Infinite dimensional hamiltonian systems and nonlinear wave equation: periodic orbits with long minimal period

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Thesis submitted for the degree of Doctor Philosophiae

Academic Year 2004-2005

Al mio papà

Ringraziamenti

Sembra incredibile ma questo momento è arrivato... Anzitutto voglio ringraziare Luca. Tra poco diventerà mio marito e quindi il nostro legame va ben oltre quanto si possa raccontare in questa pagina (e poi siamo gelosissimi delle nostre cose!). Quello che desidero far sapere a quanti leggeranno è che non so se sarei riuscita a scrivere una tesi così bella (lo dico perché ho ricevuto tanti complimenti!) senza il suo aiuto. Lavorare insieme è stato divertentissimo, inclusi gli infuocati litigi per l'inglese e le notazioni. Tutto ha contribuito a farci innamorare sempre di più. La seconda persona da nominare è senza dubbio Luigi. A lui devo molto di più che il semplice titolo di Dottore di Ricerca, infatti è grazie a Luigi che ho conosciuto Luca e, secondo me, l'incontro non è stato casuale. Per me lui e Luca sono i matematici più bravi del mondo, soprattutto perché uniscono alla passione per la matematica l'amore per la vita. È stato molto naturale assegnare a Luigi un posto speciale vicino a noi sull'altare. In questa pagina dei ringraziamenti, un posto timido nella posizione ma importante nel significato come la persona di cui parlo, va al mio papà che ha sempre creduto che io facessi la "maestra" e di questo andava fierissimo: lo ringrazio per avermi insegnato che le cose più belle del mondo sono le più semplici. Vorrei anche ringraziare le mie sorelle Lorella e Rosita, la mia mamma Gina e le mie splendide nipotine Erika, Alessandra e Francesca. Nonostante non abbiano la più pallida idea di cosa io abbia studiato in questi tre anni e tutt'ora si chiedano il perché io l'abbia fatto, sono state sempre felici della mia scelta e mi hanno sempre aiutato a superare i problemi logistici. Non è da sottovalutare che sono state comprensive anche nei momenti in cui ero molto presa dallo studio (quelli in cui non viene niente!) e in cui loro stesse *sembravano* passare in secondo piano. In particolare grazie alle mie tre fantastiche nipotine che mi hanno sempre regalato un sorriso, un abbraccio e tante coccole. Ringrazio i miei cognatini Gaetano e Gianni, se lo meritano per mille motivi ma il più importante per me è che vogliono tanto bene alle mie sorelle. Grazie ai miei vecchi compagni di casa Cristina e Pietro per la loro sincera e profonda amicizia, per averci sopportato anche loro nei periodi "no" durante la stesura della tesi e soprattutto Cristina per essermi stata vicina in un momento difficile per la mia famiglia (anche lei sarà accanto a me e Luca sull'altare quindi non serve dire altro). Naturalmente grazie anche a Francesca, mamma Fiorella e papà Emanuele che mi hanno accolto con tanto amore nella loro casa. Ringrazio Max per i suoi continui incoraggiamenti e per quanto ha apprezzato tutto il mio lavoro. Non posso non ringraziare la compagna di ufficio di Luca, Francesca, che ha dovuto dividere la stanza in tre invece che in due! Grazie a Giampiero e Giampaolo per l'amicizia e le utili consulenze informatiche. Infine grazie a Bia per la compagnia.

Contents

Introduction

Wave propagation and vibration phenomena are governed by a partial differential equation known as the *wave equation*. Such equation involves functions depending on one time variable and a given number of spatial variables. According to the number of spatial variables three, two or one, we have the following examples from Physics: the equation governing the electromagnetic field in the space is the three–dimensional wave equation, the (small) oscillations of a stretched membrane are described by the two–dimensional wave equation and finally the one–dimensional wave equation governs the (small) vibrations of a stretched string.
We will study the last one, namely the "guitar string equation". Suppose that the length of the string is L and that when the string is in equilibrium it occupies the portion of the x–axis from $x = 0$ to $x = L$. We assume that the string vibrates in a horizontal plane, the (x, u)–plane, and that each point of the string moves only along a line perpendicular to the x–axis (parallel to the u–axis). The function $u(t, x)$ denotes the displacement at the instant t of the point of the string located, when in equilibrium, at x. Under the additional assumption that $\partial_x u$ is small, namely the vibrations of the string are small in amplitude, it can be shown that $u(t, x)$ must satisfy the partial differential equation

$$\rho\,\partial_{tt}u - \tau\,\partial_{xx}u = 0\,, \tag{1}$$

where τ is the tension of the string and ρ is its linear density. We have assumed that τ and ρ are constant. Introducing the velocity $c := \sqrt{\tau/\rho}$, and changing the time–scale $t \to ct$, equation (1) becomes

$$u_{tt} - u_{xx} = 0\,. \tag{2}$$

Equation (2) is linear and it is well understood. On the other hand, in this thesis we consider the equation

$$u_{tt} - u_{xx} = F(u) \tag{3}$$

where the term F is added to take into account also *nonlinear* phenomena.

The wave equation, as in (3), describes processes without dissipation of energy, namely, it is a *hamiltonian* partial differential equation. This means that, in suitable coordinates it can be seen as an *infinite dimensional hamiltonian system*. This property is crucial and allows us to study the problem as a hamiltonian dynamical system trying to extend all the well developed machinery of the finite dimensional case.

In this thesis we are not interested in the initial value problem for the equation (3), we will rather search solutions for all times. More precisely we will look for *periodic* solutions, namely motions for which the string comes back to its initial position after a suitable interval of time.

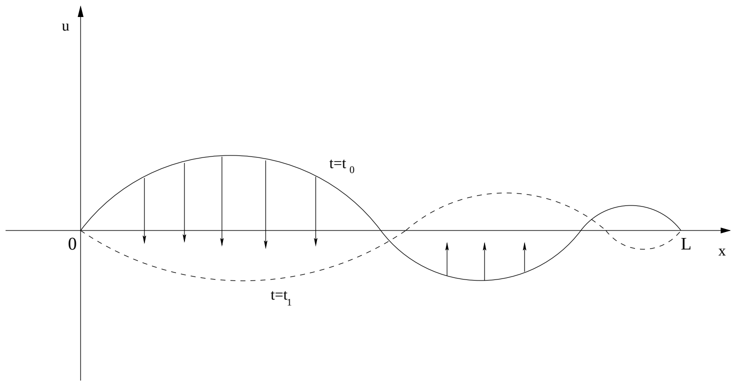

Figure 1: The guitar string from time t_0 to time t_1.

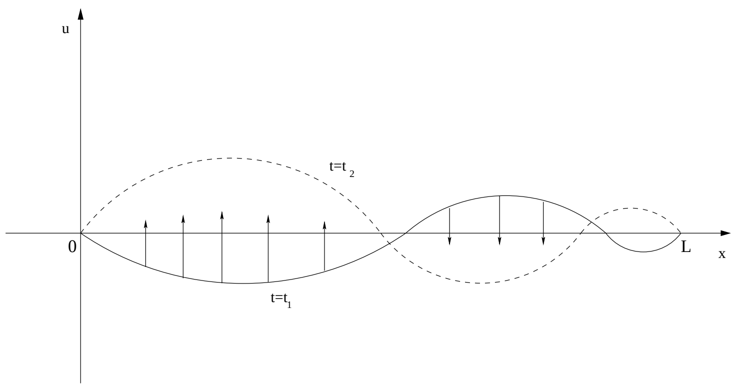

Figure 2: The guitar string from time t_1 to time $t_2 = t_0$.

Why searching periodic orbits

For finite dimensional hamiltonian system the importance of periodic solutions in order to understand the dynamics, was first highlighted by Poincaré. He wrote (talking about the three body problem)

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable."

Even if the periodic orbits form a zero measure set in the phase space, *"en effet, il ya une probabilité nulle pour que les conditions initiales du mouvement soient précisément celles qui correspondent à une solutions périodique,"*

Poincaré remarked their importance formulating the following conjecture:

"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13[1] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."* ([Po], Tome 1, ch. III, a. 36).

This conjecture encouraged the systematic study of periodic orbits of Poincaré himself followed by Lyapunov, Birkhoff, Moser, Weinstein, etc. A partial proof of this conjecture, in a generic sense (in the C^2 category of hamiltonian functions), was given by Pugh and Robinson in [PuRo83]: the periodic orbits are dense in every compact and regular energy surface. However, for specific systems, this conjecture is still open and very difficult to prove.
An intermediate step is the search of periodic orbits in the vicinity of invariant submanifolds. In this line, Birkhoff and Lewis [Bir31], [BirL34], [L34], showed in the thirties the existence of infinitely many periodic solutions close to elliptic periodic orbits. In the eighties, Conley and Zehnder [ConZ], proved the same result for maximal KAM tori; as a consequence the closure of the periodic orbits has positive measure in the phase space. Recently in [BeBiV04], the existence of periodic orbits clustering lower dimensional (elliptic) invariant tori was proved.

This thesis follows this line, using methods and techniques of the above papers, to prove existence of periodic orbits of the nonlinear wave equation close to the origin, which is an (elliptic) equilibrium of the associated infinite dimensional hamiltonian system.

In pursuit of periodic solutions for the nonlinear wave equation

The history of the quest of periodic in time solutions for the nonlinear wave equation is wide and involves many refined techniques from different fields in modern analysis.

The first real breakthrough was due to Rabinowitz [R67]. He wrote the problem as a *variational* one and showed the existence of periodic solutions under monotonicity assumption on the nonlinearity. Many authors, e.g. Brezis, Coron, Nirenberg etc., have used and developed Rabinowitz's variational methods to obtain related results. The advantage of such techniques relies on the fact that they are *global*, placing few restriction on the strength of the nonlinearity and allowing to obtain "large" amplitude solutions. On the other hand the variational techniques require a very strong restriction on the allowed periods of the solutions: the periods must be rational multiples of the length of the

[1] The Hamilton's equations.

spatial interval. Such restriction springs from the inability of variational methods to deal with "small divisors", which arise when the period is irrationally related to the length of the spatial interval.

A completely different approach, which uses the fact that the wave equation is an *infinite dimensional hamiltonian system*, was developed at the end of the eighties by Wayne, Craig, Kuksin, Pöschel, Bourgain etc. Such techniques, based on superconvergent (Newton's) methods, as the KAM theory or the Nash–Moser Implicit Function Theorem, allow to extend well known methods and results from the finite dimensional case and are the natural ways to deal with the lack of regularity due to the "small divisors" problem. Such techniques are somewhat complementary to the variational ones allowing us to obtain periods, which are irrationally related to the length of the spatial interval. However, unlike the variational methods, they are local, perturbative in nature and, therefore, restricted to equations with weak nonlinearity or, equivalently, to solutions of small amplitude.
The KAM theory deals with perturbations of integrable hamiltonian systems. Formal perturbation theories essentially date back to the nineteenth century, when especially Poincaré used them in various problems arising in Celestial Mechanics. The convergence of formal series expansions was unresolved until Kolmogorov, Arnold and Moser showed how it could be "accelerated" by a superconvergent Newton's scheme. In its "classical" form, KAM theory applies to systems with finitely many degrees of freedom and is devoted to the quest of quasi–periodic motions, in which the "small divisors" naturally appear. The presence of "small divisors" in searching periodic solutions is instead typical of the infinite dimensional situation.

The problem

We now describe the equation that we have studied in this thesis and its hamiltonian structure. Let us consider the nonlinear wave equation on the interval $[0, \pi]$ with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + \mu u + f(u) = 0 \\ \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{4}$$

where $\mu > 0$ and $f(0) = f'(0) = 0$.
Equation (4) can be studied as an infinite dimensional hamiltonian system (see section 4.1 for more details). Denoting $v = u_t$ the Hamiltonian is

$$H(v,u) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) dx\,,$$

where $g = \int_0^u f(s)ds$. The Hamilton's equations are

$$u_t = \frac{\partial H}{\partial v} = v\,, \qquad v_t = -\frac{\partial H}{\partial u} = u_{xx} - \mu u - f(u)\,.$$

Introducing coordinates $q = (q_1, q_2, \ldots)$, $p = (p_1, p_2, \ldots)$ through the relations

$$v(x) = \sum_{i\geq 1} \sqrt{\omega_i} p_i \chi_i(x)\,, \qquad u(x) = \sum_{i\geq 1} \frac{q_i}{\sqrt{\omega_i}}\,\chi_i(x)\,, \tag{5}$$

where $\chi_i(x) := \sqrt{2/\pi}\sin ix$ and $\omega_i := \sqrt{i^2+\mu}$, the Hamiltonian takes the form

$$H = \frac{1}{2}\sum_{i\geq 1}\omega_i(q_i^2+p_i^2) + \text{higher order terms}\,. \tag{6}$$

Since we are interested here in small amplitude periodic solutions, the higher order terms in (6), may be, in first approximation, neglected and we can consider only the quadratic Hamiltonian $\Lambda := \sum_{i\geq 1}\omega_i(q_i^2+p_i^2)/2$. The origin is an elliptic equilibrium point for Λ and the Λ–orbits are the superpositions of the harmonic oscillations $q_i(t) = A_i\cos\big(\omega_i t+\varphi_i\big)$ of the basic modes χ_i, where $A_i \geq 0$, $\varphi_i \in \mathbb{R}$ and ω_i are, respectively, the amplitude, the phase and the frequency of the i^{th} harmonic oscillator $\omega_i(q_i^2+p_i^2)/2$, $i\geq 1$.
Analogously, neglecting the term $f(u)$ in (4), every solution of the linear equation $u_{tt} - u_{xx} + \mu u = 0$, is of the form

$$u(t,x) = \sum_{i\geq 1} a_i\cos(\omega_i t+\varphi_i)\sin ix\,, \tag{7}$$

with $a_i = A_i\sqrt{2/\pi\omega_i} \geq 0$. These solutions, in general, are periodic if only one basic mode is excited, namely if $a_i = 0$ for any $i \neq i_0$, for a suitable $i_0 \geq 1$, while $a_{i_0} > 0$. If at least two basic modes are excited, the situation changes: except a countable set of $\mu > 0$, for any $\mathcal{I} := \{i_1,\ldots,i_N\} \subset \mathbb{N}^+$, $N \geq 2$, the vector $\omega := (\omega_{i_1},\ldots,\omega_{i_N})$ is rationally independent (see Lemma 4.2.1 for a proof) and, therefore, any solution of the form $\sum_{i\in\mathcal{I}} a_i\cos(\omega_i t+\varphi_i)\sin ix$ is quasi–periodic. Then, if at least two amplitudes in (7) are different from zero, the solution $u(t,x)$ cannot be periodic. One can conclude that, except a countable set of $\mu > 0$, the only periodic solutions of $u_{tt} - u_{xx} + \mu u = 0$ are of the form $u(t,x) = a_{i_0}\cos(\omega_{i_0}t+\varphi_{i_0})\sin i_0 x$, for $i_0 \geq 1$.

By the light of these considerations, a natural way to find periodic solutions of (4), see for example [LinSh88],[W90],[Ku93],[CW93],[Su98],[Bou99],[B00], [BeBo03], [GM04],[GMPr04],[BeBo05] (and references therein), is to extend the Lyapunov Center Theorem (see pg.31) for finite dimensional hamiltonian systems in a neighborhood of an elliptic equilibrium. Namely, for any fixed $i_0 \geq 1$, one constructs a family of small amplitude periodic orbits of the Hamiltonian H bifurcating from the i_0^{th} basic mode. This can be done since, for μ far away from zero, the linear frequency ω_{i_0} is not resonant with the other ones. The frequencies $\tilde{\omega}$'s of the solutions will be close to the linear frequency $\omega := \omega_{i_0}$ and the corresponding periods $2\pi/\tilde{\omega}$'s will be close to the linear period $2\pi/\omega$.

Main results

Here we look for solutions having *large* minimal period. Such solutions are interesting as an example of the complexity of the dynamics and since they come up only as a nonlinear phenomenon.

A classical way to find long-period orbits, close to an elliptic equilibrium point[2] in finite dimensional systems, was carried out by Birkhoff and Lewis in [BirL34] (see also [L34], [Mo77]). Their procedure consists in putting the system in fourth order Birkhoff normal form: the truncated Hamiltonian obtained by neglecting the five or higher order terms is integrable. If the so called "twist" condition on the action–to–frequency map holds, there exist infinitely many resonant tori on which the motion of the truncated Hamiltonian is periodic. By the Implicit Function Theorem and topological arguments, Birkhoff and Lewis showed the existence of a sequence of resonant tori accumulating at the origin with the property that at least two periodic orbits bifurcate from each of them.

As we have just said, in this thesis we apply the Birkhoff–Lewis procedure to the nonlinear wave equation seen as an infinite dimensional hamiltonian system. Such extension to the hamiltonian PDEs was recently carried out in [BBe05] for the beam equation and the NLS (see remark 4.2.10 below for comparison).

We point out that, in the infinite dimensional case, one meets two difficulties that do not appear in the finite dimensional one: the generalization of the Birkhoff normal form and a small divisors problem.
Concerning the first difficulty, we consider only a "seminormal form", following [P96a]. We suppose that f is a real analytic odd function $f = \sum_{m\geq 3} f_m u^m$ with $f_3 \neq 0$ and fix a *finite* subset $\mathcal{I} \subset \mathbb{N}$. Then we put the Hamiltonian H in (6) in the form

$$H = \Lambda + \bar{G} + \hat{G} + K$$

where $\Lambda = \sum_{i\geq 1} \omega_i (p_i^2 + q_i^2)/2$, $\bar{G} + \hat{G}$ is the fourth order term with $\bar{G}$ depending only on the "actions" $\mathcal{J}_i := (p_i^2 + q_i^2)/2$, $i \in \mathbb{N}^+$, $\hat{G}$ depending only on p_i, q_i, $i \notin \mathcal{I}$, and K is the sixth order term. However, to put the Hamiltonian in normal form, the linear frequencies ω_i's must satisfy a suitable non resonance condition (see Lemma 4.1.8) (which deteriorates for μ going to zero).
The truncated Hamiltonian $\Lambda + \bar{G} + \hat{G}$ possesses the $2N$–dimensional invariant manifold $\{p_i = q_i = 0,\ i \notin \mathcal{I}\}$, which is foliated by N–dimensional invariant tori. Due to the "twist" property of $\bar{G}$, which follows from $f_3 \neq 0$, the linear frequencies of such tori are an open set of $\mathbb{R}^N$. We focus on completely resonant frequencies $\tilde{\omega} := (\tilde{\omega}_{i_1}, \ldots, \tilde{\omega}_{i_N})$, namely on $\tilde{\omega}$'s such that there exist T's with $\tilde{\omega}T/2\pi \in \mathbb{Z}^N$. Then the $(\Lambda + \bar{G} + \hat{G})$–flow on the associated N–dimensional tori is periodic with periods T's. Such lower dimensional tori are highly degenerate. Hence, in order to show the persistence of periodic orbits for the whole Hamiltonian H, we have to impose some non–degeneracy conditions to avoid resonances between the torus frequencies and the ones of the normal oscillations. This is the point in which the small divisors problem appears.

[2]As we have just said above, [BirL34] actually considers a neighborhood of an elliptic, non constant, periodic orbit, but the scheme is essentially the same for elliptic equilibria.

The estimate on the small divisors is the crucial step. In this thesis we developed two different strategies to overcome this problem according to the fact that the frequency $\omega := T/2\pi$ is rational or irrational. In the former case, imposing a strong condition on the small divisors, the proof of existence of T–periodic solutions can be obtained by the (standard) Implicit Function Theorem in Banach spaces and topological arguments. In the latter case (which is the most difficult one), we impose only a Diophantine–type condition on the small divisors and reach the proof by the Nash–Moser Implicit Function Theorem and "symmetry arguments".

1) Long time periodic solutions with rational frequency

As we have just said, in our first result, we impose a strong condition on the small divisors (see (4.48)), avoiding KAM analysis (see remark 4.2.13). For such (technical) reason we can consider only periods T which are multiple of 2π (as in the classical variational approach of Rabinowitz, Brezis, Nirenberg etc.) and we also need

$$\mu^2 T \ll 1 .$$

Therefore we consider the "mass" $\mu > 0$ as a *small* parameter and we make our analysis perturbative with respect to it. On the other hand, for $\mu \to 0$ the frequencies ω_i tend to the *completely resonant* ones, namely $\omega_i \to i$, and the above described normal form *degenerates*, in the sense that its domain of definition shrinks to zero while the remainder term K blows up.
The set $\mathcal{T}$ of the periods T's verifying the above properties is *finite* but its cardinality goes to infinity, when μ goes to zero. μ^2–close to $\omega = (\omega_{i_1}, \dots, \omega_{i_N})$, we construct a set of completely resonant frequencies $\tilde{\omega} = \tilde{\omega}(T) \in \mathbb{R}^N$, parametrized by the periods $T \in \mathcal{T}$, with $\tilde{\omega} T/2\pi \in \mathbb{Z}^N$. At the same time, we construct a set of actions $\tilde{\mathcal{J}}_i = \tilde{\mathcal{J}}_i(T) \approx \mu^2$, $i \in \mathcal{I}$, parametrized by $T \in \mathcal{T}$, such that $\tilde{\omega}_i$ is the image of $\tilde{\mathcal{J}}_i$ through the action–to–frequency map $\partial_{\mathcal{J}_i} H$ on $\{p_i = q_i = 0, \ i \notin \mathcal{I}\}$. We will prove the existence of T–periodic solutions of H, μ^2–close to the T–periodic solutions of the truncated Hamiltonian $\Lambda + \bar{G} + \hat{G}$ defined as:

$$\begin{cases} p_i^*(t) = \sqrt{2\tilde{\mathcal{J}}_i} \sin(\tilde{\omega}_i t + \varphi_i), \quad q_i^*(t) = \sqrt{2\tilde{\mathcal{J}}_i} \cos(\tilde{\omega}_i t + \varphi_i), & \text{for } i \in \mathcal{I} \\ p_i^*(t) = q_i^*(t) = 0, & \text{for } i \in \mathcal{I}^c , \end{cases} \tag{8}$$

where the angles φ_i, $i \in \mathcal{I}$, have to be determined.
We perform a Lyapunov–Schmidt reduction as in [BeBiV04],[BBe05], splitting the problem into two equations: the kernel (or bifurcation) equation on the N–dimensional torus $\{\varphi_{i_1}, \dots, \varphi_{i_N}\}$ and the range equation on its orthogonal space. We first solve the infinite dimensional range equation by the Contraction Mapping Theorem using the above estimate on the small divisors and controlling the blow–up of the remainder term K for μ going to zero. Critical points of the action functional restricted to the solutions of the range equation satisfy the kernel equation. Since the restricted action functional results to be

defined on a N–dimensional torus, existence of critical points and, therefore, of solutions of H follows.

Before stating our result, we return to the PDE–formulation. Recalling (5) and (8), for every $T \in \mathcal{T}$, we find T–periodic solutions of (4), μ^2–close to the T–periodic "pseudo–solution"

$$\tilde{u}(t,x) := \mu \sum_{i \in \mathcal{I}} a_i \cos(\tilde{\omega}_i t + \varphi_i) \sin ix \,,$$

where $\mu a_i := \sqrt{\tilde{\mathcal{J}}_i / \pi \omega_i}$, $a_i \approx 1$. We note that the minimal period of $\tilde{u}$ is $\tilde{T}^{\min} = T/\gcd(k_{i_1}, \ldots, k_{i_N})$, where $k_i := \tilde{\omega}_i T/2\pi \in \mathbb{N}$, for $i \in \mathcal{I}$. Since $\tilde{\omega}$ is μ^2–close to the rationally independent[3] frequency ω, $\tilde{T}^{\min}$ will be large for μ small. Being $u - \tilde{u} \approx \mu^2$, we obtain an analogous estimate on the minimal period $T^{\min}$ of u. Not all the solutions of (4), corresponding to different T's belonging to $\mathcal{T}$, have to be geometrically distinct. However, by the very precise estimate $u - \tilde{u} \approx \mu^2$, we prove that the total number of geometrically distinct solutions found here is also large for μ small.
We now state our first result, that was announced in [BDG05] and proved in [BDG05II].

Theorem 1. *Let f be a real analytic, odd function of the form $f(u) = \sum_{m \geq 3} f_m u^m$, $f_3 \neq 0$. Let $N \geq 2$ and $\mathcal{I} := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+$. Then there exists a constant $0 < c < 1$ such that, if $0 < \mu \leq c$, there exist at least c/μ geometrically distinct smooth periodic solutions $u(t,x)$ of (4), verifying*

$$\sup_{t \in \mathbb{R}\,,\ x \in [0,\pi]} \left| u(t,x) - \mu \sum_{i \in \mathcal{I}} a_i \, \cos(\tilde{\omega}_i t + \varphi_i) \, \sin ix \right| \leq c^{-1} \mu^2 \,, \tag{9}$$

for suitable $a_i \geq c$, $\varphi_i \in \mathbb{R}$ and $\tilde{\omega}_i \in \mathbb{R}$

$$|\tilde{\omega}_i - \omega_i| \leq c^{-1} \mu^2 \qquad \left(\omega_i := \sqrt{i^2 + \mu} \right). \tag{10}$$

The minimal period $T^{\min}$ of any solution belongs to $\pi\mathbb{Q}$ and satisfies

$$\frac{c}{\mu} \leq T^{\min} \leq \frac{\text{const}}{\mu^2} \,. \tag{11}$$

Remark. *The solutions $u(t,x)$ found in Theorem 1 are infinitely differentiable. Actually, they are analytic in the spatial variable. Estimate (9) can be improved and, in particular, one can obtain analogous estimates on the derivative of u of any order $k \geq 1$. However, in this case, the constant c will depend on k. See also Remark 4.2.20 for further details. Arguing as in the proof of Theorem 2 we could obtain solutions analytic also in the time–variable.*

[3] Except a countable set of μ's.

2) Long time periodic solutions with irrational frequency

Two limitations are evident in the statement of Theorem 1: the periods cannot go to infinity and must be rational multiples of π. These restrictions are connected and follow by the strong condition on the small divisors that we have imposed (see (4.48)). As we have just said, such strong condition allows us to solve the range equation by the standard Implicit Function Theorem. To avoid the limitations on the period, one has to weak the condition on the small divisors imposing a Diophantine–type condition (known as "first order Melnikov condition") and solve the range equation by a Nash–Moser Implicit Function Theorem. Then one has to face the difficult task of inverting the linearized operator. One proceeds with the quite standard "superconvergent" iteration algorithm. One has anyhow to solve the bifurcation equation on a *Cantor set* (resulting by the excision procedure performed to control the small divisors). As we will discuss on Remark 4.2.13, this point is, in general, very difficult.
Summarizing, there are two main difficulties in extending Theorem 1:

(i) solving the bifurcation equation on a Cantor set,

(ii) inverting the linearized operator.

We now briefly discuss our second result laying emphasis on how we have overcome these difficulties.
At the beginning we proceed as for Theorem 1, fixing $\mathcal{I} \subset \mathbb{N}^+$ and putting the system in Birkhoff Normal Form with respect to the $\mathcal{I}$–modes. We look for periodic solutions close to the N–dimensional tori filled by T–periodic orbits of the truncated Hamiltonian $\Lambda + \bar{G} + \hat{G}$. However in this case we consider μ *fixed*. This means that we do not need μ small and we can consider any μ positive. In particular, instead of μ, the perturbative parameter will be here the frequency $\varpi := 2\pi/T$.
To overcome problem (i) we use a different strategy with respect to the one used in Theorem 1; in particular we do not really perform a Lyapunov–Schmidt reduction. The idea is based on the following simple observation: if the nonlinear term f in (4) is *odd*, the associated Hamiltonian H in (6) satisfies the symmetry property $H(-p, q) = H(p, q)$. This symmetry is preserved by the Birkhoff Normal Form. Then we look for solutions in the subspace of functions $p(t)$ odd and $q(t)$ even. Using the symmetries we first solve the equations of motions for $\{p_i, q_i\}_{i\in\mathcal{I}}$ for any value of the "parameters" $\{p_i, q_i\}_{i\in\mathcal{I}^c}$ and thereafter we have to solve the resulting equations for $\{p_i, q_i\}_{i\in\mathcal{I}^c}$. In the language of Theorem 1, this procedure corresponds to prove by symmetry that the bifurcation equation on the N–dimensional torus $\{\varphi_{i_1}, \ldots, \varphi_{i_N}\}$, with $\{\varphi_i\}_{i\in\mathcal{I}}$ introduced in (8), has solution for $\varphi_i = 0$ for any $i \in \mathcal{I}$. We stress that this "symmetry trick" works only for f odd; a way to face the bifurcation equation on a Cantor set in the case f not odd was given in [BeBo05] for "Lyapunov type" orbits (namely $T \approx 2\pi$).

Regarding problem (ii) we remark that, in this case, the linearized operator is a *first* order *not symmetric* operator (and not a second order symmetric one, as usual). We develop the linearized operator in time–Fourier expansion and split it into a "diagonal part" and an "off–diagonal part". Two problems appear causing serious difficulties in the spectral analysis and in controlling the influence of small divisors: the diagonal part is not symmetric and the off–diagonal one is not a Toepliz operator. This last problem comes from the fact that the linearized expression of $\{p_i, q_i\}_{i\in\mathcal{I}}$ as function of $\{p_i, q_i\}_{i\in\mathcal{I}^c}$ is not of Toepliz type.
We now state our second result[4] in which we prove existence of periodic solutions of (4) with f odd and $\mu > 0$ fixed. The frequencies of these orbits belong to an unbounded Cantor set of asymptotically full measure. Finally we prove that the origin is an accumulation point of periodic orbits of longer and longer minimal period, extending the Birkhoff–Lewis Theorem to the nonlinear wave equation.

Theorem 2. *Fix $\mu > 0$ and let f be a real analytic, odd function of the form $f(u) = \sum_{m\geq 3} f_m u^m$, $f_3 \neq 0$. Let $N \geq 2$ and $\mathcal{I} := \{i_1, \ldots, i_N\} \subset \mathbb{N}^+$. Then there exists a constant $0 < c < 1$ and a set $\mathcal{C} \subset (0, c]$ satisfying*

$$\lim_{r\to 0^+} \frac{\mathrm{meas}\big(\mathcal{C}\cap(0,r]\big)}{r} = 1\,, \tag{12}$$

such that for all $\varpi = 2\pi/T \in \mathcal{C}$ there exists a T–periodic analytic solution $u(t,x)$ of (4), satisfying

$$\sup_{t\in\mathbb{R}\,,\ x\in[0,\pi]} \left| u(t,x) - \sqrt{\varpi}\sum_{i\in\mathcal{I}} a_i\,\cos(\tilde{\omega}_i t)\,\sin ix \right| \leq c^{-1}\varpi\,, \tag{13}$$

for suitable $a_i \geq c$, $\tilde{\omega}_i \in \mathbb{R}$

$$|\tilde{\omega}_i - \omega_i| \leq c^{-1}\varpi \qquad \big(\omega_i := \sqrt{i^2+\mu}\,\big)\,. \tag{14}$$

Moreover, fix $0 < \rho < 1/2$, then, except a zero measure set of μ's, the minimal period $T^{\min}$ of the T–periodic orbit satisfies

$$T^{\min} \geq \mathrm{const}\, T^{\rho}\,. \tag{15}$$

Remark. *We can also prove an estimate similar to (13) in a suitable analytic norm. The Cantor set $\mathcal{C}$ is formed by Diophantine irrational numbers.*

Theorems 1 and 2 are the first results about periodic solutions of *large minimal period* for the autonomous nonlinear wave equation (for different–type results on large minimal period, in the forced case, see [T87]).
We also remark the following substantial difference between the periodic solutions found by analogy of the Lyapunov Center Theorem and the ones we

[4] A paper on this subject is in preparation, see [BDGIII].

construct. The "Lyapunov type" orbits are obtained as the continuation of one linear mode to the nonlinear system; they involve only one of the linear harmonic oscillator, the amplitudes on the other modes being much smaller (except in the resonant case $\mu = 0$, discussed in [BP01], [BeBo03], [BeBo04], [BeBo05]). On the other hand, the periodic solutions constructed here involve $N \geq 2$ harmonic oscillators, oscillating with the same order of magnitude, and are a *truly nonlinear phenomenon*, as they do not have any analogue in the linear case, where all periodic orbits are the oscillation of only one basic mode and do not have long minimal period (see also [BeBiV04], [BBe05]).

Scheme of the thesis

PART I.
We discuss, at first, some preliminaries regarding finite dimensional hamiltonian systems. In particular, we study the behavior of such systems in a neighborhood of an elliptic equilibrium point. We introduce the Birkhoff Normal Form and the notion of non–resonance between the linear frequencies. Therefore we report the classical theorems of Lyapunov, Weinstein, Moser, on the continuation of the periodic solutions of the linearized system to the nonlinear one. We describe the hamiltonian structure of some partial differential equations and study their basic properties as infinite dimensional systems. In particular we consider the wave equation, the Schrödinger equation and the beam equation.
Thereafter, we review most of known results about existence of periodic solutions of such hamiltonian PDEs. In particular we treat in details some results that we consider interesting for the used methods and techniques. Some of these tools will be used in the proof of our main results.
For completeness, we discuss quasi–periodic and almost–periodic solutions for hamiltonian PDEs. We report and briefly discuss the main known results.

PART II.
We prove existence and multiplicity of small amplitude periodic solutions with large period for the nonlinear wave equation with small "mass". Such solutions bifurcate from resonant finite dimensional invariant tori of the fourth order Birkhoff seminormal form of the associated hamiltonian system. We also prove that the number of geometrically distinct solutions and their minimal periods tend to infinity when the "mass" tends to zero.

PART III.
We discuss a Nash–Moser algorithm to prove existence of periodic solutions of the nonlinear wave equation with periods belonging to an unbounded Cantor set of asymptotically full measure. We also give an estimate from below on the minimal periods.

ACKNOWLEDGEMENTS. I wish to thank Luigi Chierchia and Luca Biasco for supervising this thesis. I also thank Massimiliano Berti for his interest in my work and useful discussions.

Part I

Preliminaries and known results

Chapter 1

Hamiltonian dynamical systems

The aim of this chapter is to study some partial differential equations (PDEs) as evolution equations in suitable functional spaces. In particular, we will consider some PDEs that can be seen as infinite dimensional hamiltonian systems. The general form of such equations is the following. Let $\mathcal{P}$ be a Hilbert space of functions defined on some domain D, typically $\mathcal{P}$ will be a Sobolev space. Let us denote by $\langle \cdot\,, \cdot \rangle$ the scalar product in $L^2(D)$ and by J a non degenerate antisymmetric operator. Then, the evolution equation we consider is

$$\partial_t w(t) = J\nabla H\big(w(t)\big)\,, \tag{1.1}$$

where $w \in \mathcal{P}$ and ∇H denotes the gradient of the hamiltonian function $H : \mathcal{P} \to \mathbb{R}$ with respect to the $L^2(D)$–scalar product.
This hamiltonian formalism can be generalized in the following way. On $\mathcal{B}$, which can be a Banach or a Hilbert manifold, let us suppose to have a symplectic 2–form $\omega : T_w(\mathcal{B}) \times T_w(\mathcal{B}) \longrightarrow \mathbb{R}$, where $T_w(\mathcal{B})$ is the tangent space to $\mathcal{B}$ in w, with the following properties: (i) $d\omega = 0$, (ii) ω is non degenerate in any point of $\mathcal{B}$. We note that if $\mathcal{B}$ is finite dimensional, then it has even dimension. Given a hamiltonian function $H : \mathcal{B} \to \mathbb{R}$, we define the associated hamiltonian vector field $X_H(w)$ on $T_w(\mathcal{B})$, by $\omega[X_H, Y] := dH[Y]$, for any $Y \in T_w(\mathcal{B})$. If $\mathcal{B}$ is infinite dimensional, neither the Hamiltonian H, nor the forms ω and dH, nor the vector field X_H are necessarily bounded; so one has to consider the restriction to their definition domains. In this case one has to weaken the condition (ii) above. The Hamilton's equations are

$$\partial_t w(t) = X_H\big(w(t)\big)\,. \tag{1.2}$$

If $\mathcal{B}$ is an Hilbert space, by the Riesz Theorem, the symplectic form ω, admits a representation in terms of the scalar product:

$$\omega[X, Y] = \langle X, JY\rangle\,, \tag{1.3}$$

where J is a suitable non degenerate skew–symmetric operator. On the other hand, given a non degenerate anti–symmetric operator J on a Hilbert space $\mathcal{B}$, the 2–form defined in (1.3) satisfies (i) and (ii) above.

The "energy" H is constant along the motion since

$$\frac{d}{dt}H(w(t)) = dH[\partial_t w] = \omega[X_H, \partial_t w] = \omega[X_H, X_H] = \langle X_H, JX_H \rangle = 0\,,$$

by (1.2), (1.3) and by the skew-symmetry of J.

Let us consider the following example given by the wave equation

$$u_{tt}(t,x) - u_{xx}(t,x) = 0 \tag{1.4}$$

with Dirichlet boundary conditions on $[0,\pi]$, $u(t,0) = u(t,\pi) = 0$. We will work in the Hilbert space

$$\mathcal{B} = L^2([0,\pi]) \times L^2([0,\pi]) \ni (v,u) = w\,,$$

endowed with the scalar product $\langle w_1, w_2 \rangle := \int_0^\pi (v_1 v_2 + u_1 u_2) dx$. We identify $T_w(\mathcal{B})$ and $\mathcal{B}$. Here the symplectic form ω is defined by (1.3) where $J(v,u) := (-u, v)$, namely $\omega[(h_1,k_1),\,(h_2,k_2)] := \int_0^\pi (h_2 k_1 - h_1 k_2)\, dx$. The Hamiltonian is

$$H(w) = H(v,u) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} \right) dx\,.$$

Here H is an unbounded operator from $\mathcal{B}$ into $\mathbb{R}$. Its definition domain is the phase space $\mathcal{P} := L^2([0,\pi]) \times H_0^1([0,\pi]) \subset \mathcal{B}$. Let $Y = (h,k)$, then

$$\langle X_H, JY \rangle = \omega[X_H, Y] = dH[Y] = \int_0^\pi (vh + u_x k_x)\, dx = \int_0^\pi (vh - u_{xx} k)\, dx\,,$$

so that the hamiltonian vector field $X_H(w) = (u_{xx}, v)$ is defined for $u \in H^2 \cap H_0^1$. The Hamilton's equations are

$$\dot{v} = u_{xx}\,, \qquad\qquad \dot{u} = v\,. \tag{1.5}$$

Then a weak solution $w : t \mapsto \mathcal{P}$ of the Hamilton's equations (1.5), is a weak solution of $u_{tt} = u_{xx}$.

1.0.1 Linearized equation

This thesis deals with periodic solutions. The simplest case is the equilibrium point. We say that w_* is an *equilibrium point* for H if $\partial_w H(w_*)$=0. It's not restrictive to suppose that H has an equilibrium point in $w = 0$, namely $\partial_w H(0)$=0. Moreover, since (1.2) is not affected by adding a constant, we also suppose $H(0) = 0$.
Developing the Hamiltonian H in a neighborhood of its equilibrium point $w = 0$, we have

$$H(w) = H_2(w) + N(w)\,,$$

where

$$H_2(w) := \frac{1}{2}\langle w, \partial_w^2 H(0) w \rangle \tag{1.6}$$

is the quadratic term and $N(w)$ contains all the terms of order three or higher. The small amplitude motions generated by the Hamilton's equations (1.1) are described, in first approximation, by the linearized equation

$$\partial_t w = J\partial_w^2 H(0)w = J\partial_w^2 H_2(w)\,. \tag{1.7}$$

Since we are interested in (small amplitude) periodic solutions, we will study in particular hamiltonian systems for which the linearized equation (1.7) possesses periodic solutions. A typical case is the elliptic equilibrium.

Definition 1.0.1. *The stationary point $w = 0$ is elliptic for the hamiltonian system* (1.1) *if all the eigenvalues of $J\partial_w^2 H(0)$ are pure imaginary.*

Indeed, if $\mathrm{i}\lambda_*$, with $\lambda_* \in \mathbb{R}$, is an eigenvalue of $J\partial_w^2 H(0)$ and w_* is one of its corresponding eigenvectors, then, for all $a \in \mathbb{R}$, the solution $w(t) = e^{\mathrm{i}\lambda_* t} a w_*$ of $\dot{w} = J\partial_w^2 H(0)w$, with initial datum $w(0) = aw_*$, is $2\pi/\lambda_*$–periodic.

1.0.2 Harmonic oscillator

We are going to show a simple example of a hamiltonian system with an elliptic equilibrium point. Let us define a quadratic Hamiltonian H_2 on the Hilbert space $\mathcal{B} := \ell^2 \times \ell^2$,

$$H_2(p,q) := \sum_{j\geq 1} \omega_j \, \frac{p_j^2 + q_j^2}{2}\,,$$

with $p = (p_1, \ldots)$, $q = (q_1, \ldots)$ and $w = (p,q) \in \mathcal{B}$. The equations of motion are

$$\partial_t p_j(t) = -\omega_j q_j(t)\,, \qquad \partial_t q_j(t) = \omega_j p_j(t)\,, \qquad t \in \mathbb{R}\,, \tag{1.8}$$

and describe infinitely many harmonic oscillators of frequency ω_j. Equations (1.8) can be written in the form[1]

$$\dot{w} = J\partial_w^2 H(0)w\,, \tag{1.9}$$

where $J(p,q) := (-q,p)$. The eigenvalues of

$$J\partial_w^2 H(0) = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix},$$

where $\Omega := \mathrm{diag}[\omega_j]$, are $\pm\mathrm{i}\omega_j$, $j \geq 1$. Therefore $w = 0$ is an elliptic equilibrium point. The equations (1.8) have solutions

$$\begin{aligned} p_j(t) &= \cos(\omega_j t)p_j(0) - \sin(\omega_j t)q_j(0)\,, \qquad j \geq 1\,, \\ q_j(t) &= \sin(\omega_j t)p_j(0) + \cos(\omega_j t)q_j(0)\,, \qquad j \geq 1\,. \end{aligned}$$

Let us, briefly, discuss the periodicity of such solutions. Every component $(p_j(t), q_j(t))$ is periodic in time, with frequency ω_j, while the solution itself can

[1] From now on we denote by " $\cdot$ " the derivative with respect to time.

be periodic, quasi–periodic or almost periodic. It is periodic if there exists a frequency $\omega \in \mathbb{R}$ such that, for all $j \geq 1$, with $(p_j(0), q_j(0)) \not\equiv (0,0)$, there exists $k_j \in \mathbb{Z}$ with

$$\omega = \frac{\omega_j}{k_j} .$$

The solution is quasi-periodic if there exists a finite base of frequencies $\omega_1, \ldots,$ ω_n such that, for all $j \geq 1$, $(p_j(0), q_j(0)) \not\equiv (0,0)$,

$$\omega_j = \sum_{\ell=1}^{n} \omega_\ell \, k_{\ell j} , \tag{1.10}$$

for suitable $k_{\ell j} \in \mathbb{Z}$. Finally, the solution is almost periodic if there exists an infinite base of frequencies such that (1.10) holds.

It is natural to wonder if periodic, quasi–periodic and almost–periodic solutions persist under perturbation due to cubic or higher order terms.
As regards the finite–dimensional case, the existence of periodic orbits is proved by the Lyapunov Center Theorem (in the non–resonant case) and the theorems of A. Weinstein and J. Moser (in the resonant case). On the other hand, the existence of quasi–periodic solutions is proved by the KAM Theorem (by Kolmogorov, Arnold and Moser) and by Melnikov's Theorem. We note that almost periodic solutions cannot exist in the finite dimensional case.
In the following we deal with periodic solutions. Quasi–periodic and almost–periodic solutions will be discussed in the Appendix.

1.1 Finite–dimensional hamiltonian systems: some classical results

Let us consider the linearized equation (1.7). Suppose that $\partial_w^2 H(0)$ is non degenerate. If the phase space $\mathcal{P}$ has finite dimension, the spectra of $J\partial_w^2 H(0)$ is finite and it can be decomposed in two subsets:

(i) a finite sequence of eigenvalues λ_ℓ with real part different from zero, for $\ell = 1, 2, \ldots, \ell_0$;

(ii) couples of pure imaginary eigenvalues $\pm \mathrm{i}\omega_j$, for $j = 1, \ldots, j_0$.

If w_0 belongs to the eigenspace associated to an eigenvalue of the first type, then the solution with initial datum w_0, namely $t \longmapsto \exp(tJ\partial_w^2 H(0))w_0$ is never periodic, except in the case $w_0 = 0$. On the other hand, if w_0 belongs to the eigenspace associated to one of the eigenvalues of the second type, then every solution is periodic with period $2\pi/\omega_j$.

It is natural wondering if periodic solutions of (1.7) persist under perturbations due to cubic or higher order terms. The answer is, in general, negative as the following (non-hamiltonian) example shows. Let be $(p,q) \in \mathbb{R}^2$, consider the following nonlinear system

$$\begin{cases} \dot p = -q + (p^2+q^2)p \\ \dot q = p + (p^2+q^2)q \,. \end{cases} \tag{1.11}$$

The eigenvalues of the linearized system $\dot p = -q$, $\dot q = p$ are $\pm \mathrm{i}$, so $(p,q)=(0,0)$ is an elliptic equilibrium point. Any solution $\big(p(t),q(t)\big)$ of (1.11), satisfies the following

$$\frac{d}{dt}\big(p^2(t)+q^2(t)\big) = 2\big(p^2(t)+q^2(t)\big)\,.$$

Suppose that $\big(p(t),q(t)\big)$ is a T–periodic solution of (1.11), than

$$0 = \int_0^T \frac{d}{dt}\big(p^2(t)+q^2(t)\big)\,dt = \int_0^T 2\big(p^2(t)+q^2(t)\big)\,dt\,,$$

from which it follows that $\big(p(t),q(t)\big) \equiv 0$.

We have just said that the study of the oscillations of a system in the neighborhood of an equilibrium point or a periodic motion usual begins with linearization. After this, the main properties of the oscillations in the original system can frequently be determined using the theory of normal forms of Poincaré-Birkhoff. This theory is an analogue of perturbation theory. Here the linearized system plays the role of the unperturbed system with respect to the original one.

To fix ideas, let us consider a nonlinear perturbation of a $2n$-dimensional integrable hamiltonian system, in a neighborhood of an elliptic equilibrium point (e.g. the origin) such that all the eigenvalues of $J\partial^2_w H(0)$ are distinct.
The following classical result is due to Williamson [Wi36].

Proposition 1.1.1. *If the eigenvalues are all distinct and purely imaginary, then the quadratic part of the Hamiltonian H can be reduced to the normal form*

$$H_2 = \frac{1}{2}\,\omega_1(p_1^2+q_1^2) + \ldots + \frac{1}{2}\omega_n(p_n^2+q_n^2)\,.$$

Choosing coordinates $w =: (p,q) = (p_1,\ldots,p_n,q_1,\ldots,q_n)$ in the phase space $\mathbb{R}^n \times \mathbb{R}^n$, we can write the Hamiltonian in the following way:

$$H = \sum_{j=1}^n \omega_j \frac{p_j^2+q_j^2}{2} + N(p,q)\,, \qquad \omega_j \in \mathbb{R}\,, \tag{1.12}$$

where the nonlinearity N verifies $N(p,q) = o(|p|^2+|q|^2)$. The Hamilton's equations of the linearized system are (1.8) and describe n decoupled harmonic oscillators with frequencies ω_j.

Birkhoff Normal Form

A standard way to investigate finite dimensional hamiltonian systems, close to an elliptic equilibrium point, is putting them in the so called Birkhoff Normal Form of hamiltonian mechanics, which allows to view hamiltonian systems near an equilibrium as small perturbations of integrable systems.
Let us suppose that, in the linear approximation, the equilibrium point of a hamiltonian system is elliptic and the characteristic frequencies $\omega_1, \ldots, \omega_n$ are different one from each other. Thus, the quadratic term of the hamiltonian reduces, by a canonical linear transformation, to the form

$$H_2 = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2} .$$

Definition 1.1.2. *The characteristic frequencies $\omega_1, \ldots, \omega_n$ satisfies a resonance relation of order K, if there exist integer numbers, not all vanishing, for which*

$$k_1\omega_1 + \ldots + k_n\omega_n = 0, \qquad |k_1| + \ldots + |k_n| = K .$$

Definition 1.1.3. *We call Birkhoff Normal Form of order s, a polynomial of degree s in the canonical coordinates (P_j, Q_j) which is a polynomial of degree $[s/2]$ in the "action" variables $I_j = (P_j^2 + Q_j^2)/2$.*

Theorem 1.1.4. *Let us suppose that the characteristic frequencies ω_j do not verify any resonance relation of order s or lower. Then, there exists a canonical system of coordinates, in a neighborhood of the equilibrium point, such that the Hamilton's function reduces to the Birkhoff Normal Form of order s, up to terms of order $s+1$:*

$$H(p, q) = H_s(P, Q) + R , \qquad R = O(|P| + |Q|)^{s+1} .$$

PROOF. Following [A], we give a sketch of the proof. Let us choose the canonical complex coordinates (here and in the following $\mathrm{i} := \sqrt{-1}$),

$$z_j = \frac{1}{\sqrt{2}}\,(p_j + \mathrm{i}q_j) , \qquad \bar{z}_j = \frac{1}{\sqrt{2}}\,(p_j - \mathrm{i}q_j) ,$$

in which the Hamiltonian H_2 takes the form $\sum_{j=1}^{n} \omega_j z_j \bar{z}_j$. We proceed by induction over N. We note that, if there are no terms of order lower than N, except those ones in the normal form, then a canonical transformation with generating function $Pq + S_N(P, q)$ (where S_N is a polynomial of degree N), changes only the terms of order higher or equal than N in the Taylor's expansion of the Hamiltonian. Denote by $H_{\alpha\beta}$ and $S_{\alpha\beta}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\sum_{j=1}^{n}(\alpha_j + \beta_j) = N$, the coefficients of the monomial

$$z^\alpha \bar{z}^\beta = z_1^{\alpha_1} \ldots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \ldots \bar{z}_n^{\beta_n}$$

of H and S_N respectively. After the canonical transformation generated by $Pq + S_N(P, q)$, the coefficient of the monomial $z^\alpha \bar{z}^\beta$ of the new Hamiltonian is

$$H_{\alpha\beta} + \mathrm{i} s_{\alpha\beta} \sum_{j=1}^{n} \omega_j(\beta_j - \alpha_j)\,. \tag{1.13}$$

Thanks to the non–resonance hypothesis, the sum in (1.13) is different from zero if $\alpha \not\equiv \beta$. Therefore, defining $s_{\alpha\beta} := \mathrm{i}H_{\alpha\beta}/\sum_{j=1}^{n}\omega_j(\beta_j - \alpha_j)$, we can eliminate from the new Hamiltonian all the terms of order N that are not monomials of the form $(z\bar{z})^\alpha$.

□

1.1.1 Periodic orbits

Lyapunov Center Theorem

By the light of the example in (1.11) one has to make some hypotheses to prove that periodic orbits of the linearized system can be continued to periodic solutions of the nonlinear one.
Let us consider $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. We will assume the following:

(H0) $H(0) = 0$, $\nabla H(0) = 0$ and $\partial_w^2 H(0) > 0$ (namely $\partial_w^2 H(0)$ is positive-definite).

We note that (H0) implies that the origin is an elliptic equilibrium for H. Indeed, thanks to $\partial_w^2 H(0) > 0$, the level sets of the quadratic Hamiltonian H_2, defined in (1.6), are ellipsoids. As a consequence, all the solutions of the linearized system (1.7) are bounded. Therefore the eigenvalues of $J\partial_w^2 H(0)$ must be purely imaginary.
The Lyapunov Center Theorem ([Ly07]) assumes the following non–resonance hypothesis:

(H1) $J\partial_w^2 H(0)$ has n pairs of purely imaginary simple eigenvalues $\pm \mathrm{i}\omega_k$, $k = 1, 2, \dots, n$, such that ω_i/ω_j is not an integer for all $i \neq j$.

Theorem 1.1.5. *Suppose that $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ satisfies the hypotheses (H0), (H1). Then for all $\varepsilon > 0$ small enough the hamiltonian system associated to H has n geometrically distinct periodic orbits on the surface $H(w) = \varepsilon$. More precisely, the surface $H(w) = \varepsilon$ carries n distinct periodic orbits whose periods tend to $2\pi/\omega_k$, $k = 1, 2, \dots, n$.*

Weinstein Theorem

The Weinstein Theorem allows to deal with hamiltonian systems in which the non–resonance hypothesis (H1) of the Lyapunov Center Theorem is not satisfied (see also [Mo76], where an analogous result for non necessarily hamiltonian systems is proved).

Theorem 1.1.6 ([We73])**.** *Let be $H \in C^2\big(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\big)$ defined on a neighborhood of the origin, satisfying (H0). Then for each sufficiently small energy $\varepsilon > 0$, there are at least n periodic orbits, with frequencies close to those ones of the linearized system, on the surface $H(w) = \varepsilon$, $w \in \mathbb{R}^n \times \mathbb{R}^n$.*

1.2 Infinite–dimensional hamiltonian systems: three models

As it is well known, the following partial differential equations can be considered as an infinite–dimensional hamiltonian system:

(i) nonlinear wave equation (NLW);

(ii) nonlinear Schrödinger equation (NLS);

(iii) nonlinear beam equation;

(iv) Korteweg–de Vries equation (KdV);

(v) Burgers's equation;

(vi) Euler's equation of the hydrodynamics waves;

(vii) Boussinesq system;

(viii) Laplace equation on a cylinder;

(ix) Fermi, Pasta and Ulam system.

In this thesis we are interested in three of these hamiltonian systems: the nonlinear wave equation, the nonlinear Schrödinger equation and the nonlinear beam equation. In the following we will describe their hamiltonian structure, we will write the associated linearized system and compute its eigenvalues, showing, thus, that those systems have an elliptic equilibrium point at the origin.

1.2.1 Nonlinear wave equation

Let us consider, to fix ideas, the nonlinear wave equation

$$u_{tt} - u_{xx} + \mu u + f(t,x,u) = 0\,, \tag{1.14}$$

where $\mu \in \mathbb{R}$ and

$$f(t,x,u) = o(|u|)$$

is the nonlinearity. One can be interested in searching solutions of (1.14), with Dirichlet boundary conditions, on the interval $[0,\pi]$,

$$u(t,0) = u(t,\pi) = 0 \qquad t \in \mathbb{R}\,, \tag{1.15}$$

or with spatial periodic conditions

$$u(t,x) = u(t,x+2\pi)\,, \qquad t,x \in \mathbb{R}\,. \tag{1.16}$$

The equation (1.14) is a hamiltonian PDE and, in the case of conditions (1.15), the associated Hamiltonian is

$$H_{Dir}(v,u,t) = \int_0^{\pi} \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu\frac{u^2}{2} + g(u) \right) dx\,,$$

where $g(u) := \int_0^u f(\tilde{u})d\tilde{u}$, defined on the phase space

$$\mathcal{P}_{Dir} := L^2([0,\pi]) \times H_0^1([0,\pi]) \ni (v,u)\,.$$

In the case of conditions (1.16), the associated Hamiltonian is

$$H_{per}(v,u,t) = \int_0^{2\pi} \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu\frac{u^2}{2} + g(u) \right) dx\,,$$

and it is defined on the phase space

$$\mathcal{P}_{per} := L^2([0,2\pi]) \times H_{per}^1([0,2\pi]) \ni (v,u)\,.$$

The equations of motion for the previous Hamiltonians are

$$\begin{cases} \dot{v} = -\partial_u H(u,v,t) = u_{xx} - \mu u - f(u), \\ \dot{u} = \partial_v H(u,v,t) = v\,, \end{cases}$$

where $(\partial_v H, \partial_u H)$ is the gradient of H with respect to the L^2–scalar product. Here the symplectic structure is defined as in (1.3) by the operator $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Let us consider (1.14), for u small. The equation (1.7), namely the linearized system for this hamiltonian PDE, is

$$\begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} u_{xx} - \mu u \\ v \end{pmatrix}$$

and the associated quadratic Hamiltonian, for Dirichlet b.c.[2], is

$$H_2(v,u) = \int_0^{\pi} \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu\,\frac{u^2}{2} \right) dx\,.$$

Let us prove that the origin is an elliptic equilibrium point for this Hamiltonian, both in the case of periodic and Dirichlet b.c..

Consider, at first, the case of periodic b.c.. Using the Fourier transform, we can introduce new coordinates $p := (\ldots, p_{-1}, p_0, p_1, \ldots)$, $q := (\ldots, q_{-1},\, q_0,$

[2]In the same way, one can write the Hamiltonian for periodic b.c..

$q_1, \ldots)$, $(p, q) \in \ell^2_b \times \ell^2_b$, where ℓ^2_b is the Hilbert space of all *bi*-infinite, square integrable sequences, such that

$$v(x) = \sum_{j\in\mathbb{Z}} p_j \sqrt{\omega_j} \exp(\mathrm{i}jx)\,, \qquad u(x) = \sum_{j\in\mathbb{Z}} \frac{q_j}{\sqrt{\omega_j}} \exp(\mathrm{i}jx)\,, \tag{1.17}$$

where $\{e^{\mathrm{i}jx}\}_{j\in\mathbb{Z}}$ are a base of eigenvectors for the Sturm–Liouville operator[3]

$$-\frac{d^2}{dx^2} + \mu \tag{1.18}$$

with periodic b.c., and

$$\omega_j^2 := j^2 + \mu\,,$$

are the associated eigenvalues. By the Parseval identity, the Hamiltonian H_2 takes the form

$$H_2(v, u) = \sum_{j\in\mathbb{Z}} \omega_j \frac{p_j^2 + q_j^2}{2}\,.$$

The linearized equation in the coordinates (p, q), then, reads

$$\begin{cases} \dot{p}_j = -\omega_j q_j\,, & j \in \mathbb{Z}\,, \\ \dot{q}_j = \omega_j p_j \end{cases} \tag{1.19}$$

that corresponds to a harmonic oscillator, of frequencies $\{\omega_j = \sqrt{j^2 + \mu}\,\}_{j\in\mathbb{Z}}$. One can easily verify that $(p, q) = (0, 0)$ is an elliptic equilibrium point for the nonlinear wave equation.

In the case of Dirichlet boundary conditions, the eigenvectors of the Sturm–Liouville operator (1.18) are $\{\sin jx\}_{j\geq 1}$, so one has

$$v(x) = \sqrt{\frac{2}{\pi}} \sum_{j\geq 1} p_j \sqrt{\omega_j} \sin(jx)\,, \qquad u(x) = \sqrt{\frac{2}{\pi}} \sum_{j\geq 1} \frac{q_j}{\sqrt{\omega_j}} \sin(jx)\,,$$

where $p := (p_1, \ldots)$, $q := (q_1, \ldots)$, $(p, q) \in \ell^2 \times \ell^2$.
As for the periodic b.c., the coordinates $(q_1, \ldots)$ must verify the equation

$$\ddot{q}_j + (j^2 + \mu) q_j = 0\,,$$

that corresponds to a harmonic oscillator with energy $\omega_j(p_j^2 + q_j^2)/2$ and period $2\pi/\omega_j$, $j \geq 1$.

Let us note that the behavior of the frequencies ω_j has a deep influence on the dynamic of the system. For this reason, it is natural wondering if there exists a resonance relation between the frequencies. For example, for the

[3]We assume for simplicity that all the eigenvalues are positive, though this is not necessary.

Dirichlet boundary conditions, one can verify the existence of a multi–index $k = (\ldots, k_{-1}, k_0, k_1, \ldots) \in \mathbb{Z}^\infty$, with $|k| < \infty$, such that, being $\omega := (\omega_1, \ldots)$,

$$\langle \omega, k \rangle = \sum_{j \geq 1} \omega_j k_j = 0\,.$$

In Lemma 4.2.1 it is proved that, except a countable set of $\mu > 0$, there are no resonance relations. Therefore, generally, there exists an infinite number of rationally independent frequencies and most of the solutions of the linearized equation are almost periodic.
Finally, we can summarize: in the case of periodic boundary conditions, the eigenvectors of the operator $\Box := \partial_t^2 - \partial_x^2$, are $\{e^{\mathrm{i}\omega_j t} e^{\mathrm{i}jx}\}_{j\in\mathbb{Z}}$, the associated eigenvalues are also parametrized by $j \in \mathbb{Z}$, and there always exists at least a $1:1$ resonance relation between the frequencies, being $\omega_j = \omega_{-j}$. On the other hand, in the case of Dirichlet b.c., the eigenvectors are $\{e^{\mathrm{i}\omega_j t} \sin jx\}_{j\geq 1}$, every frequencies are simple and, generally, non resonant.
Moreover, if $\mu = 0$, the linearized equation is

$$\Box u = u_{tt} - u_{xx} = 0\,, \tag{1.20}$$

the frequencies are $\omega_j = |j| \in \mathbb{N}$, and there exists an infinite number of resonance relations. It's no difficult to verify that, in this case, all the solutions of (1.20) has rational frequency.

1.2.2 Nonlinear Schrödinger equation

Let us consider the nonlinear Schrödinger equation

$$iu_t - u_{xx} + \mu u + f(|u|^2)u = 0\,, \quad u \in \mathbb{C}\,. \tag{1.21}$$

Let be μ a parameter, $\mu \in \mathbb{R}$, and f the nonlinearity. Absorbing a constant into μ one can assume $f(0) = 0$. Let us study this equation as a hamiltonian system on some suitable space. For example, for Dirichlet b.c.(boundary conditions), one has $\mathcal{P} = H_0^1([0,\pi)]$, the complex valued L^2–functions on $[0,\pi]$ with an L^2–derivative and vanishing boundary values. For $u \in \mathcal{P}$, the Hamiltonian will be

$$H_{NLS}(u,\bar{u}) = \int_0^\pi \left(\frac{|u_x|^2}{2} + \mu\frac{|u|^2}{2} + \frac{1}{2}\,g(|u|^2) \right) dx\,,$$

where $g(s) = \int_0^s f(z)dz$, then, equation (1.21) can be written in the hamiltonian form

$$\partial_t u = \mathrm{i}\partial_{\bar{u}} H_{NLS} = J\partial_{\bar{u}} H_{NLS}\,. \tag{1.22}$$

Let us note that here the operator J, that defines a symplectic structure, corresponds to the product for i. We note that, as for the nonlinear wave equation, one can deal with periodic b.c., adapting the definition of the phase space and of the associated Hamiltonian. Let us rewrite the Hamiltonian in infinitely many coordinates in the Hilbert space ℓ^2 of all complex valued square integrable sequences $q := (q_1, \ldots)$. Make the ansatz

$$u(t,x) = \mathcal{S}q = \sum_{j\geq 1} q_j(t)\chi_j(x)\,, \quad j \geq 1$$

where $\chi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx$, $j \geq 1$ are the basic modes for the linear equation $iu_t - u_{xx} + \mu u = 0$ for Dirichlet boundary conditions. Let $\omega_j = j^2 + \mu$ be their respective frequencies. We note that the eigenvalues of the Sturm-Liouville operator are simple for Dirichlet b.c. for $j \geq 1$, while for the periodic b.c., $\omega_j = \omega_{-j}$ for $j \in \mathbb{Z}$. One then obtains the Hamiltonian

$$H(q,\bar{q}) == \frac{1}{2}\sum_{j\geq 1} \omega_j |q_j|^2 + \frac{1}{2}\int_0^\pi g(|\mathcal{S}q|^2)\, dx\,.$$

The equation of motion are

$$\dot{q}_j = \mathrm{i}\frac{\partial H}{\partial \bar{q}_j}\,, \quad j \geq 1\,. \tag{1.23}$$

They are classical hamiltonian equations of motion for the real and imaginary part of $q_j = x_j + \mathrm{i}y_j$. In order to prove that the origin is an elliptic equilibrium point for the hamiltonian system (1.22), one can decompose q_j in its real and imaginary part, namely $q_j = x_j + \mathrm{i}y_j$, and equation (1.23) becomes

$$\partial_t \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_j x_j \\ \omega_j y_j \end{pmatrix}\,. \tag{1.24}$$

In this way, one has a standard harmonic oscillator, with frequencies ω_j, from which it follows that the point $q = 0$ is elliptic.

1.2.3 Nonlinear beam equation

Consider the beam equation

$$u_{tt} + u_{xxxx} + \mu u + f(u) = 0 \tag{1.25}$$

with hinged boundary conditions

$$u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0\,, \tag{1.26}$$

with $\mu \in \mathbb{R}$ and f a generic nonlinearity. Choosing for example the phase space $\mathcal{P} = H^2([0,\pi]) \times L^2([0,\pi])$ one can write the associated Hamiltonian:

$$H(u,v) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_{xx}^2}{2} + \frac{\mu u^2}{2} + g(u) \right) dx\,,$$

where $g(u) := \int_0^u f(s)\,ds$, with equations of motion

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u_{xxxx} - \mu u - f(u)\,. \end{cases}$$

Using the Fourier transform, we can introduce new coordinates $p := (p_1, \ldots)$, $q := (q_1, \ldots)$, $(p, q) \in \ell^2 \times \ell^2$, through the relations

$$v(x) = \sum_{j \geq 1} p_j \sqrt{\omega_j} \chi_j(x)\,, \qquad u(x) = \sum_{j \geq 1} \frac{q_j}{\sqrt{\omega_j}} \chi_j(x)\,.$$

where $\omega_j^2 = j^4 + \mu$ and $\chi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx$. As for the nonlinear wave equation, the coordinates q_j, for $j \geq 1$, must satisfy the equation $\ddot{q}_j + \omega_j^2 q_j = 0$ and therefore the origin is an elliptic equilibrium point.

Chapter 2

Periodic solutions: methods and known results

This chapter deals with the search of periodic solutions for some hamiltonian partial differential equations. The used methods are of two types: variational ones, which are global and shared also with other classes of partial differential equations (e.g. elliptic equations), and the perturbative ones, which are local and typical of the finite dimensional hamiltonian systems. We will describe both of them, even if we are mainly interested in the second one.
We will deal with some extensions of the classical fundamental theorems for finite dimensional hamiltonian systems, stated in Chapter 1, to the infinite–dimensional case. We want to explain how tools and techniques, which are well proven in the world of finite dimensional hamiltonian systems, may be applied in the world of infinite–dimensional evolution's equations with hamiltonian structure.
For example, thinking of the analogy with the Lyapunov Center Theorem for finite dimensional hamiltonian systems in the neighborhood of an elliptic fixed point, one expects to obtain a family of periodic orbits bifurcating from the orbits of the linearized equations whenever no integer multiple of the frequency of the solution being perturbed coincides with the frequency of any other normal mode. Obviously, there is a deep difference if the problem has infinitely many degrees of freedom. In the finite dimensional case there are smooth families of periodic solutions bifurcating from the solution of the linear problem. Indeed, for finitely many frequencies $\omega_1, \ldots, \omega_n$, if $k\omega_1 \neq \omega_j$, for $j = 2, 3, \ldots, n$ and $k \in \mathbb{Z}$, one has $k\omega \neq \omega_j$, for every ω close to ω_1. It means that one finds solutions for an interval of frequencies. On the other hand, for infinitely many frequencies, there exists, in general, a dense set of ω's for which $k\omega = \omega_j$, for some j. Here a peculiarity of the infinite–dimensional case appears: the small divisors problem arises in searching periodic solutions and not only for quasi–periodic ones as in the finite–dimensional situation. To overcome this problem, one has to excise the resonances, obtaining a Cantor set of frequencies, and thus one finds a solution for any frequency that lies in this Cantor set.
In particular, we will investigate the existence of periodic solutions for the

following partial differential equations seen as infinite–dimensional hamiltonian systems:

- nonlinear wave equation;
- nonlinear Schrödinger equation;
- nonlinear beam equation.

At first we briefly discuss about the matters that one meets in looking for periodic solutions of the nonlinear wave equation and about the types of periodic solutions that one chooses to search. After, we will report most of the results about the above mentioned hamiltonian PDEs.
In the statement of these results, we respect most of the notations of the authors, to simplify the reading.

2.1 Nonlinear wave equation

To fix idea, we treat the problem (1.14) with Dirichlet boundary conditions, namely

$$\begin{cases} u_{tt} - u_{xx} + \mu u + f(t,x,u) = 0\,, & \mu \in \mathbb{R}\,, \\ u(t,0) = u(t,\pi) = 0\,. \end{cases} \tag{2.1}$$

where $\partial_u f(t,x,0) = 0$.
We consider two cases for f in (2.1):

(a) *free vibrations*, namely $f = f(x,u)$;

(b) *forced vibrations*, namely $f = f(t,x,u) = f(t+T_0,x,u)$, $T_0 > 0\,.$

The main difference in searching T-periodic solutions between case (a) and (b) is that in the forced case it is natural to look for a solution u of the same period as the forcing term, namely $T = T_0$, while in the free case the period T is a priori unknown.
Another fundamental dichotomy consists in searching periodic solutions with period $T \in \pi\mathbb{Q}$ or $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$. To understand this point, it is useful to look at the behavior of the d'Alembertian operator $\Box$, of its kernel and of its invertibility properties.
The linear equation $\Box u = 0$ has only periodic solutions with period belonging to the set $\pi\mathbb{Q}$; indeed the kernel K of the operator $\Box$, acting on functions[1] $L^2(\Omega)$, where $\Omega := \mathbb{T} \times [0,\pi]$ and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, is

$$K := \left\{ \ v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) \ \ : \ \ \hat{v} \in L^2(\mathbb{T}) \right\}\ .$$

Obviously $L^2(\Omega)$ can be decomposed into the two orthogonal closed subspaces $L^2(\Omega) = K \oplus K^{\perp}$. Since for all $h \in K^{\perp}$ the equation $\Box u = h$ has a unique

[1] We can define a *weak* solution $u \in L^2(\Omega)$ of $\Box u = h \in L^2(\Omega)$ asking that $\int_\Omega u \Box \phi = \int_\Omega h\phi$ for any $\phi \in H^1_0(\Omega)$.

solution $u \in K^\perp$, it results that $\Box$ is invertible on the co-kernel $K^\perp$, namely the map

$$\Box^{-1} : K^\perp \longrightarrow K^\perp$$

is a well-defined bounded (even compact) linear operator.
We now pass to the non homogeneous linear equation $\Box u = h$, where

$$h = \sum_{\ell \in \mathbb{Z}, j \geq 1} h_{\ell j} e^{i\omega \ell t} \sin(jx)$$

is T–periodic with frequency $\omega := 2\pi/T$. Then we look for T–periodic solutions

$$u = \sum_{\ell \in \mathbb{Z}, j \geq 1} u_{\ell j} e^{i\omega \ell t} \sin(jx) \,.$$

The coefficients of u are easily determined:

$$u_{\ell j} = \frac{h_{\ell j}}{j^2 - \omega^2 \ell^2} \,. \tag{2.2}$$

Such expression make sense only if the divisors $j^2 - \omega^2\ell^2 = (j - \omega\ell)(j + \omega\ell)$ are different from zero. It is immediate to see that two cases occur: $\omega \in \mathbb{Q}$ or $\omega \in \mathbb{R} \setminus \mathbb{Q}$, corresponding to $T \in \pi\mathbb{Q}$ or $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$, respectively. In the case $T \in \pi\mathbb{Q}$, we have that $\omega = p/q$, $\gcd(p, q) = 1$. We note that the set $\{j^2 - \omega^2\ell^2\}_{\ell \in \mathbb{Z}, j \geq 1}$ has not accumulation points. Hence, if $\ell = \pm nq$, $j = np$ for $n \in \mathbb{N}^+$, the divisor is equal to zero; otherwise it is bounded away from zero. On the other hand in the case $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$, the divisors $j^2 - \omega^2\ell^2$ are different from zero but, except the ω's belonging to the zero measure set

$$\left\{ \omega > 0 \quad \text{s.t.} \quad \exists\, \gamma > 0 \quad \text{with} \quad |\omega\ell - j| \geq \frac{\gamma}{\ell} \,,\ \forall\, \ell \in \mathbb{Z} \,,\ j \geq 1 \right\} ,$$

accumulate to zero.
Summing up, the behavior of the self-adjoint operator $\Box$, under $T-$periodic conditions, is the following:

(1) $T \in \pi\mathbb{Q}$; the kernel is *infinite dimensional* while, on the co-kernel, $\Box^{-1}$ exists bounded;

(2) $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$; there is no kernel but a *small divisors problem* appears because $\Box^{-1}$ is unbounded.

We now pass to consider the operator $\Box + \mu$, $\mu \neq 0$. The divisors in (2.2) become $\omega_j^2 - \omega^2\ell^2 = j^2 + \mu - \omega^2\ell^2$. Again we distinguish the cases $T \in \pi\mathbb{Q}$ and $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$.
As we have just said, for $T \in \pi\mathbb{Q}$, the set $\{j^2 - \omega^2\ell^2\}_{\ell \in \mathbb{Z}, j \geq 1}$ has not accumulation points. If μ is irrational, the equation $j^2 + \mu - \omega^2\ell^2 = 0$ has no solutions, the divisors $\omega_j^2 - \omega^2\ell^2$ are bounded away from zero and therefore $\Box + \mu$ is invertible with bounded inverse. If μ is rational, the equation $j^2 + \mu - \omega^2\ell^2 = 0$

can have at most a finite number of solutions and hence $\Box + \mu$ can have a non trivial kernel but only finite dimensional. Indeed, let $\omega = p/q$, $\gcd(p,q) = 1$, the equation $q^2 j^2 + q^2\mu - p^2\ell^2 = 0$ has at most a finite number of solutions $\ell \in \mathbb{Z}$, $j \geq 1$. This follows from the fact that the equation $\bar{\ell}^2 - \bar{j}^2 = \bar{\mu}$ has at most a finite number of solutions $\bar{\ell} \in \mathbb{Z}$, $\bar{j} \geq 1$. In fact, considering the case $\bar{\mu} > 0$ and $\bar{\ell} > 0$, taking $\bar{\ell} = \bar{j} + h$, with $h > 0$, the equation $2\bar{j}h + h^2 = \bar{\mu}$ has clearly a finite number of solutions. Summarizing, for $T \in \pi\mathbb{Q}$, as for $\mu = 0$, the small divisors problem does not occur.
The case $T \in \pi(\mathbb{R} \setminus \mathbb{Q})$ is more involved and, in general, as for $\mu = 0$, the small divisors appear.

Summarizing, we can say that, besides the "physical" dichotomy between free and forced vibrations, there is also a "mathematical" dichotomy between rational and irrational frequencies. In conclusion we will consider four different cases: free vibrations with rational frequency, forced vibrations with rational frequency, free vibrations with irrational frequency, forced vibrations with irrational frequency. As we are going to see in the following, the "mathematical" dichotomy turns out to be predominant concerning methods and tools.
Regarding the search of periodic solutions with rational frequency, one takes advantage of the absence of small divisors and of the invertibility of the linear operator on the range. Problems come out, rather, from the presence of the kernel that can be even infinite dimensional. One can apply standard tools of Functional Analysis, as variational methods and topological arguments. The analysis is not restricted to a perturbative setting, in particular one requires *global* hypotheses (as e.g. monotonicity) and *no small* solutions are expected. These variational techniques were used by Rabinowitz, Brezis, Coron, Nirenberg, etc.
As regards the search of periodic solution with irrational frequency, the appearing of the small divisors is the most difficult problem. Whether one decides to bypass it, choosing the frequency in a suitable set of zero measure, see [BP01], [BeBo03], or to solve it by the KAM theory (see the works by Bourgain, Pöschel, Kuksin, Craig, Wayne), one obtains a small amplitude solution. Indeed, in both the cases, one uses local methods: the standard Implicit Function Theorem or the Nash-Moser one. These methods are definitively *perturbative* in nature.

We now give a fast review of what is known in the literature about periodic solutions of the nonlinear wave equation, describing in more details some works that we consider interesting for the performed techniques. Some of these techniques will be used in the third chapter of this thesis to prove our main result.

2.2 Free vibrations with rational frequency

♡ [**R78**] We say that $f : \mathbb{R} \to \mathbb{R}$ is *strictly monotone increasing* if

$$z_1 > z_2 \quad \Longrightarrow \quad f(z_1) > f(z_2) \tag{2.3}$$

and *superlinear at* 0 *and* ∞ if
(i) $f(z) = o(|z|)$ at $z = 0$;
(ii) exist z_0 and $\theta \in [0, 1/2]$ constants such that

$$F(z) = \int_0^z f(s)\, ds \leq \theta z f(z) \qquad \text{for} \qquad |z| \geq z_0\,.$$

Theorem 2.2.1. *Let $f \in C^k(\mathbb{R}, \mathbb{R})$, $k \geq 2$, $f(0) = 0$, strictly monotone increasing and superlinear at* 0 *and* ∞. *Then there is a nontrivial solution C^k of the equation*

$$\begin{cases} u_{tt} - u_{xx} + f(x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0\,. \end{cases} \tag{2.4}$$

This result will be discussed in more details on page 43.

♡ [**BrCoN80**] Let be $f : \mathbb{R} \to \mathbb{R}$ a continuous nondecreasing function such that $f(0) = 0$. We assume that there exist constants $\alpha > 0$ and C satisfying

$$\frac{1}{2} t f(t) - \int_0^t f(\tau)\, d\tau \geq \alpha |f(t)| - C \qquad \text{for all } t \tag{2.5}$$

and

$$\lim_{|t| \to \infty} \frac{f(t)}{t} = \infty\,. \tag{2.6}$$

Theorem 2.2.2. *Let f satisfy* (2.5),(2.6) *there exists a nontrivial (weak) solution $u \in L^\infty$ of* (2.4).

The result is achieved via a duality argument and Mountain Pass Lemma.

2.2.1 [R78]

Let us consider the partial differential equation

$$u_{tt} - u_{xx} + f(u) = 0\,, \qquad x \in [0, \pi],\, t \in \mathbb{R} \tag{2.7}$$

with Dirichlet boundary conditions

$$u(t, 0) = 0 = u(t, \pi) \qquad t \in \mathbb{R}\,,$$

which is technically slightly simpler than (2.4). The author is looking for T–periodic solutions. For definiteness, suppose one is searching a 2π–periodic solution of (2.7), with f depending only on u. Let be $\Omega := [0, 2\pi] \times [0, \pi]$ and consider the associated functional

$$\Phi(u) = \int_\Omega \left(\frac{u_t^2}{2} - \frac{u_x^2}{2} - F(u) \right) dx\, dt \tag{2.8}$$

where $F(z) = \int_0^z f(s) ds$. The integrand for (2.8) is the Lagrangian for the problem. Critical points of the functional Φ, defined on a suitable class of T–periodic functions, are weak solutions of (2.7), with Dirichlet boundary conditions. Under suitable conditions on f, that are:

(i) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$;

(ii) f is strictly monotone increasing, i.e. $z_1 > z_2$ implies $f(z_1) > f(z_2)$;

(iii) f is superlinear at 0 and ∞, i.e.

   (a) $f(z) = o(|z|)$ at $z = 0$;

   (b) there are constants $\bar{z} > 0$ and $\theta \in \left[0, \frac{1}{2}\right]$ s.t.

$$F(z) = \int_0^z f(s)ds \leq \theta z f(z), \qquad \text{for } |z| \leq \bar{z}\,.$$

Let be $\Box := \partial_{tt} - \partial_{xx}$ and $K(\Box)$ the null space of $\Box$. It's not difficult to see that the closure in L^2 of $K(\Box)$ is

$$K := \left\{ p(t+x) - p(t-x) \text{ s.t. } p \in L^2(\mathbb{T}) \right\}.$$

Let be $K^\perp$ the orthogonal complement of K in L^2.

Theorem 2.2.3. *Let f satisfies (i)-(iii), $f \in C^k$, $k \geq 2$. Then there exists a $u = v + w \in (C^k \cap K) \oplus (C^{k+1} \cap K^\perp)$ nontrivial classical solution of*

$$\begin{cases} \Box u + f(u) = 0 \qquad x \in [0, \pi], t \in \mathbb{R} \\ u(t, 0) = u(t, \pi) = 0 \\ u(x, t + 2\pi) = u(x, t)\,. \end{cases} \tag{2.9}$$

To prove this theorem, the author chooses to consider a modified Lagrangian that corresponds to this modified problem for $u = v + w$, $\beta > 0$:

$$\begin{cases} \Box u - \beta v_{tt} + f(u) = 0 \qquad x \in [0, \pi], t \in \mathbb{R} \\ u(t, 0) = u(t, \pi) = 0 \\ u(x, t + 2\pi) = u(x, t)\,. \end{cases} \tag{2.10}$$

One wants to solve (2.10) for any β and then, sending $\beta \to 0$, to obtain a solution of (2.9). Let us consider the lagrangian functional associated to the system (2.10):

$$I(u) := \int_\Omega \left(\frac{u_t^2}{2} - \frac{u_x^2}{2} - \beta \frac{v_t^2}{2} - F(u) \right) dx\, dt.$$

Then $I(u)$ is defined for all $u \in H^1$ and it is not difficult to verify that any critical point of I is a weak solution of (2.10).
I is not bounded from above, nether from below in H^1 so there is no hope of getting a solution of (2.10) maximizing I on finite-dimensional subspaces of H^1.
By restricting I to suitable finite-dimensional subspaces E_n of admissible functions of L^2, corresponding critical points, u_n, of I_{E_n} are obtained showing that

I possesses on E_n the Mountain Pass geometry. Hence, one can show that these critical points tend to a nontrivial classical solution $u = u(\beta)$ of (2.10), with L^∞ and H^1 estimates that depend on β. Finally the solutions $u = u(\beta)$ will be used to get a weak solution of (2.9). One gets at first an upper bound for $\|u(\beta)\|_{L^\infty}$ independent of β. Then one shows that $\{u(\beta)\}$ is equicontinuous in $C(\Omega)$ from which it follows that (2.9) has a weak solution.

2.3 Forced vibrations with rational frequency

♡ [**R67**] Let us consider the equation

$$\begin{cases} u_{tt} - u_{xx} = \varepsilon f(t,x,u) \qquad x \in [0,\pi]\,, \quad u, \varepsilon \in \mathbb{R}\,, \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.11}$$

with $f(t,x,u)$ 2π-periodic in time.
Using minimizing variational techniques, Rabinowitz proved the following

Theorem 2.3.1. *Let $f \in C^k$ be strictly monotone, namely $\partial_u f \geq \beta > 0$. Then, for all $|\varepsilon|$ small enough, there exists a solution $u \in H^k$ of* (2.11).

This result will be discussed in more details on page 46.

♡ [**R71**] Let us consider (2.11) with $f(t,x,u) := u^{2k+1} + \tilde{f}(t,x,u)$, $k \geq 1$.

Theorem 2.3.2. *Suppose $\tilde{f}$ continuous and monotone, namely $\tilde{f}(t,x,u_1) \leq \tilde{f}(t,x,u_2)$ for all t, x and $u_1 \leq u_2$. Then* (2.11) *has two curves of solutions $(\varepsilon, u_\varepsilon) \subset \mathbb{R}^+ \times C$ and $(\varepsilon, u_\varepsilon) \subset \mathbb{R}^- \times C$, which are unbounded (in $\mathbb{R} \times C$) and meets only at $(0, u_0)$.*

The solutions are found by a topological degree argument.

♡ [**Co83**] Let be $g : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function. In this paper the author studies the forced vibrations:

$$\begin{cases} u_{tt} - u_{xx} + g(u) = f(t,x) \\ u(t,0) = u(t,\pi) = 0 \\ u(t + 2\pi, x) = u(t,x)\,. \end{cases} \tag{2.12}$$

The main result is stated in the following:

Theorem 2.3.3. *If $f \in L^2(\mathbb{T} \times [0,\pi])$ is 2π–periodic in t and if*

$$f(\pi + t, \pi - x) = f(t,x) \;\; a.e. \;\; x \in (0,\pi),\, t \in \mathbb{R}\,,$$

$$-1 < \liminf_{|u| \to \infty} \frac{g(u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(u)}{u} < 3\,,$$

then there exists $u \in H^1(\mathbb{T} \times [0,\pi])$ solution of (2.12).

Coron finds weak solutions assuming the above additional symmetry on f and restricting to the space of functions satisfying $u(t,x) = u(t+\pi, \pi - x)$, where the kernel of the d'Alembertian operator $\Box$ reduces to 0. He also deals with the autonomous case $f \equiv 0$, proving for the first time existence of nontrivial solutions for nonmonotone nonlinearities.

♡ [**BeBi**] In this paper the authors consider the problem

$$\Box u = \varepsilon f(t,x,u) \tag{2.13}$$

with Dirichlet boundary conditions

$$u(t,0) = u(t,\pi) = 0 \tag{2.14}$$

where ε is a small parameter and the nonlinear forcing term $f(t,x,u)$ is 2π-periodic in time. They look for nontrivial 2π-periodic in time solutions $u(t,x)$ of (2.13),(2.14), i.e. satisfying

$$u(t+2\pi, x) = u(t,x)\,. \tag{2.15}$$

Let be $\Omega := \mathbb{T} \times (0,\pi)$.

Theorem 2.3.4. *Let $f(t,x,u) = \beta u^{2k} + h(t,x)$, $\beta \in \mathbb{R} \setminus \{0\}$, and $h \in N^{\perp}$ satisfies $h(t,x) > 0$ (or $h(t,x) < 0$) a.e. in Ω. Then, for ε small enough, there exists at least one weak solution $u \in H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega})$ of (2.13)-(2.14)-(2.15) with $\|u\|_{H^1(\Omega)} + \|u\|_{C^{1/2}(\overline{\Omega})} \leq C|\varepsilon|$. If, moreover, $h \in H^j(\Omega) \cap C^{j-1}(\overline{\Omega})$, $j \geq 1$, then $u \in H^{j+1}(\Omega) \cap C_0^j(\overline{\Omega})$ with $\|u\|_{H^{j+1}(\Omega)} + \|u\|_{C^j(\overline{\Omega})} \leq C|\varepsilon|$ and therefore, for $j \geq 2$, u is a classical solution.*

Here N is the kernel of the operator $\Box$ in L^2, $H^1(\Omega)$ is the usual Sobolev space and $C_0^{1/2}(\overline{\Omega})$ is the space of all the 1/2-Hölder continuous functions $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying (2.14), endowed with norm

$$\|u\|_{H^1(\Omega)} + \|u\|_{C^{1/2}(\overline{\Omega})}\ .$$

The approach of the authors is based on a variational Lyapunov-Schmidt reduction. It turns out that the infinite dimensional bifurcation equation exhibits an intrinsic lack of compactness. They solve it via a minimization argument and a-priori estimate methods inspired by the regularity theory of [R67]. The major result of this paper is that the usual monotonicity hypothesis on f is bypassed.

2.3.1 [R67]

The author seeks a nontrivial solution of the equation

$$\Box u := u_{tt} - u_{xx} = \varepsilon f(t,x,u) \tag{2.16}$$

where $(t, x) \in \Omega = \mathbb{T} \times (0, \pi)$ under the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \; . \tag{2.17}$$

Here ε is a small parameter, while $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ so it means that one is looking for time periodic solution

$$u(t + 2\pi, x) = u(t, x) \tag{2.18}$$

of the one spatial dimensional wave equation with a time periodic forcing term $f(t, x, u) = f(t+2\pi, x, u)$. The kernel N of the operator $\Box$ acting on functions $L^2(\Omega)$ satisfying (2.17),(2.18) is the following closed subset of $L^2(\Omega)$

$$N := \left\{ v(t, x) = \hat{v}(t + x) - \hat{v}(t - x) \; : \; \hat{v} \in L^2(\mathbb{T}) \, , \quad \text{with} \int_0^{2\pi} \hat{v} = 0 \right\} \, .$$

Obviously $L^2(\Omega)$ can be decomposed into the two orthogonal closed subspaces $L^2(\Omega) = N \oplus N^\perp$.

Since $\Box$ (with conditions (2.17),(2.18)) is a self-adjoint operator we have that its range in $L^2(\Omega)$ is exactly $N^\perp$. Recall that, given $f \in N^\perp$, there exists a unique function $w := \mathcal{G} f \in N^\perp$ (and verifying (2.17),(2.18)) such that $\Box w = f$; moreover $\mathcal{G}$ is a compact self-adjoint operator

$$\mathcal{G} \; : \; N^\perp \longrightarrow N^\perp \cap H^1(\Omega) \cap C^{1/2}(\overline{\Omega}) =: W \tag{2.19}$$

with

$$\| \mathcal{G} f \|_W \leq c \, \| f \|_{L^2} \tag{2.20}$$

where

$$\| w \|_W := \| w \|_{H^1} + \| w \|_{C^{1/2}}$$

and c is a suitable positive constant.

One can look for weak solution of $\Box u = f$ of the form $u = v + w$ with $v \in V := N \cap H^1(\Omega)$ and $w \in W$. Denoting by Π_N and $\Pi_{N^\perp}$ the projection from L^2 onto N and $N^\perp$ respectively, one has to solve the *kernel equation*

$$\Pi_N \, f(v + w) = 0 \tag{2.21}$$

and the *range equation*

$$w = \varepsilon \mathcal{G} \, \Pi_{N^\perp} \, f(v + w) \, . \tag{2.22}$$

The range equation (2.22) is solved by the Fixed Point Theorem.

Proposition 2.3.5. *Let $f = f(t, x, u)$ and $f_u = \partial_u f$ be continuous on $\overline{\Omega} \times \mathbb{R}$. Then $\forall \, R > 0$, there exists a constant $C_0(R) > 1$ such that $\forall \, |\varepsilon| \leq \varepsilon_0(R) := 1/2C_0(R)$ and $\forall \, v \in V$, $\|v\|_{H^1} \leq R$ there exists a unique $w = w(v, \varepsilon) \in W$ with*

$$\| w \|_W \leq C_0(R) \varepsilon \tag{2.23}$$

such that

$$w = \varepsilon \mathcal{G} \Pi_{N^\perp} f(v+w) \tag{2.24}$$

where $f(u)$ is the Nemitskii operator[2] *associated to f. Moreover v_n, $v_* \in V$, $\|v_n\|_{H^1}$, $\|v_*\|_{H^1} \le R$*

$$v_n \xrightarrow{L^2} v_* \quad \Longrightarrow \quad w(v_n) \xrightarrow{W} w(v_*) \, . \tag{2.25}$$

The kernel equation

After solving the range equation one inserts $w = w(v)$ into the kernel equation (2.21), hence one has to find solutions of

$$\Pi_N f(v + w(v)) = 0 \ . \tag{2.26}$$

Solving (2.26), since V is dense in N with the L^2-norm, is equivalent to find a v such that

$$\int_\Omega f(v+w(v))\phi = 0 \qquad \forall \, \phi \in V \ . \tag{2.27}$$

Solutions of $\Box u = f$ (satisfying also (2.17),(2.18)) are critical points of the *Lagrangian action functional*

$$\Psi : \Big\{ u \in H^1(\Omega) \quad \text{s.t.} \quad u(t,0) = u(t,\pi) = 0 \Big\} \longrightarrow \mathbb{R} \, ,$$

defined by

$$\Psi(u) := \int_\Omega \left(\frac{u_t^2}{2} - \frac{u_x^2}{2} + \varepsilon F(u) \right) dt \, dx \, ,$$

where $F(u) := \int_0^u f(\xi) d\xi$. Then one notes that (2.27) is the Euler-Lagrange equation of the *reduced Lagrangian action functional*

$$\begin{aligned} \Phi(v) \ &:= \ \Psi(v + w(v)) \\ &= \ \varepsilon \int_\Omega F(v+w(v)) - \frac{1}{2} f(v+w(v)) w(v) \qquad v \in V \, . \end{aligned} \tag{2.28}$$

since

$$D_v \Phi(v)[\phi] = \varepsilon \int_\Omega f(v + w(v))\phi \ , \qquad \forall v, \phi \in V \, . \tag{2.29}$$

Recollecting, one has that solving the kernel equation (2.26) is equivalent to finding critical points of $\Phi(v)$.
By (2.25) one has that the functional Φ is L^2-continuous in the following sense $v_n, v_* \in V, \ \ \|v_n\|_{H^1}, \|v_*\|_{H^1} \le R$,

$$v_n \xrightarrow{L^2} v_* \qquad \Longrightarrow \qquad \Phi(v_n) \longrightarrow \Phi(v_*) \, . \tag{2.30}$$

[2] Namely $[f(u)](t,x) := f(t,x,u(t,x))$.

In practice one looks for an extremal point (actually a minimum) of $\Phi(v)$ in the ball

$$B_R := \left\{ v \in V,\ \|v\|_{H^1} \leq R \right\} ,$$

where R is a positive (large) constant which will be fixed later.
Take a minimizing sequence $v_n \in B_R$, $\Phi(v_n) \to \inf_{B_R} \Phi$. Due to the H^1-boundness of $\{v_n\}_{n\in\mathbb{N}}$, one can suppose that

$$v_n \overset{H^1}{\rightharpoonup} \bar{v}, \qquad \text{with} \qquad \|\bar{v}\|_{H^1} \leq R\,.$$

By the Rellich theorem $v_n \overset{L^2}{\to} \bar{v}$. From (2.30), one obtains that $\bar{v}$ is a minimum point of Φ restricted to B_R. Obviously, since one is seeking free critical points, one has to prove that $\|\bar{v}\|_{H^1} < R$, namely $\bar{v}$ is an *inner* minimum for Φ in B_R. Following [R67], one calls $\phi \in V$ an *admissible variation* in the point $\bar{v}$, if $\bar{v} + \theta\phi \in B_R$ for every scalar $\theta < 0$ sufficiently close to 0. A sufficient condition for $\phi \in V$ to be an admissible variation is the positivity of the scalar product

$$\langle \bar{v}, \phi \rangle_{H^1} > 0 \ . \tag{2.31}$$

It is obvious by definition that, for any admissible variation $\phi \in V$,

$$0 \geq D_v\Phi(\bar{v})[\phi] = \int_\Omega f(\bar{v} + w(\bar{v}))\phi \ . \tag{2.32}$$

By now, one wants to choose a suitable admissible variation in order to obtain a L^∞–estimate and a H^1–estimate.
By (2.32) one has, since $w = O(\varepsilon)$, that

$$\int_\Omega f(\bar{v})\phi \leq \int_\Omega O(\varepsilon)\phi\,.$$

Using that $f_u \geq \beta > 0$, the next inequality holds, if $\bar{v}\phi \geq 0$,

$$\int_\Omega f(\bar{v})\phi = \int_\Omega f(0)\phi + f_u(z)\bar{v}\phi \geq \int_\Omega f(0)\phi + \beta\bar{v}\phi$$

where z is an intermediate point, and finally one obtains

$$\beta \int_\Omega \bar{v}\phi \leq \left| \int_\Omega f(0)\phi \right| + \left| \int_\Omega O(\varepsilon)\phi \right| . \tag{2.33}$$

Let, for $M > 0$,

$$q(\lambda) := q_M(\lambda) := \begin{cases} 0\ , & \text{if} \quad |\lambda| \leq M \\ \lambda - M\ , & \text{if} \quad \lambda \geq M \\ \lambda + M\ , & \text{if} \quad \lambda \leq M\ . \end{cases}$$

One can note that

$$\lambda q(\lambda) \geq M|q(\lambda)|\,. \tag{2.34}$$

If $\bar{v}(t,x) = \bar{v}_+(t,x) - \bar{v}_-(t,x) = \hat{\bar{v}}(t+x) - \hat{\bar{v}}(t-x)$, one has that

$$\phi := q_+ - q_- := q(\bar{v}_+) - q(\bar{v}_-) \in V\,.$$

It is an admissible variation and $\bar{v}\phi \geq 0$. One can easily check that if $p, q \in L^1(\mathbb{T})$ then

$$\int_\Omega p(t+x)q(t-x) = \int_0^{2\pi} p(s)ds \int_0^{2\pi} q(s)ds\,. \tag{2.35}$$

Using (2.35) and that $\int_0^{2\pi} \hat{v} = 0$, one has

$$\begin{aligned} \int \bar{v}\phi &= \int_\Omega (\bar{v}_+ - \bar{v}_-)(q_+ - q_-) = \int_\Omega (\bar{v}_+ q_+ - \bar{v}_- q_-) \\ &\geq M \int_\Omega |q(\bar{v}_+)| + |q(\bar{v}_-)| \end{aligned} \tag{2.36}$$

where the last inequality holds because of (2.34). So

$$\beta \int \bar{v}\phi \leq \left| \int_\Omega f(0)\phi \right| \leq C \int_\Omega |\phi| \leq C \int_\Omega |q(\bar{v}_+)| + |q(\bar{v}_-)|$$

and using (2.36), one has, if $|\varepsilon| \leq \varepsilon_1(R)$,

$$M \leq \frac{C}{\beta}\,.$$

Choosing $M := (1/2)\sup|\hat{\bar{v}}|$ and since $\sup|\hat{v}| \geq \frac{1}{2}\sup|\bar{v}|$,

$$M \approx \sup|\bar{v}| = \|\bar{v}\|_{L^\infty}\,.$$

Now one wants to define a suitable admissible function ϕ to obtain an H^1–estimate. For $f \in H^1(\Omega)$ we define the difference quotient of size $h \in \mathbb{R} \setminus \{0\}$

$$D_h f(t,x) := \frac{f(t+h,x) - f(t,x)}{h}\,.$$

One can check that if $f, g \in H^1(\Omega)$, $h \in \mathbb{R} \setminus \{0\}$ the following properties hold

(i) integration by parts:

$$\int_\Omega f(D_{-h}g) = -\int_\Omega (D_h f)g\ ;$$

(ii) estimate on the difference quotient

$$\|D_h f\|_{L^2(\Omega)} \leq \|f_t\|_{L^2(\Omega)}\ ;$$

(iii) convergence

$$D_h f \xrightarrow{L^2} f_t\ , \qquad \text{as} \quad h \longrightarrow 0\ . \tag{2.37}$$

Let us choose

$$\phi := -D_{-h}D_h\bar{v}\,.$$

Integrating by parts,

$$\beta \int \bar{v}\phi = \beta \int_{\Omega} (D_h\bar{v})(D_h\bar{v}) = \beta \int_{\Omega} (D_h\bar{v})^2 = \beta\, \|D_h\bar{v}\|_{L^2}^2 \tag{2.38}$$

and in the same way

$$\left| \int_{\Omega} f(0)\phi \right| = \left| \int_{\Omega} D_h f(0) D_h\bar{v} \right| \leq \mathrm{const}\|D_h\bar{v}\|_{L^2} \tag{2.39}$$

where the last inequality follows from the Cauchy-Schwartz estimate. Using (2.38), (2.39), it follows that

$$\|D_h\bar{v}\|_{L^2} \leq \frac{\mathrm{const}}{\beta}\,, \qquad\qquad \forall\, h \in \mathbb{R}\setminus\{0\}$$

and from the convergence (2.37) one has, if $|\varepsilon| \leq \varepsilon_2(R)$,

$$\|\bar{v}_t\|_{L^2} \leq \frac{\mathrm{const}}{\beta}$$

for a suitable C. Choosing

$$R := R_0 = 2\frac{\mathrm{const}}{\beta}\,,$$

one obtains the H^1–estimate and Theorem 2.3.1 follows.

2.4 Free vibrations with irrational frequency

In this case the presence of a small divisors problem is the crucial difficulty. Obviously, for finite dimensional systems, this problem does not arise and the existence of periodic orbits, close to an elliptic equilibrium point of a completely resonant hamiltonian systems, has been proved by Weinstein [We73], Moser [Mo76] and Fadell-Rabinowitz [FR78] using a Lyapunov-Schmidt splitting. This procedure reduces the problem to solve two equations: the *kernel equation* solved by variational methods and the *range equation*, solved by the Implicit Function Theorem. The following results are proved by an extension of this technique to the infinite–dimensional case. In particular, the small divisors problem appearing in the range equation is usually solved by a super–convergent Newton method.

We distinguish between Dirichlet and periodic boundary conditions.

Dirichlet boundary conditions

♡ [**W90**] Consider the general nonlinear equation

$$u_{tt} - u_{xx} + f(x,u) = 0\,.$$

Expanding $f(x,u)$ in a Taylor series about $u \equiv 0$, one obtains

$$f(x,u) = f(x,0) + f_u(x,0)u + \text{higher order terms}\,.$$

It's not restrictive to impose $f(x,0) = 0$, that physically means that there is no force acting when the string is at rest. By rescaling u, one introduces the small parameter ε and considers the following equation

$$u_{tt} - u_{xx} + v(x)u(x,t) + \varepsilon u^3(x,t) = 0\,, \tag{2.40}$$

where the potential $f_u(x,0) =: v(x)$ lies in the subspace

$$E := \left\{ v \in L^2\big([0,\pi]\big) \;\middle|\; \int_0^\pi v(x)\,dx = 0,\ v(x) = v(\pi - x) \right\} \cap L^2_+\,, \tag{2.41}$$

and $L^2_+ \subset L^2([0,\pi])$ is the open subset of all potentials $v \in L^2([0,\pi])$ with strictly positive Dirichlet eigenvalues. By fixing the average of v to be zero, one avoids to shift the eigenfrequencies of the $\varepsilon = 0$ case of (2.40) by a fixed amount.
Let be $\{\mu_j\}_{j=1}^\infty$ the eigenvalues and $\{\phi_j\}_{j=1}^\infty$ the corresponding eigenfunctions of the operator $-d^2/dx^2 + v(x)$ and with Dirichlet b.c., then

$$u_j^0(t,x) = \sin\big(\sqrt{|\mu_j|}\,t\big)\phi_j(x)$$

is a periodic solution of (2.40) for $\varepsilon = 0$, if $\mu_j > 0$. The following result states that these solutions persist when $\varepsilon > 0$.

Theorem 2.4.1. *There are sets $\mathcal{E}(j) \subset E$ such that if $v \in \mathcal{E}(j)$, there is a constant $\varepsilon_0(v,j) > 0$, such that whenever $|\varepsilon| < \varepsilon_0(v,j)$, (2.40) has a weak periodic solution $u_j^\varepsilon(x,t)$ whose frequency vector differs from $\sqrt{\mu_j}$ by $O(\varepsilon)$. There is a natural probability measure, P, defined on the subset of E, for which the spectrum of $-d^2/dx^2 + v(x)$ is positive and $\mathcal{E}(j)$ has measure one with respect to P.*

This theorem does not establish that the solutions constructed actually have periods that are irrational. However, the existence of solutions with irrational period is stated in the following result and here it is guaranteed also that one obtains a large number of solutions, which is typical for the KAM methods.

Theorem 2.4.2. *Suppose to restrict our attention to the subset of E for which the spectrum of $-d^2/dx^2 + v(x)$ is positive. Then, for almost every potential v, with respect to the P measure, there is a constant $\varepsilon_0(v) > 0$, such that if $|\varepsilon| < \varepsilon_0(v)$, there is an interval of the real line, $A(v)$, and a subset $B \subset A(v)$ with*

$$\text{meas}B \geq (1 - O(1/|\log\varepsilon|))\text{meas}\,A(v)$$

such that for every point $\Omega \in B$, (2.40) has a periodic orbit with frequency Ω.

For what concerns the techniques, the author replaces the scalar parameter μ of the Klein–Gordon equation $u_{tt} - u_{xx} + \mu u = 0$ with Dirichlet boundary conditions, by some potential function $v \in L^2([0, \pi])$. By this choice, he introduces infinitely many parameters into the system, which may be adjusted and thus substitutes the standard nondegeneracy condition.

♡ [**CW93**] Consider equation (2.4) for f odd analytic, namely $f(x, u) = -f(-x, -u)$, and π-periodic in x with Taylor's expansion in u

$$f(x, u) = f_1(x)u + f_2(x)u^2 + \dots .$$

Calling $\{\psi_j(x)\}_{j\geq 1}$ the eigenvectors of the Sturm-Liouville operator $-d^2/dx^2 + f_1(x)$, we can write the solutions of the linearized equation of (2.4), that is

$$v_{tt} - v_{xx} + f_1(x)v = 0 ,$$

in the form

$$v(x, t) = \sum_{j\geq 1} r_j \cos(\omega_j t + \xi_j)\psi_j(x)$$

which are parametrized by the amplitudes r_j and the phases ξ_j.

Theorem 2.4.3. *Among this class of nonlinearity f there is an open dense set such that there exist small amplitude time-periodic solutions of* (2.4)*. That is, there exists parameters (Ω, r, ξ) and solutions $u(x, t; r)$ satisfying*

$$|u(x, t; r) - r \cos(\Omega t + \xi)\psi_j(x)| < \mathrm{const}\, r^2$$

$$|\Omega - \omega_j| < \mathrm{const}\, r^2$$

for r in a set of positive measure.

Thinking of the analogy with the Lyapunov Center Theorem for finite dimensional hamiltonian systems in a neighborhood of an elliptic equilibrium point, the authors construct a smooth curve bifurcating from the solution of the linear equation. In the paper it is proved that there exists a Cantor set of frequencies of positive measure such that, for any point on the curve whose frequency lies in this Cantor set, one has a solution of (2.4). The proof is obtained by a version of Newton's methods.

♡ [**B00**] Let us consider a finite dimensional hamiltonian system, with Hamiltonian H such that $H(0) = 0$, with an elliptic equilibrium point at the origin. Let ω_j be the frequencies of the linear oscillations about such an equilibrium. Pick up one frequency, for example ω_1. The Lyapunov Center Theorem (see Theorem 1.1.5) states that, if

$$\omega_1 \ell - \omega_j \neq 0 , \qquad \ell \geq 1, j \neq 1 ,$$

then there exists a periodic orbit with frequency close to ω_1. The result contained in this paper is an extension of the Lyapunov Center Theorem to some partial differential equations. The stated theorem is not really new but the proof is so simple that it makes the paper very interesting. Let us consider the nonlinear wave equation

$$u_{tt} - u_{xx} + \mu u = \psi(u) \tag{2.42}$$

where $\psi(u) = \psi_0 u^r + \psi^{(1)}(u)$, with some $r > 2$, satisfying a suitable nondegeneracy condition. Moreover $\psi^{(1)}$ admits two Lipschitz derivatives, such that $\psi^{(1)}(0) = \big(\psi^{(1)}\big)'(0) = 0$ and fulfills the inequality

$$\sup_{|x|<2\varepsilon, |y|<2\varepsilon} \left| \left(\psi^{(1)}\right)''(x) - \left(\psi^{(1)}\right)''(y) \right| \leq \text{const}\, \varepsilon^{r-2} |x-y| \,.$$

In particular the above inequality holds if $\psi^{(1)}$ has a zero of order $r+1$ in the origin. To go on, we need some definitions:

Definition 2.4.4. *The frequency ω will be said to be γ strongly nonresonant with $(\omega_2, \omega_3, \ldots)$ if the following inequality holds*

$$|\omega \ell - \omega_j| \geq \frac{\gamma}{\ell}\,, \quad \ell \geq 1,\, j \geq 2\,.$$

Let us fix the parameter μ in such a way that the linear frequencies have the following (γ-NR)–property, with some γ:

(γ-NR) : We will say that $(\omega_1, \omega_2, \ldots)$ has the (γ-NR)–property if there exists a closed subset $W(\gamma) \subset \mathbb{R}$ such that any $\omega \in W(\gamma)$ is γ strongly nonresonant with $(\omega_2, \omega_3, \ldots)$ and moreover ω_1 is an accumulation point of both $W(\gamma) \cap (-\infty, \omega_1]$ and $W(\gamma) \cap [\omega_1, +\infty)$.

Using continuous fractions, one can prove the following

Proposition 2.4.5. *Let be $\omega_j = \sqrt{j^2 + \mu}$, $j \geq 1$, to be the square roots of the Dirichlet eigenvalues of $-d^2/dx^2 + \mu$ on $[0, \pi]$, then property (γ-NR) holds for μ belonging to a non-countable subset of $\mathbb{R}$. Moreover, for any μ in such a set, the family of ω's which are γ–strongly nonresonant with $(\omega_2, \omega_3, \ldots)$ and accumulate at ω_1, is not countable.*

Using the proposition 2.4.5 (for a detailed proof see [Sc], Th.5C and pg.23), the author reaches the following result:

Theorem 2.4.6. *With the above assumptions on the nonlinearity $\psi(u)$, there exists a family $\{u_\varepsilon\}_{\varepsilon \in \mathcal{E}}$ of periodic solutions of* (2.42)*, parametrized by a suitable set $\mathcal{E} \subset \mathbb{R}$ having an accumulation point at the origin. In addition, u_ε has frequency ω_ε which is γ strongly nonresonant with $(\omega_2, \omega_3, \ldots)$ and satisfies*

$$|\omega_\varepsilon - \omega_1| \leq \text{const}\, \varepsilon^{r-1}, \qquad \left\| u_\varepsilon\left(\frac{t}{\omega_\varepsilon}\right) - \varepsilon \sin x \cos t \right\|_{H^1} \leq \text{const}\, \varepsilon^r\,.$$

Moreover $u_\varepsilon \in H^1\big([0, 2\pi/\omega_\varepsilon]; H^1[0, \pi]\big)$.

The proof is based on a Lyapunov–Schmidt decomposition. One fixes a frequency ω close to ω_1 and looks for a periodic solution of frequency ω. By the splitting, one considers the 1–dimensional bifurcation equation on the kernel of the linear operator

$$L_{\omega_1} := \omega_1^2 \frac{d^2}{dt^2} - \frac{d^2}{dx^2} + \mu$$

and an infinite dimensional range equation on its orthogonal in a suitable space of functions. In solving this last equation a small divisors problem arises. The eigenvalues of the linear operator L_ω are

$$-\omega^2\ell^2 + \omega_j^2 = (-\omega\ell + \omega_j)(\omega\ell + \omega_j)\,, \qquad \ell \geq 0\,, \quad (\ell, j) \neq (1,1) \tag{2.43}$$

so that, if ω satisfies the γ strongly nonresonant condition with $(\omega_2, \omega_3, \ldots)$, then (2.43) are bounded away from zero. This choice of imposing a strong nonresonance on the frequencies is inspired by the trick used by R. De La Llave [DL00] to obtain a variational proof of existence of orbits in some nonlinear wave equations. Thanks to this excising of resonances, one can apply a standard Implicit Function Theorem to solve the range equation.

♡ **[BP01]** For $\gamma > 0$ and $n \geq 1$, let be

$$\mathcal{W}_\gamma^n := \left\{ \omega \in \mathbb{R} \;\middle|\; |\omega\ell - k_j| \geq \frac{\gamma}{\ell}, \quad \forall\, (\ell, j) : n\ell \neq k_j \right\}. \tag{2.44}$$

Such set has zero measure, is uncountable and accumulates at 1 both from the right and from the left.

Theorem 2.4.7. *Suppose that f in* (2.7) *satisfies $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$. For any $n \geq 1$ and any positive $\gamma < 1/3$ there exists a family $\{u_\varepsilon^n(t)\}_{\varepsilon \in \mathcal{E}\gamma^n}$ of periodic solutions of* (2.7)*, with Dirichlet b.c. with the following properties:*

(1) the set $\mathcal{E}_\gamma^n \subset \mathbb{R}^+$ is uncountable and has zero as an accumulation point;

(2) u_ε^n is periodic with minimal period $2\pi/\omega_\varepsilon^n$ and is of class $H^1([0, 2\pi/\omega_\varepsilon], H^1)$;

(3) there exists a strictly positive ω_*^n such that the frequency map ω_ε^n

$$\mathcal{E}_\gamma^n \ni \varepsilon \mapsto \omega_\varepsilon^n \in W^n\,, \quad \text{with} \quad W^n := \begin{cases} W_\gamma^n \cap [n, n + \omega_*^n) & \text{if} \quad \alpha < 0; \\ W_\gamma^n \cap (n - \omega_*^n, n] & \text{if} \quad \alpha > 0; \end{cases}$$

is monotone and one to one; moreover it fulfills the following estimate

$$|n - \omega_\varepsilon^n| \leq \text{const}\, \varepsilon^2\,;$$

(4) u_ε^n is close to the solution $\varepsilon v_n(x,t)$ of the linearized system, precisely

$$\sup_{t \in \mathbb{R}} \left\| u_\varepsilon^n(t) - \varepsilon v_n(\cdot, \omega_\varepsilon^n t) \right\|_{H^1} \leq \text{const}\, \varepsilon^2\,.$$

$2\pi/\omega$-periodic map

$$\tilde{u}_\omega : t \in \mathbb{R} \longrightarrow [\tilde{u}_\omega(t)](\cdot) \in H^1([0,\pi])$$

$\tilde{u}_\omega \in H^1\big([0, 2\pi/\omega], H^1([0,\pi])\big)$*, such that* $u_\omega(t,x) := [\tilde{u}_\omega(t)](x)$ *is a* $2\pi/\omega$ *time-periodic solution of* (2.4).

In this paper the authors construct small amplitude solutions close to a completely resonant elliptic equilibrium point. The proof is achieved performing a Lyapunov-Schmidt decomposition. The authors solve first the range equation imposing the strong diophantine condition on the frequency in (2.44), and then the kernel equation by averaging methods.

♡ **[BeBo03], [BeBo04]** Let us consider $\mathcal{W}_\gamma$ as (2.44).

Theorem 2.4.8. *Let* $f(u) = au^p + o(u^p)$ $(a \neq 0)$*,* $p \geq 2$*. Then* $\forall\, \omega \in \mathcal{W}_\gamma$ *satisfying* $|\omega - 1| \leq const$ *and* $\omega < 1$ $(\omega > 1$ *if* p *is odd and* $a > 0)$ *equation*

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.45}$$

possesses at least one $2\pi/\omega$*-periodic, even in time solution.*

This result will be discussed in more details on page 59.

♡ **[GMPr04]** Let be g an analytic 2π-periodic in (t,x) function, let us define

$$\|g(t,x)\|_r := \sum_{(n,m)\in\mathbb{Z}^2} g_{n,m} e^{r(|n|+|m|)},$$

where $g_{n,m}$ are the Fourier coefficients with respect to t and x. Let $v(t,x) := \eta(t+x) - \eta(t-x)$, with $\eta(\xi)$ the odd 2π-periodic solution of the equation

$$\ddot{\eta} = -\eta^3 - \frac{3}{2\pi}\eta \int_0^{2\pi} \eta^2(\xi)\, d\xi\,.$$

Theorem 2.4.9. *Consider equation* (2.7) *with* $f(u) = au^3 + O(u^5)$ *an odd analytic function,* $a \neq 0$*. There exist a positive constant* ε_0 *and a set* $\mathcal{E} \in [0, \varepsilon_0]$ *satisfying*

$$\lim_{\varepsilon \to 0} \frac{\text{meas}\,(\mathcal{E} \cap [0,\varepsilon])}{\varepsilon} = 1$$

such that for all $\varepsilon \in \mathcal{E}$*, by choosing* $\omega_\varepsilon = \sqrt{1-\varepsilon}$*, there exist* $2\pi/\omega_\varepsilon$*-periodic solutions* u_ε *of* (2.7) *analytic in* (t,x)*, with*

$$\left\| u_\varepsilon(t,x) - \sqrt{\varepsilon/|a|}\, v(\omega_\varepsilon t, x) \right\|_{k'} \leq \text{const}\, \varepsilon^{\frac{3}{2}}$$

where, calling $k/2$ *the radius of analyticity of* η*, is* $0 < k' < k$*.*

This result will be discussed in more details on page 63.

♡ **[BeBo05]** Let us consider problem (2.45). Assume that the nonlinearity f satisfies

(H) $f(x,u) = \sum_{k\geq p} a_k(x)u^k$, $p \geq 2$, and $a_k \in H^1((0,\pi),\mathbb{R})$ verify $\sum_{k\geq p} \|a_k(x)\|_{H^1}\, \rho^k < \infty$ for some $\rho > 0$,

such that it is analytic in u but only H^1 in x. Let be

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{\ell\in\mathbb{Z}} e^{i\ell t} u_\ell(x) \quad \text{s.t.} \quad u_\ell \in H^1_0((0,\pi),\mathbb{R}),$$
$$u_\ell(x) = u_{-\ell}(x)\ \forall \ell \in \mathbb{Z}, \text{ and } \|u\|^2_{\sigma,s} := \sum_{\ell\in\mathbb{Z}} e^{2\sigma|l|}(\ell^{2s}+1)\|u\|^2_{H^1} < \infty \Big\}.$$

The main result of the paper is stated in the following

Theorem 2.4.10. *Let f satisfies assumption (H) and*

$$f(x,u) = \begin{cases} a_2u^2 + \sum_{k\geq 4} a_k(x)u^k & a_2 \neq 0 \\ or & \\ a_3(x)u^3 + \sum_{k\geq 4} a_k(x)u^k & \langle a_3\rangle := (1/\pi)\int_0^\pi a_3(x)dx \neq 0\,. \end{cases}$$

Then, $s > 1/2$ being given, there exist $\delta_0 > 0$, $\bar\sigma > 0$ and a C^∞–curve $[0,\delta_0) \ni \delta \to u(\delta) \in X_{\bar\sigma/2,s}$ with the following properties:

(i) $\|u(\delta) - \delta\bar v\|_{\bar\sigma/2,s} = O(\delta^2)$ for some $\bar v \in V \cap X_{\bar\sigma,s}$, $\bar v \neq \{0\}$, where V is the kernel of the operator $\Box$;

(ii) there exists a Cantor set $\mathcal{C} \subset [0,\delta_0)$ of asymptotically full measure, i.e. satisfying

$$\lim_{\eta\to 0^+} \frac{\text{meas}(\mathcal{C}\cap(0,\eta))}{\eta} = 1\,,$$

such that, $\forall\,\delta \in \mathcal{C}$, $u(\delta)$ is a 2π–periodic, even in time, classical solution of the re-scaled problem

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0\,, \end{cases}$$

with respectively

$$\omega = \omega(\delta) = \begin{cases} \sqrt{1-2\delta^2} \\ or \\ \sqrt{1+2\delta^2\,\text{sign}\langle a_3\rangle}\,. \end{cases}$$

As a consequence, $\forall\,\delta \in \mathcal{C}$, $\tilde u(\delta)(t,x) := u(\delta)(\omega(\delta)t,x)$ is a $2\pi/\omega(\delta)$–periodic, even in time, classical solution of equation (2.45).

This result will be discussed in more details on page 65.

Periodic boundary conditions

♡ **[CW93]** In this paper the same result as for the Dirichlet boundary conditions described on pg. 53 but without oddness assumptions on f is proved.

♡ **[Bou95]** Consider the periodic wave equation in dimension d

$$u_{tt} - \Delta u + \rho u + u^3 = 0\,, \tag{2.46}$$

where Δ is the Laplacian on the d–torus $\Pi^d \times \mathbb{R}$ which is λ–periodic in t. Replacing u by δu, the equation becomes

$$u_{tt} - \triangle u + \rho u + \delta^2 u^3 = 0$$

which appears as a perturbation of the linear equation

$$u_{tt} - \Delta u + \rho u = 0\,. \tag{2.47}$$

Fix $m_0 \in \mathbb{Z}^d \backslash \{0\}$. The author shows the persistence for the perturbed equation (2.46), of the solution

$$u_0 = p_0 \cos(\langle m_0, x\rangle + \lambda_0 t)$$

of (2.47) with $\lambda_0 = (|m_0|^2 + \rho)^{1/2}$. The perturbed solution has the form

$$u(x,t) = \sum_{m\in\mathbb{Z}^d,\, n\in\mathbb{Z}_+} \hat{u}(m,n)\cos(\langle m, x\rangle + n\lambda t)$$

where $\hat{u}(m_0, 1) = p_0$ and

$$\sum_{(m,n)\neq(m_0,1)} |\hat{u}(m,n)| e^{(|m|+|n|)^c} < O(\delta).$$

Here $c > 0$ is some constant.

Theorem 2.4.11. *Let $\rho \in \mathbb{R}$ satisfy a condition of the form*

$$\left|\sum_{j=0}^{r} a_j \rho^j\right| > \Big(\sum |a_j|\Big)^{-c(r)}, \qquad \forall\, \{a_j\} \in \mathbb{Z}^{r+1} \setminus \{0\}\,.$$

Here $r := r(d)$. Consider the periodic wave equation (2.46) *in dimension d. Fix $m_0 \in \mathbb{Z}^d \setminus \{0\}$. There is a Cantor set C of positive measure in an interval $[0, \delta]$ and for $p_0 \in C$ a solution of* (2.46) *of the form*

$$u = p_0 \cos(\langle m_0, x\rangle + \lambda t) + O(p_0^3)$$

where

$$\lambda^2 = \lambda(p_0)^2 = |m_0|^2 + \rho + \frac{3}{4}p_0^2 + O(p_0^4)\,.$$

♡ **[Bou99]**: for nonlinearities

$$f(x,u) = u^3 + \sum_{j=4}^{d} a_j(x) u^j$$

where a_j are trigonometric cosine polynomials in x, existence of periodic solutions of the resonant wave equation with frequencies in a full measure set is proved. The small divisors problem is solved here by a KAM iterative scheme.

2.4.1 [BeBo03]

In [BeBo03], the authors look for time-periodic solutions of the problem

$$\begin{cases} u_{tt} - u_{xx} + f(u) = 0 \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{2.48}$$

where $f(u) = au^p$, with $p \geq 2$ and $a \neq 0$. As we have just said, the homogeneous equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

has periodic solutions if and only if the period is a rational multiple of 2π. For definiteness one can look for periodic solutions of (2.48) with period $2\pi/\omega$ close to 2π. Hence, after a time scaling $t \to t/\omega$ ($\omega \approx 1$), problem (2.48) is equivalent to

$$\begin{cases} \mathcal{L}_\omega u = f(u) \\ u(t,0) = u(t,\pi) = 0 \\ u(t+2\pi,x) = u(t,x), \end{cases} \tag{2.49}$$

where $\mathcal{L}_\omega u := -\omega^2 u_{tt} + u_{xx}$. The authors look for solutions $u : \Omega \longrightarrow \mathbb{R}$ of (2.49), where $\Omega = \mathbb{T} \times (0,\pi)$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, in the Banach space

$$X := \Big\{ u \in H^1(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R}) \text{ s.t. } u(t,0) = u(t,\pi) = 0,\ u(-t,x) = u(t,x) \Big\}.$$

They choose the space X of even in time functions because equation (2.48) is reversible. In X one can introduce the norm

$$\|u\|_X := |u|_\infty + \|u\|,$$

where $\|\cdot\|$ is the norm on $H^1(\Omega)$ associated to the scalar product

$$(u,w) := \int_\Omega (u_t w_t + u_x w_x)\, dt\, dx\,.$$

The kernel K of the operator $\mathcal{L}_1 = -\partial_{tt} + \partial_{xx}$ acting on functions $L^2(\Omega)$, with Dirichlet boundary conditions, is the following closed subset of $L^2(\Omega)$

$$K := \left\{ v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) \text{ s.t. } \hat{v} \in L^2(\mathbb{T})\,, \quad \text{with } \int_0^{2\pi} \hat{v} = 0 \right\}.$$

Obviously $L^2(\Omega)$ can be decomposed into the two orthogonal closed subspaces $L^2(\Omega) = K \oplus K^\perp$. By the light of this decomposition, one can perform a Lyapunov-Schmidt reduction and look for solutions of (2.49) of the form $u = v + w$ with $v \in V$ and $w \in W$, where[3] $V \oplus W = X$ and

$$\begin{aligned} V &:= X \cap K = \Big\{ v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) \ \text{ s.t. } \ \hat{v}(\cdot) \in H^1(\mathbb{T}),\ \hat{v}\,\text{odd} \Big\} \\ &= \Big\{ v(t,x) = \sum_{j\geq 1} \xi_j \cos(jt)\sin(jx) \ \text{ s.t. } \ \xi_j \in \mathbb{R},\ \sum_{j \geq 1} j^2 \xi_j^2 < +\infty \Big\}, \\ W &:= X \cap K^\perp = \Big\{ \sum_{\ell \geq 0, j \geq 1} w_{\ell j} \cos(\ell t) \sin(jx) \in X \ \text{ s.t. } \ w_{jj} = 0 \ \forall j \geq 1 \Big\}. \end{aligned}$$

Denoting by Π_V and Π_W the projections onto V and W respectively, (2.49) is equivalent to the following system of two equations (called respectively the (Q) and the (P) equations)

$$\begin{cases} \mathcal{L}_\omega v = \Pi_V f(v+w)\,, & (Q) \\ \mathcal{L}_\omega w = \Pi_W f(v+w)\,. & (P) \end{cases}$$

One first solve the (P) equation by an Implicit Function Theorem. Here a small divisors problem appears. In fact, as we have just said in section 2.1, writing f in Fourier series,

$$f(t,x) = \sum_{\ell \geq 0, j \geq 1, \ell \neq j} f_{\ell j} \cos(\ell t) \sin(jx)$$

then

$$\mathcal{L}_\omega^{-1} f = \sum_{\ell \geq 0, j \geq 1, \ell \neq j} \frac{f_{\ell j}}{\omega^2 \ell^2 - j^2} \cos(\ell t)\sin(jx).$$

In order to estimate the small divisors $\omega^2 \ell^2 - j^2$, one can assume that

$$\omega \in \mathcal{W} := \bigcup_{\gamma > 0} \mathcal{W}_\gamma\,,$$

where $\mathcal{W}_\gamma$ is the set of strongly non-resonant frequencies introduced in [BP01]

$$\mathcal{W}_\gamma := \Big\{ \omega \in \mathbb{R} \ \Big| \ |\omega \ell - j| \geq \frac{\gamma}{\ell},\ \forall j \neq \ell \Big\}.$$

As proved in Remark 2.4 of [BP01], for $\gamma < 1/3$, the set $\mathcal{W}_\gamma$ is uncountable, has zero measure and accumulates to $\omega = 1$ both from the left and from the right. Since

$$|\omega^2\ell^2 - j^2| = |j + \omega\ell||j - \omega\ell| \geq |j + \omega \ell| \frac{\gamma}{\ell} \geq \text{const}\,\gamma\,, \qquad \forall\, \omega \in \mathcal{W}_\gamma\,,$$

[3] We note that the H^1-norm is equivalent to the X-norm on V.

the operator $\mathcal{L}_\omega^{-1} : W \longrightarrow W$ is bounded and $\|\mathcal{L}_\omega^{-1}\| \leq \text{const}/\gamma$. Using the Implicit Function Theorem one finds $w(v) \in W$, for any v in a sufficiently small ball $\tilde{V} \subset V$, such that

$$\mathcal{L}_\omega w(v) = \Pi_W f(v + w(v)).$$

Inserting $w(v)$ in the (Q) equation, nothing remains but to solve the infinite dimensional equation

$$\mathcal{L}_\omega v = \Pi_V f(v + w(v)). \tag{2.50}$$

In order to find solutions of (2.50), one can consider the *Lagrangian Action Functional* $\Psi_\omega : X \longrightarrow \mathbb{R}$,

$$\Psi_\omega(u) := \int_0^{2\pi}\int_0^{\pi} \left(\frac{\omega^2}{2}u_t^2 - \frac{u_x^2}{2} - F(u)\right) dx\, dt\,,$$

where $F(u) := \int_0^u f(s)\, ds$. Critical points of Ψ are weak solutions of (2.49). Moreover, defining the *reduced Lagrangian Action Functional* $\Phi_\omega : \tilde{V} \longrightarrow \mathbb{R}$, by

$$\Phi_\omega(v) := \Psi_\omega(v + w(v))\,,$$

where

$$\Psi_\omega(v + w(v)) = \\ \int_0^{2\pi}\int_0^{\pi} \left(\frac{\omega^2}{2}\left(v_t + (w(v))_t\right)^2 - \frac{1}{2}\left(v_x + (w(v))_x\right)^2\right) - F(v + w(v))\, dx\, dt\,,$$

one can easily check that critical points of Φ_ω are weak solutions of the equation (2.50), because it is the Euler-Lagrange equation of Φ_ω. In fact, one can show that Φ_ω is in $C^1(\tilde{V}, \mathbb{R})$ and for every $h \in V$, it results

$$D\Phi_\omega(v)[h] = \mathcal{D}\Psi(v + w(v))\big[h + dw(v)[h]\big] = \mathcal{D}\Psi(v + w(v))[h]\,, \tag{2.51}$$

since $dw(v)h \in W$ and $w(v)$ solves the (P) equation. By (2.51), for $v, h \in V$,

$$\begin{aligned} \mathcal{D}\Phi_\omega(v)[h] &= \int_\Omega \left(\omega^2 v_t h_t - v_x h_x - \Pi_V f(v + w(v)h\right) dt\, dx \\ &= \varepsilon(v, h) - \int_\Omega \left(\Pi_V f(v + w(v))h\right) dt\, dx\,, \end{aligned} \tag{2.52}$$

where $\varepsilon := (\omega^2 - 1)/2$. By (2.52), a critical point $v \in \tilde{V} \subset V$ of Φ_ω is a solution of the equation (2.50). In this way, one can reduce the problem of finding non-trivial solutions of the infinite dimensional (Q) equation, to find critical points (different from zero) of the reduced action functional Φ_ω, that can be developed as

$$\Phi_\omega(v) = \frac{\varepsilon}{2}\|v\|^2 + \int_\Omega \left(\frac{1}{2}f(v + w(v))w(v) - F(v + w(v))\right) dt\, dx\,.$$

Let us develop the reduced action functional in a neighborhood of $w = 0$,

$$\Phi_\omega(v) = \frac{\varepsilon}{2}\|v\|^2 - \int_\Omega \Big(F(v) - \frac{1}{2} f(v)w(v) \Big)\, dt\, dx + O(w^2)\,.$$

If $f(u) = u^p$, p integer, the functional Φ_ω is of the form

$$\Phi_\omega = \frac{\varepsilon}{2}\|v\|^2 - G(v) + R(v)\,,$$

where

$$\begin{aligned} G(v) &:= \int_\Omega F(v) = \frac{1}{p+1}\int_\Omega v^{p+1}\,, \\ R(v) &:= \int_\Omega -F(v+w(v)) + F(v) + \frac{1}{2} f(v+w(v))w(v)\,. \end{aligned}$$

One has to consider the two cases:

- p odd $\Longrightarrow G(v) \neq 0\,, \quad \forall\, v \neq 0\,,$
- p even $\Longrightarrow G(v) \equiv 0\,, \quad \forall\, v \in V\,.$

In the first case we note that neglecting the reminder R, the homogeneous functional

$$\Phi^*(v) := \frac{\varepsilon}{2}\|v\|^2 - G(v)$$

is defined on *all* V and satisfies the Mountain Pass geometry choosing ε *positive*. In fact, $\Phi^*(0) = 0$ is a local minimum and

(i) $\exists\, \delta > 0$ such that $\inf\limits_{\|v\|=\delta} \Phi^*(v) > 0$;

(ii) $\Phi^*(tv) \longrightarrow -\infty$ as $t \to +\infty$,

where (ii) holds since $G(v) > 0$ for $v \neq 0$. Since the reminder R is defined only in $\tilde{V}$, one cannot directly apply the Mountain Pass Theorem. To bypass this problem, one has to suitably extend R in the whole space V, in such a way that the Mountain Pass geometry is even satisfied and the so obtained solution belongs to $\tilde{V}$. This a priori estimate can be obtained taking ε small enough. In the second case the situation is more difficult because $G \equiv 0$. For this reason, one has to develop the reminder R (in power of w) obtaining

$$R(v) = \frac{1}{2}\int_\Omega f(v)w(v) + \dots\,.$$

From the (P) equation we have, using that $v^p \in W$ if p is even,

$$\begin{aligned} w &= \mathcal{L}_\omega^{-1}\Pi_W(v+w)^p = \mathcal{L}_\omega^{-1}\Pi_W(v)^p + \dots \\ &= \mathcal{L}_\omega^{-1}(v)^p + \dots = \mathcal{L}^{-1}(v)^p + \big(\mathcal{L}_\omega^{-1} - \mathcal{L}^{-1}\big)(v)^p + \dots\,. \end{aligned}$$

One can check that $\left(\mathcal{L}_\omega^{-1} - \mathcal{L}^{-1}\right)$ has bounded and small norm for $\omega \approx 1$, so $w \approx \mathcal{L}^{-1} v^p$ and

$$R(v) = \frac{1}{2}\int_\Omega f(v)w(v) + \ldots = \frac{1}{2}\int_\Omega v^p \mathcal{L}^{-1} v^p + \ldots .$$

Since one can prove that

$$\int_\Omega v^p \mathcal{L}^{-1} v^p < 0 \qquad \forall\, v \neq 0\,,$$

the situation is the same of the odd case, choosing ε *negative*.

2.4.2 [GMPr04]

In [GMPr04], the problem is (2.48) with odd and analytic nonlinearity

$$f(u) = \mu u^3 + O(u^5) \qquad \mu \in \mathbb{R}, \mu > 0\,. \tag{2.53}$$

After a scaling $u \to \sqrt{\varepsilon} u$, for ε small, one can search $2\pi/\omega_\varepsilon$–periodic solutions in t of the problem

$$\begin{cases} u_{tt} - u_{xx} = \varepsilon u^3 + O(\varepsilon^2) \\ u(0,t) = u(\pi, t) = 0\,, \end{cases} \tag{2.54}$$

where $O(\varepsilon^2)$ denotes an analytic function of u and ε of order at least two in ε and $\omega_\varepsilon := \sqrt{1-\varepsilon}$, such that $\omega_\varepsilon = 1$ for $\varepsilon = 0$. After a time–scaling $t \to t/\omega$, the problem reduces to search 2π–periodic solutions of the equation[4]

$$(1-\varepsilon)u_{tt} - u_{xx} = \varepsilon u^3\,.$$

Now, the authors perform a Lyapunov-Schmidt reduction. Consider a solution u of the form $u = v + w$ where $v \in K$, with $K := \ker \Box$ and $w \in K^\perp$. We obtain the equivalent system

$$\begin{cases} w = \varepsilon \mathcal{L}_\varepsilon^{-1} \Pi_{K^\perp}(v+w)^3 \\ -v_{tt} = \Pi_K (v+w)^3\,, \end{cases} \tag{2.55}$$

where $\mathcal{L}_\varepsilon := (1-\varepsilon^2)\partial_t^2 - \partial_x^2$ while Π_K and $\Pi_{K^\perp}$ denote the projections, respectively, on the kernel and on its orthogonal. Let us expand by series

$$v = \sum_{k=0}^{\infty} \varepsilon^k v_k\,, \qquad w = \sum_{k=0}^{\infty} \varepsilon^k w_k\,. \tag{2.56}$$

From the first equation one obtains that $w_0 = 0$, from the second one we have

$$-v_{0tt} = \Pi_K v_0^3\,. \tag{2.57}$$

[4]From now on, we omit the term $O(\varepsilon^2)$ for simplicity of exposition.

We need some properties of Π_K. It is known (see [Lo69]), that if ψ is a function of $(t,\, x)$ then

$$\Pi_K \psi(t,x) = p(t+x) - p(t-x)$$

where

$$p(y) = \frac{1}{2\pi} \int_0^\pi \big[\psi(y-s,s) - \psi(y+s,s)\big]\, ds\,.$$

In the case $\psi(t,x) = A(t+x)$, one has

$$p(y) = \frac{1}{2\pi} \int_0^\pi \big[A(y) - A(y+2s)\big]\, ds = \frac{1}{2}A(y) - \frac{1}{2}\langle A\rangle$$

where

$$\langle A\rangle := \frac{1}{2\pi} \int_0^{2\pi} A(s)\, ds\,.$$

Hence,

$$\Pi_K A(t+x) = \frac{1}{2}\big[A(t+x) - A(t-x)\big]\,.$$

In the same way, if one takes $\psi(t,x) = B(t-x)$, then has

$$p(y) = \frac{1}{2\pi} \int_0^\pi \big[B(y-2s) - B(y)\big]\, ds = -\frac{1}{2}B(y) + \frac{1}{2}\langle B\rangle$$

so that

$$\Pi_K B(t-x) = -\frac{1}{2}\big[B(t+x) - B(t-x)\big]\,.$$

Finally, let us consider $\psi(t,x) = A(t+x)B(t-x)$. We have

$$\begin{aligned} p(y) &= \frac{1}{2\pi} \int_0^\pi \big[A(y)B(y-2s) - A(y+2s)B(y)\big]\, ds \\ &= -\frac{1}{2}A(y)\langle B\rangle + \frac{1}{2}\langle A\rangle B(y) \end{aligned}$$

so that

$$\Pi_K A(t+x)B(t-x) = \frac{\langle B\rangle}{2}\big(A(t+x) - A(t-x)\big) + \frac{\langle A\rangle}{2}\big(B(t-x) - B(t+x)\big)\,.$$

Since $v_0 \in K$, we write

$$v_0 = a(t+x) - a(t-x) := a_+(t,x) - a_-(t,x)$$

where a is 2π-periodic function, from which it follows that

$$v_0^3 = a_+ - a_-^3 - 3a_+^2 a_- + 3a_-^2 a_+\,.$$

Applying the above properties of Π_K, we have that

$$\Pi_K\, a_+^3 \;=\; \frac{1}{2}\big[a^3(t+x) - a^3(t-x)\big]$$

$$
\begin{aligned}
\Pi_K\, a_-^3 &= -\frac{1}{2}\big[a^3(t+x) - a^3(t-x)\big]\\
\Pi_K\, a_+^2 a_- &= -\frac{\langle a^2\rangle}{2}\, a(t+x) + \frac{\langle a^2\rangle}{2}\, a(t-x)\\
\Pi_K\, a_-^2 a_+ &= \frac{\langle a^2\rangle}{2}\, a(t+x) - \frac{\langle a^2\rangle}{2}\, a(t-x)\,.
\end{aligned}
$$

The equation (2.57) becomes

$$
\Pi_K\, v_0^3 = \Pi_K\, (a_+ - a_-)^3 a^3(t+x) - a^3(t-x) + 3\big[\langle a^2\rangle a(t+x) - \langle a^2\rangle a(t-x)\big]\,,
$$

and, finally, one obtains the equation that a must satisfy

$$
-\ddot{a}(t+x) + \ddot{a}(t-x) = a^3(t+x) - a^3(t-x) + 3\langle a^2\rangle\big[a(t+x) - a(t-x)\big]\,.
$$

Computing it for $t = x =: \xi/2$ we have

$$
-\ddot{a}(\xi) = a^3(\xi) + 3\langle a^2\rangle a(\xi)\,, \tag{2.58}
$$

which has a unique 2π-periodic odd solution.
The authors now consider the first order in ε in (2.55)

$$
\begin{cases} w_1 = \mathcal{L}_\varepsilon^{-1}\Pi_K^\perp v_0^3 \\ -v_{1tt} = \Pi_K(3v_0^2 w_1) \end{cases}
$$

and for the second order, they have

$$
\begin{cases} w_2 = \mathcal{L}_\varepsilon^{-1}\Pi_K^\perp (3v_0^2 w_1) \\ -v_{2tt} = \Pi_K(3v_0 w_1^2) \end{cases}
$$

and so on. In general, by induction, one solves, at first, the range equation at order k, which is of the form

$$
w_k = W_k(w_0, \ldots, w_{k-1}, v_0, \ldots, v_{k-1})\,,
$$

finding w_k, and, thereafter, the kernel equation

$$
v_k = V_k(w_0, \ldots, w_k, v_0, \ldots, v_{k-1})\,,
$$

finding v_k. In this way, one determines all coefficients in (2.56). Finally, nothing remains but to prove the convergence of the series in (2.56). This is obviously the most difficult (and technical) part and it is carried out in [GMPr04] by the Lindstedt's series.

2.4.3 [BeBo05]

In [BeBo05] the authors look for small amplitude, $2\pi/\omega$–periodic in time solutions of equation

$$
\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{2.59}
$$

with nonlinearity

$$f(x,u) = a_p(x)u^p + O(u^{p+1})\,, \quad p \ge 2\,,$$

for all frequencies ω in some Cantor set of positive measure. Note that any solution of (2.59) can be written as

$$v(x,t) = \sum_{j\ge 1} a_j \cos(jt+\theta_j)\sin jx$$

and it is 2π-periodic in time. Normalizing the period, one reduces to the problem

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.60}$$

and looks for 2π-periodic in time solutions. The functional space in which one works is the real Hilbert space

$$\begin{aligned} X_{\sigma,s} \;:=\; & \Big\{ u(t,x) = \sum_{\ell\in\mathbb{Z}} e^{i\ell t} u_\ell(x) \quad \text{s.t.} \quad u_\ell \in H^1_0((0,\pi),\mathbb{R}), \\ & u_\ell(x) = u_{-\ell}(x)\ \forall \ell\in\mathbb{Z}, \ \text{ and } \ \|u\|^2_{\sigma,s} := \sum_{\ell\in\mathbb{Z}} e^{2\sigma|\ell|}(\ell^{2s}+1)\|u\|^2_{H^1} < \infty \Big\}. \end{aligned}$$

For $\sigma > 0$, $s \ge 0$, the space $X_{\sigma,s}$ is the space of all even, 2π–periodic in time functions with values in $H^1_0((0,\pi),\mathbb{R})$, which have a bounded analytic extension in the complex strip $|\operatorname{Im} u| < \sigma$ with trace function on $|\operatorname{Im} u| = \sigma$ belonging to $H^s\big(\mathbb{T}, H^1_0((0,\pi),\mathbb{C})\big)$. A weak solution $u \in X_{\sigma,s}$ of (2.60) is a classical solution because the map $x \longmapsto u_{xx}(t,x) = \omega^2 u_{tt}(t,x) - f(x,u(t,x))$ belongs to $H^1_0(0,\pi)$ for all $t\in\mathbb{T}$ and hence $u(t,\cdot) \in H^3(0,\pi) \subset C^2([0,\pi])$.

The Lyapunov–Schmidt reduction

After the scaling $u \to \delta u$ one obtains:

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^{p-1} g_\delta(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

where

$$g_\delta(x,u) := \frac{f(x,\delta u)}{\delta^p} = a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \dots\,.$$

One performs a Lyapunov-Schmidt reduction searching a solution $u = v + w$, with

$$u \in X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s})\,,$$

where

$$W := \Big\{ u(t,x) = \sum_{\ell\in\mathbb{Z}} e^{i\ell t} w_\ell(x)\,,\ w_\ell \in X_{0,s}\,,\ \int_0^\pi w_\ell(x)\sin(\ell x)\,dx = 0\,,\ \forall \ell\in\mathbb{Z} \Big\}\,,$$

and

$$V := \Big\{ v(t,x) = \sum_{\ell \geq 1} 2\cos(\ell t) u_\ell sin(\ell x)\,,\ u_\ell \in \mathbb{R}\,,\ \sum_{\ell \geq 1} \ell^2 |u_\ell|^2 < +\infty \Big\}$$
$$= \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \ \text{ s.t. } \ \eta \in H^1(\mathbb{T}, \mathbb{R})\,,\, \eta \ \text{ odd} \Big\}\,.$$

In particular, v and w are solutions of the equations

$$\begin{cases} -\frac{(\omega^2-1)}{2} \Delta v = \delta^{p-1} \Pi_V g_\delta(x, v+w) & (Q) \\ L_\omega w = \delta^{p-1} \Pi_W g_\delta(x, v+w) & (P) \end{cases} \tag{2.61}$$

where $-\Delta v = v_{tt} + v_{xx}$, $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}$ and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projection respectively on V and W.

0th order bifurcation equation

In order to find non–trivial solutions of (2.61) one has to impose that

$$\Pi_V(a_p(x) v^p) \not\equiv 0\,, \tag{2.62}$$

that is equivalent to the condition

$$\exists v \in V \quad \text{s.t.} \quad \int_\Omega a_p(x) v^{p+1}(t,x)\, dt dx \neq 0\,, \qquad \Omega = \mathbb{T} \times (0,\pi) \tag{2.63}$$

which is verified if and only if $a_p(\pi - x) \not\equiv (-1)^p a_p(x)$. If condition (2.63) is satisfied, one chooses

$$|\varepsilon| := \delta^{p-1}\,, \qquad \varepsilon := \frac{\omega^2 - 1}{2}\,,$$

and reduces to the problem

$$\begin{cases} -\Delta v = \Pi_V g(\delta, x, v+w) & (Q) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v+w) & (P) \end{cases} \tag{2.64}$$

where

$$g(\delta, x, u) := s^* \frac{f(x, \delta u)}{\delta^p} = s^* \big(a_p(x) u^p + \delta a_{p+1}(x) u^{p+1} + \dots \big)$$

and $s^* := \text{sign}(\varepsilon)$, namely $s^* = 1$ if $\omega > 1$ and $s^* = -1$ if $\omega < 1$. The *0th–order bifurcation equation* for $\delta = 0$ is

$$-\Delta v = s^* \Pi_v (a_p(x) v^p) \tag{2.65}$$

which is the Euler-Lagrange equation of the functional $\Phi_0 : V \to \mathbb{R}$

$$\Phi_0(v) = \|v\|^2_{H^1} - s^* \int_\Omega a_p(x) \frac{v^{p+1}}{p+1}\, dx dt$$

where $\|v\|_{H^1}^2 := \int_\Omega v_t^2 + v_x^2 \, dxdt$. By the Mountain-Pass theorem, there exists at least one trivial critical point of Φ_0, namely a solution of (2.65), choosing $s^* = 1$ ($s^* = -1$), so that

$$\exists v \in V \quad \text{s.t.} \quad \int_\Omega a_p(x) v^{p+1}(t,x) \, dtdx > 0 \ \ (< 0) \, .$$

For this reason, the authors need a further discussion about the relation between the frequency and the amplitude (see [BeBo05], section 1 for details), choosing, at the end $\omega = \omega(\delta) = \sqrt{1 + 2s^* \delta^{p-1}}$. To fix idea we will search periodic solutions with frequency $\omega > 1$, namely we fix, once for all, $\omega = \sqrt{1 + 2\delta^{p-1}}$ and prove that the equation

$$-\Delta v = \Pi_v (a_p(x) v^p)$$

has a nontrivial solution if the nonlinearity f satisfies the assumption (H) and (2.62).

Solution of the (Q)–equation

The natural approach to solve (2.64), is to find a solution first of the (P)–equation, for example with a KAM procedure, and then, inserting $w(\delta, v)$, to solve the (Q)–equation. Since the bifurcation equation (Q) is infinite-dimensional, one meets a difficulty because of the lack of regularity in solving the (P)–equation. However, in this work, the authors bypass the problem, reducing the bifurcation equation to a finite-dimensional one, thanks to the compactness of the operator $(-\Delta)^{-1}$. Indeed, one performs a reduction to a finite dimensional equation on a subspace of V of dimension N independent of ω. Introduce another decomposition $V = V_1 \oplus V_2$ where

$$\begin{aligned} V_1 &:= \left\{ v \in V \,\middle|\, v(t,x) = \sum_{\ell=1}^{N} 2\cos(\ell t) u_\ell \, \sin(\ell x), \ u_\ell \in \mathbb{R} \right\} \\ V_2 &:= \left\{ v \in V \,\middle|\, v(t,x) = \sum_{\ell \geq N+1} 2\cos(\ell t) u_\ell \, \sin(\ell x), \ u_\ell \in \mathbb{R} \right\} . \end{aligned}$$

Setting $v := v_1 + v_2$, with $v_1 \in V_1$, $v_2 \in V_2$, (2.64) becomes

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q1) \\ -\Delta v_2 = \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) & (P) \end{cases}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$, $(i = 1, 2)$ denote the projectors on V_i $(i = 1, 2)$.
One first solve the $(Q2)$–equation by the Contraction Mapping Theorem provided to have chosen N large enough and $0 < \sigma < \log 2/N$ small enough, depending on the nonlinearity f but independent of δ. The crucial tool to solve the $(Q2)$–equation is the regularizing property of the operator Δ^{-1}. In this way, one obtains a solution $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s+2}$, for $w \in W \cap X_{\sigma,s}$.

Solution of the (P)–equation

One reduced to solve the (P)–equation, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \tag{2.66}$$

where

$$\Gamma(\delta, v_1, w)(t,x) = g\big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\big)\,. \tag{2.67}$$

The solution of (2.66) is obtained by a KAM procedure for (δ, v_1) belonging to a Cantor set of parameters.
Consider the orthogonal splitting $W = W^{(n)} \oplus W^{(n)\perp}$ where

$$W^{(n)} := \Big\{ w \in W \ \text{ s.t. } \ w = \sum_{|\ell| \le L_n} e^{\mathrm{i}\ell t} w_\ell(x) \Big\}$$

and

$$W^{(n)^\perp} := \Big\{ w \in W \ \text{ s.t. } \ w = \sum_{|\ell| > L_n} e^{\mathrm{i}\ell t} w_\ell(x) \Big\}$$

and L_n are integers numbers such that $L_n = L_0 2^n$ for $L_0 \in \mathbb{N}$ large enough. Denoting by

$$P_n \,:\, W \longrightarrow W^{(n)}\,, \qquad P_n^\perp \,:\, W \longrightarrow W^{(n)\perp}$$

the projection on $W^{(n)}$ and on $W^{(n)^\perp}$ respectively, one can prove the following estimate

Lemma 2.4.12. *Let be $0 \le \sigma' \le \sigma$, then*

$$\|w\|_{\sigma',s} \le e^{-L_n(\sigma - \sigma')} \|w\|_{\sigma,s}\,, \qquad \forall w \in W^{(n)\perp} \cap X_{\sigma,s}\,.$$

PROOF. From the definition of the norm

$$\begin{aligned} \|w\|^2_{\sigma',s} &= \sum_{|\ell|>L_n} e^{2\sigma'|\ell|}(\ell^{2s}+1)\|w_\ell\|^2_{H^1} = \sum_{|\ell|>L_n} e^{-2(\sigma-\sigma')|\ell|} e^{2\sigma|\ell|}(\ell^{2s}+1)\|w_\ell\|^2_{H^1} \\ &\le e^{-2L_n(\sigma-\sigma')}\|w\|^2_{\sigma,s}\,. \end{aligned}$$

□

Let us consider the linearized operator $\mathcal{L}_n(\delta, v_1, w) \,:\, W^{(n)} \longrightarrow W^{(n)}$ defined by

$$\mathcal{L}_n\,(\delta, v_1, w)[h] := L_\omega(h) - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \tag{2.68}$$

where w denote the approximate solution at a given step of the KAM scheme. The next subject is crucial for the proof of the convergence of the iterative scheme. Here the small divisors problem arises.

Decomposition of $\mathcal{L}_n$

Using (2.67), we can write the linearized operator as

$$\begin{aligned}
&\mathcal{L}_n(\delta, v_1, w)[h] := L_\omega(h) - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\
&\quad = \quad L_\omega(h) - \varepsilon P_n \Pi_W \Big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \big(h + \partial_w v_2(\delta, v_1, w)[h] \big) \Big) \\
&\quad = \quad L_\omega(h) - \varepsilon P_n \Pi_W \big(a(t,x) h \big) - \varepsilon P_n \Pi_W \big(a(t,x) \partial_w v_2(\delta, v_1, w)[h] \big) \qquad (2.69)
\end{aligned}$$

where, for brevity, we denote

$$a(t,x) := \partial_u g\Big(\delta, x, v_1 + w + v_2(\delta, v_1, w)\big)\big(h + \partial_w v_2(\delta, v_1, w)[h]\Big) .$$

In order to invert the operator $\mathcal{L}_n$, the authors choose to perform a Fourier expansion in time, so that the operator L_ω is diagonal

$$L_\omega\bigg(\sum_{|k| \le L_n} e^{\mathrm{i}kt} h_k \bigg) = \sum_{|k| \le L_n} e^{\mathrm{i}kt} (\omega^2 k^2 + \partial_{xx}) h_k \,. \qquad (2.70)$$

Moreover, the operator $h \to P_n \Pi_W \big(a(t,x) h \big)$ is the composition of the multiplication for the function $a(t,x) = \sum_{\ell \in \mathbb{Z}} e^{\mathrm{i}\ell t} a_\ell(x)$ with the projections Π_W and P_n. The multiplication operator in Fourier expansion is described by a Toepliz matrix (see [PT]),

$$a(t,x) h(t,x) = \sum_{|k| \le L_n, \ell \in \mathbb{Z}} e^{\mathrm{i}\ell t} a_{\ell - k}(x) h_k(x) \,,$$

and, from the definition of the projections Π_W and P_n, one gets

$$\begin{aligned}
P_n \Pi_W \big(a(t,x) h \big) &= \sum_{|k|, |\ell| \le L_n} e^{\mathrm{i}\ell t} \pi_\ell (a_{\ell - k}(x) h_k) \\
&= \sum_{|k| \le L_n} e^{\mathrm{i}kt} \pi_k (a_0(x) h_k) + \sum_{|k|, |\ell| \le L_n, k \ne \ell} e^{\mathrm{i}\ell t} \pi_\ell (a_{\ell - k}(x) h_k) \,,
\end{aligned}$$

where

$$\sum_{|k| \le L_n} e^{\mathrm{i}kt} \pi_k (a_0(x) h_k) = P_n \Pi_W (a_0(x) h) \qquad (2.71)$$

is the diagonal term, with $a_0(x) := \frac{1}{2\pi} \int_0^{2\pi} a(t,x) dt$, and

$$\sum_{|k|, |\ell| \le L_n, k \ne \ell} e^{\mathrm{i}\ell t} \pi_\ell (a_{\ell - k}(x) h_k) = P_n \Pi_W (\bar{a}(t,x) h) \qquad (2.72)$$

is the *off–diagonal Toepliz* term, with

$$\bar{a}(t,x) := a(t,x) - a_0(x)$$

is a function with zero average in time. Above, we have denoted by π_k the L^2–orthogonal projection

$$\pi_k : H_0^1\big((0,\pi);\mathbb{R}\big) \longrightarrow \langle \sin(kx)\rangle^{\perp},$$

defined by

$$(\pi_k f)(x) := f(x) - \left(\frac{2}{\pi}\int_0^\pi f(x)\sin kx\, dx\right)\sin(kx),$$

where

$$\langle \sin(kx)\rangle^{\perp} := \left\{ f \in H_0^1\big((0,\pi);\mathbb{R}\big) \quad \text{s.t.} \quad \int_0^\pi f(x)\sin kx\, dx = 0 \right\}.$$

By (2.69), (2.70), (2.71), (2.72), one can decompose

$$\mathcal{L}_n(\delta, v_1, w) = D - \mathcal{M}_1 - \mathcal{M}_2$$

where D, $\mathcal{M}_1$, $\mathcal{M}_2$ are the linear operators defined by

$$\begin{cases} Dh & := & L_\omega(h) - \varepsilon P_n \Pi_W(a_0(x)h) \\ \mathcal{M}_1 h & := & \varepsilon P_n \Pi_W(\bar{a}(t,x)h) \\ \mathcal{M}_2 h & := & \varepsilon P_n \Pi_W(a(t,x)\partial_w v_2[h]). \end{cases} \tag{2.73}$$

Diagonalization of $D : W^{(n)} \longrightarrow W^{(n)}$

By (2.70) and (2.71), we have that the operator D is diagonal in time-Fourier series. We can write, for all $h \in W^{(n)}$, the k^{th} time Fourier coefficient of Dh as

$$(Dh)_k = (\omega^2 k^2 + \partial_{xx})h_k - \varepsilon \pi_k(a_0(x)h_k).$$

Denoting S_k the operator

$$S_k u := -\partial_{xx} + \varepsilon \pi_k(a_0(x)u),$$

one can show that it possesses an orthonormal bases of eigenvectors, in a suitable norm, with simple and positive eigenvalues $\lambda_{k,j}$ for $j \geq 1$, $j \neq k$. As it is shown in [PT], the asymptotic estimate on the eigenvalues holds

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\left(\frac{\varepsilon \|a_0\|_{H^1}}{j}\right)$$

where $M := M(\delta, v_1, w)$ is the mean value of $a_0(x)$ on $(0,\pi)$. From the diagonalization of S_k, the diagonalization of $D : W^{(n)} \longrightarrow W^{(n)}$ follows, in a suitable base of eigenvectors of $W^{(n)}$, with associated eigenvalues $\omega^2 k^2 - \lambda_{k,j}$, $j \geq 1$, $j \neq k$.

Inversion of $\mathcal{L}_n : W^{(n)} \longrightarrow W^{(n)}$

We need some definitions.

Definition 2.4.13 (First order Melnikov conditions)**.** *Let be* $0 < \gamma < 1$ *and* $1 < \tau < 2$. *We define*

$$\Delta_n^{\gamma,\tau}(v_1, w) := \left\{ \delta \in [0, \delta_0) \;\middle|\; |\omega\ell - j| \geq \frac{\gamma}{(\ell + j)^\tau}\,,\; \left|\omega\ell - j - \varepsilon\frac{M}{2j}\right| \geq \frac{\gamma}{(\ell + j)^\tau}\,, \right.$$
$$\left. \forall\, \ell \in \mathbb{N},\, j \geq 1,\, \ell \neq j,\, \frac{1}{3\varepsilon} < \ell,\, \ell \leq L_n,\, j \leq 2L_n \right\}, \tag{2.74}$$

where $M := M(\delta, v_1, w)$.

Resuming, in order to invert $\mathcal{L}_n$, one first proves that, if $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$, the diagonal linear operator D is invertible. Thereafter, one shows that the *off–diagonal Toepliz* operator $\mathcal{M}_1$ and $\mathcal{M}_2$ are small enough with respect to D, so they can be considered as a perturbation of the operator D. In particular, in this estimates a small divisors problem arises. Defining

$$\alpha_k := \min_{j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|\,,$$

the authors prove the following

Lemma 2.4.14. *Let be* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$*, with* δ_0 *small. There exists* $c > 0$ *such that,* $\forall\, \ell \neq k$,

$$\frac{1}{\alpha_k \alpha_\ell} \leq C \frac{|k - \ell|^{2\frac{\tau-1}{\beta}}}{\gamma^2 |\varepsilon|^{\tau-1}}$$

where $\beta := \frac{2-\tau}{\tau}$.

In other words, choosing $\delta \in \Delta_n^{\gamma,\tau}$, one excises the parameters (δ, v_1) for which the eigenvalues of $\mathcal{L}_n\,(\delta, v_1, w)$ are zero or too small. The invertibility of $\mathcal{L}_n$ follows from (2.73), Lemma 2.4.14 and it is stated in the following

Lemma 2.4.15 (Invertibility)**.** *If* w *is sufficiently small and* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ *for some* $0 < \gamma < 1$, $1 < \tau < 2$, *then the operator* $\mathcal{L}_n\,(\delta, v_1, w)$ *is invertible and, for any* $h \in W^{(n)}$*, the inverse operator satisfies*

$$\|\mathcal{L}_n^{-1}(\delta, v_1, w)[h]\|_{\sigma,s} \leq \frac{C}{\gamma}\,(L_n)^{\tau-1} \|h\|_{\sigma,s}\,,$$

for some positive constant $C > 0$.

Solution of the $Q1$–equation

Finally, one has to solve the finite dimensional $Q1$–equation. Therefore, showing a "Whitney–differentiability" property, one defines a smooth functional $\Psi : [0, \delta_0) \times V_1 \to \mathbb{R}$ for which one proves the existence of critical points $v_1(\delta)$ of $\Psi(\delta, \cdot)$, for every $\delta > 0$ by the Mountain Pass Theorem. The functional $\Psi(\delta, \cdot)$ is such that, any critical point $v_1 \in V_1$ of $\Psi(\delta, \cdot)$, with (δ, v_1) that belong to the Cantor set of parameters for which the (P)–equation is exactly solved, gives rise to a solution of (2.60). The critical points v_1's such that (δ, v_1) belong to the above Cantor set has *gaps*, except for a zero measure set of δ's. For this reason, the authors require a non-degeneracy condition, namely that the path $\delta \mapsto v_1(\delta) \in C^1$.

2.5 Forced vibrations with irrational frequency

♡ **[PlYu89]** Let us consider the problem of finding 2π–periodic solution of the rescaled equation

$$\begin{cases} \omega^2 u_{tt} - u_{xx} = \varepsilon f(t, x, u) \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \tag{2.75}$$

where ε is a small parameter and f is 2π–periodic in time. Assume that the following conditions are satisfied:
(i) There is a positive constant $c_0 < 1$, such that, for each integer pair $n = (n_1, n_2) \neq 0$, the inequality

$$\big|\omega|n_1| - |n_2|\big| \geq \frac{c_0}{(1 + |n_1|)^{1+\alpha}}$$

holds, with $0 < \alpha < (\sqrt{85} - 9)4^{-1}$.
(ii) There are positive constants k and c_1 and an integer $r \geq 7$ such that f is defined in $G = \mathbb{R} \times (0, \pi) \times [-k, k]$, $D_u^i D_t^j \in C(G)$, $i + j \leq r$, and $D_u f \in C^1(G)$. Moreover, if $\|u\|_{C(G)} \leq k$, then

$$\left| \int_0^{2\pi} \int_0^{\pi} f_u(u, t, x)\, dt dx \right| \geq c_1\,.$$

Let us introduce the parameter λ:

$$\varepsilon =: \lambda h(u), \qquad h(u) := 2\pi^2 \left\{ \int_0^{2\pi} \int_0^{\pi} f_u(u, t, x)\, dt dx \right\}^{-1}. \tag{2.76}$$

The set Λ of admissible values of λ is chosen so that

$$|n_2^2 - \omega^2 n_1^2 - \lambda c_2| \geq c_2 (1 + |n_1|)^{-\alpha}\,, \qquad \forall\, n \neq 0\,, \qquad |\lambda| \leq c_2\,.$$

Λ is compact and has zero measure.

Theorem 2.5.1. *If conditions (i) and (ii) are satisfied, there are positive constant δ and c, such that, if $\lambda \in \Lambda \cap [-\delta, \delta]$, then* (2.75) *has a solution $u(\lambda) \in C^2$ corresponding to the values of the parameter $\varepsilon(\lambda)$ given by* (2.76)*. If $\lambda_i \in \Lambda \cap [-\delta, \delta]$, then*

$$\|u(\lambda_1) - u(\lambda_2)\|_{H^2} \le c\|\lambda_1 - \lambda_2\|$$

and

$$c^{-1}|\lambda_1 - \lambda_2| \le |\varepsilon(\lambda_1) - \varepsilon(\lambda_2)| \le c|\lambda_1 - \lambda_2|\,, \qquad \varepsilon(0) = 0\,.$$

2.6 Nonlinear Schrödinger equation

♡ [**KuP96**] Consider the Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - \mu u - f(|u|^2)u \tag{2.77}$$

with Dirichlet boundary conditions and a real analytic nonlinearity with $f(0) = 0$, but not necessarily $f'(0) \neq 0$. Denote by ϕ_j and λ_j, respectively, the basic modes and their frequencies for the linear equation $\mathrm{i}u_t = u_{xx} - \mu u$ with Dirichlet boundary conditions. By the standard Implicit Function Theorem the authors prove the existence of solutions of the form $u(t,x) = u_1(t)u_2(x)$.

Theorem 2.6.1. *Suppose that f is real analytic near 0 with $f(r) = O(r^s)$ for some $s \ge 1$. Then for every $j \ge 1$ there exist $r_j > 0$ and an embedded disc*

$$\mathcal{E}_j = \left\{u(t,x) = rv_j(x,r)e^{\mathrm{i}\mu_j(r)t}\,, 0 \le r < r_j\right\}$$

of real analytic, time periodic solutions of (2.77)*, where*

$$v_j = \phi_j + O(r^{2s})\,, \qquad \mu_j = \lambda_j + O(r^{2s})$$

are real analytic in r and x, and in r respectively.

♡ [**GPr05**] Consider the nonlinear Schrödinger equation on the interval $[0,\pi]$, given by

$$\begin{cases} -\mathrm{i}u_t + u_{xx} = \varphi(|u|^2)u \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.78}$$

where $\varphi(y)$ is any analytic function $\varphi(y) = \Phi y + O(y^2)$ with $\Phi \neq 0$. The authors consider the problem of existence of resonant periodic solutions for (2.78), i.e. solutions arising from superpositions of several unperturbed harmonics. For $\varepsilon > 0$, rescale $u \to \sqrt{\varepsilon/\Phi}u$ in (2.78),obtaining

$$\begin{cases} -\mathrm{i}u_t + u_{xx} = \varepsilon|u|^2u + O(\varepsilon^2) \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{2.79}$$

where $O(\varepsilon^2)$ denotes an analytic function of $u, \bar{u}$ and ε of order at least 2 in ε, and we define $\omega_\varepsilon = 1 + \varepsilon$. If $\varphi = 0$, every solution of (2.78) can be written as

$$u(t,x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n e^{\mathrm{i} n^2 t} e^{\mathrm{i} n x} , \qquad a_{-n} = -a_n .$$

Since one is looking for solutions of the form

$$u_\varepsilon(t,x) = \sum_{(n,m) \in \mathbb{Z}^2} e^{\mathrm{i} n \omega t + \mathrm{i} m x} u_{\varepsilon,n,m} , \tag{2.80}$$

$u_{\varepsilon,n,m} \in \mathbb{R}$, which is analytic both in x and t, and periodic in t, one uses the norm

$$\|F(t,x)\|_k = \sum_{(n,m) \in \mathbb{Z}^2} F_{n,m} e^{k(|n|+|m|)}$$

for analytic functions.

Theorem 2.6.2. *Consider the equation* (2.78) *where* $\varphi(y) = \Phi y + O(y^2)$ *is an analytic function, with* $\Phi \neq 0$*, or more generally, the equation*

$$\begin{cases} -\mathrm{i} u_t + u_{xx} = f(u, \bar{u}) \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.81}$$

where $f(u,\bar{u})$ *is any real analytic function, odd under the transformation* $(u,\bar{u}) \to (-u,-\bar{u})$*, such that* $f(u,\bar{u}) = \Phi |u|^2 u + O(|u|^5)$ *with* $\Phi \neq 0$*. For all* $N \geq 2$*, there are sets of* N *positive integers* $\mathcal{M}_+$ *and set of real amplitudes* $\{a_m\}_{m \in \mathcal{M}_+}$*, such that the following holds. Define*

$$a(t,x) = \sum_{m \in \mathcal{M}_+} e^{\mathrm{i} m^2 t + \mathrm{i} m x} a_m ,$$

and set $u_0(t,x) = a(t,x) - a(t,-x)$*. There are a positive constant* ε_0 *and a set* $\mathcal{E} \in [0,\varepsilon_0]$*, both depending on the set* $\mathcal{M}_+$*, satisfying*

$$\lim_{\varepsilon \to 0} \frac{\mathrm{meas}\left(\mathcal{E} \cap [0,\varepsilon]\right)}{\varepsilon} = 1 ,$$

such that for all $\varepsilon \in \mathcal{E}$*, there exists a* $2\pi/\omega_\varepsilon$*–periodic solution of* (2.78)*, analytic in* (t,x) *and of the form* (2.80) *with*

$$\left\| u_\varepsilon(t,x) - \sqrt{\varepsilon/\Phi}\, u_0(\omega_\varepsilon t, x) \right\|_k \leq \mathrm{const}\, \varepsilon^{3/2} ,$$

with $k = k(\varepsilon_0) > 0$*.*

The proof is achieved by the Lindstedt series method.

2.7 Nonlinear beam equation

♡ **[B00]** The same result, stated in Theorem 2.4.6 for the nonlinear wave equation, holds for the nonlinear beam equation

$$u_{tt} + u_{xxxx} - \alpha u_{xx} + \beta u = \psi(u)$$

with hinged boundary condition (1.26) on $[0, \pi]$ and α, β positive parameters. The author considers a nonlinearity that satisfies a suitable non degeneracy condition, for example ψ an odd polynomial.
Fix one of the parameter, for example $\alpha \geq 0$ and a linear mode. If β belongs to a full measure subset of $[0, \infty]$, the linear frequencies have the (γ-NR)–property stated on pg. 54. Then there exists a sequence of periodic orbits close to the linear mode and accumulating to zero.

♡ **[BBe05]** Consider the beam equation (1.25), with hinged boundary conditions

$$u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0\,, \tag{2.82}$$

where the nonlinearity $f(u)$ is a real analytic, odd function of the form

$$f(u) = au^3 + \sum_{k \geq 5} f_k u^k\,, \qquad a \neq 0\,. \tag{2.83}$$

Theorem 2.7.1. *Suppose that μ does not belong to a suitable finite subset of $[0, L]$ with an arbitrary L. Then equation (1.25)-(1.26)-(2.83) possesses infinitely many small amplitude periodic solutions u_j accumulating to $u = 0$. The period T_j and the amplitude A_j of any u_j go, respectively, to infinity and to zero as $j \to \infty$; moreover $T_j \approx A_j^{-2}$.*

In the following we discuss such result in details since some ideas of the proof will be used in the next part of this thesis.

2.7.1 Birkhoff-Lewis type solutions

This paper is inspired to the Birkhoff's one in which he shows the existence of periodic orbits that accumulate to the origin for a hamiltonian finite-dimensional system in a neighborhood of an elliptic equilibrium point. It used the so called Birkhoff Normal Form, which allows to view hamiltonian systems near an equilibrium as small perturbations of integrable systems, see Theorem 1.1.4.
In [BeBiV04], it is proved under suitable non–resonance and non–degeneracy "twist" conditions, a general Birkhoff–Lewis type result showing the existence of infinitely many periodic solutions, with larger and larger minimal period, accumulating onto elliptic lower dimensional invariant tori.

The result of Bambusi and Berti is an extension to the infinite-dimensional case of some nonlinear PDEs with smoothing nonlinearity, for example the nonlinear beam equation and the nonlinear Schrödinger equation.
Let us consider a real analytic Hamiltonian in a neighborhood of an elliptic equilibrium point

$$H(p,q) = H_2 + N = \sum_{i\geq 1} \omega_i \frac{p_i^2 + q_i^2}{2} + N(p,q)$$

where $N(p,q) = o(|p|^2 + |q|^2)$. Introduce the complex coordinates

$$z_i = \frac{1}{\sqrt{2}}(p_i + \mathrm{i}q_i)\,, \quad \bar{z}_i = \frac{1}{\sqrt{2}}(p_i - \mathrm{i}q_i)\,,$$

that live in the complex Hilbert space

$$\begin{aligned} \mathcal{H}^{a,s} &= \mathcal{H}^{a,s}(\mathbb{C}) \\ &:= \left\{ z = (z_1, \ldots),\ z_i \in \mathbb{C}\,,\ i \geq 1 \ \text{t.c.}\ \|z\|^2_{a,s} = \sum_{i\geq 1} |z_i|^2 i^{2s} e^{2ai} < \infty \right\}, \end{aligned}$$

with symplectic structure $\mathrm{i}\sum_{i\geq 1} dz_i \wedge d\bar{z}_i = \sum_{i\geq 1} dp_i \wedge dq_i$. Fixing $s \geq 0$ and $a \geq 0$, one can study the system in the phase space

$$\mathcal{P}^{a,s} := \mathcal{H}^{a,s} \times \mathcal{H}^{a,s} \ni (z, \bar{z})\,.$$

The Hamiltonian becomes

$$H = \sum_{i\geq 1} \omega_i |z_i|^2 + P(z,\bar{z})\,,$$

where P has a zero of third order at the origin. The Hamilton's equations write

$$\dot{z}_j = \mathrm{i}\omega_j z_j + \mathrm{i}\partial_{\bar{z}} P\,, \qquad \dot{\bar{z}}_j = -\mathrm{i}\omega_j \bar{z}_j - \mathrm{i}\partial_{z_j} P\,. \tag{2.84}$$

The Hamiltonian H is real analytic. It means that H is a function of z and $\bar{z}$, real analytic in the real and imaginary part of z; we denote by $\mathcal{A}(\ell^{a,s}, \ell^{a,s+1})$ the class of all real analytic maps from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$.
In order to extend the result of [BeBiV04] to the infinite-dimensional case one meets two difficulties: the first one is the generalization of the Birkhoff normal form and the second one is a small divisors problem. To bypass the first problem one considers a *seminormal* form. Fix $n \geq 2$ and separate the first n variables $z_1, \ldots, z_n$ from the infinite second ones. From now on, we will denote by $\hat{z}$ the infinite vector obtained bereaving $z = (z_1, z_2, \ldots)$ of its first n–components, namely $\hat{z} := (z_{n+1}, \ldots)$. Moreover, we denote by $\omega_1, \ldots, \omega_n$ the first n frequencies and by $\Omega_{n+1}, \ldots$ the second infinite ones. Let us impose a *second order Melnikov* condition on the frequencies

$$|\omega \cdot k + \Omega \cdot \ell| \neq 0, \quad |\ell| \leq 2,\ 0 < |k| + |\ell| \leq 5 \tag{2.85}$$

and a linear asymptotic behavior on the frequencies

$$\exists\, a > 0,\ d \geq 1 \ \text{s.t.}\ \omega_j \sim a j^d. \tag{2.86}$$

Moreover, assume the following smoothing property on the nonlinearity:

(S) there exists a neighborhood of the origin $\mathcal{U} \subset \mathcal{P}_{a,s}$ and $d \geq 0$ such that the vectorfield X_P associated to the nonlinearity is analytic.

Next, we transform the Hamiltonian H in (4.15) into some partial Birkhoff normal form of order four so that, in a small neighborhood of the origin, it appears as a perturbation of a nonlinear integrable system. One obtains

$$H = H_0 + \bar{G} + \hat{G} + K$$

with

$$\bar{G} = \frac{1}{2} \sum_{\min(i,j) \leq n} \bar{G}_{i,j} |z_i|^2 |z_j|^2 ,$$

with $\bar{G}_{i,j} = \bar{G}_{j,i}$, $\hat{G} = O(\|\hat{z}\|^3_{a,s})$, $K = O(\|z\|^6_{a,s})$. Moreover, it results that $X_{\bar{G}}, X_{\hat{G}}, X_K$ satisfy the smoothing property (S). Let rewrite the Hamiltonian in the form

$$H = \omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \frac{1}{2} AI \cdot I + (BI, \hat{z}\bar{\hat{z}}) + \hat{G} + K \tag{2.87}$$

where $I := (|z_1|^2, \dots, |z_n|^2)$ are the actions, A is the $n \times n$ matrix

$$A := (\bar{G}_{ij})_{1 \leq i,j \leq n} \tag{2.88}$$

and B is the $\infty \times n$ matrix

$$B := (\bar{G}_{ij})_{1 \leq j \leq n < i} .$$

One can introduce, now, action-angle variables for the first n modes $z_1, \dots, z_n$ by $z_j = |z_j| e^{\mathrm{i}\phi_j} = \sqrt{I_j} e^{\mathrm{i}\phi_j}$, for $j = 1, \dots, n$. Performing the scaling for $\eta \approx 0$, $I \to \eta^2 I$, $\hat{z} \to \eta \hat{z}$, $\phi \to \phi$, one gets[5]

$$H(I, \phi, \hat{z}, \bar{\hat{z}}) = \omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \eta^2 \left(\frac{1}{2}(AI, I) + (BI, \hat{z}\bar{\hat{z}}) \right) + \eta \hat{G} + \eta^4 K. \tag{2.89}$$

We will still denote by $\mathcal{P}^{a,s} := \mathbb{R}^n \times \mathbb{T}^n \times \mathcal{H}^{a,s} \times \mathcal{H}^{a,s}$ the phase space. For the integrable Hamiltonian

$$\omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \eta^2 \left(\frac{1}{2}(AI, I) + (BI, \hat{z}\bar{\hat{z}}) \right) \tag{2.90}$$

[5] With abuse of notations, we denote $\hat{G}$ and K the scaled terms.

one can write the equation of motion

$$\begin{cases} \dot{I} = 0 \\ \dot{\phi} = \omega + \eta^2 AI + \eta^2 B^t z\bar{z} \\ \dot{z}_j = \mathrm{i}(\Omega + \eta^2 BI)_j z_j\,, \qquad\qquad j \geq n+1\,. \end{cases}$$

Let us note that $\{\hat{z} = 0\}$ is an invariant manifold for the Hamiltonian (2.90) and it is filled up by n-dimensional invariant tori

$$\mathcal{T}(I_0) := \left\{ I = I_0,\ \phi \in \mathbb{T}^n,\ \hat{z} = 0 \right\} \tag{2.91}$$

on which the motion is linear with frequency

$$\tilde{\omega} = \omega + \eta^2 AI_0\,.$$

These tori are completely resonant and foliated by infinitely many T-periodic orbits if

$$\tilde{\omega} = \frac{2\pi k}{T}\,, \qquad\qquad k \in \mathbb{Z}^n. \tag{2.92}$$

If the action to frequency map is invertible, that is, if the non-degeneracy condition

$$\det A \neq 0$$

holds, one can choose

$$\begin{aligned} k \quad &:= \quad k(T) = \left[\frac{\omega T}{2\pi}\right], \\ I_0 \quad &:= \quad I_0(T) = \frac{2\pi}{\eta^2 T} A^{-1}\left(\left[\frac{\omega T}{2\pi}\right] - \frac{\omega T}{2\pi}\right), \end{aligned} \tag{2.93}$$

such that (2.92) is verified. Here $[\,\cdot\,]$ denotes the integer part. Taking $\eta^{-2} \leq T \leq 2\eta^{-2}$ we have that $|I_0| \leq \mathrm{const}$. Then the torus $\mathcal{T}(I_0)$ is foliated by the T–periodic motions

$$\left\{ I(t) = I_0,\ \phi(t) = \phi_0 + \tilde{\omega} t,\ \hat{z}(t) = 0 \right\}.$$

Not all these orbits will persist in the dynamic of the complete hamiltonian system (2.89). In this paper, it is shown that, under suitable assumptions, at least n geometrically distinct T–periodic orbits persist.

Inversion of the linear operator

We are looking for periodic orbits of the complete system near the periodic orbits of (2.91), that is for a solution $\zeta_0 + \zeta$ where

$$\zeta_0 = (\tilde{\omega} t, I_0, 0, 0) \qquad \text{and} \qquad \zeta = (\psi, J, \hat{z}, \bar{\hat{z}})$$

where $(\psi, J, \hat{z}, \bar{\hat{z}})$ are T-periodic small functions valued in the phase space $\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{H}^{a,s} \times \mathcal{H}^{a,s}$. One can write the equation that $\zeta = (\psi, J, \hat{z}, \bar{\hat{z}})$ satisfies

$$L\zeta = N(\zeta)\,,$$

calling N the nonlinearity and

$$L\zeta = L(\psi, J, \hat{z}, \bar{\hat{z}}) := (\dot{J}, \dot{\psi} - \eta^2 AJ, \dot{z}_j - \mathrm{i}\tilde{\Omega}_j z_j)\,, \quad j \geq n+1\,,$$

the linear operator, where

$$\tilde{\Omega}_j := \tilde{\Omega}_j(I_0) := \big(\Omega + \eta^2 BI_0\big)_j \tag{2.94}$$

are the *shifted elliptic frequencies*. At this point, one performs a finite-dimensional reduction, splitting the space in *range* + *kernel*, showing, in particular, that the kernel is exactly the resonant torus and the range is composed by the T-periodic functions $(\tilde{\psi}, \tilde{J}, \tilde{z})$ with $\int_0^T \tilde{\psi} = 0$. Let us consider the kernel of the operator L

$$\begin{cases} \dot{J} = 0 \\ \dot{\psi} - \eta^2 AJ = 0 \\ \dot{z}_j - \mathrm{i}\tilde{\Omega}_j z_j = 0\,, \qquad j \geq n+1\,. \end{cases}$$

From the first equation, one has $J(t) = J(0) =: J_0$, for all t and from the second one $\psi(t) = \psi(0) + \eta^2 AJ_0 t =: \psi_0 + \eta^2 AJ_0 t$. Solving the third equation and imposing the periodicity condition, one finds that

$$z_j(T) = e^{\mathrm{i}\tilde{\Omega}_j T} z_j(0) = z_j(0)\,.$$

There are two possibilities: $z_j(0) = 0$ or $z_j(0) \neq 0$. In the first case $z_j(t) = 0$, for all t, in the second one $\tilde{\Omega}_j T = 2\pi\ell$ with $\ell \in \mathbb{Z}$ and $j \geq n+1$. In order to avoid to have other periodic solutions near the invariant torus, one has to impose the condition on T

$$\tilde{\Omega}_j T - 2\pi\ell \neq 0\,, \qquad \ell \in \mathbb{Z},\ j \geq n+1\,. \tag{2.95}$$

If (2.95) holds, we obtain that every T-periodic solution $(\psi, J, \hat{z}, \bar{\hat{z}})$ near the torus $\mathcal{T}(I_0)$ has $J_0 \approx 0$, $\psi(t) = \psi_0$, namely belongs to $\ker L = \big\{\psi = \psi_0,\ J = 0,\ \hat{z} = 0\big\}$.
For what concerns the range of the operator L it is composed by the functions verifying $\tilde{\zeta} = (\tilde{\psi}, \tilde{J}, \tilde{z})$

$$\begin{cases} \dot{J} = \tilde{\psi} \\ \dot{\psi} = \eta^2 AJ + \tilde{J} \\ \dot{z}_j = \mathrm{i}\tilde{\Omega}_j z_j + \tilde{z}_j\,, \qquad j \geq n+1\,. \end{cases}$$

Since J has to be T-periodic, one has from the first equation that

$$\int_0^T \tilde{\psi}(t)dt = 0\,.$$

Define the constants

$$\begin{aligned} \alpha(\tilde{J},\tilde{\psi}) &:= -\frac{1}{T}\left(\int_0^T\int_0^s \tilde{J}(\theta)\,d\theta\,ds + \frac{1}{\eta^2}A^{-1}\int_0^T \tilde{\psi}(\theta)\,d\theta\right), \\ \beta(\tilde{z}) &:= \mathcal{M}^{-1}e^{\mathrm{i}\Omega T}\int_0^T e^{-\mathrm{i}\Omega\theta}\tilde{z}(\theta)\,d\theta \end{aligned}$$

where $\mathcal{M} := \left(\mathbb{I} - e^{\mathrm{i}\tilde{\Omega}T}\right)$ (here $\mathbb{I}$ is the identity), then

$$L^{-1}\begin{pmatrix}\tilde{J}\\ \tilde{\psi}\\ \tilde{z}\end{pmatrix} := \begin{pmatrix} \alpha(\tilde{J},\tilde{\psi}) + \int_0^t \tilde{J}(s)\,ds \\ \\ \eta^2 A\alpha t + \eta^2 A\int_0^t\int_0^s \tilde{J}(\theta)\,d\theta\,ds + \int_0^t \tilde{\psi}(\theta)\,d\theta \\ \\ e^{\mathrm{i}\Omega t}\left(\beta(\tilde{z}) + \int_0^t e^{-\mathrm{i}\Omega\theta}\tilde{z}(\theta)\,d\theta\right) \end{pmatrix}.$$

However, in the definition of β above, the small divisors $1 - e^{\mathrm{i}\tilde{\Omega}_j T}$ appear. We have

$$\begin{aligned} |1 - e^{\mathrm{i}\tilde{\Omega}_j T}| &= |1 - \cos\tilde{\Omega}_j T - \mathrm{i}\sin\tilde{\Omega}_j T| \\ &\geq |\sin\tilde{\Omega}_j T| \geq \frac{1}{2}\min_{\ell}|\tilde{\Omega}_j T - 2\pi\ell| \end{aligned}$$

and, by (2.93),

$$\begin{aligned} \tilde{\Omega}_j(I_0)T - 2\pi\ell &= \left(\Omega_j + \eta^2 B I_0\right)_j T - 2\pi\ell \\ &= \Omega_j T - \left(BA^{-1}\,2\pi\,\langle\frac{\omega T}{2\pi}\rangle\right)_j, \qquad j \geq n+1 \end{aligned}$$

where $\langle a\rangle := a - [a]$ is the fractionary part of $a \in \mathbb{R}$. Assuming that

$$\Omega_j - \left(BA^{-1}\omega\right)_j \neq 0, \qquad j \geq n+1, \tag{2.96}$$

and $\omega_j \sim j^d$, $d > 1$ one can prove that there exists $\delta > 0$ and $1 < \tau \leq d$ such that the following non-resonance condition between the frequencies

$$|\tilde{\Omega}_j T - 2\pi\ell| \geq \frac{\delta}{j^\tau}, \quad \forall\ell\in\mathbb{Z}, \forall j \geq n+1, \tau > 1, \tag{2.97}$$

is satisfied for a full measure set of T.
One concludes that L^{-1} is well defined and "looses τ–derivatives".

Lyapunov–Schmidt reduction

By means of a Lyapunov-Schmidt reduction, one can write

$$\zeta = \zeta_K + \zeta_R = (\phi_0, 0, 0) + \zeta_R$$

where $\zeta_K := \Pi_K \zeta$ ($\zeta_R := \Pi_R \zeta$) and Π_K (Π_R) is the projector on the kernel (on the range respectively). The range equation reads

$$L\zeta_R = \Pi_R N(\zeta_0 + \zeta_K + \zeta_R)\,. \tag{2.98}$$

Since L is invertible as operator from the range to the range, the idea is to solve first the range equation (2.98) for any fixed ϕ_0, finding a solution $\zeta_R(t) = \zeta_R(\phi_0)(t)$ by the Contraction Mapping Theorem, and thereafter the kernel equation $\Pi_K N(\zeta_0 + \zeta_R + \zeta_K) = 0$ for $\zeta_R = \zeta_R(\phi_0)$, determining ϕ_0 by a variational argument. We define

$$\Phi(\zeta_R; \phi_0) := L^{-1}\Pi_R N(\zeta_0 + \zeta_K + \zeta_R)\,.$$

The operator $L^{-1}\Pi_R$ looses τ derivatives because of (2.97). By the smoothing property (S) of the nonlinearity

$$N : \mathcal{H}^s \longrightarrow \mathcal{H}^{s+d}\,,$$

we have that Φ acts in this way:

$$\Phi : \mathcal{H}^s \longrightarrow \mathcal{H}^{s+d} \longrightarrow \mathcal{H}^{s+d-\tau} \subset \mathcal{H}^s$$

where the last relation holds if $d \geq \tau$. Hence, by (2.95), one can solve the range equation by the Implicit Function Theorem, obtaining

$$\zeta_R = \zeta_R(\phi_0).$$

As the solutions of the Hamilton's equations

$$\begin{cases} \dot{J} & = & -\eta^4 \partial_\phi K(I_0 + J, \phi_0 + \tilde{\omega}t + \psi, \hat{z}) - \eta \partial_\phi \hat{G} \\ \dot{\psi} - \eta^2 AJ & = & \eta^2 B^t \hat{z}\bar{\hat{z}} + \eta^4 \partial_I K(I_0 + J, \phi_0 + \tilde{\omega}t + \psi, \hat{z}) + \eta \partial_I \hat{G} \\ \dot{\hat{z}}_j - \mathrm{i}\tilde{\Omega}_j \hat{z}_j & = & \mathrm{i}\eta^2 (BJ)_j \hat{z}_j + \mathrm{i}\eta \partial_{\bar{\hat{z}}_j} \hat{G} + \mathrm{i}\eta \partial_{\bar{\hat{z}}_j} K(I_0 + J, \phi_0 + \tilde{\omega}t + \psi, \hat{z})\,, \end{cases}$$

$j \geq n+1$, are critical points of the action functional

$$S(I, \phi, \hat{z}) := \int_0^T \left(I \cdot \dot{\phi} - \mathrm{i} \sum_{j \geq n+1} z_j \dot{\bar{z}}_j - \widetilde{\mathcal{H}}(I, \phi, \hat{z}, \bar{\hat{z}}) \right) dt\,,$$

so the solutions of the "reduced" kernel equation

$$\Pi_K N\big(\zeta_0 + \zeta_K + \zeta_R(\phi_0)\big) = 0$$

are critical points of the *reduced action functional* $S\big(\zeta_0 + \zeta_K + \zeta_R(\phi_0)\big)$. The kernel equation is defined on a n-dimensional torus, hence solutions ζ_K can be found by a topological argument.
We have just proved the following theorem

Theorem 2.7.2. *Consider the hamiltonian system* (2.84) *and fix a positive* n. *Assume that* (2.85), (2.86), *(S) hold. Let be* A *defined in* (2.87) *and* (2.88), *suppose* $\det A \neq 0$ *and that* (2.96) *is satisfied for all* $j \geq n+1$. *Finally assume* $d > 1$ *in* (2.86). *Then for any fixed* $\eta \ll 1$ *there exist at least* n *distinct periodic orbits* $z^{(1)}(t), \ldots, z^{(n)}(t)$ *with the following properties:*

(i) $\|z^m(t)\|_{a,s} \leq \text{const}\,\eta$, *for* $m = 1, \ldots, n$ *and* $t \in \mathbb{R}$;

(ii) $\|\Pi_{>n} z^m(t)\|_{a,s} \leq \text{const}\,\eta^2$, *for* $m = 1, \ldots, n$ *and* $t \in \mathbb{R}$; *here* $\Pi_{>n}$ *denotes the projector on the modes with index larger than* n;

(iii) the period T *of* z^m *does not depend on* m *and fulfills* $\eta^{-2} \leq T \leq 2\eta^{-2}$.

The nonlinear beam equation

The previous theorem can be applied to some semilinear PDEs, for example the nonlinear Schrödinger equation and the beam equation. Consider the beam equation

$$u_{tt} + u_{xxxx} + \mu u = f(u)\,, \qquad \mu \in [0, L] \tag{2.99}$$

with hinged boundary conditions

$$u(t,0) = u(t,\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0\,, \tag{2.100}$$

where the nonlinearity $f(u)$ is a real analytic odd function of the form

$$f(u) := au^3 + O(u^5)\,, \qquad a \neq 0\,. \tag{2.101}$$

Applying Theorem 2.7.2 to the beam equation the Theorem 2.7.1 follows.

Remark 2.7.3. *We remark that an analogous proof holds also for the nonlinear Schrödinger equation but not for the nonlinear wave equation. Indeed, in the wave equation the hypothesis (S) of smoothing on the nonlinearity is missing, that is the operator* N *gets only one derivative* ($d = 1$) *but it is not enough to have the bound (2.97)* ($d \geq \tau > 1$). *This fact follows from the asymptotic behavior of the frequencies of the associated linear operator, that are* $\omega_j = \sqrt{j^2 + \mu} \sim j$ *for the NLW,* $\omega_j = \sqrt{j^4 + \mu} \sim j^2$ *for the beam equation and* $\omega_j = j^2 + \mu \sim j^2$ *for the NLS. See also Remark 4.2.13 for further details.*

Chapter 3

Some known results about quasi–periodic and almost–periodic solutions

This chapter is devoted to the description of some results about quasi–periodic and almost–periodic solutions for the nonlinear wave, the nonlinear Schrödinger and the beam equations. We insert in the beginning a brief section regarding the finite dimensional case in which we state the well known Kolmogorov and Melnikov Theorems on the conservation of invariant tori of maximal and lower dimension, respectively. Techniques and tools used in the finite dimensional situation have been implemented by many authors to prove existence of quasi–periodic and almost–periodic solutions of hamiltonian PDEs. We report some of such interesting results.

3.1 Quasi–periodic orbits in finite dimension

The celebrated KAM (Kolmogorov–Arnold–Moser) theory is a perturbation theory for quasi–periodic motions in hamiltonian systems. It is treated in exhaustive way, e.g., in [A],[A88],[Mo73]. Here we briefly discuss two of its major results: the Kolmogorov and Melnikov Theorems.

We now give a definition of quasi–periodic function, which holds also in the infinite dimensional situation.

Definition 3.1.1. *Let be E a Banach space and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. A function $u(t)$ with n frequencies is called quasi–periodic if there exists a continuous map $U : \mathbb{T}^n \longrightarrow E$ and a n-dimensional vector ω such that $u(t) \equiv U(\omega t)$.*

Kolmogorov Theorem

Let us consider the perturbation of an integrable Hamiltonian $H(I, \phi) = h(I)+ \varepsilon\mathcal{H}(I, \phi)$ in the phase space $\mathcal{I} \times \mathbb{T}^N$, where $\mathcal{I}$ is a bounded N–dimensional domain and ε is a small parameter. The system generated by the integrable Hamiltonian h has invariant tori of the form

$$\{I\} \times \mathbb{T}^N \,, \quad I \in \mathcal{I} \,,$$

on which the motion is quasi–periodic with frequency $\omega(I) = \partial_I h(I)$. A torus for which $\omega(I)$ is rationally independent is called nonresonant. Each phase trajectory on such a torus fills it densely. The other tori are called resonant and are foliated by smaller dimensional tori. The unperturbed Hamiltonian h is non-degenerate if the Hessian of h, $\partial^2 h$, is invertible. In a nondegenerate system the nonresonant tori are a dense full measure set, while the resonant tori are a dense zero measure set.
The Kolmogorov's Theorem ([K54]) states that, if h is nondegenerate and ε is small enough, most nonresonant invariant tori of h persist for the perturbed system generated by H, being only slightly deformed. More precisely, for $0 < \rho < 1$, there exists a subset $\mathcal{I}_\varepsilon$ of $\mathcal{I}$ such that $\text{meas}(\mathcal{I} \setminus \mathcal{I}_\varepsilon) \to 0$ as $\varepsilon \to 0$, and, for $I \in \mathcal{I}_\varepsilon$, there exists a map $\Phi_I : \mathbb{T}^N \longrightarrow \mathcal{I} \times \mathbb{T}^N$ and a frequency ω_I with $|\omega_I - \omega(I)| \leq \text{const}\, \varepsilon$, such that

$$t \longmapsto \Phi_I(\phi + \omega_I t)$$

is a solution of the hamiltonian system generated by H. Moreover $\text{dist}\big(\Phi_I(\phi), (I, \phi)\big) \leq \varepsilon^\rho$.
For other versions and important improvements of this theorem, see [Mo67], [Mo73], [P82], [SZ89].

Melnikov Theorem

The Lyapunov Center Theorem, see theorem 1.1.5, states the persistence of nondegenerate one–dimensional invariant tori under hamiltonian perturbations (namely of periodic solutions), and the Kolmogorov theorem states the persistence of most of the invariant N–tori of integrable system with N degrees of freedom. It is natural wondering if most of invariant tori of an intermediate dimension n, $1 < n < N$, survive under perturbations. For what concern perturbations of a linear hamiltonian system with $N = n + m$ degrees of freedom the question is the following. In the phase space

$$\mathcal{I} \times \mathbb{T}^n \times \mathbb{R}^{2m} = \{(I, \phi, z), \ z = (p, q)\} \,,$$

where $\mathcal{I} \subset \mathbb{R}^n$, $n \geq 2$ and $m \geq 1$, let us consider the Hamiltonian

$$H(I, \phi, z) = \omega \cdot I + Az \cdot z + \varepsilon \mathcal{H}(I, \phi, z) \,,$$

where A is a symmetric linear operator in $\mathbb{R}^{2m}$,$\omega \in \mathcal{O} \subset \mathbb{R}^n$ is a parameter $\mathcal{H} = \mathcal{H}(I, \phi, z)$ is an analytic perturbation and ε is a small parameter. The hamiltonian equations write

$$\dot{I} = -\varepsilon \partial_\phi \mathcal{H} \,, \quad \dot{\phi} = \omega + \varepsilon \partial_I \mathcal{H} \,, \quad \dot{z} = J(Az + \varepsilon \partial_z \mathcal{H}) \,, \tag{3.1}$$

where, as usual, $J(p,q) = (-q,p)$. For $\varepsilon = 0$ the system (3.1) has invariant n–dimensional tori $T_I = \{I\} \times \mathbb{T}^n \times \{0\}$, $I \in \mathcal{I}$. It is natural wondering if these tori persist in the system (3.1) for $\varepsilon > 0$.
Let be Ω the spectrum of the operator JA, $\Omega := \{\Omega_1, \ldots, \Omega_{2m}\}$. We suppose that the tori are nondegenerate, namely $0 \notin \Omega$, $\Omega_j \neq \Omega_k$, $\forall\, j \neq k$. The simplest case is when the tori are hyperbolic, namely there are not purely imaginary eigenvalues. The persistence of these tori was proved in [Mo73].
The case of elliptic tori, namely $\Omega \subset \mathrm{i}\mathbb{R} \setminus \{0\}$, $\Omega_j \neq \Omega_k$, $\forall\, j \neq k$, is more difficult. The statement of the persistence of the elliptic torus T_I for most ω was published by Melnikov in [M65]. The complete proof of the theorem was given later by Eliasson [E88], Pöschel [P89] and by Kuksin [Ku87], [Ku88].

3.2 Quasi–periodic solutions of some PDEs

As we have just seen, the KAM theory is very powerful in order to construct families of quasi–periodic solutions in the finite dimensional case. In this section, we state some results about quasi–periodic solutions of some PDEs, namely finite dimensional invariant tori of an infinite dimensional hamiltonian system.
The standard techniques for searching quasi–periodic solutions, are perturbative and their aim is to continue finite dimensional invariant tori with quasi–periodic motions under the influence of an infinite dimensional perturbation. In particular, the standard procedure is using the Birkhoff Normal Form of hamiltonian mechanics, see pg. 30. Such normal forms enable one to apply an infinite dimensional extension of the Melnikov's Theorem for finite dimensional almost integrable hamiltonian systems, which was developed by Kuksin, see [Ku93], by Wayne in [W90] and by Pöschel in [P89], [P96b]. In this way, one can establish the existence of large families of time-quasi-periodic solutions which are linearly stable.

3.2.1 Wave equation

Dirichlet boundary conditions

♡ **[W90]** Let us consider the equation

$$u_{tt} - u_{xx} + v(x)u + \varepsilon u^3 = 0\,, \tag{3.2}$$

where the potential $v(x)$ lies in the subspace

$$E := \left\{ v \in L^2\big([0,\pi]\big) \;\middle|\; \int_0^\pi v(x)\,dx = 0,\ v(x) = v(\pi - x) \right\} \cap L^2_+\,, \tag{3.3}$$

where $L^2_+ \subset L^2([0,\pi])$ is the open subset of all potentials $v \in L^2([0,\pi])$ with strictly positive Dirichlet eigenvalues.

Let be $\{\mu_j\}_{j=1}^{\infty}$ the eigenvalues and $\{\phi_j\}_{j=1}^{\infty}$ the corresponding eigenfunctions of the operator $-d^2/dx^2 + v(x)$ and with Dirichlet b.c.. Denoting by $u_j^0(t,x) := \sin\left(\sqrt{|\mu_j|}\,t\right)\phi_j(x)$ the periodic solution of (3.2) for $\varepsilon = 0$, then

$$u_N(t,x) := \sum_{j=1}^{N} u_j^0(t,x)$$

is a quasi-periodic solution of (3.2) for $\varepsilon = 0$, if $\{\sqrt{\mu_j}\}_{j=1}^{N}$ are rationally independent. The following result states that these quasi–periodic solutions persist when $\varepsilon \neq 0$.

Theorem 3.2.1. *There are sets $\mathcal{F}(N) \subset E$ such that if $v \in \mathcal{F}(N)$, there is a constant $\varepsilon_0(v,N) > 0$ such that whenever $\varepsilon < \varepsilon_0(v,N)$, (3.2) has a weak, quasi-periodic solution $u_N^{\varepsilon}(x,t)$ whose frequency vector differs from that of $u_N(x,t)$ by $O(\varepsilon)$. The set $\mathcal{F}(N)$ has measure one with respect to a natural probability measure P (which is the same of Theorem 2.4.1).*

By a weak, quasi–periodic solution, one means a non–zero, continuous function, quasi–periodic in time, which is a distributional solution of (3.2). The following statement guarantees that one obtains a large number of solutions, which is typical for the KAM methods.

Theorem 3.2.2. *Suppose to restrict our attention to the subset of E for which the spectrum of $-d^2/dx^2 + v(x)$ is positive. Then, for almost every potential v, with respect to the P measure, there is a constant $\varepsilon_0(v) > 0$, such that if $|\varepsilon| < \varepsilon_0(v)$, there exists a set of positive, N–dimensional Lebesgue measure such that for every point $\tilde{\Omega}$ in this set one has a quasi–periodic solution with frequency vector $\tilde{\Omega}$.*

The proof is achieved seeking to extend the KAM Theorem to a infinite dimensional setting, namely, perturbations of completely integrable partial differential equations. In this theorem, the author replaces the scalar parameter μ of the Klein–Gordon equation $u_{tt} - u_{xx} + \mu u = 0$ with Dirichlet boundary conditions, by some potential function $v \in L^2([0,\pi])$. By this choice, he introduces infinitely many parameters into the system, which may be adjusted and thus substitute the standard nondegeneracy condition. Finally, the author finds a Cantor families of small amplitude quasi–periodic solutions for a Cantor set of potentials v, via KAM theory.

♡ **[P96a]** Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + \mu u + f(u) = 0\,, \qquad \mu > 0\,, \tag{3.4}$$

where f is a real analytic, odd function of u of the form

$$f(u) = au^3 + \sum_{k\geq 5} f_k u^k, \qquad a \neq 0\,. \tag{3.5}$$

Every solution of the linear system with Dirichlet boundary condition is of the form

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = I_j \cos(\lambda_j t + \phi_j^0)\,,$$

namely it is the superposition of the harmonic oscillations of the basic modes $\phi_j = \sqrt{2/\pi}\sin jx$ and frequencies $\lambda_j = \sqrt{j^2+\mu}$, for $j = 1, \ldots$, with amplitudes $I_j \geq 0$ and initial phases ϕ_j^0. Choosing finitely many modes

$$J = \{j_1 < j_2 < \ldots < j_n\} \subset \mathbb{N}\,,$$

there is an invariant $2n$–dimensional linear subspace E_J, such that

$$E_J := \left\{(q_1\phi_{j_1} + \ldots + q_n\phi_{j_n}, p_1\phi_{j_1} + \ldots + p_n\phi_{j_n}\right\} = \bigcup_{I\in\mathbb{R}_+^n} \mathcal{T}_J(I)\,, \tag{3.6}$$

where $\mathbb{R}_+^n : \left\{I \in \mathbb{R}^n : I_j > 0 \text{ for } 1 \leq j \leq n\right\}$, is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_J(I) = \left\{q_j^2 + \lambda_j^{-2}p_j^2 = I_j \ \text{ for } 1 \leq j \leq n\right\}. \tag{3.7}$$

In particular, E_J is completely foliated into rotational tori with frequencies $\lambda_{j_1}, \ldots, \lambda_{j_n}$.
The main theorem is the following:

Theorem 3.2.3. *For all $\mu > 0$ and for each index set $J = \{j_1 < \ldots < j_n\}$ with $n \geq 2$, satisfying*

$$\min_{1\leq i<n}(j_{i+1} - j_i) \leq n - 1\,,$$

there exists a Cantor manifold $\mathcal{E}_J$ of real analytic, linearly stable, diophantine n–dimensional tori for the nonlinear wave equation given by a Lipschitz continuous embedding $\Phi : \mathcal{T}_J[\mathcal{C}] \longrightarrow \mathcal{E}_J$, which is a higher order perturbation of the inclusion map $\Phi_0 : E_J \hookrightarrow \mathcal{P}$, restricted to $\mathcal{T}_J[\mathcal{C}]$. The Cantor set $\mathcal{C}$ has full density at the origin, and $\mathcal{E}_J$ has a tangent space at the origin equal to E_J. Moreover, $\mathcal{E}_J$ is contained in the space of real analytic functions on $[0,\pi]$.

This result is achieved by transforming the associated Hamiltonian into Birkhoff Semi–Normal Form of order four with respect to any finite number of basic modes, by a symplectic change of coordinates see Proposition 4.1.6.
The approach and the result is the analogue for the nonlinear Schrödinger equation on the same interval, carried out in [KuP96], see pag. 92. A fundamental difference between these cases, is that no complete normal form of order four is available for the nonlinear wave equation, due to asymptotic resonances among the frequencies. However, since sufficiently many nonresonance condition are satisfied, there is a real analytic symplectic coordinate transformation that put, at least the interaction of the first n–modes, into a nonlinear integrable normal form up to order four. To this Hamiltonian, KAM theory may be applied to continue invariant tori of the integrable system.

♡ **[LiY]** Let be $\mathcal{P} = H_0^1([0,\pi]) \times L^2([0,\pi])$. Consider the wave equation

$$u_{tt} - u_{xx} + \mu u + f(u) = 0\,, \quad \mu > 0\,,$$

where

$$f(u) = au^{2r+1} + \sum_{k \geq r+1} f_{2k+1} u^{2k+1}\,, \qquad a \neq 0\,,\ r \in \mathbb{N}\,.$$

Recalling the definitions (3.6) and (3.7), we write the main result of this paper:

Theorem 3.2.4. *For almost all $\mu > 0$ and each index set $\mathcal{I} = \{n_1 < \ldots < n_b\}$ with $b \geq 2$, satisfying*

$$sn_i \neq n_j \ \text{ for any } s = 1, 2, \ldots, r,\ i < j,\ i, j \in \{1, \ldots, b\}\,,$$

the wave equation above possesses a local, positive measure, $2b$–dimensional invariant Cantor manifold $\mathcal{E}_{\mathcal{I}}$ given by a Whitney smooth embedding $\Phi : \mathcal{T}_{\mathcal{I}}[\mathcal{C}] \longrightarrow \mathcal{E}_{\mathcal{I}}$, which is a higher order perturbation of the inclusion map $\Phi_0 : E_{\mathcal{I}} \hookrightarrow \mathcal{P}$, restricted to $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$. Moreover, the Cantor manifold $\mathcal{E}_{\mathcal{I}}$ is foliated by real analytic, linearly stable, b dimensional invariant tori carrying quasi–periodic solutions.

The proof is based on infinite dimensional KAM theorem, partial normal form and scaling skills, following [P96a]. We note that this result can be applied also to the nonlinear beam equation.

Periodic boundary conditions

♡ **[Bou94]** The same result of the nonlinear Schrödinger equation, see pag. 92 below, but with $g(x) \neq 0$.

♡ **[ChY00]** Let us consider the equation

$$u_{tt} - u_{xx} + V(x)u + f(u) = 0 \tag{3.8}$$

with periodic b.c.; here $f(u)$ is a real analytic function satisfying $f(0) = f'(0) = 0$ and the periodic potential V belongs to

$$E_{per} := \left\{ V \in L^2(\mathbb{T}) \;\middle|\; \int_0^{2\pi} V(x)\,dx = 0 \right\} \cap L^2_+(\mathbb{T})\,, \tag{3.9}$$

where $L^2_+(\mathbb{T}) \subset L^2(\mathbb{T})$ is the open subset of all potentials $V \in L^2([0,\pi])$ with strictly positive periodic eigenvalues. Let μ_n and $\phi_n(x)$, $n \in \mathbb{N}$, be the positive eigenvalues and the eigenfunctions of the operator $d^2/dx^2 - V(x)$. Fix $n_1 < \ldots < n_d \in \mathbb{N}$, $d \geq 1$ and a compact subset $\mathcal{O} \subset \mathbb{R}^d$. Consider a family of real analytic potentials $V(x;\xi)$ parametrized by $\xi = (\xi_1, \ldots, \xi_d) \in \mathcal{O}$ where

$$\xi_i = \sqrt{\mu_i} \quad i = 1, \ldots, d.$$

Theorem 3.2.5. *Consider the equation*

$$u_{tt} - u_{xx} + V(x;\xi)u + f(u) = 0 \tag{3.10}$$

with V real analytic and $\xi \in \mathcal{O}$. Then for any $0 < \gamma \ll 1$, there exists a subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with $meas(\mathcal{O} \setminus \mathcal{O}_\gamma) \longrightarrow 0$ as $\gamma \to 0$, such that (3.10) *with $\xi \in \mathcal{O}_\gamma$ has a family of small amplitude (proportional to some power of γ), analytic quasi-periodic solutions of the form*

$$u(t,x) = \sum_{n\geq 1} u_n(\omega'_1 t, \dots, \omega'_d t)\phi_n(x)\,,$$

where $u_n : \mathbb{T}^d \longrightarrow \mathbb{R}$ and $\omega'_1, \dots, \omega'_d$ are close to $\omega_1 \equiv \xi_1, \dots, \omega_d \equiv \xi_d$.

We remark that looking for solutions with periodic boundary conditions is technically more difficult than looking for solutions which satisfy the Dirichlet b.c.. This technical reason is related to the multiplicity of the spectrum of the associated Sturm-Liouville operator $-d^2/dx^2 + \mu$, as we have just said in section 1.2. Because of this multiplicity, another small divisors problem arises, hinged with the so called normal frequencies. The proof is based on an infinite dimensional KAM theorem.

♡ [**BePr05**]: Consider the completely resonant forced equation

$$u_{tt} - u_{xx} + f(\omega_1 t, u) = 0 \tag{3.11}$$

where the nonlinear forcing term $f(\omega_1 t, u)$ is 2π–periodic in time and vanishes for $u = 0$ with a zero of order at least 2. The authors deal with the case $\omega_1 = n/m \in \mathbb{Q}$ and $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$. Assume that f satisfies assumption

(H) $f(\omega_1 t, u) = \sum_{k=2d-1}^{\infty} a_k(\omega_1 t)u^k, d \in \mathbb{N}^+,\ d \geq 2$, and $a_k(\omega_1 t) \in H^1(\mathbb{T})$ verify $\sum_{k=2d-1}^{\infty} |a_k(\omega_1 t)|_{H^1}\ \rho^k < \infty$ for some $\rho > 0$. The $a_k(\omega_1 t)$ are not all identically constant.

Theorem 3.2.6. *Let $\omega_1 = n/m \in \mathbb{Q}$. Assume that f satisfies assumption* (H) *and $a_{2d-1}(\omega_1 t) > 0$. Let be $\mathcal{B}_\gamma$ the uncountable zero measure Cantor set*

$$\mathcal{B}_\gamma := \left\{ \varepsilon \in \mathbb{R} \ :\ |\ell_1 + \varepsilon \ell_2| > \frac{\gamma}{|\ell_2|}\,,\ \forall \ell_1, \ell_2 \in \mathbb{Z} \setminus \{0\} \right\}$$

where $0 < \gamma < 1/6$. Then equation (3.11) *admits a quasi–periodic solution $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ with two frequencies $(\omega_1, \omega_2) = (\omega_1, 1 + \varepsilon)$, $\forall \varepsilon \in \mathcal{B}_\gamma$.*

A similar result can be extended to the case $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$, assuming the condition on the coefficient $\int_0^{2\pi} a_{2d-1}(\varphi)d\varphi \neq 0$.

3.2.2 Nonlinear Schrödinger equation

If one is interested in quasi-periodic solutions of the nonlinear Schrödinger equation the tools are the same for what concerns the nonlinear wave equation. Some aspects, such as the transformation into normal form, are even simpler than in the wave equation. Indeed, as we have just said, no complete normal form of order four is available for the nonlinear wave equation, due to asymptotic resonances among the frequencies. For the NLS there exists a complete normal form up to order four.

Dirichlet boundary conditions

♡ [**KuP96**] Let us consider equation

$$iu_t - u_{xx} + mu + f(|u|^2)u = 0\,, \quad m \in \mathbb{R} \tag{3.12}$$

with f real analytic in some neighborhood of the origin in $\mathbb{C}$. Assume that $f(0) = 0$ and $f'(0) \neq 0$ (non degeneracy condition).

Theorem 3.2.7. *For all $m \in \mathbb{R}$, for all $n \in \mathbb{N}$ and for f real analytic and non-degenerate, there exists a full density Cantor manifold of real analytic linearly stable, diophantine n-dimensional tori for equation* (3.12)*.*

The proof is carried out performing a complete Birkhoff normal form up to order four on the first n-modes and then using KAM theory to continue tori. We have discussed this method in few more details for the nonlinear wave equation, see [P96a].

Periodic boundary conditions

♡ [**Bou94**] Let us consider the 1D NLS of the form

$$iu_t - u_{xx} + g(x)u + \varepsilon\partial_{\bar{u}} H(u, \bar{u}) = 0\,, \tag{3.13}$$

with g a real analytic periodic function on $\mathbb{T}$ and H a polynomial expression in $u, \bar{u}$. Let us denote by ω_n, $n \in \mathbb{Z}$ the spectrum of the operator $-\partial_{xx} + g(x)$ and $\{\psi_n\}$ the corresponding eigenvectors. One is looking for quasi-periodic solutions with frequencies (λ_1, λ_2) of the form

$$u(t, x) = \sum_{m_1, m_2, n} \hat{u}(m_1, m_2, n)\, e^{i(m_1\lambda_1 + m_2\lambda_2)t}\psi_n(x)\,, \qquad m_1, m_2 \in \mathbb{Z}\,.$$

The equation of the corresponding Fourier coefficients is

$$\big(-(m_1\lambda_1 + m_2\lambda_2) + \omega_n \big)\, \hat{u}(m_1, m_2, n) + \varepsilon\widehat{\partial_{\bar{u}} H}(m_1, m_2, n) = 0\,.$$

Theorem 3.2.8. *Choose $n_1, n_2 \in \mathbb{Z}$ and let us denote $\lambda := (\lambda_1, \lambda_2)$. There exists a continuation of the unperturbed solution*

$$u(t, x) = a_1 e^{i\lambda_1 t}\psi_{n_1}(x) + a_2 e^{i\lambda_2 t}\psi_{n_2}(x)$$

of the perturbed equation (3.13)*, for λ outside a set of measure that tends to 0 for $\varepsilon \to 0$. The perturbed solution u satisfies in particular*

$$\sum_{m,n} |\hat{u}(m, n)| e^{(|m|+|n|)^c} < \infty$$

for some $c > 0$.

♡ **[Bou98]** Consider the 2D nonlinear Schrödinger equation of the form

$$iu_t + Au + \varepsilon \partial_{\bar{u}} H(u, \bar{u}) = 0 \tag{3.14}$$

where A is a self-adjoint operator and H is polynomial (or real-analytic). Let us assume

$$A := -\Delta + M_\sigma$$

where

$$M_\sigma e^{\mathrm{i}n\cdot x} := \sigma_n e^{\mathrm{i}n\cdot x}\,, \quad \sigma_n \in \mathbb{R}\,,\ n \in \mathbb{Z}^2\,, \tag{3.15}$$

$x = (x_1, x_2) \in \mathbb{R}^2$ and $\Delta = \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2}$. We note that the operator A has eigenvalues

$$\mu_n := |n|^2 + \sigma_n$$

with eigenfunctions $e^{\mathrm{i}n\cdot x}$. Fix $b \in \mathbb{N}$, $b \geq 1$ and some specific indices $n_1, \ldots, n_b \in \mathbb{Z}^2$. Let $\sigma = (\sigma_1, \ldots, \sigma_b) \in \mathbb{R}^b$ and $\lambda = (\lambda_1, \ldots, \lambda_b) \in \mathbb{R}^b$ with

$$\lambda_j := \mu_{n_j} = |n_j|^2 + \sigma_j\,, \qquad j = 1, \ldots, b\,.$$

and

$$\mathcal{S} := \{\, (n_j, e_j) \in \mathbb{Z}^2 \times \mathbb{Z}^b \ \mid j = 1, \ldots, b \,\}$$

the *resonant set* (here $e_j = (0, \ldots, 1, \ldots, 0)$). For any $a = (a_1, \ldots, a_b) \in \mathbb{R}^b$, $a_{n_j e_j} := a_j$,

$$u_0(t, x) := \sum_{j=1}^{b} a_j e^{i(\lambda_j t + n_j\cdot x)} = \sum_{(n,k)\in\mathcal{S}} a_{nk} e^{i(\lambda\cdot k\, t + n\cdot x)} \tag{3.16}$$

is a quasi-periodic (for λ rationally independent) solution of (3.14) for $\varepsilon = 0$.

Theorem 3.2.9. *Let us suppose $|a|$ suitably small (depending only on the analyticity domain of H). Then, for ε sufficiently small, there exists a set $\Sigma_\varepsilon(|a|)$ with*

$$\mathrm{meas}\big(\Sigma_\varepsilon(|a|)\big)^{\mathrm{c}} \to 0 \qquad \text{as } \varepsilon \to 0\,,$$

such that, if $\sigma \in \Sigma_\varepsilon(|a|)$, (3.14) has a solution

$$u^\varepsilon(t,x) = \sum_{n\in\mathbb{Z}^2, k\in\mathbb{Z}^b} u^\varepsilon_{nk} e^{i(\lambda'\cdot k\, t + n\cdot x)}$$

with perturbed frequency λ' ($|\lambda' - \lambda| \leq \text{const}\, \varepsilon$). The solution satisfies

$$u^\varepsilon_{n_j e_j} = a_j\,, \qquad j = 1,\ldots,b$$

and

$$\sum_{(n,k)\notin\mathcal{S}} e^{(|n|+|k|)^c} |u^\varepsilon_{nk}| \leq \sqrt{\varepsilon}\,,$$

for some $c > 0$.

3.2.3 Beam equation

♡ **[GeY03]** Consider 1D beam equation (1.25) with boundary condition (1.26), with f real analytic and odd function of the form (2.83).

Theorem 3.2.10. *For all $m > 0$ outside a zero measure set, there exists a Cantor manifold of real analytic, linearly stable and Diophantine n-tori.*

The existence is proved following [KuP96].

3.3 Almost periodic solutions

A function $u(t)$ is *almost-periodic* with frequency $\omega := (\omega_1, \omega_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ with $\omega_1, \omega_2, \ldots$ rationally independent iff

$$u(t) = \sum_{k\in\mathbb{Z}^{\mathbb{N}}_0} U_k e^{ik\cdot\omega t}\,, \tag{3.17}$$

where $\mathbb{Z}^{\mathbb{N}}_0$ is the space of all integer sequences $k := (k_1, k_2, \ldots)$, $k_j \in \mathbb{Z}$, $j \geq 1$ with only finitely many nonzero components.

We now give an example of construction of almost–periodic orbits for a infinite dimensional "second order hamiltonian system". We discuss it here as a model problem mimicking some of the basic features coming out in the study of hamiltonian PDEs.

♡ **[ChPe94]** Let us consider the Euler-Lagrange equation on $\mathbb{T}^{\mathbb{Z}^d}$

$$\ddot{x}_i = f_i(x(t)) \qquad i \in \mathbb{Z}^d\,.$$

Definition 3.3.1. *A continuous function f is a $g-$gradient if for any finite $I \subset \mathbb{Z}^d$ there exists a $C^1(\mathbb{T}^{|I|}, \mathbb{R})$ function $V^{|I|}(x)$ such that*

$$f^{[I]}_i(x) = \partial_{x_i} V^{(I)}(x) \qquad \forall i \in I\,,\ \forall x \in \mathbb{T}^{|I|}\,.$$

Theorem 3.3.2. *Let be f a uniformly weakly real analytic $g-$gradient. Assume that for some finite $I \subset \mathbb{Z}^d$ the equation*

$$\ddot{x}^{(I)} = \partial_{x^{(I)}} V^{(I)}(x^{(I)}) \qquad x^{(I)} \in \mathbb{T}^{|I|}$$

admits a non-degenerate real analytic solutions with a $(\gamma - \tau)$-Diophantine frequency $\omega^{(I)} \in \mathbb{R}^{|I|}$ (i.e. $x^{(I)}(t) = \omega^{(I)}(t) + u^{(I)}(\omega^{(I)}t)$ for a suitable real analytic function $u^{(I)} : \mathbb{T}^{|I|} \to \mathbb{R}^{|I|}$ such that $\det(Id + \partial u^{(I)}) \neq 0$). Then there exist uncountably many $\omega-$almost periodic solutions with $\omega_i = \omega_i^{(I)}\,, \quad \forall i \in I$. All such frequencies ω are $\gamma-$Diophantine in the sense that for all $J \subset I$ with $|J \setminus I| = m$ it is

$$\left| \sum_{j \in J} \omega_j n_j \right| \geq \frac{1}{\gamma (\sum_{j\in J} |n_j|)^{\tau+m}} \qquad\qquad \forall (\{n_j\}_{j\in J}) \in \mathbb{Z}^{|J|} \setminus \{0\}\,.$$

Remark 3.3.3. *The frequencies ω_j will be such that $|\omega_j|$ grows rapidly as $j \to \infty$ (like $(|j|!)^c$).*

3.3.1 Wave equation

♡ **[Bou96]** Let us consider the wave equation

$$u_t - u_{xx} + V(x)u + \varepsilon f(u) = 0 \tag{3.18}$$

under Dirichlet boundary conditions. Here V is a periodic real analytic potential and $f(u)$ an odd polynomial function of u, $f(u) = O(|u|^3)$. Denote $\{\mu_j\}$ and $\{\varphi_j\}$ the eigenvalues and the eigenfunctions of the operator $(-\frac{d^2}{dx^2} + V(x))$. Let us write $\mu_j = \lambda_j^2$ and consider λ_j as parameters. For $\varepsilon = 0$ the unperturbed solution u_0 of (3.18) will be

$$u_0(x) = \sum_{j=1}^{\infty} a_j \varphi_j(x) \cos \lambda_j t\,.$$

Theorem 3.3.4. *Given a sequence $\{a_j\}$ of positive reals such that $|a_j| \to 0$ sufficiently fast, exists an almost periodic solution of (3.18)*

$$u_\varepsilon(x,t) = \sum_{j=1}^{\infty} \sum_{n \in \Pi_\infty \mathbb{Z}} \hat{u}(j,n) \varphi_j(x) e^{in\cdot\lambda' t}\,.$$

Here $\Pi_\infty \mathbb{Z}$ stands for the space of finite sequences of integers $n = \{n_k\}$,

$$\begin{aligned} \hat{u}(j,n) &= \hat{u}(j,-n)\,, \\ \lambda'_j &= \lambda_j + O\left(\frac{\varepsilon}{j}\right) \quad \textit{(uniformly in j), is the perturbed frequency,} \end{aligned}$$

$$\hat{u}(j, e_j) \;=\; \hat{u}(j, -e_j) = \frac{1}{2}\, a_j \qquad (e_j = j - \mathit{unit\,vector\,in}\ \Pi_\infty \mathbb{Z})\,,$$

and

$$\sum_{(j,n)\notin S} e^{j^c + \sum_k w_k |n_k|^c} |\hat{u}(j,n)| < \sqrt{\varepsilon}$$

where

$$S = \big\{(j, \pm e_j)\big\}$$

is the resonant set, $c > 0$ *and* $\{w_k\}$ *are increasing weights, depending on the decay of sequence* $\{a_k\}$.

3.3.2 Nonlinear Schrödinger equation

♡ **[P02]** Let us consider the NLS

$$iu_t = u_{xx} - V(x)u - N(u)\,, \qquad 0 \le x \le \pi \tag{3.19}$$

with Dirichlet b.c. depending on a potential $V \in E \subset L^2([0,\pi])$ where E is defined in (3.3). Here, given f a real analytic function in a neighborhood of $0 \in \mathbb{C}$ with $f(0) = 0$, the nonlinearity is

$$N(u) := \Psi\Big(f\big(|\Psi u|^2\big)\Big)$$

where $\Psi : u \mapsto \psi * u$ is a convolution operator with an *even* function ψ on $\mathbb{R}$, which is smoothing of order $\sigma > 0$, namely

$$\Psi : H_0^s([0,\pi]) \longrightarrow H_0^{s+\sigma}([0,\pi])\,, \qquad \|\Psi u\|_{H^{s+\sigma}} \le c_s \|u\|_{H^s}\,,$$

for all $0 \le s \le 1$ with $\sigma < 1/4$. Since ψ is even, the Dirichlet problem on $[0,\pi]$ is equivalent to the *periodic* problem with period 2π within the space of all *odd* functions.

Theorem 3.3.5. *For "almost all" potentials* $V \in E$ *equation* (3.19) *admits uncountably many analytic, almost periodic solutions of the form* (3.17) *in every neighborhood of* $u \equiv 0$ *in* $H_0^1([0,\pi])$.

Here "almost all" potentials means that the complementary set of potentials in E has measure zero with respect to a large class of (natural) probability measure.

Part II

Periodic orbits with rational frequency

Chapter 4

Long time periodic orbits for the NLW

4.1 Hamiltonian setting and Birkhoff normal form

We study the equation (4) as an infinite dimensional hamiltonian system with coordinates u and $v = u_t$. Denoting $g = \int_0^u f(s)ds$, the Hamiltonian is

$$H(v, u) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) dx \, .$$

The equations of motion are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = u_{xx} - \mu u - f(u) \, .$$

Let us rewrite the Hamiltonian in infinite coordinates $(p, q) \in \ell^{a,s} \times \ell^{a,s}$, where

$$\begin{aligned} \ell^{a,s} &= \ell^{a,s}(\mathbb{R}) \\ &:= \left\{ q = (q_1, \ldots),\, q_i \in \mathbb{R} \,,\ i \geq 1 \ \text{s.t.}\ \|q\|_{a,s}^2 = \sum_{i \geq 1} |q_i|^2 i^{2s} e^{2ai} < \infty \right\} \end{aligned} \tag{4.1}$$

by the transformation

$$v = \mathcal{S}' p = \sum_{i \geq 1} \sqrt{\omega_i} p_i \chi_i \, , \quad u = \mathcal{S} q = \sum_{i \geq 1} \frac{q_i}{\sqrt{\omega_i}} \chi_i \, , \tag{4.2}$$

with $\omega_i = \sqrt{i^2 + \mu}$ and $\chi_i = \sqrt{2/\pi} \sin ix$. We get

$$H = \Lambda + G = \frac{1}{2} \sum_{i \geq 1} \omega_i (q_i^2 + p_i^2) + \int_0^\pi g(\mathcal{S} q) \, dx \, , \tag{4.3}$$

where we denote with Λ the integrable part and with G the not integrable one. The equations of motion are:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\omega_i q_i - \frac{\partial G}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} = \omega_i p_i \tag{4.4}$$

with respect to the symplectic structure $\sum dp_i \wedge dq_i$ on $\ell^{a,s} \times \ell^{a,s}$.

For $k \in \mathbb{N}$, we consider the space $C^k(\mathbb{R}, \ell^{a,s})$ of all the functions $\mathbb{R} \ni t \longmapsto q(t) \in \ell^{a,s}$ with finite norm

$$\|q\|_{C^k(\mathbb{R},\ell^{a,s})} := \sum_{h=0}^{k} \sup_{t\in\mathbb{R}} \|\partial_t^h q(t)\|_{a,s}\,. \tag{4.5}$$

Similarly for a C^k function $\mathbb{R} \ni t \longmapsto \big(p(t), q(t)\big) \in \ell^{a,s} \times \ell^{a,s}$ we consider the norm

$$\|(p,q)\|_{C^k(\mathbb{R},\ell^{a,s}\times\ell^{a,s})} := \|p\|_{C^k(\mathbb{R},\ell^{a,s})} + \|q\|_{C^k(\mathbb{R},\ell^{a,s})}$$

Lemma 4.1.1. *Let us assume that $a > 0$ and s arbitrary. Let be $\mathbb{R} \ni t \longmapsto \big(p(t), q(t)\big) \in \ell^{a,s} \times \ell^{a,s}$ a solution of* (4.4) *of class C^k, $2 \le k \le \infty$, then*

$$u(t,x) := \sum_{i\ge 1} \frac{q_i(t)}{\sqrt{\omega_i}}\,\chi_i(x) \tag{4.6}$$

is a classical solution of (4) *of class C^k.*

PROOF. We first prove that $u(t,x)$ is C^k and moreover, for any fixed $t \in \mathbb{R}$, the function $x \mapsto \partial_t^h u(t,x)$, $h < k+1$, is real analytic with analytic extension in the complex strip $|\mathrm{Im}\, x| < a$. Since $q \in C^k(\mathbb{R}, \ell^{a,s})$, we have that for all $t \in \mathbb{R}$

$$\sum_{i\ge 1} |q_i(t)|^2 i^{2s} e^{2ai} = \|q(t)\|_{a,s}^2 \le \|q\|^2_{C^k(\mathbb{R},\ell^{a,s})} < \infty\,. \tag{4.7}$$

For any fixed $\tilde{a} < a$ by (4.7), Cauchy–Schwartz inequality and

$$\sup_{|\mathrm{Im}\, x|\le\tilde{a}} |\chi_i(x)| \le \sup_{|\mathrm{Im}\, x|\le\tilde{a}} |\sin ix| \le e^{\tilde{a}i}\,,$$

we get, for $i_0 \ge 1$,

$$\begin{aligned} \sup_{|\mathrm{Im}\, x|\le\tilde{a}} \left|\sum_{i\ge i_0} \frac{q_i(t)}{\sqrt{\omega_i}}\,\chi_i(x)\right|^2 &\le \sum_{i\ge i_0} |q_i(t)|^2 i^{2s} e^{2ai} \sum_{i\ge i_0} \omega_i^{-1} i^{-2s} e^{-2(a-\tilde{a})i} \\ &\le \|q\|^2_{C^k(\mathbb{R},\ell^{a,s})} \sum_{i\ge i_0} \omega_i^{-1} i^{-2s} e^{-2(a-\tilde{a})i} \quad \overset{i_0\to\infty}{\longrightarrow} \quad 0\,, \end{aligned}$$

from which we have that, for any $t \in \mathbb{R}$, the series in (4.6) uniformly converges to a 2π–periodic real analytic odd function with analytic extension on the complex strip $|\mathrm{Im}\, x| < a$. Moreover, again by Cauchy–Schwartz inequality, we get

$$|\partial_x^h u(t,x)| \le \|q(t)\|_{a,s}^2 \sum_{i\ge 1} \omega_i^{-1} i^{2(h-s)} e^{-2ai}\,, \qquad \forall\,(t,x)\in\mathbb{R}\times[0,\pi]\,,\ \ h\in\mathbb{N}\,,$$

and, from (4.7), we have that

$$\forall\, h\in\mathbb{N}\ \ \exists\, c_{h,a,s}>0 \ \text{ s.t.} \qquad \sup_{(t,x)\in\mathbb{R}\times[0,\pi]} |\partial_x^h u(t,x)| \le c_{h,a,s}\,. \tag{4.8}$$

By similar arguments one proves that for any $x\in[0,\pi]$ the function

$$\mathbb{R}\ni t\mapsto u(t,x)=\sum_{i\ge 1}\frac{q_i(t)}{\sqrt{\omega_i}}\,\chi_i(x)$$

is continuous since the series uniformly converges on $\mathbb{R}$ (and the functions $t\mapsto q_i(t)$ are continuous since $q\in C^0(\mathbb{R},\ell^{a,s})$). The continuity in both variables follows by (4.8) and the continuity in t for x_0 fixed:

$$\begin{aligned} |u(t,x)-u(t_0,x_0)| &\le |u(t,x)-u(t,x_0)|+|u(t,x_0)-u(t_0,x_0)| \\ &\le c_{1,a,s}|x-x_0|+|u(t,x_0)-u(t_0,x_0)|\,. \end{aligned}$$

Arguing as above one proves that, for any fixed $x\in[0,\pi]$, the series

$$\sum_{i\ge 1}\frac{1}{\sqrt{\omega_i}}\,\partial_t q_i(t)\chi_i(x)$$

uniformly converges for $t\in\mathbb{R}$; hence we can differentiate inside the summation in (4.6) obtaining

$$\partial_t u(t,x)=\sum_{i\ge 1}\frac{1}{\sqrt{\omega_i}}\,\partial_t q_i(t)\chi_i(x)\,.$$

Carrying on the above arguments we finally have that $u\in C^k$ and that

$$\partial_t^h u(t,x)=\sum_{i\ge 1}\frac{1}{\sqrt{\omega_i}}\,\partial_t^h q_i(t)\chi_i(x)\,, \qquad h<k+1\,, \tag{4.9}$$

is, for any fixed $t\in\mathbb{R}$, a 2π–periodic real analytic odd function with analytic extension on the complex strip $|\mathrm{Im}\, x|<a$.
We now prove that u, defined in (4.6), is a classical solution of (4). From (4.2) and (4.3),

$$\frac{\partial G}{\partial q_i}=\frac{1}{\sqrt{\omega_i}}\int_0^\pi f(u)\chi_i\,dx\,, \tag{4.10}$$

hence, we have, by (4.9) and (4.4),

$$u_{tt} \;=\; \sum_{i\ge 1}\frac{\ddot q_i(t)}{\sqrt{\omega_i}}\,\chi_i(x)$$

$$
\begin{aligned}
&= \sum_{i\geq 1} \frac{1}{\sqrt{\omega_i}} \left(-\omega_i^2 q_i - \omega_i \frac{\partial G}{\partial q_i} \right) \chi_i \\
&= -\sum_{i\geq 1} \frac{q_i(t)}{\sqrt{\omega_i}} \left(\mu - \partial_{xx}\right) \chi_i(x) - \sum_{i\geq 1} \chi_i \int_0^\pi f(u)\chi_i \, dx \, ,
\end{aligned}
$$

because ω_i^2 are the eigenvalues of the operator $\mu - \partial_{xx}$. Moreover, being χ_i, for $i \geq 1$, a complete orthonormal basis for the L^2 functions on $[0, \pi]$, we obtain

$$
u_{tt} = -(\mu - \partial_{xx})u - \sum_{i\geq 1} \chi_i \int_0^\pi f(u)\chi_i \, dx = -(\mu - \partial_{xx})u - f(u) \, .
$$

□

Let us note that χ_i, for $i \geq 1$, are a complete orthonormal basis for the L^2 functions on $[0, \pi]$, but not for all analytic function on $[0, \pi]$. To overcome this problem, let us consider ℓ_b^2 and L^2, respectively, the Hilbert spaces of all *bi*-infinite, square integrable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi, \pi]$. To identify the two spaces we can consider the inverse discrete Fourier transform,

$$
\mathcal{F} : \, \ell_b^2 \longrightarrow L^2 \, , \qquad q \longmapsto \left[\mathcal{F}q\right](x) := \frac{1}{\sqrt{2\pi}} \sum_{i\in\mathbb{Z}} q_i e^{\mathrm{i}ix} \, ,
$$

which defines an isometry between the two spaces. Let $a \geq 0$ and $s \geq 0$. The subspaces $\ell_b^{a,s} \subset \ell_b^2$ contain all bi-infinite sequences whose norm is defined by

$$
\|q\|_{a,s}^2 := \sum_{i\in\mathbb{Z}} |q_i|^2 |i|_*^{2s} e^{2a|i|} \, ,
$$

where $|i|_* := \max\left\{|i|, 1\right\}$. In this way we obtain, through the Fourier transform $\mathcal{F}$, the subspaces $W^{a,s} \subset L^2$ endowed with the norm

$$
\|\mathcal{F}q\|_{a,s} = \|q\|_{a,s} \, .
$$

For $a > 0$, the subspaces $W^{a,s}$ consist of all 2π-periodic functions which are analytic and bounded in the complex strip $|\mathrm{Im} z| < a$ with trace functions on $|\mathrm{Im} z| = a$ belonging to the standard Sobolev space H^s. In this way, we obtain an orthonormal basis for all analytic functions on $[0, \pi]$.
The following two results were proved in [P96a], for completeness we give here the proofs:

Lemma 4.1.2. *For $a \geq 0$ and $s > 1/2$, the space $\ell_b^{a,s}$ is a Hilbert algebra with respect to convolution of the sequences, $(q * p)_j := \sum_k q_{j-k} p_k$,*

$$
\|q * p\,\|_{a,s} \leq \mathrm{const} \, \|q\|_{a,s} \, \|p\,\|_{a,s}
$$

with a constant depending only on s.

PROOF. Let $\gamma_{jk} := \big(|j-k|_*|k|_*/|j|_*\big)$. By the Schwarz inequality,

$$\left|\sum_{k\in\mathbb{Z}} x_k\right|^2 = \left|\sum_{k\in\mathbb{Z}} \frac{\gamma_{jk}^s x_k}{\gamma_{jk}^s}\right|^2 \le \left(\sum_{k\in\mathbb{Z}} \frac{1}{\gamma_{jk}^{2s}}\right)\left(\sum_{k\in\mathbb{Z}} \gamma_{jk}^{2s}|x_k|^2\right), \qquad \forall\, j\,.$$

On the other hand, from the definition,

$$\frac{1}{\gamma_{jk}} \le \frac{|j-k|_* + |k|_*}{|j-k|_*|k|_*} = \frac{1}{|j-k|_*} + \frac{1}{|k|_*}\,,$$

so that

$$\sum_{k\in\mathbb{Z}} \frac{1}{\gamma_{jk}^{2s}} \le \sum_{k\in\mathbb{Z}} \left(\frac{1}{|j-k|_*} + \frac{1}{|k|_*}\right)^{2s} \le 4^s \sum_{k\in\mathbb{Z}} \frac{1}{|k|_*^{2s}} := C^2 < \infty\,, \qquad \forall\, j\,.$$

In conclusion,

$$\left|\sum_{k\in\mathbb{Z}} x_k\right|^2 \le C^2 \sum_{k\in\mathbb{Z}} \gamma_{jk}^{2s}|x_k|^2\,. \tag{4.11}$$

Hence, for all $a = 0$, we obtain

$$\begin{aligned}
\|q*p\|_{a,s}^2 &= \sum_{j\in\mathbb{Z}} |j|_*^{2s} \left|\sum_{k\in\mathbb{Z}} q_{j-k}p_k\right|^2 \\
&\le C^2 \sum_{j\in\mathbb{Z}} |j|_*^{2s} \sum_{k\in\mathbb{Z}} \gamma_{jk}^{2s}|q_{j-k}p_k|^2 \\
&= C^2 \sum_{j,k\in\mathbb{Z}} |j-k|_*^{2s}|q_{j-k}|^2|k|_*^{2s}|p_k|^2 = C^2\|q\|_{a,s}^2\|p\|_{a,s}^2\,.
\end{aligned}$$

The case $a > 0$ is a simple variation of the last estimate.

□

Lemma 4.1.3. *For $a \ge 0$ and $s > 0$, the gradient $G_q := \left(\frac{\partial G}{\partial q_1}, \frac{\partial G}{\partial q_2}, \dots\right)$ is a real analytic map from a neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$. Moreover*

$$\|G_q\|_{a,s+1} = O(\|q^3\|_{a,s})\,.$$

PROOF. Let be $q \in \ell^{a,s}$. From (4.2), we have $u = \mathcal{S}q \in \ell^{a,s+1/2}$ on $[-\pi,\pi]$ with $\|u\|_{a,s+1/2} \le \|q\|_{a,s}$ for $a \ge 0$. By the algebra property and the analityticity of f, the function $f(u)$ also belongs to $\ell^{a,s+1/2}$ with

$$\|f(u)\|_{a,s+1/2} \le \text{const}\,\|u\|_{a,s+1/2}^3$$

in a sufficiently small neighborhood of the origin. From (4.2), we have the estimate

$$\|G_q\|_{a,s+1} \le \|f(u)\|_{a,s+1/2} \le \text{const}\,\|u\|_{a,s+1/2}^3 \le \text{const}\,\|q\|_{a,s}^3\,,$$

that is $G_q \in \ell^{a,s+1}$. The regularity of G_q follows from the regularity of its components and its local boundedness, as it is proved in the following two Lemmata, see [PT].

□

Lemma 4.1.4. *Let be E and F Banach complex spaces. Let be U an open subset of E. Let be $f : U \longrightarrow H$. Then are equivalent:*

(i) f is analytic on U, namely it is continuously differentiable on U;

(ii) f is locally bounded and weakly analytic on U, namely for each $x \in U$, $h \in E$ and $L \in F^*$, the function

$$z \longrightarrow Lf(x + zh)$$

is analytic in some neighborhood of the origin in $\mathbb{C}$ in the usual sense of one complex variable;

(iii) f is infinitely differentiable on U and is represented by the Taylor series in a neighborhood of each point in U.

PROOF. $(i) \Longrightarrow (ii)$ Suppose f is analytic. Hence it is a differentiable map and it implies that it is continuous and locally bounded. The map $Lf(x+zh)$ for all $L \in F^*$ is continuously differentiable in z. Hence, f is weakly analytic. $(ii) \Longrightarrow (i)$ Suppose f is weakly analytic and locally bounded. The idea is to show that f is continuous in order to use Cauchy's formula. Fix $x \in U$ and choose $r > 0$ small such that

$$\sup_{\|h\| \leq r} \|f(x + h)\| = M < \infty.$$

By a chain rule in the Cauchy's formula,

$$Lf(x + zh) - Lf(x) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{Lf(x + \xi h)}{(\xi - z)} \frac{z}{\xi} \, d\xi \qquad (4.12)$$

uniformly for $|z| < 1$, $\|h\| < r$. Hence, denoting by $\|L\|$ the norm operator of $L \in F^*$, for $|z| < 1/2$, we obtain from (4.12)

$$\frac{|Lf(x + zh) - Lf(x)|}{|z|} \leq 2M\|L\| \,,$$

for $|z| < 1$, $\|h\| < r$. It follows the continuity of f. Now we can apply the Cauchy's formula

$$f(x + zh) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(x + \xi h)}{\xi - z} \, d\xi$$

for $|z| < 1$ and $\|h\| < r$. It follows that f has directional derivatives in every direction h, namely

$$\delta_x(h) = \lim_{z \to 0} \frac{1}{z} \left[\frac{f(x + zh) - f(x)}{z} \right] .$$

Since

$$\left\| \frac{1}{z} \left[f(\xi + zh) - f(\xi) \right] - \delta_\xi(h) \right\| = \left\| \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(x+\xi h)}{\xi^2(\xi - z)} \, d\xi \right\| \le 2M|z|$$

for $|z| < 1/2$, the limit is uniform in $\|\xi - x\| < r/2$ and $\|h\| < r/2$, thus

$$\delta_x(h) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(x+\xi h)}{\xi^2} \, d\xi \,.$$

From this it follows that f is continuously differentiable on U, hence analytic.

□

Lemma 4.1.5. *Let be U an open subset of a Banach complex space E and H a Hilbert space with orthonormal basis e_n, $n \ge 1$. Let be $f : U \longrightarrow H$. Then f is an analytic map on U if and only if f is locally bounded and each "coordinate function"*

$$f_n = \langle f, e_n \rangle : U \longrightarrow \mathbb{C}$$

is analytic on U.

PROOF. Let be f locally bounded and f_n analytic on U. The idea is to show that f is weakly analytic to use the previous theorem. Let be $L \in H^*$. By the Riesz Theorem, there exists a unique element $\ell \in H$ such that $L\phi = \langle \phi, \ell \rangle$ for all $\phi \in H$. Write ℓ in the basis of H

$$\ell = \sum_{n \ge 1} \lambda_n e_n$$

and choose

$$\ell_m = \sum_{n=1}^{m} \lambda_n e_n \,.$$

In this way we have L as the limit of functionals operators L_m in the operator norm, namely

$$\sup_{\|\phi\| \le 1} \|(L - L_m)(\phi)\| \longrightarrow 0 \,, \qquad m \to \infty \,.$$

Given $x \in U$, let us choose $r > 0$ such that f is bounded in the ball centered in x of radius r. Fixed $h \in E$ such that $\|h\| < r$. By hypotheses, the functions

$$z \longmapsto L_m f(x + zh) = \sum_{n=1}^{m} \lambda_n f_n(x + zh) \,, \qquad m \ge 1 \,, |z| < 1$$

are analytic and goes uniformly to $z \longmapsto Lf(x + zh)$, since f is bounded. Thus f is analytic in $|z| < 1$. It follows that f is weakly analytic and locally bounded. By theorem (4.1.4) the thesis follows.

□

By the light of these considerations, we have the real analytic Hamiltonian H in (4.3), defined in some neighborhood of the origin in the Hilbert space $\ell^{a,s} \times \ell^{a,s}$ with standard symplectic structure $\sum_{i\geq 1} dp_i \wedge dq_i$. The parameters a and s may be fixed arbitrarily; in particular we take $a > 0$ and $s > 1$. The term G is independent of p, so the associated hamiltonian vectorfield,

$$X_G := \sum_{i\geq 1} \left(\frac{\partial G}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial G}{\partial q_i}\frac{\partial}{\partial p_i} \right),$$

is *smoothing* of order 1, that is it defines a real analytic map from $\ell^{a,s} \times \ell^{a,s}$ into $\ell^{a,s+1} \times \ell^{a,s+1}$. In particular, for the nonlinearity u^3 one finds

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4\, dx = \frac{1}{4}\sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l$$

with

$$G_{ijkl} = \frac{1}{\sqrt{\omega_i\omega_j\omega_k\omega_l}} \int_0^\pi \chi_i\chi_j\chi_k\chi_l \, dx\,.$$

In [P96a] it is proved that $G_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$, for some combination of plus and minus signs. In particular

$$G_{iijj} = \frac{1}{2\pi}\frac{2+\delta_{ij}}{\omega_i\omega_j}\,. \tag{4.13}$$

From now on, we focus our attention on the nonlinearity $f(u) = u^3$, since terms of order five or more do not make any difference.

4.1.1 Partial Birkhoff normal form

For the rest of this paper we introduce the complex coordinates

$$\mathrm{z}_i = \frac{1}{\sqrt{2}}(q_i + \mathrm{i}p_i)\,, \quad \bar{\mathrm{z}}_i = \frac{1}{\sqrt{2}}(q_i - \mathrm{i}p_i)\,, \tag{4.14}$$

that live in the now *complex* Hilbert space

$$\begin{aligned} \ell^{a,s} &= \ell^{a,s}(\mathbb{C}) \\ &:= \left\{ \mathrm{z} = (\mathrm{z}_1, \ldots),\ \mathrm{z}_i \in \mathbb{C}\,,\ i \geq 1 \ \text{ t.c. } \ \|\mathrm{z}\|^2_{a,s} = \sum_{i\geq 1} |\mathrm{z}_i|^2 i^{2s} e^{2ai} < \infty \right\}, \end{aligned}$$

with symplectic structure $-\mathrm{i}\sum_{i\geq 1} d\mathrm{z}_i \wedge d\bar{\mathrm{z}}_i = \sum_{i\geq 1} dp_i \wedge dq_i$. The Hamiltonian becomes

$$H = \Lambda + G = \sum_{i\geq 1} \omega_i |\mathrm{z}_i|^2 + G(\mathrm{z}, \bar{\mathrm{z}})\,, \tag{4.15}$$

where, with abuse of notation, we have still denoted by G the function $G(\mathrm{z}, \bar{\mathrm{z}}) = G(p, q)$. The Hamilton's equations write $\dot{\mathrm{z}} = -\mathrm{i}\partial_{\bar{\mathrm{z}}} H$, $\dot{\bar{\mathrm{z}}} = \mathrm{i}\partial_{\mathrm{z}} H$. The Hamiltonian H is real analytic. Real analytic means that H is a function of z and

$\bar{z}$, real analytic in the real and imaginary part of z; we denote by $A(\ell^{a,s}, \ell^{a,s+1})$ the class of all real analytic maps from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$.

Notation. Given a finite multi–index $\mathcal{I}$, we will denote by $\hat{z}$ the infinite vector obtained by excising $z = (z_1, z_2, \ldots)$ of its $\mathcal{I}$–components, namely $\hat{z} := (\ldots, z_{i_1-1}, z_{i_1+1}, \ldots, z_{i_j-1}, z_{i_j+1}, \ldots, z_{i_N-1}, z_{i_N+1}, \ldots) = (z_i)_{i\in\mathcal{I}^c}$, where $\mathcal{I}^c := \mathbb{N}^+ \setminus \mathcal{I}$. The symbol "$\vee$" will mean "or" in the sense of the Latin "vel". Fix $a > 0$, $s > 1$. We will denote by const > 0 and $0 < c_i < 1$, $i = 1, 2, \ldots$ suitable constants depending only on $\mathcal{I}, a, s$; moreover $y = O(x)$ means that $|y| \leq \mathrm{const}\, x$. In the following, we will often omit the explicit expressions for $\bar{z}$, since they can be derived by the analogous ones for z.

Next, following [P96a], we transform the Hamiltonian H in (4.15) into some partial Birkhoff normal form of order four so that, in a small neighborhood of the origin, it appears as a perturbation of a nonlinear integrable system. However, as we have just said in the introduction, the normal form degenerates when μ is close to zero, in the sense that its domain shrinks to zero and the remainder blows up. Then, we need a quantitative version of the Birkhoff normal form, in which we explicitly investigate the dependence on μ, for μ small. Such analysis is not available in literature.

Proposition 4.1.6 (Birkhoff normal form)**.** *Let be $0 < \mu < 1$, $\mathcal{I} \subset \mathbb{N}^+$. There exists a real analytic, close to the identity, symplectic change of coordinates $z := \Gamma(z_*)$ defined in $B_r \subset \ell^{a,s}$ into $B_{2r} \subset \ell^{a,s}$ with*

$$r := c_1\sqrt{\mu}\,, \tag{4.16}$$

verifying

$$\|z - z_*\|_{a,s+1} = O\left(\frac{\|z_*\|^3_{a,s}}{\mu}\right), \tag{4.17}$$

transforming the Hamiltonian $H = \Lambda + G$ in (4.15) *in seminormal form up to order six. That is*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K\,,$$

where

$$X_{\bar{G}}, X_{\hat{G}}, X_K \in A(\ell^{a,s}, \ell^{a,s+1}), \tag{4.18}$$

$$\bar{G} = \frac{1}{2}\sum_{i\vee j\,\in\mathcal{I}} \bar{G}_{ij}|z_{*i}|^2|z_{*j}|^2$$

with uniquely determined coefficients $\bar{G}_{ij} = (3/8\pi)\,(4 - \delta_{ij}/\omega_i\omega_j)$, and

$$|\hat{G}| = O(\|\hat{z}_*\|^4_{a,s})\,, \qquad |K| = O\left(\frac{\|z_*\|^6_{a,s}}{\mu}\right).$$

Remark 4.1.7. *It is worth pointing out that the Hamiltonian $\Lambda + \bar{G}$ is integrable with integrals $|\mathrm{z}_{*i}|^2$, $i = 1, 2, \ldots$. Moreover, although the fourth order term $\hat{G}$ is not integrable, it only depends on $\hat{\mathrm{z}}_* := (\mathrm{z}_{*i})_{i \in \mathcal{I}^c}$, namely it is independent of the $\mathcal{I}$–modes.*

PROOF. Let us introduce another set of coordinates $(\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots)$ in $\ell_b^{a,s}$ defined by $\mathrm{z}_{*i} = w_i$ and $\bar{\mathrm{z}}_{*i} = w_{-i}$. The Hamiltonian becomes

$$\begin{aligned} H &= \Lambda + G \\ &= \sum_{i \geq 1} \omega_i \, \mathrm{z}_{*i} \bar{\mathrm{z}}_{*i} + \frac{1}{16} \sum_{i,j,k,\ell} G_{ijk\ell} (\mathrm{z}_{*i} + \bar{\mathrm{z}}_{*i}) \cdots (\mathrm{z}_{*\ell} + \bar{\mathrm{z}}_{*\ell}) \\ &= \sum_{i \geq 1} \omega_i \, w_i w_{-i} + \frac{1}{16} {\sum_{i,j,k,\ell}}' G_{ijk\ell} \, w_i w_j w_k w_\ell \,, \end{aligned}$$

where the prime means that the summation is over all nonzero integers. The coefficients are defined for arbitrary integers by setting $G_{ijk\ell} = G_{|i|,|j|,|k|,|\ell|}$. We notice that $G_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$, for some combination of plus and minus signs. The transformation Γ is obtained as the time-1-map of the flow of the hamiltonian vectorfield X_F given by a Hamiltonian

$$F = {\sum_{i,j,k,\ell}}' F_{ijk\ell} \, w_i w_j w_k w_\ell \,, \tag{4.19}$$

with coefficients

$$\mathrm{i} F_{ijk\ell} = \begin{cases} \dfrac{G_{ijk\ell}}{16(\omega'_i + \omega'_j + \omega'_k + \omega'_\ell)} & \text{for } (i,j,k,\ell) \in \mathcal{L}_\mathcal{I} \setminus \mathcal{N}_\mathcal{I} \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.20}$$

Here $\omega'_i = \operatorname{sign} i \cdot \omega_{|i|}$,

$$\mathcal{L}_\mathcal{I} = \Big\{ (i,j,k,\ell) \in \mathbb{Z}^4 \quad \text{s.t.} \quad |i| \vee |j| \vee |k| \vee |\ell| \in \mathcal{I} \Big\} \,,$$

and $\mathcal{N}_\mathcal{I} \subset \mathcal{L}_\mathcal{I}$ is the subset of all $(i,j,k,\ell) = (p, -p, q, -q)$ or some permutation of it. For these indices the denominator $\omega'_i + \omega'_j + \omega'_k + \omega'_\ell$ vanishes identically in μ. The following Lemma on the small divisors is proved in [P96a]. We give the proof for completeness.

Lemma 4.1.8. *Let be i, j, k, ℓ non-zero integers, such that $i \pm j \pm k \pm l = 0$, but $(i,j,k,\ell) \neq (p, -p, q, -q)$, then*

$$|\omega'_i + \omega'_j + \omega'_k + \omega'_\ell| \geq \frac{\mathrm{const}\, \mu}{(M^2 + \mu)^{3/2}} \,, \quad M = \min(|i|, \ldots, |\ell|) \,,$$

with some absolute const > 0. *Hence, the denominators in* (4.20) *are uniformly bounded away from zero on every compact μ-interval on* $(0, \infty)$.

PROOF. We may restrict to positive integers such that $i \le j \le j \le k \le \ell$. The condition $i \pm j \pm k \pm l = 0$ then reduces to two possibilities, either $i - j - k + \ell = 0$ or $i + j + k - \ell = 0$. We have to study divisors of the form $\pm\omega_i \pm \omega_j \pm \omega_k \pm \omega_\ell$ for all combinations of plus and minus signs. To do this, we distinguish them according to their number of plus and minus signs. To simplify the notation, let us call $\delta_{++-+} = \omega_i + \omega_j - \omega_k + \omega_\ell$.

Case 0 : *No minus sign.* This is trivial.

Case 1 : *One minus sign.* We have the following terms: $\delta_{++-+}, \delta_{+-++}, \delta_{-+++} \ge \delta_{+++-}$, so it is enough to study the last one. Define $\delta := \delta_{+++-}$. Let us consider δ as a function of μ. It results

$$\delta(\mu) = \sqrt{i^2 + \mu} + \sqrt{j^2 + \mu} + \sqrt{k^2 + \mu} - \sqrt{\ell^2 + \mu}$$

and hence

$$\delta(0) = i + j + k - \ell \ge 0\,, \qquad \delta'(\mu) = \frac{1}{2}\left(\frac{1}{\omega_i} + \frac{1}{\omega_i} + \frac{1}{\omega_i} - \frac{1}{\omega_i}\right) \ge \frac{1}{\omega_i} > 0\,.$$

Being ω_i increasing with μ, it follows that

$$\delta \ge \delta'(\mu)\mu \ge \frac{\mu}{\sqrt{i^2 + \mu}}\,.$$

Case 2 : *Two minus signs.* We have the following terms: $\delta_{-+-+}, \delta_{--++} \ge \delta_{+--+}$, and all other cases reduce to these ones by inverting the signs. Thus, let us consider only the case $\delta = \delta_{+--+}$. The function $f(t) = \sqrt{t^2 + \mu}$ is monotone increasing and convex for $t \ge 0$. Hence, using the mean value theorem, we have the estimate $\omega_\ell - \omega_k \ge \omega_{\ell-p} - \omega_{k-p}$ for every $0 \le p \le k$. In the case $i + j + k = \ell$ we obtain $\omega_\ell - \omega_k \ge \omega_{\ell-k+i} - \omega_i = \omega_{j+2i} - \omega_i$, hence,

$$\delta \ge \omega_{j+2i} - \omega_j \ge 2(\omega_{j+1} - \omega_j) \ge 2if'(j) \ge \frac{i}{\sqrt{1 + \mu}}\,,$$

by the monotonicity of f'. In the other case $i - j - k = \ell$ we have $j - i = \ell - k$, thus we obtain $\omega_\ell - \omega_k \ge \omega_{j+1} - \omega_{i+1}$ and $\omega_{j+1} - \omega_j \ge \omega_{i+2} - \omega_{i+1}$. Finally we get the estimate

$$\delta \ge \omega_{j+1} - \omega_{i+1} - \omega_j + \omega_i \ge \omega_{i+2} - 2\omega_{i+1} + \omega_i \ge f''(i+2)$$

by the monotonicity of f'', and

$$\delta \ge \left.\frac{\mu}{\sqrt{t^2 + \mu}^{\,3}}\right|_{i+2} \ge \frac{\mathrm{const}\,\mu}{\sqrt{i^2 + \mu}^{\,3}}\,.$$

Case 3 : *Three and four minus signs.* These ones reduce to the case 1 and 0 respectively.

$\square$

We note that F is real. Indeed

$$\begin{aligned}
\bar{F} &= \sum_{i,j,k,\ell}{}' \overline{F}_{ijk\ell}\, \bar{w}_i \bar{w}_j \bar{w}_k \bar{w}_\ell \\
&= \sum_{i,j,k,\ell}{}' \mathrm{i} \frac{G_{ijk\ell}}{\omega'_i + \omega'_j + \omega'_k + \omega'_\ell}\, w_{-i} w_{-j} w_{-k} w_{-\ell} \\
&= -\sum_{i,j,k,\ell}{}' \mathrm{i} \frac{G_{ijk\ell}}{\omega'_i + \omega'_j + \omega'_k + \omega'_\ell}\, w_i w_j w_k w_\ell = F
\end{aligned}$$

where we used that $G_{ijk\ell} = G_{-i,-j,-k,-\ell} = G_{|i|,|j|,|k|,|\ell|}$ and $\omega'_{-i} = -\omega'_i$. Expanding at $t = 0$ with the Taylor's formula we obtain

$$\begin{aligned}
H \circ \Gamma &= H \circ X_F^t \big|_{t=1} \\
&= H + \{H, F\} + \int_0^1 (1-t) \big\{ \{H, F\}, F \big\} \circ X_F^t \, dt \\
&= \Lambda + G + \{\Lambda, F\} + \{G, F\} + \int_0^1 (1-t) \big\{ \{H, F\}, F \big\} \circ X_F^t \, dt \,,
\end{aligned}$$

where $\{\cdot, \cdot\}$ denote the Poisson brackets. We can compute

$$\{\Lambda, F\} = -\mathrm{i} \sum_{i,j,k,\ell}{}' (\omega'_i + \omega'_j + \omega'_k + \omega'_\ell) F_{ijk\ell}\, w_i w_j w_k w_\ell \,,$$

thus

$$G + \{\Lambda, F\} = \frac{1}{16} \left(\sum_{(i,j,k,\ell) \in \mathcal{N}_\mathcal{I}} + \sum_{(i,j,k,\ell) \notin \mathcal{L}_\mathcal{I}} G_{ijk\ell}\, w_i w_j w_k w_\ell \right) = \bar{G} + \hat{G} \,,$$

where $\hat{G}$ is independent of the $\mathcal{I}$–coordinates.
In the variables z_*,$\bar{\mathrm{z}}_*$ we find, from (4.13) and counting the multiplicities, that

$$\bar{G} = \frac{1}{2} \sum_{i \vee j \in \mathcal{I}} \bar{G}_{ij} |\mathrm{z}_{*i}|^2 |\mathrm{z}_{*j}|^2$$

with uniquely determined coefficients

$$\bar{G}_{ij} = \begin{cases} 24 G_{iijj} = \dfrac{3}{2\pi} \dfrac{1}{\omega_i \omega_j} & \text{for } i \neq j \,, \\ 12 G_{iiii} = \dfrac{9}{8\pi} \dfrac{1}{\omega_i \omega_j} & \text{for } i = j \,. \end{cases} \tag{4.21}$$

Hence, we have $H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$ where

$$K = \{G, F\} + \int_0^1 (1-t) \big\{ \{H, F\}, F \big\} \circ X_F^t \, dt \tag{4.22}$$

is composed by all the terms of order six or more.

CLAIM. The vectorfield of the Hamiltonian F is analytic, that is

$$X_F \in \mathcal{A}(\ell_b^{a,s}, \ell_b^{a,s+1}) . \tag{4.23}$$

In fact, from lemma (4.1.8), it results

$$\begin{aligned}
\left|\frac{\partial F}{\partial w_\ell}\right| &\leq \sum\nolimits'_{\pm i \pm j \pm k = l} |F_{ijk\ell}|\,|w_i w_j w_k| \\
&\leq \frac{\text{const}}{\mu\sqrt{\ell}} \sum\nolimits'_{\pm i \pm j \pm k = l} \frac{|w_i w_j w_k|}{\sqrt{|ijk|}} \\
&\leq \frac{\text{const}}{\mu\sqrt{\ell}} \sum\nolimits'_{i+j+k=l} \tilde{w}_i \tilde{w}_j \tilde{w}_k = \frac{\text{const}}{\mu\sqrt{\ell}} (\tilde{w} * \tilde{w} * \tilde{w})_\ell \,,
\end{aligned}$$

where $\tilde{w}_i = \dfrac{|w_i| + |w_{-i}|}{\sqrt{|i|}}$. If $w \in \ell_b^{a,s}$ then $\tilde{w} \in \ell_b^{a,s+1/2}$, which is a Hilbert algebra for $s > 0$ by the lemma (4.1.2), thus $\tilde{w} * \tilde{w} * \tilde{w}$ also belongs to $\ell_b^{a,s+1/2}$. Therefore $F_w \in \ell_b^{a,s+1}$ with

$$\|F_w\|_{a,s+1} \leq \frac{\text{const}}{\mu} \|\tilde{w}*\tilde{w}*\tilde{w}\|_{a,s+1/2} \leq \frac{\text{const}}{\mu} \|\tilde{w}\|^3_{a,s+1/2} \leq \frac{\text{const}}{\mu} \|w\|^3_{a,s} . \tag{4.24}$$

The analyticity of F_w follows from the analyticity of each components function and its local boundedness, proving (4.23). From (4.23) and (4.24) it follows that the hamiltonian flow X_F^t is well defined, in a sufficiently small neighborhood of the origin in $\ell^{a,s}$, for all $0 \leq t \leq 1$; in particular, by (4.24), for $\frac{\|w\|_{a,s}}{2} = \|\mathrm{z}_*\|_{a,s} \leq r$ the map $\Gamma := X_F^1$ verifies

$$\|\Gamma(\mathrm{z}_*) - \mathrm{z}_*\|_{a,s+1} \leq \frac{\text{const}}{\mu} \|\mathrm{z}_*\|^3_{a,s} \leq \text{const}\, c_1^3 \sqrt{\mu} \leq c_1 \sqrt{\mu} = r \tag{4.25}$$

taking c_1 small enough in (4.16). In the same way (taking c_1 small enough),

$$\|D\Gamma - \mathbb{I}\|^{\text{op}}_{a,s+1,s} \leq \frac{\text{const}}{\mu} r^2 = \text{const}\, c_1^2 \leq \frac{1}{2} ,$$

where the operator norm $\|\cdot\|^{\text{op}}_{a,r,s}$, is defined by

$$\|\cdot\|^{\text{op}}_{a,r,s} = \sup_{w \neq 0} \frac{\|Aw\|_{a,r}}{\|w\|_{a,s}} .$$

Accordingly, $\Gamma : \ell^{a,s} \supset B_r \to B_{2r} \subset \ell^{a,s}$ is a real analytic, symplectic change of coordinates and (4.17) follows from (4.25); moreover, since $\|D\Gamma - \mathbb{I}\|^{\text{op}}_{a,s+1,s+1} \leq \|D\Gamma - \mathbb{I}\|^{\text{op}}_{a,s+1,s}$, $D\Gamma$ defines an isomorphism of $B_r \subset \ell^{a,s+1}$. It follows that with $X_H \in \mathcal{A}(\ell^{a,s}, \ell^{a,s+1})$, also

$$D\Gamma^{-1} X_H \circ \Gamma = X_{H\circ\Gamma} \in \mathcal{A}(\ell^{a,s}, \ell^{a,s+1}) .$$

The same holds for the Lie bracket: the boundedness of $\|DX_F\|^{\text{op}}_{a,s+1,s}$ implies that

$$[X_F, X_H] = X_{\{H,F\}} \in \mathcal{A}(\ell^{a,s}, \ell^{a,s+1}) .$$

These two facts show that $X_K \in \mathcal{A}(\ell^{a,s}, \ell^{a,s+1})$. The analogue claims for $X_{\bar{G}}$ and $X_{\hat{G}}$ are obvious.
Finally, recalling (4.22), we can write

$$K = \{G, F\} + \int_0^1 (1-t)\big[\big\{\{\Lambda, F\}, F\big\} + \big\{\{G, F\}, F\big\}\big] \circ X_F^t \, dt . \tag{4.26}$$

It results

$$\{G, F\} = O\left(\frac{\|w\|^6_{a,s}}{\mu}\right) ,$$

and

$$\big\{\{\Lambda, F\}, F\big\} = O\left(\frac{\|w\|^6_{a,s}}{\mu}\right) , \tag{4.27}$$

since $\{\Lambda, F\} = \bar{G} + \hat{G} - G = O(\|w\|^4_{a,s})$. Moreover

$$\big\{\{G, F\}, F\big\} = O\left(\frac{\|w\|^8_{a,s}}{\mu^2}\right) , \tag{4.28}$$

hence, by (4.26)-(4.28),

$$|K| = O\left(\frac{\|w\|^6_{a,s}}{\mu}\right) , \qquad \text{for} \quad \|w\|_{a,s} \leq \text{const}\,\sqrt{\mu} .$$

□

Since we are looking for small amplitude solutions it is convenient to introduce the small parameter $0 < \eta < 1$ and perform the following rescaling

$$\mathrm{z}_* =: \eta z , \quad \bar{\mathrm{z}}_* =: \eta \bar{z} , \qquad H \circ \Gamma \longrightarrow \eta^{-2}\big(H \circ \Gamma\big) =: \mathcal{H} , \tag{4.29}$$

by which the Hamiltonian reads

$$\mathcal{H}(z, \bar{z}; \eta) = \Lambda + \eta^2(\bar{G} + \hat{G}) + \eta^4 \widetilde{K}(z, \bar{z}; \eta) , \qquad \|z\|_{a,s}, \|\bar{z}\|_{a,s} \leq c_1 \frac{\sqrt{\mu}}{\eta} , \tag{4.30}$$

where

$$\widetilde{K}(z, \bar{z}; \eta) := \eta^{-2} K(\eta z, \eta \bar{z}) , \qquad |\widetilde{K}| = O\left(\frac{\|z\|^6_{a,s}}{\mu}\right) . \tag{4.31}$$

We now introduce action–angle variables $(I, \phi) \in \mathbb{R}^N_+ \times \mathbb{T}^N$ on the $\mathcal{I}$–modes by the following symplectic change of variables

$$z_i := \sqrt{I_i}(\cos\phi_i - \mathrm{i}\sin\phi_i) , \qquad \bar{z}_i := \sqrt{I_i}(\cos\phi_i + \mathrm{i}\sin\phi_i) , \qquad i \in \mathcal{I} . \tag{4.32}$$

The action $I := (I_i)_{i\in\mathcal{I}}$, $I_i := z_i\bar z_i$, is defined for

$$|I| \le c_2 \frac{\mu}{\eta^2}\,. \tag{4.33}$$

We note that $\sum_{i\in\mathcal{I}} dI_i \wedge d\phi_i = -\mathrm{i}\sum_{i\in\mathcal{I}} dz_i\wedge d\bar z_i = \sum_{i\in\mathcal{I}} dp_i\wedge dq_i$ and the *phase space* is[1]

$$\mathcal{P}_{a,s} := \mathbb{R}^N_+ \times \mathbb{T}^N \times \ell^{a,s} \ni (I,\phi,\hat z)\,. \tag{4.34}$$

In these variables the Hamiltonian becomes

$$\begin{aligned}\widetilde{\mathcal{H}} &= \widetilde{\mathcal{H}}(I,\phi,\hat z,\bar{\hat z};\eta) \\ &= \omega\cdot I + \Omega\cdot \hat z\bar{\hat z} + \eta^2\left[\frac{1}{2}(AI,I) + (BI,\hat z\bar{\hat z}) + \hat G(\hat z,\bar{\hat z})\right] + \eta^4 \widetilde K(I,\phi,\hat z,\bar{\hat z};\eta)\,,\end{aligned} \tag{4.35}$$

where

$$\omega := (\omega_{i_1},\ldots,\omega_{i_N})\,,$$

and

$$\Omega := (\ldots,\omega_{i_1-1},\omega_{i_1+1},\ldots,\omega_{i_j-1},\omega_{i_j+1},\ldots,\omega_{i_N-1},\omega_{i_N+1},\ldots)\,,$$

$\Omega\cdot\hat z\bar{\hat z}$ is short for $\sum_{i\in\mathcal{I}^c}\omega_i\hat z_i\bar{\hat z}_i$, A is the $N\times N$ matrix

$$A = A_{\mathcal{I}} := \big(\bar G_{ij}\big)_{i,j\in\mathcal{I}}$$

and B is the $\infty\times N$ matrix

$$B = B_{\mathcal{I}} := \big(\bar G_{ij}\big)_{i\in\mathcal{I}^c,\,j\in\mathcal{I}}\,.$$

Moreover $(\cdot,\cdot)$ denotes the standard scalar product and we have denoted again by $\widetilde K$ the function $\widetilde K(I,\phi,\hat z,\bar{\hat z};\eta) = \widetilde K(z,\bar z;\eta)$. Recalling (4.21), we have

$$A = \frac{3}{8\pi}\begin{pmatrix} \frac{3}{\omega_{i_1}^2} & \frac{4}{\omega_{i_1}\omega_{i_2}} & \cdots & \frac{4}{\omega_{i_1}\omega_{i_N}} \\ \frac{4}{\omega_{i_2}\omega_{i_1}} & \frac{3}{\omega_{i_2}^2} & \cdots & \frac{4}{\omega_{i_2}\omega_{i_N}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{4}{\omega_{i_N}\omega_{i_1}} & \frac{4}{\omega_{i_N}\omega_{i_2}} & \cdots & \frac{3}{\omega_{i_N}^2}\end{pmatrix}\,,$$

$$B = \frac{3}{8\pi}\begin{pmatrix} \vdots & \vdots & \vdots \\ \frac{4}{\omega_{i_j-1}\omega_{i_1}} & \cdots & \frac{4}{\omega_{i_j-1}\omega_{i_N}} \\ \frac{4}{\omega_{i_j+1}\omega_{i_1}} & \cdots & \frac{4}{\omega_{i_j+1}\omega_{i_N}} \\ \vdots & \vdots & \vdots \end{pmatrix}\,.$$

Defining the matrices

$$D = D_{\mathcal{I}} := \mathrm{diag}\,[\omega] \in \mathrm{Mat}_{N\times N}\,, \qquad E = E_{\mathcal{I}} := \mathrm{diag}\,[\Omega] \in \mathrm{Mat}_{\infty\times\infty}\,, \tag{4.36}$$

[1] Clearly here $\ell^{a,s} = \ell^{a,s}(\mathbb{C})$.

we can rewrite A and B as

$$A = \frac{3}{8\pi}\, D^{-1}\tilde{A}D^{-1}\,, \qquad B = \frac{3}{2\pi}\, E^{-1}\tilde{B}D^{-1}\,, \tag{4.37}$$

where

$$\tilde{A} := \begin{pmatrix} 3 & 4 & \dots & 4 \\ 4 & 3 & \dots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & \dots & 3 \end{pmatrix} \in \mathrm{Mat}_{N\times N}\,, \qquad \tilde{B} = \begin{pmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ \vdots & \vdots & \vdots \end{pmatrix} \in \mathrm{Mat}_{\infty\times\infty}\,. \tag{4.38}$$

We note that the matrix A is invertible, since

$$\det \tilde{A} =: d_N = 3d_{N-1} + (-1)^N(1-N)4^2 = (-1)^N(1-4N) \neq 0 \qquad (d_N \text{ is odd}). \tag{4.39}$$

Moreover

$$\tilde{A}^{-1} = \frac{1}{4N-1}\begin{pmatrix} 5-4N & 4 & \dots & 4 \\ 4 & 5-4N & \dots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & \dots & 5-4N \end{pmatrix}. \tag{4.40}$$

4.2 Long-period orbits

Let us consider the Hamiltonian $\omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}}$. Such Hamiltonian is linear in the action variables I and $\hat{z}_i\bar{\hat{z}}_i$, $i \in \mathcal{I}^c$. Except a countable set of $\mu > 0$, it has no periodic solutions of the form $\hat{z}_i(t) \equiv \bar{\hat{z}}_i(t) \equiv 0$, $i \in \mathcal{I}^c$ and $I(t) \not\equiv 0$. Indeed, the following Lemma holds

Lemma 4.2.1. *Except a countable set of $\mu > 0$, for any $\mathcal{I} = \{i_1 < \ldots < i_N\} \subset \mathbb{N}^+$, $N \geq 1$, the vector $\omega = (\omega_{i_1}, \ldots, \omega_{i_N})$, $\omega_i = \sqrt{i^2+\mu}$, is rationally independent.*

PROOF. For any $n \in \mathbb{Z}^N \setminus \{0\}$ let us define $E_n := \big\{\mu > 0 \quad \text{s.t.} \quad \omega \cdot n = 0\big\}$. We claim that E_n is at the most countable. Indeed, for $\mu > -1$, let us consider the analytic function

$$f_n(\mu) := \sum_{j=1}^{N} \sqrt{i_j^2 + \mu} \cdot n_j = \omega \cdot n\,.$$

It is enough to show that f_n is not identically zero, so that the set of its zeros is at the most countable. Suppose, by contradiction, that $f_n(\mu) \equiv 0$ for any $\mu > -1$, then $d^k f_n/d\mu^k(0) = 0$, for any $k \geq 1$, and therefore $\sum_{j=1}^N n_j\, i_j^{1-2k} = 0$, for any $k \geq 1$. Hence, multiplying for i_1^{2k-1}, we have

$$n_1 + \sum_{j=2}^{N} n_j \left(\frac{i_1}{i_j}\right)^{2k-1} = 0\,, \qquad \forall\, k \geq 1\,.$$

Noting that $i_j > i_1$, for any $j \geq 2$, and taking the limit for $k \to \infty$, we get $n_1 = 0$. In the same way one can prove that $n_2 = 0$ and, by induction, that $n_1 = \ldots = n_N = 0$.

$\square$

4.2.1 Geometrical construction

The absence of periodic solutions of the linear Hamiltonian $\omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}}$ leads us to find periodic solutions of the Hamiltonian $\widetilde{\mathcal{H}}$ in (4.35) close to the ones of the quadratic and integrable Hamiltonian

$$\widetilde{\mathcal{H}}_{int} = \omega \cdot I + \Omega \cdot \hat{z}\bar{\hat{z}} + \eta^2 \left[\frac{1}{2}(AI, I) + (BI, \hat{z}\bar{\hat{z}}) \right], \tag{4.41}$$

in which $\hat{G}$ and $\widetilde{K}$ have been neglected. The advantage of considering $\widetilde{\mathcal{H}}_{int}$ consists in the presence of the "twist" term $\eta^2 \frac{1}{2}(AI, I)$, which enables us to "modulate the frequency". The equations of motion for $\widetilde{\mathcal{H}}_{int}$

$$\begin{cases} \dot{I} = 0 \\ \dot{\phi} = \omega + \eta^2 AI + \eta^2 B^t \hat{z}\bar{\hat{z}} \\ \dot{\hat{z}}_i = -\mathrm{i}(\Omega + \eta^2 BI)_i \, \hat{z}_i \qquad i \in \mathcal{I}^c \, , \end{cases} \tag{4.42}$$

can be easily integrated:

$$\begin{cases} I(t) = I_0 \\ \phi(t) = \phi_0 + \tilde{\omega} t + \eta^2 B^t \hat{z}_0 \bar{\hat{z}}_0 t \\ \hat{z}_i(t) = e^{-\mathrm{i}\tilde{\Omega}_i t} \, (\hat{z}_0)_i \, , \qquad i \in \mathcal{I}^c \, , \end{cases} \tag{4.43}$$

where

$$\tilde{\omega} := \tilde{\omega}(I_0, \eta) = \omega + \eta^2 A I_0 \tag{4.44}$$

is the vector of the *shifted linear frequencies* and

$$\tilde{\Omega}_i := \tilde{\Omega}_i(I_0, \eta) = \Omega_i + \eta^2 (BI_0)_i = \omega_i + \eta^2 (BI_0)_i \, , \qquad i \in \mathcal{I}^c \, , \tag{4.45}$$

are the *shifted elliptic frequencies*. Consequently $\{\hat{z} = 0\}$ is, for (4.41), an invariant manifold which is completely foliated by the N-dimensional invariant tori

$$\mathcal{T}(I_0) := \left\{ I = I_0, \, \phi \in \mathbb{T}^N, \, \hat{z} = 0 \right\} .$$

On $\mathcal{T}(I_0)$ the flow

$$t \longmapsto (I_0, \phi_0 + \tilde{\omega} t, 0)$$

is T-periodic, $T > 0$, if and only if

$$\tilde{\omega}(I_0, \eta)\tau =: k \in \mathbb{Z}^N \, , \tag{4.46}$$

where

$$\tau := \frac{T}{2\pi}$$

is the rescaled period. Hence, if (4.46) holds, the torus $\mathcal{T}(I_0)$ is completely resonant and supports the infinitely many T-periodic orbits of the family

$$\mathcal{F} := \big\{ I(t) = I_0, \quad \phi(t) = \phi_0 + \tilde{\omega} t, \quad \hat{z}(t) = 0 \big\} . \tag{4.47}$$

The family $\mathcal{F}$ will not persist in its entirety for the Hamiltonian $\widetilde{\mathcal{H}}$. However, we claim that if the period T is "sufficiently non-resonant" with the shifted elliptic frequencies, we can prove the persistence of at least N geometrically distinct T-periodic solutions of $\widetilde{\mathcal{H}}$ close to $\mathcal{F}$. More precisely, the required non-resonance condition is

$$\Big| \ell - \tilde{\Omega}_i(I_0, \eta)\tau \Big| \geq \frac{\text{const}}{i} \qquad \forall\, \ell \in \mathbb{Z}, \ \forall\, i \in \mathcal{I}^c . \tag{4.48}$$

We now consider the periodicity condition (4.46). As we have said in the introduction, we construct a set of actions and of completely resonant frequencies, parametrized by the (rescaled) periods τ's. Such actions and frequencies are related by the action–to–frequency map $I \to (\partial_I \widetilde{\mathcal{H}}_{int})_{|\hat{z}=0} = \omega + \eta^2 A I$. Indeed, since A is invertible we can choose I_0 and k as functions of τ and η so that (4.46) is always satisfied

$$I_0 \ := \ \frac{1}{\eta^2 \tau} A^{-1} \left(\kappa - \{\omega\tau\} \right) , \tag{4.49}$$
$$k \ := \ [\omega\tau] + \kappa , \tag{4.50}$$

where $[(x_1, \ldots, x_N)] := ([x_1], \ldots, [x_N])$, $\{(x_1, \ldots, x_N)\} := (\{x_1\}, \ldots, \{x_N\})$ and

$$\kappa := (\kappa_i)_{i \in \mathcal{I}} \in \mathbb{Z}^N , \qquad \kappa_i := i^{-1} \tilde{\kappa} , \qquad \tilde{\kappa} := 10 \prod_{j \in \mathcal{I}} j . \tag{4.51}$$

Here the functions $[\cdot] : \mathbb{R} \to \mathbb{Z}$ and $\{\cdot\} : \mathbb{R} \to [0, 1)$ denote the integer part and the fractional part respectively.
In order to have $I_0 \approx 1$ in (4.49), we choose the parameter η, which is related to the amplitude of the solution, as a function of the rescaled period τ such that

$$\eta^2 \tau = 1 \qquad \text{namely} \qquad \eta := 1/\sqrt{\tau} . \tag{4.52}$$

Consequently we can express I_0, k and $\tilde{\Omega}_i$ in (4.49),(4.50),(4.45) as functions of τ only

$$I_0 \ := \ I_0(\tau) = A^{-1} \left(\kappa - \{\omega\tau\} \right) \tag{4.53}$$
$$k \ := \ k(\tau) = [\omega\tau] + \kappa , \tag{4.54}$$
$$\tilde{\Omega}_i \ := \ \tilde{\Omega}_i(\tau) = \omega_i + \eta^2 \big(B I_0(\tau) \big)_i , \qquad \text{for} \quad i \in \mathcal{I}^c . \tag{4.55}$$

We point out that the constant vector κ defined in (4.51) has been added to have $(I_0)_i > 0$ in view of (4.34). In particular the following lemma holds.

Lemma 4.2.2. *If μ is small enough, then $(I_0)_i > \pi\omega_i$, $\forall\, i \in \mathcal{I}$.*

PROOF. By (4.37) and (4.53) we get

$$I_0 = \frac{8\pi}{3} D\tilde{A}^{-1} D\big(\kappa - \{\omega t\}\big)\,.$$

Recalling (4.36) and (4.40) we have, $\forall\, i \in \mathcal{I}$,

$$\begin{aligned}
(I_0)_i &= \frac{8\pi\omega_i}{3(4N-1)} \left((1-4N)\big(\omega_i\kappa_i - \omega_i\{\omega_i t\}\big) + 4\sum_{j\in\mathcal{I}} \big(\omega_j\kappa_j - \omega_j\{\omega_j t\}\big) \right) \\
&= \frac{8\pi\omega_i}{3(4N-1)} \left((1-4N)\big(\tilde{\kappa} - i\{\omega_i t\}\big) + 4\sum_{j\in\mathcal{I}} \big(\tilde{\kappa} - j\{\omega_j t\}\big) + O(\mu) \right)
\end{aligned}$$

since, recalling (4.51), $\omega_i\kappa_i = i^{-1}\omega_i\tilde{\kappa}$ and $1 < i^{-1}\omega_i = i^{-1}\sqrt{i^2+\mu} < 1+\mu$. Taking μ small enough we get

$$\begin{aligned}
(I_0)_i &= \frac{8\pi\omega_i}{3(4N-1)} \left(\tilde{\kappa} + (4N-1)i\{\omega_i t\} - 4\sum_{j\in\mathcal{I}} j\{\omega_j t\} + O(\mu) \right) \\
&\geq \frac{8\pi\omega_i}{3(4N-1)} \left(\tilde{\kappa} - 4\sum_{j\in\mathcal{I}} j + O(\mu) \right) \\
&= \frac{8\pi\omega_i}{3(4N-1)} \left(10\prod_{j\in\mathcal{I}} j - 4\sum_{j\in\mathcal{I}} j + O(\mu) \right) \\
&\geq \frac{8\pi\omega_i}{3(4N-1)} \left(2\prod_{j\in\mathcal{I}} j + O(\mu) \right) \\
&\geq \frac{8\pi\omega_i}{3(4N-1)} \big(2N + O(\mu)\big) \\
&\geq \frac{8\pi\omega_i}{3(4N-1)} \frac{3N}{2} > \pi\omega_i\,,
\end{aligned}$$

where we used that $2\prod_{j\in\mathcal{I}} j \geq \sum_{j\in\mathcal{I}} j$ and $\prod_{j\in\mathcal{I}} j \geq N$. $\square$

We note that by the choice of η made in (4.52), it results that $|I_0| \leq$ const. Then I_0 belongs to the domain of definition of the Hamiltonian $\widetilde{\mathcal{H}}$, namely it verifies (4.33), making the hypothesis

$$\mu\tau \geq c_3^{-1}\,. \tag{4.56}$$

We finally remark that, by (4.52) and (4.55), the quantities that we have to estimate in the crucial non–resonance condition (4.48) write

$$\ell - \tilde{\Omega}_i\tau = \ell - \tau\omega_i - \Big(BA^{-1}\big(\kappa - \{\omega\tau\}\big)\Big)_i\,, \qquad \ell \in \mathbb{Z}\,, i \in \mathcal{I}^c\,. \tag{4.57}$$

In this form, (4.48) clearly appears as a non resonance condition between the frequency of the torus ω and the normal ones $\{\omega_i\}_{i\in\mathcal{I}^c}$.

4.2.2 Small divisors estimate

In order to estimate the quantities in (4.57) we will perform the expansion $\tau\omega_i = \tau\sqrt{i^2+\mu} = i\tau + \frac{\mu\tau}{2i} + O(\mu^2\tau)$ requiring that $\mu^2\tau$ is small, namely

$$\mu^2\tau \le c_4 \, .$$

The other aspect of such request is that, for any fixed μ, we only have a finite number of (rescaled) periods τ. Moreover, we note that the smallness of $\mu^2\tau$ implies that of μ since $\mu \le c_3\mu^2\tau$ by (4.56). Hence, since for μ close to zero the Birkhoff Normal Form degenerates, it is not obvious that we can make $\mu^2\tau$ small for some fixed $\tau \ge 1$. Moreover, even if it is not necessary, we limit for simplicity to consider $\tau \sim \mu^{-2}$, namely $c_4/2\mu^2 \le \tau \le c_4/\mu^2$. Again for technical reasons, we will need that τ is an *integer* and that $\mu\tau$ is far away from even integers. Let us define

$$\mathcal{T}_\mu := \left\{ \tau \in \mathbb{N}^+ \, , \quad \frac{c_4}{2\mu^2} \le \tau \le \frac{c_4}{\mu^2} \, , \quad \text{s.t.} \quad \mu\tau \in \mathcal{N} \right\} \, , \tag{4.58}$$

where

$$\mathcal{N} := \left\{ x > 0 \;\; \text{s.t.} \;\; |x - 2m| \ge \frac{1}{2} \, , \quad \forall\, m \in \mathbb{Z} \right\} = \bigcup_{n>0 \text{ odd}} \left[n - \frac{1}{2}, n + \frac{1}{2} \right] \, . \tag{4.59}$$

The constant c_4 will be choose suitably small in the following:

Proposition 4.2.3. *If μ is small enough and $\tau \in \mathcal{T}_\mu$, then the following estimate holds*

$$\left|\ell - \tilde{\Omega}_i \tau\right| \ge \frac{c_5}{i} \qquad\qquad \forall\, \ell \in \mathbb{Z} \, , \; i \in \mathcal{I}^c \, . \tag{4.60}$$

PROOF. We will prove that

$$\left|\ell - \tau\sqrt{i^2+\mu} - \big(BI_0(\tau)\big)_i\right| \ge \frac{1}{6(4N-1)\, i} \qquad \forall\, \ell \in \mathbb{Z} \, , i \in \mathcal{I}^c \, . \tag{4.61}$$

Recalling (4.57) the crucial estimate (4.60) follows from (4.61) taking $c_5 := \frac{1}{6(4N-1)}$. We first consider the term $BI_0(\tau)$. From (4.37) we have $BA^{-1} = 4(E^{-1}\tilde{B}\tilde{A}^{-1}D)$, while, by (4.40), $\tilde{B}\tilde{A}^{-1} = d^{-1}\tilde{B}$ where $d := 4N-1$. Recalling (4.53), we get

$$BI_0(\tau) = BA^{-1}\big(\kappa - \{\omega\tau\}\big) = 4d^{-1}\, E^{-1}\tilde{B}D\big(\kappa - \{\omega\tau\}\big)$$

and, in particular, for $i \in \mathcal{I}^c$,

$$\begin{aligned} (BI_0)_i &= \frac{4}{d}\Big(E^{-1}\tilde{B}D\big(\kappa - \{\omega\tau\}\big)\Big)_i = \frac{4}{d\omega_i}\Big(\tilde{B}D\big(\kappa - \{\omega\tau\}\big)\Big)_i \\ &= \frac{4}{d\omega_i}\left(\hat{\kappa} - \sum_{h\in\mathcal{I}} \omega_h\{\omega_h\tau\}\right) \, , \end{aligned} \tag{4.62}$$

where

$$\hat{\kappa} := \left(\tilde{B}D\kappa\right)_i = \left(\tilde{B}\begin{pmatrix} \vdots \\ \dfrac{\omega_h}{h}\,\tilde{\kappa} \\ \vdots \end{pmatrix}\right)_i = \sum_{h\in\mathcal{I}} \frac{\omega_h}{h}\,\tilde{\kappa} = \tilde{\kappa}N + O(\mu)\,. \tag{4.63}$$

Now we need the following lemma proved on page 122.

Lemma 4.2.4. *Let* $0 < \mu < 1$*. Then* $\forall\, h \in \mathbb{N}^+$ *there exist* $\delta_h^{(k)} > 0$, $k = 1, 2, 3$, *such that*

$$\begin{aligned} \omega_h &= h + \delta_h^{(1)}\,, & \delta_h^{(1)} &< \frac{\mu}{2h} \qquad (4.64)\\ &= h + \frac{\mu}{2h} - \delta_h^{(2)}\,, & \delta_h^{(2)} &< \frac{\mu^2}{8h^3}\,. \qquad (4.65) \end{aligned}$$

Moreover, if $\tau \in \mathbb{N}^+$*, there exists* $n_h := n_h(\mu,\tau) \in \mathbb{Z}$ *such that*

$$\left|\omega_h\{\omega_h\tau\} - \frac{\mu\tau}{2} - n_h\right| \le \frac{\mu}{2h} + \frac{\mu^2\tau}{8h^2}\,, \qquad \forall\, h \in \mathbb{N}^+\,. \tag{4.66}$$

By the elementary inequality $0 < 1 - (1+x)^{-1} < x$, $\forall\, x > 0$, we get, using (4.64),

$$0 < \frac{1}{i} - \frac{1}{\omega_i} = \frac{1}{i}\left(1 - \frac{1}{1 + \delta_i^{(1)}/i}\right) < \frac{\delta_i^{(1)}}{i^2} < \frac{\mu}{2i^3}\,,$$

namely

$$\frac{1}{\omega_i} - \frac{1}{i} = O\left(\frac{\mu}{i^3}\right)\,. \tag{4.67}$$

Since $|\{\omega_h\tau\}| \le 1/2$, substituting (4.67) in (4.62), we get

$$(BI_0)_i = \frac{4}{di}\left(\hat{\kappa} - \sum_{h\in\mathcal{I}} \omega_h\{\omega_h\tau\}\right) + O\left(\frac{\mu}{i^3}\right)$$

hence, by (4.66) and (4.63)

$$(BI_0)_i = \frac{4}{di}\left(\tilde{\kappa}N - \sum_{h\in\mathcal{I}}\left(\frac{\mu\tau}{2} + n_h\right)\right) + O\left(\frac{\mu}{i} + \frac{\mu^2\tau}{i}\right)\,. \tag{4.68}$$

Moreover, since by (4.65) we get

$$\tau\sqrt{i^2+\mu} = \tau i + \frac{\mu\tau}{2i} + O\left(\frac{\mu^2\tau}{i^3}\right)\,,$$

using (4.68), we have

$$\ell - \tau\sqrt{i^2+\mu} - \big(BI_0(\tau)\big)_i =$$

$$= \ell - \tau i - \frac{\mu\tau}{2i} + \frac{4}{di}\left(\sum_{h\in\mathcal{I}}\left(\frac{\mu\tau}{2} + n_h\right) - \tilde{\kappa}N\right) + O\left(\frac{\mu}{i} + \frac{\mu^2\tau}{i}\right). \quad (4.69)$$

From the hypothesis $\tau \in \mathcal{T}_\mu$ and for μ small enough, it follows that

$$\frac{\mu}{i} + \frac{\mu^2\tau}{i} < \frac{2c_4}{i}.$$

Hence, by (4.69) and choosing c_4 small enough, in order to prove (4.61), it is sufficient to show that

$$\left|\ell - \tau i - \frac{\mu\tau}{2i} + \frac{4}{di}\left(\sum_{h\in\mathcal{I}}\left(\frac{\mu\tau}{2} + n_h\right) - \tilde{\kappa}N\right)\right| \geq \frac{1}{4(4N-1)\,i}. \quad (4.70)$$

Now, since $d = 4N - 1$ and $\sum_{h\in\mathcal{I}} 1 = N$, (4.70) is equivalent to

$$\left|\mu\tau + 2\left(id(\ell - \tau i) - 4\tilde{\kappa}N + 4\sum_{h\in\mathcal{I}} n_h\right)\right| \geq \frac{1}{2}. \quad (4.71)$$

Since

$$2\left(id(\ell - \tau i) - 4\tilde{\kappa}N + 4\sum_{h\in\mathcal{I}} n_h\right)$$

is an even integer, (4.71) follows by hypothesis $\tau \in \mathcal{T}_\mu$. $\square$

PROOF OF LEMMA 4.2.4. Since $\omega_h = h\sqrt{1+x}$ with $x := \mu/h^2$, $0 < x < 1$, (4.64), (4.65) directly follow by the elementary inequalities

$$1 < \sqrt{1+x}\,, \qquad -\frac{x^2}{8} < \sqrt{1+x} - 1 - \frac{x}{2} < 0\,,$$

holding for any $0 < x < 1$. We now prove (4.66). Being $|\{\omega_h\tau\}| \leq 1$, we have

$$|\omega_h\{\omega_h\tau\} - h\{\omega_h\tau\}| = |(\omega_h - h)\{\omega_h\tau\}| \leq |\omega_h - h| \leq \frac{\mu}{2h}\,, \quad (4.72)$$

where in the last inequality we have used (4.64). Moreover by (4.65)

$$\{\omega_h\tau\} = \omega_h\tau - [\omega_h\tau] = h\tau + \frac{\mu\tau}{2h} - \delta_h^{(2)}\tau - [\omega_h\tau]\,,$$

from which we get

$$\left|h\{\omega_h\tau\} - \frac{\mu\tau}{2} - n_h\right| \leq \frac{\mu^2\tau}{8h^2}\,, \qquad \text{where} \quad n_h := h^2\tau - h[\omega_h\tau]\,.$$

Then the proof follows by the previous inequality and (4.72). $\square$

We now give a lower estimate on the cardinality of $\mathcal{T}_\mu$.

Lemma 4.2.5. *For μ small enough*

$$\sharp\ \mathcal{T}_\mu \geq \frac{c_4}{6\mu^2}\,.$$

PROOF. We first claim that

$$\sharp\ \left\{\tau \in \mathbb{N}\,, \quad \text{s.t.} \quad \mu\tau \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]\right\} \geq \left[\frac{1}{\mu}\right] \tag{4.73}$$

for any n odd. Indeed if $\tau_0 := \min\{\tau \in \mathbb{N} \text{ s.t. } \mu\tau \geq n - 1/2\}$, then $\mu\tau_0 < \mu + n - 1/2$ and therefore $\mu\tau_0 + \mu m \leq n + 1/2$ for any $0 \leq m \leq -1 + 1/\mu$, proving (4.73). Since

$$\sharp\ \left\{\ n \text{ odd} \quad \text{s.t.} \quad \frac{c_4}{2\mu} + \frac{1}{2} \leq n \leq \frac{c_4}{\mu} - \frac{1}{2}\ \right\} \geq \frac{c_4}{4\mu} - 2 \geq \frac{c_4}{5\mu}\,,$$

then

$$\sharp\ \mathcal{T}_\mu \geq \frac{c_4}{5\mu}\left[\frac{1}{\mu}\right] \geq \frac{c_4}{6\mu^2}\,,$$

for μ small enough. $\square$

4.2.3 Functional setting

Since the problem is hamiltonian, any T–periodic solution of the Hamilton's equations for $\widetilde{\mathcal{H}}$ in (4.35), namely

$$\begin{cases} \dot{I} &= -\eta^4\partial_\phi\widetilde{K}(I,\phi,\hat{z},\bar{\hat{z}}) \\ \dot{\phi} &= \omega + \eta^2 AI + \eta^2 B^t z\bar{z} + \eta^4\partial_I\widetilde{K}(I,\phi,\hat{z},\bar{\hat{z}}) \\ \dot{\hat{z}}_i &= \mathrm{i}(\Omega + \eta^2 BI)_i\,\hat{z}_i + \mathrm{i}\eta^2\partial_{\bar{z}_i}\hat{G}(\hat{z},\bar{\hat{z}}) + \mathrm{i}\eta^4\partial_{\bar{z}_i}\widetilde{K}(I,\phi,\hat{z},\bar{\hat{z}})\,, \qquad i \in \mathcal{I}^c\,. \end{cases} \tag{4.74}$$

is a critical point of the lagrangian action functional

$$S(I,\phi,\hat{z}) = \int_0^T \left(I\cdot\dot{\phi} - \mathrm{i}\sum_{i\in\mathcal{I}^c} z_i\dot{\bar{z}}_i - \widetilde{\mathcal{H}}(I,\phi,\hat{z},\bar{\hat{z}})\right)dt\,, \tag{4.75}$$

in the space of T–periodic, $\mathcal{P}_{a,s}$–valued curves $\big(I(t),\phi(t),\hat{z}(t)\big)$.

In particular we are looking for periodic orbits of the Hamiltonian $\widetilde{\mathcal{H}}$ near the family $\mathcal{F}$ defined in (4.47), namely we seek solutions of the form

$$\begin{cases} I(t) &= I_0 + J(t) \\ \phi(t) &= \phi_0 + \tilde{\omega}t + \psi(t) \\ \hat{z}(t) &= 0 + w(t)\,, \end{cases} \tag{4.76}$$

where I_0 was defined in (4.53), $\phi_0 \in \mathbb{T}^N$ is a parameter to determine. Recalling (4.74), the equations that $\zeta(t) = (J(t),\psi(t),w(t))$ and $\phi_0 \in \mathbb{T}^N$ must satisfy are

$$\begin{cases} \dot{\psi} - \eta^2 AJ &= \eta^2 B^t w\bar{w} + \eta^4\partial_I\widetilde{K}(I_0 + J,\phi_0 + \tilde{\omega}t + \psi, w, \bar{w}) \\ \dot{J} &= -\eta^4\partial_\phi\widetilde{K}(I_0 + J,\phi_0 + \tilde{\omega}t + \psi, w, \bar{w}) \\ \dot{w}_i - \mathrm{i}\tilde{\Omega}_i w_i &= \mathrm{i}\eta^2(BJ)_i w_i + \mathrm{i}\eta^2\partial_{\bar{z}_i}\hat{G}(w,\bar{w}) + \mathrm{i}\eta^4\partial_{\bar{z}_i}\widetilde{K}\,, \quad i \in \mathcal{I}^c\,. \end{cases} \tag{4.77}$$

We will look for $\zeta(t)$ as a T-periodic curve taking values in the covering space $\mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s}$ (that for simplicity we will still denote by $\mathcal{P}_{a,s}$) with $\int_0^T \psi(t)dt = 0$. For $\zeta = (J, \psi, w) \in \mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s}$ we define the norm[2]

$$\|\zeta\|_{\mathcal{P}_{a,s}} = \|(J, \psi, w)\|_{\mathcal{P}_{a,s}} := |J| + |\psi| + \|w\|_{a,s} \, .$$

With such norm $\mathcal{P}_{a,s}$ is a Banach algebra, recalling that $s > 1$ and Lemma 4.1.2.
In particular we will look for H^k–solutions $\zeta(t)$ in the Banach space

$$\overline{H}^k_{T,a,s} := \left\{ \zeta \in H^k_{T,a,s} \, , \quad \int_0^T \psi(t)dt = 0 \right\}$$

where $k \in \mathbb{N}$, $T > 0$,

$$H^k_{T,a,s} := \left\{ \zeta \in H^k(\mathbb{R}, \mathcal{P}_{a,s}) \, , \quad \zeta(t+T) = \zeta(t) \right\}$$

and $H^k(\mathbb{R}, \mathcal{P}_{a,s})$ is the Sobolev space of the functions $\zeta : \mathbb{R} \longrightarrow \mathcal{P}_{a,s}$ with k weak derivatives (for $k = 0$, $H^0(\mathbb{R}, \mathcal{P}_{a,s}) = L^2(\mathbb{R}, \mathcal{P}_{a,s})$).
The space $H^k_{T,a,s}$ is endowed with the norm

$$\|\zeta\|_{H^k_{T,a,s}} := \sum_{h=0}^{k} T^h \|\partial_t^h \zeta\|_{T,a,s} \, ,$$

where

$$\begin{aligned}
\|\zeta\|_{T,a,s} &:= |J|_{L^2,T} + |\psi|_{L^2,T} + \|w\|_{L^2,T,a,s} + \|\bar{w}\|_{L^2,T,a,s} \, , \\
|J|^2_{L^2,T} &:= \frac{1}{T} \int_0^T |J(t)|^2 dt \, , \qquad |\psi|^2_{L^2,T} := \frac{1}{T} \int_0^T |\psi(t)|^2 dt \, , \\
\|w\|^2_{L^2,T,a,s} &:= \frac{1}{T} \int_0^T \|w(t)\|^2_{a,s} dt \, .
\end{aligned}$$

Note that $H^k_{T,a,s} = \overline{H}^k_{T,a,s} \oplus \mathbb{R}^N$.

Remark 4.2.6. *With the above definitions the following result holds*

$$\|\zeta(t)\|_{\mathcal{P}_{a,s}} \leq \|\zeta\|_{H^1_{T,a,s}} \, , \qquad \forall \, t \in \mathbb{R} \, . \tag{4.78}$$

Hence the spaces $H^k_{T,a,s}$, for $k \geq 1$, are Banach algebras and the $H^k_{T,a,s}$–norm of the product of any component of a vector ζ with any component of a vector ζ' is bounded by $\|\zeta\|_{H^k_{T,a,s}} \|\zeta'\|_{H^k_{T,a,s}}$.

We will consider the system (4.77) as a functional equation in $H^k_{T,a,s}$.
To simplify notations we rewrite (4.77) in the form

$$L\zeta = N(\zeta; \phi_0) \tag{4.79}$$

[2]Here $|\cdot|$ is the standard euclidian norm on $\mathbb{R}^N$, $\|w\|^2_{a,s} = \sum_{i \in \mathcal{I}^c} |w_i|^2 i^{2s} e^{2ai}$.

where L is the linear operator

$$L\zeta = L(J,\psi,w) := (\dot{\psi} - \eta^2 AJ, \dot{J}, \dot{w}_i + \mathrm{i}\tilde{\Omega}_i w_i) \tag{4.80}$$

and N is the nonlinearity

$$N(\zeta;\phi_0) := \big(N_I(\zeta;\phi_0), N_\phi(\zeta;\phi_0), N_{\hat{z}}(\zeta;\phi_0)\big) \tag{4.81}$$

defined by

$$\begin{aligned} N_I &:= \eta^2 B^t w\bar{w} + \eta^4 \partial_I \tilde{K}(I_0 + J, \phi_0 + \tilde{\omega} t + \psi, w) \\ N_\phi &:= -\eta^4 \partial_\phi \tilde{K}(I_0 + J, \phi_0 + \tilde{\omega} t + \psi, w) \\ (N_{z_i}) &:= \mathrm{i}\eta^2 (BJ)_i w_i + \mathrm{i}\eta^2 \partial_{\bar{z}_i} \hat{G}(w) + \mathrm{i}\eta^4 \partial_{\bar{z}_i} \tilde{K}(I_0 + J, \phi_0 + \tilde{\omega} t + \psi, w)\,, \quad i \in \mathcal{I}^c\,. \end{aligned} \tag{4.82}$$

We note that by (4.18) and remark 4.2.6 we get that $\forall\, \phi_0 \in \mathbb{T}^N$

$$N(\cdot;\phi_0) \in C^\infty\big(H^k_{T,a,s}, H^k_{T,a,s+1}\big) \qquad \forall\, k \ge 1\,. \tag{4.83}$$

Since A is invertible and the non-resonance condition (4.48) holds by Proposition 4.2.3, the kernel of the linear operator L is

$$\mathcal{K} = \Big\{\zeta(t) = (J(t),\psi(t),w(t)) \ \text{ s.t. } \ \psi(t) \equiv \text{const}\,,\ J(t) \equiv 0\,, w(t) \equiv 0\Big\}\,.$$

On the other hand the range of L is composed by the curves $\tilde{\zeta}(t) := (\tilde{J}(t), \tilde{\psi}(t), \tilde{w}(t))$ with $\int_0^T \tilde{\psi} = 0$ as we will show in the next subsection concerning the inversion of L.

4.2.4 Inversion of the linear operator

Recall that $\tau = T/(2\pi) = \eta^{-2}$. By the theory of the symmetric operators and since A is invertible, it possesses an orthonormal basis of eigenvectors $e^{(1)}, \ldots, e^{(N)} \in \mathbb{R}^N$ with respective eigenvalues $\nu_1, \ldots, \nu_N \in \mathbb{R} \setminus \{0\}$. In these coordinates we can write

$$\begin{aligned} \tilde{J}(t) &= \sum_{j=1}^N \tilde{J}^{(j)}(t) e^{(j)} = \sum_{j=1}^N e^{(j)} \sum_{\ell\in\mathbb{Z}} \tilde{J}^{(j)}_\ell \exp(\mathrm{i}\ell t/\tau)\,, \\ \tilde{\psi}(t) &= \sum_{j=1}^N \tilde{\psi}^{(j)}(t) e^{(j)} = \sum_{j=1}^N e^{(j)} \sum_{\ell\in\mathbb{Z}} \tilde{\psi}^{(j)}_\ell \exp(\mathrm{i}\ell t/\tau)\,, \\ \tilde{w}_i(t) &= \sum_{\ell\in\mathbb{Z}} \tilde{w}_{i\ell} \exp(\mathrm{i}\ell t/\tau)\,, \qquad\qquad i \in \mathcal{I}^c\,. \end{aligned}$$

We define the linear operator

$$\mathcal{L}\tilde{\zeta} = \mathcal{L}(\tilde{J},\tilde{\psi},\tilde{w}) := (J,\psi,w) = \zeta \tag{4.84}$$

in the following way:

$$J(t) \;:=\; \tau \sum_{j=1}^{N} e^{(j)} \left(-\frac{1}{\nu_j} \tilde{J}_0^{(j)} + \sum_{\ell \neq 0} \frac{1}{\mathrm{i}\ell} \tilde{\psi}_\ell^{(j)} \exp(\mathrm{i}\ell t/\tau) \right), \tag{4.85}$$

$$\psi(t) \;:=\; \tau \sum_{j=1}^{N} e^{(j)} \sum_{\ell \neq 0} \frac{1}{\mathrm{i}\ell} \left(\frac{\nu_j}{\mathrm{i}\ell} \tilde{\psi}_\ell^{(j)} + \tilde{J}_\ell^{(j)} \right) \exp(\mathrm{i}\ell t/\tau), \tag{4.86}$$

$$w_i(t) \;:=\; \tau \sum_{\ell \in \mathbb{Z}} \frac{1}{\mathrm{i}(\ell - \tilde{\Omega}_i \tau)} \tilde{w}_{i\ell} \exp(\mathrm{i}\ell t/\tau) \qquad i \in \mathcal{I}^c. \tag{4.87}$$

The following proposition states that $\mathcal{L}$ is the inverse of L and also gives an upper bound of its norm. However, because of the small divisors $\ell - \tilde{\Omega}_i \tau$, appearing in (4.87), it turns out that $\mathcal{L}$ "loses one spatial derivative". On the other hand, $\mathcal{L}$ also "gains one time derivative", if one gives up "two spatial derivatives". The estimate (4.60) will be crucial.

Proposition 4.2.7. *Suppose $\tau \in \mathcal{T}_\mu$ and μ small enough. Take $k \in \mathbb{N}^+$. If $\tilde{\zeta} \in \overline{H}^k_{T,a,s+1}$ then $\zeta = \mathcal{L}\tilde{\zeta} \in \overline{H}^k_{T,a,s} \cap \overline{H}^{k+1}_{T,a,s-1}$ and*

$$\|\zeta\|_{H^k_{T,a,s}} + \|\zeta\|_{H^{k+1}_{T,a,s-1}} \leq c_6^{-1}\, \tau\, \|\tilde{\zeta}\|_{H^k_{T,a,s+1}}. \tag{4.88}$$

Moreover

$$L\mathcal{L}\tilde{\zeta} = L\zeta = \tilde{\zeta}. \tag{4.89}$$

PROOF. First we note that the average of ψ is zero as it follows by (4.86). We now prove (4.88). Since

$$\frac{1}{T} \int_0^T \left| \sum_{\ell \in \mathbb{Z}} a_\ell \exp(\mathrm{i}\ell t/\tau) \right|^2 dt = \sum_{\ell \in \mathbb{Z}} |a_\ell|^2 \tag{4.90}$$

and (being $\{e^{(j)}\}_{1 \leq j \leq N}$ an orthonormal basis) $\left| \sum_{j=1}^N b_j e^{(j)} \right|^2 = \sum_{j=1}^N |b_j|^2$, we have that

$$f(t) = \sum_{j=1}^{N} e^{(j)} \sum_{\ell \in \mathbb{Z}} f_\ell^{(j)} \exp(\mathrm{i}\ell t/\tau)$$

implies

$$T|f(t)|^2 dt = \sum_{j=1}^{N} \sum_{\ell \in \mathbb{Z}} |f_\ell^{(j)}|^2.$$

Hence, if $\tilde{C} := 2\max_{1 \leq j \leq N}\{|\nu_j|^2, 1/|\nu_j|^2\}$, from (4.86) we get

$$|\psi|^2_{L^2,T} \;\leq\; \tilde{C}\tau^2 \sum_{j=1}^{N} \sum_{\ell \neq 0} \ell^{-2} \left(|\tilde{\psi}_\ell^{(j)}|^2 + |\tilde{J}_\ell^{(j)}|^2 \right)$$

$$
\begin{aligned}
&\leq \tilde{C}\tau^2 \left(|\tilde{\psi}|^2_{L^2,T} + |\tilde{J}|^2_{L^2,T} \right) , && (4.91)\\
|\partial_t^h \psi|^2_{L^2,T} &\leq \tilde{C}\tau^{2(1-h)} \sum_{j=1}^{N} \sum_{\ell\neq 0} \ell^{2(h-1)} \left(|\tilde{\psi}_\ell^{(j)}|^2 + |\tilde{J}_\ell^{(j)}|^2 \right) \\
&\leq \tilde{C}\tau^2 \left(|\partial_t^{h-1}\tilde{\psi}|^2_{L^2,T} + |\partial_t^{h-1}\tilde{J}|^2_{L^2,T} \right) , \qquad \text{for } h \geq 1 . && (4.92)
\end{aligned}
$$

Similar estimates hold for J defined in (4.85), namely:

$$
\begin{aligned}
|J|^2_{L^2,T} &\leq \tilde{C}\tau^2 \left(\sum_{j=1}^{N} |\tilde{J}_0^{(j)}|^2 + \sum_{j=1}^{N} \sum_{\ell\neq 0} \ell^{-2} |\tilde{\psi}_\ell^{(j)}|^2 \right) \\
&\leq \tilde{C}\tau^2 \left(|\tilde{J}|^2_{L^2,T} + |\tilde{\psi}|^2_{L^2,T} \right) , && (4.93)\\
|\partial_t^h J|^2_{L^2,T} &= |\partial_t^{h-1}\tilde{\psi}|^2_{L^2,T} , \qquad \text{for } h \geq 1 . && (4.94)
\end{aligned}
$$

We now go on to estimate w defined in (4.87) in which the small divisors $\ell - \tilde{\Omega}_j\tau$ appear. By (4.90) we have that if $w(t) = \big(w_i(t)\big)_{i\in\mathcal{I}^c}$ with $w_i(t) = \sum_{\ell\in\mathbb{Z}} w_{i\ell} \exp(\mathrm{i}\ell t/\tau)$ then

$$
\begin{aligned}
\|w\|^2_{L^2,T,a,s} &= \frac{1}{T}\int_0^T \|w(t)\|^2_{a,s} = \frac{1}{T}\int_0^T \sum_{i\in\mathcal{I}^c} i^{2s} e^{2ai} |w_i(t)|^2 dt \\
&= \sum_{i\in\mathcal{I}^c} i^{2s} e^{2ai} \frac{1}{T}\int_0^T |w_i(t)|^2 dt = \sum_{i\in\mathcal{I}^c} i^{2s} e^{2ai} \sum_{\ell\in\mathbb{Z}} |w_{i\ell}|^2 .
\end{aligned}
$$

Hence, recalling (4.87), we get

$$
\begin{aligned}
\|\partial_t^h w\|^2_{L^2,T,a,s} &= \tau^2 \sum_{i\in\mathcal{I}^c} i^{2s} e^{2ai} \sum_{\ell\in\mathbb{Z}} \frac{|\ell|^{2h}}{\tau^{2h}} \frac{|\tilde{w}_{i\ell}|^2}{|\ell - \tilde{\Omega}_i\tau|^2} \\
&\leq \frac{\tau^2}{c_5^2} \sum_{i\in\mathcal{I}^c} i^{2(s+1)} e^{2ai} \sum_{\ell\in\mathbb{Z}} \frac{|\ell|^{2h}}{\tau^{2h}} |\tilde{w}_{i\ell}|^2 && (4.95)\\
&= \frac{\tau^2}{c_5^2} \|\partial_t^h \tilde{w}\|^2_{L^2,T,a,s+1} && (4.96)
\end{aligned}
$$

by the crucial estimate (4.60). Moreover we claim that by (4.60)

$$
\frac{|\ell|}{|\ell - \tilde{\Omega}_i\tau|} \leq \frac{4\tau i^2}{c_5} . \qquad (4.97)
$$

To prove (4.97) we distinguish two cases $\ell \leq 2\tilde{\Omega}_i\tau$ and $\ell > 2\tilde{\Omega}_i\tau$. In the first case we have by (4.60)

$$
\frac{|\ell|}{|\ell - \tilde{\Omega}_i\tau|} \leq \frac{|\ell| i}{c_5} \leq \frac{2\tilde{\Omega}_i\tau i}{c_5} \leq \frac{4\tau i^2}{c_5}
$$

since $\tilde{\Omega}_i \le 2i$. On the other hand, if $\ell > 2\tilde{\Omega}_i\tau$ we have $|\ell - \tilde{\Omega}_i\tau| \ge |\ell|/2$ which implies

$$\frac{|\ell|}{|\ell - \tilde{\Omega}_i\tau|} \le 2 \le \frac{4\tau i^2}{c_5}$$

and (4.97) follows. Using (4.97) we get

$$\begin{aligned}
\|\partial_t^h w\|^2_{L^2,T,a,s-1} &= \tau^2 \sum_{i\in\mathcal{I}^c} i^{2(s-1)} e^{2ai} \sum_{\ell\in\mathbb{Z}} \frac{|\ell|^{2h}}{\tau^{2h}} \frac{|\tilde{w}_{i\ell}|^2}{|\ell - \tilde{\Omega}_i\tau|^2} \\
&\le \frac{16\tau^2}{c_5^2} \sum_{i\in\mathcal{I}^c} i^{2(s+1)} e^{2ai} \sum_{\ell\in\mathbb{Z}} \frac{|\ell|^{2(h-1)}}{\tau^{2(h-1)}} |\tilde{w}_{i\ell}|^2 \\
&= \frac{16\tau^2}{c_5^2} \|\partial_t^{h-1}\tilde{w}\|^2_{L^2,T,a,s+1} \, .
\end{aligned} \tag{4.98}$$

Therefore (4.88) follows from (4.91)–(4.96) and (4.98).
Finally we note that (4.89) directly follows from the definition of $\mathcal{L}$ given in (4.84)–(4.87).
We also remark that the constant c_6 does not depend on k, a, s. $\square$

4.2.5 Lyapunov-Schmidt reduction

From the previous section it results that the kernel $\mathcal{K}$ and the range $\mathcal{R}$ of the linear operator L are $\{\tilde{\psi} \equiv \text{const}\}$ and $\{\int_0^T \tilde{\psi} = 0\}$ respectively. For $\tilde{\zeta} = (\tilde{J}, \tilde{\psi}, \tilde{w})$ let us define the projections

$$\Pi_{\mathcal{K}}\tilde{\zeta} := \Big(0, \langle\tilde{\psi}\rangle, 0\Big) \, , \qquad \Pi_{\mathcal{R}}\tilde{\zeta} := \Big(\tilde{J}, \tilde{\psi} - \langle\tilde{\psi}\rangle, \tilde{w}\Big) \, ,$$

where $\langle\tilde{\psi}\rangle := \int_0^T \tilde{\psi}$. In such a way the equation $L\zeta = N(\zeta;\phi_0)$ decomposes into the equation on the Kernel

$$0 = \Pi_{\mathcal{K}} N(\zeta;\phi_0) \, , \quad \text{namely} \quad \big\langle N_\phi(\zeta,\phi_0)\big\rangle = 0 \, , \tag{4.99}$$

and the one on the range

$$L\zeta = \Pi_{\mathcal{R}} N(\zeta;\phi_0) \tag{4.100}$$

respectively. The idea is to solve first the range equation for any fixed ϕ_0, finding a solution $\zeta(t) = \zeta_{\phi_0}(t)$ by the Contraction Mapping Theorem, and thereafter the Kernel equation (4.99) for $\zeta = \zeta_{\phi_0}$, namely the *finite dimensional* equation

$$\big\langle N_\phi(\zeta_{\phi_0};\phi_0)\big\rangle = 0 \, , \tag{4.101}$$

determining ϕ_0 by a variational argument.

4.2.6 Range equation

We rewrite the range equation (4.100) in a fixed–point form:

$$\zeta = \Phi(\zeta; \phi_0)$$

with

$$\Phi(\zeta; \phi_0) := \mathcal{L}\Pi_{\mathcal{R}} N(\zeta; \phi_0)\,.$$

By Proposition 4.2.7, the operator $\mathcal{L}$ "looses one derivative", but, by the smoothing property (4.83), the nonlinearity N gains exactly *one* derivative. In particular, we have that, for any $\phi_0 \in \mathbb{T}^N$ fixed,

$$\Phi(\cdot; \phi_0) \in C^\infty\big(\overline{H}^k_{T,a,s}, \overline{H}^k_{T,a,s}\big) \qquad \forall\, k \geq 1\,. \tag{4.102}$$

In the following Lemma we prove that Φ is a contraction on a suitable closed ball of $\overline{H}^1_{T,a,s}$.

Lemma 4.2.8. *Suppose $\tau \in \mathcal{T}_\mu$ and μ small enough. For any $\phi_0 \in \mathbb{T}^N$ the map $\Phi(\cdot; \phi_0)$ is a contraction on the closed ball of radius $\rho := c_7^{-1}\mu$ of $\overline{H}^1_{T,a,s}$.*

PROOF. Let $\zeta, \zeta', h \in \overline{H}^1_{T,a,s}$ and $\|\zeta\|_{H^1_{T,a,s}}, \|\zeta'\|_{H^1_{T,a,s}} \leq \rho$. From (4.81), (4.82) we get the following estimates on the nonlinearity

$$\|N(\zeta)\|_{H^1_{T,a,s+1}} \leq c_8^{-1}\left(\eta^2\rho^2 + \frac{\eta^4}{\mu}\right) \tag{4.103}$$

$$\|DN(\zeta)[h]\|_{H^1_{T,a,s+1}} \leq c_8^{-1}\left(\eta^2\rho + \frac{\eta^4}{\mu}\right)\|h\|_{H^1_{T,a,s}}\,. \tag{4.104}$$

Using Proposition 4.2.7 and (4.103), we obtain

$$\|\Phi(\zeta)\|_{H^1_{T,a,s}} \leq (c_6c_8)^{-1}\left(\rho^2 + \frac{\eta^2}{\mu}\right) = (c_6c_8)^{-1}\rho^2 + \rho/2 \leq \rho\,,$$

taking $c_7 := c_4c_6c_8/4$ and μ small enough. Hence Φ maps the ball in itself. Nothing remains but to show that Φ is a contraction. By (4.104) we get

$$\|N(\zeta) - N(\zeta')\|_{H^1_{T,a,s+1}} \leq c_8^{-1}\left(\eta^2\rho + \frac{\eta^4}{\mu}\right)\|\zeta - \zeta'\|_{H^1_{T,a,s}}\,,$$

by Proposition 4.2.7 we have

$$\|\Phi(\zeta) - \Phi(\zeta')\|_{H^1_{T,a,s}} \leq (c_6c_8)^{-1}\left(\rho + \frac{\eta^2}{\mu}\right)\|\zeta - \zeta'\|_{H^1_{T,a,s}}\,.$$

Since, for μ small enough,

$$(c_6c_8)^{-1}\left(\rho + \frac{\eta^2}{\mu}\right) \leq \left(\frac{1}{c_6c_8} + \frac{1}{2}\right)\frac{\mu}{c_7} < 1\,,$$

Φ is a contraction. $\square$

By the Contraction Mapping Theorem and noting that the dependence of the nonlinearity N and, therefore, of Φ on the parameter ϕ_0 is smooth, we conclude that there exists a smooth function $\mathbb{T}^N \ni \phi_0 \longmapsto \zeta_{\phi_0} \in \overline{H}^1_{T,a,s}$ solving $\zeta_{\phi_0} = \Phi(\zeta_{\phi_0}; \phi_0)$. By (4.89), ζ_{ϕ_0} also solves the range equation (4.100) as the following corollary states.

Corollary 4.2.9. *Suppose $\tau \in \mathcal{T}_\mu$ and μ small enough. Then there exists a smooth function $\mathbb{T}^N \ni \phi_0 \longmapsto \zeta_{\phi_0} \in \overline{H}^1_{T,a,s}$ solving* (4.100) *and satisfying*

$$\|\zeta_{\phi_0}\|_{H^1_{T,a,s}} \leq \frac{\mu}{c_7} \,.$$

Remark 4.2.10. *In [BBe05], instead of* (4.48)*, it is imposed the weaker "diophantine–type" condition $|\ell - \tilde{\Omega}_i \tau| \geq \mathrm{const}\, i^{-\sigma}$, $\sigma > 1$, on the small divisors. Then the operator $\mathcal{L}$ "looses σ derivatives". If the nonlinearity N is smoothing of order $d > 1$, namely it "gains d derivatives", taking $1 < \sigma < d$, the Contraction Mapping Theorem can still be used in solving the range equation for almost every rescaled period τ. In particular, it results $d = 2$ for the beam equation and $d > 1$ for the NLS. Since we have exactly $d = 1$ for the wave equation, in order to have a positive measure set of rescaled periods, a KAM analysis is necessary (see remark 4.2.13).*

4.2.7 Kernel equation

Once we have solved the range equation (4.100) finding the smooth function

$$\mathbb{T}^N \ni \phi_0 \longmapsto \zeta_{\phi_0} =: (J_{\phi_0}, \psi_{\phi_0}, w_{\phi_0}) \in \overline{H}^1_{T,a,s} \,,$$

we have to solve the "reduced" kernel equation (4.101) yet. As the solutions of the Hamilton's equations (4.74) are critical points of the action functional S defined in (4.75), so the solutions of the "reduced" kernel equation (4.101) are critical points of the *reduced action functional*

$$\mathcal{S}(\phi_0) := S(I_{\phi_0}, \phi_{\phi_0}, \hat{z}_{\phi_0}) = \int_0^T \Big(I_{\phi_0} \dot{\phi}_{\phi_0} - \mathrm{i} \hat{z}_{\phi_0} \dot{\bar{\hat{z}}}_{\phi_0} - \widetilde{\mathcal{H}}(\phi_{\phi_0}, I_{\phi_0}, \hat{z}_{\phi_0}, \bar{\hat{z}}_{\phi_0}) \Big)\, dt \,, \tag{4.105}$$

where

$$I_{\phi_0}(t) := I_0 + J_{\phi_0}(t) \,, \qquad \phi_{\phi_0}(t) := \phi_0 + \tilde{\omega} t + \psi_{\phi_0}(t) \,, \qquad \hat{z}_{\phi_0}(t) := w_{\phi_0}(t) \,. \tag{4.106}$$

Actually we are claiming that

$$\big\langle N_\phi(\zeta_{\phi_0}; \phi_0) \big\rangle = 0$$

or equivalently

$$\partial_{\phi_0} \mathcal{S}(\phi_0) = 0 \,. \tag{4.107}$$

Indeed (4.107) is a corollary of (4.35), (4.82) and of the following

Lemma 4.2.11. *The reduced action functional satisfies the following*

$$\partial_{\phi_0}\mathcal{S}(\phi_0) = -T\big\langle \partial_\phi \widetilde{\mathcal{H}}(I_{\phi_0}, \phi_{\phi_0}, \hat{z}_{\phi_0}, \bar{\hat{z}}_{\phi_0})\big\rangle\,.$$

PROOF. We have

$$\begin{aligned}
\partial_{\phi_0}\mathcal{S}(\phi_0) &= \int_0^T \Big[\big(\dot{\phi}_{\phi_0} - \partial_I\widetilde{\mathcal{H}}\big)\partial_{\phi_0}I_{\phi_0} + I_{\phi_0}\partial_{\phi_0}\dot{\phi}_{\phi_0} - \partial_\phi\widetilde{\mathcal{H}}\ \partial_{\phi_0}\phi_{\phi_0} \\
&\qquad -\big(\mathrm{i}\dot{\bar{w}}_{\phi_0} + \partial_{\hat{z}}\widetilde{\mathcal{H}}\big)\partial_{\phi_0}w_{\phi_0} - \mathrm{i}w_{\phi_0}\partial_{\phi_0}\dot{\bar{w}}_{\phi_0} - \partial_{\bar{\hat{z}}}\widetilde{\mathcal{H}}\ \partial_{\phi_0}\bar{w}_{\phi_0}\Big]\,dt \\
&= I_{\phi_0}\partial_{\phi_0}\phi_{\phi_0}\Big|_0^T - \mathrm{i}w_{\phi_0}\partial_{\phi_0}\bar{w}_{\phi_0}\Big|_0^T - \int_0^T \big\langle\partial_\phi\widetilde{\mathcal{H}}\big\rangle\partial_{\phi_0}\phi_{\phi_0}\ dt\,,
\end{aligned}$$

by an integration by parts and since ζ_{ϕ_0} satisfies the range equation (4.100). Moreover, since ζ_{ϕ_0} is periodic and $\int_0^T \psi_{\phi_0} = 0$, we get

$$\begin{aligned}
I_{\phi_0}\partial_{\phi_0}\phi_{\phi_0}\Big|_0^T &= I_{\phi_0}(0)\partial_{\phi_0}\big(\phi_{\phi_0}(T) - \phi_{\phi_0}(0)\big) = I_{\phi_0}\partial_{\phi_0}\tilde{\omega}T = 0\,, \\
w_{\phi_0}\partial_{\phi_0}\bar{w}_{\phi_0}\Big|_0^T &= w_{\phi_0}(0)\partial_{\phi_0}\big(\bar{w}_{\phi_0}(T) - \bar{w}_{\phi_0}(0)\big) = 0 \\
\int_0^T \phi_{\phi_0}(t)\,dt &= \int_0^T \big(\phi_0 + \tilde{\omega}t\big)\,dt + \int_0^T \psi_{\phi_0}(t)\,dt = \phi_0 T + \tilde{\omega}\frac{T^2}{2}\,.
\end{aligned}$$

Finally

$$\partial_{\phi_0}\mathcal{S}(\phi_0) = -\int_0^T \big\langle\partial_\phi\widetilde{\mathcal{H}}\big\rangle\partial_{\phi_0}\phi_{\phi_0}\ dt = -\big\langle\partial_\phi\widetilde{\mathcal{H}}\big\rangle\partial_{\phi_0}\int_0^T \phi_{\phi_0}(t)\,dt = -\big\langle\partial_\phi\widetilde{\mathcal{H}}\big\rangle T\,.$$

□

4.2.8 Existence

By (4.107) every critical point $\phi_0 \in \mathbb{T}^N$ of the reduced action functional $\mathcal{S}$ defined in (4.105) solves the "reduced" kernel equation (4.101) and, therefore, the curve $\big(I_{\phi_0}(t), \phi_{\phi_0}(t), \hat{z}_{\phi_0}(t)\big)$ defined in (4.106) is a solution of the Hamilton's equations (4.74). In particular we expect the existence of at least N geometrically distinct T–periodic solutions, namely solutions not obtained one from each other simply by time translations.

Indeed let us consider the restriction of $\mathcal{S}$ to the plane $E := [\tilde{\omega}]^\perp$. The set $\mathbb{Z}^N \cap E$ is a lattice of E, hence $\mathcal{S}$ can be defined on the quotient space $\Gamma := E/(\mathbb{Z}^N \cap E) \sim \mathbb{T}^{N-1}$. Due to the invariance of $\mathcal{S}$ with respect to the time shift, any critical point of $\mathcal{S}_{|\Gamma}\colon \Gamma \longrightarrow \mathbb{R}$ is also a critical point of $\mathcal{S}\colon \mathbb{T}^N \longrightarrow \mathbb{R}$. By the Lusternik-Schnirelman category theory, since $\operatorname{cat}\Gamma = \operatorname{cat}\mathbb{T}^{N-1} = N$, we can define the N min-max critical values $\mathrm{c}_1 \le \mathrm{c}_2 \le \ldots \le \mathrm{c}_N$ for the reduced action functional $\mathcal{S}_{|\Gamma}$. If the critical levels c_i are all distinct, the corresponding T-periodic solutions are surely geometrically distinct, since their actions c_i are different. On the other hand, if some min-max critical levels coincide, then $\mathcal{S}_{|\Gamma}$

possesses infinitely many critical points. Although not all the corresponding T–periodic solutions are necessarily geometrically distinct, since a periodic solution can cross Γ at most a finite number of times, the existence of infinitely many geometrically distinct orbits follows (see [BeBiV04] for further details).

Proposition 4.2.12. *Suppose $\tau \in \mathcal{T}_\mu$ and μ small enough. Then the system* (4.74) *possesses (at least) N geometrically distinct T–periodic solutions*

$$\big(I_{\phi_0^{(j)}}(t), \phi_{\phi_0^{(j)}}(t), \hat{z}_{\phi_0^{(j)}}(t)\big) = \big(I_0(\tau), \phi_0^{(j)} + \tilde{\omega}t, 0\big) + \zeta_{\phi_0^{(j)}}(t)\,, \qquad \zeta_{\phi_0^{(j)}} \in \overline{H}^1_{T,a,s}\,, \tag{4.108}$$

parametrized by suitable $\phi_0^{(j)} \in \mathbb{T}^N$, $1 \leq j \leq N$. Moreover

$$\|\zeta_{\phi_0^{(j)}}\|_{H^1_{T,a,s}} \leq \frac{\mu}{c_7}\,, \qquad \forall\, 1 \leq j \leq N\,. \tag{4.109}$$

Remark 4.2.13. *In* (4.48) *we have imposed a strong condition on the small divisors in order to use the standard Contraction Mapping Theorem in solving the range equation. The other side of such request is that we are able to consider only a finite number of periods. The natural way to deal with the small divisors problem* (4.48)*, in order to obtain a positive measure set of periods, should be a KAM analysis. The range equation should be solved by a Nash–Moser Implicit Function Theorem. Thereafter, one should prove that, for any fixed value of the perturbative parameter η, the bifurcation equation $0 = \Pi_{\mathcal{K}} N(\zeta_{\phi_0}; \phi_0)$ has solution for ϕ_0 belonging to a suitable η–dependent Cantor set $\mathcal{C}_\eta$ (see [BeBo05],[GMPr04] where the extension of the Weinstein's Theorem [We73] for completely resonant wave equation is considered). A standard way to proceed is to develop the reduced action functional in powers of the perturbative parameter and to prove that the first non trivial term in the development has a non–degenerate critical point. Such non–degeneracy is used to show that, for any η in a suitable positive measure set, there exists a critical point $\phi_0 = \phi_0(\eta)$ of the whole reduced action functional, belonging to $\mathcal{C}_\eta$. However, in the present case, while the perturbative parameter η goes to zero, the period T goes to infinity. Then, in the computation of $\mathcal{S}$ in* (4.105)*, one has to average over an infinite time. As a consequence, all the low order terms in the development of $\mathcal{S}$ vanish; the first non trivial terms appear only at a very high order in η and it is difficult to deal with them.*
A different approach will be used in the third part of this thesis where it is proved that, if the nonlinear term f in (4) *is odd, then equation* (4.101) *has solution $\phi_0 = 0$ by symmetry.*

4.2.9 Regularity

The solutions $\big(I_0(\tau), \phi_0^{(j)} + \tilde{\omega}t, 0\big) + \zeta_{\phi_0^{(j)}}(t)$ of the system (4.74) described in (4.108) belong to $H^1_{T,a,s}$ and verify the estimate (4.109). With the same procedure, for any fixed $k \geq 1$, we can also find solutions in $H^k_{T,a,s}$ verifying

$$\|\zeta_{\phi_0^{(j)}}\|_{H^k_{T,a,s}} \leq \frac{\mu}{c_7}\,. \tag{4.110}$$

Indeed we can solve the range equation (4.100) in $\overline{H}^k_{T,a,s}$ adapting Lemma 4.2.8. However, in this case, the constant c_7 *depends on* k.
On the other hand, giving up the $H^k_{T,a,s}$–estimate in (4.110), it is anyhow possible to prove, by a bootstrap argument, that the solutions (4.108) of Proposition 4.2.12 actually belong to $H^k_{T,a,s}$ for any $k \geq 1$. In particular we show that ζ_{ϕ_0} of Corollary 4.2.9 belongs to $\overline{H}^k_{T,a,s}$.
Indeed, $\zeta_{\phi_0} \in \overline{H}^1_{T,a,s}$ solves the fixed point equation

$$\zeta_{\phi_0} = \Phi(\zeta_{\phi_0};\phi_0) = \mathcal{L}\Pi_{\mathcal{R}} N(\zeta_{\phi_0};\phi_0)\,.$$

Since $N(\zeta_{\phi_0};\phi_0) \in H^1_{T,a,s+1}$ by (4.83), then $\zeta_{\phi_0} \in \overline{H}^2_{T,a,s-1}$ by Proposition 4.2.7. Noting that $\overline{H}^k_{T,a,s-1} \subset \overline{H}^k_{T,\tilde{a},s}$ for any $k \geq 1$ and $0 < \tilde{a} < a$, we have $\zeta_{\phi_0} \in \overline{H}^2_{T,\tilde{a},s}$. Defining $a_k := a\left(\frac{1}{2} + \frac{1}{2^k}\right)$, we prove that $\zeta_{\phi_0} \in \overline{H}^k_{T,a_k,s}$ for any $k \geq 1$ and finally, $\zeta_{\phi_0} \in \overline{H}^k_{T,a/2,s}$ for any $k \geq 1$. However, in this fashion, the $H^k_{T,a,s}$–estimates deteriorate while k increases.
Summarizing, by the Sobolev immersions, we have that $\zeta_{\phi_0} \in C^k(\mathbb{R},\mathcal{P}_{a/2,s})$ for any $k \geq 1$. We have shown the following:

Corollary 4.2.14. *The solutions* (4.108) *of* (4.74) *belong to* $C^\infty(\mathbb{R},\mathcal{P}_{a/2,s})$.

4.2.10 Minimal period

Lemma 4.2.15. *Let* $h,k \in \mathbb{N}^+$, $h < k$. *We have*

$$[\omega_k\tau - k\tau] \leq [\omega_h\tau - h\tau] < \frac{\mu\tau}{2h}\,. \tag{4.111}$$

PROOF. We first prove that

$$\omega_k - k < \omega_h - h\,, \tag{4.112}$$

which implies $\omega_k\tau - k\tau < \omega_h\tau - h\tau$ and the first inequality in (4.111). Dividing by k, (4.112) is equivalent to

$$f(x) := \sqrt{x^2 + \mu/k^2} - x - \sqrt{1+\mu/k^2} + 1 > 0\,, \qquad \text{for} \quad 0 < x < 1\,,$$

where $x := h/k$. Since $f(1) = 0$ and $f'(x) = x(x^2+\mu/k^2)^{-1/2} - 1 < 0$ for $0 < x < 1$, we get $f(x) > 0$ and (4.112) follows. Then the second inequality in (4.111) directly follows from (4.64). $\square$

Lemma 4.2.16. *Let be $T^{\min}$ the minimal period of a T-periodic orbit* (4.108) *of Proposition 4.2.12. If $N \geq 2$ then*

$$T^{\min} \geq \frac{c_9}{\mu} \,. \tag{4.113}$$

PROOF. Let $\big(I(t), \phi(t), \hat{z}(t)\big)$ be a T-periodic solution of Proposition 4.2.12. We know that $\phi(T) - \phi(0) = 2\pi k$ with $k \in \mathbb{Z}^N$ defined in (4.54). Denoting by $T_\phi^{\min} \leq T^{\min}$ the minimal period of $\phi(t)$, we have that there exist $n \in \mathbb{N}^+$ such that $nT_\phi^{\min} = T$ and $\tilde{k} \in \mathbb{Z}^N$ such that $\phi(T_\phi^{\min}) - \phi(0) = 2\pi\tilde{k}$, verifying $n\tilde{k} = k$. Hence we deduce that n divides $g := \gcd(k_{i_1}, \ldots, k_{i_N})$ and we get that

$$T^{\min} \geq T_\phi^{\min} = \frac{T}{n} \geq \frac{T}{g} \,. \tag{4.114}$$

We claim that

$$\tilde{g} := \gcd(k_{i_1}, k_{i_2}) < \frac{\mu\tau i_2}{i_1} \,. \tag{4.115}$$

Then the Lemma follows by (4.114) and (4.115) noting that $\tilde{g} \geq g$ and recalling that $T = 2\pi\tau$ (with $c_9 = 2\pi i_1/i_2$).
We now prove (4.115). Since $\tau \in \mathbb{N}$ we have $k_i = [\omega_i \tau] + \kappa_i = i\tau + [\omega_i\tau - i\tau] + \kappa_i$ for all $i \in \mathcal{I}$ and

$$\begin{aligned} i_2 k_{i_1} - i_1 k_{i_2} &= i_2[\omega_{i_1}\tau - i_1\tau] - i_1[\omega_{i_2}\tau - i_2\tau] + i_2\kappa_{i_1} - i_1\kappa_{i_2} \\ &\geq (i_2 - i_1)[\omega_{i_1}\tau - i_1\tau] + \left(\frac{i_2}{i_1} - \frac{i_1}{i_2}\right)\tilde{\kappa} > 0 \end{aligned} \tag{4.116}$$

by (4.111) and recalling (4.51). Moreover, since $\tilde{g} = \gcd(k_{i_1}, k_{i_2})$ there exist $h_1, h_2 \in \mathbb{N}$ such that $k_{i_1} = h_1\tilde{g}$ and $k_{i_2} = h_2\tilde{g}$. From (4.116) we have that $i_2 k_{i_1} - i_1 k_{i_2} > 0$ and therefore

$$i_2 k_{i_1} - i_1 k_{i_2} = \big(i_2 h_1 - i_1 h_2\big)\tilde{g} \geq \tilde{g} \,. \tag{4.117}$$

Finally by (4.111)

$$\begin{aligned} i_2 k_{i_1} - i_1 k_{i_2} &= i_2[\omega_{i_1}\tau - i_1\tau] - i_1[\omega_{i_2}\tau - i_2\tau] + i_2\kappa_{i_1} - i_1\kappa_{i_2} \\ &< i_2[\omega_{i_1}\tau - i_1\tau] + \frac{i_2}{i_1}\tilde{\kappa} \leq \frac{i_2}{i_1}\left(\frac{\mu\tau}{2} + \tilde{\kappa}\right) \leq \frac{i_2\mu\tau}{i_1} \,. \end{aligned} \tag{4.118}$$

for μ small enough. Then (4.115) follows from (4.117) and (4.118). $\square$

4.2.11 Distinct orbits

Take $\tau, \tau' \in \mathcal{T}_\mu$, For μ small enough, τ, τ' satisfy the hypotheses of Proposition 4.2.12. Therefore, let be

$$\big(I(t), \phi(t), \hat{z}(t)\big) = \big(I_0(\tau), \phi_0 + \tilde{\omega}t, 0\big) + \zeta(t) \,,$$

with $\phi_0 \in \mathbb{T}^N$, $\zeta \in \overline{H}^1_{T,a,s}$, $T = 2\pi\tau$, and

$$\big(I'(t), \phi'(t), \hat{z}'(t)\big) = \big(I_0(\tau'), \phi'_0 + \tilde{\omega} t, 0\big) + \zeta'(t)\,,$$

with $\phi'_0 \in \mathbb{T}^N$, $\zeta' \in \overline{H}^1_{T',a,s}$, $T' = 2\pi\tau'$, two solutions of (4.74) found in Proposition 4.2.12; we recall that by (4.109)

$$\|\zeta\|_{H^1_{T,a,s}} \leq \frac{\mu}{c_7}\,, \qquad\qquad \|\zeta'\|_{H^1_{T',a,s}} \leq \frac{\mu}{c_7}\,. \tag{4.119}$$

Suppose that they are geometrically the same solution, namely, up to a time translation,

$$\big(I(t), \phi(t), \hat{z}(t)\big) = \big(I'(t), \phi'(t), \hat{z}'(t)\big)\,, \qquad \forall\, t \in \mathbb{R}. \tag{4.120}$$

We claim that

$$|I_0(\tau) - I_0(\tau')| \leq \frac{2\mu}{c_7}\,. \tag{4.121}$$

Indeed, using $I(t) = I'(t)$,

$$\begin{aligned}
|I_0(\tau) - I_0(\tau')| \;&=\; |J(t) - J'(t)| \leq |J(t)| + |J'(t)| \\
&\leq\; \|\zeta(t)\|_{\mathcal{P}_{a,s}} + \|\zeta'(t)\|_{\mathcal{P}_{a,s}} \leq \|\zeta\|_{H^1_{T,a,s}} + \|\zeta'\|_{H^1_{T',a,s}} \leq \frac{2\mu}{c_7}\,,
\end{aligned}$$

recalling (4.78) and (4.119).
Moreover we claim that (4.120) implies also

$$\{\omega_{i_1}(\tau' - \tau)\} < \frac{\mu}{c_{10}} \qquad \text{or} \qquad \{\omega_{i_1}(\tau - \tau')\} < \frac{\mu}{c_{10}}\,, \tag{4.122}$$

with $c_{10} := c_7/2\|A\|$. Indeed, since $|A^{-1}v| \geq \|A\|^{-1}|v|$ for any $v \in \mathbb{R}^N$, recalling (4.53) and choosing $v := \{\omega\tau'\} - \{\omega\tau\}$, we have $I_0(\tau) - I_0(\tau') = A^{-1}v$ and therefore

$$|I_0(\tau) - I_0(\tau')| \geq \|A\|^{-1}|\{\omega\tau'\} - \{\omega\tau\}| \geq \|A\|^{-1}|\{\omega_{i_1}\tau'\} - \{\omega_{i_1}\tau\}|\,. \tag{4.123}$$

Noting that[3]

$$|\{\omega_{i_1}\tau'\} - \{\omega_{i_1}\tau\}| = \{\omega_{i_1}(\tau' - \tau)\} \qquad \text{or} \qquad |\{\omega_{i_1}\tau'\} - \{\omega_{i_1}\tau\}| = \{\omega_{i_1}(\tau - \tau')\}\,,$$

(4.122) follows by (4.121) and (4.123).

[3]Indeed, if $\{x\} \geq \{y\}$ then

$$|\{x\} - \{y\}| = \{x\} - \{y\} = \big\{\{x\} - \{y\}\big\} = \big\{x - y - [x] + [y]\big\} = \{x - y\}\,,$$

on the other hand, if $\{y\} \geq \{x\}$ then $|\{x\} - \{y\}| = \{y - x\}$.

Lemma 4.2.17. *Let*

$$\mathcal{M} := \left\{ n \in \mathbb{Z}\,, \quad |n| \le \frac{c_4}{2\mu^2}\,, \quad \text{s.t.} \quad \{\omega_{i_1} n\} \le \frac{\mu}{c_{10}} \right\}. \tag{4.124}$$

Then

$$\sharp\,\mathcal{M} \le \frac{10 c_4}{c_{10}\mu}\,.$$

PROOF. We first note that, for μ small enough, we get

$$\left[\frac{2c_4}{i_1\mu}\right]\left[\frac{i_1}{\mu}\right] \ge \left(\frac{2c_4}{i_1\mu} - 1\right)\left(\frac{i_1}{\mu} - 1\right) \ge \frac{c_4}{\mu^2} + 1$$

and, therefore,

$$\left[-\frac{c_4}{2\mu^2}, \frac{c_4}{2\mu^2}\right] \subseteq \bigcup_{1 \le m \le \left[\frac{2c_4}{i_1\mu}\right]} \left[\left[-\frac{c_4}{2\mu^2}\right] + (m-1)\left[\frac{i_1}{\mu}\right], \left[-\frac{c_4}{2\mu^2}\right] + m\left[\frac{i_1}{\mu}\right]\right). \tag{4.125}$$

Now we claim that

$$\sharp\left(\mathcal{M} \cap \big[\bar{n}, \bar{n} + [i_1/\mu]\big)\right) \le \frac{4i_1}{c_{10}} + 1\,, \qquad \forall\, \bar{n} \in \mathbb{Z}\,. \tag{4.126}$$

Then from (4.125) and (4.126) we have

$$\sharp\,\mathcal{M} \le \left(\frac{4i_1}{c_{10}} + 1\right)\left[\frac{2c_4}{i_1\mu}\right] \le \frac{5i_1}{c_{10}}\frac{2c_4}{i_1\mu} \le \frac{10c_4}{c_{10}\mu}$$

and the lemma follows.
Nothing remains but to prove (4.126). If $\mathcal{M} \cap \big[\bar{n}, \bar{n} + [i_1/\mu]\big) = \emptyset$, then (4.126) is trivially true. Otherwise let

$$n_0 := \min\left(\mathcal{M} \cap \big[\bar{n}, \bar{n} + [i_1/\mu]\big)\right).$$

By definition $n \notin \mathcal{M}$ for any $\bar{n} \le n < n_0$. Moreover by (4.64),(4.65) we have

$$\{\omega_{i_1} n\} = \{\delta^{(1)}_{i_1} n\} \qquad \text{with} \quad \frac{\mu}{4i_1} < \delta^{(1)}_{i_1} < \frac{\mu}{2i_1}\,. \tag{4.127}$$

We now prove that

$$n \in \mathcal{M} \cap \big[\bar{n}, \bar{n} + [i_1/\mu]\big) \qquad \Longrightarrow \qquad n = n_0 + n'\,, \qquad 0 \le n' < 4i_1/c_{10}\,, \tag{4.128}$$

from which (4.126) follows. By definition of n_0 it is obvious that $n' \ge 0$. Let us consider $n' \in \mathbb{N}$ such that

$$4i_1/c_{10} \le n' < \bar{n} - n_0 + [i_1/\mu]\,.$$

For such n' we will show that $\{\omega_{i_1}(n_0+n')\} > \mu/c_{10}$. We have that

$$\{\omega_{i_1}(n_0+n')\} = \{\delta^{(1)}_{i_1} n_0 + \delta^{(1)}_{i_1} n'\} = \{\{\delta^{(1)}_{i_1} n_0\} + \delta^{(1)}_{i_1} n'\}\,. \tag{4.129}$$

By (4.127)

$$\frac{\mu}{c_{10}} = \frac{\mu}{4i_1}\frac{4i_1}{c_{10}} < \{\delta^{(1)}_{i_1} n_0\} + \delta^{(1)}_{i_1} n' < \{\omega_{i_1} n_0\} + \frac{\mu}{2i_1} n' < \frac{\mu}{c_{10}} + \frac{1}{2} \le 1\,. \tag{4.130}$$

Therefore

$$\{\delta^{(1)}_{i_1} n_0\} + \delta^{(1)}_{i_1} n' = \big\{\{\delta^{(1)}_{i_1} n_0\} + \delta^{(1)}_{i_1} n'\big\} = \{\omega_{i_1}(n_0+n')\}$$

from (4.129). Finally, by (4.130),

$$\{\omega_{i_1}(n_0+n')\} > \frac{\mu}{c_{10}}\,.$$

Hence $n_0 + n' \notin \mathcal{M}$ and (4.128) follows. $\square$

Lemma 4.2.18. *Fix $\tau \in \mathcal{T}_\mu$. For μ small enough*

$$\sharp\Big\{\ \tau' \in \mathcal{T}_\mu \ \text{ s.t. } \ (4.120) \ \text{ holds }\Big\} \le \frac{10c_4}{c_{10}\mu}\,.$$

PROOF. If (4.120) holds then τ and τ' verify (4.122). Hence $\tau - \tau' \in \mathcal{M}$, defined in (4.124). We conclude by Lemma 4.2.17. $\square$

By Lemma 4.2.5 and Lemma 4.2.18 we conclude that the number of geometrically distinct solutions found in Proposition (4.2.12) is greater then

$$\frac{c_4}{6\mu^2}\left(\frac{10c_4}{c_{10}\mu}\right)^{-1} = \frac{c_{10}}{60\mu}\,.$$

Actually, since in Proposition 4.2.12, to any τ correspond N geometrically distinct orbits, then the total number of geometrically distinct solutions is greater then c_{11}/μ with $c_{11} := c_{10}N/60$.

Corollary 4.2.19. *The total number of geometrically distinct solutions found in Proposition* (4.2.12) *is greater then c_{11}/μ.*

4.2.12 Proof of Theorem 1

Suppose that $\tau \in \mathcal{T}_\mu$, μ small enough and consider a solution found in proposition 4.2.12. Such solution is of the form

$$\big(I(t),\phi(t),\hat{z}(t)\big) = \big(I_0(\tau),\tilde{\omega}t+\phi_0,0\big)+\zeta(t) = \big(I_0(\tau)+J(t),\tilde{\omega}t+\phi_0+\psi(t),w(t)\big) \tag{4.131}$$

with

$$|J(t)| + |\psi(t)| + \|w(t)\|_{a,s} \le \frac{\mu}{c_7}\,, \qquad \forall\, t \in \mathbb{R}\,, \tag{4.132}$$

by (4.109) and (4.78). We now want to rewrite such solution in the z's variables defined in (4.29); by (4.32) and recalling that $\hat{z}_i = z_i$ for $i \in \mathcal{I}^c$ we get

$$\begin{cases} z_i(t) &= \sqrt{I_i(t)}\big(\cos\phi_i(t) - \mathrm{i}\sin\phi_i(t)\big)\,, & \text{for } i \in \mathcal{I} \\ z_i(t) &= w_i(t)\,, & \text{for } i \in \mathcal{I}^c\,. \end{cases}$$

In the z_*'s variables of Proposition 4.1.6 we have

$$\begin{cases} \mathrm{z}_{*i}(t) &= \eta\sqrt{I_i(t)}\big(\cos\phi_i(t) - \mathrm{i}\sin\phi_i(t)\big)\,, & \text{for } i \in \mathcal{I} \\ \mathrm{z}_{*i}(t) &= \eta w_i(t)\,, & \text{for } i \in \mathcal{I}^c\,, \end{cases}$$

by (4.29) and

$$\sup_{t\in\mathbb{R}} \|\mathrm{z}_*(t)\|_{a,s} = O(\eta)\,. \tag{4.133}$$

We define $\check{z} := (\check{z}_i)_{i\geq 1}$ by

$$\check{z}_i(t) := \begin{cases} \sqrt{(I_0)_i}\Big(\cos\big(\tilde{\omega}t + (\phi_0)_i\big) - \mathrm{i}\sin\big(\tilde{\omega}t + (\phi_0)_i\big)\Big)\,, & \text{for } i \in \mathcal{I} \\ 0\,, & \text{for } i \in \mathcal{I}^c\,. \end{cases}$$

By (4.132) we get

$$\sup_{t\in\mathbb{R}} \|\mathrm{z}_*(t) - \eta\check{z}(t)\|_{a,s} = O\left(\eta\mu\right)\,. \tag{4.134}$$

Concerning the z's variables defined in (4.14) we have, recalling (4.17), (4.133) and (4.134),

$$\sup_{t\in\mathbb{R}} \|\mathrm{z}(t) - \eta\check{z}(t)\|_{a,s} = O\left(\eta\mu + \eta^3\right) = O\left(\eta\mu\right) = O\big(\mu^2\big)\,, \tag{4.135}$$

since

$$\eta^2 = 1/\tau \leq 2\mu^2/c_4\,, \tag{4.136}$$

recalling (4.52) and (4.58). Regarding the q's variables defined in (4.2) we have, recalling (4.14) and (4.135),

$$\sup_{t\in\mathbb{R}} \|q(t) - \eta\check{q}(t)\|_{a,s} = O\big(\mu^2\big)\,, \tag{4.137}$$

where $\check{q} := (\check{q}_i)_{i\geq 1}$ and

$$\check{q}_i(t) := \begin{cases} \sqrt{2(I_0)_i}\cos\big(\tilde{\omega}t + (\phi_0)_i\big)\,, & \text{for } i \in \mathcal{I} \\ 0\,, & \text{for } i \in \mathcal{I}^c\,. \end{cases}$$

We note that, by Corollary 4.2.14, the solution in (4.131) belongs to $C^\infty(\mathbb{R}, \mathcal{P}_{a/2,s})$ and, therefore, $q \in C^\infty(\mathbb{R}, \ell^{a/2,s})$. Finally, by Lemma 4.1.1, we have that

$$u(t,x) := \sum_{i\geq 1} q_i(t)\sqrt{\frac{2}{\pi\omega_i}}\,\sin ix$$

belongs to $C^\infty\big(\mathbb{R}\times[0,\pi],\mathbb{R}\big)$ and is a solution of (4). Defining

$$\tilde u(t,x) := \eta\sum_{i\in\mathcal{I}} 2\sqrt{\frac{(I_0)_i}{\pi\omega_i}}\cos\big(\tilde\omega t + (\phi_0)_i\big)\sin ix\,, \tag{4.138}$$

we have, for any $t\in\mathbb{R}$ and $x\in[0,\pi]$,

$$\begin{aligned}|u(t,x)-\tilde u(t,x)| &= \left|\sum_{i\ge1}\Big(q_i(t)-\eta\check q_i(t)\Big)\sqrt{\frac{2}{\pi\omega_i}}\sin ix\right| \\ &\le \sum_{i\ge1}\big|q_i(t)-\eta\check q_i(t)\big|\sqrt{\frac{2}{\pi\omega_i}} \\ &\le c_{12}\|q(t)-\eta\check q(t)\|_{a,s}\,,\end{aligned}$$

where, in the last line, we have used the Cauchy-Schwarz inequality. Therefore, by (4.137), we get

$$\sup_{t\in\mathbb{R},\,x\in[0,\pi]}|u(t,x)-\tilde u(t,x)| = O\big(\mu^2\big)\,. \tag{4.139}$$

Define, for $i\in\mathcal{I}$,

$$a_i := 2\frac{\eta}{\mu}\sqrt{\frac{(I_0)_i}{\pi\omega_i}}\,. \tag{4.140}$$

Since $\eta = 1/\sqrt{\tau}\ge\mu/\sqrt{c_4}$ (recall (4.52) and (4.58)), by Lemma 4.2.2 we get

$$a_i \ge \frac{2}{\sqrt{c_4}}\,.$$

Defining $\varphi_i := (\phi_0)_i$ for $i\in\mathcal{I}$, (9) follows by (4.138),(4.139) and (4.140). Estimate (11) follows from Lemma 4.2.16, while (10) follows from (4.44) and (4.136). Finally, the statement about the total number of geometrically distinct solutions follows from Corollary 4.2.19.

Remark 4.2.20. *We can improve estimate* (9) *or, equivalently,* (4.139)*. Indeed, for any fixed $k\ge1$ and ζ in* (4.131)*, by* (4.110)*, we have $\|\zeta\|_{H^k_{T,a,s}}\le$ $\mathrm{const}(k)\,\mu$, where $\mathrm{const}(k)$ is a suitable large constant depending on $\mathcal{I}$, a,s and k. Arguing as above and using the Sobolev immersion $H^k\subset C^{k-1}$, we get $\|q-\eta\check q\|_{C^{k-1}(\mathbb{R},\ell^{a,s})}\le\mathrm{const}(k)\,\mu^2$. Therefore $\sup_{\mathbb{R}\times[0,\pi]}\big|\partial_t^h(u-\tilde u)\big|\le\mathrm{const}(k)\,\mu^2$ for any $h\le k-1$. Since the estimates on the x–derivatives directly follows by the analyticity, we conclude that $\|u-\tilde u\|_{C^k(\mathbb{R}\times[0,\pi])}\le\mathrm{const}(k)\,\mu^2$. We remark that, if one needs the previous C^k–estimate, the constant c in Theorem 1 must depend on k.*

Part III

Periodic orbits with irrational frequency

Chapter 5

A Birkhoff–Lewis type theorem for the NLW by a Nash–Moser algorithm

In this Chapter we will prove Theorem 2. Many aspects of the proof are similar to the ones of Theorem 1; we will not repeat them here. In particular the hamiltonian setting, the Birkhoff Normal Form and the geometrical construction[1] are essentially the same apart from the fact that we work here with analytic in time functions and consider μ fixed. On the other hand we will focus on the new aspects: using symmetry to solve the bifurcation equation and analysis of small divisors.
We have been inspired by [BeBo05] in facing many problems of the following.

Now we give a scheme of this Chapter.

5.1 Analytic norms

We define the functional spaces in which the solutions live: these are the Hilbert algebras of time-periodic *analytic* curves taking values in the phase space or in suitable subspaces. We prove some technical lemmata on the composition of analytic functions. We also introduce a special subspace of the space of linear operators defined on the above Hilbert spaces. Such subspace contains all the "product type" linear operators, namely the Toepliz operator, but also other kinds of linear operators that are not product operators; we will denote this class of operators "quasi–product operators".

5.2 Symmetry of the Hamiltonian

We consider the change of variables defined in Proposition 4.1.6 putting the Hamiltonian of the wave equation in (4.15) in Birkhoff Normal Form. In particular we prove that it is symmetric in the variables $z, \bar{z}$. This implies that, if the "old" Hamiltonian is symmetric, namely $H(z, \bar{z}) = H(\bar{z}, z)$, then the "new" Hamiltonian in Birkhoff Normal Form is still symmetric. Note that if $f(u)$ in (4) is odd, the associated Hamiltonian is symmetric.

[1] Namely Section 4.1 and Subsection 4.2.1.

Thereafter we introduce real coordinates $(I, \phi, \ \hat{p}, \hat{q})$ and look for solution $(I(t), \phi(t), \hat{p}(t), \hat{q}(t)) = (I_0 + \eta J(t), \tilde{\omega} t + \eta\psi(t), \eta p(t), \eta q(t))$, with J, q even and ψ, p odd. If one is looking for solutions of this particular form and the Hamiltonian possesses the above symmetry property, then, in the language of Part II, the kernel equation is automatically solved. This means that, being ψ odd, the bifurcation equation (4.99) is solved taking $\phi_0 = 0$ (see Proposition 5.2.7).

5.3 Solution of J and ψ

We find J(even) and ψ(odd) as functions of the parameters p, q by the Fixed Point Theorem using the symmetry of the Hamiltonian.

5.4 Linearized equation

We consider the nonlinear equation for (p, q) obtained substituting the expression for $J = J(p, q)$ and $\psi = \psi(p, q)$ into the equations of motion. We prove that its linearized operator is a "quasi–product operator".

5.5 Nash–Moser scheme

We set out the Nash–Moser scheme, introduce the first order Melnikov condition on the excision of resonant frequencies and state the crucial Lemma 5.5.4 on the inversion of the linearized operator close to the origin. Moreover we perform the Nash–Moser iteration and find time–periodic solutions of the nonlinear wave equation for periods belonging to the non–resonant set $\mathcal{C}$ obtained by the above excision procedure.
Nothing remains but to prove the invertibility of the linear operator carried out in Sections 5.6–5.9 and to estimate the measure of $\mathcal{C}$ (see Section 5.10).

5.6 Evaluating the linearized operator

We perform a suitable change of variables after which the linearized operator is decomposed into two terms. The first one is a diagonal term in time–Fourier expansion, while the second one is an off–diagonal "quasi–product operator".

5.7 Diagonal term

We diagonalize *in space* the time–Fourier components of the above diagonal *in time* operator. The difficulties arise from the fact that we deal with a *not symmetric* operator. This fact requires a not standard spectral analysis and a "weighted asymmetric diagonalization" (see Subsection 5.7.3).

5.8 Estimate on the off-diagonal term

Using a suitable lemma on small divisors, which will be proved in Section 5.9, we show that the off–diagonal "quasi–product operator" in the diagonalizing basis introduced in Section 5.7 is small. Therefore the linearized operator (in the above basis) decomposes into an invertible term plus a small perturbation and it is, hence, invertible.

5.9 Small divisors

We carry out the analysis of small divisors. The difficulties derive from the fact that the small divisors originate from a not symmetric first order operator.

5.10 Measure estimates

We finally prove that the set $\mathcal{C}$ of non resonant frequencies obtained in Section 5.5 by an excision procedure has large measure.

5.11 Minimal period

We prove a lower bound estimate on the minimal periods.

5.1 Analytic norms

Let E be a Hilbert space. Given $T > 0$, $\alpha > 0$ and $\sigma > \frac{1}{2}$, let us define

$$H_E^{\alpha,\sigma} := H_{T-\mathrm{per}}^{\alpha,\sigma}(\mathbb{R}, E)\,. \tag{5.1}$$

Let us write a T–periodic $h : \mathbb{R} \longrightarrow E$ as $h(t) = \sum_{k\in\mathbb{Z}} h_k e^{\mathrm{i}kt/\tau}$ where $\tau = T/2\pi$, and $h_k \in E$. $H_E^{\alpha,\sigma}$ is a Hilbert space with norm

$$\|h\|^2_{H_E^{\alpha,\sigma}} := \sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\|h_k\|_E^2 \tag{5.2}$$

where $|k|_* := \max\{|k|, 1\}$.

Remark 5.1.1. *We remark that in this section we use the complex exponential while in the following sections we will use the real notations with cosines and sines. We hope this fact will not create confusion.*

5.1.1 Some technical lemmata

Lemma 5.1.2. *Let be $\varsigma > 1$. Then $(x_1 + \ldots + x_m)^\varsigma \le m^{\varsigma-1}(x_1^\varsigma + \ldots + x_m^\varsigma)$ for $x_i \ge 0$, $i = 1, \ldots, m$.*

PROOF. Let be $x = (x_1, \ldots, x_m)$, $f(x) := (x_1 + \ldots + x_m)^\varsigma$ and $g(x) := x_1^\varsigma + \ldots + x_m^\varsigma$. By the homogeneity of g, it is enough to show that $\max_S f \le m^{\varsigma-1}$, where $S := \{x_i \ge 0, g(x) = 1\}$. Denoting λ the Lagrange multiplier, we have $(x_1 + \ldots + x_m)^{\varsigma-1} = \lambda x_i^{\varsigma-1}$, for all i. It follows that $x_i = r =$ const and then, being $g(x) = 1$, one has $mr^\varsigma = 1$, namely $r = \frac{1}{m^{1/\varsigma}}$. It results $f\big(\frac{1}{m^{1/\varsigma}}, \ldots, \frac{1}{m^{1/\varsigma}}\big) = m^{\varsigma-1}$.

$\square$

Lemma 5.1.3. *Let be $\varsigma > 1$, $m \in \mathbb{N}$, $j \in \mathbb{Z}$. Then*

$$\sum_{k_1,\ldots,k_m\in\mathbb{Z}} \left(\frac{|j|_*}{|j - k_1 - \ldots - k_m|_*|k_1|_* \cdots |k_m|_*}\right)^\varsigma \le (m+1)^\varsigma B_\varsigma^m\,, \tag{5.3}$$

where

$$B_\varsigma := \sum_{k\in\mathbb{Z}} \frac{1}{|k|_*^\varsigma} < \infty\,. \tag{5.4}$$

PROOF. By Lemma 5.1.2 we have that

$$\begin{aligned}
|j|_*^\varsigma &\le \big(|j-k_1-\ldots-k_m|_* + |k_1|_* + \ldots + |k_m|_*\big)^\varsigma \\
&\le (m+1)^{\varsigma-1}\big(|j-k_1-\ldots-k_m|_*^\varsigma + |k_1|_*^\varsigma + \ldots + |k_m|_*^\varsigma\big) \qquad (5.5)
\end{aligned}$$

and, using (5.5), one can write

$$\begin{aligned}
\sum_{k_1,\ldots,k_m\in\mathbb{Z}} & \left(\frac{|j|_*}{|j-k_1-\ldots-k_m|_*|k_1|_*\cdots|k_m|_*}\right)^\varsigma \\
&\le (m+1)^{\varsigma-1}\sum_{k_2,\ldots,k_m}\frac{1}{|k_2|_*^\varsigma\cdots|k_m|_*^\varsigma} \\
&\quad \cdot\sum_{k_1}\frac{|j-k_1-\ldots-k_m|_*^\varsigma+|k_1|_*^\varsigma+\ldots+|k_m|_*^\varsigma}{|j-k_1-\ldots-k_m|_*^\varsigma|k_1|_*^\varsigma} \\
&= (m+1)^{\varsigma-1}\sum_{k_2,\ldots,k_m}\frac{1}{|k_2|_*^\varsigma\cdots|k_m|_*^\varsigma}\cdot\left(\sum_{k_1}\frac{1}{|k_1|_*^\varsigma}\right. \\
&\quad \left.+\sum_{k_1}\frac{1}{|j-k_1-\ldots-k_m|_*^\varsigma}+\sum_{k_1}\frac{|k_2|_*^\varsigma+\ldots+|k_m|_*^\varsigma}{|j-k_1-\ldots-k_m|_*^\varsigma|k_1|_*^\varsigma}\right) \\
&\le (m+1)^{\varsigma-1}\sum_{k_2,\ldots,k_m}\frac{1}{|k_2|_*^\varsigma\cdots|k_m|_*^\varsigma} \\
&\quad \cdot\left(2B_\varsigma+\sum_{k_1}\frac{|k_2|_*^\varsigma+\ldots+|k_m|_*^\varsigma}{|j-k_1-\ldots-k_m|_*^\varsigma|k_1|_*^\varsigma}\right) \\
&\le (m+1)^{\varsigma-1}\left(2B_\varsigma^m+\sum_{k_1}\frac{1}{|k_1|_*^\varsigma}\right. \\
&\quad \left.\cdot\sum_{k_2,\ldots,k_m}\frac{|k_2|_*^\varsigma+\ldots+|k_m|_*^\varsigma}{|j-k_1-\ldots-k_m|_*^\varsigma|k_2|_*^\varsigma\cdots|k_m|_*^\varsigma}\right) \\
&= (m+1)^{\varsigma-1}\left(2B_\varsigma^m+(m-1)\sum_{k_1}\frac{1}{|k_1|_*^\varsigma}\right. \\
&\quad \left.\cdot\sum_{k_3,\ldots,k_m}\sum_{k_2}\frac{1}{|j-k_1-\ldots-k_m|_*^\varsigma|k_3|_*^\varsigma\cdots|k_m|_*^\varsigma}\right) \\
&= (m+1)^{\varsigma-1}\left[2B_\varsigma^m+(m-1)B_\varsigma^m\right]=(m+1)^\varsigma B_\varsigma^m\,,
\end{aligned}$$

getting the estimate (5.3).

□

From now on, we will denote by $\|\cdot\|_{\mathrm{op}}$ the standard operatorial norm.

Lemma 5.1.4. *Let be $L_n : E\times\cdots\times E(n \text{ times}) \longrightarrow F$ a linear and continuous operator. Let be $h^{(1)},\ldots,h^{(n)} \in H_E^{\alpha,\sigma}$, with $\alpha > 0$, $\sigma > 1/2$ and $g(t) :=$*

$L_n[h^{(1)}(t), \cdots, h^{(n)}(t)]$. *Then* $g \in H_F^{\alpha,\sigma}$ *and, in particular,*

$$\|g\|_{H_F^{\alpha,\sigma}} \leq n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|L_n\|_{\mathrm{op}} \|h^{(1)}\|_{H_E^{\alpha,\sigma}} \ldots \|h^{(n)}\|_{H_E^{\alpha,\sigma}} ,$$

where

$$\|L_n\|_{\mathrm{op}} := \sup_{\|h_i\|_E = 1\,, 1 \leq i \leq n} \|L_n[h^{(1)}, \ldots, h^{(n)}]\|_F \,.$$

PROOF. Let be $h^{(j)}(t) = \sum_k e^{\mathrm{i}kt/\tau} h_k^{(j)}$, $j = 1, \ldots, n$. Hence,

$$\begin{aligned} g(t) &= L_n[h^{(1)}(t), \ldots, h^{(n)}(t)] \\ &= L_n\Big[\sum_{k_1} e^{\mathrm{i}k_1 t/\tau} h_{k_1}^{(1)}, \ldots, \sum_{k_n} e^{\mathrm{i}k_n t/\tau} h_{k_n}^{(n)}\Big] \\ &= \sum_j e^{\mathrm{i}jt/\tau} \sum_{k_1+\ldots+k_n=j} L_n[h_{k_1}^{(1)}, \ldots, h_{k_n}^{(n)}] \\ &= \sum_j e^{\mathrm{i}jt/\tau} \sum_{k_1,\ldots,k_{n-1}} L_n[h_{k_1}^{(1)}, \ldots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\ldots-k_{n-1}}^{(n)}] \,. \end{aligned}$$

Being $g = \sum_j g_j e^{\mathrm{i}jt/\tau}$ it follows that

$$g_j = \sum_{k_1,\ldots,k_{n-1}} L_n[h_{k_1}^{(1)}, \ldots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\ldots-k_{n-1}}^{(n)}]$$

and, from the triangular inequality,

$$\begin{aligned} \|g_j\|_F &\leq \sum_{k_1,\ldots,k_{n-1}} \|L_n[h_{k_1}^{(1)}, \ldots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\ldots-k_{n-1}}^{(n)}]\|_F \\ &\leq \|L_n\|_{\mathrm{op}} \sum_{k_1,\ldots,k_{n-1}} \|h_{k_1}^{(1)}\|_E \ldots \|h_{j-k_1-\ldots-k_{n-1}}^{(n)}\|_E \,. \end{aligned} \tag{5.6}$$

By the definition of the norm (5.2) and by using (5.6), one gets

$$\begin{aligned} \|g\|^2_{H_F^{\alpha,\sigma}} &= \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \|g_j\|_F^2 \\ &\leq \|L_n\|^2_{\mathrm{op}} \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \bigg(\sum_{k_1,\ldots,k_{n-1}} \|h_{k_1}^{(1)}\|_E \cdots \|h_{j-k_1-\ldots-k_{n-1}}^{(n)}\|_E \bigg)^2 . \end{aligned}$$

Define $\vec{k} := (k_1, \ldots, k_{n-1})$ and $x_{\vec{k}} := \|h_{k_1}^{(1)}\|_E \cdots \|h_{k_{n-1}}^{(n-1)}\|_E \|h_{j-k_1-\ldots-k_{n-1}}^{(n)}\|_E$, then

$$\|g\|^2_{H_F^{\alpha,\sigma}} \leq \|L_n\|^2_{\mathrm{op}} \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \Big(\sum_{\vec{k}} x_{\vec{k}} \Big)^2 . \tag{5.7}$$

In order to estimate $\sum_{\vec{k}} x_{\vec{k}}$, let us define

$$\gamma_{\vec{k}} := \left(\frac{|k_1|_* \cdots |k_{n-1}|_* |j - k_1 - \ldots - k_{n-1}|_*}{|j|_*} \right)^\sigma .$$

We have, using the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\Big(\sum_{\vec{k}} x_{\vec{k}}\Big)^2 &= \Big(\sum_{\vec{k}} \frac{1}{\gamma_{\vec{k}}}\,\gamma_{\vec{k}} x_{\vec{k}}\Big)^2 \le \sum_{\vec{k}} \Big(\frac{1}{\gamma_{\vec{k}}}\Big)^2 \sum_{\vec{k}} \big(\gamma_{\vec{k}} x_{\vec{k}}\big)^2 \\
&\le n^{2\sigma} B_{2\sigma}^{n-1} \sum_{\vec{k}} \big(\gamma_{\vec{k}} x_{\vec{k}}\big)^2 , \qquad (5.8)
\end{aligned}
$$

where in the last inequality we used Lemma 5.1.3, with $\varsigma = 2\sigma > 1$ and $m = n-1$. Being

$$
\begin{aligned}
\sum_{\vec{k}} \big(\gamma_{\vec{k}} x_{\vec{k}}\big)^2 = {} & \frac{1}{|j|_*^{2\sigma}} \sum_{k_1} \big(|k_1|_*^{2\sigma} \|h^{(1)}_{k_1}\|_E^2\big) \cdot \sum_{k_2} \big(|k_2|_*^{2\sigma} \|h^{(2)}_{k_2}\|_E^2\big) \cdots \\
& \cdots \sum_{k_{n-1}} \big(|k_{n-1}|_*^{2\sigma} \|h^{(n-1)}_{k_{n-1}}\|_E^2\big) |j-k_1-\ldots-k_{n-1}|_*^{2\sigma} \|h^{(n)}_{j-k_1-\ldots-k_{n-1}}\|_E^2 ,
\end{aligned}
$$

from (5.7) and (5.8), it follows that

$$
\begin{aligned}
\|g\|^2_{H_F^{\alpha,\sigma}} \le {} & \|L_n\|^2_{\mathrm{op}} n^{2\sigma} B_{2\sigma}^{n-1} \sum_{k_1} \big(|k_1|_*^{2\sigma} \|h^{(1)}_{k_1}\|_E^2\big) \cdots \sum_{k_{n-1}} \big(|k_{n-1}|_*^{2\sigma} \|h^{(n-1)}_{k_{n-1}}\|_E^2\big) \\
& \cdot \sum_j \Big(|j-k_1-\ldots-k_{n-1}|_*^{2\sigma} \|h^{(n)}_{j-k_1-\ldots-k_{n-1}}\|_E^2\Big) e^{2\alpha|j|} \\
\le {} & \|L_n\|^2_{\mathrm{op}} n^{2\sigma} B_{2\sigma}^{n-1} \left(\sum_{k_1} e^{2\alpha|k_1|} |k_1|_*^{2\sigma} \|h^{(1)}_{k_1}\|_E^2\right) \cdots \\
& \cdots \left(\sum_{k_{n-1}} e^{2\alpha|k_{n-1}|} |k_{n-1}|_*^{2\sigma} \|h^{(n-1)}_{k_{n-1}}\|_E^2\right) \\
& \cdot \left(\sum_j e^{2\alpha|j-k_1-\ldots-k_{n-1}|} |j-k_1-\ldots-k_{n-1}|_*^{2\sigma} \|h^{(n)}_{j-k_1-\ldots-k_{n-1}}\|_E^2\right) \\
= {} & \|L_n\|^2_{\mathrm{op}} n^{2\sigma} B_{2\sigma}^{n-1} \|h^{(1)}\|^2_{H_E^{\alpha,\sigma}} \cdots \|h^{(n)}\|^2_{H_E^{\alpha,\sigma}} ,
\end{aligned}
$$

where we used that $e^{2\alpha|j|} \le e^{2\alpha|k_1|} \cdots e^{2\alpha|j-k_1-\ldots-k_{n-1}|}$ and, in the last equality, that

$$
\|h^{(n)}\|^2_{H_E^{\alpha,\sigma}} := \sum_j e^{2\alpha|j-k_1-\ldots-k_{n-1}|} |j-k_1-\ldots-k_{n-1}|_*^{2\sigma} \|h^{(n)}_{j-k_1-\ldots-k_{n-1}}\|_E^2
$$

by the definition of the norm.

□

Lemma 5.1.5. *Let be $f : E \longrightarrow F$ analytic for $\|x\|_E < r_0$,*

$$
f(x) = \sum_{n \ge n_0} \frac{1}{n!}\, d^n f(0)[x, \ldots, x] \, . \qquad (5.9)
$$

Then

$$\|d^n f(0)\|_{\rm op} \le M_{r_1} \left(\frac{n}{r_1}\right)^n , \qquad \forall\ 0 < r_1 < r_0 ,$$

where $M_{r_1} := \max_{\|x\|_E \le r_1} \|f(x)\|_F$.

PROOF. Consider the map $d^n f(0)$ defined by[2]

$$d^n f(0)[h_1, \ldots, h_n] = \frac{1}{(2\pi {\rm i})^n} \int_{|\zeta_1|=\varepsilon} \cdots \int_{|\zeta_n|=\varepsilon} \frac{f(\zeta_1 h_1 + \ldots + \zeta_n h_n)}{\zeta_1^2 \cdots \zeta_n^2} \, d\zeta_1 \cdots d\zeta_n \, .$$

Defining $\varepsilon := r_1/n$, we have, if $\|h_i\|_E = 1$, for all $1 \le i \le n$,

$$\|\zeta_1 h_1 + \ldots + \zeta_n h_n\|_E \le \varepsilon(\|h_1\|_E + \ldots + \|h_n\|_E) = \varepsilon n = r_1 < r_0 \, .$$

Therefore one gets

$$\begin{aligned} \|d^n f(0)\|_{\rm op} \ &:= \sup_{\|h_i\|_E=1\,,1\le i\le n} \|d^n f(0)[h_1, \ldots, h_n]\|_F \\ &\le \ \frac{1}{\varepsilon^n} \, \|f(\zeta_1 h_1 + \ldots + \zeta_n h_n)\|_F \\ &\le \ \frac{M_{r_1}}{\varepsilon^n} = M_{r_1} \left(\frac{n}{r_1}\right)^n . \end{aligned}$$

□

Theorem 5.1.6. *Let be* $f : E \longrightarrow F$, $f(x) = \sum_{n\ge n_0} \frac{1}{n!} d^n f(0)[x, \ldots, x]$, *analytic for* $\|x\|_E < r_0$, *such that* $f(0) = 0$, *namely* $n_0 \ge 1$. *There exist* $M > 0$ *and* $r > 0$ *such that if* $h \in H_E^{\alpha,\sigma}$ *with* $\alpha > 0$, $\sigma > 1/2$ *and* $\|h\|_{H_E^{\alpha,\sigma}} < r$ *then* $f(h) \in H_F^{\alpha,\sigma}$ *and*

$$\|f(h)\|_{H_F^{\alpha,\sigma}} \le M \sum_{n\ge n_0} \left(\frac{\|h\|_{H_E^{\alpha,\sigma}}}{r}\right)^n ,$$

where $f(h) : \mathbb{R} \longrightarrow F$, $t \longmapsto f(h(t))$ *and* $[f(h)](t) := f(h(t))$. *Moreover, defining*

$$f_* : H_E^{\alpha,\sigma} \longrightarrow H_F^{\alpha,\sigma} ,$$

by $\big(f_*(h)\big)(t) := f(h(t))$, *then*

$$f_* \in \mathcal{A}\big(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}\big) \, .$$

Remark 5.1.7. *Notice that* M *and* r *do not depend on* σ.

Remark 5.1.8. *Notice that, if* $h, \tilde h \in H_E^{\alpha,\sigma}$ *with* $L_n = d^n f(0)$, *then, if* $g_n := d^n f(0)[h, \ldots, h, \tilde h]$, *one has*

$$g_n(t) := L_n f(0)[h(t), \ldots, h(t), \tilde h(t)] \, ,$$

from which it follows that, by using Lemma 5.1.4,

$$\|g_n\|_{H_F^{\alpha,\sigma}} \le n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|d^n f(0)\|_{\rm op} \|h\|_{H_E^{\alpha,\sigma}}^{n-1} \|\tilde h\|_{H_E^{\alpha,\sigma}} \, .$$

[2]See [PT] pg. 136, 137.

PROOF. For $h \in H_E^{\alpha,\sigma}$, by Lemma 5.1.4 with $L_n := d^n f(0)$,

$$\begin{aligned}
\|f(h)\|_{H_F^{\alpha,\sigma}} &= \Big\| \sum_{n\geq n_0} \frac{1}{n!}\, d^n f(0)[h,\dots,h] \Big\|_{H_F^{\alpha,\sigma}} \\
&\leq \sum_{n\geq n_0} \frac{1}{n!} \|d^n f(0)[h,\dots,h]\|_{H_F^{\alpha,\sigma}} \\
&\leq \sum_{n\geq n_0} \frac{1}{n!}\, n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|d^n f(0)\|_{\mathrm{op}} \|h\|^n_{H_E^{\alpha,\sigma}} \\
&\leq \sum_{n\geq n_0} \left(\frac{1}{n!}\, n^\sigma B_{2\sigma}^{\frac{n-1}{2}} M_{r_1} \frac{n^n}{r_1^n} \right) \|h\|^n_{H_E^{\alpha,\sigma}} \,,
\end{aligned}$$

where, in the last inequality, we used Lemma 5.1.5. Therefore, we can choose $M := M_{r_1}$ and $r < r_1$ suitably small.
Now, let us prove that f_* is continuously differentiable. We have

$$Df_* : H_E^{\alpha,\sigma} \longrightarrow \mathcal{L}\big(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}\big)\,.$$

Noting that $f' : E \longrightarrow \mathcal{L}(E,F)$, one has $Df_*(h) = f' \circ h$, namely for every $\tilde h \in H_E^{\alpha,\sigma}$, $\big(Df_*(h)\big)[\tilde h] \in H_F^{\alpha,\sigma}$, is defined by $\Big(\big(Df_*(h)\big)[\tilde h]\Big)(t) := \big(f'(h(t))\big)[\tilde h(t)]$. Recalling (5.9), we have

$$\big(f'(x)\big)[\tilde x] := \sum_{n\geq n_0} \frac{1}{n!}\, n\, d^n f(0)[x,\dots,x,\tilde x] \tag{5.10}$$

and hence, using the symmetry of $d^n f(0)$,

$$\big(f'(x)\big)[\tilde x] = \frac{d}{ds}\Big|_{s=0} f(x+s\tilde x) := \sum_{n\geq n_0} \frac{1}{(n-1)!}\, d^n f(0)[x,\dots,x,\tilde x]\,. \tag{5.11}$$

Denoting

$$g_n(t) := d^n f(0)[h(t),\dots,h(t),\tilde h(t)]\,,$$

we obtain that $Df_*(h)$ is bounded, indeed

$$\begin{aligned}
\big\|Df_*(h)\big\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})} &= \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \Big\| (Df_*(h))[\tilde h] \Big\|_{H_F^{\alpha,\sigma}} \\
&= \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\geq n_0} \frac{1}{(n-1)!}\, g_n \Big\|_{H_F^{\alpha,\sigma}} \\
&\leq \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \sum_{n\geq n_0} \frac{1}{(n-1)!} \|g_n\|_{H_F^{\alpha,\sigma}} \\
&\leq \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \sum_{n\geq n_0} \frac{n^\sigma B_{2\sigma}^{\frac{n-1}{2}}}{(n-1)!} \|d^n f(0)\|_{\mathrm{op}} \|h\|^{n-1}_{H_E^{\alpha,\sigma}} \|\tilde h\|_{H_E^{\alpha,\sigma}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\geq n_0} \frac{n^\sigma B_{2\sigma}^{\frac{n-1}{2}}}{(n-1)!} M_{r_1}\Big(\frac{n}{r_1}\Big)^n \|h\|_{H_E^{\alpha,\sigma}}^{n-1} \\
&\leq M \sum_{n\geq n_0-1} \Big(\frac{\|h\|_{H_E^{\alpha,\sigma}}}{r}\Big)^n,
\end{aligned}
$$

where we used Lemma 5.1.5 for $0 < r < r_1$ sufficiently small and Remark 5.1.8. Nothing remains but to prove that $Df_* : H_E^{\alpha,\sigma} \longrightarrow \mathcal{L}\big(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}\big)$ is continuous. Let be $\tilde{h}, h_1, h_2 \in H_E^{\alpha,\sigma}$. We have

$$
\begin{aligned}
&\big\|Df_*(h_1) - Df_*(h_2)\big\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})} \\
&= \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \big\|Df_*(h_1)[\tilde{h}] - Df_*(h_2)[\tilde{h}]\big\|_{H_F^{\alpha,\sigma}} \\
&= \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\geq n_0} \frac{1}{(n-1)!} \Big(d^n f(0)[h_1,\ldots,h_1,\tilde{h}] \\
&\quad -d^n f(0)[h_2,\ldots,h_2,\tilde{h}]\Big)\Big\|_{H_F^{\alpha,\sigma}} \\
&= \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\geq n_0} \frac{1}{(n-1)!} \Big(d^n f(0)[h_1-h_2,h_1,\ldots,h_1,\tilde{h}] \\
&\quad +d^n f(0)[h_2,h_1,\ldots,h_1,\tilde{h}] - d^n f(0)[h_2,\ldots,h_2,\tilde{h}]\Big)\Big\|_{H_F^{\alpha,\sigma}} \\
&= \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\geq n_0} \frac{1}{(n-1)!} \Big(d^n f(0)[h_1-h_2,h_1,\ldots,h_1,\tilde{h}] \\
&\quad +d^n f(0)[h_2,h_1-h_2,\ldots,h_1,\tilde{h}] + d^n f(0)[h_2,h_2,h_1,\ldots,h_1,\tilde{h}] \\
&\quad -d^n f(0)[h_2,\ldots,h_2,\tilde{h}]\Big)\Big\|_{H_F^{\alpha,\sigma}} \\
&= \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\geq n_0} \frac{1}{(n-1)!} \Big(d^n f(0)[h_1-h_2,h_1,\ldots,h_1,\tilde{h}] \\
&\quad +d^n f(0)[h_2,h_1-h_2,h_1,\ldots,h_1,\tilde{h}] + \ldots \\
&\quad +d^n f(0)[h_2,\ldots,h_2,h_1-h_2,h_1,\ldots,h_1,\tilde{h}] + \ldots \\
&\quad -d^n f(0)[h_2,\ldots,h_2,h_1-h_2,\tilde{h}]\Big)\Big\|_{H_F^{\alpha,\sigma}} \\
&\leq \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \sum_{n\geq n_0} \frac{1}{(n-1)!} n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|d^n f(0)\|_{\mathrm{op}} \|\tilde{h}\|_{H_E^{\alpha,\sigma}} \\
&\quad \cdot\Big(\|h_1\|_{H_E^{\alpha,\sigma}}^{n-2} + \ldots + \|h_2\|_{H_E^{\alpha,\sigma}}^{j-1}\|h_1\|_{H_E^{\alpha,\sigma}}^{n-j-1} + \ldots + \|h_2\|_{H_E^{\alpha,\sigma}}^{n-2}\Big)\|h_1-h_2\|_{H_E^{\alpha,\sigma}} \\
&\leq \sup_{\|\tilde{h}\|_{H_E^{\alpha,\sigma}}=1} \sum_{n\geq n_0} \frac{1}{(n-1)!} n^\sigma B_{2\sigma}^{\frac{n-1}{2}} M_{r_1}\Big(\frac{n}{r_1}\Big)^n (n-1) r^{n-2} \|h_1-h_2\|_{H_E^{\alpha,\sigma}}
\end{aligned}
$$

where we used Lemma 5.1.5 and that $\|h_i\| \leq r$, for $i = 1, 2$, taking $r < r_1$ sufficiently small. $\square$

5.1.2 Special norms of linear operators

Definition 5.1.9. *Let E, F Hilbert spaces. Fix $T = 2\pi\tau > 0$ and consider $H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}$ defined in (5.1). Let us define the following subspace of $\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})$:*

$$\begin{aligned}\mathcal{L}^{\alpha,\sigma}(E,F) \;:=\; \Big\{ & L \in \mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}) \qquad \text{s.t.} \\ & L\Big[\sum_\ell e^{\mathrm{i}\ell t/\tau} x_\ell\Big] = \sum_k e^{\mathrm{i}kt/\tau} \sum_\ell L_{k\ell}[x_\ell]\,, \quad \vert\!\vert\!\vert L \vert\!\vert\!\vert < \infty \Big\}\end{aligned}$$

with $L_{k\ell} \in \mathcal{L}(E,F)$ and

$$\vert\!\vert\!\vert L \vert\!\vert\!\vert^2 := \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|L_{k\ell}\|^2_{\mathcal{L}(E,F)} < \infty\,. \tag{5.12}$$

We will call $\mathcal{L}^{\alpha,\sigma}(E,F)$ the subspace of "quasi–product operators" (for this terminology see Remark 5.1.13).

We are going to show some properties of $\mathcal{L}^{\alpha,\sigma}(E,F)$.

Proposition 5.1.10. *$\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ is a norm and $\mathcal{L}^{\alpha,\sigma}(E,F)$ is complete with respect to it.*

PROOF. Show first the triangular inequality. Let be $L^{(1)}$, $L^{(2)}$ linear operators of $\mathcal{L}^{\alpha,\sigma}(E,F)$. Then

$$\begin{aligned}\vert\!\vert\!\vert L^{(1)} + L^{(2)} \vert\!\vert\!\vert^2 &= \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|L^{(1)}_{k\ell} + L^{(2)}_{k\ell}\|^2 \\ &\le \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \\ &\quad \Big(\|L^{(1)}_{k\ell}\|^2 + \|L^{(2)}_{k\ell}\|^2 + 2\|L^{(1)}_{k\ell}\|\,\|L^{(2)}_{k\ell}\|\Big) \\ &\le \vert\!\vert\!\vert L^{(1)} \vert\!\vert\!\vert^2 + \vert\!\vert\!\vert L^{(2)} \vert\!\vert\!\vert^2 \\ &\quad +2\sup_\ell \sum_k \Big(e^{\alpha|k-\ell|}|k-\ell|_*^{\sigma}\|L^{(1)}_{k\ell}\|\Big)\Big(e^{\alpha|k-\ell|}|k-\ell|_*^{\sigma}\|L^{(2)}_{k\ell}\|\Big) \\ &\le \vert\!\vert\!\vert L^{(1)} \vert\!\vert\!\vert^2 + \vert\!\vert\!\vert L^{(2)} \vert\!\vert\!\vert^2 + 2\sup_\ell \Big(\sum_k e^{\alpha|k-\ell|}|k-\ell|_*^{\sigma}\|L^{(1)}_{k\ell}\|\Big) \\ &\quad \cdot\Big(\sum_k e^{\alpha|k-\ell|}|k-\ell|_*^{\sigma}\|L^{(2)}_{k\ell}\|\Big) \\ &\le \vert\!\vert\!\vert L^{(1)} \vert\!\vert\!\vert^2 + \vert\!\vert\!\vert L^{(2)} \vert\!\vert\!\vert^2 + 2\vert\!\vert\!\vert L^{(1)} \vert\!\vert\!\vert\,\vert\!\vert\!\vert L^{(2)} \vert\!\vert\!\vert \\ &\le \Big(\vert\!\vert\!\vert L^{(1)} \vert\!\vert\!\vert + \vert\!\vert\!\vert L^{(2)} \vert\!\vert\!\vert\Big)^2,\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Let us show that $\mathcal{L}^{\alpha,\sigma}(E,F)$ endowed with the norm $\vert\!\vert\!\vert\cdot\vert\!\vert\!\vert$ is complete. Let be $L^{(n)}$ a Cauchy sequence, namely for all $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that, for all $m > n \geq N$, it results that $\|\!| L^{(n)} - L^{(m)} \|\!| \leq \varepsilon$. It follows that for all k, ℓ, $L^{(n)}_{k\ell}$ is a Cauchy sequence. Hence, being $\mathcal{L}(E, F)$ complete, $L^{(n)}_{k\ell} \to L_{k\ell}$ for a suitable $L_{k\ell} \in \mathcal{L}(E, F)$. We want to show that $L : x \longmapsto \sum_k e^{\mathrm{i}kt/\tau} \sum_\ell L_{k\ell}[x_\ell]$ belongs to $\mathcal{L}(E, F)$ and that $L^{(n)}$ tends to L. Fix $M > 0$ such that

$$\sup_{|\ell| \leq M} \sum_{|k| \leq M} e^{2\alpha|k-\ell|} |k - \ell|_*^{2\sigma} \| L^{(n)}_{k\ell} - L^{(m)}_{k\ell} \|^2 \leq \|\!| L^{(n)} - L^{(m)} \|\!|^2 \leq \varepsilon^2 .$$

Taking the limit for $m \to \infty$, we obtain

$$\sup_{|\ell| \leq M} \sum_{|k| \leq M} e^{2\alpha|k-\ell|} |k - \ell|_*^{2\sigma} \| L^{(n)}_{k\ell} - L_{k\ell} \|^2 \leq \varepsilon^2 ,$$

and hence taking the sup on $M > 0$ one obtains

$$\|\!| L^{(n)} - L \|\!|^2 \leq \varepsilon^2 ,$$

namely

$$\|\!| L^{(n)} - L \|\!| \leq \varepsilon .$$

Moreover

$$\|\!| L \|\!| \leq \|\!| L^{(n)} - L \|\!| + \|\!| L^{(n)} \|\!| \leq \varepsilon + \|\!| L^{(n)} \|\!| < \infty .$$

Thus we get that $\mathcal{L}^{\alpha,\sigma}(E, F)$ with the norm $\|\!| \cdot \|\!|$ is complete. □

Proposition 5.1.11. *Let be $\mathcal{L}^{\alpha,\sigma}(E, F)$ as in Definition 5.1.9 endowed with the norm $\|\!| \cdot \|\!|$. One has*

$$\| \cdot \|_{\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})} \leq \mathrm{const} \|\!| \cdot \|\!| .$$

PROOF. Let be $x \in H_E^{\alpha,\sigma}$ with $|x|^2_{H_E^{\alpha,\sigma}} = \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} |x_k|_E^2 < \infty$ and be $L \in \mathcal{L}^{\alpha,\sigma}(E, F) \subseteq \mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})$. We have that, by using (4.11),

$$\begin{aligned}
\|L\|^2_{\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})} &= \sup_{|x|_{H_E^{\alpha,\sigma}} = 1} \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} |(Lx)_k|_F^2 \\
&= \sup \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \Big| \sum_\ell L_{k\ell} x_\ell \Big|_F^2 \\
&= \sup \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \Big(\sum_\ell \|L_{k\ell}\| \, |x_\ell|_E \Big)^2 \\
&\leq \mathrm{const} \sup \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \sum_\ell \frac{|k - \ell|_*^{2\sigma} |\ell|_*^{2\sigma}}{|k|_*^{2\sigma}} \|L_{k\ell}\|^2 |x_\ell|_E^2 \\
&\leq \mathrm{const} \sup \sum_\ell e^{2\alpha|\ell|} |\ell|_*^{2\sigma} |x_\ell|_E^2 \\
&\quad\ \sum_k e^{2\alpha|k-\ell|} |k - \ell|_*^{2\sigma} \|L_{k\ell}\|^2_{\mathcal{L}(E,F)} \\
&= \mathrm{const} \|\!| L \|\!| ,
\end{aligned}$$

where we used that $|x|^2_{H_E^{\alpha,\sigma}} = 1$.

□

Proposition 5.1.12. *Let E, F, G Hilbert spaces. Let be $L^{(1)} \in \mathcal{L}^{\alpha,\sigma}(E, F)$ and $L^{(2)} \in \mathcal{L}^{\alpha,\sigma}(F, G)$. Then $L^{(2)} \circ L^{(1)} \in \mathcal{L}^{\alpha,\sigma}(E, G)$ and moreover*

$$\|L^{(2)} \circ L^{(1)}\| \le \mathrm{const} \|L^{(1)}\| \, \|L^{(2)}\| \, .$$

PROOF. Let

$$L^{(1)}x = \sum_\ell e^{\mathrm{i}\ell t/\tau} \big(L^{(1)}x\big)_\ell = \sum_\ell e^{\mathrm{i}\ell t/\tau} \Big(\sum_j L^{(1)}_{\ell j}[x_j] \Big) .$$

Moreover

$$\begin{aligned} L^{(2)}[L^{(1)}x] &= \sum_k e^{\mathrm{i}kt/\tau} \sum_\ell L^{(2)}_{k\ell} \big[(L^{(1)}x)_\ell\big] \\ &= \sum_k e^{\mathrm{i}kt/\tau} \sum_\ell L^{(2)}_{k\ell} \Big[\sum_j L^{(1)}_{\ell j}[x_j] \Big] \\ &= \sum_k e^{\mathrm{i}kt/\tau} \sum_j \sum_\ell L^{(2)}_{k\ell} \Big[L^{(1)}_{\ell j}[x_j] \Big] \\ &= \sum_k e^{\mathrm{i}kt/\tau} \sum_j \Big(\sum_\ell L^{(2)}_{k\ell} \circ L^{(1)}_{\ell j} \Big)[x_j] \, . \end{aligned}$$

Thus we obtain, by using (4.11),

$$\begin{aligned} \|L^{(2)} \circ L^{(1)}\|^2 &= \sup_j \sum_k e^{2\alpha|j-k|} |j-k|_*^{2\sigma} \Big\| \sum_\ell L^{(2)}_{k\ell} L^{(1)}_{\ell j} \Big\|^2_{\mathcal{L}(E,G)} \\ &\le \sup_j \sum_k e^{2\alpha|j-k|} |j-k|_*^{2\sigma} \Big(\sum_\ell \Big\| L^{(2)}_{k\ell} L^{(1)}_{\ell j} \Big\|^2_{\mathcal{L}(E,G)} \Big) \\ &\le \sup_j \sum_k e^{2\alpha|j-k|} |j-k|_*^{2\sigma} \Big(\sum_\ell \big\| L^{(2)}_{k\ell} \big\|^2_{\mathcal{L}(F,G)} \big\| L^{(1)}_{\ell j} \big\|^2_{\mathcal{L}(E,F)} \Big) \\ &\le \mathrm{const} \sup_j \sum_k e^{2\alpha|k-\ell|} e^{2\alpha|\ell-j|} |j-k|_*^{2\sigma} \\ &\quad \sum_\ell \frac{|k-\ell|_*^{2\sigma} |\ell-j|_*^{2\sigma}}{|j-k|_*^{2\sigma}} \big\| L^{(2)}_{k\ell} \big\|^2_{\mathcal{L}(F,G)} \big\| L^{(1)}_{\ell j} \big\|^2_{\mathcal{L}(E,F)} \\ &\le \mathrm{const} \sup_j \sum_\ell e^{2\alpha|\ell-j|} |\ell-j|_*^{2\sigma} \big\| L^{(1)}_{\ell j} \big\|^2_{\mathcal{L}(E,F)} \\ &\quad \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \big\| L^{(2)}_{k\ell} \big\|^2_{\mathcal{L}(F,G)} \\ &\le \mathrm{const} \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \big\| L^{(2)}_{k\ell} \big\|^2_{\mathcal{L}(F,G)} \\ &\quad \sup_j \sum_\ell e^{2\alpha|\ell-j|} |\ell-j|_*^{2\sigma} \big\| L^{(1)}_{\ell j} \big\|^2_{\mathcal{L}(E,F)} \\ &= \mathrm{const} \|L^{(1)}\|^2 \, \|L^{(2)}\|^2 \, , \end{aligned}$$

from which the thesis follows.

□

Remark 5.1.13. *We note that* $H^{\alpha,\sigma}_{\mathcal{L}(E,F)} \subset \mathcal{L}^{\alpha,\sigma}(E,F)$ *in the sense that, if* $L \in H^{\alpha,\sigma}_{\mathcal{L}(E,F)}$ *where* $L(t) = \sum_k e^{\mathrm{i}kt/\tau} L_k$, $L_k \in \mathcal{L}(E,F)$, *then the "product operator", namely the Toepliz operator,* L^*, *defined as*

$$\big(L^*[x]\big)(t) := \big(L(t)\big)[x(t)] = \sum_k e^{\mathrm{i}kt/\tau} \sum_\ell L_{k-\ell}[x_\ell]\,,$$

belongs to $\mathcal{L}^{\alpha,\sigma}(E,F)$ *and*

$$\|L^*\| = \|L\|_{H^{\alpha,\sigma}_{\mathcal{L}(E,F)}}\,.$$

Indeed

$$\begin{aligned}\|L^*\|^2 &= \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|L_{k-\ell}\|^2_{\mathcal{L}(E,F)} \\ &= \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \|L_k\|^2_{\mathcal{L}(E,F)} = \|L\|^2_{H^{\alpha,\sigma}_{\mathcal{L}(E,F)}}\,.\end{aligned}$$

Thus we have

$$H^{\alpha,\sigma}_{\mathcal{L}(E,F)} \subset \mathcal{L}^{\alpha,\sigma}(E,F) \subset \mathcal{L}\big(H^{\alpha,\sigma}_E, H^{\alpha,\sigma}_F\big)\,.$$

5.2 Symmetry of the Hamiltonian

We performed a partial Birkhoff Normal Form, see Proposition 4.1.6, defining a symplectic change of coordinates

$$\mathrm{z} = \Gamma(\mathrm{z}_*) := \big(\Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*), \Gamma_2(\mathrm{z}_*, \bar{\mathrm{z}}_*)\big)\,, \tag{5.13}$$

such that $H \circ \Gamma = H \circ X^t_F\big|_{t=1}$. The transformation Γ is obtained as the map at time $t = 1$ of the flow of the hamiltonian vectorfield X_F, given by the Hamiltonian

$$F = \sum_{i,j,k,\ell}{}' F_{ijk\ell}\, w_i w_j w_k w_\ell\,, \tag{5.14}$$

with coefficients (4.20), where $w_i := \mathrm{z}_{*i}$ and $w_{-i} := \bar{\mathrm{z}}_{*i}$. Therefore $\Gamma(t) = \big(\Gamma_1(t), \Gamma_2(t)\big)$ satisfies the equations

$$\begin{cases} \dot{\Gamma}_1 = -\mathrm{i}\partial_{\bar{\mathrm{z}}_*} F(\Gamma_1, \Gamma_2) \\ \dot{\Gamma}_2 = \mathrm{i}\partial_{\mathrm{z}_*} F(\Gamma_1, \Gamma_2)\,, \end{cases} \tag{5.15}$$

with initial datum $\Gamma(0) = (\mathrm{z}_*, \bar{\mathrm{z}}_*)$. It follows that $(\mathrm{z}, \bar{\mathrm{z}}) = (\Gamma_1(1), \Gamma_2(1)) = \Gamma(\mathrm{z}_*, \bar{\mathrm{z}}_*)$.

Lemma 5.2.1. *Let be* F *a Hamiltonian as in* (5.14). *Then*

$$\partial_{\bar{\mathrm{z}}_*} F(\mathrm{z}_*, \bar{\mathrm{z}}_*) = \partial_{\mathrm{z}_*} F(\mathrm{z}_*, \bar{\mathrm{z}}_*)\,, \qquad \overline{\partial_{\mathrm{z}_*} F(\mathrm{z}_*, \bar{\mathrm{z}}_*)} = \partial_{\bar{\mathrm{z}}_*} F(\mathrm{z}_*, \bar{\mathrm{z}}_*)\,.$$

PROOF. Define, recalling (4.20),

$$\tilde{F}_{i,j,k,\ell} := \frac{G_{ijk\ell}}{16(\omega'_i + \omega'_j + \omega'_k + \omega'_\ell)} \tag{5.16}$$

such that $\tilde{F}_{i,j,k,\ell}$ are real and they satisfies $\tilde{F}_{i,j,k,\ell} = -\tilde{F}_{-i,-j,-k,-\ell}$. From (5.14) and (4.20), we have

$$F(\bar{\mathrm{z}}_*, \mathrm{z}_*) = -\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell}\, w_i w_j w_k w_\ell \tag{5.17}$$

where $w_i = \mathrm{z}_{*i}$ for $i \geq 1$ and $w_i = \bar{\mathrm{z}}_{*i}$ for $i \leq -1$. Therefore, since $\partial_{\bar{\mathrm{z}}_{*n}} = \partial_{w_{-n}}$,

$$\begin{aligned}
\partial_{\bar{\mathrm{z}}_{*n}} F(\mathrm{z}_*, \bar{\mathrm{z}}_*) &= \partial_{w_{-n}} \left(-\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell}\, w_i w_j w_k w_\ell \right) \\
&= -\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell} \Big(\delta_{-n,i}\, w_j w_k w_\ell + \ldots + \delta_{-n,\ell}\, w_i w_j w_k \Big) \\
&= -\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{-i,-j,-k,-\ell} \Big(\delta_{n,i}\, w_{-j} w_{-k} w_{-\ell} + \ldots + \delta_{n,\ell}\, w_{-i} w_{-j} w_{-k} \Big) \\
&= \mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell} \Big(\delta_{n,i}\, \bar{w}_j \bar{w}_k \bar{w}_\ell + \ldots + \delta_{n,\ell}\, \bar{w}_i \bar{w}_j \bar{w}_k \Big) \\
&= \overline{-\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell} \left(\delta_{n,i} w_j w_k w_\ell + \ldots + \delta_{n,\ell} w_i w_j w_k \right)}
\end{aligned}$$

where we used $\delta_{-n,-i} = \delta_{n,i}$ and $w_{-i} = \bar{w}_i$. In the same way, one has $\partial_{\mathrm{z}_{*n}} = \partial_{w_n}$ and hence

$$\partial_{\mathrm{z}_{*n}} F(\mathrm{z}_*, \bar{\mathrm{z}}_*) = -\mathrm{i} \sum_{i,j,k,\ell}{}' \tilde{F}_{i,j,k,\ell} \Big(\delta_{n,i}\, w_j w_k w_\ell + \ldots + \delta_{n,\ell}\, w_i w_j w_k \Big),$$

from which it follows that

$$\overline{\partial_{\bar{\mathrm{z}}_{*n}} F(\mathrm{z}_*, \bar{\mathrm{z}}_*)} = \partial_{\mathrm{z}_{*n}} F(\mathrm{z}_*, \bar{\mathrm{z}}_*).$$

□

Now we can prove the following

Lemma 5.2.2. *Let be* $\Gamma(t) = \big(\Gamma_1(t), \Gamma_2(t)\big)$ *the symplectic change of coordinates as in* (5.13), $\Gamma = X_F^t\big|_{t=1}$ *with* F *as in* (5.14). *Then*

$$\overline{\Gamma_1(t)} = \Gamma_2(t), \qquad \forall\, t.$$

PROOF. It results that $\overline{\Gamma_1(0)} = \bar{\mathrm{z}}_* = \Gamma_2(0)$. Moreover, by Lemma 5.2.1,

$$\dot{\overline{\Gamma_1(t)}} = \overline{\dot{\Gamma}_1(t)} = \mathrm{i}\partial_{\bar{\mathrm{z}}_*} F(\Gamma_1, \Gamma_2) = \mathrm{i}\partial_{\mathrm{z}_*} F(\Gamma_1, \Gamma_2) = \dot{\Gamma}_2(t),$$

and the thesis follows.

□

Lemma 5.2.3. *Let be F as in* (5.14). *Then*

$$\partial_{\bar{z}_*} F(\bar{z}_*, z_*) = -\partial_{z_*} F(z_*, \bar{z}_*)\,, \qquad \partial_{z_*} F(\bar{z}_*, z_*) = -\partial_{\bar{z}_*} F(z_*, \bar{z}_*)\,.$$

PROOF. We claim that

$$F(\bar{z}_*, z_*) = -F(z_*, \bar{z}_*)\,. \tag{5.18}$$

Indeed, from (5.17) and recalling $\tilde{F}_{i,j,k,\ell} = -\tilde{F}_{-i,-j,-k,-\ell}$, one has

$$\begin{aligned} F(\bar{z}_*, z_*) &= -\mathrm{i}{\sum_{i,j,k,\ell}}' \tilde{F}_{i,j,k,\ell}\, w_{-i} w_{-j} w_{-k} w_{-\ell} \\ &= -\mathrm{i}{\sum_{i,j,k,\ell}}' \tilde{F}_{-i,-j,-k,-\ell}\, w_i w_j w_k w_\ell \\ &= \mathrm{i}{\sum_{i,j,k,\ell}}' \tilde{F}_{i,j,k,\ell}\, w_i w_j w_k w_\ell \\ &= -F(z_*, \bar{z}_*)\,. \end{aligned}$$

Therefore, we get

$$\partial_{\bar{z}_*} F(\bar{z}_*, z_*) = \frac{d}{dz_*} F(\bar{z}_*, z_*) = \frac{d}{dz_*}(z_*, \bar{z}_*) = -\partial_{z_*} F(z_*, \bar{z}_*)\,.$$

In the same way, one can prove the second equality of the thesis.

□

Lemma 5.2.4. *Let be* $\Gamma(t) = \big(\Gamma_1(t), \Gamma_2(t)\big)$ *the symplectic change of coordinates as in* (5.13), $\Gamma = X_F^t\big|_{t=1}$ *with F as in* (5.14). *Then*

$$\Gamma_1(\bar{z}_*, z_*) = \Gamma_2(z_*, \bar{z}_*)\,, \qquad \Gamma_2(\bar{z}_*, z_*) = \Gamma_1(z_*, \bar{z}_*)\,. \tag{5.19}$$

PROOF. Define α, β such that $\Gamma_1(\bar{z}_*, z_*) =: \alpha(1)$, $\Gamma_2(\bar{z}_*, z_*) =: \beta(1)$ where $\alpha(0) = \bar{z}_*$, $\beta(0) = z_*$ and

$$\begin{cases} \dot{\alpha} = -\mathrm{i}\partial_{\bar{z}_*} F(\alpha, \beta) \\ \dot{\beta} = \mathrm{i}\partial_{z_*} F(\alpha, \beta)\,. \end{cases} \tag{5.20}$$

By Lemma 5.2.3, we have that $\big(\Gamma_2(t), \Gamma_1(t)\big)$ satisfies the system (5.20), indeed

$$\begin{cases} \dot{\Gamma}_2 = \mathrm{i}\partial_{\bar{z}_*} F(\Gamma_1, \Gamma_2) = -\mathrm{i}\partial_{\bar{z}_*} F(\Gamma_2, \Gamma_1) \\ \dot{\Gamma}_1 = -\mathrm{i}\partial_{\bar{z}_*} F(\Gamma_1, \Gamma_2) = \mathrm{i}\partial_{z_*} F(\Gamma_2, \Gamma_1)\,. \end{cases}$$

Being $\alpha(0) = \bar{z}_* = \Gamma_2(0)$ and $\beta(0) = z_* = \Gamma_1(0)$, it follows that $\alpha(t) = \Gamma_2(t)$ and $\beta(t) = \Gamma_1(t)$ for all t. Then, $\alpha(1) = \Gamma_2(1) = \Gamma_2(z_*, \bar{z}_*)$ and $\beta(1) = \Gamma_1(1) = \Gamma_1(z_*, \bar{z}_*)$, from which the thesis follows.

□

Now we can prove the following theorem

Theorem 5.2.5. *Let be $\Gamma(t) = \big(\Gamma_1(t), \Gamma_2(t)\big)$ a symplectic change of coordinates such that* $\mathrm{z} := \Gamma(\mathrm{z}_*)$ *and be* $\mathcal{H} := H \circ \Gamma = H \circ X_F^t\big|_{t=1}$ *with F as in* (5.14) *and H such that* $H(\mathrm{z}, \bar{\mathrm{z}}) = H(\bar{\mathrm{z}}, \mathrm{z})$. *Then*

$$\mathcal{H}(\bar{\mathrm{z}}_*, \mathrm{z}_*) = \mathcal{H}(\mathrm{z}_*, \bar{\mathrm{z}}_*)\,.$$

PROOF. We have, by Lemma 5.2.2 and Lemma 5.2.4,

$$\begin{aligned}\mathcal{H}(\bar{\mathrm{z}}_*, \mathrm{z}_*) &= H\big(\Gamma_1(\bar{\mathrm{z}}_*, \mathrm{z}_*), \Gamma_2(\bar{\mathrm{z}}_*, \mathrm{z}_*)\big) = H\big(\Gamma_2(\mathrm{z}_*, \bar{\mathrm{z}}_*), \Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*)\big) \\ &= H\big(\overline{\Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*)}, \Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*)\big) = H\big(\Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*), \overline{\Gamma_1(\mathrm{z}_*, \bar{\mathrm{z}}_*)}\big) = \mathcal{H}(\mathrm{z}_*, \bar{\mathrm{z}}_*)\,,\end{aligned}$$

where we used also the symmetry of the Hamiltonian H.

□

We proved that after the Birkhoff Normal Form, the Hamiltonian $\mathcal{H}$ is still symmetric, namely $\mathcal{H}(\bar{\mathrm{z}}_*, \mathrm{z}_*) = \mathcal{H}(\mathrm{z}_*, \bar{\mathrm{z}}_*)$. Let us introduce the new coordinates (p, q) and the parameter $\eta \in \mathbb{R}$, by

$$\mathrm{z}_{*i} := \frac{\eta}{\sqrt{2}}(\mathrm{q}_i + \mathrm{i}\mathrm{p}_i)\,, \quad \bar{\mathrm{z}}_{*i} := \frac{\eta}{\sqrt{2}}(\mathrm{q}_i - \mathrm{i}\mathrm{p}_i)\,. \tag{5.21}$$

In the variables (p, q), defining $H^*(\mathrm{p}, \mathrm{q}) = H^*(\mathrm{p}, \mathrm{q}; \eta) := \mathcal{H}\big(\eta(\mathrm{q} + \mathrm{i}\mathrm{p})/\sqrt{2}, \eta(\mathrm{q} - \mathrm{i}\mathrm{p})/\sqrt{2}\big)$, one has

$$\begin{aligned}H^*(-\mathrm{p}, \mathrm{q}) &= \mathcal{H}\big(\eta(\mathrm{q} - \mathrm{i}\mathrm{p})/\sqrt{2}, \eta(\mathrm{q} + \mathrm{i}\mathrm{p})/\sqrt{2}\big) \\ &= \mathcal{H}\big(\eta(\mathrm{q} + \mathrm{i}\mathrm{p})/\sqrt{2}, \eta(\mathrm{q} - \mathrm{i}\mathrm{p})/\sqrt{2}\big) = H^*(\mathrm{p}, \mathrm{q})\,,\end{aligned}$$

that is

$$H^*(-\mathrm{p}, \mathrm{q}) = H^*(\mathrm{p}, \mathrm{q})\,, \tag{5.22}$$

namely H^* is even in the p-variable. Now we introduce action–angle variables $(I, \phi) \in \mathbb{R}_+^N \times \mathbb{T}^N$ on the $\mathcal{I}$–modes by the following symplectic change of coordinates

$$\begin{aligned}\frac{\sqrt{2}}{\eta}\,\mathrm{z}_{*i} &= \mathrm{q}_i + \mathrm{i}\mathrm{p}_i := \sqrt{I_i}(\cos\phi_i + \mathrm{i}\sin\phi_i)\,, \qquad i \in \mathcal{I}\,, \\ \frac{\sqrt{2}}{\eta}\,\bar{\mathrm{z}}_{*i} &= \mathrm{q}_i - \mathrm{i}\mathrm{p}_i := \sqrt{I_i}(\cos\phi_i - \mathrm{i}\sin\phi_i)\,, \qquad i \in \mathcal{I}\,.\end{aligned}$$

We note that

$$\sum_{i\in\mathcal{I}} dI_i \wedge d\phi_i = -\mathrm{i}\Big(\sum_{i\in\mathcal{I}} d\mathrm{z}_{*i} \wedge \, d\bar{\mathrm{z}}_{*i}\Big)/\eta^2 = \sum_{i\in\mathcal{I}} d\mathrm{p}_i \wedge d\mathrm{q}_i\,.$$

Denote by $(\hat{p}, \hat{q})$ the infinite vector obtained by excising (p, q) of its $\mathcal{I}$–components, namely $(\hat{p}, \hat{q}) = (\mathrm{p}_i, \mathrm{q}_i)_{i\in\mathcal{I}^c}$. Such $(\hat{p}, \hat{q})$ have not to be confused with the (p, q) used in chapter 4. With a little abuse of notation, we denote

$$\begin{aligned}\tilde{H}(I, \phi, \hat{p}, \hat{q}) &:= \tilde{H}(I, \phi, \hat{p}, \hat{q}; \eta) \\ &:= H^*(\sqrt{I}\sin\phi, \sqrt{I}\cos\phi, \hat{p}, \hat{q}; \eta) = H^*(\mathrm{p}, \mathrm{q}; \eta)\,. \end{aligned} \tag{5.23}$$

5.2.1 Space of the solutions

Consider $w \in W_{s,\alpha,\sigma} := H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}$, with $a, s, \alpha, \sigma > 0$ parameters, endowed with the norm

$$\|w\|^2_{s,\alpha,\sigma} := \sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\|w_k\|^2_{a,s}\,,$$

where $\|w_k\|_{a,s}$ is the norm in $\ell^{a,s} \times \ell^{a,s}$ defined in (4.1). Note that $W_{s,\alpha,\sigma}$ is nothing else but $H^{\alpha,\sigma}_E$ defined in (5.1) with $E = \ell^{a,s} \times \ell^{a,s}$.
We are looking for a solution $w(t) = \big(I(t), \phi(t), \hat{p}(t), \hat{q}(t)\big) \in \mathbb{R}^N_+ \times \mathbb{T}^N \times W_{s,\alpha,\sigma}$ in the space of the T–periodic functions

$$\mathcal{X} := \Big\{ w(t) = \big(I(t), \phi(t), \hat{p}(t), \hat{q}(t)\big),\ \text{s.t.} \tag{5.24}$$
$$I(t) = I(-t),\ \phi(t) = -\phi(-t),\ \hat{p}(t) = -\hat{p}(-t),\ \hat{q}(t) = \hat{q}(-t) \Big\}\,.$$

Let be

$$\hat{q}(t) = \sum_{k\geq 0} \hat{q}_k \cos kt/\tau\,, \qquad \hat{q}_k \in \ell^{a,s}\,,$$

and

$$\hat{p}(t) = \sum_{k\geq 1} \hat{p}_k \sin kt/\tau\,, \qquad \hat{p}_k \in \ell^{a,s}\,.$$

Proposition 5.2.6. *Consider the Hamiltonian $\tilde{H}$ defined in* (5.23) *such that H^* satisfies* (5.22)*. Then*

$$\partial_\phi \tilde{H}(I, -\phi, -\hat{p}, \hat{q}) = -\partial_\phi \tilde{H}(I, \phi, \hat{p}, \hat{q})\,.$$

PROOF. From (5.23) and from (5.22),

$$\tilde{H}(I, \phi, \hat{p}, \hat{q}) = H^*(\mathrm{p}, \mathrm{q}) = H^*(-\mathrm{p}, \mathrm{q}) = \tilde{H}(I, -\phi, -\hat{p}, \hat{q})\,,$$

namely

$$\tilde{H}(I, \phi, \hat{p}, \hat{q}) = \tilde{H}(I, -\phi, -\hat{p}, \hat{q})\,. \tag{5.25}$$

Deriving by ϕ the equation (5.25), the thesis follows.

□

Proposition 5.2.7. *Let be $w^*(t) = (I_0, \tilde{\omega}t, 0, 0)$. Then, the following holds*

$$\int_0^T \partial_\phi \tilde{H}(w^*(t) + w(t))\,dt = 0\,, \qquad \forall\, w(t) \in \mathcal{X}(t)\,.$$

PROOF. By Proposition 5.2.6 and using that $w(t)$ is T–periodic on the torus, one has

$$\int_0^T \partial_\phi \tilde{H}(w^*(t) + w(t))\,dt \;=\; -\int_0^{-T} \partial_\phi \tilde{H}(w^*(-\tilde{t}) + w(-\tilde{t}))\,d\tilde{t}$$

$$
\begin{aligned}
&= \int_{-T}^{0} \partial_\phi \tilde{H}(w^*(-\tilde{t}) + w(-\tilde{t}))\, d\tilde{t} \\
&= \int_{-T}^{0} \partial_\phi \tilde{H}\big(I_0 + I(\tilde{t}), -\tilde{\omega}\tilde{t} - \phi(\tilde{t}), -\hat{p}(\tilde{t}), \hat{q}(\tilde{t})\big)\, d\tilde{t} \\
&= -\int_{-T}^{0} \partial_\phi \tilde{H}\big(I_0 + I(\tilde{t}), \tilde{\omega}\tilde{t} + \phi(\tilde{t}), -\hat{p}(\tilde{t}), \hat{q}(\tilde{t})\big)\, d\tilde{t} \\
&= -\int_{-T}^{0} \partial_\phi \tilde{H}(w^*(\tilde{t}) + w(\tilde{t}))\, d\tilde{t} \\
&= -\int_{0}^{T} \partial_\phi \tilde{H}\big((I_0, \tilde{\omega}t - \tilde{\omega}T, 0, 0) + w(t-T)\big)\, dt \\
&= -\int_{0}^{T} \partial_\phi \tilde{H}(w^*(t) + w(t))\, dt\,.
\end{aligned}
$$

□

5.3 Solution of J and ψ

Let us write explicitly the Hamiltonian in the variables $(I, \phi, \hat{p}, \hat{q})$. Recalling (4.35), we have

$$
\tilde{H}(I, \phi, \hat{p}, \hat{q}) = \omega I + \Omega \frac{\hat{p}^2 + \hat{q}^2}{2} + \eta^2 \left[\frac{1}{2} A I \cdot I + \frac{1}{2} B I \cdot (\hat{p}^2 + \hat{q}^2) + \hat{G}(\hat{p}, \hat{q}) \right] + \eta^4 \tilde{K},
$$

where, with abuse of notation, $\hat{G}(\hat{p}, \hat{q}) = \hat{G}(\hat{z}, \bar{\hat{z}})$ and $\tilde{K} = \widetilde{K}(I, \phi, \hat{p}, \hat{q}) = \widetilde{K}(I, \phi, \hat{z}, \bar{\hat{z}})$. The equations of motion are

$$
\left\{
\begin{array}{lcl}
\dot{\phi} &=& \omega + \eta^2 A I + \eta^2 B^t (\hat{p}^2 + \hat{q}^2)/2 + \eta^4 \partial_I \widetilde{K}(I, \phi, \hat{p}, \hat{q}) \\
\dot{I} &=& -\eta^4 \partial_\phi \widetilde{K}(I, \phi, \hat{p}, \hat{q}) \\
\dot{\hat{p}} &=& (\Omega + \eta^2 B I)\, \hat{p} + \eta^2 \partial_{\hat{p}} \hat{G}(\hat{p}, \hat{q}) + \eta^4 \partial_{\hat{p}} \widetilde{K}(I, \phi, \hat{p}, \hat{q})\,, \\
\dot{\hat{q}} &=& -(\Omega + \eta^2 B I)\, \hat{q} - \eta^2 \partial_{\hat{q}} \hat{G}(\hat{p}, \hat{q}) - \eta^4 \partial_{\hat{q}} \widetilde{K}(I, \phi, \hat{p}, \hat{q})\,.
\end{array}
\right. \tag{5.26}
$$

We are looking for a solution (J, ψ, p, q) such that

$$
\left\{
\begin{array}{lcl}
I(t) &=& I_0 + \eta J(t) \\
\phi(t) &=& \tilde{\omega} t + \eta \psi(t) \\
\hat{p}(t) &=& \eta p(t)\,, \\
\hat{q}(t) &=& \eta q(t)\,.
\end{array}
\right. \tag{5.27}
$$

The equations that (J, ψ, p, q) must satisfy are

$$
\left\{
\begin{array}{lcl}
\dot{\psi} - \eta^2 A J &=& \eta^3 B^t (p^2 + q^2)/2 + \eta^3 \partial_I \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q) \\
\dot{J} &=& -\eta^3 \partial_\phi \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q) \\
\dot{q} - \tilde{\Omega} p &=& \eta^3 B J p + \eta^4 \partial_p \tilde{G}(p, q) + \eta^3 \partial_{\hat{p}} \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q)\,, \\
\dot{p} + \tilde{\Omega} q &=& -\eta^3 B J q - \eta^4 \partial_q \tilde{G}(p, q) - \eta^3 \partial_{\hat{q}} \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q)\,,
\end{array}
\right. \tag{5.28}
$$

where $\tilde{G}(p,q) = \tilde{G}(p,q;\eta)$ is the analytic function defined as $\hat{G}(\eta p, \eta q)/\eta^4$. Define $N := (N_I, N_\phi, N_p, N_q)$ such that

$$\begin{cases} \dot{\psi} - \eta^2 AJ &= N_I \\ \dot{J} &= N_\phi \\ \dot{q} - \tilde{\Omega} p &= N_p\,, \\ \dot{p} + \tilde{\Omega} q &= N_q\,, \end{cases}$$

namely

$$\begin{aligned} N:\ \mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s} \times \ell^{a,s} \times \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s+1} \times \ell^{a,s+1} \\ \big((J,\psi,p,q),t;\eta\big) &\longmapsto (N_I, N_\phi, N_p, N_q)\,. \end{aligned}$$

From (4.18), (5.21), (5.23), (5.27) we have that

$$N \in \mathcal{A}(\mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s} \times \ell^{a,s} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^N \times \mathbb{R}^N \times \ell^{a,s+1} \times \ell^{a,s+1})\,. \tag{5.29}$$

We can also see N as a functional acting on $H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N\times\ell^{a,s}\times\ell^{a,s}}$; indeed by Theorem 5.1.6 we can define

$$N_* \in \mathcal{A}(H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N\times\ell^{a,s}\times\ell^{a,s}}, H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N\times\ell^{a,s+1}\times\ell^{a,s+1}}) \tag{5.30}$$

by $(N_*(h))(t) = N(h(t))$, with $h(t) = (J(t), \psi(t), p(t), q(t))$.
In order to find solutions of (5.28), we start solving the first two equations in J and ψ with (p,q) as parameters by the (standard) Implicit Function Theorem. Therefore we substitute the obtained expressions for $J = J(p,q)$ and $\psi = \psi(p,q)$ into the last two equations and solve the new resulting equations in (p,q) by a Nash–Moser Implicit Function Theorem.

We want to solve the equation

$$\begin{cases} \dot{\psi} - \eta^2 AJ &= \tilde{J}(J,\psi) \\ \dot{J} &= \tilde{\psi}(J,\psi)\,. \end{cases} \tag{5.31}$$

where

$$\begin{cases} \tilde{J} &= \tilde{J}(J,\psi;p,q) = \eta^3 B^t(p^2+q^2)/2 + \eta^3 \partial_I \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q) \\ \tilde{\psi} &= \tilde{\psi}(J,\psi;p,q) = -\eta^3 \partial_\phi \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q)\,, \end{cases} \tag{5.32}$$

with $I_0 = I_0(\eta)$ as in (4.53). We write (5.31) in the form

$$\mathrm{L}(J,\psi) = \big(\tilde{J}(J,\psi), \tilde{\psi}(J,\psi)\big) \tag{5.33}$$

where L is the linear operator $\mathrm{L}(J,\psi) := (\dot{\psi} - \eta^2 AJ, \dot{J})$. We note that

$$\int_0^T \tilde{\psi} = 0\,. \tag{5.34}$$

In fact

$$\int_0^T \tilde{\psi} = -\eta^3 \int_0^T \partial_\phi \widetilde{K}(I_0 + \eta J, \tilde{\omega} t + \eta\psi, \eta p, \eta q)\, dt = 0\,, \tag{5.35}$$

as it follows from Proposition 5.2.7 and being $\partial_\phi \tilde{H} = \partial_\phi \tilde{K}$.
Let us define the linear operator $\daleth \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N \times \mathbb{R}^N)$ acting on the functions of the form

$$\tilde{J} := \sum_{\ell \geq 0} \tilde{J}_\ell \cos \ell t/\tau\,, \qquad \tilde{\psi} := \sum_{\ell \geq 1} \tilde{\psi}_\ell \sin \ell t/\tau$$

(note that the average of $\tilde{\psi}$ is zero), by

$$\daleth \begin{pmatrix} \tilde{J} \\ \tilde{\psi} \end{pmatrix} := \begin{pmatrix} J \\ \psi \end{pmatrix}\,,$$

where

$$\begin{aligned} J &:= -\tau A^{-1} \tilde{J}_0 - \sum_{\ell \geq 1} \frac{\tau}{\ell} \tilde{\psi}_\ell \cos \ell t/\tau\,, \\ \psi &:= \sum_{\ell \geq 1} \frac{\tau}{\ell} \left(\tilde{J}_\ell - \frac{A}{\ell} \tilde{\psi}_\ell \right) \sin \ell t/\tau\,, \end{aligned}$$

In the norm defined in (5.12), we have

$$|\!|\!|\daleth|\!|\!| = \left(\sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|\daleth_{k\ell}\|^2 \right)^{1/2} \leq \mathrm{const} \sup_\ell \frac{\tau}{\ell} \leq \mathrm{const}\tau\,, \tag{5.36}$$

where $\daleth_{k\ell} = 0$ if $k \neq \ell$ and $\daleth_{k\ell} = \daleth_\ell$ if $k = \ell$ and

$$\daleth_\ell := \begin{pmatrix} 0 & -\dfrac{\tau}{\ell}\mathbb{I} \\ & \\ \dfrac{\tau}{\ell}\mathbb{I} & -A\dfrac{\tau}{\ell^2}\mathbb{I} \end{pmatrix}\,.$$

It is immediate to see that in the space of the functions satisfying (5.34) $\daleth$ is the inverse of L. Therefore (5.33) becomes

$$(J, \psi) = \Phi(J, \psi; p, q) := \daleth\big[\tilde{J}(J, \psi; p, q), \tilde{\psi}(J, \psi; p, q)\big] \tag{5.37}$$

Theorem 5.3.1 (Fixed Point Theorem)**.** *Fix $(p, q) \in H^{\alpha,\sigma}_{\ell^{a,s} \times \ell^{a,s}}$ parameters. Then $\Phi(\cdot, \cdot; p, q)$ is a contraction on the ball $\bar{B}_{C\eta}^{H^{\alpha,\sigma}_{\mathbb{R}^N \times \mathbb{R}^N}}$ of radius $C\eta$ of the space $H^{\alpha,\sigma}_{\mathbb{R}^N \times \mathbb{R}^N}$ for C a sufficiently large constant. Therefore there exist $J, \psi \in \mathcal{A}\big(H^{\alpha,\sigma}_{\ell^{a,s} \times \ell^{a,s}}, H^{\alpha,\sigma}_{\mathbb{R}^N}\big)$ such that*

$$\big(J(p, q), \psi(p, q)\big) = \daleth\Big[\tilde{J}\big(J(p, q), \psi(p, q); p, q\big), \tilde{\psi}\big(J(p, q), \psi(p, q); p, q\big)\Big]\,. \tag{5.38}$$

PROOF. From (5.37) and (5.32) it follows that

$$\big\|\Phi(J,\psi)\big\|_{H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N}} < C\eta\,,$$

hence Φ is a contraction in the ball of radius $C\eta$. Moreover, by an analogous of Proposition 4.2.7, let be (J,ψ) and (J',ψ') solutions of (5.37). We have, by (5.32) and (5.36),

$$\begin{aligned}\big\|\Phi(J,\psi)-\Phi(J',\psi')\big\|_{H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N}} &\le \mathrm{const}\tau\big\|\mathrm{N}(J,\psi)-\mathrm{N}(J',\psi')\big\|_{H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N}}\\ &\le \mathrm{const}\tau\|D\mathrm{N}(J,\psi)\|\,\|(J,\psi)-(J',\psi')\|_{H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N}}\\ &\le \mathrm{const}\eta\|(J,\psi)-(J',\psi')\|_{H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N}}\,,\end{aligned}$$

where $\|D\mathrm{N}(J,\psi)\|_{\mathcal{L}(H^{\alpha,\sigma}_{\mathbb{R}^N,\mathbb{R}^N}\times H^{\alpha,\sigma}_{\mathbb{R}^N\times\mathbb{R}^N})}<\infty$. Thus Φ is a contraction for η sufficiently small.

□

Solving (5.37) by the Contraction Mapping Theorem we obtain $J=J(p,q)$ and $\psi=\psi(p,q)$ of order $O(\eta)$ that satisfy the first two equations in (5.28). Inserting $J=J(p,q)$ and $\psi=\psi(p,q)$ in the third and in the fourth equations, one obtains

$$\begin{cases}\dot q-\tilde\Omega p =: \varepsilon N_1(p,q)\\ \dot p+\tilde\Omega q =: -\varepsilon N_2(p,q)\,,\end{cases}\tag{5.39}$$

where

$$\varepsilon := \eta^3\,,\tag{5.40}$$

$$\begin{aligned}N_1(p,q) \;:=\;& \mathrm{diag}\big(BJ(p,q)\big)p+\eta\partial_p\tilde G(p,q)\\ &+\partial_{\hat p}\tilde K(I_0+\eta J(p,q),\tilde\omega t+\eta\psi(p,q),\eta p,\eta q)\end{aligned}\tag{5.41}$$

and

$$\begin{aligned}N_2(p,q) \;:=\;& \mathrm{diag}\big(BJ(p,q)\big)q+\eta\partial_q\tilde G(p,q)\\ &+\partial_{\hat q}\tilde K(I_0+\eta J(p,q),\tilde\omega t+\eta\psi(p,q),\eta p,\eta q)\,.\end{aligned}\tag{5.42}$$

We can rewrite the equations of motion (5.39) in the form

$$L(p,q)+\varepsilon\mathcal{N}(p,q)=0$$

where $\mathcal{N}=(-N_1,N_2)$ and

$$L(p,q):=\begin{pmatrix}\dot q-\tilde\Omega p\\ \dot p+\tilde\Omega q\end{pmatrix}.\tag{5.43}$$

Note that by (5.30) and Theorem 5.3.1

$$\mathcal{N}\in\mathcal{A}(H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}},H^{\alpha,\sigma}_{\ell^{a,s+1}\times\ell^{a,s+1}})\,.\tag{5.44}$$

5.4 Linearized equation

Let be $(p_0, q_0) \in H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}$ close to the origin. Linearizing the operator (5.39) in a neighborhood of (p_0, q_0) and denoting by (p, q) the increments, we obtain

$$\mathcal{L}\left[\begin{array}{c} p \\ q \end{array}\right] := \left(\begin{array}{c} \dot{q} - \tilde{\Omega}p - \varepsilon\partial_p N_1(p_0,q_0)[p] - \varepsilon\partial_q N_1(p_0,q_0)[q] \\ \dot{p} + \tilde{\Omega}q + \varepsilon\partial_p N_2(p_0,q_0)[p] + \varepsilon\partial_q N_2(p_0,q_0)[q] \end{array}\right) . \tag{5.45}$$

Let

$$w_0 := (I_0 + \eta J(p_0,q_0), \tilde{\omega}t + \eta\psi(p_0,q_0), \eta p_0, \eta q_0) . \tag{5.46}$$

We have

$$\begin{aligned} \partial_p N_1(p_0,q_0)[p] \;&=\; \operatorname{diag}\big(BJ(p_0,q_0)\big)p + \operatorname{diag}\big(B\hat{J}_p\big)p_0 + \eta\partial^2_{pp}\tilde{G}(p_0,q_0)p \\ &\quad + \eta\partial^2_{\hat{p}I}\tilde{K}(w_0)\hat{J}_p + \eta\partial^2_{\hat{p}\phi}\tilde{K}(w_0)\hat{\psi}_p + \eta\partial^2_{\hat{p}\hat{p}}\tilde{K}(w_0)p , \end{aligned}$$

where

$$\hat{J}_p := \partial_p J(p_0,q_0)[p] \qquad \text{and} \qquad \hat{\psi}_p := \partial_p\psi(p_0,q_0)[p] .$$

Analogously,

$$\begin{aligned} \partial_q N_1(p_0,q_0)[q] \;&=\; \operatorname{diag}\big(B\hat{J}_q(p_0,q_0)\big)p_0 + \eta\partial^2_{pq}\tilde{G}(p_0,q_0)q \\ &\quad + \eta\partial^2_{\hat{p}I}\tilde{K}(w_0)\hat{J}_q + \eta\partial^2_{\hat{p}\phi}\tilde{K}(w_0)\hat{\psi}_q + \eta\partial^2_{\hat{p}\hat{q}}\tilde{K}(w_0)q , \end{aligned}$$

where

$$\hat{J}_q := \partial_q J(p_0,q_0)[q] \qquad \text{and} \qquad \hat{\psi}_q := \partial_q\psi(p_0,q_0)[q] .$$

In the same way

$$\begin{aligned} \partial_p N_2(p_0,q_0)[p] \;&=\; \operatorname{diag}\big(B\hat{J}_p(p_0,q_0)\big)q_0 + \eta\partial^2_{pq}\tilde{G}(p_0,q_0)p \\ &\quad + \eta\partial^2_{\hat{q}I}\tilde{K}(w_0)\hat{J}_p + \eta\partial^2_{\hat{q}\phi}\tilde{K}(w_0)\hat{\psi}_p + \eta\partial^2_{\hat{p}\hat{q}}\tilde{K}(w_0)p . \end{aligned}$$

Finally

$$\begin{aligned} \partial_q N_2(p_0,q_0)[q] \;&=\; \operatorname{diag}\big(BJ(p_0,q_0)\big)q + \operatorname{diag}\big(B\hat{J}_q\big)q_0 + \eta\partial^2_{qq}\tilde{G}(p_0,q_0)q \\ &\quad + \eta\partial^2_{\hat{q}I}\tilde{K}(w_0)\hat{J}_q + \eta\partial^2_{\hat{q}\phi}\tilde{K}(w_0)\hat{\psi}_q + \eta\partial^2_{\hat{q}\hat{q}}\tilde{K}(w_0)q . \end{aligned}$$

Shortly, we can rewrite (5.45) in the following way

$$\mathcal{L}\left[\begin{array}{c} p \\ q \end{array}\right] := \left(\begin{array}{c} \dot{q} - \tilde{\Omega}p \\ \dot{p} + \tilde{\Omega}q \end{array}\right) + \varepsilon\Lambda\left[\begin{array}{c} p \\ q \end{array}\right] , \tag{5.47}$$

where

$$\begin{aligned} \Lambda(p_0,q_0)\left[\begin{array}{c} p \\ q \end{array}\right] \;&:=\; D\mathcal{N}(p_0,q_0)\left[\begin{array}{c} p \\ q \end{array}\right] \\ &:=\; \left(\begin{array}{c} -\partial_p N_1(p_0,q_0)[p] - \partial_q N_1(p_0,q_0)[q] \\ \partial_p N_2(p_0,q_0)[p] + \partial_q N_2(p_0,q_0)[q] \end{array}\right) . \end{aligned} \tag{5.48}$$

Our aim is to prove that Λ belongs to $\mathcal{L}^{\alpha,\sigma}(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s+1} \times \ell^{a,s+1})$ (recall Definition 5.1.9). Note that

$$p \longmapsto \operatorname{diag}\big(BJ(p_0, q_0)\big)p \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \ell^{a,s+1}) \tag{5.49}$$

is a product operator. In fact, $J(p_0, q_0) \in H^{\alpha,\sigma}_{\mathbb{R}^N}$ and, from the definition of B one has

$$(J, p) \longmapsto \operatorname{diag}(BJ)p \in \mathcal{L}(\mathbb{R}^N \times \ell^{a,s}, \ell^{a,s+1}),$$

from which it follows that

$$\operatorname{diag}\big(BJ(p_0, q_0)\big) \in H^{\alpha,\sigma}_{\mathcal{L}(\ell^{a,s},\ell^{a,s+1})}.$$

Moreover, by the same arguments and by (4.18), we have that

$$\hat{J} \longmapsto \operatorname{diag}(B\hat{J})p_0 \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N, \ell^{a,s+1}); \tag{5.50}$$

$$p \longmapsto \partial^2_{pp} G(p_0, q_0)p \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \ell^{a,s+1}); \tag{5.51}$$

$$\hat{J} \longmapsto \partial^2_{\hat{p}I} \tilde{K}(w_0)\hat{J} \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N, \ell^{a,s+1}); \tag{5.52}$$

$$\hat{\psi} \longmapsto \partial^2_{\hat{p}\phi} \tilde{K}(w_0)\hat{\psi} \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N, \ell^{a,s+1}); \tag{5.53}$$

$$p \longmapsto \partial^2_{\hat{p}\hat{p}} \tilde{K}(w_0)p \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \ell^{a,s+1}), \tag{5.54}$$

and, in particular, they are all "product operators" in the sense of Remark 5.1.13. But $\hat{J}_p$ and $\hat{\psi}_p$ are not product operators. Moreover we note that the following proposition holds and we will prove it later:

Proposition 5.4.1. *It results that*

$$p \longmapsto \partial_p J(p_0, q_0)[p] \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \mathbb{R}^N)$$

and, analogously,

$$p \longmapsto \partial_p \psi(p_0, q_0)[p] \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \mathbb{R}^N).$$

By using Proposition 5.4.1 and (5.49)-(5.54) it follows that

Proposition 5.4.2. *Let be $r > 0$ small enough. Then*

$$\Lambda = \Lambda(p_0, q_0) \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s+1} \times \ell^{a,s+1}),$$

and $\|\Lambda\| \le C(r)$ for all $\|(p_0, q_0)\|_{H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}} \le r$.

PROOF. The thesis follows by using (5.49)-(5.54) and Theorem 5.1.12.

$\square$

Nothing remains but to prove Proposition 5.4.1. Deriving (5.38) in the point (p_0, q_0) with respect to p we can implicitly deduce the expression for $\hat{J}_p$ and $\hat{\psi}_p$:

$$\begin{pmatrix} \hat{J}_p \\ \hat{\psi}_p \end{pmatrix} = \beth \begin{pmatrix} \partial_J \tilde{J}(w_\sharp)[\hat{J}_p] + \partial_\psi \tilde{J}(w_\sharp)[\hat{\psi}_p] + \partial_p \tilde{J}(w_\sharp)[p] \\ \partial_J \tilde{\psi}(w_\sharp)[\hat{J}_p] + \partial_\psi \tilde{\psi}(w_\sharp)[\hat{\psi}_p] + \partial_p \tilde{\psi}(w_\sharp)[p] \end{pmatrix}$$

$$= \quad M_1 \begin{pmatrix} \hat{J}_p \\ \hat{\psi}_p \end{pmatrix} + M_2[p]$$

where

$$M_1 \begin{pmatrix} \hat{J} \\ \hat{\psi} \end{pmatrix} := \gimel \begin{pmatrix} \partial_J \tilde{J}(w_\sharp)[\hat{J}] + \partial_\psi \tilde{J}(w_\sharp)[\hat{\psi}] \\ \partial_J \tilde{\psi}(w_\sharp)[\hat{J}] + \partial_\psi \tilde{\psi}(w_\sharp)[\hat{\psi}] \end{pmatrix}, \tag{5.55}$$

$$M_2[p] := \gimel \begin{pmatrix} \partial_p \tilde{J}(w_\sharp)[p] \\ \partial_p \tilde{\psi}(w_\sharp)[p] \end{pmatrix}, \tag{5.56}$$

and $w_\sharp := \big(J(p_0, q_0), \psi(p_0, q_0); p_0, q_0\big)$. Hence, being

$$(\mathbb{I} - M_1) \begin{bmatrix} \hat{J}_p \\ \hat{\psi}_p \end{bmatrix} = M_2[p]\,,$$

one gets

$$\begin{pmatrix} \hat{J}_p \\ \hat{\psi}_p \end{pmatrix} = \sum_{m=0}^{\infty} M_1^m \big[M_2[p]\big]\,.$$

We need two technical lemmata.

Lemma 5.4.3. *Let be M_1 as in* (5.55)*. It results*

$$M_1, M_1^m, \sum_{m=0}^{\infty} M_1^m \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N \times \mathbb{R}^N)\,,$$

and $\|M_1\| \ll 1$.

PROOF. We have that $\gimel \in \mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N \times \mathbb{R}^N)$ and we have just seen that $\|\gimel\| \leq \mathrm{const}\tau$ in (5.36). Moreover, being (recall (5.32))

$$\big(\partial_J \tilde{J}(w_\sharp)\big)[\hat{J}](t) = \eta^4 \partial^2_{II} \tilde{K}(w_0(t))\hat{J}(t)$$

a product operator of norm $\|\cdot\| \leq \mathrm{const}\eta^4$, we obtain that $\|M_1\| \leq \mathrm{const}\eta^2$. Since $\mathcal{L}^{\alpha,\sigma}(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N \times \mathbb{R}^N)$ is a Banach space, one gets that $\sum_{m=0}^{\infty} M_1^m$ converges.

□

Lemma 5.4.4. *Let be M_2 as in* (5.56)*. Then*

$$p \longmapsto M_2[p] \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \mathbb{R}^N \times \mathbb{R}^N)$$

and $\|M_2\| \leq \mathrm{const}\eta$.

PROOF. We have that (recall (5.32))

$$\big(\partial_p \tilde{J}(w_\sharp)\big)[p](t) = \eta^3 B^t p_0(t) p(t) + \eta^4 \partial^2_{I\hat{p}} \tilde{K}(w_0(t)) p(t)\,,$$

is a product operator. This operator belongs to $\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}, \mathbb{R}^N \times \mathbb{R}^N)$ with norm $\|\cdot\| \leq \mathrm{const}\eta$ bounded as above. □

For the other components we should prove analogous Lemmata by using that the following

$$\begin{array}{rcl}\big(\partial_J\tilde{\psi}(w_\sharp)\big)[\hat{J}](t) &=& -\eta^4\partial^2_{\phi I}\tilde{K}(w_0(t))\hat{J}(t)\\ \big(\partial_\psi\tilde{J}(w_\sharp)\big)[\hat{\psi}](t) &=& \eta^4\partial^2_{\phi I}\tilde{K}(w_0(t))\hat{\psi}(t)\\ \big(\partial_\psi\tilde{J}(w_\sharp)\big)[\hat{\psi}](t) &=& -\eta^4\partial^2_{\phi\phi}\tilde{K}(w_0(t))\hat{\psi}(t)\end{array}$$

and

$$\begin{array}{rcl}\big(\partial_p\tilde{\psi}(w_\sharp)\big)[p](t) &=& -\eta^4\partial^2_{\phi\hat{p}}\tilde{K}(w_0(t))p(t)\,,\\ \big(\partial_q\tilde{J}(w_\sharp)\big)[q](t) &=& \eta^3 B^t q_0(t)q(t)+\eta^4\partial^2_{I\hat{q}}\tilde{K}(w_0(t))q(t)\,,\\ \big(\partial_q\tilde{\psi}(w_\sharp)\big)[q](t) &=& -\eta^4\partial^2_{\phi\hat{q}}\tilde{K}(w_0(t))q(t)\end{array}$$

are product operators.
Now we can easily write the proof of the Proposition.
Proof [of Proposition 5.4.1]. It follows directly from Lemma 5.4.3 and Lemma 5.4.4.

□

By Proposition 5.4.2 we can write the operator (5.45) in the following form

$$\mathcal{L}\begin{pmatrix} p\\ q\end{pmatrix} =: \begin{pmatrix} \displaystyle\sum_{k\geq 1}\Big(-\frac{k}{\tau}\,q_k-\tilde{\Omega}p_k-\varepsilon\sum_{\ell\geq 1}\big(\Lambda^{11}_{k,\ell}p_\ell+\Lambda^{12}_{k,\ell}q_\ell\big)\Big)\sin(kt/\tau)\\ \displaystyle\sum_{k\geq 0}\Big(\frac{k}{\tau}\,p_k+\tilde{\Omega}q_k+\varepsilon\sum_{\ell\geq 1}\big(\Lambda^{21}_{k,\ell}p_\ell+\Lambda^{22}_{k,\ell}q_\ell\big)\Big)\cos(kt/\tau)\end{pmatrix} \tag{5.57}$$

for suitable

$$\Lambda^{ij}_{k,\ell}=\Lambda^{ij}_{k,\ell}(p_0,q_0)\in\mathcal{L}(\ell^{a,s},\ell^{a,s+1})\,,\qquad i,j\in\{1,2\}\,, \tag{5.58}$$

satisfying

$$\sup_\ell\sum_k e^{2\alpha|k-\ell|}|k-\ell|_*^{2\sigma}\Big(\|\Lambda^{11}_{k,\ell}\|^2+\|\Lambda^{12}_{k,\ell}\|^2+\|\Lambda^{21}_{k,\ell}\|^2+\|\Lambda^{22}_{k,\ell}\|^2\Big)<\infty \tag{5.59}$$

for any $\|(p_0,q_0)\|_{H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}}\leq r$, r small enough.

5.5 Nash–Moser scheme

Let us fix an increasing sequence $L_n\to\infty$. Let us consider the orthogonal splitting $W_{s,\alpha,\sigma}=W^{(n)}_{s,\alpha,\sigma}\oplus W^{(n)\perp}_{s,\alpha,\sigma}$ where

$$W^{(n)}_{s,\alpha,\sigma}:=\Big\{(p,q)\text{ s.t. }p=\sum_{k=1}^{L_n}p_k\sin(kt/\tau)\,,\ q=\sum_{k=0}^{L_n}q_k\cos(kt/\tau)\,,\,p_k,q_k\in\ell^{a,s}\Big\} \tag{5.60}$$

and

$$W^{(n)\perp}_{s,\alpha,\sigma} := \left\{ (p,q) \text{ s.t. } p = \sum_{k>L_n} p_k \sin(kt/\tau)\,,\ q = \sum_{k>L_n} q_k \cos(kt/\tau)\,,\, p_k, q_k \in \ell^{a,s} \right\}.$$

Define the operators

$$P_n :\ W_{s,\alpha,\sigma} \to W^{(n)}_{s,\alpha,\sigma}\,, \quad P_n^{\perp} :\ W_{s,\alpha,\sigma} \to W^{(n)\perp}_{s,\alpha,\sigma} \tag{5.61}$$

acting on $\xi := (p,q)$ in this way:

$$P_n \xi := \big((p_1,q_1),(p_2,q_2),\ldots,(p_{L_n},q_{L_n}),0,0,\ldots\big)$$

and, analogously,

$$P_n^{\perp}\xi := \big(0,\ldots,0,(p_{L_n+1},q_{L_n+1}),(p_{L_n+2},q_{L_n+2}),\ldots\big)\,.$$

Define also

$$P_n^{n+1}\xi := (P_{n+1}-P_n)\xi = \big(0,\ldots,0,(p_{L_n+1},q_{L_n+1}),\ldots,(p_{L_{n+1}},q_{L_{n+1}},),0,\ldots\big)\,.$$

Consider $W_{s,\alpha,\sigma}$ endowed with the norm

$$\|\xi\|^2_{s,\alpha,\sigma} := \sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\big(\|p_k\|^2_{a,s} + \|q_k\|^2_{a,s}\big)\,, \tag{5.62}$$

where $\|\cdot\|_{a,s}$ is the norm of $\ell^{a,s}$ defined in (4.1).

Lemma 5.5.1. *Let be $\alpha > \alpha'$, then*

$$\|\xi\|_{s,\alpha',\sigma} \le e^{-(\alpha-\alpha')\,(L_n+1)}\|\xi\|_{s,\alpha,\sigma}\,, \qquad \forall\, \xi \in W^{(n)\perp}_{s,\alpha,\sigma}\,.$$

PROOF. From the definition of the norm (5.62) and since $\xi \in W^{(n)\perp}_{s,\alpha,\sigma}$, one has

$$\begin{aligned}
\|\xi\|^2_{s,\alpha',\sigma} &= \sum_{k>L_n} e^{2\alpha' k}|k|_*^{2\sigma}\big(\|p_k\|^2_{a,s} + \|q_k\|^2_{a,s}\big) \\
&= \sum_{k>L_n} e^{2\alpha k}|k|_*^{2\sigma}\big(\|p_k\|^2_{a,s} + \|q_k\|^2_{a,s}\big)e^{-2(\alpha-\alpha')k} \\
&\le e^{-2\,(\alpha-\alpha')(L_n+1)}\|\xi\|^2_{s,\alpha,\sigma}\,.
\end{aligned}$$

□

Lemma 5.5.2. *Let $\alpha > \alpha'$ and $\sigma' > \sigma$. Then*

$$\|h\|_{s,\alpha',\sigma'} \le c(\sigma,\sigma')\left(\frac{1}{\alpha-\alpha'}\right)^{(\sigma'-\sigma)}\|h\|_{s,\alpha,\sigma}\,. \tag{5.63}$$

PROOF. Let us estimate

$$\begin{aligned}
\|h\|^2_{s,\alpha',\sigma'} &= \sum_{k=1}^{L_n} e^{2\alpha' k} k^{2\sigma'} \|h_k\|^2_{a,s} = \sum_{k=1}^{L_n} e^{2\alpha k} k^{2\sigma} \|h_k\|^2_{a,s} e^{2k(\alpha'-\alpha)} k^{2(\sigma'-\sigma)} \\
&\le \sum_{k=1}^{L_n} e^{2\alpha k} k^{2\sigma} \|h_k\|^2_{a,s} \max_k \left(e^{2k(\alpha'-\alpha)} k^{2(\sigma'-\sigma)} \right) \\
&\le \|h\|^2_{\alpha,\sigma} e^{2(\sigma-\sigma')} \left(\frac{\sigma'-\sigma}{\alpha-\alpha'} \right)^{2(\sigma'-\sigma)} \\
&= \big(c(\sigma,\sigma')\big)^2 \|h\|^2_{s,\alpha,\sigma} \left(\frac{1}{\alpha-\alpha'} \right)^{2(\sigma'-\sigma)} ,
\end{aligned} \tag{5.64}$$

where $c(\sigma,\sigma') = e^{(\sigma-\sigma')}(\sigma'-\sigma)^{(\sigma'-\sigma)}$.

□

Let us consider the projection P_n on the space $W^{(n)}_{s,\alpha,\sigma}$, defined in (5.60), of the linearized operator $\mathcal{L}(p,q)$ defined in (5.57), for $(p,q) \in W^{(n)}_{s,\alpha,\sigma}$, namely

$$\mathcal{L}^{(n)}(p,q) := P_n \mathcal{L}(p,q) . \tag{5.65}$$

5.5.1 Melnikov condition and invertibility

For any $\xi_0 = (p_0,q_0) \in H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}$, we define the *not symmetric* operator $\mathcal{M}_k = \mathcal{M}_k(p_0,q_0) : \ \ell^{a,s}\times\ell^{a,s} \longrightarrow \ell^{a,s+1}\times\ell^{a,s+1}$

$$\mathcal{M}_k = \mathcal{M}_k(p_0,q_0) := \begin{pmatrix} \mathcal{M}^{11}_k & \mathcal{M}^{12}_k \\ \mathcal{M}^{21}_k & \mathcal{M}^{22}_k \end{pmatrix} , \tag{5.66}$$

where

$$\begin{aligned}
2\mathcal{M}^{11}_k &:= \Lambda^{11}_{k,k} + \Lambda^{12}_{k,k} + \Lambda^{21}_{k,k} + \Lambda^{22}_{k,k} \\
2\mathcal{M}^{12}_k &:= \Lambda^{11}_{k,k} - \Lambda^{12}_{k,k} + \Lambda^{21}_{k,k} - \Lambda^{22}_{k,k} \\
2\mathcal{M}^{21}_k &:= -\Lambda^{11}_{k,k} - \Lambda^{12}_{k,k} + \Lambda^{21}_{k,k} + \Lambda^{22}_{k,k} \\
2\mathcal{M}^{22}_k &:= -\Lambda^{11}_{k,k} + \Lambda^{12}_{k,k} + \Lambda^{21}_{k,k} - \Lambda^{22}_{k,k} .
\end{aligned} \tag{5.67}$$

Let us define $\gamma := \omega^\epsilon$ with $0 < \epsilon < \nu - 1$, $1 < \nu < 2$. We summarize the set of parameters that we are using:

$$\varpi := 2\pi T^{-1} = \tau^{-1} = \eta^2 , \quad \varepsilon := \omega^{3/2} = \eta^3 , \quad \gamma := \omega^\epsilon , \quad 0 < \epsilon < \nu - 1 < 1. \tag{5.68}$$

Definition 5.5.3 (First order Melnikov conditions). *Let $1 < \nu < 2$, $n \in \mathbb{N}$, $\varpi_* > 0$ and $\xi_0 = (p_0,q_0) \in H^{\alpha,\sigma}_{\ell^{a,s}\times\ell^{a,s}}$. Let*

$$\big(\mathcal{M}^{22}_k\big)_{jj}(\xi_0) := \langle \mathcal{M}^{22}_k(\xi_0) e_j, e_j \rangle_{\ell^{a,s}\times\ell^{a,s}}$$

with e_j the standard orthonormal basis of $\ell^{a,s} \times \ell^{a,s}$. Define the open set

$$\Delta_n^\nu(\xi_0) \;:=\; \Big\{\varpi \in (0,\varpi_*) \text{ s.t. } \Big|\varpi k - \tilde\Omega_j + \varepsilon\big(\mathcal{M}_k^{22}(\xi_0)\big)_{jj}\Big| > \frac{\varpi^{1+\epsilon}\gamma_n}{|j|^\nu}\,, \quad |\varpi k - j| > \frac{\varpi^{1+\epsilon}}{|j|^\nu}\,,\ \forall\, j \geq 1,\ \forall\, 1 \leq k \leq L_n\Big\}\,,$$

where

$$\gamma_n = \left(1 + \frac{1}{2^{n+1}}\right). \tag{5.69}$$

For the reminder of this section, we will denote, for brevity, $W^{(n)}_{s,\alpha,\sigma}$ by $W^{(n)}_{\alpha,\sigma}$ since all the considerations are independent of the parameter s. In the same way we denote $\|\cdot\|_{s,\alpha,\sigma} = \|\cdot\|_{\alpha,\sigma}$ where there is no ambiguity about the parameter s.
Let us choose $L_n = 4^n$. Define $\alpha_0 = \bar\alpha$.

$$\alpha_n = \alpha_{n-1} - \frac{\alpha_0}{2^{n+1}}\,.$$

It results that

$$\alpha_n = \alpha_0\left(1 - \sum_{j=2}^{n+1}\frac{1}{2^j}\right) \tag{5.70}$$

so that $\alpha_n \to \alpha_0/2$.
The following Lemma is crucial for the iterative scheme. Its proof is the real core of the issue and it will be proved in the following sections.

Lemma 5.5.4 (Invertibility)**.** *If ξ is "sufficiently" small then for all $n \geq 1$ and $\varpi \in \Delta_n^\nu(\xi)$ there exist operators $\mathcal{G}^{(n)}(\varpi,\cdot) : W^{(n)}_{s,\alpha_n,\sigma} \longrightarrow W^{(n)}_{s,\alpha_n,\sigma}$ such that, for any $h \in W^{(n)}_{s,\alpha_n,\sigma}$, it results*

$$\left\|\mathcal{G}^{(n)}(\varpi,\xi)[h]\right\|_{\alpha_n,\sigma} \leq \begin{cases} C_0\,\|h\|_{\alpha_n,\sigma}, & \text{if } L_n \leq \tau/2; \\[2mm] \dfrac{C_0}{\gamma\,\tau^{\nu-1}}\,(L_n)^\nu\,\|h\|_{\alpha_n,\sigma}\,, & \text{if } L_n > \tau/2. \end{cases} \tag{5.71}$$

Moreover

$$\mathcal{L}^{(n)}\mathcal{G}^{(n)}h = h\,, \qquad \forall\, h \in W^{(n)}_{s,\alpha_n,\sigma}\,, \tag{5.72}$$

where $\mathcal{L}^{(n)} := P_n\mathcal{L}$.

5.5.2 Iteration

From (5.70), we have that

$$\alpha_{n+1} - \alpha_i \;=\; \alpha_0\Big(\sum_{j=2}^{i+1}\frac{1}{2^j} - \sum_{j=2}^{n+2}\frac{1}{2^j}\Big) = -\alpha_0\sum_{j=i+2}^{n+2}\frac{1}{2^j}$$

$$= \quad -\frac{\alpha_0}{2^{i+2}}\sum_{j=0}^{n-i}\frac{1}{2^j} \le -\frac{\alpha_0}{2^{i+2}}\,. \tag{5.73}$$

Let $\xi_n = \sum_{i=1}^n h_i$ for certain $h_i \in W^{(i)}_{\alpha_i,\sigma}$ satisfying

$$\|h_i\|_{\alpha_i,\sigma} \le c_0\eta^3\, e^{-\chi^i\alpha_0/8}\,, \quad 1<\chi<2. \tag{5.74}$$

Since, by (5.63) and (5.74),

$$\begin{aligned}\|\xi_n\|_{\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}} &\le \sum_{i=1}^n \|h_i\|_{\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}} \le c(\nu,\alpha_0,\sigma)\sum_{i=1}^n 2^{(i+2)\frac{2\nu+1}{\nu+1}}\|h_i\|_{\alpha_i,\sigma}\\ &\le c(\nu,\alpha_0,\sigma)c_0\,\eta^3\sum_{i=1}^n\left(2^{i+2}\right)^{\frac{2\nu+1}{\nu+1}}e^{-\chi^i\alpha_0/8}\,, \end{aligned}\tag{5.75}$$

from (5.75), it follows that

$$\|\xi_n\|_{\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}} \le \mathrm{const}(\nu,\alpha_0,\sigma)\,c_0\eta^3\,. \tag{5.76}$$

Now, by Theorem 5.1.6, if $\|\xi_n\|_{\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}} \le r$ then,

$$\begin{aligned}\left\|\mathcal{N}(\xi_n)\right\|_{s+1,\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}} &\le M\sum_{m\ge 0}\left(\frac{\|\xi_n\|_{s,\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}}}{r}\right)^m\\ &\le \mathrm{const}\,,\end{aligned}\tag{5.77}$$

where const $=$ const$(a,s,\nu,\alpha_0,\sigma)$ and, in the same way,

$$\left\|D\mathcal{N}(\xi_n)\right\|_{\mathrm{op}} \le \mathrm{const}(a,s,\nu,\alpha_0,\sigma) \tag{5.78}$$

where the operatorial norm is on the space

$$\mathcal{L}^{\alpha_{n+1},\sigma+\frac{2\nu+1}{\nu+1}}(\ell^{a,s}\times\ell^{a,s},\ell^{a,s+1}\times\ell^{a,s+1})$$

and it is bounded by Proposition 5.4.2 (recall (5.48)).

Lemma 5.5.5 (Iterative Lemma)**.** *Let be $\xi_0 = 0$, $A_0 := (0,\varpi_*)$. Then there exists a sequence of open sets $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_0$ and a $\xi_n = \xi_n(\varpi)$, $\xi_n \in W^{(n)}_{\alpha_n,\sigma}$ defined for $\varpi \in A_n$ with*

$$A_n := \left\{\varpi \in A_{n-1} \;\; s.t. \;\; \varpi \in \Delta^\nu_n(\xi_{n-1})\right\}. \tag{5.79}$$

$\xi_n(\varpi) = \sum_{i=1}^n h_i(\varpi)$, where $h_n \in W^{(n)}_{\alpha_n,\sigma}$ is defined as

$$h_n := -\mathcal{G}^{(n)}\left[L\xi_{n-1} + \varepsilon P_n\mathcal{N}(\xi_{n-1})\right], \tag{5.80}$$

$h_0 = \xi_0 = 0$ and

$$\|h_n\|_{\alpha_n,\sigma} \le c_0\,\varepsilon e^{-\chi^n\alpha_0/8}\,, \qquad 1<\chi<2\,, \tag{5.81}$$

for a large enough constant c_0. Moreover h_n is differentiable with respect to η in the open set A_n with

$$\|\partial_\varpi h_n\|_{\alpha_n,\sigma} \le c_0\,\varpi^{-1/2}\,e^{-\tilde\chi^n\alpha_0/8}\,, \qquad 1<\tilde\chi<\chi\,. \tag{5.82}$$

Remark 5.5.6. *Note that, from* (5.81) *this estimate on the* ξ_n *follows*

$$\|\xi_n\|_{\alpha_n,\sigma} \leq c_0\,\varepsilon \sum_{i=1}^{n} e^{-\chi^i\alpha_0/8} \leq \mathrm{const}\,\varepsilon$$

and from (5.82), *we have that*

$$\|\partial_\varpi \xi_n\|_{\alpha_n,\sigma} \leq \sum_{i=1}^{n} \|\partial_\varpi h_i\|_{\alpha_{n+1},\sigma} \leq \mathrm{const}\,\varpi^{-1/2}\,.$$

PROOF. Notice that, by (5.72) and (5.80),

$$-\mathcal{L}^{(n)} h_n = L\xi_{n-1} + \varepsilon P_n \mathcal{N}(\xi_{n-1})\,. \tag{5.83}$$

We are going to prove that (5.81) and (5.82) holds for h_{n+1} by induction. Using that

$$P_{n+1}\mathcal{N}(\xi_n) - P_{n+1}\mathcal{N}(\xi_{n-1}) = P_{n+1}D\mathcal{N}(\xi_{n-1})[h] + R(h_n)$$

with $R(h_n) = O(\|h_n\|^2_{\alpha_n,\sigma})$, one has

$$\begin{aligned}
h_{n+1} &= -\mathcal{G}^{(n+1)}\big[L\xi_n + \varepsilon P_{n+1}\mathcal{N}(\xi_n)\big]\\
&= -\mathcal{G}^{(n+1)}\Big[L\xi_n + \varepsilon P_{n+1}\mathcal{N}(\xi_{n-1}) + \varepsilon P_{n+1}D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon R(h_n)\Big]\\
&= -\mathcal{G}^{(n+1)}\Big[L\xi_{n-1} + Lh_n + \varepsilon P_{n+1}\mathcal{N}(\xi_{n-1}) + \varepsilon P_{n+1}D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon R(h_n)\Big]\\
&= -\mathcal{G}^{(n+1)}\Big[L\xi_{n-1} + \varepsilon P_n\mathcal{N}(\xi_{n-1}) + Lh_n + \varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon R(h_n)\\
&\qquad -\varepsilon P_n\mathcal{N}(\xi_{n-1}) + \varepsilon P_{n+1}\mathcal{N}(\xi_{n-1})\\
&\qquad -\varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon P_{n+1}D\mathcal{N}(\xi_{n-1})[h_n]\Big]\\
&= -\mathcal{G}^{(n+1)}\Big[-\varepsilon P_n\mathcal{N}(\xi_{n-1}) + \varepsilon P_{n+1}\mathcal{N}(\xi_{n-1})\\
&\qquad -\varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon P_{n+1}D\mathcal{N}(\xi_{n-1})[h_n] + \varepsilon R(h_n)\Big]\\
&= -\varepsilon\mathcal{G}^{(n+1)}\Big[P_n^{n+1}\mathcal{N}(\xi_{n-1}) + P_n^{n+1}D\mathcal{N}(\xi_{n-1})[h_n] + R(h_n)\Big]\,,
\end{aligned} \tag{5.84}$$

where we used (5.83) and that

$$\mathcal{L}^{(n)} h_n = Lh_n + \varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n]\,, \tag{5.85}$$

from the definition of $\mathcal{L}^{(n)}$. Notice that, from (5.73),

$$(L_n + 1)(\alpha_n - \alpha_{n+1}) = (4^n + 1)\frac{\alpha_0}{2^{n+2}} \geq \frac{\alpha_0}{4}2^n\,. \tag{5.86}$$

By the Lemma of invertibility 5.5.4 and by Lemma 5.5.1 we have

$$\|h_{n+1}\|_{\alpha_{n+1},\sigma} \;\;\leq\;\; \varepsilon\|\mathcal{G}^{(n+1)}\|_{\alpha_{n+1},\alpha_{n+1}}\Big(\mathrm{const}\,\|h_n\|^2_{\alpha_{n+1},\sigma} + \|P_n^{n+1}\mathcal{N}(\xi_{n-1})\|_{\alpha_{n+1},\sigma}$$

$$
\begin{aligned}
&\quad +\|P_n^{n+1} D\mathcal{N}(\xi_{n-1})\|_{\alpha_{n+1},\sigma}\|h_n\|_{\alpha_{n+1},\sigma}\Big) \\
&\leq \operatorname{const}\varepsilon\|\mathcal{G}^{(n+1)}\|_{\alpha_{n+1},\alpha_{n+1}}\Big(c_0^2\,\varepsilon e^{-\chi^n\alpha_0/4} \\
&\quad +e^{-(L_n+1)(\alpha_n-\alpha_{n+1})} \\
&\quad +c_0\,\varepsilon e^{-(L_n+1)(\alpha_n-\alpha_{n+1})-\chi^n\alpha_0/8}\Big) \\
&\leq \operatorname{const}\varepsilon\|\mathcal{G}^{(n+1)}\|_{\alpha_{n+1},\alpha_{n+1}}\Big(c_0^2\varepsilon e^{-\chi^n\alpha_0/4}+e^{-2^n\alpha_0/4} \\
&\quad +c_0\varepsilon e^{(-\chi^n-2^{n+1})\alpha_0/8}\Big), \qquad (5.87)
\end{aligned}
$$

where we used the estimates on the nonlinearity (5.77) and (5.78). We now choose c_0 large enough and subsequently ϖ small enough such that

$$
\operatorname{const} C_0\left(c_0^2\,\varepsilon e^{-\chi^n\alpha_0/4}+e^{-2^n\alpha_0/4}+c_0\,\varepsilon e^{(-\chi^n-2^{n+1})\alpha_0/8}\right)\leq c_0 e^{-\chi^{n+1}\alpha_0/8}\,, \quad (5.88)
$$

where C_0 was defined in Lemma 5.5.4. Now fix n_0 such that

$$
\operatorname{const} C_0(L_n)^\nu\left(e^{-\chi^n\alpha_0/4}+e^{-2^n\alpha_0/4}+e^{(-\chi^n-2^{n+1})\alpha_0/8}\right)\leq e^{-\chi^{n+1}\alpha_0/8}\,, \quad (5.89)
$$

for all $n\geq n_0$. Such n_0 exists since $\chi<2$, $L_n=4^n$ and recalling that $1/(\gamma\tau^{\nu-1})=\varpi^{\nu-1-\epsilon}\ll 1$. Taking ϖ small enough (equivalently τ large enough) such that $L_{n_0}\leq\tau/2$, we have that, for all $n\geq n_0$, by (5.89),

$$
\begin{aligned}
&\operatorname{const}\frac{C_0}{\gamma\tau^{\nu-1}}(L_n)^\nu\left(c_0^2\,\varepsilon e^{-\chi^n\alpha_0/4}+e^{-2^n\alpha_0/4}+c_0\,\varepsilon e^{(-\chi^n-2^{n+1})\alpha_0/8}\right) \\
&\qquad \leq c_0 e^{-\chi^{n+1}\alpha_0/8}\,, \qquad (5.90)
\end{aligned}
$$

choosing c_0 large enough and subsequently ϖ small enough. Finally, from Lemma 5.5.4, (5.87), (5.88) and (5.90), it follows that

$$
\|h_{n+1}\|_{\alpha_{n+1},\sigma}\leq c_0\,\varepsilon e^{-\chi^{n+1}\alpha_0/8}\,,
$$

proving by induction (5.81).

Now we obtain a similar estimate for the derivative of h_n with respect to η. In particular we want to prove

$$
\|\partial_\eta h_n\|_{\alpha_n}\leq c_0 e^{-\tilde{\chi}^n\alpha_0/8}\,, \qquad 1<\tilde{\chi}<\chi\,. \quad (5.91)
$$

Using (5.72) in (5.84) we get

$$
-\mathcal{L}^{(n+1)}h_{n+1}=\varepsilon\big(P_n^{n+1}\mathcal{N}(\xi_{n-1})+P_n^{n+1}D\mathcal{N}(\xi_{n-1})[h_n]+R(h_n)\big) \quad (5.92)
$$

where

$$
R(h_n)=P_{n+1}\big(\mathcal{N}(\xi_n)-\mathcal{N}(\xi_{n-1})-D\mathcal{N}(\xi_{n-1})[h_n]\big)=O(h_n^2)\,. \quad (5.93)
$$

Deriving (5.93) with respect to η, we have

$$\begin{aligned}\partial_\eta\big(R(h_n)\big) &= P_{n+1}\Big(\partial_\eta N(\xi_n)+D\mathcal{N}(\xi_n)[\partial_\eta\xi_n]-D\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1}]\\ &\quad -\partial_\eta\mathcal{N}(\xi_{n-1})-\partial_\eta D\mathcal{N}(\xi_{n-1})[h_n]-D^2\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1},h_n]\\ &\quad -D\mathcal{N}(\xi_{n-1})[\partial_\eta h_n]\Big)\\ &= P_{n+1}\Big(D\mathcal{N}(\xi_n)[\partial_\eta h_n]-D\mathcal{N}(\xi_{n-1})[\partial_\eta h_n]\\ &\quad +D\mathcal{N}(\xi_n)[\partial_\eta\xi_{n-1}]-D\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1}]-D^2\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1},h_n]\\ &\quad +\partial_\eta\mathcal{N}(\xi_n)-\partial_\eta\mathcal{N}(\xi_{n-1})-\partial_\eta D\mathcal{N}(\xi_{n-1})[h_n]\Big)\,. \end{aligned} \tag{5.94}$$

From (5.94) we obtain

$$\begin{aligned}\|\partial_\eta\big(R(h_n)\big)\|_{\alpha_{n+1}} &\le \mathrm{const}\|h_n\|_{\alpha_n}\|\partial_\eta h_n\|_{\alpha_n}+\mathrm{const}\,\|h_n\|^2_{\alpha_n}+\mathrm{const}\varepsilon^{-1}\,\|h_n\|^2_{\alpha_n}\\ &\le \mathrm{const}\,c_0^2\,\varepsilon e^{-\chi^n\alpha_0/8-\tilde\chi^n\alpha_0/8}\,. \end{aligned} \tag{5.95}$$

Deriving (5.92) with respect to η and recalling that $\eta^2=\varpi$ and $\eta^3=\varepsilon$, we get

$$\begin{aligned}-\partial_\eta\mathcal{L}^{(n+1)}\partial_\eta h_{n+1} &= \big(\partial_\eta\mathcal{L}^{(n+1)}\big)h_{n+1}\\ &\quad +P_n^{n+1}\Big[3\varpi\mathcal{N}(\xi_{n-1})+\varepsilon\partial_\eta\mathcal{N}(\xi_{n-1})\\ &\quad +\varepsilon D\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1}]+3\varpi D\mathcal{N}(\xi_{n-1})[h_n]\\ &\quad +\varepsilon\partial_\eta D\mathcal{N}(\xi_{n-1})[h_n]+\varepsilon D^2\mathcal{N}(\xi_{n-1})[\partial_\eta\xi_{n-1},h_n]\Big]\\ &\quad +3\varpi R(h_n)+\varepsilon\partial_\eta\big(R(h_n)\big)\,. \end{aligned} \tag{5.96}$$

From the definition of the linear operator L in (5.43), using (4.55) and noting that by (4.53)

$$\partial_\eta I_0 = O(1/\eta^3)\,, \tag{5.97}$$

one has

$$\partial_\eta L = O(\eta)+O(1/\eta)\,,$$

hence we obtain

$$\partial_\eta\mathcal{L}^{(n+1)} = O(\eta)+O(1/\eta)\,,$$

namely

$$\|\big(\partial_\eta\mathcal{L}^{(n+1)}\big)h_{n+1}\|_{\alpha_{n+1}} \le \mathrm{const}c_0\varpi e^{-\chi^{n+1}\alpha_0/8}\,. \tag{5.98}$$

By (5.93) and (5.95) we get

$$\|3\varpi R(h_n)+\varepsilon\partial_\eta\big(R(h_n)\big)\|_{\alpha_{n+1}} \le \mathrm{const}\,c_0^2\,\varepsilon^2 e^{-\chi^n\alpha_0/8-\tilde\chi^n\alpha_0/8}\,. \tag{5.99}$$

We now observe that the dominant term into the square brackets in (5.96) is $\varepsilon\partial_\eta\mathcal{N}(\xi_{n-1})$, that can be estimated using (5.97):

$$\|\varepsilon\partial_\eta\mathcal{N}(\xi_{n-1})\|_{\alpha_n} \le \mathrm{const}\,. \tag{5.100}$$

Using (5.98),(5.99),(5.100), (5.86) and Lemma 5.5.1, by (5.96) we have

$$\big\|\mathcal{L}^{(n+1)}\partial_\eta h_{n+1}\big\|_{\alpha_{n+1}} \le \mathrm{const}\big(c_0\varpi e^{-\chi^{n+1}\alpha_0/8} + e^{-2^n\alpha_0/4} + c_0^2\,\varepsilon^2 e^{-\chi^n\alpha_0/8-\tilde\chi^n\alpha_0/8}\big). \tag{5.101}$$

By (5.72) and (5.101) we get

$$\begin{aligned}\|\partial_\eta h_{n+1}\|_{\alpha_{n+1}} \ \le\ & \mathrm{const}\|\mathcal{G}^{(n+1)}\|_{\alpha_{n+1},\alpha_{n+1}}\Big(c_0\varpi e^{-\chi^{n+1}\alpha_0/8}\\ & + e^{-2^n\alpha_0/4} + c_0^2\,\varepsilon^2 e^{-\chi^n\alpha_0/8-\tilde\chi^n\alpha_0/8}\Big)\,.\end{aligned} \tag{5.102}$$

Arguing as above, we take c_0 large enough and subsequently ϖ small enough by (5.71) and (5.102) we obtain (5.91) by induction.
Finally, noting that $\partial_\varpi\eta = \frac{1}{2\sqrt{\varpi}}$, we get (5.82). □

Corollary 5.5.7 (Solution of the equations of motion)**.** *Define*

$$\mathcal{C} := \bigcap_{n\ge 0} A_n\,.$$

Suppose that $\varpi \in \mathcal{C}$. *Then* $\sum_{i\ge 0} h_i(\varpi)$ *normally converges in* $W^{(n)}_{\alpha_0/2,\sigma}$ *to a solution* $\xi(\varpi)$ *of equation*

$$L\xi + \varepsilon\mathcal{N}(\xi;\varpi) = 0$$

with $\|\xi(\varpi)\|_{\alpha_0/2,\sigma} \le \mathrm{const}\,\varepsilon^3 = \mathrm{const}\varpi^{3/2}$.

PROOF. By (5.81)

$$\begin{aligned}\|\xi_{n+m}-\xi_n\|_{\alpha_0/2,\sigma} &= \left\|\sum_{i=1}^{n+m} h_i - \sum_{i=1}^{n} h_i\right\|_{\alpha_0/2,\sigma} = \left\|\sum_{i=n+1}^{n+m} h_i\right\|_{\alpha_0/2,\sigma}\\ &\le \sum_{i=n+1}^{\infty} c_0\varepsilon e^{-\chi^i\alpha_0/8}\,.\end{aligned}$$

It follows that $\|\xi_n - \xi\|_{\alpha_0/2,\sigma} \to 0$. From (5.85) and (5.83), one has

$$\mathcal{L}^{(n)}h_n = -Lh_n - \varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n] = L\xi_{n-1} + \varepsilon P_n\mathcal{N}(\xi_{n-1})\,.$$

Taking the limit for $n\to\infty$ in the space $W^{(n)}_{s-1,\alpha_0/2,\sigma}$, one gets that

$$L\xi_{n-1} + \varepsilon P_n\mathcal{N}(\xi_{n-1}) \longrightarrow L\xi + \varepsilon\mathcal{N}(\xi)\,.$$

Now we have to prove that

$$-Lh_n - \varepsilon P_n D\mathcal{N}(\xi_{n-1})[h_n] \longrightarrow 0$$

in the same space. Indeed,

$$\big\|Lh_n\big\|_{s-1,\alpha_0/2,\sigma} \le \|L\|_{s,s-1}\|h_n\|_{s,\alpha_0/2,\sigma} \longrightarrow 0\,,$$

and, in the same way

$$\big\|P_n D\mathcal{N}(\xi_{n-1})[h_n]\big\| \le \|L\|_{s,s-1}\|h_n\|_{s,\alpha_0/2,\sigma} \longrightarrow 0\,.$$

□

5.6 Evaluating the linearized operator

Consider the projection on the space of functions (5.60) of the linearized operator in (5.57), namely (5.65). We obtain

$$\mathcal{L}^{(n)}\begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} \displaystyle\sum_{k=1}^{L_n}\Big(-\frac{k}{\tau}q_k - \tilde{\Omega}p_k\Big)\sin(kt/\tau) \\ \displaystyle\sum_{k=0}^{L_n}\Big(\frac{k}{\tau}p_k + \tilde{\Omega}q_k\Big)\cos(kt/\tau) \end{pmatrix} \tag{5.103}$$
$$+\varepsilon\begin{pmatrix} \displaystyle -\sum_{k=1}^{L_n}\Big(\sum_{\ell=1}^{L_n}\Lambda^{11}_{k,\ell}p_\ell + \Lambda^{12}_{k,\ell}q_\ell\Big)\sin(kt/\tau) \\ \displaystyle\sum_{k=0}^{L_n}\Big(\sum_{\ell=1}^{L_n}\Lambda^{21}_{k,\ell}p_\ell + \Lambda^{22}_{k,\ell}q_\ell\Big)\cos(kt/\tau) \end{pmatrix}.$$

We want to invert the operator

$$\mathcal{L}^{(n)}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}, \tag{5.104}$$

where $\tilde{p} = \sum_{k=1}^{L_n}\tilde{p}_k\sin kt/\tau$ and $\tilde{q} = \sum_{k=1}^{L_n}\tilde{q}_k\cos kt/\tau$. We obtain the following relation between the k^{th}–coefficients,

$$\begin{pmatrix} \displaystyle -\Big(\frac{k}{\tau}q_k + \tilde{\Omega}p_k\Big) - \varepsilon\sum_{\ell=1}^{L_n}\Big(\Lambda^{11}_{k,\ell}p_\ell + \Lambda^{12}_{k,\ell}q_\ell\Big) \\ \displaystyle\Big(\frac{k}{\tau}p_k + \tilde{\Omega}q_k\Big) + \varepsilon\sum_{\ell=1}^{L_n}\Big(\Lambda^{21}_{k,\ell}p_\ell + \Lambda^{22}_{k,\ell}q_\ell\Big) \end{pmatrix} = \begin{pmatrix} \tilde{p}_k \\ \tilde{q}_k \end{pmatrix}, \tag{5.105}$$

for $1 \le k \le L_n$. Thus, we reduce to invert (5.105).
Consider $(a,b) \in \hat{W}^{(n)}_{s,\alpha,\sigma}$, where

$$\hat{W}^{(n)}_{s,\alpha,\sigma} := \left\{(a,b)\ \text{ s.t. }\ \|(a,b)\|^2_{\hat{W}^{(n)}_{s,\alpha,\sigma}} := \sum_{k=1}^{L_n}\Big(\|a_k\|^2_{a,s} + \|b_k\|^2_{a,s}\Big)e^{2k\alpha}k^{2\sigma} < \infty\right\}, \tag{5.106}$$

with $a_k, b_k \in \ell^{a,s}$. Now, let us transform the operator $\mathcal{L}^{(n)}(p,q)$, in (5.105) into the operator $\hat{\mathcal{L}}^{(n)}$ whose first component is the sum of the components $\tilde{q}_k - \tilde{p}_k$ while the second one is the sum $\tilde{q}_k + \tilde{p}_k$. Moreover, introduce the following change of coordinates

$$a_k := p_k + q_k\,, \qquad b_k := p_k - q_k\,. \tag{5.107}$$

In this coordinates, $\mathcal{L}^{(n)}$ reads

$$\left[\hat{\mathcal{L}}^{(n)}\begin{pmatrix} a \\ b \end{pmatrix}\right]_k := \begin{pmatrix} \dfrac{k}{\tau}a_k + \tilde{\Omega}a_k \\ \\ \dfrac{k}{\tau}b_k - \tilde{\Omega}b_k \end{pmatrix}$$

$$
+\frac{\varepsilon}{2}\begin{pmatrix} \displaystyle\sum_{\ell=1}^{L_n}\Big[\big(\Lambda^{21}_{k,\ell}+\Lambda^{11}_{k,\ell}\big)(a_\ell+b_\ell)+\big(\Lambda^{22}_{k,\ell}+\Lambda^{12}_{k,\ell}\big)(a_\ell-b_\ell)\Big] \\ \\ \displaystyle\sum_{\ell=1}^{L_n}\Big[\big(\Lambda^{21}_{k,\ell}-\Lambda^{11}_{k,\ell}\big)(a_\ell+b_\ell)+\big(\Lambda^{22}_{k,\ell}-\Lambda^{12}_{k,\ell}\big)(a_\ell-b_\ell)\Big] \end{pmatrix},
$$

namely

$$
\left[\hat{\mathcal{L}}^{(n)}\begin{pmatrix} a \\ b \end{pmatrix}\right]_k = \begin{pmatrix} \left(\dfrac{k}{\tau}+\tilde{\Omega}+\varepsilon\hat{\Lambda}^{11}_{k,k}\right)a_k+\varepsilon\hat{\Lambda}^{12}_{k,k}b_k \\ \\ \varepsilon\hat{\Lambda}^{21}_{k,k}a_k+\left(\dfrac{k}{\tau}-\tilde{\Omega}+\varepsilon\hat{\Lambda}^{22}_{k,k}\right)b_k \end{pmatrix} \tag{5.108}
$$

$$
+\varepsilon\begin{pmatrix} \displaystyle\sum_{\ell=1\,,\ell\neq k}^{L_n}\Big[\hat{\Lambda}^{11}_{k,\ell}a_\ell+\hat{\Lambda}^{12}_{k,\ell}b_\ell\Big] \\ \\ \displaystyle\sum_{\ell=1\,,\ell\neq k}^{L_n}\Big[\hat{\Lambda}^{21}_{k,\ell}a_\ell+\hat{\Lambda}^{22}_{k,\ell}b_\ell\Big] \end{pmatrix}, \qquad 1\le k\le L_n\,. \tag{5.109}
$$

where, for simplicity,

$$
\begin{aligned}
2\hat{\Lambda}^{11}_{k,\ell} &:= \Lambda^{21}_{k,\ell}+\Lambda^{11}_{k,\ell}+\Lambda^{22}_{k,\ell}+\Lambda^{12}_{k,\ell} \\
2\hat{\Lambda}^{12}_{k,\ell} &:= \Lambda^{21}_{k,\ell}+\Lambda^{11}_{k,\ell}-\Lambda^{22}_{k,\ell}-\Lambda^{12}_{k,\ell} \\
2\hat{\Lambda}^{21}_{k,\ell} &:= \Lambda^{21}_{k,\ell}-\Lambda^{11}_{k,\ell}+\Lambda^{22}_{k,\ell}-\Lambda^{12}_{k,\ell} \\
2\hat{\Lambda}^{22}_{k,\ell} &:= \Lambda^{21}_{k,\ell}-\Lambda^{11}_{k,\ell}-\Lambda^{22}_{k,\ell}+\Lambda^{12}_{k,\ell}\,.
\end{aligned} \tag{5.110}
$$

In this way, (5.105) reads

$$
\hat{\mathcal{L}}^{(n)}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\tilde{b} \\ \tilde{a} \end{pmatrix},
$$

where $\tilde{a},\tilde{b}$ are such that $\tilde{a}_k=\tilde{p}_k+\tilde{q}_k$ and $\tilde{b}_k=\tilde{p}_k-\tilde{q}_k$. Thus, we reduce to invert

$$
\hat{\mathcal{L}}^{(n)}\begin{pmatrix} a \\ b \end{pmatrix} = \hat{\mathcal{L}}^{(n)}_1\begin{pmatrix} a \\ b \end{pmatrix}+\varepsilon\,\hat{\mathcal{L}}^{(n)}_2\begin{pmatrix} a \\ b \end{pmatrix} \tag{5.111}
$$

where $\hat{\mathcal{L}}^{(n)}_1$ is the right hand side of (5.108), while $\hat{\mathcal{L}}^{(n)}_2$ is defined in (5.109). Denote

$$
y := \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \tilde{y} := \begin{pmatrix} -\tilde{b} \\ \tilde{a} \end{pmatrix}. \tag{5.112}
$$

where $(a,b)=\big((a_1,b_1),\ldots,(a_{L_n},b_{L_n})\big)\in(\ell^{a,s})^{2L_n}$, with $\|(a,b)\|_{\hat{W}^{(n)}_{s,\alpha,\sigma}}$ defined in (5.106). Hence we can write the k^{th}-coefficient in (5.111) as

$$
\big[\hat{\mathcal{L}}^{(n)}(y)\big]_k =: D_k y+\varepsilon\big[\check{\mathcal{L}}^{(n)}\big]_k y\,, \qquad 1\le k\le L_n\,, \tag{5.113}
$$

where

$$D_k := \frac{k}{\tau}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + \begin{pmatrix} \tilde{\Omega} & 0 \\ 0 & -\tilde{\Omega} \end{pmatrix} + \varepsilon \mathcal{M}_k \,, \tag{5.114}$$

with $\mathcal{M}_k$ defined in (5.66) (recall (5.67) and (5.110)) and

$$\big[\check{\mathcal{L}}^{(n)}\big]_k y := \sum_{\ell=1\,,\ell\neq k}^{L_n} \begin{pmatrix} \hat{\Lambda}^{11}_{k,\ell} & \hat{\Lambda}^{12}_{k,\ell} \\ \hat{\Lambda}^{21}_{k,\ell} & \hat{\Lambda}^{22}_{k,\ell} \end{pmatrix} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} =: \sum_{\ell=1\,,\ell\neq k}^{L_n} \hat{\Lambda}_{k,\ell}\, y_\ell \,, \qquad 1 \le k \le L_n. \tag{5.115}$$

We have, $1 \le k, \ell \le L_n$,

$$\big[\hat{\mathcal{L}}^{(n)}\big]_{k\ell} = D_k \delta_{k\ell} + \varepsilon(1-\delta_{k\ell})\,\hat{\Lambda}_{k,\ell} =: \big(D^{(n)}\big)_{k\ell} + \varepsilon\big(T^{(n)}\big)_{k\ell}\,. \tag{5.116}$$

The idea to proceed to invert $\hat{\mathcal{L}}^{(n)}$ is to invert first the operator $D^{(n)}$ defined in (5.116). Thereafter, we will give an estimate of the operator $T^{(n)}$ and show that it is a little perturbation of $D^{(n)}$.

5.7 Diagonal term

In order to invert the operator $D^{(n)}$, we now analyze the spectral properties of the operators

$$S_k := \mathbf{\Omega} + \varepsilon \mathcal{M}_k \tag{5.117}$$

where[3]

$$\mathbf{\Omega} := \begin{pmatrix} \tilde{\Omega} & 0 \\ 0 & -\tilde{\Omega} \end{pmatrix} \tag{5.118}$$

and $\mathcal{M}_k \in \mathcal{L}(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s+1} \times \ell^{a,s+1})$. This spectral analysis will enable us to diagonalize $D^{(n)}$ (see subsection 5.7.3). Unfortunately it results that $\mathcal{M}_k$ is not symmetric and this fact makes the spectral analysis very involved.

Remark 5.7.1. *The operator $\mathcal{M}_k$ defined in* (5.66), (5.67) *and* (5.57) *is not symmetric. This follows by the fact that the term $\Lambda^{11}_{k,k} - \Lambda^{22}_{k,k}$ is different from zero, as one can calculate. For example, $\Lambda^{11}_{k,k}$ contains the term $\frac{1}{T}\int_0^T \partial^2_{pp}\tilde{K}(w_0)$, with w_0 defined in* (5.46)*, which is different from zero since $\partial^2_{pp}\tilde{K}(w_0(t))$ is even; on the other hand $\Lambda^{22}_{k,k}$ contains the term $\frac{1}{T}\int_0^T \partial^2_{qq}\tilde{K}(w_0) \neq 0$ and, as one can verify, $\frac{1}{T}\int_0^T \partial^2_{pp}\tilde{K}(w_0) \neq \frac{1}{T}\int_0^T \partial^2_{qq}\tilde{K}(w_0)$.*

5.7.1 Diagonalization of S_k

Theorem 5.7.2. *Let be $r, \rho > 0$. Let be X, Y, Z Banach spaces. Let be $X_0 := \overline{B_r(x_0)}$, $Y_0 := \overline{B_\rho(y_0)}$, $F : Y_0 \times X_0 \longrightarrow Z$ continuous such that $F(y_0, x_0) = 0$*

[3] We introduce ε as a conventional parameter. In these notations, $\mathcal{M}_k$ will not depend on it.

and $\partial_y F : Y_0 \times X_0 \longrightarrow \mathcal{L}(Y, Z)$ continuous. Suppose that $T := \big[\partial_y F(y_0, x_0)\big]^{-1}$ exists, $T : Z \longrightarrow Y$, $T \in \mathcal{L}(Z, Y)$ and

$$\sup_{x\in X_0} \|F(y_0, x)\|_Z \le \frac{\rho}{2\|T\|}\,, \qquad \sup_{(y,x)\in Y_0\times X_0} \|\mathbb{I} - T\partial_y F(y, x)\|_{\mathcal{L}(Y,Y)} \le \frac{1}{2}\,.$$

Then, there exists a unique $g : X_0 \longrightarrow Y_0$ such that $F(g(x), x) \equiv 0$, $g(x_0) = y_0$.

Let be $\tilde{\Omega} = \mathrm{diag}\{\tilde{\Omega}_i\}_{i\in\mathcal{I}^c}$ with $\tilde{\Omega}_i$ the shifted elliptic frequencies in (2.94).

Lemma 5.7.3. *Let be $\tilde{\Omega}_i$ as in* (2.94). *Then*

$$\left|\frac{i}{\tilde{\Omega}_i - \tilde{\Omega}_j}\right| \le \mathrm{const}\, j\,, \qquad \forall\, j, i \in \mathcal{I}^c\,.$$

PROOF. It results that $|\tilde{\Omega}_i - \tilde{\Omega}_j| \ge |j - i|/2$. Hence

$$\frac{1}{|\tilde{\Omega}_i - \tilde{\Omega}_j|} \le \frac{2}{|j - i|}\,.$$

If $i < j$ the thesis follows because $i, j \ge 1$. Otherwise, if $i \ge j$, denoting $k := i - j$, we have to show that $k + j \le \mathrm{const}\, kj$. Since $j \le kj$ and $k \le kj$, the thesis follows with const $= 2$.

$\square$

Remark 5.7.4. *From now on we fix $s > 3$.*

Remark 5.7.5 (Notations)**.** *In this section, from now on, we will denote for more clarity the norm and the scalar product in $\ell^{a,s} \times \ell^{a,s}$ by $|\cdot|_s$ and $\langle\cdot,\cdot\rangle_s$ respectively. We will also denote by $\|\cdot\|_{s_1,s_2}$ the operatorial norm of a linear operator from $\ell^{a,s_1} \times \ell^{a,s_1}$ to $\ell^{a,s_2} \times \ell^{a,s_2}$.*

Theorem 5.7.6. *Let be $S_k = \mathbf{\Omega} + \varepsilon\mathcal{M}_k$ with $\mathcal{M}_k : \ell^{a,s} \times \ell^{a,s} \longrightarrow \ell^{a,s+1} \times \ell^{a,s+1}$, as in* (5.66). *Then, there exists $\varepsilon_0 > 0$, independent of k, such that, $\forall\, |\varepsilon| \le \varepsilon_0$ and $\forall\, k, j$, there exist eigenvalues $\lambda_{kj}(\varepsilon)$ and associated eigenvectors $\varphi_{kj}(\varepsilon) \in \ell^{a,s} \times \ell^{a,s}$ with $|\varphi_{kj}|_s = 1$, such that $S_k\varphi_{kj} = \lambda_{kj}\varphi_{kj}$.*

PROOF. We will prove this theorem using the quantitative version of the Implicit Function Theorem stated in Theorem 5.7.2.
Fix j. Let us define the function $F : Y_0 \times X_0 \to Z$, acting as

$$F(v, \lambda; \varepsilon) := \left(-\mathbf{\Omega} v - \varepsilon\mathcal{M}_k v + \lambda v, \frac{|v|_s^2 - 1}{2}\right)\,, \qquad v \in \ell^{a,s}\,,$$

where Y_0 is the closed ball of radius ρ contained in the space $Y := \ell^{a,s} \times \ell^{a,s} \times \mathbb{R}$, X_0 is the closed ball of radius r contained in the space $X := \mathbb{R}$ and, finally, $Z := \ell^{a,s-1} \times \ell^{a,s-1} \times \mathbb{R}$. We want to apply the Implicit Function Theorem

to the function F in a neighborhood of $(y_0, x_0) := (e_j, \tilde{\Omega}_j, 0)$. We have that $\partial_y F(e_j, \tilde{\Omega}_j, 0) \in \mathcal{L}(Y, Z)$ and $T := \big[\partial_y F(e_j, \tilde{\Omega}_j, 0)\big]^{-1} \in \mathcal{L}(Z, Y)$.
Let us recall that, if e_j is a basis of $\ell^{a,s} \times \ell^{a,s}$ then $\tilde{e}_j := je_j$ is a basis of $\ell^{a,s-1} \times \ell^{a,s-1}$. Let be v a vector, denote by $v_j := \langle e_j, v\rangle_s$ the j^{th}–component in the space $\ell^{a,s} \times \ell^{a,s}$ and by $\tilde{v}_j := \langle e_j, v\rangle_s$ the j^{th}–component in the space $\ell^{a,s-1} \times \ell^{a,s-1}$.
Compute

$$\begin{aligned}
\partial_y F(e_j, \tilde{\Omega}_j, 0)[w, \mu] &= \big(-\boldsymbol{\Omega} w + \tilde{\Omega}_j w + \mu e_j, \langle e_j, w\rangle_s \big) \\
&= \big(\mu e_j + \sum_{i \neq j} (\tilde{\Omega}_j - \tilde{\Omega}_i) w_i e_i, w_j\big) \\
&= \left(\frac{\mu}{j} \tilde{e}_j + \sum_{i \neq j} \frac{\tilde{\Omega}_j - \tilde{\Omega}_i}{i} w_i \tilde{e}_i, w_j \right) ,
\end{aligned}$$

and, being $T \circ \partial_y F = \mathbb{I}$, denoting $\partial_y F(e_j, \tilde{\Omega}_j, 0)[w, \mu] = (\tilde{w}, \tilde{\mu})$, one obtains

$$T[\tilde{w}, \tilde{\mu}] = \left(\tilde{\mu} e_j + \sum_{i \neq j} \frac{i \tilde{w}_i e_i}{\tilde{\Omega}_j - \tilde{\Omega}_i}, j \tilde{w}_j \right) . \tag{5.119}$$

Hence, by Lemma 5.7.3,

$$\begin{aligned}
\|T\|^2_{s-1,s} &= \sup_{|\tilde{w}|_{s-1} + |\tilde{\mu}|^2 = 1} \left(\left| \tilde{\mu} e_j + \sum_{i \neq j} \frac{i \tilde{w}_i e_i}{\tilde{\Omega}_j - \tilde{\Omega}_i} \right|^2_s + |j \tilde{w}_j|^2 \right) \\
&= \sup_{|\tilde{w}|_{s-1} + |\tilde{\mu}|^2 = 1} \left(|\tilde{\mu}|^2 + \sum_{i \neq j} \left(\frac{i \tilde{w}_i e_i}{\tilde{\Omega}_j - \tilde{\Omega}_i} \right)^2 + j^2 \tilde{w}_j^2 \right) \\
&\le \sup_{|\tilde{w}|_{s-1} + |\tilde{\mu}|^2 = 1} \left(|\tilde{\mu}|^2 + \mathrm{const}\, j^2 \sum_i \tilde{w}_i^2 \right) \le \mathrm{const}\, j^2 ,
\end{aligned}$$

where in the last inequality we used that $\sum_i \tilde{w}_i^2 = |\tilde{w}|^2_{s-1}$. It follows that

$$\|T\|_{s-1,s} \le \mathrm{const}\, j . \tag{5.120}$$

We have to choose the radius ρ such that

$$\sup_{x \in X_0} \|F(y_0, x)\|_Z \le \frac{\rho}{2\|T\|_{s-1,s}} .$$

It results $F(y_0, x) = F(e_j, \tilde{\Omega}_j; \varepsilon) = (-\varepsilon \mathcal{M}_k e_j, 0)$ and $\|F\|_Z = \varepsilon |\mathcal{M}_k e_j|_{s-1}$. In particular, denoting by $M = \mathrm{diag}(j)$, $j \ge 1$,

$$\begin{aligned}
|\mathcal{M}_k e_j|_{s-1} &= |\mathcal{M}_k M M^{-1} e_j|_{s-1} \le \|\mathcal{M}_k\|_{s-2,s-1} \|M\|_{s-1,s-2} \\
&\le |M^{-1} e_j|_{s-1} = \frac{\|\mathcal{M}_k\|_{s-2,s-1}}{j^2} ,
\end{aligned}$$

where we used that $|M^{-1}e_j|_{s-1} = |e_j/j|_{s-1} = |\tilde{e}_j/j^2|_{s-1} = 1/j^2$. Hence

$$\|F\|_Z \leq \frac{\varepsilon\|\mathcal{M}_k\|_{s-2,s-1}}{j^2} \leq \frac{\rho}{2\|T\|_{s-1,s}}\,,$$

where the last inequality holds if, by (5.120),

$$\rho := \text{const}\,\frac{\varepsilon}{\varepsilon_0 j}\,, \tag{5.121}$$

where

$$\varepsilon_0 := \text{const}\min\left\{\frac{1}{\|\mathcal{M}_k\|_{s,s}}, \frac{1}{\|\mathcal{M}_k\|_{s-2,s-1}}\right\}. \tag{5.122}$$

We note that $\|\mathcal{M}_k\|_{s,s}$ and $\|\mathcal{M}_k\|_{s-2,s-1}$ are bounded by a constant *independent* of k (recall (5.59),(5.66),(5.67)). Now, we have to compute $(-\mathbb{I} + T\partial_y F)(v,\lambda;\varepsilon)[w,\mu]$. First, compute

$$\begin{aligned}\partial_y F(v,\lambda;\varepsilon)[w,\mu] &= \big(-\tilde{\Omega}w + \lambda w + \mu v - \varepsilon\mathcal{M}_k w, \langle v,w\rangle_s\big)\\ &= \left(\sum_i\left(\frac{\lambda-\tilde{\Omega}_i}{i}w_i + \frac{\mu v_i}{i}\right)\tilde{e}_i - \varepsilon\sum_i\langle\mathcal{M}_k w,\tilde{e}_i\rangle_{s-1}\tilde{e}_i, \langle v,w\rangle_s\right).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&-(w,\mu) + T\partial_y F(v,\lambda;\varepsilon)[w,\mu] =\\ &= \Bigg(-w + \langle v,w\rangle_s e_j + \sum_{i\neq j}\frac{(\lambda-\tilde{\Omega}_i)w_i + \mu v_i - \varepsilon i\langle\mathcal{M}_k w,\tilde{e}_i\rangle_{s-1}}{\tilde{\Omega}_j-\tilde{\Omega}_i}e_i,\\ &\quad -\mu + (\lambda-\tilde{\Omega}_j)w_j + \mu v_j - \varepsilon j\langle\mathcal{M}_k w,\tilde{e}_j\rangle_{s-1}\Bigg)\\ &= \Bigg(\langle(v-e_j),w\rangle_s e_j + \sum_{i\neq j}\frac{(\lambda-\tilde{\Omega}_i)w_i + \mu v_i - \varepsilon\langle\mathcal{M}_k w,e_i\rangle_s}{\tilde{\Omega}_j-\tilde{\Omega}_i}e_i,\\ &\quad \mu(v_j-1) + (\lambda-\tilde{\Omega}_j)w_j - \varepsilon\langle\mathcal{M}_k w,e_i\rangle_s\Bigg),\end{aligned}$$

where we used that $\langle\mathcal{M}_k w,\tilde{e}_j\rangle_{s-1} = \langle\mathcal{M}_k w,e_j\rangle_s$. We write

$$\begin{aligned}\big\|\mathbb{I} - T\partial_y F(v,\lambda;\varepsilon)\big\|^2_{\mathcal{L}(Y,Y)} &= \sup_{|\tilde{w}|_{s-1}+|\tilde{\mu}|^2=1}\Big|-(w,\mu) + T\partial_y F(v,\lambda;\varepsilon)[w,\mu]\Big|^2_s\\ &\leq |v-e_j|^2_s|w|^2_s\\ &+ \text{ct}\sum_{i\neq j}\frac{|\lambda-\tilde{\Omega}_i|^2w_i^2 + |\mu|^2v_i^2 + \varepsilon^2|\langle\mathcal{M}_k w,e_i\rangle_s|^2}{(\tilde{\Omega}_j-\tilde{\Omega}_i)^2}\\ &+ |\mu|^2|1-v_j|^2 + |\lambda-\tilde{\Omega}_j|^2w_j^2 + \varepsilon^2|\langle\mathcal{M}_k w,e_j\rangle_s|^2\\ &\leq \text{const}\,\rho^2|w|^2_s + \text{const}\,|\mu|^2\rho^2\end{aligned}$$

$$
\begin{aligned}
&+ \quad \varepsilon^2 \|\mathcal{M}_k\|_{s,s}^2 \left(1 + \mathrm{const} \sum_{i \neq j} \frac{1}{(j-i)^2}\right) \\
&\leq \quad \mathrm{const}\left(\rho^2 + \varepsilon^2 \|\mathcal{M}_k\|_{s,s}^2\right),
\end{aligned}
$$

where we used that $|\tilde{\Omega}_j - \tilde{\Omega}_i| \geq \mathrm{const} > 0$ and $\sum_{i \neq j} \frac{1}{(j-i)^2} \leq \mathrm{const}$. By (5.122) we have that

$$
\left\|\mathbb{I} - T\partial_y F(v, \lambda; \varepsilon)\right\|_{\mathcal{L}(Y,Y)}^2 \leq \frac{1}{4},
$$

and, from the Implicit Function Theorem 5.7.2, the thesis follows.

□

Remark 5.7.7. *We notice that $\varphi_{k1}, \ldots, \varphi_{kj}, \ldots$ is a "basis" for $\ell^{a,s} \times \ell^{a,s}$ but it is not orthonormal, in the sense that, given $y \in \ell^{a,s} \times \ell^{a,s}$, namely given $(y_1, \ldots, y_j, \ldots)$ with $\sum_i y_i^2 = |y|_s^2 < \infty$ such that $y = \sum_i y_i e_i$, then there exist $(\hat{y}_1, \ldots, \hat{y}_j, \ldots)$, with $\sum_i \hat{y}_i^2 < \infty$ such that $y = \sum_i \hat{y}_i \varphi_{ki}$ and viceversa.*

In particular, from Theorem 5.7.6, using (5.121), we obtain the following asymptotic estimate for the eigenvalues

$$
|\lambda_{kj}(\varepsilon) - \tilde{\Omega}_j| \leq \rho \leq \mathrm{const}\frac{\varepsilon}{j}, \qquad j \in \mathcal{I}^c, \tag{5.123}
$$

and for the eigenvectors

$$
|\varphi_{kj}(\varepsilon) - e_j|_s \leq \rho \leq \mathrm{const}\frac{\varepsilon}{j}, \qquad j \in \mathcal{I}^c. \tag{5.124}
$$

Now we give an improvement to these estimates.

5.7.2 Asymptotic estimate for the eigenvalues

By Theorem 5.7.6 we found eigenvalues φ_{kj} and eigenvectors λ_{kj} of the operator S_k such that $S_k\varphi_{kj} = \lambda_{kj}\varphi_{kj}$. Deriving this expression on ε and recalling (5.117), we obtain

$$
\mathcal{M}_k\varphi_{kj} + S_k\varphi'_{kj} = \lambda'_{kj}\varphi_{kj} + \lambda_{kj}\varphi'_{kj}. \tag{5.125}
$$

Since $\langle \varphi_{kj}, \varphi'_{kj}\rangle_s = 0$, multiplying by φ_{kj} we have, $\lambda'_{kj} = \langle \mathcal{M}_k\varphi_{kj}, \varphi_{kj}\rangle_s + \langle S_k\varphi'_{kj}, \varphi_{kj}\rangle_s$, Let us note that $\langle S_k\varphi_{kj}, \varphi'_{kj}\rangle_s = \lambda_{kj}\langle \varphi_{kj}, \varphi'_{kj}\rangle_s = 0$, hence, by using (5.117),

$$
\begin{aligned}
\langle S_k\varphi'_{kj}, \varphi_{kj}\rangle_s &= \langle S_k\varphi'_{kj}, \varphi_{kj}\rangle_s - \langle S_k\varphi_{kj}, \varphi'_{kj}\rangle_s \\
&= \langle \boldsymbol{\Omega}\varphi'_{kj}, \varphi_{kj}\rangle_s - \langle \boldsymbol{\Omega}\varphi_{kj}, \varphi'_{kj}\rangle_s \\
&\quad +\varepsilon\big(\langle \mathcal{M}_k\varphi'_{kj}, \varphi_{kj}\rangle_s - \langle \mathcal{M}_k\varphi_{kj}, \varphi'_{kj}\rangle_s\big),
\end{aligned}
$$

from which it follows that

$$
\lambda'_{kj} = \mathcal{M}_k\langle \varphi_{kj}, \varphi_{kj}\rangle_s + \varepsilon\big(\langle \mathcal{M}_k\varphi'_{kj}, \varphi_{kj}\rangle_s - \langle \mathcal{M}_k\varphi_{kj}, \varphi'_{kj}\rangle_s\big). \tag{5.126}
$$

Let us consider (5.125). Define Π_{kj} the projection on $E_{kj} := \langle \varphi_{kj} \rangle^{\perp}$. It follows that

$$\Pi_{kj}(S_k - \lambda_{kj})\varphi'_{kj} = -\Pi_{kj}\mathcal{M}_k \varphi_{kj}\,. \tag{5.127}$$

Let be

$$B_{kj} := \Big(\Pi_{kj}(S_k - \lambda_{kj})\Big)\big|_{E_{kj}}\,, \qquad B_{kj}\ : E_{kj} \to E_{kj}\,. \tag{5.128}$$

Notice that

$$E_{kj} := \left\{ y = \sum_i \hat{y}_i \varphi_{ki} \ \text{ s.t. } \ \hat{y}_j = -\sum_{i \neq j} \hat{y}_i d_{i,j} \right\}\,, \tag{5.129}$$

where $d_{i,j} := \langle \varphi_{ki}, \varphi_{kj} \rangle_s$. In particular, we can write

$$\begin{aligned} \langle \varphi_{ki}, \varphi_{kj} \rangle_s &= \langle e_i, e_j \rangle_s + \langle \varphi_{ki} - e_i, e_j \rangle_s + \langle e_i, \varphi_{kj} - e_j \rangle_s \\ &\quad + \langle \varphi_{kj} - e_j, \varphi_{ki} - e_i \rangle_s \\ &= \delta_{ij} + O\left(\frac{\varepsilon}{i}\right) + O\left(\frac{\varepsilon}{j}\right) + O\left(\frac{\varepsilon^2}{ij}\right) = \delta_{ij} + O(\varepsilon)\,, \end{aligned}$$

namely

$$d_{ij} = \delta_{ij} + O(\varepsilon)\,. \tag{5.130}$$

Lemma 5.7.8. *The operator B_{kj} defined in* (5.128) *is invertible and*

$$\|B_{kj}\|_{s,s} \leq \text{const}\,.$$

PROOF. Let be b a vector of $\ell^{a,s} \times \ell^{a,s}$. By Remark 5.7.7 we can write $b = \sum_i \hat{b}_i \varphi_{ki}$ and by the definition of Π_{kj}, we have $\Pi_{kj} b := b - \langle b, \varphi_{kj} \rangle_s \varphi_{kj}$. It follows that

$$\Pi_{kj}\Big(\sum_i \hat{b}_i \varphi_{ki}\Big) = \sum_i \hat{b}_i \varphi_{ki} - \Big(\sum_{i \neq j} \hat{b}_i d_{i,j}\Big)\varphi_{kj}\,. \tag{5.131}$$

Let be $y, b \in E_{kj}$, we have to invert

$$\Pi_{kj}(S_k - \lambda_{kj}) y = b\,. \tag{5.132}$$

Since

$$y = \sum_i \hat{y}_i \varphi_{ki} - \Big(\sum_{i \neq j} \hat{y}_i d_{i,j}\Big)\varphi_{kj}\,,$$

we obtain

$$\begin{aligned} (S_k - \lambda_{kj}) y &= \sum_i \hat{y}_i (S_k - \lambda_{kj})\varphi_{ki} - \Big(\sum_{i \neq j} \hat{y}_i d_{i,j}\Big)(S_k - \lambda_{kj})\varphi_{kj} \\ &= \sum_{i \neq j} \hat{y}_i (\lambda_{ki} - \lambda_{kj})\varphi_{ki}\,. \end{aligned}$$

Since $b \in E_{kj}$, it is of the form (5.131). Hence, we get, from (5.132),

$$\begin{aligned} B_{kj}y &= \Pi_{kj}(S_k - \lambda_{kj})y = \sum_i \hat{y}_i(\lambda_{ki} - \lambda_{kj})\varphi_{ki} - \Big(\sum_{i\neq j} \hat{y}_i(\lambda_{ki} - \lambda_{kj})d_{i,j}\Big)\varphi_{kj} \\ &= \sum_i \hat{b}_i\varphi_{ki} - \Big(\sum_{i\neq j} \hat{b}_i d_{i,j}\Big)\varphi_{kj} \,. \end{aligned}$$

Thus

$$\hat{y}_i = \frac{\hat{b}_i}{\lambda_{ki} - \lambda_{kj}}\,, \qquad \forall\, i \neq j \quad \text{and} \quad \hat{y}_j = -\sum_{i\neq j} \hat{y}_i d_{i,j}\,. \tag{5.133}$$

From (5.133), it follows that $|\hat{y}_i| \leq \text{const}\,|\hat{b}_i|$ for all $i \neq j$, being $|\lambda_{ki} - \lambda_{kj}| \geq$ const > 0, as it follows by (5.123). Moreover, using (5.130) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |\hat{y}_j| &\leq \sum_{i\neq j} \frac{|\hat{b}_i|}{|\lambda_{ki} - \lambda_{kj}|}\,|d_{i,j}| \leq \text{const}\,\varepsilon \sum_{i\neq j} \frac{|\hat{b}_i|}{|\lambda_{ki} - \lambda_{kj}|} \\ &\leq \text{const}\,\varepsilon\Big(\sum_{i\neq j} \frac{1}{|\lambda_{ki} - \lambda_{kj}|^2}\Big)^{1/2}\Big(\sum_{i\neq j} |\hat{b}_i|^2\Big)^{1/2} \\ &\leq \text{const}\,\varepsilon\Big(\sum_{i\neq j} |\hat{b}_i|^2\Big)^{1/2}. \end{aligned}$$

Therefore, by Proposition 5.7.16, defining $B_{kj}^{-1}b := y = \sum_i \hat{y}_i\varphi_{ki}$, one obtains

$$|B_{kj}^{-1}b|_s = |y|_s = \Big|\sum_i \hat{y}_i\varphi_{ki}\Big| \leq 2\Big(\sum_i \hat{y}_i^2\Big)^{1/2} \leq \text{const}\,\Big(\sum_i \hat{b}_i^2\Big)^{1/2} \leq \text{const}\,|b|_s\,,$$

from which it follows that

$$\|B_{kj}^{-1}\|_{s,s=} \sup_{|b|_s=1} |B_{kj}b|_s \leq \text{const}\,.$$

□

Lemma 5.7.9. *Let be $\mathcal{M}_k$ the operator defined in* (5.66) *and let be e_j an orthonormal basis of $\ell^{a,s} \times \ell^{a,s}$, then*

$$|\mathcal{M}_k e_j|_s \leq \frac{\text{const}}{j}\,, \qquad j \in \mathcal{I}^c\,.$$

PROOF. Since

$$\|M\mathcal{M}_k\|_{s,s} = \|\mathcal{M}_k M\|_{s,s} \leq \|\mathcal{M}_k\|_{s,s} < \infty\,, \tag{5.134}$$

we get

$$
\begin{aligned}
|\mathcal{M}_k e_j|_s &= |\mathcal{M}_k M M^{-1} e_j|_s \le \|\mathcal{M}_k M\|_{s,s} |M^{-1} e_j|_s \\
&= \|\mathcal{M}_k\|_{s,s} |e_j\, j^{-1}|_s = \frac{\|\mathcal{M}_k\|_{s,s}}{j} \le \frac{\text{const}}{j}\,,
\end{aligned}
$$

with constant *independent* of k (recall (5.59),(5.66),(5.67)).

□

Lemma 5.7.10. *Let be ε suitably small such that Theorem 5.7.6 holds. Let be $\{\varphi_{kj}\}_{j\in\mathcal{I}^c}$ the eigenvectors of the operator S_k. Then*

$$
|\varphi'_{kj}|_s \le \frac{\text{const}}{j}\,, \qquad j \in \mathcal{I}^c\,.
$$

PROOF. By Lemma 5.7.8 and from (5.127), it follows that

$$
\varphi'_{kj} = -B_{kj}^{-1} \Pi_{kj} \mathcal{M}_k \varphi_{kj}\,. \tag{5.135}
$$

Since, by hypothesis, Theorem 5.7.6 holds, φ_{kj} satisfy (5.124). Hence, from (5.135), using Lemma 5.7.8, we get

$$
\begin{aligned}
|\varphi'_{kj}|_s &\le \|B_{kj}^{-1}\|_{s,s} |\mathcal{M}_k \varphi_{kj}|_s \le \text{const}\left(|\mathcal{M}_k e_j|_s + |\mathcal{M}_k(\varphi_{kj} - e_j)|_s\right) \\
&\le \text{const}\left(\frac{1}{j} + \frac{\varepsilon}{j}\right) \le \frac{\text{const}}{j}\,.
\end{aligned}
$$

where in the last inequality we used Lemma 5.7.9.

□

Recall (5.126), we are going to study the asymptotic behavior of the eigenvalues λ_{kj}.

Proposition 5.7.11. *Let be ε suitably small such that Theorem 5.7.6 holds. Let be $\{\lambda_{kj}\}_{j\in\mathcal{I}^c}$ the eigenvalues associated to the eigenvectors $\{\varphi_{kj}\}_{j\in\mathcal{I}^c}$ of the operator S_k. Then*

$$
\left|\lambda'_{kj} - \langle \mathcal{M}_k e_j, e_j\rangle_s\right| \le \text{const}\frac{\varepsilon}{j^2}\,, \qquad j \in \mathcal{I}^c\,, \tag{5.136}
$$

with constant independent of k.

PROOF. Recall the expression for λ'_{kj} in (5.126). Using the linearity of the scalar product, one obtains

$$
\begin{aligned}
\langle \mathcal{M}_k \varphi_{kj}, \varphi_{kj}\rangle_s &= \langle \mathcal{M}_k e_j, e_j\rangle_s + \langle \mathcal{M}_k e_j, \varphi_{kj} - e_j\rangle_s + \big\langle \mathcal{M}_k(\varphi_{kj} - e_j\rangle_s, e_j\big\rangle_s \\
&\quad + \big\langle \mathcal{M}_k(\varphi_{kj} - e_j), \varphi_{kj} - e_j\big\rangle_s.
\end{aligned} \tag{5.137}
$$

By (5.124) and Lemma 5.7.9, we have

$$
\left|\big\langle \mathcal{M}_k e_j, \varphi_{kj} - e_j\big\rangle_s\right| \le |\mathcal{M}_k e_j|_s \frac{\text{const}\,|\varepsilon|}{j} \le \frac{\text{const}\,|\varepsilon|}{j^2}\,. \tag{5.138}
$$

Recalling that $M = \mathrm{diag}(j)$, by (5.124), (5.134) and Lemma 5.7.9, we get

$$\begin{aligned}
\Big|\big\langle \mathcal{M}_k(\varphi_{kj}-e_j), e_j\big\rangle_s\Big| &= \Big|\big\langle M^{-1}M\mathcal{M}_k(\varphi_{kj}-e_j), e_j\big\rangle_s\Big| \\
&= \Big|\big\langle M\mathcal{M}_k(\varphi_{kj}-e_j), M^{-1}e_j\big\rangle_s\Big| \\
&\le \|M\mathcal{M}_k\|_{s,s}|\varphi_{kj}-e_j|_s|M^{-1}e_j|_s \\
&\le \frac{\mathrm{const}\,|\varepsilon|}{j^2}. \qquad (5.139)
\end{aligned}$$

In the same way

$$\Big|\big\langle \mathcal{M}_k(\varphi_{kj}-e_j), \varphi_{kj}-e_j\big\rangle_s\Big| \le \|\mathcal{M}_k\|_{s,s}|\varphi_{kj}-e_j|_s|\varphi_{kj}-e_j|_s \le \frac{\mathrm{const}\,\varepsilon^2}{j^2}. \qquad (5.140)$$

Hence by (5.137), (5.138), (5.139), we get

$$\langle \mathcal{M}_k\varphi_{kj}, \varphi_{kj}\rangle_s = \langle \mathcal{M}_k e_j, e_j\rangle_s + O\Big(\frac{|\varepsilon|}{j^2}\Big). \qquad (5.141)$$

By (5.134), Lemma 5.7.9 and Lemma 5.7.10, we obtain

$$\begin{aligned}
\big|\langle \mathcal{M}_k\varphi'_{kj}, \varphi_{kj}\rangle_s\big| &= \Big|\big\langle M^{-1}M\mathcal{M}_k\varphi'_{kj}, \varphi_{kj}\big\rangle_s\Big| \\
&= \Big|\big\langle M\mathcal{M}_k\varphi'_{kj}, M^{-1}\varphi_{kj}\big\rangle_s\Big| \le \|M\mathcal{M}_k\|_{s,s}|\varphi'_{kj}|_s|M^{-1}\varphi_{kj}|_s \\
&\le \frac{\mathrm{const}}{j}\Big(|Me_j|_s + |M^{-1}(\varphi_{kj}-e_j)|_s\Big) \\
&\le \frac{\mathrm{const}}{j}\left(\frac{1}{j} + \frac{\mathrm{const}\,|\varepsilon|}{j}\right) \le \frac{\mathrm{const}}{j^2}. \qquad (5.142)
\end{aligned}$$

Analogously, by Lemma 5.7.9, Lemma 5.7.10 and by the linearity of the scalar product, one has

$$\begin{aligned}
\big|\langle \mathcal{M}_k\varphi_{kj}, \varphi'_{kj}\rangle_s\big| &= |\mathcal{M}_k\varphi_{kj}|_s|\varphi'_{kj}|_s \\
&\le \frac{\mathrm{const}}{j}|\mathcal{M}_k\varphi_{kj}|_s \le \frac{\mathrm{const}}{j}\Big(|\mathcal{M}_k e_j|_s + |\mathcal{M}_k(\varphi_{kj}-e_j)|_s\Big) \\
&\le \frac{\mathrm{const}}{j}\left(\frac{1}{j} + \frac{\mathrm{const}\,|\varepsilon|}{j}\right) \le \frac{\mathrm{const}}{j^2}. \qquad (5.143)
\end{aligned}$$

Hence from (5.126), by (5.141), (5.142), (5.143) we reach the final estimate (5.136).

□

From (5.136), the asymptotic estimate for the eigenvalues follows:

$$\big|\lambda_{kj} - \tilde{\Omega}_j - \varepsilon\langle \mathcal{M}_k e_j, e_j\rangle_s\big| \le \mathrm{const}\frac{\varepsilon^2}{j^2}, \qquad j \in \mathcal{I}^c, \qquad (5.144)$$

with constant independent of k.

In order to prove the invertibility of the linear operator $D^{(n)}$, nothing remains but to prove that its eigenvalues are different from zero. We reach an estimate from below for the eigenvalues, in particular it is stated in the following lemma. Let us define

$$\mu_{k,j} := \frac{k}{\tau} - \lambda_{kj}\,.$$

Lemma 5.7.12. *Let be $\eta \in \Delta_n^\nu(\xi)$, $1 < \nu < 2$. Then*

$$\alpha_k := \min_{j\geq 1} |\mu_{k,j}| \geq \begin{cases} \text{const}, & \text{if } k \leq \tau/2; \\ \dfrac{\gamma\,\tau^{\nu-1}}{|k|^\nu}, & \text{if } k > \tau/2. \end{cases} \tag{5.145}$$

PROOF. From the first Melnikov condition in (5.5.3), being $\varpi = 1/\tau$, we have, since λ_{kj} satisfy (5.144) and $\nu < 2$,

$$|\mu_{k,j}| \;=\; |\varpi k - \lambda_{kj}| = \left|\varpi k - \tilde{\Omega}_j - \varepsilon\langle \mathcal{M}_k e_j, e_j\rangle_s + O\left(\frac{\varepsilon^2}{j^2}\right)\right| \geq \frac{\gamma_n \varpi}{j^\nu}\,,$$

where γ_n has been defined in (5.69). We have that the $\alpha_k := \min_{j\geq 1}|\mu_{k,j}|$ will be achieved at $j = j(k)$, i.e. $j \approx \varpi k$. Therefore, we get

$$\min_{j\geq 1}|\mu_{k,j}| \geq \frac{\gamma_n}{\varpi^{1-\nu}|k|^\nu}\,.$$

□

Let be e_i the canonical basis of $\ell^{a,s}\times\ell^{a,s}$. Let us define $V_k : \ell^{a,s}\times\ell^{a,s} \longrightarrow \ell^{a,s}\times\ell^{a,s}$ the operator acting on $y = \sum y_i e_i$, as

$$V_k\Big(\sum_i y_i e_i\Big) := \sum_i y_i \varphi_{ki}\,. \tag{5.146}$$

Lemma 5.7.13. *V_k in* (5.146) *are well defined.*

PROOF. Let be $y^{(n)} := \sum_{i=1}^n y_i\varphi_{ki}$. We have to show that $y^{(n)}$ is a Cauchy–sequence, namely, if $n > m$, then $|y^{(n)} - y^{(m)}|_s \to 0$ when $m \to \infty$. By (5.124) and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |y^{(n)} - y^{(m)}|_s \;&=\; \Big|\sum_{i=m+1}^n y_i\varphi_{ki}\Big|_s = \Big|\sum_{i=m+1}^n y_i e_i + \sum_{i=m+1}^n y_i(\varphi_{ki} - e_i)\Big|_s \\ &\leq\; \Big|\sum_{i=m+1}^n y_i e_i\Big|_s + \sum_{i=m+1}^n |y_i||\varphi_{ki} - e_i|_s \\ &\leq\; \Big|\sum_{i=m+1}^n y_i e_i\Big|_s + \Big(\sum_{i=m+1}^n |y_i|^2\Big)^{1/2}\Big(\sum_{i=m+1}^n |\varphi_{ki} - e_i|_s^2\Big)^{1/2} \\ &\leq\; \Big(\sum_{i=m+1}^n y_i^2\Big)^{1/2} + \text{const}\Big(\sum_{i=m+1}^n y_i^2\Big)^{1/2}\Big(\sum_{i=m+1}^n \frac{\varepsilon^2}{i^2}\Big)^{1/2} \end{aligned}$$

$$= \quad (1+\mathrm{const}\,\varepsilon)\Big(\sum_{i=m+1}^{n} y_i^2\Big)^{1/2} \longrightarrow 0 \qquad \text{as} \;\; m\to\infty\,.$$

$\square$

Proposition 5.7.14. *The operators V_k in* (5.146) *are linear and continuous. Moreover, V_k are invertible and*

$$\|V_k^{-1}\|_{s,s} \leq \mathrm{const}\,.$$

PROOF. First we prove the boundness of V_k. Let be $y = \sum y_i e_i$. Using the Parseval identity and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned}
\|V_k\|_{s,s} &= \sup_{|y|_s=1} |V_k y|_s = \sup_{\sum y_i^2=1} \Big|\sum_i y_i\varphi_{ki}\Big|_s \\
&\leq \sup_{\sum y_i^2=1} \bigg(\Big|\sum_i y_i e_i\Big|_s + \Big|\sum_i y_i(\varphi_{ki}-e_i)\Big|_s\bigg) \\
&\leq \sup_{\sum y_i^2=1} \bigg(1+\varepsilon\Big(\sum_i y_i^2\Big)^{1/2}\Big(\sum_i \frac{1}{i^2}\Big)^{1/2}\bigg) \leq 1+\mathrm{const}\varepsilon \leq 2\,,
\end{aligned}$$

where we used (5.124). Moreover V_k are invertible. Indeed, defining $\tilde V_k := V_k - \mathbb{I}$, namely $V_k = \mathbb{I} + \tilde V_k$, then

$$\begin{aligned}
\|\tilde V_k\|_{s,s} &= \sup_{|y|_s=1} \Big|\sum_i y_i(\varphi_{ki}-e_i)\Big|_s \leq \sup_{|y|_s=1} \sum_i |y_i||\varphi_{ki}-e_i|_s \\
&\leq \mathrm{const}\,\varepsilon \sup_{|y|_s=1} \sum_i \frac{|y_i|}{i} \leq \mathrm{const}\,\varepsilon\Big(\sum_i y_i^2\Big)^{1/2}\Big(\sum_i \frac{1}{i^2}\Big)^{1/2} \leq \mathrm{const}\,\varepsilon\,.
\end{aligned}$$

$\square$

Remark 5.7.15. *From Remark 5.7.7, it follows that $y = V_k\hat y$ where $\hat y := \sum_i \hat y_i e_i$ and $\hat y = V_k^{-1} y$.*

The following proposition holds:

Proposition 5.7.16. *Let be $y \in \ell^{a,s}\times\ell^{a,s}$ such that $y = \sum_i \hat y_i \varphi_{ki}$. Then*

$$\frac{1}{2}\Big(\sum_i \hat y_i^2\Big)^{1/2} \leq |y|_s \leq 2\Big(\sum_i \hat y_i^2\Big)^{1/2}.$$

PROOF. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|y|_s &= \Big|\sum_i \hat y_i\varphi_{ki}\Big|_s \leq \Big|\sum_i \hat y_i e_i\Big|_s + \Big|\sum_i \hat y_i(\varphi_{ki}-e_i)\Big|_s \\
&\leq \Big(\sum_i \hat y_i^2\Big)^{1/2} + \Big(\sum_i \hat y_i^2\Big)^{1/2} \mathrm{const}\,\varepsilon\Big(\sum_i \frac{1}{i^2}\Big)^{1/2}
\end{aligned}$$

$$\leq \quad (1+\mathrm{const}\,\varepsilon)\Big(\sum_i \hat{y}_i^2\Big)^{1/2} \leq 2\Big(\sum_i \hat{y}_i^2\Big)^{1/2}. \tag{5.147}$$

Moreover

$$|y|_s \geq \Big|\sum_i \hat{y}_i e_i\Big|_s - \Big|\sum_i \hat{y}_i(\varphi_{ki}-e_i)\Big|_s \geq (1-\mathrm{const}\,\varepsilon)\Big(\sum_i \hat{y}_i^2\Big)^{1/2}. \tag{5.148}$$

The thesis follows from (5.147) and (5.148).

□

5.7.3 "Weighted asymmetric diagonalization" of $D^{(n)}$

Let us define the operator

$$U^{(n)} := \mathrm{diag}_{\ell\leq L_n}\begin{pmatrix} \mathrm{diag}_j\Big(\mathrm{sign}\left(\frac{\ell}{\tau}+\lambda_{\ell j}\right)\Big) & 0 \\ 0 & \mathrm{diag}_j\Big(\mathrm{sign}\left(\frac{\ell}{\tau}-\lambda_{\ell j}\right)\Big)\end{pmatrix}.$$

Lemma 5.7.17. *We have that* $U^{(n)}: \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}$*,* $\|U^{(n)}\|_{\mathrm{op}} = 1$*, moreover* $U^{(n)}$ *is invertible with* $(U^{(n)})^{-1} = U^{(n)}$*.*

We now introduce the parameter $1/2 < \delta < 1$.

Remark 5.7.18. *Such parameter will be chosen in the estimate on the small divosors, see* (5.191)*. We need this parameter* δ *to balance the asymmetry of the operator* $D^{(n)}$*.*

Define

$$\mathcal{D}_k := \begin{pmatrix} \mathrm{diag}_j\left[\frac{k}{\tau}+\lambda_{kj}\right] & 0 \\ 0 & \mathrm{diag}_j\left[\frac{k}{\tau}-\lambda_{kj}\right]\end{pmatrix},$$

and the operators

$$Z^{(n)} := \mathrm{diag}_{k\leq L_n} V_k|\mathcal{D}_k|^{-(1-\delta)}, \qquad \tilde{Z}^{(n)} := \mathrm{diag}_{k\leq L_n}|\mathcal{D}_k|^{-\delta}V_k^{-1}, \tag{5.149}$$

where V_k are defined in (5.146) and, for any $\kappa\in\mathbb{R}$,

$$|\mathcal{D}_k|^{-\kappa} := \begin{pmatrix} \mathrm{diag}_j\left[\left|\frac{k}{\tau}+\lambda_{kj}\right|^{-\kappa}\right] & 0 \\ 0 & \mathrm{diag}_j\left[\left|\frac{k}{\tau}-\lambda_{kj}\right|^{-\kappa}\right]\end{pmatrix}. \tag{5.150}$$

Since, by (5.145), $\min_{k \le L_n,\ j \ge 1} |k/\tau - \lambda_{kj}| \ge \mathrm{const}(L_n)$, and, by (5.144), $|\frac{k}{\tau} \pm \lambda_{kj}| \sim j$ for j large, we have

$$Z^{(n)} : \hat{W}^{(n)}_{s-1+\delta,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}\,, \tag{5.151}$$

and

$$\tilde{Z}^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s+\delta,\alpha_n,\sigma}\,. \tag{5.152}$$

Moreover we define the operators

$$(Z^{(n)})^{-1} := \mathrm{diag}_{k \le L_n} |\mathcal{D}_k|^{1-\delta} V_k^{-1}\,, \qquad (\tilde{Z}^{(n)})^{-1} := \mathrm{diag}_{k \le L_n} V_k |\mathcal{D}_k|^{\delta}\,. \tag{5.153}$$

We have

$$(Z^{(n)})^{-1} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s-1+\delta,\alpha_n,\sigma} \tag{5.154}$$

and

$$(\tilde{Z}^{(n)})^{-1} : \hat{W}^{(n)}_{s+\delta,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}\,. \tag{5.155}$$

By definition we get that $(Z^{(n)})^{-1} Z^{(n)}$ and $(\tilde{Z}^{(n)})^{-1} \tilde{Z}^{(n)}$ coincide with the identity on $W^{(n)}_{s,\alpha_n,\sigma}$. Let us consider now the operators $D^{(n)} Z^{(n)} : \hat{W}^{(n)}_{s-1+\delta,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s-1,\alpha_n,\sigma}$ and $(\tilde{Z}^{(n)})^{-1} U^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s-\delta,\alpha_n,\sigma}$. Noting that (5.146) implies $D_k^{(n)} V_k = V_k \mathcal{D}_k$, we conclude that $D^{(n)} Z^{(n)}$ restricted to $\hat{W}^{(n)}_{s,\alpha_n,\sigma}$ coincides with $(\tilde{Z}^{(n)})^{-1} U^{(n)}$.

Lemma 5.7.19 ("Diagonalization")**.** $D^{(n)} Z^{(n)} |_{\hat{W}^{(n)}_{s,\alpha_n,\sigma}} = (\tilde{Z}^{(n)})^{-1} U^{(n)}$.

5.8 Off-diagonal term

Remark 5.8.1. *In this section we deal with the operator $T^{(n)}$ defined in* (5.116)*. We point out that $T^{(n)}$ is not a Toepliz operator but only a "quasi–product operator" according to Definition 5.1.9.*

Define

$$\tilde{T}^{(n)} := \tilde{Z}^{(n)} T^{(n)} Z^{(n)} \tag{5.156}$$

namely

$$\left(\tilde{T}^{(n)}\right)_{k\ell} := |\mathcal{D}_k|^{-\delta} V_k^{-1} \hat{\Lambda}_{k,\ell} V_\ell\, |\mathcal{D}_\ell|^{-(1-\delta)}\,. \tag{5.157}$$

We note that

$$\tilde{T}^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}\,.$$

Then we define

$$\Xi^{(n)} := U^{(n)} + \varepsilon \tilde{T}^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}\,.$$

We want to prove that $\Xi^{(n)}$ is invertible. We need the following *crucial* lemma, that will be proved on pg. 193.

Lemma 5.8.2. $\|\tilde{T}^{(n)}\|_{\rm op} \le {\rm const}\,\gamma^{-1}\tau$ *with constant independent of* n.

By Lemmata 5.7.17 and 5.8.2, $\Xi^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}$ is invertible for ε small enough (but independent of n). So we can define

$$\hat{\mathcal{G}}^{(n)} := Z^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)} : \hat{W}^{(n)}_{s,\alpha_n,\sigma} \longrightarrow \hat{W}^{(n)}_{s,\alpha_n,\sigma}\,. \tag{5.158}$$

It results that $\hat{\mathcal{G}}^{(n)}$ is the (right) inverse of $\hat{\mathcal{L}}^{(n)}$. Namely, even if $\hat{\mathcal{L}}^{(n)}\hat{\mathcal{G}}^{(n)}$ is a priori defined only from $W^{(n)}_{s,\alpha_n,\sigma}$ into $W^{(n)}_{s-1,\alpha_n,\sigma}$, it results that

$$\hat{\mathcal{L}}^{(n)}\hat{\mathcal{G}}^{(n)}y = y\,, \qquad \forall\, y \in W^{(n)}_{s,\alpha_n,\sigma}\,. \tag{5.159}$$

Recalling that $\hat{\mathcal{L}}^{(n)} = D^{(n)} + \varepsilon T^{(n)}$, we get, by Lemma 5.7.19

$$\begin{aligned}
\hat{\mathcal{L}}^{(n)}\hat{\mathcal{G}}^{(n)} &= \Big(D^{(n)} + \varepsilon T^{(n)}\Big)\Big(Z^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)}\Big)\\
&= D^{(n)}Z^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)} + \varepsilon T^{(n)}Z^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)}\\
&= \big(\tilde{Z}^{(n)}\big)^{-1}U^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)} + \varepsilon\big(\tilde{Z}^{(n)}\big)^{-1}\tilde{Z}^{(n)}T^{(n)}Z^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)}\\
&= \big(\tilde{Z}^{(n)}\big)^{-1}\big(U^{(n)} + \varepsilon\tilde{Z}^{(n)}T^{(n)}Z^{(n)}\big)\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)}\\
&= \big(\tilde{Z}^{(n)}\big)^{-1}\Xi^{(n)}\big(\Xi^{(n)}\big)^{-1}\tilde{Z}^{(n)}\\
&= \big(\tilde{Z}^{(n)}\big)^{-1}\tilde{Z}^{(n)} = \mathbb{I}
\end{aligned}$$

from which (5.159) follows.

Proof of Lemma 5.5.4 It follows by (5.158), (5.149), (5.150), Lemma 5.7.12 and (5.107). □

5.8.1 Estimate on $\tilde{T}^{(n)}$

We are going to prove the invertibility of $\hat{\mathcal{L}}^{(n)} : W^{(n)}_{s,\alpha,\sigma} \to W^{(n)}_{s,\alpha,\sigma}$ showing that $\varepsilon\tilde{T}^{(n)}$ is a small perturbation of $U^{(n)}$, namely that $\tilde{T}^{(n)}$ is bounded.

Lemma 5.8.3. *Let be* $\hat{\Lambda}_{k,\ell}$ *as in* (5.110) *and* $|\mathcal{D}_k|^{-\delta}$, $|\mathcal{D}_\ell|^{-(1-\delta)}$ *as in* (5.150), $1/2 < \delta < 1$. *Let* V_k *as in* (5.146). *For* $k \neq \ell$

$$\Big\||\mathcal{D}_k|^{-\delta}V_k^{-1}\hat{\Lambda}_{k,\ell}\,V_\ell|\mathcal{D}_\ell\,|^{-(1-\delta)}\Big\|_{s,s} \le \||\mathcal{D}_k|^{-\delta}M^{-1}\|_{s,s}\,\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s}\|\hat{\Lambda}_{k,\ell}\|_{s,s+1}\,. \tag{5.160}$$

proof. Let us write, recalling $S_k = \mathbf{\Omega} + \varepsilon\mathcal{M}_k$,

$$\begin{aligned}
&|\mathcal{D}_k|^{-\delta}\,V_k^{-1}\hat{\Lambda}_{k,\ell}\,V_\ell|\mathcal{D}_\ell\,|^{-(1-\delta)}\\
&\qquad = \big(|\mathcal{D}_k|^{-\delta}M^{-1}\big)\big(M\mathbf{\Omega}^{-1}\big)\big(\mathbf{\Omega}S_k^{-1}\big)\big(S_kV_k^{-1}S_k^{-1}\big)\big(S_k\hat{\Lambda}_{k,\ell}\big)\,V_\ell|\mathcal{D}_\ell\,|^{-(1-\delta)}\,.
\end{aligned}$$

Hence, we have

$$\big\||\mathcal{D}_k|^{-\delta}V_k^{-1}\hat{\Lambda}_{k,\ell}\,V_\ell|\mathcal{D}_\ell\,|^{-(1-\delta)}\big\|_{s,s} \;\le\; \big\||\mathcal{D}_k|^{-\delta}M^{-1}\big\|_{s,s}\big\|M\mathbf{\Omega}^{-1}\big\|_{s,s}\big\|\mathbf{\Omega}S_k^{-1}\big\|_{s,s}$$

$$
\begin{aligned}
&\cdot\big\|S_kV_k^{-1}S_k^{-1}\big\|_{s,s}\big\|S_k\hat{\Lambda}_{k,\ell}\big\|_{s,s}\\
&\cdot\big\|V_\ell\big\|_{s,s}\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}. \qquad (5.161)
\end{aligned}
$$

We are going to estimate the terms of the product in the inequality (5.161). It results

$$
\big\|M\boldsymbol{\Omega}^{-1}\big\|_{s,s}=\big\|\mathrm{diag}_j\{j\tilde{\Omega}_j^{-1}\}\big\|_{s,s}\leq 1\,. \qquad (5.162)
$$

Since $\|\mathcal{M}_k\boldsymbol{\Omega}^{-1}\|_{s,s}\leq 1$ and

$$
\boldsymbol{\Omega}S_k^{-1}=\big(S_k\boldsymbol{\Omega}^{-1}\big)^{-1}=\Big[\big(\boldsymbol{\Omega}+\varepsilon\mathcal{M}_k\big)\boldsymbol{\Omega}^{-1}\Big]^{-1}=\big(\mathbb{I}+\varepsilon\mathcal{M}_k\boldsymbol{\Omega}^{-1}\big)^{-1}\,,
$$

it follows that

$$
\big\|\boldsymbol{\Omega}S_k^{-1}\big\|_{s,s}\leq \mathrm{const}\,. \qquad (5.163)
$$

Let us prove, now, that the operator $S_kV_kS_k^{-1}$ is invertible from $W^{(n)}_{s,\alpha,\sigma}$ in $W^{(n)}_{s,\alpha,\sigma}$. We have that

$$
S_kV_kS_k^{-1}=S_k\mathbb{I}S_k^{-1}+S_k(V_k-\mathbb{I})S_k^{-1}=\mathbb{I}+S_k(V_k-\mathbb{I})S_k^{-1}\,. \qquad (5.164)
$$

From (5.164), we have to show that $\|S_k(V_k-\mathbb{I})S_k^{-1}\|_{s,s}\leq \mathrm{const}\,\varepsilon$. Since, by (5.163),

$$
\|S_k^{-1}\|_{s,s}=\|\boldsymbol{\Omega}^{-1}(S_k\boldsymbol{\Omega}^{-1})^{-1}\|_{s,s}\leq \mathrm{const}\,,
$$

it is enough to show that

$$
\|S_k(V_k-\mathbb{I})\|_{s,s}\leq \mathrm{const}\,\varepsilon\,. \qquad (5.165)
$$

Indeed, let be $y\in W^{(n)}_{s+1,\alpha,\sigma}$, $y=\sum_i y_ie_i$ such that $\sum_i i^2y_i^2=1$. Hence, by (5.144) and (5.124), using (5.146),

$$
\begin{aligned}
\Big|S_k(V_k-\mathbb{I})y\Big|_s &= \Big|S_kV_ky-S_ky\Big|_s=\Big|S_kV_k\sum_i y_ie_i-S_k\sum_i y_ie_i\Big|_s\\
&= \Big|S_k\sum_i y_i\varphi_{ki}-S_k\sum_i y_ie_i\Big|_s=\Big|\sum_i y_i\lambda_i\varphi_{ki}-S_k\sum_i y_ie_i\Big|_s\\
&= \Big|\sum_i y_i\lambda_ie_i+\sum_i y_i\lambda_i(\varphi_{ki}-e_i)-S_k\sum_i y_ie_i\Big|_s\\
&= \Big|\sum_i y_i\lambda_ie_i-\sum_i y_i\tilde{\Omega}_ie_i\\
&\quad -\varepsilon\mathcal{M}_k\sum_i y_ie_i+\sum_i y_i\lambda_i(\varphi_{ki}-e_i)\Big|_s\\
&= \Big|\sum_i y_i(\lambda_i-\tilde{\Omega}_i)e_i\Big|_s-\varepsilon\big|\mathcal{M}_ky\big|_s+\Big|\sum_i y_i\lambda_i(\varphi_{ki}-e_i)\Big|_s\\
&\leq \Big(\sum_i y_i^2(\lambda_i-\tilde{\Omega}_i)^2\Big)^{1/2}+\varepsilon\big|\mathcal{M}_ky\big|_s+\mathrm{ct}\Big(\sum_i|\varphi_{ki}-e_i|_s^2\Big)^{1/2}
\end{aligned}
$$

$$\leq \quad \mathrm{const}\,\varepsilon\,,$$

where we used that $\sum_i y_i^2\lambda_i^2 \leq \mathrm{const}$. Therefore, (5.165) is true. The last estimate we need is

$$\left\|S_k\hat{\Lambda}_{k,\ell}\right\|_{s,s} \leq \mathrm{const}\,\|\hat{\Lambda}_{k,\ell}\|_{s,s+1}\,, \tag{5.166}$$

obtained by (5.117). Finally, by (5.161), (5.162), (5.163), (5.165), (5.166) and Lemma 5.7.14, the inequality (5.160) in the thesis follows.

□

Lemma 5.8.3 will be used in Section 5.9 to prove the following fundamental lemma on small divisors.

Lemma 5.8.4 (Small divisors)**.** *Let be $\varepsilon = \eta^3$. Let be $\eta = \sqrt{\varpi}$, $\varpi \in \Delta_n^\nu(\xi)$. There exists a constant $C > 0$ such that, for all $\ell \neq k$, $1 < \nu < 2$, $\delta = \nu/(\nu{+}1)$,*

$$\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s}\left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} \leq C\,\frac{\tau}{\gamma}\,|\ell-k|^{1-\nu\delta+(\nu-1)/\beta} = C\,\frac{\tau}{\gamma}\,|\ell-k|^{\frac{2\nu+1}{\nu+1}}\,,$$

where $\beta = (\nu-1)/\nu$.

PROOF. See Section 5.9.

Using Lemma 5.8.4 we give the following

Proof of Lemma 5.8.2. We have, from (5.157), (5.160) and Lemma 5.8.4,

$$\begin{aligned}
\left\|\big(\tilde{T}^{(n)}w\big)_k\right\|_{a,s} &= \left\|\sum_{\ell=1\,,\ell\neq k}^{L_n}|\mathcal{D}_k|^{-\delta}V_k^{-1}\hat{\Lambda}_{k,\ell}\,V_\ell|\mathcal{D}_\ell\,|^{-(1-\delta)}w_\ell\right\|_{s,s}\\
&\leq \sum_{\ell=1\,,\ell\neq k}^{L_n}\left\||\mathcal{D}_k|^{-\delta}V_k^{-1}\hat{\Lambda}_{k,\ell}\,V_\ell|\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{a,s}\|w_\ell\|_{a,s}\\
&\leq \sum_{\ell=1\,,\ell\neq k}^{L_n}\left\||\mathcal{D}_k|^{-\delta}M^{-1}\right\|_{s,s}\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s}\left\|\hat{\Lambda}_{k,\ell}\right\|_{s,s+1}\|w_\ell\|_{a,s}\\
&\leq \sum_{\ell=1\,,\ell\neq k}^{L_n}\gamma^{-1}\tau|\ell-k|^{\frac{2\nu+1}{\nu+1}}\|\hat{\Lambda}_{k,\ell}\|_{s,s+1}\|w_\ell\|_{a,s}\,,
\end{aligned}$$

from which it follows that

$$\left\|\big(\tilde{T}^{(n)}w\big)_k\right\|_{a,s} \leq \sum_{\ell=1\,,k\neq\ell}^{L_n}\gamma^{-1}\tau|\ell-k|^{\frac{2\nu+1}{\nu+1}}\|\hat{\Lambda}_{k,\ell}\|_{s,s+1}\|w_\ell\|_{a,s}\,. \tag{5.167}$$

Hence, using (4.11) and (5.167),

$$\left\|T^{(n)}w\right\|_{\alpha,\sigma}^2 \;=\; \sum_{k=1}^{L_n}e^{2k\alpha}k^{2\sigma}\left\|(\tilde{T}^{(n)}w)_k\right\|_{a,s}^2$$

$$
\begin{aligned}
&\leq \frac{\tau^2}{\gamma^2}\sum_{k=1}^{L_n} e^{2k\alpha}k^{2\sigma}\Bigg(\sum_{\ell=1\,,k\neq\ell}^{L_n}|\ell-k|^{\frac{2\nu+1}{\nu+1}}\|\hat\Lambda_{k,\ell}\|_{s,s+1}\|w_\ell\|_{a,s}\Bigg)^2\\
&\leq \mathrm{ct}\,\frac{\tau^2}{\gamma^2}\sum_{k=1}^{L_n} e^{2k\alpha}k^{2\sigma}\sum_{\ell=1\,,k\neq\ell}^{L_n}\frac{|\ell-k|^{2\sigma}\ell^{2\sigma}}{k^{2\sigma}}\,|k-\ell|^{2\,\frac{2\nu+1}{\nu+1}}\|\hat\Lambda_{k,\ell}\|^2_{s,s+1}\|w_\ell\|^2_{a,s}\\
&\leq \mathrm{ct}\,\frac{\tau^2}{\gamma^2}\sum_{\ell=1}^{L_n} e^{2\ell\alpha}\ell^{2\sigma}\|w_\ell\|^2_{a,s}\sum_{k=1\,,k\neq\ell}^{L_n}|\ell-k|^{2\left(\sigma+\frac{2\nu+1}{\nu+1}\right)}e^{2|\ell-k|\alpha}\|\hat\Lambda_{k,\ell}\|^2_{s,s+1}
\end{aligned}
$$

where in the last inequality we used $k\leq \ell+|k-\ell|$. We can estimate

$$
\sum_k e^{2|\ell-k|\alpha}|\ell-k|^{2(\sigma+\frac{2\nu+1}{\nu+1})}\|\Lambda^{ij}_{\ell-k}\|^2_{s,s+1}\leq \|\Lambda^{ij}\|^2_{H^{\alpha,\sigma+\frac{2\nu+1}{\nu+1}}_{\mathcal{L}(\ell^{a,s},\ell^{a,s+1})}}\,.
$$

In the same way

$$
\begin{aligned}
&\sum_k e^{2|\ell-k|\alpha}\,|\ell-k|^{2(\sigma+\frac{2\nu+1}{\nu+1})}\|\Lambda^{ij}_{k+\ell}\|^2_{s,s+1}\\
&\qquad\leq \sum_k e^{2|k+\ell|\alpha}|k+\ell|^{2(\sigma+\frac{2\nu+1}{\nu+1})}\|\Lambda^{ij}_{k+\ell}\|^2_{s,s+1}\\
&\qquad= \|\Lambda^{ij}\|^2_{H^{\alpha,\sigma+\frac{2\nu+1}{\nu+1}}_{\mathcal{L}(\ell^{a,s},\ell^{a,s+1})}}\,.
\end{aligned}
$$

Hence

$$
\left\|T^{(n)}w\right\|^2_{\alpha,\sigma}\leq \mathrm{const}\,\frac{\tau^2}{\gamma^2}\,\|\Lambda^{ij}\|^2_{H^{\alpha,\sigma+\frac{2\nu+1}{\nu+1}}_{\mathcal{L}(\ell^{a,s},\ell^{a,s+1})}}\|w\|^2_{\alpha,\sigma}\,. \tag{5.168}
$$

The thesis follows from estimate (5.76) and Proposition 5.4.2.

□

5.9 Small divisors

Let be i_0 and j_0 the smallest integers such that

$$
\min_i\left|\frac{k}{\tau}-\lambda_{ki}\right|=\left|\frac{k}{\tau}-\lambda_{ki_0}\right|=|\mu_{k,i_0}|\,, \tag{5.169}
$$

and

$$
\min_j\left|\frac{\ell}{\tau}-\lambda_{\ell j}\right|=\left|\frac{\ell}{\tau}-\lambda_{\ell j_0}\right|=|\mu_{\ell,j_0}|\,. \tag{5.170}
$$

Remark 5.9.1. *If $i_0=1$ then $k<3\tau$. Indeed, by contradiction, if $k>3\tau$, then $|k/\tau-\lambda_{k1}|\geq 2$ and, for $i^*\approx k/\tau$, $|k/\tau-\lambda_{ki^*}|\leq 1$, namely i^* is a point in which it takes a value smaller than the minimum. If $i_0>1$ then $i_0\approx k/\tau$.*

Remark 5.9.2. *If $k>\tau/4$, then $i_0\approx k/\tau$.*

5.9.1 Some technical lemmata

Lemma 5.9.3. *Let be* $1 < \nu < 2$, $\varpi \in \Delta_n^\nu$ *and* $\ell/\tau \geq 1/4$. *Then*

$$\left|\frac{\ell}{\tau} - j\right| \geq \mathrm{cost}\frac{\tau^{\nu-1}}{\ell^\nu}\,, \quad j \neq 0\,.$$

PROOF. If $\ell > \tau/4$, $j \approx \ell/\tau$ then, for the Melnikov condition in (5.5.3)

$$\left|\frac{\ell}{\tau} - j\right| \geq \frac{\gamma}{\tau} j^\nu \geq \mathrm{const}\,\gamma\frac{\tau^\nu}{\tau\ell^\nu} = \mathrm{const}\,\gamma\frac{\tau^{\nu-1}}{\ell^\nu}\,.$$

□

Lemma 5.9.4. *Let be* $|\mathcal{D}_\ell|^{-(1-\delta)}$ *as in* (5.150). *Then*

$$\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s} \leq \max\begin{cases} \dfrac{\ell^{\nu(1-\delta)}}{\gamma^{(1-\delta)}\tau^{(\nu-1)(1-\delta)}}, & \text{if } \ell > \tau/2; \\ \\ \mathrm{const}, & \text{if } \ell \leq \tau/2. \end{cases} \tag{5.171}$$

PROOF. From the definition of the operator $|\mathcal{D}_\ell|^{-(1-\delta)}$ in (5.150), it follows that

$$\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s} \leq \max_j \left\{|\mu_{\ell,j}|^{-(1-\delta)}\right\}\,. \tag{5.172}$$

Moreover, from the *first Melnikov condition* in (5.5.3), we have

$$|\mu_{\ell,j}| \geq |\mu_{\ell,j_0}| \geq \frac{\gamma}{\tau j_0^\nu}$$

for $j_0 > 1$ and using that $j_0 \approx \ell/\tau$, we get the estimate

$$|\mu_{\ell,j}|^{(1-\delta)} \geq \frac{\gamma^{(1-\delta)}}{\tau^{(1-\nu)(1-\delta)}\ell^{\nu(1-\delta)}}\,. \tag{5.173}$$

In particular if $\ell \leq \tau/2$ we have

$$|\mu_{\ell,j}|^{(1-\delta)} \geq \frac{\gamma^{(1-\delta)}}{\tau^{(1-\delta)}} \geq \mathrm{const}\,. \tag{5.174}$$

From (5.173) and (5.174), the thesis follows. □

Lemma 5.9.5. *Suppose that* $\ell > \tau/2$ *and*

$$\left|\frac{\ell}{\tau} - j_0\right| \geq \mathrm{const}\,\frac{\tau^{\nu-1}}{\ell^{\nu\beta}}\,, \qquad \beta < \frac{\nu-1}{\nu}\,.$$

Then, the following estimate holds

$$\frac{1}{2}\left|\frac{\ell}{\tau} - j_0\right| \geq \frac{\mu}{j_0}\,, \qquad \text{if} \quad \beta < \frac{\nu-1}{\nu}\,. \tag{5.175}$$

PROOF. We want to show that

$$\mathrm{const}\,\frac{\tau^{\nu-1}}{\ell^{\nu\beta}} \geq \frac{\mu}{j_0}\,. \tag{5.176}$$

It is true, since $\beta < (\nu-1)/\nu$, $\ell \geq \tau/4$ and by Remark 5.9.2. Indeed

$$\frac{\tau^{\nu-1}}{\ell^{\nu\beta}} \approx \frac{\tau^{\nu\beta}}{j_0^{\nu\beta}} = \mathrm{const}\,\frac{\tau^{\nu-1-\nu\beta}}{j_0^{\nu\beta}} \geq \mathrm{const}\,\frac{\mu}{j_0}\,,$$

since $\nu\beta < \nu - 1$ implies $\nu\beta < 1$, for $1 < \nu < 2$.

□

Lemma 5.9.6. *The following estimate holds:*

$$|\mu_{k,1}|^\delta = \left|\frac{k}{\tau} - \lambda_{k1}\right|^\delta \geq \begin{cases} k^\delta/\tau^\delta, & \textit{if } k \geq 2\tau; \\ \gamma^\delta/\tau^\delta, & \textit{if } \tau/2 < k < 2\tau; \\ \mathrm{const}, & \textit{if } k \leq \tau/2. \end{cases} \tag{5.177}$$

PROOF. We have that, for $k \leq \tau/2$, k/τ is at most $1/2$ hence $|\mu_{k,1}| \geq \mathrm{const}$. For $\tau/2 < k < 2\tau$, we can use the diophantine estimate in (5.145) and we obtain $|\mu_{k,1}| \geq \gamma/\tau$. Finally, for $k > 2\tau$, being λ_{k1} negligible with respect to k/τ, we have $|\mu_{k,1}| \geq \mathrm{const}\, k/\tau$. Therefore (5.177) follows.

□

Let be i_0 such that (5.169) holds. We have that

$$\|M^{-1}|\mathcal{D}_k|^{-\delta}\|_{s,s} \leq \max\left\{\frac{1}{i_0|\mu_{k,i_0}|^\delta}, \frac{1}{|\mu_{k,1}|^\delta}\right\}. \tag{5.178}$$

Lemma 5.9.7. *Let be i_0 such that* (5.169) *holds. Then*

$$\left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} \leq \begin{cases} \mathrm{const}\,\dfrac{\tau^\delta}{\gamma^\delta}\left(\dfrac{\tau}{k}\right)^{1-\nu\delta}, & k > \tau/2 \\ \mathrm{const}\,, & k \leq \tau/2\,. \end{cases} \tag{5.179}$$

PROOF. If $k \leq \tau/2$ then $i_0 = 1$, hence from (5.177) the thesis follows. If $k > \tau/2$, from (5.178), we have to show that

$$\frac{1}{i_0|\mu_{k,i_0}|^\delta} \leq \mathrm{const}\,\frac{\tau^\delta}{\gamma^\delta}\left(\frac{\tau}{k}\right)^{1-\nu\delta}, \tag{5.180}$$

and

$$\frac{1}{|\mu_{k,1}|^\delta} \leq \mathrm{const}\,\frac{\tau^\delta}{\gamma^\delta}\left(\frac{\tau}{k}\right)^{1-\nu\delta}. \tag{5.181}$$

We note that (5.180) and (5.181) coincide if $i_0 = 1$. In the case $i_0 = 1$, if $\tau/2 \le k \le 2\tau$, (5.181) directly follows from (5.177). For $k > 2\tau$, we have, using that $\delta > 1/2 > 1 - \nu\delta$ and from (5.177),

$$\mathrm{const}\,\frac{\tau^\delta}{k^\delta} \le \mathrm{const}\,\frac{\tau^\delta}{\gamma^\delta}\left(\frac{\tau}{k}\right)^{1-\nu\delta},$$

from which (5.181) follows.
If $i_0 > 1$, we have that $i_0 \approx k/\tau$, therefore,

$$i_0|\mu_{k,i_0}|^\delta = i_0\left|\frac{k}{\tau} - \lambda_{ki_0}\right|^\delta \ge i_0\left(\frac{\gamma}{\tau i_0^\nu}\right)^\delta = \frac{\gamma^\delta}{i_0^{\nu\delta-1}\tau^\delta} \ge \mathrm{const}\,\frac{\gamma^\delta}{\tau^\delta}\left(\frac{\tau}{k}\right)^{\nu\delta-1},$$

that is just (5.180).

□

Lemma 5.9.8. *Be $\ell, k \ge 1$ such that $|\ell - k| < [\max(\ell, k)]^\beta$ and $\ell, k > \tau/2$. Then*

$$|\ell - k| \le \mathrm{const}\,\ell^\beta.$$

PROOF. If $k \le \ell$ then $|\ell - k| < \ell^\beta$. If $k > \ell$, then $|\ell - k| = k - \ell \le k^\beta$, that is $k - k^\beta \le \ell$. Moreover $k - k^\beta \ge k/2$ if $k > \tau/2$. So that $k \le 2\ell$ definitively. Then, (5.9.8) directly follows. □

Remark 5.9.9. *Note that, for $\ell, k \ge 1$, $\ell \ne k$, the following estimate holds*

$$\frac{\ell}{k} \le \frac{|\ell - k| + k}{k} \le 1 + |\ell - k|,$$

hence

$$\frac{\ell}{k} \le 2|\ell - k|. \tag{5.182}$$

5.9.2 Proof of Lemma 5.8.4

Recall (5.179) and (5.171). We distinguish four different cases.
FIRST CASE: $|\ell - k| \ge [\max(\ell, k)]^\beta$.
If $\ell > \tau/2$, we have, from (5.179) and (5.171),

$$\begin{aligned}
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} &\le \frac{\ell^{\nu(1-\delta)}}{\gamma\tau^{(\nu-1)(1-\delta)-\delta-1+\nu\delta}}\frac{1}{k^{1-\nu\delta}}\\
&= \frac{\ell^{\nu(1-\delta)}}{\gamma\tau^{\nu-2}}\frac{1}{k^{1-\nu\delta}}\\
&= \frac{\ell^{\nu-1}}{\gamma\tau^{\nu-2}}\left(\frac{\ell}{k}\right)^{1-\nu\delta}\\
&\le \frac{|\ell - k|^{1-\nu\delta}}{\gamma\tau^{\nu-2}}\,|\ell - k|^{(\nu-1)/\beta}
\end{aligned}$$

$$= \frac{1}{\gamma} |\ell - k|^{(1-\nu\delta)+(\nu-1)/\beta} \tau^{2-\nu} ,$$

where we used (5.182) and the hypothesis $|\ell - k| \geq \ell^\beta$, while, if $\ell \leq \tau/2$,

$$\left\| |\mathcal{D}_\ell|^{-(1-\delta)} \right\|_{s,s} \left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \frac{\text{const}}{\gamma^\delta} \frac{\tau^{\delta+1-\nu\delta}}{k^{1-\nu\delta}} \leq \text{const} \frac{\tau}{\gamma^\delta} . \qquad (5.183)$$

SECOND CASE: $0 < |\ell - k| < [\max(\ell, k)]^\beta$ and $\ell \leq \tau/2$ or $k \leq \tau/2$.
Let be $0 \leq \ell \leq \tau/2$. If $\ell = 0$, $\lambda_{\ell,j} = \lambda_{0,j}$, then

$$\begin{aligned} \min_{j\geq 1} |\lambda_{0,j}| &= \min_{j\geq 1} |\lambda_j| \\ &= \min_{j\geq 1} \left| \left[\sqrt{j^2+m} + \eta^2 (BI_0)_j + O\left(\frac{\varepsilon}{j}\right) \right] \right| \geq \text{const} . \end{aligned}$$

Let be $0 < \ell \leq \tau/2$. From (5.171) and from (5.179), we get

$$\left\| |\mathcal{D}_\ell|^{-(1-\delta)} \right\|_{s,s} \left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \text{const} , \qquad (5.184)$$

if $k \leq \tau/2$ and, if $k > \tau/2$, from (5.179), we get

$$\left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \frac{\tau^{1+\delta-\nu\delta}}{\gamma^\delta k^{1-\nu\delta}} \leq \text{const} \frac{\tau^\delta}{\gamma^\delta} ,$$

therefore, it follows that

$$\left\| |\mathcal{D}_\ell|^{-(1-\delta)} \right\|_{s,s} \left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \text{const} \frac{\tau^\delta}{\gamma^\delta} . \qquad (5.185)$$

In the case $k \leq \tau/2$, since, from (5.179),

$$\left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \text{const} ,$$

from (5.171) and using Remark 5.9.9, we obtain

$$\begin{aligned} \left\| |\mathcal{D}_\ell|^{-(1-\delta)} \right\|_{s,s} &\leq \text{const} \left(\frac{\ell}{k} \right)^{\nu(1-\delta)} \frac{k^{\nu(1-\delta)}}{\tau^{(\nu-1)(1-\delta)} \gamma^{1-\delta}} \\ &\leq \frac{|\ell - k|^{\nu(1-\delta)}}{\gamma^{1-\delta}} \tau^{1-\delta} , \end{aligned}$$

and therefore

$$\left\| |\mathcal{D}_\ell|^{-(1-\delta)} \right\|_{s,s} \left\| M^{-1} |\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \text{const} \frac{|\ell - k|^{\nu(1-\delta)}}{\gamma^{1-\delta}} \tau^{1-\delta} . \qquad (5.186)$$

THIRD CASE: $0 < |\ell - k| < [\max(\ell, k)]^\beta$, $\ell, k > \tau/2$, $0 < |\ell - k| \leq \tau |j_0 - i_0|/4$, where i_0 and j_0 are such that (5.169) and (5.170) hold.

To reach the final estimate, let us consider previously

$$\begin{aligned}
|\mu_{\ell,j_0} - \mu_{k,i_0}| &= \left|\frac{\ell}{\tau} - \lambda_{\ell j_0} - \frac{k}{\tau} + \lambda_{k i_0}\right| \\
&\geq |\lambda_{k i_0} - \lambda_{\ell j_0}| - \frac{|\ell - k|}{\tau} \geq \frac{1}{4}|j_0 - i_0|\,,
\end{aligned}$$

where, in the last inequality, we used that $|\lambda_{ki} - \lambda_{\ell j}| \geq |j - i|/2$. Since $|j_0 - i_0| \geq 1$, we get the estimate

$$|\mu_{\ell,j_0} - \mu_{k,i_0}| \geq \frac{1}{4}\,.$$

It follows that

$$|\mu_{\ell,j_0}| \geq \frac{1}{8} \qquad \text{or} \qquad |\mu_{k,i_0}| \geq \frac{1}{8}\,.$$

In the case $|\mu_{\ell,j_0}| \geq 1/8$, from (5.179), we have, being $k > \tau/2$,

$$\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s} \left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} \leq \text{const}\,\frac{\tau^\delta}{\gamma^\delta}\left(\frac{\tau}{k}\right)^{1-\nu\delta} \leq \frac{\tau^\delta}{\gamma^\delta}\,. \tag{5.187}$$

In the case $|\mu_{k,i_0}| \geq 1/8$, one has, generally,

$$\min_i\left\{i\left|\frac{k}{\tau} - \lambda_{ki}\right|^\delta\right\} = \min\left\{\left|\frac{k}{\tau} - \lambda_{k1}\right|^\delta, i_0\left|\frac{k}{\tau} - \lambda_{k i_0}\right|^\delta\right\}. \tag{5.188}$$

and in particular, we have $|\mu_{k,i_0}| = |k/\tau - \lambda_{k i_0}| \geq 1/8$; hence

$$\min_i\left\{i\left|\frac{k}{\tau} - \lambda_{ki}\right|^\delta\right\} \geq \begin{cases} \text{const}\,k^\delta/\tau^\delta, & \text{if } k > 2\tau; \\[2ex] \gamma^\delta/\tau^\delta, & \text{if } \tau/2 < k \leq 2\tau\,. \end{cases} \tag{5.189}$$

Indeed, if $i_0 = 1$ we have that

$$\min_i\left\{i\left|\frac{k}{\tau} - \lambda_{ki}\right|^\delta\right\} = \left|\frac{k}{\tau} - \lambda_{k1}\right|^\delta,$$

hence (5.189) follows from (5.177). On the other hand, if $i_0 > 1$, then, from Remark 5.9.1, $i_0 \approx k/\tau$, hence

$$i_0\left|\frac{k}{\tau} - \lambda_{k i_0}\right|^\delta \geq \frac{i_0}{8^\delta} \approx \text{const}\,\frac{k}{\tau} \geq \text{const}\left(\frac{k}{\tau}\right)^\delta$$

where we used that $\delta < 1$. Thus (5.189) follows. Therefore, from (5.179), we get

$$\left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} \leq \begin{cases} \text{const}\,\tau^\delta/k^\delta, & \text{if } k > 2\tau; \\[2ex] \tau^\delta/\gamma^\delta, & \text{if } \tau/2 < k \leq 2\tau. \end{cases} \tag{5.190}$$

Finally, from (5.171) and (5.190), we reach the estimate

$$
\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s}\|M^{-1}|\mathcal{D}_k|^{-\delta}\|_{s,s} \le \begin{cases} \dfrac{\ell^{\nu(1-\delta)}}{k^\delta\,\gamma^{(1-\delta)}\tau^{\nu(1-\delta)-1}}\,, & \text{if } k > 2\tau; \\[2ex] \dfrac{\ell^{\nu(1-\delta)}}{\gamma\tau^{\nu(1-\delta)-1}}\,, & \text{if } \tau/2 < k \le 2\tau. \end{cases}
$$

In the first case $k > 2\tau$, fixing

$$
\delta := \nu/(\nu+1)\,, \tag{5.191}
$$

we have $\nu(1-\delta) = \delta$ and therefore

$$
\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s}\|M^{-1}|\mathcal{D}_k|^{-\delta}\|_{s,s} \le \left(\frac{\ell}{k}\right)^\delta \frac{1}{\gamma^{1-\delta}\tau^{\nu(1-\delta)-1}} \le \frac{|\ell-k|^\delta}{\gamma^{1-\delta}}\,\tau^{1-\delta} \tag{5.192}
$$

where in the last inequality we used (5.182). In the second case, $\tau/2 < k \le 2\tau$, we have

$$
\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s}\|M^{-1}|\mathcal{D}_k|^{-\delta}\|_{s,s} \le \frac{k^{\nu(1-\delta)}}{\gamma\tau^{\nu(1-\delta)-1}}\left(\frac{\ell}{k}\right)^{\nu(1-\delta)} \le \text{const}\,\frac{\tau}{\gamma}\,|\ell-k|^\delta\,.
$$

In conclusion,

$$
\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s}\|M^{-1}|\mathcal{D}_k|^{-\delta}\|_{s,s} \le \text{const}\,\frac{\tau}{\gamma}\,|\ell-k|^\delta\,. \tag{5.193}
$$

Fourth case: $0 < |\ell-k| < [\max(\ell,k)]^\beta$, $\ell, k > \tau/2$, $|\ell-k| > \tau|j_0-i_0|/4$.
Let us distinguish the two cases $i_0 = j_0$ and $i_0 \ne j_0$.
In the case $i_0 = j_0$, it results

$$
|\mu_{\ell,j_0} - \mu_{k,i_0}| = \left|\frac{\ell}{\tau} - \lambda_{\ell j_0} - \frac{k}{\tau} + \lambda_{k i_0}\right| \ge \frac{|\ell-k|}{\tau} \ge \frac{1}{\tau}\,,
$$

thus $|\mu_{\ell,j_0}| \ge 1/2\tau$ or $|\mu_{k,i_0}| \ge 1/2\tau$. Let be first $|\mu_{\ell,j_0}| \ge 1/2\tau$. Hence

$$
\||\mathcal{D}_\ell|^{-(1-\delta)}\|_{s,s} \le \text{const}\,\tau^{(1-\delta)}\,.
$$

In this case one obtains, from (5.179), using that $k > \tau/2$,

$$
\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s}\left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} \le \text{const}\,\frac{\tau^{1-\delta}\tau^\delta}{\gamma^\delta}\left(\frac{\tau}{k}\right)^{1-\nu\delta} \le \text{const}\,\frac{\tau}{\gamma^\delta}\,. \tag{5.194}
$$

On the other hand, if $|\mu_{k,i_0}| \ge 1/2\tau$, we have to distinguish the case $i_0 = 1$ from the case $i_0 > 1$.
If $i_0 = 1$, one has, from (5.177),

$$
\min_i\left\{i\left|\frac{k}{\tau} - \lambda_{ki}\right|^\delta\right\} \ge \left|\frac{k}{\tau} - \lambda_{ki_0}\right|^\delta = \left|\frac{k}{\tau} - \lambda_{k1}\right|^\delta \ge \text{const}\,\frac{\gamma^\delta}{\tau^\delta}\,, \tag{5.195}
$$

therefore, from (5.171),

$$
\begin{aligned}
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} &\le \text{ const } \frac{\tau^\delta}{\gamma}\,\frac{\ell^{1-\nu\delta}}{\tau^{(\nu-1)(1-\delta)}} \\
&= \text{ const } \frac{\ell^{1-\nu\delta}}{\gamma\,\tau^{\nu(1-\delta)-1}}\,.
\end{aligned}
$$

Hence this case reduce to the third case with $\tau/2 < k \le 2\tau$, by using (5.182) the final estimate is

$$
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} \le \text{const}\,\frac{\tau}{\gamma}\,|\ell-k|^\delta\,.
$$

If $i_0 > 1$, from Remark 5.9.1, one has $i_0 \approx k/\tau$. In this case, (5.189) and (5.188) hold with $i_0 > 1$. Hence we have

$$
\min_i\left\{i\left|\frac{k}{\tau}-\lambda_{ki}\right|^\delta\right\} \ge \begin{cases} k^\delta/\tau^\delta, & \text{if } k \ge \tau^{1/(1-\delta)}; \\ k/\tau^{\delta+1}, & \text{if } 2\tau < k < \tau^{1/(1-\delta)}; \\ \gamma^\delta/\tau^\delta, & \text{if } \tau/2 < k \le 2\tau. \end{cases} \tag{5.196}
$$

In the case $k \ge \tau^{1/(1-\delta)}$, we obtain

$$
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} \le \text{const}\,\frac{1}{\gamma^{1-\delta}\,\tau^{\nu(1-\delta)-1}}\,\frac{\ell^{\nu(1-\delta)}}{k^\delta}\,, \tag{5.197}
$$

thus, this case reduces to the third one with $k > 2\tau$.
In the case $2\tau < k < \tau^{1/(1-\delta)}$, one gets, using that $\nu(1-\delta) = \delta$,

$$
\begin{aligned}
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} &\le \text{ const } \frac{\tau^{1+\delta}}{\gamma^{1-\delta}\,\tau^{(\nu-1)(1-\delta)}}\,\frac{\ell^{\nu(1-\delta)}}{k} \\
&= \text{ const } \frac{\tau^{2-\delta-1+\delta}}{\gamma^{1-\delta}}\left(\frac{\ell}{k}\right)^\delta \\
&\le \text{ const } \frac{\tau}{\gamma^{1-\delta}}\,|\ell-k|^\delta
\end{aligned} \tag{5.198}
$$

In the last case, $\tau/2 < k \le 2\tau$, from (5.171),

$$
\big\||\mathcal{D}_\ell|^{-(1-\delta)}\big\|_{s,s}\big\|M^{-1}|\mathcal{D}_k|^{-\delta}\big\|_{s,s} \le \text{const}\,\frac{\tau^\delta}{\gamma}\,\frac{\ell^{\nu(1-\delta)}}{\tau^{(\nu-1)(1-\delta)}} = \frac{\ell^{\nu(1-\delta)}}{\gamma\,\tau^{\nu(1-\delta)-1}}\,, \tag{5.199}
$$

thus, this case reduces to the third one with $\tau/2 < k \le 2\tau$.
On the other hand, in the case $i_0 \neq j_0$, it results $|\ell-k| > \tau/4$, therefore, from Lemma 5.9.3,

$$
\left|\frac{\ell}{\tau}-j_0-\frac{k}{\tau}+i_0\right| = \left|\frac{|\ell-k|}{\tau}-(j_0-i_0)\right| \ge \text{const}\,\frac{\tau^{\nu-1}}{|\ell-k|^\nu}\,.
$$

We have two cases, from Lemma 5.9.8,

$$\left|\frac{\ell}{\tau} - j_0\right| \geq \text{const}\,\frac{\tau^{\nu-1}}{|\ell-k|^\nu} \geq \text{const}\,\frac{\tau^{\nu-1}}{\ell^{\nu\beta}}\,, \tag{5.200}$$

or

$$\left|\frac{k}{\tau} - i_0\right| \geq \text{const}\,\frac{\tau^{\nu-1}}{|\ell-k|^\nu} \geq \text{const}\,\frac{\tau^{\nu-1}}{k^{\nu\beta}}\,. \tag{5.201}$$

Consider at first (5.200). Since $\varepsilon = \eta^3$, from the asymptotic estimates for the eigenvalues it results that

$$\begin{aligned}
|\mu_{\ell,j_0}| &= \left|\frac{\ell}{\tau} - \lambda_{\ell j_0}\right| = \left|\frac{\ell}{\tau} - \sqrt{{j_0}^2+\mu} - \eta^2(BI_0)_{j_0} + O\left(\frac{\varepsilon}{j}\right)\right| \\
&= \left|\frac{\ell}{\tau} - j_0 + O\left(\frac{\mu}{j}\right) + O\left(\frac{\eta^2}{j}\right) + O\left(\frac{\varepsilon}{j}\right)\right| \\
&\geq \left|\frac{\ell}{\tau} - j_0\right| - \frac{\mu}{j_0}\,.
\end{aligned} \tag{5.202}$$

Using Lemma 5.9.5, we obtain

$$|\mu_{\ell,j_0}| \geq \left|\frac{\ell}{\tau} - j_0\right| - \frac{\mu}{j_0} \geq \frac{1}{2}\left|\frac{\ell}{\tau} - j_0\right| \geq \text{const}\,\frac{\tau^{\nu-1}}{|\ell-k|^\nu}\,. \tag{5.203}$$

Hence, we have

$$\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s} \leq \frac{|\ell-k|^{\nu(1-\delta)}}{\tau^{(\nu-1)(1-\delta)}}\,. \tag{5.204}$$

Therefore it follows that, using (5.179) and $k > \tau/2$, by (5.204),

$$\begin{aligned}
\left\||\mathcal{D}_\ell|^{-(1-\delta)}\right\|_{s,s}\left\|M^{-1}|\mathcal{D}_k|^{-\delta}\right\|_{s,s} &\leq \text{const}\,\frac{|\ell-k|^\delta}{\gamma^\delta\,\tau^{(\nu-1)(1-\delta)-\delta}}\left(\frac{\tau}{k}\right)^{1-\nu\delta} \\
&\leq \text{const}\,\frac{|\ell-k|^\delta}{\gamma^\delta\,\tau^{\nu(1-\delta)-1}}\,.
\end{aligned} \tag{5.205}$$

Let us consider, now, (5.201).
Using Lemma 5.9.5, we obtain

$$|\mu_{k,i_0}| \geq \left|\frac{k}{\tau} - i_0\right| - \frac{\mu}{i_0} \geq \frac{1}{2}\left|\frac{k}{\tau} - i_0\right| \geq \text{const}\,\frac{\tau^{(\nu-1)}}{|\ell-k|^\nu}\,. \tag{5.206}$$

Moreover $i_0 \approx k/\tau$ for $i_0 \geq 1$ by Remark 5.9.1 and being $3\tau > k \geq \tau/2$, therefore

$$i_0\left|\frac{k}{\tau} - \lambda_{ki_0}\right|^\delta \geq \frac{k}{\tau}\,\text{const}\,\frac{\tau^{(\nu-1)\delta}}{|\ell-k|^{\nu\delta}}\,.$$

Thus it follows that

$$\min_i\left\{i\left|\frac{k}{\tau} - \lambda_{ki}\right|^\delta\right\} \geq \frac{k}{\tau}\,\frac{\tau^{(\nu-1)\delta}}{|\ell-k|^{\nu\delta}}$$

hence

$$\left\| M^{-1}|\mathcal{D}_k|^{-\delta} \right\|_{s,s} \leq \frac{\tau}{k} \frac{|\ell - k|^{\nu\delta}}{\tau^{(\nu-1)\delta}}\,. \tag{5.207}$$

From (5.207) and using that $\nu(1-\delta) = \delta$, it follows the estimate

$$\begin{aligned}
\| |\mathcal{D}_\ell|^{-(1-\delta)} \|_{s,s} \| M^{-1} |\mathcal{D}_k|^{-\delta} \|_{s,s} &\leq \frac{\tau}{k} \frac{\ell^{\nu(1-\delta)}}{\gamma^{1-\delta}\, \tau^{(\nu-1)(1-\delta)}} \frac{|\ell - k|^{\nu\delta}}{\tau^{(\nu-1)\delta}} \\
&\leq \frac{\ell^{\nu(1-\delta)}\, |\ell - k|^{\nu\delta}\, \tau^{(2-\nu)}}{\gamma^{1-\delta}\, k^{1-\delta} k^{\delta}} \\
&= \left(\frac{\ell}{k}\right)^{\delta} \frac{|\ell - k|^{\nu\delta}\, \tau^{(2-\nu)}}{\gamma^{1-\delta}\, k^{1-\delta}} \\
&\leq \frac{|\ell - k|^{(\nu+1)\delta}\, \tau^{(2-\nu)}}{\gamma^{1-\delta}\, k^{1-\delta}} \\
&= \frac{|\ell - k|^{\nu}\, \tau^{(2-\nu-1+\delta)}}{\gamma^{1-\delta}} \left(\frac{\tau}{k}\right)^{1-\delta} \\
&\leq \text{const}\, |\ell - k|^{\nu}\, \frac{\tau^{1-\nu+\delta}}{\gamma^{1-\delta}}\,,
\end{aligned} \tag{5.208}$$

where we used (5.182) and $k > \tau/2$. □

5.10 Measure estimates

In this section we will give an estimate from below of the measure of the set $\mathcal{C}$ of admissible frequencies. We first need the following standard result on the measure of Diophantine numbers.

Lemma 5.10.1. *Let*

$$D := \left\{ \varpi > 0 \text{ s.t. } \left| \varpi\ell - j \right| > \frac{\varpi^{1+\epsilon}}{j^{\nu}},\ \forall\, \ell, j \geq 1 \right\}, \qquad 1 < \nu < 2\,, \tag{5.209}$$

then

$$\lim_{\varpi_0 \to 0^+} \frac{\text{meas}\big((0, \varpi_0] \cap D\big)}{\varpi_0} = 1\,. \tag{5.210}$$

PROOF. Let be

$$D_{\ell j} := \left\{ \varpi > 0 \text{ s.t. } \left| \frac{\ell}{\varpi^{\epsilon}} - \frac{j}{\varpi^{1+\epsilon}} \right| \leq \frac{1}{j^{\nu}} =: \delta_j \right\}.$$

Let be $\varpi := \varpi_{\ell,j} + \tilde{\varpi} \in D_{\ell,j}$ with $\varpi_{\ell,j} := j/\ell$ satisfying $\varpi_{\ell,j}\ell - j = 0$. We have to estimate the measure of the translated set

$$\tilde{D}_{\ell j} := \left\{ \tilde{\varpi} \text{ s.t. } \left| \frac{\ell}{(j/\ell + \tilde{\varpi})^{\epsilon}} - \frac{j}{(j/\ell + \tilde{\varpi})^{1+\epsilon}} \right| \leq \delta_j \right\}. \tag{5.211}$$

Define

$$f(\tilde{\varpi}) := \frac{\ell}{(j/\ell+\tilde{\varpi})^{\epsilon}} - \frac{j}{(j/\ell+\tilde{\varpi})^{1+\epsilon}} = \frac{\tilde{\varpi}\ell^{2+\epsilon}}{(j+\ell\tilde{\varpi})^{1+\epsilon}}\,.$$

We have that $\operatorname{meas}(\tilde{D}_{\ell j}) \sim \delta_j/|\partial_{\tilde{\varpi}} f(0)|$. Compute

$$\partial_{\tilde{\varpi}} f(\tilde{\varpi}) = \frac{\ell^{2+\epsilon}(j+\ell\tilde{\varpi})^{1+\epsilon} - (1+\epsilon)\tilde{\varpi}\ell^{3+\epsilon}(j+\ell\tilde{\varpi})^{\epsilon}}{(j+\ell\tilde{\varpi})^{2(1+\epsilon)}} = \frac{\ell^{2+\epsilon}(j+\epsilon\ell\tilde{\varpi})}{(j+\ell\tilde{\varpi})^{2+\epsilon}}\,,$$

thus

$$\partial_{\tilde{\varpi}} f(0) = \frac{\ell^{2+\epsilon}}{j^{1+\epsilon}}$$

and

$$\operatorname{meas}(D_{\ell j}) \le \operatorname{meas}(\tilde{D}_{\ell j}) \sim \delta_j \frac{j^{1+\epsilon}}{\ell^{2+\epsilon}} = \frac{1}{\ell^{2+\epsilon} j^{\nu-1-\epsilon}}\,. \tag{5.212}$$

Being, by the definition of $D_{\ell j}$, $D_{\ell j} = \emptyset$ if $\ell < j/2\varpi_0$, one has

$$(0,\varpi_0) \cap D^c = \bigcup_{j\ge 1} \bigcup_{\ell \ge j/2\varpi_0} D_{\ell j}\,.$$

Therefore by (5.212),

$$\begin{aligned}
\operatorname{meas}\big((0,\varpi_0)\cap D^c\big) &\le \text{const} \int_1^{\infty}\int_{j/2\varpi_0}^{\infty} \frac{1}{j^{\nu-1-\epsilon}\ell^{2+\epsilon}}\, d\ell\, dj \\
&= \text{const}\,{\varpi_0}^{1+\epsilon} \int_1^{\infty} \frac{1}{j^{\nu}}\, dj = \text{const}\,{\varpi_0}^{1+\epsilon}\,.
\end{aligned}$$

The thesis follows.

□

Proposition 5.10.2. *Let* $A_n := \big\{\varpi \in A_{n-1} \ \ s.t. \ \ \varpi \in \Delta_n^{\nu}(\xi_{n-1})\big\}$, $A_0 = (0,\varpi_*)$, *as in* (5.79). *Define*

$$\mathcal{C} := \bigcap_n A_n\,.$$

Then

$$\lim_{\varpi_0\to 0^+} \frac{\operatorname{meas}\big(\mathcal{C}\cap(0,\varpi_0)\big)}{\varpi_0} = 1\,. \tag{5.213}$$

PROOF. For any $\varpi_0 > 0$ fixed we note that

$$\mathcal{C}\cap(0,\varpi_0) \supset D\cap\tilde{\mathcal{C}} \tag{5.214}$$

where D is defined in (5.209) and $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(\varpi_0) \subset (0,\varpi_0)$ is defined in such a way that

$$(0,\varpi_0)\setminus\tilde{\mathcal{C}} := \bigcup_{n\ge 0} B^n \tag{5.215}$$

where

$$B^n = B^n(\varpi_0) := \bigcup_{j\geq 1, \ell\leq L_{n+1}} F^n_{\ell j} \tag{5.216}$$

and

$$\begin{aligned} F^n_{\ell j} &= F^n_{\ell j}(\varpi_0) := \Big\{ \varpi \in (0,\varpi_0) \quad \text{s.t.} \\ &\left| \frac{k}{\varpi^\epsilon} - \frac{i}{\varpi^{1+\epsilon}} + \frac{\varepsilon}{\varpi^{1+\epsilon}} \big(\mathcal{M}^{22}_k(\xi_{n-1})\big)_{ii} \right| > \frac{\gamma_n}{i^\nu}, \quad \forall\, i \geq 1,\, k \leq L_n, \\ &\left| \frac{\ell}{\varpi^\epsilon} - \frac{j}{\varpi^{1+\epsilon}} + \frac{\varepsilon}{\varpi^{1+\epsilon}} \big(\mathcal{M}^{22}_\ell(\xi_n)\big)_{jj} \right| \leq \frac{\gamma_{n+1}}{j^\nu} \Big\}. \end{aligned} \tag{5.217}$$

Using (5.210) and (5.214), we note that (5.213) is proved once we show that

$$\lim_{\varpi_0\to 0^+} \frac{\text{meas}\,(\tilde{\mathcal{C}})}{\varpi_0} = 1\,,$$

or, equivalently,

$$\lim_{\varpi_0\to 0^+} \frac{\text{meas}\,\big((0,\varpi_0)\setminus\tilde{\mathcal{C}}\big)}{\varpi_0} = \lim_{\varpi_0\to 0^+} \frac{\text{meas}\,\big(\bigcup_{n\geq 0} B^n\big)}{\varpi_0} = 0\,. \tag{5.218}$$

Nothing remains but to prove (5.218). Since

$$\text{meas}\Big(\bigcup_{n\geq 0} B^n\Big) \leq \sum_{n\geq 0} \text{meas}\big(B^n\big) \leq \sum_{n\geq 0}\sum_{j\geq 1}\sum_{\ell\leq L_{n+1}} \text{meas}\,\big(F^n_{\ell j}\big)\,, \tag{5.219}$$

we are going to estimate the measure of $F^n_{\ell j}$. We immediately observe that

$$F^n_{\ell j} \subset \left(\frac{j}{2\ell}, \frac{2j}{\ell}\right), \tag{5.220}$$

since, for $\varpi \leq j/(2\ell)$ or $\varpi \geq 2j/\ell$,

$$\begin{aligned} \left| \frac{\ell}{\varpi^\epsilon} - \frac{j}{\varpi^{1+\epsilon}} + \frac{\varepsilon}{\varpi^{1+\epsilon}} \big(\mathcal{M}^{22}_\ell(\xi_n)\big)_{jj} \right| &\geq \frac{\ell}{2\varpi^\epsilon} - \frac{\varepsilon}{\varpi^{1+\epsilon}} \left| \big(\mathcal{M}^{22}_\ell(\xi_n)\big)_{jj} \right| \\ &\geq \frac{\ell}{2\varpi^\epsilon} - \text{const}\,\varpi^{\frac{1}{2}-\epsilon} \geq \frac{\ell}{4\varpi^\epsilon} > 2 \\ &> \frac{\gamma_{n+1}}{j^\nu}, \end{aligned}$$

taking ϖ small enough. Moreover we note that

$$F^n_{\ell j} = \emptyset \qquad \text{if} \quad \ell \leq L_n\,. \tag{5.221}$$

Indeed

$$\left| \frac{\ell}{\varpi^\epsilon} - \frac{j}{\varpi^{1+\epsilon}} + \frac{\varepsilon}{\varpi^{1+\epsilon}} \big(\mathcal{M}^{22}_\ell(\xi_n)\big)_{jj} \right| \geq \left| \frac{\ell}{\varpi^\epsilon} - \frac{j}{\varpi^{1+\epsilon}} + \frac{\varepsilon}{\varpi^{1+\epsilon}} \big(\mathcal{M}^{22}_\ell(\xi_{n-1})\big)_{jj} \right|$$

$$
\begin{aligned}
& -\frac{\varepsilon}{\varpi^{1+\epsilon}}\left|\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}-\big(\mathcal{M}_\ell^{22}(\xi_{n-1})\big)_{jj}\right| \\
\geq\ & \frac{\gamma_n}{j^\nu}-\frac{\varepsilon}{\varpi^{1+\epsilon}}\left\|D\big(\mathcal{M}_\ell^{22}\big)_{jj}\right\|\|\xi_n-\xi_{n-1}\| \\
\geq\ & \frac{\gamma_{n+1}}{j^\nu}\,, \qquad (5.222)
\end{aligned}
$$

where the last inequality holds if

$$
\frac{\varepsilon}{\varpi^{1+\epsilon}}\left\|D\big(\mathcal{M}_\ell^{22}\big)_{jj}\right\|\|\xi_n-\xi_{n-1}\| \leq \frac{\gamma_n-\gamma_{n+1}}{j^\nu}=\frac{1}{j^\nu\,2^{n+2}}\,,
$$

or, equivalently, since it must be verified only for $j\leq 2\varpi\,L_n$ (recall (5.220)),

$$
\frac{\varepsilon}{\varpi^{1+\epsilon}}\left\|D\big(\mathcal{M}_\ell^{22}\big)_{jj}\right\|\|\xi_n-\xi_{n-1}\| \leq \frac{1}{2^{n+2+\nu}}\,\frac{1}{\varpi^\nu\,L_n^\nu}\,. \qquad (5.223)
$$

Being $h_n=\xi_n-\xi_{n-1}$, we have

$$
\frac{\varepsilon}{\varpi^{1+\epsilon}}\left\|D\big(\mathcal{M}_\ell^{22}\big)_{jj}\right\|\|h_n\| \leq \mathrm{const}\,\frac{\varepsilon^2}{\varpi^{1+\epsilon}}\,e^{-\chi^n\alpha_0/8}=\mathrm{const}\,\varpi^{2-\epsilon}e^{-\chi^n\alpha_0/8}\,.
$$

Finally taking ϖ small enough we have

$$
\mathrm{const}\,\varpi^{2+\nu-\epsilon}e^{-\chi^n\alpha_0/8}\leq\frac{1}{2^nL_n^\nu}\,,
$$

from which we get (5.223) and therefore (5.221).
We now estimate the measure of $F^n_{\ell j}$ for $L_n<\ell\leq L_{n+1}$. By (5.217) and (5.220), we have that

$$
\begin{aligned}
F^n_{\ell j}\ \subseteq\ \tilde{F}^n_{\ell j} := \Bigg\{ & \varpi\in(0,\varpi_0)\quad \text{s.t.}\quad \varpi\in\left(\frac{j}{2\ell},\frac{2j}{\ell}\right), \\
& \left|\frac{\ell}{\varpi^\epsilon}-\frac{j}{\varpi^{1+\epsilon}}+\frac{\varepsilon}{\varpi^{1+\epsilon}}\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\right|\leq\frac{\gamma_{n+1}}{j^\nu}\Bigg\}. \qquad (5.224)
\end{aligned}
$$

Let us introduce

$$
f^n_{\ell j}(\varpi):=\frac{\ell}{\varpi^\epsilon}-\frac{j}{\varpi^{1+\epsilon}}+\frac{\varepsilon}{\varpi^{1+\epsilon}}\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\,.
$$

Recalling that $\varepsilon=\varpi^{3/2}$, $\partial_\varpi\varepsilon=3\sqrt{\varpi}/2$, we derive $f^n_{\ell j}$ with respect to ϖ

$$
\begin{aligned}
\partial_\varpi f^n_{\ell j}(\varpi)\ =\ \varpi^{-2-\epsilon}\Bigg[& (1+\epsilon)j-\varepsilon\varpi\ell+\frac{3}{2}\varpi^{3/2}\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj} \\
& +\varpi^{5/2}\Big(D_\xi\big(\mathcal{M}_\ell^{22}\big)_{jj}\Big)[\partial_\varpi\xi_n]+\varpi^{5/2}\Big(\partial_\varpi\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\Big) \\
& -\varpi^{3/2}(1+\epsilon)\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\Bigg]. \qquad (5.225)
\end{aligned}
$$

Recalling Remark 5.5.6 and noting that[4] $\partial_\varpi\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj} \approx \mathrm{const}/\varpi^2$, we have that

$$\varpi^{3/2}\big|\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\big| \leq \mathrm{const}\,\varpi^{3/2}\,, \tag{5.226}$$

$$\varpi^{5/2}\Big|\Big(D_\xi\big(\mathcal{M}_\ell^{22}\big)_{jj}\Big)[\partial_\varpi\xi_n]\Big| \leq \mathrm{const}\,\varpi^2\,, \tag{5.227}$$

and

$$\varpi^{5/2}\big|\big(\partial_\varpi\big(\mathcal{M}_\ell^{22}(\xi_n)\big)_{jj}\big)\big| \leq \mathrm{const}\,\varpi^{1/2}\,. \tag{5.228}$$

Hence, from (5.225),(5.226),(5.227),(5.228) we get

$$m := \min_{\varpi\in\left(\frac{j}{2\ell},\frac{2j}{\ell}\right)} \partial_\varpi f_{\ell j}^n(\varpi) \geq \min_{\varpi\in\left(\frac{j}{2\ell},\frac{2j}{\ell}\right)} \frac{j}{2\varpi^{2+\epsilon}} \geq \mathrm{const}\,\frac{\ell^{2+\epsilon}}{j^{1+\epsilon}}\,. \tag{5.229}$$

Recalling (5.220), (5.229) and (5.224) we have

$$\mathrm{meas}(F_{\ell j}^n) \leq \mathrm{meas}(\tilde{F}_{\ell j}^n) \leq \frac{\gamma_{n+1}}{2mj^\nu} \leq \frac{\mathrm{const}}{j^{\nu-1-\epsilon}\ell^{2+\epsilon}}\,. \tag{5.230}$$

Recalling (5.217) and (5.221) we have that

$$F_{\ell j}^n \neq \emptyset \qquad \Longleftrightarrow \qquad M := \max\big(j/2\varpi_0, L_n\big) \leq \ell \leq L_{n+1}\,, \quad 1 \leq j \leq 2\varpi_0 L_{n+1}$$

Therefore, by (5.219), we have

$$\begin{aligned}
\mathrm{meas}\Big(\bigcup_{n\geq 0} B^n\Big) &\leq \sum_{n:2\varpi_0 L_{n+1}\geq 1} \sum_{j\geq 1}^{2\varpi_0 L_{n+1}} \sum_{\ell\geq M}^{L_{n+1}} \mathrm{meas}(F_{\ell j}) \\
&\leq \mathrm{const} \sum_{n:2\varpi_0 L_{n+1}\geq 1} \sum_{j\geq 1}^{2\varpi_0 L_{n+1}} \sum_{\ell\geq M}^{L_{n+1}} \frac{1}{j^{\nu-1-\epsilon}\,\ell^{2+\epsilon}} \\
&\leq \mathrm{const} \sum_{n:2\varpi_0 L_{n+1}\geq 1} \int_1^{2\varpi_0 L_{n+1}} \frac{1}{j^{\nu-1-\epsilon}} \int_M^\infty \frac{1}{\ell^{2+\epsilon}}\,d\ell\,dj \\
&\leq \mathrm{const} \sum_{n:2\varpi_0 L_{n+1}\geq 1} \int_1^{2\varpi_0 L_{n+1}} \frac{1}{M^{1+\epsilon} j^{\nu-1-\epsilon}}\,dj \\
&= \mathrm{const} \sum_{n:\frac{1}{8}\leq 4^n\varpi_0\leq\infty} \int_1^{2\varpi_0 L_{n+1}} \frac{1}{M^{1+\epsilon} j^{\nu-1-\epsilon}}\,dj
\end{aligned} \tag{5.231}$$

recalling that $L_n = 4^n$. We split the integral (5.231) into

$$\int_1^{2\varpi_0 L_{n+1}} = \int_1^{2\varpi_0 L_n} + \int_{2\varpi_0 L_n}^{2\varpi_0 L_{n+1}}\,.$$

[4] We recall that, by (4.53), $\partial_\varpi I_0 = O(\varpi^{-2})$, since $\tau = \varpi^{-1}$.

For the first term, since $1 \le j \le 2\varpi_0 L_n$, we have that $M = L_n$, therefore

$$
\begin{aligned}
\int_1^{2\varpi_0 L_n} \frac{1}{j^{\nu-1-\epsilon} M^{1+\epsilon}}\, dj &= \frac{1}{L_n^{1+\epsilon}} \int_1^{2\varpi_0 L_n} \frac{1}{j^{\nu-1-\epsilon}}\, dj \\
&\le \text{const}\, \varpi_0^{1+\epsilon} \left(\frac{1}{\varpi_0 L_n} \right)^{\nu-1} ; \qquad (5.232)
\end{aligned}
$$

for the second term, since $M = j/2\varpi_0$,

$$
\begin{aligned}
\int_{2\varpi_0 L_n}^{2\varpi_0 L_{n+1}} \frac{1}{j^{\nu-1-\epsilon} M^{1+\epsilon}}\, dj &\le \text{const}\, \varpi_0^{1+\epsilon} \int_{2\varpi_0 L_n}^{\infty} \frac{1}{j^{\nu}}\, dj \\
&\le \text{const}\, \varpi_0^{1+\epsilon} \left(\frac{1}{\varpi_0 L_n} \right)^{\nu-1} . \qquad (5.233)
\end{aligned}
$$

Therefore by (5.232) and (5.233), we can estimate (5.231),

$$
\begin{aligned}
\sum_{n:\frac{1}{8} \le 4^n \varpi_0 \le \infty} \int_1^{2\varpi_0 L_{n+1}} \frac{1}{M^{1+\epsilon} j^{\nu-1-\epsilon}}\, dj &\le \text{const}\, \varpi_0^{1+\epsilon} \sum_{n:\frac{1}{8} \le 4^n \varpi_0 \le \infty} \left(\frac{1}{\varpi_0 4^n} \right)^{\nu-1} \\
&\le \text{const}\, \varpi_0^{1+\epsilon} \int_{\frac{1}{8}}^{\infty} \frac{dt}{t^{\nu}} \\
&\le \text{const}\, \varpi_0^{1+\epsilon} . \qquad (5.234)
\end{aligned}
$$

Therefore by (5.231) and (5.234) we have

$$
\text{meas} \Big(\bigcup_{n \ge 0} B^n \Big) \le \text{const}\, \varpi_0^{1+\epsilon}
$$

and (5.218) follows.

$\square$

5.11 Minimal period

Lemma 5.11.1. *Let $i_1 < i_2 \in \mathbb{N}^+$ and $\varsigma > 1$. Then there exists $\Upsilon \subset (0, 1]$, with* $\text{meas}\big(\Upsilon\big) = 1$ *such that for all $\mu \in \Upsilon$ the vector*

$$
\bar{\omega}_\mu := \left(\sqrt{i_1^2 + \mu}, \sqrt{i_2^2 + \mu} \right)
$$

is Diophantine, namely

$$
\big| \bar{\omega}_\mu \cdot h \big| > \frac{c(\mu)}{|h|^{\varsigma}} , \qquad \forall h \in \mathbb{Z}^2 \setminus \{0\} .
$$

PROOF. Define, forall $\bar\gamma > 0$ and $h \in \mathbb{Z}^2 \setminus \{0\}$,

$$A_h^{\bar\gamma} = A_{h_1,h_2}^{\bar\gamma} := \left\{ \mu \in (0,1] \text{ s.t. } \left|\bar\omega_\mu \cdot h\right| \le \frac{\bar\gamma}{|h|^\varsigma} \right\}$$

and

$$\Upsilon^{\bar\gamma} := \left\{ \mu \in (0,1] \text{ s.t. } \left|\bar\omega_\mu \cdot h\right| > \frac{\bar\gamma}{|h|^\varsigma}\,, \quad \forall h \in \mathbb{Z}^2 \setminus \{0\} \right\}. \tag{5.235}$$

We note that if $A_{h_1,h_2}^{\bar\gamma} \neq \emptyset$, then $\tilde c|h_1| \le |h_2| \le \bar c|h_1|$, $\bar c < 1$. Moreover

$$\Upsilon^{\bar\gamma} = (0,1] \setminus \bigcup_{h \in \mathbb{Z}^2 \setminus \{0\}} A_h^{\bar\gamma}\,. \tag{5.236}$$

We will prove that

$$\text{meas}(\Upsilon^{\bar\gamma}) = 1 - O(\bar\gamma)\,, \tag{5.237}$$

Note that $\text{meas}(A_h^{\bar\gamma}) \le \text{const}\,\frac{\bar\gamma}{|h|^{\varsigma+1}}$. In fact, we can write

$$|h\cdot\bar\omega_\mu| = \left|h_1\sqrt{i_1^2+\mu} + h_2\sqrt{i_2^2+\mu}\right| = \frac{|h_1^2(i_1^2+\mu) + h_2^2(i_2^2+\mu)|}{|h_1|\sqrt{i_1^2+\mu} + |h_2|\sqrt{i_2^2+\mu}|}\,.$$

Hence $\mu \in A_h^{\bar\gamma}$ implies that

$$|f(\mu)| := |h_1^2 i_1^2 - h_2^2 i_2^2 + \mu(h_1^2 - h_2^2)| \le \text{const}\,\frac{\bar\gamma}{|h|^{\varsigma-1}}\,.$$

Moreover, being

$$|f'(\mu)| = |h_1^2 - h_2^2| \approx |h_1|^2 \approx |h|^2\,,$$

one has

$$\text{meas}(A_h^{\bar\gamma}) < \text{const}\,\frac{\bar\gamma}{|h|^{\varsigma-1}}\,\frac{1}{|h|^2} = \text{const}\,\frac{\bar\gamma}{|h|^{\varsigma+1}}\,.$$

Therefore we have that

$$\begin{aligned}
\text{meas} \bigcup_{h \in \mathbb{Z}^2\setminus\{0\}} (A_h^{\bar\gamma}) &\le \sum_{h \in \mathbb{Z}^2\setminus\{0\}} \text{meas}(A_h^{\bar\gamma}) \\
&\approx \int_1^\infty \int_{\tilde c h_1}^{\bar c h_1} \frac{\bar\gamma}{(h_1+h_2)^{\varsigma+1}}\, dh_1 dh_2 \\
&= \bar\gamma \int_1^\infty \int_{\tilde c}^{\bar c} \frac{h_1}{(1+\xi)^{\varsigma+1} h_1^{\varsigma+1}}\, d\xi dh_1 \\
&= \bar\gamma \int_1^\infty \frac{1}{h_1^\varsigma}\, dh_1 \int_{\tilde c}^{\bar c} \frac{1}{(1+\xi)^{\varsigma+1}}\, d\xi = \text{const}_\varsigma\, \bar\gamma\,.
\end{aligned}$$

Then (5.237) follows by (5.236). Choosing $\Upsilon := \bigcup_{\bar\gamma>0} \Upsilon^{\bar\gamma}$, we get, by (5.237),

$$\text{meas}(\Upsilon) = \lim_{\bar\gamma\to 0} \text{meas}(\Upsilon^{\bar\gamma}) = 1\,.$$

□

Lemma 5.11.2. *Fix $\rho < 1/2$. There exists $\Upsilon \subset (0,1]$, with $\mathrm{meas}\big(\Upsilon\big) = 1$ such that, if $\mu \in \Upsilon$, $N \geq 2$ and $T^{\min}$ is the minimal period of a T-periodic solution of (5.26), then*

$$T^{\min} \geq \mathrm{const}\, T^{\rho}\,, \tag{5.238}$$

with $\mathrm{const} = \mathrm{const}(\mu, \mathcal{I})$.

PROOF. Let $\big(I(t), \phi(t), \hat{p}(t), \hat{q}(t)\big)$ a T-periodic solution of (5.26). We know that $\phi(T) = 2\pi k$ with $k \in \mathbb{Z}^N$ defined in (4.54). Denoting by $T_{\phi}^{\min} \leq T^{\min}$ the minimal period of $\phi(t)$, we have that there exist $n \in \mathbb{N}^+$ such that $nT_{\phi}^{\min} = T$ and $\tilde{k} \in \mathbb{Z}^N$ such that $\phi(T_{\phi}^{\min}) = 2\pi\tilde{k}$, verifying $n\tilde{k} = k$. We get that

$$T^{\min} \geq T_{\phi}^{\min} = \frac{T}{n}\,. \tag{5.239}$$

We have, by using (5.27) and (4.44),

$$2\pi n\tilde{k} = 2\pi k = \phi(T) = \tilde{\omega}T + O(\eta^2) = \omega T + O(\eta^2 T) = \omega T + O(1)\,. \tag{5.240}$$

Consider the vector $\bar{\omega}_{\mu} := (\omega_{i_1}, \omega_{i_2})$. One has $2\pi n(\tilde{k}_1, \tilde{k}_2) = \bar{\omega}_{\mu} T + O(1)$ and taking $h = (h_1, h_2) = (\tilde{k}_2, -\tilde{k}_1)$, one gets

$$0 = 2\pi n(\tilde{k}_1, \tilde{k}_2)\cdot h = \bar{\omega}_{\mu}\cdot hT + O(|h|)\,. \tag{5.241}$$

From Lemma 5.11.1, we have that, except for μ in a zero measure set,

$$|\bar{\omega}_{\mu}\cdot h| \geq \frac{c(\mu)}{|h|^{\varsigma}}\,, \qquad \varsigma > 1\,,$$

from which it follows that

$$O(|h|) = |\bar{\omega}_{\mu}\cdot h| T \geq \frac{c(\mu)T}{|h|^{\varsigma}}$$

and hence $|h|^{\varsigma+1} \geq \mathrm{const}\, T$, namely

$$|h| \geq \mathrm{const}\, T^{1/(\varsigma+1)} = \mathrm{const}\, T^{\rho} \tag{5.242}$$

where

$$\rho := \frac{1}{1+\varsigma}\,.$$

From (5.240), from (5.242) and being $|h| = |\tilde{k}|$, we have

$$\begin{aligned} T^{\min} &\geq \frac{T}{n} \geq 2\pi\frac{|\tilde{k}|}{|\omega|} - O\Big(\frac{1}{n|\omega|}\Big) = \frac{2\pi|h|}{|\omega|} - O\Big(\frac{1}{n|\omega|}\Big) \\ &\geq \mathrm{const}\,\frac{T^{\rho}}{|\omega|} - O\Big(\frac{1}{n|\omega|}\Big) \geq \mathrm{const}\, T^{\rho}\,. \end{aligned}$$

□

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