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Probabilistic analysis of quasi-one-dimensional disordered quantum systems

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Chapter 1

Introduction

1.1 Motivations

This thesis is dedicated to the mathematical study of disordered quantum systems. The rise of quantum mechanics was a formidable boost for the development of mathematical concepts throughout the 20th century, and the mathematical tools at the core of quantum mechanics (namely, linear operators in Hilbert spaces) were modified and adapted to any sort of physical situation by analogy with previously studied classical counterparts. The necessity of models for material with impurities led several physicists in the 50s, notably P. W. Anderson [And], to investigate the diffusion of quantum particles in random potentials, and to try to explain why metals with impurities develop insulating properties.

1.1.1 What is Anderson Localisation?

Due to Heisenberg's famous Uncertainty Principle, quantum particles are, in a sense, never fully "localised": their position is described by a probability density rather than, as in classical mechanics, by a vector in the space. Furthermore, they are affected by two major phenomena that are not seen in classical mechanics: tunnelling and resonance. The first pertains to the existence of a positive probability, however small, for a quantum particle to overcome "potential barriers" (i.e., to pass through zones where the potential energy is higher than the kinetic energy of the particle; this is impossible in classical mechanics.). The second phenomenon, on the other hand, has to do with the partially undulatory nature

of quantum "particles". If an obstacle manages to deflect the particle and prevent it from tunnelling, the particle might superpose with its own trajectory, creating an interference which might be either constructive or destructive.
To sum up, predicting whether a quantum particle's wave function scattering through a material will force the particle to get trapped in a small region or not is not an intuitive, nor an easy task (whatever "being trapped" means in quantum mechanics). So, the three questions that arise naturally are the following:

1. What does it actually mean for a quantum particle to be "localised"? Is there any precise mathematical definition?

2. What causes localisation phenomena? Is it there a way to prove localisation mathematically?

3. How do these notions manifest themselves in disordered systems?

The first question is the most articulated one, as there are two accepted "mathematical signatures" of localisation: namely, if the wave functions at all energies decay exponentially outside a certain ball of finite radius independent on the energy is certainly a signature of localisation, as the probability distribution of the position of the particle in the material would be *light tailed*, meaning in fact that the event of finding the particle away from a finite box *at a given instant* would be exceedingly rare.
Some authors state/prove spectral localisation for a model by stating/proving that the underlying operator has, almost surely, *pure point spectrum*. This is already sometimes considered, due to the RAGE theorem (see Appendix A and the discussion at the end of this paragraph), a very weak form of localisation.

Definition 1. (Spectral localisation)
Let H be a selfadjoint operator. We say that H has ***spectral localisation*** *if its spectrum is pure point.*

Definition 2. (Exponential spectral localisation)
Let H be a selfadjoint operator with domain $\mathscr{D} \subseteq L^2(\Omega)$, and let $\gamma > 0$ be a positive number. We say that H has ***exponential spectral localisation*** *with* ***localisation length*** $1/\gamma$ *if for all $(E, \psi) \in \mathcal{E}[H]$*

$$\limsup_{|x|\to\infty} \frac{1}{|x|} \log \|\psi(x)\| \leqslant -\gamma \ ,$$

where

$$\mathcal{E}[H] = \{(E, \psi) \in \mathbb{R} \times \mathscr{D} \ : \ \|\psi\| = 1 \ , \ H\psi = E\psi\}$$

is the set of eigenpairs of H.

This the notion of spectral localisation captures in some sense the essence of localisation, but it is somewhat unsatisfactory in that it shows that the support of the wave function is localised only at any given instant. It could move around and go anywhere with the time evolution. For this reason, a stronger form of localisation has been introduced.

Definition 3. (Dynamical localisation)
Let H *be as above. For a fixed* $x \in \Omega$*, consider the family of probability measures indexed by* t

$$\mathbb{P}^t_x(A) := |\langle \, P_I \exp\{-iHt\}\, \delta_x, \ \mathbb{1}_A \, \rangle|^2$$

where $A \subseteq \Omega$ *is a measurable set and* P_I *is the* spectral projector *to the interval* I*. We say that* H *has* ***dynamical localisation*** *if the family of probability measures* $\left\{\mathbb{P}^t_x\right\}_{t>0}$ *is tight, that is, for any fixed* $x \in \Omega$*,* $\epsilon > 0$ *there exists a compact set* M_ϵ *such that*

$$\limsup_{t\to\infty} \mathbb{P}^t_x(M_\epsilon) \geqslant 1 - \epsilon$$

see e.g. [Dur].

Definition 4. (Exponential dynamical localisation)
Let H *be a selfadjoint operator ,*γ *be a positive number , and let* $I \subseteq \mathbb{R}$ *be an interval as above . We say that* H *has* ***exponential dynamical localisation*** *on* I *with localisation length* $1/\gamma$ *if, for all* $x \in \Omega$

$$\sup_{t>0} \limsup_{|y|\to\infty} \frac{1}{|x-y|} \log |\langle \, P_I \exp\{-iHt\}\, \delta_x, \ \delta_y \, \rangle|^2 \leqslant -\gamma$$

The notion of exponential dynamical localisation is considered the strongest notion of localisation one can prove for a quantum model, as it states that the probability to find in a distant state y a particle observed on the site x *after any amount of time* is exponentially small in the distance between the two sites. It is worth noticing that, while the fact dynamical localisation is a stronger notion than spectral localisation is somewhat obvious in terms of "physical intuition", its mathematical proof is highly non-trivial. In fact, the statement that exponential dynamical localisation implies exponential spectral localisation (which in

turns implies almost sure pure point spectrum) is the consequence of an important theorem, namely the Ruelle-Amrein-Georgescu-Enss (RAGE) theorem (see [AW], Theorem 2.6, or paragraph A.0.2 in Appendix A).

1.2 Panorama on Anderson localisation for random Schrödinger operators

The simplest model of an electron moving through a disordered material is the **random Schrödinger operator model** (sometimes referred in the mathematical community as *Anderson model*, despite this name being associated to a completely different model in the condensed matter physics community; see for example [Hew], 1.4-1.5).

Definition 5. (Lattice Anderson model)
Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a countable graph, denote by $H_0 := \Delta_{\mathbb{G}}$ the graph Laplacian

$$(\Delta_{\mathbb{G}}\psi)(x) = \sum_{y \in \mathcal{V} \,:\, \{x,y\} \in \mathcal{E}} (\psi(y) - \psi(x))$$

and let $\{V_x\}_{x \in \mathcal{V}}$ be a i.i.d. random field indexed by the vertices of $\mathbb{G}$.
*We will refer to a lattice Schrödinger operator $H_\omega := H_0 + V_\omega$ as the **Anderson model** on $\mathbb{G}$.*

The Anderson model on $\mathbb{Z}^d$ is an example of an *ergodic* operator, therefore, by a theorem of Pastur[Pas], its spectrum is almost surely equal to a deterministic set (see Appendix B).

1.2.1 Panorama on Anderson localisation in 1d

Since Anderson's paper in the late fifties [And], almost twenty years had to pass before the first rigorous mathematical result could be proven. After that, many proofs of spectral localisation and dynamical localisation for the Anderson and related models in one dimension appeared ([GMP],[KS], [CKM], [JZ], [BDFGVWZ], [GK]).
Despite their differences, most of the the approaches to 1d Anderson localisation rely on the **transfer matrix method**. This approach is based on the observation that any eigenfunction

ψ of H can be completely "reconstructed" knowing only its values at 0 and 1. In fact

$$\begin{pmatrix} \psi_E(x+1) \\ \psi_E(x) \end{pmatrix} = T_x(E)T_{x-1}(E)\cdots T_1(E)\begin{pmatrix} \psi_E(1) \\ \psi_E(0) \end{pmatrix} \tag{1.1}$$

where

$$T_k(E) := \begin{pmatrix} E - V_k & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

as can be checked by manipulating the eigenvalue equation for H.
The analysis of an eigenfunction's asymptotic behaviour boils down to the study of the positivity of the *Lyapunov exponent* $\gamma(E)$ associated with the above random matrix product:

$$\gamma(E) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\,\|T_n(E)T_{n-1}(E)\cdots T_1(E)\|_{\mathrm{op}} \tag{1.3}$$

It turns out that, if $\gamma(E)$ is positive, then the eigenfunction corresponding to the generalised energy E decays exponentially with rate at least $\gamma(E)$ (see e.g. [JZ]). Such a limit exists by sub-additivity, and it is an almost sure limit by a theorem by Furstenberg and Kesten ([FK]). It isn't particularly difficult to check the positivity of $\gamma(E)$ for any *fixed* energy E, even in the case of a singular potential, thanks to a criterion due to Furstenberg .

Theorem 1. (Furstenberg, [Fur])
Let $\{T_n\}_{n\in\mathbb{N}} \in \mathrm{SL}(2,\mathbb{R})$ *be a sequence of i.i.d. random matrices having determinant one, with common distribution* μ*. Suppose that*

(i) $\mathbb{E}\log^+ \|T_0\|_{\mathrm{op}} < \infty$*;*

(ii) The semigroup $\langle \mathrm{supp}(\mu)\rangle$ *generated by the support of the distribution of* T_0 *is not compact in* $\mathrm{SL}(d,\mathbb{R})$*;*

(iii) There is no finite set S *of points in the projective space* $\mathbb{RP}^1$ *such that* $gS \subseteq S$ *simultaneously for all* $g \in \mathrm{supp}(\mu)$*.*

Then $\gamma(E) > 0$ *almost surely.*

However, the real problem is to make sure that , for any given realisation of the potential, *all* the eigenfunctions simultaneously decay exponentially. While the function on $\mathbb{Z}$ generated by applying the transfer cocycle $\Phi_n(E)$ to a pair of initial states $\left(\begin{smallmatrix} \psi_1 \\ \psi_0 \end{smallmatrix}\right)$ satisfies

the eigenvalue equation at energy E, there's no guarantee that it lies in $\ell^2(\mathbb{Z})$, i.e., that it is an actual eigenfunction (and E an actual eigenvalue) of the operator. While the set of eigenvalues (and of an eigenfunction basis) of the operator is necessarily countable, the transfer matrix mechanism would generate such a "candidate eigenfunction" for any real value of the energy. The fact that is not possible a priori to tell which ones of these candidate eigenfunctions actually lie in $\ell^2(\mathbb{Z})$ prevents the use of a union bound in combination with the Furstenberg Theorem.

A first result result in the direction of spectral localisation for the Anderson model on $\ell^2(\mathbb{Z})$ was first proven in [GMol], where they proved that, if H is the Anderson Hamiltonian on $\mathbb{R}$,

$$(H\psi)(t,\omega) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi(t) + q(t,\omega)\psi(t) \tag{1.4}$$

(where the potential q is a stochastic process satisfying certain assumptions) or the one dimensional discrete model

$$(H\psi)(x,\omega) := \psi(x+1) + \psi(x-1) + V(x,\omega)\psi(x)\,. \tag{1.5}$$

where the potential obeys some assumptions. The authors proved that in either of these two situations $\overline{\sigma_{\mathrm{pp}}(H)} = \sigma(H)$, that is, the pure point spectrum of the 1d Anderson Hamiltonian is dense in its spectrum.

The first complete proof of spectral localisation has been obtained in [GMP] for the Hamiltonian (1.4) where $q(\omega,t)$ is a random, piecewise continuous function which is bounded uniformly in ω and t a.s., obtained as a "projection" of a continuous stochastic process on a Riemannian manifold as follows: Let (M,g) be a Riemannian manifold, and let $\boldsymbol{h}$ be a vector field on M. Consider the Markov-Feller process Q_t on M having infinitesimal generator $\Delta_M + \boldsymbol{h}$ (where Δ_M is the Laplace-Beltrami operator on M), and let $F : M \to \mathbb{R}$ be a non constant smooth function. Then, we define $q(\omega,t) := F(Q_t)$. Goldsheid, Molchanov and Pastur showed in 1979 that for such a potential, the random Schr̈odinger operator (1.4) has pure point spectrum with probability 1 . In their paper, they also, discuss the possibility of adapting their proof to the i.i.d. Anderson model on $\ell^2(\mathbb{Z})$ provided that the distribution is sufficiently regular. Three years later, Molchanov proved exponential spectral localisation for the same model [Mol], and Carmona proved that the rate of exponential decay for the eigenfunctions is bounded by the Lyapunov exponent [Car].

After this first result in one dimension, in 1980, Kunz and Souillard proved Anderson localisation on the whole spectrum for the one dimensional Anderson model with i.i.d. potential assuming the probability distribution of the potential to be absolutely continuous, with density function $f \in L^\infty((1+|x|)^{1+\alpha})$ for some $\alpha \in (0,1)$, where

$$L^\infty((1+|x|)^{1+\alpha}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } \sup_{x\in\mathbb{R}} |f(x)(1+|x|)^{1+\alpha}| < \infty \right\}$$

This result was then generalised in 1987 when Carmona, Klein and Martinelli proved spectral localisation for the i.i.d. model on $\ell^2(\mathbb{Z})$ under the sole assumptions that the distribution of the potential is not a delta function, and has a finite fractional moment [CKM]. In their proof, they combined the transfer matrix method with a multiscale analysis, a tool introduced some years earlier to prove localisation for the Anderson model in several dimension at high disorder or/and extreme energies (see the next paragraph).

1.2.2 Panorama on Anderson localisation in $d > 1$

After localisation in $1d$, approaches localisation for the Anderson model in several dimensions were developed, usually working for eigenvalues in a neighbourhood of the ground state (extreme energies), or for potentials of the form $\lambda \cdot V$ with λ very large (large disorder). In 1983, Frölich and Spencer proved a localisation result for the lattice Anderson model on $\mathbb{Z}^d$ in this setting , introducing in the main proof the subsequently widely used *multiscale method* [FS]. Their proof, much like several subsequent proofs (including that of the main result of this thesis) consists basically in bounding (in norm) the operator's eigenfunctions in term of the *Green function*, and then show that the (x,y)-entry of the Green function is exponentially small in $|x-y|$. The multiscale analysis comes into play in this last, crucial step. While having been employed mostly to prove localisation results at the bottom of the spectrum or at high disorder in higher dimensions, multiscale analysis has also been used in the aforementioned work by Carmona, Klein and Martinelli.

In the early nineties, Aizenman and Molchanov derived a simple and useful criterion to show localisation for the Anderson Model in $\mathbb{Z}^d$, namely the *fractional moments criterion*. The criterion consists in checking that the model's Green function satisfies the following condition for some $s \in (0,1)$, some constant A_μ^s possibly depending on s and μ_s and for every $x, y \in \mathbb{Z}^d$:

$$\sup_{\eta>0} \mathbb{E}|G_{E+i\eta}[H_\omega(x,y)]|^s \leqslant A_\mu^s \mathrm{e}^{-\mu_s \|x-y\|}$$

If such a condition is satisfied, then H has spectral [AM] and dynamical [Aiz] localisation with localisation length $1/\mu_s$. Also, in [AM] and and subsequent papers, further criteria to verify the above condition were developed, e.g., the *finite volume criterion* [ASFH]. In [AM] they also managed to check that such a condition is satisfied by the Anderson model on $\mathbb{Z}^d$ with a regular enough potential, for sufficiently high disorder or at energies near the bottom of the spectrum.
The fractional moment method presents both advantages and disadvantages with respect to multiscale analysis: on one hand it usually requires stricter regularity conditions on the potential, on the other hand it usually allows to prove stronger results, such as localisation in the Anderson model on the tree graph under some assumption on the potential's density (which wouldn't be possible with multiscale analysis because the exponential growth of the tree graph defeats the decay given by the induction step), see e.g. [AW], chapter 16.

A recent, interesting new approach to Anderson localisation was introduced by Filoche and Mayboroda in [FM]. The method has been introduced in the context of the *continuous* Anderson model $\Delta + V$ acting on a domain in $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ and is based on the notion of *landscape function*. This notion also applied to a far wider class of operators of the form $\mathcal{L} + V$ where $\mathcal{L}$ is a *second order elliptic operator*, defined on the domain $\mathring{H}^m(\Omega)$ where

$$\mathring{H}^m(\Omega) := \left\{ \phi : \forall \epsilon > 0,\ \exists f^\epsilon \in C_0^\infty \ \text{ s.t. } \ \|\nabla^m(\phi - f^\epsilon)\|_{L^2(\Omega)} \leqslant \epsilon \right\}$$

equipped with the norm $\|f\|_{\mathring{H}^m(\Omega)} := \|\nabla^m f\|_{L^2(\Omega)}$, the *homogeneous Sobolev space*. Consider the eigenvalue problem

$$\mathcal{L}\psi_E = E\psi_E \ \text{ in } \Omega,\ \psi_E|_{\partial\Omega} = 0,\ \psi_E \in \mathscr{D} \ .$$

We say that u is a *landscape function* for this problem if it solves the problem

$$\mathcal{L}u = 1 \ \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u \in \mathring{H}^m(\Omega)$$

The relevance of landscape function in solving eigenvalue problems comes from the following inequality, valid for a bounded subset $\Omega \subset \mathbb{R}^d$: if $\mathcal{L}$ is a elliptic operator, selfadjoint on $\mathring{H}^m(\Omega)$ for Ω bounded, ψ_E be an eigenfunction of $\mathcal{L}$ corresponding to an eigenvalue E, and u is a landscape function for the corresponding eigenvalue problem, then

$$\frac{|\psi_E(x)|}{\|\psi_E\|_{L^\infty(\Omega)}} \leqslant Eu(x) \quad \forall x \in \Omega$$

that is, u bounds from above *all* the eigenfunctions of $\mathcal{L}$!

1.3 The strip

An intermediate case between $d = 1$ and $d > 1$ is the case of operators whose domain is a dense domain in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^W = \ell^2(\mathbb{Z} \times [1, W])$, that is, they can be interpreted as a particle moving on the *strip* $\mathbb{Z} \times [1, W]$. Such models are interesting on their own since they model bidimensional disordered media extending in one dimension much more than in the other. Furthermore, models of this kind can be studied by 1d methods for any W, yet their asymptotics for large W could potentially offer insights on higher dimensional models in the future, even if it must be said that this approach hasn't provided significant insights so far.

The transfer matrix for the Anderson model on the strip is the $2W \times 2W$ matrix

$$T_i(E) = Q(\mathbb{1},\ E\mathbb{1} - V_i) := \begin{pmatrix} E\mathbb{1} - V_i & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in \mathrm{Sp}(2W, \mathbb{R}) \tag{1.6}$$

where $\mathbb{1}$ is the $W \times W$ identity matrix and V_i is a random tridiagonal matrix with all ones on the external diagonals and i.i.d. random variables on the main diagonal. While well defined for this random matrix product, it turns out that the Lyapunov exponent as defined in 1.3 does not control the eigenfunctions' decay rate on a strip of width $W > 1$. In order to capture the correct decay rate, it is necessary to introduce the notion of *Lyapunov spectrum*. According to the Multiplicative Ergodic Theorem (first proven in [OS]; see also [GM], [CK]) , for a product $\Phi_n = T_1 \cdots T_n$ of i.i.d. invertible random matrices $T_i \in \mathrm{GL}(W, \mathbb{R})$, the limit

$$\Phi_\infty := \lim_{n\to\infty} \sqrt[2n]{\Phi_n \Phi_n^*}$$

exists as a random matrix, and its spectrum is deterministic. The i-th Lyapunov exponents γ_i of Φ_n is then the deterministic number defined as the logarithm of the i-th largest eigenvalue of Φ_∞. The diagonal matrix $\mathrm{diag}(\gamma_1, \dots, \gamma_W)$ is often referred as the Lyapunov spectrum of Φ_n.

It turns out that, for the Anderson model on a strip of width W, it is the smallest positive Lyapunov exponent $\gamma_W(E)$ of $\Phi_n(E) := T_1(E) \cdots T_n(E)$, with $T_i(E)$ as in (1.6), to prescribe the exponential decay of the eigenfunctions. The intuitive reason for this is the following: by a consequence of the Multiplicative Ergodic Theorem (see [GM] or [CK]), there exists a random flag of vector spaces $\{0\} = \mathscr{V}_0 \subset \mathscr{V}_1 \subset \dots \subset \mathscr{V}_{2W} = \mathbb{C}^{2W}$, $\dim(\mathscr{V}_i) = i$, such that if $\psi \in \mathscr{V}_i \ominus \mathscr{V}_{i-1}$, then $\|\Phi_n(E)\psi\|$ decays exponentially at rate $\gamma_i(E)$. The fact that

a generalised eigenfunction has necessarily to be polynomially bounded (a consequence of Sch'nol's theorem, see Appendix A.0.3) forces $\psi := \binom{\psi_1}{\psi_0}$ to lie in $\mathscr{V}_W \subset \mathbb{R}^{2W}$. If we consider the transfer matrix product in the opposite direction $\Phi_{-n}(E) = \Phi_{[-n,0]}(E) := T_{-n} \cdots T_0$, then by construction $\Phi_n(E)$ and $\Phi_{-n}(E)$ are independent for every $E \in \mathbb{R}$. Therefore, their associated flags are independent as well and the probability that the corresponding W dimensional spaces have nonempty intersection is 0, at least if the distribution of the flags is regular enough. Since the transfer matrices depend on the real parameter E, it is not surprising that for some selected values of E (which turn out to be the generalised eigenvalues of the operator) the two W dimensional spaces do in fact intersect. The vectors belonging to the intersection in such an occurence turn out to be those giving rise to the generalised eigenfunctions when evolved by the transfer matrix product. Since such vectors lie in both the W-dimensional elements of their respective random flags, the decay rate of the resulting eigenfunction is controlled by $\gamma_W(E)$.

A property of the Lyapunov exponents that will be very important for our analysis is their continuity in the energy. In particular, it is possible to prove that, under quite general assumptions, $\gamma_W(E)$ is locally continuous as a function of E. A first result in of this kind has been proven in 1983 by Furstenberg and Kifer [FKif], where they showed that the first Lyapunov exponent $\gamma_1(\mu_k)$ of a sequence of products of i.i.d. random matrices in $\mathrm{GL}(W, \mathbb{R})$ sampled according to measures μ_n converges to the first Lyapunov exponent $\gamma_1(\mu)$ of a random matrix product as above where the individual matrices are sampled according to a measure μ if the support of μ acts strongly irreducibly on $\mathbb{R}^W$ and $\mu_n \to \mu$ in the "probabilists'" weak topology (that analysts call weak-* topology). A more general continuity result for all the Lyapunov exponents for general random linear cocycles has been proven in 2014 by Duarte and Klein (see [DK], Theorem 3.1).

Models of this kind have been studied since 1980. In [G80] Goldsheid considered an i.i.d. random Schrödinger operator on the strip with absolutely continuously distributed potential and proved spectral localisation (discrete spectrum) for such a model. Exponential spectral localisation was then tackled by Lacroix under absolute continuity assumptions for the distribution of the potential [L83, L84, L90].

1.3.1 Singular potentials

Random Schrödinger operators with singular potentials have long been studied in the mathematics and physics literature on the Anderson model. Lifshitz proved his famous anomalous asymptotics for the spectral measure near the edge of the spectrum [Lif] in the case of a tight binding model with a Bernoulli potential, modeling the motion of a quantum particle in a disordered alloy composed by two types of atoms randomly placed on the vertices of a lattice. A similar model with the addition of hoppings was later studied by Kirkpatrik and Eggarter [KE].

The analysis of random Schrödinger operators on the strip with singular potentials has the additional problem, compared to the pure one dimensional system, of showing the positivity of the smallest Lyapunov exponent. While Guivarc'h and Raugi [GR] proved that if the distribution of a random matrix contains an open set in $\mathrm{Sp}(2W,\mathbb{R})$, then the Lyapunov spectrum of the associated i.i.d. product is simple (which, in case of a $2W$-dimensional simplectic matrix, implies the positivity of γ_W), and it isn't difficult to prove that if the distribution of the V_i's contains an open set in $\mathbb{R}^W$, then the distribution of T_i's as in (1.6) contains an open set in $\mathrm{Sp}(2W,\mathbb{R})$, the case of potentials with a purely atomic distribution remained elusive.

Nevertheless, in 1989, a new criterion for the simplicity of the Lyapunov spectrum was proven by Goldsheid and Margulis [GM]. The criterion is true for any semisimple Lie group, and it says that if the support the distribution of a $n\times n$ random matrix acts irreducibly on $\mathbb{R}^n$ and its Zariski closure coincides with the whole group, then its Lyapunov spectrum is simple.

Using this criterion, exponential spectral localisation on the strip for singular potentials was eventually obtained in 1990 by Klein, Lacroix and Speis [KLS], where they employed, as in [CKM], a combination of the transfer matrix approach and multiscale analysis.

1.4 The 2018 resurgence: from multiscale to single scale

During 2018, three different papers appeared where alternative proofs of 1d Anderson localisation allowing for singular potentials came out, all three avoiding multiscale analysis ([JZ], [BDFGVWZ], [GK]). All of the three papers assume that the support of the potential is bounded, and the proofs appearing in [BDFGVWZ] and [JZ] have notable points of

similarity. Conversely, [GK] does not even prove localisation as their main result, but rather deduce it from techniques they used to show the almost sure existence of a dense set of energies for which the Lyapunov exponent associated to the transfer cocycle of the Anderson model (see the next paragraph, and the discussion in paragraph 2.1.1) is 0.
In the next paragraph we will focus on the proof in [JZ], as it largely inspired the proof of the main result in this thesis.

1.4.1 The Jitomirskaya-Zhu proof

In [JZ], Jitomirskaya and Zhu proved spectral and dynamical localisation for the classical 1d Anderson model:

Theorem 2. ([JZ], Theorem 2.1)
Let $H = \psi(x+1) + \psi(x-1) + V_x\psi(x)$ be the random Schrödinger operator on $\mathbb{Z}$, where $\{V_x\}_{x\in\mathbb{Z}}$ are i.i.d., non-deterministic bounded random variables. Then, for any generalised eigenvalue E of H ,its corresponding generalised eigenfunction decays exponentially with rate at least $\gamma(E)$.

The spectral localisation was proven with a single-scale proof which largely inspired the proof of the main result of this thesis. Therefore, it will be summarised in this section. It is a classical transfer matrix proof where the problem of proving the eigenfunction decay simultaneously for all the energies is not addressed, as in [CKM], via multiscale analysis. In their paper, Jitomirskaya and Zhu solve this problem by analysing the deviations of the eigenfunctions' decay from the prescribed exponential rate, uniformly in a small energy interval, using a large deviation bound for the entries of the random matrix product in (1.1), that has been first proven by Tsay in [Tsay], basing it on a former result by Le Page in [LP].
They then define a set of exceptional sites x such that the large deviation bound proven for the random matrix product is violated by the Green function on large box of side-length n centred at x. They then prove spectral localisation by proving that for n large enough, sites $2n, 2n+1$ almost surely don't belong to such an exceptional set (compare this to Proposition 3) . The results follows then by a simple bound on eigenfunctions in term of the Green function (compare to (2.21)).

1.5 Beyond Schrödinger: random hoppings with and without on-site disorder

In its relatively short history, most of the research on Anderson localisation has been concentrated on random Schrödinger operators, but in recent times different and/or more general models have also been considered. One such extension is given by disordered classical and quantum mechanical models with *random hoppings*. A prototypical example of such a model is the *disordered linear chain* studied in 1953 by Dyson in [Dys], which models a chain of unit masses connected by springs with random elastic coefficients. While Dyson didn't address localisation in that paper, he studied the DOS of the random operator on $\ell^2(\mathbb{Z})$

$$(H\psi)(x) = i\lambda_x\psi(x+1) - i\lambda_{x-1}\psi(x-1)\,. \tag{1.7}$$

where $\{\lambda_j\}_{j\in\mathbb{Z}}$ are i.i.d. random variables related to the elastic coefficients of the springs, and are called the (random) hoppings of H. Note that in this model there is no diagonal potential, i.e. there is no coefficient multiplying $\psi(x)$.

Another natural random hopping extension of the Anderson model in pure one dimension is the model described by the random Hamiltonian acting on a dense domain of $\ell^2(\mathbb{Z})$

$$(H_\omega\psi)(x) = l_{x-1}(\omega)\psi(x-1) + l_x(\omega)\psi(x+1) + V_x(\omega)\psi(x)$$

where $\{l_\omega(x)\}_{x\in\mathbb{Z}}$, $\{V(x)\}_{x\in\mathbb{Z}}$ are two independent sequences of i.i.d. random variables. This model is sometimes known as the *random Jacobi matrix* model, and it is basically an extension of (1.8) where a random diagonal potential is added . Spectral and dynamical localisation for this model have been proven by Rangamani in [Ran] under the assumptions that $\mathbb{E}[l_0^\eta] + \mathbb{E}[l_0^{-\eta}] + \mathbb{E}[V_0^\eta] < \infty$ and that the distribution of V_0 is not a delta function, using a similar argument to that in [JZ].

Another interesting model with random hoppings is the *Wegner orbital model*, defined by the Hamiltonian acting on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^W$

$$(H_\omega\psi)(x) = L_{x-1}\psi(x-1) + L_x^\intercal\psi(x+1) + V_x\psi(x) \tag{1.8}$$

where $\{V_x\}_{x\in\mathbb{Z}}$ is an i.i.d. sequence of $W\times W$ GOE matrices and $\{L_x\}_{x\in\mathbb{Z}}$ is an i.i.d. sequence of Ginibre matrices. The main object of this thesis, the model (2.1), is an extension of this model where the hopping matrices and the potential matrices are not necessarily

Gaussian.
Recently, Chapman and Stolz proved exponential dynamical localisation for a random operator on the strip of width 2 with random hoppings obtained as an effective one-particle Hamiltonian for the XY quantum spin chain in a random magnetic field [CS]. Very recently, Jacob Shapiro studied another pure random hopping model [Sha], that is, a model with no diagonal potential and random hoppings like the model studied by Dyson (1.7), but where the hopping are not necessarily hermitian matrices like in (1.8), with the difference of being complex. This model is relevant to the study of topological insulators with chiral symmetry. His results and their possible extension with the methods presented in this thesis is discussed in Section 4.2.

1.6 Original results

The main results in this thesis are Theorem 3 and Theorem 4, appearing in the preprint [MS1]. Chapter 2, where the proof of such theorems is presented, follows [MS1] with some adjustments.
These two theorems pertain to exponential spectral and dynamical localisation for a quasi-one-dimensional random operator which is a generalisation of (1.8) where the distributions of the hoppings and of the potential are assumed to be very general and to obey only some relaxed technical conditions (see assumptions (A)-(C) in the next chapter).

Theorem. *Let H_ω be a random operator of the form(1.8), where the matrix coefficients have very general distributions obeying some relaxed conditions (nondegeneracy of the potential, existence of fractional moments for the distribution of the norm of the matrix coefficients). Then H_ω exhibits exponential spectral localisation on the entire spectrum.*

Theorem. *Let H_ω be a random operator satisfying the conditions of the above Theorem. Then it also has exponential dynamical localisation on any interval $I \subseteq \mathbb{R}$.*

As for (1.8), the model we consider describes the behaviour of a quantum particle with W internal degrees of freedom moving in a one dimensional medium and subject to a random potential. The relaxed conditions on the matrix coefficients of the operators allow, in particular, for the analysis of a large class of singular potentials and singular hoppings. Models with singular hoppings have been considered in the physics literature, e.g., the

dual random dimer model (DRDM) is a model with random hoppings for which both the potential and the hoppings have a singular distribution [CV]. Deterministic hoppings different from the identity matrix are not allowed (the absence of disorder in the hoppings adds considerable difficulties in the proof of the simplicity of the Lyapunov spectrum whose overcoming requires the use advanced algebraic geometry except in special cases such as when the hoppings are deterministically equal to the identity matrix), we plan to cover them in a future work (in a very advanced state of progress). A strategy to obtain such an extension is outlined in Chapter 4 .

The proof of exponential spectral localisation (Theorem 3) is strongly inspired to [JZ], in particular it makes no use of multiscale analysis. In our more general case, however, the proof presents considerable additional technical difficulty, particularly in the analysis of the smallest singular value of a random matrix-valued rational function. As for dynamical localisation (Theorem 4), its proof is substantially different from the one in [JZ]. Instead of exploiting the subarmonicity of the Lyapunov exponents, our proof makes use of an integral representation for the eigenfunction correlator used e.g. in [ESS], combined with a statement on resonant energies (Proposition 3) proven earlier in the paper as a fundamental Lemma to obtain spectral localisation . In the case of $L_0 \equiv \mathbb{1}$ deterministically (Anderson model on the strip) this result complements those in [KLS] in that they prove only exponential spectral localisation.

Other substantial results appear in Chapter 3, where a result by Ge and Zhao [GZ] on the positivity of the disorder-averaged dynamical decay exponent for the 1d Anderson model is re-proven and generalised with a shorter and arguably simpler proof, and a question posed by Jitomirkskaya, Krüger and Liu [JKL], on whether the disorder-averaged dynamical decay exponent and the almost sure dynamical decay exponent for the 1d Anderson model coincide (as they did prove in the same paper is the case for the supercritical almost Mathieu operator) , is answered negatively. This latter results will appear in an upcoming paper written in collaboration with Sasha Sodin.

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[1] from the Venetian language, "great friend". I invite all the readers to call the locals this word during their future visits of Venice, it'll make you instantly popular.

[2] In the Sorani Kurdish language, "thank you". A must-know word for those who will visit the suggestive city of Erbil, one of the most ancient continuously inhabited settlements in the world, and want to befriend the locals.

Chapter 2

Anderson localisation for quasi-one-dimensional random operators

2.1 Introduction

2.1.1 The operator, transfer matrices and Lyapunov exponents

Let $W \geqslant 1$. Let $\{L_x\}_{x\in\mathbb{Z}}$ be a sequence of identically distributed $W \times W$ random matrices in $\mathrm{GL}(W, \mathbb{R})$, and let $\{V_x\}_{x\in\mathbb{Z}}$ be a sequence of identically distributed $W \times W$ real symmetric matrices, so that $\{L_x\}_{x\in\mathbb{Z}}, \{V_x\}_{x\in\mathbb{Z}}$ are jointly independent. Denote by $\mathcal{L}$ the support of L_0 and by $\mathcal{V}$ – the support of V_0. Throughout this paper we assume that

(A) there exists $\eta > 0$ such that

$$\mathbb{E}(\|V_0\|^\eta + \|L_0\|^\eta + \|L_0^{-1}\|^\eta) < \infty \, ;$$

(B) the Zariski closure of the group generated by $\mathcal{L}\,\mathcal{L}^{-1}$ in $\mathrm{GL}(W, \mathbb{R})$ intersects $\mathcal{L}$ (this holds for example when $\mathbb{1} \in \mathcal{L}$);

(C) $\mathcal{V}$ is irreducible (i.e. has no non-trivial common invariant subspace), and $\mathcal{V} - \mathcal{V}$ contains a matrix of rank one.

We are concerned with the spectral properties of the random operator H acting on (a dense subspace of) $\ell^2(\mathbb{Z} \to \mathbb{C}^W)$ via

$$(H\psi)(x) = L_x\psi(x+1) + V_x\psi(x) + L_{x-1}^{\intercal}\psi(x-1) \, , \quad x \in \mathbb{Z} \, . \tag{2.1}$$

This model is the general Hamiltonian describing a quantum particle with W internal degrees of freedom in random potential and with nearest-neighbour random hopping. Some extensively studied special cases of this model include:

- The *block Anderson model* ($L_x \equiv \mathbb{1}$) ; this is in turn a generalisation of the Anderson model on the strip $\mathbb{Z}\times[1, W]$, and, more generally, on $\mathbb{Z}\times\Gamma$, where Γ is any connected finite graph (the assumption that Γ is connected ensures that $\mathcal{V}$ is irreducible)

- The *Wegner orbital model*: here L_x are i.i.d. Ginibre matrices, and V_x are i.i.d GOE matrices.

Fix $E \in \mathbb{R}$. If $\psi : \mathbb{Z} \to \mathbb{C}^W$ is a formal solution of the equation

$$L_x\psi(x+1) + V_x\psi(x) + L_{x-1}^{\intercal}\psi(x-1) = E\psi(x)\,, \quad x \geqslant 1\,,$$

then

$$\begin{pmatrix} \psi(x+1) \\ \psi(x) \end{pmatrix} = T_x \begin{pmatrix} \psi(x) \\ \psi(x-1) \end{pmatrix}, \tag{2.2}$$

where the one-step transfer matrix $T_x \in \mathrm{GL}(2W, \mathbb{R})$ is given by

$$T_x = \begin{pmatrix} L_x^{-1}(E\mathbb{1} - V_x) & -L_x^{-1}L_{x-1}^{\intercal} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.3}$$

The multi-step transfer matrices $\Phi_{x,y} \in \mathrm{GL}(2W, \mathbb{R})$, $x, y \in \mathbb{Z}$, are defined by

$$\Phi_{x,y} = \begin{cases} T_{x-1}\cdots T_y\,, & x > y \\ \mathbb{1}\,, & x = y \\ T_x^{-1}\cdots T_{y-1}^{-1}\,, & x < y\,, \end{cases} \tag{2.4}$$

so that

$$\Phi_{x,y}\begin{pmatrix} \psi(y) \\ \psi(y-1) \end{pmatrix} = \begin{pmatrix} \psi(x) \\ \psi(x-1) \end{pmatrix}. \tag{2.5}$$

In particular, $T_x = \Phi_{x+1,x}$. We abbreviate $\Phi_N = \Phi_{N,0}$. The Lyapunov exponents $\gamma_j(E)$, $1 \leqslant j \leqslant 2W$, are defined as

$$\gamma_j(E) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log s_j(\Phi_N(E))\,,$$

where s_j stands for the j-th singular value. It is known [FK] that (for fixed E) this limit in expectation is also an almost sure limit. The cocycle $\{\Phi_{x,y}\}$ is conjugate to a symplectic one (see Section 2.3.1), and hence

$$\gamma_j(E) = -\gamma_{2W+1-j}(E)\,, \quad j = 1, \cdots, W\,.$$

Further, as we shall see in Section 2.3.2, using the work of Goldsheid [G95] to verify the conditions of the Goldsheid–Margulis theorem [GM] on the simplicity of the Lyapunov spectrum, that

$$\gamma_1(E) > \gamma_2(E) > \cdots > \gamma_W(E) > 0\,.$$

We also mention that the Lyapunov exponents $\gamma_j(E)$ are continuous functions of E.

2.1.2 The main results

Theorem 3. (Exponential spectral localisation for the quasi-one-dimensional random operator)
Assume (A)–(C). Then the spectrum of H is almost surely pure point.
Moreover, if

$$\mathcal{E}[H] = \left\{(E,\psi) \in \mathbb{R} \times \ell^2(\mathbb{Z} \to \mathbb{C}^W) \;:\; \|\psi\| = 1\,,\; H\psi = E\psi\right\}$$

is the collection of eigenpairs of H, then

$$\mathbb{P}\left\{\forall (E,\psi) \in \mathcal{E}[H] \;\; \limsup_{x\to\pm\infty} \frac{1}{|x|} \log \|\psi(x)\| \leqslant -\gamma_W(E)\right\} = 1\,, \tag{2.6}$$

i.e. each eigenfunction decays exponentially, with the rate lower-bounded by the slowest Lyapunov exponent.

Remark 1. *It is believed that the lower bound is sharp, i.e. the rate of decay can not be faster than the slowest Lyapunov exponent. See [GS2] for a discussion and partial results in this direction.*

The property of having pure point spectrum with exponentially decaying eigenfunctions is a manifestation of Anderson localisation of the random operator H. The mathematical work on Anderson localisation in one dimension was initiated by Goldsheid, Molchanov and Pastur [GMP], who considered the case $W = 1$, $L_x \equiv 1$ and established the pure point

nature of the spectrum under the assumption that the distribution of V_x is regular enough (absolutely continuous with bounded density). A different proof of the result of [GMP] was found by Kunz and Souillard [KS]. Under the same assumptions, the exponential decay of the eigenfunctions was established by Molchanov [Mol]. The case of singular distributions was treated by Carmona, Klein, and Martinelli [CKM].

The case $W > 1$ was first considered by Goldsheid [G80], who established the pure point nature of the spectrum for the case of the Schrödinger operator on the strip, i.e. when $L_x \equiv \mathbb{1}$, V_x is tridiagonal with the off-diagonal entries equal to 1 and the diagonal ones independent and identically distributed, under the assumption that the distribution of the diagonal entries of V_x is regular. In the same setting, Lacroix [L83, L84, L90] proved that the eigenfunctions decay exponentially. The case of the Anderson model on a strip with general (possibly, singular) distributions was settled by Klein–Lacroix–Speis [KLS], who established localisation in the strong form (2.6).

Unlike the earlier, more direct arguments treating regular distributions, the works [CKM, KLS] allowing singular distributions involve a multi-scale argument (as developed in the work of Fröhlich–Spencer [FS] on localisation in higher dimension); the theory of random matrix products is used to verify the initial hypothesis of multi-scale analysis. Recently, proofs of the result of [CKM] avoiding multi-scale analysis were found by Bucaj et al. [BDFGVWZ], Jitomirskaya and Zhu [JZ], and Gorodetski and Kleptsyn [GK]; the general one-dimensional case (allowing for random hopping) was settled by Rangamani [Ran]. Our Theorem 3 can be seen as a generalisation of these works: we give a relatively short and single-scale proof of localisation which applies to arbitrary $W \geqslant 1$, and allows for rather general distributions of V_0 and L_0 (under no regularity assumptions on the distribution of the potential). In particular, we recover and generalise the result of [KLS].

In fact, we prove a stronger result pertaining to the eigenfunction correlators, introduced by Aizenman [Aiz] (see further the monograph of Aizenman–Warzel [AW]). If $\Lambda \subset \mathbb{Z}$ is a finite set, denote by H_Λ the restriction of H to $\ell^2(\Lambda \to \mathbb{C}^W)$, i.e.

$$H_\Lambda = P_\Lambda H P_\Lambda^* ,$$

where $P_\Lambda : \ell^2(\mathbb{Z} \to \mathbb{C}^W) \to \ell^2(\Lambda \to \mathbb{C}^W)$ is the coordinate projection. If $I \subset \mathbb{R}$ is a compact interval, denote

$$Q_I^\Lambda(x,y) = \sup \left\{ \| f(H_\Lambda)_{x,y} \| \ : \ \operatorname{supp} f \subset I \ , \ |f| \leqslant 1 \right\} \ , \quad Q_I(x,y) = \sup_{a \leqslant x,y \leqslant b} Q_I^{[a,b]}(x,y) \ .$$

Here $\|f(H_\Lambda)_{x,y}\|$ is the operator norm of the (x, y) block of $f(H_\Lambda)$, and the functions f in the supremum are assumed to be, say, Borel measurable.

Theorem 4. (Exponential dynamical localisation for the quasi-one-dimensional random operator)
Assume (A)–(C). For any compact interval $I \subset \mathbb{R}$,

$$\mathbb{P}\left\{\limsup_{x\to\pm\infty} \frac{1}{|x|} \log Q_I(x, y) \leqslant -\inf_{E\in I} \gamma_W(E)\right\} = 1\,. \tag{2.7}$$

It is known (see [AW]) that Theorem 4 implies Theorem 3. By plugging in various choices of f, it also implies dynamical localisation with the sharp rate of exponential decay, the exponential decay of the Fermi projection, et cet. We chose to state Theorem 3 as a separate result rather than a corollary of Theorem 4 since its direct proof is somewhat shorter than that of the latter.

We refer to Jitomirskaya–Zhu [JZ] and Ge-Zhao [GZ] for earlier results on dynamical localisation for $W = 1$.

2.1.3 Main ingredients of the proof

Similarly to many of the previous works, including [CKM, KLS] and also the recent works [BDFGVWZ, JZ, GK], the two main ingredients of the proof of localisation are a large deviation estimate and a Wegner-type estimate. We state these in the generality required here. Let $I \subset \mathbb{R}$ be a compact interval, and let $F \subset \mathbb{R}^{2W}$ be a Lagrangian subspace (see Section 2.3, Definition 6). Denote by $\pi_F : \mathbb{R}^{2W} \to F$ the orthogonal projection onto F.

Proposition 1. *Assume (A)–(C). For any $\epsilon > 0$ there exist $C, c > 0$ such that for any $E \in I$ and any Lagrangian subspace $F \subset \mathbb{R}^{2W}$*

$$\mathbb{P}\left\{\left|\frac{1}{N} \log s_W(\Phi_N(E)\pi_F^*) - \gamma_W(E)\right| \geqslant \epsilon\right\} \leqslant C\mathrm{e}^{-cN}\,. \tag{2.8}$$

The proof is essentially given in [KLS]; we outline the necessary reductions in Section 2.3.1. The second proposition could be also proved along the lines of the special case considered in [KLS]; we present an alternative (arguably, simpler) argument in Section 2.3.3.

Proposition 2. *Assume (A)–(C). For any $\epsilon > 0$ there exist $C, c > 0$ such that for any $E \in I$ and $N \geqslant 1$*

$$\begin{aligned} \mathbb{P}\left\{\|G_E[H_{[-N,N]}](i,i)\| \leqslant \mathrm{e}^{-\epsilon N}\right\} &\leqslant C\mathrm{e}^{-cN} \quad (i \in [-N,N]) \\ \mathbb{P}\left\{\|G_E[H_{[-N,N]}](i,i\pm 1)\| \leqslant \mathrm{e}^{-\epsilon N}\right\} &\leqslant C\mathrm{e}^{-cN} \quad (i, i\pm 1 \in [-N,N]) \end{aligned} \tag{2.9}$$

Klein, Lacroix and Speis [KLS] use (special cases of) Propositions 1 and 2 to verify the assumptions required for multi-scale analysis. We deduce Theorems 3 and 4 directly from these propositions. In this aspect, our general strategy is similar to the cited works [BDFGVWZ, JZ, GK]. However, several of the arguments employed in these works rely on the special features of the model for $W = 1$; therefore our implementation of the strategy differs in several crucial aspects.

2.2 Proof of the main theorems

2.2.1 Resonant sites; the main technical proposition

Let $\tau > 0$ be a (small) number. We say that $x \in \mathbb{Z}$ is (τ, E, N)-non-resonant ($x \notin \mathrm{Res}(\tau, E, N)$) if

$$\begin{cases} \|L_x\| \leqslant \mathrm{e}^{\tau N}\,, \\ \|G_E[H_{[x-N,x+N]}](x, x\pm N)\| \leqslant \mathrm{e}^{-(\gamma_W(E)-\tau)N}\,, \end{cases} \tag{2.10}$$

and (τ, E, N)-resonant ($x \in \mathrm{Res}(\tau, E, N)$) otherwise. The following proposition is the key step towards the proof of Theorems 3 and 4.

Proposition 3. *Assume (A)–(C). Let $I \subset \mathbb{R}$ be a compact interval, and let $\tau > 0$. There exist $C, c > 0$ such that for any $N \geqslant 1$*

$$\mathbb{P}\left\{\max_{E\in I} \mathrm{diam}(\mathrm{Res}(\tau, E, N) \cap [-N^2, N^2]) > 2N\right\} \leqslant C\mathrm{e}^{-cN}\,.$$

The remainder of this section is organised as follows. In Section 2.2.2, we express the Green function in terms of the transfer matrices. Using this expression and Propositions 1 and 2, we show that the probability that $x \in \mathrm{Res}(\tau, E, N)$ (for a fixed $E \in \mathbb{R}$) is exponentially small. In Section 2.2.3, we rely on this estimate to prove Proposition 3. Then we use this proposition to prove Theorem 3 (Section 2.2.4) and Theorem 4 (Section 2.2.5).

2.2.2 Reduction to transfer matrices

Fix $N \geqslant 1$. Consider the $W \times W$ matrices

$$\begin{aligned} \Psi_i^+ &= (\mathbb{1}\ 0)\, \Phi_{i,N+1} \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} = (0\ \ \mathbb{1})\, \Phi_{i+1,N+1} \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix}, \\ \Psi_i^- &= (\mathbb{1}\ 0)\, \Phi_{i,-N} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix} = (0\ \ \mathbb{1})\, \Phi_{i+1,-N} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix}. \end{aligned} \tag{2.11}$$

The Green function of $H_{[-N,N]}$ can be expressed in terms of these matrices using the following proposition, which holds deterministically for any H of the form (2.1). A similar expression has been employed already in [G80].

Proposition 4. *If $E \notin \sigma(H_{[-N,N]})$, then:*

1. $$\begin{pmatrix} \Psi^{\pm}_{\pm 1} \\ \Psi^{\pm}_{0} \end{pmatrix} G_E[H_{[-N,N]}](0, \pm N) = \begin{pmatrix} G_E[H_{[-N,N]}](0, \pm 1) \\ G_E[H_{[-N,N]}](0, 0) \end{pmatrix};$$

2. *for any $i \in [-N, N]$,*

$$G_E[H_{[-N,N]}](i,i) = \left(\Psi^+_{i+1}(\Psi^+_i)^{-1} - \Psi^-_{i+1}(\Psi^-_i)^{-1}\right)^{-1} L_i^{-1}.$$

Proof. Abbreviate $G_E = G_E[H_{[-N,N]}]$, and set $G_E(i,j) = 0$ for $j \notin [-N, N]$. The matrices $G_E(i,j)$, $-N \leqslant j \leqslant N$, are uniquely determined by the system of equations

$$L_j G_E(i, j+1) + (V_j - E\mathbb{1}) G_E(i,j) + L_{j-1}^{\intercal} G_E(i, j-1) = \delta_{j,i}\mathbb{1}, \quad -N \leqslant j \leqslant N. \tag{2.12}$$

We look for a solution of the form

$$G_E(i,j) = \begin{cases} \Psi_j^- \alpha_i^-, & j \leqslant i \\ \Psi_j^+ \alpha_i^+, & j \geqslant i, \end{cases} \tag{2.13}$$

where

$$\Psi_i^- \alpha_i^- - \Psi_i^+ \alpha_i^+ = 0 \tag{2.14}$$

$$\Psi_{i+1}^- \alpha_i^- - \Psi_{i+1}^+ \alpha_i^+ = -L_i^{-1}. \tag{2.15}$$

The first equation ensures that (2.13) defines $G_E(i,i)$ consistently, while the second one guarantees that (2.12) holds for $j = i$. For the other values of j, (2.12) follows from the construction of the matrices $\Psi_j^{\pm}$.

The solution to (2.14)–(2.15) is explicitly found by elimination:

$$\alpha_i^- = (\Psi_i^-)^{-1}\Psi_i^+\alpha_i^+ , \quad \alpha_i^+ = -(\Psi_{i+1}^-(\Psi_i^-)^{-1}\Psi_i^+ - \Psi_{i+1}^+)^{-1}L_0^{-1} .$$

This implies the second part of the claim. For the first part, note that for $j \geqslant 0$

$$G_E(0,j) = \Psi_j^+\alpha_0^+ = \Psi_j^+(\Psi_0^+)^{-1}G_E(0,0) = \Psi_j^+(\Psi_1^+)^{-1}G_E(0,1) .$$

Observing that $\Psi_N^+ = \mathbb{1}$, we conclude that

$$G_E(0,N) = (\Psi_0^+)^{-1}G_E(0,0) = (\Psi_1^+)^{-1}G_E(0,1) ,$$

as claimed. Similarly,

$$G_E(0,-N) = (\Psi_0^-)^{-1}G_E(0,0) = (\Psi_{-1}^-)^{-1}G_E(0,-1) . \qquad \square$$

2.2.3 Proof of Proposition 3

,

Fix a small $\tau > 0$. Without loss of generality I is short enough to ensure that

$$\max_{E\in I}\gamma_W(E) - \min_{E\in I}\gamma_W(E) \leqslant \frac{\tau}{2}$$

(this property is valid for short intervals due to the continuity of γ_W which, in turn, follows from the locally uniform large deviation bound and Theorem 3.1 in [DK] ; the statement for larger intervals I follows by compactness). Fix such I (which will be suppressed from the notation), and let

$$\gamma = \frac{1}{2}(\max_{E\in I}\gamma_W(E) + \min_{E\in I}\gamma_W(E)) , \quad \text{so that} \quad \sup_{E\in I}|\gamma_W(E) - \gamma| \leqslant \frac{\tau}{4} .$$

For $x \in \mathbb{Z}$, let

$$\mathrm{Res}^*(\tau,x,N) = \left\{ E\in I \ : \ \max_{\pm}\|G_E[H_{[x-N,x+N]}](x,x\pm N)\|_{1,\infty} \geqslant \mathrm{e}^{-(\gamma(E)-\frac{\tau}{2})N} \right\} ,$$

where $\|A\|_{1,\infty} = \max_{1\leqslant\alpha,\beta\leqslant W} |A_{\alpha,\beta}|$. For N large enough ($N \geqslant N_0(\tau)$),

$$\left(\|L_x\| \leqslant \mathrm{e}^{\tau N}\right) \wedge (E \notin \mathrm{Res}^*(\tau, x, N)) \Longrightarrow x \notin \mathrm{Res}(\tau, E, N) \,.$$

By (A) and the Chebyshev inequality

$$\mathbb{P}\left\{\exists x \in [-N^2, N^2] \,:\, \|L_x\| \geqslant \mathrm{e}^{\tau N}\right\} \leqslant (2N^2+1)\frac{\mathbb{E}\|L_0\|^\eta}{e^{\tau\eta N}} \leqslant C_1\mathrm{e}^{-c_1 N} \,.$$

Hence the proposition boils down to the following statement:

$$|x-y| > 2N \Longrightarrow \mathbb{P}\left\{\mathrm{Res}^*(\tau, x, N) \cap \mathrm{Res}^*(\tau, y, N) \neq \varnothing\right\} \leqslant C\mathrm{e}^{-cN} \,. \tag{2.16}$$

The proof of (2.16) rests on two propositions. The first one is deterministic:

Proposition 5. $\mathrm{Res}^*(\tau, x, N)$ *is the union of at most* $C_W N$ *disjoint closed intervals.*

Proof. By Cramer's rule, for each $\alpha, \beta \in \{1, \cdots, W\}$ and $\pm$ the function

$$g^{\pm}_{\alpha,\beta} \,:\, E \mapsto (G_E[H_{[x-N,x+N]}](x, x \pm N))_{\alpha,\beta}$$

is the ratio of two polynomials of degree $\leqslant W(2N+1)$. Hence the level set

$$\left\{E \,:\, |g^{\pm}_{\alpha,\beta}(E)| = \mathrm{e}^{-(\gamma-\frac{\tau}{2})N}\right\}$$

is of cardinality $\leqslant W(2N+1)$ (note that the $\leqslant W(2N+1)$ discontinuity points of $g^{\pm}_{\alpha,\beta}$ are poles, hence they can not serve as the endpoints of the superlevel sets of this function). Hence our set

$$\left\{E \,:\, |g^{\pm}_{\alpha,\beta}(E)| \geqslant \mathrm{e}^{-(\gamma-\frac{\tau}{2})N}\right\}$$

is the union of at most $\leqslant W(2N+1)/2$ closed intervals, and $\mathrm{Res}^*(\tau, x, N)$ is the union of at most

$$2\,\frac{W(W+1)}{2}\,\frac{W(2N+1)}{2} \leqslant C_W N$$

closed intervals. □

Proposition 6. *Assume (A)–(C). For any compact interval* $I \subset \mathbb{R}$ *there exist* $C, c > 0$ *such that for any* $N \geqslant 1$ *and any* $E \in I$,

$$\mathbb{P}\left\{E \in \mathrm{Res}^*(\tau, x, N)\right\} \leqslant C\mathrm{e}^{-cN} \,.$$

Proof. According to Proposition 4,

$$\|G_E[H_{[-N,N]}](0,\pm N)\| \leqslant \left\{ s_W \begin{pmatrix} \Psi^{\pm}_{\pm 1} \\ \Psi^{\pm}_{0} \end{pmatrix} \right\}^{-1} \left\| \begin{pmatrix} G_E[H_{[-N,N]}](0,\pm 1) \\ G_E[H_{[-N,N]}](0,0) \end{pmatrix} \right\| ;$$

hence

$$\begin{aligned} &\mathbb{P}\left\{ \|G_E[H_{[-N,N]}](0,\pm N)\| \geqslant \mathrm{e}^{-(\gamma_W(E)-\frac{\tau}{4})N)} \right\} \\ &\leqslant \mathbb{P}\left\{ s_W \begin{pmatrix} \Psi^{\pm}_{\pm 1} \\ \Psi^{\pm}_{0} \end{pmatrix} \leqslant \mathrm{e}^{(\gamma_W(E)-\frac{\tau}{8})N} \right\} + \mathbb{P}\left\{ \left\| \begin{pmatrix} G_E[H_{[-N,N]}](0,\pm 1) \\ G_E[H_{[-N,N]}](0,0) \end{pmatrix} \right\| \geqslant \mathrm{e}^{\frac{\tau}{8}N} \right\} . \end{aligned}$$

By Propositions 1 and 2, both terms decay exponentially in N, locally uniformly in E. □

Now we can prove (2.16). By Proposition 5 both $\mathrm{Res}^*(\tau, x, N)$ and $\mathrm{Res}^*(\tau, y, N)$ are unions of at most $C_W N$ closed intervals. If these two sets intersect, then either one of the edges of the intervals composing the first one lies in the second one, or vice versa. The operators $H_{[x-N,x+N]}$ and $H_{[y-N,y+N]}$ are independent due to the assumption $|x-y| > 2N$, hence by Proposition 6

$$\mathbb{P}\left\{\mathrm{Res}^*(\tau, x, N) \cap \mathrm{Res}^*(\tau, y, N) \neq \varnothing\right\} \leqslant 4C_W N \times C\mathrm{e}^{-cN} \leqslant C_1 \mathrm{e}^{-c_1 N} .$$

This concludes the proof of (2.16) and of Proposition 3. □

2.2.4 Spectral localisation: proof of Theorem 3

The proof or localisation is based on Schnol's lemma, which we now recall (see [Han] for a version applicable in the current setting). A function $\psi : \mathbb{Z} \to \mathbb{C}^W$ is called a generalised eigenfunction corresponding to a generalised eigenvalue $E \in \mathbb{R}$ if

$$L_x\psi(x+1) + V_x\psi(x) + L_{x-1}^{\intercal}\psi(x-1) = E\psi(x) , \quad x \geqslant 0 \tag{2.17}$$

$$\limsup_{|x|\to\infty} \frac{1}{|x|} \log \|\psi(x)\| = 0 . \tag{2.18}$$

Schnol's lemma asserts that any spectral measure of H is supported on the set of generalised eigenvalues. Thus we need to show that (with full probability) any generalised eigenpair (E, ψ) satisfies

$$\limsup_{|x|\to\infty} \frac{1}{|x|} \log \|\psi(x)\| \leqslant -\gamma_W(E) . \tag{2.19}$$

Fix a compact interval $I \subset \mathbb{R}$, and $\tau > 0$. Consider the events

$$\mathcal{G}_M(I,\tau) = \left\{ \forall E \in I \ \forall N \geqslant M \ \operatorname{diam}(\operatorname{Res}(\tau,E,N) \cap [-N^2, N^2]) \leqslant 2N \right\} .$$

By Proposition 3 and the Borel–Cantelli lemma,

$$\mathbb{P}\left(\bigcup_{M \geqslant 1} \mathcal{G}_M(I,\tau) \right) = 1 .$$

We shall prove that on any $\mathcal{G}_M(I,\tau)$ every generalised eigenpair (E,ψ) with $E \in I$ satisfies

$$\limsup_{|x|\to\infty} \frac{1}{|x|} \log \|\psi(x)\| \leqslant -\gamma_W(E) + 3\tau . \tag{2.20}$$

From (2.17), we have for any x

$$\begin{aligned} \psi(x) = &- G_E[H_{[x-N,x+N]}](x, x-N) L^{\intercal}_{-N-1} \psi(x-N-1) - \\ &- G_E[H_{[x-N,x+N]}](x, x+N) L_N \psi(x+N+1) . \end{aligned} \tag{2.21}$$

If $x \notin \operatorname{Res}(\tau,E,N)$, this implies

$$\begin{aligned} \|\psi(x)\| &\leqslant \mathrm{e}^{-(\gamma_W(E)-2\tau)N} (\|\psi(x-N)\| + \|\psi(x+N)\|) \\ &\leqslant 2\mathrm{e}^{-(\gamma_W(E)-2\tau)N} \max(\|\psi(x-N)\|, \|\psi(x+N)\|) , \end{aligned}$$

whence $f_\tau(x) := \mathrm{e}^{-\tau|x|}\|\psi(x)\|$ satisfies

$$f_\tau(x) \leqslant 2\mathrm{e}^{-(\gamma_W(E)-3\tau)N} \max(f_\tau(x-N), f_\tau(x+N))) . \tag{2.22}$$

The function f_τ is bounded due to (2.18), hence it achieves a maximum at some $x_\psi \in \mathbb{Z}$. For

$$N > \log 2/(\gamma_W(E) - 3\tau) ,$$

(2.22) can not hold at $x = x_\psi$, thus on $\mathcal{G}_M(I,\tau)$ we have for all $N \geqslant N_0 = \max(M, \log 2/(\gamma_W(E) - 3\tau), |x_\psi|)$:

$$\operatorname{Res}(\tau,E,N) \cap [-N^2, N^2] \subset [x_\psi - 2N, x_\psi + 2N] \subset [-3N, 3N] .$$

Thus (2.22) holds whenever x, N are such that $3N < |x| \leqslant N^2$ and $N \geqslant N_0$.

For each $x \in \mathbb{Z}$, let $N(x)$ be such that $N^2/10 \leqslant |x| \leqslant N^2/5$. If $|x|$ is large enough, $N(x) \geqslant N_0$. Applying (2.22) $\lfloor |x|/N \rfloor - 4$ times, we obtain

$$f_\tau(x) \leqslant (2\mathrm{e}^{-(\gamma_W(E)-3\tau)N})^{\lfloor x/N \rfloor - 4} \times \max f_\tau \leqslant \mathrm{e}^{-(\gamma_W(E)-3\tau)|x| + C\sqrt{|x|}} \times \max f_\tau ,$$

which implies (2.20). □

2.2.5 Eigenfunction correlator: proof of Theorem 4

Fix a compact interval $I \subset \mathbb{R}$, and let $\gamma = \min_{E\in I} \gamma_W(E)$. The proof of (2.7) relies on the following fact from [ESS, Lemma 4.1], based on an idea from [AW]:

$$Q_I^\Lambda(x,y) \leqslant \lim_{\epsilon\to+0} \frac{\epsilon}{2} \int_I \|G_E[H_\Lambda](x,y)\|^{1-\epsilon} dE \leqslant W \,. \tag{2.23}$$

Our goal is to bound on this quantity uniformly in the interval $\Lambda \supset \{x,y\}$. Without loss of generality we can assume that $x = 0$. Choose N such that $N^2/10 \leqslant |y| \leqslant N^2/5$. By Proposition 3, for any $\tau \in (0,\gamma)$,

$$\mathbb{P}\left\{\Xi_{\tau,N}\right\} \geqslant 1 - Ce^{-cN} \,.$$

where

$$\Xi_{\tau,N} := \left\{\forall E \in I \,\mathrm{diam}(\mathrm{Res}(\tau,E,N)) \cap [-N^2,N^2] \leqslant 2N\right\}$$

We show that on $\Xi_{\tau,N}$

$$Q_I^\Lambda(0,y) \leqslant \mathrm{e}^{-(\gamma-2\tau)|y|} \,, \quad |y| > C_0(\gamma-\tau) \,. \tag{2.24}$$

Expand the Green function $G_E[H_\Lambda](0,y)$ as follows. First, iterate the resolvent identity

$$\begin{aligned} G_E[H_\Lambda](x,y) = {} & G_E[H_{[x-N,x+N]}](x,x-N) L_{x-N-1}^\intercal G_E[H_\Lambda](x-N-1,y) \\ & + G_E[H_{[x-N,x+N]}](x,x+N) L_{x+N} \quad G_E[H_\Lambda](x+N+1,y) \end{aligned}$$

starting from $x = 0$ at most $|y|/N$ times, or until the first argument of $G_E[H_\Lambda]$ reaches the set $\mathrm{Res}(\tau,E,N)$. Then apply the identity

$$\begin{aligned} G_E[H_\Lambda](x,u) = {} & G_E[H_\Lambda](x,u-N-1) L_{u-N-1} G_E[H_{[u-N,u+N]}](u-N,u) \\ & + G_E[H_\Lambda](x,u+N+1) L_{u+N}^\intercal \quad G_E[H_{[u-N,u+N]}](u+N,u) \end{aligned}$$

starting from $u = y$ at most $|y|/N$ times, or until the second argument of $G_E[H_\Lambda]$ reaches the set $\mathrm{Res}(\tau,E,N)$. The resulting expansion has $\leqslant 2^{2|y|/N}$ addends, each of which has the form

$$\begin{aligned} & G_E[H_{[x_0-N,x_0+N]}](x_0,x_1) \cdots G_E[H_{[x_{k-1}-N,x_{k-1}+N]}](x_{k-1},x_k) \\ & \quad G_E[H_\Lambda](x_k,y_\ell) \\ & \quad G_E[H_{[y_{\ell-1}-N,y_{\ell-1}+N]}](y_\ell,y_{\ell-1}) \cdots G_E[H_{[y_0-N,y_0+N]}](y_1,y_0) \,, \end{aligned} \tag{2.25}$$

where $x_0 = 0$, $x_{j+1} = x_j \pm N$, $y_0 = y$, $y_{j+1} = y_j \pm N$, and $k + \ell \geqslant |y|/N - 4$. All the terms in the first and third line of (2.25) are bounded in norm by $e^{-(\gamma-\tau)N} < 1/2$, hence

$$\|G_E[H_\Lambda](0,y)\| \leqslant 64 \left(4e^{-(\gamma-\tau)N}\right)^{|y|/N-4} \sum_{u,v \leqslant 2|y|} \|G_E[H_\Lambda](u,v)\| \,.$$

Now we raise this estimate to the power $1-\epsilon$ and integrate over $E \in I$:

$$\frac{\epsilon}{2}\int_I \|G_E[H_\Lambda](0,y)\|^{1-\epsilon} dE \leqslant 64^{1-\epsilon} \left(4e^{-(\gamma-\tau)N}\right)^{(1-\epsilon)(|y|/N-4)} \sum_{u,v \leqslant 2|y|} \frac{\epsilon}{2}\int_I \|G_E[H_\Lambda](u,v)\|^{1-\epsilon} dE \,.$$

It remains to let $\epsilon \to +0$ while making use of the two inequalities in (2.23). □

2.3 Properties of transfer matrices

2.3.1 Preliminaries

Denote

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in \mathrm{GL}(2W, \mathbb{R}) \,.$$

A matrix $Q \in \mathrm{GL}(2W, \mathbb{R})$ is called symplectic, $Q \in \mathrm{Sp}(2W, \mathbb{R})$, if $Q^\intercal J Q = J$.

The matrices T_x are, generally speaking, not symplectic. However, the cocycle $\{\Phi_{x,y}\}_{x,y\in\mathbb{Z}}$ is conjugate to a symplectic one. Indeed, observe that

Proposition 7. *If $L \in \mathrm{GL}(W, \mathbb{R})$ and Z is $W \times W$ real symmetric, then*

$$Q(L,Z) = \begin{pmatrix} L^{-1}Z & -L^{-1} \\ L^\intercal & 0 \end{pmatrix}$$

is symplectic.

Denote $D_x = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & L_x^\intercal \end{pmatrix}$, then

$$\widetilde{T}_x(E) := D_x T_x(E) D_{x-1}^{-1} = Q(L_x, E\mathbb{1} - V_x) \in \mathrm{Sp}(2W, \mathbb{R}) \,.$$

Thus also

$$\widetilde{\Phi}_{x,y}(E) = D_{x-1}\Phi_{x,y}(E)D_{y-1}^{-1} = \begin{cases} \widetilde{T}_{x-1}(E)\cdots\widetilde{T}_y(E) \,, & x > y \\ \mathbb{1} \,, & x = y \\ \widetilde{T}_x^{-1}(E)\cdots T_{y-1}^{-1}(E) \,, & x < y \end{cases} \in \mathrm{Sp}(2W, \mathbb{R}) \,.$$

Symplectic actions cannot be irreducible on symmetric powers (see e.g.[BL], IV.3. for a detailed discussion of this issue). In fact, if $e_1, \dots, e_{2W}$ is the canonical basis of $\mathbb{R}^{2W}$, the subspace $\sum_{i,j} e_i \wedge e_j \subset \mathbb{R}^{2W} \wedge \mathbb{R}^{2W}$ is invariant with respect of the action $T \wedge T$, $T \in \mathrm{Sp}(2W)$ of the whole symplectic group. Therefore, for a more quantitative study of the Lyapunov spectrum of symplectic actions it will be necessary to restrict to a special subspaces of $\wedge^a \mathbb{R}^{2W}$ called **isotropic Grassmannians**.

Definition 6. (Isotropic Grassmannian)
The isotropic Grassmannian $\mathbf{L}_a(\mathbb{R}^{2W})$ *is the subspace*

$$\mathbf{L}_a(\mathbb{R}^{2W}) := \left\{ \bigwedge_{i=1}^{a} Te_i,\ T \in \mathrm{Sp}(2W, \mathbb{R}) \right\} =$$

$$= \left\{ \bigwedge_{i=1}^{a} v_i \ \text{ s.t. } v_i \in \mathbb{R}^{2W}, v_i^{\intercal} J v_j = 0 \quad \forall 0 \leqslant i, j \leqslant a \right\}$$

An element of the isotropic Grassmannian is called an *isotropic subspace*. In the notable case of $a = W$, $\mathbf{L}_W(\mathbb{R}^{2W})$ is called the **Lagrangian Grassmannian** and its elements *Lagrangian subspaces*. For future reference, here is an equivalent definition of a Lagrangian subspace: a vector subspace $V \subseteq \mathbb{R}^{2W}$ is Lagrangian if

$$\{\phi \in \mathbb{R}^{2W} \ : \ \phi J \psi^{\intercal} = 0 \ \ \forall \psi \in V\} = V$$

2.3.2 Simplicity of the Lyapunov spectrum and large deviations

Goldsheid and Margulis showed [GM] that if g_j are independent, identically distributed random matrices in $\mathrm{Sp}(2W, \mathbb{R})$, and the group generated by the support of g_1 is Zariski dense in $\mathrm{Sp}(2W, \mathbb{R})$ (see Definition 17, Appendix B), then the Lyapunov spectrum of a random matrix product $\{g_N \cdots g_1\}$ is simple, i.e.

$$\gamma_1 > \cdots > \gamma_W > 0 \, .$$

Goldsheid showed [G95] that if $\mathcal{V}$ is irreducible and $\mathcal{V} - \mathcal{V}$ contains a rank-one matrix, then for any $E \in \mathbb{R}$ the group generated by $Q(\mathbb{1}, E\mathbb{1} - V)$, $V \in \mathcal{V}$, is Zariski dense in $\mathrm{Sp}(2W, \mathbb{R})$.

Corollary 5. *Assume (A)–(C). Then for any* $E \in \mathbb{R}$

$$\gamma_1(E) > \cdots > \gamma_W(E) > 0 \, .$$

Proof. Observe that

$$Q(L, E\mathbb{1} - V) = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^{\intercal} \end{pmatrix} Q(\mathbb{1}, E\mathbb{1} - V) \,,$$

whence

$$Q(\widehat{L}, E\mathbb{1} - V)^{-1} Q(L, E\mathbb{1} - V) = \begin{pmatrix} \widehat{L}L^{-1} & 0 \\ 0 & \widehat{L}^{-\intercal}L^{\intercal} \end{pmatrix} .$$

If the Zariski closure of the group generated by $\mathcal{L}\mathcal{L}^{-1}$ intersects $\mathcal{L}$, then the Zariski closure of the group generated by $\{Q(L, E\mathbb{1} - V)\}_{L\in\mathcal{L}, V\in\mathcal{V}}$ contains that of the group generated by $\{Q(\mathbb{1}, E\mathbb{1} - V)\}_{V\in\mathcal{V}}$. □

Having the corollary at hand, we deduce from [KLS, Proposition 2.7] applied to the matrices $\widetilde{\Phi}_N(E)$:

Proposition 8. *Assume (A)–(C). For any $\epsilon > 0$ there exist $C, c > 0$ such that for any $E \in I$ and $1 \leqslant j \leqslant W$*

$$\mathbb{P}\left\{ \left| \frac{1}{N} \log s_j(\widetilde{\Phi}_N(E)) - \gamma_j(E) \right| \geqslant \epsilon \right\} \leqslant C\mathrm{e}^{-cN} \,. \tag{2.26}$$

and any Lagrangian subspace $F \subset \mathbb{R}^{2W}$

$$\mathbb{P}\left\{ \left| \frac{1}{N} \log s_j(\widetilde{\Phi}_N(E)\pi_F^*) - \gamma_j(E) \right| \geqslant \epsilon \right\} \leqslant C\mathrm{e}^{-cN} \,. \tag{2.27}$$

Proof. The estimate (2.27) is a restatement of [KLS, Proposition 2.7]. Since the Lagrangian Grassmannian is a compact manifold, it has a δ-net; therefore (2.26) can be deduced from (2.27) through a union bound on such a net. One can also find a proof of this statement (in a much more general setting) in [DK]. □

Note that Proposition 1 follows from (2.27).

Now fix ϵ and a Lagrangian subspace F, and let

$$\Omega_\epsilon^F[\widetilde{\Phi}_N] = \left\{ \max_{j=1}^{W} \left[|\frac{1}{N} \log s_j(\widetilde{\Phi}_N(E)) - \gamma_j| + |\frac{1}{N} \log s_j(\widetilde{\Phi}_N \pi_F^*) - \gamma_j| \right] \leqslant \frac{\epsilon}{100W} \right\} . \tag{2.28}$$

According to Proposition 8,

$$\mathbb{P}(\Omega_\epsilon^F[\widetilde{\Phi}_N]) \geqslant 1 - C(\epsilon, E)\mathrm{e}^{-c(\epsilon,E)N} \,,$$

where the constants are locally uniform in E. Let

$$\widetilde{\Phi}_N(E) = U_N(E)\Sigma_N(E)V_N(E)^\intercal$$

be the singular value decomposition of $\widetilde{\Phi}_N(E)$. Assume that the singular values on the diagonal of $\Sigma_N(E)$ are arranged in non-increasing order; the choice of the additional degrees of freedom is not essential for the current discussion. Denote

$$F_+ = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \,:\, x \in \mathbb{R}^W \right\} \subset \mathbb{R}^{2W} \,, \quad F_- = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \,:\, y \in \mathbb{R}^W \right\} \subset \mathbb{R}^{2W} \,.$$

Proposition 9. *Let $F \subset \mathbb{R}^{2W}$ be a Lagrangian subspace. For N large enough (depending on ϵ), one has (deterministically) on the event $\Omega_\epsilon^F[\widetilde{\Phi}_N(E)]$ defined in (2.28)*

$$s_W(\pi_{F_+} V_N(E)^\intercal \pi_F^*) \geqslant \mathrm{e}^{-\frac{\epsilon}{25}N} \,.$$

Remark 2. *For future reference, we record the dual version of the claim: on $\Omega_\epsilon^F[\widetilde{\Phi}_N(E)^\intercal]$*

$$s_W(\pi_F^* U_N(E) \pi_{F_+}) \geqslant \mathrm{e}^{-\frac{\epsilon}{25}N} \,.$$

To prove this lemma, we need *Horn's multiplicative inequality* and *Weyl inequalities*.

Theorem 6. (Horn; see e.g. [Zhan], Theorem 2.6.)
Let $A, B \in \mathrm{Mat}(n, \mathbb{R})$ be any two matrices. Then, for any $1 \geqslant k \geqslant n$

$$\prod_{i=1}^{k} s_i(AB) \leqslant \prod_{i=1}^{k} s_i(A) \prod_{i=1}^{k} s_i(B)$$

Theorem 7. (Multiplicative Weyl inequalities, see e.g. [Zhan])
Let $A, B \in \mathrm{Mat}(n, \mathbb{R})$ be two matrices. Then

$$s_{i+j-1}(AB^\intercal) \leqslant s_i(A)\, s_j(B)$$

Theorem 8. (Additive Weyl inequalities, see e.g. [HJ])
Let again $A, B \in \mathrm{Mat}(n, \mathbb{R})$ be two matrices. Then

$$s_{i+j-1}(AB) \leqslant s_i(A) + s_j(B)$$

Proof. We abbreviate $\Sigma = \Sigma_N(E)$, $V = V_N(E)$, and $\gamma_j = \gamma_j(E)$. On the other hand, the constants with ϵ not explicitly present in the notation will be uniform in $\epsilon \to +0$.

Clearly, $s_j(\pi_{F_+} V^\intercal \pi_F^*) \leqslant \|\pi_{F_+} V^\intercal \pi_F^*\| \leqslant 1$. Hence it will suffice to show that on $\Omega_\epsilon^F[\widetilde{\Phi}_N(E)]$

$$\prod_{k=1}^{W} s_k(\pi_{F_+} V^\intercal \pi_F^*) \geqslant \mathrm{e}^{-\epsilon N} \,. \tag{2.29}$$

Let Σ^+ be the diagonal matrix obtained by setting the (k,k) matrix entries of Σ to zero for $k > W$ and let $\widehat{\Sigma}^+ = \pi_{F_+} \Sigma^+ \pi_{F_+}^*$. On $\Omega_\epsilon^F[\widetilde{\Phi}_N(E)]$ we have

$$\|\Sigma - \Sigma^+\| \leqslant \|(\widehat{\Sigma}^+)^{-1}\| \leqslant \exp(-cN)$$

with $c > 0$ uniform in $\epsilon \to +0$.
Thus $s_j(\widetilde{\Phi}_N \pi_F^*) = s_j(\Sigma V^\intercal \pi_F^*)$ satisfies

$$|s_j(\widetilde{\Phi}_N \pi_F^*) - s_j(\Sigma^+ V^\intercal \pi_F^*)| \leqslant \mathrm{e}^{-cN} \,.$$

In fact,

$$\begin{aligned} &|s_j(\widetilde{\Phi}_N \pi_F^*) - s_j(\Sigma^+ V_N^\intercal \pi_F^*)| = \\ &= |s_j(\Sigma V_N^\intercal \pi_F^*) - s_j(\Sigma^+ V_N^\intercal \pi_F^*)| \leqslant \\ &\leqslant \|\Sigma V_N^\intercal \pi_F^* - \Sigma^+ V_N^\intercal \pi_F^*\| \leqslant \\ &\leqslant \|\Sigma - \Sigma^+\| \|V_N^\intercal \pi_F^*\| \leqslant \\ &\leqslant \mathrm{e}^{-cN} \end{aligned}$$

where the third line is justified by additive Weyl inequalities.
Observing that $s_j(\Sigma^+ V^\intercal \pi_F^*) = s_j(\widehat{\Sigma}^+ \pi_{F_+} V^\intercal \pi_F^*)$, and that

$$s_j(\widetilde{\Phi}_N \pi_F^*) \geqslant \mathrm{e}^{(\gamma_j - \frac{\epsilon}{100W})N}$$

on Ω_ϵ^F , we get (for sufficiently large N):

$$\begin{aligned} s_j(\widehat{\Sigma}^+ \pi_{F_+} V^\intercal \pi_F^*) &\geqslant \mathrm{e}^{(\gamma_j - \frac{\epsilon}{50W})N} \,, \\ \prod_{k=1}^{j} s_k(\widehat{\Sigma}^+ \pi_{F_+} V^\intercal \pi_F^*) &\geqslant \exp\Big\{ \Big(\sum_{k=1}^{j} \gamma_k - \frac{\epsilon}{50} \Big) N \Big\} \,. \end{aligned}$$

On the other hand, by Horn's inequalities,

$$\prod_{k=1}^{j} s_k(\widehat{\Sigma}^+ \pi_{F_+} V^\intercal \pi_F^*) \leqslant \prod_{k=1}^{j} s_k(\widehat{\Sigma}^+) \times \prod_{k=1}^{j} s_k(\pi_{F_+} V^\intercal \pi_F^*) \leqslant$$

$$\leqslant \exp\left\{ \Big(\sum_{k=1}^{j} \gamma_k + \frac{\epsilon}{100} \Big) N \right\} \prod_{k=1}^{j} s_k(\pi_{F_+} V^\intercal \pi_F^*) \,,$$

whence

$$\prod_{k=1}^{j} s_k(\pi_{F_+} V^\intercal \pi_F^*) \geqslant \mathrm{e}^{-\frac{\epsilon}{25} N} \,, \quad 1 \leqslant j \leqslant W \,,$$

thus concluding the proof of (2.29) and of the claim. □

2.3.3 Wegner-type estimate: proof of Proposition 2

First, it suffices to show that for any $i \in [-N, N]$

$$\mathbb{P}\left\{ \| G_E[H_{[-N,N]}](i,i) \| \geqslant \mathrm{e}^{\epsilon N} \right\} \leqslant C_\epsilon \mathrm{e}^{-c_\epsilon N} \,.$$

By Proposition 4,

$$G_E[H_{[-N,N]}](i,i) = \left(\Psi^+_{i+1} (\Psi^+_i)^{-1} - \Psi^-_{i+1} (\Psi^-_i)^{-1} \right)^{-1} L_i^{-1} \,,$$

where

$$\begin{pmatrix} \Psi^+_{i+1} \\ \Psi^+_i \end{pmatrix} = \Phi_{i+1,N+1} \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & L_i^{-\intercal} \end{pmatrix} \widetilde{\Phi}_{i+1,N+1} \begin{pmatrix} 0 \\ L_N^\intercal \end{pmatrix} ,$$

$$\begin{pmatrix} \Psi^-_{i+1} \\ \Psi^-_i \end{pmatrix} = \Phi_{i+1,-N} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & L_i^{-\intercal} \end{pmatrix} \widetilde{\Phi}_{i+1,-N} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix} .$$

Hence

$$G_E[H_{[-N,N]}](i,i) = L_i^{-\intercal} \left(X^+ - X^- \right)^{-1} L_i^{-1} \,,$$

where

$$X^+ = (\widetilde{\Phi}_{i+1,N+1})_{12} (\widetilde{\Phi}_{i+1,N+1})_{22}^{-1} \,, \quad X^- = (\widetilde{\Phi}_{i+1,-N})_{11} (\widetilde{\Phi}_{i+1,-N})_{21}^{-1} \,,$$

and the subscripts 11 and 21 represent extracting the corresponding $W \times W$ blocks from a $2W \times 2W$ matrix (i.e. $Y_{11} = \pi_{F_+} Y \pi_{F_+}^*$, $Y_{21} = \pi_{F_-} Y \pi_{F_+}^*$). Both matrices $X^\pm$ are Hermitian:

for X^-, this follows from the symplectic property of the transfer matrix, whereas for X^+ one needs to observe that the expression does not change if L_N is initially set to be the identity matrix.

Without loss of generality we can assume that $i \geqslant 0$. We shall prove that

$$\mathbb{P}\left\{s_W(X^+ - X^-) \leqslant \mathrm{e}^{-\epsilon N} \mid X^+\right\} \leqslant C_\epsilon \mathrm{e}^{-c_\epsilon N} .$$

To this end, denote

$$F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2W} \ : \ y = -X^+ x \right\} .$$

F is a Lagrangian subspace as

$$\begin{pmatrix} v^\intercal & -v^\intercal X^+ \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} u \\ -X^+ u \end{pmatrix} = 0 \quad \forall v \in \mathbb{R}^W ,$$

where we used that X^+ is a symmetric matrix, which is true in virtue of the fact that the ratio of the 21 and the 11 blocks of a symplectic matrix is symmetric (see e.g. [DJ]). In the notation of Proposition 9, consider the transfer matrix $\widetilde{\Phi}_{i+1,-N}$, and let

$$\Omega_\epsilon = \Omega_\epsilon^F[\widetilde{\Phi}^*] \cap \Omega_\epsilon^{F_+}[\widetilde{\Phi}^*] \cap \Omega_\epsilon^{F_-}[\widetilde{\Phi}^*] \cap \Omega_\epsilon^{F_+}[\widetilde{\Phi}]$$

(note that $\widetilde{\Phi}_{i+1,-N}$ is independent of X^+ and thus also of F). It suffices to show that on Ω_ϵ

$$s_W(X^+ - X^-) \geqslant \mathrm{e}^{-\frac{\epsilon}{2} N} . \tag{2.30}$$

Let us write the singular value decomposition of $\widetilde{\Phi} = \widetilde{\Phi}_{i+1,-N}$ in block form:

$$\begin{pmatrix} \widetilde{\Phi}_{11} & \widetilde{\Phi}_{12} \\ \widetilde{\Phi}_{21} & \widetilde{\Phi}_{22} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \widehat{\Sigma}^+ & \\ & \widehat{\Sigma}^- \end{pmatrix} \begin{pmatrix} V_{11}^\intercal & V_{21}^\intercal \\ V_{12}^\intercal & V_{22}^\intercal \end{pmatrix}$$

whence on Ω_ϵ

$$\|\widetilde{\Phi}_{11} - U_{11}\widehat{\Sigma}^+ V_{11}^\intercal\| , \|\widetilde{\Phi}_{21} - U_{21}\widehat{\Sigma}^+ V_{11}^\intercal\| \leqslant \mathrm{e}^{-cN} .$$

Further, by Proposition 9 we have on Ω_ϵ:

$$s_W(U_{11}) , s_W(U_{21}) , s_W(V_{22}) \geqslant \mathrm{e}^{-\frac{\epsilon}{25} N} . \tag{2.31}$$

Let us show that

$$\|X^- - U_{11}U_{21}^{-1}\| \leqslant \mathrm{e}^{-c'N} . \tag{2.32}$$

To this end, start with the relation

$$X^- = (U_{11}\widehat{\Sigma}^+ V_{11}^\intercal + E_1)(U_{21}\widehat{\Sigma}^+ V_{11}^\intercal + E_2)^{-1}, \quad \|E_1\|, \|E_2\| \leqslant \mathrm{e}^{-cN} .$$

In view of the bound

$$s_W(U_{21}\widehat{\Sigma}^+ V_{11}^\intercal) \geqslant s_W(U_{21})\, s_W(\widehat{\Sigma}^+)\, s_W(V_{11}^\intercal) \geqslant \mathrm{e}^{c_1 N} ,$$

we can set $E_2' = E_2(U_{21}\widehat{\Sigma}^+ V_{11}^\intercal)^{-1}$ and rewrite

$$(U_{21}\widehat{\Sigma}^+ V_{11}^\intercal + E_2)^{-1} = (U_{21}\widehat{\Sigma}^+ V_{11}^\intercal)^{-1}(1 + E_2') , \quad \|E_2'\| \leqslant \mathrm{e}^{-c_2 N} ,$$

as

$$\|E_2'\| := \|E_2(U_{11}\widehat{\Sigma}^+ V_{11}^\intercal)^{-1}\| \leqslant \frac{\|E_2\|}{s_W(U_{11})\, s_W(\widehat{\Sigma}^+)\, s_W(V_{11}^\intercal)} \leqslant \mathrm{e}^{-c_2 N}$$

for some $c_2 > 0$, which implies (2.32).

Now, since F is a Lagrangian subspace, $x - X^+ y = 0$ for $\binom{x}{y} \in F^\perp$, whence for any $\binom{x}{y} \in \mathbb{R}^{2W}$

$$x - X^+ y = (\mathbb{1} \;\mid\; -X^+)\pi_F^* \pi_F \begin{pmatrix} x \\ y \end{pmatrix}$$

(where the first term is a 1×2 block matrix). Therefore we have, by another application of Proposition 9:

$$\begin{aligned} s_W(U_{11} - X^+ U_{21}) = s_W((\mathbb{1} \;\mid\; -X^+)\pi_F^* \pi_F U \pi_{F_+}^*) \geqslant \\ \geqslant s_W((\mathbb{1} \;\mid\; -X^+)\pi_F^*)\, s_W(\pi_F U \pi_{F_+}^*) \geqslant \\ \geqslant s_W(\pi_F U \pi_{F_+}^*) \geqslant \\ \geqslant \mathrm{e}^{-\frac{\epsilon}{25} N} . \end{aligned}$$

This, together with (2.32) and (2.31), concludes the proof of (2.30), and of Proposition 2. □

Now we consider the elements $G_E[H_{[-N,N]}](i, i \pm 1)$. We have:

$$G_E[H_{[-N,N]}](i, i \pm 1) = \Psi_{i+1}^{\pm}(\Psi_i^{\pm})^{-1} G_E[H_{[-N,N]}](i, i) .$$

The norm of $G_E[H_{[-N,N]}](i, i)$ is controlled by (2.3.3), whereas $\Psi_{i+1}^{\pm}(\Psi_i^{\pm})^{-1} = L_i^{-1} X^{\pm}$ are controlled using (2.32) and Claim 9.

2.4 On generalisations

The assumptions (A)–(C) in Theorems 3 and 4 can probably be relaxed. Instead of a finite fractional moment in (A), it should be sufficient to assume the existence of a sufficiently high logarithmic moment:

$$\mathbb{E}(\log_+^A \|V_0\| + \log_+^A \|L_0\| + \log_+^A \|L_0^{-1}\|) < \infty$$

for a sufficiently large $A > 1$. To carry out the proof under this assumption in place of (A), one would need appropriate versions of large deviation estimates for random matrix products.

As we saw in the previous section, the rôle of the assumptions (B)–(C) is to ensure that the conditions of the Goldsheid–Margulis theorem [GM] are satisfied. That is, our argument yields the following:

Theorem 9. *Let $I \subset \mathbb{R}$ be a compact interval. Assume (A) and that for any $E \in I$ the group generated by*

$$\{Q(L, E\mathbb{1} - V)\}_{L\in\mathcal{L}, V\in\mathcal{V}}$$

is Zariski-dense in $\mathrm{Sp}(2W, \mathbb{R})$. *Then:*

1. *The spectrum of H is I is almost surely pure point, and*

$$\mathbb{P}\left\{\forall(E,\psi) \in \mathcal{E}[H] \;\; E \in I \Longrightarrow \limsup_{x\to\pm\infty} \frac{1}{|x|} \log \|\psi(x)\| \leqslant -\gamma_W(E)\right\} = 1\,; \tag{2.33}$$

2. *for any compact subinterval $I' \subset I$ (possibly equal to I) one has:*

$$\mathbb{P}\left\{\limsup_{x\to\pm\infty} \frac{1}{|x|} \log Q_I(x,y) \leqslant -\inf_{E\in I} \gamma_W(E)\right\} = 1\,. \tag{2.34}$$

As we saw in the previous section, the second condition of this theorem is implied by our assumptions (B)–(C). Most probably, weaker assumptions should suffice, and, in fact, we believe that the conclusions of Theorems 3 and 4 hold as stated without the assumption (B). A proof would require an appropriate generalisation of the results of Goldsheid [G95].

Another interesting class of models appears when $V_x \equiv 0$. The complex counterpart of this class, along with a generalisation in which the distribution of L_x depends on the

parity of x, has recently been considered by Shapiro [Sha], in view of applications to topological insulators. An interesting feature of such models is that the slowest Lyapunov exponent $\gamma_W(E)$ may vanish at $E = 0$. This circle of questions (in particular, the positivity of the smallest Lyapunov exponent and Anderson localisation) is studied in [Sha] under the assumption that the distribution of L_0 in $\mathrm{GL}(W, \mathbb{C})$ is regular. In order to extend the results of [Sha] (for matrices with complex entries) to singular distributions, one would first need an extension of [GM] to the hermitian symplectic group.

Returning to the (real) setting of the current paper, assume that (B)–(C) are replaced with

(B′) the group generated by $\mathcal{L}$ is Zariski-dense in $\mathrm{GL}(W, \mathbb{R})$;

(C′) $V_x \equiv 0$.

Along the arguments of [Sha], one can check that the conditions of [GM] hold for any $E \neq 0$. From Theorem 9, one deduces that the conclusion of Theorem 3 holds under the assumptions (A), (B′), (C′), whereas the conclusion (2.34) of Theorem 4 holds for compact intervals I not containing 0. If $\gamma_W(0) = 0$, (2.34) is vacuous for $I \ni 0$. If $\gamma_W(0) > 0$, (2.34) is meaningful and probably true for such intervals, however, additional arguments are required to establish the large deviation estimates required for the proof.

Finally, we note that Theorem 9 remains valid if the independence assumption is relaxed as follows: $\{(V_x, L_x)\}_{x\in\mathbb{Z}}$ are jointly independent (i.e. we can allow dependence between V_x and the corresponding L_x).

Chapter 3

Annealed dynamical decay

3.1 Introduction

As far as now, we have dealt with the analysis of eigenfunctions under *quenched disorder*, that is, we looked if we could describe universal behaviours in the eigenfunctions valid for almost any realisation of the disorder. In this section, we will consider the *annealed* behaviour, that is, the behaviour of the eigenfunctions averaged over the disorder. In particular, we will focus on the **annealed dynamical decay**.

Definition 7. *Let $H(\omega)$ be a random selfadjoint operator, and let I be an interval. The* annealed dynamical decay *of H on I is the quantity*

$$\gamma_I^{\mathbb{E}} := -\limsup_{x\to\infty} \frac{1}{|x|} \log \mathbb{E} Q_I(0,x)$$

where $Q_I(x,y)$ is the eigenfunction correlator of $H(\omega)$, defined in paragraph 2.1.2.

Such definition was introduced in [JK], where it is suggested that proving the positivity of $\gamma^{\mathbb{E}}$ would be simpler than proving that the eigenfunction correlator is almost surely exponentially bounded. Later in the same work, they prove the positivity for the supercritical almost-Mathieu operator at diophantine frequencies. In a more recent paper [JKL], they prove that for same operator $\gamma^{\mathbb{E}}$ actually coincides with the quenched dynamical decay, that is, the Lyapunov exponent γ. They then refer to a work of Ge and Zhao [GZ] where it is proven that $\gamma^{\mathbb{E}} > 0$ for the one dimensional Anderson model , and ask whether it is true that $\gamma^{\mathbb{E}} = \gamma$ for the Anderson operator as well.

After re-proving and extending Ge and Zhao's result with an arguably simpler proof, this chapter will be devoted to prove that for a class of potential and for large disorder, $\gamma^{\mathbb{E}}$ gets much smaller than the quenched dynamical decay . This warns against the use of $\gamma^{\mathbb{E}}$ as a quantity to look to discern the dynamical localisation length of a typical realisation of a random quantum system.

3.2 A generalised Ge-Zhao theorem

In this section, we will prove the following result:

Theorem 10. (Generalised Ge-Zhao theorem)
Let I be a compact interval, and let $\gamma^{\mathbb{E}}(H)$ be the annealed dynamical decay of the operator (2.1) subject to conditions (A),(B) and (C), in the previous chapter.
Then $\gamma^{\mathbb{E}}(H) > 0$.

A proof of this result appeared in [GZ] in the case of the Anderson model on $\mathbb{Z}$. Their proof relies heavily on arguments developed in [JZ] . In the following, we provide a proof of it in this more general setting relying on the methods developed in the previous chapter.

Proof. To prove the above result, we will make use of (2.23) and of (3). Take a finite restriction $H_{[-N,N]}$ of H and let $x \in [-N, N]$, $K := \left\lfloor \frac{x}{4} \right\rfloor$ and $\tau := \frac{1}{2} \inf_{E\in\sigma(H)} \gamma_W(E)$. By Proposition 3 the probability of the event

$$\mathcal{R} := \{\exists E \in \mathbb{R} \text{ s.t. } 0 \in \mathrm{Res}(\tau, E, K), x \in \mathrm{Res}(\tau, E, K)\}$$

is exponentially small in K. Furthermore, on the event $\Omega \setminus \mathcal{R}$, $\mathbb{R} = R_0 \cup R_y$, with

$$R_\bullet = \{E \in \mathbb{R} \text{ s.t. } \bullet \notin \mathrm{Res}(\tau, E, K)\}$$

Supposing that we are on the event $\Omega \setminus \mathcal{R}$, we perform a two-boxes resolvent expansion on $H_{[-N,N]}$,

$$\begin{aligned}
\|\underline{G}[H_{[-N,N]}](0,x)\|_{\mathrm{op}} &\leqslant \|\underline{G}[H_{[-K,K]}](0,K)\|_{\mathrm{op}} \|L_K\|_{\mathrm{op}} \|\underline{G}[H_{[-N,N]}](K+1,x)\|_{\mathrm{op}} + \\
&+ \|\underline{G}[H_{[-K,K]}](-K,0)\|_{\mathrm{op}} \|L_{-K}\|_{\mathrm{op}} \|\underline{G}[H_{[-N,N]}](-x,-1-K)\|_{\mathrm{op}} \leqslant \\
&\leqslant \mathrm{e}^{-(\gamma_W(E)-\tau)K} \left[\|\underline{G}[H_{[-N,N]}](K+1,x)\|_{\mathrm{op}} \mathrm{e}^{\tau K} + \|\underline{G}[H_{[-N,N]}](-x,-1-K)\|_{\mathrm{op}} \mathrm{e}^{\tau K} \right] \leqslant \\
&\leqslant \mathrm{e}^{-\tau K} \left[\|\underline{G}[H_{[-N,N]}](K+1,x)\|_{\mathrm{op}} + \|\underline{G}[H_{[-N,N]}](-x,-1-K)\|_{\mathrm{op}} \right]
\end{aligned}$$

since $0 < \tau < \gamma_W(E)$.
By (2.23), on $\Omega \setminus \mathcal{R}$,

$$Q(0,x) \leqslant 2\mathrm{e}^{-\tau K} \lim_{\epsilon \downarrow 0} \left(\int_{R_0} + \int_{R_y} \right) \|\underline{G}[H_{[-N,N]}](K+1,x)\|_{\mathrm{op}}^{1-\epsilon} \, \mathrm{d}E \leqslant 4\mathrm{e}^{-\tau K}$$

Decomposing the expectation along $\mathcal{R}$ and its complement, and using Proposition 3, we conclude that

$$\begin{aligned} \mathbb{E}Q(0,x) = \mathbb{E}[Q(0,x)\,|\,\mathcal{R}]\mathbb{P}\{\mathcal{R}\} + \mathbb{E}[Q(0,x)\,|\,\Omega \setminus \mathcal{R}]\mathbb{P}(\Omega \setminus \mathcal{R}) \leqslant \\ \leqslant C\mathrm{e}^{-cK} + 4\mathrm{e}^{-\tau K} \end{aligned}$$

and that $\gamma^{\mathbb{E}}(H) \geqslant \min\left\{\frac{c}{4}, \frac{\tau}{4}\right\} > 0$. □

3.3 Non Lyapunov decay for pure 1d models

In [JKL], Jitomirskaya, Krüger and Liu show that for the Almost-Mathieu operator H_α with diophantine frequency α,

$$\gamma^{\mathbb{E}} = \inf_{E \in \sigma(H_\alpha)} \gamma(E)$$

and asked whether this statement is true also for the Anderson model. In this section, we will prove that this isn't the case by exhibiting two notable counterexamples, one with singular potential and another one with absolutely continuous potential. In order to do so, we need to recall first two very important results in spectral theory that will play a major rôle in this proof.

3.3.1 Preliminaries: Combes-Thomas bounds

The Combes-Thomas bound basically affirms that operator eigenfunctions decay exponentially fast away from the spectrum and provides a quantitative estimate on the rate of decay.

Theorem 11. (Combes, Thomas, [CT], see also [AW], p.159)
Let H be an operator on a countable metric graph $\mathbb{G}$, such that, for some $\alpha > 0$

$$S_\alpha := \sup_x \sum_{y \in \mathbb{G}} |H(x,y)|(\mathrm{e}^{\alpha \operatorname{dist}(x,y)} - 1) < \infty$$

Then, for all $z \in \mathbb{C} \setminus \sigma(H)$ *and* $\mu \leqslant \alpha$ *such that* $S_\mu < \Delta(z) := \mathrm{dist}(z, \sigma(H))$

$$G_z[H](x,y) \leqslant \frac{1}{\Delta(z) - S_\mu} \mathrm{e}^{-\mu\,\mathrm{dist}(x,y)}$$

Furthermore, if $\Re(z) < \inf \sigma(H)$ *or* $\Re(z) > \sup \sigma(H)$*, then calling* $\Delta_{\Re}(z) := \mathrm{dist}(\Re(z), \sigma(H))$ *we get*

$$G_z[H](x,y) \leqslant \frac{2}{\Delta_{\Re}(z)} \exp\left\{ -\frac{\alpha\sqrt{\Delta_{\Re}(z)}}{\sqrt{2S_\alpha}} \mathrm{dist}(x,y) \right\}$$

3.3.2 Preliminaries : Wegner estimate

The analysis of the density of states of a random Schrödinger operator plays a fundamental rôle in proving that the Lyapunov exponent of a certain family of random Schrödinger operator on $\ell^2(\mathbb{Z})$ with absolutely continuous random potential, that will be used as a counterexample to the JKL conjecture in paragraph 3.3.4, grows logarithmically in the intensity of the disorder.

Definition 8. (Integrated Density of States)
Let H *be a selfadjoint operator on* $\mathbb{R}^d$*, and let* $\{\Lambda_L\}_{L\in\mathbb{N}}$ *be a sequence of boxes invading* $\mathbb{R}^d$*. Then the* ***Integrated Density of States****, or IDS of* H *is the measure*

$$N(E) := \lim_{N\to\infty} \frac{1}{L^d} \#\{\text{eigenvalues of } H \text{ in } (-\infty, E] \text{ counted with their multiplicities}\},$$

A crucial bound on the Radon-Nykodim derivative $\frac{\mathrm{d}N(E)}{\mathrm{d}E}$ of the DOS in the case of random Schrödinger operators whose potential has bounded density was given by Franz Wegner in 1981:

Theorem 12. (Wegner, [Weg])
Let $H = H_0 + V$ *be a random Schrödinger operator with potential* V *having bounded density* ρ_V*. Then*

$$\frac{\mathrm{d}N(E)}{\mathrm{d}E} \leqslant \|\rho_V\|_\infty$$

3.3.3 Bernoulli-Anderson model at large coupling

In 1987, Martinelli and Micheli proved the following theorem on the asymptotic behaviour in the coupling for the Lyapunov exponent for one dimensional random Schrödinger

operators with Bernoulli potential (this model is also known as Bernoulli-Anderson model, or binary alloy model).

Theorem 13. (Martinelli-Micheli, [MM])
Let $V_0 \overset{\mathrm{d}}{=} \mathrm{Be}(p)$ *be a generalised Bernoulli random variable taking values* 1 *with probability* p *and* 0 *with probability* $1-p$, $p \in (0,1)$ *and let* $H^a := H_0 + aV$ *be a random Schrödinger operator on* $\ell^2(\mathbb{Z})$ *with potential* $V^a := \{V_i^a\}_{i\in\mathbb{Z}}$. *Then*

$$\inf_{E\in\sigma(H^a)} \gamma_E(H^a) > k(p)\log(a)$$

where $k(p)$ *is a strictly positive function on* $(0,1)$.

We will now use this result to prove that H^a is, for a large enough, a counterexample to JKL conjecture. Since, by the above theorem, $\gamma_E(H^a)$ grows logarithmically, we will show, conversely, that $\gamma^{\mathbb{E}}(H^a)$ is bounded from above.

Proposition 10. *Let* $H^a := H_0 + aV$, *a be a random Schrödinger operator on* $\ell^2(\mathbb{Z})$ *with i.i.d. Bernoulli potential as above. Then, for a large enough,*

$$\inf_{E\in\sigma(H^a)} \gamma_E(H^a) > \gamma^{\mathbb{E}}(H^a)\,.$$

Proof. We will prove that $\gamma^{\mathbb{E}}(H^a)$ is bounded in a (but might depend on p). This, and the logarithmic divergence in a of $\gamma_E(H^a)$ given Martinelli-Micheli theorem will prove the result.
Take a large box $\Lambda_{K,x} := [-(K-1)x, Kx]$, for some large constant K to be specified later, and consider the event $\Omega_{K,x} := \{V_y^a = 0 \ \ \forall y \in \Lambda_{K,x}\}$.
We claim that, on $\Omega_{K,x}$, $Q(0,x) \geqslant \mathrm{e}^{-cx}$. Since $\mathbb{P}(\Omega_{K,x}) = p^{2Kx+1}$, this this would imply that

$$\gamma^{\mathbb{E}} \leqslant -\lim_{x\to\infty} \frac{1}{x}\log p^{2Kx+1}\mathrm{e}^{-cx} \leqslant 2K\log(p^{-1}) + c \tag{3.1}$$

Observe that for any $\delta > 0$

$$Q(0,x) \geqslant \delta\,|G_\delta[H](0,x)| \geqslant \delta|G_\delta[H - 2\cdot\mathbb{1}](0,x)|\,. \tag{3.2}$$

Let $T = H_0$ be the free Laplacian (obtained by setting $V_x \equiv 0$ in (2.1)), and let $T_{K,x}$ be the

restriction of T to the finite volume $[-Kx, (K+1)x]$. Then

$$
\begin{aligned}
&|G_\delta[H - 2\cdot\mathbb{1}](0,x)| \geqslant \\
&\quad \geqslant |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,-Kx)||G_\delta[H - 2\cdot\mathbb{1}](-Kx-1,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,(K+1)x)||G_\delta[H - 2\cdot\mathbb{1}]((K+1)x+1,x)| \geqslant \\
&\quad \geqslant |G_\delta[T](0,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,-Kx)||G_\delta[H - 2\cdot\mathbb{1}](-Kx-1,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,(K+1)x)||G_\delta[H - 2\cdot\mathbb{1}]((K+1)x+1,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,-Kx)||G_\delta[T - 2\cdot\mathbb{1}](-Kx-1,x)| - \\
&\qquad - |G_\delta[T_{K,x} - 2\cdot\mathbb{1}](0,(K+1)x)||G_\delta[T - 2\cdot\mathbb{1}]((K+1)x+1,x)| \,.
\end{aligned}
$$

By Combes-Thomas inequality (Theorem 11), the four addends subtracted from the Green function decay exponentially, in particular

$$
|G_\delta[H](0,x)| \geqslant |G_\delta[T](0,x)| - \frac{C}{\delta}\,\mathrm{e}^{-c_1|2K+1|\sqrt{\delta}x} \tag{3.3}
$$

and therefore to prove the proposition is sufficient to show that the free Green function decays sufficiently slowly. In order to do so, we solve "by hands" the free Green function eigenvalue equation plugging the ansatz $g(x) := \alpha\,\mathrm{e}^{-\beta|x|}$ in the equation

$$
g(x-1) + g(x+1) - (2+\delta)g(x) = \delta_{0,x}
$$

satisfied by the free Green function $G_\delta[T - 2\cdot\mathbb{1}](0,x) = g(x)$. By separating the $x = 0$ and $x \neq 0$ cases we get

$$
\begin{cases}
\alpha(2\cosh\beta - 2 - \delta)\,\mathrm{e}^{-\beta|x|} = 0 \quad \text{for } x \neq 0 \\
\alpha(2\,\mathrm{e}^{\beta} - 2 - \delta) = 1
\end{cases}
$$

from which we deduce that

$$
\begin{aligned}
\beta &= \operatorname{arccosh}\left(1 + \frac{\delta}{2}\right), \\
\alpha &= \left(2\exp\left\{\operatorname{arccosh}\left(1+\frac{\delta}{2}\right)\right\} - 2 - \delta\right)^{-1}
\end{aligned}
$$

Therefore we get, using $|ab^{-1}| \geqslant \frac{|a|}{|b|}$,

$$|g(x)| \geqslant \left|2\exp\left\{\operatorname{arccosh}\left(1+\frac{\delta}{2}\right)\right\} - 2 - \delta\right|^{-1} \exp\left\{\operatorname{arccosh}\left(1+\frac{\delta}{2}\right)|x|\right\}$$
$$\geqslant Ce^{-c\sqrt{\delta}}$$

The latter estimate combined with 3.3, this implies the thesis. □

3.3.4 Absolutely continuous potential

Looking at the proof in the previous paragraph, one might be tempted to think that the result is true because Bernoulli random variables have an atom at zero. While this property is in fact crucial in the proof presented above, the discrepancy between the Lyapunov exponent and the annealed dynamical decay exposed above holds true also for certain bounded, absolutely continuous potentials provided that the value of the disorder strength parameter a is sufficiently large.

Proposition 11. *Let $R > 0$ and let $\{V_x^R\}_{x\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with a.c. density supported on $[-R, R]$. Let $V_x^a := V_x^R + V_x^{\mathrm{Be}}$, where $\{V_x^{\mathrm{Be}}\}_{x\in\mathbb{Z}}$ being a sequence of i.i.d. Bernoulli random variables of parameter $p \in (0,1)$ independent on the V_x^R, and let $H^a := H_0 + V^a$.*
Then, again

$$\gamma(H^a) := \inf_{E\in\sigma(H^a)} \gamma_E(H^a) > \gamma^{\mathbb{E}}(H^a)\,.$$

Remark 3. *Note that V^a is actually an absolutely continuous potential. It is, in fact a "smoothed out" version of a Bernoulli.*

In order to prove this Lemma we need an analogous of the Martinelli-Micheli theorem valid for absolutely continuous potentials. Avron, Craig and Simon in [ACS] proved something close to what we need:

Theorem 14. (Avron-Craig-Simon, [ACS])
Let $H^a = H_0 + aV$ be a random Schrödinger operator random potential V having bounded, absolutely continuous density. Then

$$\gamma(H^a) \geqslant \log(a) - C$$

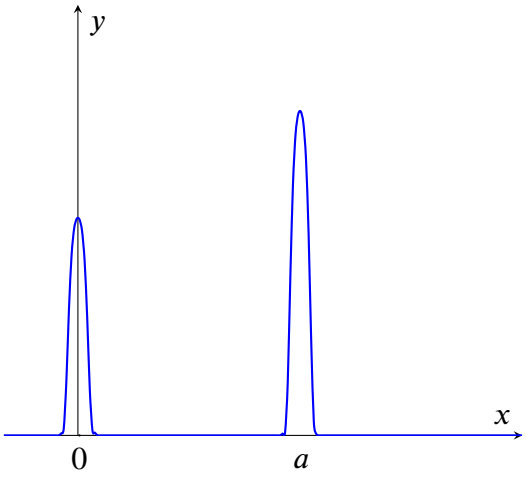

Figure 3.1: A potential V_a constructed from a Bernoulli with $p = 2/5$.

We cannot use directly this result in our case as V^a is not a potential of the form aV, but it is not too difficult to adapt their proof to our slightly different case.

Proof of Proposition 11. We start by adapting the proof in [ACS] to V^a.
Thouless' formula for the Lyapunov exponent of a Schrödinger operator H states that

$$\gamma_E(H) = \int \log|E - E'| \, \mathrm{d}\kappa_H(E')$$

where κ_H denotes the integrated density of states of H.
By decomposing the logarithm in the Thouless formula in its positive and negative parts, $\log(x) = \log^+(x) - \log^-(x)$, we get that

$$\begin{aligned}
&\int \log|E - E'| \, \mathrm{d}\kappa_{H^a}(E') = \\
&= \int \log_+|E - E'| \, \mathrm{d}\kappa_{H^a}(E') - \int \log_-|E - E'| \, \mathrm{d}\kappa_{H^a}(E') \geqslant \\
&\geqslant \int_{a/2}^{\infty} \log_+|E - E'| \, \mathrm{d}\kappa_{H^a}(E') - \int_0^{\infty} \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\} \, \mathrm{d}t
\end{aligned}$$

We claim that

$$\int_0^{\infty} \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\} \, \mathrm{d}t \leqslant C \, ; \tag{3.4}$$

for some $C > 0$ uniform in a, and that

$$\int_{a/2}^{\infty} \log_+|E - E'| \, \mathrm{d}\kappa_{H^a}(E') \geqslant c \log a \, . \tag{3.5}$$

for some positive constant c.
The first bound (3.4) is proven by using Wegner inequality (Theorem 12):

$$\int_0^\infty \kappa_{H^a}\{E' : |E - E'| \leqslant \mathrm{e}^{-t}\}\,\mathrm{d}t \leqslant \|V_a\|_\infty \int_0^\infty \mathrm{e}^{-t}\mathrm{d}t \leqslant \\ \leqslant 1 + \frac{R}{a}$$

The second bound is proven by showing that the density of states has mass above $a/2$ bounded below by a constant independent on a. To do this we employ some elementary perturbation theory. Restrict H^a to an interval $[-N, N]$. Then $H^a_{[-N,N]}$ is a large matrix on which we can apply Gershgorin theorem to assert that, for all $i \in [-N, N]$

$$V_i - 2 \leqslant \lambda_i(H^a_{[-N,N]}) \leqslant V_i + 2$$

This implies that, for $a > 2 + 2R$, and for any N, $\mathbb{P}\left\{\lambda_i(H^a_{[-N,N]}) \geqslant \frac{a}{2}\right\} = p$. Taking the limit for $N \to \infty$, we eventually deduce that, $\kappa_{H^a}\left(\left[\frac{a}{2}, \infty\right]\right) = p > 0$. Hence, for a large enough

$$\int_{a/2}^\infty \log_+ |E - E'|\,\mathrm{d}\kappa_{H^a}(E') \geqslant \log_+ |E - a|\,\kappa_{H^a}\left\{\left[\frac{a}{2} - \epsilon, a + 4 + \epsilon\right]\right\} \geqslant \\ \geqslant \frac{p}{4}\log(a)$$

Now we have to lower bound the operator's Green function uniformly in a. Let $T^R = T_0 + V^R$ and call $T^R_{K,x}$ the restriction of T^R to the box $\Lambda_{K,x} := [-(K-1)x, Kx]$, analogously as before. Like in the proof of Proposition 10, we bound the eigenfunction correlator of H^a with the Green function of a positive operator obtained by shifting H^a, then use the same double resolvent expansion conditioned to the event $\Omega_{K,x} := \{V^{\mathrm{Be}}_x = 1 \ \ \forall x \in \Lambda_{K,x}]$. The Combes-Thomas bound yields

$$Q(0,x) \geqslant \delta\, G_\delta[H^a - (2 + \|V^R\|_\infty)\mathbb{1}](0,x) \geqslant \\ \geqslant \delta\, G_\delta[T^R - 2\cdot\mathbb{1}](0,x) - C_3\mathrm{e}^{-c_3(2K+1)\sqrt{\delta}|x|}$$

Eventually, the only thing left for us to show is that

$$G_\delta[T^R - (2+R)\mathbb{1}](0,x) \geqslant \widetilde{C}\mathrm{e}^{-\tilde{c}\sqrt{\delta+M}|x|}\ .$$

for some positive constants M and $\tilde{c}$.
Let's now write the Neumann series for $G_\delta[T^R-(2+R)\mathbb{1}]$. We get the following inequalities:

$$\begin{aligned} G_\delta[T^R-(2+R)\mathbb{1}] &= \frac{1}{\mathbb{1}-T_0(2+R+\delta-V^R)^{-1}}\cdot\frac{1}{2+R+\delta-V^R} = \\ &= \left(\sum_{n=0}^{\infty} T_0^n(2+R+\delta-V^R)^{-n}\right)\cdot\frac{1}{2+R+\delta-V^R} \geqslant \\ &\geqslant \frac{1}{2+\delta}\sum_{n=0}^{\infty} T_0^n(2+\delta)^{-n} = \\ &= G_\delta[T_0+2\cdot\mathbb{1}] \end{aligned}$$

where the inequality between the second and the third line is intended entrywise and follows from the fact that we substituted a multiplication operator by a function with a uniform lower bound on such a function. In particular

$$G_\delta[(T^R-(2+R)\mathbb{1})](0,x) \geqslant G_\delta(T_0-2\cdot\mathbb{1})](0,x) \geqslant \widetilde{C}\mathrm{e}^{-\tilde{c}\sqrt{\delta+R}\,|x|}\,.$$

Fixing a $\delta\in(0,1)$ and a large K such that $2K+1\geqslant\left\lceil\frac{\tilde{c}}{c_3}\left(1+\frac{R}{\delta}\right)^{1/2}\right\rceil$ and setting $\epsilon = c_3(2K+1)\sqrt{\delta}-\tilde{c}\sqrt{\delta+R}>0$, we get eventually that

$$Q(0,x)\geqslant\delta\, G_\delta[(T^R-(2+R)\mathbb{1})](0,x)\geqslant \mathrm{e}^{-\epsilon|x|}\quad\text{on }\Omega_{K,x}\,.$$

By (3.1)

$$\gamma^{\mathbb{E}}\leqslant 2K\log(p^{-1})+\epsilon\,.$$

independently on a, which implies the thesis. □

In conclusion, we believe that the validity of this kind of result is not limited to the above narrow class of potentials and that is, in fact, true for any bounded potential at sufficiently large coupling.

Conjecture 1. *Let V be* any *bounded i.i.d. random potential, and let $H^a := H_0+aV$. Then*

$$\gamma(H^a)>\gamma^{\mathbb{E}}(H^a)\,.$$

for a large enough.

Chapter 4

Perspectives and related models

4.1 Localisation for deterministic hopping and minimal conditions on the diagonal disorder (a work in progress)

A project in advanced state of progress pertains to the extension proof of Anderson localisation for the quasi-one-dimensional random operator studied in the previous chapter without requiring the assumption iv), necessary there. In particular, the distribution of the hopping matrices $\{L_i\}_{i\in\mathbb{Z}}$ can be any measure on $\mathrm{GL}(\mathbb{R}, W)$. It can even be a Dirac delta, allowing $\{L_i\}_{i\in\mathbb{Z}} \equiv L$, where L is a deterministic matrix. The structure of the proof will be strongly inspired by the proof by Goldsheid of the analogous theorem for Schrödinger matrices [G95].This more general case presents some notable additional difficulties, forcing major differences in the structure of the proof.

Conjecture 2. *Suppose that the support of the on-site disorder $\mathcal{V}$ obeys the conditions* (A *and* C *at the beginning of Paragraph 2.1.1 . Then the Zariski closure of the support $\mathcal{O}$ of the transfer matrix ensemble coincides with the whole symplectic group* $\mathrm{Sp}(2W, \mathbb{R})$.

As an immediate corollary of a positive resolution this conjecture one gets the following strengthening of Theorems 3 and 4:

Theorem 15. *Let $\{V_i\}_{i\in\mathbb{Z}} \in \mathrm{Sym}(W, \mathbb{R})$ be a sequence of i.i.d. random matrices satisfying the assumptions* A,C *of Paragraph 2.1.1, and let $\{L_i\}_{i\in\mathbb{Z}} \in \mathrm{GL}(W, \mathbb{R})$ be* any *sequence of*

i.i.d. random matrices. Suppose also that L_i is independent of V_j if $i \neq j$.
Then the conclusions of Theorems 3 and 4 hold true.

Notice that, in addition of generalising the distribution of the hopping matrices, this result allows us to relax the independence assumption between the hopping and the on-site potential by allowing dependence between the hopping and the potential *on the same site*. This is possible because the transfer matrix T_i at site i depends only on L_i and V_i (thus allowing for a dependence between the two, as it doesn't compromises the independence of the transfer matrices).
In this section we will sketch the strategy of the proof, that at the time of writing still has some technical issues. The proof is based on the notion of *Chevalley replicas*:

4.1.1 Chevalley replicas

In Paragraph 2.3.2, we proved that the Lyapunov spectrum for the symplectic transfer ensemble is almost surely simple under the assumption iv) because this assumption allows us to deduce it easily from Theorem 1.6 in [G95]. One of the main issues we found in directly replicating his proof in our more general case, and without requiring assumption iv), is the following: while in the case of Schrödinger matrices is relatively easy to show that, calling Z a symmetric matrix of the form $E\mathbb{1} - V, V \in \mathcal{V}$, the matrix

$$\widetilde{Z} := \begin{pmatrix} \mathbb{1} & 0 \\ Z & \mathbb{1} \end{pmatrix} \tag{4.1}$$

lies in $\overline{\langle \mathcal{O} \rangle}^{\text{Zar}}$, as it can be done passing through the respective Lie algebras and using basic Lie operations,things get much more complicated with the introduction of transfer matrices. Standard Lie theory seems to be no longer sufficient in this case, as there seems to be no way to show that $\widetilde{Z} - \mathbb{1} \in \text{Lie}\overline{\langle \mathcal{O} \rangle}^{\text{Zar}}$ directly by taking commutators and conjugating known elements of the Lie algebra by the transfer matrix. At some point one has necessarily pass through algebraic geometry.
Let G be a linear algebraic group, and let $\mathfrak{g}$ be its Lie algebra. We will see $\mathfrak{g}$ as equipped with the push-forward of the Zariski topology through the Lie functor associating to each Lie group the corresponding Lie algebra , and call it an **algebraic Lie algebra**. Let $M \in \mathfrak{g}$ be an arbitrary element of $\mathfrak{g}$, and let $\mathcal{A}(M)$ be the set of algebraic Lie subalgebras of $\mathfrak{g}$.

We define a neighbourhood of M in the above described topology via

$$\mathbf{a}(M) := \bigcap_{\substack{\mathfrak{h}\in\mathcal{A}(M)\\ \mathfrak{h}\ni M}} \mathfrak{h}$$

We call an element $M' \in \mathbf{a}(M)$ a **Chevalley replica** (or, more simply, a **replica**) of M. The set of replicas of a diagonal matrix has been characterized by Chevalley:

Theorem 16. (Chevalley)
Let $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, be a diagonal matrix, λ denoting the vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, and let $\mathbf{\Lambda}$ be the lattice

$$\mathbf{\Lambda}_D := \{\, \alpha \in \mathbb{Z}^n \ \textit{such that}\ \ \alpha \cdot \lambda = 0 \,\}$$

Then, any other diagonal matrix $D' = \mathrm{diag}(\lambda')$ such that for every $\alpha \in \mathbf{\Lambda}_D$, $\alpha \cdot \lambda' = 0$ is a replica of D. Furthermore, for any invertible matrix $M \in \mathrm{GL}(n, \mathbb{R})$, $M D' M^{-1}$ is a replica of $M D M^{-1}$.

From this theorem it follows that any matrix of the form $\Lambda \oplus -\Lambda$, with Λ being an invertible diagonal matrix, is a replica of the matrix $\mathbb{1} \oplus -\mathbb{1}$. Therefore, if any matrix of such a form is in the Lie algebra, then also $\mathbb{1} \oplus -\mathbb{1}$ is in there, and

$$\frac{1}{2}\widetilde{T}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\widetilde{T}^{-1} - \frac{1}{2}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix} = \begin{pmatrix}0 & 0\\ Z & 0\end{pmatrix} \in \mathrm{Lie}\overline{\langle\mathcal{O}\rangle}^{\mathrm{Zar}}$$

The main technical problem in this approach is to prove that some $\Lambda \oplus -\Lambda$ is in $\mathrm{Lie}\overline{\langle\mathcal{O}\rangle}^{\mathrm{Zar}}$ or that the Lie algebra contains any diagonal matrix, even non invertible. Even knowing that $\boldsymbol{s}(X), \boldsymbol{n}(X) \in \mathbf{a}(X)$ for any matrix $X \in \mathfrak{g}$, where $\boldsymbol{s}(X), \boldsymbol{n}(X)$ are, respectively, the semisimple and nilpotent components in the Jordan-Chevalley decomposition of X, $X = \boldsymbol{s}(X) + \boldsymbol{n}(X), [\boldsymbol{s}(X), \boldsymbol{n}(X)] = 0$, the problem of diagonalising in $\mathrm{Lie}\overline{\langle\mathcal{O}\rangle}^{\mathrm{Zar}}$ remains to date the main obstacle to the proof of Anderson localisation for the model (2.1) with deterministic hoppings .

4.2 Pure random hopping models

In this thesis we addressed the problem of proving Anderson localisation for a class of random operators with i.i.d. random on-site potential and random hoppings assuming

minimal disorder on the potential and no conditions on the hoppings other than being invertible matrices (they could have no disorder and be deterministic). However, we didn't cover the equally interesting case where the on-site potential is deterministically equal to 0 and the only randomness is thus given by the hopping matrices.
A model of this type has been considered by Shapiro in [Sha]. In particular, he considered an Hamiltonian acting on a domain dense in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^W$ of the form

$$(H\psi)(x) := L_{x+1}^* \psi(x+1) + L_x \psi(x-1) .$$

where L_x^* are i.i.d. random matrices in $\mathrm{GL}(W, \mathbb{C})$. Apart from the onsite potential set to be identically 0, his model differs from the model considered in this thesis in the fact that the hopping matrices are complex. Its transfer matrix lies in the *Hermitian symplectic group*

$$\mathrm{Sp}^*(2W, \mathbb{C}) := \{A \in \mathrm{Mat}(2W, \mathbb{C}) \text{ s. t. } AJA^* = J\}$$

where, as usual,

$$J := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} .$$

In [Sha], Shapiro proves, using the fractional moments method, that this model exhibits *incomplete Anderson localisation*: the model is localised at all energies except possibly 0. His proof of the positivity of the W-th Lyapunov exponent assumes the relatively strong assumption that the support of the measure sampling the hopping matrices contains an open set (thus ruling out pure point measures which, as discussed in Paragraph 1.3.1, were difficult to handle for the classical Anderson model on the strip before the publication of Goldhseid-Margulis theorem) . An interesting research direction could thus be trying to use the methods developed in this thesis and those developed in the work in progress mentioned in the former paragraph to generalise Shapiro's result to more general distributions of hopping matrices.
However, the fact that the Hermitian symplectic group is not semisimple causes troubles in the use of Goldsheid-Margulis Theorem as it is true only for products of random matrices drawn from semisimple Lie groups. Therefore, in order to use Goldsheid-Margulis and the "replica method" sketched in the previous paragraph , it might be necessary to "retro-engineer" the proof of Goldsheid-Margulis to adapt it to this specific case.

4.3 $\mathscr{P}\mathscr{T}$-symmetry breaking and the non-hermitian Su-Schrieffer-Heeger model

A fascinating, relatively recent development in quantum mechanics is **non-hermitian quantum mechanics**. The reason why quantum mechanics requires its observables to be Hermitian operators is that such operators have real spectrum. However, hermiticity is not the only condition capable to force an operator's spectrum on the real axis. Bender and Boettcher in [BB] realised that invariance under the so-called $\mathscr{P}\mathscr{T}$- symmetry has the same effect. Furthermore, $\mathscr{P}\mathscr{T}$- symmetric operators have spectrum bounded from below. Those two properties come in a somewhat strange fashion in non-hermitian quantum models ; in fact, they usually hold true as long as some parameter "measuring the non-hermiticity" is below a critical parameter, and above such a parameter the spectrum get complex. This phenomenon, observed in several such models, is called $\mathscr{P}\mathscr{T}$- *symmetry breaking*.

Definition 9. (Parity exchange operator)
*The **parity exchange operator** $\mathscr{P}$ is unitary operator defined by the action on the Heisenberg algebra generated by position, momentum operators, and central operators $\hat{p}$, $\hat{x}$, $i\mathbb{1}$, $[\hat{p},\hat{x}] = i\mathbb{1}$*

$$\mathscr{P}\hat{p}\mathscr{P} = -\hat{p},\ \mathscr{P}\hat{x}\mathscr{P} = -\hat{x},\ \mathscr{P}i\mathbb{1}\mathscr{P} = i\mathbb{1}$$

Definition 10. (Time reversal operator)
*The **time reversal operator** $\mathscr{T}$ is the anti-unitary operator defined by the action on the Heisenberg algebra as above*

$$\mathscr{T}\hat{p}\mathscr{T} = -\hat{p},\ \mathscr{T}\hat{x}\mathscr{T} = \hat{x},\ \mathscr{T}i\mathbb{1}\mathscr{T} = -i\mathbb{1}$$

Definition 11. *A selfadjoint operator H is said to be $\mathscr{P}\mathscr{T}$-symmetric if*

$$\mathscr{P}\mathscr{T}H\mathscr{P}\mathscr{T} = H\ .$$

$\mathscr{P}\mathscr{T}$-symmetry breaking has been numerically observed for an important $\mathscr{P}\mathscr{T}$-symmetric model, the **non-hermitian Su-Schrieffer-Heeger model** [ZLC]

Definition 12. (Non-hermitian Su-Schrieffer-Heeger Hamiltonian with random hoppings)
The random, non-hermitian Su-Schrieffer-Heeger Hamiltonian *is the operator*

$$(H^{\mathrm{HHS}}\psi)(x) := t_{x+1 \bmod 2}\,\psi(x+1) + t_{x \bmod 2}\,\psi(x-1) + (i\gamma\mathbb{1}_{\{x=1 \bmod 2\}} - i\gamma\mathbb{1}_{\{x=-1 \bmod 2\}})\psi(x)$$

where $\{t_x\}_{x\in 2\mathbb{Z}}$, $\{t_x\}_{x\in 2\mathbb{Z}+1}$ *are two mutually independent i.i.d. sequences of random variables with possibly different distributions, and* γ *is a real deterministic parameter.*

An example of an operator for which phase transition similar to a $\mathscr{PT}$-symmetry breaking has been rigorously proven ([GK], [GS])is the *Hatano-Nelson Hamiltonian*[HN], defined by

$$(H^g\psi)(x) := \mathrm{e}^{-g}\psi(x+1) + \mathrm{e}^{g}\psi(x-1) + V_x\psi(x), \qquad g > 0$$

Theorem 17. (Goldsheid-Khoruzhenko-Sodin, [GK],[GS])
Let $H^g_{[1,N]}$ *be the restriction of above described model to a finite box of size* N*, and let* $\lambda^g_1, \dots, \lambda^g_N$ *be its eigenvalues . Furthermore, let* $\gamma(\lambda^0_j)$ *be the Lyapunov exponent (1.3) associated with the hermitian operator.*
Then for any N*,*

$$\lim_{N\to\infty} \mathbb{P}\left\{\Im\lambda^g_j = 0\ \forall 1 \leqslant j \leqslant N \,\middle|\, |g| \leqslant \gamma(\lambda^0_j)\right\} = 1\ ;$$

$$\lim_{N\to\infty} \mathbb{P}\left\{\exists 1 \leqslant j \leqslant N \ \text{ s.t. } \Im\lambda^g_j = 0 \,\middle|\, |g| > \gamma(\lambda^0_j)\right\} = 0\ .$$

It is worth noticing that this theorem has been proven using a transfer matrix approach.

Some questions addressed by Khoruzhenko, Goldsheid and Sodin for the Hatano-Nelson model can naturally be translated to the non-hermitian Su-Schrieffer-Heeger model:

1. Is it possible to rigorously prove the $\mathscr{PT}$-symmetry breaking for the non-hermitian SSH model?

2. In the broken symmetry phase, is it possible to determine, as Goldsheid and Khoruzhenko did for the HN model, explicit equations for curves on the complex plane containing all the eigenvalues ?

3. In the $\mathscr{PT}$-symmetric phase, is it true that the real eigenvalues of the non-hermitian SSH Hamiltonian are close to those of the corresponding Hermitian model (that is, with $\gamma = 0$) as it was shown by Goldsheid and Sodin for the HN model?

Appendices

Appendix A

Some notions from spectral theory and probability

In this chapter we review some classical results from spectral theory which played a crucial rôle at some point in this thesis

A.0.1 Ergodic operators

Let $\mathscr{H}$ be a Hilbert space $\{\Omega, \mathscr{M}, \mu\}$ be a measure space (we are mostly interested in the case when μ is a probability measure) and let G be a subgroup of the group of all the maps $g : \Omega \to \Omega$ whose action is *measure preserving*, i.e., such that for any measurable set $S \in \mathscr{M}$ and any $g \in G$, $\mu(gS) = \mu(S)$. Let $U : G \to \mathscr{H}$ be an unitary, irreducible(in the sense that there is no finite dimensional subspace $\mathscr{V} \subseteq \mathscr{H}$ such that $U\mathscr{V} \subseteq \mathscr{V}$), representation fo G.

Definition 13. (Ergodic family of operators)
A family of selfadjoint operators $H(\omega), \omega \in \Omega$ is said to be ergodic if:

(i) The subspace $\mathscr{D} := \bigcup_{\omega\in\Omega} \mathscr{D}(H(\omega))$, where $\mathscr{D}(H)$ is the selfadjointness domain of H, is a dense in $\mathscr{H}$.

(ii) For any $T \in G$, $H(T\omega) = U_T H(\omega) U_T^*$.

Lattice Scrödinger operators with i.i.d. random potential can be shown to be ergodic. Ergodic families of operators share completely their whole spectrum.

Theorem 18. (Pastur,[Pas])
Let H be a selfadjoint, ergodic operator. Then $\sigma_{\mathrm{pp}}(H)$, $\sigma_{\mathrm{ac}}(H)$ and $\sigma_{\mathrm{sc}}(H)$ are almost surely equal to deterministic sets.

A.0.2 RAGE Theorem

Theorem 19. (RAGE Theorem)
Let H be a selfadjoint operator acting on a (dense domain in a) Hilbert space $\mathcal{H} = \mathcal{H}^{\mathrm{pp}} \oplus \mathcal{H}^{\mathrm{c}}$, with $\mathcal{H}^{\mathrm{pp}}$ being the pure point spectral subspace and $\mathcal{H}^{\mathrm{c}}$ being the continuous spectral subspace of $\mathcal{H}$. Moreover, let A_L be a sequence of compact operators such that

$$\lim_{L\to\infty} \|A_L - \mathbb{1}_{\mathcal{H}}\|_{\mathrm{op}} = 0$$

Then

$$\mathcal{H}^{\mathrm{c}} = \left\{ \psi \in \mathcal{H} \ \text{ s.t. } \lim_{\substack{L\to\infty \\ T\to\infty}} \frac{1}{T} \int_0^T \|A_L \mathrm{e}^{-iHt}\psi\|^2 \mathrm{d}t = 0 \right\}$$

$$\mathcal{H}^{\mathrm{pp}} = \left\{ \psi \in \mathcal{H} \ \text{ s.t. } \lim_{L\to\infty} \sup_{t\in\mathbb{R}} \|(\mathbb{1} - A_L)\mathrm{e}^{-iHt}\psi\| = 0) \right\}$$

A.0.3 Sch'nol's lemma and its descendants

A crucial aspect of Jitomirskaya- Zhu proof, as well as the localisation proof presented here, is the possibility of looking for (de)localised states only among the *generalised eigenfunctions* of H.

Definition 14. *Let H be a selfadjoint operator. A* generalised eigenpair *(E, ψ) is a pair composed by a function ψ and a real number E such that:*

(i) $H\psi = E\psi$;

(ii) For x large enough, $|\psi(x)| \leqslant |x|^{\alpha}$ for some $\alpha \in \mathbb{R}$.

The difference between a generalised eigenpair and a regular eigenpair is that the generalised eigenfunction is not required to belong to the domain of H (usually an L^2 space, or a dense subset of one), but it is sufficient that it is *polynomially bounded*.
The following important result by Sch'nol [Sch] tells us that any element of $\sigma(H)$, including

its non pure point part) is a generalised eigenvalue, and in particular that the existence of an function exponentially increasing function satisfying the eigenvalue equation for H is not possible.

Theorem 20. (Sch'nol, [Sch])
Let $H := \Delta + V$ be a Schrödinger operator. Then the set of generalised eigenvalues contains the support of the spectral measure of H.

While the classical Sch'nol's theorem was sufficient to implement the single scale approach to localisation in the Schrödinger case treated in [JZ], it isn't for our model. Luckily, in 2017 Rui Han proved a far reaching generalisation to *long range* selfadjoint operators including our block-tridiagonal case [Han].

Theorem 21. (Han, [Han])
Let H be a selfadjoint operator on $\mathbb{Z}^d$ of the form

$$(H\psi)(x) := \sum_{y \in \mathbb{Z}^d} a(x,y)\psi(x-y)$$

where $a(x,y) \leqslant (1+\|y\|)^{-d/2}$ uniformly in x.
Then $\overline{\mathcal{G}_\epsilon} = \sigma(H)$, where

$$\mathcal{G}_\epsilon := \left\{ E \in \mathbb{R} \text{ s.t. } \exists\, \psi_E(x) \leqslant (1+\|x\|)^{\frac{d}{2}+\epsilon},\ H\psi_E = E\psi_E \right\}$$

Appendix B

Some notions on algebraic groups

B.1 Algebraic groups

In this section we will recall some notions from algebraic geometry, and in particular, from the theory of algebraic groups required to understand the assumptions of the Goldsheid-Margulis theorem, and the proof of Theorem 3 in the simplified scenario where we assume condition iv). More advanced notions on algebraic groups required to understand the proof of Theorem 2 (and thus of Theorem 3 without assuming condition iv)) have been explained in section 4.1.1

B.1.1 Zariski topology on a Lie group

We start by recalling the notions of a topological group and of Lie group, which in some sense are analogues to that of an algebraic group:

Definition 15. (Topological group)
*A **topological group** G is a group which is also a topological space such that the maps $g \mapsto gh$ and $g \mapsto g^{-1}$ are homemomorphisms.*

Definition 16. (Lie group)
*A **Lie group** G is a group which is also a smooth manifold such that the maps $g \mapsto gh$ and $g \mapsto g^{-1}$ are analytic diffeomorphisms.*

Then, we introduce the most important topology in algebraic geometry:

Definition 17. (Zariski topology on $\mathbb{R}^n$)
*The **Zariski topology** on $\mathbb{R}^n$ is the topology characterised by the following closed sets: a set S is closed in this topology if there exists a finite set of polynomials $P_1, \dots, P_k$ of finite degree such that*

$$S = \{x := (x_1, \dots, x_n) \in \mathbb{R}^n \text{ s.t. } P_1(x_1, \dots, x_n) = 0, \dots, P_k(x_1, \dots, x_n) = 0\}$$

that is, a set is closed iff it is the set of common zeroes of a family of polynomials.
*A closed set in the Zariski topology is also called an **algebraic set**.*

It is not possible to make a Lie group out of a topological group equipped with the Zariski topology because the Zariski topology is highly *non Hausdorff*, thus it doesn't admit any smooth manifold (or even topological manifold) structure. However, it is possible to construct a meaningful manifold structure on Zariski topological spaces, commonly referred as an *algebraic variety*. In order to rigorously define it, we need to define an algebraic analogue of the concept of *atlas* in the context of differentiable manifolds: the concept of *sheaf*.

Definition 18. (Sheaf)
*A **sheaf** on a topological space $(X, \mathcal{T})$ is a map associating to each open set $U \in \mathcal{T}$ a family of functions $\mathcal{F}(U) = \{f : U \to S\}$ having the following two properties:*

(i) if $V \in \mathcal{T}, V \subseteq U$, and $f \in \mathcal{F}(U)$, then the restricted map $f|_V$ lies in $\mathcal{F}(V)$.

(ii) if $U \in \mathcal{T}$, $\{U_i\}_{i \in I}, U_i \subseteq U$ is an open covering of U, and $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for any $i, j \in I$ then there exists a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

*A sheaf such that S is a ring and and for all $U \in \mathcal{T}$, $\mathcal{F}(U)$ is a ring (with the natural ring structure inherited from the space of all functions $\{f : U \to S\}$) is called a **sheaf of rings**.*

Definition 19. (Algebraic variety)
*An **algebraic variety** is the datum $(X, \mathcal{O}_X)$ of a Zariski topological space X and a sheaf of rings $\mathcal{O}_X$ such that*

1. *There exists a covering $\{U_i\}_{i \in I}, U_i \subseteq X$ of algebraic sets such that $\mathcal{O}_{U_i}$ are sheaves of rings.*

2. *The diagonal* $\Delta \subseteq X \times X, \Delta := \{(x,x), x \in X\}$ *is closed in the product Zariski topology.*

Definition 20. (Regular function)
Let S *be a Zariski closed set. A* ***regular function*** f *is a function from* $f : S \to \mathbb{F}^n$ *such that there exists a n-variables polynomial* $p : \mathbb{F}^n \to \mathbb{F}^n$ *such that* $f = p|_S$.

The space of all the regular function on S can be shown to be an $\mathbb{F}$- algebra, which will be denoted by $\mathscr{A}(S)$

Definition 21. (Algebraic morphism)
A ***morphism*** *of algebraic varieties* X, Y *is a continuous map* $u : X \to Y$ *such that for any open subset* $V \subseteq Y$ *and any* $f \in \mathscr{A}(Y)$*, also the map* $f \circ u \in \mathscr{A}(Y)$.

Now we can finally define what is an algebraic group in an analogous way to how we defined topological and Lie groups.

Definition 22. (Algebraic group)
An ***algebraic group*** *is an algebraic variety endowed with a group structure such that the maps* $g \mapsto gh$ *and* $g \mapsto g^{-1}$ *are algebraic morphisms.*

B.1.2 Zariski closure of a linear semigroup

Goldsheid-Margulis Theorem reduces the problem of proving of the a.s. simplicity of the Lyapunov spectrum of a random matrix product to checking the semisimplicity, as a Lie group, of the Zariski closure of the free linear semigroup generated by the distribution of its support. Here we precisely define what this semigroup is. It is not difficult to prove that its Zariski closure is a group (see [BQ]).

Definition 23. (Linear semigroup)
Let $\mathbb{F}$ *be a field. A* ***linear semigroup*** Γ *is a subset of* $\mathrm{GL}(n, \mathbb{F})$ *such that ,if* $A, B \in \Gamma$ *then* $AB \in \Gamma$.
If $S \subseteq \mathrm{GL}(n, \mathbb{F})$ *is a general set of matrices, the* ***free semigroup*** *generated by* S*, is the smallest linear semigroup containing* S.

Notice that a semigroup lacks in general the neuter element and the elements' inverses to be a group. However, taking the Zariski closure of a linear semigroup one gets a (algebraic) group.

Lemma 22. ([BQ], Lemma 5.15.)
Let $\Gamma \in \mathrm{GL}(n, \mathbb{F})$ *be a linear semigroup. Then* $\overline{\Gamma}^{\mathrm{Zar}}$ *is a group.*

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