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Group schemes of order p^2 and extension of $\mathbb{Z}/p^2\mathbb{Z}$-torsors

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Introduction

Notation and Conventions. A discrete valuation ring (in the sequel d.v.r.) of unequal characteristic is a discrete valuation ring of characteristic zero with residue field of characteristic $p > 0$. If, for $n \in \mathbb{N}$, there is a distinguished primitive p^n-th root of unity ζ_n in a d.v.r., we write $\lambda_{(n)} := \zeta_n - 1$. Moreover, for any $i \leq n$, we suppose $\zeta_{i-1} = \zeta_i^p$. All the schemes and rings will be assumed nœtherian. If not otherwise specified the cohomology is computed in the fppf topology.

An important problem of research is the following.

Problem. *Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group. Let us suppose that G acts on a smooth projective k-curve C. Is it possible to find a complete d.v.r R of unequal characteristic with residue field k and a lifting $\mathcal{C}$ of C over R endowed with a G-action, such that the G-action lifts the action on the special fiber?*

If the answer is positive we will say that the action on C is liftable to characteristic 0. There are some known results. If $(|G|, p) = 1$, the answer is positive for any action (SGA I). If $|G| > 84(g(C) - 1)$ any action is not liftable because of a trivial contradiction with the use of the Hurwitz bound for the automorphism group in characteristic 0. If G is cyclic of order pm with $(m, p) = 1$ any action is liftable, as proved in [**45**]. This result has induced Oort and Sekiguchi to state the following conjecture.

Conjecture. *Any action of a cyclic group on C is liftable to characteristic* 0.

It has been proved that this problem is actually local: it is equivalent to lift $\mathrm{Spec}(\widehat{\mathcal{O}_{C,y}})$, for any closed point $y \in C$, with an action of I_y, the inertia group at y. The first proof was given by Green-Matignon ([**20**, III 1.3]) in the case of covers whose inertia are cyclic of order not divisible by p^3. It has been proved in general by Bertin-Mezard ([**9**, 3.3.4]), using deformation theory, and by Chinburg-Guralnick-Harbater ([**14**, 2.2]). Therefore one can consider a local version of the problem.

Problem (Local). *Given a k-linear faithful action $\varphi : G \hookrightarrow Aut_k(k[[y]])$, does there exists a lifting of φ to an R-linear action $\varphi_R : G \hookrightarrow Aut_R(R[[y]])$, for some R as above?*

Green and Matignon have proved in [**20**] the conjecture for groups of order p^2m, with $(m, p) = 1$. The conjecture is yet open for G of order divisible by p^3.

It is interesting in general to find groups for which the problem has positive answer for any action. Such groups are called *Oort groups*, analogously groups for which the local problem has positive answer for any action are called *local Oort groups*. There are some works, for instance [**8**], [**20**], [**12**], and [**14**], which study the problem of characterizing such groups. It seems reasonable to conjecture that, for $p > 2$, among the cyclic by-p-groups the Oort groups are the cyclic groups and dihedral groups of order $2p^n$. If $p > 2$, in [**14**] it has been proved that if a cyclic-by-p group G (i.e. an extension of a prime-to-p cyclic group by a p-group) is an Oort group then G is cyclic or dihedral. Moreover if G is a local Oort group, then G is a cyclic or dihedral group. If $p = 2$, it happens that the list of possible Oort groups includes A_4 too. In fact, Bouw has proved (unpublished) that A_4 is an Oort group.

In order to solve this kind of problems it is important to construct automorphisms of $Spec(R[[y]])$ with R a complete d.v.r. of unequal characteristic with algebraically closed residue field k. This took Green and Matignon to study systematically the automorphisms of order p of the formal disc $Spec(R[[y]])$. Let K be the field of fractions of R and $\bar{K}$ an algebraic closure of K. We define $D = \{z \in \bar{K} | v_K(z) > 0\}$. We recall that $\mathcal{D}_K = \mathrm{Spec}(R[[Z]] \otimes K)$ is naturally in a bijective correspondence with $D/Gal(\bar{K}, K)$. For details see, for instance, [**26**, lemma 1.3]. In [**21**], Green and Matignon are interested in the study of automorphisms which specialize on k to automorphisms of order p. It is possible to show that any such automorphism has necessarily fixed points in $\mathcal{D}_K$. After a finite extension of R it is possible to assume that the fixed points are R-rational. We consider the minimal semi-stable model $\mathcal{M}_{\mathcal{D}_K}$ of $\mathcal{D}_K$ in which any fixed point specializes in distinct smooth points (this can be achieved by successive formal blow-ups). The special fibre is an oriented tree of projective lines attached to the original generic point $(\pi) \in \mathrm{Spec}(R[[Z]])$. The main result presented in loc. cit. is associating to any automorphism of the formal disc with fixed points some differential forms and a "Hurwitz datum" defined on the exceptional curves of the minimal semi-stable model. How do these differential forms arise? Before explaining it we recall the definition of torsor under a group scheme.

DEFINITION. *Let G be a faithfully flat and locally of finite type group scheme over a scheme X. Let us suppose that G acts on an X-scheme Y. We say that Y is a G-torsor if there is a covering $\{U_i \longrightarrow X\}$ for the fppf topology on X such that, for every i, Y_{U_i} is isomorphic, as G_{U_i}-scheme, to G_{U_i}.*

Using standard methods it is possible to show that, for any commutative group scheme G over X, the G-torsors over X are classified by the group $H^1(X, G)$. If G is not commutative there is the same classification but $H^1(X, G)$ is only a pointed set. (See [**32**, III.4]). Since it is not easy to work in the fppf topology, it is convenient to have other descriptions of torsors, as we will see.

We now come back to explain how Green and Matignon associate to an automorphism σ of the formal disc with fixed points some differential forms. We use the above notation. Let $\mathcal{M}_{\mathcal{D}_K}$ be the minimal semi-stable model associated to σ.

Let z be a double point of the special fiber of $\mathcal{M}_{\mathcal{D}_K}$ and ξ the generic point of an irreducible component which contains z. The automorphism σ induces an action of $\mathbb{Z}/p\mathbb{Z}$ on

$$X = \mathrm{Spec}(\widehat{(\mathcal{O}_{\mathcal{M}_{\mathcal{D}_K},z})_\xi}) = \mathrm{Spec}(R[[T]]\{T^{-1}\}),$$

such that $X_K \longrightarrow X_K/(\mathbb{Z}/p\mathbb{Z})$ is a $\mathbb{Z}/p\mathbb{Z}$-torsor. By $R[[T]]\{T^{-1}\}$ we mean the algebra of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$ such that $\lim\limits_{i\to-\infty} a_i = 0$. It is possible to show that there is a group scheme G of order p on R and an action of G on X such that G is generically isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (with the same action) and $X \longrightarrow X/G$ is a G-torsor (see later in the introduction for some references about this result). On the special fiber we have that $X_k \longrightarrow X/G_k$ is an α_p, μ_p or $\mathbb{Z}/p\mathbb{Z}$-torsor. The following result is well known (see [**32**, III.4.14]).

THEOREM. I.1.2 *Let X be a smooth scheme over a perfect field k of characteristic p, then there are natural isomorphisms*

$$d : H^1(X,\alpha_p) \longrightarrow H^0(X_{Zar}, B\Omega^1_X) = \{\omega \in H^0(X,\Omega^1_X) |\ \omega \textit{ is locally exact}\},$$

$$\mathrm{dlog} : H^1(X,\mu_p) \longrightarrow H^0(X_{Zar}, \Omega^1_{X,log}) = \{\omega \in H^0(X,\Omega^1_X) |\ \omega \textit{ is locally logarithmic}\}.$$

So, to any α_p-torsor and μ_p-torsor, we can associate a differential form on X_k. Using this description, Henrio ([**25**]) constructed a Hurwitz space, i.e. a certain graph (the dual graph of the special fiber) with an attached datum which is strictly linked to the differential forms found by Green and Matignon. In loc. cit. the author proved that this Hurwitz space classifies automorphisms of the formal disc with fixed points. Bouw and Wewers ([**12**]), using Henrio's work, have proved that D_{2p}, with $p > 2$, is an Oort group. The case $p = 2$ has been proved by Pagot ([**34**]).

The main motivation of this work is to find a Hurwitz space for automorphisms of order p^2 with fixed points of the formal disc. This would be useful to face the problem of lifting tocharacteristic zero for dihedral groups of order $2p^2$. One can see that for automorphisms of order p^2 there are two kinds of problems:

(1) There is no interpretation through differential forms of torsors under finite group schemes of order p^2.
(2) Let X be as above. Given a $(\mathbb{Z}/p^2\mathbb{Z})_K$-torsor $Y_K \longrightarrow X_K$ does there exists an action of a group scheme G on X, extending that of $(\mathbb{Z}/p^2\mathbb{Z})_K$, such that $X \longrightarrow X/G$ is a G-torsor? If it is not always true when does it happen?

We point out that, after answering these questions, it becomes reasonable to construct a Hurwitz space for automorphisms of the formal disc of order p^2 which give a structure of torsor to $\mathrm{Spec}(R[[T]]) \longrightarrow \mathrm{Spec}(R[[T]])/G$ where G is the group of order p^2 generated by the automorphisms we are considering.

We will give answers to these problems in a more general setting. In the first chapter we will generalize the theorem I.1.2 in two directions:

i) in characteristic p for the group schemes μ_{p^n} and α_{p^n};
ii) in unequal characteristic: for some finite group schemes of order p^n.

In §I.2 we recall the definition of the De Rham Witt complex $W_\bullet\Omega^\bullet_X$, for a scheme X over a perfect field k of characteristic p. This satisfies the equalities $W_1\Omega^\bullet_X = \Omega^\bullet_X$ and $W_\bullet\Omega^0_X = W_\bullet\mathcal{O}_X$. There are defined the differential maps

$$\mathrm{d}_n : \mathcal{O}_X \longrightarrow W_n\Omega^1_X$$
$$\mathrm{dlog}_n : \mathcal{O}^*_X \longrightarrow W_n\Omega^1_X.$$

We define $W_n\Omega^1_{X,log}$ as the image sheaf of the $\mathbb{Z}$-module map dlog_n and $BW_n\Omega^1_X$ as the sheaf image of d_n. They are both sheaves of $\mathbb{Z}$-modules. For $n \geq 1$ the above maps induce the morphisms

$$\mathrm{d}\,Log_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{et}, W_n\Omega^1_{X,log}),$$
$$\mathrm{d}_n : H^1(X, \alpha_{p^n}) \longrightarrow H^0(X_{et}, BW_n\Omega^1_X).$$

Then we prove the following theorem.

THEOREM. I.3.2 *Let X be a smooth scheme over a perfect field of characteristic $p > 0$. Then, for any $n \geq 1$*

$$\mathrm{d}\,Log_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{et}, W_n\Omega^1_{X,log}),$$
$$\mathrm{d}_n : H^1(X, \alpha_{p^n}) \longrightarrow H^0(X_{et}, BW_n\Omega^1_X).$$

are isomorphisms.

It is possible to show that $H^0(X_{et}, BW_n\Omega^1_X) = H^0(X_{Zar}, BW_n\Omega^1_X)$. We remark that for $n = 1$ the isomorphisms coincide with those of I.1.2. We now consider a faithfully flat morphism $g : X \longrightarrow \mathrm{Spec}(R)$ with geometrically integral generic fiber. For any integer $n \geq 0$ we will call $\Omega^1_{X/R,log,n}$ the sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules $\Omega^1_{X/R,log}/p^n\Omega^1_{X/R,log}$. When there is no ambiguity we write $\Omega^1_{log,n}$.

THEOREM. I.4.6 *Notation as above. For any $n \geq 1$, there is an isomorphism*

$$\mathrm{d}\,Log_n : H^1(X, \mu_{p^n})/H^1(\mathrm{Spec}(R), \mu_{p^n}) \longrightarrow H^0(X_{Zar}, \Omega^1{}_{\mathrm{log},n}).$$

If X_k is smooth we show (see I.4.8) that the isomorphisms of I.3.2 and I.4.6 are compatible, when restricted to the special fiber. We remark that the hypothesis that X_k is smooth can be weakened: see I.3.12.

Now, let us suppose that R is a d.v.r. of unequal characteristic with no further assumption on the residue field. Let us consider, for any $n \geq 1$ and $\lambda \in R$ with $v(p) \geq p^{n-1}(p-1)v(\lambda)$, the group schemes $G_{\lambda,n}$ (for its definition see §I.5). If R contains a primitive p^n-th root of unity ζ_n then the above inequality is equivalent to $v(\lambda) \leq v(\lambda_{(n)})$. We recall that

$$\begin{array}{ll} (G_{\lambda,n})_k \simeq \mu_{p^n}, & \text{if } v(\lambda) = 0; \\ (G_{\lambda,n})_k \simeq \alpha_{p^n}, & \text{if } 0 < p^{n-1}(p-1)v(\lambda) < v(p); \\ (G_{\lambda,n})_k \simeq \alpha_{p^{n-1}} \times \mathbb{Z}/p\mathbb{Z}, & \text{if } p^{n-1}(p-1)v(\lambda) = v(p). \end{array}$$

We use these groups to find a filtration of $H^1(X, \mu_{p^n})$.

PROPOSITION. I.5.6 *Let X be an integral normal faithfully flat R-scheme. Let $i_0 = \max\{i | v(p) \geq p^{n-1}(p-1)v(\pi^i)\}$. Then, for any n, we have the following filtration*

$$0 \subseteq H^1(X, G_{\pi^{i_0},n}) \subseteq H^1(X, G_{\pi^{i_0-1},n}) \subseteq \ldots \subseteq H^1(X, G_{\pi,n}) \subseteq H^1(X, \mu_{p^n}),$$

where π is the uniformizer of R.

For smooth proper curves over R and $n = 1$ it is essentially the filtration constructed by Saïdi ([**41**, 5.2]). Adding hypothesis on the d.v.r. we obtain also a relative version of the above filtration.

PROPOSITION. I.5.9 *Let us suppose that R has perfect residue field. Let X be a normal integral faithfully flat R-scheme with integral special fiber. We have the following filtration*

$$(1) \qquad 0 \subseteq H^1(X, G_{\pi^{i_0},n})/H^1(R, G_{\pi^{i_0},n}) \subseteq \ldots \subseteq H^1(X, \mu_{p^n})/H^1(R, \mu_{p^n})$$

where π is a fixed uniformizer.

In the case where X is an abelian scheme and $n = 1$ this filtration coincide with that of Andreatta-Gasbarri ([**4**]). Andreatta-Gasbarri moreover proved that their filtration coincides with the Bloch-Kato filtration, defined in [**10**, §1]. Let X be as in the above proposition. We construct a filtration (I.5.12)

$$(2) \qquad 0 \subseteq H^0(X_{Zar}, \Omega^1_{log^{\pi^{i_0}},n}) \subseteq \ldots \subseteq H^0(X_{Zar}, \Omega^1_{log^{\pi},n}) \subseteq H^0(X_{Zar}, \Omega^1_{log,n}),$$

where $\Omega^1_{log^{\pi^i},n}$ are the spaces of differential forms defined in §I.5.2. We prove that this filtration coincides with that of I.5.9 under the isomorphism of I.4.6. More precisely we have the following result.

THEOREM. I.5.13 *Let us suppose that R has perfect residue field. Consider $\lambda \in R$ such that $v(p) \geq p^{n-1}(p-1)v(\lambda)$ and let X be a normal integral faithfully flat R-scheme with integral special fiber. Then there is an isomorphism*

$$H^1(X, G_{\lambda,n})/H^1(R, G_{\lambda,n}) \overset{\mathrm{d}\,Log_n^{\lambda}}{\longrightarrow} H^0(X_{Zar}, \Omega^1_{log^{\lambda},n}).$$

Moreover $\mathrm{d}\,Log_n^{\lambda}$ is compatible with the filtrations (1) *and* (2).

If $0 < p^{n-1}(p-1)v(\lambda) < v(p)$ we prove that this isomorphism is a deformation of the isomorphism I.3.2 given in characteristic p.

In the second and third chapter we study the extendibility of $\mathbb{Z}/p^2\mathbb{Z}$-torsors. We suppose that R is a d.v.r. of unequal characteristic with no further assumption on the residue field. We write $S = \mathrm{Spec}(R)$. We suppose that R contains a primitive p^n-root of unity ζ_n any time we talk about $\mathbb{Z}/p^n\mathbb{Z}$. In that case $\mathbb{Z}/p^n\mathbb{Z}$ is isomorphic to μ_{p^n} on the generic fiber. Let G be an abstract group. Let X be a scheme over R and $Y_K \to X_K$ a G_K-torsor. We remark that, since the characteristic of K is 0, any finite group scheme is étale; so, up to an extension of R, any group scheme over K is an abstract group. Moreover, let us assume that Y is the normalization of X in Y_K. The following question arises naturally.

COARSE QUESTION. *Is it always possible to find a model $\mathcal{G}$ of $(G)_K$ over R together with an action on Y such that $Y \longrightarrow X$ is a $\mathcal{G}$-torsor and the action of G coincides with that of $(G)_K$ on the generic fiber?*
If the answer is positive we will say that $Y_K \longrightarrow X_K$ is *strongly extendible*. First, we will give a weaker answer for any commutative group scheme G_K.

PROPOSITION. III.3.10 *Let X be a normal and faithfully flat scheme over R with integral fibers. Let G be any commutative group-scheme over K and f_K : $Y_K \longrightarrow X_K$ a G-torsor. Let Y be the normalization of X in Y_K. Suppose that Y_k is integral. Up to an extension of R, there exist a (commutative) group-scheme G' and a G'-torsor $Y' \longrightarrow X$ over R which extends f_K.*

The point is that we do not require Y' to coincide with Y, i.e. we do not require Y' to be normal. In such a case we speak about *weak extension*. Clearly strong extension implies weak extension. We remark that in the above result it is necessary to extend R. Indeed we can give an example of a $\mathbb{Z}/p\mathbb{Z}$-torsor not weakly extendible if we do not extend R. We guess that it is really necessary to extend R only if Y_k is not reduced. Under the same hypothesis as above we have a more precise statement for $G = \mathbb{Z}/p^n\mathbb{Z}$. A $(\mathbb{Z}/p^n\mathbb{Z})_K$-torsor can indeed be extended to a μ_{p^n}-torsor over any R which contains a primitive p^n-th root of unity (see III.3.8).

It is well known that the *coarse question* has a positive answer if $(|G|, p) = 1$. Let us now suppose $G = \mathbb{Z}/p\mathbb{Z}$. For this group, strong extension has been proved in some cases. For details see [**37**, 1.2.2] when X is the spectrum of a d.v.r, [**25**, 1.6] and [**21**, 1.1] for formal affine curves and [**41**, 2.4] for formal curves in a more general setting. In particular the statement is true if $X = \mathrm{Spec}(R[[T]]\{T^{-1}\})$, as remarked above. See also the paper of Abramovich ([**2**]) for some results in dimension 2. In the present thesis we will study also higher dimension. Let us suppose $X = \mathrm{Spec}(A)$ with A a faithfully flat and factorial R-algebra, complete with respect to the π-adic topology or X a normal local faithfully flat R-scheme. We remark that A factorial is equivalent to say that X is normal and $Cl(X) = 0$ (see [**24**, 6.2, 6.11]). Moreover we suppose X_k integral. Now, let $Y_K \to X_K$ be a nontrivial $\mathbb{Z}/p\mathbb{Z}$-torsor and Y the normalization of X in Y_K. Let us suppose that Y_k is integral. From III.3.8, cited above, it easily follows that $[Y_K]$, the class in $H^1(X_K, \mathbb{Z}/p\mathbb{Z})$ of the $\mathbb{Z}/p\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$, is induced, by restriction, by an element of $H^1(X, \mu_{p^n})$. For simplicity we think $[Y_K] \in H^1(X, \mu_{p^n})$. Then, by I.5.6, there exists a j such that $[Y_K] \in H^1(X, G_{\pi^j,1}) \setminus H^1(X, G_{\pi^{j+1},1})$. We then get the following result.

THEOREM. III.4.2 *Let us suppose that R contains a primitive p-th root of unity. Let us consider*

$$Y_K \longrightarrow X_K,$$

a nontrivial $\mathbb{Z}/p\mathbb{Z}$-torsor as above. If $[Y_K] \in H^1(X, G_{\pi^j,1}) \setminus H^1(X, G_{\pi^{j+1},1})$ then Y is a $G_{\pi^j,1}$-torsor. Moreover the valuation of the different of the extension $\mathcal{O}_{X,(\pi)} \subseteq \mathcal{O}_{Y,(\pi)}$ is $(p-1)(v(\lambda_{(1)}) - j)$.

In this work we have moreover studied the case $G = \mathbb{Z}/p^2\mathbb{Z}$. Before explaining in more details our results in such a case we come back for a moment to the general situation, presented above. We observe that any G-action on Y_K can be extended to a G-action $\mu : G \times Y \longrightarrow Y$. One phenomena which may occur is that the reduced action of G on the special fibre is not faithful. To solve this problem Romagny ([**40**]) has introduced the notion of effective models. We remark that a very close notion, called Raynaud's group scheme, has been introduced by Abramovich ([**2**]).

DEFINITION. *Let G be a finite flat group scheme over R. Let Y be a flat scheme over R. Let $\mu\colon G \times Y \to Y$ be an action, faithful on the generic fibre. An* effective model *for μ is a finite flat R-group scheme $\mathcal{G}$ acting on Y, dominated by G compatibly (with the actions), such that $\mathcal{G}$ acts faithfully on Y (i.e. the map $\mathcal{G} \longrightarrow \mathcal{A}ut(Y)$ is injective).*

We recall that to say that G dominates $\mathcal{G}$ means that there exists an R-morphism $G \longrightarrow \mathcal{G}$ which is an isomorphism when restricted to the generic fibers. In particular the effective model $\mathcal{G}$ is a model of G_K, i.e. a finite and flat R-group scheme with generic fiber isomorphic to G_K. Moreover if the effective model exists it is unique, as Romagny has proved. The same author proved its existence in the case that Y is of finite type (see §III.1 for the precise statement). The uniqueness of the effective model easily implies that, using the above notation, $Y_K \longrightarrow X_K$ is strongly extendible if and only if Y is a $\mathcal{G}$-torsor. Hence the coarse question can be reformulated in the following way.

QUESTION. *Which is the effective model (if it exists) $\mathcal{G}$ for a G-action? When is Y a $\mathcal{G}$-torsor?*

We now treat the case $G = \mathbb{Z}/p^2\mathbb{Z}$. As remarked above, the effective model of a $\mathbb{Z}/p^2\mathbb{Z}$-action is a model of $(\mathbb{Z}/p^2\mathbb{Z})_K$. The second chapter is devoted to the classification of $(\mathbb{Z}/p^2\mathbb{Z})_K$-models. We now explain more in details what we do.

Let K be a field of characteristic 0 which contains a primitive p^n-th root of unity. We remark that this implies $\mu_{p^n} \simeq \mathbb{Z}/p^n\mathbb{Z}$. We recall the following exact sequence

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 1,$$

so-called the Kummer sequence. We stress that the Kummer sequence can be written also as follows

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m^n \xrightarrow{\theta_n} \mathbb{G}_m^n \longrightarrow 1$$

where $\theta_n((T_1, \ldots, T_n)) = (1 - T_1^p, T_1 - T_2^p, \ldots, T_{n-1} - T_n^p)$. Let k be a field of characteristic $p > 0$. The following exact sequence

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow W_n(k) \xrightarrow{\mathrm{F}-1} W_n(k) \longrightarrow 0,$$

where $W_n(k)$ is the group scheme of Witt vectors of length n, is called the Artin-Schreier-Witt sequence. Let now R be a d.v.r. of unequal characteristic which contains a p^n-th root of unity. It has been proved, independently, by Oort-Sekiguchi-Suwa ([**45**]) and Waterhouse ([**58**]) the existence of an exact sequence of group schemes over R which unifies the above two sequences for $n = 1$. Later Green-Matignon ([**20**]) and Sekiguchi-Suwa([**53**]) have, independently, constructed explicitly a unifying exact sequence for $n = 2$. This means that it has been found an exact sequence

$$0 \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{W}'_2 \longrightarrow 0 \tag{3}$$

that coincides with the Kummer sequence on the generic fiber and with the Artin-Schreier-Witt sequence on the special fiber. The case $n > 2$ is treated in [**44**] and [**52**]. In the second chapter we will generalize this construction. First, we consider the case $n = 1$ and we prove that any $\mathbb{Z}/p\mathbb{Z}$-model is isomorphic to $G_{\lambda,1}$ for some $\lambda \in R$. This is a well known result. A proof was already given, for instance, by Romagny in his PhD thesis. Moreover by definition we have that $G_{\lambda,1}$ is the kernel of an isogeny, between smooth R-group schemes, which is generically isomorphic to the Kummer sequence. Next we will consider the case $n = 2$. Analogously, we will prove that for any model of $(\mathbb{Z}/p^2\mathbb{Z})_K$ there exists an exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow 0,$$

with $\mathcal{E}_1, \mathcal{E}_2$ smooth R-group schemes, which coincides with the Kummer sequence on the generic fiber. We will describe explicitly all such isogenies and their kernels. Moreover we will give a classification of models of $\mathbb{Z}/p^2\mathbb{Z}$.

We now explain more precisely the classification we have obtained. First of all we show that any model of $\mathbb{Z}/p^2\mathbb{Z}$ is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ for some $\mu, \lambda \in R \setminus \{0\}$. Then the first step is to investigate on $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$.

We suppose $p > 2$. Let us define the group

$$rad_{p,\lambda}(< 1 + \mu S >) := \Big\{ (F(S), j) \in \mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \times \mathbb{Z}/p\mathbb{Z} \text{ such that}$$
$$F(S)^p(1+\mu S)^{-j} = 1 \in \mathrm{Hom}(G_{\mu,1|S_{\lambda^p}}, \mathbb{G}_{m|S_{\lambda^p}}) \Big\} / < 1 + \mu S, 0 > .$$

There is a conflict of notation since S denote $\mathrm{Spec}(R)$, too. But it should not cause any problem. For any $(F, j) \in rad_{p,\lambda}(< 1 + \mu S >)$ we will explicitly define in II.3.4 an extension $\mathcal{E}^{(\mu,\lambda;F,j)}$ of $G_{\mu,1}$ by $G_{\lambda,1}$. We will give a description of $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$.

THEOREM. II.3.35. *Suppose that $\lambda, \mu \in R$ with $v(\lambda_{(1)}) \geq v(\lambda), v(\mu)$. There exists a (natural) exact sequence*

$$0 \longrightarrow rad_{p,\lambda}(< 1 + \mu S >) \xrightarrow{\beta} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \longrightarrow$$
$$\longrightarrow \ker\left(H^1(S, G^\vee_{\mu,1}) \longrightarrow H^1(S_\lambda, G^\vee_{\mu,1}) \right),$$

where β is defined by

$$(F, j) \longmapsto \mathcal{E}^{(\mu,\lambda;F,j)}.$$

In particular $rad_{p,\lambda}(< 1 + \mu S >) \simeq \{\mathcal{E}^{(\mu,\lambda;F,j)}\}$.

From this result it follows that, up to an extension of R, any group scheme of order p^2 is of the form $\mathcal{E}^{(\mu,\lambda;F,j)}$. Moreover we can determine all the models of $(\mathbb{Z}/p^2\mathbb{Z})_K$.

THEOREM. II.3.53 *Let us suppose $p > 2$. Let G be a finite and flat R-group scheme such that $G_K \simeq (\mathbb{Z}/p^2\mathbb{Z})_K$. Then $G \simeq \mathcal{E}^{(\pi^m,\pi^n;F,1)}$ for some $v(\lambda_{(1)}) \geq m \geq n \geq 0$. Moreover $F(S) = E_p(aS) := \sum_{i=0}^{p-1} \frac{a^i}{i!} S^i$ with $pa - j\pi^m \equiv \frac{p}{\pi^{m(p-1)}} a^p \mod \pi^{np}$. Finally m, n and $a \in R/\pi^n R$ are unique.*

In fact this statement is slightly weaker respect than that presented inside the thesis. We indeed remark that we can explicitly find all the solutions a of the equation $pa - j\pi^m \equiv \frac{p}{\pi^{m(p-1)}} a^p \mod \pi^{np}$ if $m \geq n$(see II.3.46).

The above result gives us candidates for being effective models for a $\mathbb{Z}/p^2\mathbb{Z}$-action. In the second part of the third chapter we study the problem of the extendibility of $\mathbb{Z}/p^2\mathbb{Z}$-torsors. We suppose that $X = \mathrm{Spec}(A)$ with A a factorial R-algebra, complete with respect to the π-adic topology, or X a normal local R-scheme. We moreover suppose X is essentially semireflexive (see §III.1 for the definition). This implies in particular that $X \longrightarrow \mathrm{Spec}(R)$ is faithfully flat. For instance we can consider $X = \mathrm{Spec}(R[[T]]\{T^{-1}\})$ or $X = \mathrm{Spec}(R[[T]])$. Let us consider a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$ and let Y be the normalization of X in Y_K. It is possible to show that there is a factorization

$$h : Y \xrightarrow{h_2} Y_1 \xrightarrow{h_1} X \tag{4}$$

with h_1 and h_2 degree p morphisms. We define γ_i such that the valuation of the different $\mathcal{D}(h_i)$ of h_i (localized in the special fiber) is

$$v(\mathcal{D}(h_i)) = v(p) - (p-1)\gamma_i$$

for $i = 1, 2$. So we can apply III.4.2 to h_i. Moreover, by I.5.6, we have that $[Y_K] \in H^1(X, G_{\pi^j,2}) \setminus H^1(X, G_{\pi^{j+1},2})$ for some $j \leq v(\lambda_{(2)})$. Finally we can prove that $Y = \mathrm{Spec}(B)$ with

$$B = B_1[T_2]/\left(\frac{(1+\pi^{\gamma_2}T_2)^p - 1}{\pi^{p\gamma_2}} - \frac{(1+\pi^{pj}g_0)H(T_1)^{-p}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}\right)$$

and

$$B_1 = A[T_1]/(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1),$$

for some $H(T_1) \in B_1^*$ and $g_0, f_1 \in A$. $H(T_1)$ and g_0 are uniquely determined $\mod \pi^{\gamma_2}$, while f_0 is uniquely determined $\mod \pi^{\gamma_1}$.

Now, given $H(T_1) = \sum_{k=0}^{p-1} a_k {T_1}^k \in B_1^*$, let us consider its formal derivative $H'(T_1)$. For any $m \geq \gamma_1$, we will say that $a \in \pi R$ satisfies $(\Delta)_m$ if

$$aH(T) \equiv \pi^{m-\gamma_1} H'(T) \mod \pi^{\gamma_2}.$$

DEFINITION. We will call *effective threshold* the number

$$\kappa = \min\{m \geq \gamma_1 | \exists a \in \pi R \text{ which satisfies } (\Delta)_m\}.$$

For any $m \geq \gamma_1$ there exists at most one $a \in \pi R$ which satisfies $(\Delta)_m$. We will call it α_m. We will prove that we can construct a group scheme $\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)}$.

DEFINITION. Using the previous notation we say that the *degeneration type* of $Y_K \longrightarrow X_K$ is $(j, \gamma_1, \gamma_2, \kappa)$.

Then we will prove the following theorem.

THEOREM III.5.7 *Let $X := \operatorname{Spec} A$ be as above. Let $Y_K \to X_K$ be a connected $\mathbb{Z}/p^2\mathbb{Z}$-torsor. Let Y be the normalization of X in Y_K and let us suppose that Y_k is integral. If Y_K has $(j, \gamma_1, \gamma_2, \kappa)$ as degeneration type then its effective model is*

$$\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)}.$$

We remark that the existence of the effective model in this case was not assured by the above cited result of Romagny about the existence of effective models. Indeed we do not assume Y to be of finite type.

Then we give a criterion for Y to have a structure of torsor under some finite and flat group scheme G.

COROLLARY III.5.12. *Under the hypothesis of theorem* III.5.7 *$Y \longrightarrow X$ is a G-torsor under some finite and flat group scheme G if and only if $\kappa = \gamma_1$.*

Moreover, we will give an example in which Y is not a G-torsor under some finite and flat group scheme. We observe that in the case R of equal characteristic the extension of $\mathbb{Z}/p^2\mathbb{Z}$-torsors had been studied by Saïdi ([**43**]). We remark that, in that case, there is no criterion to determine if a cover is a torsor under some group scheme. Finally we observe that Romagny [**40**] and Saïdi [**42**] have given, in equal characteristic, an example in which $Y_K \longrightarrow X_K$ is not strongly extendible.

We finally try to determine 4-uples of positive integer numbers that can be degeneration types. Since R contains ζ_2 then $v(p) = p(p-1)v(\zeta_2 - 1)$

DEFINITION. Any 4-uple $(j, \gamma_1, \gamma_2, \kappa) \in \mathbb{N}^4$ with the following properties:

i) $\max\{\gamma_1, \gamma_2\} \leq \kappa \leq \frac{v(p)}{p-1}$;
ii) $\gamma_2 \leq p(\kappa - \gamma_1 + j) \leq p\gamma_2$;
iii) if $\kappa < p\gamma_2$ then $\gamma_1 - j = \frac{v(p)}{p}$; if $\kappa \geq p\gamma_2$ then $0 \leq p(\gamma_2 - j) \leq v(p) - p\gamma_1 + \kappa$;
iv) $pj \leq \gamma_1$

will be called an *admissible degeneration type.*

DEFINITION. Any admissible degeneration type which is the degeneration type attached to a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $Y_K \to X_K$ as above, will be called *realizable.*

And finally we prove the following statement.

THEOREM III.6.7. *Any admissible degeneration type $(j, \gamma_1, \gamma_2, \kappa)$ with $\kappa = \gamma_1$ is realizable.*

If $\kappa > \gamma_1$ we have examples of admissible degeneration types $(j, \gamma_1, \gamma_2, \kappa)$ that are not realizable.

Contents

CHAPTER I

Differential forms and torsors under some finite group schemes

I.1. Classical results

Let X be a scheme over a perfect field k of characteristic $p>0$. For any $\mathbb{F}_p$-scheme Y we will denote by $\mathrm{F}_Y : Y \longrightarrow Y$ the *absolute Frobenius* of Y. Let us define $X^{(p)}$ the scheme obtained by the base change F_k, i.e.

$$\begin{array}{ccc} X^{(p)} & \xrightarrow{q} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{F}_k} & \mathrm{Spec}(k) \end{array}$$

is cartesian. We remark that q is an isomorphism (but not as k-schemes) and F_X factors in $q \circ \mathrm{F}_{X/k}$, where the k-morphism

$$\mathrm{F}_{X/k} : X \longrightarrow X^{(p)}$$

is called *relative Frobenius*. Let $\Omega^\bullet_{X/k} = (\wedge\Omega^1_{X/k}, d)$ be the De Rham complex. The differential d is $\mathcal{O}_{X^{(p)}}$-linear (through $\mathrm{F}_{X/k} : \mathcal{O}_{X^{(p)}} \longrightarrow (\mathrm{F}_{X/k})_*\mathcal{O}_X$). Hence

$$Z\Omega^i_{X/k} = \ker \mathrm{d} : \Omega^i_{X/k} \longrightarrow \Omega^{i+1}_{X/k}$$

and

$$B\Omega^i_{X/k} = \mathrm{d}(\Omega^{i-1}_{X/k})$$

are sheaves of $\mathcal{O}_{X^{(p)}}$-modules. Then $H^*(\Omega^\bullet_{X/k})$, the cohomology of the complex $\Omega^\bullet_{X/k}$, is a graded $\mathcal{O}_{X^{(p)}}$-algebra.

We recall that that there exists a unique homomorphism of graded $\mathcal{O}_X$-algebras

$$C^{-1}_{X/k} : \Omega^\bullet_{X/k} \longrightarrow q_*(H^*(\Omega^\bullet_{X/k})) \tag{I.5}$$

which is equal to $\mathrm{F}_{X/k}$ in degree zero and such that, for any section s of $\mathcal{O}_X$,

$$C^{-1}_{X/k}(\mathrm{d}\, s) = [s^{p-1}\, \mathrm{d}\, s] \in H^1(\Omega^\bullet_{X/k})$$

Cartier has proved the following result:

THEOREM I.1.1. *If X/k is smooth then $C^{-1}_{X/k}$ is an isomorphism.*

PROOF. See [**29**, 7.2] for a proof. □

The inverse of (I.5) defines an $\mathcal{O}_X$-linear morphism $C_{X/k} : q_* Z\Omega^\bullet_{X/k} \longrightarrow \Omega^\bullet_{X/k}$, called *Cartier's operation*.

Let us consider the morphism of sheaves of abelian groups

$$\mathrm{d}\,log : \mathcal{O}_X^* \longrightarrow \Omega^1_{X/k}$$

given by $\mathrm{d}\,log(s) = \frac{\mathrm{d}\,s}{s}$. We call $\Omega^1_{log,X}$ the image sheaf. If we look at the isomorphism $C_{X/k}$ in degree zero we have the following exact sequences of sheaves on X

$$\text{(I.6)} \qquad \begin{aligned} 0 \longrightarrow \mathcal{O}_X^{*p} \longrightarrow \mathcal{O}_X^* \xrightarrow{\mathrm{d}\,log} Z\Omega^1_{X/k} \\ 0 \longrightarrow \mathcal{O}_X^{p} \longrightarrow \mathcal{O}_X \xrightarrow{d} Z\Omega^1_{X/k} \end{aligned}$$

We now recall that, for any commutative group scheme G over X, the group $H^1(X, G)$ classifies the G-torsors over X. If G is not commutative there is the same classification but $H^1(X, G)$ is only a pointed set (see [**32**, III.4]). Let us consider the natural continuous map of sites $f : X_{fl} \longrightarrow X_{et}$. It is known that $\mathrm{R}^i f_* \mathbb{G}_m = \mathrm{R}^i f_* \mathbb{G}_a = 0$ for $i > 0$ (see the proof of [**32**, III.3.9]). Considering the exact sequences associated to the exact sequences in X_{fl}

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \longrightarrow 1$$

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{p} \mathbb{G}_a \longrightarrow 0$$

and, applying the functor associated to $f : X_{fl} \longrightarrow X_{et}$, we obtain in X_{et}

$$1 \longrightarrow \mu_p \longrightarrow \mathcal{O}_X^* \xrightarrow{\mathrm{F}_X} \mathcal{O}_X^* \longrightarrow \mathrm{R}^1 f_* \mu_p \longrightarrow 0,$$

$$0 \longrightarrow \alpha_p \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{F}_X} \mathcal{O}_X \longrightarrow \mathrm{R}^1 f_* \alpha_p \longrightarrow 0$$

and

$$\mathrm{R}^i f^* \mu_p = \mathrm{R}^i f_* \alpha_p = 0$$

for any $i > 1$. In particular

$$\text{(I.7)} \qquad \begin{aligned} \mathcal{O}_X^*/(\mathcal{O}_X^*)^p \simeq \mathrm{R}^1 f_* \mu_p \\ \mathcal{O}_X/(\mathcal{O}_X)^p \simeq \mathrm{R}^1 f_* \alpha_p. \end{aligned}$$

Moreover, in the étale topology the sheaves μ_p and α_p are the zero sheaves since X is reduced. So $H^i(X_{et}, \mu_p) = H^i(X_{et}, \alpha_p) = 0$ for any i. Then the Leray spectral sequences

$$H^i(X_{et}, \mathrm{R}^j f_* \mu_p) \Rightarrow H^{i+j}(X, \mu_p)$$

$$H^i(X_{et}, \mathrm{R}^j f_* \alpha_p) \Rightarrow H^{i+j}(X, \alpha_p)$$

give the isomorphisms

$$H^1(X, \mu_p) \xrightarrow{\delta} H^0(X_{et}, \mathrm{R}^1 f_* \mu_p)$$

$$H^1(X, \alpha_p) \xrightarrow{\delta} H^0(X_{et}, \mathrm{R}^1 f_* \alpha_p).$$

We now have, by (I.6), in the étale topology, the following isomorphisms

$$\mathcal{O}_X^*/(\mathcal{O}_X^*)^p \overset{\mathrm{d}\,log}{\underset{\simeq}{\longrightarrow}} \Omega^1_{X,log},$$

$$\mathcal{O}_X/(\mathcal{O}_X)^p \overset{d}{\underset{\simeq}{\longrightarrow}} B\Omega^1_{X/k},$$

which give the isomorphisms

$$\mathrm{d}\,Log := \mathrm{d}\,log \circ \delta : H^1(X, \mu_p) \longrightarrow H^0(X_{et}, \Omega^1_{X,log}),$$
$$\mathrm{d} := d \circ \delta : H^1(X, \alpha_p) \longrightarrow H^0(X_{et}, B\Omega^1_{X/k})$$

using the identifications of (I.7). But $H^0(X_{et}, \Omega^1_{X,log}) = H^0(X_{Zar}, \Omega^1_{X,log})$ and $H^0(X_{et}, B\Omega^1_{X/k}) = H^0(X_{Zar}, B\Omega^1_{X/k})$. This is the proof (taken from [**32**, III.4.14]) of the following result.

Theorem I.1.2. *Let X be a smooth scheme over a perfect field k of characteristic p, then the maps*

$$\mathrm{d}\,Log : H^1(X, \mu_p) \longrightarrow H^0(X_{Zar}, \Omega^1_{X,log}),$$
$$\mathrm{d} : H^1(X, \alpha_p) \longrightarrow H^0(X_{Zar}, B\Omega^1_{X/k})$$

are isomorphisms.

We remark that the previous statement means

$$H^1(X, \alpha_p) = \{\omega \in H^0(X_{Zar}, \Omega^1_X) |\ \omega \text{ is locally exact}\},$$
$$H^1(X, \mu_p) = \{\omega \in H^0(X_{Zar}, \Omega^1_X) |\ \omega \text{ is locally logarithmic}\}.$$

The purpose of this chapter is to generalize the previous theorem for α_{p^n}-torsors and μ_{p^n}-torsors. Moreover we obtain similar statements for some group schemes over a d.v.r. of unequal characteristic.

I.2. De Rham-Witt V-pro-complexes

For reference see [**27**] where the author uses this complex to give a more explicit description of crystalline cohomology. For any scheme X over $\mathbb{F}_p$, a V-De Rham pro-complex is a projective system

$$M_\bullet = \{(M_n)_{n\in\mathbb{Z}},\ R : M_{n+1} \longrightarrow M_n\}$$

of sheaves of differential graded algebras (dga) over X, and a family of additive maps

$$V : (M_n^i \longrightarrow M_{n+1}^i)_{n\in\mathbb{Z}}$$

such that $RV = VR$ and

(V1) $M_n = 0$ for $n \leq 0$, M_1^0 is an $\mathbb{F}_p$-algebra, $M_n^0 = W_n(M_1^0)$, R and V are the usual operators over $W_n(M_1^0)$;

(V2) For any n, i, j and $x \in M_n^i$ and $y \in M_n^j$ we have

$$V(xdy) = V(x)dV(y);$$

(V3) For any $x \in M_1^0$ and $y \in M_n^0$ we have

$$Vyd[x] = V(y[x^{p-1}]d[x]).$$

NOTATION: for any $A \in Ob(\mathbb{F}_p\text{-alg}(X))$, where $\mathbb{F}_p$-alg(X) denote the category of $\mathbb{F}_p$-algebras over X, and $x \in A$ we denote by $[x] \in W(A)$ the Teichmüller representant $(x, 0, 0, \dots)$. And $[x]_{\leq n}$ is the image in $W_n\Omega^\bullet_X$.

The De Rham V-pro-complexes form a category $VDR(X)$. An arrow $f : M_\bullet \longrightarrow N_\bullet$ of $VDR(X)$ is, by definition, a homomorphism of projective systems of dga's $(f_n : M_n \longrightarrow N_n)_{n\in\mathbb{N}}$ such that $f_{n+1}V = Vf_n$ and $f_n^0 = W_n(f_1^0)$, for any n. [**27**, I 1.3] ensures that there exists a left adjoint (called $W_\bullet\Omega^\bullet_X$) of the forgetful functor

$$VDR(X) \longrightarrow \mathbb{F}_p\text{-alg}(X)$$
$$M_\bullet \longrightarrow M_1^0$$

This means

$$\mathrm{Hom}_{VDR(X)}(W_\bullet\Omega^\bullet_A, M_\bullet) = \mathrm{Hom}_{\mathbb{F}_p\text{-alg}(X)}(A, M_1^0),$$

for any $A \in \text{ob } \mathbb{F}_p\text{-alg}(X)$ and $M_\bullet \in \text{ob } VDR(X)$.

Moreover [**27**, I 1.3] also says that the homomorphism $\pi_n : \Omega^\bullet_{W_n(A)} \longrightarrow W_n\Omega^1_A$, such that π_n^0 is the identity, is surjective and $\pi_1 : \Omega^\bullet_A \longrightarrow W_1\Omega^\bullet_A$ is an isomorphism. So we can think of any element of $W_n\Omega^1_A$ as a differential form with coefficients in $W_n(A)$. In the case that X is defined over a perfect field of characteristic p then, by [**27**, I 1.6], $W_n\Omega^\bullet_A$ is naturally a sheaf of $W_n(k)$-dga's.

I.2.1. Operator F. We recall that $W_\bullet(\mathcal{O}_X)$ has a *Frobenius* endomorphism denoted

$$\mathrm{F} : W_\bullet\mathcal{O}_X \longrightarrow W_\bullet\mathcal{O}_X.$$

For $\mathbb{F}_p$-schemes it is defined by $(x_0, \dots, x_n) \mapsto (x_0^p, \dots, x_n^p)$. It extends the absolute Frobenius $F_X : \mathcal{O}_X \longrightarrow \mathcal{O}_X$.

In the following we recall the definition of the operator F on the de Rham-Witt complex and its principal properties.

THEOREM I.2.1. *Let X be an $\mathbb{F}_p$-scheme. Then the homomorphism of projective systems of rings $R\mathrm{F} = \mathrm{F}R : W_\bullet\mathcal{O}_X \longrightarrow W_{\bullet-1}\mathcal{O}_X$ (where $W_i\mathcal{O}_X = 0$ if $i \leq 0$) can be uniquely extended to a homomorphism of projective systems of graded algebras*

$$\mathrm{F} : W_\bullet\Omega^\bullet_X \longrightarrow W_{\bullet-1}\Omega^\bullet_X,$$

such that

a) *for any $n \geq 2$ and for any $x \in \mathcal{O}_X$*

$$\mathrm{F}\,\mathrm{d}[x] = [x^{p-1}]_{\leq n}\,\mathrm{d}[x]_{\leq n-1};$$

b) *for any $n \leq 1$, we have*

$$\mathrm{F}\,\mathrm{d}V = \mathrm{d} : W_n\mathcal{O}_X \longrightarrow W_n\Omega^1_X.$$

PROOF. [**27**, I 2.17]. □

Proposition I.2.2. F *has the following properties:*

i) $\mathrm{F}\,V = V\,\mathrm{F} = p : W_n\Omega^i_X \longrightarrow W_n\Omega^i_X$;
ii) $\mathrm{d}\,\mathrm{F} = p\,\mathrm{F}\,\mathrm{d} : W_n\Omega^i_X \longrightarrow W_{n-1}\Omega^{i+1}_X$ *(i.e.* $p^i\,\mathrm{F}$ *is a morphism of dga);*
iii) $\mathrm{F}\,\mathrm{d}\,V = \mathrm{d} : W_n\Omega^i_X \longrightarrow W_n\Omega^{i+1}_X$;
iv) *if we denote by* $F : W_n\Omega^\bullet_X \longrightarrow W_n\Omega^\bullet_X$ *the endomorphism defined functorially by* $\mathrm{F} : W_n\mathcal{O}_X \longrightarrow W_n\mathcal{O}_X$*, we have, for any* i*, the following commutative diagram*

$$\begin{array}{ccc} W_n\Omega^i_X & \xrightarrow{F} & W_n\Omega^i_X \\ & \searrow^{p^i\,\mathrm{F}} & \downarrow \\ & & W_{n-1}\Omega^i_X \end{array}$$

Proof. These properties are all proved in [**27**, I 2.18]. □

Moreover it is possible to prove the following result.

Proposition I.2.3. *If* X *is a smooth scheme over a perfect scheme* T *of characteristic* p *then, for any* n*, we have*

$$Ker(\mathrm{F}^n\,\mathrm{d} : W_{n+1}\Omega^i_X \longrightarrow \Omega^{i+1}_X) = F(W_{n+2}\Omega^i_X).$$

Proof. See the proof of [**27**, I 3.11]. □

Remark I.2.4. We can think of the operator F as a generalization of the inverse Cartier operator $C^{-1} : \Omega^i_X \longrightarrow \Omega^i_X/\,\mathrm{d}\,\Omega^{i-1}_X$. In fact it is possible to show ([**27**, 3.3]) that if X is smooth and T is perfect, F induces an homomorphism

$$\mathrm{F} : W_n\Omega^i_X \longrightarrow W_n\Omega^i_X/\,\mathrm{d}\,V^{n-1}\Omega^{i-i}_X$$

which coincides with C^{-1} for $n = 1$.

I.2.2. The canonical filtration of $W_\bullet\Omega^\bullet_X$.

In this paragraph and in the next one we suppose that X is a connected smooth scheme over a perfect base T of characteristic p. Then, for any $n, r \in \mathbb{Z}$, we define

$$Fil^r W_n\Omega^\bullet_X = \begin{cases} W_n\Omega^\bullet_X, & \text{if } n \leq 0 \text{ or } r \leq 0; \\ Ker R^{n-r} : W_n\Omega^\bullet_X \longrightarrow W_r\Omega^\bullet_X, & \text{if } 1 \leq r < n; \\ 0, & \text{if } r \geq n\ . \end{cases}$$

The following result characterizes, in another way, the previous filtration

Proposition I.2.5. *For any* n *and* $0 \leq i \leq n$ *we have*

$$Ker(p^i : W_n\Omega^\bullet_X \longrightarrow W_n\Omega^\bullet_X) = Fil^{n-i}W_n\Omega^\bullet_X.$$

Proof. [**27**, I 3.4]. □

I.2.3. Two exact sequences. We consider the following morphisms

$$\begin{aligned} \mathrm{dlog}_n : \mathcal{O}_X^* &\longrightarrow W_n\Omega_X^1 \\ x &\longmapsto \frac{d[x]_{\leq n}}{[x]_{\leq n}} \end{aligned}$$

and

$$\begin{aligned} \mathrm{d} : \mathcal{O}_X &\longrightarrow W_n\Omega_X^1 \\ x &\longmapsto d[x]_{\leq n} \end{aligned}$$

Then

PROPOSITION I.2.6. *For any $n \geq 1$, the sequence*

$$0 \longrightarrow \mathcal{O}_X^* \xrightarrow{\mathrm{F}_X} \mathcal{O}_X^* \xrightarrow{\mathrm{dlog}_n} W_n\Omega_X^1$$

is exact in the étale topology.

PROOF. [**27**, I 3.23.2]. □

We can prove the following

PROPOSITION I.2.7. *For any $n \geq 1$ the sequence*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{F}_X^n} \mathcal{O}_X \xrightarrow{\mathrm{d}} W_n\Omega_X^1$$

is exact in the étale topology.

REMARK I.2.8. Here there is an abuse of notation. In fact we recall that the operator F in degree zero is equal to $R\,\mathrm{F} = \mathrm{F}\,R : W_n\mathcal{O}_X \longrightarrow W_{n-1}\mathcal{O}_X$ with F the usual Frobenius.

PROOF. By I.2.5 we see that $p^n W_n\Omega_X^1 = 0$. So

$$\mathrm{d}[\mathrm{F}^n(x)]_{\leq n} = \mathrm{d}\,\mathrm{F}^n([x]_{\leq 2n}) \overset{\text{I.2.2}(ii)}{=} p^n\,\mathrm{F}^n\,\mathrm{d}[x]_{\leq 2n} = 0.$$

We now prove $Ker\,\mathrm{d} \subseteq Im\,\mathrm{F}^n$ by induction. For $n = 1$ the result derives by the well known Cartier isomorphism. Now suppose $\mathrm{d}[x]_{\leq n} = 0$ with $x \in \mathcal{O}_X$. Then $R(\mathrm{d}[x]_{\leq n}) = \mathrm{d}[x]_{\leq n-1} = 0$. So, by inductive hypothesis, we have, locally, $x = \mathrm{F}^{n-1}(y)$, for some $y \in \mathcal{O}_X$. So

$$\mathrm{d}[x]_{\leq n} = \mathrm{d}([\mathrm{F}^{n-1}(y)]_{\leq n}) = \mathrm{d}\,\mathrm{F}^{n-1}([y]_{\leq 2n-1}) = p^{n-1}\,\mathrm{F}^{n-1}\,\mathrm{d}[y]_{\leq 2n-1} = 0,$$

which implies, by I.2.5, $\mathrm{F}^{n-1}\,\mathrm{d}[y]_{\leq 2n-1} \subseteq Fil^1 W_n\Omega_X^1$. Hence

$$R^{n-1}(\mathrm{F}^{n-1}\,\mathrm{d}[y]_{\leq 2n-1}) = \mathrm{F}^{n-1}\,\mathrm{d}[y]_{\leq n} = 0,$$

which implies $[y]_{\leq n} = \mathrm{F}(\alpha)$ with $\alpha \in W_{n+1}\mathcal{O}_X$, by I.2.3. Then, locally, $y = F(z)$ for some $z \in \mathcal{O}_X$ and so $x = F^n(z)$, locally.

□

I.3. Differential forms and torsors in characteristic p

DEFINITION I.3.1. For any scheme over a perfect field of characteristic p we define $W_n\Omega^1_{X,log}$ as the image sheaf of the $\mathbb{Z}$-module map dlog_n. Moreover we define $BW_n\Omega^1_X$ as the image sheaf of d_n. They are both sheaves of $\mathbb{Z}$-modules.

Following the strategy of the proof of I.1.2, we give a description of μ_{p^n}-torsors and α_{p^n}-torsors in the smooth case in terms of De Rham-Witt differential forms, in a way similar to I.1.2.

Let X be a smooth scheme over a perfect field of characteristic p. Let us consider the natural continuous map of sites $f : X_{fl} \longrightarrow X_{et}$.

Let us consider the exact sequences in X_{fl}

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 1$$

$$0 \longrightarrow \alpha_{p^n} \longrightarrow \mathbb{G}_a \xrightarrow{p^n} \mathbb{G}_a \longrightarrow 0.$$

We obtain the following isomorphisms, mutatis mutandis, as in the proof of I.1.2

$$\begin{aligned} \mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n} &\simeq \mathrm{R}^1 f_*\mu_{p^n} \\ \mathcal{O}_X/(\mathcal{O}_X)^{p^n} &\simeq \mathrm{R}^1 f_*\alpha_{p^n} \end{aligned} \tag{I.8}$$

and

$$H^1(X, \mu_{p^n}) \xrightarrow{\delta_n} H^0(X_{et}, \mathrm{R}^1 f_*\mu_{p^n})$$

$$H^1(X, \alpha_{p^n}) \xrightarrow{\delta_n} H^0(X_{et}, \mathrm{R}^1 f_*\alpha_{p^n}).$$

We now have, by I.2.6 and I.2.7, in the étale topology, the following isomorphisms

$$\mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n} \xrightarrow[\simeq]{\mathrm{d}\,log_n} W_n\Omega^1_{X,log},$$

$$\mathcal{O}_X/(\mathcal{O}_X)^{p^n} \xrightarrow[\simeq]{d} BW_n\Omega^1_X,$$

which give the isomorphisms

$$\mathrm{d}\,Log_n := \mathrm{d}\,log_n \circ \delta_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{et}, W_n\Omega^1_{X,log}),$$

$$\mathrm{d}_n := d \circ \delta_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{et}, BW_n\Omega^1_X),$$

using the identifications of (I.8). We remark that $\mathrm{d}\,Log_n$ and d_n are defined even if X is not smooth, but in these cases they are not in general isomorphisms. So we have proved the following result.

THEOREM I.3.2. *Let X be a smooth scheme over a perfect field of characteristic p. Then, for any $n \geq 1$ the maps*

$$\mathrm{d}\,Log_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{et}, W_n\Omega^1_{X,log}),$$

$$\mathrm{d}_n : H^1(X, \alpha_{p^n}) \longrightarrow H^0(X_{et}, BW_n\Omega^1_X)$$

are isomorphisms.

Moreover $\mathrm{d}\,Log_n$ *(resp.* d*) is compatible with the natural restrictions map*

$$r_{n,m} : H^{i+1}(X, \mu_{p^n}) \longrightarrow H^{i+1}(X, \mu_{p^m})$$

and

$$r'_{n,m} : H^i(X_{et}, \Omega^1{}_{\log,n}) \longrightarrow H^i(X_{et}, \Omega^1{}_{\log,m})$$

with $n \geq m$ $\big($*resp.*

$$r_{n,m} : H^{i+1}(X, \alpha_{p^n}) \longrightarrow H^{i+1}(X, \alpha_{p^m})$$

and

$$r'_{n,m} : H^i(X_{et}, \Omega^1{}_{\log,n}) \longrightarrow H^i(X_{et}, \Omega^1{}_{\log,m})$$

with $n \geq m$ $\big)$. *In other words*

$$\mathrm{d}\,Log_m \circ r_{n,m} = r'_{n,m} \circ \mathrm{d}\,Log_n,$$

$$\textit{(resp.} \qquad \mathrm{d} \circ r_{n,m} = r'_{n,m} \circ \mathrm{d}\ ).$$

REMARK I.3.3. For $n = 1$ we obtain theorem I.1.2 (with the same proof).

PROOF. The statements about compatibilities are clear since all maps are functorial. □

We now give a more explicit description of $\mathrm{d}\,Log_n$. By Kummer theory we know that $H^1(X, \mu_{p^n})$ is the set (modulo isomorphism) of pairs (L, ψ) where L is an invertible sheaf on X and ψ is an isomorphism $\mathcal{O}_X \longrightarrow L^{\otimes n}$. This means that any torsor $Z \longrightarrow X$ is determined by an affine covering (for the Zariski topology) $(U_i = \mathrm{Spec}(A_i))_{i \in I}$ of X, a cocycle $\{f_{ij}\}$ in $H^1(X, \mathcal{O}_X^*)$ and $g_i \in H^0(U_i, \mathcal{O}_{U_i}^*)$ such that $f_{ij}^{p^n} = \frac{g_i}{g_j}$ for any i, j. So locally (for the Zariski topology) it is of type $Z_i = \mathrm{Spec}(A_i[z_i]/(z_i^{p^n} - g_i))$, with $z_i = f_{ij} z_j$. And $Z \longrightarrow X$ is trivial if and only if, up to refining the covering, there are $\{h_i\} \in H^0(U_i, \mathcal{O}_{U_i}^*)$ such that $f_{ij} = \frac{h_i}{h_j}$ and $a \in H^0(X, \mathcal{O}_X^*)$ such that $a_{|U_i}^{p^n} = \frac{h_i^{p^n}}{g_i}$.

This means that $Z \longrightarrow X$ is a trivial torsor for the flat topology if and only if any $Z_i \longrightarrow U_i$ is trivial, i.e. there exists $\{\gamma_i\}$ such that $\gamma_i^{p^n} = g_i$. The map which we have defined is

$$\begin{array}{ccccc} \mathrm{d}\,Log_n : H^1(X, \mu_{p^n}) & \xrightarrow{\delta_n} & H^0(X_{et}, \mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n}) & \xrightarrow{\mathrm{d}\,log_n} & H^0(X_{et}, W_n\Omega^1_{X,log}) \\ (\{f_{ij}\}, \{g_i\}) & \longmapsto & \{g_i\} & \longmapsto & \omega = \{\dfrac{\mathrm{d}\,[g_i]_{\leq n}}{[g_i]_{\leq n}}\} \end{array}$$

In a similar way it is possible to give explicitly d_n .

REMARK I.3.4. Since the $W_n\Omega^i_X$ are quasi-coherent sheaves over the scheme $W_n(X)$, with $\Gamma(U, W_n\Omega^i_X) = W_n\Omega^i_U$ for any U/X étale ([**27**, I 1.13, 1.14]), then by [**32**, III.3.7], we have $H^j(X_{et}, W_n\Omega^i_X) = H^j(X_{Zar}, W_n\Omega^i_X)$ for any i, j. In particular, since $W_n\Omega^1_{X,log}, BW_n\Omega^1_X \subseteq W_n\Omega^1_X$, then $H^0(X_{et}, W_n\Omega^1_{X,log}) = H^0(X_{Zar}, W_n\Omega^1_{X,log})$ and $H^0(X_{et}, BW_n\Omega^1_X) = H^0(X_{Zar}, BW_n\Omega^1_X)$.

As a corollary we reobtain a result of Illusie and Raynaud (cfr [**27**, 5.7.2] and [**27**, 5.8.3]).

Corollary I.3.5. *For any smooth scheme X over a perfect field k there is an isomorphism*

$$H^1(X, \mathbb{Z}_p(1)) := \varprojlim H^1(X, \mu_{p^n}) \simeq H^0(X_{Zar}, W\Omega^1_{X,log}),$$

where $W\Omega^1_{X,log} = \varprojlim W_n\Omega^1_{X,log}$.

Proof. $\mathrm{d}\,Log_n$ is compatible with the natural restrictions map $r_{n,m}$ and $r'_{n,m}$, defined in the above theorem, i.e.

$$\mathrm{d}\,Log_m \circ r_{n,m} = r'_{n,m} \circ \mathrm{d}\,Log_n \qquad \text{for } n \geq m.$$

Passing to inverse limits, the thesis is immediate. □

I.3.1. Normal case. In this section we want to generalize I.3.2 to μ_{p^n}-torsors over some particular normal schemes.

Proposition I.3.6. *Let X be a normal integral scheme. For any finite and flat commutative group scheme G over X,*

$$i^* : H^1(X, G) \longrightarrow H^1(\mathrm{Spec}(K(X)), G_{|K(X)})$$

is injective, where $i : \mathrm{Spec}(K(X)) \longrightarrow X$ is the generic point.

Proof. A sketch of the proof has been suggested to us by F. Andreatta. Consider a G-torsor $f : Y \longrightarrow X$ such that $i^*f : i^*Y \longrightarrow \mathrm{Spec}(K(X))$ is trivial. This means there exists a section s of i^*f. We consider the scheme Y_0 which is the closure of $s(\mathrm{Spec}(K(X)))$ in Y. Then $f_{|Y_0} : Y_0 \longrightarrow X$ is a finite birational morphism with X a normal integral scheme. So, by Zariski's Main Theorem ([**30**, 4.4.6]), we have that $f_{|Y_0}$ is an open immersion. On the other hand, since $f_{|Y_0}$ is finite, then it is proper. In particular it is closed. Hence it is an isomorphism. So we have a section of f and Y is a trivial G-torsor. □

Remark I.3.7. If G is not finite the proposition is not true. Take, for instance, a scheme X with $H^1(X, \mathcal{O}^*_X) \neq 0$. This one classifies $\mathbb{G}_m$-torsors over X. But $H^1(\mathrm{Spec}(K(X)), \mathcal{O}^*_{\mathrm{Spec}(K(X))}) = 0$.

Corollary I.3.8. *Let X be a normal integral scheme. Let $f : Y \longrightarrow X$ be a morphism with a rational section and let $g : G \longrightarrow G'$ be a map of finite and flat commutative group schemes over X, which is an isomorphism over* $\mathrm{Spec}(K(X))$. *Then*

$$f^*g_* : H^1(X, G) \longrightarrow H^1(Y, G'_Y)$$

is injective.

Proof. By hypothesis $\mathrm{Spec}(K(X)) \longrightarrow X$ factors through $f : Y \longrightarrow X$. If $i : \mathrm{Spec}(K(X)) \longrightarrow X$, we have

$$i_* : H^1(X, G) \longrightarrow H^1(X, G') \longrightarrow H^1(Y, G'_Y) \longrightarrow H^1(\mathrm{Spec}(K(X)), G_{K(X)}).$$

Therefore, by the previous proposition, it follows that

$$H^1(X, G) \longrightarrow H^1(Y, G'_Y)$$

is injective. $\square$

REMARK I.3.9. The previous corollary can be applied, for instance, to the case $f = \mathrm{id}_X$ or to the case $f : U \longrightarrow X$ an open immersion and $g = \mathrm{id}_G$. Roberts ([**38**, p. 692]) has proved the corollary in the case $f = \mathrm{id}_X$, with $X = \mathrm{Spec}(A)$ and A the integer ring of a local number field.

We now prove a result of purity for μ_n-torsors.

PROPOSITION I.3.10. *Let X be a separated, locally factorial and integral scheme. And let $i : U \subseteq X$ be an open such that $codim(X \setminus U) \geq 2$. The map*

$$i^* : H^1(X, \mu_n) \longrightarrow H^1(U, \mu_n)$$

is an isomorphism, for $n \geq 2$.

PROOF. The injectivity comes from the previous corollary. We now prove the surjectivity. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
H^0(X, \mathcal{O}_X^*) & \xrightarrow{n} & H^0(X, \mathcal{O}_X^*) & \longrightarrow & H^1(X, \mu_n) & \longrightarrow & H^1(X, \mathcal{O}_X^*) & \xrightarrow{n} & H^1(X, \mathcal{O}_X^*) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(U, \mathcal{O}_X^*) & \xrightarrow{n} & H^0(U, \mathcal{O}_X^*) & \longrightarrow & H^1(U, \mu_n) & \longrightarrow & H^1(U, \mathcal{O}_X^*) & \xrightarrow{n} & H^1(U, \mathcal{O}_X^*)
\end{array}$$

induced by exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1$$

in the flat site. Since X is normal and the complementary of U has codimension at least two, the first two vertical maps are isomorphisms. By [**24**, II.6.5] we have $Cl(X) \simeq Cl(U)$. But, since X is separated and locally factorial, then $Cl(X) \simeq Pic(X)$ and $Cl(U) \simeq Pic(U)$ and so the last two vertical maps are isomorphisms. By the five lemma we have the thesis. $\square$

REMARK I.3.11. The previous proposition is not true if the scheme X is only normal. The following example has been suggested to us by M. Roth. For instance we consider an ordinary elliptic curve $E \subseteq \mathbb{P}^2$ over an algebraically closed field k of characteristic p. It is projectively normal since it is an hypersurface. Then the projective cone X over E is normal. Let q be the singular point of X. By Kummer Theory we know there is a surjective map

$$H^1(X \setminus q, \mu_{p^n}) \longrightarrow \ker(p^n : Pic(X \setminus q) \longrightarrow Pic(X \setminus q)).$$

On the other hand $Pic(X \setminus q) = Cl(X \setminus q)$ since $X \setminus q$ is regular and it is well known that $Cl(X) = Cl(X \setminus q) = Cl(E) = Pic(E)$.

We call X' the affine cone over C. It is known that $Cl(X') = Cl(\mathcal{O}_{X,q})$. Moreover since $\mathcal{O}_{X,q}$ is local then $Pic(\mathcal{O}_{X,q}) = Pic(X') = 0$. So any Cartier divisor of X restricted to X' is trivial, which means that any Cartier divisor of X is contained in the hyperplane section at infinity C. So we have that $Pic(X)$

is generated by the hyperplane section at infinity. This means that $Pic(X) \simeq \mathbb{Z}$. Moreover we also know that since X is projective and k is algebraically closed

$$H^1(X, \mu_{p^n}) \simeq \ker(p^n : Pic(X) \longrightarrow Pic(X)).$$

So $H^1(X, \mu_{p^n}) = 0$.

Since E is ordinary, we can take a μ_{p^n}-torsor Y over $X \setminus q$ with an associated non-trivial line bundle D' of p^n-torsion. Therefore it is not possible to extend it to all X.

COROLLARY I.3.12. *Let X be a separated, locally factorial and integral scheme over a perfect field of characteristic p such that the set of regular points, $Reg(X)$, is open. Then we have the isomorphism*

$$\mathrm{d}\, Log_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{Zar}, W_n\Omega^1_{X,log}).$$

REMARK I.3.13. The condition on $Reg(X)$ is satisfied for instance by excellent schemes, e.g. an algebraic variety over a field or $\mathrm{Spec}(A)$ with A a complete and local ring.

PROOF. Since X is normal then $codim(X \setminus Reg(X)) \geq 2$ ([**30**, 4.2.24]). We set $U = Reg(X)$. By above proposition $H^1(X, \mu_{p^n}) \simeq H^1(U, \mu_{p^n})$. Then we have the following commutative diagram

$$\begin{array}{ccc} H^1(X, \mu_{p^n}) & \xrightarrow{\mathrm{d}\, Log_n} & H^0(X_{Zar}, W_n\Omega^1_{X,log}) \\ \downarrow \simeq & & \downarrow \\ H^1(U, \mu_{p^n}) & \xrightarrow{(\mathrm{d}\, Log_n)_U} & H^0(U_{Zar}, W_n\Omega^1_{X,log}) \end{array}$$

The map $(\mathrm{d}\, Log_n)_U$ is an isomorphism and the second vertical map is injective since U is an open dense of X. An easy verification shows that $\mathrm{d}\, Log_n$ is an isomorphism. $\square$

I.3.2. Non-normal case. We now want to explain why it is not possible to have a similar statement in general for non-normal schemes. The main reason is that I.3.6 is not true in general.

EXAMPLE I.3.14. Consider $X = \mathrm{Spec}(k[x, y]/(x^p - y^{p+1})) = \mathrm{Spec}(A)$ and Y the α_p-torsor $\mathrm{Spec}(A[T]/(T^p - y))$. Generically this torsor is trivial since we have $y = (\frac{x}{y})^p$. But Y is not trivial since y is not a p-power in A.

On the other hand we know that any (also generalized) differential form which is trivial on an open dense is trivial over the whole scheme. This is so an obstruction to classify α_{p^n}-torsor and μ_{p^n}-torsor as differential forms as we did in the smooth case.

I.4. Differential forms and μ_{p^n}-torsors in unequal characteristic

By now, R will be a d.v.r. of unequal characteristic. While K will indicate its fraction field and k its residue field. We now want to classify μ_{p^n}-torsors by opportune differential forms. We remark that $H^1(X, \mu_{p^n})$ is a $\mathbb{Z}/p^n\mathbb{Z}$-module. In characteristic p we have seen that there are natural sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules. In mixed characteristic we will define opportune sheaves. For any faithfully flat morphism of schemes $X \longrightarrow \mathrm{Spec}(R)$, let us consider the map of sheaves of $\mathbb{Z}$-modules

$$\mathrm{dlog} : \mathcal{O}_X^* \longrightarrow \Omega^1_{X/R}$$

defined by

$$f \longmapsto \frac{\mathrm{d}\, f}{f}.$$

We call $\Omega^1_{log,X}$ the image sheaf.

Definition I.4.1. For any integer $n \geq 0$ we will call $\Omega^1_{X/R,log,n}$ the sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules $\Omega^1_{X/R,log}/p^n\Omega^1_{X/R,log}$. When there will be no possibility of confusion we will write $\Omega^1_{log,n}$.

One of the main ingredients in characteristic p was I.2.6. We now will prove a similar statement over a d.v.r. with fraction field of characteristic zero.

Proposition I.4.2. *For any faithfully flat morphism $g : X \longrightarrow R$ with geometrically integral generic fiber the following sequence*

$$1 \longrightarrow g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}_X^*)^{p^n} \longrightarrow \mathcal{O}_X^* \xrightarrow{\mathrm{d}\,log_n} \Omega^1_{X/R,log,n} \longrightarrow 0$$

is exact.

Remark I.4.3. Let $g : X \longrightarrow Y$ be a morphism with Y a Dedekind scheme and the generic fiber X_η integral. Then it is faithfully flat if and only if X is integral and g is surjective. This follows from [**30**, 4.3.8] and [**30**, 4.3.10].

Proof. First, we prove the following result.

Proposition I.4.4. *Under the hypothesis of the above proposition*

$$1 \longrightarrow g^{-1}\mathcal{O}_{\mathrm{Spec}(R)} \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_{X/R}$$

is exact.

Proof. Since $char(K) = 0$ then any subextension $L \subseteq K(X)$ is separable. Since $K(X)/L$ is separable, it is formally smooth ([**22**, 19.6.1]) and it follows that there is an exact sequence

$$0 \to \Omega^1_{L/K} \otimes_L K(X) \to \Omega^1_{K(X)/K} \to \Omega^1_{K(X)/L} \to 0 \tag{I.9}$$

(see [**22**, 20.5.7]).

Now let x be a point of X. It is known that $(g^{-1}\mathcal{O}_{\mathrm{Spec}(R)})_x = \mathcal{O}_{\mathrm{Spec}(R),g(x)}$. Since g is faithfully flat then $\mathcal{O}_{\mathrm{Spec}(R),g(x)} \longrightarrow \mathcal{O}_{X,x}$ is injective. Moreover, by definition of relative differentials, it follows that

$$\mathcal{O}_{\mathrm{Spec}(R),g(x)} \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\mathrm{d}} (\Omega^1_{X/R})_x$$

is the zero morphism. We now prove that $\ker \mathrm{d} = \mathcal{O}_{\mathrm{Spec}(R),g(x)}$. We denote by B the local ring $\mathcal{O}_{\mathrm{Spec}(R),g(x)}$. We remark that $B = K$ or R. Let $f \in \mathcal{O}_{X,x}$ be such that $\mathrm{d}f = 0 \in (\Omega^1_{X/R})_x = \Omega^1_{\mathcal{O}_{X,x}}/B$. Then $\mathrm{d}f = 0 \in \Omega^1_{K(X)/K}$. We consider $L = K(f) \subseteq K(X)$. So by (I.9) we have $\mathrm{d}f = 0$ in $\Omega^1_{K(f)/K}$. This means

$$\Omega^1_{K(f)/K} = 0.$$

In particular $K(f)/K$ is a finite extension, thus

$$f \in K(X) \cap \bar{K}.$$

But since X has geometrically integral generic fiber, then

$$K(X) \cap \bar{K} = K$$

by [**23**, 4.5.9]. So $f \in K \cap \mathcal{O}_{X,x}$. If x is in the generic fibre then $B = K$ so $f \in B$ and we are done. If x is in the special fibre then $B = R$ and $K \cap \mathcal{O}_{X,x}$ is a proper sub-R-algebra of K, thus of the form $\frac{1}{\pi^s}R$ for some $s \geq 0$. If $f \in R = B$ we are done, otherwise there exists $n \geq 0$ such that $\pi^n f \in R^*$. Since $B \longrightarrow \mathcal{O}_{X,x}$ is faithfully flat it follows, by [**30**, 1.2.17], that $\pi\mathcal{O}_{X,x} \neq \mathcal{O}_{X,x}$. In particular $\pi \notin \mathcal{O}^*_{X,x}$ and so $n = 0$, which implies $f \in B$. □

We now can prove the exactness of the sequence of the statement. Clearly $g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}^*_X)^{p^n} \longrightarrow \mathcal{O}^*_X$ is injective and $\mathrm{dlog}(g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}^*_X)^{p^n}) = 0$.

Now let x be a point of X. Then $(g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}\mathcal{O}^{*p^n}_X)_x = \mathcal{O}^*_{\mathrm{Spec}(R),g(x)}(\mathcal{O}_{X,x})^{*p^n}$. If $g(x) = \mathrm{Spec}(B)$ as above then $\mathrm{dlog}\, f = 0 \in (\Omega^1_{X/R,log,n})_x$ means that $\mathrm{dlog}(f) = \mathrm{dlog}\, h^{p^n}$ for some $h \in \mathcal{O}^*_{X,x}$, i.e.

$$d\left(\frac{f}{h^{p^n}}\right) = 0.$$

This means, by (I.4.4), $f = ah^{p^n}$, with $a \in B$. This implies

$$\ker \mathrm{dlog} = g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}^*_X)^{p^n}.$$

The surjectivity of $\mathrm{d}\,log_n$ it follows from the definition of $\Omega^1_{X/R,log,n}$. □

Proposition I.4.5. *Let $Y \longrightarrow \mathrm{Spec}(R)$ be a morphism with geometrically integral fibres and G a finite flat R group scheme. If we denote by $f : Y_{fl} \longrightarrow Y_{Zar}$ the natural continuous map of sites, then*

$$\delta : H^1(Y, G) \longrightarrow H^0(Y_{Zar}, R^1 f_* G),$$

induced by the Leray spectral sequence

$$H^i(Y_{Zar}, \mathrm{R}^j f_* G) \Rightarrow H^{i+j}(Y, G),$$

is an isomorphism.

Proof. Since f_*G is G restricted to the Zariski site, the Leray spectral sequence above gives

$$0 \longrightarrow H^1(Y_{Zar}, G) \longrightarrow H^1(Y, G) \xrightarrow{\delta} H^0(Y_{Zar}, \mathrm{R}^1 f_* G) \longrightarrow H^2(Y_{Zar}, G) \tag{I.10}$$

in low degrees. We now prove that

$$H^i(Y_{Zar}, G) = 0 \tag{I.11}$$

for $i > 0$. Indeed we will prove that G is a constant sheaf over Y_{Zar}. In particular it is flasque since Y is an irreducible topological space. Then, by [**24**, III.2.5], we can conclude $H^i(Y_{Zar}, G) = 0$ for $i > 0$.

Let us fix a presentation $R[G] = R[T_1, \dots, T_n]/(P_1, \dots, P_m)$. An element of $H^0(U_{Zar}, G)$ is an R-algebra homomorphism $R[G] \to H^0(U, \mathcal{O}_U) \subset K(Y)$. This is given by the images $x_1, \dots, x_n$ of the generators of $R[G]$, satisfying $P_i(x_1, \dots, x_n) = 0$ for $1 \le i \le m$. Since $R[G]$ is finite over R, then $R[x_1, \dots, x_n]$ is finite also so the elements x_j are integral. Since $K(Y) \cap \bar{K} = K$ we get $x_j \in K$, and since R is integrally closed in fact $x_j \in R$. Finally

$$H^0(U_{Zar}, G) = \{(x_1, \dots, x_n) \in R^a \text{ s. t. } P_i(x_1, \dots, x_n) = 0 \text{ for } 1 \le i \le m\} = G(R)\ ,$$

and this does not depend on U. □

With abuse of notation, we will denote by f both the natural continuous map of sites $X_{fl} \longrightarrow X_{Zar}$ and $\mathrm{Spec}(R)_{fl} \longrightarrow \mathrm{Spec}(R)_{Zar}$. It will be clear by the context which we are considering. Since $R^1 f_* \mathbb{G}_{m,X} = R^1 f_* \mathbb{G}_{m,R} = 0$, (see the proof of [**32**, III 4.9]), reasoning as in the proof of I.3.2 we can conclude that

$$\begin{aligned} &\mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n} \simeq \mathrm{R}^1 f_* \mu_{p^n,X}, \\ &\mathcal{O}_{\mathrm{Spec}(R)}^*/(\mathcal{O}_{\mathrm{Spec}(R)}^*)^{p^n} \simeq \mathrm{R}^1 f_* \mu_{p^n,\mathrm{Spec}(R)}. \end{aligned} \tag{I.12}$$

We define, using I.4.5 and this identification,

$$\mathrm{d}\, Log_n := \mathrm{d}\, log_n \circ \delta_n : H^1(X, \mu_{p^n}) \longrightarrow H^0(X_{Zar}, \Omega^1{}_{\log,n}).$$

THEOREM I.4.6. *Let $g : X \longrightarrow \mathrm{Spec}(R)$ be a faithfully flat morphism with geometrically integral generic fiber. Then, for any $n \ge 1$, there is an isomorphism*

$$\mathrm{d}\, Log_n : H^1(X, \mu_{p^n})/H^1(\mathrm{Spec}(R), \mu_{p^n}) \longrightarrow H^0(X_{Zar}, \Omega^1{}_{\log,n}).$$

Moreover $\mathrm{d}\, Log_n$ is compatible with the restriction maps $r_{n,m} : H^1(X, \mu_{p^n}) \longrightarrow H^1(X, \mu_{p^m})$ and $r'_{n,m} : H^0(X_{Zar}, \Omega^1{}_{\log,n}) \longrightarrow H^0(X_{Zar}, \Omega^1{}_{\log,m})$ with $n \ge m$, i.e.

$$\mathrm{d}\, Log_m \circ r_{n,m} = r'_{n,m} \circ \mathrm{d}\, Log_n.$$

PROOF. The map g induces a morphism of sheaves on X

$$g^{-1}(\mathrm{R}^1 f_* \mu_{p^n,\mathrm{Spec}(R)}) \longrightarrow \mathrm{R}^1 f_* \mu_{p^n,X},$$

which is the natural map

$$g^{-1}(\mathcal{O}_{\mathrm{Spec}(R)}^*/(\mathcal{O}_{\mathrm{Spec}(R)}^*)^{p^n}) \longrightarrow \mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n}$$

under the identifications of (I.12). We prove it is injective. Let x be a point of X. Let $g(x) = \mathrm{Spec}(B)$ with $B = R$ or $B = K$. We have to prove that

$$B^*/(B^*)^{p^n} \longrightarrow \mathcal{O}_{X,x}^*/(\mathcal{O}_{X,x}^*)^{p^n}$$

is injective. Suppose $z \in B^* \cap \mathcal{O}_{X,x}^{*p^n}$. Then $z = y^{p^n}$ for some $y \in \mathcal{O}_{X,x}^*$. Then $y \in \bar{K} \cap K(X)$. But, since $X \longrightarrow R$ has geometrically integral generic fiber, then, by [**23**, 4.5.9], we have $\bar{K} \cap K(X) = K$. So $y \in K$ and $y^{p^n} \in B^*$. Therefore $y \in B^*$.

We now have in X_{Zar}, by I.4.2, an exact sequence

$$1 \longrightarrow g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}^*_X)^{p^n}/(\mathcal{O}^*_X)^{p^n} \longrightarrow \mathcal{O}^*_X/(\mathcal{O}^*_X)^{p^n} \xrightarrow{\mathrm{dlog}_n} \Omega^1{}_{\log,n} \longrightarrow 0. \tag{I.13}$$

So we have

$$\mathrm{R}^1 f_*\mu_{p^n,X}/\mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)} \simeq \mathcal{O}^*_X/g^{-1}\mathcal{O}^*_{\mathrm{Spec}(R)}(\mathcal{O}^*_X)^{p^n} \overset{\mathrm{dlog}_n}{\underset{\simeq}{\longrightarrow}} \Omega^1{}_{\log,n},$$

which gives the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathrm{Spec}(R)_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)}) \longrightarrow H^0(X_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,X}) \xrightarrow{\mathrm{dlog}_n} \\ \longrightarrow H^0(X_{Zar}, \Omega^1{}_{\log,n}) \longrightarrow H^1(\mathrm{Spec}(R)_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)}). \end{aligned} \tag{I.14}$$

We claim that

$$H^1(\mathrm{Spec}(R)_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)}) = 0. \tag{I.15}$$

Therefore, using I.4.5 and (I.14),

$$0 \longrightarrow H^1(\mathrm{Spec}(R), \mu_{p^n}) \longrightarrow H^1(X, \mu_{p^n}) \xrightarrow{\mathrm{dlog}_n} H^0(X_{Zar}, \Omega^1{}_{\log,n}) \longrightarrow 0. \tag{I.16}$$

We have so proved that $dLog_n$ is an isomorphism. We now prove (I.15). By

$$1 \longrightarrow (\mathcal{O}^*_{\mathrm{Spec}(R)})^{p^n} \longrightarrow \mathcal{O}^*_{\mathrm{Spec}(R)} \longrightarrow \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)} \longrightarrow 0, \tag{I.17}$$

it follows

$$\begin{aligned} \longrightarrow H^1(\mathrm{Spec}(R)_{Zar}, (\mathcal{O}^*_{\mathrm{Spec}(R)})^{p^n}) \longrightarrow H^1(\mathrm{Spec}(R)_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)}) \longrightarrow \\ \longrightarrow H^2(\mathrm{Spec}(R)_{Zar}, (\mathcal{O}^*_{\mathrm{Spec}(R)})^{p^n}) \end{aligned}$$

Considering the exact sequence in X_{Zar}

$$1 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{O}^*_X \longrightarrow (\mathcal{O}^*_X)^{p^n} \longrightarrow 1,$$

and taking the long associated cohomology sequence, we obtain

$$H^i(\mathrm{Spec}(R)_{Zar}, \mathcal{O}^*_X) = H^i(\mathrm{Spec}(R)_{Zar}, (\mathcal{O}^*_X)^{p^n}) \tag{I.18}$$

for $i > 0$. But, since R is a d.v.r. then $Pic(\mathrm{Spec}(R)) = 0$. So, using the next lemma,

$$H^i(\mathrm{Spec}(R)_{Zar}, (\mathcal{O}^*_{\mathrm{Spec}(R)})^{p^n}) = H^i(\mathrm{Spec}(R)_{Zar}, \mathcal{O}^*_{\mathrm{Spec}(R)}) = 0$$

for $i = 1, 2$. Then

$$H^1(\mathrm{Spec}(R)_{Zar}, \mathrm{R}^1 f_*\mu_{p^n,\mathrm{Spec}(R)}) = 0.$$

LEMMA I.4.7. *We have that*

$$H^2(\mathrm{Spec}(R)_{Zar}, \mathbb{G}_m) = 0.$$

PROOF. Any group of cohomology is computed here in the Zariski topology. Let $\mathcal{F}$ be a sheaf (for the Zariski topology) on a scheme X. We define, for any $n \in \mathbb{N}$, the sheaf $\underline{H}^n(\mathcal{F})$ as the sheaf associated to the presheaf

$$U \mapsto H^n(U, \mathcal{F}_{|U}).$$

Moreover we denote by $\check{H}^n(X, \mathcal{F})$ the n^{th}-group of Čech cohomology. The following spectral sequence (see [**32**, III 2.7])

$$\check{H}^i(X, \underline{H}^j(\mathcal{F})) \Rightarrow H^n(X, \mathcal{F})$$

induces the exact sequence (since $\check{H}^0(X, \underline{H}^j(\mathcal{F})) = 0$ for any $j > 0$ by [**32**, III 2.9])

$$0 \longrightarrow \check{H}^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \check{H}^1(X, \underline{H}^1(\mathcal{F})) \longrightarrow \check{H}^3(X, \mathcal{F}).$$

Suppose now $X = \mathrm{Spec}(R)$ and $\mathcal{F} = \mathbb{G}_m$. We remark that, since

$$H^1((\mathrm{Spec}(R), \mathbb{G}_m) = H^1(\mathrm{Spec}(K), \mathbb{G}_m) = 0,$$

$\underline{H}^1(\mathbb{G}_m)$ is the zero sheaf, which implies $\check{H}^1(\mathrm{Spec}(R), \underline{H}^1(\mathbb{G}_m)) = 0$. Hence

$$\check{H}^2(\mathrm{Spec}(R), \mathbb{G}_m) \longrightarrow H^2(\mathrm{Spec}(R), \mathbb{G}_m)$$

is an isomorphism. But, since we are working in Zariski topology, Čech cohomology can be computed using alternating cochains (see [**24**, III.4]). But since in $\mathrm{Spec}(R)$ there are only two open sets, then, for any $i > 1$, $\check{H}^i(X, \mathbb{G}_m) = 0$, too. Hence $H^2(\mathrm{Spec}(R), \mathbb{G}_m) = 0$. □

The explicit description of $\mathrm{d}\,Log_n$ is as follows. By Kummer theory we know that $H^1(X, \mu_{p^n})$ is the set (modulo isomorphism) of pairs (L, ψ) where L is an invertible sheaf on X and ψ is an isomorphism $\mathcal{O}_X \longrightarrow L^{\otimes n}$. This means that any torsor $Z \longrightarrow X$ is determined by an affine covering (for the Zariski topology) $(U_i = \mathrm{Spec}(A_i))_{i \in I}$ of X, a cocycle $\{f_{ij}\}$ in $H^1(X, \mathcal{O}_X^*)$ and $g_i \in H^0(U_i, \mathcal{O}_{U_i}^*)$, such that $f_{ij}^{p^n} = \frac{g_i}{g_j}$ for any i, j. So locally (for the Zariski topology) it is of type $Z_i = \mathrm{Spec}(A_i[z_i]/(z_i^{p^n} - g_i))$, with $z_i = f_{ij} z_j$. And $Z \longrightarrow X$ is trivial if and only if, refining the covering if necessary, there are $\{h_i\} \in H^0(U_i, \mathcal{O}_{U_i}^*)$ such that $f_{ij} = \frac{h_i}{h_j}$ and $a \in H^0(X, \mathcal{O}_X^*)$ such that $a_{|U_i}^{p^n} = \frac{g_i}{h_i^{p^n}}$.

This means that $Z \longrightarrow X$ is a trivial torsor for the flat topology if and only if any $Z_i \longrightarrow U_i$ is trivial, i.e. there exists $\{\gamma_i\}$ such that $\gamma_i^{p^n} = g_i$. The map which we have defined is

$$\begin{array}{lclcl} \mathrm{d}\,Log_n : H^1(X, \mu_{p^n}) & \xrightarrow{\delta_n} & H^0(X_{Zar}, \mathcal{O}_X^*/(\mathcal{O}_X^*)^{p^n}) & \xrightarrow{\mathrm{d}\,log_n} & H^0(X_{Zar}, {\Omega^1}_{\log^n}) \\ (\{f_{ij}\}, \{g_i\}) & \longmapsto & \{g_i\} & \longmapsto & \omega = \{\dfrac{\mathrm{d}\,g_i}{g_i}\} \end{array}$$

The compatibility condition is clear by construction.

□

PROPOSITION I.4.8. *If, in addition to the hypothesis of the theorem, X_k is a locally factorial and separated scheme such that the set of regular points, $Reg(X)$, is open, then the map* $\mathrm{d}\,Log_n$ *is compatible with the restriction. This means that there exists a map* $res'_n : H^0(X_{Zar}, \Omega^1_{log,n}) \longrightarrow H^0(X_{k,Zar}, W_n\Omega^1_{X_k,log})$ *such that*

$$\begin{array}{ccc} H^1(X,\mu_{p^n})/H^1(R,\mu_{p^n}) & \xrightarrow{\mathrm{d}\,Log_n} & H^0(X_{Zar},\Omega^1_{log,n}) \\ \downarrow i_n^* & & \downarrow res'_n \\ H^1(X_k,\mu_{p^n}) & \xrightarrow{\mathrm{d}\,Log_n} & H^0(X_{k,Zar}, W_n\Omega^1_{X_k,log}) \end{array}$$

commutes (i^* is the pull-back of $i : X_k \longrightarrow X$).

PROOF. By hypothesis in the special fiber we have I.3.12. For the functoriality of (I.10) we have that δ_n commutes with i^*. We now define

$$res'_n : H^0(X_{Zar}, \Omega^1_{log,n}) \longrightarrow H^0(X_{Zar}, (W_n\Omega^1_{X_k,log})$$
$$\{\frac{\mathrm{d}\, g_i}{g_i}\} \longmapsto \{\frac{\mathrm{d}\,[\bar{g}_i]}{[\bar{g}_i]}\},$$

which clearly commutes with $\mathrm{d}\,log_n$. □

I.5. Differential forms and $G_{\lambda,n}$-torsors

For any $\lambda \in R$ define the group scheme

$$\mathcal{G}^{(\lambda)} = \mathrm{Spec}(R[T, \frac{1}{1+\lambda T}])$$

The R-group scheme structure is given by

$$T \longrightarrow 1\otimes T + T\otimes 1 + \lambda T\otimes T \qquad \text{comultiplication}$$
$$T \longrightarrow 0 \qquad \text{counit}$$
$$T \longrightarrow -\frac{T}{1+\lambda T} \qquad \text{coinverse}$$

We observe that if $\lambda = 0$ then $\mathcal{G}^{(\lambda)} \simeq \mathbb{G}_a$. It is possible to prove that $\mathcal{G}^{(\lambda)} \simeq \mathcal{G}^{(\mu)}$ if and only if $v(\lambda) = v(\mu)$ and the isomorphism is given by $T \longrightarrow \frac{\lambda}{\mu}T$. Moreover it is easy to see that, if $\lambda \in \pi R \setminus \{0\}$, then $\mathcal{G}^{(\lambda)}_k \simeq \mathbb{G}_a$ and $\mathcal{G}^{(\lambda)}_K \simeq \mathbb{G}_m$. It has been proved by Waterhouse and Weisfeiler, in [**59**, 2.5], that any deformation, as a group scheme, of $\mathbb{G}_a$ to $\mathbb{G}_m$ is isomorphic to $\mathcal{G}^{(\lambda)}$ for some $\lambda \in \pi R \setminus \{0\}$. If $\lambda \in R \setminus \{0\}$ we can define the morphism

$$\alpha^\lambda : \mathcal{G}^{(\lambda)} \longrightarrow \mathbb{G}_m$$

given, on the level of Hopf algebras, by $x \longmapsto 1 + \lambda x$: it is an isomorphism on the generic fiber. If $v(\lambda) = 0$ then α^λ is an isomorphism.

We now define some finite and flat group schemes of order p^n. Let $\lambda \in R$ satisfy the condition

$$(*) \qquad v(p) \geq p^{n-1}(p-1)v(\lambda).$$

Then the map

$$\begin{aligned} \psi_{\lambda,n} : & \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{G}^{(\lambda^{p^n})} \\ & T \longrightarrow P_{\lambda,n}(T) := \frac{(1+\lambda T)^{p^n}-1}{\lambda^{p^n}} \end{aligned}$$

is an isogeny of degree p^n. Let

$$G_{\lambda,n} := \mathrm{Spec}(R[T]/P_{\lambda,n}(T))$$

be its kernel. It is a commutative finite flat group scheme over R of rank p^n. It is possible to prove that

$$\begin{aligned} (G_{\lambda,n})_k &\simeq \mu_{p^n} && \text{if } v(\lambda)=0 \\ (G_{\lambda,n})_k &\simeq \alpha_{p^n} && \text{if } p^{n-1}(p-1)v(\lambda) < v(p); \\ (G_{\lambda,n})_k &\simeq \alpha_{p^{n-1}} \times \mathbb{Z}/p\mathbb{Z} && \text{if } p^{n-1}(p-1)v(\lambda) = v(p). \end{aligned}$$

We observe that α^λ is compatible with $\psi_{\lambda,n}$, i.e the following diagram is commutative

(I.19)
$$\begin{array}{ccc} \mathcal{G}^{(\lambda)} & \xrightarrow{\alpha^\lambda} & \mathbb{G}_m \\ {\scriptstyle \psi_{\lambda,n}}\downarrow & & \downarrow{\scriptstyle p^n} \\ \mathcal{G}^{(\lambda^{p^n})} & \xrightarrow{\alpha^{\lambda^{p^n}}} & \mathbb{G}_m \end{array}$$

Then it induces a map

$$\alpha^{\lambda,n} : G_{\lambda,n} \longrightarrow \mu_{p^n}$$

which is an isomorphism on the generic fiber. And if $v(\lambda)=0$ then $\alpha^{\lambda,n}$ is an isomorphism.

We remark that

$$\mathrm{Hom}(G_{\lambda,n}, G_{\lambda',n}) = \begin{cases} 0, & \text{if } v(\lambda) < v(\lambda'); \\ \mathbb{Z}/p^n\mathbb{Z}, & \text{otherwise.} \end{cases}$$

If $v(\lambda) \geq v(\lambda')$ the morphisms are given by

$$\begin{aligned} & G_{\lambda,n} \longrightarrow G_{\lambda',n} \\ & T \longmapsto \frac{(1+\lambda T)^i - 1}{\lambda'} \end{aligned}$$

for $i=0,\ldots,p^n-1$. It follows easily that $G_{\lambda,n} \simeq G_{\lambda',n}$ if and only if $v(\lambda)=v(\lambda')$.

In the following any time we will speak about $G_{\lambda,n}$ it will be assumed that λ satisfies $(*)$. If R contains a primitive p^n-th root of unity ζ_n then, since

$$v(p) = p^{n-1}(p-1)v(\lambda_{(n)}),$$

the condition $(*)$ is equivalent to $v(\lambda) \leq v(\lambda_{(n)})$.

Remark I.5.1. We report here an useful remark taken from [**5**]. Let R be a complete local Λ_p-algebra with $\Lambda_p = \mathbb{Z}[\zeta_{p-1}, \frac{1}{p(p-1)}] \cap \mathbb{Z}_p \subseteq \mathbb{Q}_p$. In [**33**] it has been proved that there exists a 1-1 correspondence between isomorphism classes of finite and flat group schemes of order p and isomorphism classes of pairs $(a,c) \in R^2$

such that $ac = p$. Two pairs (a,c) and (a',c') are said isomorphic if there exist $u \in R^*$ such that $a' = u^{p-1}a$ and $c' = u^{p-1}c$. Denote by $G_{(a,c)}$ the group scheme associated to a pair (a,c), as above. As an R-scheme, it is given by $G_{(a,c)} = \mathrm{Spec}(R[T]/(T^p - aT))$. Under the further assumption that cw_{p-1} admits a $p-1$-th root β in R we have the isomorphism $G_{(a,c)} \longrightarrow G_{\frac{\beta}{1-p},1}$ defined at the level of the underlying Hopf algebra by $x \to \sum_{i=1}^{p-1} \beta^{i-1} \frac{y^i}{w^i}$, where $w_1, \dots, w_{p-1}$ are the universal constants defined in [**33**].

I.5.1. A filtration of $H^1(X, \mu_{p^n})$. Let X be a faithfully flat R-scheme. Let us consider the exact sequence on the étale site X_{et}

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\alpha^\lambda} \mathbb{G}_m \longrightarrow i_*\mathbb{G}_m \longrightarrow 0,$$

where i denotes the closed immersion $X_\lambda = X \otimes_R (R/\lambda R) \hookrightarrow X$ (see [**51**, 1.2]. The associated long exact sequence is the following

$$\begin{aligned} \text{(I.20)} \quad 0 \longrightarrow H^0(X, \mathcal{G}^{(\lambda)}) \longrightarrow H^0(X, \mathbb{G}_m) \longrightarrow H^0(X_\lambda, \mathbb{G}_m) \longrightarrow \\ \longrightarrow H^1(X, \mathcal{G}^{(\lambda)}) \longrightarrow H^1(X, \mathbb{G}_m) \longrightarrow H^1(X_\lambda, \mathbb{G}_m). \end{aligned}$$

Now by the exact sequence of group schemes

$$0 \longrightarrow G_{\lambda,n} \xrightarrow{i} \mathcal{G}^{(\lambda)} \xrightarrow{\psi_{\lambda,n}} \mathcal{G}^{(\lambda^{p^n})} \longrightarrow 0$$

we have the exact sequence of groups

(I.21)

$$H^0(X, \mathcal{G}^{(\lambda)}) \xrightarrow{(\psi_{\lambda,n})_*} H^0(X, \mathcal{G}^{(\lambda^{p^n})}) \longrightarrow H^1(X, G_{\lambda,n}) \xrightarrow{i_*} H^1(X, \mathcal{G}^{(\lambda)}) \longrightarrow H^1(X, \mathcal{G}^{(\lambda^{p^n})}).$$

DEFINITION I.5.2. We define for any $n \geq 1$

$$H^1(X, G_{\lambda,n})^{loc} := \ker(H^1(X, G_{\lambda,n}) \xrightarrow{i_*} H^1(X, \mathcal{G}^{(\lambda)})).$$

LEMMA I.5.3. *Let X be a flat R-scheme such that $H^0(X, \mathcal{O}_X)^* \longrightarrow H^0(X_\lambda, \mathcal{O}_{X_\lambda})^*$ is surjective and $Pic(X) = 0$. Then*

$$H^1(X, \mathcal{G}^{(\lambda)}) = 0$$

and

$$H^1(X, G_{\lambda,n}) = H^1(X, G_{\lambda,n})^{loc}$$

REMARK I.5.4. This result will be applied to the case X a local scheme or $X = \mathrm{Spec}(A)$ with A factorial π-adically complete R-algebra. For X local see the proposition I.5.5 below. If A is factorial we recall that, by [**24**, 6.2, 6.11], it follows that A factorial implies X normal and $Pic(X) = 0$. Moreover it is easy to see that the restriction map $H^0(X, \mathcal{O}_X)^* \longrightarrow H^0(X_\lambda, \mathcal{O}_{X_\lambda})^*$ is surjective. We also observe that if $v(\lambda) = 0$, which corresponds to the case $G_{\lambda,n} \simeq \mu_{p^n}$, then $H^0(X, \mathcal{O}_X)^* \longrightarrow H^0(X_\lambda, \mathcal{O}_{X_\lambda})^*$ is always surjective, since $X_\lambda = \emptyset$.

PROOF. Since $H^1(X,\mathbb{G}_m) = Pic(X) = 0$, it follows by (I.20) and by the fact that $H^0(X,\mathcal{O}_X)^* \longrightarrow H^0(X_\lambda,\mathcal{O}_{X_\lambda})^*$ is surjective that $H^1(X,\mathcal{G}^{(\lambda)}) = 0$. Then by (I.21) it follows that

$$H^1(X,G_{\lambda,n}) = H^1(X,G_{\lambda,n})^{loc}.$$

□

From this lemma follows the following result which says that Hilbert's Theorem 90, as stated in [**32**, III 4.9], is true for any $\mathcal{G}^{(\lambda)}$.

PROPOSITION I.5.5. *Let X be a faithfully flat R-scheme. Let $f : X_{fl} \longrightarrow X_{Zar}$ be the natural continuous morphism of sites. Then, if $\lambda \neq 0$, $R^1f_*(\mathcal{G}^{(\lambda)}) = 0$. In particular $H^1(X_{fl},\mathcal{G}^{(\lambda)}) = H^1(X_{Zar},\mathcal{G}^{(\lambda)})$.*

PROOF. It is sufficient to prove that $H^1(\mathrm{Spec}(A),\mathcal{G}^{(\lambda)}) = 0$ for any local ring A flat over R. This has been proved in [**51**, 1.3] with the same proof that we give here. This comes from the above lemma, just noting that, since A is local, then $H^0(\mathrm{Spec}(A),\mathbb{G}_m) \longrightarrow H^0(\mathrm{Spec}(A/\lambda A),\mathbb{G}_m)$ is surjective and $\mathrm{Pic}(\mathrm{Spec}(A)) = 0$ (see [**32**, III 4.9]). Now, since $R^1f_*(\mathcal{G}^{(\lambda)}) = 0$ it follows, by the Leray spectral sequence, that

$$H^1(X_{fl},\mathcal{G}^{(\lambda)}) = H^1(X_{Zar},\mathcal{G}^{(\lambda)}).$$

□

If we have a morphism $Y \longrightarrow X$ then we have a commutative diagram

$$\begin{array}{ccc} H^1(X,G_{\lambda,n}) & \longrightarrow & H^1(Y,G_{\lambda,n}) \\ \downarrow & & \downarrow \\ H^1(X,\mathcal{G}^{(\lambda)}) & \longrightarrow & H^1(Y,\mathcal{G}^{(\lambda)}) \end{array}.$$

This induces a morphism

$$H^1(X,G_{\lambda,n})^{loc} \longrightarrow H^1(Y,G_{\lambda,n})^{loc}. \qquad \text{(I.22)}$$

Using I.4.5 and the exact sequence

$$0 \longrightarrow G_{\lambda,n} \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{G}^{(\lambda^{p^n})} \longrightarrow 0$$

it is possible to prove, similarly to (I.16), that $H^1(\mathrm{Spec}(R),G_{\lambda,n}) \longrightarrow H^1(X,G_{\lambda,n})$ is injective. So by (I.22) and I.5.3, applied to $\mathrm{Spec}(R)$, it follows that we have a little more, i.e.

$$H^1(\mathrm{Spec}(R),G_{\lambda,n}) \hookrightarrow H^1(X,G_{\lambda,n})^{loc}.$$

We now construct a filtration of $H^1(X,\mu_{p^n})$. If X is proper over R and $n = 1$, it coincides with that of [**41**, 5.2] and [**4**].

PROPOSITION I.5.6. *Let X be a normal integral faithfully flat R-scheme, and let $i_0 = \max\{i|v(p) \geq p^{n-1}(p-1)v(\pi^i)\}$. Then, for any n, we have the following filtration*

$$0 \subseteq H^1(X,G_{\pi^{i_0},n}) \subseteq H^1(X,G_{\pi^{i_0-1},n}) \subseteq \ldots \subseteq H^1(X,G_{\pi,n}) \subseteq H^1(X,\mu_{p^n}).$$

PROOF. For any λ, μ with $v(\lambda) \geq v(\mu)$ we have a morphism $\mathcal{G}^{(\lambda)} \longrightarrow \mathcal{G}^{(\mu)}$ defined by $x \longrightarrow \frac{\lambda}{\mu}x$ and compatible with $\psi_{\lambda,n}$ and $\psi_{\mu,n}$. So it induces a morphism $G_{\lambda,n} \longrightarrow G_{\mu,n}$ such that

$$\begin{array}{ccc} G_{\lambda,n} & \longrightarrow & G_{\mu,n} \\ & {\scriptstyle\psi_{\lambda,n}} \searrow \quad \swarrow {\scriptstyle\psi_{\mu,n}} & \\ & \mu_{p^n} & \end{array}$$

commutes. We obtain the following commutative diagram

$$\begin{array}{ccc} H^1(X, G_{\lambda,n}) & \longrightarrow & H^1(X, G_{\mu,n}) \\ & {\scriptstyle\psi_{\lambda,n}} \searrow \quad \swarrow {\scriptstyle\psi_{\mu,n}} & \\ & H^1(X, \mu_{p^n}) & \end{array}$$

and, applying I.3.8, we have that $\psi_{\lambda,n}$ and $\varphi_{\lambda,n}$ are injective. In this way we obtain the desired filtration. □

REMARK I.5.7. Let A be an integrally closed faithfully flat R-algebra. Then we can apply the above proposition. The injection $H^1(X, G_{\lambda,n}) \longrightarrow H^1(X, \mu_{p^n})$ induces the injection $H^1(X, G_{\lambda,n})^{loc} \longrightarrow H^1(X, \mu_{p^n})^{loc}$. This means that the map

$$\alpha^{\lambda^{p^n}}(A) : \mathcal{G}^{(\lambda^{p^n})}(A)/\psi_{\lambda,n_*}(\mathcal{G}^{(\lambda)}(A)) \longrightarrow A^*/(A^*)^{p^n}$$

is injective. Explicitly this means that, for any $x \in A^*$, $x^{p^n} \equiv 1 \mod \lambda^{p^n}$ if and only if $x \equiv 1 \mod \lambda$.

Before constructing a relative version of the above filtration we give locally, for the Zariski topology, equations for $G_{\lambda,n}$-torsors. If $Y \longrightarrow X$ is a G-torsor we will denote by $[Y]$ its class in $H^1(X, G)$. Now let $Y \longrightarrow X$ be a $G_{\lambda,n}$-torsor. Let us consider the exact sequence (I.21). By I.5.5 we can take a covering $\{U_i = \mathrm{Spec}(A_i)\}$ of X by affine subschemes such that the class $(i_*[Y])_{|U_i}$ is trivial, where $i : G_{\lambda,n} \longrightarrow \mathcal{G}^{(\lambda)}$. This means that

$$Y_{|U_i} = \mathrm{Spec}\left(A_i[T_i]/(\frac{(1+\lambda T_i)^{p^n} - 1}{\lambda^{p^n}} - f_i)\right) \tag{I.23}$$

for some $f_i \in A_i$ such that $1 + \lambda^{p^n} f_i \in A_i^*$. Moreover $1 + \lambda T_i = f_{ij}(1 + \lambda T_j)$ with $\{f_{ij} = 1 + \lambda g_{ij}\} = i_*[Y] \in H^1(X, \mathcal{G}^{(\lambda)})$.

REMARK I.5.8. Andreatta and Gasbarri have given a description of $G_{\lambda,n}$-torsors from which they deduced that a $G_{\lambda,n}$-torsor is locally (I.23). From this fact they deduced that

$$H^1(X_{fl}, \mathcal{G}^{(\lambda)}) = H^1(X_{Zar}, \mathcal{G}^{(\lambda)}).$$

See [**5**].

PROPOSITION I.5.9. *Let us suppose that R has perfect residue field. Let X be a normal integral faithfully flat R-scheme with reduced special fiber. We have the following filtration*

$$0 \subseteq H^1(X, G_{\pi^{i_0},n})/H^1(R, G_{\pi^{i_0},n}) \subseteq \ldots \subseteq H^1(X, \mu_{p^n})/H^1(R, \mu_{p^n}).$$

PROOF. We remark that it is sufficient to prove that

$$H^1(X, G_{\lambda,n})/H^1(\mathrm{Spec}(R), G_{\lambda,n}) \longrightarrow H^1(X, \mu_{p^n})/H^1(\mathrm{Spec}(R), \mu_{p^n})$$

is injective. Indeed, if it is true, we obtain the thesis reasoning as in the proof of I.5.6. First, we suppose that $X = \mathrm{Spec}(A)$ is an affine scheme. We prove that

$$H^1(X, G_{\lambda,n})^{loc}/H^1(\mathrm{Spec}(R), G_{\lambda,n}) \longrightarrow H^1(X, \mu_{p^n})^{loc}/H^1(\mathrm{Spec}(R), \mu_{p^n})$$

is injective. Let $[Y]$ be an element of $H^1(X, G_{\lambda,n})^{loc}$; it is given by $f \in \mathcal{G}^{(\lambda^{p^n})}(A)$. Let us suppose that $(\alpha^\lambda)_*([Y]) \in H^1(\mathrm{Spec}(R), \mu_{p^n})$. This means that there exists $g \in A^*$ such that $(1 + \lambda^{p^n} f)g^{p^n} = a \in R^*$. In particular

$$g^{p^n} \equiv a \mod \lambda^{p^n} A. \tag{I.24}$$

We remark that

$$(\alpha^\lambda)_*([Y]) = [a] \in H^1(\mathrm{Spec}(R), \mu_{p^n}).$$

If $[a] \in H^1(\mathrm{Spec}(R), G_{\lambda_{(n)},n})$ then, since $v(\lambda_{(n)}) \geq v(\lambda)$, $[a] \in H^1(\mathrm{Spec}(R), G_{\lambda,n})$ and we are done. We now suppose $[a] \in H^1(\mathrm{Spec}(R), G_{\pi^r,n}) \setminus H^1(\mathrm{Spec}(R), G_{\pi^{r+1},n})$ for some r with $v(p) > p^{n-1}(p-1)r$. If $r \geq v(\lambda)$, reasoning as above we are done. We now consider the case $r < v(\lambda)$. We will prove that this can not happen. Up to a multiplication by a p^n-power in R, which does not change the class of $[a]$, we can suppose $a = 1 + \pi^{p^n r} a_0$ with $a_0 \not\equiv 0 \mod \pi^{p^r}$. Since A is an integral domain, by the Theorem of Krull ([**30**, 1.3.13]), it follows that A is separated with respect to the π-adic topology. Then there exists $r' \in \mathbb{N}$ such that $g = 1 + \pi^{r'} g_0$ and $g_0 \not\equiv 0 \mod \pi A$. Since the residue field k is perfect we have that there exists $b \in R$ such that $a_0 \equiv b^{p^n} \mod \pi$. So $(1 - \pi^r b)^{p^n}(1 + \pi^{p^n r} a_0) \equiv 1 \mod \pi^{p^n r+1}$. Therefore we can suppose that $a_0 \equiv 0 \mod \pi$. We now have, using (I.24),

$$g^{p^n} = (1 + \pi^{r'} g_0)^{p^n} \equiv 1 + \pi^{p^n r} a_0 \mod \lambda^{p^n}$$

and, on the other hand, as it is easy to see,

$$g^{p^n} \equiv 1 + \pi^{p^n r'} g_0^{p^n} \mod \pi^{p^n r'+1}.$$

We now compare the last two equations. If $r' \geq v(\lambda)$ then

$$g^{p^n} \equiv 1 \equiv 1 + \pi^{p^n r} a_0 \mod \lambda^{p^n}.$$

Since $a_0 \not\equiv 0 \mod \pi^{p^n}$ then $r \geq v(\lambda)$. This is a contradiction since we have supposed $r < v(\lambda)$. Hence $r' < v(\lambda)$. Comparing again the above two equations we have

$$1 + \pi^{p^n r} a_0 \equiv 1 + \pi^{p^n r'} g_0^{p^n} \mod \lambda^{p^n}.$$

Since X_k is reduced then $g_0^{p^n} \not\equiv 0 \mod \pi$. Moreover $a_0 \not\equiv 0 \mod \pi^{p^n}$, so it follows $r' = r$ and $g_0^{p^n} \equiv a_0 \mod \pi$. But $a_0 \equiv 0 \mod \pi$. Hence we have $g_0^{p^n} \equiv 0 \mod \pi$, which is contradiction.

We now suppose that X is possibly not affine. Let $[Y]$ be an element of $H^1(X, G_{\lambda,n})$ such that $(\psi_{\lambda,n})_*[Y] \in H^1(\mathrm{Spec}(R), \mu_{p^n})$. By I.5.5 we can take an affine subscheme U of X faithfully flat over R such that $[Y_{|U}] \in H^1(U, G_{\lambda,n})^{loc}$. Therefore, by what we just proved, we have that $[Y_{|U}] \in H^1(\mathrm{Spec}(R), G_{\lambda,n})$. This means that there exists a $G_{\lambda,n}$-torsor $R' \longrightarrow R$ such that $Y_{|U} \simeq U \times_R R'$. Therefore $Y \longrightarrow X$ and $X \times_R R'$ are two $G_{\lambda,n}$-torsors which are isomorphic on the open U. By I.3.8 it follows that they are isomorphic. So $[Y] \in H^1(\mathrm{Spec}(R), G_{\lambda,n})$. □

For any $n \in \mathbb{N}$ and $\lambda \in R$ the natural map

$$\varphi_{n,i} : G_{\lambda,n} \longrightarrow G_{\lambda^{p^i},n-i}$$

induces a map

$$\varphi_{n,i} : H^1(X, G_{\lambda,n}) \longrightarrow H^1(X, G_{\lambda^{p^i},n-i}), \tag{I.25}$$

which associates to a $G_{\lambda,n}$-torsor Y the $G_{\lambda^{p^i},n-i}$-torsor $Y/\ker\varphi_{n,i}$. This map is compatible with the filtration.

I.5.2. Deformation between d and dlog. Let $g : X \longrightarrow \mathrm{Spec}(R)$ be a faithfully flat morphism. We consider the map of sheaves of $\mathbb{Z}$-modules on X_{Zar}

$$\delta log^\lambda : \mathcal{G}^{(\lambda)} \longrightarrow \Omega^1_{X/R},$$

defined on each open set by

$$a \longmapsto \frac{\mathrm{d}\, a}{1 + \lambda a}.$$

Then the following diagram

$$\begin{array}{ccc} \mathcal{G}^{(\lambda)} & \xrightarrow{\delta log^\lambda} & \Omega^1_{X/R} \\ \downarrow{\scriptstyle \alpha^\lambda} & & \downarrow{\scriptstyle \lambda} \\ \mathbb{G}_m & \xrightarrow{\mathrm{d}\, log} & \Omega^1_{X/R} \end{array} \tag{I.26}$$

commutes. If X is flat over R then λ is injective for $\lambda \neq 0$.

On the generic fiber over R the vertical arrows of the previous diagram are isomorphism. So, on the generic fiber, δlog^λ is essentially $\mathrm{d}\, log$. While on the special fiber we have

$$\delta log^\lambda_{|k} = \mathrm{d} : \mathcal{G}^{(\lambda)}_k \simeq \mathbb{G}_a \longrightarrow \Omega^1_{X_k/k}.$$

Moreover we remark that, if $\mu \mid \lambda$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{G}^{(\mu)} \\ \downarrow{\scriptstyle \delta log^\lambda} & & \downarrow{\scriptstyle \delta log^\mu} \\ \Omega^1_{X/R} & \xrightarrow{\frac{\lambda}{\mu}} & \Omega^1_{X/R} \end{array} \tag{I.27}$$

where the first horizontal map $\mathcal{G}^{(\lambda)} \longrightarrow \mathcal{G}^{(\mu)}$ is given by $S \longrightarrow \frac{\lambda}{\mu}S$. From now on, $\lambda \in R$ satisfies condition $(*)$, i.e. $v(p) \geq p^{n-1}(p-1)v(\lambda)$. We now define the sheaf of differential forms candidate for classifying $G_{\lambda,n}$-torsors.

Definition i.5.10. Let $\Omega^1_{log^\lambda,X/R}$ be the image of δlog^λ. When there is no ambiguity we will simply call it $\Omega^1_{log^\lambda}$. We define

$$\delta : \Omega^1_{log^\lambda} \longrightarrow \Omega^1_{log^{\lambda^{p^n}}}$$

the map such that the following diagram commutes

$$\begin{array}{ccc} \Omega^1_{log^\lambda} & \xrightarrow{\delta} & \Omega^1_{log^{\lambda^{p^n}}} \\ \downarrow \lambda & & \downarrow \lambda^{p^n} \\ \Omega^1_{log,X/R} & \xrightarrow{p^n} & \Omega^1_{log,X/R} \end{array}$$

And for any n, we consider the subsheaf of $\mathbb{Z}$-modules $\delta(\Omega^1_{log^\lambda})$ of $\Omega^1_{log^{\lambda^{p^n}}}$. We define the sheaf of $\mathbb{Z}$-modules

$$\Omega^1_{log^\lambda,n} := \Omega^1_{log^{\lambda^{p^n}}}/\delta(\Omega^1_{log^\lambda}).$$

We have in fact that $\Omega^1_{log^\lambda,n}$ is a sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules since $p^n\Omega^1_{log^{\lambda^{p^n}}} \subset \delta(\Omega^1_{log^\lambda})$. Indeed $\lambda^{p^n-1}\Omega^1_{log^{\lambda^{p^n}}} \subset \Omega^1_{log^\lambda}$, because

$$\lambda^{p^n-1}\frac{da}{1+\lambda^{p^n}a} = \frac{d(\lambda^{p^n-1}a)}{1+\lambda(\lambda^{p^n-1}a)} \quad ;$$

so $p^n\Omega^1_{log^{\lambda^{p^n}}} = \delta\lambda^{p^n-1}\Omega^1_{log^{\lambda^{p^n}}} \subset \delta\Omega^1_{log^\lambda}$.

Remark i.5.11. We observe that δlog^λ depends only on $v(\lambda)$. Indeed if $v(\lambda) = v(\lambda')$ then $\lambda = c\lambda'$ with $c \in R^*$. So δlog^λ and $\delta log^{\lambda'}$ are the same maps up the isomorphism $c : \Omega^1_{X/Y} \longrightarrow \Omega^1_{X/Y}$.

i.5.3. The theorem. Let X be a normal faithfully flat R-scheme with geometrically integral generic fiber and geometrically reduced special fiber. We will construct a filtration of $H^0(X_{Zar}, \Omega^1_{log,n})$. By i.4.6, $H^0(X_{Zar}, \Omega^1_{log,n})$ is isomorphic to $H^1(X, \mu_{p^n})/H^1(\mathrm{Spec}(R), \mu_{p^n})$. Moreover we constructed a filtration of $H^1(X, \mu_{p^n})/H^1(\mathrm{Spec}(R), \mu_{p^n})$ in i.5.9. We will prove that these filtrations are the same. Let us consider the map of sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules

$$\lambda^{p^n} : \Omega^1_{log^\lambda,n} \longrightarrow \Omega^1_{log,n}.$$

If $\mu \mid \lambda$ then, using (i.27), it is easy to see that the following diagram commutes

$$\begin{array}{ccc} \Omega^1_{log^\lambda,n} & \xrightarrow{(\frac{\lambda}{\mu})^{p^n}} & \Omega^1_{log^\mu,n} \\ & \searrow^{\lambda^{p^n}} \quad \swarrow^{\mu^{p^n}} & \\ & \Omega^1_{log,n} & \end{array}$$

This implies that the following diagram

$$
\text{(I.28)} \qquad \begin{array}{ccc}
H^0(X, \Omega^1_{log^\lambda,n}) & \xrightarrow{(\frac{\lambda}{\mu})^{p^n}} & H^0(X, \Omega^1_{log^\mu,n}) \\
& {\scriptstyle \lambda^{p^n}} \searrow \quad \swarrow {\scriptstyle \mu^{p^n}} & \\
& H^0(X, \Omega^1_{log,n}) &
\end{array}
$$

commutes, too.

Proposition I.5.12. *Notation as above. Let* $i_0 = \max\{i | v(p) \geq p^{n-1}(p-1)v(\pi^i)\}$. *Then there is a filtration*

$$0 \subseteq H^0(X_{Zar}, \Omega^1_{log^{\pi^{i_0}},n}) \subseteq \ldots \subseteq H^0(X_{Zar}, \Omega^1_{log^{\pi},n}) \subseteq H^0(X_{Zar}, \Omega^1_{log,n}).$$

Proof. We remark that, by (I.28), it is sufficient to prove that, for any $\lambda \in R$ which satisfies $(*)$, then $H^0(X_{Zar}, \Omega^1_{log^\lambda,n}) \subseteq H^0(X_{Zar}, \Omega^1_{log,n})$. We will prove a little more. Indeed we will show that

$$\lambda^{p^n} : \Omega^1_{log^\lambda,n} \longrightarrow \Omega^1_{log,n}$$

is an inclusion of sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules. But it suffices to prove this in the category of presheaves. We are so reduced to proving that for any open set $U \subseteq X$, then

$$\lambda^{p^n} : H^0(U_{Zar}, \Omega^1_{log^{\lambda^{p^n}}})/\delta(H^0(U_{Zar}, \Omega^1_{log^\lambda})) \subseteq H^0(U_{Zar}, \Omega^1_{log,U/R})/p^n H^0(U_{Zar}, \Omega^1_{log,U/R}).$$

is injective. Let $\omega \in H^0(U_{Zar}, \Omega^1_{log^{\lambda^{p^n}}})$ be such that

$$\lambda^{p^n}\omega = p^n\eta$$

with $\eta \in H^0(U_{Zar}, \Omega^1_{log,U/R})$. Let $\{U_i = \mathrm{Spec}(A_i)\}$ be a covering by affine open sets of U such that

$$\lambda^{p^n}\omega_{|U_i} = \mathrm{dlog}\,(1 + \lambda^{p^n} f_i)$$

for some $f_i \in \mathcal{G}^{(\lambda^{p^n})}(A_i)$ and

$$\eta_{|U_i} = \mathrm{dlog}\, g_i$$

for some $g_i \in A_i^*$. Therefore

$$\lambda^{p^n}\omega_{|U_i} = \mathrm{dlog}\,(1 + \lambda^{p^n} f_i) = p^n\,\mathrm{dlog}\, g_i.$$

Hence

$$\mathrm{dlog}(\frac{(1 + \lambda^{p^n} f_i)}{g_i^{p^n}}) = 0,$$

which implies, by I.4.4, that $(1 + \lambda^{p^n} f_i)g_i^{p^n} = r_i$ for some $r_i \in R^*$. Using the same argument as in the proof of I.5.9 we obtain that $g_i = 1 + \lambda h_i$ for some $h_i \in \mathcal{G}^{(\lambda)}(A_i)$. This is the point where we use the fact that X_k is reduced. So we have shown that

$$\eta_{|U_i} = \mathrm{dlog}(1 + \lambda h_i) = \lambda\delta log^\lambda(h_i).$$

We remark that if we have $\delta log^\lambda(h_i)$ and $\delta log^\lambda(h_j)$ then

$$\eta_{|U_i \cap U_j} = \lambda\delta log^\lambda(h_i)_{|U_j} = \lambda\delta log^\lambda(h_j)_{|U_i}.$$

So

$$\lambda\delta log^{\lambda}(h_i) = \lambda\delta log^{\lambda}(h_j)$$

over $U_i \cap U_j$. Since X is flat over R and X_k is geometrically reduced then $\Omega^1_{X/R}$ is flat over R; hence

$$\delta log^{\lambda}(h_i) = \delta log^{\lambda}(h_j).$$

We call ω' the element of $H^0(U_{Zar}, \Omega^1_{log^{\lambda}})$ given, on the covering $\{U_i\}$, by $\{\delta log^{\lambda}(h_i)\}$. Then

$$p^n\lambda(\omega') = \lambda^{p^n}\omega,$$

which means, by definition of δ,

$$\delta(\omega') = \omega,$$

as we wanted. □

Now by

$$0 \longrightarrow G_{\lambda,n} \longrightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\psi_{\lambda,n}} \mathcal{G}^{(\lambda^{p^n})} \longrightarrow 0$$

we have

$$R^1f_*G_{\lambda,n} \simeq \mathcal{G}^{(\lambda^{p^n})}/\psi_{\lambda,n}(\mathcal{G}^{(\lambda)}).$$

So, by I.4.5,

$$H^0(X_{Zar}, R^1f_*G_{\lambda,n}) \simeq H^1(X, G_{\lambda,n}).$$

We can now define

$$\mathrm{d}\, Log_n^{\lambda} : H^1(X, G_{\lambda,n}) \longrightarrow H^0(X_{Zar}, \Omega^1_{log^{\lambda},n})$$

as the map induced by

$$\delta log_n^{\lambda} : \mathcal{G}^{(\lambda^{p^n})}/\psi_{\lambda,n}(\mathcal{G}^{(\lambda)}) \longrightarrow \Omega^1_{log^{\lambda},n}.$$

THEOREM I.5.13. *Let us suppose k is perfect. Let X be a normal faithfully flat R-scheme of with geometrically integral generic fiber and geometrically reduced special fiber. Then there is an isomorphism*

$$H^1(X, G_{\lambda,n})/H^1(R, G_{\lambda,n}) \xrightarrow{\mathrm{d}\, Log_n^{\lambda}} H^0(X_{Zar}, \Omega^1_{log^{\lambda},n}).$$

Moreover

a) $\mathrm{d}\, Log_n^{\lambda}$ *is compatible with the filtrations of* I.5.9 *and* I.5.12;

b) $\mathrm{d}\, Log_n^{\lambda}$ *is compatible with the natural restriction maps*

$$r^{\lambda}_{n,m} : H^1(X, G_{\lambda,n}) \longrightarrow H^1(X, G_{\lambda,m})$$

and

$$r'^{\lambda}_{n,m} : H^0(X_{Zar}, \Omega^1_{log^{\lambda},n}) \longrightarrow H^0(X_{Zar}, \Omega^1_{log^{\lambda},m})$$

with $n \geq m$, i.e.

$$\mathrm{d}\, Log_m \circ r_{n,m} = r'_{n,m} \circ \mathrm{d}\, Log_n;$$

c) *if $Y \longrightarrow \mathrm{Spec}(R)$ satisfies the conditions of the theorem and there is an R-morphism $h : Y \longrightarrow X$, then $\mathrm{d}\, Log_n^{\lambda}$ commutes with pull-backs.*

PROOF. Clearly $H^1(R, G_{\lambda,n}) \subseteq \ker \mathrm{d}\, Log_n^\lambda$. Now, by I.4.6, I.5.6, (I.26) and I.5.12 (the last one will be useful only at the end of the proof) we have the following commutative diagram

$$\text{(I.29)}\quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & H^1(X, G_{\lambda,n})/H^1(R, G_{\lambda,n}) & \xrightarrow{\mathrm{d}\, Log_n^\lambda} & H^0(X_{Zar}, \Omega^1_{log^\lambda,n}) & & \\ & & \downarrow{\scriptstyle \psi_{\lambda,n}} & & \downarrow & & \\ 0 & \longrightarrow & H^1(X, \mu_{p^n})/H^1(R, \mu_{p^n}) & \xrightarrow{\mathrm{d}\, Log_n} & H^0(X_{Zar}, \Omega^1{}_{\log,n}) & \longrightarrow & 0 \end{array}$$

which proves that $dLog_n^\lambda$ is injective, too.

We now show the surjectivity of $\mathrm{d}\, Log_n^\lambda$. Let $\omega \in H^0(X_{Zar}, \Omega^1_{log^\lambda,n})$. There is a covering by affine open sets $\{U_i = \mathrm{Spec}(A_i)\}$ of X such that $\omega_{|U_i} = \delta log^{\lambda^{p^n}} f_i$ for some $f_i \in \mathcal{G}^{(\lambda)}(U_i)$. Now $\lambda^{p^n}\omega$ gives an element of $H^0(X_{Zar}, \Omega^1{}_{\log,n})$ so, by I.4.6, it determines uniquely a class $[Y]$ of $H^1(X, \mu_{p^n})/H^1(R, \mu_{p^n})$. A representant, $Y \longrightarrow X$, is defined in the following way

$$Y_{|U_i} = \mathrm{Spec}(A[T_i]/(T_i^{p^n} - (1 + \lambda^{p^n} f_i))),$$

with $T_i = f_{ij} T_j$ for some $f_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. Since $f_{ij}^{p^n} = \frac{1+\lambda^{p^n} f_i}{1+\lambda^{p^n} f_j}$ then, by I.5.7, $f_{ij} = 1 + \lambda h_{ij}$ for $h_{ij} \in \mathcal{G}^{(\lambda)}(U_i \cap U_j)$. Now, for any i, we consider $W \longrightarrow X$ such that $W_{|U_i} = \mathrm{Spec}(A_i[Z_i]/(\frac{(1+\lambda Z_i)^{p^n}-1}{\lambda^{p^n}} - f_i)$ and $Z_i = f_{ij} Z_j + h_{ij}$. If $G_{\lambda,n}$ is, locally, $\mathrm{Spec}(A_i[T_i]/(\frac{(1+\lambda T_i)^{p^n}-1}{\lambda^{p^n}})$ with $T_i = T_j$ we define the $G_{\lambda,n}$-action $m : G_{\lambda,n} \times W \longrightarrow W$ locally by $Z_i \longrightarrow T_i \otimes 1 + 1 \otimes Z_i + \lambda T_i \otimes Z_i$. This action makes W a $G_{\lambda,n}$-torsor over X. Moreover $\psi_{\lambda,n}([W]) = [Y]$ by construction. If we now look at (I.29), it is simple to verify that $\mathrm{d}\, Log_n^\lambda([W]) = \omega$ by a simple diagram chasing.

The statements about compatibilities are clear by construction.

□

We now show the compatibility with restrictions on the special fiber. If indeed $0 < p^{n-1}(p-1)v(\lambda) < v(p)$ then, as remarked at the beginning of the section, $G_{\lambda,n|k} \simeq \alpha_{p^n}$. So, if X_k is smooth, we can ask if there is compatibility between the isomorphisms of I.5.13 and I.3.2. The answer is in the following proposition.

PROPOSITION I.5.14. *If we add to the hypothesis of the theorem that $0 < p^{n-1}(p-1)v(\lambda) < v(p)$, k is perfect and X_k is smooth, then there exists a map*

$$res_n : H^0(X_{Zar}, \Omega^1_{log^\lambda,n}) \longrightarrow H^0(X_{Zar}, BW_n\Omega^1_{X_k})$$

such that the following diagram

$$\begin{array}{ccc} H^1(X, G_{\lambda,n})/H^1(R, G_{\lambda,n}) & \xrightarrow{\mathrm{d}\, Log_n^\lambda} & H^0(X_{Zar}, \Omega^1_{log^\lambda,n}) \\ \downarrow{\scriptstyle i_n^*} & & \downarrow{\scriptstyle res_n} \\ H^1(X_k, \alpha_{p^n}) & \xrightarrow{\mathrm{d}} & H^0(X_{k,Zar}, BW_n\Omega^1_{X_k}) \end{array}$$

commutes (i^* is the pull-back of $i : X_k \longrightarrow X$).

REMARK I.5.15. In the case $v(\lambda) = 0$ the compatibility has been already treated in I.4.8. We recall that if $v(\lambda) = 0$ then $G_{\lambda,n} \simeq \mu_{p^n}$.

PROOF. We define res_n in the following way

$$res_n : H^0(X_{Zar}, \Omega^1_{log^\lambda,n}) \longrightarrow H^0(X_{Zar}, (BW_n\Omega^1_{X_k})$$
$$\{\delta log^\lambda(g_i)\} \longmapsto \{\mathrm{d}\, \underline{\bar{g}_i}\},$$

where $\overline{g}$ is the restriction on the special fibre of $g \in \mathcal{O}_X(U)$. The map res_1 is nothing else but the restriction map. By construction of $\mathrm{d}\, Log_n^\lambda$ and d the diagram commutes. □

CHAPTER II

Models of $\mathbb{Z}/p^2\mathbb{Z}$ over a d.v.r. of unequal characteristic

Let R be any discrete valuation ring with fraction field K, uniformizer π and residue field k of characteristic $p > 0$. We will moreover suppose, till the end of the thesis, that R is of unequal characteristic if not otherwise specified. We will write $S = \mathrm{Spec}(R)$. As remarked in the introduction there will be a conflict of notation since S will denote the indeterminate of some polynomials, too. But it should not cause any problem. The aim of this chapter is the classification of finite and flat R-group schemes of order p^2 which are isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})_K$ on the generic fiber, i.e. models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. As asserted in the introduction, we will prove that for any such group scheme G, there exists an exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow 0,$$

with $\mathcal{E}_1, \mathcal{E}_2$ smooth R-group schemes, which coincides with the Kummer sequence on the generic fiber. We will describe explicitly all such isogenies. The explicit description of the models of $(\mathbb{Z}/p^2\mathbb{Z})_K$ presented here will be used in the third chapter to study the problem of the extension of $\mathbb{Z}/p^2\mathbb{Z}$-torsors.

II.1. Néron blow-ups

We recall here the definition of Néron blow-up. For details see [**11**, Ch. 3], [**47**] and [**59**]. In this section R is a not necessarily of unequal characteristic.

Definition II.1.1. Let X be a flat affine R-scheme of finite type and $R[X]$ its coordinate ring. Let Y be a closed subscheme of X_k defined by some proper ideal $I(Y)$ of $R[X]$. Then $\pi \in I(Y)$. We define the *Néron blow-up* (or *dilatation*) of Y in X by

$$X^Y := \mathrm{Spec}(A[\pi^{-1}I(Y)]).$$

Then X^Y is a flat affine R-scheme of finite type and the R-homomorphism $R[X] \subseteq R[\pi^{-1}I(Y)]$ induces a morphism

$$X^Y \longrightarrow X,$$

which gives an isomorphism on the generic fiber.

The Néron-blow up is explicitly given as follows: let $I = (\pi, f_1, \ldots, f_k)$ with $f_i \in R$. Then

$$R[X^Y] = R[X][\pi^{-1}f_1, \ldots, \pi^{-1}f_k].$$

So X^Y is the open set of $x \in \text{Proj}(\oplus_{i\geq 0} I^i)$ (the classical blow-up of X in Y), where I_x is generated by π. Clearly it is possible to give the definition for schemes in general (see [**11**, Ch. 3]).

In the following we are interested in the case where X is an affine flat group scheme G and Y a subgroupscheme H of G_k. We recall the following definitions.

DEFINITION II.1.2. Let $\varphi : G \longrightarrow H$ be a morphism of flat R-group schemes which is an isomorphism restricted to the generic fibers. Then it is called a *model map*.

DEFINITION II.1.3. Let H_K be a group scheme over K. Any flat R-group scheme G such that $G_K \simeq H_K$ is called a *model* of H_K.

It is possible to prove that G^H is a group scheme and $G^H \longrightarrow G$ is a model map ([**59**, 1.1]). We recall the following results:

PROPOSITION II.1.4. *The canonical map $G^H \longrightarrow G$ sends the special fiber into H. Moreover G^H has the following universal property: any model map $G' \longrightarrow G$ sending the special fiber into H factors uniquely through G^H.*

PROOF. [**59**, 1.2]. □

THEOREM II.1.5. *Any model map between affine group schemes is isomorphic to a composite of Néron blow-ups.*

PROOF. [**59**, 1.4]. □

EXAMPLE II.1.6. Let us consider the group scheme $G_{\mu,1} = \text{Spec}(R[S]/(\frac{(1+\mu S)^p-1}{\mu^p}))$ with $v(p) > (p-1)v(\mu)$. The only possible subgroup of $(G_{\mu,1})_k$ which gives a non-trivial blow-up is $H = e$. Then $I(H) = (\pi, S)$ if $v(\mu) > 0$ and $I(H) = (\pi, S-1)$ otherwise. It is easy to see that, in both cases,

$$G^e_{\mu,1} = G_{\mu\pi,1}.$$

So if there exists a model map $G \longrightarrow G_{\mu,1}$ then, using II.1.5, $G \simeq G_{\lambda,1}$ for some λ.

II.2. Models of $(\mathbb{Z}/p\mathbb{Z})_K$

As remarked in I.5.1 every finite and flat R group scheme of order p, up to an extension of R, is of type $G_{\lambda,1}$ for some $\lambda \in R$. For $(\mathbb{Z}/p\mathbb{Z})_K$-models we have a more precise statement, which is well known to specialists. Two proofs are for instance given in [**39**, 1.4.4, 3.2.2]. The second one is essentially that we present here. We remark that if G is a model of $(\mathbb{Z}/m\mathbb{Z})_K$ and R contains a primitive m-th root of unity then there are the following model maps

$$\mathbb{Z}/m\mathbb{Z} \longrightarrow G \longrightarrow \mu_m.$$

Indeed the first one is the normalization map, while the second one is the dual morphism of the normalization $\mathbb{Z}/m\mathbb{Z} \longrightarrow G^\vee$ (see also [**35**, 2.2.3] for a more general result).

PROPOSITION II.2.1. *Let us suppose that R contains a primitive p-th root of unity. If G is a finite and flat R-group scheme such that $G_K \simeq \mathbb{Z}/p\mathbb{Z}$ then $G \simeq G_{\lambda,1}$ for some $\lambda \in R \setminus \{0\}$.*

PROOF. As remarked above we have an R-model map

$$\varphi : G \longrightarrow \mu_p.$$

By II.1.5 it is a composition of Néron blow-ups. Then, by II.1.6, it follows that $G \simeq G_{\lambda,1}$ for some $\lambda \in R \setminus \{0\}$. □

II.3. Models of $(\mathbb{Z}/p^2\mathbb{Z})_K$

In this section we study models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. Throughout the section we suppose that R contains a primitive p^2-th root of unity. First of all we prove that any such group is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ for some $\mu, \lambda \in R \setminus \{0\}$.

LEMMA II.3.1. *Let G be a finite and flat R-group scheme of order p^2 such that G_K is a constant group. Then G is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ for some $\mu, \lambda \in R \setminus \{0\}$.*

PROOF. If G_K is a constant group then G_K is isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})_K$ or to $(\mathbb{Z}/p\mathbb{Z})_K \times (\mathbb{Z}/p\mathbb{Z})_K$. We consider the factorization

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})_K \longrightarrow G_K \longrightarrow (\mathbb{Z}/p\mathbb{Z})_K \longrightarrow 0.$$

We take the closure G_1 of $(\mathbb{Z}/p\mathbb{Z})_K$ in G. Then G_1 is a model of $(\mathbb{Z}/p\mathbb{Z})_K$. So by II.2.1 it follows that $G_1 \simeq G_{\lambda,1}$ for some $\lambda \in R \setminus \{0\}$. $G/G_{\lambda,1}$ is a model of $(\mathbb{Z}/p\mathbb{Z})_K$, too. So, again by II.2.1, we have $G/G_{\lambda,1} \simeq G_{\mu,1}$ for some $\mu \in R \setminus \{0\}$. We are done. □

So we study, first of all, the group $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$.

II.3.1. Extensions of group schemes. We here recall some generalities on extensions of group schemes. For more details see [**15**, III.6].

Let G and H be group schemes on S. We moreover suppose that H is commutative and that G acts on H. Let us denote

$$\mathrm{Ext}^0_S(G,H) = \{\varphi \in \mathrm{Hom}_{Sch_S}(G,H) | \varphi(gg') = \varphi(g) + g(\varphi(g')) \quad \text{for any local sections} \quad g, g' \text{ of } G\}.$$

We are interested in the case that G acts trivially on H. In this situation

$$\mathrm{Ext}^0_S(G,H) = \mathrm{Hom}_{gr}(G,H).$$

Now $H \mapsto \mathrm{Ext}^0(G,H)$ is a left exact functor from the category of fppf-sheaves of G-modules on S to that of abelian groups. Let $\mathrm{Ext}^\bullet_S(G,H)$ denote the left derived functor of $H \mapsto \mathrm{Ext}^0_S(G,H)$. It is known that $\mathrm{Ext}^1_S(G,H)$ is isomorphic to the group of equivalence classes of extensions of G by H (see [**15**, III 6.2]).

Recall that an extension of G by H is by definition an exact sequence of fppf-sheaves of groups

$$0 \longrightarrow H \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 0,$$

such that $i(j(g)h) = gi(h)g^{-1}$ for any local sections h of H and g of E.

Consider two extensions $(E) : 0 \longrightarrow H \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 0$ and $(F) : 0 \longrightarrow H \xrightarrow{i} F \xrightarrow{j} G \longrightarrow 0$. They are equivalent if there exists a morphism of group schemes $f : E \longrightarrow F$ which makes the following diagram

$$\begin{array}{ccccccccc} (E):0 & \longrightarrow & H & \xrightarrow{i} & E & \xrightarrow{j} & G & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \| & & \\ (F):0 & \longrightarrow & H & \xrightarrow{i} & F & \xrightarrow{j} & G & \longrightarrow & 0 \end{array}$$

commute. Clearly such an f is an isomorphism of group schemes. If G and H are flat affine groups over S, then it is the same for E.

We now recall the definitions of pushforward and pull-back of extensions. Let G and H be as above and $\varphi : G' \longrightarrow G$ a morphism of group-schemes. Then φ induces a morphism

$$\varphi^* : \mathrm{Ext}^1_S(G, H) \longrightarrow \mathrm{Ext}^1_S(G', H).$$

It is explicitly given as follows. Let

$$(E) : 0 \longrightarrow H \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 0$$

be an extension of G by H. Then $\varphi^*[E]$ is defined by the diagram

$$\begin{array}{ccccccccc} \varphi^*[E]:0 & \longrightarrow & H & \xrightarrow{i} & E' & \xrightarrow{j} & G' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\ (E):0 & \longrightarrow & H & \xrightarrow{i} & E' & \xrightarrow{j} & G & \longrightarrow & 0 \end{array}$$

where the right square is cartesian.

Now consider a group scheme H' together with a G-action. If $\psi : H \longrightarrow H'$ is a morphism which preserves the G-action then it induces a morphism

$$\psi_* : \mathrm{Ext}^1_S(G, H) \longrightarrow \mathrm{Ext}^1_S(G, H'),$$

which we can explicitly describe as follows. Let

$$(E) : 0 \longrightarrow H \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 0$$

be an extension of G by H. Then $\psi_*[E]$ is defined by the diagram

$$\begin{array}{ccccccccc} (E):0 & \longrightarrow & H & \xrightarrow{i} & E & \xrightarrow{j} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow & & \| & & \\ \psi_*[E]:0 & \longrightarrow & H' & \xrightarrow{i} & E' & \xrightarrow{j} & G & \longrightarrow & 0 \end{array}$$

where the left square is cocartesian.

Next we recall the Hochschild cohomology. Let G be a presheaf of groups on $Sch_{|S}$ and F a presheaf of G-modules on $Sch_{|S}$. We define a complex $\{C^n(G, F), \delta^n\}$ as follows: $C^n(G, F)$ denotes the set of morphisms of schemes from G^n to H and the boundary map

$$\delta^n : C^n(G, F) \longrightarrow C^{n+1}(G, F)$$

is defined by

$$(\delta^n f)(g_0, g_1, \ldots, g_n) = g_0 f(g_1, \ldots, g_n) + \\ + \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, g_1, \ldots, g_i g_{i+1}, \quad \ldots, g_n) + (-1)^{n+2} f(g_0, g_1, \ldots, g_{n-1}).$$

Put

$$Z^n(G,F) = \ker(\delta^n : C^n(G,F) \longrightarrow C^{n+1}(G,F)),$$
$$B^n(G,F) = Im(\delta^{n-1} : C^{n-1}(G,F) \longrightarrow C^n(G,F)),$$

and

$$H_0^n(G,F) = Z^n(F,G)/B^n(G,F).$$

For our purposes we are interested in the second group of cohomology. The following result is indeed well known.

Proposition II.3.2. *Let G and H be group schemes over S. Given an action of G on H then $H_0^2(G,H)$ is isomorphic to the group of equivalence classes of extensions of G by H which have a scheme-theoretic section.*

Proof. [**15**]. □

II.3.2. Sekiguchi-Suwa Theory. Here is a very partial review of results of [**47**], [**49**] and [**53**]. Let $\mu, \lambda \in \pi R \backslash \{0\}$. For any $\lambda' \in R\backslash\{0\}$ set $S_{\lambda'} = \mathrm{Spec}(R/\lambda' R)$.

What we call Sekiguchi-Suwa theory is their description of $\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ and $\mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathcal{G}^{(\lambda)})$ through Witt vectors.

Let $Y = \mathrm{Spec}(R[T_1, \ldots, T_m]/(F_1, \ldots, F_n))$ be an affine R-scheme of finite type. We recall that, for any R-scheme X we have that $\mathrm{Hom}_{Sch}(X,Y)$ is in a bijective correspondence with the set

$$\{(a_1, \ldots, a_m) \in H^0(Y, \mathcal{O}_Y)^m | F_1(a_1, \ldots, a_m) = 0, \ldots, F_n(a_1, \ldots, a_m) = 0\}.$$

With an abuse of notation we will identify these two sets. If X and Y are R-group schemes we will also identify $\mathrm{Hom}_{gr}(X,Y)$ with a subset of

$$\{(a_1, \ldots, a_m) \in H^0(Y, \mathcal{O}_Y)^m | F_1(a_1, \ldots, a_m) = 0, \ldots, F_n(a_1, \ldots, a_m) = 0\}.$$

We now fix presentations for the group schemes $\mathbb{G}_m$ and $\mathcal{G}^{(\lambda)}$ with $\lambda \in \pi R$. Indeed we write $\mathbb{G}_m = \mathrm{Spec}(R[S, 1/S])$ and $\mathcal{G}^{(\lambda)} = \mathrm{Spec}(R[S, 1/1+\lambda S])$. Before illustrating the Sekiguchi-Suwa theory we see what happens when $\mu \in R^*$. In this case $\mathcal{G}^{(\mu)} \simeq \mathbb{G}_m$, and we have the following well known lemma.

Lemma II.3.3. *For any $\lambda \in \pi R$ we have*

$$\mathrm{Hom}_{gr}(\mathbb{G}_{m|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{S^i \in R[S, 1/S] | i \in \mathbb{Z}\}.$$

In particular if $v(\lambda_1) \geq v(\lambda_2) > 0$, the restriction map

$$\mathrm{Hom}_{gr}(\mathbb{G}_{m|S_{\lambda_1}}, \mathbb{G}_{m|S_{\lambda_1}}) \longrightarrow \mathrm{Hom}_{gr}(\mathbb{G}_{m|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}})$$

is an isomorphism.

Moreover for the extensions group we have

PROPOSITION II.3.4. *For any $\lambda \in \pi R \setminus \{0\}$, any S-action of $\mathbb{G}_m$ on $\mathcal{G}^{(\lambda)}$ is trivial. Moreover*

$$\mathrm{Ext}^1(\mathbb{G}_m, \mathcal{G}^{(\lambda)}) = 0$$

PROOF. See [**49**, I 1.6, II 1.4]. □

We also want to recall what happens to the extensions group when $\lambda \in R^*$, i.e. $\mathcal{G}^{(\lambda)} \simeq \mathbb{G}_m$.

PROPOSITION II.3.5. *For any $\mu \in R \setminus \{0\}$, any action of $\mathcal{G}^{(\mu)}$ on $\mathbb{G}_m$ is trivial. Moreover*

$$\mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathbb{G}_m) = 0.$$

PROOF. See [**49**, I 1.5, I 2.7] □

We now consider the case $\mu, \lambda \in \pi R \setminus \{0\}$. We observe that by definitions we have that

$$\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{F(S) \in (R/\lambda R[S, \frac{1}{1+\mu S}])^* | F(S)F(T) = F(S+T+\mu ST)\}$$

Any action of $\mathcal{G}^{(\mu)}$ on $\mathcal{G}^{(\lambda)}$ is trivial ([**49**, I 1.6]). We now consider the map

$$\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \xrightarrow{\alpha} \mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathcal{G}^{(\lambda)})$$

given by

$$F \longmapsto \mathcal{E}^{(\mu,\lambda;F)},$$

where

$$\mathcal{E}^{(\mu,\lambda;F)}$$

is a smooth affine commutative group defined as follows: let $\tilde{F}(S) \in R[S]$ be a lifting of $F(S)$, then

$$\mathcal{E}^{(\mu,\lambda;F)} = \mathrm{Spec}(R[S_1, S_2, \frac{1}{1+\mu S_1}, \frac{1}{\tilde{F}(S_1)+\lambda S_2}])$$

(1) law of multiplication

$$\begin{aligned} S_1 &\longmapsto S_1 \otimes 1 + 1 \otimes S_1 + \mu S_1 \otimes S_1 \\ S_2 &\longmapsto S_2 \otimes \tilde{F}(S_1) + \tilde{F}(S_1) \otimes S_2 + \lambda S_2 \otimes S_2 + \\ &\qquad \frac{\tilde{F}(S_1) \otimes \tilde{F}(S_1) - \tilde{F}(S_1 \otimes 1 + 1 \otimes S_1 + \mu S_1 \otimes S_1)}{\lambda} \end{aligned}$$

(2) unit

$$S_1 \longmapsto 0$$

$$S_2 \longmapsto \frac{1-\tilde{F}(0)}{\lambda}$$

(3) inverse

$$S_1 \longmapsto -\frac{S_1}{1+\mu S_1}$$
$$S_2 \longmapsto \frac{\frac{1}{\tilde{F}(S_1)+\lambda S_2} - \tilde{F}(-\frac{S_1}{1+\mu S_1})}{\lambda}$$

We moreover define the following homomorphisms of group schemes

$$\mathcal{G}^{(\lambda)} = \mathrm{Spec}(R[S, (1+\lambda S)^{-1}]) \longrightarrow \mathcal{E}^{(\mu,\lambda;F)}$$

by

$$S_1 \longmapsto 0$$
$$S_2 \longmapsto S + \frac{1-\tilde{F}(0)}{\lambda}$$

and

$$\mathcal{E}^{(\mu,\lambda;F)} \longrightarrow \mathcal{G}^{(\mu)} = \mathrm{Spec}(R[S, \frac{1}{1+\mu S}])$$

by

$$S \longrightarrow S_1.$$

It is easy to see that

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{E}^{(\mu,\lambda;F)} \longrightarrow \mathcal{G}^{(\mu)} \longrightarrow 0 \tag{II.30}$$

is exact. A different choice of the lifting $\tilde{F}(S)$ gives an isomorphic extension. We recall the following theorem.

THEOREM II.3.6. *For any* $\lambda, \mu \in \pi R \setminus \{0\}$, *the map*

$$\alpha : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathcal{G}^{(\lambda)})$$

is a surjective morphism of groups. And $\ker(\alpha)$ *is generated by the class of* $1+\mu S$. *In particular any extension of* $\mathcal{G}^{(\mu)}$ *by* $\mathcal{G}^{(\lambda)}$ *is commutative.*

PROOF. [**48**, §3]. □

We now define some spaces which had been used by Sekiguchi and Suwa to describe $\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ and, by the above result, $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$. See [**53**] for details.

DEFINITION II.3.7. For any ring A, let $W_n(A)$ be the ring of Witt vectors of length n and $W(A)$ the ring of infinite Witt vectors. We define

$$\widehat{W}_n(A) = \Big\{(a_0, \dots, a_n) \in W_n(A) | a_i \text{ is nilpotent for any } i \text{ and }$$
$$a_i = 0 \text{ for all but a finite number of } i\Big\}$$

and

$$\widehat{W}(A) = \Big\{(a_0, \dots, a_n, \dots) \in W(A) | a_i \text{ is nilpotent for any } i \text{ and}$$
$$a_i = 0 \text{ for all but a finite number of } i\Big\}.$$

We recall the definition of the so-called Witt-polynomial: for any $r \geq 0$ it is

$$\Phi_r(T_0, \dots, T_r) = T_0^{p^r} + pT_1^{p^{r-1}} + \dots + p^r T_r.$$

Then the following maps are defined:

- Verschiebung

$$V : W_n(A) \longrightarrow W_{n+1}(A)$$
$$(a_0, \dots, a_n) \longmapsto (0, a_0, \dots, a_n)$$

- Generalization of Frobenius

$$\mathrm{F} : W_{n+1}(A) \longrightarrow W_n(A)$$
$$(a_0, \dots, a_n) \longmapsto (F_0(\mathbf{T}), F_1(\mathbf{T}), \dots, F_n(\mathbf{T}))$$

where the polynomials $F_r(\mathbf{T}) = F_r(T_0, \dots, T_r) \in \mathbb{Q}[T_0, \dots, T_{r+1}]$ are defined inductively by

$$\Phi_r(F_0(\mathbf{T}), F_1(\mathbf{T}), \dots, F_r(\mathbf{T})) = \Phi_{r+1}(T_0, \dots, T_{r+1}).$$

If $p = 0 \in A$ then F is the usual Frobenius. The subring $\widehat{W}(A)$ is stable respect to these maps.

For any morphism $G : \widehat{W}(A) \longrightarrow \widehat{W}(A)$ we will set $\widehat{W}(A)^G := \ker G$. And for any $a \in A$ we denote the element $(a, 0, 0, \dots, 0, \dots) \in W(A)$ by $[a]$.

We recall the following standard result about Witt vectors.

LEMMA II.3.8. *Let $S_r[\mathbf{T}, \mathbf{U}] \in \mathbb{Z}[\mathbf{T}, \mathbf{U}]$ such that, if $\boldsymbol{a}, \boldsymbol{b} \in W(A)$, then*

$$\boldsymbol{a} + \boldsymbol{b} = (S_0[\boldsymbol{a}, \boldsymbol{b}], \dots, S_r[\boldsymbol{a}, \boldsymbol{b}], \dots)$$

If T_i and U_i have weight p^i then $S_r[\mathbf{T}, \mathbf{U}]$ is isobaric of weight p^r.

The following lemma will be useful later.

LEMMA II.3.9. *Let $\lambda \in R$. If $\boldsymbol{a} = (a_0, a_1, \dots), \boldsymbol{b} = (b_0, b_1, \dots) \in \widehat{W}(R/\lambda R)^{\mathrm{F}}$ then*

$$\boldsymbol{a} + \boldsymbol{b} = (a_0 + b_0, a_1 + b_1, \dots, a_i + b_i, \dots)$$

PROOF. We suppose that $\boldsymbol{a} + \boldsymbol{b} = (c_0, c_1, \dots, c_i, \dots)$. By the previous lemma we have that $c_r(\boldsymbol{a}, \boldsymbol{b})$ is isobaric of weight p^r. It is a standard result that

$$c_r(\boldsymbol{a}, \boldsymbol{b}) = a_r + b_r + c'_r((a_0, a_1, \dots, a_{r-1}), (b_0, b_1, \dots, b_{r-1})).$$

for some polynomial $c'_r(S_0, \dots, S_{r-1}, T_0, \dots, T_{r-1})$. Clearly $c'_r(\boldsymbol{a}, \boldsymbol{b})$ is isobaric of weight p^r, too. Hence $\deg(c'_r) \geq p$.

Let $\tilde{a}_i, \tilde{b}_i \in R$ be liftings of a_i and b_i, respectively. For any $r \geq 1$, up to changing $\boldsymbol{a}$ with $\boldsymbol{b}$, we can suppose that $v(\tilde{a}_k) = \min\{v(\tilde{a}_i), v(\tilde{b}_i) | i = 0, \dots, r-1\}$, for some

$0 \leq k \leq r-1$. Since $\deg c'_r \geq p$ then $v(c'_r(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}})) \geq pv(\tilde{a}_k)$. But $v(\tilde{a}_k^p) \geq v(\lambda)$ since $\mathrm{F}(\boldsymbol{a}) = 0$. Hence $c'_r(\boldsymbol{a}, \boldsymbol{b}) = 0 \in R/\lambda R$. So

$$\boldsymbol{a} + \boldsymbol{b} = (a_0 + b_0, a_1 + b_1, \ldots, a_i + b_i, \ldots)$$

□

We now recall the definition of the Artin-Hasse exponential series

$$E_p(T) := \exp\left(\sum_{r\geq 0} \frac{T^{p^r}}{p^r}\right) = \prod_{r=0}^{\infty} \exp\left(\frac{T^{p^r}}{p^r}\right) \in \mathbb{Z}_{(p)}[[T]].$$

Sekiguchi and Suwa introduced a deformation of the Artin-Hasse exponential map in [**53**]. By the well known formula $\lim_{\lambda\to 0}(1+\lambda x)^{\frac{\alpha}{\lambda}} = \exp(\alpha x)$, it can be seen that $(1+\lambda x)^{\frac{\alpha}{\lambda}}$ is a deformation of $\exp(\alpha x)$. From this point of view they defined the formal power series $E_p(U, \Lambda; T) \in \mathbb{Q}[U, \Lambda][[T]]$ by

$$E_p(U, \Lambda; T) := (1 + \Lambda T)^{\frac{U}{\Lambda}} \prod_{r=1}^{\infty} (1 + \Lambda^{p^r} T^{p^r})^{\frac{1}{p^r}((\frac{U}{\Lambda})^{p^r} - (\frac{U}{\Lambda})^{p^{r-1}})}$$

They proved that $E_p(U, \Lambda; T)$ has in fact its coefficients in $\mathbb{Z}_{(p)}[U, \Lambda]$. It is possible to show ([**53**, 2.4]) that

$$E_p(U, \Lambda; T) = \begin{cases} \prod_{(i,p)=1} E_p(U\Lambda^{i-1}T^i)^{\frac{(-1)^{i-1}}{i}}, & \text{if } p > 2; \\ \prod_{(i,2)=1} E_p(U\Lambda^{i-1}T^i)^{\frac{1}{i}} \left[\prod_{(i,2)=1} E_p(U\Lambda^{2i-1}T^{2i})^{\frac{1}{i}}\right]^{-1}, & \text{if } p = 2. \end{cases}$$

Let A be a $\mathbb{Z}_{(p)}$-algebra and $a, \lambda \in A$. We define $E_p(a, \mu; T)$ as $E_p(U, \Lambda; T)$ evaluated at $U = a$ and $\Lambda = \mu$.

EXAMPLE II.3.10. It is easy to see that $E_p(a, 0; T) = E_p(aT)$ and $E_p(\mu, \mu; T) = 1 + \mu T$. Moreover if $a^p = \mu^{p-1}a \in A$ then $(\frac{a}{\mu})^{p^r} - (\frac{a}{\mu})^{p^{r-1}} = 0$ for $r \geq 1$. Hence

$$E_p(a, \mu; T) = (1 + \mu T)^{\frac{a}{\mu}} = 1 + \sum_{i=1}^{p-1} \frac{\prod_{k=0}^{i-1}(a - k\mu)}{i!} T^i.$$

In particular if $\mu = 0$ and $a^p = 0 \in A$ then

$$E_p(a, 0; T) = \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i.$$

If $\boldsymbol{a} = (a_0, a_1, a_2, \ldots) \in W(A)$ we define the formal power series

$$E_p(\boldsymbol{a}, \mu; T) = \prod_{k=0}^{\infty} E_p(a_k, \mu^{p^k}; T^{p^k}). \tag{II.31}$$

The following result gives an explicit description of $\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|A}, \mathbb{G}_{m|A})$.

THEOREM II.3.11. *Let A be a $\mathbb{Z}_{(p)}$-algebra and $\mu \in A$ a nilpotent element. The homomorphism*

$$\xi^0_A : \widehat{W}(A)^{\mathrm{F}-[\mu^{p-1}]} \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|A}, \mathbb{G}_{m|A})$$
$$\boldsymbol{a} \longmapsto E_p(\boldsymbol{a}, \mu; S)$$

is bijective.

PROOF. [**53**, 2.19.1]. □

And II.3.6 and II.3.11 give the following:

COROLLARY II.3.12. *For any $\lambda, \mu \in \pi R \setminus \{0\}$ the map*

$$\alpha \circ \xi^0_{R/\lambda R} : \widehat{W}(R/\lambda R)^{\mathrm{F}-[\mu^{p-1}]}/ < 1 + \mu T > \longrightarrow \mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathbb{G}_m)$$
$$\boldsymbol{a} \longmapsto \mathcal{E}^{(\lambda,\mu;E_p(\boldsymbol{a},\lambda;S))}$$

is an isomorphism.

We now describe some natural maps through these identifications. Consider the isogeny

$$\psi_{\mu,1} : \mathcal{G}^{(\mu)} \longrightarrow \mathcal{G}^{(\mu^p)}.$$

Let us now suppose that $p > 2$. Then we have that, if $p^2 \equiv 0 \mod \lambda$,

$$\psi^*_{\mu,1} : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu^p)}{}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$$

is given by

$$\boldsymbol{a} \longmapsto [\frac{p}{\mu^{p-1}}]\boldsymbol{a} + V(\boldsymbol{a}) \tag{II.32}$$

(see [**53**, 1.4.1 and 3.8]).

For $p = 2$ the situation is slightly different. Let us define a variant of the Verschiebung as follows. Define polynomials

$$\tilde{V}_r(\mathbf{T}) = \tilde{V}_r(T_0, \ldots, T_r) \in \mathbb{Q}[T_0, \ldots, T_r]$$

inductively by $\tilde{V}_0 = 0$ and

$$\Phi_r(\tilde{V}_0(\mathbf{T}), \ldots, \tilde{V}_r(\mathbf{T})) = p^{p^r}\Phi_{r-1}(T_0, \ldots, T_{r-1})$$

for $r \geq 1$. Then we have that (with possibly $2^2 \not\equiv 0 \mod \lambda$)

$$\psi^*_{\mu,1} : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu^2)}{}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$$

is given by

$$\boldsymbol{a} \longmapsto [\frac{2}{\mu}]\boldsymbol{a} + V(\boldsymbol{a}) + \tilde{V}(\boldsymbol{a})$$

(see [**53**, 3.8]).

For simplicity, to avoid to use this description of $\psi^*_{\mu,1}$, we will consider sometimes only the case $p > 2$.

Consider the morphism

$$\text{(II.33)}\qquad \begin{aligned} p : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) &\longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}{}_{|S_{\lambda^p}}, \mathbb{G}_{m|S_{\lambda^p}}) \\ F(S) &\longmapsto F(S)^p \end{aligned}$$

This morphism is such that

$$\psi_{\lambda,1_*} \circ \alpha = \alpha \circ p.$$

Let $\boldsymbol{a} \in (\widehat{W}(R/\lambda R))^{\mathrm{F}-[\mu^{p-1}]}$. Take any lifting $\tilde{\boldsymbol{a}} \in W(R)$. Using the identifications of II.3.11 the morphism p above is given by

$$\text{(II.34)}\qquad \boldsymbol{a} \longmapsto p\tilde{\boldsymbol{a}}$$

(see [**53**, 4.6]). We will sometimes simply write $p\boldsymbol{a}$.

II.3.3. Two exact sequences. The main tools which we will use to calculate the extensions of $G_{\lambda,1}$ by $G_{\mu,1}$ are two exact sequences. We recall them in this subsection. See (II.35) and (II.38) below. First of all we prove that any action of $G_{\mu,1}$ on $G_{\lambda,1}$ is trivial.

LEMMA II.3.13. *Let $\varphi : G \longrightarrow H$ be an S-morphism of affine S-groups. Assume that G is flat over S. Then $\varphi = 0$ if and only if the generic fiber $\varphi_K = 0$.*

PROOF. [**49**, 1.1]. □

LEMMA II.3.14. *Every action of $G_{\mu,1}$ on $G_{\lambda,1}$ is trivial.*

PROOF. Giving an action of $G_{\lambda,1}$ on $G_{\mu,1}$ is the same as giving a morphism $G_{\mu,1} \longrightarrow Aut_R(G_{\lambda,1})$. If we consider the generic fiber we have a morphism

$$\mu_{p,K} \longrightarrow Aut_K(\mu_{p,K}).$$

The last one is the étale group scheme $(\mathbb{Z}/p\mathbb{Z})^*_K$. It is a group scheme of order $p-1$. So any morphism $\mu_{p,K} \longrightarrow Aut_K(\mu_{p,K})$ is trivial. Applying II.3.13 we have the thesis. □

In the following, all the actions will be supposed trivial. Applying now the functor Ext to the following exact sequence of group schemes

$$(\Lambda): \qquad 0 \longrightarrow G_{\lambda,1} \xrightarrow{i} \mathcal{G}^{(\lambda)} \xrightarrow{\psi_{\lambda,1}} \mathcal{G}^{(\lambda^p)} \longrightarrow 0,$$

we obtain

$$\text{(II.35)}\qquad \begin{aligned} 0 \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) \xrightarrow{\delta'} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{i_*} \\ \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)}) \xrightarrow{\psi_{\lambda,1_*}} \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}). \end{aligned}$$

We remark that δ' is injective since

$$\psi_{\lambda,1_*} : \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda)}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)})$$

is the zero morphism. Indeed since G is flat over R, then by II.3.13,

$$\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda)}) \hookrightarrow \mathrm{Hom}_{gr}(\mu_{p,K}, \mathbb{G}_{mK}) \simeq \mathbb{Z}/p\mathbb{Z}.$$

And it is easy to verify that

$$\text{(II.36)} \qquad \mathrm{Hom}_{gr}(G_{\mu,1},\mathcal{G}^{(\lambda)}) = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } \lambda \mid \mu \\ 0, & \text{if } \lambda \nmid \mu \end{cases}$$

Let us write $G_{\mu,1} = \mathrm{Spec}(R[S]/(\frac{(1+\mu S)^p-1}{\mu^p}))$. If $\lambda \mid \mu$ the group is formed by the morphisms given by $\sigma_i : S \longmapsto \frac{(1+\mu S)^i-1}{\lambda}$ with $i \in \mathbb{Z}/p\mathbb{Z}$. The map $(\psi_{\lambda,1})_* : \mathrm{Hom}_{gr}(G_{\mu,1},\mathcal{G}^{(\lambda)}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1},\mathcal{G}^{(\lambda^p)})$ is moreover nothing else but the multiplication by p. So it is clearly zero.

The map

$$\text{(II.37)} \qquad \delta' : \mathrm{Hom}_{gr}(G_{\mu,1},\mathcal{G}^{(\lambda^p)}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1},G_{\lambda,1})$$

is defined by

$$\sigma_i \longmapsto (\sigma_i)^*(\Lambda),$$

where $(\sigma_i)^*(\Lambda)$ is explicitly

$$\mathrm{Spec}(R[S_1,S_2]/(\frac{(1+\mu S_1)^p-1}{\mu^p}, \frac{(1+\lambda S_2)^p-(1+\mu S_1)^i}{\lambda^p}),$$

with the maps

$$\begin{aligned} G_{\lambda,1} &\longrightarrow \sigma_i^*(\Lambda) \\ S_1 &\longmapsto 0 \\ S_2 &\longmapsto S \end{aligned}$$

and

$$\begin{aligned} \sigma_i^*(\Lambda) &\longrightarrow G_{\mu,1} \\ S &\longmapsto S_1 \end{aligned}$$

The structure of group scheme on $\sigma_i^*(\Lambda)$ is the unique one which makes the map

$$\begin{aligned} \sigma_i^*(\Lambda) &\longrightarrow \mu_{p^2} = \mathrm{Spec}(R[Z_1,Z_2]/(Z_1^p-1, Z_2^p-Z_1^i)) \\ Z_1 &\longmapsto 1+\mu S_1 \\ Z_2 &\longmapsto 1+\lambda S_2 \end{aligned}$$

a morphism of group schemes.

As remarked in [**49**, 4.4], there is the following long exact sequence

(II.38)

$$\begin{aligned} 0 \longrightarrow \; & \mathrm{Hom}_{gr}(G_{\mu,1},\mathcal{G}^{(\lambda')}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1},\mathbb{G}_m) \xrightarrow{r_{\lambda'}} \mathrm{Hom}_{gr}(G_{\mu,1|S_{\lambda'}},\mathbb{G}_{m|S_{\lambda'}}) \xrightarrow{\delta} \\ & \longrightarrow \mathrm{Ext}^1(G_{\mu,1},\mathcal{G}^{(\lambda')}) \xrightarrow{\alpha_*^{\lambda'}} \mathrm{Ext}^1(G_{\mu,1},\mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(G_{\mu,1|S_{\lambda'}},\mathbb{G}_{m|S_{\lambda'}}). \end{aligned}$$

We so have

$$\text{(II.39)} \qquad \ker \alpha_*^{\lambda'} \simeq Im\delta \simeq \mathrm{Hom}_{gr}(G_{\mu,1|S_{\lambda'}},\mathbb{G}_{m|S'_\lambda})/r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1},\mathbb{G}_m)).$$

We remark that by (II.36), setting $\lambda'=1$, it follows that

$$\mathrm{Hom}_{gr}(G_{\mu,1},\mathbb{G}_m) \simeq \mathbb{Z}/p\mathbb{Z}$$

and the group is formed by the morphisms $S \longmapsto (1+\mu S)^i$. While, by (II.36), $\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda')}) \simeq \mathbb{Z}/p\mathbb{Z}$ if $\lambda'|\mu$ and it is 0 otherwise. Hence, by (II.38), if $\lambda'|\mu$ then $r_{\lambda'}$ is the zero morphism, otherwise $r_{\lambda'}$ is an isomorphism. Hence, by (II.39),

$$\ker \alpha_*^{\lambda'} \simeq \mathrm{Hom}_{gr}(G_{\mu,1_{|S_{\lambda'}}}, \mathbb{G}_{m|S_{\lambda'}})/ < 1+\mu S > . \tag{II.40}$$

In the following we give a more explicit description of the main ingredients of the exact sequences (II.35) and (II.38).

II.3.4. Explicit description of $\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$. First we consider the simplest cases. If $\lambda \in \pi R$,

$$\mathrm{Hom}_{gr}(\mu_{p_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) = \{S^i \in (R/\lambda R)[S,1/S] | i \in \mathbb{Z}/p\mathbb{Z}\}. \tag{II.41}$$

While if $\lambda \in R^*$ we have $S_\lambda = \emptyset$ and $\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) = \{1\}$.

Now we study $\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$ for $\mu, \lambda \in \pi R \setminus \{0\}$.

PROPOSITION II.3.15. *Let $\lambda, \mu \in \pi R \setminus \{0\}$. The map*

$$i^* : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$$

induced by

$$i : G_{\mu,1} \hookrightarrow \mathcal{G}^{(\mu)}$$

is surjective. If $p > 2$, $\xi^0_{R/\lambda}$, defined in II.3.11, induces an isomorphism

$$\widehat{W}(R/\lambda R)^{\mathrm{F}-[\mu^{p-1}]}/\left\{[\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b}) | \boldsymbol{b} \in \widehat{W}(R/\lambda R)^{\mathrm{F}-[\mu^{p(p-1)}]}\right\} \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$$

PROOF. We have by definitions that

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) = \{F(S) \in \left(R/\lambda R[S]/(\frac{(1+\mu S)^p - 1}{\mu^p})\right)^* | \\ F(S)F(T) = F(S+T+\mu ST)\}$$

and

$$\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{F(S) \in \left(R/\lambda R[S, \frac{1}{1+\mu S}]\right)^* | \\ F(S)F(T) = F(S+T+\mu ST)\}.$$

Since $(G_{\mu,1})_k$ is isomorphic to α_p or $\mathbb{Z}/p\mathbb{Z}$ then the group $\mathrm{Hom}_{gr}((G_{\mu,1})_k, \mathbb{G}_{m,k})$ is trivial. So $F(S) \equiv 1 \mod \pi$. Moreover any $F(S) \in (R/\lambda R)[S]/(\frac{(1+\mu S)^p - 1}{\mu^p})$ such that $F(S) \equiv 1 \mod \pi$ is invertible. The same is true in $\mathrm{Hom}_{gr}({\mathcal{G}^{(\mu)}}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ since $\mathcal{G}^{(\mu)}_k \simeq \mathbb{G}_a$. We now say that F satisfies condition ($\sharp$) if

$F(S) \equiv 1 \mod \pi$;
$F(S)F(T) = F(S+T+\mu ST)$.

Then

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) = \{F(S) \in R/\lambda R[S]/(\frac{(1+\mu S)^p - 1}{\mu^p})|F(S) \text{ satifies } (\sharp)\}$$

and

$$\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{F(S) \in R/\lambda R[S, \frac{1}{1+\mu S}]|F(S) \text{ satifies } (\sharp)\}.$$

Any $F \in R/\lambda R[S]/(\frac{(1+\mu S)^p-1}{\mu^p})$ can be represented by a polynomial of degree $p-1$. And if it satisfies $(\sharp)$, it also satisfies $(\sharp)$ in $R/\lambda R[S, \frac{1}{1+\mu S}]$.

So

$$i^* : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$$

is surjective.

Now, by the exact sequence

$$(\Lambda') \qquad 0 \longrightarrow G_{\mu,1} \xrightarrow{i} \mathcal{G}^{(\mu)} \xrightarrow{\psi_{\mu,1}} \mathcal{G}^{(\mu^p)} \longrightarrow 0$$

over S_λ, we have the long exact sequence of cohomology

$$0 \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu^p)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \xrightarrow{\psi^*_{\mu,1}} \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \xrightarrow{i^*}$$
$$\longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) \xrightarrow{\delta''} \mathrm{Ext}^1(\mathcal{G}^{(\mu^p)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \dots$$

By II.3.11 we have that

$$\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \simeq \widehat{W}(R/\lambda R)^{F-[\mu^{p-1}]}$$

and, by (II.32),

$$\psi^*_{\mu,1}(\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu^p)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})) \simeq \left\{[\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b})|\boldsymbol{b} \in \widehat{W}(R/\lambda R)^{\mathrm{F}-[\mu^{p(p-1)}]}\right\}.$$

Therefore the proposition is proved. □

We now give a more explicit description of $\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$.

PROPOSITION II.3.16. *If $\lambda, \mu \in R$ with $v(p) \geq (p-1)v(\mu) > 0$ and $v(p) \geq v(\lambda)$, then*

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) = \{E_p(a, \mu; S) = 1 + \sum_{i=1}^{p-1} \frac{\prod_{k=0}^{i-1}(a - k\mu)}{i!} S^i$$
$$\text{s.t. } a \in R/\lambda R \text{ and } a^p = \mu^{p-1} a \in R/\lambda R\}$$

REMARK II.3.17. In [**47**, 3.5], an inductive formula for the coefficients of the polynomials $F(T) \in \mathrm{Hom}(\mathcal{G}^{(\mu)}{}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ is given. If we consider only polynomials of degree less or equal to $p-1$, it coincides with (II.44). But for the reader's convenience, we prefer to give here a direct proof of this formula.

REMARK II.3.18. If $v(\mu) \geq v(\lambda)$ then

$$\mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{\sum_{i=0}^{p-1} \frac{a^i}{i!} T^i | a^p = 0\} = \{E_p(aT) | a^p = 0\}$$

PROOF. As seen in II.3.15

$$\mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) = \{F(S) = \sum_{i=0}^{p-1} a_i S^i \in R/\lambda R[S]/(\frac{(1+\mu S)^p - 1}{\mu^p})$$
$$\text{s.t. } F(S) \equiv 1 \mod \pi \text{ and } F(S)F(T) = F(S+T+\mu ST)\}.$$

Now

(II.42)
$$\begin{aligned} F(S+T+\mu ST) &= \sum_{i=0}^{p-1} a_i (S+T+\mu ST)^i \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j} \binom{j}{k} \mu^{i-j} a_i S^{k+i-j} T^{i-k} \\ &= \sum_{r=0}^{p-1} \sum_{l=0}^{p-1} \sum_{\max\{r,l\} \leq i \leq r+l} \binom{i}{2i-(r+l)} \binom{2i-(r+l)}{i-l} \mu^{r+l-i} a_i S^r T^l \end{aligned}$$

and

$$F(S)F(T) = \sum_{r=0}^{p-1} \sum_{l=0}^{p-1} a_r a_l S^r T^l. \tag{II.43}$$

So we have the equality if and only if

$$\begin{aligned} a_r a_l &= \sum_{\max\{r,l\} \leq i \leq r+l} \binom{i}{2i-(r+l)} \binom{2i-(r+l)}{i-l} \mu^{r+l-i} a_i \\ &= \sum_{\max\{r,l\} \leq i \leq r+l} \frac{i!}{(r+l-i)!(i-l)!(i-r)!} \mu^{r+l-i} a_i \end{aligned} \tag{II.44}$$

for any $0 \leq r, l \leq p-1$. Clearly $a_0 = 1$.

We now have the following lemma:

LEMMA II.3.19. *For any $\mu, \lambda \in \pi R \backslash \{0\}$, the following statements are equivalent*

i) $a_r = \frac{\prod_{k=0}^{r-1}(a_1 - k\mu)}{r!}$ *for any* $1 \leq r \leq p-1$ *and* $\prod_{k=0}^{p-1}(a_1 - k\mu) = 0$*;*
ii) $a_{r-1}a_1 = (r-1)\mu a_{r-1} + r a_r$ *for any* $1 \leq r \leq p-1$*;*
iii) $a_r a_l = \sum_{\max\{r,l\} \leq i \leq r+l} \frac{i!}{(r+l-i)!(i-l)!(i-r)!} \mu^{r+l-i} a_i$ *for any* $1 \leq l, r \leq p-1$.

PROOF. In the following we use the convention that $a_i = 0$ if $i > p-1$.
i) $\Leftrightarrow$ *ii)*. It is clear that

$$a_{r-1}a_1 = (r-1)\mu a_{r-1} + r a_r$$

is equivalent to $a_r = a_{r-1}\frac{a_1-\mu(r-1)}{r}$, if $r < p$, and $a_{p-1}(a_1 - \mu(p-1)) = 0$. An easy induction shows that this is equivalent to

$$a_r = \frac{\prod_{k=0}^{r-1}(a_1 - k\mu)}{r!}$$

if $r < p - 1$ and

$$\prod_{k=0}^{p-1}(a_1 - k\mu) = 0.$$

ii) $\Leftarrow$ *iii)*. It is obvious.

ii) $\Rightarrow$ *iii)*. We will prove it by induction on l. By hypothesis $a_{r-1}a_1 = (r-1)\mu a_{r-1} + ra_r$ for any r. We now suppose that *iii)* is true for $k \le l-1$ for any r. Then we will prove it is also true for l for any r. We can clearly suppose $l \le r$, otherwise, up to a change of l with r, we can conclude by induction. We have

$$\begin{aligned}
a_r a_l &\overset{ii)}{=} a_r(a_{l-1}\frac{a_1-\mu(l-1)}{l}) \\
&= (a_r a_{l-1})\frac{a_1-\mu(l-1)}{l} \\
&\overset{induct.}{=} \left(\sum_{r\le i\le r+l-1}\frac{i!}{(r+l-i-1)!(i-l-1)!(i-r)!}\mu^{r+l-i-1}a_i\right)\frac{a_1-\mu(l-1)}{l} \\
&\overset{induct.}{=} \sum_{r\le i\le r+l-1}\frac{i!}{(r+l-i-1)!(i-l-1)!(i-r)!}\mu^{r+l-i-1} \\
&\quad (\frac{\mu(i-l+1)a_i + (i+1)a_{i+1}}{l}) \\
&= \frac{r!}{l!(r+1-l)!}\mu^l(r+1-l)a_r + \\
&\quad + \sum_{r+1\le i\le r+l-1}\left(\frac{i!}{(r+l-i)!(i-l)!(i-r-1)!l}\mu^{r+l-i} + \right. \\
&\quad \left. + \frac{i!(i-l+1)}{(r+l-i)!(i-l+1)!(i-r)!l}\mu^{r+l-i}\right)a_i + \frac{(r+l-1)!(r+l)}{r!l!}a_{r+l} \\
&= \sum_{r\le i\le r+l}\frac{i!}{(r+l-i)!(i-l)!(i-r)!}\mu^{r+l-i}a_i.
\end{aligned}$$

□

We come back to the proof of the proposition. In $R/\lambda R$ the condition

$$\prod_{k=0}^{p-1}(a_1 - k\mu) = 0$$

is equivalent to $a_1^p = \mu^{p-1}a_1$. Indeed we have the following equality in $\mathbb{Z}/p\mathbb{Z}[S]$

$$\prod_{k=0}^{p-1}(S - k) = S^p - S,$$

since these polynomials have the same zeros. Since $p = 0 \in R/\lambda R$, then

$$\prod_{k=0}^{p-1}(a_1 - k\mu) = a_1^p - \mu^{p-1}a_1.$$

By the lemma and II.3.10 the thesis follows. $\square$

We now essentially rewrite II.3.19 in a more expressive form.

COROLLARY II.3.20. *Let $\lambda, \mu \in \pi R \setminus \{0\}$ and let $F(S) = \sum_{i=0}^{p-1} a_i S^i \in R/\lambda R[S]$ be a polynomial of degree less than or equal to $p-1$. Then the following statements are equivalent*

(i) $F(S)F(T) - a_0^2 = F(S + T + \mu ST) - a_0$

(ii) $F(S)a_1 = F'(S)(1 + \mu S)$ *where F' is the formal derivative of F.*

REMARK II.3.21. Let us suppose $v(\mu) \geq v(\lambda)$. This corollary, together with II.3.16, says that the solution of the differential equation in $R/\lambda R[S]/(\frac{(1+\mu S)^p - 1}{\mu^p})$

$$\begin{cases} F'(S) = aF(S), \\ F(0) = 1 \end{cases}$$

has as unique solution $F(S) = E_p(aS) = \sum_{i=0}^{p-1} \frac{a^i}{i!} S^i$ and $a^p = 0$.

PROOF. By (II.44), we have that II.3.19(iii) is equivalent to

$$F(S)F(T) - a_0^2 = F(S + T + \mu ST) - a_0. \tag{II.45}$$

If we put $l = 1$ in (II.44), we obtain the coefficient of T in both members of (II.45). This means that II.3.19(ii) is equivalent to

$$F(S)a_1 = F'(S)(1 + \mu S).$$

Then their equivalence comes from II.3.19.

$\square$

When $v(\mu) \geq v(\lambda)$, putting together II.3.15 and II.3.16, we have a simpler description of $\mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda})$.

COROLLARY II.3.22. *Let $p > 2$. Let $\lambda, \mu \in R$ with $v(p) \geq (p-1)v(\mu) > 0$ and $v(\mu) \geq v(\lambda) > 0$. Then we have the following isomorphism of groups*

$$(\xi^0_{R/\lambda R})_p : (R/\lambda R)^{\mathrm{F}} \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda})$$

given by

$$a \longmapsto E_p(aS).$$

Moreover the restriction map

$$i^* : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \simeq \widehat{W}(R/\lambda R)^{\mathrm{F}} \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \simeq (R/\lambda R)^{\mathrm{F}}$$

is given, in terms of Witt vectors, by

$$\boldsymbol{a} = (a_0, a_1, \dots, 0, 0, 0, \dots) \longmapsto \sum_{i=0}^{\infty}(-1)^i(\frac{p}{\mu^{p-1}})^i a_i$$

PROOF. We first remark that the restriction of the Teichmüller map

$$T : (R/\lambda R)^{\mathrm{F}} \longrightarrow \widehat{W}(R/\lambda R)^{\mathrm{F}},$$

given by

$$a \longmapsto [a],$$

is a morphism of groups. This follows from II.3.9. Moreover, if we consider the isomorphism

$$\xi^0_{R/\lambda R} : \widehat{W}(R/\lambda R)^{\mathrm{F}} \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$$

and

$$i^* : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}),$$

we have

$$i^* \circ \xi^0_{R/\lambda R} \circ T = (\xi^0_{R/\lambda R})_p.$$

So $(\xi^0_{R/\lambda R})_p$ is a morphism of groups. It is surjective by II.3.16 and, by II.3.15, its kernel is

$$T((R/\lambda R)^{\mathrm{F}}) \cap \left\{ [\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b}) | \boldsymbol{b} \in \widehat{W}(R/\lambda R)^{\mathrm{F}} \right\}.$$

Let us now suppose that there exists $\boldsymbol{b} = (b_0, b_1, \dots) \in \widehat{W}(R/\lambda R)^{\mathrm{F}}$ and $a \in (R/\lambda R)^{\mathrm{F}}$ such that $[\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b}) = [a]$. It follows by the definition of Witt vector ring that

$$[\frac{p}{\mu^{p-1}}]\boldsymbol{b} = (\frac{p}{\mu^{p-1}} b_0, \dots, (\frac{p}{\mu^{p-1}})^{p^j} b_j, \dots), \tag{II.46}$$

and

$$[a] - V(\boldsymbol{b}) = (a_0, -b_0, -b_1, \dots). \tag{II.47}$$

Since $\boldsymbol{b} \in \widehat{W}(R/\lambda R)$, there exists $r \geq 0$ such that $b_j = 0$ for any $j \geq r$. Moreover, comparing (II.46) and (II.47) it follows

$$(\frac{p}{\mu^{p-1}})^{p^j} b_{j+1} = -b_j \quad \text{for } j \geq 0$$

$$(\frac{p}{\mu^{p-1}})^p b_0 = a.$$

Hence $b_j = a = 0$ for any $j \geq 0$. It follows that $(\xi^0_{R/\lambda R})_p$ is injective.

We now prove the second part of the statement. First of all we remark that for any $\boldsymbol{a} = (a_0, \dots, a_j, \dots) \in \widehat{W}(R/\lambda R)^{\mathrm{F}}$ we have

$$\boldsymbol{a} = \sum_{j=0}^{\infty} V^j([a_j]).$$

It is clear that for any $a \in R/\lambda R$ we have $i^*([a]) = a$. While, by II.3.15, it follows that $i^*V(\boldsymbol{b}) = -i^*([\frac{p}{\mu^{p-1}}]\boldsymbol{b})$ for any $\boldsymbol{b} \in \widehat{W}(R/\lambda R)^{\mathrm{F}}$. Hence $i^*V^j(\boldsymbol{b}) =$

$(-1)^j i^*([(\frac{p}{\mu^{p-1}})^j]\boldsymbol{b})$ for any $j \geq 1$. From these facts it follows that

$$\begin{aligned} i^*(\boldsymbol{a}) &= i^*(\sum_{j=0}^{\infty} V^j([a_j])) \\ &= \sum_{j=0}^{\infty}(i^*(V^j([a_j]))) \\ &= \sum_{j=0}^{\infty}(-1)^j(\frac{p}{\mu^{p-1}})^j a_j. \end{aligned}$$

□

II.3.5. Explicit description of δ. The map

$$\delta : \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)})$$

can also be explicitly described. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) & \longrightarrow & \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \delta & & \\ \mathrm{Ext}^1(\mathcal{G}^{(\mu)}, \mathcal{G}^{(\lambda)}) & \xrightarrow{i^*} & \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)}) & & \end{array}$$

where the first horizontal map is surjective by II.3.15. So, given

$$F(S) \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}),$$

we can choose a representant in $\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ which we denote again by $F(S)$ for simplicity. Then δ is defined by

$$F(S) \longmapsto \widetilde{\mathcal{E}}^{(\mu,\lambda;F)} := i^*(\mathcal{E}^{(\mu,\lambda;F)}) = i^*(\alpha(F(S))).$$

If $\tilde{F}(S) \in R[S]$ is any lifting then it is defined, as a scheme, by

$$\widetilde{\mathcal{E}}^{(\mu,\lambda;F)} = \mathrm{Spec}\left(R[S_1, S_2, (\tilde{F}(S_1) + \lambda S_2)^{-1}]/\frac{(1+\mu S_1)^p - 1}{\mu^p}\right).$$

This extension does not depend on the choice of the lifting since the same is true for $\mathcal{E}^{(\mu,\lambda;F)}$.

So, by (II.38), we see that $\ker(\alpha_*^\lambda) \subseteq \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)})$ is nothing else but the group $\{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\}$. We recall that, by (II.39), for any $\lambda' \in R$,

$$\ker \alpha_*^{\lambda'} \simeq \mathrm{Hom}_{gr}(G_{\mu,1_{|S_{\lambda'}}}, \mathbb{G}_{m|S_{\lambda'}})/r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)).$$

We have therefore proved the following proposition.

PROPOSITION II.3.23. *Let* $\lambda, \mu \in R \setminus \{0\}$ *with* $v(p) \geq (p-1)v(\mu)$. *Then* δ *induces an isomorphism*

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})/r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) \longrightarrow \{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\}$$
$$F(S) \longmapsto \widetilde{\mathcal{E}}^{(\mu,\lambda;F)}$$

REMARK II.3.24. As remarked in §II.3.3, if $\lambda' \mid \mu$ then

$$r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) = 0,$$

otherwise

$$r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) \simeq < 1 + \mu S > \simeq \mathbb{Z}/p\mathbb{Z}.$$

EXAMPLE II.3.25. a) Let us suppose $v(\mu) = 0$ and $v(\lambda) > 0$. Since, by (II.41) and the previous remark,

$$\mathrm{Hom}_{gr}(\mu_{p|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \simeq r_\lambda(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) \simeq \mathbb{Z}/p\mathbb{Z}$$

then $\{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\} = 0$.

b) Let us suppose $v(\lambda) = 0$. Since $S_\lambda = \emptyset$, then $\{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\} = 0$.

II.3.6. Interpretation of $\mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m)$. First of all, we briefly recall a useful spectral sequence . Let $\mathcal{E}xt^i(G, H)$ denote the fppf-sheaf on $Sch_{/S}$, associated to the presheaf $X \longmapsto \mathrm{Ext}^i\,(G \times_S X, H \times_S X)$. Then we have a spectral sequence

$$E_2^{ij} = H^i(S, \mathcal{E}xt^j(G, H)) \Rightarrow \mathrm{Ext}^{i+j}(G, H),$$

which in low degrees gives

$$\begin{aligned} 0 \longrightarrow H^1(S, \mathcal{E}xt^0(G, H)) \longrightarrow \mathrm{Ext}^1(G, H) \longrightarrow H^0(S, \mathcal{E}xt^1(G, H)) \longrightarrow \\ \longrightarrow H^2(S, \mathcal{E}xt^0(G, H)) \longrightarrow \mathrm{Ext}^2(G, H). \end{aligned} \tag{II.48}$$

Moreover $H^1(S, \mathcal{E}xt^0(G, H))$ is isomorphic to the subgroup of $\mathrm{Ext}^1(G, H)$ formed by the extensions E which split over some faithfully flat affine S-scheme of finite type (cf. [**15**, III 6.3.6]). We suppose that G acts trivially on H, then $\mathcal{E}xt^0(G, H) = \mathcal{H}om_{gr}(G, H)$. We will consider the case $H = \mathbb{G}_m$ and G a finite flat group scheme. In this case, $\mathcal{E}xt^0(G, \mathbb{G}_m)$ is by definition the Cartier dual of G, denoted by $G^\vee$. We recall the following result which will play a role in the description of extensions of $G_{\mu,1}$ by $G_{\lambda,1}$ (see II.3.35 below).

THEOREM II.3.26. *Let G be a commutative finite flat group scheme over S. Then the canonical map*

$$H^1(S, G^\vee) \longrightarrow \mathrm{Ext}^1(G, \mathbb{G}_m)$$

is bijective.

PROOF. This is a Theorem of S.U. Chase. For a proof see [**56**]. We stress that he proves $\mathcal{E}xt^1(G, \mathbb{G}_m) = 0$, then he applies (II.48). We remark that he proves everything in the fpqc site. However the same proof works in the fppf site. □

We apply this result to $G = G_{\lambda,1}$. We have that the map

$$H^1(S, G^{\vee}_{\lambda,1}) \longrightarrow \mathrm{Ext}^1(G_{\lambda,1}, \mathbb{G}_m)$$

is an isomorphism.

By II.3.26 and I.3.6 we obtain the following result.

COROLLARY II.3.27. *Let G be a commutative finite flat group scheme over S. The restriction map*

$$\mathrm{Ext}^1(G, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(G_K, \mathbb{G}_{m|K})$$

is injective.

Let us consider a commutative finite and flat group scheme G of order n. We also consider the n^{th} power map $n : \mathbb{G}_m \longrightarrow \mathbb{G}_m$. It induces a morphism $n_* : \mathrm{Ext}^1(G, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(G, \mathbb{G}_m)$. We have the following commutative diagram

$$\begin{array}{ccc} H^1(S, G^{\vee}) & \longrightarrow & \mathrm{Ext}^1(G, \mathbb{G}_m) \\ \downarrow n_* & & \downarrow n_* \\ H^1(S, G^{\vee}) & \longrightarrow & \mathrm{Ext}^1(G, \mathbb{G}_m), \end{array}$$

where the horizontal maps are isomorphisms by II.3.35. We remark that $n_* : H^1(S, G^{\vee}) \longrightarrow H^1(S, G^{\vee})$ is the zero morphism since the map $n_* : G^{\vee} \longrightarrow G^{\vee}$, induced by $n : \mathbb{G}_m \longrightarrow \mathbb{G}_m$, is the zero morphism. This proves the following lemma.

LEMMA II.3.28. *Let G be a commutative finite and flat group scheme of order n. Then*

$$n_* : \mathrm{Ext}^1(G, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(G, \mathbb{G}_m)$$

is the zero morphism.

II.3.7. Description of $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$. We finally have all the ingredients to give a description of the group $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$. In particular we will focus on the extensions which are isomorphic, as group schemes, to $\mathbb{Z}/p^2\mathbb{Z}$ on the generic fiber.

First of all we remark that if $v(\mu_1) = v(\mu_2)$ and $v(\lambda_1) = v(\lambda_2)$ then

$$\mathrm{Ext}^1(G_{\mu_1,1}, G_{\lambda_1,1}) \simeq \mathrm{Ext}^1(G_{\mu_2,1}, G_{\lambda_2,1}).$$

Indeed we know, by hypothesis, that there exist two isomorphisms $\psi_1 : G_{\lambda_1,1} \longrightarrow G_{\lambda_2,1}$ and $\psi_2 : G_{\mu_2,1} \longrightarrow G_{\mu_1,1}$. Then we have that

$$(\psi_1)_* \circ (\psi_2)^* : \mathrm{Ext}^1(G_{\mu_1,1}, G_{\lambda_1,1}) \longrightarrow \mathrm{Ext}^1(G_{\mu_2,1}, G_{\lambda_2,1})$$

is an isomorphism.

We now recall what happens if $v(\mu) = v(\lambda) = 0$. In this case we have the following result.

PROPOSITION II.3.29. *Let A be a d.v.r or a field. Then there exists an exact sequence*

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Ext}^1_A(\mu_p, \mu_p) \longrightarrow H^1(\mathrm{Spec}(A), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

PROOF. The proposition is proved in [**46**, 3.7] when A a is d.v.r. The same proof works when A is a field. □

Let us define the extension of μ_p by μ_p

$$\mathcal{E}_{i,A} = \mathrm{Spec}(A[S_1, S_2]/(S_1^p - 1, \frac{S_2^p}{S_1^i} - 1)).$$

It is the kernel of the morphism $(\mathbb{G}_m)^2 \longrightarrow (\mathbb{G}_m)^2$ given by $(S_1, S_2) \longrightarrow (S_1^p, S_1^{-1}S_2^p)$ Then the group $\mathbb{Z}/p\mathbb{Z}$ in the above proposition is formed by the extensions $\mathcal{E}_{i,A}$.

DEFINITION II.3.30. Let $F \in \mathrm{Hom}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$, $j \in \mathbb{Z}/p\mathbb{Z}$ such that

$$F(S)^p(1+\mu S)^{-j} = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}).$$

Let $\tilde{F}(S) \in R[S]$ be a lifting of F. We denote by $\mathcal{E}^{(\mu,\lambda;F,j)}$ the subgroup scheme of $\mathcal{E}^{(\mu,\lambda;F)}$ given on the level of schemes by

$$\mathcal{E}^{(\mu,\lambda;F,j)} = \mathrm{Spec}\left(R[S_1, S_2]/\Big(\frac{(1+\mu S_1)^p - 1}{\mu^p}, \frac{(\tilde{F}(S_1) + \lambda S_2)^p(1+\mu S_1)^{-j} - 1}{\lambda^p}\Big)\right).$$

We moreover define the following homomorphisms of group schemes

$$G_{\lambda,1} \longrightarrow \mathcal{E}^{(\mu,\lambda;F,j)}$$

by

$$S_1 \longmapsto 0$$

$$S_2 \longmapsto S + \frac{1 - \tilde{F}(0)}{\lambda}$$

and

$$\mathcal{E}^{(\mu,\lambda;F,j)} \longrightarrow G_{\mu,1}$$

by

$$S \longrightarrow S_1.$$

It is easy to see that

$$0 \longrightarrow G_{\lambda,1} \longrightarrow \mathcal{E}^{(\mu,\lambda;F)} \longrightarrow G_{\mu,1} \longrightarrow 0$$

is exact. A different choice of the lifting $\tilde{F}(S)$ gives an isomorphic extension. It is easy to see that $(\mathcal{E}^{(\mu,\lambda;F,j)})_K \simeq (\mathbb{Z}/p^2\mathbb{Z})_K$, as a group scheme, if $j \neq 0$ and $(\mathcal{E}^{(\mu,\lambda;F,0)})_K \simeq (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})_K$.

REMARK II.3.31. In the above definition the integer j is uniquely determined by $F \in \mathrm{Hom}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$ if and only if $\lambda^p \nmid \mu$.

From the exact sequence over S_λ

$$0 \longrightarrow G_{\mu,1} \xrightarrow{i} \mathcal{G}^{(\mu)} \xrightarrow{\psi_{\mu,1}} \mathcal{G}^{(\mu^p)} \longrightarrow 0$$

we have that
(II.49)

$$\ker\left(i^*:\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda},\mathbb{G}_{m|S_\lambda})\longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}},\mathbb{G}_{m|S_\lambda})\right)=\psi_{\mu,1_*}\mathrm{Hom}_{gr}(\mathcal{G}^{(\mu^p)}_{|S_\lambda},\mathbb{G}_{m|S_\lambda})$$

So let $F(S)\in\mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}},\mathbb{G}_{m|S_{\lambda^p}})$. By II.3.15 we can choose a representant of $F(S)$ in $\in\mathrm{Hom}(\mathcal{G}^{(\mu)}{}_{|S_{\lambda^p}},\mathbb{G}_{m|S_{\lambda^p}})$ which we denote again $F(S)$ for simplicity. Therefore, by (II.49), we have that $F(S)^p(1+\mu S)^{-j}=1\in\mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}},\mathbb{G}_{m|S_{\lambda^p}})$ is equivalent to saying that there exists $G\in\mathrm{Hom}(\mathcal{G}^{(\mu^p)}{}_{|S_{\lambda^p}},\mathbb{G}_{m|S_{\lambda^p}})$ with the property that $F(S)^p(1+\mu S)^{-j}=G(\frac{(1+\mu S)^p-1}{\mu^p})\in\mathrm{Hom}(\mathcal{G}^{(\mu)}{}_{|S_{\lambda^p}},\mathbb{G}_{m|S_{\lambda^p}})$. This implies that $\mathcal{E}^{(\mu,\lambda;F,j)}$ can be seen as the kernel of the isogeny

$$\begin{aligned}\psi^j_{\mu,\lambda,F,G}:\mathcal{E}^{(\mu,\lambda;F)}&\longrightarrow\mathcal{E}^{(\mu^p,\lambda^p;G)}\\ S_1&\longmapsto\frac{(1+\mu S_1)^p-1}{\mu^p}\\ S_2&\longmapsto\frac{(\tilde{F}(S_1)+\lambda S_2)^p(1+\mu S_1)^{-j}-\tilde{G}(\frac{(1+\mu S_1)^p-1}{\mu^p})}{\lambda^p}\end{aligned}$$

where $\tilde{F},\tilde{G}\in R[T]$ are liftings of F and G.

As remarked in II.3.18, if $v(\mu)\geq v(\lambda)$ we can suppose

$$\tilde{F}(S)=\sum_{i=0}^{p-1}\frac{a^i}{i!}S^i$$

with $a^p\equiv 0 \mod \lambda$.

EXAMPLE II.3.32. This example has been the main motivation for our definition of the group schemes $\mathcal{E}^{(\mu,\lambda;F,j)}$. Let us define

$$\eta=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\lambda^k_{(2)}.$$

We remark that $v(\eta)=v(\lambda_{(2)})$. We consider

$$F(S)=E_p(\eta S)=\sum_{k=1}^{p-1}\frac{(\eta S)^k}{k!}$$

It has been shown in [**53**, §5] that, using our notation,

$$\mathbb{Z}/p^2\mathbb{Z}\simeq\mathcal{E}^{(\lambda_{(1)},\lambda_{(1)};E_p(\eta S),1)}.$$

A similar description of $\mathbb{Z}/p^2\mathbb{Z}$ was independently found by Green and Matignon ([**20**]).

EXAMPLE II.3.33. It is easy to see that the group scheme $G_{\lambda,2}$ is isomorphic to

$$\mathcal{E}^{(\lambda^p,\lambda;1,1)}.$$

Moreover if we have an extension of type $\mathcal{E}^{(\mu,\lambda;F,j)}$ with $F(S)=1$ then $v(\mu)\geq pv(\lambda)$. Indeed we have that

$$\mathcal{E}^{(\mu,\lambda;1,j)} = \operatorname{Spec}\left(A[S_1,S_2]/\left(\frac{(1+\mu S_1)^p-1}{\mu^p}, \frac{(1+\lambda S_2)^p(1+\mu S_1)^{-j}-1}{\lambda^p}\right)\right).$$

Since $(1+\lambda S_2)^p = 1 \in \operatorname{Hom}_{gr}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_\lambda^p})$ then $(1+\lambda S_2)^p(1+\mu S_1)^{-j} = 1 \in \operatorname{Hom}_{gr}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}})$ if and only if $v(\mu)\geq pv(\lambda)$. In particular we remark that, in such a case, $v(\lambda)\leq v(\lambda_{(2)})$. Otherwise

$$pv(\lambda) > pv(\lambda_{(2)}) = v(\lambda_{(1)}) \geq v(\mu),$$

which is not possible.

Let us define, for any $\mu,\lambda\in R$ with $v(\mu),v(\lambda)\leq v(\lambda_{(1)})$, the group

$$rad_{p,\lambda}(<1+\mu S>) := \Big\{(F(S),j)\in \operatorname{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})\times \mathbb{Z}/p\mathbb{Z} \text{ such that}$$

$$F(S)^p(1+\mu S)^{-j} = 1 \in \operatorname{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}})\Big\}/ < (1+\mu S, 0) > .$$

We define

$$\beta : rad_{p,\lambda}(<1+\mu S>) \longrightarrow \operatorname{Ext}^1(G_{\mu,1}, G_{\lambda,1})$$

by

$$(F(S),j) \longmapsto \mathcal{E}^{(\mu,\lambda;F(S),j)}$$

We remark that the image of β is the set $\{\mathcal{E}^{(\mu,\lambda;F(S),j)}\}$.

LEMMA II.3.34. *β is a morphism of groups. In particular the set $\{\mathcal{E}^{(\mu,\lambda;F(S),j)}\}$ is a subgroup of $\operatorname{Ext}^1(G_{\mu,1}, G_{\lambda,1})$.*

PROOF. Let $i: G_{\lambda,1}\longrightarrow \mathcal{G}^{(\lambda)}$. We remark that

$$i_*(\beta(F,j)) = i_*(\mathcal{E}^{(\mu,\lambda;F(S),j)}) = \tilde{\mathcal{E}}^{(\mu,\lambda;F)} = \delta(F)$$

for any $(F,j)\in rad_{p,\lambda}(<1+\mu S>)$. Moreover by construction

$$(\mathcal{E}^{(\mu,\lambda;F(S),j)})_K = (\mathcal{E}_j)_K \in \operatorname{Ext}^1(\mu_{p,K},\mu_{p,K}).$$

Let $(F_1,j_1),(F_2,j_2)\in rad_{p,\lambda}(<1+\mu S>)$. Then

$$i_*(\beta(F_1,j_1)+\beta(F_2,j_2)-\beta(F_1+F_2,j_1+j_2)) = \delta(F_1)+\delta(F_2)-\delta(F_1+F_2) = 0 \tag{II.50}$$

since δ is a morphism of groups. And

$$(\beta(F_1,j_1)+\beta(F_2,j_2)-\beta(F_1+F_2,j_1+j_2))_K = \mathcal{E}_{j_1,K}+\mathcal{E}_{j_2,K}-\mathcal{E}_{j_1+j_2,K} = 0, \tag{II.51}$$

since $\mathbb{Z}/p\mathbb{Z}\simeq \operatorname{Ext}^1(\mu_{p,K},\mu_{p,K})$ through the map $j\mapsto \mathcal{E}_{j,K}$. By (II.50) it follows that

$$\beta(F_1,j_1)+\beta(F_2,j_2)-\beta(F_1+F_2,j_1+j_2)\in \ker i_*.$$

and then, by (II.35) and (II.37), we have

$$\beta(F_1, j_1) + \beta(F_2, j_2) - \beta(F_1 + F_2, j_1 + j_2) = (\sigma_j)^*\Lambda.$$

for some $j \in \mathbb{Z}/p\mathbb{Z}$. By (II.51) it follows that

$$((\sigma_j)^*\Lambda)_K = \mathcal{E}_{j,K} = 0,$$

therefore $j = 0$. So β is a morphism of groups. The last assertion is clear. □

We now give a description of $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$.

THEOREM II.3.35. *Suppose that $\lambda, \mu \in R$ with $v(\lambda_{(1)}) \geq v(\lambda), v(\mu)$. The following sequence*

$$0 \longrightarrow rad_{p,\lambda}(< 1 + \mu S >) \xrightarrow{\beta} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{\alpha_*^\lambda \circ i_*} \longrightarrow \ker\Big(H^1(S, G_{\mu,1}^\vee) \longrightarrow H^1(S_\lambda, G_{\mu,1}^\vee)\Big)$$

is exact. In particular β induces an isomorphism $rad_{p,\lambda}(< 1 + \mu S >) \simeq \{\mathcal{E}^{(\mu,\lambda;F,j)}\}$.

PROOF. Using (II.38) and II.3.23, we consider the following commutative diagram
(II.52)

$$\begin{array}{ccccccccc}
0 \longrightarrow & \{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\} & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)}) & \xrightarrow{\alpha_*^\lambda} & \mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \\
& \downarrow \widetilde{\psi_{\lambda,1_*}} & & \downarrow \psi_{\lambda,1_*} & & \downarrow p_* & & \downarrow p_* \\
0 \longrightarrow & \{\widetilde{\mathcal{E}}^{(\mu,\lambda^p;G)}\} & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) & \xrightarrow{\alpha_*^{\lambda^p}} & \mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda})
\end{array}$$

The map $\widetilde{\psi_{\lambda,1_*}}$, induced by $\psi_{\lambda,1_*} : \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda^p)})$, is given by $\widetilde{\mathcal{E}}^{(\mu,\lambda;F)} \longmapsto \widetilde{\mathcal{E}}^{(\mu,\lambda^p;F^p)}$. Now, since $G_{\mu,1}$ is of order p then, $p_* : \mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m) \to \mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m)$ is the zero map (see II.3.28). Moreover, by (II.48) and II.3.26, we have the following situation

$$\begin{array}{ccccc}
0 \longrightarrow & H^1(S, G_{\mu,1}^\vee) & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1}, \mathbb{G}_m) & \longrightarrow 0 \\
& \downarrow & & \downarrow & \\
0 \longrightarrow & H^1(S_\lambda, G_{\mu,1}^\vee) & \longrightarrow & \mathrm{Ext}^1(G_{\mu,1|S_\lambda}, \mathbb{G}_{m|S_\lambda}) &
\end{array}$$

which implies that $Im(\alpha_*^\lambda) \simeq \ker(H^1(S, G_{\mu,1}^\vee) \longrightarrow H^1(S_\lambda, G_{\mu,1}^\vee))$.

So applying the snake lemma to (II.52) we obtain
(II.53)

$$0 \longrightarrow \ker(\widetilde{\psi_{\lambda,1_*}}) \xrightarrow{\widetilde{\delta}} \ker(\psi_{\lambda,1_*}) \xrightarrow{\alpha_*^\lambda} \ker\Big(H^1(S, G_{\mu,1}^\vee) \longrightarrow H^1(S_\lambda, G_{\mu,1}^\vee)\Big).$$

We now divide the proof in some steps.

Connection between $\ker(\widetilde{\psi_{\lambda,1_*}})$ and $rad_{p,\lambda}(< 1 + \mu S >)$. We are going to give the connection in the form of the isomorphism (II.57) below. We recall that, by

(II.35), $i : G_{\lambda,1} \longrightarrow \mathcal{G}^{(\lambda)}$ induces an isomorphism

(II.54) $$i_* : \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})/\delta'(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)})) \longrightarrow \ker(\psi_{\lambda,1_*});$$

for the definition of δ' see (II.37).

By II.3.23 we have an isomorphism

$$\delta : \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})/r_{\lambda'}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) \longrightarrow \{\widetilde{\mathcal{E}}^{(\mu,\lambda;F)}\}$$

Through this identification we can identify $\ker(\widetilde{\psi_{\lambda,1_*}})$ with

(II.55) $$\Big\{F(S) \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) | \exists i \in r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) \text{ such that } F(S)^p(1+\mu S)^{-i} = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}})\Big\}/ < 1+\mu S > .$$

Moreover

(II.56)
$$\widetilde{\delta} : \ker(\widetilde{\psi_{\lambda,1_*}}) \hookrightarrow \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})/\delta'(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)})) \subseteq \mathrm{Ext}^1(G_{\mu,1}, \mathcal{G}^{(\lambda)})$$

is defined by $\widetilde{\delta}(F) = \delta(F) = \widetilde{\mathcal{E}}^{(\mu,\lambda;F)}$.

We now define a morphism of groups

$$\iota : \ker(\widetilde{\psi_{\lambda,1_*}}) \longrightarrow r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m))$$

as follows: for any $F(S) \in \ker(\widetilde{\psi_{\lambda,1_*}})$, $\iota(F) = i_F$ is the unique $i \in r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m))$ such that $F(S)^p(1+\mu S)^{-i} = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}})$. The morphism of groups

(II.57) $$\begin{aligned} \ker(\widetilde{\psi_{\lambda,1_*}}) \times \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) &\longrightarrow rad_{p,\lambda}(< 1+\mu S >) \\ (F, j) &\longmapsto (F, i_F + j) \end{aligned}$$

is an isomorphism. We prove only the surjectivity since the injectivity is clear. Now, if $\lambda^p \nmid \mu$ then $\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) = 0$ and $r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) = \mathbb{Z}/p\mathbb{Z}$. So, if $(F, j) \in rad_{p,\lambda}(< 1+\mu S >)$, then $j \in r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m))$. So $i_F = j$. Hence $(F, 0) \mapsto (F, i_F) = (F, j)$. While if $\lambda^p \mid \mu$ then $\mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) = \mathbb{Z}/p\mathbb{Z}$ and $r_{\lambda^p}(\mathrm{Hom}_{gr}(G_{\mu,1}, \mathbb{G}_m)) = 0$. Hence

$$\ker(\widetilde{\psi_{\lambda,1_*}}) = \Big\{F(S) \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) | F(S)^p = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}})\Big\}.$$

Let us now take $(F, j) \in rad_{p,\lambda}(< 1+\mu S >)$. This means that

$$F(S)^p = (1+\mu S)^j = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}).$$

Therefore $F(S) \in \ker(\widetilde{\psi_{\lambda,1_*}})$ and $i_F = 0$. So

$$(F, j) \longmapsto (F, i_F + j) = (F, j).$$

Interpretation of β. We now define the morphism of groups

$$\varrho : \ker(\widetilde{\psi_{\lambda,1_*}}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$$
$$F \longmapsto \beta(F, i_F) = \mathcal{E}^{(\mu,\lambda;F,i_F)}$$

We recall the definition of δ' given in (II.37):

$$\delta' : \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$$

is defined by $\delta'(\sigma_i) = \sigma_i^*(\Lambda)$. Then, under the isomorphism (II.57), we have

$$\beta = \varrho + \delta' : \ker(\widetilde{\psi_{\lambda,1_*}}) \times \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) \longrightarrow \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$$

Injectivity of β. First of all we observe that $\widetilde{\delta}$ factors through ϱ, i.e.

(II.58) $$\widetilde{\delta} = i_* \circ \varrho : \ker(\widetilde{\psi_{\lambda,1_*}}) \xrightarrow{\varrho} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{i_*} \ker(\psi_{\lambda,1_*}).$$

Indeed

$$i_* \circ \varrho(F) = i_*(\mathcal{E}^{(\mu,\lambda;F,i_F)}) = \widetilde{\mathcal{E}}^{(\mu,\lambda,F)} = \widetilde{\delta}(F).$$

In particular, since $\widetilde{\delta}$ is injective, ϱ is injective, too.

We now prove that $\beta = \varrho + \delta'$ is injective, too. By (II.54),

$$i_* \circ \delta' = 0.$$

Now, if $(\varrho + \delta')(F, \sigma_i) = 0$, then $\varrho(F) = -\delta'(\sigma_i)$. So

$$\widetilde{\delta}(F) = i_*(\varrho(F)) = i_*(-\delta'(\sigma_i)) = 0.$$

But $\widetilde{\delta}$ is injective, so $F = 1$. Hence $\delta'(\sigma_i) = 0$. But by (II.35), also δ' is injective. Then $\sigma_i = 0$.

Calculation of $Im\beta$. We finally prove $Im(\varrho+\delta') = \ker(\alpha_*^\lambda \circ i_*)$. Since $\widetilde{\delta} = i_* \circ \varrho$, $\alpha_*^\lambda \circ \widetilde{\delta} = 0$ and $i_* \circ \delta' = 0$ then

$$\alpha_*^\lambda \circ i_* \circ (\varrho + \delta') = \alpha_*^\lambda \circ i_* \circ \varrho + \alpha_*^\lambda \circ (i_* \circ \delta') = \alpha_*^\lambda \circ i_* \circ \varrho = \alpha_*^\lambda \circ \widetilde{\delta} = 0.$$

So $Im(\varrho + \delta') \subseteq \ker(\alpha_*^\lambda \circ i_*)$. On the other hand, if $E \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ is such that $\alpha_*^\lambda \circ i_*(E) = 0$, then, by (II.53), there exists $F \in \ker(\widetilde{\psi_{\lambda,1_*}})$ such that $i_*(E) = \widetilde{\delta}(F) = i_*(\varrho(F))$. Hence, by (II.54), $E - \varrho(F) \in Im(\delta')$. Therefore $Im(\varrho + \delta') = \ker(\alpha_*^\lambda \circ i^*)$. Moreover since $i_* : \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \longrightarrow \ker(\psi_{\lambda,1_*})$ is surjective then $Im(\alpha_*^\lambda) = Im(\alpha_*^\lambda \circ i_*)$. We have so proved, using also (II.53), that the following sequence

$$0 \longrightarrow \ker(\widetilde{\psi_{\lambda,1_*}}) \times \mathrm{Hom}_{gr}(G_{\mu,1}, \mathcal{G}^{(\lambda^p)}) \xrightarrow{\varrho+\delta'} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{\alpha_*^\lambda \circ i_*}$$
$$\longrightarrow \ker\left(H^1(S, G_{\mu,1}^\vee) \longrightarrow H^1(S_\lambda, G_{\mu,1}^\vee)\right)$$

is exact. Finally, by definitions, it follows that

$$\beta(rad_{p,\lambda}(< 1 + \mu S >)) = \{\mathcal{E}^{(\mu,\lambda;F,j)}\}.$$

$\square$

EXAMPLE II.3.36. Let us suppose $v(\lambda)=0$. In such a case $rad_{p,\lambda}(< 1+\mu T >) = \mathbb{Z}/p\mathbb{Z}$. Hence by the theorem we have

$$0 \longrightarrow \{\mathcal{E}^{(\mu,\lambda;1,j)} | j \in \mathbb{Z}/p\mathbb{Z}\} \longrightarrow \mathrm{Ext}^1(G_{\mu,1},\mu_p) \longrightarrow H^1(S, G^\vee_{\mu,1}) \longrightarrow 0.$$

EXAMPLE II.3.37. Let us now suppose $v(\mu)=0$ and $v(\lambda)>0$. In such a case

$$\mathrm{Hom}_{gr}(\mu_{p|S_\lambda}, \mathbb{G}_{m|S_\lambda}) =< 1+\mu T > .$$

Hence it is easy to see that

$$rad_{p,\lambda}(< 1+\mu T >) = 0.$$

Therefore, by the theorem,

$$\mathrm{Ext}^1(\mu_p, G_{\lambda,1}) \longrightarrow \ker\Big(H^1(S,\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(S_\lambda,\mathbb{Z}/p\mathbb{Z})\Big)$$

is an isomorphism.

COROLLARY II.3.38. *Under the hypothesis of the theorem, any extension* $E \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ *is of type* $\mathcal{E}^{(\mu,\lambda;F,j)}$*, up to an extension of* R*. In particular any extension is commutative.*

PROOF. Let $E \in \mathrm{Ext}^1\,(G_{\mu,1}, G_{\lambda,1})$. Suppose that $\alpha^\lambda_*(i_*E) = [S']$, with $S' \longrightarrow S$ a $G^\vee_{\mu,1}$-torsor. We consider the integral closure S'' of S in S'_K. Up to a localization (in the case $S'' \longrightarrow S$ is étale), we can suppose S'' local. So $S'' = \mathrm{Spec}(R'')$ where R'' is a noetherian local integrally closed ring of dimension 1, i.e. a d.v.r. (see [**7**, 9.2]). Since $S''_K \simeq S'_K$, then $S'_K \times_K S''_K$ is a trivial E_K-torsor over S''_K. By I.3.6 we have that $S' \times_S S''$ is a trivial E-torsor trivial over S''. So, if we make the base change $f : S'' \to S$, then $\alpha^\lambda_*(i_*(E_{S'})) = 0$. By II.3.35, this implies that E'' is of type

$$\mathcal{E}^{(\mu,\lambda,F,j)}.$$

Hence any $E \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ is a commutative group scheme over an extension R' of R. So it is a commutative group scheme over R.

□

By II.3.2 the extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ over K, which are extensions of abstract groups, are classified by $H^2_0(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$ (see for instance [**46**, 2.7]). This group is formed by $\mathcal{E}_{j,K}$ with $j \in \mathbb{Z}/p\mathbb{Z}$. If $j \neq 0$ we have that $\mathcal{E}_{j,K}$ is isomorphic, as a group scheme, to $\mathbb{Z}/p^2\mathbb{Z}$, while if $j=0$ it is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. We also define the following morphism of extensions

$$\begin{aligned} \alpha_{\mu,\lambda} : \mathcal{E}^{(\mu,\lambda;F,j)} &\longrightarrow \mathcal{E}_{j,R} \\ S_1 &\longmapsto 1+\mu S_1 \\ S_2 &\longmapsto F(S_1)+\lambda S_2. \end{aligned} \tag{II.59}$$

It is an isomorphism on the generic fiber. Now, by the theorem, we get that $\mathcal{E}^{(\mu,\lambda;F,j)}$ are the only extensions which are isomorphic to $\mathcal{E}_{j,K}$ on the generic fiber.

COROLLARY II.3.39. *The extensions of type* $\mathcal{E}^{(\mu,\lambda;F,j)}$ *are the only extensions* $\mathcal{E} \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ *which are isomorphic, as extensions, to* $\mathcal{E}_{j,K}$ *on the generic fiber. In particular they are the unique finite and flat R-group schemes of order* p^2 *which are models of constant groups. More precisely, they are isomorphic on the generic fiber, as group schemes, to* $\mathbb{Z}/p^2\mathbb{Z}$ *if* $j \neq 0$ *and to* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ *if* $j = 0$.

PROOF. As above remarked any $\mathcal{E}^{(\mu,\lambda;F,j)}$ has the properties of the statement. We now prove that they are the unique extensions of $G_{\mu,1}$ by $G_{\lambda,1}$ to have these properties. Let $\mathcal{E} \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ be such that $\mathcal{E}_K \simeq \mathcal{E}_{j,K}$ as group schemes. By II.3.26, II.3.29 and II.3.35 we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1}) & \xrightarrow{\alpha_*^\lambda \circ i_*} & \ker\Big(H^1(S, G_{\mu,1}^\vee) \longrightarrow H^1(S_\lambda, G_{\mu,1}^\vee)\Big) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_K(\mu_p, \mu_p) & \xrightarrow{\alpha_*^\lambda \circ i_*} & \mathrm{Ext}^1_K(\mu_p, \mathbb{G}_m) \simeq H^1(\mathrm{Spec}(K), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \end{array}$$

where the vertical maps are the restrictions to the generic fiber. Suppose now that $\mathcal{E}_K$ is of type $\mathcal{E}_{j,K}$. By II.3.29 it follows that $\alpha_*^\lambda \circ i_*(\mathcal{E}_K) = 0$. Since the above diagram commutes, this means that $(\alpha_*^\lambda \circ i_*(\mathcal{E}))_K = 0$. By I.3.6 we have that the second vertical map of the diagram is injective. This means that

$$\alpha_*^\lambda \circ i_*(\mathcal{E}) = 0.$$

So II.3.35 implies that $\mathcal{E}$ is of type $\mathcal{E}^{(\mu,\lambda;F,j)}$. Now, if G is a model of a constant group, by II.3.1 we have that G is an extension $\mathcal{E}$ of $G_{\mu,1}$ by $G_{\lambda,1}$. Moreover, since $\mathcal{E}_K$ is a constant group, then $\mathcal{E}_K \in \mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$. Therefore $\mathcal{E}_K \simeq \mathcal{E}_j$ for some j. So, by what we just proved, $\mathcal{E}$ is of type $\mathcal{E}^{(\mu,\lambda;F,j)}$. The last assertion is clear. □

II.3.8. $\mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ **and the Sekiguchi-Suwa theory.** We now give a description of $\mathcal{E}^{(\mu,\lambda;F,j)}$ through the Sekiguchi-Suwa theory. We study separately the cases $\lambda \nmid \mu$ and $\lambda | \mu$.

COROLLARY II.3.40. *Let* $\mu, \lambda \in R$ *be with* $v(\lambda_{(1)}) \geq v(\lambda) > v(\mu)$. *Then, no* $\mathcal{E} \in \mathrm{Ext}^1(G_{\mu,1}, G_{\lambda,1})$ *is a model of* $(\mathbb{Z}/p^2\mathbb{Z})_K$. *Moreover, if* $p > 2$ *and* $v(\mu) > 0$, *the group* $\{\mathcal{E}^{(\mu,\lambda;F,j)}\}$ *is isomorphic to*

$$\Big\{ \boldsymbol{a} \in \widehat{W}(R/\lambda R)^{\mathrm{F}-[\mu^{p-1}]} | \exists \boldsymbol{b} \in \widehat{W}(R/\lambda^p R)^{\mathrm{F}-[\mu^{p(p-1)}]} \text{ such that}$$

$$p\boldsymbol{a} = [\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b}) \in \widehat{W}(R/\lambda^p R) \Big\} \Big/ < [\mu], \Big\{ [\frac{p}{\mu^{p-1}}]\boldsymbol{b} + V(\boldsymbol{b}) | \boldsymbol{b} \in \widehat{W}(R/\lambda^p R)^{\mathrm{F}-[\mu^{p(p-1)}]} \Big\} >,$$

through the map

$$\boldsymbol{a} \longmapsto \mathcal{E}^{(\mu,\lambda;E_p(\boldsymbol{a},\mu;S),0)}.$$

REMARK II.3.41. We know by II.3.16, II.3.15 and II.3.10 that any element of the set defined above can be chosen of the type $[a]$ for some $a \in (R/\lambda R)^{\mathrm{F}-[\mu^{p-1}]}$. So, if we have two elements as above of the form $[a]$ and $[b]$ then $[a] + [b] = [c]$ for some $c \in (R/\lambda R)^{\mathrm{F}\,-[\mu^{p-1}]}$. We are not able to describe explicitly this element. If we were able to do it we could have a simpler description of the above set, as it happens in the case $v(\mu) \geq v(\lambda)$. We will see this in II.3.42.

PROOF. We now prove the first statement. We remark that by II.3.39 it is sufficient to prove the statement only for the extensions in $\{\mathcal{E}^{(\mu,\lambda;F,j)}\}$. Let us consider the restriction map

$$r : \mathrm{Hom}_{gr}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}) \longrightarrow \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}).$$

The morphism $p : \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}_{|S_\lambda}, \mathbb{G}_{m|S_\lambda}) \longrightarrow \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}{}_{|S_{\lambda^p}}, \mathbb{G}_{m|S_{\lambda^p}})$ defined in (II.33) is given by $F(S) \mapsto F(S)^p$ and induces a map

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) \stackrel{p}{\longrightarrow} \mathrm{Hom}_{gr}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}).$$

Then

$$\mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) \stackrel{p}{\longrightarrow} \mathrm{Hom}_{gr}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}) \stackrel{r}{\longrightarrow} \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$$

is the trivial morphism. Indeed

$$(r \circ p)(F(S)) = F(S)^p \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}),$$

which is zero by definition of group scheme morphisms and by the fact that $G_{\mu,1}$ has order p. Now let us take

$$F(S) \in rad_{p,\lambda}(< 1 + \mu S >) \simeq \{\mathcal{E}^{(\mu,\lambda;F,j)}\}.$$

By definition

$$F(S)^p(1+\mu S)^{-j} = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}),$$

for some $j \in \mathbb{Z}/p\mathbb{Z}$. Hence

$$r(F(S)^p(1+\mu S)^{-j}) = (1+\mu S)^{-j} = 1 \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}).$$

If $(\mathcal{E}^{(\mu,\lambda;F,j)})_K \simeq (\mathbb{Z}/p^2\mathbb{Z})_K$ then $j \neq 0$. Therefore

$$(1+\mu S)^{-j} = 1 \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda})$$

means $v(\mu) \geq v(\lambda)$. So, if $v(\mu) < v(\lambda)$, necessarily $j = 0$. Hence

$$rad_{p,\lambda}(< 1 + \mu S >) := \Big\{ F(S) \in \mathrm{Hom}_{gr}(G_{\mu,1_{|S_\lambda}}, \mathbb{G}_{m|S_\lambda}) \text{ such that } F(S)^p = 1 \in \mathrm{Hom}(G_{\mu,1_{|S_{\lambda^p}}}, \mathbb{G}_{m|S_{\lambda^p}}) \Big\} / < 1 + \mu S > .$$

Therefore by II.3.35, II.3.15 and (II.34) we have the thesis. □

COROLLARY II.3.42. *Let us suppose $p > 2$. Let $\mu, \lambda \in R \setminus \{0\}$ be with $v(\lambda_{(1)}) \geq v(\mu) \geq v(\lambda)$. Then, $\{\mathcal{E}^{(\mu,\lambda;F,j)}\}$ is isomorphic to the group*

$$\Phi_{\mu,\lambda} := \Big\{ (a,j) \in (R/\lambda R)^{\mathrm{F}} \times \mathbb{Z}/p\mathbb{Z} \text{ such that } pa - j\mu = \frac{p}{\mu^{p-1}} a^p \in R/\lambda^p R \Big\},$$

through the map

$$(a,j) \longmapsto \mathcal{E}^{(\mu,\lambda;\sum_{i=0}^{p-1}\frac{a^i}{i!}S^i,j)}.$$

REMARK II.3.43. It is clear that if $(0,j) \in \Phi_{\mu,\lambda}$, with $j \neq 0$, then $\mu \equiv 0 \mod \lambda^p$.

PROOF. By II.3.15, II.3.22, (II.34) and II.3.36 (for the case $v(\lambda)=0$) it follows that $rad_{p,\lambda}(<1+\mu S>)$ is isomorphic to

(II.60)

$$\Big\{(a,j) \in (R/\lambda R)^{\mathrm{F}} \times \mathbb{Z}/p\mathbb{Z} | \exists \boldsymbol{b} \in \widehat{W}(R/\lambda^p R)^{\mathrm{F}} \text{ such that } p[a]-j[\mu] = [\frac{p}{\mu^{p-1}}]\boldsymbol{b}+V(\boldsymbol{b}) \in \widehat{W}(R/\lambda^p R)\Big\}.$$

Let a, j and $\boldsymbol{b}=(b_0,b_1,\dots)$ be as above.

By [**53**, 5.10],

$$p[a] \equiv (pa, a^p, 0, \dots) \mod p^2.$$

Since $[\mu] \in \widehat{W}(R/\lambda^p R)^{\mathrm{F}}$ it follows by II.3.9 that

$$j[\mu] = [j\mu] \tag{II.61}$$

and

$$p[a]-j[\mu] = (pa-j\mu, a^p, 0, 0, \dots, 0, \dots) \in \widehat{W}(R/\lambda^p R). \tag{II.62}$$

We recall that

$$[\frac{p}{\mu^{p-1}}]\boldsymbol{b} = (\frac{p}{\mu^{p-1}}b_0, \dots, (\frac{p}{\mu^{p-1}})^{p^i}b_i, \dots),$$

then, again by II.3.9, we have

$$[\frac{p}{\mu^{p-1}}]\boldsymbol{b}+V(\boldsymbol{b}) = (\frac{p}{\mu^{p-1}}b_0, (\frac{p}{\mu^{p-1}})^p b_1 + b_0, \dots, (\frac{p}{\mu^{p-1}})^{p^{i+1}} b_{i+1} + b_i, \dots). \tag{II.63}$$

Since $\boldsymbol{b} \in \widehat{W}(R/\lambda^p R)$ there exists $r \geq 0$ such that $b_i = 0$ for any $i \geq r$. Moreover, comparing (II.62) and (II.63), it follows

$$(\frac{p}{\mu^{p-1}})^{p^i} b_{i+1} + b_i = 0 \quad \text{for } i \geq 1$$

$$(\frac{p}{\mu^{p-1}})^p b_1 + b_0 = a^p$$

$$(\frac{p}{\mu^{p-1}})^p b_0 = pa - j\mu.$$

So $b_i = 0$ if $i \geq 1$, $b_0 = a^p$ and $pa - j\mu = \frac{p}{\mu^{p-1}}a^p$. □

EXAMPLE II.3.44. Let us suppose $\mu = \lambda = \lambda_{(1)}$. Then $G_{\lambda_{(1)}} \simeq \mathbb{Z}/p\mathbb{Z}$. By II.3.32, II.3.42 and II.3.35 we have that

$$\{(k\eta, k) | k \in \mathbb{Z}/p\mathbb{Z}\} \subseteq \Phi_{\lambda_{(1)},\lambda_{(1)}} \simeq rad_{p,\lambda_{(1)}}(<1+\lambda_{(1)}S>).$$

On the other hand by II.3.35 and II.3.39 it follows that $rad_{p,\lambda_{(1)}}(< 1+\lambda_{(1)}S >) \simeq H^2_0(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$. Therefore $\{(k\eta,k)|k\in\mathbb{Z}/p\mathbb{Z}\} \simeq rad_{p,\lambda_{(1)}}(< 1+\lambda_{(1)}S >)$.

We now concentrate on to the case $v(\mu)\geq v(\lambda)$, which is the unique case, as proved in II.3.40, where extensions of $G_{\mu,1}$ by $G_{\lambda,1}$ could be models of $\mathbb{Z}/p^2\mathbb{Z}$, as group schemes. Our task is to find explicitly all the solutions $(a,j)\in(R/\lambda R)^{\mathrm{F}}$ of the equation $pa-j\mu=a^p\in R/\lambda^pR$. By II.3.42 this means finding explicitly all the extensions of type $\mathcal{E}^{(\mu,\lambda;F,j)}$. Let us consider the restriction map

$$r:\{\mathcal{E}^{(\mu,\lambda;F,j)}\}\longrightarrow \mathrm{Ext}^1_K(\mu_p,\mu_p)\simeq\mathbb{Z}/p\mathbb{Z}.$$

We remark that it coincides with the projection

$$p_2:\left\{(a,j)\in(R/\lambda R)^{\mathrm{F}}\times\mathbb{Z}/p\mathbb{Z} \text{ such that } pa-j\mu=\frac{p}{\mu^{p-1}}a^p\in R/\lambda^pR\right\}\longrightarrow\mathbb{Z}/p\mathbb{Z}$$

So there is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ which is a model of $(\mathbb{Z}/p^2\mathbb{Z})_K$ if and only if p_2 is surjective. First of all we describe explicitly the kernel of the above map.

LEMMA II.3.45. *We have*

$$\ker p_2=\Big\{(a,0)\in R/\lambda R\times\mathbb{Z}/p\mathbb{Z} \ \textit{s. t., for any lifting } \tilde{a}\in R,$$
$$pv(\tilde{a})\geq\max\{pv(\lambda)+(p-1)v(\mu)-v(p),v(\lambda)\}\Big\}$$

In particular p_2 is injective if and only if $v(\lambda)\leq 1$ or $v(p)-(p-1)v(\mu)<p$.

PROOF. Let $(a,0)\in\ker p_2\cap R/\lambda R\times\mathbb{Z}/p\mathbb{Z}$. By the definitions we have that

$$pa=\frac{p}{\mu^{p-1}}a^p\in R/\lambda^pR.$$

Let $\tilde{a}\in R$ be a lift of a. Since $v(\mu)\geq v(\lambda)$, if $a\neq 0$ then $v(\tilde{a})<v(\mu)$. Hence

$$v(p)+v(\tilde{a})>pv(\tilde{a})+v(p)-(p-1)v(\mu) \tag{II.64}$$

Therefore

$$pa=\frac{p}{\mu^{p-1}}a^p\in R/\lambda^pR$$

if and only if

$$\frac{p}{\mu^{p-1}}a^p=0\in R/\lambda^pR,$$

if and only if

$$pv(\tilde{a})+v(p)-(p-1)v(\mu)\geq pv(\lambda).$$

We remark that $a^p=0\in R/\lambda R$ means $pv(\tilde{a})\geq v(\lambda)$. So we have proved the first assertion. Now if $v(\lambda)\leq 1$ or $v(p)-(p-1)v(\mu)<p$ it is easy to see that there are no nonzero elements in $\ker p_2$. While if $v(\lambda)>1$ and $v(p)-(p-1)v(\mu)\geq p$, take $a\in R/\lambda R$ with a lifting $\tilde{a}\in R$ of valuation $v(\lambda)-1$. Therefore

$$p(v(\lambda)-1)\geq\max\{pv(\lambda)-v(p)+(p-1)v(\mu),v(\lambda)\}.$$

Hence $(a,0) \in \ker p_2$. $\square$

We remark that $\ker p_2$ depends only on the valuations of μ and λ. So we can easily compute $\Phi_{\mu,\lambda}$, too.

PROPOSITION II.3.46. *Let us suppose $p > 2$. Let $\mu, \lambda \in R\backslash\{0\}$ be with $v(\lambda_{(1)}) \geq v(\mu) \geq v(\lambda)$.*

a) *If $v(\mu) < pv(\lambda)$ then p_2 is surjective if and only if $pv(\mu) - v(\lambda) \geq v(p)$. And, if p_2 is surjective, $\Phi_{\mu,\lambda}$ is isomorphic to the group*
$$\{(j\eta\frac{\mu}{\lambda_{(1)}} + \alpha, j) | (\alpha, 0) \in \ker(p_2) \text{ and } j \in \mathbb{Z}/p\mathbb{Z}\}$$
For the definition of η see II.*3.32.*

b) *If $v(\mu) \geq pv(\lambda)$ then p_2 is surjective and $\Phi_{\mu,\lambda}$ is isomorphic to*
$$\{(\alpha, j) | (\alpha, 0) \in \ker(p_2) \text{ and } j \in \mathbb{Z}/p\mathbb{Z}\} \simeq \ker p_2 \times \mathbb{Z}/p\mathbb{Z}.$$

c) *If p_2 is not surjective then p_2 is the zero morphism. So $\Phi_{\mu,\lambda} = \ker p_2$.*

REMARK II.3.47. Let us suppose $v(\mu) < pv(\lambda)$. Let $(b, j) \in \Phi_{\mu,\lambda}$ with $j \neq 0$. By II.3.43, then $b \neq 0$. Let $\tilde{b} \in R$ be any of its lifting. Then $v(\tilde{b}) = v(\eta\frac{\mu}{\lambda_{(1)}}) = v(\mu) - \frac{v(p)}{p}$. Indeed, by the theorem, we have $\tilde{b} = \eta\frac{\mu}{\lambda_{(1)}} + \alpha$ for some $\alpha \in R/\lambda R$ with $v(\tilde{\alpha}) > v(\eta\frac{\mu}{\lambda_{(1)}}) = v(\mu) - \frac{v(p)}{p}$, where $\tilde{\alpha} \in R$ is any lifting of α.

PROOF. a) First, we suppose that p_2 is surjective. This is equivalent to saying that

$$pa - j\mu = \frac{p}{\mu^{p-1}}a^p \in R/\lambda R \tag{II.65}$$

has a solution $a \in (R/\lambda R)^{\mathrm{F}}$ if $j \neq 0$. Since $v(\mu) < v(p)$, by (II.64) it follows that

$$v(\mu) = v(p) - (p-1)v(\mu) + pv(\tilde{a}), \tag{II.66}$$

with $\tilde{a} \in R$ a lifting of a. Since $a \in (R/\lambda R)^{\mathrm{F}}$ we have $pv(\tilde{a}) \geq v(\lambda)$. Hence, by (II.66), $pv(\mu) - v(\lambda) \geq v(p)$.

Conversely let us suppose that $pv(\mu) - v(\lambda) \geq v(p)$. We know by II.3.32 and II.3.39 that
$$p\eta - \lambda_{(1)} = \frac{p}{\lambda_{(1)}^{p-1}}\eta^p \in R/\lambda_{(1)}^p R.$$
We recall that $v(\eta) = v(\lambda_{(2)})$. Since $pv(\lambda_{(1)}) - v(\lambda_{(1)}) + \mu \geq p\mu \geq p\lambda$, if we divide the above equation by $\frac{\lambda_{(1)}}{\mu}$ we obtain
$$p\eta\frac{\mu}{\lambda_{(1)}} - \mu = \frac{p}{\mu^{(p-1)}}(\frac{\mu}{\lambda_{(1)}}\eta)^p \in R/\lambda^p R.$$
We remark that $\eta\frac{\mu}{\lambda_{(1)}} \in (R/\lambda R)^{\mathrm{F}}$, since, by hypothesis, $v(\big(\eta\frac{\mu}{\lambda_{(1)}}\big)^p) = pv(\mu) - v(p) \geq v(\lambda)$. Clearly $j\eta\frac{\mu}{\lambda_{(1)}}$ is a solution of (II.65) for any $j \in \mathbb{Z}/p\mathbb{Z}$.

In particular it follows that, if p_2 is surjective, $\Phi_{\mu,\lambda}$ is isomorphic to the group

$$\{(j\eta\frac{\mu}{\lambda_{(1)}}+\alpha, j)|(\alpha,0)\in\ker(p_2) \text{ and } j\in\ \mathbb{Z}/p\mathbb{Z}\}$$

b) If $v(\mu)\geq pv(\lambda)$ then we have that $\mu=0\in R/\lambda^pR$. We remark that $(0,j)\in\Phi_{\mu,\lambda}$. This implies that p_2 is surjective and that $(\alpha,j)\in R/\lambda R\times\mathbb{Z}/p\mathbb{Z}\cap\Phi_{\mu,\lambda}$ if and only if $(\alpha,0)\in\ker(p_2)$.

c) Since p_2 is a morphism of groups with target $\mathbb{Z}/p\mathbb{Z}$ then the image of p_2 is a subgroup of $\mathbb{Z}/p\mathbb{Z}$. Then the image of p_2 is trivial or it is equal to $\mathbb{Z}/p\mathbb{Z}$. The assertion follows.

□

EXAMPLE II.3.48. Let us suppose $v(\mu)=v(\lambda_{(1)})$, i.e. $G_{\mu,1}\simeq\mathbb{Z}/p\mathbb{Z}$. For simplicity we will suppose $\mu=\lambda_{(1)}$. Then p_2 is an isomorphism. Indeed in this case $\ker(p_2)=0$ by II.3.45 and it is surjective by II.3.46(a)-(b). This means that, in this case, any extension $\mathcal{E}^{(\lambda_{(1)},\lambda;F,j)}$ is uniquely determined by the induced extension over K. Let us now consider the map

$$\mathrm{Ext}^1(G_{\lambda_{(1)},1},G_{\lambda_{(1)},1})\longrightarrow\mathrm{Ext}^1(G_{\lambda_{(1)},1},G_{\lambda,1})$$

induced by the map $\mathbb{Z}/p\mathbb{Z}\simeq G_{\lambda_{(1)},1}\longrightarrow G_{\lambda,1}$ given by $S\mapsto\frac{\lambda_{(1)}}{\lambda}S$. It is easy to see that $\mathcal{E}^{(\lambda_{(1)},\lambda;F,j)}$ is the image of $\mathcal{E}^{(\lambda_{(1)},\lambda_{(1)};E_p(\eta S),j)}$ through the above map. Indeed from the above proposition we have that $F(S)\equiv E_p(\eta S)\mod\lambda$. We remark that if $pv(\lambda)\leq v(\lambda_{(1)})$ then $\eta\equiv0\mod\lambda$, indeed in such a case $v(\lambda)\leq v(\lambda_{(2)})=v(\eta)$.

II.3.9. Classification of models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. By the previous subsections we have a classification of extensions of $G_{\mu,1}$ by $G_{\lambda,1}$ whose generic fibre is isomorphic, as group scheme, to $\mathbb{Z}/p^2\mathbb{Z}$. But this classification is too fine for our tasks. We want here to forget the structure of extension. We are only interested in the group scheme structure. We observe that it can happen that two non isomorphic extensions are isomorphic as group schemes. We here study when it happens.

First of all we recall what the model maps between models of $\mathbb{Z}/p\mathbb{Z}$ are. Let us suppose $\varrho,\tilde{\varrho}\in R$ with $v(\varrho),v(\tilde{\varrho})\leq v(\lambda_{(1)})$. Since $G_{\varrho,1}$ is flat over R, by II.3.13 it follows that the restriction map

$$\mathrm{Hom}_{gr}(G_{\varrho,1},G_{\tilde{\varrho},1})\longrightarrow\mathrm{Hom}_{gr}((G_{\varrho,1})_K,(G_{\tilde{\varrho},1})_K)\simeq\mathbb{Z}/p\mathbb{Z}$$

in an injection. It follows easily by (II.36) that

$$\mathrm{Hom}_{gr}(G_{\varrho,1},G_{\tilde{\varrho},1})=\begin{cases}\mathbb{Z}/p\mathbb{Z}, & \text{if } v(\varrho)\geq v(\tilde{\varrho});\\ 0, & \text{if } v(\varrho)<v(\tilde{\varrho}),\end{cases}$$

where, in the first case, the morphisms are given by $S\longmapsto\frac{(1+\varrho S)^r-1}{\tilde{\varrho}}$ with $r\in\mathbb{Z}/p\mathbb{Z}$. We remark that, if $v(\varrho)=v(\tilde{\varrho})$ and $r\neq0$, these morphisms are isomorphisms.

We now recall that by II.3.1, II.3.39, II.3.42 and II.3.40 any model of $(\mathbb{Z}/p^2\mathbb{Z})_K$ is of the form $\mathcal{E}^{(\mu,\lambda;F,j)}$ such that $j\neq0$, $v(\lambda_{(1)})\geq v(\mu)\geq v(\lambda)$ and $F(S)=\sum_{i=0}^{p-1}\frac{a^i}{i!}S^i$ with $(a,j)\in\Phi_{\mu,\lambda}$. See II.3.46 for the explicit description of $\Phi_{\mu,\lambda}$. For $i=1,2$ let us consider $\mathcal{E}^{(\mu_i,\lambda_i;F_i,j_i)}$, models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. First of all we remark that there is an injection

$$r_K : \mathrm{Hom}(\mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)},\mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)}) \longrightarrow \mathrm{Hom}_K(\mathcal{E}_{j_1,K},\mathcal{E}_{j_2,K})$$

given by

$$f \longmapsto (\alpha_{\mu_2,\lambda_2})_K \circ f_K \circ (\alpha_{\mu_1,\lambda_1})_K^{-1}$$

See (II.59) for the definition $\alpha_{\mu,\lambda}$. We recall that

$$\mathrm{Hom}(\mathcal{E}_{j_1},\mathcal{E}_{j_2}) \simeq \mathrm{Hom}_K(\mathcal{E}_{j_1,K},\mathcal{E}_{j_2,K}).$$

and the elements are the morphisms

$$\psi_{r,s} : \mathcal{E}_{j_1} \longrightarrow \mathcal{E}_{j_2},$$

which, on the level of Hopf algebras, are given by

$$S_1 \longmapsto S_1^{\frac{rj_1}{j_2}} \tag{II.67}$$

$$S_2 \longmapsto S_1^s S_2^r, \tag{II.68}$$

for some $r \in \mathbb{Z}/p\mathbb{Z}$ and $s \in \mathbb{Z}/p\mathbb{Z}$. Moreover the map

$$\mathrm{Hom}(\mathcal{E}_{j_1},\mathcal{E}_{j_2}) \longrightarrow \mathbb{Z}/p^2\mathbb{Z}$$
$$\psi_{r,s} \longmapsto r + \frac{p}{j_1}s$$

is an isomorphism. So $\mathrm{Hom}(\mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)},\mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)})$ is a subgroup of $\mathbb{Z}/p^2\mathbb{Z}$ through the map r_K. We remark that the unique nontrivial subgroup of $\mathrm{Hom}(\mathcal{E}_{j_1},\mathcal{E}_{j_2})$ is $\{\psi_{0,s} | s \in \mathbb{Z}/p\mathbb{Z}\}$. Finally we have that any morphism $\mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)} \longrightarrow \mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)})$ is given by

$$\begin{aligned} S_1 &\longrightarrow \frac{(1+\mu_1 S_1)^{\frac{rj_1}{j_2}} - 1}{\mu_2} \\ S_2 &\longrightarrow \frac{(F_1(S_1)+\lambda_1 S_2)^r(1+\mu_1 S_1)^s - F_2(\frac{(1+\mu_1 S_1)^{\frac{rj_1}{j_2}}-1}{\mu_2})}{\lambda_2}, \end{aligned} \tag{II.69}$$

for some $r, s \in \mathbb{Z}/p\mathbb{Z}$. With abuse of notation we call it $\psi_{r,s}$. We remark that the morphisms $\psi_{r,s} : \mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)} \longrightarrow \mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)}$ which are model maps correspond, by (II.67), to $r \neq 0$. In such a case $\psi_{r,s}$ is a morphism of extensions, i.e. there exist morphisms $\psi_1 : G_{\lambda,1} \longrightarrow G_{\lambda,2}$ and $\psi_2 : G_{\mu,1} \longrightarrow G_{\mu,2}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_{\lambda_1,1} & \longrightarrow & \mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)} & \longrightarrow & G_{\mu_1,1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_{r,s}} & & \downarrow{\scriptstyle\psi_2} & & \\ 0 & \longrightarrow & G_{\lambda_2,1} & \longrightarrow & \mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)} & \longrightarrow & G_{\mu_2,1} & \longrightarrow & 0 \end{array} \tag{II.70}$$

commutes. More precisely ψ_1 is given by $S \mapsto \frac{(1+\lambda_1 S)^r - 1}{\lambda_2}$ and ψ_2 by $S \mapsto \frac{(1+\mu S_1)^{\frac{rj_1}{j_2}}-1}{\mu_2}$.

We now calculate $\mathrm{Hom}(\mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)},\mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)})$.

PROPOSITION II.3.49. *For $i = 1, 2$, if $F_i(S) = E_p(a_iS) = \sum_{k=0}^{p-1} \frac{a_i^k}{k!} S^i$ and $\mathscr{E}_i = \mathcal{E}^{(\mu_i,\lambda_i;F_i,j_i)}$ are models of $(\mathbb{Z}/p^2\mathbb{Z})_K$ we have*

$$\mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2) = \begin{cases} 0, & \text{if } v(\mu_1) < v(\lambda_2); \\ \{\psi_{r,s}\} \simeq \mathbb{Z}/p^2\mathbb{Z}, & \text{if } v(\mu_2) \leq v(\mu_1),\ v(\lambda_2) \leq v(\lambda_1) \\ & \text{and } a_1 \equiv \frac{j_1}{j_2}\frac{\mu_1}{\mu_2} a_2 \mod \lambda_2; \\ \{\psi_{0,s}\} \simeq \mathbb{Z}/p\mathbb{Z}, & \text{otherwise.} \end{cases}$$

PROOF. It is immediate to see that $\psi_{0,s} \in \mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2)$, with $s \neq 0$, if and only if $v(\mu_1) \geq v(\lambda_2)$. We now see conditions for the existence of $\psi_{r,s}$ with $r \neq 0$. If it exists, in particular, we have two morphisms $G_{\mu_1,1} \longrightarrow G_{\mu_2,1}$ and $G_{\lambda_1,1} \longrightarrow G_{\lambda_2,1}$. This implies $v(\mu_1) \geq v(\mu_2)$ and $v(\lambda_1) \geq v(\lambda_2)$. Moreover we have that

$$F_1(S_1)^r(1+\mu_1S_1)^s = F_2(\frac{(1+\mu_1S_1)^{\frac{rj_1}{j_2}} - 1}{\mu_2}) \in \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}).$$

Since $v(\mu_1) \geq v(\mu_2) \geq v(\lambda_2)$, we have

$$F_1(S_1)^r = F_2(\frac{(1+\mu_1S_1)^{\frac{rj_1}{j_2}} - 1}{\mu_2}) \in \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}). \tag{II.71}$$

If we define the morphism of groups

$$[\frac{\mu_1}{\mu_2}]^* : \mathrm{Hom}(G_{\mu_2,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}) \longrightarrow \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}})$$
$$F(S_1) \longmapsto F(\frac{\mu_1}{\mu_2}S_1)$$

then

$$\begin{aligned} F_2(\frac{(1+\mu_1S_1)^{\frac{rj_1}{j_2}} - 1}{\mu_2}) &= [\frac{\mu_1}{\mu_2}]^*\left(F_2\left(\frac{(1+\mu_1S_1)^{\frac{rj_1}{j_2}} - 1}{\mu_1}\right)\right) \\ &= [\frac{\mu_1}{\mu_2}]^*(F_2(S_1))^{\frac{rj_1}{j_2}} \\ &= F_2(\frac{\mu_1}{\mu_2}(S_1))^{\frac{rj_1}{j_2}}. \end{aligned}$$

Therefore we have

$$F_1(S_1)^r = (F_2(\frac{\mu_1}{\mu_2}S_1))^{\frac{rj_1}{j_2}} \in \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}). \tag{II.72}$$

Every element of $\mathrm{Hom}(G_{\mu_1,1|S_\lambda}, \mathbb{G}_{m|S_\lambda})$ has order p. Let t be an inverse for r modulo p. Then raising the equality to the t^{th}-power we obtain

$$F_1(S_1) = (F_2(\frac{\mu_1}{\mu_2}S_1))^{\frac{j_1}{j_2}} \in \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}).$$

By II.3.22 this means

$$a_1 \equiv \frac{j_1}{j_2}\frac{\mu_1}{\mu_2}a_2 \mod \lambda_2.$$

It is conversely clear that, if $v(\mu_1) \geq v(\mu_2)$, $v(\lambda_1) \geq v(\lambda_2)$ and

$$F_1(S_1) = (F_2(\frac{\mu_1}{\mu_2}S_1))^{\frac{j_1}{j_2}} \in \mathrm{Hom}(G_{\mu_1,1|S_{\lambda_2}}, \mathbb{G}_{m|S_{\lambda_2}}),$$

then (II.69) defines a morphism of group schemes.

□

We have the following result which gives a criterion to determine the class of isomorphism, as a group scheme, of an extension of type $\mathcal{E}^{(\mu,\lambda;F,j)}$.

Corollary II.3.50. *For $i = 1, 2$, let $F_i(S) = E_p(aS) = \sum_{k=0}^{p-1} \frac{a_i^k}{k!} S^k$ and let $\mathscr{E}_i = \mathcal{E}^{(\mu_i,\lambda_i;F_i,j_i)}$ be models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. Then they are isomorphic if and only if $v(\mu_1) = v(\mu_2)$, $v(\lambda_1) = v(\lambda_2)$ and $a_1 \equiv \frac{j_1}{j_2}\frac{\mu_1}{\mu_2} a_2 \mod \lambda_2$. Moreover if it happens then any model map between them is an isomorphism.*

Proof. By the proposition we have that a model map $\psi_{r,s} : \mathcal{E}^{(\mu_1,\lambda_1,F_1,j_1)} \longrightarrow \mathcal{E}^{(\mu_2,\lambda_2,F_2,j_2)}$ exists if and only if $v(\mu_1) \geq v(\mu_2)$, $v(\lambda_1) \geq v(\lambda_2)$ and $a_1 \equiv \frac{j_1}{j_2}\frac{\mu_1}{\mu_2} a_2 \mod \lambda_2$. It is a morphism of extensions as remarked before the proposition. Let us consider the commutative diagram (II.67). Then $\psi_{r,s}$ is an isomorphism if and only if ψ_i is an isomorphism for $i = 1, 2$. By the discussion made at the beginning of this section this is equivalent to requiring $v(\mu_1) = v(\mu_2)$ and $v(\lambda_1) = v(\lambda_2)$. This also proves the last assertion. □

We remarked that if $v(\mu_1) = v(\mu_2)$ and $v(\lambda_1) = v(\lambda_2)$ then

$$\mathrm{Ext}^1(G_{\mu_1,1}, G_{\lambda_1,1}) \simeq \mathrm{Ext}^1(G_{\mu_2,1}, G_{\lambda_2,1}).$$

The following is a more precise statement for extensions of type $\mathcal{E}^{(\mu,\lambda;F,j)}$.

Corollary II.3.51. *Let $\mathcal{E}^{(\mu_1,\lambda_1;E_p(aS),j)} \in \mathrm{Ext}^1(G_{\mu_1,1}, G_{\lambda_1,1})$ be a model of $\mathbb{Z}/p^2\mathbb{Z}$. Then for any μ_2, λ_2 such that $v(\mu_1) = v(\mu_2)$ and $v(\lambda_1) = v(\lambda_2)$ we have*

$$\mathcal{E}^{(\mu_1,\lambda_1;E_p(aS),j)} \simeq \mathcal{E}^{(\mu_2,\lambda_2;E_p(\frac{a}{j}\frac{\mu_2}{\mu_1}S),1)}$$

as group schemes.

Proof. Firstly we prove that there exists the group scheme $\mathcal{E}^{(\mu_2,\lambda_2;E_p(\frac{a}{j}\frac{\mu_2}{\mu_1}S),1)}$. By II.3.42 we have that $a \in (R/\lambda R)^{\mathrm{F}}$ and

$$pa - j\mu_1 = \frac{p}{\mu_1^{p-1}} a^p \mod \lambda_1^p. \tag{II.73}$$

Then, multiplying (II.73) by $\frac{\mu_2}{\mu_1}\frac{1}{j}$, we have

$$p\frac{a\mu_2}{j\mu_1} - \mu_2 \equiv \frac{p}{\mu_2^{p-1}} \left(\frac{a\mu_2}{j\mu_1}\right)^p \mod \lambda_2^p.$$

Hence $\mathcal{E}^{(\mu_2,\lambda_2;E_p(\frac{a}{j}\frac{\mu_2}{\mu_1}S),1)}$ is a group scheme (see again II.3.42). Then by the above proposition we can conclude that

$$\mathcal{E}^{(\mu_1,\lambda_1;E_p(aS),j)} \simeq \mathcal{E}^{(\mu_2,\lambda_2;E_p(\frac{a}{j}\frac{\mu_2}{\mu_1}S),1)}$$

as group schemes.

□

EXAMPLE II.3.52. Let $\mu, \lambda \in R$ be such that $v(\mu) = v(\lambda) = v(\lambda_{(1)})$. We now want to describe $\mathbb{Z}/p^2\mathbb{Z}$ as $\mathcal{E}^{(\mu,\lambda;F,1)}$. We recall that we defined

$$\eta = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda_{(2)}^k$$

By II.3.32 and the previous corollary we have

$$\mathbb{Z}/p^2\mathbb{Z} \simeq \mathcal{E}^{(\mu,\lambda;E_p(\eta\frac{\mu}{\lambda_{(1)}}S),1)}.$$

We conclude the section with the complete classification of $(\mathbb{Z}/p^2\mathbb{Z})_K$-models. The following theorem summarizes the results of this chapter.

THEOREM II.3.53. *Let us suppose $p > 2$. Let G be a finite and flat R-group scheme such that $G_K \simeq (\mathbb{Z}/p^2\mathbb{Z})_K$. Then $G \simeq \mathcal{E}^{(\pi^m,\pi^n;E_p(aS),1)}$ for some $v(\lambda_{(1)}) \geq m \geq n \geq 0$ and $(a,1) \in \Phi_{\pi^m,\pi^n}$. Moreover m, n and $a \in R/\pi^n R$ are unique.*

REMARK II.3.54. The explicit description of the set Φ_{π^m,π^n} has been given in II.3.46 and II.3.45.

PROOF. By II.3.1, II.3.39, II.3.40, II.3.42 and II.3.51 any model of $(\mathbb{Z}/p^2\mathbb{Z})_K$ is of type $\mathcal{E}^{(\pi^m,\pi^n;E_p(aS),1)}$ with $m \geq n$ and $(a,1) \in \Phi_{\pi^m,\pi^n}$. By II.3.50, it follows that,

$$\mathcal{E}^{(\pi^{m_1},\pi^{n_1},E_p(a_1S),1)} \simeq \mathcal{E}^{(\pi^{m_2},\pi^{n_2},E_p(a_2S),1)}$$

as group schemes if and only if $m_1 = m_2$, $n_1 = n_2$ and $a_1 = a_2 \in R/\pi^{n_1}R$. □

II.4. Torsors under $\mathcal{E}^{(\mu,\lambda;F(S),j)}$

We now give an explicit description of $\mathcal{E}^{(\mu,\lambda;F,j)}$-torsors. It will be useful in the next chapter for the study of the extensions (in the sense of III.2) of $(\mathbb{Z}/p^2\mathbb{Z})_K$-torsors. We have seen that there is the following exact sequence

$$0 \longrightarrow \mathcal{E}^{(\mu,\lambda;F(S),j)} \xrightarrow{\iota} \mathcal{E}^{(\mu,\lambda;F(S))} \xrightarrow{\psi^j_{\mu,\lambda,F,G}} \mathcal{E}^{(\mu,\lambda;G(S))} \longrightarrow 0$$

for $G(S) \in \mathrm{Hom}_{gr}(G^{(\mu^p)}{}_{|S_{\lambda^p}}, \mathbb{G}_{m|S_{\lambda^p}})$ such that $F(S)^p(1+\mu S)^{-j} = G(\frac{(1+\mu S)^p-1}{\mu^p}) \in \mathrm{Hom}_{gr}(\mathcal{G}^{(\mu)}{}_{|S_{\lambda^p}}, \mathbb{G}_{m|S_{\lambda^p}})$. The associated long exact sequence is

$$\ldots \longrightarrow H^0(X, \mathcal{E}^{(\mu,\lambda;F(S))}) \xrightarrow{(\psi^j_{\mu,\lambda,F,G})_*} H^0(X, \mathcal{E}^{(\mu,\lambda;G(S))}) \xrightarrow{\delta} H^1(X, \mathcal{E}^{(\mu,\lambda;F(S),j)}) \xrightarrow{\iota_*}$$
$$\longrightarrow H^1(X, \mathcal{E}^{(\mu,\lambda;F(S))}) \longrightarrow H^1(X, \mathcal{E}^{(\mu,\lambda;F(S))}) \longrightarrow \ldots$$

LEMMA II.4.1. *Let X be a faithfully flat R-scheme and let $f : X_{fl} \longrightarrow X_{Zar}$ be the natural continuous morphism of sites. For any R-group scheme $\mathcal{E}^{(\mu,\lambda;F)}$ we have $R^1 f_*(\mathcal{E}^{(\mu,\lambda;F)}) = 0$. In particular $H^1(X_{fl}, \mathcal{E}^{(\mu,\lambda;F)}) = H^1(X_{Zar}, \mathcal{E}^{(\mu,\lambda;F)})$.*

PROOF. Let us consider the exact sequence (II.30), in the fppf topology,

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{E}^{(\mu,\lambda;F)} \longrightarrow \mathcal{G}^{(\mu)} \longrightarrow 0.$$

If we apply the functor f_* we obtain

$$\dots \longrightarrow R^1 f_* \mathcal{G}^{(\lambda)} \longrightarrow R^1 f_*(\mathcal{E}^{(\mu,\lambda;F)}) \longrightarrow R^1 f_* \mathcal{G}^{(\mu)} \longrightarrow \dots$$

By I.5.5 it follows that $R^1 f_*(\mathcal{G}^{(\lambda)}) = R^1 f_*(\mathcal{G}^{(\mu)}) = 0$. Hence $R^1 f_*(\mathcal{E}^{(\mu,\lambda;F)}) = 0$. Using the Leray spectral sequence we conclude that $H^1(X_{fl}, \mathcal{E}^{(\mu,\lambda;F)}) = H^1(X_{Zar}, \mathcal{E}^{(\mu,\lambda;F)})$. □

We remark that

$$\begin{aligned} H^0(X, \mathcal{E}^{(\mu,\lambda;F(S))}) = \{(f_1, f_2) \in H^0(X, \mathcal{O}_X) \times H^0(X, \mathcal{O}_X) |\\ 1 + \mu f_1 \in H^0(X, \mathcal{O}_X)^* \text{ and } \tilde{F}(f_1) + \lambda f_2 \in H^0(X, \mathcal{O}_X)^*\}, \end{aligned}$$

for some lift $\tilde{F}$ of F, and

$$\begin{aligned} H^0(X, \mathcal{E}^{(\mu,\lambda;G(S))}) = \{(f_1, f_2) \in H^0(X, \mathcal{O}_X) \times H^0(X, \mathcal{O}_X) |\\ 1 + \mu^p f_1 \in H^0(X, \mathcal{O}_X)^* \text{ and } \tilde{G}(f_1) + \lambda^p f_2 \in H^0(X, \mathcal{O}_X)^*\}, \end{aligned}$$

for some lift $\tilde{G}$ of G. The map $(\psi^j_{\mu,\lambda,F,G})_* : H^0(X, \mathcal{E}^{(\mu,\lambda;F(S))}) \longrightarrow H^0(X, \mathcal{E}^{(\mu,\lambda;G(S))})$ is given by

$$(f_1, f_2) \longmapsto \left(\frac{(1+\mu f_1)^p - 1}{\mu^p}, \frac{(\tilde{F}(f_1) + \lambda f_2)^p (1+\mu f_1)^{-j} - \tilde{G}(\frac{(1+\mu f_1)^p - 1}{\mu^p})}{\lambda^p}\right)$$

Now let us suppose that $X = \operatorname{Spec}(A)$. We describe explicitly the map δ. Let $\tilde{F}(S), \tilde{G}(S)$ be liftings of $F(S)$ and $G(S)$ in $R[S]$. Then

$$H^0(X, \mathcal{E}^{(\mu,\lambda;G(S))}) = \{(f_1, f_2) \in A \times A | 1 + \mu^p f_1 \in A^* \text{ and } \tilde{G}(f_1) + \lambda^p f_2 \in A^*\}.$$

and $\delta((f_1, f_2))$ is, as a scheme,

(II.74)

$$Y = \operatorname{Spec} A[T_1, T_2] / \left(\frac{(1+\mu T_1)^p - 1}{\mu^p} - f_1, \frac{(\tilde{F}(T_1) + \lambda T_2)^p (1+\mu T_1)^{-j} - \tilde{G}(f_1)}{\lambda^p} - f_2 \right).$$

and the $\mathcal{E}^{(\mu,\lambda;F(S),j)}$-action over Y is given by

$$\begin{aligned} T_1 &\longmapsto S_1 + T_1 + \mu S_1 T_1 \\ T_2 &\longmapsto S_2 \tilde{F}(T_1) + \tilde{F}(S_1) T_2 + \lambda S_2 T_2 + \\ &\qquad \frac{\tilde{F}(S_1)\tilde{F}(T_1) - \tilde{F}(S_1 + T_1 + \mu S_1 T_1)}{\lambda} \end{aligned}$$

Now let X be any scheme. If $Y \longrightarrow X$ is a $\mathcal{E}^{(\mu,\lambda;F(S),j)}$-torsor then, by II.4.1 there exists a Zariski covering $\{U_i = \operatorname{Spec}(A_i)\}$ such that $(\iota_*)_{U_i}([Y]) = 0$, for any i. This means that $Y_{|U_i} = \operatorname{Spec} A_i[T_1, T_2] / \left(\frac{(1+\mu T_1)^p - 1}{\mu^p} - f_{1,i}, \frac{(\tilde{F}(T_1) + \lambda T_2)^p (1+\mu T_1)^{-j} - \tilde{G}(f_1)}{\lambda^p} - f_{2,i} \right)$ for some $f_{1,i}, f_{2,i}$ as above. By a standard argument we can see that the cocycle $\iota_*([Y]) \in H^1(X_{Zar}, \mathcal{E}^{(\mu,\lambda;F)})$ permits to patch together the torsors $Y_{|U_i}$ to obtain Y.

In the next chapter we will consider the case X affine and we are interested only to $\mathcal{E}^{(\mu,\lambda;F,j)}$-torsors of the form (II.74).

II.5. Reduction on the special fiber of the models of $(\mathbb{Z}/p^2\mathbb{Z})_K$

In the following we study the special fibers of the extensions of type $\mathcal{E}^{(\lambda,\mu,;F,j)}$ with $v(\mu) \geq v(\lambda)$. In particular, by II.3.40, this includes the extensions which are models of $(\mathbb{Z}/p^2\mathbb{Z})_K$ as group schemes. We study separately the different cases which can occur.

II.5.1. Case $\mathbf{v}(\mu) = \mathbf{v}(\lambda) = \mathbf{0}$. We have $(G_{\lambda,1})_k \simeq (G_{\mu,1})_k \simeq \mu_p$. The extensions of type $\mathcal{E}^{(\lambda,\mu,;F,j)}$ are the extensions $\mathcal{E}_i$ with $i \in \mathbb{Z}/p\mathbb{Z}$. The special fibers of the extensions $\mathcal{E}_i$ with $i \in \mathbb{Z}/p\mathbb{Z}$ are clearly $\mathcal{E}_{i,k}$. See also II.3.29.

II.5.2. Case $\mathbf{v}(\lambda_{(1)}) \geq \mathbf{v}(\mu) > \mathbf{v}(\lambda) = \mathbf{0}$. In such a case we have $(G_{\lambda,1})_k \simeq \mu_p$. It is immediate by the definitions that any extension $\mathcal{E}^{(\mu,\lambda;1,j)}$ is trivial on the special fiber.

II.5.3. Case $\mathbf{v}(\lambda_{(1)}) > \mathbf{v}(\mu) \geq \mathbf{v}(\lambda) > \mathbf{0}$. Then $(G_{\mu,1})_k \simeq (G_{\lambda,1})_k \simeq \alpha_{p,k}$.

First, we recall some results about extensions of group schemes of order p over a field k. See [**15**, III §6 7.7.] for a reference.

THEOREM II.5.1. *Let us suppose that α_p acts trivially on α_p over k. The exact sequence $0 \longrightarrow \alpha_p \to \mathbb{G}_a \xrightarrow{\mathrm{F}} \mathbb{G}_a \to 0$ induces the following split exact sequence*

$$0 \longrightarrow \mathrm{Hom}_k(\alpha_p, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^1(\alpha_p, \alpha_p) \longrightarrow \mathrm{Ext}^1(\alpha_p, \mathbb{G}_a) \longrightarrow 0.$$

It is also known that

$$\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_a) \simeq H_0^2(\mathbb{G}_a, \mathbb{G}_a) \longrightarrow H_0^2(\alpha_p, \mathbb{G}_a) \simeq \mathrm{Ext}^1(\alpha_p, \mathbb{G}_a).$$

is surjective. Since $\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_a) \simeq H_0^2(\mathbb{G}_a, \mathbb{G}_a)$ is freely generated as a right $k[\mathrm{F}]$-module by $C_i = \frac{X^{p^i} + X^{p^i} - (X+Y)^{p^i}}{p^i}$ and $D_i = XY^{p^i}$ for all $i \in \mathbb{N} \setminus \{0\}$, it follows that $H_0^2(\alpha_p, \mathbb{G}_a) \simeq \mathrm{Ext}^1(\alpha_p, \mathbb{G}_a)$ is freely generated as right k-module by the class of the cocycle $C_1 = \frac{X^p + X^p - (X+Y)^p}{p}$. So $\mathrm{Ext}^1(\alpha_p, \mathbb{G}_a) \simeq k$.

Moreover it is easy to see that $\mathrm{Hom}_k(\alpha_p, \mathbb{G}_a) \simeq k$. The morphisms are given by $T \mapsto aT$ with $a \in k$. By these remarks we have that the isomorphism

$$\mathrm{Hom}_k(\alpha_p, \mathbb{G}_a) \times \mathrm{Ext}^1(\alpha_p, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^1(\alpha_p, \alpha_p),$$

deduced from II.5.1, is given by

$$(\beta, \gamma C_1) \mapsto E_{\beta,\gamma}.$$

The extension $E_{\beta,\gamma}$ is so defined:

$$E_{\beta,\gamma} = \mathrm{Spec}(k[S_1, S_2]/(S_1^p, S_2^p - \beta S_1))$$

(1) law of multiplication

$$S_1 \longmapsto S_1 \otimes 1 + 1 \otimes S_1$$
$$S_2 \longmapsto S_2 \otimes 1 + 1 \otimes S_2 + \gamma \frac{S_1^p \otimes 1 + 1 \otimes S_1^p - (S_1 \otimes 1 + 1 \otimes S_1)^p}{p}$$

(2) unit

$$S_1 \longmapsto 0$$
$$S_2 \longmapsto 0$$

(3) inverse

$$S_1 \longmapsto -S_1$$
$$S_2 \longmapsto -S_2$$

It is clear that all such extensions are commutative. In [**53**, 4.3.1] the following result was proved.

PROPOSITION II.5.2. *Let $\lambda, \mu \in \pi R \backslash \{0\}$. Then $[\mathcal{E}_k^{(\mu,\lambda;E_p(\boldsymbol{a},\mu,S))}] \in H_0^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ coincides with the class of*

$$\sum_{k=1}^{\infty} \frac{(\mathrm{F} - [\mu^{p-1}])(\tilde{\boldsymbol{a}})}{\lambda} C_k,$$

where $\tilde{\boldsymbol{a}}$ is a lifting of $\boldsymbol{a} \in \widehat{W}(R/\lambda R)$.

We deduce the following corollary about the extensions of α_p by $\mathbb{G}_a$.

COROLLARY II.5.3. *Let $\lambda, \mu \in \pi R \setminus \{0\}$. Then $[\tilde{\mathcal{E}}_k^{(\mu,\lambda;E_p(\boldsymbol{a},\mu,S))}] \in H_0^2(\alpha_{p,k}, \mathbb{G}_{a,k})$ coincides with the class of*

$$\frac{(\mathrm{F} - [\mu^{p-1}])(\tilde{\boldsymbol{a}})}{\lambda} C_1,$$

where $\tilde{\boldsymbol{a}}$ is a lifting of $\boldsymbol{a} \in \widehat{W}(R/\lambda R)$.

PROOF. This follows from the fact that $\mathcal{E}_k^{(\mu,\lambda;E_p(\boldsymbol{a},\mu,S))} \mapsto \tilde{\mathcal{E}}_k^{(\mu,\lambda;E_p(\boldsymbol{a},\mu,S))}$ through the map

$$\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_a) \simeq H_0^2(\mathbb{G}_a, \mathbb{G}_a) \longrightarrow H_0^2(\alpha_p, \mathbb{G}_a) \simeq \mathrm{Ext}^1(\alpha_p, \mathbb{G}_a).$$

□

Let us take an extension $\mathcal{E}^{(\mu,\lambda;E_p(aS),j)}$. Let $\tilde{a}$ be a lifting of a. We have that on the special fiber this extension is given as a scheme by

$$\mathcal{E}_k^{(\mu,\lambda;E_p(aS),j)} = \mathrm{Spec}(k[S_1, S_2]/(S_1^p, S_2^p - (-\frac{(\sum_{i=0}^{p-1} \frac{\tilde{a}^i}{i!} S^i)^p (1 + \mu S_1)^{-j} - 1}{\lambda^p}))).$$

By II.3.42 we know that

$$pa - j\mu - \frac{p}{\mu^{p-1}} a^p = 0 \in R/\lambda^p$$

In the proof of the same corollary we have seen that

$$p[a]-j[\mu]-[\frac{p}{\mu^{p-1}}a^p]-V([a^p])=[pa-j\mu-\frac{p}{\mu^{p-1}}a^p]\in\widehat{W}(R/\lambda^pR).$$

By the definitions we have the following equality in $\mathrm{Hom}(\mathcal{G}^{(\mu)}_{|S_{\lambda^p}},\mathbb{G}_{m|S_{\lambda^p}})$

$$\xi^0_{R/\lambda^pR}(p[a]-j[\mu]-[\frac{p}{\mu^{p-1}}a^p]-V([a^p]))=E_p(aS_1)^p(1+\mu S_1)^{-j}\left(E_p\left(a^p\left(\frac{(1+\mu S_1)^p-1}{\mu}\right)\right)\right)^{-1}.$$

Moreover we have

$$\xi^0_{R/\lambda^pR}([pa-j\mu-\frac{p}{\mu^{p-1}}a^p])=E_p((pa-j\mu-\frac{p}{\mu^{p-1}}a^p)S_1)$$

So we have that

$$\begin{aligned}(\sum_{i=0}^{p-1}\frac{\tilde{a}^i}{i!}S_1^i)^p(1+\mu S_1)^{-j}-1&\equiv\sum_{i=0}^{p-1}\frac{(pa-j\mu-\frac{p}{\mu^{p-1}}a^p)^iS_1^i}{i!}-1\\&\equiv 0\mod\lambda^p\left(R[S_1]/(\frac{(1+\mu S_1)^p-1}{\mu^p})\right).\end{aligned}$$

Hence

$$\begin{aligned}\frac{(\sum_{i=0}^{p-1}\frac{\tilde{a}^i}{i!}S_1^i)^p(1+\mu S_1)^{-j}-1}{\lambda^p}&\equiv\frac{\sum_{i=0}^{p-1}\frac{(pa-j\mu-\frac{p}{\mu^{p-1}}a^p)^i}{i!}S_1^i-1}{\lambda^p}\\&\equiv\frac{(pa-j\mu-\frac{p}{\mu^{p-1}}a^p)}{\lambda^p}S_1\mod\pi.\end{aligned}$$

On the other hand $\mathcal{E}_k^{(\mu,\lambda;E_p(aS),j)}\mapsto\tilde{\mathcal{E}}_k^{(\mu,\lambda;E_p(aS))}$ through the map $\mathrm{Ext}^1(\alpha_p,\alpha_p)\to\mathrm{Ext}^1(\alpha_p,\mathbb{G}_a)$.

Therefore $\mathcal{E}_k^{(\mu,\lambda;E_p(aS),j)}\simeq E_{\beta,\gamma}$ with $\beta=(\frac{p\tilde{a}-j\mu-\frac{p}{\mu^{p-1}}\tilde{a}^p}{\lambda^p}\mod\pi)$ and $\gamma=(\frac{\tilde{a}^p}{\lambda}\mod\pi)$. We have so proved the following result.

PROPOSITION II.5.4. *Let $\lambda,\mu\in\pi R$ be such that $v(\lambda)\le v(\mu)<v(\lambda_{(1)})$. Then $[\mathcal{E}_k^{(\mu,\lambda;E_p(aS),j)}]\in\mathrm{Ext}^1_k(\alpha_p,\alpha_p)$ coincides with the class of*

$$\left(-\frac{p\tilde{a}-j\mu-\frac{p}{\mu^{p-1}}\tilde{a}^p}{\lambda^p},\frac{\tilde{a}^p}{\lambda}C_1\right),$$

where $\tilde{a}$ is a lifting of $a\in R/\lambda R$.

II.5.4. Case $\mathbf{v}(\lambda_{(1)})=\mathbf{v}(\mu)>\mathbf{v}(\lambda)>0$.

In this situation we have

$$(G_{\mu,1})_k\simeq\mathbb{Z}/p\mathbb{Z}\quad\text{and}\quad(G_{\lambda,1})_k\simeq\alpha_p.$$

PROPOSITION II.5.5. *Let $\lambda,\mu\in\pi R$ be such that $v(\mu)=v(\lambda_{(1)})>v(\lambda)$. Then $\mathcal{E}_k^{(\mu,\lambda;F,j)}$ is the trivial extension.*

PROOF. We can suppose $\mu = \lambda_{(1)}$. From II.3.48 it follows that $\mathcal{E}^{(\lambda_{(1)},\lambda;F,j)}$ is in the image of the morphism

$$\mathrm{Ext}^1(G_{\lambda_{(1)},1}, G_{\lambda_{(1)},1}) \longrightarrow \mathrm{Ext}^1(G_{\lambda_{(1)},1}, G_{\lambda,1})$$

induced by the map $\mathbb{Z}/p\mathbb{Z} \simeq G_{\lambda_{(1)},1} \longrightarrow G_{\lambda,1}$ given by $S \mapsto \frac{\lambda_{(1)}}{\lambda}S$. But this morphism is the zero morphism on the special fiber. So we are done. □

II.5.5. Case $\mathbf{v}(\lambda_{(1)}) = \mathbf{v}(\mu) = \mathbf{v}(\lambda)$. We have

$$(G_{\mu,1})_k \simeq \mathbb{Z}/p\mathbb{Z} \quad \text{and } (G_{\lambda,1})_k \simeq \mathbb{Z}/p\mathbb{Z}.$$

For simplicity we will consider the case $\mu = \lambda = \lambda_{(1)}$. We recall the following result.

PROPOSITION II.5.6. *Let suppose that $\mathbb{Z}/p\mathbb{Z}$ acts trivially on $\mathbb{Z}/p\mathbb{Z}$ over k. The exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \overset{\mathrm{F}-1}{\to} \mathbb{G}_a \to 0$ induces the following exact sequence*

$$\mathrm{Hom}_{gr}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) \simeq k \overset{\mathrm{F}-1}{\longrightarrow} \mathrm{Hom}_{gr}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) \simeq k \longrightarrow \mathrm{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) \simeq k \overset{\mathrm{F}-1}{\longrightarrow} \mathrm{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) \simeq k$$

PROOF. [15] □

We observe that $\ker\left(\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) \overset{\mathrm{F}-1}{\longrightarrow} \mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a)\right) \simeq \mathbb{Z}/p\mathbb{Z}$. It is possible to describe more explicitly $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$. We recall that $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a) = H^2_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_a)$ is freely generated as a right k-module by the class of the cocycle $C_1 = \frac{X^p + X^p - (X+Y)^p}{p}$.

There is an isomorphism, induced by the maps of II.5.6,

$$k/(\mathrm{F}-1)(k) \times \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}),$$

given by

$$(a, b) \mapsto E_{a,b}.$$

The extension $E_{a,b}$ is so defined: let $\bar{a} \in k$ a lifting of a,

$$E_{a,b} = \mathrm{Spec}(k[S_1, S_2]/(S_1^p - S_1, S_2^p - S_2 - \bar{a}S_1))$$

(1) law of multiplication

$$S_1 \longmapsto S_1 \otimes 1 + 1 \otimes S_1$$
$$S_2 \longmapsto S_2 \otimes 1 + 1 \otimes S_2 + b\frac{S_1^p \otimes 1 + 1 \otimes S_1^p - (S_1 \otimes 1 + 1 \otimes S_1)^p}{p}$$

(2) unit

$$S_1 \longmapsto 0$$
$$S_2 \longmapsto 0$$

(3) inverse

$$S_1 \longmapsto -S_1$$
$$S_2 \longmapsto -S_2$$

We remark that the extensions which are isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ as group schemes are the extensions $E_{0,b}$ with $b \neq 0$. By II.3.44 we have that any extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ is given by $\mathcal{E}^{(\lambda_{(1)},\lambda_{(1)};E_p(j\eta S),j)}$. We now study its reduction on the special fiber.

PROPOSITION II.5.7. *For any $j \in \mathbb{Z}/p\mathbb{Z}$,*

$$[\mathcal{E}_k^{(\lambda_{(1)},\lambda_{(1)};E_p(j\eta S),j)}] = E_{0,j} \in \mathrm{Ext}^1_k(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}).$$

PROOF. As group schemes, $\mathcal{E}^{(\lambda_{(1)},\lambda_{(1)};E_p(j\eta S),j)} \simeq \mathbb{Z}/p^2\mathbb{Z}$, if $j \neq 0$, and $\mathcal{E}^{(\lambda_{(1)},\lambda_{(1)};1,0)} \simeq \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ otherwise. In particular $\mathcal{E}_k^{(\lambda_{(1)},\lambda_{(1)};E_p(j\eta S),j)}$ has a scheme-theoretic section. It is easy to see that $\mathcal{E}_k^{(\lambda_{(1)},\lambda_{(1)};E_p(j\eta S),j)} \simeq E_{0,b}$ with

$$b = (-j\frac{\eta^p}{\lambda_{(1)}(p-1)!} \mod \pi) = j,$$

since $\frac{\eta^p}{\lambda_{(1)}} \equiv \frac{\lambda_{(2)}^p}{\lambda_{(1)}} \equiv 1 \mod \pi$ and $(p-1)! \equiv -1 \mod \pi$ (Wilson Theorem). □

CHAPTER III

Extension of torsors

III.1. Effective models

Here we recall some definitions and results about effective models which will play a key role in our results about extensions of $\mathbb{Z}/p^2\mathbb{Z}$-torsors. For more details see [**40**], which is the source where most of the material of this paragraph has been taken.

In this section $char(K)$ is not necessarily 0. Let Y_K be a flat K-scheme endowed with a faithful action of a finite group scheme G. Given an R-model Y to which the action extends, it may happen that the reduced action on the special fibre acquires a kernel, especially if p divides $|G|$. The effective models are the objects which solve this problem.

In [**35**, § 2], the relation of *domination* between models of a group scheme has been introduced.

Definition III.1.1. Let G_1 and G_2 be finite flat group schemes over R with an isomorphism $u_K\colon G_{1,K} \longrightarrow G_{2,K}$. We say that G_1 *dominates* G_2 and we write $G_1 \geq G_2$, if we are given an R-morphism $u\colon G_1 \longrightarrow G_2$ which restricts to u_K on the generic fibre. The map u is also called a *model map*. If moreover G_1 and G_2 act on Y, we say that G_1 dominates G_2 *compatibly* (with the actions) if $\mu_1 = \mu_2 \circ (u \times \mathrm{id})$.

We now recall the definition of a faithful action.

Definition III.1.2. Let G be a group scheme which acts on a scheme Y over a scheme T. This action is *faithful* if the induced morphism of sheaves of groups, in the fppf topology of T,

$$G \longrightarrow \mathcal{A}ut_T(Y)$$

is injective.

We recall here the definition of effective model given by Romagny.

Definition III.1.3. Let G be a finite flat group scheme over R. Let Y be a flat scheme over R. Let $\mu\colon G \times Y \longrightarrow Y$ be an action, faithful on the generic fibre. An *effective model* for μ is a finite flat R-group scheme $\mathcal{G}$ acting on Y, dominated by G compatibly, such that $\mathcal{G}$ acts faithfully on Y.

Example III.1.4. Let X be a flat scheme over R and $\mathcal{G}$ a finite and flat group scheme over R. Let $Y \longrightarrow X$ be a $\mathcal{G}$-torsor over R. Then $\mathcal{G}$ is already an effective model. Indeed let us suppose that $\mathcal{G} \longrightarrow \mathcal{A}ut_R(Y)$ is not injective. Then there exists a faithfully flat morphism $U \longrightarrow \mathrm{Spec}(R)$ and $g \in \mathcal{G}(U) \setminus \{0\}$ such that

$$Y \times_R U \xrightarrow{g \times id} \mathcal{G} \times Y \times_R U \xrightarrow{\mu \times \mathrm{id}} Y \times_R Y \times_R U$$

is equal to $\Delta \times \mathrm{id} : Y \times_R U \longrightarrow Y \times_R Y \times_R U$ where $\Delta : Y \longrightarrow Y \times_R Y$ is the diagonal morphism. By the definition of $\mathcal{G}$-torsor $\mathcal{G} \times_R Y \times_R U \xrightarrow{\mu\times\mathrm{id}} Y \times_R Y \times_R U$ is an isomorphism. Then

$$Y \times_R U \xrightarrow{g\times id} \mathcal{G} \times_R Y \times_R U \xrightarrow{\mu\times\mathrm{id}} Y \times_R Y \times_R U \xrightarrow{(\mu\times\mathrm{id})^{-1}} \mathcal{G} \times_R Y \times_R U$$

is the zero section, against assumptions.

We report here some results about effective models.

PROPOSITION III.1.5. *An effective model is unique up to unique isomorphism, if it exists.*

PROOF. [**40**, 1.1.2]. □

We will constantly use the following crucial remark in the next sections.

REMARK III.1.6. Let G be a finite and flat group scheme over R and Y a flat scheme over R. Let $\mu\colon G \times Y \longrightarrow Y$ be an action. Moreover we suppose that $Y_K \longrightarrow Y_K/G_K$ is a G_K-torsor. Then by III.1.5 we have that the effective model $\mathcal{G}$ whose action extends that of G_K is unique if it exists. By III.1.4 this means that if there exists a model G' of G_K, compatible with the action, such that Y is a G'-torsor, then G' is the effective model for μ.

We recall that an action $\mu : G \times X$ is admissible if X can be covered by G-stable open affine subschemes. We here state the main results about effective models obtained in [**40**].

PROPOSITION III.1.7. *Let G be a finite flat group scheme over R. Let X be a flat scheme over R and $\mu : G \times X \longrightarrow X$ an admissible action, faithful on the generic fiber. Assume there exists an effective model $\mathcal{G}$. Then*

(i) *If H is a finite flat subgroup of G the restriction of the action to H has an effective model $\mathcal{H}$ which is the schematic image of H in $\mathcal{G}$. If H is normal in G, then $\mathcal{H}$ is normal in $\mathcal{G}$.*
(ii) *The identity of X induces an isomorphism $X/G \simeq X/\mathcal{G}$.*
(iii) *Assume that there exists an open subset $U \subseteq X$ which is schematically dense in any fiber of $X \longrightarrow \mathrm{Spec}(R)$ such that $\mathcal{G}$ acts freely on U. Then for any normal subgroup $H \lhd G$ the effective model of G/H acting on X/H is $\mathcal{G}/\mathcal{H}$.*

PROOF. [**40**, 1.1.3]. □

THEOREM III.1.8. *Let X be a flat R-scheme and $\mu\colon G \times X \longrightarrow X$ an action. Assume that X is covered by G-stable open affines U_i with function ring separated for the π-adic topology, such that G acts faithfully on the generic fibre $U_{i,K}$. Denote by $\mathcal{F}$ the family of all closed subschemes $Z \subset X$ which are finite and flat over R. Assume that $\mathcal{F}_k$ is schematically dense in X_k. Then there exists an effective model for the action of G.*

PROOF. [**40**, 1.2.2]. □

For algebraic schemes we have the following:

COROLLARY III.1.9. *Let G be a finite flat group scheme over R. Let Y be a flat scheme of finite type over R and let $\mu\colon G\times Y \longrightarrow Y$ be an action. We assume that Y is covered by G-stable open affines U_i with function ring separated for the π-adic topology, such that G acts faithfully on the generic fibre $U_{i,K}$. Then, if Y has reduced special fibre, there exists an effective model for the action of G.*

PROOF. [**40**, 1.2.3]. □

We remark that the condition about the separatedness of the function rings of U_i is assured if, for instance, we assume Y integral. This follows from the Theorem of Krull ([**30**, 1.3.13]).

If we add some hypothesis on Y then we have an useful criterion to see if a group scheme which acts on Y is the effective model for the action.

Recall that a module M over a ring A is called *semireflexive* if the canonical map from M to its bidual is injective. Equivalently M is a submodule of some product module A^I. Indeed, consider the set $I = \mathrm{Hom}(M, A)$ and the morphism $a : M \to A^I$ mapping x to the collection of values $(f(x))_{f\in I}$ for all linear forms f. By definition, if M is semireflexive then for each nonzero $x \in M$ there exists a linear form such that $f(x) \neq 0$, so a is injective. The converse is easy.

REMARK III.1.10. A semireflexive module over a d.v.r. R is faithfully flat and separated with respect to the π-adic topology. Indeed since $M \subseteq R^I$ for some set I then M is torsion free, hence flat over R. Moreover let $x \in \cap\pi^m M$. Then for any linear form f we have $f(x) \in \cap\pi^m R = 0$. Since M is semireflexive this implies $\cap\pi^m M = 0$. So M is separated with respect to the π-adic topology. But, over a d.v.r, being flat and separated with respect to the π-adic topology implies faithfully flat. Indeed since $\cap\pi^m M = 0$ in particular $M \neq \pi M$. So M is faithfully flat (see [**30**, 1.2.17]). We do not know if the converse is true too, i.e. if any (faithfully) flat R-module separated with respect to the π-adic topology is semireflexive.

EXAMPLE III.1.11. Any free R-module over R is semireflexive, e.g. $R[T_1,\dots,T_n]$. Other examples, not free, are $R[[T_1,\dots,T_n]]$ or $R[[T_1,\dots,T_n]]\{T_1^{-1},\dots,T_n^{-1}\}$.

The following lemma will be useful in §III.5.

LEMMA III.1.12. *Let R be a d.v.r. Let A be an R-algebra which is semireflexive as an R-module and let B be a flat R-algebra. If there exists a finite R-morphism of modules*

$$A \longrightarrow B,$$

such that B_K is semireflexive as an A_K-module, then B is semireflexive as an R-module.

REMARK III.1.13. In particular any finite and flat R-algebra is semireflexive as an R-module.

PROOF. Let us consider B_K. It is a vector space over K, so in particular it is semireflexive over K. Since A and B are flat over R the natural maps $A \longrightarrow A_K$ and

$B \longrightarrow B_K$ are injective. We now prove that B is semireflexive over A. Let $b \in B$ and let us take an A_K-linear form $f : B_K \longrightarrow A_K$ such that $f(b) \neq 0$. It exists since B_K is semireflexive over A_K. Let $b_1, \dots, b_m$ generators of B as an A-module and let $n \in \mathbb{N}$ such that $\pi^n f(b_i) \in A$ for $i = 1, \dots, m$. Then we have $\pi^n f(B) \subseteq A$. Moreover $\pi^n f(b) \neq 0$, since A is flat over R by III.1.10. So $\pi^n f : B \longrightarrow A$ is a linear form with $\pi^m f(b) \neq 0$. Then B is semireflexive as an A-module. But A is semireflexive over R. Therefore B is semireflexive over R. Indeed for any $b \in B$ let us take an A-linear form $g : B \longrightarrow A$ with $g(b) \neq 0$. Moreover let us consider an R-linear form $h : A \longrightarrow R$ such that $h(g(b)) \neq 0$. Then $h \circ g : B \longrightarrow R$ is an R-linear form with $h \circ g(b) \neq 0$. Hence B is semireflexive over R. □

DEFINITION III.1.14. We will say that a morphism of schemes $f : X \to T$ is *essentially semireflexive* if there exists a cover of T by open affine subschemes T_i, an affine faithfully flat T_i-scheme T'_i for all i, and a cover of $X'_i = X \times_T T'_i$ by open affine subschemes X'_{ij}, such that the function ring of X'_{ij} is semireflexive as a module over the function ring of S'_i.

This is a generalization of the definition of an essentially free morphism given in [**1**]. The proofs of the following two lemmas have been suggested to us by Romagny.

LEMMA III.1.15. *Let X be essentially semireflexive and separated over T. Let G be a T-group scheme acting on $X \to T$. Then the kernel of the action is representable by a closed subscheme of G.*

PROOF. Proceeding like in [**1**] we are reduced to proving the analogue of [**1**, 6.4]. Then the proof given in [**1**] works in our case, because the only property of free modules that is used in the proof is that they are semireflexive. □

The next lemma is the reason because we are interested in essentially semireflexive schemes. Indeed in such a case we have an useful criterion to check if a finite group scheme is an effective model.

LEMMA III.1.16. *Assume furthermore that $T = \mathrm{Spec}(R)$ where R is a discrete valuation ring, and G is finite and flat over T. Then the action of G is faithful if and only the action of G_k on the special fibre is faithful.*

PROOF. Only the *if* part needs a proof. Let I_G be the augmentation ideal of G and let J be the ideal defining the kernel H of the action. Since H is a subgroup-scheme of H and H_k is trivial then

$$J \subseteq I_G \quad \text{and } I_G + \pi R[G] = J + \pi R[G] \tag{III.75}$$

Moreover, since $R[G]/I_G$ is flat over R then

$$I_G \cap \pi R[G] = \pi I_G. \tag{III.76}$$

We now claim that

$$I_G = J + \pi I_G.$$

Clearly $J + \pi I_G \subseteq I_G$. We now prove the converse. Let $a \in I_G$, then from (III.75) it follows $a = b + \pi c$ for some $b \in J$ and $c \in R[G]$. Since $J \subseteq I_G$ then

$\pi c \in I_G \cap R[G]$. Therefore by (III.76) we have $c \in J$. Hence $I_G \subseteq J + \pi I_G$. We have so proved $I_G = J + \pi I_G$. Then I_G/J is an R-module of finite type and $(I_G/J) \otimes k = I_G/(J + \pi I_G) = 0$, so $I_G/J = 0$ by Nakayama's lemma. Hence the kernel is trivial. □

III.2. Presentation of the problem

We now recall the problem which we will study in this chapter. In the following, K is of characteristic zero. Let G be a finite abstract group. Let X be a faithfully flat scheme over R and $Y_K \longrightarrow X_K$ a G_K-torsor. We remark that, since K is of characteristic 0, any finite group scheme is étale. Moreover let us consider Y the normalization of X in Y_K. A natural question is:
is it always possible to find a model $\mathcal{G}$ of $(G)_K$ over R together with an action on Y such that Y is a $\mathcal{G}$-torsor and the action of $\mathcal{G}$ coincides with that of $(G)_K$ on the generic fiber?

We will say that the G-torsor $Y_K \longrightarrow X_K$ can be *strongly extended* if the previous question has positive answer. We also consider another notion of extension of torsors. We say that $Y_K \longrightarrow X_K$ can be *weakly extended* if there exists a model $\mathcal{G}$ of $(G)_K$ over R together with an action over a scheme Y', with $Y'_K \simeq Y_K$, such that Y' is a $\mathcal{G}$-torsor and the action of G coincides with that of $(G)_K$ on the generic fiber. It is clear that strong extension implies weak extension. The converse is not true as we will see. The point is that in general Y' is not equal to Y. For instance if X is normal, which is the case which we will study, Y is normal (see III.3.7), but in general Y' is not normal.

Using the theory of effective models the above question can be reformulated in another way. First of all we observe that, since Y is the normal closure of X in Y_K, any G-action on Y_K can be extended to a G- action on Y. Then we can ask what the effective model of this action is, if it exists. Moreover, by III.1.6, we see that the previous question can be rewritten in the following way:
does there always exist an effective model $\mathcal{G}$ for the action of G which makes Y a $\mathcal{G}$-torsor?

III.3. Weak extension of torsors under commutative group schemes

The aim of this section is to prove a result of weak extension for torsors under commutative group schemes over normal schemes with some hypothesis.

III.3.1. Preliminary results. We here state some results which will be useful in what follows.

PROPOSITION III.3.1. *Let $i = 1, 2$. Let Z_i be a faithfully flat S-scheme and let G_i be an affine flat S-group scheme, together with an admissible action, over a faithfully flat Z_i-schemes Y_i. Moreover we suppose that $Y_2 \longrightarrow Z_2$ is a G_2-torsor and that there exists a morphism*

$$\varphi_K : (G_1)_K \longrightarrow (G_2)_K.$$

Let us suppose we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & Y_2 \\ \downarrow & & \downarrow \\ Z_1 & \longrightarrow & Z_2 \end{array}$$

of S-schemes such that f_K is an isomorphism compatible with the actions. Then there exists a unique morphism

$$\varphi : G_1 \longrightarrow G_2$$

which extends φ_K and such that f is compatible with the actions.

PROOF. For $i = 1, 2$ we call $\sigma_i : G_i \times_R Y_i \longrightarrow Y_i$ the actions. Since $Y_2 \longrightarrow Z_2$ is a G_2-torsor then $\sigma_2 \times \mathrm{id}$ is an isomorphism. So by

$$G_1 \times_R Y_1 \xrightarrow{\sigma_1 \times \mathrm{id}} Y_1 \times_{Z_1} Y_1 \xrightarrow{f \times f} Y_2 \times_{Z_2} Y_2 \xrightarrow{(\sigma_2 \times \mathrm{id})^{-1}} G_2 \times_R Y_2$$

we obtain a morphism

$$G_1 \times_R Y_1 \longrightarrow G_2 \times_R Y_2.$$

If we compose it with the projection $p_1 : G_2 \times_R Y_2 \longrightarrow G_2$ we obtain a morphism

$$G_1 \times_R Y_1 \longrightarrow G_2.$$

Moreover we consider the projection

$$p_2 : G_1 \times_R Y_1 \longrightarrow Y_1.$$

Therefore we have a map

$$\varphi_{Y_1} : G_1 \times_R Y_1 \longrightarrow G_2 \times_R Y_1. \tag{III.77}$$

We now prove that it is compatible with φ_K, i.e. φ_{Y_1} and φ_K induce the same morphism $G_1 \times_R (Y_1)_K \longrightarrow G_2 \times_R (Y_1)_K$. We observe that φ_{Y_1} and φ_K induce two morphisms, $(\varphi_{Y_1})_K$ and $(\varphi_K)_{Y_1}$ respectively, which are compatible with f_K. For any $\psi : G_1 \times_R (Y_1)_K \longrightarrow G_2 \times_R (Y_1)_K$, to be compatible with f_K means that the following diagram

$$\begin{array}{ccc} G_1 \times_R (Y_1)_K & \xrightarrow{\sigma_1} & (Y_1)_K \\ {\scriptstyle (\mathrm{id} \times f_K) \circ \psi} \downarrow & & \downarrow {\scriptstyle f_K} \\ G_2 \times_R (Y_2)_K & \xrightarrow{\sigma_2} & (Y_2)_K \end{array}$$

commutes. Hence $\sigma_2 \circ (\mathrm{id} \times f_K) \circ \psi = f_K \circ \sigma_1$. So we have

$$\sigma_2 \circ (\mathrm{id} \times f_K) \circ (\varphi_{Y_1})_K = \sigma_2 \circ (\mathrm{id} \times f_K) \circ (\varphi_K)_{Y_1} = f_K \circ \sigma_1.$$

Since $(Y_2)_K \longrightarrow (Z_2)_K$ is a $(G_2)_K$-torsor then

$$(\mathrm{id} \times f_K) \circ (\varphi_{Y_1})_K = (\mathrm{id} \times f_K) \circ (\varphi_K)_{Y_1}.$$

For $i = 1, 2$, let p_i be the projections from $G_2 \times_R (Y_1)_K$ and let p'_i be the projections from $G_2 \times_R (Y_2)_K$. Then

$$p_1 \circ (\varphi_{Y_1})_K = p'_1 \circ (\mathrm{id} \times f_K) \circ (\varphi_{Y_1})_K = p'_1 \circ (\mathrm{id} \times f_K) \circ (\varphi_K)_{Y_1} = p_1 \circ (\varphi_K)_{Y_1}.$$

and

$$f_K \circ p_2 \circ (\varphi_{Y_1})_K = p_2' \circ (\mathrm{id} \times f_K) \circ (\varphi_{Y_1})_K = p_2' \circ (\mathrm{id} \times f_K) \circ (\varphi_K)_{Y_1} = f_K \circ p_2 \circ (\varphi_K)_{Y_1}.$$

Since f_K is an isomorphism then $p_2 \circ (\varphi_{Y_1})_K = p_2 \circ (\varphi_K)_{Y_1}$. Hence $(\varphi_{Y_1})_K = (\varphi_K)_{Y_1}$, i.e. φ_K is compatible with φ_{Y_1}. By the next descent lemma we have a unique morphism of schemes $\varphi : G_1 \longrightarrow G_2$ which extends φ_K and φ_{Y_1}. Since G_2 is flat over R and φ_K is a morphism of group schemes it follows easily from II.3.13 that φ is a morphism of group schemes. By construction it is clear that, through φ, the morphism f preserves the actions. □

We now prove the descent lemma used in the previous proof.

LEMMA III.3.2. *Let R be a d.v.r, $S = \mathrm{Spec}(R)$, and let $S' \to S$ be a faithfully flat morphism of schemes. Let X_1, X_2 be affine S-schemes with X_2 flat over S. Given two morphisms $\varphi_K : (X_1)_K \to (X_2)_K$ and $\varphi_{S'} : X_1 \times_S S' \to X_2 \times_S S'$ that coincide on S'_K, there is a unique morphism $\varphi : X_1 \to X_2$ that extends them.*

PROOF. Up to restricting ourselves to an affine subscheme of S', we can suppose $S' = \mathrm{Spec}(A)$. For $i = 1, 2$, let us consider $X_i = \mathrm{Spec}(B_i)$. In terms of function rings we have two morphisms

$$\varphi_A^\sharp := \varphi_{S'}^\sharp(X_2 \times_S S') : B_2 \otimes_R A \longrightarrow B_1 \otimes_R A \tag{III.78}$$

and

$$\varphi_K^\sharp := \varphi_K^\sharp((X_2')_K) : B_2 \otimes_R K \longrightarrow B_1 \otimes_R K. \tag{III.79}$$

Moreover, by compatibility, it follows that the above morphisms induce the same map

$$\varphi_{A_K}^\sharp := (\varphi_A^\sharp) \otimes \mathrm{id}_K = (\varphi_K^\sharp) \otimes \mathrm{id}_{A_K} : B_2 \otimes_R A \otimes_R K \longrightarrow B_1 \otimes_R A \otimes_R K.$$

First of all we prove the uniqueness of φ. Since X_2 is flat over S then the inclusion $(X_2)_K \longrightarrow X_2$ induces an injection $B_2 \hookrightarrow B_2 \otimes_R K$. Therefore if (any) φ exists then it is given, on the level of function rings, by the restriction of $\varphi_K^\sharp$ to B_2. Therefore it is unique.

We now prove the existence of φ. First of all we have the following commutative diagram with the obvious maps

$$\begin{array}{ccccc} & & B_i \otimes_R A & & \\ & \nearrow & & \searrow & \\ B_i & & & & B_i \otimes_R A \otimes_R K \\ & \searrow & & \nearrow & \\ & & B_i \otimes_R K & & \end{array}$$

Since $S' \longrightarrow S$ is flat in particular the induced map $R \longrightarrow A$ is injective. Moreover B_2 is a flat R-algebra, then for $i = 2$ all the maps of the above diagram are injective. We remark that (III.78) and (III.79) imply $\varphi_{A_K}^\sharp(B_2) \subseteq (B_1 \otimes_R K) \cap (B_1 \otimes_R A)$. We claim that $\varphi_{A_K}^\sharp(B_2) \subseteq B_1$. Let us suppose that there exists $b \in B_2$ such that

$\varphi^\sharp_{A_K}(b) \notin B_1$. Then there exists $n \geq 1$ such that $\pi^n \varphi^\sharp_{A_K}(b) \in B_1$ and $\pi^{n-1}\varphi^\sharp_{A_K}(b) \notin B_1$. Hence

$$\pi^n \varphi^\sharp_{A_K}(b) \in B_1 \cap \pi(B_1 \otimes_R A).$$

We remark that since $S' \longrightarrow S$ is surjective then $S'_k = \mathrm{Spec}(A/\pi A)$ is nonempty. Now, since any scheme over a field is flat,

$$B_1/\pi B_1 \longrightarrow (B_1 \otimes_R A)/\pi(B_1 \otimes_R A) \simeq B_1/\pi B_1 \otimes_k A/\pi A$$

is injective. Therefore

$$B_1 \cap \pi(B_1 \otimes_R A) = \pi B_1,$$

which implies $\pi^{n-1}\varphi^\sharp_{A_K}(b) \in B_1$. This is a contradiction. So $\varphi^\sharp_{A_K}$ induces a morphism

$$\varphi^\sharp : B_2 \longrightarrow B_1.$$

We have so proved that $\varphi_K : (Y_1)_K \longrightarrow (Y_2)_K$ is extendible to a morphism $\varphi : Y_1 \longrightarrow Y_2$. □

The previous descent lemma has as consequence the following result, which however will not be used in the rest of the thesis.

PROPOSITION III.3.3. *Let G be an affine flat and commutative S-group scheme. Then $H^1(S, G) \longrightarrow H^1(K, G_K)$ is injective.*

REMARK III.3.4. For $X = S$ this result is stronger than I.3.6 (we have removed the hypothesis G finite over S).

PROOF. Let $f : Y \longrightarrow S$ be a G-torsor. This means that there exists a faithfully flat S-scheme T such that $Y_T := Y \times_X T \longrightarrow T$ is trivial. (For instance we can chose $T = Y$). Then it has a section $\varphi_T : T \longrightarrow Y_T$. Moreover let us suppose that $Y \longrightarrow S$ is trivial as G_K-torsor on X_K. Then there is a section $\varphi_K : \mathrm{Spec}(K) \longrightarrow Y_K$ of $Y_K \longrightarrow \mathrm{Spec}(K)$. Since G is affine then $f : Y \longrightarrow S$ is an affine morphism. So Y is affine. From the previous lemma the thesis follows. □

LEMMA III.3.5. *Let X, Y be integral flat schemes over S. Moreover let us suppose that X is normal. If $f : Y \longrightarrow X$ is an integral dominant R-morphism then f_k is schematically dominant, i.e. $f^\sharp_k : \mathcal{O}_{X_k} \longrightarrow f_*\mathcal{O}_{Y_k}$ is injective. In particular if Y_k is integral then X_k is integral, too.*

PROOF. Since any integral morphism is affine by the definition it is enough to prove the lemma in the affine case. So we can suppose $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ with an integral injection $A \hookrightarrow B$. We will prove that $A_k \hookrightarrow B_k$. This is equivalent to proving $\pi B \cap A = \pi A$. One inclusion is obvious. Now let $a \in \pi B \cap A$, then $a = \pi b$ with $b \in B$. We remark that $b = \frac{a}{\pi} \in A_K \cap B$ is integral over A. But A is integrally closed by hypothesis. Therefore $b \in A$.

Now let us suppose that B_k is an integral domain. Then $A_k \hookrightarrow B_k$ implies that A_k is an integral domain, too. □

LEMMA III.3.6. *Let R be a d.v.r. with field of fractions K and residue field k. Let X be a flat R-scheme. If X_K is normal and X_k reduced, then X is normal.*

PROOF. For a proof see [**30**, 4.1.18]. □

III.3.2. Weak extension. We now consider a normal separated and faithfully flat R-scheme X with integral fibers. Let $Y_K \longrightarrow X_K$ be a connected G_K-torsor, for some finite group-scheme G_K over K, and Y the normalization of X in Y_K. We remark that Y_K is normal, too. In particular Y_K is integral, hence Y is integral. We denote by g the morphism $Y \longrightarrow X$. We observe that $g : Y \longrightarrow X$ is finite (see [**30**, 4.1.25]), in particular it is affine. We remark that Y has the following properties.

LEMMA III.3.7. *Let $f : Z \longrightarrow X$ be an integral morphism of R-flat schemes such that there exists an X_K-isomorphism $h_K : Y_K \longrightarrow Z_K$. Then there exists an X-morphism $h : Y \longrightarrow Z$, which uniquely extends h_K, such that*

$$g = f \circ h.$$

Moreover

i) *Y is normal;*

ii) *if Z is normal, too, then h is an isomorphism;*

PROOF. As remarked above $Y \longrightarrow X$ is affine. So, first, we suppose $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $Z = \mathrm{Spec}(C)$. By hypothesis we can suppose

$$A \subseteq C \subseteq C_K$$

with C integral over A and $C_K = B_K$. But, since B is the integral closure of A in B_K, then $C \subseteq B$. So we have

$$A \subseteq C \subseteq B.$$

If we rewrite all this in terms of schemes, it is easy to see that this is equivalent to the existence of h. The uniqueness of h_K derives from the fact that $X = \mathrm{Spec}(A)$ is a separated scheme (over $\mathrm{Spec}(R)$), X_K is an open dense of X and Y is reduced.

We now prove that Y is normal, i.e. B is integrally closed. Since Y_K is normal then B_K is integrally closed. So if $b \in Frac(B)$ is integral over B then $b \in B_K$. On the other hand B is integral over A, then b is integral over A. But B is the integral closure of A in B_K, therefore $b \in B$.

We now consider the general case. Let $X = \cup U_i$ with U_i affine open subschemes. Since f and g are affine morphisms then $f^{-1}(U_i)$ and $g^{-1}(U_i)$ are affine open subschemes of Y and Z, respectively. By hypothesis,

$$h_{iK} := h_{|(f^{-1}(U_i))_K} : (f^{-1}(U_i))_K \longrightarrow (g^{-1}(U_i))_K$$

is an isomorphism. Then, by what we just proved in the affine case, we have, for any i, an unique morphism $h_i : f^{-1}(U_i) \longrightarrow g^{-1}(U_i)$ which extends h_{iK}. Now, since X is separated over R and the morphisms $Y \longrightarrow X$ and $Z \longrightarrow X$ are separated, then Y and Z are separated over R too. Hence $f^{-1}(U_i) \cap f^{-1}(U_j)$ and $g^{-1}(U_i) \cap g^{-1}(U_i)$ are affine (see [**30**, 3.3.6]). So, again by the uniqueness of h in the affine case, we have $h_i = h_j$ on $f^{-1}(U_i) \cap f^{-1}(U_j)$. So there exists an X-morphism $h : Y \longrightarrow Z$ which extends h_K. Moreover, since normality is a local property, Y is normal. We have so proved i), too.

ii) If Z is normal then, by definition of the integral closure of X in Y_K, we have that h is an isomorphism.

□

We now prove a result of weak extension of $\mathbb{Z}/m\mathbb{Z}$-torsors.

PROPOSITION III.3.8. *Let $m \geq 1$ be an integer. Let X be a normal and faithfully flat scheme over R with integral fibers. Let $f_K : Y_K \longrightarrow X_K$ be a connected $\mathbb{Z}/m\mathbb{Z}$-torsor. Let Y be the normalization of X in Y_K. Suppose that Y_k is integral. If R contains a primitive m-th root of unity, there exists a unique μ_m-torsor Y' over X which extends f_K.*

PROOF. First, we consider the affine case $X = \mathrm{Spec}(A)$ and

$$Y_K = \mathrm{Spec}(A_K[Z]/(Z^m - f))$$

with $f \in A_K^*$. Since $Y \longrightarrow X$ is an affine morphism then $Y = \mathrm{Spec}(B)$ for some normal and finite A-algebra B. Multiplying f by an m^{th} power of π if necessary, which does not change the μ_m-torsor Y_K, we can suppose $f \in A$ and $0 \leq v_{X,\pi}(f) < m$ (since X and Y are normal and X_k and Y_k are integral, then $\mathcal{O}_{Y,(\pi)}$ and $\mathcal{O}_{X,(\pi)}$ are both d.v.r.; moreover $\mathcal{O}_{Y,(\pi)}/\mathcal{O}_{X,(\pi)}$ has index of ramification 1). We call $Y' = \mathrm{Spec}(A[Z]/(Z^m - f))$. We prove that Y' is a μ_m-torsor over X, i.e. $f \in A^*$. Since Y' is flat over R and Y_K is connected and normal then Y' is integral. Moreover $Y' \longrightarrow X$ is an integral morphism, so Y' is dominated by Y. So $Z \in B$. Now by $Z^m = f$ in B, we have

$$m v_{Y,\pi}(Z) = v_{Y,\pi}(f).$$

But, since $v_{Y,\pi}(f) = v_{X,\pi}(f) < m$, then $v_{Y,\pi}(f) = v_{X,\pi}(f) = 0$. And since $f \in A_K^*$ there exists $g \in A \setminus \pi A$ and $l \in \mathbb{N}$ such that $f\frac{g}{\pi^l} = 1$. So $fg = \pi^l$. But X_k is integral. Then $l = 0$, which implies $f \in A^*$.

We now consider the general case. By Kummer theory we have that there exists a covering $\{U_i = \mathrm{Spec}(A_i)\}$ of X, $f_i \in H^0(U_{i,K}, \mathcal{O}^*_{U_{i,K}})$ and $\{g_{ij}\} \in H^1(X_K, \mathcal{O}^*_{X_K})$ such that $(Y_K)_{U_i} = \mathrm{Spec}(A_{i,K}[T_i]/(T_i^m - f_i))$ and $g_{ij}^m = \frac{f_i}{f_j}$. As proved in the affine case, we can suppose $f_i \in A_i^*$. So $\frac{f_i}{f_j} \in H^0(U_{ij}, \mathcal{O}^*_{U_{ij}})$. But, since U_{ij} is normal, $g_{ij} \in H^0(U_{ij}, \mathcal{O}^*_{U_{ij}})$. So $\{Y_i' = \mathrm{Spec}(A_i[T_i]/(T_i^m - f_i))\}$ is a μ_m-torsor which extends the $\mathbb{Z}/m\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$. The uniqueness comes from I.3.8. □

REMARK III.3.9. We remark that Y does not usually coincide with Y'. This means that Y' is possibly not normal.

COROLLARY III.3.10. *Let X be as above and let G_K be any commutative group-scheme over K. Let us consider a connected G_K-torsor $f_K : Y_K \longrightarrow X_K$. Let Y be the normalization of X in Y_K and let us suppose that Y_k is integral. Possibly after an extension of R, there exists a (commutative) group-scheme G' and a G'-torsor $Y' \longrightarrow X$ over R which extends f_K.*

REMARK III.3.11. As we will see in the proof, the extension of R depends only on the group G. We do not know if it is really necessary to extend R.

PROOF. First of all we remark that, since K is of characteristic 0, then G is étale. So, up to an extension of R, we can suppose that G is an abstract group. Now, since G is commutative, by the classification of abelian groups we have that $G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_r\mathbb{Z}$ for some $m_1, \dots, m_r \in \mathbb{N}$. We moreover assume that R contains a primitive m_i-th root of unity for $i = 1, \dots, r$. We remark that, since we are only adding roots of unity, Y_k is again reduced. We firstly state the following lemma.

LEMMA III.3.12. *Let $G_1, \dots, G_r$ be finite and flat group schemes over a scheme X. Let $Y_i \longrightarrow X$ be a G_i-torsor for any i. Then $\tilde{Y} = Y_1 \times_X \cdots \times_X Y_r$ is a $G_1 \times \cdots \times G_r$-torsor, with the action induced by those of G_i.*

PROOF. We prove the lemma for $r = 2$. The lemma follows by induction. First of all we remark that, since $Y_1 \longrightarrow X$ and $Y_2 \longrightarrow X$ are faithfully flat the $\tilde{Y} \longrightarrow X$ is faithfully flat, too. We now call σ_i, for $i = 1, 2$, the action of G_i on Y_i. Then, since Y_i is a G_i-torsor,

$$\sigma_i \times \mathrm{id} : G_i \times Y_i \longrightarrow Y_i \times_X Y_i$$

is an isomorphism. We call $\sigma = \sigma_1 \times \sigma_2$ the action of $G_1 \times G_2$ on $Y_1 \times_X Y_2$. We now prove that $Y_1 \times_X Y_2$ is a $G_1 \times G_2$-torsor. We consider the morphism

$$(G_1 \times G_2) \times (Y_1 \times_X Y_2) \stackrel{\sigma\times\mathrm{id}}{\longrightarrow} (Y_1 \times_X Y_2) \times_X (Y_1 \times_X Y_2).$$

But there are natural isomorphisms

$$g_1 : (G_1 \times G_2) \times (Y_1 \times_X Y_2) \longrightarrow (G_1 \times Y_1) \times_X (G_2 \times Y_2)$$

and

$$g_2 : (Y_1 \times_X Y_2) \times_X (Y_1 \times_X Y_2) \longrightarrow (Y_1 \times_X Y_1) \times_X (Y_2) \times_X Y_2),$$

such that

$$\begin{array}{ccc} (G_1 \times G_2) \times (Y_1 \times_X Y_2) & \xrightarrow{\sigma\times\mathrm{id}} & (Y_1 \times_X Y_2) \times_X (Y_1 \times_X Y_2) \\ \downarrow g_1 & & \downarrow g_2 \\ (G_1 \times Y_1) \times (G_2 \times Y_2) & \xrightarrow{(\sigma_1\times\mathrm{id})\times(\sigma_2\times\mathrm{id})} & (Y_1 \times_X Y_1)_{\times_X}(Y_2 \times_X Y_2) \end{array}$$

commutes. Since the second horizontal morphism is an isomorphism, then $\sigma \times \mathrm{id}$ is an isomorphism, too. So $Y_1 \times_X Y_2$ is a $G_1 \times G_2$-torsor.

□

We now come back to the proof of the corollary. We call $G_i = \mathbb{Z}/m_i\mathbb{Z}$ for $i = 1, \dots, r$. Moreover we call $\tilde{G}_i = G_1 \times \cdots \times \hat{G}_i \times \cdots \times G_r$. Let us define $(Y_i)_K = Y_K/(\tilde{G}_i)$, then $(Y_i)_K$ is a G_i-torsor. For any i, we call σ_i the action of G_i induced by that of G on $(Y_i)_K$. Hence

$$\sigma_i \times \mathrm{id} : G_i \times (Y_i)_K \longrightarrow (Y_i)_K \times_{X_K} (Y_i)_K$$

is an isomorphism. By the above lemma we have that $(Y_1)_K \times_{X_K} \cdots \times_{X_K} (Y_r)_K$ is a G-torsor. Moreover the natural map

$$q : Y_K \longrightarrow (Y_1)_K \times_{X_K} \cdots \times_{X_K} (Y_r)_K$$

preserves the G-actions, therefore it is a morphism of G-torsors. But, as it is well known, any morphism of G-torsors is an isomorphism of schemes; hence q is an isomorphism.

For $i = 1, \dots, r$, we denote by Y_i the normalization of X in $(Y_i)_K$. Since Y is integral it easily follows that Y_i is integral for any i. Since $Y \longrightarrow Y_i$ is an integral morphism, we have by III.3.5, that $(Y_i)_k$ is integral. So, by III.3.8, for any $i = 1, \dots, r$, there exists a μ_{m_i}-torsor $Y'_i \longrightarrow X$ which extends $(Y_i)_K \longrightarrow X_K$. Now let us consider $Y' = Y'_1 \times_X \cdots \times_X Y'_r$. Using the above lemma again, it follows that Y' is a $\mu_{m_1} \times \cdots \times \mu_{m_r}$-torsor. □

Remark III.3.13. We want to stress the spirit of this corollary. Let X be an integral scheme faithfully flat over R and x an R-point of X. Let us consider the fundamental group schemes of Gasbarri $\pi(X, x)$ over R(see [**17**]) and $\pi(X_K, x_K)$. Then Antei, in [**6**], proved that the natural morphism

$$\varphi : \pi(X_K, x_K) \longrightarrow \pi(X, x)_K$$

is a quotient morphism and that $\ker(\varphi) = 0$ if and only if, for any reductive group scheme G_K over K (i.e. $\pi(X_K, x_K) \longrightarrow G$ is a quotient morphism), any G_K-torsor $Y_K \longrightarrow X_K$ is weakly extendible to a G-torsor $Y \longrightarrow X$ over R, for some model G of G_K. So the previous proposition gives us information about $\ker(\varphi)$. In the next remark there is an example in which φ is not injective.

Remark III.3.14. The hypothesis on Y_k can be weakened to Y_k irreducible. Indeed it has been proved by Epp([**16**]) that up to an extension of R it is possible to suppose Y_k reduced. But we remark that if Y_k is not reduced then it is necessary to extend R. For instance take $X = \mathrm{Spec}(R[Z, 1/Z])$ and $Y_K = \mathrm{Spec}(K[Z, 1/Z][T]/(T^p - \pi Z))$ as $\mathbb{Z}/p\mathbb{Z}$-torsor over X_K. It is not too hard to see that $Y = \mathrm{Spec}(R[Z, 1/Z][T]/(T^p - \pi Z))$ is normal (see for example [**30**, 8.2.6]), so it is the normalization of X in Y_K. Moreover the action of $\mu_p = \mathrm{Spec}(R[S]/(S^p - 1))$ over Y given by $T \longrightarrow ST$ is clearly faithful. So μ_p is the effective model. Using III.3.1 it follows that, if $Y_K \longrightarrow X_K$ is weakly extendible by a G'-torsor, then there is a model map $\mu_p \longrightarrow G'$. Hence $G' \simeq \mu_p$, since μ_p does not dominate any group scheme except itself. We now claim that there is no μ_p-torsor $Y' \longrightarrow X$ which extends $Y_K \longrightarrow X_K$. We remark that by the Kummer theory we have

$$\dots \longrightarrow H^0(X, \mathbb{G}_m) \xrightarrow{p} H^0(X, \mathbb{G}_m) \longrightarrow H^1(X, \mu_p) \longrightarrow Pic(X) \longrightarrow \dots$$

Since $\mathrm{Pic}(R[Z, 1/Z]) = 0$ we would have $Y' = \mathrm{Spec}(R[Z, 1/Z][T]/(T^p - f))$, for some $f \in R[Z, 1/Z]^*$ such that there exists $g \in K[Z, 1/Z]^*$ with $fg^p = \pi Z$. If $g = \frac{g_0}{\pi^m}$, with $m \geq 0$ and $g_0 \notin \pi R[Z, 1/Z]$, then $fg_0^p \equiv 0 \mod \pi R[Z, 1/Z]$. Since X_k is integral this is a contradiction, by the definition of f and g_0. In particular $Y_K \longrightarrow X_K$ is also not strongly extendible.

III.4. Strong extension of $\mathbb{Z}/p\mathbb{Z}$-torsors

Before studying the problem of extension of $\mathbb{Z}/p\mathbb{Z}$-torsors we consider for a moment the case of $\mathbb{Z}/p^n\mathbb{Z}$-torsors. What we say will be useful in the next section, too. Let us suppose that R contains a primitive p^n-th root of unity. Let us consider $X = \mathrm{Spec}(A)$ with A a faithfully flat and factorial R-algebra, complete with respect to the π-adic topology, or X a normal local faithfully flat R-scheme. We moreover suppose that X_k is integral. By I.5.3 and I.5.4 it follows that

$$H^1(X, \mu_{p^n}) = A^*/(A^*)^p.$$

Therefore any μ_{p^n}-torsor $Y' \longrightarrow X$ is of type $Y' = \mathrm{Spec}(A[T]/(T^p - f)) \longrightarrow X$ for some $f \in A^*$. By the filtration of $H^1(X, \mu_{p^n})$ described in I.5.6, it follows that $[Y'] \in H^1(X, G_{\lambda,n})$ for some $\lambda \in R$. But, again by I.5.3 and I.5.4, we have that

$$H^1(X, G_{\lambda,n}) = \mathcal{G}^{(\lambda^{p^n})}(A)/(\psi_{\lambda,n})_*(\mathcal{G}^{(\lambda)}(A)).$$

The following lemma explains in terms of the element $f \in A^*$ the meaning of $[Y]$ being in $H^1(X, G_{\lambda,n})$.

Lemma III.4.1. *Notation as above. Let us denote by $Y = \mathrm{Spec}(B)$ the normalization of X in Y'_K. We moreover suppose that Y_k is reduced. Then, using the filtration of I.5.6, $[Y'] \in H^1(X, G_{\pi^j,n})$ if and only if there exists $g \in A^*$ such that $fg^{p^n} = 1 + \pi^{jp^n} f_0$ for some $f_0 \in A$.*

Let us suppose moreover $j < v(\lambda_{(n)})$. If $[Y'] \in H^1(X, G_{\pi^j,n}) \setminus H^1(X, G_{\pi^{j+1},n})$ and $fg^{p^n} = 1 + \pi^{jp^n} f_0$, for some $f_0 \in A$ and $g \in A^*$, then f_0 is not a p^n-th power mod π.

Proof. The first assertion follows from the fact that the injection

$$H^1(X, G_{\pi^j,n}) \subseteq H^1(X, \mu_{p^n})$$

corresponds to the injection

$$\mathcal{G}^{(\pi^{p^n j})}(A)/(\psi_{\pi^j,n})_*\mathcal{G}^{(\pi^j)}(A) \longrightarrow A^*/(A^*)^{p^n}.$$

We now prove the second statement. Let us suppose that $[Y'] \in H^1(X, G_{\pi^j,n}) \setminus H^1(X, G_{\lambda_{(n)},n})$. We take any $g \in A^*$ such that $fg^{p^n} = 1 + \pi^{jp^n} f_0$, for some $f_0 \in A$. If $[Y'] \notin H^1(X, G_{\pi^{j+1},n})$, then, by what just proved, $f_0 \not\equiv 0 \mod \pi^{p^n}$. In fact we will prove $f_0 \not\equiv 0 \mod \pi$ in A. Since the torsor $Y_1 = \mathrm{Spec}(A[T]/(\frac{(1+\pi^j T)^{p^n} - 1}{\pi^{jp^n}} - f_0))$, associated to $[Y'] \in H^1(X, G_{\pi^j,n})$, is integral over X and its generic fiber is isomorphic to Y_K, then, by III.3.7, the morphism $Y \longrightarrow X$ factors through Y_1. Moreover $Y \longrightarrow Y_1$ is a dominant morphism between integral affine schemes, hence $T \in B \setminus \{0\}$. The fact that $f_0 \not\equiv 0 \mod \pi^{p^n} A$ implies $T \not\equiv 0 \mod \pi B$. Otherwise, if $T = \pi T_0$ for some $T_0 \in B$, then $T^{p^n} \equiv 0 \mod \pi^{p^n} B$. And, since $j + 1 \leq v(\lambda_{(n)})$, we have

$$f_0 = \frac{(1 + \pi^{j+1} T_0)^{p^n} - 1}{\pi^{jp^n}} \equiv 0 \mod \pi^{p^n} B.$$

So, by III.3.5, $f_0 \equiv 0 \mod \pi^{p^n} A$ against the assumptions. Therefore $T \not\equiv 0 \mod \pi B$. Now if $f_0 \equiv 0 \mod \pi A$, then, since $j < v(\lambda_{(n)})$, $T^{p^n} \equiv 0 \mod \pi B$. But, as we just proved, $T \not\equiv 0 \mod \pi B$, which contradicts the fact that Y_k is reduced. So $f_0 \not\equiv 0 \mod \pi A$. We finally prove that f_0 is not a p^n-th power $\mod \pi$. Indeed, if $f_0 \equiv g_0^{p^n} \mod \pi$ for some $g_0 \in A \setminus \pi A$ then $f \equiv (1 + \pi^j g_0)^{p^n} \mod \pi^{jp^n+1}$.

By the hypothesis on A, $1+\pi^j g_0$ is invertible. Multiplying f by $(1+\pi^j g_0)^{-p^n}$, we can suppose $f \equiv 1 \mod \pi^{jp^n+1}$, which implies, by what we have just proved, $[Y'] \in H^1(X, G_{\pi^{j+1},n})$; this contradicts the hypothesis of maximality of j. □

We now consider the case $n = 1$ and we prove the strong extension. Let X be as above. Let

$$Y_K \longrightarrow X_K$$

be a nontrivial $\mathbb{Z}/p\mathbb{Z}$-torsor. We remark that A_K is factorial, too. So $Pic(A_K) = 0$, which implies $H^1(X_K, \mu_p) \simeq A_K^*/(A_K^*)^{p^n}$, by I.5.3 and I.5.4. Therefore $Y_K = \operatorname{Spec}(A_K[T]/(T^{p^n} - f))$ with $f \in A_K^*$. So the class $[Y_K] \in H^1(X_K, \mu_{p^n})$ corresponds to the class of $[f]$ in $A_K^*/(A_K^*)^p$. Let Y be the normalization of X in Y_K. We suppose that Y_k is integral. There exists, by III.3.8, a unique μ_p-torsor $Y' \longrightarrow X$ such that $Y'_K \longrightarrow X_K$ is isomorphic to the $\mathbb{Z}/p\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$. So Y_K defines uniquely an element $[Y'] \in H^1(X, \mu_p)$. We remark that this means that, up to a multiplication by a p^n-th power, we can suppose $f \in A^*$. The proof of the following theorem is close to that of Henrio ([**25**, 1.6]) for formal curves, but rewritten in an other language.

Theorem III.4.2. *Notation as above. If* $[Y_K] \in H^1(X, G_{\pi^\gamma,1}) \setminus H^1(X, G_{\pi^{\gamma+1},1})$ *then* Y *is a* $G_{\pi^\gamma,1}$*-torsor. Moreover the valuation of the different of the extension* $\mathcal{O}_{Y,(\pi)}/\mathcal{O}_{X,(\pi)}$ *is* $(p-1)(v(\lambda_{(1)}) - \gamma)$*.*

Proof. We have $Y_K = \operatorname{Spec}(A_K[T]/(T^p - f)$ for some $f \in A_K^*$. As remarked above, we can suppose $f \in A^*$. Moreover $Y' = \operatorname{Spec}(A[T]/(T^p - f))$ is the μ_p-torsor which extends Y_K. We now study the different cases which may occur.

If $\gamma = 0$ then f is not a p-power $\mod \pi$ (otherwise, by III.4.1, up to a multiplication by a p^{th}-power, we can suppose $f \equiv 1 \mod \pi$ and so $\gamma > 0$). So Y' is normal by III.3.6. Since Y' is integral over X and its generic fiber is isomorphic to Y_K then, by III.3.7, the morphism $Y \longrightarrow X$ factors through $Y' \longrightarrow X$, i.e. we have

$$Y \longrightarrow Y' \longrightarrow X.$$

But, since it is normal, Y' coincides with the integral closure of X in Y_K, i.e. $Y \simeq Y'$.

If $\gamma = v(\lambda_{(1)})$ then it is an étale torsor and we are done.

If $v(\lambda_{(1)}) > \gamma > 0$ we can suppose, by III.4.1, $f = 1 + \pi^{p\gamma} f_0$ with $f_0 \not\equiv 0 \mod \pi$ in A. Let us consider the $G_{\pi^\gamma,1}$-torsor

$$Y_1 = \operatorname{Spec}(A[T]/(\frac{(1+\pi^\gamma T)^p - 1}{\pi^{p\gamma}} - f_0)).$$

By III.4.1 $f_0 \not\equiv g_0^p \mod \pi$ for any $g_0 \in A \setminus \pi A$. So $(Y_1)_k$ is reduced and, by III.3.6, Y_1 is normal. Moreover we have, by III.3.7, the following factorization

$$Y \longrightarrow Y_1 \longrightarrow X.$$

But, since Y_1 is normal, it follows that $Y_1 \simeq Y$ by III.3.7. The statement about the valuation of the different is clear.

□

III.5. Strong extension of $\mathbb{Z}/p^2\mathbb{Z}$-torsors

III.5.1. Setup and degeneration types. From now on we will suppose that R contains a primitive p^2-th root of unity. Therefore we have $(\mathbb{Z}/p^2\mathbb{Z})_K \simeq (\mu_{p^2})_K$. We moreover suppose $p > 2$. Let $X := \operatorname{Spec} A$ be with the same hypothesis used in §III.4. We moreover assume that X is essentially semireflexive. Let $h_K : Y_K \longrightarrow X_K$ be a connected $\mathbb{Z}/p^2\mathbb{Z}$-torsor. Then we consider the factorization

$$Y_K \xrightarrow{(h_2)_K} (Y_1)_K \xrightarrow{(h_1)_K} X$$

with both $(h_1)_K, (h_2)_K$ nontrivial $\mathbb{Z}/p\mathbb{Z}$-torsors. Let $Y_1 = \operatorname{Spec}(B_1)$ be the normalization of X in $(Y_1)_K$ and $Y = \operatorname{Spec}(B)$ the normalization of X in Y_K. We moreover suppose that Y_k is integral. By well known facts about integral closure, we have that Y is the integral closure of Y_1 in Y_K. So we have the factorization

$$h : Y \xrightarrow{h_2} Y_1 \xrightarrow{h_1} X$$

with h_1 and h_2 degree p morphisms. Since X is normal, by III.3.7 it follows that Y_1 and Y are normal schemes. By III.3.5 we have that, since Y_k is integral, then $(Y_1)_k$ is integral, too.

Now, by III.3.8, we can extend Y_K to a μ_{p^2}-torsor Y' over X. So we can suppose $Y_K = \operatorname{Spec}(A_K[T]/(T^{p^2} - f))$ for some $f \in A^*/(A^*)^{p^2}$. We can also write

$$Y_K = \operatorname{Spec}(A[T_1, T_2]/(T_1^p - f, \frac{T_2^p}{T_1} - 1)).$$

Therefore we have

$$(Y_1)_K = \operatorname{Spec}(A_K[T_1]/(T_1^p - f))$$

and

$$Y_K = \operatorname{Spec}((B_1)_K[T_2]/(\frac{T_2^p}{T_1} - 1)).$$

We remark that B_K is finite and free as an A_K-module. In particular it is semireflexive over A_K. From III.1.12 it follows that Y is an essentially semireflexive scheme over $\operatorname{Spec}(R)$. Therefore we can apply III.1.16 to check if a group scheme is an effective model for the $\mathbb{Z}/p^2\mathbb{Z}$-action on Y. We now want to attach to the cover $Y_K \to X_K$ four invariants. We have seen in the previous section that there exists an invariant, which we called γ, that is sufficient to solve the problem of strong extension of $(\mathbb{Z}/p\mathbb{Z})_K$-torsors. So the first two invariants are simply the invariants γ which arise from the two $(\mathbb{Z}/p\mathbb{Z})_K$-torsors $Y_K \longrightarrow (Y_1)_K$ and $(Y_1)_K \longrightarrow X_K$. We are now more precise. By the above discussion it follows that h_1 satisfies hypothesis of §III.4, hence we can apply III.4.2. Then, if we define $\gamma_1 \leq v(\lambda_{(1)})$ such that

$$[(Y_1)_K] \in H^1(X, G_{\pi^{\gamma_1},1}) \setminus H^1(X, G_{\pi^{\gamma_1+1},1}),$$

it follows that $Y_1 \longrightarrow X$ is a $G_{\pi^{\gamma_1},1}$-torsor. We stress that γ_1 is also determined by the valuation of the different $\mathcal{D}(h_1)$ of $h_1 : \operatorname{Spec}(\mathcal{O}_{Y_1,(\pi)}) \longrightarrow \operatorname{Spec}(\mathcal{O}_{X,(\pi)})$. We indeed have

$$v(\mathcal{D}(h_1)) = v(p) - (p-1)\gamma_1.$$

We can apply III.4.2 also to the morphism $h_2 : Y \longrightarrow Y_1$. Then, if we define $\gamma_2 \leq v(\lambda_{(1)})$ such that

$$[(Y)_K] \in H^1(Y_1, G_{\pi^{\gamma_2},1}) \setminus H^1(Y_1, G_{\pi^{\gamma_2+1},1}),$$

it follows that $Y \longrightarrow Y_1$ is a $G_{\pi^{\gamma_1},1}$-torsor. The invariant γ_2 is determined by the different of $h_2 : \operatorname{Spec}(\mathcal{O}_{Y,(\pi)}) \longrightarrow \operatorname{Spec}(\mathcal{O}_{Y_1,(\pi)})$, too. Indeed

$$v(\mathcal{D}(h_2)) = v(p) - (p-1)\gamma_2.$$

The third invariant is linked to the filtration I.5.6 of $H^1(X, \mu_{p^2})$. It is the integer $j \leq v(\lambda_{(2)})$ such that $[Y_K] \in H^1(X, G_{\pi^j,2}) \setminus H^1(X, G_{\pi^{j+1},2})$. We observe that there exists a $G_{\pi^j,2}$-torsor Y'' which extends $Y_K \longrightarrow X_K$. By III.3.7 we have morphisms $Y \longrightarrow Y''$ and $Y_1 \longrightarrow Y''/G_{\pi^j,1}$ such that the following diagram commutes

(III.80)
$$\begin{array}{ccc} Y & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y''/G_{\pi^j,1} \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

By definition of j, up to a multiplication of f by an element of $(A_K^*)^{p^2}$, which does not change the μ_{p^2}-torsor on the generic fiber, we can suppose $f = 1 + \pi^{jp^2} f_0$ with $f_0 \in A$. And, if $j < v(\lambda_{(2)})$, by III.4.1 f_0 is not a p^2-power $\mod \pi$.

Before introducing the last invariant we describe explicitly the scheme Y. By definition of γ_1 and by the proof of III.4.2, there exists $g \in A^*$ such that $fg^{-p} = 1 + \pi^{p\gamma_1} f_1$ with $f_1 \in A$. By III.4.1 it follows that, if $\gamma_1 < v(\lambda_{(1)})$, f_1 is not a p^{th}-power $\mod \pi$. Therefore $Y_1 = \operatorname{Spec}(B_1)$ with

$$B_1 = A[T_1]/(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1).$$

If $Y_1' \longrightarrow Y_1$ is the μ_p-torsor which extends $Y_K \longrightarrow (Y_1)_K$ then Y is the normalization of Y_1 in

$$(Y_1')_K = \operatorname{Spec}((B_1)_K[T_2]/(\frac{T_2^p}{1+\pi^{\gamma_1}T_1} - g)).$$

Then, reasoning as above, there exists $H(T_1) \in B_1^*$, such that

$$g(1+\pi^{\gamma_1}T_1)(H(T_1))^{-p} \equiv 1 \mod \pi^{\gamma_2}B_1.$$

Therefore $Y = \operatorname{Spec}(B)$ with

$$B = B_1[T_2]/\left(\frac{(1+\pi^{\gamma_2}T_2)^p - 1}{\pi^{p\gamma_2}} - \frac{(1+\pi^{pj}g_0)H(T_1)^{-p}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}\right).$$

We remark that $Y \longrightarrow X$ is a faithfully flat morphism. Indeed, since $Y \longrightarrow Y_1$ and $Y_1 \longrightarrow X$ are, respectively, a $G_{\pi^{\gamma_2},1}$-torsor and a $G_{\pi^{\gamma_1},1}$-torsor, they are in particular faithfully flat morphisms. Therefore, by transitivity of flatness, it follows that $Y \longrightarrow X$ is faithfully flat, too.

We remark that the definition of g and $H(T_1)$ depends on the choice of the representant f of $[Y_K] \in H^1(X, \mu_{p^2})$. We now see how they vary as f varies. We stress that we require $f \equiv 1 \mod \pi^{p^2 j}$. Let us substitute $a^{p^2} f$ to f, with $a \in A^*$ and $a^{p^2} f \equiv 1 \mod \pi^{p^2 j}$. Since X_k is integral it follows from I.5.7 that $a^{p^2} f \equiv 1 \mod \pi^{p^2 j}$ is equivalent to $a \equiv 1 \mod \pi^j$. Now it is immediate to see that we have to substitute $a^p g$ to g and $H(T_1)$ by $a^{-1} H(T_1)$. We now prove that, for a fixed f, the elements g and $H(T_1)$ are uniquely determined in a certain sense.

LEMMA III.5.1. *Using above notation we have that any $g, H(T_1)$ as above are of the form*

$$g = 1 + \pi^{jp} g_0 \text{ with } g_0 \not\equiv 0 \mod \pi A$$
$$H(T_1) = 1 + \pi^j H_1(T_1) \text{ with } H_1(T_1) \notin A \mod \pi B_1.$$

If $j < v(\lambda_{(2)})$ then g_0 is not a p^{th}-power $\mod \pi A$ and $H_1(T_1)$ is not a p^{th}-power $\mod \pi B_1$. Moreover, for any representant $f = 1 + \pi^{p^2 j} f_0$ of $[Y_K] \in H^1(X, \mu_{p^2})$, the element g is uniquely determined $\mod \pi^{\gamma_1}$ and $H(T_1)$ is uniquely determined $\mod \pi^{\gamma_2}$. Finally, if $j > 0$, up to a change of the representant f, we can suppose $H(0) = 1$.

PROOF. Since A is separated with respect to the π-adic topology and $g^p \equiv f \mod \pi$, we can suppose that $g = 1 + \pi^h g_0$ with $g_0 \not\equiv 0 \mod \pi$. Moreover as remarked we can suppose $f = 1 + \pi^{jp^2} f_0$ with f_0 not a p^2-th power $\mod \pi$. By definition of g

$$(1 + \pi^{jp^2} f_0) \equiv (1 + \pi^h g_0)^p \mod \pi^{p\gamma_1}.$$

Then

$$(1 + \pi^{jp^2} f_0) \equiv (1 + \pi^{hp} g_0^p + p\pi^h g_1) \mod \pi^{p\gamma_1}. \tag{III.81}$$

Since $pj < v(\lambda_{(1)})$ then $h < v(\lambda_{(1)})$. Otherwise $f_0 \equiv 0 \mod \pi$ and in particular f_0 would be a $p^2 - th$ power. This contradicts the hypothesis on f_0. Hence $v(p) > (p-1)h$, which implies $v(p) + h > ph$. So, since X_k is integral, by (III.81) it follows that

$$h = jp.$$

Hence by (III.81) we obtain $f_0 \equiv g_0^p \mod \pi$. Therefore g_0 is not a p^{th}-power $\mod \pi$, otherwise f_0 would be a p^2-th power $\mod \pi$ against hypothesis on f_0. Let us suppose there exists $g' \in A^*$ such that

$$g^p \equiv g'^p \mod \pi^{p\gamma_1}.$$

Reasoning as above it is easy to show that

$$(\frac{g}{g'}) \equiv 1 \mod \pi^{\gamma_1}.$$

Since $g^{-1} \equiv 1 \mod \pi^{pj} B_1$ then $H(T_1)^p \equiv 1 \mod \pi^{jp} B_1$. Hence, in a similar way as above we have $H(T_1) = 1 + \pi^j H_1(T_1)$ with $H_1(T_1) \not\equiv 0 \mod \pi B_1$. Now let us suppose there exists $a \in A$ such that $H_1(T_1) \equiv a \mod B_1$. Then it follows

$$1 + \pi^{pj} a^p \equiv 1 + \pi^{jp} g_0 \mod \pi^{pj+1}$$

This means

$$a^p \equiv g_0 \mod \pi$$

which contradicts what we have just proved. As above it is easy to prove the statements about the reduction $\mod \pi$ of $H_1(T_1)$ and the uniqueness of $H(T_1)$ $\mod \pi^{\gamma_2}$. We now prove the last statement. We assume that $j > 0$. Let us suppose that $H(0) \neq 1$. By what we just proved we know that $H(0) \equiv 1 \mod \pi^j A$. So, since A is π-adically complete or A is local, it follows that $H(0) \in A^*$. Hence $H(T_1) = H(0)H_0(T_1)$ with $H_0(T_1) \in B_1^*$ and $H_0(0) = 1$. If we change f into $fH(0)^{p^2}$, by the discussion before the lemma we have that we have to replace $H(T_1)$ with $H_0(T_1)$. So we are done.

□

Now, given $H(T_1) = \sum_{k=0}^{p-1} a_k {T_1}^k \in B_1^*$, let us consider $H'(T_1)$ as its formal derivative. Using the above lemma we suppose $a_0 = 1$ if $j > 0$. For any $m \geq \gamma_1$, we will say that $a \in \pi R$ satisfies $(\triangle)_m$ if

$$aH(T_1) \equiv \pi^{m-\gamma_1} H'(T_1) \mod \pi^{\gamma_2}.$$

We finally give the definition of the fourth invariant.

Definition III.5.2. We will call *effective threshold* the number

$$\kappa = \min\{m \geq \gamma_1 | \exists a \in \pi R \text{ which satisfies } (\triangle)_m\}.$$

If we take $m \geq \gamma_1 + \gamma_2$ and $a = 0$ we see that such a minimum exists.

Lemma III.5.3. *For any $m \geq \kappa$ there exists an unique solution, $\mod \pi^{\gamma_2}$, of $(\triangle)_m$. We will call $\alpha_m \in \pi R$ any of its lifting. If $H(0) = a_0 \equiv 0 \mod \pi A$ then $\alpha_m \equiv 0 \mod \pi^{\gamma_2}$.*

Remark III.5.4. By III.4.1 it follows that the case $H(0) \equiv 0 \mod \pi$ can only happen if $j = 0$.

Proof. Let us firstly suppose $a_0 \not\equiv 0 \mod \pi A$. If b_i, for $i = 1, 2$, are solutions of $(\triangle)_m$ it follows that for any $m \geq \gamma_1$ we have in particular $b_i a_0 = \pi^{m-\gamma_1} a_1$ $\mod \pi^{\gamma_2}$. Therefore

$$a_0(b_1 - b_2) \equiv 0 \mod \pi^{\gamma_2}.$$

But $a_0 \notin \pi A$ and X_k is integral, therefore $b_1 \equiv b_2 \mod \pi^{\gamma_2}$.

We now consider the case $a_0 \in \pi A$. Since $H(T) \in (R/\lambda R[T])^*$ and $a_0 \in \pi A$ then there exists $0 < i \leq p-1$ such that $a_i \notin \pi A$. Let $\bar{i}$ be the least integer with this property.

Let a be a solution solution of $(\triangle)_m$ and suppose that $a \not\equiv 0 \mod \pi^{\gamma_2}$. In particular

$$aa_{\bar{i}} = (\bar{i}+1)a_{\bar{i}+1}\pi^{m-\gamma_1}$$

and

$$aa_{\bar{i}-1} = \bar{i}a_{\bar{i}}\pi^{m-\gamma_1}.$$

Therefore, by the minimality of $\bar{i}$ and by the fact that $a \not\equiv 0 \mod \pi^{\gamma_2}$,

$$v(a) \geq v(\pi^{m-\gamma_1}) > v(a)$$

which is a contradiction. Therefore, if $a_0 \in \pi A$, then $a \equiv 0 \mod \pi^{\gamma_2}$. □

DEFINITION III.5.5. Using the previous notation we say that the degeneration type of $Y_K \longrightarrow X_K$ is $(j, \gamma_1, \gamma_2, \kappa)$.

We now give some restrictions on the degeneration type.

LEMMA III.5.6. *We have the following relations.*

i) $pj \leq \gamma_1$,
ii) $j \leq \gamma_2$,
iii) $\gamma_1 \leq \kappa \leq \gamma_1 + \gamma_2 - j$. *In particular* $\gamma_2 = j$ *implies* $\kappa = \gamma_1$.

PROOF. Let us consider the diagram (III.80).

i) We recall that $Y_1 \longrightarrow X$ is a $G_{\pi^{\gamma_1},1}$-torsor and $Y''/G_{\pi^j,1} \longrightarrow X$ is a $G_{\pi^{pj},1}$-torsor. So by III.3.1 we have a morphism $G_{\pi^{\gamma_1},1} \longrightarrow G_{\pi^{pj},1}$. Therefore $\gamma_1 \geq pj$
ii) We recall that $Y \longrightarrow Y_1$ is $G_{\pi^{\gamma_2},1}$-torsor and $Y'' \longrightarrow Y''/G_{\pi^j,1}$ is a $G_{\pi^j,1}$-torsor. Again by III.3.1 we have a morphism $G_{\pi^{\gamma_2},1} \longrightarrow G_{\pi^j,1}$. Therefore $\gamma_2 \geq j$.
iii) By III.5.1 $H'(T) \equiv 0 \mod \pi^j$. Therefore if we take $m = \gamma_1 + \gamma_2 - j$ then
$$\pi^{m-\gamma_1} H'(T) \equiv 0 \mod \pi^{\gamma_2}.$$
Therefore $a = 0$ satisfies $(\triangle)_m$. This implies $\kappa \leq \gamma_1 + \gamma_2 - j$. Now, if $\gamma_2 = j$, then $\kappa \leq \gamma_1$. But, by definition of κ, we have $\kappa \geq \gamma_1$. Hence $\kappa = \gamma_1$.

□

III.5.2. The main theorem. We here prove the main theorem of the chapter.

THEOREM III.5.7. *Let us consider* $X = \mathrm{Spec}(A)$ *with* A *a factorial* R*-algebra, complete with respect to the* π*-adic topology, or* X *a normal local* R*-scheme. We moreover assume* X *essentially semireflexive (see §III.1). Let* $Y_K \longrightarrow X_K$ *be a connected* $\mathbb{Z}/p^2\mathbb{Z}$*-torsor and let* Y *be the normalization of* X *in* Y_K*. Let us suppose that* Y_k *is integral. If* Y_K *has* $(j, \gamma_1, \gamma_2, \kappa)$ *as degeneration type then its effective model is*
$$\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)}.$$
Moreover if $\alpha_\kappa \not\equiv 0 \mod \pi^{\gamma_2}$ *then* $v(\alpha_\kappa) = \kappa - \gamma_1 + j$. *Otherwise* $\kappa - \gamma_1 + j = \gamma_2$.

PROOF. As we proved in the previous section $Y = \mathrm{Spec}(B)$ with
$$B = B[T_1, T_2]/\left(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1, \frac{(1+\pi^{\gamma_2}T_2)^p - 1}{\pi^{p\gamma_2}} - \frac{(1+\pi^{pj}g_0)H(T_1)^{-p}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}\right).$$
By the definition of integral closure of X in Y_K the $\mathbb{Z}/p^2\mathbb{Z}$-action on Y_K can be extended to an action on Y. We now explicitly describe this action. If we set
$$\eta_\pi = \frac{\pi^{v(\lambda_{(1)})}}{\lambda_{(1)}}\eta = \frac{\pi^{v(\lambda_{(1)})}}{\lambda_{(1)}} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda_{(2)}^k$$

then we can write, by II.3.52,
(III.82)

$$\mathbb{Z}/p^2\mathbb{Z} = \operatorname{Spec}(A[S_1,S_2]/\Bigg(\frac{(1+\pi^{v(\lambda_{(1)})}S_1)^p-1}{\pi^{pv(\lambda_{(1)})}}, \frac{\frac{(E_p(\eta_\pi S_1)+\pi^{v(\lambda_{(1)})}S_2)^p}{1+\pi^{v(\lambda_{(1)})}S_1}-1}{\pi^{pv(\lambda_{(1)})}}\Bigg)).$$

Since Y_K is a μ_{p^2}- torsor, on the generic fiber the action is given by

$$1+\pi^{\gamma_1}T_1 \longmapsto (1+\pi^{v(\lambda_{(1)})}S_1)(1+\pi^{\gamma_2}T_1)$$
$$(1+\pi^{\gamma_2}T_2)H(T_1) \longmapsto (E_p(\eta_\pi S_1)+\pi^{v(\lambda_{(1)})}S_2)(1+\pi^{\gamma_2}T_2)H(T_1),$$

so it is globally given by

$$T_1 \longmapsto \pi^{v(\lambda_{(1)})-\gamma_1}S_1 + T_1 + \pi^{v(\lambda_{(1)})}S_1T_1$$
$$T_2 \longmapsto \frac{(E_p(\eta_\pi S_1)+\pi^{v(\lambda_{(1)})}S_2)\left(\frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{v(\lambda_{(1)})-\gamma_1}S_1+T_1+\pi^{v(\lambda_{(1)})}S_1T_1)}\right)-1}{\pi^{\gamma_2}}$$

The proof of the theorem is obtained as a consequence of several lemmas.

LEMMA III.5.8. *If an effective model $\mathcal{G}$ for the action of $\mathbb{Z}/p^2\mathbb{Z}$ exists then it is of the form $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ with $v(\lambda_{(1)}) \geq m \geq \max\{\gamma_2,\gamma_1\}$.*

PROOF. Since the effective model is a model of $(\mathbb{Z}/p^2\mathbb{Z})_K$, by II.3.53, it follows that the effective model $\mathcal{G}$ is of the form $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ with $v(\lambda_{(1)}) \geq m \geq \gamma_2$. Moreover $\mathcal{G}/G_{\pi^{\gamma_2},1} \simeq G_{\pi^m,1}$ has an X-action over Y_1. But $Y_1 \longrightarrow X$ is a $G_{\pi^{\gamma_1},1}$-torsor. So, by III.3.1, we have a model map $G_{\pi^m,1} \longrightarrow G_{\pi^{\gamma_1},1}$. Then $m \geq \gamma_1$. □

Let us now consider a group scheme of type $\mathcal{E}^{(\pi^m,\pi^{\gamma_2},F,1)}$. We consider the normalization map $\varphi : \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathcal{E}^{(\pi^m,\pi^{\gamma_2},F,1)}$. We give necessary and sufficient conditions to have an action of $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ on Y compatible with φ. By II.3.53 we have that

$$F(S) = E_p(aS) \in ((R/\pi^{\gamma_2}R)[S]/(\frac{(1+\pi^{\gamma_1}S)^p-1}{\pi^{p\gamma_1}}))^*$$

for some $a \in R/\pi^{\gamma_2}R$. In the following we take a lifting $\tilde{a} \in R$ of $a \in R/\pi^{\gamma_2}R$ and we consider $\tilde{F}(S) = \sum_{i=0}^{p-1} \frac{\tilde{a}^i}{i!}S^i \in R[S]$ as a lifting of $F(S)$.

LEMMA III.5.9. *There exists an action of $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ on Y compatible with φ if and only if*

$$\tilde{F}(S)H(T) - H(\pi^{m-\gamma_1}S + T + \pi^m ST) \equiv 0 \mod \pi^{\gamma_2}$$

PROOF. Let us suppose that such an action exists. Reasoning as above, it is possible to show that the action is given by

$$T_1 \longmapsto \pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1T_1$$
$$T_2 \longmapsto \frac{(\tilde{F}(S_1)+\pi^{\gamma_2}S_2)\left(\frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}\right)-1}{\pi^{\gamma_2}}$$

Then, in particular, $\dfrac{(\tilde{F}(S_1)+\pi^{\gamma_2}S_2)\left(\frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}\right)-1}{\pi^{\gamma_2}}$ belongs to

$$B\otimes A[S_1,S_2]/\left(\frac{(1+\pi^m S_1)^p-1}{\pi^{mp}},\frac{\frac{(\tilde{F}(S_1)+\pi^{\gamma_2}S_2)^p}{1+\pi^m S_1}-1}{\pi^{p\gamma_2}}\right)$$

So we have

$$\frac{(\tilde{F}(S_1)+\pi^{\gamma_2}S_2)\left(\frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}\right)-1}{\pi^{\gamma_2}}=$$

$$\frac{\tilde{F}(S_1)H(T_1)-H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}{\pi^{\gamma_2}H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}+T_2\frac{\tilde{F}(S_1)H(T_1)}{H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}+$$
$$+S_2\frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)}.$$

This implies

$$\tilde{F}(S_1)H(T_1)-H(\pi^{m-\gamma_1}S_1+T_1+\pi^m S_1T_1)\equiv 0 \mod \pi^{\gamma_2}.$$

But it is clear that this condition is also sufficient to define the wanted action. □

The next lemma, together with III.5.9, links the definition of the effective threshold with the existence of an action of a model of $\mathbb{Z}/p^2\mathbb{Z}$ on Y.

LEMMA III.5.10. *Let $\tilde{b}\in\pi R$. Let us consider $\tilde{G}(S)=\sum_{i=0}^{p-1}\frac{\tilde{b}^i}{i!}S^i\in R[S]$. The following statements are equivalent.*

(i) $\tilde{G}(S)H(T)\equiv H(\pi^{m-\gamma_1}S+T+\pi^m ST) \mod \pi^{\gamma_2}$*;*

(ii) $\tilde{b}H(T)\equiv\pi^{m-\gamma_1}H'(T) \mod \pi^{\gamma_2}$ *where H' is the formal derivative of H.*

Moreover they imply the following assertions.

(1) *Let us consider $R[G_{\pi^m,1}]=R[S]/(\frac{(1+\pi^m S)^p-1}{\pi^{mp}})$. Then*

$$\tilde{G}(S)\in\operatorname{Hom}(G_{\pi^m,1|S_\pi^{\gamma_2}},\mathbb{G}_{m|S_\pi^{\gamma_2}})$$

and

$$\tilde{G}(S)^p\equiv 1+\pi^m S \mod \pi^{\gamma_2}R[G_{\mu,1}].$$

(2) *If $m>\gamma_1$ then*

$$\frac{\tilde{G}(S)H(T)-H(\pi^{m-\gamma_1}S+T+\pi^m ST)}{\pi^{\gamma_2}}\equiv\frac{\tilde{b}H(T)-\pi^{m-\gamma_1}H'(T)}{\pi^{\gamma_2}} \mod \pi$$

PROOF. *(i)* ⇒ *(ii)*. Let us suppose

$$\tilde{G}(S)H(T)\equiv H(\pi^{m-\gamma_1}S+T+\pi^m ST) \mod \pi^{\gamma_2}.$$

We consider both members as polynomials in S with coefficients in $R[T]$. Then if we compare the coefficients of S we obtain (ii).

$(i) \Leftarrow (ii)$. Let $H^{(k)}(T)$ denote the k^{th} formal derivative of $H(T)$. We remark that (i) is equivalent to

$$\tilde{b}^k H(T) \equiv (\pi^{m-\gamma_1})^k H^{(k)}(T) \mod \pi^{\gamma_2}$$

for $0 \leq k \leq p-1$. We prove a little more. We prove that

$$\tilde{b}^k H(T) \equiv (\pi^{m-\gamma_1})^k H^{(k)}(T) \mod \pi^{\gamma_2+\min\{(k-1)v(\tilde{b}),(k-1)(m-\gamma_1)\}} \tag{III.83}$$

For $k=0$ it is obvious. Let us now suppose it is true for k, we prove it for $k+1$. If we multiply (III.83) by $\tilde{b}$ we obtain

$$\tilde{b}^{k+1} H(T) \equiv \tilde{b}(\pi^{m-\gamma_1})^k H^{(k)}(T) \mod \pi^{\gamma_2+\min\{(k-1)v(\tilde{b}),(k-1)(m-\gamma_1)\}+v(\tilde{b})}. \tag{III.84}$$

Moreover if we differentiate the equation (ii) k times, we obtain

$$\tilde{b} H^{(k)}(T) \equiv \pi^{m-\gamma_1} H^{(k+1)}(T) \mod \pi^{\gamma_2}. \tag{III.85}$$

Multiplying (III.85) by $\pi^{k(m-\gamma_1)}$ we obtain

$$\tilde{b}\pi^{k(m-\gamma_1)} H^{(k)}(T) \equiv (\pi^{m-\gamma_1})^{(k+1)} H^{(k+1)}(T) \mod \pi^{\gamma_2+k(m-\gamma_1)}. \tag{III.86}$$

Then (III.84) and (III.86) give

$$\tilde{b}^{k+1} H(T) \equiv (\pi^{m-\gamma_1})^{k+1} H^{(k+1)}(T) \mod \pi^{\gamma_2+\min\{kv(\tilde{b}),k(m-\gamma_1)\}}.$$

as we wanted. So (i) and (2) are proved. Let us now suppose (i) true.

(1) We recall that $H(T) = \sum_{i=0}^{p-1} a_i T^i \in (A[T]/(\frac{(1+\pi^{\gamma_1}T)^p-1}{\pi^{\gamma_1 p}} - f_1))^* = B_1^*$. If $a_0 \in \pi A$ then $\tilde{b} \equiv 0 \mod \pi^{\gamma_2}$ by III.5.3. Let us now suppose that $a_0 \notin \pi A$. We now think $H(S) \in A[S]/(\frac{(1+\pi^{\gamma_1}S)^p-1}{\pi^{\gamma_1 p}}) = R[G_{\pi^{\gamma_1},1}]$. We consider the morphism $\psi_{\pi^m,\pi^{\gamma_1}} : G_{\pi^m,1} \longrightarrow G_{\pi^{\gamma_1},1}$. Then

$$\psi^*_{\pi^m,\pi^{\gamma_1}}(H(S)) \equiv H(\pi^{m-\gamma_1}S)$$

On the other side if we compare the coefficients of T in (i) we obtain

$$H(\pi^{m-\gamma_1}S) = a_0\tilde{G}(S) \mod \pi^{\gamma_2}.$$

Therefore

$$\psi^*_{\pi^m,\pi^{\gamma_1}}(H(S)) \equiv a_0\tilde{G}(S) \mod \pi^{\gamma_2}.$$

Let us now consider $\mathrm{id} \times \psi_{\pi^m,\pi^{\gamma_1}} : G_{\pi^m,1} \times G_{\pi^m,1} \longrightarrow G_{\pi^m,1} \times G_{\pi^{\gamma_1},1}$. Hence if we apply $\mathrm{id} \times \psi^*_{\pi^m,\pi^{\gamma_1}}$ to (i) we obtain

$$a_0\tilde{G}(S)\tilde{G}(T) \equiv a_0\tilde{G}(S+T+\pi^m ST) \mod \pi^{\gamma_2}$$

which implies, since $a_0 \notin \pi A$ and $G_{\pi^m,1} \times G_{\pi^m,1}$ is flat over A,

$$\tilde{G}(S)\tilde{G}(T) \equiv \tilde{G}(S+T+\pi^m ST) \mod \pi^{\gamma_2}.$$

This means $\tilde{G}(S) \in \mathrm{Hom}_{gr}(G_{\pi^m,1|S_{\pi^{\gamma_2}}}, \mathbb{G}_{m|S_{\pi^{\gamma_2}}})$. Moreover we know that

$$H(T)^p \equiv g^{-1}(1+\pi^{\gamma_1}T) \mod \pi^{\gamma_2} B_1. \tag{III.87}$$

Hence

$$(H(T)\tilde{G}(S))^p \equiv g^{-1}(1+\pi^{\gamma_1}T)\tilde{G}(S)^p \mod \pi^{\gamma_2}(R[G_{\pi^m,1}] \otimes B_1).$$

Moreover it is easy to see that

$$R[G_{\pi^m,1}]\otimes B_1 = R[S,T]/\left(\frac{((1+\pi^{\gamma_1}T)(1+\pi^m S))^p-1}{\pi^{p\gamma_1}}-f_1, \frac{(1+\pi^m S)^p-1}{\pi^{mp}}\right),$$

then we can substitute $\frac{(1+\pi^{\gamma_1}T)(1+\pi^m S)-1}{\pi^{\gamma_1}}$ to T in (III.87) and we obtain

$$(H(\pi^{m-\gamma_1}S+T+\pi^m ST))^p \equiv g^{-1}(1+\pi^m S)(1+\pi^{\gamma_1}T) \mod \pi^{\gamma_2}(R[G_{\pi^m,1}]\otimes B_1)$$

By hypothesis we have that

$$\tilde{G}(S)H(T) \equiv H(\pi^{m-\gamma_1}S+T+\pi^m ST) \mod \pi^{\gamma_2}$$

and therefore

$$g^{-1}(1+\pi^{\gamma_1}T)\tilde{G}(S)^p \equiv g^{-1}(1+\pi^m S)(1+\pi^{\gamma_1}T) \mod \pi^{\gamma_2}(A_1\otimes B_1).$$

This implies

$$\tilde{G}(S)^p \equiv (1+\pi^m S) \mod \pi^{\gamma_2}A_1.$$

□

We are now able to find a candidate to be the effective model.

LEMMA III.5.11. *If an effective model for the $\mathbb{Z}/p^2\mathbb{Z}$-action exists it must be the group scheme $\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$. In particular $(\alpha_\kappa,1)\in\Phi_{\pi^\kappa,\pi^{\gamma_2}}$. Moreover $\gamma_2\le\kappa\le v(\lambda_{(1)})$.*

PROOF. Since, as we have seen, $\mathbb{Z}/p^2\mathbb{Z}$ acts on Y then, by III.5.9 and the previous lemma it follows that η_π satisfies $(\triangle)_{v(\lambda_{(1)})}$. Therefore $\kappa\le v(\lambda_{(1)})$.

By III.5.8 it follows that the effective model is of the form $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ for some $m\le v(\lambda_{(1)})$ and $F\in \mathrm{Hom}_{gr}(G_{\pi^m,1|S_{\pi^{\gamma_2}}},\mathbb{G}_{m|S_{\pi^{\gamma_2}}})$. By III.5.9 and III.5.10 we have that if a group scheme $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};F,1)}$ acts on Y then $F=E_p(\alpha_m S)$ with $\alpha_m\in\pi R$ which satisfies $(\triangle)_m$. Conversely if $m\le v(\lambda_{(1)})$ and $\alpha_m\in\pi R$ satisfies $(\triangle)_m$ then by III.5.9 and III.5.10(1) we can construct the group scheme $\mathcal{E}^{(\pi^m,\pi^{\gamma_2};E_p(\alpha_m S),1)}$ and it acts on Y. We remark that by III.5.3 the equation $(\triangle)_m$ has (unique) solution if and only if $m\ge\kappa$. Moreover, for any $v(\lambda_{(1)})\ge m'\ge m$ there exists a model map $\mathcal{E}^{(\pi^{m'},\pi^{\gamma_2};E_p(\alpha_{m'}S),1)}\longrightarrow\mathcal{E}^{(\pi^m,\pi^{\gamma_2};E_p(\alpha_m S),1)}$. Indeed by definition of m, we have that there exists $\alpha_m\in\pi R$ such that

$$\alpha_m H(T)\equiv\pi^{m-\gamma_1}H'(T) \mod \pi^{\gamma_2}.$$

Therefore

$$\pi^{m'-m}\alpha_m H(T)\equiv\pi^{m'-\gamma_1}H'(T) \mod \pi^{\gamma_2}.$$

But we know that

$$\alpha_{m'}H(T)\equiv\pi^{m'-\gamma_1}H'(T) \mod \pi^{\gamma_2}.$$

And, as seen in III.5.3, the solution of the above equation is a unique mod π^{γ_2}. Therefore $\pi^{m'-m}\alpha_m\equiv\alpha_{m'} \mod \pi^{\gamma_2}$. So, by II.3.49, there exists a model map

$$\mathcal{E}^{(\pi^{m'},\pi^{\gamma_2};E_p(\alpha_{m'}S),1)}\longrightarrow\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_m S),1)}.$$

We recall that for any $m \geq \kappa$ the action of $\mathcal{E}^{(\pi^{m'},\pi^{\gamma_2};E_p(\alpha_{m'}S),1)}$ is given by

$$\begin{aligned} T_1 &\longmapsto \pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1 T_1 \\ T_2 &\longmapsto \frac{F(S_1)H(T_1) - H(\pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1 T_1)}{\pi^{\gamma_2} H(\pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1 T_1)} + \\ &\qquad + T_2 \frac{F(S_1)H(T_1)}{H(\pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1 T_1)} + \frac{(1+\pi^{\gamma_2}T_2)H(T_1)}{H(\pi^{m-\gamma_1}S_1 + T_1 + \pi^m S_1 T_1)} \end{aligned}$$

The above model map is compatible with the actions on Y. In particular we have, for any $v(\lambda_{(1)}) \geq m > \kappa$, a model map

$$\mathcal{E}^{(\pi^m,\pi^{\gamma_2};E_p(\alpha_m S),1)} \longrightarrow \mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}.$$

compatible with the actions. Since the above model map is not an isomorphism, there is a non trivial kernel $\tilde{H}$ of the morphism restricted to the special fiber. Since the map is compatible with the actions then $\tilde{H} \subseteq (\mathcal{E}^{(\pi^m,\pi^{\gamma_2};E_p(\alpha_m S),1)})_k$ acts trivially on Y_k. So

$$\mathcal{E}^{(\pi^m,\pi^{\gamma_2};E_p(\alpha_m S),1)}$$

is not the effective model of the $\mathbb{Z}/p^2\mathbb{Z}$-action if $m > \kappa$. Hence if an effective model exists it must be $\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$. Since the group $\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$ exists it follows, by II.3.53, that $\kappa \geq \gamma_2$ and $(\alpha_\kappa, 1) \in \Phi_{\pi^\kappa,\pi^{\gamma_2}}$. □

We remark that if X was of finite type then Y would be of finite type. So applying the theorem of existence of effective models III.1.9 we would have finished. We now prove that $\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$ is the effective model for the action of $\mathbb{Z}/p^2\mathbb{Z}$ in the general case. By construction the action is faithful on the generic fiber. We now check the faithfulness on the special fiber. Let us suppose that the map

$$\mathcal{G}_k = (\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)})_k \longrightarrow Aut_k(Y_k)$$

has nontrivial kernel $\tilde{K}$. Since the action of $(G_{\pi^{\gamma_2},1})_k$ on Y_k is faithful by definition of γ_2 then $\tilde{K} \times_{\mathcal{G}_k} (G_{\pi^{\gamma_2},1})_k$ is the trivial group scheme. Therefore $\tilde{K}$ is a group scheme of order p and

$$(\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)})_k \simeq (G_{\pi^{\gamma_2},1})_k \times_k \tilde{K}$$

Since $(\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)})_k$ is an extension of $(G_{\pi^\kappa,1})_k$ by $(G_{\pi^{\gamma_2},1})_k$ we have that $\tilde{K} \simeq (G_{\pi^\kappa,1})_k$. We distinguish two cases.

$\boxed{\kappa = \gamma_1.}$ The action $(G_{\pi^{\gamma_1},1})_k \times_k Y_k/(G_{\pi^{\gamma_2},1})_k \longrightarrow Y_k/(G_{\pi^{\gamma_2},1})_k$ is induced by the action $\tilde{K} \times_k Y_k \longrightarrow Y_k$, which is trivial by definition of $\tilde{K}$. But by definition of γ_1, $(G_{\pi^{\gamma_1},1})_k$ acts faithfully on $Y_k/(G_{\pi^{\gamma_2},1})_k$. So $\tilde{K}$ is trivial and the action of $\mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$ is faithful on the special fiber.

$\boxed{\kappa > \gamma_1.}$ We remark that necessarily $\gamma_2 > 0$. Indeed if $\gamma_2 = 0$ then, by III.5.6(i) and (iii) necessarily $\kappa = \gamma_1$. It is also clear that $\kappa > 0$. Now, by $\gamma_2 > 0$ and III.5.10(2), it follows that the action on the special fiber is given by the reduction mod π of

$$T_1 \longmapsto T_1$$
$$T_2 \longmapsto \frac{\alpha_\kappa H(T_1) - \pi^{\kappa-\gamma_1} H'(T_1)}{\pi^{\gamma_2} H(T_1)} S_1 + T_2 + S_2$$

We remark that if $j > 0$ then $H(T_1) \equiv 1 \mod \pi$. We now prove that

$$\frac{\alpha_\kappa H(T_1) - \pi^{\kappa-\gamma_1} H'(T_1)}{\pi^{\gamma_2} H(T_1)} S_1 \not\equiv b S_1 \mod \pi \tag{III.88}$$

for any $b \in R$. Let us suppose $\frac{\alpha_\kappa H(T_1) - \pi^{\kappa-\gamma_1} H'(T_1)}{\pi^{\gamma_2} H(T_1)} S_1 \equiv b S_1 \mod \pi$ with $b \in R$. Then

$$\frac{\alpha_\kappa H(T_1) - \pi^{\kappa-\gamma_1} H'(T_1)}{\pi^{\gamma_2}} S_1 \equiv b H(T_1) S_1 \mod \pi$$

with $b \in R$. Therefore

$$\frac{(\alpha_\kappa - b\pi^{\gamma_2}) H(T_1) - \pi^{\kappa-\gamma_1} H'(T_1)}{\pi^{\gamma_2}} \equiv 0 \mod \pi.$$

It clearly follows that

$$\frac{\alpha_\kappa - b\pi^{\gamma_2}}{\pi} H(T_1) \equiv \pi^{\kappa-1-\gamma_1} H'(T_1) \mod \pi^{\gamma_2}$$

Then $\alpha_{\kappa-1} = \frac{\alpha_\kappa - b\pi^{\gamma_2}}{\pi}$ satisfies $(\triangle)_{\kappa-1}$; it is easy to see that this implies $\alpha_{\kappa-1} \in \pi R$. The minimality of κ is contradicted. So we have proved (III.88).

We now consider three different cases. If $\gamma_2, \kappa < v(\lambda_{(1)})$ then

$$(\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)})_k \simeq \alpha_p \times_k \alpha_p = \mathrm{Spec}(k[S_1, S_2]/(S_1^p, S_2^p)).$$

Its subgroups of order p different from $(G_{\pi^{\gamma_2},1})_k$ are the subgroups $S_2 + bS_1 = 0$ with $b \in k$. If $\gamma_2 < \kappa = v(\lambda_{(1)})$, then

$$(\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)})_k \simeq \alpha_p \times_k \mathbb{Z}/p\mathbb{Z} = \mathrm{Spec}(k[S_1, S_2]/(S_1^p, S_2^p - S_2))$$

and the only subgroup isomorphic to $\tilde{K} \simeq \mathbb{Z}/p\mathbb{Z}$ is $S_2 = 0$. Finally if $\gamma_2 = \kappa = v(\lambda_{(1)})$ then

$$(\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)})_k \simeq \mathbb{Z}/p\mathbb{Z} \times_k \mathbb{Z}/p\mathbb{Z} = \mathrm{Spec}(k[S_1, S_2]/(S_1^p - S_1, S_2^p - S_2))$$

and the only subgroups isomorphic to $\tilde{K} \simeq \mathbb{Z}/p\mathbb{Z}$ different from $(G_{\pi^{\gamma_2},1})_k$ are the subgroups $S_2 + bS_1 = 0$ with $b \in \mathbb{F}_p$. In any case, by (III.88) the action restricted to any subgroup of $(\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)})_k$ is not trivial.

We now prove the last sentence of the theorem. We have, by definition,

$$\alpha_\kappa H(T_1) \equiv \pi^{\kappa-\gamma_1} H'(T_1) \mod \pi^{\gamma_2} \tag{III.89}$$

Moreover $H(T_1) \in B_1^*$ and, if we consider $H'(T_1) \in B_{1(\pi)}$, we have

$$v(H'(T_1)) = j, \tag{III.90}$$

by III.5.1. If $\alpha_\kappa \equiv 0 \mod \pi^{\gamma_2}$ then, by (III.89) and (III.90), it follows $\pi^{\kappa-\gamma_1+j} \equiv 0 \mod \pi^{\gamma_2}$. Therefore $\kappa - \gamma_1 + j \geq \gamma_2$. So, by III.5.6(iv), we have $\kappa - \gamma_1 + j = \gamma_2$.

While, if $\alpha_\kappa \not\equiv 0 \mod \pi^{\gamma_2}$, it follows by (III.89) that $v(\alpha_k) = \kappa - \gamma_1 + j$. The theorem is proved. □

We here give a criterion to determine when Y has a structure of torsor.

COROLLARY III.5.12. *Let us suppose we are in the hypothesis of the theorem. Then $Y \longrightarrow X$ is a G-torsor under some finite and flat group scheme G if and only if $\kappa = \gamma_1$. Moreover $\kappa = \gamma_1$ if and only if $\gamma_1 \geq \gamma_2$ and $H(T) \equiv E_p(aT) \mod \pi^{\gamma_2}$, for some $a \in \pi A$ such that $a^p \equiv 0 \mod \pi^{\gamma_2}$. In such a case $G = \mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};H,1)}$.*

REMARK III.5.13. The degeneration type of any $\mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$-torsor is

$$(v(\alpha_\kappa), \gamma_1, \gamma_2, \gamma_1).$$

This follows from III.5.7 and III.5.12.

PROOF. We remarked in III.1.6 that if $Y \longrightarrow X$ is a G-torsor for some finite and flat group scheme then G must coincide with the effective model $\mathcal{G}$ of $\mathbb{Z}/p^2\mathbb{Z}$ acting on Y. In other words $Y \longrightarrow X$ is a G-torsor if and only if it is a $\mathcal{G}$-torsor. By the theorem we have that the effective model for the $\mathbb{Z}/p^2\mathbb{Z}$-action is $\mathcal{E}^{(\pi^\kappa,\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$. Moreover there is the following exact sequence

$$0 \longrightarrow G_{\pi^{\gamma_2},1} \xrightarrow{i} \mathcal{G} \xrightarrow{p} G_{\pi^\kappa,1} \longrightarrow 0$$

By III.1.7 (i) we have that $G_{\pi^{\gamma_2},1}$ is the effective model of the action of $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{Z}/p^2\mathbb{Z}$ on Y. Now if $Y \longrightarrow X$ is a $\mathcal{G}$-torsor then it satisfies the hypothesis of III.1.7 (iii), then $G_{\pi^\kappa,1}$ is the effective model of the action of $\mathbb{Z}/p^2\mathbb{Z}/\mathbb{Z}/p\mathbb{Z}$ on Y_1. But by the definition of γ_1 we have that $Y_1 \longrightarrow X$ is a $G_{\pi^{\gamma_1},1}$-torsor. Then, again by III.1.6, we have $G_{\pi^\kappa,1} \simeq G_{\pi^{\gamma_1},1}$, which implies $\kappa = \gamma_1$.

Let us now suppose that $\kappa = \gamma_1$. We recall that

$$Y = \mathrm{Spec}(A[T_1,T_2]/(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1, \frac{(1+\pi^{\gamma_2}T_2)^p - 1}{\pi^{p\gamma_2}} - \frac{g^{-1}H(T_1)^{-p}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}))$$

Moreover by definition of κ we have $\alpha_\kappa H(T) \equiv H'(T) \mod \pi^{\gamma_2}$. Then by II.3.21 it follows that $H(T) \equiv E_p(\alpha_\kappa T) \mod \pi^{\gamma_2}$. We recall that we suppose, using III.5.1, $H(0) = 1$. Now let us substitute $T_2 H(T_1)$ to T_2. Then we obtain

$$Y = \mathrm{Spec}(A[T_1,T_2]/(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1, \frac{(H(T_1)+\pi^{\gamma_2}T_2)^p(1+\pi^{\gamma_1}T_1)^{-1} - g^{-1}}{\pi^{p\gamma_2}}))$$

By definition of $\mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$ there exists $G \in \mathrm{Hom}(\mathcal{G}^{(\pi^{\gamma_1})}{}_{|S_{\pi^{p\gamma_2}}}, \mathbb{G}_{m|S_{\pi^{p\gamma_2}}})$ such that

$$E_p(\alpha_\kappa S)^p(1+\pi^{\gamma_1}S)^{-1} = G(\frac{(1+\pi^{\gamma_1}S)^p - 1}{\pi^{p\gamma_1}}) \in \mathrm{Hom}(\mathcal{G}^{(\pi^{\gamma_1})}{}_{|S_{\pi^{p\gamma_2}}}, \mathbb{G}_{m|S_{\pi^{p\gamma_2}}}).$$

We remark that, if we think $E_p(\alpha_\kappa T_1), G(T_1) \in B_1^*$, the previous equation gives $E_p(\alpha_\kappa T_1)^p(1+\pi^{\gamma_1}T_1)^{-1} \equiv G(f_1) \mod \pi^{\gamma_2}B_1$ On the other hand we have that $H(T_1)^p(1+\pi^{\gamma_1}T_1)^{-1} \equiv E_p(\alpha_\kappa T_1)^p(1+\pi^{\gamma_1}T_1)^{-1} \equiv g^{-1} \mod \pi^{p\gamma_2}B_1$. Therefore we have

$$g^{-1} \equiv G(f_1) \mod \pi^{p\gamma_2}A$$

i.e. $g^{-1} = G(f_1) + \pi^{p\gamma_2} f_2$ for some $f_2 \in A$. Hence, by §II.4, $Y \longrightarrow X$ is a $\mathcal{E}^{(\pi^{\gamma_1}, \pi^{\gamma_2}; E_p(\alpha_\kappa S), 1)}$-torsor.

We now have, by definition of κ, that $\kappa = \gamma_1$ if and only if there exists $\alpha_\kappa \in A$ such that

$$\alpha_\kappa H(T_1) \equiv H'(T_1) \mod \pi^{\gamma_2} \tag{III.91}$$

We remark that, since $\kappa \geq \gamma_2$, $\kappa = \gamma_1$ only if $\gamma_1 \geq \gamma_2$. In such a case, by II.3.21, $H(T_1)$ satisfies (III.91) if and only if there exist $\alpha_\kappa \in \pi A$ such that $\alpha_\kappa^p \equiv 0 \mod \pi^{\gamma_2}$ and $H(T_1) \equiv E_p(\alpha_\kappa T_1) \mod \pi^{\gamma_2}$.

□

COROLLARY III.5.14. *If $\gamma_1 < \gamma_2$ then $Y \longrightarrow X$ has no structure of torsor.*

PROOF. We know by III.5.11 that $\kappa \geq \gamma_2$. So by hypothesis we have $\kappa > \gamma_1$. By the above corollary the thesis follows. □

REMARK III.5.15. Unfortunately we have no example of coverings with $\gamma_1 < \gamma_2$. So we don't know if this case can really occur.

EXAMPLE III.5.16. We here give an example, for any $p \geq 3$, where $Y \longrightarrow X$ is not a G-torsor under any group scheme G. We suppose that $A_k^* \neq A_k^{*p}$. Let $X = \mathrm{Spec}(A)$ be as above. Take γ_1, γ_2 such that $v(p) > p\gamma_1 > p^2\gamma_2 > 0$. In particular we have $v(p) > (p-1)\gamma_1 + p\gamma_2$. Let us consider any $a_1 \in A^*$. Moreover take $f_1 \in A^*$ and $f_2 \in A$ such that they are not a p^{th}-power $\mod \pi$. Moreover let us consider $g^{-1} = a_1^p(f_1 + \pi^{p\gamma_2} f_2) \in A^*$. For instance we can take $A = R[[Z]]\{Z^{-1}\}$, $\gamma_1 = p+1$, $\gamma_2 = 1$, $a_1 = 1$, $f_1 = f_2 = Z$ and $g^{-1} = Z(1+\pi^{p\gamma_2})$. So the $\mathbb{Z}/p^2\mathbb{Z}$-torsor over $\mathrm{Spec}(A_K)$ is $\mathrm{Spec}(A_K[T]/(T^{p^2} - \frac{1+\pi^{p(p+1)}Z}{Z^p(1+\pi^p)^p}))$.

Then we consider $Y_1 = \mathrm{Spec}(B_1) = \mathrm{Spec}(A[T]/(\frac{(1+\pi^{\gamma_1}T_1)^p - 1}{\pi^{p\gamma_1}} - f_1))$. Since f_1 is not a p^{th}-power $\mod \pi$ then Y_1 is normal (see III.3.6). We remark that by hypothesis we have that $T_1^p \equiv f_1 \mod \pi^{p\gamma_2+1}$. We now take $H(T) = a_1 T_1 \in B_1$. Then we have, by construction,

$$\frac{H(T_1)^p - g^{-1}(1+\pi^{\gamma_1}T_1)}{\pi^{p\gamma_2}} \equiv a_1^p f_2 \mod \pi.$$

So we consider

$$Y = \mathrm{Spec}(B_1[T_2]/(\frac{(1+\pi^{\gamma_2}T_2)^p - 1}{\pi^{\pi^{p\gamma_2}}} - \frac{H(T_1)^{-p} g^{-1}(1+\pi^{\gamma_1}T_1) - 1}{\pi^{p\gamma_2}}))$$

Then

$$Y_k = \mathrm{Spec}(A[T_1, T_2]/(T_1^p - f_0, T_2^p - f_1))$$

Hence Y_k is integral and Y normal. We remark that

$$Y_K \simeq \mathrm{Spec}(A_K[T]/(T^{p^2} - (1+\pi^{\gamma_1} f_1) g^{-p})).$$

Since $a_1 \notin \pi A$ then, by III.5.12, we have that $Y \longrightarrow X$ has no structure of torsor.

The degeneration type of Y_K is $(0, \gamma_1, \gamma_2, \gamma_1 + \gamma_2)$. Indeed $H'(T_1) = a_1$. So

$$a(a_1 T_1) \equiv \pi^{\kappa - \gamma_1} a_1 \mod \pi^{\gamma_2}$$

if and only if $a \equiv 0 \mod \pi^{\gamma_2}$ and $\kappa - \gamma_1 \geq \gamma_2$. This means $\kappa = \gamma_1 + \gamma_2$ since $\kappa \leq \gamma_1 + \gamma_2 - j$ and $j = 0$. The effective model is

$$G = \mathcal{E}^{(\pi^{\gamma_1+\gamma_2}, \pi^{\gamma_2}; 1, 1)}.$$

Since $\kappa = \gamma_1 + \gamma_2 > \gamma_1$ then Y is not a G-torsor by the previous corollary.

III.6. Realization of degeneration types

We have shown in the above section that the degeneration type has to satisfy some restrictions. We here want to study the problem of determining which are the elements of $\mathbb{N}^4$ which can be degeneration type of some cover $Y \longrightarrow X$.

DEFINITION III.6.1. Any 4-uple $(j, \gamma_1, \gamma_2, \kappa) \in \mathbb{N}^4$ with the following properties:

i) $\max\{\gamma_1, \gamma_2\} \leq \kappa \leq v(\lambda_{(1)})$;
ii) $\gamma_2 \leq p(\kappa - \gamma_1 + j) \leq p\gamma_2$;
iii) if $\kappa < p\gamma_2$ then $\gamma_1 - j = v(\lambda_{(1)}) - v(\lambda_{(2)}) = \frac{v(p)}{p}$; if $\kappa \geq p\gamma_2$ then $0 \leq p(\gamma_2 - j) \leq v(p) - p\gamma_1 + \kappa$;
iv) $pj \leq \gamma_1$;

will be called an *admissible degeneration type.*

REMARK III.6.2. We remark that if $\kappa < p\gamma_2$ then j is uniquely determined from γ_1 and moreover i) and iii) imply iv). The first assertion follows from iii). For the second we note that, if $\kappa < p\gamma_2$, multiplying iii) by p we have $p\gamma_1 - j = (p-1)v(\lambda_{(1)})$, since $p\lambda_{(2)} = \lambda_{(1)}$. Therefore, by i), we have $\gamma_1 - pj = (p-1)(v(\lambda_{(1)}) - \gamma_1) \geq 0$.

Moreover we remark that

$$\kappa - \gamma_1 + j \leq \min\{\gamma_2, v(\lambda_{(2)})\}.$$

By ii) we have only to prove $\kappa - \gamma_1 + j \leq v(\lambda_{(2)})$. Moreover, since $\lambda_{(1)} \geq \kappa \geq p\gamma_2$ implies $\gamma_2 \leq v(\lambda_{(2)})$, we have only to consider the case $\kappa < p\gamma_2$. But by iii) and i) it follows that

$$\kappa - \gamma_1 + j = \kappa - \frac{v(p)}{p} \leq v(\lambda_{(1)}) - v(\lambda_{(1)}) + v(\lambda_{(2)}) = v(\lambda_{(2)}).$$

LEMMA III.6.3. *Any degeneration type $(j, \gamma_1, \gamma_2, \kappa)$ attached to a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$ is admissible.*

PROOF. i) comes from definitions and III.5.11. While iv) has been proved in III.5.6(i). We now prove ii). By III.5.7 it follows that the effective model of the action of $\mathbb{Z}/p^2\mathbb{Z}$ on Y is $\mathcal{E}^{(\pi^\kappa, \pi^{\gamma_1}; E_p(\alpha_\kappa), 1)}$ with $v(\alpha_\kappa) = \kappa - \gamma_1 + j$, if $\alpha_\kappa \neq 0$; and $\kappa - \gamma_1 + j = \gamma_2$ if $\alpha_\kappa = 0$. Since, by III.5.11, $(\alpha_\kappa, 1) \in \Phi_{\pi^\kappa, \pi^{\gamma_2}}$ then

$$\alpha_\kappa^p \equiv 0 \mod \pi^{\gamma_2}. \qquad \text{(III.92)}$$

Hence we have

$$\gamma_2 \leq p(\kappa - \gamma_1 + j).$$

By III.5.6(iii) it follows that $\kappa - \gamma_1 + j \leq \gamma_2$. This proves ii).

Let us now suppose $\kappa < p\gamma_2$. Since $(\alpha_\kappa, 1) \in \Phi_{\pi^\kappa,\pi^{\gamma_2}}$, by II.3.42 and II.3.47, we have that

$$\kappa - \gamma_1 + j = \kappa - \frac{v(p)}{p}$$

which implies $\gamma_1 - j = \frac{v(p)}{p}$. While, if $\kappa \geq p\gamma_2$, by II.3.42 we have that

$$pv(\alpha_\kappa) = p(\kappa - \gamma_1 + j) \geq p\gamma_2 + (p-1)\kappa - v(p)$$

which gives

$$p(\gamma_2 - j) \leq v(p) - p\gamma_1 + \kappa.$$

We remark that $\gamma_2 - j \geq 0$ comes from III.5.6. Hence iii) is proved. □

Definition III.6.4. Any admissible degeneration type which is the degeneration type attached to a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$ as in §III.5, will be called *realizable.*

We now see, as a consequence of theorem III.5.7, what happens in some particular cases. Moreover we observe that, by III.5.14, one can imagine to find a generically $\mathbb{Z}/p^2\mathbb{Z}$-torsor with no global structure of torsor in some easier cases. For instance $\gamma_1 < v(\lambda_{(1)})$ and $\gamma_2 = v(\lambda_{(1)})$. But the following result shows in particular that any such admissible degeneration types are not realizable.

Proposition III.6.5. *Let us suppose Y_K has $(j, \gamma_1, \gamma_2, \kappa)$ as degeneration type.*

i) *If $j < v(\lambda_{(2)})$ then $pj = \gamma_1$ if and only if Y is a $G_{\pi^j,2}$-torsor. Moreover the degeneration type is (j, pj, j, pj). In particular Y is a μ_{p^2}-torsor if and only if $\gamma_1 = 0$, i.e. $v(\mathcal{D}(h_1)) = v(p)$.*

ii) *$j = v(\lambda_{(2)})$ if and only if Y is an $\mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{\gamma_2};E_p(\eta_\pi S),1)}$-torsor. Necessarily $\gamma_2 \geq v(\lambda_{(2)})$ and the degeneration type is $(v(\lambda_{(2)}), v(\lambda_{(1)}), \gamma_2, v(\lambda_{(1)}))$.*

iii) *$\gamma_2 = j$ if and only if Y is a $\mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};1,1)}$-torsor. Necessarily $\gamma_1 \geq p\gamma_2$ and the degeneration type is $(\gamma_2, \gamma_1, \gamma_2, \gamma_1)$. In particular Y is a $\mathcal{E}^{(\pi^{\gamma_1},1;1,1)}$-torsor if and only if $\gamma_2 = 0$, i.e. $v(\mathcal{D}(h_2)) = v(p)$.*

iv) *Y is a $\mathbb{Z}/p^2\mathbb{Z}$-torsor if and only if $\gamma_2 = v(\lambda_{(1)})$, i.e. $v(\mathcal{D}(h_2)) = 0$. And the degeneration type is $(v(\lambda_{(2)}), v(\lambda_{(1)}), v(\lambda_{(1)}), v(\lambda_{(1)}))$.*

v) *If $\gamma_1 = v(\lambda_{(1)})$, i.e. $v(\mathcal{D}(h_1)) = 0$, then $j = \min\{\gamma_2, v(\lambda_{(2)})\}$. So we are in the case (ii) or (iii).*

Proof.

i) Let us suppose $\gamma_1 = pj$. By the previous lemma $(j, \gamma_1, \gamma_2, \kappa)$ is an admissible degeneration type. If $\kappa < p\gamma_2$ then by III.6.1(iii) it follows

$$(p-1)j = \gamma_1 - j = \frac{v(p)}{p} = (p-1)v(\lambda_{(2)}),$$

but this is in contradiction with $j < v(\lambda_{(2)})$. Hence $\kappa \geq pv(\gamma_2)$. Therefore, by III.6.1(ii),

$$(p-1)\gamma_2 \leq \kappa - \gamma_2 \leq \gamma_1 - j = (p-j)j.$$

But, by III.6.1(iii), $\gamma_2 \geq j$. Hence $\gamma_2 = j$. So, by III.5.6(iii), $\kappa = \gamma_1$. Then, by III.5.12, we have that Y is a $\mathcal{E}^{(\pi^j,\pi^{pj};1,1)}$-torsor. But, as we have seen in the example II.3.33,

$$\mathcal{E}^{(\pi^j,\pi^{pj};1,1)} \simeq G_{\pi^j,2}$$

Conversely, as remarked in III.5.13 (j, pj, j, pj) is the degeneration type of a $G_{\pi^j,2}$-torsor.

We now observe that, in particular, Y is a μ_{p^2}-torsor if and only if $\gamma_1 = pj = 0$. But since $pj \leq \gamma_1$ (see III.5.6(i)) then it is true if and only if $\gamma_1 = 0$, as stated.

ii) Let us suppose $j = v(\lambda_{(2)})$. By III.5.6 we have $\gamma_1 = pj = v(\lambda_{(1)})$ and $\kappa = v(\lambda_{(1)})$. Therefore by the theorem we have that $\mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}$ is the effective model. In particular there is a model map

$$\mathbb{Z}/p^2\mathbb{Z} \simeq \mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{v(\lambda_{(1)})};E_p(\eta_\pi S),1)} \longrightarrow \mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{\gamma_2};E_p(\alpha_\kappa S),1)}.$$

Hence by II.3.49 it follows that

$$\alpha_\kappa \equiv \eta_\pi \mod \pi^{\gamma_2}.$$

So, by III.5.12, Y is a $\mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{\gamma_2};E_p(\eta_\pi S),1)}$-torsor. Conversely if Y is a $\mathcal{E}^{(\pi^{v(\lambda_{(1)})},\pi^{\gamma_2};E_p(\eta_\pi S),1)}$-torsor then, by III.5.13, the degeneration type is

$$(v(\eta_\pi), v(\lambda_{(1)}), \gamma_2, v(\lambda_{(1)})).$$

So $j = v(\eta_\pi) = v(\lambda_{(2)})$. We observe that $j = v(\lambda_{(2)}) \leq \gamma_2$ by III.5.6.

iii) By III.5.6(iii) we have $\kappa = \gamma_1$. Therefore $\kappa = \gamma_1 + \gamma_2 - j$. Hence, by III.5.7, it follows that Y is a $\mathcal{E}^{(\pi_1^{\gamma},\pi_2^{\gamma};1,1)}$-torsor. By II.3.33 it follows that $\gamma_1 \geq p\gamma_2$. And the degeneration type is $(\gamma_2, \gamma_1, \gamma_2, \gamma_1)$. Now if $\gamma_2 = 0$ then $j = 0$ and we have the last sentence.

iv) By III.5.13 it follows that a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $Y \longrightarrow X$ has

$$(v(\lambda_{(2)}), v(\lambda_{(1)}), v(\lambda_{(1)}), v(\lambda_{(1)}))$$

as degeneration type. Now let us suppose $\gamma_2 = v(\lambda_{(1)})$. Since $\kappa \geq \gamma_2$ then $\kappa = v(\lambda_{(1)})$. Therefore the effective model for Y is $\mathbb{Z}/p^2\mathbb{Z}$, since it is a model of $\mathbb{Z}/p^2\mathbb{Z}$ which is an extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ (see II.3.49). Let σ be a generator of $\mathbb{Z}/p^2\mathbb{Z}$. Since $\gamma_2 = v(\lambda_{(1)})$ then, by III.4.2, $Y \longrightarrow Y_1$ is a $<\sigma^p>$-torsor. In particular $<\sigma^p>$ has no inertia at the generic point of the special fiber. This implies that $\mathbb{Z}/p^2\mathbb{Z} =<\sigma>$ has no inertia at the generic point of the special fiber, too. Let us now consider the action of $<\sigma>/<\sigma^p>$ on $Y_1 = Y/<\sigma^p>$. If $\sigma_{|(Y_1)_k} = \mathrm{id}$ then we will have the following commutative diagram

$$\begin{array}{ccc} Y_k & \xrightarrow{\sigma} & Y_k \\ & \searrow \quad \swarrow & \\ & (Y_k)/<\sigma^p> & \end{array}$$

This is a contradiction, since $\sigma_{|Y_k} \neq \mathrm{id}$. So $<\sigma>/<\sigma^p>$ has no inertia at the generic fiber therefore $Y_1 \longrightarrow X$ is a $\mathbb{Z}/p\mathbb{Z}$-torsor, by III.4.2. Hence $\gamma_1 = \kappa = \lambda_{(1)}$. Which implies, by III.5.12, that $Y \longrightarrow X$ is a $\mathbb{Z}/p^2\mathbb{Z}$-torsor.

v) Since $\gamma_1 \leq \kappa \leq v(\lambda_{(1)})$, if $\gamma_1 = v(\lambda_{(1)})$ then $\kappa = \gamma_1$. So iii) of the definition of admissible degeneration type gives
$$j = v(\lambda_{(2)})$$
if $\gamma_1 < p\gamma_2$ (i.e. $\gamma_2 > v(\lambda_{(2)})$), and
$$j = \gamma_2$$
if $\gamma_1 \geq p\gamma_2$ (i.e. $\gamma_2 \leq v(\lambda_{(2)})$).

□

REMARK III.6.6. Let us suppose that $p|v(p)$. Then, for instance,
$$(j, \frac{v(p)}{p} + j, v(\lambda_{(1)}), v(\lambda_{(1)}))$$
is admissible for $0 \leq j \leq v(\lambda_{(2)})$ but is not realizable, if $j \neq v(\lambda_{(2)})$, by the point iv) of the proposition.

We have so seen that in general not all the degeneration types are realizable. But we now see that it is true for admissible degeneration types with $\kappa = \gamma_1$. They are degeneration types attached to $(\mathbb{Z}/p^2\mathbb{Z})_K$-torsors which are strongly extendible.

THEOREM III.6.7. *Any admissible degeneration type $(j, \gamma_1, \gamma_2, \kappa)$ with $\kappa = \gamma_1$ is realizable.*

PROOF. We recall that in this case to be an admissible degeneration type means

i) $\gamma_1 \leq v(\lambda_{(1)})$;
ii) $\gamma_2 \leq pj \leq p\gamma_2$;
iii) if $\gamma_1 < p\gamma_2$ then $\gamma_1 - j = v(\lambda_{(1)}) - v(\lambda_{(2)}) = \frac{v(p)}{p}$; if $\gamma_1 \geq p\gamma_2$ then $p(\gamma_2 - j) \leq (p-1)(v(\lambda_{(1)}) - \gamma_1)$;
iv) $pj \leq \gamma_1$;

We remark that (iv) is in fact implied by the others. Indeed let us suppose that $pj > \gamma_1$. Then by (ii) we have $p\gamma_2 \geq pj > \gamma_1$. But we know by III.6.2 that if $p\gamma_2 > \gamma_1$ then $pj \geq \gamma_1$.

Since $\kappa = \gamma_1$, it follows, by III.5.12, that if $(j, \gamma_1, \gamma_2, \kappa)$ is realizable it is the degeneration type of a $\mathcal{E}^{(\gamma_1,\gamma_2;E_p(\alpha_{\gamma_1}S),1)}$-torsor, with $v(\alpha_{\gamma_1}) = j$ if $\alpha_{\gamma_1} \neq 0$. For any γ_1, γ_2 as in the degeneration type, by II.3.53 and II.3.46, there exists a group scheme $\mathcal{E}^{(\pi^{\gamma_1},\pi^{\gamma_2};E_p(aS),1)}$. If $a \neq 0$ then we can choose a such that $v(\tilde{a}) = j$, if $\tilde{a} \in R$ is a lifting of a. In fact if $\gamma_1 < p\gamma_2$ it is automatic, by II.3.47 and iii), that $v(\tilde{a}) = j$. We call $a = \alpha_{\gamma_1}$.

We now construct a normal $\mathcal{E}^{(\gamma_1,\gamma_2;E_p(\alpha_{\gamma_1}S),1)}$-torsor. First of all we remark that if $\gamma_1 = 0$ then $\gamma_2 = 0$ then $\mathcal{E}^{(\gamma_1,\gamma_2;E_p(\alpha_{\gamma_1}S),1)} \simeq \mu_{p^2}$. So if we take $Y = \mathrm{Spec}(A[T]/(T^{p^2} - f)$ with f not a p-power $\mod \pi$ then Y_k is integral.

We now suppose $\gamma_1 > 0$. As seen in §II.4, for any $f_1, f_2 \in A$ such that $1+\pi f_1 \in A^*/(A^*)^p$ and $E_p(\alpha_{\gamma_1} f_1) + f_2 \in A^*/(A^*)^p$ we can define the $\mathcal{E}^{(\gamma_1,\gamma_2;E_p(\alpha_{\gamma_1}S),1)}$-torsor (see §II.4)

$$Y = \operatorname{Spec}\left(A[T_1,T_2]/(\frac{(1+\pi^{\gamma_1}T_1)^p-1}{\pi^{p\gamma_1}}-f_1, \frac{(E_p(\alpha_{\gamma_1}T_1)+\pi^{\gamma_2}T_2)^p(1+\pi^{\gamma_1}T_1)^{-k}-E_p(\alpha_{\gamma_1}^p f_1)}{\pi^{p\gamma_2}}-f_2)\right)$$

We have only to find f_1 and f_2 such that Y has integral special fiber. If $\gamma_1 = v(\lambda_{(1)})$ then $Y_1 \longrightarrow X$ is a nontrivial $\mathbb{Z}/p\mathbb{Z}$-torsor, so the special fiber is integral. Otherwise take f_1 such that f_1 is not a p^{th}-power $\mod \pi$. Then we have that the special fiber of

$$Y_1 = \operatorname{Spec}(B_1) = \operatorname{Spec}\left(A[T_1]/(\frac{(1+\pi^{\gamma_1}T_1)^p-1}{\pi^{p\gamma_1}}-f_1)\right)$$

is integral. We now consider

$$Y = \operatorname{Spec}\left(B_1[T_2]/(\frac{(E_p(\alpha_{\gamma_1}T_1)+\pi^{\gamma_2}T_2)^p(1+\pi^{\gamma_1}T_1)^{-1}-E_p(\alpha_{\gamma_1}f_1)}{\pi^{p\gamma_2}}-f_2)\right).$$

If $\gamma_2 = v(\lambda_{(2)})$ then $Y \longrightarrow Y_1$ is a $\mathbb{Z}/p\mathbb{Z}$-torsor so Y_k is integral. Let us suppose $\gamma_2 < v(\lambda_{(2)})$. The special fiber is

$$Y_k = \operatorname{Spec}\left((B_1)_k[T_2]/(T_2^p - \frac{E_p(\alpha_{\gamma_1}T_1)^{-p}(1+\pi^{\gamma_1}T_1)E_p(\alpha_{\gamma_1}f_1)-1}{\pi^{p\gamma_2}}-f_2)\right)$$

If $G(T_1) = \frac{E_p(\alpha_{\gamma_1}T_1)^{-p}(1+\pi^{\gamma_1}T_1)E_p(\alpha_{\gamma_1}f_1)-1}{\pi^{p\gamma_2}} - f_2$ is not a p^{th}-power then Y_k is reduced. While if $G(T_1)+f_2 \equiv G_1(T_1)^p \mod \pi$ for some $G_1(T_1) \in B_1^*$ then we substitute f_2+f_3 to f_2 with f_3 not a p^{th}-power $\mod \pi A$. Indeed, if

$$G(T_1)+f_2+f_3 \equiv G_2(T_1)^p \mod \pi B_1$$

for some $G_2(T_1) \in B_1^*$, then

$$f_3 \equiv (G_2(T_1)-G_1(T_1))^p \mod \pi B_1.$$

But by III.3.5 it follows that f_3 is a p-power $\mod \pi A$, against hypothesis on f_3.

Finally we verify that Y has $(j,\gamma_1,\gamma_2,\gamma_1)$ as degeneration type. Since $\kappa = \gamma_1$, by III.5.7 we have that the degeneration type is $(v(\alpha_{\gamma_1}),\gamma_1,\gamma_2,\gamma_1)$ if $\alpha_{\gamma_1} \neq 0$ and $(\gamma_2,\gamma_1,\gamma_2,\gamma_1)$ if $\alpha_{\gamma_1} = 0$. But since we have chosen α such that $\alpha_{\gamma_1} = 0$ and $j = \gamma_2$ or $v(\alpha_{\gamma_1}) = j$ and $\alpha_{\gamma_1} \neq 0$ then we have the thesis.

□

Bibliography

[1] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux* – Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin, 1962/1964.

[2] D. Abramovich et appendix by Jonathan Lubin – "Raynaud's group-scheme and reduction of coverings", 2003, arXiv:math/0304352.

[3] S. Anantharaman – "Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1", Sur les groupes algébriques, Soc. Math. France, Paris, 1973, p. 5–79. Bull. Soc. Math. France, Mém. 33.

[4] F. Andreatta et C. Gasbarri – "The canonical subgroup for families of abelian varieties", *Compos. Math.* **143** (2007), no. 3, p. 566–602.

[5] ———, "Torsors under some group schemes of order p^n", *J. Algebra* **318** (2007), no. 2, p. 1057–1067.

[6] M. Antei – "Extension de torseurs", preprint, 2007.

[7] M. F. Atiyah et I. G. Macdonald – *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[8] J. Bertin – "Obstructions locales au relèvement de revêtements galoisiens de courbes lisses", *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 1, p. 55–58.

[9] J. Bertin et A. Mézard – "Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques", *Invent. Math.* **141** (2000), p. 195–238.

[10] S. Bloch et K. Kato – "p-adic étale cohomology", *Inst. Hautes Études Sci. Publ. Math.* (1986), no. 63.

[11] S. Bosch, W. Lütkebohmert et M. Raynaud – *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[12] I. I. Bouw et S. Wewers – "The local lifting problem for dihedral groups", *Duke Math. J.* **134** (2006), no. 3, p. 421–452.

[13] A. Chambert-Loir – "Cohomologie cristalline: un survol", *Exposition. Math.* **16** (1998), no. 4, p. 333–382.

[14] T. Chinburg, R. Guralnick et D. Harbater – "Oort groups and lifting problems", 2007, arXiv:math/0709.0284.

[15] M. Demazure et P. Gabriel – *Groupes algébriques*, vol. I, North-Holland Publishing company, 1970.

[16] H. Epp – "Eliminating wild ramification", *Inventiones Mathematicae* **19** (1973), p. 235–249.

[17] C. Gasbarri – "Heights of vector bundles and the fundamental group scheme of a curve", *Duke Mathematical Journal* **117** (2003), no. 2, p. 287–311.

[18] B. Green – "Automorphisms of formal power series rings over a valuation ring", Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun., vol. 33, Amer. Math. Soc., Providence, RI, 2003, p. 79–87.

[19] ———, "Realizing deformations of curves using Lubin-Tate formal groups", *Israel J. Math.* **139** (2004), p. 139–148.

[20] B. Green et M. Matignon – "Liftings of Galois covers of smooth curves", *Compositio Math.* **113** (1998), no. 3, p. 237–272.

[21] ———, "Order p automorphisms of the open disc of a p-adic field", *Journal of the American Mathematical society* **12** (1999), no. 1, p. 269–303.

[22] A. Grothendieck – "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I", *Inst. Hautes Études Sci. Publ. Math.* (1964), no. 20, p. 259.

[23] ———, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II", *Inst. Hautes Études Sci. Publ. Math.* (1965), no. 24.

[24] R. Hartshorne – *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, New York-Heidelberg, 1977.

[25] Y. Henrio – "Arbres de Hurwitz et automorphismes d'ordre p des disques et des couronnes p-adiques formels", 2000, arXiv:math/0011098.

[26] ———, "Relèvement galoisien des revêtements de courbes nodales", *Manuscripta Math.* **106** (2001), no. 2, p. 131–150.

[27] L. Illusie – "Complexe de de Rham-Witt et cohomologie cristalline", *Ann. Sci. École Norm. Sup. (4)* **12** (1979), no. 4, p. 501–661.

[28] ———, "Frobenius et dégénérescence de Hodge", Introduction à la théorie de Hodge, Panor. Synthèses, vol. 3, Soc. Math. France, 1996, p. 113–168.

[29] N. M. Katz – "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin", *Inst. Hautes Études Sci. Publ. Math.* (1970), no. 39, p. 175–232.

[30] Q. Liu – *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2002.

[31] M. Matignon – "p-groupes abéliens de type $(p,\cdots,p)$ et disques ouverts p-adiques", *Manuscripta Math.* **99** (1999), no. 1, p. 93–109.

[32] J. S. Milne – *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[33] F. Oort et J. Tate – "Group schemes of prime order", *Ann. Sci. École Norm. Sup. (4)* **3** (1970), p. 1–21.

[34] G. Pagot – "$\mathbb{F}_p$-espaces vectoriels de formes différentielles logarithmiques sur la droite projective", *J. Number Theory* **97** (2002), no. 1, p. 58–94.

[35] M. Raynaud – "Schémas en groupes de type $(p,\ldots,p)$", *Bull. Soc. Math. France* **102** (1974), p. 241–280.

[36] ———, "p-groupes et réduction semi-stable des courbes", The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, p. 179–197.

[37] ———, "Spécialisation des revêtements en caractéristique $p>0$", *Ann. Sci. École Norm. Sup. (4)* **32** (1999), no. 1, p. 87–126.

[38] L. G. Roberts – "The flat cohomology of group schemes of rank p", *American Journal of Mathematics* **95** (1973), no. 3, p. 688–702.

[39] M. Romagny – "Sur quelques aspects des champs de revêtements de courbes algébriques", Thèse, Institut Fourier, Univ. Grenoble 1, 2002.

[40] ———, "Effective model of a finite group action", 2006, arXiv:math/0601639.

[41] M. Saïdi – "Torsors under finite and flat group schemes of rank p with Galois action", *Math. Z.* **245** (2003), no. 4, p. 695–710.

[42] ———, "On the existence of a torsor structure for Galois covers", 2004, arXiv:math/0403389.

[43] ———, "On the degeneration of étale $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$-torsors in equal characteristic $p>0$", *Hiroshima Math. J.* **37** (2007), no. 2, p. 315–341.

[44] T. Sekiguchi – "On the unification of Kummer and Artin-Schreier-Witt theories", (2001).

[45] T. Sekiguchi, F. Oort et N. Suwa – "On the deformation of Artin-Schreier to Kummer", *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 3, p. 345–375.

[46] T. Sekiguchi et N. Suwa – "Some cases of extensions of group schemes over a discrete valuation ring. II", *Bull. Fac. Sci. Engrg. Chuo Univ. Ser. I Math.* **32** (1989), p. 17–35.

[47] ———, "A case of extensions of group schemes over a discrete valuation ring", *Tsukuba J. Math.* **14** (1990), no. 2, p. 459–487.
[48] ———, "On the deformations of Witt groups to tori II", *J. Algebra* **138** (1991), p. 273–297.
[49] ———, "Some cases of extensions of group schemes over a discrete valuation ring. I", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **38** (1991), no. 1, p. 1–45.
[50] ———, "Théories de Kummer-Artin-Schreier-Witt", *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), no. 2, p. 105–110.
[51] ———, "Théorie de Kummer-Artin-Schreier et applications", *J. Théor. Nombres Bordeaux* **7** (1995), no. 1, p. 177–189.
[52] ———, "On the unified Kummer-Artin-Schreier-Witt theory", *Mathématiques Pures de Bordeaux C.N.R.S* (1999), Prepublication.
[53] ———, "A note on extensions of algebraic and formal groups. IV. Kummer-Artin-Schreier-Witt theory of degree p^2", *Tohoku Math. J. (2)* **53** (2001), no. 2, p. 203–240.
[54] ———, "A note on extensions of algebraic and formal groups. V", *Japan. J. Math. (N.S.)* **29** (2003), no. 2, p. 221–284.
[55] C. S. Seshadri – "L'operation de Cartier, applications", *Secr. math* **4** (1960), exp. n°6 in Variétés de Picard (Seminarie Chevalley 1958/59).
[56] W. C. Waterhouse – "Principal homogeneous spaces and group scheme extensions", *Transactions of the American Mathematical Society* **153** (1971), p. 181–189.
[57] ———, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York, 1979.
[58] ———, "A unified Kummer-Artin-Schreier sequence", *Math. Ann.* **277** (1987), no. 3.
[59] W. C. Waterhouse et B. Weisfeiler – "One-dimensional affine group schemes", *J. Algebra* **66** (1980), no. 2, p. 550–568.