care should be taken in that the authors use a slightly different definition of coordinates: u is now $1/\sqrt{2}$ the retarded time, and r is $\sqrt{2}$ the radius of the asymptotically flat 2-sphere.

is leading-order twist-free and affine, but still non-geodesic:

$$\omega_{(n)} := \mathrm{Im}(\mu) = \bar{m}^\mu m^\nu \nabla_\nu n_\mu = O(r^{-2}), \qquad \gamma = O(r^{-2}), \qquad \nu = \frac{\psi_3^0}{r} + O(r^{-2}). \tag{5.92}$$

The non-geodesicity at leading order depends on one of the asymptotic complex projections of the Weyl tensor, in turn given by the radiative data $\psi_3^0 = \delta\dot{\bar{\sigma}}^0$.[30] Since n^μ is not geodesic, it is also not hypersurface orthogonal, in spite of being twist-free at lowest order: the radiative term $\delta\dot{\bar{\sigma}}^0$ prevents the identification of a null hypersurface normal to (5.91) (except in the very special case of completely isotropic radiation at all times). Consequently, there is no unique definition of shear for the congruence it generates. Using the NP formalism, it is natural to consider the shear along the 2d space-like hypersurface spanned by m^μ, and define

$$\sigma_{(n)} := -\lambda = -\bar{m}^\mu \bar{m}^\nu \nabla_\nu n_\mu = \frac{\lambda^0}{r} + O(r^{-2}). \tag{5.93}$$

At the same lowest order $O(r^{-1})$, the algebraic Bianchi identity (5.71) can be solved to give

$$\lambda^0 = \dot{\bar{\sigma}}^0, \tag{5.94}$$

which relates the transverse asymptotic shear to the λ-shear of n^μ. Hence, the radiative data at future null infinity correspond to a shear of a non-geodesic vector field 'aligned' with $\mathscr{I}^+$. The fact that the vector is non-geodesic shows that the asymptotic $2+2$ problem can not be formulated in real spacetime. On the other hand, this is how close one can get, in terms of the interpretation of the main constraint-free data, in bridging between the local 2+2 characteristic initial-value problem, and the asymptotic one.

5.6 Concluding remarks

Here we have presented and discussed many aspects of the canonical structure of General Relativity in real connection variables on null hypersurfaces. We have clarified the geometric structure of the Hamiltonian analysis presented in [Alexandrov and Speziale, 2015], explaining the role of the various constraints and their geometric effect on a null congruence. We have seen how the Lorentz transformations of the null tetrad are generated canonically, and how to restrict them so to adapt the tetrad to a $2+2$ foliation, and compare the connection Hamiltonian analysis to the metric one. Restricting our analysis to the Bondi gauge we have identified constraint-free data in connection variables. The metric canonical conjugated pair 'conformal 2d metric/shear' it is replaced in the first order formalism by a pair 'densitized dyad/ null rotation components of the connection', with the null rotations becoming the shear on-shell of the light-cone secondary simplicity constraints. In the presence of torsion, the connection can pick up additional terms that contribute to the shear, twist and expansion of the congruence, leading to modifications of Sachs' optical and Raychaudhuri's equations.
Even in the absence of torsion, the on-shell-ness is not automatically preserved under

[30] This can be seen solving at first order in $1/r$ the NP components $R_{n\bar{m}nl}$ and $R_{n\bar{m}nm}$ of the Riemann tensor, see e.g. (310i) and (310m) of [Chandrasekhar, 1985].

retarded time evolution, but it requires tertiary constraints, unusual in canonical formulations of GR. We have shown that the tertiary constraints encode Sachs' propagating equations thanks to a specific algebraic Bianchi identity, the same that allows one to switch the interpretation of the radiative data at $\mathscr{I}^+$ from the transverse asymptotic shear σ^0 to the 'shear' λ^0 of a non-geodetic null vector aligned with $\mathscr{I}^+$. The identification of the connection constraint-free data as null rotations means that the degrees of freedom form a group, albeit non-compact, hence one could try to use loop quantum gravity quantization techniques without introducing the Immirzi parameter [**?**]. Some of the corner data, which we did not investigate here, have already be shown to lead to a quantization of the area [Speziale and Zhang, 2014, Wieland, 2017b, Freidel and Perez, 2015]. A quantization of the connection description of the radiative degrees of freedom can lead to new insights both for loop quantum gravity and for asymptotic quantisations based on a Fock space.

We infer that the connection formalism can provide a new angle on some of the open questions on the dynamics of null hypersurfaces in GR. We left some open questions that are interesting to investigate: the symplectic potential and Dirac brackets among physical data without the Bondi gauge, the inclusion of boundary terms and identification of the BMS generators in this Hamiltonian language.

Chapter 6

Conclusion

In this thesis, we discussed different aspects of the derivation of conserved charges in the first order formulation of GR and we explored different formalisms to described such conserved charges. The basic element of our analysis was the Einstein-Cartan action, which describes the gravitational field in terms of tetrads variables and affine connection and could be generalized to the presence of torsional degrees of freedoms. To derive conserved quantities, we first introduced the two common approaches, the Hamiltonian methods which is based on a phase-space splitting and it is not covariant, and the Noether theorem which applies to every Lagrangian density but in this case applied in a subtle way as the symmetries that we are considering are local gauge symmetries. We have then discussed the covariant phase-space formalism and we showed how the Hamiltonian and Noether charges can be derived from the symplectic structures and which are the ambiguities between the different definitions.

The study of these ambiguities have an important role if one consider some sort of boundary degrees of freedom. In particular, it has been shown that the boundary term which breaks the gauge invariance leads to an interesting description in terms of a conformal field theory on the boundary [Wieland, 2017a, Freidel and Perez, 2018, Freidel et al., 2017b]. In this interpretation the redundancy fixed at the boundary is not a gauge invariance but could be interpreted as a physical degree of freedom, for different choice of the boundary gauge fixing we could select different physics.

In this context, our work can be viewed as a further step in understanding more deeply the connection between the gauge invariance of the symplectic potential and the surface charges and we hope that may aid to have a clear distinction between different methods to derive them.

Another interesting analysis that could be carried on starting from our gauge invariant symplectic potential regards the study of BMS charges, the surface charges associated with the symmetries of asymptotically flat four dimensional spacetimes at null infinity. In this direction a first direct implementation of our work is to study how the surface charges defined in finite region trough the covariant-phase space formalism can have a correspondence with the BMS surface charges. More in the specific, one could ask if the extension from Poincare charges to BMS charges at null infinity is just a consequence of the asymptotically-flat limit or could have some physical consequence also in not idealized (not isolated) systems.

In the analysis of conserved charges an appealing study is the thermodynamical interpretation of gravitational field, which is based on the derivation of a conserved energy-momentum tensor off-shell of the fields equations. We treated this problem in the tetrad and connection variables starting from the Jacobson analysis [Jacobson, 1995], to see which were the main differences respect to the metric formulation and if they could lead to an interesting insight of the first order tetrad formulation. Two interesting features arised in this case: the first one concerns derivation of the first law of thermodynamics for BH, while the second one was related to find a proper conservation law for the energy-momentum tensor valid also not on-shell of the equations of motion. The first issue is related with previous analysis of the symplectic potential as to prove the first law we needed an energy-momentum tensor. In this analysis we also pointed out that the derivation of the first law from the two tetrad symplectic potential is equivalent, i.e it is invariant under the cohomology ambiguity in the symplectic potential. This statement was necessary as we were considering a specific physical application, that do not depend on the reformulation of the theoretical frameworks. In particular we proved that it suffices to take into account the non-trivial Lorentz charges that can be present to make the two formulations equivalent.
For the thermodynamical argument, on the other hand, one needs to identify a conserved energy-momentum tensor without using the field equations, since these are to be derived. This analysis was more intriguing as we were able to identify a conserved energy tensor, conserved not on-shell of the Einstein equations, for the pure gravitational field, but we weren't able to find a conserved tensor for the torsion fields. This remains a crucial open question in order to truly extend if Jacobson?s argument could be extend to theories with independent metric and connection and to fully understand the thermodynamical interpretation of the gravitational fields.
In the conclusive part of the dissertation we addressed the problem of the identification of degrees of freedom selected by the boundary terms in the first order formalism using the tetrad and connection variables on a null foliation. These variables has a central role in this analysis as they allows to easily re-write the fields equations in terms of the Newman-Penrose spin-coefficients and the Weyl tensor, which has a well-known physical interpretation in the study of gravitational radiation.The use of null hypersurfaces is crucial to Sachs' identification of constraint-free data in metric variables on a double null foliations, which is the only case where one can exactly identify the non-perturbative physical degrees of freedom without imposing any symmetries on the physical spacetime.
The problem of the identification of free-initial data is also a well-known problem in numerical relativity. Current simulations still have some limitations: for example the initial data present the problem of "junk" radiation that can become relevant in cases where either the linear momentum or the spin of the black holes are high. This effect has been shown to affect considerably, for example, the high energy black hole collisions [Cook, 2000], requiring a specific grid construction to overcome this difficulty. As consequence a better knowledge on how to improve the initial data construction would have a great impact for the gravitational wave modelling as will improve a more the precision of waveforms produced in numerical simulations.
This study has an important consequence also for approaches to quantum gravity. In loop quantum gravity for example, there exists a compelling kinematical picture of quantum spacetime, where the smooth manifold of GR is replaced by a collection of

quanta of space, and geometric operators with discrete spectra and non-commutativity properties. This picture holds at the kinematical level, namely prior to the imposition of the quantum version of the Hamiltonian diffeomorphism constraints, and it is not proved that the same quantum geometry would also describe the physical Hilbert space of the theory, defined on-shell on the constraints.

Appendix I

A.1 Exterior calculus of forms

On a differentiable manifold, the exterior derivative extends the concept of the differential of a function to differential forms of higher degree and it allows for a natural, metric-independent generalization of Stokes' theorem, Gauss' theorem, and Green's theorem from vector calculus.
The exterior derivative of a differential form of degree k is a differential form of degree k+1. Consider a smooth function f (a 0-form), then the exterior derivative of f is the differential of f, df . This is the unique 1-form such that for every smooth vector field X, $df(X) = d_X f$, where $d_X f$ is the directional derivative of f in the direction of X and for any 0-form (smooth function) f it satisfies $d(df) = 0$. These properties can be extended to any k-form α and more generally we can define the exterior derivative as the unique linear mapping from k-forms to (k + 1)-forms satisfying the following properties:

1. df is the differential of f, for 0-forms f (smooth functions);

2. $d(d\alpha) = 0$ for any k-form α;

3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k(\alpha \wedge d\beta)$.

The main advantage of the exterior calculus form is that it allows to work in an arbitrary manifold without worrying of the coordinate system's choice. Of course one can always come back to work a local coordinate system $(x_1, ..., x_n)$, where the coordinate differentials $dx_1, ..., dx_n$ form a basis of the space of one-forms, each associated with a coordinate. Using the one-forms basis and denoting $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ with an abuse of notation as dx^I (where I is a multi-index $I = (i_1, ..., i_k)$) the exterior derivative of a k-form ω can be expressed as

$$d\omega = \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I. \tag{A.1}$$

In particular, for a 1-form ω, the components of $d\omega$ in local coordinates are

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i. \tag{A.2}$$

Appendix II

A.2 Index jugglers

In this Appendix we prove Proposition 1, namely that the matter Noether identities (3.27) on-shell of the matter field equations, plus the torsion field equation (3.12b), imply the conservation law for the effective energy-momentum tensor (3.29), reported here for convenience

$$d_{\omega(e)}\left[\star\tau_I+\frac{1}{16\pi}P_{IJKL}(e^J\wedge d_{\omega(e)}C^{KL}+e^J\wedge C^{KM}\wedge C_M{}^L)\right]=0, \tag{A.1}$$

namely,

$$d_{\omega(e)}\star\tau_I=\frac{1}{8\pi}P_{IJKL}\Big(e^J\wedge C^K{}_M\wedge F^{ML}(e)+e^J\wedge d_{\omega(e)}C^{KM}\wedge C_M{}^L\Big). \tag{A.2}$$

To prove this identity, we start from (3.27b). On the left-hand side, we split the connection into Levi-Civita plus contorsion, see (3.13), obtaining

$$d_\omega\star\tau_I=d_{\omega(e)}\star\tau_I-(C^{JK}\lrcorner e_I)e_K\wedge\star\tau_J+T^J\lrcorner e_I\wedge\star\tau_J \tag{A.3}$$

where we used

$$T^I=C^{IJ}\wedge e_J\quad\rightarrow\quad C_I{}^J=-(C^{JK}\lrcorner e_I)e_K+T^J\lrcorner e_I. \tag{A.4}$$

In the second term of the right-hand side of (A.3) we use the second Noether identity (3.27a), whereas the last term cancels the corresponding one on the right-hand side of (3.27b), which then reads

$$\begin{aligned}d_{\omega(e)}\star\tau_I&=\frac{1}{2}F^{JK}(\omega)\lrcorner e_I\wedge\star\sigma_{JK}-\frac{1}{2}(C^{JK}\lrcorner e_I)d_\omega\star\sigma_{JK}\\&=\frac{1}{16\pi}\left[F^{JK}(\omega)\lrcorner e_I\wedge-(C^{JK}\lrcorner e_I)d_\omega\right]P_{JKLM}e^L\wedge C^M{}_N\wedge e^N\\&=\frac{1}{16\pi}\left[\big(F^{JK}(e)+d_{\omega(e)}C^{JK}\big)\lrcorner e_I\wedge-(C^{JK}\lrcorner e_I)d_{\omega(e)}\right]P_{JKLM}e^L\wedge C^M{}_N\wedge e^N.\end{aligned} \tag{A.5}$$

In the second equality above we eliminated the torsion source using the corresponding field equation (3.17). In the third equality we expanded the curvature using the contorsion, see (3.15), and observed that the piece quadratic in C cancels the contorsion part of the exterior derivative in the last term.[1]

[1] Following the same steps but without eliminating the torsion source in favour of the contorsion one gets (3.30) in the main text, which does not use the torsion field equations but only the Noether identities.

Having performed these simplifications, our goal is to show the equivalence of the right-hand sides of (A.2) and (A.5). This will follow from the equivalence of the terms with the Riemann tensor $F^{IJ}(e)$, and the equivalence of the terms involving the Levi-Civita exterior derivatives. Both are consequences of trivial algebraic symmetries. Let us show them one by one. We notice in advance the following useful cycling identities:

$$P_{IJKL} e^K \wedge F^{LM} \wedge e_M = -P_{ABC[I} e^A \wedge F^{BC} \wedge e_{J]}, \tag{A.6}$$

$$P_{IJKL} F^{KM} \wedge C_M{}^L = -P_{ABC[I} F^{AB} \wedge C^C{}_{J]}, \tag{A.7}$$

which are easy to check.

To show the equivalence of the terms with the curvature, we start hooking a cotetrad vector field on a trivially vanishing 5-form,

$$\begin{aligned}
0 &= \Big(P_{JKLM} F^{JK}(e) \wedge e^L \wedge C^M{}_N \wedge e^N\Big) \lrcorner e_I \\
&= P_{JKLM} F^{JK}(e) \lrcorner e_I \wedge e^L \wedge C^M{}_N \wedge e^N + P_{JKIM} F^{JK}(e) \wedge C^M{}_N \wedge e^N \\
&\quad - P_{JKLM} (C^M{}_N \lrcorner e_I) F^{JK}(e) \wedge e^L \wedge e^N + P_{JKLM} F^{JK}(e) \wedge e^L \wedge C^M{}_I.
\end{aligned} \tag{A.8}$$

Of these four terms, the third vanishes identically: its $1/\gamma$ part directly through the algebraic Bianchi identities for the Riemann tensor, the other part because of the antisymmetry in the *LP* indices. The second and fourth terms recombine giving the left-hand side of (A.7), hence (A.8) gives

$$2P_{IJKL} F^{KM}(e) \wedge C_M{}^L \wedge e^J = P_{JKML} \left(F^{JK}(e) \lrcorner e_I\right) \wedge e^L \wedge C^M{}_N \wedge e^N, \tag{A.9}$$

which proves the equality of the curvature terms of (A.2) and (A.5).

The equivalence of the $d_{\omega(e)} C$ terms follows analogously. We hook the following 5-form,

$$\begin{aligned}
0 &= \Big(P_{JKLM} C^{JK} \wedge e^L \wedge d_{\omega(e)} C^M{}_N \wedge e^N\Big) \lrcorner e_I \\
&= P_{JKLM} (C^{JK} \lrcorner e_I) e^L \wedge d_{\omega(e)} C^M{}_N \wedge e^N - P_{JKIM} C^{JK} \wedge d_{\omega(e)} C^M{}_N \wedge e^N \\
&\quad + P_{JKLM} C^{JK} \wedge e^L \wedge d_{\omega(e)} C^M{}_N \lrcorner e_I \wedge e^N + P_{JKLM} C^{JK} \wedge e^L \wedge d_{\omega(e)} C^M{}_I.
\end{aligned} \tag{A.10}$$

Using an identity like (A.7), the second and fourth term give

$$P_{JKM[I} C^{JK} \wedge d_{\omega(e)} C^M{}_{N]} \wedge e^N = -2P_{IJKL} \wedge e^J \wedge d_{\omega(e)} C^K{}_M \wedge C^{ML}. \tag{A.11}$$

For the third term we have

$$\begin{aligned}
P_{JKLM} C^{JK} \wedge e^L \wedge d_{\omega(e)} C^M{}_N \lrcorner e_I \wedge e^N &= -P_{JKLM} d_{\omega(e)} C^{MN} \lrcorner e_I \wedge e^L \wedge C^{JK} \wedge e_N \\
&= P_{JKLM} d_{\omega(e)} C^{JK} \lrcorner e_I \wedge e^L \wedge C^M{}_N \wedge e^N,
\end{aligned} \tag{A.12}$$

which follows from a similar cycling identity as before. Hence, (A.10) gives

$$\begin{aligned}
2P_{IJKL} \wedge e^J \wedge d_{\omega(e)} C^K{}_M \wedge C^{ML} &= P_{JKLM} d_{\omega(e)} C^{JK} \lrcorner e_I \wedge e^L \wedge C^M{}_N \wedge e^N \\
&\quad + P_{JKLM} (C^{JK} \lrcorner e_I) e^L \wedge d_{\omega(e)} C^M{}_N \wedge e^N,
\end{aligned} \tag{A.13}$$

which proves precisely the equivalence between the $d_{\omega(e)} C$ terms in (A.2) and (A.5).

Appendix III

A.3 Covariant Phase Space Methods

In the covariant phase-space methods one defines the symplectic potential form in the field space as the integral over the hypersurface Σ:

$$\Theta(\delta) := \int_\Sigma \theta(\delta) \tag{A.1}$$

of the boundary term $\Theta(\delta)$, obtained by varying the Lagrangian. The variation δ represents a vector field over the field space, hence $\Theta(\delta) \equiv I_\delta \Theta$ is understood as the inner product between a one form and a vector field. The pre-symplectic 2-form is the exterior derivative in field space of the symplectic potential and it is denoted by $\Omega = \delta\theta$. It can be written in terms of standard functional differentials as follows,

$$\Omega(\delta_1, \delta_2) \,=\, \delta_1[\Theta(\delta_2)] - \delta_2[\Theta(\delta_1)] - \Theta[\delta_1, \delta_2]. \tag{A.2}$$

This quantity depends a priori on the hypersurface Σ chosen to evaluate the integrals, but it can be easily shown to be closed in spacetime, $d\Omega \sim 0$, if the fields and their linear variations are on shell.

Having a pre-symplectic form at disposal, one can look for Hamiltonian generator associated with a symmetry δ_ε as in classical mechanics, starting from

$$\delta H_\varepsilon := \Omega(\delta, \delta_\varepsilon) \,=\, \delta[\Theta(\delta_\varepsilon)] - \delta_\varepsilon[\Theta(\delta)] - \Theta[\delta, \delta_\varepsilon]\,. \tag{A.3}$$

The slashed delta used in this definition is meant to highlight that the right hand side is not necessary a total function variation. When it is, we say that the expression is integrable, and refer to H_ε as the Hamiltonian generator. It would make sense to refer the generic expression (A.3) as pseudo-generator. However this distinction is rarely made, and on the contrary some literature loosely refers to (A.3) as Hamiltonian generator, even tough this makes sense only in integrable case. By inspection of (A.3), we see that

$$\delta_e ps[\Theta(\delta)] = 0, \qquad [\delta, \delta_\varepsilon] = 0, \tag{A.4}$$

are sufficient conditions for integrability. A necessary and sufficient condition is

$$\int_{\partial\Sigma} i_\varepsilon \omega(\delta_1, \delta_2) = 0, \tag{A.5}$$

where $\omega(\delta_1, \delta_2)$ is the integrand of (A.2).

It can be shown by explicit calculation that for a gauge and diffeomorphism symmetries the integral of (A.3) is exact. The generator is then a surface integral, if Σ has a signle boundary, and this is referred to as Hamiltonian charge.

Appendix IV

A.4 Spin coefficients

We use $\chi^i = (1,0,0)$ and $v^{\pm} := (v^2 \pm iv^3)/\sqrt{2}$. For the tetrad derivatives we have

$$D = l^\mu \nabla_\mu = \frac{1}{N} E_1^a \nabla_a, \qquad \Delta = n^\mu \nabla_\mu = \frac{1}{2}(\nabla_t - (N^a + \frac{N}{2} E_1^a)\nabla_a), \tag{A.1}$$

$$\delta = m^\mu \nabla_\mu = E_-^a \nabla_a, \qquad \bar{\delta} = m^\mu \nabla_\mu = E_+^a \nabla_a\,. \tag{A.2}$$

For the spin coefficients we use the standard notation consistent with our mostly plus signature (which carries an opposite sign as to the notation with mostly minus signature) and use an apex $\circ$ to keep track of the fact that the connection ω_μ^{IJ} is off-shell. We then have

$$\overset{\circ}{\alpha} := -\frac{1}{2}(n^\mu \bar{\delta} l_\mu + m^\mu \bar{\delta} \bar{m}_\mu) = \frac{1}{2} E_+^a \eta_a^1 - \frac{i}{2} r_{1+} - \frac{1}{4}\omega^+ - \frac{1}{2}\bar{\delta} \ln N \tag{A.3}$$

$$\overset{\circ}{\beta} := -\frac{1}{2}(n^\mu \delta l_\mu + m^\mu \delta \bar{m}_\mu) = \frac{1}{2} E_-^a \eta_a^1 - \frac{i}{2} r_{1-} + \frac{1}{4}\omega^- - \frac{1}{2}\delta \ln N \tag{A.4}$$

$$\overset{\circ}{\gamma} := -\frac{1}{2}(n^\mu \Delta l_\mu + m^\mu \Delta \bar{m}_\mu) = -\frac{1}{4} N E_1^a \eta_a^1 - \frac{1}{2} N^a \eta_a^1 + \frac{i}{4} N r_{11} + \frac{i}{2} N^a E_a^j r_{1j} + \frac{i}{4} N^a (E_a^l \varepsilon_{1lm} \omega^m) + \tag{A.5}$$

$$+ \frac{1}{2}(\omega_0^{01} - i\omega_0^{23}) - \frac{1}{2}\Delta \ln N$$

$$\overset{\circ}{\varepsilon} := -\frac{1}{2}(n^\mu D l_\mu + m^\mu D \bar{m}_\mu) = \frac{1}{2N} E_1^a \eta_a^1 - \frac{i}{2N} r_{11} - \frac{1}{2} D \ln N \tag{A.6}$$

$$\overset{\circ}{\kappa} := -m^\mu D l_\mu = -\frac{1}{N^2} E_1^a \eta_a^- \tag{A.7}$$

$$\overset{\circ}{\tau} := -m^\mu \Delta l_\mu = \frac{1}{2} E_1^a \eta_a^- + \frac{N^a}{N}\eta_a^- - \frac{\sqrt{2}}{N}(\omega_0^{0-} - \omega_0^{1-}) \tag{A.8}$$

$$\overset{\circ}{\sigma} := -m^\mu \delta l_\mu = -\frac{1}{N} E_-^a \eta_a^- \tag{A.9}$$

$$\overset{\circ}{\rho} := -m^\mu \bar{\delta} l_\mu = -\frac{1}{N} E_+^a \eta_a^- \tag{A.10}$$

$$\overset{\circ}{\mu} := \bar{m}^\mu \delta n_\mu = \frac{1}{2} N\Big(E_-^a \eta_a^+ - \omega^1 - i r_{22} - i r_{33}\Big) \tag{A.11}$$

$$\mathring{\nu} := \bar{m}^\mu \Delta n_\mu = -\frac{N}{4}(NE_1^a + 2N^a)\eta_a^+ - \frac{(N)^2}{2}(\omega^+ - ir_{1-}) - \frac{N}{2}N^a(E_a^1\omega^+ - \tag{A.12}$$
$$+ E_a^+\omega^1 - 2iE_a^i r_{i+}) + \frac{1}{2\sqrt{2}}N(\omega_0^{0+} + \omega_0^{1+})$$

$$\mathring{\lambda} := \bar{m}^\mu \bar{\delta} n_\mu = \frac{1}{2}N\Big(E_+^a\eta_a^+ + 2r_{23} - ir_{22} + ir_{33}\Big) \tag{A.13}$$

$$\mathring{\pi} := \bar{m}^\mu D n_\mu = \frac{1}{2}E_1^a\eta_a^+ + \frac{1}{2}\omega^+ - ir_{1-} \tag{A.14}$$

Under the rescaling $(l^\mu, n^\mu) \mapsto (l^\mu/A, An^\mu)$ (a class III transformation),

$$\alpha \mapsto \alpha - \frac{1}{2A}\bar{\delta}A, \qquad \beta \mapsto \beta - \frac{1}{2A}\delta A, \qquad \gamma \mapsto A\gamma - \frac{1}{2A}\Delta A, \qquad \varepsilon \mapsto \frac{1}{A}\varepsilon - \frac{1}{2A}DA, \tag{A.15}$$

$$k \mapsto \frac{1}{A^2}k, \qquad \tau \mapsto \tau, \qquad \sigma \mapsto \frac{1}{A}\sigma, \qquad \rho \mapsto \frac{1}{A}\rho, \qquad \mu \mapsto A\mu, \tag{A.16}$$

$$\nu \mapsto A^2\nu, \qquad \lambda \mapsto A\lambda, \qquad \pi \mapsto \pi, \tag{A.17}$$

Hence, many factors of N disappear in the spin coefficients if we use the ADM-like normal $l_\mu^{\mathrm{ADM}} = -N\partial_\mu u$.

A.5 Congruence

The complete expression of the congruence tensor with an affine connection is

$$\nabla_0 l_0 = \omega_0^{0i}(\frac{1}{N}\mathscr{X}_{ij}E_a^j N^a - \chi_i) + \frac{1}{N}\omega_0^{ij}\chi_j E_a^i N^a + \frac{1}{N}\partial_0 N, \qquad \nabla_a l_b = \frac{1}{N}\mathscr{X}_{ij}\eta_a^i E_b^j,$$
$$\nabla_0 l_a = \frac{1}{N}(\omega_0^{0j}\mathscr{X}_{ij}E_a^i + \omega_0^{ij}\chi^j E_a^i), \qquad \nabla_a l_0 = \eta_a^i(\frac{1}{N}\mathscr{X}_{ij}N^a E_a^j - \chi^i) + \partial_a \ln N \tag{A.18}$$

with projection $B_{\mu\nu} = \perp^\rho{}_\mu \perp^\sigma{}_\nu \nabla_\rho l_\sigma$ given by

$$B_{00} := \frac{1}{N}q^c{}_b N^a N^b \eta_a^M E_c^M, \qquad B_{0a} = \frac{1}{N}q^b{}_c N^c \eta_b^M E_a^M, \qquad B_{a0} = \frac{1}{N}q^b{}_a N^c \eta_b^M E_c^M,$$
$$B_{ab} = \frac{1}{N}q^c{}_a q^d{}_b \mathscr{X}_{ij}\eta_a^i E_b^j. \tag{A.19}$$

A.6 Tetrad transformations and gauge fixings

At the Hamiltonian level, the Lorentz transformations are generated by the Gauss constraint $\mathscr{G}_{IJ}$, usually decomposed into spatial rotations L_i and boosts K_i, whose canonical form from (5.26) reads

$$L_i := \frac{1}{2}\varepsilon_{ijk}\mathscr{G}^{jk} = \partial_a(\varepsilon_{ijk}\widetilde{E}_j^a\chi^k) - \varepsilon_{ij}{}^k\eta_a^j\widetilde{E}_k^a - \varepsilon_{ij}{}^k\tilde{\omega}^j\chi_k,$$
$$K_i := \mathscr{G}_{0i} = \partial_a\widetilde{E}_i^a + (\widetilde{E}_i^a\chi_j - \widetilde{E}_j^a\chi_i)\eta_a^j - \mathscr{X}_{ij}\tilde{\omega}^j. \tag{A.20}$$

Since we are working on a null hypersurface, it is convenient to introduce the subgroups ISO(2) stabilising the null directions $x^I_\pm = (\pm 1, \chi^i)$, with generators $T^I := 1/2\varepsilon^{IJKL}x_{+J}J_{KL}$ and $\hat{T}^I := -1/2\varepsilon^{IJKL}x_{-J}J_{KL}$. Both groups are 3-dimensional and contain the helicity generator L_χ, plus two independent pairs of 'translations', $T^i_\perp := \varepsilon^{ijk}\chi_j T_k$ stabilising x^I_+, and $\hat{T}^i_\perp := \varepsilon^{ijk}\chi_j \hat{T}_k$ stabilising x^I_-. Taking both sets and the radial boost K_χ we obtain the complete the Lorentz algebra, expressed in terms of canonical variables in (5.56).

For ease of notation and to make the formulas more transparent, we fix from now on $\chi^i = (1,0,0)$, as we did in most of the main text. We use the orthogonal internal indices $M = 2,3$, and write the canonical form of the generators as follows,

$$\begin{aligned} L_1 &= \varepsilon_{1MN}\widetilde{E}^{aM}\eta^N_a, & T_M &= -\varepsilon_{1Mi}\widetilde{E}^a_1\eta^i_a \\ K_1 &= \partial_a\widetilde{E}^a_1 - \widetilde{E}^a_M\eta^M_a, & \hat{T}_M &= -\varepsilon_{1Mi}(\widetilde{E}^a_1\eta^i_a - 2\partial_a\widetilde{E}^a_i - 2\widetilde{E}^a_i\eta^1_a + 2\tilde{\omega}^i). \end{aligned} \tag{A.21}$$

To compute the action on the tetrad, we use the brackets (5.29). First of all, $\hat{T}_M$ change the internal null direction χ^i:

$$\{\hat{T}_M, \chi_N\} = -\varepsilon_{1MN}. \tag{A.22}$$

Since the direction is gauge-fixed by (5.28) in the action, these constraints are second class.

The stabilisers T_M are first class, and can be used to put the triad in (partially) lower triangular form:

$$\{T_M, \widetilde{E}^a_i\} = -\tfrac{1}{2}\varepsilon_{1Mi}\widetilde{E}^a_1, \tag{A.23}$$

so we can always reach $E^r_M = 0$ with these transformations, and $E^1_A = 0$ follows from the invertibility of the triad. The radial boost K_χ can be used to fix $E^r_1 = 1$, since

$$\{K_1, \widetilde{E}^r_1\} = 0, \qquad \{K_\chi, E\} = \tfrac{1}{2}E, \qquad \{K_\chi, E^r_1\} = -\tfrac{1}{2}E^r_1. \tag{A.24}$$

The triad so gauge-fixed reads

$$E^i_a = \begin{pmatrix} 1 & 0 \\ E^M_r & E^M_A \end{pmatrix}, \qquad E^a_i = \begin{pmatrix} 1 & 0 \\ E^A_1 & E^A_M \end{pmatrix}, \tag{A.25}$$

where E^M_A is the 2d dyad with inverse E^A_M, and $E^A_1 = -E^A_M E^M_r$. In this gauge, $d\phi^1 = dr$, so the coordinates are adapted to the $2+2$ foliation. Furthermore, $E = \sqrt{\gamma}$ and so $\sqrt{-g} = NE = N\sqrt{\gamma}$. Finally, the helicity rotation L_1, acting as

$$\{L_1, \widetilde{E}^a_i\} = \tfrac{1}{2}\varepsilon_{1Mi}\widetilde{E}^a_M, \tag{A.26}$$

can be used to put to zero one off-diagonal component of the dyad and thus complete the triangular gauge of the triad.

Using hypersurface diffeomorphisms instead, we can put the triad in (partially) upper-triangular form:

$$\mathscr{D}_a = 2\partial_b(\eta^i_a\widetilde{E}^b_i) - 2\widetilde{E}^b_i\partial_a\eta^i_b + 2\tilde{\omega}^i\partial_a\chi_i, \qquad \{D(\vec{N}), \widetilde{E}^a_i\} = \pounds_{\vec{N}}\widetilde{E}^a_i, \tag{A.27}$$

so we can use $\mathscr{D}_A$ to fix $E^A\chi = 0$, and $\mathscr{D}_r$ to fix $E^r_1 = 1$. This gives

$$E^i_a = \begin{pmatrix} 1 & E^1_A \\ 0 & E^M_A \end{pmatrix}, \qquad E^a_i = \begin{pmatrix} 1 & E^r_M \\ 0 & E^A_M \end{pmatrix}, \tag{A.28}$$

with $E^r_M = -E^A_M E^1_A$. In this gauge the hypersurface coordinates are not adapted to the $2+1$ foliation (the level sets $r=$constant do not span the 2d space-like surfaces), on the other hand the tangent to the null directions is now the coordinate vector ∂_r.

For clarity, the various conditions that can be fixed using the various constraints are summarised in the table below, where by r_{gf} we mean the final gauge fixing on r, for instance affine or areal.

$\mathscr{H}$ $g^{00}=0$	$\mathscr{D}_A$ $E^A\chi = 0 \Leftrightarrow E^M_r = 0$	$\mathscr{D}_r$ $E^r\chi = 1$ aut r_{gf}	
$\hat{T}^i_\perp$ $\chi^i = (1,0,0)$	$T^i_\perp$ $E^r_M = 0 \Leftrightarrow E_A\chi = 0$	K_χ r_{gf} aut $E^r\chi = 1$	L_χ δ

Notice that if one does not fix the upper or lower triangular form of the triad, the inverse of the 2d dyad if of course not given by the corresponding entries of the inverse triad. A general parametrisation of the triad in terms of the dyad can be easily written as follows,

$$E^i_a = \begin{pmatrix} \hat{M} & \mathscr{E}^M_A f_M \\ \mathscr{E}^M_A \gamma^A & \mathscr{E}^M_A \end{pmatrix}, \qquad E^a_i = \frac{1}{M}\begin{pmatrix} 1 & -f_M \\ -\gamma^A & M\mathscr{E}^A_M + \gamma^A f_M \end{pmatrix}. \tag{A.29}$$

Here $\mathscr{E}^M_A$ is the dyad and $\mathscr{E}^A_M$ its inverse, $E = \mathscr{E}M$ and $M = \hat{M} - \gamma^A \mathscr{E}^M_A f_M$ is a $2+1$ lapse function. Then $\gamma_{AB} = \mathscr{E}^M_A \mathscr{E}_{MB}$ and

$$q_{ab} = \begin{pmatrix} \gamma_{AB}\gamma^A\gamma^B & \gamma_{AB}\gamma^B \\ \gamma_{BA}\gamma^A & \gamma_{AB} \end{pmatrix}. \tag{A.30}$$

The Bondi gauge sets $\gamma^A = 0$, namely $q_{ra} = 0$.

A.7 $2+2$ foliations and NP tetrads

We collect here various useful formulas relating the tetrad formalism to the $2+2$ foliation of [d'Inverno and Smallwood, 1980] and [Torre, 1986]. As briefly explained in Section 5.2.2, the $2+2$ foliation is induced by two closed 1-forms, $n^\alpha := d\phi^\alpha$ locally, $\alpha = 0,1$. These define a 'lapse matrix' $N_{\alpha\beta}$, as the inverse of $N^{\alpha\beta} := n^\alpha_\mu n^{\beta\mu}$, and a dual basis of vectors $n^\mu_\alpha := N_{\alpha\beta} g^{\mu\nu} n^\beta_\nu$. Note that n^μ_0 and n^μ_1 are tangent respectively to the hypersurfaces $\phi^1 = const$ and $\phi^0 = const$. We assume $\det N_{\alpha\beta} < 0$, so that the codimension-2 leaves $\{S\}$ are space-like. The projector on $\{S\}$ is $\perp^\mu{}_\nu := \delta^\mu_\nu - N_{\alpha\beta} n^{\alpha\mu} n^\beta_\nu$, and the covariant induced metric $\gamma_{\mu\nu} := \perp_{\mu\nu}$. The 2d spaces $\{T\}$ tangent to n^μ_α are not integrable in generic spacetimes, since $\perp^\mu{}_\nu [n_0, n_1]^\nu \neq 0$. This non-integrability is often referred to as twist in the literature. On the other hand, the orthogonal 2d spaces foliate spacetime by construction, and we can introduce shift vectors to relate the tangent vectors to coordinate vectors, $b^\mu_\alpha = (\partial_{\phi^\alpha})^\mu - n^\mu_\alpha$.

To write the metric explicitly, we take coordinates (ϕ^α, σ^A) adapted to the foliation, then

$$g_{\mu\nu} = \begin{pmatrix} N_{\alpha\beta} + \gamma_{AB} b^A_\alpha b^B_\beta & \gamma_{BC} b^C_\alpha \\ \gamma_{AC} b^C_\beta & \gamma_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} N^{\alpha\beta} & -N^{\alpha\beta} b^B_\beta \\ -N^{\beta\alpha} b^A_\alpha & \gamma^{AB} + N^{\alpha\beta} b^A_\alpha b^B_\beta \end{pmatrix}. \tag{A.31}$$

For a null foliation, we fix one diffeomorphism requiring $N_{11} = 0 = N^{00} = g^{00}$, so that the first normal is null, and $N^{01} = 1/N_{01}$, $N^{11} = -N_{00}/N_{01}^2$. The norm of n^1 is N^{11} and we leave it free (it can be both time-like or space-like without changing the fact that the orthogonal spaces $\{S\}$ are space-like), but notice that we can always switch to a null frame $(n^0, \tilde{n}^1)$ with

$$\tilde{n}^1 = N_{01} n^1 + \frac{1}{2} N_{00} n^0, \qquad \|\tilde{n}^1\|^2 = 0, \qquad n^0_\mu \tilde{n}^{1\mu} = 1. \tag{A.32}$$

This can be used to define the first two vectors of a NP tetrad adapted to the foliation, via $l_\mu := -n^0_\mu$, $n_\mu := \tilde{n}^1_\mu$, so that the 2d space-like induced metrics coincide

$$\gamma_{\mu\nu} = g_{\mu\nu} - N_{\alpha\beta} n^\alpha_\mu n^\beta_\nu = g_{\mu\nu} + 2 l_{(\mu} n_{\nu)}. \tag{A.33}$$

Notice that acting with a Lorentz transformation preserving l, we have

$$n^\mu \mapsto n^\mu + \bar{a} m^\mu + a \bar{m}^\mu + |a|^2 l^\mu, \qquad m^\mu \mapsto m^\mu + a l^\mu; \tag{A.34}$$

one thus obtains a new covariant 2d metric, still space-like and transverse to l_μ, but not associated with the $2+2$ foliation any longer. In terms of the NP tetrad, the non-integrability of the time-like spaces is measured by the two spin coefficients τ and π,

$$m_\mu [l, n]^\mu = \tau + \bar{\pi}. \tag{A.35}$$

A.7.1 Adapting a NP tetrad

We can also reverse the procedure: start from an arbitrary NP tetrad, and adapt it to a $2+2$ foliation. To that end, recall first that

$$[m, \bar{m}]^\nu = (\mu - \bar{\mu}) l^\mu + (\rho - \bar{\rho}) n^\mu - (\alpha - \bar{\beta}) m^\mu + (\bar{\alpha} - \beta) \bar{m}^\mu, \tag{A.36}$$

so the general non-integrability of $(m, \bar{m})$ is given by non-vanishing $\mathrm{Im}(\rho)$ and $\mathrm{Im}(\mu)$. To adapt the NP to the $3+1$ null foliation, we choose $l := -d\phi^0$. This fixes 3 Lorentz transformation, and implies $\mathrm{Im}(\rho) = 0 = \kappa$ and $\tau = \bar{\alpha} + \beta$. We can also fix the SO(2) helicity rotation requiring $\varepsilon = \bar{\varepsilon}$. This leaves us with two tetrad transformations left. To have a $2+2$ foliation induced by the tetrad, we need

$$\mu - \bar{\mu} = 2 m^\mu \bar{m}^\nu \partial_{[\nu} n_{\mu]} = 0. \tag{A.37}$$

This is achieved if in coordinates (ϕ^α, σ^A) adapted to the foliation $m^\mu = (0, 0, m^A)$, hence $n_\mu = (c_\alpha, 0, 0)$ by orthogonality; this fixes the remaining two tetrad freedoms (And if we fix radial diffeomorphisms to have $N^{01} = -1$, this gauge also implies $\pi = \alpha + \bar{\beta}$). Inverting this linear system we find

$$d\phi^0 = -l, \qquad d\phi^1 = \frac{c_0}{c_1} l + \frac{1}{c_1} n. \tag{A.38}$$

This identifies $c_\alpha = (N_{00}/2, N_{01})$, and (A.33) follows again. For more on the characteristic initial value problem in NP formalism see e.g. [Rácz, 2014]. The use of a tetrad adapted to a $2+2$ foliation is common, e.g. [Sachs, 1962a, Hawking, 1968, Mädler and Winicour, 2016], but not universal. In particular in [Sachs, 1962a] the partial Bondi gauge is completed with $N^{11} = 0 = N_{00} = c_0$, so to have both 1-forms $d\phi^\alpha$ null.

A.7.2 The Bondi gauge and Newman-Unti tetrad

A more wide-spread tetrad description, particularly suited to study asymptotic radiation, is the one introduced by Newman and Unti [Newman and Unti, 1962], see e.g. [Newman and Tod, 1981, Adamo et al., 2009] for reviews, which is adapted to the $3+1$ null foliation and to the Bondi gauge. We take coordinates (u,r,θ,ϕ) and fix $g^{00}=0$, so that the level sets of u give a null foliation with normal $l_\mu = -\partial_\mu u$. Recall that the null hypersurfaces Σ normal to l_μ are ruled by null geodesics, with tangent vector

$$l^\mu \partial_\mu = -g^{0\mu}\partial_\mu = \frac{1}{N}\partial_r - g^{0A}\partial_A. \tag{A.39}$$

This suggests a natural $2+1$ foliation of Σ given by the level sets of a parameter along the null geodesics (affine or not). The description simplifies greatly if we gauge-fix $g^{0A}=0$, as to identify the geodesic parameter with the coordinate r, while simultaneously putting to zero the shift vector of the $r = const.$ foliation on Σ. In other words, the (partial) Bondi gauge $g^{0A}=0$ gives a physical meaning to the coordinate foliation defined by u and r by identifying it with the foliation defined by the null geodesics on Σ. In the $2+2$ language of [d'Inverno and Smallwood, 1980, Torre, 1986], with adapted coordinate $\phi^0 = u$, the gauge corresponds to a vanishing shift vector b^μ_1, so that ∂_{ϕ^1} is tangent to the null geodesics.

Let us complete the Bondi gauge choosing affine parametrization, namely $g^{01}=-1$. The metric and its inverse read

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ & g^{11} & g^{1A} \\ & & g^{AB} \end{pmatrix}, \qquad g_{\mu\nu} = \begin{pmatrix} -g^{11} + g_{AB}g^{1A}g^{1B} & -1 & g_{AB}g^{1B} \\ & 0 & 0 \\ & & g_{AB} \end{pmatrix}. \tag{A.40}$$

The Newman-Unti tetrad adapted to these coordinates is chosen identifying l_μ with the normal to the foliation, and requiring n^μ and m^μ to be parallel propagated along l^μ. It is parametrised as follows,

$$l^\mu\partial_\mu = \partial_r, \qquad n^\mu\partial_\mu = \partial_u + U\partial_r + X^A\partial_A, \qquad m^\mu\partial_\mu = \omega\partial_r + \xi^A\partial_A, \tag{A.41}$$

with $A=\zeta,\bar\zeta$ stereographic coordinates for S^2 ($\zeta = \cot\theta/2 e^{i\phi}$), and

$$g^{11} = 2(|\omega|^2 - U), \qquad g^{1A} = \omega\bar\xi^A + \bar\omega\xi^A - X^A, \qquad g^{AB} = \xi^A\bar\xi^B + \bar\xi^A\xi^B. \tag{A.42}$$

The co-tetrad is

$$l_\mu = (-1,0,0,0), \quad n_\mu = \Big(U - g_{AB}X^A(\omega\bar\xi^B + \bar\omega\xi^B), -1, g_{AB}(\omega\bar\xi^B+\bar\omega\xi^B)\Big), \quad m_\mu = (-g_{AB}\xi^A X^B, 0, g_{AB}\xi^B$$

The coefficients are a priori 9 real functions ($U\in\mathbb{R}$, $X^A\in\mathbb{R}^2$, $\omega\in\mathbb{C}$, $\xi^A\in\mathbb{C}^2$) parametrising the 6 independent components of the metric plus 3 internal components corresponding to the ISO(2) stabiliser of l^μ. The helicity subgroup generates dyad rotations $\xi^A \mapsto e^{i\delta}\xi^A$, and the translations the class I transformations (A.34). The latter in particular shift $\omega\mapsto\omega+a$, $a\in\mathbb{C}$, and can be used to put $\omega=0$, so $m^\mu=(0,0,m^A)$ with 2d space-like components only. This is the $2+2$-adapted choice described above, and corresponds to $E^r_M=0$ as in the lower-triangular form (A.25), that we also used in Section 5.5 in the main text to make easier contact with the metric Hamiltonian formalism. Alternatively, this null rotation can be used to achieve $\pi = 0$, so to make n^μ and m^μ to be parallel propagated along l^μ as demanded by Newman and Unti.

In terms of spin coefficients, we have the following simplifications: $\kappa = \mathrm{Im}(\rho) = 0$, $\tau = \bar{\alpha} + \beta$ which follow from l_μ being a gradient, $\mathrm{Re}(\varepsilon) = 0$ from fixing the radial diffeos requiring r affine parametrization, and $\pi = 0$ from the parallel transport of n^μ and m^μ. Finally $\mathrm{Im}(\varepsilon) = 0$ if we fix the helicity SO(2) rotation. This complete fixing is usually referred to as NP gauge, to be contrasted with the $2+2$-adapted gauge described above, where the condition $\pi = 0$ is replaced by $\pi = \bar{\tau}$ and $\mathrm{Im}(\mu) = 0$.

Hence, when we refer to the Newman-Unti tetrad (A.41) in NP gauge there are only 6 free functions of all 4 coordinates. The NP gauge is preserved by class I and helicity transformations with r-independent parameters.

A.8 Mappings to the χ-tetrad

In this Appendix we discuss the detailed relation between the χ-tetrad used to perform the canonical analysis in real connection variables and the results of the previous Appendix. It provides formulas completing the discussion in the main text.

At the end of Section (5.3.1) we introduced the internal 'radial gauge' (5.23), stating that it adapts the tetrad to the $2+2$ foliation and identifies the lapse function with the one used in the metric formalism. We now provide the relevant details and proofs. The χ-tetrad and its inverse are given by

$$e^I_\mu = \begin{pmatrix} \hat{N} & E^i_a \chi_i \\ N^a E^i_a & E^i_a \end{pmatrix}, \qquad e^\mu_I = \frac{1}{N} \begin{pmatrix} 1 & -\chi_i \\ -N^a & N E^a_i + N^a \chi_i \end{pmatrix}, \tag{A.43}$$

where $\chi^2 = 1$ to have a null foliation, $e = EN$ and $N = \hat{N} - N^a E^i_a \chi_i$ is the lapse function. Taking the soldered internal null directions $x_{\pm\mu} = e^I_\mu x_{\pm I}$ of (5.17), and defining m^μ to be a complex linear combination of the two orthogonal tetrad directions $\mathscr{X}^{ij} e^\mu_j$, e.g. $m^\mu := \frac{1}{\sqrt{2}}(e^\mu_2 - i e^\mu_3)$ when $\chi^i = (1,0,0)$, the basis

$$(x^\mu_+, -x^\mu_-, m^\mu, \bar{m}^\mu)$$

is a doubly-null tetrad. We then rescale it by

$$l^\mu = \frac{1}{N} x^\mu_+, \qquad n^\mu = -\frac{N}{2} x^\mu_-, \tag{A.44}$$

to define an NP tetrad adapted to the $3+1$ null foliation as described in the main text, see (5.40). In general, the 2d spaces with tangent vectors $(m^\mu, \bar{m}^\mu)$ will not be integrable. With reference to (A.36), we see that integrability requires $\mathrm{Im}(\rho) = \mathrm{Im}(\mu) = 0$. The first condition is guaranteed by the fact that l_μ is a gradient. The second can be obtained with a class I transformation, generated by the translations $\mathscr{X}_{ij} T^j$ stabilising l^μ, fixing

$$E_A \chi = 0 \quad \Leftrightarrow \quad \mathscr{X}^{ij} E^r_j = 0. \tag{A.45}$$

In this gauge

$$m^\mu = (0, 0, E^A_-), \qquad n_\mu = (N(N/2 + N^r E_r \chi), N E_r \chi, 0, 0), \tag{A.46}$$

so that the null tetrad is manifestly adapted to the $2+2$ foliation defined by the level sets of the coordinates u and r. Then $\mathrm{Im}(\mu)=0$ also follows immediately by explicit calculation of (A.37) using the fact that m^μ only has 2d surface components.

In a first-order formalism with independent connection, the statement holds in the absence of torsion. We have already seen in Section 5.4.2 that on-shell of the torsionless condition $\mathrm{Im}(\overset{\circ}{\rho})=\mathrm{Im}(\rho)$. Let us show here explicitly how $\mathrm{Im}(\overset{\circ}{\mu})$ goes on-shell. From (A.11) we have

$$\mathrm{Im}(\overset{\circ}{\mu})=-\frac{N^2}{2}\mathrm{Im}(\overset{\circ}{\rho})-\frac{N}{2}(r_{22}+r_{33}), \tag{A.47}$$

and from one of the secondary simplicity constraints (5.30) we have

$$\Psi^{11}=-r_{22}-r_{33}-\varepsilon^{1MN}\tilde{E}^a_M \underset{\sim}{E}{}^1_b \partial_a \tilde{E}^b_N. \tag{A.48}$$

The last term vanishes for $E^r_M=0=E^1_A$, hence $\mathrm{Im}(\overset{\circ}{\mu})=0$ in this gauge.

To complete the comparison with the $2+2$ formalism, let us fix the internal direction $\chi^i=(1,0,0)$, and use $M=2,3$ to refer to the orthogonal directions. Then (A.45) puts the triad in the form

$$E^i_a=\begin{pmatrix} E^1_r & 0 \\ E^M_r & E^M_A \end{pmatrix}, \qquad E^a_i=\begin{pmatrix} E^r_1 & 0 \\ E^A_1 & E^A_M \end{pmatrix}, \tag{A.49}$$

thus E^M_A is the 2d dyad and E^A_M its inverse, and we further have the equalities $E^1_r=1/E^r_1$, $E^M_r=-E^M_A E^A_1/E^r_1$. We then have $g_{AB}=q_{AB}=E^M_A E^{BM}=\gamma_{AB}=\mathscr{E}^M_A\mathscr{E}_{BM}$, consistently with the fact that the metric induced by the dyad is adapted to the coordinates by the gauge-fixing, and $q^{AB}=E^A_M E^{BM}=\gamma^{AB}$ is its inverse. Notice that the (partial) Bondi gauge $g^{0A}=0$ achieves $g^{AB}=\gamma^{AB}$, analogously to the vanishing-shift gauge for space-like foliations.

At this point $E=E_r\chi\sqrt{\gamma}$ and $\sqrt{-g}=E_r\chi N\sqrt{\gamma}$. A look at the metric shows that

$$-1/g^{01}=E_r\chi N, \tag{A.50}$$

hence, the lapse function in the metric Hamiltonian analysis of [Torre, 1986] equals the one in the connection formulation up to a factor $E_r\chi$. This ambiguity is not surprising due to the null nature of the foliation and the lack of a canonical normalization of its normal. To identify our lapse with the one in the metric formalism is sufficient to fix the radial boost K_χ as to have $E^r\chi=1$, as we did with (5.23). Then also $E_r\chi=1$ because of (A.45) and the triad takes the form (A.25). We also recover the relation $\sqrt{-g}=N\sqrt{\gamma}$ between lapse and the determinant of the metric Hamiltonian analysis. For completeness, we report below the relation between the χ-tetrad coefficients and the $2+2$ foliation with a general radial gauge. The case with coinciding lapse functions can immediately be read plugging $E^r\chi=1=E_r\chi$ in the formulas below.

The relation between the foliating normals and the adapted null co-frame is given by

$$n^0=du=-\frac{1}{N}x_{+I}e^I, \qquad n^1=dr=\frac{1}{2E_r\chi}\Big(\frac{N+2N^rE_r\chi}{N}x_{+I}+x_{-I}\Big)e^I, \qquad n^0_\mu n^{1\mu}=-\frac{1}{E_r\chi N}. \tag{A.51}$$

The dual basis, shift vectors and lapse matrix are

$$n_0^\mu = N(\tfrac{N^r}{E_r\chi}+\tfrac{N}{2})l^\mu + n^\mu = (1,0,N^rE_r\chi E^A\chi - N^A), \qquad b_0^A = N^A - N^rE_r\chi E^A\chi, \tag{A.52}$$

$$n_1^\mu = E_r\chi N l^\mu = (0,1,E_r\chi E^A\chi), \qquad b_1^A = -E_r\chi E^A\chi, \tag{A.53}$$

$$N_{\alpha\beta} = \begin{pmatrix} -N(N+2N^rE_r\chi) & -NE_r\chi \\ -NE_r\chi & 0 \end{pmatrix}, \qquad N^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{NE_r\chi} \\ -\frac{1}{NE_r\chi} & \frac{1}{N(E_r\chi)^2}(N+2N^rE_r\chi) \end{pmatrix}, \tag{A.54}$$

and the formulas for the 2d projector and covariant induced metric coincide,

$$\gamma_{\mu\nu} := g_{\mu\nu} - x_{+(\mu}x_{-\nu)} = g_{\mu\nu} - N_{\alpha\beta}n_\mu^\alpha n_\nu^\beta = \begin{pmatrix} q_{ab}N^aN^b & q_{bc}N^c \\ q_{ab}N^b & q_{ab} \end{pmatrix}. \tag{A.55}$$

From (A.25), we see also that $E_1^A = -E_M^A E_r^M$, which provides an alternative characterisation of the second shift vector in terms of E_r^M.

The non-integrability of the $\{T\}$ surfaces is the same as measured by the null dyad,

$$\perp^\mu{}_\nu [n_0,n_1]^\nu \equiv [n_0,n_1]^\mu, \qquad m_\mu [n_0,n_1]^\mu = -N(\tau+\bar\pi). \tag{A.56}$$

Having gauge-fixed $N^{00}=0$ to have du null and r affine or areal, we cannot for general metrics simultaneously take dr to be null. It can be made null on a single hypersurface $\tilde\Sigma$ defined by some fixed value of $r=r_0$, if we exploit the left-over freedom of hypersurface diffeomorphisms to fix $N^{11}=0$. This is what was done by Sachs in setting up the $2+2$ characteristic initial value problem, further fixing $N^A=0$ on the same hypersurface, so that the normal vector of $\tilde\Sigma$ at $r=r_0$ is just $n_0^\mu = n^\mu = \partial_u$, as in Fig.5.1.

A.8.1 The Bondi gauge and Newman-Unti tetrad

In Section 5.5 in the main text we discussed the Bondi gauge with a null tetrad already adapted to the $2+2$ foliation. This was motivated by the goal of recovering properties of the metric symplectic formalism. On the other hand, the Newman-Unti tetrad (A.41) mostly used in the literature is adapted to the $3+1$ null foliation only. In this Appendix we present the relation between our metric coefficients and those of (A.41) without fixing the internal 'radial gauge' (5.72). To that end, we first fix all diffeomorphisms requiring the Bondi gauge

$$\frac{1}{N}E^a\chi = (1,0,0). \tag{A.57}$$

We then fix the internal direction $\chi^i = (1,0,0)$, and adapt $l=-du=x_+/N$. This leaves the freedom of acting with the ISO(2) subgroup stabilizing the direction. Because we rescaled the canonical tetrad by N, we also gain the freedom of canonical transformations corresponding to the radial boost K_χ, which does not affect l. This additional gauge freedom should be fixed requiring $E^r\chi = 1$, implying $N=1$. We are then left with 9 free functions, 6 for the metric and 3 for the internal ISO(2) stabilising l. Comparing our tetrad (5.40) in this gauge with (A.41) we immediately identify

$$U = -\frac{1}{2} - N^r, \qquad X^A = -N^A, \qquad \omega = E_-^r, \qquad \xi^A = E_-^A. \tag{A.58}$$

The $2+2$-adapted tetrad is recovered with a class I transformation setting $\omega = E_-^M = 0$.

A.8.2 Areal r and Sachs' metric coefficients

Above we used affine r, as usual in literature using the Newman-Penrose formalism. The alternative common choice is Sachs', leaving $g_{01} = -e^{2\beta}$ free and requiring instead $\sqrt{\gamma} = r^2 f(\theta, \phi)$. Again we fix the internal direction $\chi^i = (1,0,0)$ and the radial boosts with $E^r \chi = 1$, so to have the identification of our $N > 0$ with the metric lapse $e^{2\beta}$. The triad has the form (A.28), and the metric reads

$$g_{\mu\nu} = \begin{pmatrix} -N(N+2N^r+2N^A E_A\chi)+\gamma_{AB}N^A N^B & -N & \gamma_{AB}N^B - NE_A\chi \\ & 0 & 0 \\ & & \gamma_{AB} \end{pmatrix}. \quad \text{(A.59)}$$

Comparing with (5.1) in the main text, we find

$$\beta = \frac{1}{2}\ln N, \qquad U^A = -N^A + N\gamma^{AB}E_B\chi, \qquad \frac{V}{r} = 2N^1 + N(1+\gamma^{AB}E_A\chi E_B\chi). \quad \text{(A.60)}$$

Reverting to affine r, $N = 1$ and the map from Sachs' metric coefficients to Newman-Unti's is $V/r = 2(|\omega|^2 - U)$, $U^A = X^A - \omega\bar{\xi}^A - \bar{\omega}\xi^A$.

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