QUANTUM NOISE IN THE SPIN TRANSFER TORQUE EFFECT

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It appeared to me that
there were two paths to truth

— Georges Lemaître

ACKNOWLEDGMENTS

I want to give a special thanks to the supervisor, Roberto Raimondi (Dipartimento di Matematica e Fisica – Roma Tre), and to Marco Barbieri (Dipartimento di Scienze – Roma Tre). The thesis work has been produced thanks to their patience and crucial contributions.

Due to the deadline, I could not do a further thesis revision and I am very grateful to the referees for any suggestions they will make for the final review.

I also want to thank all the people I worked with and talked about physics during these years: the people who gave me the chance to work on the plasma physics; in particular, Brunello Tirozzi, Renato Spigler, Paolo Buratti, Alessandro Cardinali and Giovanni Montani. Those who gave me the chance to work and discuss on cellular automata; in particular (besides Marco Barbieri), Alessio Serafini, Marco Genoni, Federico Centrone and Michele Avalle. The people who gave me the chance to work at the "Dipartimento di Architettura" and those I worked with; in particular, Laura Tedeschini Lalli, Valerio Talamanca, Fabio Bruni and Giuditta Bravaccino. The people who gave me the chance to work at the "LS-OSA" project; in particular (besides Roberto Raimondi), Settimio Mobilio, Ilaria De Angelis and Marco Valli. Furthermore I want to thank the people of the "laboratorio di liquidi", my colleagues and, in particular, the "NEQO" group.

Finally a special thanks to my family and friends that have always supported me and, last but not least, to Denise.

CONTENTS

1 INTRODUCTION

The central topic of the thesis work is the quantum noise in the spin transfer torque, which is a spintronics effect.[1] In particular in figure 1.1 on the following page we can see some of the most studied spintronics applications: the giant magnetoresistance (GMR), in which we have a current in a ferromagnetic layer. Here the electrons with spin parallel to the ferromagnet magnetizazion feel a lower resistance with respect to the anti-parallel spin electrons. In the spin transfer torque effect (STT), a polarized current produces a torque in the magnetization of a ferromagnet. In the spin Hall effect (SHE), we see a spin accumulation on the lateral surfaces of an electric paramagnetic conductor, due to the spin-orbit interaction. Finally, in a paramagnetic conductor, we can see in the figure a current-induced spin polarization (CISP) [41].

In particular the GMR is the effect that allows us to read the bits in the hard drives (each bit is encoded in a ferromagnetic layer), while the STT allows to write a hard disk bit.

In a typical STT device a hard ferromagnet (its magnetizazion does not rotate) polarizes a current that exerts a torque on the magnetizazion of a second ferromagnet [51] (see figure 1.2 on the next page).

In recent years, the advances in fast time-resolved measurements have showed that the magnetization dynamics of a nanomagnet crossed by a polarized current presents a stochastic behaviour at short time interval [11, 13, 14, 59].

If we want fast and small devices the stochastic behaviour becomes important. We need to study it because:

- we do not want that the noise disturbs the device working,
- we can engineer the noise to help the magnet switching, without increasing the current (to increase the current means to increase dissipative effects and then heat up the device) [32].

One of the first heuristic equation considered for the magnetization dynamics, is the Landau-Lifshitz-Gilbert equation [21] that describes the magnetization dynamics in presence of an external magnetic field $\vec{H}$:

$$\frac{\partial \vec{M}}{\partial t} = \tilde{\gamma}\,\vec{M} \times \vec{H} - \lambda\,\vec{M} \times (\vec{M} \times \vec{H}). \quad (1.1)$$

If $\lambda = 0$ we simply have the Euler equation (see the figure 2.1a on page 10); the term proportional to λ is perpendicular to both $\vec{M}$

1 "Spintronics" is a contraction between "spin" and "electronics". In particular it studies the electronic devices in which the information is carried by both the charge and the spin of electrons.

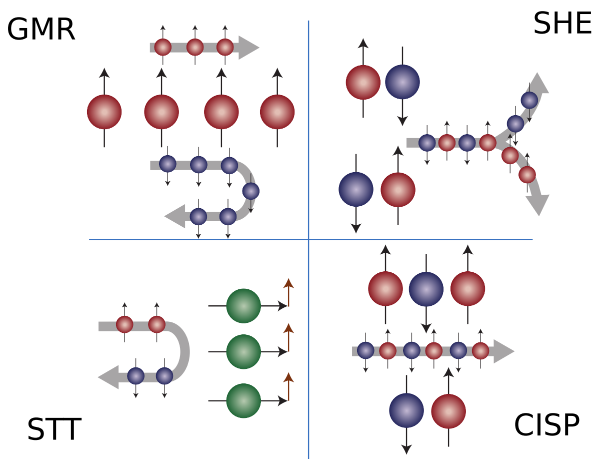


Figure 1.1: In this figure, taken from the reference [41], some of the most studied spintronics effects are condensed.

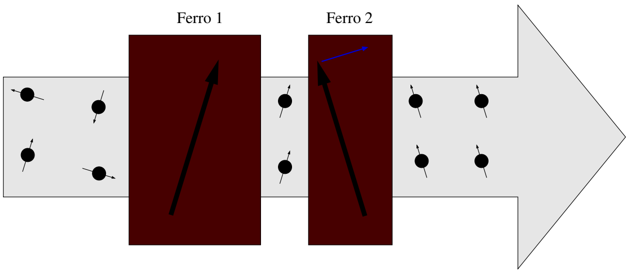


Figure 1.2: The scheme of a typical spin transfer torque device: an unpolarized current comes from left; the first ferromagnet polarizes the current. Then the polarized current exerts a torque on the second ferromagnet.

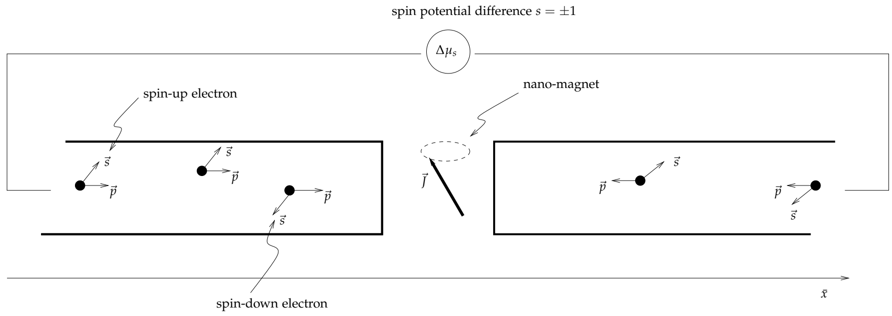


Figure 1.3: We considered a spin potential difference that induces a spin current, flowing trough the magnet.

and the precession direction and then describes the damping (the magnetization tends to align to the external magnetic field – see figure 2.1b on page 10). Finally, to take into account the thermal fluctuactions, Brown [7] introduced a stochastic magnetic field $\vec{h}(t)$ in the equation:

$$\frac{\partial \vec{M}}{\partial t} = \tilde{\gamma}\, \vec{M} \times (\vec{H} + \vec{h}(t)) - \lambda\, \vec{M} \times [\vec{M} \times (\vec{H} + \vec{h}(t))]$$

(see figure 2.1c on page 10).

Our aim in the thesis work is

- to find a microscopically derived equation that could describe the magnetization dynamics in presence of a polarized current,
- to describe the thermal and the quantum noise induced in the magnet motion by the electric current;
- to employ a formalism that is easily generalizable (electron-electron interaction, presence and interaction of several magnets, etc.).

To obtain that, we considered the setup described in figure 1.3 and we employed the Keldysh formalism.

The interaction model between the magnet and the electrons that we have considered is very simple [63]:

$$H = -\frac{\partial^2_{\bar{x}}}{2\,m} + \gamma \vec{B} \cdot \vec{J} + \delta(\bar{x})\,(\lambda_0 + \lambda\, \vec{J} \cdot \vec{s}),$$

where the Hamiltonian first term is the electron kinetic energy, $\vec{B}$ is a weak external magnetic field, $\vec{s}$ is the electron spin, $\vec{J}$ is a spin that represents the magnet degrees of freedom and the delta function fixes the magnet position at $\bar{x} = 0$. Figure 1.4 on the following page shows the potential seen by the electron.

To treat the problem with the many-body Keldysh formalism, we considered the magnet degrees of freedom as a bosonic system, thanks

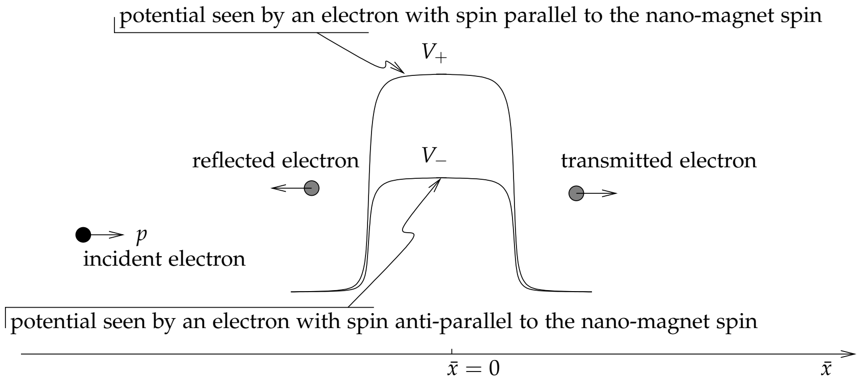
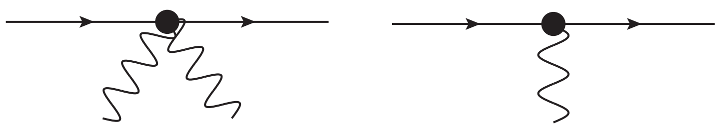


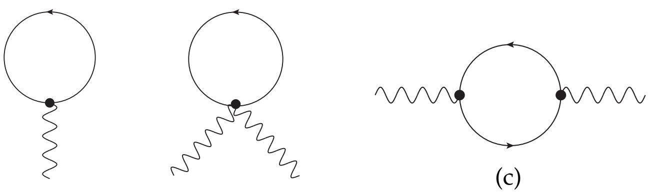
Figure 1.4: An electron that comes from left to right feels a potential $V_+ := d \cdot (\lambda_0 + J\,\lambda/2)$ and $V_- := d \cdot (\lambda_0 - J\,\lambda/2)$, depending whether its spin is parallel or anti-parallel to the magnetization, where d is the magnet length.



(a) *Two-boson vertex.* (b) *One-boson vertex.*

Figure 1.5: Feynman vertices.

to the Holstein-Primakoff bosonization (that is explained in detail in the section 5.1 on page 52). With this representation, the interaction between the magnet and the electrons can be depicted by the Feynman vertices in figure 1.5: the one-boson vertex is of the order $1/\sqrt{J}$, while the two-boson vertex is of the order $1/J$ (this is considered in the section 5.4 on page 56). For a typical nanomagnet $J \sim 10^4$ [63], so we will consider only the terms up to the $1/J$-order, that is, Feynman diagrams with a single one-boson vertex (order of magnitude $1/\sqrt{J}$) and Feynman diagrams with a couple of one-boson vertices or with a single two-boson vertex (order of magnitude $1/J$); see figures 1.6.



(a) (b) (c)

Figure 1.6: Feynman diagrams up to the $1/J$ order. The "tadpole" diagram is $1/\sqrt{J}$-order, while the other two are $1/J$-order.

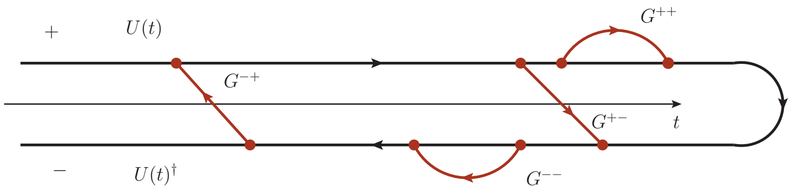


Figure 1.7: The Keldysh time contour with the propagators.

We then applied the Keldysh formalism, that allowed us to write the observables in term of functional integral:

$$\begin{aligned}\langle\hat{O}(t)\rangle &= \mathrm{tr}\left[\hat{U}^\dagger_{t,t_0}\,\hat{O}\,\hat{U}_{t,t_0}\,\hat{\rho}(t_0)\right] = \\ &= \int D[\bar{b}_+,\bar{b}_-,\bar{\psi}_+,\bar{\psi}_-,b_+,b_-,\psi_+,\psi_-]\cdot \\ &\quad \cdot O(\bar{b}_+,\bar{b}_-,\bar{\psi}_+,\bar{\psi}_-,b_+,b_-,\psi_+,\psi_-)\,e^{i\,S(\bar{b}_+,\bar{b}_-,\bar{\psi}_+,\bar{\psi}_-,b_+,b_-,\psi_+,\psi_-)},\end{aligned} \tag{1.2}$$

where b (complex numbers) refers to the magnet degrees of freedom and ψ (Grassmann numbers) to the electronic ones; S is the Keldysh action. $\hat{U}^\dagger_{t,t_0} = \hat{U}_{t_0,t}$ and then in the Keldysh formalism we have both a forward and backward in time path, with four propagators (see 1.7). In particular the indices $\pm$ for the paths $t \to b_\pm(t)$ and $t \to \psi_\pm(t)$ refer to the time direction. The Keldysh rotation reduces the propagators number from four (not independent) to three; for the bosons the Keldysh rotation is given by

$$b^{\mathrm{cl}} = \frac{b^+ + b^-}{\sqrt{2}}, \qquad b^{\mathrm{q}} = \frac{b^+ - b^-}{\sqrt{2}}, \tag{1.3a}$$

$$\bar{b}^{\mathrm{cl}} = \frac{\bar{b}^+ + \bar{b}^-}{\sqrt{2}}, \qquad \bar{b}^{\mathrm{q}} = \frac{\bar{b}^+ - \bar{b}^-}{\sqrt{2}}; \tag{1.3b}$$

note that, like in the Feynman path integral case, in the classical limit we have a single path (for both forward and backward time direction) and then $b^{\mathrm{q}} = 0$ (that justifies the name *quantum* and *classical part* for b^{q} and b^{cl} respectively); anyway the formalism will be discussed in the section 4.2 on page 33.

To obtain the magnet equation of motion, we traced over the electronic degrees of freedom, getting terms that can be represented by the Feynman diagrams in figures 1.6 on the preceding page. In this way, we found a functional integral expression for the magnet observables of the form:

$$\begin{aligned}\langle\hat{O}(t)\rangle &= \mathrm{tr}\left[\hat{U}^\dagger_{t,t_0}\,\hat{O}\,\hat{U}_{t,t_0}\,\hat{\rho}(t_0)\right] = \int D[I_1,I_2]\,e^{-\int dt\,\frac{I_1^2(t)+I_2^2(t)}{2}}\cdot \\ &\quad\cdot\int D[\bar{b}^{\mathrm{cl}},\bar{b}^{\mathrm{q}},b^{\mathrm{cl}},b^{\mathrm{q}}]\,O(\bar{b}^{\mathrm{cl}},b^{\mathrm{cl}})\,e^{i\left\{\int dt\,\bar{b}^{\mathrm{q}}\left[i\,\partial_t b^{\mathrm{cl}}+f(b^{\mathrm{cl}},\theta,I_1,I_2)\right]+\mathrm{h.c.}\right\}},\end{aligned} \tag{1.4}$$

where θ is the angle between the current polarization axis and the magnetization direction $\vec{J}$; we will discuss in the thesis the explicit form of the function f, that is the sum of different contributions: in particular, in the chapter 8 on page 69 we will calculate explicitly the contribution corresponding to the "tadpole" diagram in figure 1.6 on page 4, while in the chapter 9 on page 73 we will consider the other diagrams terms. I_1, I_2 are two auxiliary time function that allows us to linearize the Keldysh action with respect to b.[2] By performing the integration of 1.5 with respect to the $b^{q}, \bar{b}^{q}$, as described in the section 8.3 on page 71, we get:

$$\langle \hat{O}(t) \rangle = \int D[I_1, I_2]\, e^{-\int dt \, \frac{I_1^2(t)+I_2^2(t)}{2}} \int D[\bar{b}^{\mathrm{cl}}, b^{\mathrm{cl}}]\, O(\bar{b}^{\mathrm{cl}}, b^{\mathrm{cl}}) \cdot \\ \cdot \delta \left[i\, \partial_t b^{\mathrm{cl}} + f(b^{\mathrm{cl}}, \theta, I_1, I_2) \right] \delta \left[-i\, \partial_t \bar{b}^{\mathrm{cl}} + \bar{f}(b^{\mathrm{cl}}, \theta, I_1, I_2) \right], \quad (1.5)$$

where δ is the Dirac function. This means that $i\,\partial_t b^{\mathrm{cl}} + f(b^{\mathrm{cl}}, \theta, I_1, I_2) = 0$ and the complex conjugate are the equations of the motion. $t \mapsto I_i(t)$ is a generic function that in the functional integral is weighed by the factor $e^{\int -\frac{I_1^2(t)+I_2^2(t)}{2} dt}$. This is the same situation of the Martin-Siggia-Rose action: the weight is the multivariate Gaussian distribution probability, with zero mean value and unitary variance:

$$\langle I_i(t) \rangle = 0, \qquad \langle I_i(t_1)\, I_j(t_2) \rangle = \delta(t_1 - t_2)\, \delta_{ij}, \quad (1.6)$$

that is, I_i must be considered as Langevin terms in the equation of motion (see for example the discussion in the section 2.2 on page 12).

The stochastic terms presence is not surprising, since it is the typical situation of the open quantum systems: as we will summarize in the section 3.2 on page 19, when some degrees of freedom are traced over, a stochastic behaviour appears.[3]

Finally the equation of motion for the magnet, in terms of microscopical quantities, is:

$$\partial_t \vec{J} = \gamma\, \vec{B} \times \vec{J} + \left(\Re C_1 + \frac{-\cos\theta\, \Im C_2\, I_1 + \Re C_2\, I_2}{\sin\theta} \right) \hat{z}' \times \vec{J} + \\ + \left(\frac{\Im C_1}{J} + \frac{\cos\theta\, \Re C_2\, I_1 + \Im C_2\, I_2}{\sin\theta\, J} \right) \vec{J} \times (\hat{z}' \times \vec{J}) \quad (1.7)$$

where

- C_i depend on the scattering matrix and increase with the spin potential difference $\Delta\mu_s$. $\hat{z}'$ is the current polarization axis. $\hat{z}' \times \vec{J}$

2 that is done thanks to the Hubbard–Stratonovich transformation:

$$e^{-\frac{a}{2} x^2} = \sqrt{\frac{1}{2\pi a}} \int dI\, e^{-\frac{I^2}{2a} - i\,x\,I},$$

as described in the chapter 9 on page 73.

3 the same formalism can explain, for example, the decoherence, where the quantum state collapses randomly in an eigen-energy state.

and $\vec{J} \times (\hat{z}' \times \vec{J})$ are respectively a field-like and a damping-like term (compare with Landau-Lifshitz-Gilbert equation 1.1 on page 1), that produce respectively a precession around the polarization current direction $\hat{z}'$ and an alignment to it;

- C_1 is the contribution of tadpole diagram, while C_2 corresponds to the higher order corrections with respect to $1/\sqrt{J}$ (and then disappears in the macroscopic limit $J \to +\infty$). C_2 contributes also at the zero temperature (both quantum and thermal noise).

Comparing our results with the simpler model in [53] we obtained both the field-like and the damping-like terms and a more complex expression for the noise (note that the field-like and damping-like coefficients are not independent).

The thesis is organized as follow: in the first part the fundamental concepts and techniques are introduced. In particular, in the chapter 2 on page 9 the earlier phenomenological theories for the ferromagnet magnetization dynamics are presented. In the chapter 3 on page 17 some microscopic theory that describes the interaction between ferromagnetic layers and polarized currents are described. Finally in the chapter 4 on page 29 the concepts of the Keldysh technique that we need in the following are synthetized.

The second part contains the most original results of the thesis work: in the chapter 5 on page 51 it is described the many-body model that we have considered. In chapter 6 on page 59 we study the relations between the spin potential difference and the spin current. In the chapter 7 on page 63 the Keldysh action is calculated. In the chapter 8 on page 69 the terms that give rise to the classical equation of motion for the magnetization are considered, while in the chapter 9 on page 73 the quantum corrections are evaluated; from that we obtain in particular the quantum noise. Finally in the chapter 10 on page 89 we draw a possible interesting extension of the model, while the appendix A on page 93 describes some terms of the Keldysh action that are typically suppressed, but that give rise to physical interesting interpretations.

2 PRECESSION PHENOMENOLOGICAL THEORIES

In this chapter we consider the earlier phenomenological theories proposed to describe the magnetization precession in a solid. In particular, we first consider the Landau–Lifshitz–Gilbert equation proposed in 1955 by Gilbert (see e. g. the reprinted article [21]), which modifies a previous equation proposed by Landau and Lifshitz in 1935 [31]. Finally we consider the thermal fluctuactions for the Landau–Lifshitz–Gilbert equation introduced by Brown [7].

2.1 LANDAU–LIFSHITZ–GILBERT EQUATION

In a ferromagnetic material the magnetization is mainly due to the spin of the electrons (one can take into account the contribution of the electrons orbital motion by simply adjusting the value of the gyromagnetic ratio). Below the Curie temperature, the material is divided into elementary domains that are magnetized near the saturation. Then we may divide the material into n cells that are large enough to avoid to consider the microscopic fluctuactions but small enough to take into account the domain structures (that is possible, because a Weiss domain is typically composed by $10^{12} - 10^{15}$ atoms). To the i-th cell, a local magnetization field $\vec{M}_i$ is associated.

It is possible to assume for the magnetization the equation of motion:

$$\frac{d\vec{M}_i}{dt} = \gamma\,\vec{M}_i \times \vec{H}_i \tag{2.1}$$

where

$$\gamma = \frac{-|e|}{2\,m_e}\,g, \qquad g \simeq 2 \tag{2.2}$$

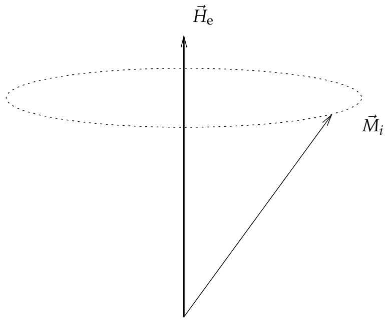
is the gyromagnetic ratio and $\vec{H}_i$ is an effective field acting on the i-th moment:

$$\vec{H}_i = -\frac{\partial U}{\partial \vec{M}_i}(\vec{M}_1, \ldots, \vec{M}_n)$$

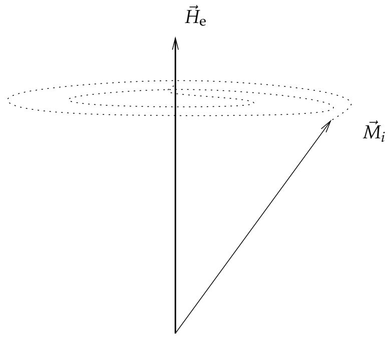
(the derivative is intended by components; for example $(\vec{H}_x)_i = -\partial_{(\vec{M}_x)_i} U$).[1]

The form of the potential U is established experimentally and for a typical ferromagnet contains five terms: the external magnetic field, the demagnetization energy (that is a self-interaction term), the exchange interaction energy (associated with the gradient in the orientation

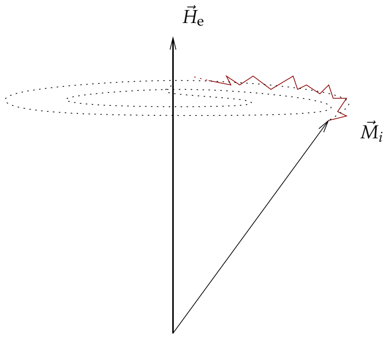
1 With the definition 2.2 of γ, $\vec{H}$ is measured in Tesla in the SI.



(a) Precession motion.



(b) Damping effect.



(c) Thermal fluctuactions.

Figure 2.1: The motion of the magnetization $\vec{M}_i$ around the external magnetic field $\vec{H}_e$.

of the magnetization), the anisotropy energy (the potential energy depends on the magnetization orientation with respect to the crystal axes), the magnetoelastic energy (deformation effects). In [21] the explicit form of each term is described; for example the external field term is:

$$U = -\sum_i \vec{M}_i \cdot \vec{H}_e,$$

where $\vec{H}_e$ is the external magnetic field. In particular it is easy to check that, in this case, equation 2.1 on the preceding page reduces to

$$\frac{d\vec{M}_i}{dt} = \gamma \vec{M}_i \times \vec{H}_e,$$

that is a simple precession motion, like in figure 2.1a. It is convenient to consider the continuous limit:

$$\frac{\vec{M}_i}{\Delta\vec{r}} \to \vec{M}(\vec{r}), \qquad \frac{U(\vec{M}_1, \ldots, \vec{M}_n)}{\Delta\vec{r}} \to U[\vec{M}(\vec{r})], \qquad \vec{H}_i \to \vec{H}(\vec{r}),$$

where $\vec{r}$ is the position in the ferromagnet, $\Delta\vec{r}$ is the infinitesimal volume and U becomes the density energy functional.

We argue that the equation of motion 2.1 on page 9 can be written in the Lagrangian form, as proposed by Gilbert:

$$\frac{d}{dt}\frac{\delta\mathcal{L}[\vec{M},\dot{\vec{M}}]}{\delta\dot{\vec{M}}} = \frac{\delta\mathcal{L}[\vec{M},\dot{\vec{M}}]}{\delta\vec{M}}, \qquad \mathcal{L}[\vec{M},\dot{\vec{M}}] = \mathcal{T}[\vec{M},\dot{\vec{M}}] - U[\vec{M}];$$

Gilbert did not fix the form of the kinetic energy $\mathcal{T}$, because he says he was not able to find an expression for $\mathcal{T}$ that would correspond to the spin of an elementary particle in quantum mechanics that made physical sense [21].[2]

If, for example, we consider the case of a fixed external magnetic field, we know from experience that the motion of $\vec{M}_i$ is not simply a precession but, after a while, $\vec{M}_i$ will be oriented along $\vec{H}_e$; that is, we have some dissipative effects that produce a damping (see figure 2.1b on the preceding page). Gilbert introduced the damping effect (for the general case $\vec{H}_i$) by adding a new term to the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\delta\mathcal{L}[\vec{M},\dot{\vec{M}}]}{\delta\dot{\vec{M}}} - \frac{\delta\mathcal{L}[\vec{M},\dot{\vec{M}}]}{\delta\vec{M}} + \frac{\delta\mathcal{R}[\dot{\vec{M}}]}{\delta\dot{\vec{M}}} = 0, \quad \mathcal{R} := \frac{\eta}{2}\int d\vec{r}\,\dot{\vec{M}}\cdot\dot{\vec{M}}$$

in complete analogy with the motion in a viscous fluid ($\mathcal{R}$ is the Rayleigh dissipation functional). Then the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\delta\mathcal{T}[\vec{M},\dot{\vec{M}}]}{\delta\dot{\vec{M}}} - \frac{\delta\mathcal{T}[\vec{M},\dot{\vec{M}}]}{\delta\vec{M}} + \left[-\vec{H} + \eta\,\dot{\vec{M}}\right] = 0 \tag{2.3a}$$

$$\Longrightarrow \qquad \frac{\partial\vec{M}}{\partial t} = \gamma\,\vec{M}\times\left[\vec{H} - \eta\,\frac{\partial\vec{M}}{\partial t}\right] \tag{2.3b}$$

(indeed note that, even without specifying the form of $\mathcal{T}$, comparing with equation 2.1 on page 9, we simply have to substitute $\vec{H}$ with $\vec{H} - \eta\,\partial_t\vec{M}$) that is the Landau-Lifshitz-Gilbert equation.

The Landau-Lifshitz-Gilbert can be put in the original Landau-Lifshitz form [31] simply by redefining the γ coefficient: indeed, by considering the cross product between $\vec{M}$ and the Landau-Lifshitz-Gilbert equation and using the relations $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})$ and $\partial_t\vec{M}\cdot\vec{M} = 0$, one obtains immediately an expression for $\vec{M}\times\partial_t\vec{M}$ in terms of $\vec{M}$ and $\vec{H}$. Substituting this expression in the Landau-Lifshitz-Gilbert equation one gets:

$$\frac{\partial\vec{M}}{\partial t} = \gamma'\,\vec{M}\times\vec{H} - \lambda\,\vec{M}\times(\vec{M}\times\vec{H}), \tag{2.4a}$$

$$\gamma' := \frac{\gamma}{1+\gamma^2\,\eta^2\,|\vec{M}|^2}, \qquad \lambda := \frac{\gamma^2\,\eta}{1+\gamma^2\,\eta^2\,|\vec{M}|^2}, \tag{2.4b}$$

where the $\gamma'\,\vec{M}\times\vec{H}$ *field* term is joined by the $-\lambda\,\vec{M}\times(\vec{M}\times\vec{H})$ damping term, which is orthogonal to both $\hat{M}$ and the field term (and then produce the damping effect of figure 2.1b on the preceding page).

2 in the next chapters we will find a Lagrangian for a magnet $\vec{M}_i$ that is directly obtained from a quantum microscopical derivation.

2.2 THERMAL FLUCTUACTIONS

Now we want to introduce the thermal fluctuactions in the $\vec{M}$ dynamics, by following the reference [7]. To do that, a couple of words on the stochastic processes are useful. In particular, the one-dimensional Brownian motion of a grain in a fluid can be modelled by:

$$\dot{v} = -\beta v + A(t) \tag{2.5}$$

where v is the grain velocity. The interaction between the grain and the fluid is given by two terms: the dynamical friction $-\beta v$ and a stochastic force per unit mass $A(t)$, such that

- $\langle A(t) \rangle = 0$;
- it is assumed that the evolution governed by the deterministic component, $\dot{v} = -\beta v$, is much slower than the evolution given by the stochastic term. This means that there exists a time interval dt for which v is practically constant in time only considering the deterministic evolution, while, in dt, a rapid variation of A occurs. Typically dt is small with respect to the resolution time and it is a good interval to discretize the time. A is due to the fluid molecules collisions,[3] which are pratically independent, if the scattering sections between molecules are small enough. Then we may consider $A(t)$ and $A(t + n\,dt)$ ($n \in \mathbb{N}$) independent random variables:

  $$\langle A(t_1)\, A(t_2) \rangle = \mu\, \delta(t_1 - t_2);$$

- $A(t)$ are Gaussian random variables. There are some good reasons to assume that: the central limit theorem, since the big number of collisions in the time dt, and the fact that, under this assumption, $v(t = +\infty)$ is Maxwellian distributed [10].

It follows that

$$W_{\Delta t} := \int_t^{t+\Delta t} A(t)\, dt,$$

is a Gaussian process (the superposition of independent Gaussian random variables is a Gaussian random variable) with

$$\langle W_{\Delta t} \rangle = 0, \qquad \langle W_{\Delta t}^2 \rangle = \mu\, \Delta t$$

and independent increments; that is a Wiener process. As known, the Wiener process does not have differentiable realizations and the meaning of the equation 2.5 is simply:

$$dv = -\beta v\, dt + W_{dt}, \tag{2.6}$$

3 each collision produces a small variation of v; typically there are few collisions between the grain and the molecules every femtosecond.

where

$$W_{\Delta t} = \int_t^{t+\Delta t} A(t)\, dt := \sum_i (W_{t_{i+1}} - W_{t_i})$$

is the Itô integral.

From equation 2.6 on the preceding page we get immediately

$$\langle dv \rangle = -\beta\, v\, dt, \tag{2.7a}$$

$$\langle dv^2 \rangle = \mu\, dt, \tag{2.7b}$$

$$\langle dv^n \rangle = O(dt^2), \qquad n > 2, \tag{2.7c}$$

where v is the velocity at t time, that follows immediately from the moments of Gaussian distribution values. The process is clearly Markovian, so we can write the Chapman-Kolmogorov equation:

$$p(v, t + dt) = \int dv'\, p(v', t)\, p_{dt}(v|v')$$

where $p(v,t)$ is the probability distribution that the grain velocity is v at time t and $p_{dt}(v|v')$ is the conditional probability distribution to find the velocity equal to v at time $t + dt$, if it was v' at time t. In particular:

$$\begin{aligned} p_{dt}(v|v') &= \langle \delta(v - dv - v') \rangle = \\ &= \left[1 + \langle dv \rangle \frac{d}{dv'} + \frac{1}{2} \langle dv^2 \rangle \frac{d^2}{d(v')^2} + O(\langle dv^3 \rangle)\right] \delta(v - v') = \\ &= \delta(v - v') - \beta\, v' \frac{d\delta(v - v')}{dv'}\, dt + \frac{1}{2}\, \mu \frac{d^2\delta(v - v')}{d(v')^2}\, dt + O(dt^2), \end{aligned}$$

where relations 2.7 have been used. By inserting in the Chapman-Kolmogorov equation, dividing by dt and taking the limit $dt \to 0$, one gets:

$$\frac{\partial p}{\partial t} = \beta \frac{\partial (p\, v)}{\partial v} + \frac{1}{2}\, \mu \frac{\partial^2 p}{\partial v^2},$$

that is the Fokker-Planck equation.[4]

More in general, if we have a stochastic equation of the form

$$d\boldsymbol{X}_t = \boldsymbol{\mu}(\boldsymbol{X}_t, t)\, dt + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\, d\boldsymbol{W}_t, \tag{2.9}$$

where $\boldsymbol{X}_t$ is an N-dimensional column vector of unknown functions, $\boldsymbol{W}_t$ is an M-dimensional column vector of independent standard

4 The generalization to the three-dimensional case is immediate:

$$\frac{\partial p}{\partial t} = \beta\, \nabla_v \cdot (p\, \vec{v}) + \frac{1}{2}\, \mu\, \nabla_v^2 p. \tag{2.8}$$

Wiener processes, $\boldsymbol{\mu}$ is called *drift vector*, $\boldsymbol{\sigma}$ is an $N \times M$-dimensional matrix and

$$\boldsymbol{D} := \frac{1}{2} \boldsymbol{\sigma} \boldsymbol{\sigma}^t$$

is called *diffusion tensor*, we have that equation 2.9 on the previous page is equivalent to the probability density equation (Fokker-Planck equation):

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial t} = - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\mu_i(\boldsymbol{x}, t)\, p(\boldsymbol{x}, t)\right] + \\ + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i\, \partial x_j} \left[D_{ij}(\boldsymbol{x}, t)\, p(\boldsymbol{x}, t)\right]. \quad (2.10)$$

Brown (nomen omen) [7] introduced the thermal fluctuactions in the Landau-Lifshitz-Gilbert equation by assuming that the interaction between the magnet $\vec{M}$ and the thermal bath is modelled by adding a stochastic field $\vec{h}(t)$:

$$\frac{\partial \vec{M}}{\partial t} = \gamma\, \vec{M} \times \left[\vec{H} + \vec{h}(t) - \eta\, \frac{\partial \vec{M}}{\partial t} \right],$$

with the same statistical properties of the force $A(t)$ in the equation 2.5 on page 12 (see figure 2.1c on page 10). In particular

$$\langle h_i(t) \rangle = 0, \qquad \langle h_i(t_1)\, h_j(t_2) \rangle = \mu_{ij}\, \delta(t_1 - t_2)$$

and Brown principally concentrated on the isotropic case $\mu_{ij} = \mu\, \delta_{ij}$.

As we will see in the next chapters, starting from a microscopical derivation, the interaction between the magnet and an electronic current at temperature T produces terms that are analogous to that introduced phenomenologically by Brown in the equation of motion for $\vec{M}$.

By applying the same steps to get the Brownian Fokker-Planck equation (here the only complication is that $\vec{M}$ moves on the sphere of radius $M_s := |\vec{M}|$), Brown obtained:

$$\frac{\partial W}{\partial t} = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left\{ \sin\theta \left[\left(h' \frac{\partial U}{\partial \theta} - g' \frac{1}{\sin\theta} \frac{\partial U}{\partial \phi} \right) W + k' \frac{\partial W}{\partial \theta} \right] \right\} + \\ + \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \left[\left(g' \frac{\partial U}{\partial \theta} + h' \frac{1}{\sin\theta} \frac{\partial U}{\partial \phi} \right) W + k' \frac{1}{\sin\theta} \frac{\partial W}{\partial \phi} \right],$$

where

$$h' = \frac{\eta}{1/\gamma^2 + \eta^2 M_s^2}, \qquad g' = \frac{1/\gamma}{M_s\, (1/\gamma^2 + \eta^2 M_s^2)}$$

and

$$W(\theta, \phi)\, d\Omega = W(\theta, \phi)\, \sin\theta\, d\theta\, d\phi = p(\theta, \phi)\, d\theta\, d\phi,$$

with p probability distribution to find $\vec{M}$ pointing in the direction (θ, ϕ) (azimuthal and polar angle of the spherical coordinate system).

We may expect that the variance μ increases with temperature; to find the relation between μ and T, Brown observed that the canonical statistical distribution

$$W_0 = A_0 \, e^{-U(\theta,\phi) \, v/(k_B \, T)}$$

(v is the volume of the magnet, since in our definition U is the volume density energy) satisfies the equilibrium ($\partial_t W = 0$) Fokker-Planck equation and gives:

$$\mu = 2 \, k_B \, T \, \eta / v.$$

Until now we have considered the phenomenological models introduced to describe the magnetization in a ferromagnet. In the next chapter we will consider the attempts to finds microscopical derivations of the magnet equation of motion in the form of the Landau-Lifshitz-Gilbert and also the attempts to consider other phenomena, like in the Slonczewski-Berger theory that takes into account the interaction between ferromagnets and electric currents.

3 MICROSCOPIC DERIVATIONS OF THE DYNAMICS

Now we are going to consider the microscopic theory that describes the interaction between ferromagnetic layers and polarized currents in the section 3.1 and, after the section 3.2 on page 19 devoted to the introduction to the open quantum systems (that is useful for the next chapters) we will consider two microscopic models (introduced in [62] and [53]) that can describe many aspects of the ferromagnet magnetization dynamics.

3.1 THE SLONCZEWSKI-BERGER TERM

On 1996, Slonczewski [51] and Berger [3] independently considered how a ferromagnetic layer can polarize an electric current and how a polarized electric current can induce a magnetization rotation on a ferromagnetic layer. This is called spin transfer torque effect and has a large number of technical applications, especially in electronic memory devices (in particular, combined with the giant magnetoresistance effect, has been the leading actor in the rapid storage capacity increase of hard disk drivers in recent years).

In particular Slonczewski considered an electrical conductor composed by 5 layers: A, B, C that are paramagnetic layers and F1, F2 ferromagnetic layers crossed by a current along ξ (see fig. 3.1 on the following page). The characteristic thickness of the layers is of the order of the nanometer, so that the interlayer exchange coupling can be neglected, but the spin relaxation length is large enough to consider the electronic current as ballistic.

The ferromagnetic layers are seen by electrons as classical potentials with two different values depending on the spin electron orientation (parallel or anti-parallel with respect to the ferromagnet magnetization).

Slonczewski solved the electron scattering problems in the two regions $\xi < 0$ and $\xi > 0$ separately and then imposed the matching conditions. To explain better, we can for example consider an electron incident from B onto F2, that in $\xi = 0$ has the spin oriented along $\vec{M}_1$ (due to the interaction with F1). If $\vec{M}_2 = \vec{M}_2(t)$ is the F2 magnetization vector, one can consider the moving reference frame $\hat{x}\,\hat{y}\,\hat{z}$ with $\hat{z} \parallel \vec{M}_2$ and $\hat{y} \parallel \vec{M}_2 \times \vec{M}_1$. Then, if θ is the angle between $\vec{M}_2$ and $\vec{M}_1$ (figure 3.1 on the next page), the electron spinor in $\xi = 0$ (the center of the B region) is $(\cos(\theta/2), \sin(\theta/2))$ (see figure 3.2 on the following page).

In particular, by assuming that the de Broglie wavelength is short compared with the typical variation length of the potential $V_\pm$ and in

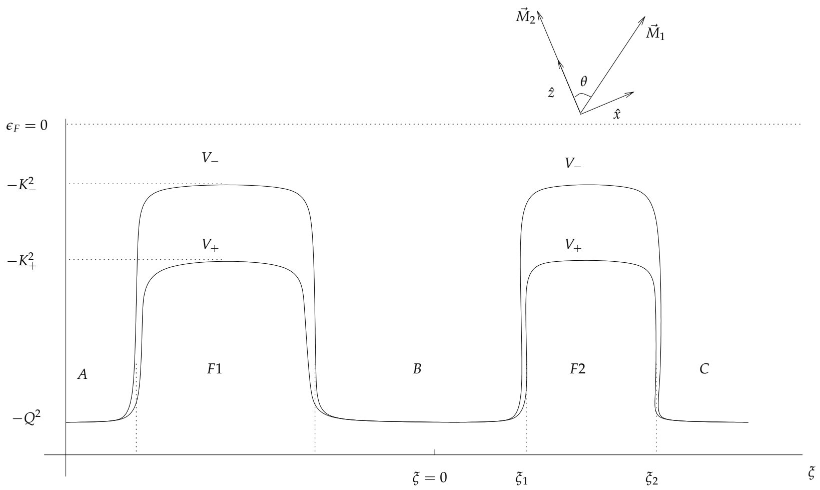


Figure 3.1: ξ is the direction of motion for electrons that come from left to right. A, B and C are paramagnetic layers, while F1, F2 are ferromagnets. An electron with spin parallel to the ferromagnet magnetization experiences a potential V_- in the regions F, while an electron with spin anti-parallel experiences a potential V_+. Electrons can have energy between $-Q^2$ and $\epsilon_F = 0$, where, for simplicity, a unit system in which $\hbar^2/(2\,m) = 1$ (and m is the electron mass) has been chosen. $\vec{M}_{1,2}$ is the F1,2 magnetization vector. This figure is present in the reference [51].

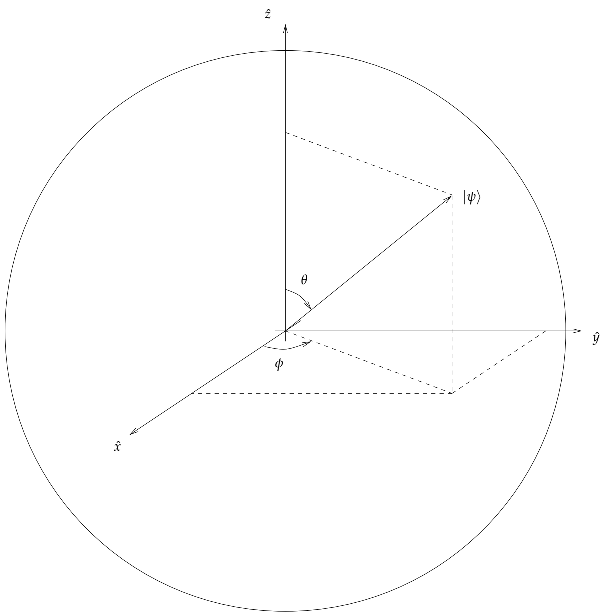


Figure 3.2: The Bloch sphere representation gives a simple geometric interpretation of the 1/2-spin systems. In particular, the generic spin state $|\psi\rangle = \alpha\,|\uparrow_z\rangle + \beta\,|\downarrow_z\rangle = \cos(\theta/2)\,|\uparrow_z\rangle + e^{i\,\phi}\,\sin(\theta/2)\,|\downarrow_z\rangle$ (where α can be chosen real, since two kets that differ for a multiplicative unitary complex number represent the same quantum state) coincides with the state $|\uparrow_{z'}\rangle$, where $\hat{z}'$ is the direction individuated by the polar angles (θ, ϕ).

the parabolic band approximation (that is the electron is considered free inside the layers), Slonczewski found the expression of the electron scattering states (and then of the electron flux) thanks to the WKB approximation.

Since the potential depends on the electron spin, after the scattering the electrons change their spin and also the ferromagnets magnetizations must change direction because of the total angular momentum conservation. This crucial consideration allowed Slonczewski to obtain the equation of motion for the two ferromagnets:

$$\frac{d\vec{M}_{1,2}}{dt} = I\, g\, \hat{M}_{1,2} \times (\hat{M}_1 \times \hat{M}_2)$$

where I is the electrons current, $\hat{M} = \vec{M}/|\vec{M}|$ and

$$g = g(\theta) := \left[-4 + \frac{(1+P)^3\,(3 + \hat{M}_1 \cdot \hat{M}_2)}{4\,P^{3/2}} \right]^{-1},$$
$$P := \frac{K_+ - K_-}{K_+ + K_-}$$

(the $K_\pm$ parameters are described in the figure 3.1 on the preceding page). In particular we must conclude that the presence of a polarized current modifies the Landau–Lifshitz–Gilbert equation.

3.2 OPEN QUANTUM SYSTEMS

Before going on and describe the models introduced in the last years to obtain a microscopic derivation of the Landau-Lifshitz-Gilbert-Slonczewski equation, a couple of words on the open quantum theory [6, 48] are useful. Indeed we are going to consider the ferromagnetic layer as a quantum system that is not isolated but interacts with an electric current and an *open quantum system* is, by definition, a quantum system that interacts with an other system, that we can call *environment*.

The open quantum theory is useful for different reasons:

- as one can easily imagine, to have a completely isolated quantum system is practically impossible (this is, for example, one of the most important issue in the quantum computing implementations - also known as decoherence problem);
- it can answer to some fundamental problems of the quantum physics; for example:
  - why can't we find the macroscopic systems in a superposition state of two distant positions (or, if you prefer, how does the quantum to classical transition work)? Why in most cases the microscopic quantum systems (like, for example, the electrons in a solid or in a molecule) are found in

an eigenstate of the Hamiltonian and not in a superposition of two eigenstates?[1]

– the quantum theory is causal, that is, if we know the state of a system at a given time (and we know the nature of the interactions), in principle we may evaluate the state of the system at any time. But when we perform a measurement, as known, a stochastic behaviour appears. But how is that possible if we assume that the whole system - quantum system plus measurement apparatus - is a bigger quantum system (and then is described by a causal dynamic equation)?

To give a look on how the open quantum theory can approach these questions, we can consider a simple example. But before, we need to introduce the formalism that will be useful also in the next chapters. In particular we know that, if we have a system in a state $|\psi_i\rangle$ with probability p_i, that is a mixed state, the mathematical representation of the state is given by the density matrix

$$\hat{\rho} = \sum_i p_i \, |\psi_i\rangle \, \langle\psi_i| \, , \tag{3.1}$$

in the sense that everything we can measure can be obtained from it:

$$\langle\hat{O}\rangle = \mathrm{tr}\left[\hat{\rho}\,\hat{O}\right] ,$$

where $\hat{O}$ is an observable. The fundamental properties of the density matrix cen be easily checked: $\hat{\rho}^\dagger = \hat{\rho}$, $\langle\psi\,|\,\hat{\rho}\,|\,\psi\rangle \geq 0$ (positivity), $\mathrm{tr}[\hat{\rho}] = 1$. A generic linear operator $\hat{\rho}$ that satisfies these three properties is a density matrix; indeed, by using the spectral theorem for self-adjoint operators, we can write the spectral representation

$$\hat{\rho} = \sum_{\lambda,k} \lambda \, |\lambda, k\rangle \, \langle\lambda, k| \tag{3.2}$$

and verify that $\lambda \geq 0$, $\sum_{\lambda,k} \lambda = 1$. A crucial point is that the expressions 3.1 and 3.2 can be different: while $\hat{\rho}$ is the same, it can happens, for example, that the states $|\psi_i\rangle$ are not mutually orthogonal. From a physical point of view, this means that we may have two mixed states prepared in different ways that are not distinguishable by simply performing a measurement on the system.

If we have a system that interacts with the environment, we need to consider the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$; it is easy to check that

$$\hat{\rho}_S := \mathrm{tr}_E[\hat{\rho}]$$

(where $\hat{\rho}$ is a density matrix on $\mathcal{H}_S \otimes \mathcal{H}_E$ and tr_E indicates the trace over the environment degrees of freedom) is an operator on the $\mathcal{H}_S$

1 Another interesting case is that the chiral molecules are always observed in chirality eigenstates, which are superpositions of different energy eigenstates; the open quantum theory can give an interpretation also for this phenomenon [25, 42].

space that satisfies the density matrixes properties and describes the state of the system S, in the sense that, if $\hat{O}_S$ is an observable that acts only on the system S, we have

$$\langle \hat{O}_S \rangle = \langle \hat{O}_S \otimes \hat{I}_E \rangle = \mathrm{tr}_S[\hat{\rho}_S \, \hat{O}_S]$$

(with a slight abuse of notation), where $\hat{I}_E$ indicates the identity operator over $\mathcal{H}_E$.

The time evolution for $\hat{\rho}_S$ is given by

$$\hat{\rho}_S(t) = \mathrm{tr}_E[\hat{U}(t, t_0) \, \hat{\rho} \, \hat{U}^\dagger(t, t_0)];$$

in particular, if at time $t_0 = 0$ the system and environment are not correlated, that is the state is of the form $\hat{\rho}(t = 0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$, and the spectral decomposition of the initial environment state is $\hat{\rho}_E(0) = \sum_i p_i \left| E_i \right\rangle \left\langle E_i \right|$, it is easy to check that

$$\hat{\rho}_S(t) = \sum_{ij} p_i \left\langle E_j \,\middle|\, \hat{U}(t) \,\middle|\, E_i \right\rangle \hat{\rho}_S(0) \left\langle E_i \,\middle|\, \hat{U}^\dagger(t) \,\middle|\, E_j \right\rangle = \\ = \sum_{ij} \hat{W}_{ij} \, \hat{\rho}_S(0) \, \hat{W}_{ij}^\dagger, \quad (3.3)$$

where the $\hat{W}_{ij} := \sqrt{p_i} \left\langle E_j \,\middle|\, \hat{U}(t) \,\middle|\, E_i \right\rangle$ operators on $\mathcal{H}_S$ are called *Kraus operators* (sometimes the name refers directly to $\left\langle E_j \,\middle|\, \hat{U}(t) \,\middle|\, E_i \right\rangle$).

Sometimes the equation of motion can be put in the form

$$\frac{d}{dt}\hat{\rho}_S(t) = \mathcal{L}[\hat{\rho}_S(t)] := -i \, [\hat{H}, \hat{\rho}_S(t)] + \mathcal{D}[\hat{\rho}_S(t)],$$

that is called *master equation*. This equation is local in time in the sense that $\hat{\rho}_S(t + dt)$ depends only on $\hat{\rho}_S(t)$. Here $\mathcal{L}$ and $\mathcal{D}$ are *superoperators* (as we will see immediately, the form of $\mathcal{D}$ is well defined under reasonable assumptions), that is linear applications that acts in the space of opearotors on $\mathcal{H}_S$. In particular, if $\mathcal{D} = 0$, we have the standard Heisenberg evolution equation and the system can be considered isolated (or the only effect of the environment is to give a particular form to $\hat{H}$). If we assume that the evolution $\hat{\rho}(t_0) \mapsto \hat{\rho}(t)$ ensures the density matrix positivity and it is trace preserving,[2] the most general form of the master equation is [6]:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i \, [\hat{H}, \hat{\rho}_S(t)] + \\ - \frac{1}{2} \sum_{\mu} \kappa_\mu \left\{ \hat{L}_\mu^\dagger \, \hat{L}_\mu \, \hat{\rho}_S(t) + \hat{\rho}_S(t) \, \hat{L}_\mu^\dagger \, \hat{L}_\mu - 2 \, \hat{L}_\mu \, \hat{\rho}_S(t) \, \hat{L}_\mu^\dagger \right\},$$

$\kappa_\mu \geq 0$, that is the *Lindblad master equation*. Here $\hat{L}_\mu$ are generic operators on $\mathcal{H}_S$, whose expression depends on the form of the interaction between the system and the environment.

2 note that, if the master equation is exact, the positivity and the trace preserving are given; but this is not necessarily true when the master equation is obtained by approximations.

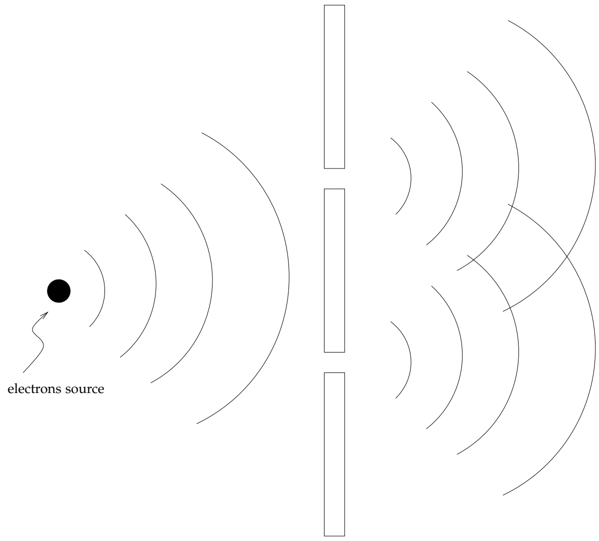


Figure 3.3: The double slit experiment.

Now we can consider the double slit experiment example: as known, if we consider an electron that crosses a screen with two slits, we will observe interference phenomena (figure 3.3). But if we light up the system (or, if you prefer, if we measure which slit the electron crosses) the interference phenomena disappears and the electron chooses a random slit to cross the screen. It is the electron Hamletic doubt (to be a particle or to be a wave?). The state of the whole system (electron plus light) is pure:

$$\begin{aligned}\rho &= |s\rangle\langle s| = \\ &= \left[\frac{1}{\sqrt{2}}|\psi_1, L_1\rangle + \frac{1}{\sqrt{2}}|\psi_2, L_2\rangle\right]\left[\frac{1}{\sqrt{2}}\langle\psi_1, L_1| + \frac{1}{\sqrt{2}}\langle\psi_2, L_2|\right],\end{aligned}$$

where $|\psi_i\rangle$ is the electron state when the electron crosses the i-th slit ($i = 1, 2$), while $|L_i\rangle$ is the light state. By tracing over the light degrees of freedom, we get

$$\begin{aligned}\rho_\mathrm{e} &= \mathrm{tr}_L\,\rho = \\ &= \frac{1}{2}\left[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right] + \frac{1}{2}\left[|\psi_1\rangle\langle\psi_2|\langle L_1 \,|\, L_2\rangle + \mathrm{h.c.}\right];\end{aligned}$$

we can imagine that a single photon cannot resolve well the electron posistion, that is $\langle l_1 \,|\, l_2\rangle \lesssim 1$. But, if the light is composed by a large number of photons, that is $|L\rangle = \bigotimes_{j=1}^{N}|l\rangle$, where j runs over the photons,

$$\langle L_1 \,|\, L_2\rangle = \prod_{j=1}^{N}\langle l_1 \,|\, l_2\rangle = \langle l_1 \,|\, l_2\rangle^N \xrightarrow{N\to\infty} 0$$

and then

$$\rho_\mathrm{e} \simeq \frac{1}{2}\left[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right],$$

that is we have a mixed state $|\psi_{1,2}\rangle$ with probability 1/2. Clearly this does not mean that the whole system state is not pure, but the reduced density matrix of electrons is not distinguishable from a mixed state if we operate measurements that involve only the electron observables. Then the coherence (namely the fundamental ingredient of the quantum computation) still exits, but somehow is spread in the environment. By using the open theory standard notation, we say that the environment (the photons) *monitors* the system, producing a *decoherence* in one of the two *robust* (with respect to the interaction system plus environment) states $|\psi_{1,2}\rangle$.

Most quantum systems experience one of the two following environment monitoring (sometimes these are called *classical* and *quantum limit* respectively):

- for macroscopic objects, the typical distance-dependent environment interaction (for example with photons) gives rise to decoherence into spatially localized wave packets (the eigenstates of the position operator are the robust states). This is for example the case of measurements, since typically the measurement apparatus is macroscopic (and not isolated) [25];

- many microsopic systems are found in energy eigenstates, even if the system-environment interaction Hamiltonian depends on observables like the position. This happens when the typical difference of the energy eigenstates of the system is greater than the energies available in the environment (the environment is able to monitor only quantities that are constants of motion) [40].

3.3 MICROSCOPIC MODELS FOR THE MAGNET DYNAMICS

In recent years, the advances in fast time-resolved measurements have showed that the magnetization dynamics of a nanomagnet crossed by a polarized current have a stochastic behaviour at short time intervals [11, 13, 14, 59]. Motivated by that, in 2012 Wang and Sham [62, 63] proposed a model that could go behind the classical point of view described by the Landau-Lifshitz-Gilbert-Slonczewski equation. Their idea was that, when the magnet is in the mesoscopic range, some quantum effects can appear.

To take into account the quantum behaviour of the ferromagnet, it is not longer modelled as an external classical field (like in the Slonczewski model), but as a quantum spin $\vec{J}$. Wang and Sham considered then the Hamiltonian for the magnet plus a single electron given by

$$H = -\frac{1}{2}\,\partial_x^2 + \delta(x)\,(\lambda_0 + \lambda\,\vec{s}\cdot\vec{J}),$$

where the electron moves along x, $K_e := -\frac{1}{2}\,\partial_x^2$ is the electron kinetic energy in a unit system in which $\hbar = m = 1$ (m is the electron mass),

$\vec{s}$ is the electron spin, the magnet is placed at $x = 0$ and $d \cdot (\lambda_0 \mp \lambda)$ correspond to the potentials $V_\pm$ in the Slonczewski model (d is the magnet thickness).

The $\hat{H}$ eigenstates can be easily evaluated by observing that $\hat{H}$ commutes with the total spin; in particular, if

$$[\hat{J} + \hat{s}]\,|\mathcal{J}, \mu\rangle = \mu\,|\mathcal{J}, \mu\rangle\,,$$
$$[\hat{J} + \hat{s}]^2\,|\mathcal{J}, \mu\rangle = \mathcal{J}\,(\mathcal{J} + 1)\,|\mathcal{J}, \mu\rangle\,,$$

we have

$$\psi_k(x) = |\mathcal{J}, \mu\rangle \begin{cases} L_\rightarrow\, e^{i\,k\,x} + L_\leftarrow\, e^{-i\,k\,x}, & x < 0 \\ R_\rightarrow\, e^{i\,k\,x} + R_\leftarrow\, e^{-i\,k\,x}, & x > 0 \end{cases} =$$
$$= |\mathcal{J}, \mu\rangle \begin{cases} L_\rightarrow\, \langle x\,|\,k\rangle + L_\leftarrow\, \langle x\,|\,-k\rangle\,, & x < 0 \\ R_\rightarrow\, \langle x\,|\,k\rangle + R_\leftarrow\, \langle x\,|\,-k\rangle\,, & x > 0 \end{cases}$$

where, since we are interested in the incoming electrons from left to right, $R_\leftarrow = 0$. To solve the $\hat{H}$ eingevectors problem means to obtain the relations that allow to evaluate the reflected wave coefficient $L_\leftarrow$ and the transmitted wave coefficient $R_\rightarrow$ as function of $L_\rightarrow$, namely the scattering matrix $\hat{\mathcal{S}}$.

By using the Clebsch-Gordan coefficients it is possible to write $|\mathcal{J}, \mu\rangle$ in terms of $|m, s\rangle$ basis (where $s = \pm$ refers to the electron spin and $m = -J, -J + 1, \ldots, J$ to the magnet spin) and then obtain the explicit form of the Kraus operators

$$\hat{\mathcal{K}}_{k,s;k',s'} := \left\langle k, s\,\middle|\,\hat{S}\,\middle|\,k', s'\right\rangle,$$

where $|k, s\rangle$ is the incoming wave, while $|k' = \pm k, s' = \pm\rangle$ is the outgoing wave. In particular Wang and Sham obtained

$$\hat{\mathcal{K}}_{k,s;\pm k,s} = \left(\xi \pm \frac{1}{2}\right) + s\,\zeta\,\hat{J}_z, \qquad \hat{\mathcal{K}}_{k,-s;\pm k,s} = \zeta\,\hat{J}_s$$

where ξ and ζ are complex number that depends on $(\lambda_0, \lambda, J, k)$ and, as usual:

$$\hat{J}_\pm := \hat{J}_x \pm i\,\hat{J}_y. \tag{3.4}$$

Before the scattering the state of the electron must be not correlated to the magnet state and Wang and Sham assumed as initial state

$$\hat{\rho}_{\text{in}} = \sum_{ss'} f_{ss'}\,|k, s\rangle\,\langle k, s'| \otimes \hat{\rho}^J_{\text{in}} = \hat{\rho}^e_{\text{in}} \otimes \hat{\rho}^J_{\text{in}}$$

so that, after the scattering the magnet state is (see equation 3.3 on page 21)

$$\hat{\rho}^J_{\text{out}} = \sum_{\pm,s,s',s''} f_{ss'}\,\hat{\mathcal{K}}_{\pm k,s'';k,s}\,\hat{\rho}^J_{\text{in}}\,\hat{\mathcal{K}}^\dagger_{\pm k,s'';k,s'}.$$

Now, assuming that the the polarized current is composed by a sequence of electrons injected one by one at equal time interval τ, with a coarse graining time scale approximation, Wang and Sham obtained the magnet master equation

$$\partial_t \hat{\rho}^J(t) \simeq \frac{\hat{\rho}^J_{\text{out}} - \hat{\rho}^J_{\text{in}}}{\tau} = \frac{1}{\tau}\left[\mathcal{T}_0(t) + \vec{S}(t)\cdot\vec{\mathcal{T}}(t)\right] \tag{3.5}$$

where, after some calculations:

$$\begin{aligned}
\vec{S} &= \text{tr}[\vec{\sigma}\,\hat{\rho}^e_{\text{in}}],\\
\mathcal{T}_0(t) &= \left(|\xi|^2 - \frac{1}{4}\right)\hat{\rho}^J(t) + |\zeta|^2\left(\hat{J}_z\,\hat{\rho}^J(t)\,\hat{J}_z + \hat{J}_+\,\hat{\rho}^J(t)\,\hat{J}_-\right) + \text{h.c.},\\
\mathcal{T}_x(t) &= 2\,\xi\,\zeta^*\,\hat{\rho}^J(t)\,\hat{J}_x + |\zeta|^2\left(\hat{J}_z\,\hat{\rho}^J(t)\,\hat{J}_+ - \hat{J}_+\,\hat{\rho}^J(t)\,\hat{J}_z\right) + \text{h.c.},\\
\mathcal{T}_y(t) &= 2\,\xi\,\zeta^*\,\hat{\rho}^J(t)\,\hat{J}_y + i\,|\zeta|^2\left(\hat{J}_+\,\hat{\rho}^J(t)\,\hat{J}_z - \hat{J}_z\,\hat{\rho}^J(t)\,\hat{J}_+\right) + \text{h.c.},\\
\mathcal{T}_z(t) &= 2\,\xi\,\zeta^*\,\hat{\rho}^J(t)\,\hat{J}_z + |\zeta|^2\left(\hat{J}_+\,\hat{\rho}^J(t)\,\hat{J}_- - \hat{J}_-\,\hat{\rho}^J(t)\,\hat{J}_+\right) + \text{h.c.}
\end{aligned}$$

and $\vec{\sigma}$ are the Pauli matrixes.

To study the quantum-classical crossover, Wang and Sham considered the spin coherent states representation. We can consider the spin coherent states as the "classical" states for spins. In particular, if $J_z\,|m = J\rangle = J\,|m = J\rangle$, that is $|m = J\rangle$ is the state perfectly aligned withe the $\hat{z}$ axis, the generic spin coherent state is obtained by rotating $|m = J\rangle$ in the polar direction $\Omega = (\theta, \phi)$:

$$|\Omega\rangle := \hat{R}(\theta,\phi)\,|m = J\rangle\,, \qquad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\,\pi, \tag{3.6}$$

where $\hat{R}(\theta,\phi)$ is the unitary operator that rotates the spin states. In the case of $\frac{1}{2}$-spin, all states are also coherent states (see figure 3.2 on page 18), while in the general case this is not true, but anyhow the set of coherent states constitutes a (non-orthonormal) basis [2]:

$$(2\,J + 1)\int \frac{d\Omega}{4\,\pi}\,|\Omega\rangle\,\langle\Omega| = 1,$$

where $d\Omega$ is the solid angle element. The operators of the form $\hat{J}_i\,|\Omega\rangle\,\langle\Omega|\,\hat{J}_j$ ($i, j = 0, +, -, z$, where $\hat{J}_0 := \hat{I}$ is the identity) are explicitly evaluated in [37] and it is possible to write

$$\hat{\rho}^J(t) = \int d\Omega\,\mathcal{P}_J(\Omega, t)\,|\Omega\rangle\,\langle\Omega| \tag{3.7}$$

and then, from a simple calculation, the master equation 3.5 can be written in the $\mathcal{P}_J$-representation:

$$\frac{\partial}{\partial t}\mathcal{P}_J(\hat{m}, t) = -\nabla\cdot(\vec{T}\,\mathcal{P}_J) + \nabla^2(\mathcal{D}\,\mathcal{P}_J), \tag{3.8}$$

where the equation is now in Cartesian coordinate system ($\hat{m}$ is the magnetization direction) and

$$\vec{T} := \mathcal{A}\,(\hat{m}\times\vec{S})\times\hat{m} + \mathcal{B}\,\hat{m}\times\vec{S}, \qquad \mathcal{D} := \mathcal{A}\,\frac{1-\hat{m}\cdot\vec{S}}{2\,J+1}, \tag{3.9a}$$

$$\mathcal{A} := (2\,J+1)\,\frac{|\zeta|^2}{\tau}, \qquad \mathcal{B} := 2\,\frac{\Im(\zeta^*\,\zeta)}{\tau}. \tag{3.9b}$$

The equation 3.8 on the preceding page is a Fokker-Planck-like equation (compare with 2.8 on page 13) for the "quasi-probability" distribution function $\mathcal{P}_J$; in particular, it is not true in general that $\mathcal{P}_J \geq 0$ in the equation 3.7 on the preceding page. The Fokker-Planck equation is composed by a drift term, with the drift vector $\vec{T}$, and a diffusion term, with the diffusion coefficient $\mathcal{D}$. As one can see from equations 3.9a, $\vec{T}$ contains a field-like term and a damping-like term (compare with equation 2.4a on page 11). These two terms are well known [44], while the diffusion term is peculiar of the "quantum" Wang and Sham model and introduces magnetization fluctuations. These fluctuations are dominants at low temperatures, while are negligible at hight temperatures: the fluctuations induced by $\mathcal{D}$ must be compared with those described in section 2.2 on page 12.

Finally Wang and Sham simplified the Fokker-Planck equation by considering the limit $J \gg 1$ and employing the WKB approximation at the lowest order:

$$\mathcal{P}_J(\hat{M},t) = e^{-J\,W(\hat{m},t)}\sum_{n=0}^{\infty}\frac{1}{J^n}\,\phi_n(\hat{m},t) \simeq e^{-J\,W(\hat{m},t)}$$

and explored some numerical solutions.

Generalizing the method proposed by Wang and Sham to more complicated situations could be quite difficult. In particular, one could want to include the electron-electron interaction, or go behind the coarse graining time scale approximation, or consider the interaction (and then the quantum noise effects) mediated by the current in presence of several magnets.[3] To obtain a good framework we then considered the Keldysh formulation and the Holstein–Primakoff semi-classical approximation (see next chapters). These methods have been already employed to study the spin transfer torque. In particular Swiebodzinski, Chudnovskiy, Dunn, and Kamenev [53] considered the simple case of two ferromagnets separated by an insulator; the first ferromagnet was considered blocked (hard ferromagnet) and its role was to polarize the current, while the second could rotate; by considering a weak tunnel effect, they obtained for the free ferromagnet magnetization the equation of motions:

$$\partial_t\vec{J} = \gamma\,[\vec{H}\times\vec{J}] - \frac{\alpha(\theta)}{J}\,[\vec{J}\times\partial_t\vec{J}] + \frac{1}{J^2}\,[\vec{J}\times[(\vec{I}_s+\vec{j}(t))\times\vec{J}]],$$

3 in particular, the comparison with the case described in [36] could be very interesting. Here a polarization laser beam go trough two atomic ensembles producing a steady-state entangled.

where $\vec{H}$ is an external magnetic field that interacts with the free layer; $\alpha(\theta)$ is the damping term coefficient and its explicit expression depends on the angle between the free layer and the blocked layer; $\vec{I}_s$ is the spin current, while $\vec{j}$ is a Gaussian stochastic current with zero mean value and

$$\langle j_l(t)\, j_m(t')\rangle = \mathcal{D}(\theta)\, \delta(t-t')\, \delta_{lm}, \qquad l,m = x,y,z.$$

As we will see, with the Wang and Sham model we will obtain a richer form for the magnet equation of motion, in particular with a more complicated expression for the stochastic part.

4 NON-EQUILIBRIUM FORMALISM

This chapter is devoted to the introduction of the Keldysh formalism. We will follow quite closely the reference [28]. Since it is a technical chapter, it can be left out if the reader is already familiar.

4.1 GENERALITIES ON MANY-BODY PHYSICS

In this section we review the basics of quantum many-body systems to introduce the formalism that will be used in the following.

4.1.1 *Creation and annihilation operators*

If we have a system of N particles and $\{\,|\alpha\rangle\,\}_\alpha$ is a basis for a single particle (for example $\alpha = (\vec{p}, s)$, where $\vec{p}$ is the momentum and s the spin), a basis for the total system is $\{\,|\alpha_1\rangle \cdots |\alpha_N\rangle\,\}$. If particles are identical, we assume that a general state must satisfy the relation:

$$|\Psi\rangle = \sum_{\alpha_1,\dots,\alpha_N} c_{\alpha_1\cdots\alpha_N} \,|\alpha_1\rangle \cdots |\alpha_N\rangle = \\ = \zeta^{\text{sign}(\pi)} \sum_{\alpha_1,\dots,\alpha_N} c_{\alpha_1\cdots\alpha_N} \left|\alpha_{\pi(1)}\right\rangle \cdots \left|\alpha_{\pi(N)}\right\rangle,$$

where π is a permutation of $1, 2, \dots, N$; ζ is 1 for bosons and -1 for fermions.

4.1.1.1 *Bosons*

Then a basis for states of identical bosons is given by $|n_{\alpha_1}, n_{\alpha_2}, \dots\rangle$, where n_{α_i} is the number of bosons in the single-particle state $|\alpha_i\rangle$.

In analogy with the harmonic oscillator, we may introduce creation and annihilation operators:

$$\hat{c}^\dagger_{\alpha_k} |\dots, n_{\alpha_k}, \dots\rangle = \sqrt{n_{\alpha_k} + 1}\, |\dots, n_{\alpha_k} + 1, \dots\rangle\,,$$
$$\hat{c}_{\alpha_k} |\dots, n_{\alpha_k}, \dots\rangle = \sqrt{n_{\alpha_k}}\, |\dots, n_{\alpha_k} - 1, \dots\rangle\,,$$

that is

$$\hat{c}_{\alpha_k} = \sum_{n_{\alpha_1},\dots,n_{\alpha_k},\dots} \sqrt{n_{\alpha_k}}\, |\dots, n_{\alpha_{k-1}}, n_{\alpha_k} - 1, n_{\alpha_{k+1}}, \dots\rangle \cdot \\ \cdot \langle \dots, n_{\alpha_{k-1}}, n_{\alpha_k}, n_{\alpha_{k+1}}, \dots|$$

and

$$\hat{c}^{\dagger}_{\alpha_k} = \sum_{n_{\alpha_1},\dots,n_{\alpha_k},\dots} \sqrt{n_k+1}\, |\dots, n_{\alpha_{k-1}}, n_{\alpha_k}+1, n_{\alpha_{k+1}}, \dots\rangle \cdot \\ \cdot \langle \dots, n_{\alpha_{k-1}}, n_{\alpha_k}, n_{\alpha_{k+1}}, \dots| ,$$

where $\hat{c}^{\dagger}$ is the hermitian conjugated of $\hat{c}$. They satisfy the commutator relations

$$[\hat{c}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}]_- := [\hat{c}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}] = \delta_{kl}, \qquad [\hat{c}_{\alpha_k}, \hat{c}_{\alpha_l}] = [\hat{c}^{\dagger}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}] = 0$$

and

$$|n_{\alpha_1}, n_{\alpha_2}, \dots\rangle = \frac{1}{\sqrt{n_{\alpha_1}!\, n_{\alpha_2}! \cdots}} (\hat{c}^{\dagger}_{\alpha_1})^{n_{\alpha_1}} (\hat{c}^{\dagger}_{\alpha_2})^{n_{\alpha_2}} \cdots |0\rangle .$$

If we have several bosonic species, since operators associated to different species commute, all the expressions obtained are still valid simply considering an extra index in α that specifies the kind of particle (for example $\alpha = (\vec{p}, s, b)$, where $\vec{p}$ is the momentum, s the spin and $b =$ gluon, Higgs', ...).

4.1.1.2 *Fermions*

A basis for states of an identical fermions system is given by

$$|n_{\alpha_1}, n_{\alpha_2}, \dots\rangle = |\alpha \mapsto n_\alpha\rangle ,$$

where n_α is the number of fermions in the state of single particle $|\alpha\rangle$.

Unlike the bosonic case, we need $\hat{c}^{\dagger}|1\rangle = 0$, from which we expect that $\{\hat{c}, \hat{c}^{\dagger}\} = 1$. Then we may introduce creation and annihilation operators:

$$\begin{aligned}
&\hat{c}^{\dagger}_{\alpha_k} |\dots, n_{\alpha_{k-1}}, 0, n_{\alpha_{k+1}}, \dots\rangle = \\
&\qquad = (-1)^{n_{\alpha_1}+\dots+n_{\alpha_{k-1}}} |\dots, n_{\alpha_{k-1}}, 1, n_{\alpha_{k+1}}, \dots\rangle , \\
&\hat{c}^{\dagger}_{\alpha_k} |n_{\alpha_1}, \dots, n_{\alpha_{k-1}}, 1, n_{\alpha_{k+1}}, \dots\rangle = 0, \\
&\hat{c}_{\alpha_k} |\dots, n_{\alpha_{k-1}}, 0, n_{\alpha_{k+1}}, \dots\rangle = 0, \\
&\hat{c}_{\alpha_k} |\dots, n_{\alpha_{k-1}}, 1, n_{\alpha_{k+1}}, \dots\rangle = \\
&\qquad = (-1)^{n_{\alpha_1}+\dots+n_{\alpha_{k-1}}} |\dots, n_{\alpha_{k-1}}, 0, n_{\alpha_{k+1}}, \dots\rangle ,
\end{aligned}$$

that is

$$\hat{c}_{\alpha_k} = \sum_{n_{\alpha_1},\dots,n_{\alpha_{k-1}},n_{\alpha_{k+1}},\dots} (-1)^{n_{\alpha_1}+\dots+n_{\alpha_{k-1}}} \cdot \\ \cdot |\dots, n_{\alpha_{k-1}}, n_{\alpha_k}=0, n_{\alpha_{k+1}}, \dots\rangle \langle \dots, n_{\alpha_{k-1}}, n_{\alpha_k}=1, n_{\alpha_{k+1}}, \dots|$$

and

$$\hat{c}^{\dagger}_{\alpha_k} = \sum_{n_{\alpha_1},\dots,n_{\alpha_{k-1}},n_{\alpha_{k+1}},\dots} (-1)^{n_{\alpha_1}+\dots+n_{\alpha_{k-1}}} \cdot \\ \cdot |\dots, n_{\alpha_{k-1}}, n_{\alpha_k}=1, n_{\alpha_{k+1}}, \dots\rangle \langle \dots, n_{\alpha_{k-1}}, n_{\alpha_k}=0, n_{\alpha_{k+1}}, \dots| .$$

They satisfy the anticommutator relations

$$[\hat{c}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}]_+ = \{\hat{c}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}\} = \delta_{kl}, \qquad \{\hat{c}_{\alpha_k}, \hat{c}_{\alpha_l}\} = \{\hat{c}^{\dagger}_{\alpha_k}, \hat{c}^{\dagger}_{\alpha_l}\} = 0$$

and

$$|n_{\alpha_1}, n_{\alpha_2}, \dots\rangle = (\hat{c}^{\dagger}_{\alpha_1})^{n_{\alpha_1}} (\hat{c}^{\dagger}_{\alpha_2})^{n_{\alpha_2}} \cdots |0\rangle \,.$$

Like in the bosons case, one can show that for several fermions all previous relation are still valid by simply adding to α an index for the fermionic species.

4.1.1.3 *Basis change*

If $\{\,|\beta\rangle\,\}_\beta$ is a different single-particle states basis, it is easy to show that

$$\hat{c}^{\dagger}_{\beta} = \sum_{\alpha} \langle \alpha \,|\, \beta\rangle\, \hat{c}^{\dagger}_{\alpha}, \qquad \hat{c}_{\beta} = \sum_{\alpha} \langle \beta \,|\, \alpha\rangle\, \hat{c}_{\alpha}.$$

For example, if $\beta = (\vec{x}, s)$ is the position-spin basis, we have

$$\hat{\psi}_s(\vec{x}) := \hat{c}_{\vec{x},s} = \sum_{\alpha} \psi_{\alpha,s}(\vec{x})\, c_{\alpha}, \qquad \psi_{\alpha}(\vec{x}) := \langle \vec{x}, s \,|\, \alpha\rangle \,,$$

where $\psi_{\alpha,s}$ is the wave-function associated to the state of single particle $|\alpha\rangle$ and $\hat{\psi}_s(\vec{x})$ is called *field operator* (in the language of second quantization). In particular $\hat{c}^{\dagger}_{\vec{x},s} = \hat{\psi}^{\dagger}_s(\vec{x})$ creates a particle in the state $|\vec{x}, s\rangle$, that is, a particle with spin s in $\vec{x}$. By considering the continuous limit

$$\sum_{\vec{x},s} \to \sum_{s} \int dx, \qquad \delta_{(\vec{x}_1,s_1),(\vec{x}_2,s_2)} \to \delta_{s_1 s_2}\, \delta(\vec{x}_1 - \vec{x}_2)$$

the (anti-)commutation relations for the $\hat{c}_{\vec{x},s}$ operators give:

$$\begin{aligned} \left[\hat{\psi}_{s_1}(\vec{x}_1), \hat{\psi}_{s_2}(\vec{x}_2)\right]_{\zeta} &= \left[\hat{\psi}^{\dagger}_{s_1}(\vec{x}_1), \hat{\psi}^{\dagger}_{s_2}(\vec{x}_2)\right]_{\zeta} = 0, \\ \left[\hat{\psi}_{s_1}(\vec{x}_1), \hat{\psi}^{\dagger}_{s_2}(\vec{x}_2)\right]_{\zeta} &= \delta(\vec{x}_1 - \vec{x}_2)\, \delta_{s_1 s_2}, \end{aligned}$$

where $\zeta = \pm 1$ for bosons and fermions.

4.1.2 *Observables*

Observables are operators that tranform $|n_{\alpha_1}, n_{\alpha_2}, \dots\rangle$ in some others $|n'_{\alpha_1}, n'_{\alpha_2}, \dots\rangle$; then we may expect that can be written in terms of creation and annihilation operators. Here we see some examples:

- $\hat{n}_{\alpha_k} := \hat{c}^{\dagger}_{\alpha_k} \hat{c}_{\alpha_k}$ counts the number of particles in the state of single particle $|\alpha_k\rangle$, indeed

$$\begin{aligned} \hat{n}_{\alpha_k} \left|\dots, n_{\alpha_{k-1}}, n_{\alpha_k}, n_{\alpha_{k+1}}, \dots\right\rangle &= \\ &= n_{\alpha_k} \left|\dots, n_{\alpha_{k-1}}, n_{\alpha_k}, n_{\alpha_{k+1}}, \dots\right\rangle . \end{aligned}$$

Furthermore, if $\hat{O}\,|\alpha\rangle = O_\alpha\,|\alpha\rangle$ for some single-particle observable $\hat{O}$, we have

$$\hat{O} = \sum_\alpha O_\alpha\,\hat{n}_\alpha;$$

- if $\hat{V}$ is a single-particle operator, that is[1] $\hat{V}\,|\alpha_1,\ldots,\alpha_n\rangle = \sum_i \hat{V}\,|\alpha_i\rangle$ (for example, in the position representation, a potential $\hat{V} = \sum_i V(\vec{x}_i)$ where $\vec{x}_i$ is the i-nth particle position; or also the kinetic energy: $\hat{T} = -\hbar^2 \sum_i \nabla^2_{\vec{x}_i}/(2\,m)$), it is quite easy to show

$$\hat{V} = \sum_{\alpha,\alpha'} \left\langle \alpha \,\middle|\, \hat{V} \,\middle|\, \alpha' \right\rangle \hat{c}^\dagger_\alpha\,\hat{c}_{\alpha'}; \tag{4.1}$$

- if $\hat{V}$ is a two-particle operator $\hat{V}\,|\alpha_1,\ldots,\alpha_n\rangle = \frac{1}{2}\sum_{i\neq j}\hat{V}\,|\alpha_i,\alpha_j\rangle$:

$$\hat{V} = \frac{1}{2}\sum_{\alpha_1,\alpha'_1,\alpha_2,\alpha'_2} \left\langle \alpha_1,\,\alpha_2 \,\middle|\, \hat{V} \,\middle|\, \alpha'_1,\,\alpha'_2 \right\rangle \hat{c}^\dagger_{\alpha_1}\,\hat{c}^\dagger_{\alpha_2}\,\hat{c}_{\alpha'_2}\,\hat{c}_{\alpha'_1}. \tag{4.2}$$

If we apply these considerations to the field operator, we get:

- $\hat{\rho}_s(\vec{x}) = \hat{\psi}^\dagger_s(\vec{x})\,\hat{\psi}_s(\vec{x})$ is the particle density operator; in particular the total particles number operator is

$$\hat{N} := \sum_s \int d^3x\,\hat{\psi}^\dagger_s(\vec{x})\,\hat{\psi}_s(\vec{x});$$

- be $\hat{V}$ a single-particle operator that assumes the diagonal form $\hat{V}_{\vec{x}}$ in the $|\vec{x}\rangle$ representation; for example, $\hat{V}$ could be a scalar potential

$$\left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \phi \right\rangle = \sum_{\vec{x}',s'} \left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \vec{x}',s' \right\rangle \left\langle \vec{x}',s' \,\middle|\, \phi \right\rangle = \\ = \sum_{s'} V_{s,s'}(\vec{x})\,\phi_{s'}(\vec{x}) = \sum_{s'} \hat{V}_{\vec{x},s,s'}\,\phi_{s'}(\vec{x})$$

from which:

$$\left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \vec{x}',s' \right\rangle = \delta(\vec{x}-\vec{x}')\,\hat{V}_{\vec{x},s,s'}; \tag{4.3}$$

or be $\hat{V}$ the momentum:

$$\left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \phi \right\rangle = \sum_{\vec{x}'} \left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \vec{x}',s' \right\rangle \left\langle \vec{x}',s' \,\middle|\, \phi \right\rangle = \\ = -i\,\hbar \sum_s \nabla_{\vec{x}}\,\phi_s(\vec{x}) = \sum_s \hat{V}_{\vec{x}}\,\phi_s(\vec{x}),$$

from which

$$\left\langle \vec{x},s \,\middle|\, \hat{V} \,\middle|\, \vec{x}',s' \right\rangle = \delta(\vec{x}-\vec{x}')\,\delta_{ss'}\,\hat{V}_{\vec{x}}. \tag{4.4}$$

1 with a slight abuse of notation we indicate with the same symbol the operator that acts on a single particle space and the operator that acts on the n-particles space.

Then combining equation 4.1 on the facing page with 4.3 on the preceding page or 4.4 on the facing page, we get

$$\hat{V} = \sum_{ss'} \int d^3x \, \hat{\psi}_s^\dagger(\vec{x}) \, \hat{V}_{\vec{x},s,s'} \, \hat{\psi}_{s'}(\vec{x});$$

and

$$\hat{\mathcal{V}}_{ss'}(\vec{x}) := \hat{\psi}_s^\dagger(\vec{x}) \, \hat{V}_{\vec{x},s,s'} \, \hat{\psi}_{s'}(\vec{x})$$

is called second-quantized density for the observable $\hat{V}$;

- in the same way, if $\hat{V}$ is, for example, a two-particle interaction potential that assume the form $V(\vec{x}_1, \vec{x}_2)$ in the $|\vec{x}\rangle$ representation, from equation 4.2 on the preceding page we get:

$$\hat{V} = \frac{1}{2} \int d^3x_1 \int d^3x_2 \, \hat{\psi}^\dagger(\vec{x}_1) \, \hat{\psi}^\dagger(\vec{x}_2) \, V(\vec{x}_1, \vec{x}_2) \, \hat{\psi}(\vec{x}_2) \, \hat{\psi}(\vec{x}_1).$$

4.2 Keldysh formalism

In this section we briefly review the key concepts of the Keldysh formalism, which allows to treat the non-equilibrium many-body problems. It is the fundamental tool used in the next chapters.

4.2.1 *Keldysh contour motivation*

As we know from quantum mechanics, everything we may measure (for example the probability of finding some particle in a space region at a given time, etc.) is the mean value of some observable. In the Schrödinger picture we have

$$\langle \hat{O}(t) \rangle = \mathrm{tr}\left[\hat{O} \, \hat{U}_{t,t_0} \, \hat{\rho}(t_0) \, \hat{U}^\dagger_{t,t_0} \right],$$

where $\hat{\rho}$ is the density matrix for a mixed state of the system and $\hat{U}$ the unitary evolution operator:

$$\hat{U}_{dt} := \hat{U}_{t+dt,t} = 1 - i \, \hat{H} \, dt = e^{-i \hat{H} \, dt},$$
$$\hat{U}_{t,t_0} = \hat{U}_{t,t-dt} \, \hat{U}_{t-dt,t-2\,dt} \cdots \hat{U}_{t_0+dt,t_0},$$

which is true only in the limit $dt \to 0$. Furthermore $\hat{U}_{t,t_0} = e^{-i \hat{H} (t-t_0)}$ if $\hat{H}$ does not depend on time (in our case, we assume in general that $\hat{H} = \hat{H}(t)$).

We know that solving exactly the evolution problem for an interacting many-body system is hopeless in the general case, but we may put $H = H_0 + H_1$, where H_0 is a single particle component of the Hamiltonian:

$$H_0 = \sum_\alpha \epsilon_\alpha \, \hat{c}^\dagger_\alpha \, \hat{c}_\alpha.$$

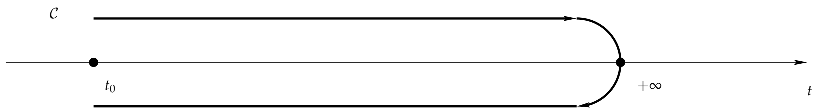


Figure 4.1: $\mathcal{C}$ is the path along the real axis that goes from $t = t_0$ to $t = +\infty$ and from $t = +\infty$ to $t = t_0$ again.

H_0 could be, for example, the free particle Hamiltonian, or a mean field approximation Hamiltonian. We may assume that α also contains an index for the particle species. If we want to expand in terms of $\hat{H}_1$, the simplest way is to use the cyclic property of traces and the unitarity of $\hat{U}$ and write

$$\langle \hat{O}(t) \rangle = \text{tr}\left[\hat{U}_{t_0,t}\, \hat{O}\, \hat{U}_{t,t_0}\, \hat{\rho}(t_0)\right] = \text{tr}\left[\hat{U}_t^-\, \hat{O}\, \hat{U}_t^+\, \hat{\rho}(t_0)\right],$$

where $\hat{U}_t^+$ is the forward and $\hat{U}_t^-$ the backward evolution operator from t_0 to t. Now we want to include $\hat{O}$ in $\hat{U}$; to do that, we define

$$\hat{H}_\pm(t) := \hat{H}(t) \pm \hat{O}\, \eta(t)/2,$$

where $\eta(t)$ is a generic time function, and we introduce two evolution operators: $\hat{U}^+$ is the forward operator between $t = t_0$ and $t = +\infty$ under the action of the Hamiltonian $\hat{H}^+$, while $\hat{U}^-$ is the backward operator from $t = +\infty$ to $t = t_0$ under the action of the Hamiltonian $\hat{H}^-$. Finally we put

$$Z[\eta] := \text{tr}[\hat{U}^-\, \hat{U}^+\, \hat{\rho}(t_0)] =: \text{tr}[\hat{U}_\mathcal{C}\, \hat{\rho}(t_0)], \tag{4.5}$$

where $\mathcal{C}$ is the Keldysh contour[2] (see figure 4.1). Then it is easy to check that[3]

$$\langle \hat{O}(t) \rangle = i\, \frac{\delta Z[\eta]}{\delta \eta(t)}\bigg|_{\eta=0}, \qquad \forall t > t_0 \tag{4.6}$$

or, if $\hat{\rho}$ is not normalized ($\text{tr}[\hat{\rho}(t_0)] \neq 1$):

$$\langle \hat{O}(t) \rangle = i\, \frac{\delta}{\delta \eta(t)} \log Z[\eta]\bigg|_{\eta=0}.$$

Now we may expand the whole operator $\hat{U}_\mathcal{C}$; but expanding operators is not so easy, since they do not commute. A way to escape from nested commutators is to transform operators in numbers by means of the path integral formulation. But before going on, it is useful to briefly compare the Keldysh contour with typical equilibrium techniques.

2 this technique was originally invented by Schwinger and then improved by Keldysh.
3 the functional differentiation rules can be easily found by discretizing the time:

$$\eta(t) \overset{\text{d}}{\to} \eta_i, \qquad \eta(t') \overset{\text{d}}{\to} \eta_{i'},$$

$$\frac{\delta Z[\eta]}{\delta \eta(t)} \overset{\text{d}}{\to} \frac{\partial Z(\eta_1, \eta_2, \ldots)}{\partial \eta_i}, \qquad \frac{\delta \eta(t)}{\delta \eta(t')} \overset{\text{d}}{\to} \frac{\partial \eta_i}{\partial \eta_{i'}} = \delta_{ii'} \overset{\text{d}}{\leftarrow} \delta(t - t').$$

4.2.1.1 *Equilibrium and non-equilibrium formalisms*

The equilibrium theory comes with a trick. Assume for example that we want to evaluate $\langle \mathrm{IGS}|\hat{O}|\mathrm{IGS}\rangle$, where $|\mathrm{IGS}\rangle$ indicates the ground state of an interacting system (this discussion can be generalized to the case of grand-canonical mixed state). Typically only the non-interacting ground state $|\mathrm{NIGS}\rangle$ is well known and so one introduces an evolution operator $\hat{U}_{t,-\infty}$ that adiabatically[4] switches on the interaction, so that

$$|\mathrm{IGS}\rangle = \hat{U}_{t,-\infty}\,|\mathrm{NIGS}\rangle\,;$$

if we assume that at the $t = +\infty$ time the interaction is adiabatically switched off we may argue that

$$\hat{U}_{+\infty,-\infty}\,|\mathrm{NIGS}\rangle = e^{i\,L}\,|\mathrm{NIGS}\rangle\,; \tag{4.7}$$

by using this trick, it is easy to check that

$$\left\langle \mathrm{IGS}\,\middle|\,\hat{O}\,\middle|\,\mathrm{IGS}\right\rangle = \frac{\left\langle \mathrm{NIGS}\,\middle|\,\hat{U}_{+\infty,t}\,\hat{O}\,\hat{U}_{t,-\infty}\,\middle|\,\mathrm{NIGS}\right\rangle}{\left\langle \mathrm{NIGS}\,\middle|\,\hat{U}_{+\infty,-\infty}\,\middle|\,\mathrm{NIGS}\right\rangle}$$

and then only the forward-path is necessary. It can be shown that the denominator produce in particular the cancellation of the disconnected vacuum Feynman diagrams. Anyway this strategy complicates the calculations in some cases; for example, in the theory of disordered systems, the phase $e^{i\,L}$ depends on the particular realization of the disorder and then the denominator must be included in every disorder averaging.

In any case this trick does not work in the non-equilibrium case since the interactions may lead the system far from the equilibrium and we cannot find a relation similar to 4.7.

4.2.2 *Coherent states*

We have seen that observables can be written in terms of creation and annihilation operators; we assume that we are dealing with observables that have zero mean value when no particles are present: $\langle n_\alpha = 0|\hat{O}|n_\alpha = 0\rangle = 0$. This means that, if we write $\hat{O}$ in terms of creation and annihilation operators, in all the creation/annihilation operators products, all creation operators are on the left with respect to the annihilation operators (the observables are *normally ordered*).

To introduce the path integral formulation we need to obtain the eigenvectors of annihilation operators; for bosons we have no problems:

$$\hat{c}_\alpha\left|\underline{\xi}\right\rangle = \xi_\alpha\left|\underline{\xi}\right\rangle, \tag{4.8}$$

4 if a system at the beginning is in an eigenstate of the initial Hamiltonian, at the end of the adiabatic evolution it will be in the corresponding eigenstate of the final Hamiltonian.

Property	Comment
$\lvert\xi\rangle = e^{\zeta \sum_\alpha \xi_\alpha \hat{c}_\alpha^\dagger} \lvert 0\rangle$	coherent states
$\langle \xi \mid \xi' \rangle = e^{\sum_\alpha \bar{\xi}_\alpha \xi'_\alpha}$	overlap relations
$\langle n_\alpha \mid \xi\rangle \langle \xi \mid n'_\alpha\rangle = \langle \zeta\, \xi \mid n'_\alpha\rangle \langle n_\alpha \mid \xi\rangle$	scalar product
$\int d[\bar{\xi},\xi]\, e^{-\sum_\alpha \bar{\xi}_\alpha \xi_\alpha} \lvert\xi\rangle \langle\xi\rvert = 1$	complet. relation
$\int d[\bar{\xi},\xi]\, e^{-\xi^\dagger M \xi + \eta^\dagger \xi + \eta \xi^\dagger} = (\det M)^{-\zeta} e^{\eta^\dagger M^{-1} \eta}$	Gaussian integrals
$\int d[\bar{\xi},\xi]\, P(\bar{\xi},\xi) = \left\lvert \frac{\partial(\bar{\xi},\xi)}{\partial(\theta^*,\theta)} \right\rvert^{\zeta} \int d[\theta^*,\theta]\, P(\bar{\xi},\xi)$	linear var. change
$d[\bar{\xi},\xi] := \prod_\alpha d\bar{\xi}_\alpha\, d\xi_\alpha / \mathcal{N}$	measure definition

Table 4.1: Some coherent states properties and useful tools. Note that $\zeta = 1$ is for bosons and $\zeta = -1$ for fermions. The integral for fermions is the Berezin integral, while for bosons $d\bar{\xi}_\alpha\, d\xi_\alpha := d\Re\xi_\alpha\, d\Im\xi_\alpha$ and the normalization is $\mathcal{N} = \pi$ for bosons and $\mathcal{N} = 1$ for fermions. In the Gaussian integrals formula we used the notation $\xi^\dagger M \xi = \sum_{\alpha\beta} \bar{\xi}_\alpha M_{\alpha\beta} \xi_\beta$ and we have to assume that all eigenvalues of M have non-negative real part.

where ξ_α are complex numbers since $\hat{c}_\alpha$ is not Hermitian. To avoid confusion in this section we use the notation $\underline{\xi} = (\xi_{\alpha_1}, \xi_{\alpha_2}, \ldots)$, but note that in table 4.1 and in the future, ξ is used for $\underline{\xi}$. It is easy to check that $\left\lvert\underline{\xi}\right\rangle$ are given by:

$$\left\lvert\underline{\xi}\right\rangle = \prod_\alpha \sum_{n_\alpha=0}^{\infty} \frac{\xi_\alpha^n}{\sqrt{n_\alpha!}} \lvert n\rangle = \prod_\alpha e^{\xi_\alpha \hat{c}_\alpha^\dagger} \lvert 0\rangle = e^{\sum_\alpha \xi_\alpha \hat{c}_\alpha^\dagger} \lvert 0\rangle$$

and are said *coherent states*; they form a set of super-complete (not orthonormal) states. Indeed it is easy to evaluate $\left\langle \underline{\xi} \middle| \underline{\xi}' \right\rangle$ (table 4.1) and the completeness relation can be obtained from the Gaussian integral (see again table 4.1):

$$\begin{aligned} &\int \frac{d\Re\xi_\alpha\, d\Im\xi_\alpha}{\pi} e^{-|\xi_\alpha|^2} (\bar{\xi}_\alpha)^n \xi_\alpha^{n'} = \\ &= \frac{\partial^{n+n'}}{\partial\eta_\alpha^n \partial(\eta_\alpha^*)^{n'}} \int \frac{d\Re\xi_\alpha\, d\Im\xi_\alpha}{\pi} e^{-|\xi_\alpha|^2 + \bar{\xi}_\alpha \eta_\alpha + \eta_\alpha^* \xi_\alpha} \Big|_{\eta_\alpha^* = \eta_\alpha = 0} \overset{G}{=} \\ &\qquad = \frac{\partial^{n+n'} e^{\eta_\alpha^* \eta_\alpha}}{\partial\eta_\alpha^n \partial(\eta_\alpha^*)^{n'}} \Big|_{\eta_\alpha^* = \eta_\alpha = 0} = n!\, \delta_{nn'} \end{aligned}$$

($\overset{G}{=}$ means $=$ because of the Gaussian integrals) and the relation $\sum_{n_{\alpha_1}, n_{\alpha_2}, \ldots} \lvert n_{\alpha_1}, n_{\alpha_2}, \ldots\rangle \langle n_{\alpha_1}, n_{\alpha_2}, \ldots\rvert = 1$.

The situation is more involved for fermions, since annihilation operators do not commute; if we assume relation 4.8 on page 35, we have

$$\hat{c}_{\alpha_1}\,\hat{c}_{\alpha_2}\left|\underline{\xi}\right\rangle = \xi_{\alpha_1}\,\xi_{\alpha_2}\left|\underline{\xi}\right\rangle = -\hat{c}_{\alpha_2}\,\hat{c}_{\alpha_1}\left|\underline{\xi}\right\rangle = -\xi_{\alpha_2}\,\xi_{\alpha_1}\left|\underline{\xi}\right\rangle$$
$$\Longrightarrow \quad \xi_{\alpha_1}\,\xi_{\alpha_2} = -\xi_{\alpha_2}\,\xi_{\alpha_1}$$

and then ξ_α cannot be complex numbers. These anti-commutation relations make sense if we extend the states Hilbert space to a larger super-Hilbert space $\mathcal{H}$ such that

$$\langle\psi\,|\,\phi\rangle \in \mathbb{G} \quad \text{and} \quad \theta_1\,|\psi\rangle + \theta_2\,|\phi\rangle \in \mathcal{H}, \quad \theta_1,\,\theta_2 \in \mathbb{G},$$

where $\mathbb{G}$ are not complex numbers but the so called *Grassmann numbers*. We need $\mathbb{G}$ to be an algebra – that is, summation and (non commutative) multiplication are defined in $\mathbb{G}$ – over $\mathbb{C}$ (that is, the product between complex number and Grassmann number is again a $\mathbb{G}$-number and complex numbers commutate with Grassmann numbers). We even have to consider $\bar{\xi}_\alpha$ defined as

$$\left\langle\underline{\xi}\right|\hat{c}^\dagger_\alpha = \left\langle\underline{\xi}\right|\bar{\xi}_\alpha.$$

At the end we may simply assume that $\{\,\bar{\xi},\,\xi\,\}_\xi$ are anti-commutant and generate the algebra, that is, $\mathbb{G}$ is the smallest algebra that contains all the products of such numbers. It is natural to require that

$$\bar{\bar{\xi}} = \xi, \qquad \overline{\xi\,\xi'\cdots} = \cdots\bar{\xi}'\,\bar{\xi}, \qquad \overline{a_1\,\xi + a_2\,\xi'} = \bar{a}_1\,\bar{\xi} + \bar{a}_2\,\bar{\xi}',$$

where $a \in \mathbb{C}$, and it is useful to observe that, for example, the Grassmann number $\theta_1 = \xi\,\xi'$ commutes with any other number $\theta_2 = a_1 + a_2\,\xi'' + a_3\,(\xi''')^*\,\xi'''' + \cdots$, since it is an even products of anti-commutant generators. We also have $\xi^2 = 0$ and Taylor expanding any analytic function we must have

$$f(\xi) = f_0 + f_1\,\xi, \qquad A(\bar{\xi},\xi') = a_0 + a_1\,\xi' + a_1^*\,\bar{\xi} + a_{12}\,\bar{\xi}\,\xi'.$$

It is possible to define the derivatives (identical to the complex number case) but one needs to pay attention to the order:

$$\frac{\partial}{\partial\xi'}(\bar{\xi}\,\xi') = \frac{\partial}{\partial\xi'}(-\xi'\,\bar{\xi}) = -\bar{\xi}.$$

To define an integral over Grassmann numbers, for which we have not a sup and an inf definitions, we don't use the Riemann approach. We may consider an algebraic definition; in particular we want

$$\int d\xi\,[a_1\,f(\xi) + a_2\,g(\xi)] = a_1\int d\xi\,f(\xi) + a_2\int d\xi\,g(\xi),$$
$$\delta(\xi - \xi') = \int d\xi''\,e^{-\xi''(\xi-\xi')},$$

where the last is motivated by the ordinary Fourier representation of the Dirac delta. In particular, Taylor expanding

$$\int d\xi'\,\delta(\xi-\xi')\,f(\xi') = \int d\xi' \int d\xi''\,[1-\xi''(\xi-\xi')]\,(f_0+f_1\,\xi') \\ \stackrel{!}{=} f_0+f_1\,\xi,$$

from which

$$\int d\xi\,1 = 0, \qquad \int d\xi\,\xi = 1.$$

Then we have the Berezin integral definition:

$$\int d\xi := \frac{\partial}{\partial\xi}, \qquad \int d\bar{\xi} := \frac{\partial}{\partial\bar{\xi}}.$$

It is easy to obtain the linear variables change rule for Berezin integrals (see table 4.1 on page 36; the difference with respect to the complex numbers case is the inverse of the Jacobian, as one may expect since Berezin integrals behave like derivatives) and then the Gaussian integral for Grassmann numbers (again table 4.1 on page 36).

Finally we may introduce the fermionic coherent states; but before it is convenient to fix the following algebra rules for the $\hat{c}_\alpha$ operators and their eigenvalues:

$$[\tilde{\xi},\tilde{c}]_+ = 0, \qquad \tilde{\xi}\,|0\rangle = |0\rangle\,\tilde{\xi}, \qquad \tilde{\xi}\,\langle 0| = \langle 0|\,\tilde{\xi}, \qquad (\tilde{\xi}\,\tilde{c})^\dagger = \tilde{c}^\dagger\,\tilde{\xi}^*$$

where $\tilde{\xi} \in \{\,\bar{\xi}_\alpha, \xi_\alpha\,\}$ and $\tilde{c} \in \{\,\hat{c}^\dagger_\alpha, \hat{c}_\alpha\,\}$ (the second rule establishes the relation between ξ and $\bar{\xi}$). As immediate consequence, $\tilde{\xi}$ commutes with fermionic even numbers states and anti-commutes with fermionic odd numbers states. With these rules it is quite easy to check that the eigenvectors are

$$\left|\underline{\xi}\right\rangle = e^{-\sum_\alpha \xi_\alpha\,\hat{c}^\dagger_\alpha}\,|0\rangle = \prod_\alpha (1-\xi_\alpha\,\hat{c}^\dagger_\alpha)\,|0\rangle$$

and to obtain the overlap relation $\left\langle \underline{\xi}' \middle| \underline{\xi} \right\rangle$ and the resolution of the identity in table 4.1 on page 36. Another useful relation, which can be easily verified, holds:

$$\left\langle n_{\alpha_1}, n_{\alpha_2}, \ldots \middle| \underline{\xi} \right\rangle \left\langle \underline{\xi} \middle| n'_{\alpha_1}, n'_{\alpha_2}, \ldots \right\rangle = \\ = \left\langle -\underline{\xi} \middle| n'_{\alpha_1}, n'_{\alpha_2}, \ldots \right\rangle \left\langle n_{\alpha_1}, n_{\alpha_2}, \ldots \middle| \underline{\xi} \right\rangle. \quad (4.9)$$

In summary, we can write normally-ordered observables for bosons and fermions in the following way:

$$\left\langle \underline{\xi} \middle| O(\underline{\hat{c}}^\dagger, \underline{\hat{c}}) \middle| \underline{\xi}' \right\rangle = \left\langle \underline{\xi} \middle| \underline{\xi}' \right\rangle O(\underline{\bar{\xi}}, \underline{\xi}') = e^{\sum_\alpha \bar{\xi}_\alpha\,\xi'_\alpha}\,O(\underline{\bar{\xi}}, \underline{\xi}'),$$

where $O(\underline{\bar{\xi}}, \underline{\xi}')$ must be a commutant Grassmann number (compare with section 4.1.2 on page 31).

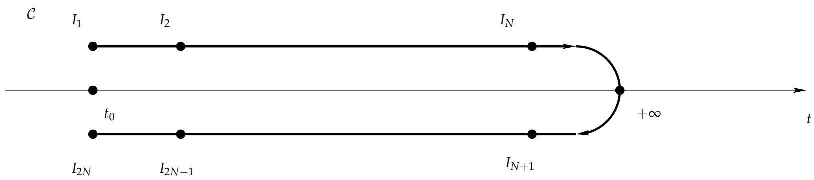


Figure 4.2: Position along the Keldysh contour of the resolution of the identies $I_j = \int d[\bar\phi_j, \bar\psi_j, \phi_j, \psi_j]\, e^{-\phi_j^\dagger \phi_j - \psi_j^\dagger \psi_j} \left|\phi_j, \psi_j\right\rangle \left\langle \phi_j, \psi_j\right| .$

4.2.3 *Keldysh path integral*

We indicate with $\hat b_\alpha$ the bosonic annihilation operators and $\hat b_\alpha |\phi\rangle = \phi_\alpha |\phi\rangle$; for fermions we use instead the notation $\hat c_\alpha |\psi\rangle = \psi_\alpha |\psi\rangle$. Then consider the definition 4.5 on page 34:

$$Z = \frac{1}{\mathrm{tr}[\hat\rho_0]} \,\mathrm{tr}[\hat\rho_0\, \hat U^- \,\hat U^+] = = \frac{1}{\mathrm{tr}[\hat\rho_0]} \sum_{n_{\alpha_1}, n_{\alpha_2}, \dots} \langle n_{\alpha_1}, n_{\alpha_2}, \dots \,|\, \hat\rho_0\, \underbrace{\hat U^-_{-dt} \cdots \hat U^-_{-dt}}_{N}\, 1 \cdot \cdot \underbrace{\hat U^+_{+dt} \cdots \hat U^+_{+dt}}_{N} \,|\, n_{\alpha_1}, n_{\alpha_2}, \dots \rangle$$

that is true in the limit $N \to \infty$. By inserting the resolution of the identity for the coherent states between the operators, as depicted in figure 4.2, by using relations 4.9 on the facing page and $\sum |n_\alpha\rangle \langle n_\alpha| = 1$ and by observing that

$$\left\langle \phi_i, \psi_i \,\middle|\, U^{\pm}_{\pm dt} \,\middle|\, \phi_{i-1}, \psi_{i-1} \right\rangle = e^{\phi_i^\dagger \phi_{i-1}}\, e^{\psi_i^\dagger \psi_{i-1}}\, e^{\mp i \hat H^{\pm}(\bar\psi_i, \bar\phi_i, \psi_{i-1}, \phi_{i-1})\, dt}$$

(where e. g. $\psi_i^\dagger \psi_{i-1} = \sum_\alpha (\bar\psi_i)_\alpha \,(\psi_{i-1})_\alpha$),[5] we get

$$Z = \frac{1}{\mathrm{tr}(\hat\rho_0)} \int D[\bar\phi, \bar\psi, \phi, \psi] \cdot \cdot \langle \phi_1, -\psi_1 \,|\, \hat\rho_0 \,|\, \phi_{2N}, \psi_{2N} \rangle \; e^{-\phi_1^\dagger \phi_1 - \psi_1^\dagger \psi_1}\, e^{i S[\bar\phi, \bar\psi, \phi, \psi]}, \quad (4.10)$$

where

$$S[\bar\phi, \bar\psi, \phi, \psi] = \sum_{j=2}^{2N} \left(i\, \phi_j^\dagger\, \frac{\phi_j - \phi_{j-1}}{dt}\, dt + i\, \psi_j^\dagger\, \frac{\psi_j - \psi_{j-1}}{dt}\, dt \right) + - i \sum_{j=2}^{N} H^+(\bar\psi_j, \bar\phi_j, \psi_{j-1}, \phi_{j-1})\, dt + + i \sum_{j=N+2}^{2N} \hat H^-(\bar\psi_j, \bar\phi_j, \psi_{j-1}, \phi_{j-1})\, dt \quad (4.11)$$

5 remember that α is an index that describes the single particle state and the particle species. For compactness we used α for both fermions and bosons; to make this notation consistent we then have to require that, when α refers to a boson, we have $\psi_\alpha = 0$ while, if it refers to a fermion, $\phi_\alpha = 0$.

and

$$D[\bar{\phi},\bar{\psi},\phi,\psi] = \prod_{j=1}^{2N} d[\bar{\phi}_j,\phi_j]\, d[\bar{\psi}_j,\psi_j].$$

Considering the limit $N \to \infty$, we have to substitute to j a continuous parameter that runs along the forward-backward contour of figure 4.2 on the previous page:

$$\begin{aligned}
&(\phi_1,\phi_2,\dots,\phi_N) \to \\
&\qquad (\phi^+(t_0),\dots,\phi^+(+\infty)) \to \phi^+(t) \qquad \text{forward-path,}\\
&(\phi_{N+1},\phi_{N+2},\dots,\phi_{2N}) \to \\
&\qquad\quad (\phi^-(+\infty),\dots,\phi^-(t_0)) \to \phi^-(t) \qquad \text{backward-path}
\end{aligned}$$

and analogous substitutions for ψ. Then we get:

$$\begin{aligned}
&S[\bar{\phi},\bar{\psi},\phi,\psi] = \\
&= \int_{t_0}^{+\infty} dt \left[\bar{\phi}^+(t)\, i\,\partial_t\, \phi^+(t) + \bar{\psi}^+(t)\, i\,\partial_t\, \psi^+(t) + \right.\\
&\left. - H^+(\bar{\psi}^+(t),\bar{\phi}^+(t),\psi^+(t),\phi^+(t))\right] + \\
&- \int_{t_0}^{+\infty} dt \left[\bar{\phi}^-(t)\, i\,\partial_t\, \phi^-(t) + \bar{\psi}^-(t)\, i\,\partial_t\, \psi^-(t) + \right.\\
&\qquad\qquad \left. - H^-(\bar{\psi}^-(t),\bar{\phi}^-(t),\psi^-(t),\phi^-(t))\right]
\end{aligned} \tag{4.12}$$

and $\int D[\bar{\phi},\bar{\psi},\phi,\psi]$ becomes a functional integration.[6]

In the following section we will use the notation

$$\begin{aligned}
\langle\cdots\rangle := \int D[\bar{\phi},\bar{\psi},\phi,\psi]\, \langle \phi^+(t_0), -\psi^+(t_0)\,|\,\hat{\rho}_0\,|\,\phi^-(t_0),\psi^-(t_0)\rangle \cdot \\
\cdot\, e^{-(\phi^+)^\dagger(t_0)\,\phi^+(t_0) - (\psi^+)^\dagger(t_0)\,\psi^+(t_0)}\, e^{i\,S[\bar{\phi},\bar{\psi},\phi,\psi]}\, \dots\, .
\end{aligned}$$

(compare with equation 4.10 on the preceding page).

4.2.4 *Two-point Green function*

If we want to perform an expansion to calculate the Keldysh path integral, a central role is played by the two-point Green function:

$$i\,\hat{G}^T(1,2) := \langle \psi^+_{\alpha_1}(t_1)\,\bar{\psi}^+_{\alpha_2}(t_2)\rangle\,, \tag{4.13a}$$

$$i\,\hat{G}^<(1,2) := \langle \psi^+_{\alpha_1}(t_1)\,\bar{\psi}^-_{\alpha_2}(t_2)\rangle\,, \tag{4.13b}$$

$$i\,\hat{G}^>(1,2) := \langle \psi^-_{\alpha_1}(t_1)\,\bar{\psi}^+_{\alpha_2}(t_2)\rangle\,, \tag{4.13c}$$

$$i\,\hat{G}^{\tilde{T}}(1,2) := \langle \psi^-_{\alpha_1}(t_1)\,\bar{\psi}^-_{\alpha_2}(t_2)\rangle\,, \tag{4.13d}$$

6 Since in the $N \to \infty$ limit we are working with distributions (such as the Dirac delta function in the footnote 3 on page 34), it would be more proper to write, for example, $H^+(\bar{\psi}^+(t),\bar{\phi}^+(t),\psi^+(t^-),\phi^+(t^-))$ instead of $H^+(\bar{\psi}^+(t),\bar{\phi}^+(t),\psi^+(t),\phi^+(t))$, to indicate that $\psi^+(t),\phi^+(t)$ must be evaluated before $\bar{\psi}^+(t),\bar{\phi}^+(t)$ (compare with the relation 4.11 on the previous page). Anyway we will follow the classical notation, but consider for example the discussion in the Kamenev book [28, p. 25] for more details.

and same definitions for bosons.[7] Furthermore we put:

$$\check{G}(1,2) := \begin{pmatrix} \hat{G}^T(1,2) & \hat{G}^<(1,2) \\ \hat{G}^>(1,2) & \hat{G}^{\tilde{T}}(1,2) \end{pmatrix}$$

and we may think to $\check{G}$ as an operator over α, t and contour region indexes, in the sense that:

$$\sum_2 \check{G}(1,2) \begin{pmatrix} \psi^+(2) \\ \psi^-(2) \end{pmatrix} = \\ = \sum_{\alpha_2,t_2} \begin{pmatrix} G^T(1,2)\, \psi^+_{\alpha_2}(t_2) + G^<(1,2)\, \psi^-_{\alpha_2}(t_2) \\ G^>(1,2)\, \psi^+_{\alpha_2}(t_2) + G^{\tilde{T}}(1,2)\, \psi^-_{\alpha_2}(t_2) \end{pmatrix},$$

where the summation over t_3 becomes an integral in the contimìnuum limit. The motivation of that will be clear soon.

Now be $\hat{d} = \hat{b}_\alpha$, $\hat{c}_\alpha$ a generic annihilation operator. We indicate with $\hat{d}_H(t)$ the time evolution of $\hat{d}$ in the Heisenberg representation; for semplicity we assume that $\hat{H}^\pm = \hat{H}$ (that is, we are not including observables in the time evolution). Then the time ordered product is defined as:

$$T[\hat{d}_H(1)\, \hat{d}^\dagger_H(2)] = \begin{cases} \hat{d}_H(1)\, \hat{d}^\dagger_H(2) & t_1 > t_2, \\ \zeta\, \hat{d}_H(2)^\dagger\, \hat{d}_H(1) & t_2 > t_1, \end{cases}$$

where ζ is 1 for bosons and -1 for fermions; in the same way, the time anti-oredered product is

$$\tilde{T}[\hat{d}_H(1)\, \hat{d}^\dagger_H(2)] = \begin{cases} \hat{d}_H(1)\, \hat{d}^\dagger_H(2) & t_2 > t_1, \\ \zeta\, \hat{d}_H(2)^\dagger\, \hat{d}_H(1) & t_1 > t_2. \end{cases}$$

Then we put

$$i\,\hat{G}^{++}(1,2) := \left\langle T[\hat{d}_H(1)\, \hat{d}^\dagger_H(2)] \right\rangle_{\rho_0}, \tag{4.14a}$$

$$i\,\hat{G}^{+-}(1,2) := \zeta \left\langle \hat{d}^\dagger_H(2)\, \hat{d}_H(1) \right\rangle_{\rho_0}, \tag{4.14b}$$

$$i\,\hat{G}^{-+}(1,2) := \left\langle \hat{d}_H(1)\, \hat{d}^\dagger_H(2) \right\rangle_{\rho_0}, \tag{4.14c}$$

$$i\,\hat{G}^{--}(1,2) := \left\langle \tilde{T}[\hat{d}_H(1)\, \hat{d}^\dagger_H(2)] \right\rangle_{\rho_0}, \tag{4.14d}$$

where $\langle \cdots \rangle_{\rho_0} = \mathrm{tr}(\rho_0 \cdots)$; note that we may easily interpret the $\hat{G}^{\pm\pm}$ as a time ordered product along the Keldysh contour 4.1 on page 34 ($\hat{G}^{++}$ is the ordered product when t_1, t_2 are both in the forward-path;

7 here the notation is: $1 = (\alpha_1, t_1)$ and $2 = (\alpha_2, t_2)$. In particular, the term "two-point" refers to (α_1, t_1) and (α_2, t_2); typically in the second quatization theory "two-point" refers to two space-temporal points, since $|\alpha\rangle = |x\rangle$.

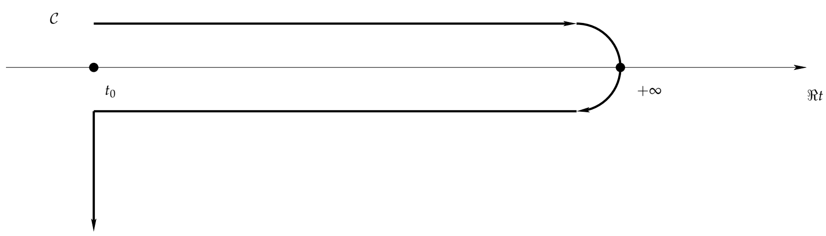


Figure 4.3: Kadanoff-Baym contour in the complex time plane.

$\hat{G}^{+-}$ is the case in which t_1 is in the forward-path and t_2 is in the backward-path, and so on). Furthermore it is useful to observe that they are not independent:

$$G^{++} + G^{--} = G^{+-} + G^{-+}$$

(remember that we are considering $\hat{H}^{\pm} = \hat{H}$ in this case) and, as we will see soon, the so called *Keldysh rotation* is a smart way to reduce their number.

By considering that $\hat{d}_H(t) = \hat{U}^{\dagger}\,\hat{d}\,\hat{U}$, writing

$$\hat{U} = \hat{U}_{dt}\cdots\hat{U}_{dt}, \qquad \hat{U}^{\dagger} = \hat{U}_{-dt}\cdots\hat{U}_{-dt}$$

and repeating the same steps of the section 4.2.3 on page 39 to put $\hat{G}^{\pm\pm}$ in a functional integral form, after a long but straightforward calculation, one obtains

$$\hat{G}^{++}(1,2) = \hat{G}^{T}(1,2), \qquad \hat{G}^{+-}(1,2) = \hat{G}^{<}(1,2),$$
$$\hat{G}^{-+}(1,2) = \hat{G}^{>}(1,2), \qquad \hat{G}^{--}(1,2) = \hat{G}^{\tilde{T}}(1,2),$$

for $t_1 \neq t_2$. Note that when $t_1 = t_2$ particular care is needed; for the bosonic case for example, it must be:

$$\langle[\hat{b}_{\alpha}, \hat{b}^{\dagger}_{\alpha}]\rangle_{\rho_0} = 1 = -i\left[\lim_{t_2\to t_1^+} G^{++}_{\alpha\alpha}(t_1,t_2) - \lim_{t_2\to t_1^-} G^{++}_{\alpha\alpha}(t_1,t_2)\right]$$

but

$$\left\langle\phi^{+}_{\alpha}(t_1)\,\bar{\phi}^{+}_{\alpha}(t_1)\right\rangle - \left\langle\bar{\phi}^{+}_{\alpha}(t_1)\,\phi^{+}_{\alpha}(t_1)\right\rangle = 0.$$

4.2.4.1 *The choice of ρ_0*

In the typical applications we may imagine that the system is at equilibrium at time t_0:

$$\hat{\rho}_0 = e^{-\beta(\hat{H} - \sum_p \mu_p \hat{N}_p)} = e^{-\beta \hat{H}_S}$$

where H contains both the free part H_0 and the interaction H_I between particles, and at time t_0 a term $H_1(t)$ that drives the system out the equilibrium is switched on. In this case, typically we cannot calculate ρ_0 exactly because of the interaction H_I and we need to expand even

$$e^{-\beta \hat{H}_S} = \hat{U}^{H_S}_{-i\,d\beta} \cdots \hat{U}^{H_S}_{-i\,d\beta},$$

where $i\,d\beta$ can be interpreted as an imaginary time step in the so called Kadanoff-Baym contour depicted in figure 4.3 on the preceding page.

Luckily in most cases we don't need that; for example, if we are not interested in transient phenomena at the initial times, we may consider $t_0 = -\infty$ and, after a sufficient long time, we may expect that the system lose the memory of the initial correlations. Then we can simply put

$$\hat{\rho}_0 = e^{-\beta\,(\hat{H}_0 - \sum_p \mu_p\,\hat{N}_p)},$$

including the interactions only in the time evolution operator: $\hat{U}_{dt} = 1 - i\,[\hat{H}_0 + \hat{H}_I + \hat{H}_1(t)]\,dt$ (see [46] for more details).

4.2.4.2 *Free Green function*

Now we want to calculate the two point Green function for the single particle Hamiltonian H_0:

$$\hat{\rho}_0 = e^{-\beta\,(\hat{H}_0 - \sum_p \mu_p\,\hat{N}_p)}, \qquad \hat{U}_{t_2,t_1} = e^{-i\,\hat{H}_0\,(t_2 - t_1)}.$$

Without loss of generality, we may assume in this section that $|\alpha\rangle$ are the single particle states that diagonalize H_0 (remember that in α it is even included an index that select the species of the particle), so that

$$H_0 = \sum_\alpha \epsilon_\alpha\,\hat{d}^\dagger_\alpha\,\hat{d}_\alpha = \sum_\alpha \epsilon_\alpha\,\hat{N}_\alpha$$

and then:

$$\hat{\rho}_0 = e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu_\alpha)\,\hat{N}_\alpha}, \qquad \hat{U}_{t_2,t_1} = e^{-i\,\epsilon_\alpha\,\hat{N}_\alpha\,(t_2 - t_1)}.$$

In particular:

- $\langle \phi | x^{\hat{N}_\alpha} | \phi' \rangle = e^{\bar{\phi}_\alpha\,\phi'_\alpha\,x}$ for bosons,[8] then:
$$\left\langle \phi \,\middle|\, e^{-\beta\,(\epsilon_\alpha - \mu_\alpha)\,\hat{b}^\dagger_\alpha\,\hat{b}_\alpha} \,\middle|\, \phi' \right\rangle = \exp\left[\bar{\phi}_\alpha\,\phi'_\alpha\,e^{-\beta\,(\epsilon_\alpha - \mu_\alpha)}\right];$$
- form fermions an analogous calculation leads to:
$$\left\langle -\psi \,\middle|\, e^{-\beta\,(\epsilon_\alpha - \mu_\alpha)\,\hat{c}^\dagger_\alpha\,\hat{c}_\alpha} \,\middle|\, \psi' \right\rangle = \exp\left[-\bar{\psi}_\alpha\,\psi'_\alpha\,e^{-\beta\,(\epsilon_\alpha - \mu_\alpha)}\right];$$

8 this relation can be proved simply by observing that $f(\hat{b}^\dagger_\alpha, \hat{b}_\alpha)\,\hat{b}_\alpha = \hat{b}_\alpha\,f(\hat{b}^\dagger_\alpha, \hat{b}_\alpha - 1)$ hence:

$$\partial_x \langle \phi | x^{\hat{b}^\dagger_\alpha \hat{b}_\alpha} | \phi' \rangle = \langle \phi | \hat{b}^\dagger\,\hat{b}\,x^{\hat{b}^\dagger_\alpha \hat{b}_\alpha - 1} | \phi' \rangle = \langle \phi | \hat{b}^\dagger\,x^{\hat{b}^\dagger_\alpha \hat{b}_\alpha}\,\hat{b}_\alpha | \phi' \rangle = \bar{\phi}_\alpha\,\phi'_\alpha\,\langle \phi | x^{\hat{b}^\dagger_\alpha \hat{b}_\alpha} | \phi' \rangle.$$

- finally:

$$H_0(\bar{\psi}, \bar{\phi}, \psi', \phi') = \sum_{\alpha} \epsilon_\alpha \left(\bar{\psi}_\alpha \, \psi'_\alpha + \bar{\phi}_\alpha \, \phi'_\alpha \right).$$

Collecting all terms, we have that equation 4.10 on page 39 is of the form:

$$Z_0 = \frac{1}{\operatorname{tr}(\hat{\rho}_0)} \int D[\bar{\phi}, \bar{\psi}, \phi, \psi] \cdot$$
$$\cdot \exp\Bigg\{ \sum_{\alpha,\alpha',t,t'} \Bigg[\begin{pmatrix} \bar{\psi}^+_\alpha(t) & \bar{\psi}^-_\alpha(t) \end{pmatrix} \check{M}_{\alpha\alpha'}(t,t') \begin{pmatrix} \psi^+_{\alpha'}(t') \\ \psi^-_{\alpha'}(t') \end{pmatrix} +$$
$$+ \begin{pmatrix} \bar{\phi}^+_\alpha(t) & \bar{\phi}^-_\alpha(t) \end{pmatrix} \check{M}_{\alpha\alpha'}(t,t') \begin{pmatrix} \phi^+_{\alpha'}(t') \\ \phi^-_{\alpha'}(t') \end{pmatrix} \Bigg] \Bigg\},$$

(where the 0-subscript indicates that we are considering the free case)[9] that is a Gaussian integral.[10] Furthermore, since $\hat{H}^\pm = \hat{H}_0$, it is $Z_0 = \operatorname{tr}(\hat{U}^\dagger \, \hat{U} \, \hat{\rho}_0) / \operatorname{tr}(\hat{\rho}_0) = 1$. But by differentiating the Gaussian integral in table 4.1 on page 36 with respect to $\partial_{\bar{\eta}_\alpha} \partial_{\eta_{\alpha'}} |_{\eta=\bar{\eta}=0}$[11] one gets immediately

$$\langle \psi_\alpha(t) \, \bar{\psi}_{\alpha'}(t') \rangle_0 = \check{M}^{-1}_{\alpha\alpha'}(t,t')$$

and analogous relation for bosons. Comparing with the definition 4.13 on page 40, we must conclude that

$$\check{M}_{\alpha_1\alpha_2}(t_1, t_2) = \check{G}_0^{-1}(1,2).$$

9 to be more explicit, the form of $\check{M}$ is

$$\check{M}_{\alpha\alpha'}(j,j') = \delta_{\alpha,\alpha'} \cdot$$
$$\cdot \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \zeta\,\rho(\epsilon_\alpha) \\ \ddots & -1 & & & & & & \\ & h_-(\epsilon_\alpha) & -1 & & & & & \\ & & h_-(\epsilon_\alpha) & -1 & & & & \\ \hline & & & 1 & -1 & & & \\ & & & & h_+(\epsilon_\alpha) & -1 & & \\ & & & & & h_+(\epsilon_\alpha) & -1 & \\ & & & & & & \ddots & \ddots \end{array} \right)$$

where $j, j' = 1, \ldots, 2N$ are the discretized time indexes, $\zeta = \pm 1$ respectively for bosons and fermions, $\rho(\epsilon_\alpha) = e^{-\beta(\epsilon_\alpha - \mu_\alpha)}$ and $h_\pm(\epsilon_\alpha) = 1 \pm i \, \epsilon_\alpha \, dt$.

10 that reflects the fact that typically we know how to perform calculations without an expansion only in the free case.

11 the terms $(\det M)^{-\zeta}$ cancel with $1/\operatorname{tr}(\hat{\rho}_0)$, since $Z_0 = 1$.

Now we can use 4.14 on page 41 to easily evaluate the free Green functions:[12]

$$\begin{aligned} i\,\hat{G}_0^<(1,2) &= \zeta\, n_{B/F}(\epsilon_{\alpha_1})\, e^{-i\,\epsilon_{\alpha_1}\,(t_1-t_2)}\,\delta_{\alpha_1,\alpha_2},\\ i\,\hat{G}_0^>(1,2) &= [\zeta\, n_{B/F}(\epsilon_{\alpha_1})+1]\, e^{-i\,\epsilon_{\alpha_1}\,(t_1-t_2)}\,\delta_{\alpha_1,\alpha_2},\\ i\,\hat{G}_0^T(1,2) &= \theta(t_1-t_2)\, i\,\hat{G}^>(1,2)+\theta(t_2-t_1)\, i\,\hat{G}^<(1,2),\\ i\,\hat{G}_0^{\tilde{T}}(1,2) &= \theta(t_2-t_1)\, i\,\hat{G}^>(1,2)+\theta(t_1-t_2)\, i\,\hat{G}^<(1,2), \end{aligned}$$

where, as usual, $\zeta = 1$ for bosons and -1 for fermions and

$$n_{B/F}(\epsilon_\alpha) := \frac{1}{e^{\beta\,(\epsilon_\alpha-\mu_\alpha)}-\zeta}$$

are the bosonic/fermionic occupation numbers.[13]

4.2.5 *Keldysh rotation*

4.2.5.1 *Bosons case*

When Feynman introduced the path integral formalism, he wanted to obtain a formulation of quantum mechanics that could lead easily to the classical limit. In particular, while a "full quantum" particle that goes from x_1 to x_2 can follow every path, in the classical limit $\hbar \to 0$ the probability that it follows a path different from the classical one goes to zero. In our case the state of the system evolves following the Keldysh countour and, in the full quantum case, it could choose a path in the forward time direction and another for the backward one. We may expect that in the classical limit, when no observables are included in the evolution operators ($H^\pm = H$), we have only one path and then:

$$\phi^+(t)-\phi^-(t) = \bar{\phi}^+(t)-\bar{\phi}^-(t) = 0,$$

12 if we want for example evaluate $G_0^<$ for bosons, we may observe that in the free case

$$\begin{aligned} (\hat{b}_\alpha)_H(t) &= \exp\left(i\sum_{\alpha'}\epsilon_{\alpha'}\,\hat{b}^\dagger_{\alpha'}\,\hat{b}_{\alpha'}\right)\hat{b}_\alpha\,\exp\left(-i\,\sum_{\alpha''}\epsilon_{\alpha''}\,\hat{b}^\dagger_{\alpha''}\,\hat{b}_{\alpha''}\right)=\\ &= \exp\left(i\,\epsilon_\alpha\,\hat{b}^\dagger_\alpha\,\hat{b}_\alpha\right)\hat{b}_\alpha\,\exp\left(-i\,\epsilon_\alpha\,\hat{b}^\dagger_\alpha\,\hat{b}_\alpha\right)=\\ &= \exp\left(i\,\epsilon_\alpha\,\hat{b}^\dagger_\alpha\,\hat{b}_\alpha\right)\exp\left(-i\,\epsilon_\alpha\,(\hat{b}^\dagger_\alpha\,\hat{b}_\alpha+1)\right)\hat{b}_\alpha = e^{i\,\epsilon_\alpha\,t}\,\hat{b}_\alpha \end{aligned}$$

and h. c. At the end we have

$$i\,\hat{G}_0^<(1,2) = \left\langle e^{-i\,\epsilon_{\alpha_2}\,t_2}\,e^{-i\,\epsilon_{\alpha_1}\,t_1}\,\hat{b}^\dagger_{\alpha_2}\,\hat{b}_{\alpha_1}\right\rangle_{\rho_0} = e^{-i\,\epsilon_{\alpha_1}\,(t_1-t_2)}\,n_B(\epsilon_{\alpha_1})\,\delta_{\alpha_1,\alpha_2}.$$

13 If $\hat{H}_0$ is not diagonal with respect to $|\alpha\rangle$ but it is still a single particle operator, that is $\hat{H}_0 = \sum_{\alpha\alpha'}\epsilon_{\alpha\alpha'}\hat{d}^\dagger_{\alpha'}\,\hat{d}_\alpha$ (for example $|\alpha\rangle = |x\rangle$ and H_0 is diagonal with respect to $|p\rangle$), the expression for Green function is a bit more complicated but the calculation is similar.

that suggests to introduce the change of variables:

$$\phi^{\text{cl}} = \frac{\phi^+ + \phi^-}{\sqrt{2}}, \qquad \phi^{\text{q}} = \frac{\phi^+ - \phi^-}{\sqrt{2}},$$
$$\bar{\phi}^{\text{cl}} = \frac{\bar{\phi}^+ + \bar{\phi}^-}{\sqrt{2}}, \qquad \bar{\phi}^{\text{q}} = \frac{\bar{\phi}^+ - \bar{\phi}^-}{\sqrt{2}}$$

(classical and quantum fields). The $\sqrt{2}$ factor guarantees that the Jacobian is unitary in the functional integral and for the free case it is easy to check that

$$Z_0 = \frac{1}{\text{tr}(\hat{\rho}_0)} \int D[\bar{\phi}^{\text{cl}}, \bar{\phi}^{\text{q}}, \phi^{\text{cl}}, \phi^{\text{q}}] \cdot \\ \cdot \exp\left[\sum_{\alpha,\alpha',t,t'} \begin{pmatrix} \bar{\phi}^{\text{cl}}_{\alpha_1}(t_1) & \bar{\phi}^{\text{q}}_{\alpha_1}(t_1) \end{pmatrix} \check{G}_0^{-1}(1,2) \begin{pmatrix} \phi^{\text{cl}}_{\alpha_2}(t_2) \\ \phi^{\text{q}}_{\alpha_2}(t_2) \end{pmatrix} \right]$$

where we must substitute the $\check{G}_0$ defined in the previous sections with

$$i\,\check{G}_0(1,2) = \begin{pmatrix} i\,\hat{G}_0^K(1,2) & i\,\hat{G}_0^R(1,2) \\ i\,\hat{G}_0^A(1,2) & 0 \end{pmatrix} = \\ = \left\{ \left\langle \phi^j_{\alpha_1}(t_1)\, \bar{\phi}^k_{\alpha_2}(t_2) \right\rangle \right\}_{j,k=\text{cl},\text{q}}$$

and

$$i\,\hat{G}_0^R(1,2) = \theta(t_1 - t_2)\, e^{-i\,\epsilon_{\alpha_1}(t_1-t_2)}\, \delta_{\alpha_1,\alpha_2},$$
$$i\,\hat{G}_0^A(1,2) = -\theta(t_2 - t_1)\, e^{-i\,\epsilon_{\alpha_1}(t_1-t_2)}\, \delta_{\alpha_1,\alpha_2},$$
$$i\,\hat{G}_0^K(1,2) = [1 + 2\,n_B(\epsilon_{\alpha_1})]\, e^{-i\,\epsilon_{\alpha_1}(t_1-t_2)}\, \delta_{\alpha_1,\alpha_2}.$$

Observe that the advanced and retarded Green functions $\hat{G}_0^{A/R}$ contain information about the spectrum, while the Keldysh Green function $\hat{G}_0^K$ contains statistical information.

The Green functions depends only on the time difference (a consequence of the time invariance present in the free-particle case) and we may Fourier tranform with respect the time

$$\hat{G}_0^R(1,2) \xrightarrow{\text{F.T.}} \left(\epsilon - \epsilon_{\alpha_1} + i\,0^+\right)^{-1} \delta_{\alpha_1,\alpha_2},$$
$$\hat{G}_0^A(1,2) \xrightarrow{\text{F.T.}} \left(\epsilon - \epsilon_{\alpha_1} - i\,0^+\right)^{-1} \delta_{\alpha_1,\alpha_2},$$
$$\hat{G}_0^K(1,2) \xrightarrow{\text{F.T.}} -2\,\pi\,i\,[1 + 2\,n_B(\epsilon)]\,\delta(\epsilon - \epsilon_{\alpha_1})\,\delta_{\alpha_1,\alpha_2}.$$

By observing that

$$1 + 2\,n_B(\epsilon) = \coth\frac{\beta\,(\epsilon - \mu)}{2}$$

and using the Sokhotski–Plemelj theorem:

$$\frac{1}{x + i\,0^{\pm}} = P\,\frac{1}{x} \mp i\,\pi\,\delta(x)$$

(where P stands for the Cauchy principal value), we get immediately

$$\hat{G}_0^K(\epsilon) = \coth\frac{\beta(\epsilon-\mu_\alpha)}{2}\left[\hat{G}_0^R(\epsilon) - \hat{G}_0^A(\epsilon)\right]$$

that constitutes the fluctuation-dissipation theorem (remember that in this case the statistical Hamiltonian coincides with the evolution Hamiltonian $\hat{H}_0$ and then we are not going out of equilibrium).

We may write the action in terms of classical and quantum field and observe that for $H^{\pm} = H$

$$S[\bar{\phi}^{\text{cl}}, \phi^{\text{cl}}, \bar{\phi}^{\text{q}} = \phi^{\text{q}} = 0] = 0, \tag{4.15}$$

since in that case the forward-path contribution cancels the backward one (see equation 4.12 on page 40).

At the end, the classical limit is obtained with the saddle point equation and requiring that the quantum field is zero:

$$\begin{cases} \frac{\delta S}{\delta \bar{\phi}^{\text{cl}}} = \frac{\delta S}{\delta \phi^{\text{cl}}} = 0 \\ \frac{\delta S}{\delta \bar{\phi}^{\text{q}}} = \frac{\delta S}{\delta \phi^{\text{q}}} = 0 \\ \bar{\phi}^{\text{q}} = \phi^{\text{q}} = 0 \end{cases} \iff \left.\frac{\delta S}{\delta \bar{\phi}^{\text{q}}}\right|_{\bar{\phi}^{\text{q}} = \phi^{\text{q}} = 0} = \left.\frac{\delta S}{\delta \phi^{\text{q}}}\right|_{\bar{\phi}^{\text{q}} = \phi^{\text{q}} = 0} = 0, \tag{4.16}$$

because of 4.15. One may ask if there are solutions of the saddle point equation

$$\begin{cases} \frac{\delta S}{\delta \bar{\phi}^{\text{cl}}} = \frac{\delta S}{\delta \phi^{\text{cl}}} = 0 \\ \frac{\delta S}{\delta \bar{\phi}^{\text{q}}} = \frac{\delta S}{\delta \phi^{\text{q}}} = 0 \end{cases}$$

for which ϕ^{q} is not identically zero. The answer is positive when $\hat{H}^+ \neq \hat{H}^-$ and some examples include tunneling, thermal activation and oscillatory contributions to the level statistics (see [28]).

4.2.5.2 *Fermion case*

The Grassmann number have no classical meaning but it is useful to introduce a Keldysh rotation for them also (we have reduced the number of Green function $\check{G}_0$ for bosons after the Keldysh rotation). In particular we are more free in the choice of the rotation and we may follow Larkin and Ovchinnikov notation:

$$\psi^1 = \frac{\psi^+ + \psi^-}{\sqrt{2}}, \qquad \psi^2 = \frac{\psi^+ - \psi^-}{\sqrt{2}},$$
$$\bar{\psi}^1 = \frac{\bar{\psi}^+ - \bar{\psi}^-}{\sqrt{2}}, \qquad \bar{\psi}^2 = \frac{\bar{\psi}^+ + \bar{\psi}^-}{\sqrt{2}}.$$

Note that, with this choice, $\bar{\psi}^i$ is not the conjugate of ψ^i as defined in the previuos sections.

After the Keldysh rotation, for the free case we obtain

$$Z_0 = \int \frac{D[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2]}{\mathrm{tr}(\hat{\rho}_0)} \cdot \\ \cdot \exp\left[\sum_{\alpha,\alpha',t,t'} \begin{pmatrix} \bar{\psi}^1_{\alpha_1}(t_1) & \bar{\psi}^2_{\alpha_1}(t_1) \end{pmatrix} \check{G}_0^{-1}(1,2) \begin{pmatrix} \psi^1_{\alpha_2}(t_2) \\ \psi^2_{\alpha_2}(t_2) \end{pmatrix} \right]$$

where

$$i\,\check{G}_0(1,2) = \begin{pmatrix} i\,\hat{G}_0^R(1,2) & i\,\hat{G}_0^K(1,2) \\ 0 & i\,\hat{G}_0^A(1,2) \end{pmatrix} = \\ = \left\{ \left\langle \phi^i_{\alpha_1}(t_1)\, \bar{\phi}^j_{\alpha_2}(t_2) \right\rangle \right\}_{i,j=1,2}$$

and

$$i\,\hat{G}_0^R(1,2) = \theta(t_1 - t_2)\, e^{-i\,\epsilon_{\alpha_1}(t_1 - t_2)}\, \delta_{\alpha_1,\alpha_2}, \tag{4.17a}$$

$$i\,\hat{G}_0^A(1,2) = -\theta(t_2 - t_1)\, e^{-i\,\epsilon_{\alpha_1}(t_1 - t_2)}\, \delta_{\alpha_1,\alpha_2}, \tag{4.17b}$$

$$i\,\hat{G}_0^K(1,2) = [1 - 2\,n_F(\epsilon_{\alpha_1})]\, e^{-i\,\epsilon_{\alpha_1}(t_1 - t_2)}\, \delta_{\alpha_1,\alpha_2}. \tag{4.17c}$$

The Larkin and Ovchinnikov choise presents some advantages for the calculations (remember that the product of lower triangular matrices is again a lower triangular matrix).

Finally the fluctuation-dissipation theorem for fermions reads

$$\hat{G}_0^K(\epsilon) = \tanh\frac{\beta\,(\epsilon - \mu_\alpha)}{2} \left[\hat{G}_0^R(\epsilon) - \hat{G}_0^A(\epsilon) \right].$$

Note that the only difference with respect to the bosonic case is that the hyperbolic tangent takes the place of the hyperbolic cotangent.

4.2.6 *Expansions and Wick theorem*

To introduce the Wick theorem, we consider the fermionic case (the bosonic one is similar). The Wick theorem for Grassmann number is easily obtained by differentiating the Gaussian integral in table 4.1 on page 36 with respect to $\partial_{\bar{\eta}_{\alpha_1}} \cdots \partial_{\bar{\eta}_{\alpha_n}} \partial_{\eta_{\alpha'_1}} \cdots \partial_{\eta_{\alpha'_n}} |_{\bar{\eta}=\eta=0}$:

$$\langle \psi_{\alpha_1} \cdots \psi_{\alpha_n}\, \bar{\psi}_{\alpha'_n} \cdots \bar{\psi}_{\alpha'_1} \rangle_0 = \\ = \det(M) \sum_\pi \mathrm{sign}(\pi)\, M^{-1}_{\alpha_{\pi(n)}\alpha'_n} \cdots M^{-1}_{\alpha_{\pi(1)}\alpha'_1},$$

where π are permutations.

We now assume that in equation 4.10 on page 39 we have $S = S_0 + S_1$ (S_0 and S_1 are commuting Grassmann numbers - compare with section 4.1.2 on page 31) and then

$$Z = \int \frac{D[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2]}{\mathrm{tr}(\hat{\rho}_0)} \langle -\psi_1 \,|\, \hat{\rho}_0 \,|\, \psi_{2N} \rangle \, e^{-\psi_1^\dagger \psi_1} \, e^{i\,S[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2]} =$$

$$= \frac{1}{\mathrm{tr}(\hat{\rho}_0)} \int D[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2] \cdot$$

$$\cdot \exp \left[\sum_{\alpha, \alpha', t, t'} \begin{pmatrix} \bar{\psi}^1_{\alpha_1}(t_1) & \bar{\psi}^2_{\alpha_1}(t_1) \end{pmatrix} \check{G}_0^{-1}(1,2) \begin{pmatrix} \psi^1_{\alpha_2}(t_2) \\ \psi^2_{\alpha_2}(t_2) \end{pmatrix} \right] \cdot$$

$$\cdot \, e^{i\,S_1[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2]} = \left\langle e^{i\,S_1[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2]} \right\rangle_0,$$

where, when $S_1 = 0$, we deal with a Gaussian integral (as we have seen, this is the case when $\hat{H}_0$ is a single particle operator). Finally we may expand the exponential:

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(i\,S_1[\bar{\psi}^1, \bar{\psi}^2, \psi^1, \psi^2] \right)^n \right\rangle_0;$$

to fix ideas, we may assume that S_1 is the Keldysh action associated to a Coulomb interaction (that is something of the form 4.2 on page 32) and then, by applying Wick theorem, we may evaluate each order of the expansion by simply knowing the two point free Green function (and associate Feynman diagrams to each term).

5 THE MANY-BODY MODEL

Motivated by quantum optics works like [36], our original idea was to try to generalize the Wang and Sham model to the case of several nanomagnets. We tried to use the same methods of Wang and Sham (namely the derivation of a master equation for the magnets, in the coarse graining time scale approximation - see section 3.3 on page 23), but that was infeasible from a mathematical point of view. Instead we found that the Keldysh formalism [16] is a promising framework as basis to generalize the Wang and Sham model in many directions (several magnets, electron-electron interaction,. . .) and produces a dynamic equation for the magnet with an easy physical interpretation. I detail the results of this original research in this chapter and the following.

Therefore we consider the Wang and Sham single-electron Hamiltonian with a weak external magnetic field $\vec{B}$:

$$\hat{H} = -\frac{\partial_{\bar{x}}^2}{2\,m} + \gamma\,\vec{B}\cdot\vec{J} + \delta(\bar{x})\left(\lambda_0 + \lambda\,\vec{s}\cdot\vec{J}\right),$$

where γ is the electron gyromagnetic ratio, m is the electron mass and we have chosen a unit system in which $\hbar = 1$. As in [63], we considered the electron moving only in the $\bar{x}$ direction. But in principle the model is easily generalizable: if the electric current flows in a lead with nanometric transverse dimensions, one can quantize the electron state along $\bar{y}$ and $\bar{z}$ and consider the eigenstates along these directions as current channels [38]. The only mathematical complication is that the magnetic scattering center in $\bar{x} = 0$ produces a mixing between the channels. Anyhow the single channel approximation is good for nanometric leads [1, p. 747].

The Keldysh formalism will allow us to treat that model directly in a many-electrons framework, but to do that we need to translate also the magnet degrees of freedom in the many-body language. How it was done in [53], we will consider the Holstein-Primakoff bosonization, which allows to consider directly a semi-classical approximation for the magnet dynamics.

5.1 HOLSTEIN-PRIMAKOFF BOSONIZATION

We consider the operators

$$\hat{S}_+ := s\left(\sqrt{2 - \frac{\hat{b}^\dagger\,\hat{b}}{s}}\right)\frac{\hat{b}}{\sqrt{s}}, \qquad \hat{S}_- := s\,\frac{\hat{b}^\dagger}{\sqrt{s}}\left(\sqrt{2 - \frac{\hat{b}^\dagger\,\hat{b}}{s}}\right),$$
$$\hat{S}_z := s - \hat{b}^\dagger\,\hat{b}$$

and

$$\hat{S}_x := \frac{S_+ + S_-}{2}, \qquad \hat{S}_y := \frac{S_+ - S_-}{2\,i}$$

(compare with the definition 3.4 on page 24), where $\hat{b}^\dagger$ and $\hat{b}$ are bosonic creation and annihilation operators, that is

$$[\hat{b}, \hat{b}^\dagger] = 1, \qquad [\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0.$$

By using these relations and Taylor expanding the square root in the $\hat{S}_\pm$ definition, one gets:

$$[\hat{S}_i, \hat{S}_j] = i\,\epsilon_{ijk}\,\hat{S}_k, \qquad i, j, k = x, y, z,$$

where ϵ_{ijk} is the Levi-Civita symbol, that is the standard commutation relations for the spin operators. This is called Holstein-Primakoff bosonization.

To have a perfect corrispondence between bosons and spin, we need only to restrict the Fock space to states that count a number of bosons between 0 and $2\,s$, since $\hat{J}_z$ has eigenvalues $-s, -s+1, \dots, s$. Furthermore the Holstein-Primakoff representation is useful for the semi-classical approximation; indeed, the zero-boson state $|0\rangle = |m = s\rangle$ is the generic spin coherent state up to a rotation (see definition 3.6 on page 25). For that state, we have

$$\left\langle 0 \,\middle|\, \hat{b} \,\middle|\, 0 \right\rangle = \left\langle 0 \,\middle|\, \hat{b}^\dagger \,\middle|\, 0 \right\rangle =$$
$$= \langle m = s \,|\, \hat{S}_x \,|\, m = s\rangle = \langle m = s \,|\, \hat{S}_y \,|\, m = s\rangle = 0$$

and for large s and slight deviation from the coherent state, that is, for states that are combination of few bosons states (i. e. for states such that $\langle \hat{S}_{x,y}\rangle \ll s$), we obtain

$$\frac{\langle \hat{b}\rangle}{\sqrt{s}}, \frac{\langle \hat{b}^\dagger\rangle}{\sqrt{s}} = O\left(\frac{1}{\sqrt{s}}\right), \qquad \langle \hat{b}^\dagger\,\hat{b}\rangle \ll s.$$

This means that we are considering states that have slight quantum and thermal fluctuations around the $\hat{z}$ axis.

We may apply all these considerations to our magnet spin $\vec{J}$: we may assume that $\hat{z}$ is, in a moving Cartesian coordinate system, the

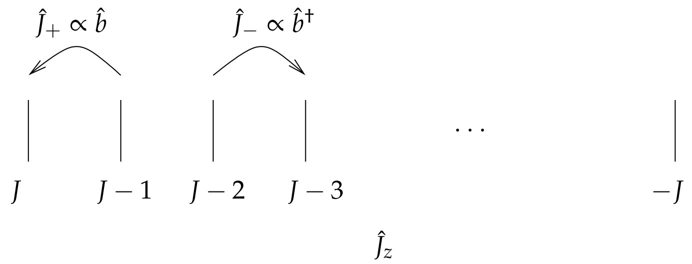


Figure 5.1: Representation of the Holstein-Primakoff bosonization in the semi-classical approximation. The zero bosons case coincides with the classical limit. To create a boson means decreasing the spin value along $\hat{z}$.

direction of the magnet in the classical limit. Then, to obtain the semi-classical approximation, we Taylor expand the relations 5.1 on the preceding page:

$$\hat{J}_+ = J\left[\sqrt{2}\,\frac{\hat{b}}{\sqrt{J}} + O\left(\frac{1}{\sqrt{J}^3}\right)\right] = \sqrt{2\,J}\,\hat{b} + O\left(\frac{1}{\sqrt{J}}\right), \tag{5.1a}$$

$$\hat{J}_- = J\left[\sqrt{2}\,\frac{\hat{b}^\dagger}{\sqrt{J}} + O\left(\frac{1}{\sqrt{J}^3}\right)\right] = \sqrt{2\,J}\,\hat{b}^\dagger + O\left(\frac{1}{\sqrt{J}}\right), \tag{5.1b}$$

$$\begin{aligned}\hat{J}_z = J\left[1 - \frac{\hat{b}^\dagger}{\sqrt{J}}\,\frac{\hat{b}}{\sqrt{J}}\right] = J\left[1 - \frac{\hat{b}^\dagger}{\sqrt{J}}\,\frac{\hat{b}}{\sqrt{J}} + O\left(\frac{1}{\sqrt{J}^3}\right)\right] = \\ = J - \hat{b}^\dagger\,\hat{b} + O\left(\frac{1}{\sqrt{J}}\right);\end{aligned} \tag{5.1c}$$

in particular we will consider the terms of order J, the terms with one bosonic (creation or annihilation) operator – that are suppressed by a factor $1/\sqrt{J}$ with respect to the J order terms – and the terms with two bosonic operators – that are suppressed by a factor $1/J$ (see figure 5.1).

5.2 THE MANY-BODY HAMILTONIAN

Now we may write the many-body Hamiltonian as:

$$\hat{H} = \hat{H}_{\mathrm{m}} + \hat{H}_{\mathrm{oe}} + \hat{H}_b,$$

where:

- the magnet part is:

$$\hat{H}_{\mathrm{m}} = \gamma\,\vec{B}\cdot\vec{J} = \frac{\gamma\sqrt{J}}{\sqrt{2}}\,B_+\,\hat{b}^\dagger + \frac{\gamma\sqrt{J}}{\sqrt{2}}\,B_-\,\hat{b} + \gamma\,B_z\left(J - \hat{b}^\dagger\,\hat{b}\right)$$

  (again $B_\pm := B_x \pm i\,B_y$);

- the part of the single-electron Hamiltonian that does not contain the bosonic operators is $\hat{H}_{\rm oe} = -\frac{\partial^2_{\bar{x}}}{2m} + (\lambda_0 + \lambda\, J\, s_z)\, \delta(\bar{x})$; it is diagonal with respect to the scattering states Ψ_{ks} given by:

$$\Psi_{|k|s}(\bar{x}) = |s\rangle \otimes N \cdot \begin{cases} e^{i\,|k|\,\bar{x}} + r_s(k)\, e^{-i\,|k|\,\bar{x}}, & \bar{x} < 0, \\ t_s(k)\, e^{i\,|k|\,\bar{x}}, & \bar{x} > 0, \end{cases} \tag{5.2a}$$

$$\Psi_{-|k|s}(\bar{x}) = |s\rangle \otimes N \cdot \begin{cases} t_s(k)\, e^{-i\,|k|\,\bar{x}}, & \bar{x} < 0, \\ e^{-i\,|k|\,\bar{x}} + r_s(k)\, e^{i\,|k|\,\bar{x}}, & \bar{x} > 0, \end{cases} \tag{5.2b}$$

that constitute a basis for electrons (respectively coming from left to right and from right to left), where

$$\hat{H}_{\rm oe}\Psi_{ks} = \epsilon_{ks}\, \Psi_{ks}, \qquad \epsilon_{ks} = \epsilon_k = \frac{k^2}{2\,m}, \qquad s = \pm \tag{5.3a}$$

$$r_s(k) = \frac{1}{-1 + i\,\frac{|k|}{m\,(\lambda_0 + \lambda\, s\, J/2)}}, \qquad t_s(k) = \frac{1}{1 + i\,\frac{m\,(\lambda_0 + \lambda\, s\, J/2)}{|k|}}, \tag{5.3b}$$

N is a real normalization constant[1] and $|s\rangle$ is the spinor with respect to the quantization axis z. Then the many-electrons free Hamiltonian is written as

$$\hat{H}_{\rm oe} = \sum_{ks} \epsilon_k\, \hat{c}^\dagger_{ks}\, \hat{c}_{ks},$$

where $\hat{c}_{ks}$ creates an electron in the state Ψ_{ks}. It is important to stress that the choice to consider an expansion based on the scattering eigenfuctions Ψ_{ks} is a key point in our argumentation that allows us to go beyond the Wang-Sham approach;

- $\hat{H}_b$ is the remaining Hamiltonian part that contains the $\hat{b}$, $\hat{b}^\dagger$ operators. It can be considered a perturbative term. Indeed, while $\hat{H}_{\rm oe}$ contains the terms λ_0 and $\lambda\, J$ that can be considered of the similar order (see figure 3.1 on page 18), it is easy to check that $\hat{H}_b$ is given by the sum of terms that contain a single bosonic operator ($\hat{b}$ or $\hat{b}^\dagger$) which is of the order $\lambda\,\sqrt{J}$ (and then suppressed by a factor $1/\sqrt{J}$ with respect to $\lambda\, J$) and a term proportional to $\hat{b}\,\hat{b}^\dagger$ which is of the order λ (and then suppressed by a factor $1/J$ with respect to $\lambda\, J$). Therefore we can say that $\hat{H}_{\rm oe}$ is zero-order in $1/\sqrt{J}$, terms with a single bosonic operators is of the first order and the term with two bosonic operators is of the second order. We neglect higher order terms, as described in formula 5.1 on the preceding page, since for typical applications $J \sim 10^4$ [63]. Anyhow we will write the explicit form of $\hat{H}_b$ in a next section.

1 for example, if we consider that electron are bounded in a region of $\bar{x}$ with dimension L and with periodic boundary conditions, we have $N = 1/\sqrt{L}$; if L is much greater with respect to the characteristic electron wave length, we may consider the continuous limit for k and $N = 1/\sqrt{2\,\pi}$.

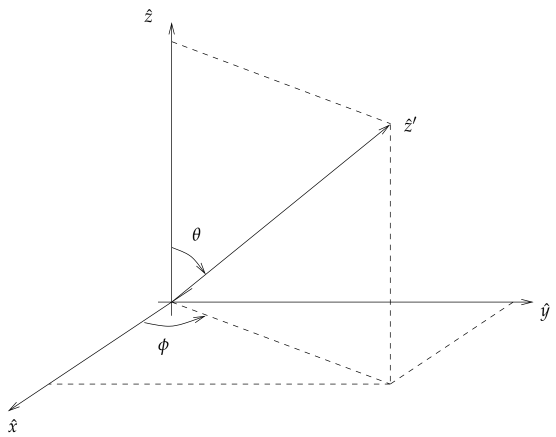


Figure 5.2: $\hat{z}$ is the quantization axis for the magnet, while $\hat{z}'$ is current polarization axis.

5.3 INITIAL STATE

The incoming current is polarized with respect to the $\hat{z}'$ axis (see figure 5.2). The spin states in the two quantization axis are related by

$$\hat{U}(\theta,\phi)\,|s\rangle = |s'\rangle\,,$$

$$\langle s_1' \,|\, \hat{U} \,|\, s_2' \rangle = \langle s_1 \,|\, \hat{U} \,|\, s_2 \rangle = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} & e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix},$$

where $\hat{U} = e^{-i\hat{s}_z\phi}\,e^{-i\hat{s}_y\theta}$. If $\hat{c}^\dagger_{ks}$ creates an electron in the state $|k,s\rangle$, $\hat{c}^\dagger_{ks'}$ creates an electron in the state $|k,s'\rangle$:

$$\hat{c}^\dagger_{ks_1} = \sum_{s_2'} \langle s_2' \,|\, s_1 \rangle\, \hat{c}^\dagger_{ks_2'} = \sum_{s_2'} \left\langle s_2' \,\middle|\, \hat{U}^\dagger \,\middle|\, s_1' \right\rangle \hat{c}^\dagger_{ks_2'}. \tag{5.4}$$

Since the H_{oe} eigenvalues do not depend on s, we may write

$$\hat{H}_{\text{oe}} = \sum_{ks'} \epsilon_k\, \hat{c}^\dagger_{ks'}\, \hat{c}_{ks'}.$$

Then we assume that the incoming electrons density matrix is given by

$$\hat{\rho}_0^{s',d} = \frac{1}{\mathcal{Z}_{s',d}} \exp\left[-\beta \sum_k \left(\epsilon_k - \mu^{s',d}\right) \hat{c}^\dagger_{d|k|,s'} \cdot \hat{c}_{d|k|,s'}\right], \quad s,d = \pm,$$

where the index d describes the direction of the electrons motion. This means that we are introducing a differential potential between the left and right regions:

$$\mu^{s',d} = \epsilon_F + e\,V_0^d + s'\,2\,\mu_B\,B_0^d,$$

where ϵ_F is the Fermi energy, V_0^d is an electric potential, B_0^d is a magnetic field and μ_B is the Bohr magneton. In particular, in the typical cases, B_0^d is the magnetic field due to the presence of hard ferromagnetic layers (see figure 5.3 on the following page).

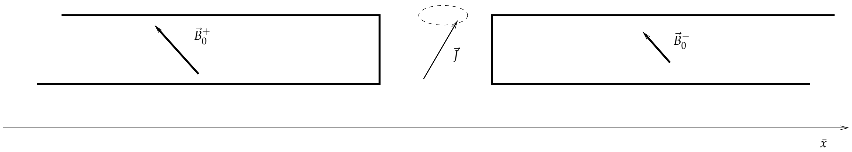


Figure 5.3: In a possible experimental setup, the nano-magnet is placed between two hard ferromagnetic contacts. In particular $\vec{B}^{\pm} \parallel \hat{z}'$ and $\vec{J} \parallel \hat{z}$.

(a) *$\hat{z}$-parallel contribution vertex. It involves two bosons and it is of the second order in $1/\sqrt{J}$ with respect to $\hat{H}_{oe}$, as described in section 5.2 on page 53.*

(b) *$\hat{z}$-perpendicular contribution vertex. It involves a single boson and it is of the first order in $1/\sqrt{J}$ with respect to $\hat{H}_{oe}$, as described in section 5.2.*

Figure 5.4: $\hat{H}_b$ vertices.

5.4 H_b HAMILTONIAN COMPONENT

The operator that annihilates (creates) an electron in x with spin s is given by

$$\hat{\psi}_s(x) = \sum_k \Psi_{ks}(x;s)\, c_{ks}, \qquad \hat{\psi}_s^\dagger(x) = \sum_k \Psi_{ks}^*(x;s)\, c_{ks}^\dagger,$$

where $\Psi_{ks}(x;s) := \langle s|\, \Psi_{ks}(x)$, from which

$$\hat{\psi}_s(0) = N \sum_k t_s(k)\, c_{ks}, \qquad \hat{\psi}_s^\dagger(0) = N \sum_k t_s^*(k)\, c_{ks}^\dagger.$$

Then

$$\hat{H}_b = \lambda \sum_{s_1 s_2} \int d\bar{x} \left[-\frac{1}{2}\, \hat{b}^\dagger\, \hat{b}\, \sigma^3_{s_2 s_1} + \frac{\sqrt{2J}}{2}\, \sigma^+_{s_2 s_1}\, \hat{b}^\dagger + \frac{\sqrt{2J}}{2}\, \sigma^-_{s_2 s_1}\, \hat{b} \right] \cdot \cdot\, \hat{\psi}^\dagger_{s_2}(\bar{x})\, \hat{\psi}_{s_1}(\bar{x})\, \delta(\bar{x});$$

integrating over $\bar{x}$ and using relation 5.4 on the previous page:

$$\hat{H}_b = \sum_{s_2' s_1' k_2 k_1} \left\{ \hat{b}^\dagger\, \hat{b}\, \mathcal{M}^{s_2 s_1}_{k_2 k_1}(\parallel)\, \hat{c}^\dagger_{k_2 s_2'}\, \hat{c}_{k_1 s_1'} + + \left[\hat{b}\, \mathcal{M}^{s_2 s_1}_{k_2 k_1}(\perp)\, \hat{c}^\dagger_{k_2 s_2'}\, \hat{c}_{k_1 s_1'} + \text{h.c.} \right] \right\} = H_{b\parallel} + H_{b\perp},$$

where

- the $\hat{z}$-parallel contribution coefficient is given by

$$\mathcal{M}^{s_2' s_1'}_{k_2 k_1}(\|) := -\frac{\lambda}{4} N^2 \cdot$$
$$\cdot \left[t_+^*(k_2)\, t_+(k_1)\, \mathcal{L}_+^{s_2' s_1'}(\|) + t_-^*(k_2)\, t_-(k_1)\, \mathcal{L}_-^{s_2' s_1'}(\|) \right],$$

  with

$$\mathcal{L}_+^{s_2' s_1'}(\|) = \begin{pmatrix} 1+\cos\theta & -\sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix},$$
$$\mathcal{L}_-^{s_2' s_1'}(\|) = \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ -\sin\theta & -1-\cos\theta \end{pmatrix}$$

  and we will indicate this contribution with the vertex in figure 5.4a on the facing page;

- while the $\hat{z}$-perpendicular contribution coefficient is

$$\mathcal{M}^{s_2' s_1'}_{k_2 k_1}(\perp) = \lambda \frac{\sqrt{2J}}{2} N^2\, t_-^*(k_2)\, t_+(k_1)\, \mathcal{L}^{s_2' s_1'}(\perp),$$

  with

$$\mathcal{L}^{s_2' s_1'}(\perp) = e^{-i\phi} \begin{pmatrix} \frac{\sin\theta}{2} & -\sin^2\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} & -\frac{\sin\theta}{2} \end{pmatrix}$$

  and we will indicate this contribution with the vertex in figure 5.4b on the preceding page.

It is relevant to note that using the scattering states makes simple the form of the free Green function, as we will see. Furthermore, in the $\hat{H}_b$ Hamiltonian, the $\hat{z}$-perpendicular part has a simple interpretation: if an electron is flipped up (the $\sigma^+ \hat{b}^\dagger$ term) a boson is created (that is the magnet spin along $\hat{z}$ is decreased) and vice versa. This is a consequence of the total angular momentum conservation.

6 THE CLASSICAL CURRENT

In this chapter we are going to calculate the electric current flowing in the magnet, without considering the quantum fluctuations (that we will call *classical current*) and assuming that the fermionic dynamics is much faster than the bosonic one.

To discard the quantum fluctuations of the magnet, we need $\hat{b}$, $\hat{b}^\dagger \to 0$. In particular, the Hamiltonian reduces to H_{0e}, except for constant terms.

The calculation is substantially similar to that proposed in [52]. The differences are that:

- in our case, the delta function modelizes a three-layer system. Instead in [52] a two-layer system is considered: the electrons move from a non-magnetic to a magnetic layer. This is obtained with a step potential model, which produces a different form for the reflection and transmission amplitudes (in particular the transmission amplitudes are real for the step potential);
- in [52] the one-dimensional approximation is not considered; anyway the interface is considered flat and then the transverse wave function component remains unchanged during the scattering.

By using the language of the scattering matrix [38], we may write the wave functions 5.2 on page 54 as:

$$\Psi_{|k|L_\rightarrow}(x) = N \times \begin{cases} |L_\rightarrow\rangle\, e^{i\,|k|\,\bar{x}} + \hat{r}(k)\,|L_\rightarrow\rangle\, e^{-i\,|k|\,\bar{x}}, & \bar{x} < 0, \\ \hat{t}(k)\,|L_\rightarrow\rangle\, e^{i\,|k|\,\bar{x}}, & \bar{x} > 0, \end{cases} \tag{6.1a}$$

$$\Psi_{-|k|R_\leftarrow}(x) = N \times \begin{cases} \hat{t}(k)\,|R_\leftarrow\rangle\, e^{-i\,|k|\,\bar{x}}, & \bar{x} < 0, \\ |R_\leftarrow\rangle\, e^{-i\,|k|\,\bar{x}} + \hat{r}(k)\,|R_\leftarrow\rangle\, e^{i\,|k|\,\bar{x}}, & \bar{x} > 0, \end{cases} \tag{6.1b}$$

where $|L_\rightarrow\rangle$ and $|R_\leftarrow\rangle$ are spinors and $\hat{t}$, $\hat{r}$ are spinor operators given by

$$\langle s_1\,|\,\hat{r}(k)\,|\,s_2\rangle = \begin{pmatrix} r_+(k) & 0 \\ 0 & r_-(k) \end{pmatrix},$$

$$\langle s_1\,|\,\hat{t}(k)\,|\,s_2\rangle = \begin{pmatrix} t_+(k) & 0 \\ 0 & t_-(k) \end{pmatrix}$$

with respect to the $\hat{z}$-quantization axis. We may transform the operators in the $\hat{z}'$-quatization axis:

$$\langle s_1' \,|\, \hat{t}(k) \,|\, s_2' \rangle = \sum_{s_3 s_4} \left\langle s_1 \,\middle|\, \hat{U}^\dagger \,\middle|\, s_3 \right\rangle \langle s_3 \,|\, \hat{t}(k) \,|\, s_4 \rangle \langle s_4 \,|\, \hat{U} \,|\, s_2 \rangle =$$
$$= \begin{pmatrix} t_+(k)\,\cos^2\frac{\theta}{2} + t_-(k)\,\sin^2\frac{\theta}{2} & -\frac{1}{2}\left[t_+(k) - t_-(k)\right]\sin\theta \\ -\frac{1}{2}\left[t_+(k) - t_-(k)\right]\sin\theta & t_-(k)\,\cos^2\frac{\theta}{2} + t_+(k)\,\sin^2\frac{\theta}{2} \end{pmatrix} \tag{6.2}$$

where θ, ϕ may be considered as time-depending parameters in our case (fermionic dynamics much faster than bosonic one), and analog formula for $\hat{r}(k)$.

In particular, $\hat{c}^\dagger_{\pm|k|s'}$ creates an electron in the single-particle state $\Psi_{\pm|k|s'}$ (given by relations 6.1 on the previous page, with $|L_\rightarrow\rangle = |s'\rangle$ and $|R_\leftarrow\rangle = |s'\rangle$ respectively).

The current must be conserved along $\bar{x}$ in our stationary case; in particular we may concentrate on the transmitted waves. The current associated to a plane wave $\Psi(\bar{x}) = A\,e^{\pm i\,|k|\,\bar{x}}$ is given by

$$I = \frac{e}{m}\,\Im\left(\Psi^*\,\partial_{\bar{x}}\Psi\right) = \pm\frac{e}{m}\,|k|\,|A|^2, \tag{6.3}$$

where $e < 0$ is the electron charge. If we are interested in the spin current, e must be substituted by $1/2$.

By using relations 6.2, we obtain the spin currents:

- for electrons with spin $|\uparrow'\rangle$ coming respectively from left to right ($+$ direction) and vice versa ($-$ direction), we have a transmitted $\uparrow'$-component:

$$I_\pm^{\uparrow'\uparrow'} = \pm\sum_{|k|} n_F^{\mu^{\uparrow'\pm}}(\epsilon_k)\,\frac{N^2\,|k|}{2\,m}\left|t_\uparrow(k)\,\cos^2\frac{\theta}{2} + t_\downarrow(k)\,\sin^2\frac{\theta}{2}\right|^2,$$

  and a transmitted $\downarrow'$-component:

$$I_\pm^{\uparrow'\downarrow'} = \pm\sum_{|k|} n_F^{\mu^{\uparrow'\pm}}(\epsilon_k)\,\frac{N^2\,|k|}{2\,m}\,|t_\uparrow(k) - t_\downarrow(k)|^2\,\frac{\sin^2\theta}{4};$$

- for electrons with spin $|\downarrow'\rangle$:

$$I_\pm^{\downarrow'\uparrow'} = \pm\sum_{|k|} n_F^{\mu^{\downarrow'\pm}}(\epsilon_k)\,\frac{N^2\,|k|}{2\,m}\,|t_\uparrow(k) - t_\downarrow(k)|^2\,\frac{\sin^2\theta}{4},$$

$$I_\pm^{\downarrow'\downarrow'} = \pm\sum_{|k|} n_F^{\mu^{\downarrow'\pm}}(\epsilon_k)\,\frac{N^2\,|k|}{2\,m}\left|t_\uparrow(k)\,\sin^2\frac{\theta}{2} + t_\downarrow(k)\,\cos^2\frac{\theta}{2}\right|^2,$$

(to simplify the notation, here we indicate the direction of electrons motion along $\bar{x}$ with $\pm$ and the spin with $\uparrow\downarrow$).

We define

$$I^{s'} := \sum_{d,s'_0} I_d^{s'_0 s'}, \qquad I_{\text{spin}} := I^{\uparrow'} - I^{\downarrow'}.$$

In the continuous limit for k (that is the linear dimension of the system along $\bar{x}$ is much greater with respect to the characteristic electron wavelength) and in the low temperature limit, it is possible to calculate all the currents by using the Sommerfeld expansion:

$$\int_0^{+\infty} d\epsilon\, n_F^{\mu}(\epsilon)\, H(\epsilon) = \int_0^{\mu} d\epsilon\, H(\epsilon) + \frac{\pi^2}{6\,\beta^2}\, H'(\mu) + O\left(\frac{1}{\beta\,\mu}\right)^4.$$

To compare with results obtained in [63], we consider the zero temperature limit and we assume that all the chemical potentials have similar values: $\mu^{s'd} \simeq \epsilon_F$ (Ohmic limit); in particular

$$n_F^{\mu}(\epsilon) = \theta(\epsilon - \mu) \simeq \theta(\epsilon - \epsilon_F) + \delta(\epsilon - \epsilon_F)\,(\mu - \epsilon_F). \tag{6.4}$$

Since the currents may be written as

$$I_{\pm}^{s'_1 s'_2} = \sum_{|k|} n_F^{\mu^{s'_1 \pm}}(\epsilon_k)\, f^{s'_1 s'_2}(k) = \int_0^{+\infty} d\epsilon\, \frac{m}{|k_\epsilon|}\, n_F^{\mu^{s'_1 \pm}}(\epsilon)\, f^{s'_1 s'_2}(k_\epsilon)$$

(the fact that f does not depend on the electron motion direction $\pm$ is a consequence of the parity symmetry), by using approximation 6.4 we have

$$I_{+}^{s_1 s'_2} + I_{-}^{s'_1 s'_2} = \frac{m}{k_F}\, f^{s'_1 s'_2}(k_F)\, \Delta\mu^{s'_1},$$

where $\Delta\mu^{s'_1} =: \mu^{s'_1 +} - \mu^{s'_1 -}$ is the difference of chemical potentials for the two electron motion directions. In particular for example

$$I_{\text{spin}} = \frac{N^2}{8}\Big\{\Delta\mu_{\text{spin}}\left[|t_\uparrow + t_\downarrow|^2 + |t_\uparrow - t_\downarrow|^2 \cos(2\,\theta)\right] + \\ + 2\,\Delta\mu_{\text{charge}}\left(|t_\uparrow|^2 - |t_\downarrow|^2\right)\cos\theta\Big\}, \tag{6.5}$$

where $t_{\uparrow\downarrow} := t_{\uparrow\downarrow}(k_F)$ and

$$\Delta\mu_{\text{spin}} := \Delta\mu^{\uparrow'} - \Delta\mu^{\downarrow'}, \qquad \Delta\mu_{\text{charge}} := \Delta\mu^{\uparrow'} + \Delta\mu^{\downarrow'};$$

this means that

- if the scattering potential does not depend on the spin, that is $t_\uparrow = t_\downarrow =: t$,

  $$I_{\text{spin}} = \frac{N^2\,|t|^2}{2}\,\Delta\mu_{\text{spin}}$$

  and there is a linear relation between the the spin current and the polarization potential;

- if the polarization potential $\Delta\mu_{\text{spin}}$ is zero, we still have a polarized current due to the magnet interaction:

$$I_{\text{spin}} = \frac{N^2}{4}\Delta\mu_{\text{charge}}\left(|t_\uparrow|^2 - |t_\downarrow|^2\right)\cos\theta.$$

The charge current is given instead by:

$$I_{\text{charge}} = 2\,e\left(I^{\uparrow'} + I^{\downarrow'}\right) = \frac{N^2\,e}{2}\left[|t_\uparrow|^2\left(\Delta\mu_{\text{charge}} + \Delta\mu_{\text{spin}}\cos\theta\right) + |t_\downarrow|^2\left(\Delta\mu_{\text{charge}} - \Delta\mu_{\text{spin}}\cos\theta\right)\right] \quad (6.6)$$

and we have a potential/current linear relation that does not depend on the magnet orientation when

- the scattering potential does not depend on the spin

$$I_{\text{charge}} = N^2\,e\,|t|^2\,\Delta\mu_{\text{charge}};$$

- the polarization potential is zero:

$$I_{\text{charge}} = N^2\,e\left(|t_\uparrow|^2 + |t_\downarrow|^2\right)\Delta\mu_{\text{charge}}.$$

We may summarize relations 6.5 on the preceding page and 6.6:

$$\begin{pmatrix} I_{\text{charge}} \\ I_{\text{spin}} \end{pmatrix} = \hat{G}(\theta)\begin{pmatrix} \Delta\mu_{\text{charge}} \\ \Delta\mu_{\text{spin}} \end{pmatrix}. \quad (6.7)$$

Note that the conductance $\hat{G}$ does not depend on ϕ, as one may expect from the symmetry of the model.

It is also useful to introduce

$$\Delta\mu^L_{\text{spin}} = \mu^{\uparrow'+} - \mu^{\downarrow'+}, \qquad \Delta\mu^R_{\text{spin}} = \mu^{\uparrow'-} - \mu^{\downarrow'-},$$

Indeed, in [63] some polarized electrons are sent only from left to right; in that case, we have to assume that $\Delta\mu^R_{\text{spin}} = 0$, which means $\Delta\mu_{\text{spin}} = \Delta\mu^L_{\text{spin}}$.

7 KELDYSH ACTION

7.1 MAGNET-ELECTRONS ACTION

The magnet-electrons Keldysh action for our system (see section 4.2 on page 33) is given by

$$S = \int_{-\infty}^{+\infty} dt \Big(\bar{b}_+ \, i\,\partial_t \, b_+ + \sum_{ks'} (\bar{\psi}_+)_{ks'} \, i\,\partial_t \, (\psi_+)_{ks'} + \\ - H[\bar{b}_+, \bar{\psi}_+, b_+, \psi_+] \Big) + \\ - \int_{-\infty}^{+\infty} dt \Big(\bar{b}_- \, i\,\partial_t \, b_- + \sum_{ks'} (\bar{\psi}_-)_{ks'} \, i\,\partial_t \, (\psi_-)_{ks'} + \\ - H[\bar{b}_-, \bar{\psi}_-, b_-, \psi_-] \Big) = \int dt \, \mathcal{L}$$

where $b_\pm$ are number that correspond to the bosonic degree of freedom and $\psi_\pm$ are Grassmann number for fermions modes.

We introduce the Keldysh rotation:

$$b^{\mathrm{cl}} = \frac{b_+ + b_-}{\sqrt{2}}, \qquad b^{\mathrm{q}} = \frac{b_+ - b_-}{\sqrt{2}},$$
$$\bar{b}^{\mathrm{cl}} = \frac{\bar{b}_+ + \bar{b}_-}{\sqrt{2}}, \qquad \bar{b}^{\mathrm{q}} = \frac{\bar{b}_+ - \bar{b}_-}{\sqrt{2}},$$

for bosons and

$$\psi_1 = \frac{\psi_+ + \psi_-}{\sqrt{2}}, \qquad \psi_2 = \frac{\psi_+ - \psi_-}{\sqrt{2}},$$
$$\bar{\psi}_1 = \frac{\bar{\psi}_+ - \bar{\psi}_-}{\sqrt{2}}, \qquad \bar{\psi}_2 = \frac{\bar{\psi}_+ + \bar{\psi}_-}{\sqrt{2}},$$

for fermions. Then the action is the sum of the components:

- from $\mathcal{L}_m = \bar{b}_+ \, i\,\partial_t \, b_+ - H_m[\bar{b}_+, b_+] - \bar{b}_- \, i\,\partial_t \, b_- + H_m[\bar{b}_-, b_-]$, we get:

$$S_m = \int dt \begin{pmatrix} \bar{b}^{\mathrm{cl}} & \bar{b}^{\mathrm{q}} \end{pmatrix} \begin{pmatrix} 0 & i\,\partial_t + \gamma \, B_z \\ i\,\partial_t + \gamma \, B_z & 0 \end{pmatrix} \begin{pmatrix} b^{\mathrm{cl}} \\ b^{\mathrm{q}} \end{pmatrix} + \\ - \int dt \left[\frac{\gamma \sqrt{J}}{\sqrt{2}} B_+ \left(\bar{b}_+ - \bar{b}_- \right) + \frac{\gamma \sqrt{J}}{\sqrt{2}} B_- \left(b_+ - b_- \right) \right] = \\ = \int dt \left[\bar{b}^{\mathrm{q}} \left(i\,\partial_t + \gamma \, B_z \right) b^{\mathrm{cl}} - \gamma \sqrt{J} \, B_+ \, \bar{b}^{\mathrm{q}} \right] + \mathrm{h.c.}, \quad (7.1)$$

where the first integral has been evaluated by parts;

- consider now

$$\mathcal{L}_{\text{oe}} = \sum_{ks'} (\bar{\psi}_+)_{ks'}\, i\,\partial_t\, (\psi_+)_{ks'} - H_m[\bar{\psi}_+, \psi_+] + \\ - \sum_{ks'} (\bar{\psi}_-)_{ks'}\, i\,\partial_t\, (\psi_-)_{ks'} + H_m[\bar{\psi}_-, \psi_-];$$

we introduce:

$$\bar{\psi} = \Big(\begin{pmatrix} \bar{\psi}_{1\uparrow'+} & \bar{\psi}_{2\uparrow'+} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{1\downarrow'+} & \bar{\psi}_{2\downarrow'+} \end{pmatrix} \\ \begin{pmatrix} \bar{\psi}_{1\uparrow'-} & \bar{\psi}_{2\uparrow'-} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{1\downarrow'-} & \bar{\psi}_{2\downarrow'-} \end{pmatrix} \Big)$$

and

$$\psi = \begin{pmatrix} \begin{pmatrix} \psi_{1\uparrow'+} \\ \psi_{2\uparrow'+} \end{pmatrix} \\ \begin{pmatrix} \psi_{1\downarrow'+} \\ \psi_{2\downarrow'+} \end{pmatrix} \\ \begin{pmatrix} \psi_{1\uparrow'-} \\ \psi_{2\uparrow'-} \end{pmatrix} \\ \begin{pmatrix} \psi_{1\downarrow'-} \\ \psi_{2\downarrow'-} \end{pmatrix} \end{pmatrix},$$

where $\uparrow'\downarrow'$ refer to spin along $\hat{z}'$, $\pm$ to the electron motion direction and each $\psi_{is'd}$ is a block indexed by the momentum $|k|$:

$$\bar{\psi}_{is'\pm} = \begin{pmatrix} \bar{\psi}_{is'\pm|k_1|} & \bar{\psi}_{is'\pm|k_2|} & \cdots \end{pmatrix}, \qquad \psi_{is'\pm} = \begin{pmatrix} \psi_{is'\pm|k_1|} \\ \psi_{is'\pm|k_2|} \\ \vdots \end{pmatrix}.$$

With this notation,

$$S_{\text{oe}} = \int dt \sum_{ks} \begin{pmatrix} \bar{\psi}_{1ks} & \bar{\psi}_{2ks} \end{pmatrix} \begin{pmatrix} i\,\partial_t - \epsilon_k & 0 \\ 0 & i\,\partial_t - \epsilon_k \end{pmatrix} \begin{pmatrix} \psi_{1ks} \\ \psi_{2ks} \end{pmatrix} \\ = \int dt\, \bar{\psi}\, \check{G}_0^{-1} \psi,$$

where

$$\check{G}_0^{-1} = \hat{I}_4 \otimes \hat{G}_0^{-1} \otimes \hat{\gamma}^{\text{cl}}$$

with $\hat{I}_4$ the 4×4-identity matrix and we use the standard notation:

$$\hat{\gamma}^{\text{cl}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{\gamma}^{\text{q}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\left(\hat{G}_0^{-1}(t) \right)_{|k_1||k_2|} = \delta_{12} \left(i\,\partial_t - \epsilon_{k_1} \right).$$

Observe that the classical gamma matrix is diagonal in the Keldysh space, while the quantum gamma matrix flips the Keldysh components;

- from $H_{b\|}$ we obtain

$$
\begin{aligned}
S_{b\|} = & -\int dt \sum_{s_2' s_1' k_2 k_1} \frac{1}{2} \mathcal{M}^{s_2' s_1'}_{k_2 k_1}(\|) \big[\left(\bar{b}^{\text{cl}} \, b^{\text{cl}} + \bar{b}^{\text{q}} \, b^{\text{q}} \right) \cdot \\
& \cdot \left(\bar{\psi}_{1 k_2 s_2'} \, \psi_{1 k_1 s_1'} + \bar{\psi}_{2 k_2 s_2'} \, \psi_{2 k_1 s_1'} \right) + \\
& + \left(\bar{b}^{\text{cl}} \, b^{\text{q}} + \bar{b}^{\text{q}} \, b^{\text{cl}} \right) \left(\bar{\psi}_{2 k_2 s_2'} \, \psi_{1 k_1 s_1'} + \bar{\psi}_{1 k_2 s_2'} \, \psi_{2 k_1 s_1'} \right) \big] = \\
& = -\frac{1}{2} \int dt \, \bar{\psi} \Big[\left(\bar{b}^{\text{cl}} \, b^{\text{cl}} + \bar{b}^{\text{q}} \, b^{\text{q}} \right) \hat{\mathcal{M}}(\|) \otimes \hat{\gamma}^{\text{cl}} + \\
& \qquad \qquad + \left(\bar{b}^{\text{cl}} \, b^{\text{q}} + \bar{b}^{\text{q}} \, b^{\text{cl}} \right) \hat{\mathcal{M}}(\|) \otimes \hat{\gamma}^{\text{q}} \Big] \psi,
\end{aligned}
$$

where

$$
\hat{\mathcal{M}}(\|) := \begin{pmatrix}
\hat{\mathcal{M}}^{\uparrow'\uparrow'}_{++}(\|) & \hat{\mathcal{M}}^{\uparrow'\downarrow'}_{++}(\|) & \hat{\mathcal{M}}^{\uparrow'\uparrow'}_{+-}(\|) & \hat{\mathcal{M}}^{\uparrow'\downarrow'}_{+-}(\|) \\
\hat{\mathcal{M}}^{\downarrow'\uparrow'}_{++}(\|) & \hat{\mathcal{M}}^{\downarrow'\downarrow'}_{++}(\|) & \hat{\mathcal{M}}^{\downarrow'\uparrow'}_{+-}(\|) & \hat{\mathcal{M}}^{\downarrow'\downarrow'}_{+-}(\|) \\
\hat{\mathcal{M}}^{\uparrow'\uparrow'}_{-+}(\|) & \hat{\mathcal{M}}^{\uparrow'\downarrow'}_{-+}(\|) & \hat{\mathcal{M}}^{\uparrow'\uparrow'}_{--}(\|) & \hat{\mathcal{M}}^{\uparrow'\downarrow'}_{--}(\|) \\
\hat{\mathcal{M}}^{\downarrow'\uparrow'}_{-+}(\|) & \hat{\mathcal{M}}^{\downarrow'\downarrow'}_{-+}(\|) & \hat{\mathcal{M}}^{'\downarrow\uparrow'}_{--}(\|) & \hat{\mathcal{M}}^{\downarrow'\downarrow'}_{--}(\|)
\end{pmatrix}
$$

and

$$
\left(\hat{\mathcal{M}}^{s_2' s_1'}_{d_2 d_1}(\|) \right)_{|k_2|,|k_1|} := \mathcal{M}^{s_2' s_1'}_{d_2 |k_2|, d_1 |k_1|}(\|), \qquad d_1, d_2 = \pm;
$$

- from $H_{b\perp}$ we obtain

$$
\begin{aligned}
& S_{b\perp} = \\
& \quad -\frac{1}{\sqrt{2}} \int dt \, \bar{\psi} \left\{ \sum_{\alpha = \text{cl,q}} \left[b^{\alpha} \, \hat{\mathcal{M}}(\perp) + \bar{b}^{\alpha} \, \hat{\mathcal{M}}^{\dagger}(\perp) \right] \otimes \hat{\gamma}^{\alpha} \right\} \psi,
\end{aligned}
$$

with analogous meaning of the symbols.

7.2 MAGNET ACTION

The total action is of the form

$$
\begin{aligned}
S &= S_m + S_{\text{oe}} + S_{b\|} + S_{b\perp} = \\
&= S_m + \int dt \, \bar{\psi} \left[\check{G}_0^{-1} + \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right] \psi = \\
&\qquad \qquad = S_m + \int dt \, \bar{\psi} \left[\check{G}_0^{-1} + \check{Q}_{b\perp} + \check{Q}_{b\|} \right] \psi;
\end{aligned}
$$

to obtain the motion law for the magnet, we have to trace over fermions:

$$\int \frac{\mathcal{D}[\bar{\psi}\,\psi]}{\prod_{s'd} \text{tr}\left[\rho_0^{s'd}\right]} \exp\left\{ i \int dt\, \bar{\psi} \left[\check{G}_0^{-1} + \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right] \psi \right\} =$$
$$= \frac{1}{\prod_{s'd} \text{tr}\left[\rho_0^{s'd}\right]} \det\left[i \left(\check{G}_0^{-1} + \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right) \right] =$$
$$= \det\left[\check{I} + \check{G}_0\, \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right] =$$
$$= e^{\text{tr} \ln\left[\check{I} + \check{G}_0\, \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}})\right]} = e^{i\, S_{m-e}},$$

where in the first identity the Gaussian integrals have been used and, in the second, the fact that $\det\left[i\, \check{G}_0^{-1}\right] = \prod_{s'd} \text{tr}\left[\rho_0^{s'd}\right]$ (see [29, p. 229]).

In the semi-classical limit, we may expand the logarithm:

$$S_{m-e} = -i\ \text{tr} \ln\left[\check{I} + \check{G}_0\, \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right] \simeq$$
$$\simeq -i\ \text{tr}\left[\check{G}_0\, \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right] +$$
$$+ \frac{i}{2}\ \text{tr}\left\{ \left[\check{G}_0\, \check{Q}(\bar{b}^{\text{cl}}, \bar{b}^{\text{q}}, b^{\text{cl}}, b^{\text{q}}) \right]^2 \right\} \simeq$$
$$\simeq -i\ \text{tr}\left[\check{G}_0\, \check{Q}_{b\perp} \right] - i\ \text{tr}\left[\check{G}_0\, \check{Q}_{b\|} \right] + \frac{i}{2}\ \text{tr}\left[\check{G}_0\, \check{Q}_{b\perp}\, \check{G}_0\, \check{Q}_{b\perp} \right], \quad (7.2)$$

where we took into account only terms up to the second order in $1/\sqrt{J}$. In particular, the first term is the lowest order term: we have a free electron propagator and a vertex with a single boson; the fermions degrees of freedom are traced over and then we can represent it with the Feynman diagram in figure 1.6a on page 4. The other two terms are corrections (both of the same order): in particular, in the second term we have a single fermionic line and a two-boson vertex (figure 1.6b on page 4), while the third term is composed by two fermionic lines and two single-boson vertices (figure 1.6c on page 4).

The Green functions matrix is given by:

$$\check{G}_0 = \begin{pmatrix} \hat{G}_{0\uparrow'+} & 0 & 0 & 0 \\ 0 & \hat{G}_{0\downarrow'+} & 0 & 0 \\ 0 & 0 & \hat{G}_{0\uparrow'-} & 0 \\ 0 & 0 & 0 & \hat{G}_{0\downarrow'-} \end{pmatrix},$$
$$\hat{G}_{0s'd} = \begin{pmatrix} \hat{G}^R_{0s'd} & \hat{G}^K_{0s'd} \\ 0 & \hat{G}^A_{0s'd} \end{pmatrix}, \qquad s' = \uparrow', \downarrow', \quad d = \pm$$

where (see equations 4.17 on page 48)

- the four retarded Green functions are equal: $\hat{G}^R_{0\uparrow'\downarrow'\pm} =: \hat{G}^R_0$, with

$$\left[\hat{G}^R_0(t', t) \right]_{|k_2||k_1|} = -i\, \delta_{21}\, \theta(t' - t)\, e^{-i\, \epsilon_{k_1}\, (t' - t)} =$$
$$= R_{k_1}(t' - t)\, \delta_{21} \xrightarrow{\text{F.T.}} \left(\epsilon - \epsilon_{k_1} + i\, 0^+ \right)^{-1} \delta_{21}, \quad (7.3)$$

- similarly for the advanced Green functions: $\hat{G}^A_{0\uparrow'\downarrow'\pm} =: \hat{G}^A_0$, with

$$\left[\hat{G}^A_0(t',t)\right]_{|k_2||k_1|} = i\,\delta_{21}\,\theta(t-t')\,e^{-i\,\epsilon_{k_1}\,(t'-t)} = \\ = A_{k_1}(t'-t)\,\delta_{21} \xrightarrow{\text{F.T.}} \left(\epsilon-\epsilon_{k_1}-i\,0^+\right)^{-1}\delta_{21}, \quad (7.4)$$

- while for $s' = \uparrow', \downarrow'$ and $d = \pm$, the Keldysh Green functions are given by

$$\left[\hat{G}^K_{0s'd}(t',t)\right]_{|k_2||k_1|} = \\ = -i\,\delta_{21}\,\left[1-2\,n_F^{s'd}(\epsilon_{k_1})\right]e^{-i\,\epsilon_{k_1}\,(t'-t)} = K^{s'd}_{k_1}(t'-t)\,\delta_{21} \to \\ \xrightarrow{\text{F.T.}} -2\,\pi\,i\,\delta_{21}\left[1-2\,n_F^{s'd}(\epsilon_{k_1})\right]\delta(\epsilon-\epsilon_{k_1}); \quad (7.5)$$

where for simplicity we wrote δ_{12} instead of $\delta_{|k_1||k_2|}$.

Note that the $\hat{G}^{A,R}_0$ Green function evaluated in the scattering states have the same simple form of the plane waves functions. Furthermore the (θ,ϕ)-dependence is in the $\mathcal{M}$ interaction matrices.

It is useful to observe that (since $\hat{H}^\pm = \hat{H}$ and the forward-path contribution cancels the backward one) we must have

$$S[\bar{b}^{\text{cl}}, b^{\text{cl}}, \bar{b}^{\text{q}} = 0, b^{\text{q}} = 0] = 0 \quad (7.6)$$

(see equations 4.15 on page 47); in particular we have no linear terms in b^{cl}.

8 LINEAR TERMS IN b

8.1 ACTION

The non vanishing linear terms in the $\bar{b}, b$-expansion (that is up to the order $1/\sqrt{J}$) are given by (see equation 7.2 on page 66):

$$
\begin{aligned}
S_1 &= -i \operatorname{tr}\left[\check{G}_0\, \check{Q}_{b\perp}\right] = \\
&= -i \operatorname{tr}\left[\check{G}_0 \left\{-\frac{1}{\sqrt{2}}\left[b^{\mathrm{q}}\, \hat{\mathcal{M}}(\perp) + \bar{b}^{\mathrm{q}}\, \hat{\mathcal{M}}^{\dagger}(\perp)\right] \otimes \hat{\gamma}^{\mathrm{q}}\right\}\right] = \\
&= \frac{i}{\sqrt{2}} \sum_{s'd} \operatorname{tr}\left[b^{\mathrm{q}}\, G^{K}_{0s'd}\, \mathcal{M}^{s's'}_{dd}(\perp)\right] + \text{h.c.} = \\
&= \frac{1}{\sqrt{2}} \int dt\, b^{\mathrm{q}} \sum_{|k_1||k_2|s'd} \delta_{21}\left[1 - 2\, n^{s'd}_F(\epsilon_{k_1})\right] \cdot \\
&\cdot \lambda\, N^2 \frac{\sqrt{2J}}{2}\, t^{*}_{\downarrow}(|k_2|)\, t_{\uparrow}(|k_1|)\, s\, e^{-i\phi}\, \frac{\sin\theta}{2} + \text{h.c.} = \\
&= -\frac{N^2 \sqrt{J}\,\lambda}{2} \int dt\, b^{\mathrm{q}}\, e^{-i\phi} \sin\theta \cdot \\
&\cdot \sum_{|k|d}\left[n^{\uparrow' d}_F(\epsilon_k) - n^{\downarrow' d}_F(\epsilon_k)\right] t^{*}_{\downarrow}(k)\, t_{\uparrow}(k) + \text{h.c.} = \\
&= -C_1^{*}\sqrt{J} \int dt\, b^{\mathrm{q}}\, e^{-i\phi} \sin\theta + \text{h.c.}; \qquad (8.1)
\end{aligned}
$$

in the zero temperature and low differential potential limits:

$$
C_1 = \frac{N^2\,\lambda\, m\, t^{*}_{\uparrow}(k_F)\, t_{\downarrow}(k_F)}{2\, k_F}\left(\Delta\mu^{L}_{\text{spin}} + \Delta\mu^{R}_{\text{spin}}\right).
$$

In particular the nanomagnet action up to the first order is given by $S_m + S_1$.

8.2 CLASSICAL LIMIT

The classical limit is obtained from $\frac{\delta S}{\delta b^{\mathrm{q}}}|_{b^{\mathrm{q}}=\bar{b}^{\mathrm{q}}=0} = \frac{\delta S}{\delta \bar{b}^{\mathrm{q}}}|_{b^{\mathrm{q}}=\bar{b}^{\mathrm{q}}=0} = 0$ (see equations 4.16 on page 47):

$$
\begin{cases}
(i\,\partial_t + \gamma\, B_z)\, b^{\text{cl}} - \gamma\sqrt{J}\, B_+ - C_1\sqrt{J}\, e^{i\phi} \sin\theta = 0, \\
(-i\,\partial_t + \gamma\, B_z)\, \bar{b}^{\text{cl}} - \gamma\sqrt{J}\, B_- - C_1^{*}\sqrt{J}\, e^{-i\phi} \sin\theta = 0.
\end{cases} \qquad (8.2)
$$

With the substitutions

$$b^{\rm cl}=\frac{b_++b_-}{\sqrt{2}}\to\frac{2}{\sqrt{2}}\,b=\frac{J_+}{\sqrt{J}},$$
$$\bar b^{\rm cl}=\frac{\bar b_++\bar b_-}{\sqrt{2}}\to\frac{2}{\sqrt{2}}\,\bar b=\frac{J_-}{\sqrt{J}},$$
$$J=J_z+O\left(\frac{1}{J}\right)\to J_z,$$

(remember that, with the $\mathcal{S}_1$ action term, we are considering perturbation terms up to $1/\sqrt{J}$ order) the classical equations become:

$$\begin{cases}\partial_t J_x=\gamma\,[\vec B\times\vec J]_x+J_z\,\sin\theta\,(\Im C_1\,\cos\phi+\Re C_1\,\sin\phi)\,,\\ \partial_t J_y=\gamma\,[\vec B\times\vec J]_y-J_z\,\sin\theta\,(\Re C_1\,\cos\phi-\Im C_1\,\sin\phi)\,,\end{cases}\tag{8.3a}$$

$$\Longleftrightarrow\begin{cases}\partial_t J_x=\gamma\,[\vec B\times\vec J]_x+\Re C_1\,[\hat z'\times\vec J]_x+\frac{\Im C_1}{J}\,[\vec J\times(\hat z'\times\vec J)]_x,\\ \partial_t J_y=\gamma\,[\vec B\times\vec J]_y+\Re C_1\,[\hat z'\times\vec J]_y+\frac{\Im C_1}{J}\,[\vec J\times(\hat z'\times\vec J)]_y,\end{cases}\tag{8.3b}$$

where, for the last passage, see figure 5.2 on page 55; the equations are completed by the condition $\vec J\cdot\partial_t\vec J=0$ (indeed $\hat J_z=J$ up to the $1/\sqrt{J}$ order), which gives rise to

$$\partial_t\vec J=\gamma\,\vec B\times\vec J+\Re C_1\,\hat z'\times\vec J+\frac{\Im C_1}{J}\,\vec J\times(\hat z'\times\vec J).\tag{8.4}$$

Note that C_1 depends on the product $t^*_\uparrow(k_F)\,t_\downarrow(k_F)$. In particular, in the Slonczewski theory [51] an electron may change its polarization flowing through a magnetic layer: this variation constitutes the spin-pumping effect and we must consider a spin-mixing conduction. Consequently, considering the total angular momentum conservation, the magnet must change its magnetization direction. Then our results are in agreement for example with [5, 56, 65].[1]

1 Indeed observe that we are considering a single channel model; furthermore, with our scattering potential, we have $t_{\downarrow\uparrow}=i\,g_{\downarrow\uparrow}\,r_{\downarrow\uparrow}$, where $g_s:=m\,(\lambda_0+s\,\lambda\,J/2)/|k|$ (see equations 5.3b on page 54).
If for example the left reservoir produces a net flux of $|\uparrow'\rangle=\cos\frac{\theta}{2}\,|\uparrow\rangle+e^{i\,\phi}\,\sin\frac{\theta}{2}\,|\downarrow\rangle$ electrons while the right reservoir is $\Delta\mu^R_{\rm spin}=0$, after the scatterings the transmitted spinor is

$$\hat t\,|\uparrow'\rangle=t_\uparrow\,\cos\frac{\theta}{2}\,|\uparrow\rangle+t_\downarrow\,e^{i\,\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle$$

(since $\hat t$ is diagonal in the $|\downarrow\uparrow\rangle$ basis) and a $\sigma^\pm$ flux is produced:

$$I_+\propto i\left[\Psi^\dagger_t\,\sigma^+\,\frac{d\Psi_t}{d\bar x}-\frac{d\Psi^\dagger_t}{d\bar x}\,\sigma^+\,\Psi_t\right]\propto e^{i\,\phi}\,\sin\theta\,t^*_\uparrow\,t_\downarrow,$$
$$I_-=I^*_+$$

(where here $\propto$ indicates "proportional by means a real factor"). These fluxes must be absorbed by the reservoirs to restore the equilibrium and, by using the conservation

One may also observe that the classical equation includes a field-like and a damping-like term, proportional respectively to $\Re C_1$ and $\Im C_1$; in the zero temperature and low differential potential limits and considering $\Delta\mu^R_{\text{spin}} = 0$, by using the inverse relations of 6.7 on page 62, we may write the first order classical equation as

$$\partial_t \vec{J} = \gamma \vec{B} \times \vec{J} + \Re C_1^{\text{spin}}(\theta)\, \vec{I}_{\text{spin}} \times \vec{J} + \Re C_1^{\text{charge}}(\theta)\, \vec{I}_{\text{charge}} \times \vec{J} + \\ + \frac{\Im C_1^{\text{spin}}(\theta)}{J}\, \vec{J} \times (\vec{I}_{\text{spin}} \times \vec{J}) + \frac{\Im C_1^{\text{charge}}(\theta)}{J}\, \vec{J} \times (\vec{I}_{\text{charge}} \times \vec{J}),$$

where both $\vec{I}_{\text{spin}}$ and $\vec{I}_{\text{charge}}$ are directed along the spin polarization axis $\hat{z}'$ and

$$C_1 = \frac{N^2\, \lambda\, m\, t^*_\uparrow(k_F)\, t_\downarrow(k_F)}{2\, k_F} \left[(\hat{G}^{-1}(\theta))_{21}\, I_{\text{charge}} + (\hat{G}^{-1}(\theta))_{22}\, I_{\text{spin}} \right]$$

$$C_1^{\text{charge}} := \frac{N^2\, \lambda\, m\, t^*_\uparrow(k_F)\, t_\downarrow(k_F)}{2\, k_F} [\hat{G}^{-1}(\theta)]_{21},$$

$$C_1^{\text{spin}} := \frac{N^2\, \lambda\, m\, t^*_\uparrow(k_F)\, t_\downarrow(k_F)}{2\, k_F} [\hat{G}^{-1}(\theta)]_{22}$$

and $\hat{G}$ is defined in 6.7 on page 62.

Finally we may also note that in the spin transfer torque effect, the electrons intercat with the magnet producing the variation in the magnetization direction and reciprocally the magnet can cause the electrons spin flipping. This situation is quite complicated with respect to the case of a simple external magnetic field. But we can simplify the system by considering the limit $\lambda \to 0$. In this case the the potential seen by the electrons does not depend on their spins and $t^*_\uparrow(k_F)\, t_\downarrow(k_F) = |t^*_\uparrow(k_F)|^2$; in particular the imaginary part of C_1 disappears and, for $\vec{B} = 0$, the magnet classical motion is simply a precession around the current polarization axis. This is not surprising: in this case the magnet cannot mix the electrons channels (producing, for example, a spin flip on an electron coming from left to right taken from the larger spin population) and the "dissipative" damping-like term disappears.

8.3 EQUATION OF MOTION

As explained in the section 4.2 on page 33, the action S describes the complete dynamics of the system and we can go behind the classical

argument of Slonczewski, they must give us the $J_\pm$ rate of change . Comparing with equation 8.2 on page 69, we find that it must actually be

$$C_1 \propto t^*_\uparrow\, t_\downarrow.$$

limit equations 4.16 on page 47. Indeed for an action linear in b^{q}, $\bar{b}^{\mathrm{q}}$, of the form

$$S = \int dt \left[\bar{b}^{\mathrm{q}}\,(i\,\partial_t + \alpha)\,b^{\mathrm{cl}} + \beta\,\bar{b}^{\mathrm{q}}\right] + \text{h.c.}, \tag{8.5}$$

(like $S_m + S_1$), if we consider an observable $\hat{O}$ that, after the Keldysh rotation, depends only on the classical field $O(\bar{b}^{\mathrm{cl}}, b^{\mathrm{cl}})$, we have:

$$\begin{aligned}
\langle\hat{O}(t)\rangle &= \int D[\bar{b},b]\,O(\bar{b}^{\mathrm{cl}}(t),b^{\mathrm{cl}}(t))\,e^{i\,S} = \\
&= \int D[\bar{b}^{\mathrm{cl}},b^{\mathrm{cl}}]\,O(\bar{b}^{\mathrm{cl}}(t),b^{\mathrm{cl}}(t))\cdot \\
&\cdot \int D[\bar{b}^{\mathrm{q}},b^{\mathrm{q}}]\,e^{i\int dt\,[\bar{b}^{\mathrm{q}}(i\,\partial_t+\alpha)b^{\mathrm{cl}}+\beta\,\bar{b}^{\mathrm{q}}+\text{h. c.}]} = \\
&= \int D[\bar{b}^{\mathrm{cl}},b^{\mathrm{cl}}]\,O(\bar{b}^{\mathrm{cl}}(t),b^{\mathrm{cl}}(t))\cdot \\
&\qquad\qquad \cdot\,\delta[(i\,\partial_t+\alpha)\,b^{\mathrm{cl}}+\beta]\,\delta[(-i\,\partial_t+\alpha^*)\,\bar{b}^{\mathrm{cl}}+\beta^*],
\end{aligned}$$

(up to multiplicative terms that we can include in $D[\bar{b},b]$) that is, the dynamics of each observable $O(\bar{b}^{\mathrm{cl}}(t), b^{\mathrm{cl}}(t))$ is exactly determined by the equations 4.16 on page 47.

In particular, $\hat{b}(t)$ and $\hat{b}^\dagger(t)$ depend only on the classical field; indeed, from equations 4.6 on page 34 and 4.10 on page 39:

$$\begin{aligned}
\langle\hat{b}(t)\rangle = i\,\frac{\delta Z}{\delta\eta(t)}\bigg|_{\eta=0} &= \\
&= \int D[\bar{b},b]\,\frac{b^+(t)+b^-(t)}{2}\,e^{i\,S} = \int D[\bar{b},b]\,\frac{b^{\mathrm{cl}}(t)}{\sqrt{2}}\,e^{i\,S}
\end{aligned}$$

(we have again included the multiplicative terms $\langle b^+(t_0)\,|\,\hat{\rho}_0\,|\,b^-(t_0)\rangle$ and $e^{-\bar{b}^+(t_0)\,b^+(t_0)}$ in the definition of $D[\bar{b},b]$) and analogously

$$\langle\hat{b}^\dagger(t)\rangle = \int D[\bar{b},b]\,\frac{\bar{b}^{\mathrm{cl}}(t)}{\sqrt{2}}\,e^{i\,S},$$

from which one gets equation 8.3 on page 70. Note that the footnote 6 on page 40 caveat must be considered in evaluating observables.

9

QUADRATIC CORRECTIONS IN b

We consider the quadratic corrections in $\bar{b}, b$. They are suppressed by a factor $1/\sqrt{J}$ with respect to the linear terms. We can consider corrections up to quadratic terms (see the expansion 5.1 on page 53).

At this order, we have Feynman diagrams with both one and two fermionic propagators. In particular, the one propagator term is (see equation 7.2 on page 66):

$$S_{2-1} = -i \text{ tr} \left[\check{G}_0 \, \check{Q}_{b\|} \right] ,$$

while the two-fermionic propagator term have the form:

$$S_{2-2} = \frac{i}{2} \text{ tr} \left[\check{G}_0 \, \check{Q}_{b\perp} \, \check{G}_0 \, \check{Q}_{b\perp} \right] .$$

Before describing the calculations, in the next section we will show what kind of corrections S_{2-1} and S_{2-2} give rise to in the equation of motion 8.4 on page 70.

In particular we will see that they produce, among others, a term that is quadratic in b^{q}, which is not of the form 8.5 on the preceding page. But to include it in our dynamics equation we will show that it is mathematically indistinguishable from a linear action provided you include some stochastic terms. This is not surprising from a physical point of view, since we have shown in the section 3.2 on page 19 that, when we trace over some degrees of freedom, a pure state can be not distinguishable from a mixed state.

9.1 CORRECTIONS TO THE MOTION EQUATION

The term with the single fermionic propagator is of the form

$$S_{2-1} = \int dt \, \tilde{B}_z^{2-1}(\theta) \, \bar{b}^{\mathrm{cl}} \, b^{\mathrm{q}} + \text{h. c.}; \tag{9.1}$$

comparing with the equation 7.1 on page 63, we see that the contribution to the equation of motion of this term can be considered as a correction (which depends on the angle θ between the magnet and the polarizzazion of the current) to the z-component of the external magnetic field. Anyway the form of the equations 8.2 on page 69 remains unchanged. As discussed in the section 8.3 on page 71, this equation is valid when we are in a (moving) frame of reference such that the number of bosons is negligeable with respect to J. If we assume that the system decoheres in a classical spin coherent state in a time that is much shorter with respect to the magnet-dynamics typical times, we can consider also $J_z = J$ at any time and then $\vec{J} \cdot \partial_t \vec{J} = 0$. This means that the contribution of the term 9.1 on the preceding page to the dynamics equation is zero (since it is parallel to $\vec{J}$ at any time).

The two-fermionic propagator action S_{2-2} gives rise to two terms: one with both classical and quantum bosonic legs $S_{\text{cl-q}}$ and one with two quantum legs $S_{\text{q-q}}$.[1]

In particular:

$$S_{\text{cl-q}} = i \int dt_1 \int dt_2 \begin{pmatrix} \bar{b}^{\text{cl}}(t_1) & \bar{b}^{\text{q}}(t_1) \end{pmatrix} \cdot \cdot \begin{pmatrix} 0 & D^A(t_1 - t_2) \\ D^R(t_1 - t_2) & 0 \end{pmatrix} \begin{pmatrix} b^{\text{cl}}(t_2) \\ b^{\text{q}}(t_2), \end{pmatrix} \tag{9.2}$$

where the D-functions depend on the electronic dynamics. From the fermions point of view, they are the spin-spin response functions of the Kubo formula.

In the typical situations the magnet dynamics is much slower than the fermionic dynamics; for example, in the reference [63] the typical flipping times for the magnet are of the order of the nanosecond, while the typical electrons Fermi energy ϵ_F is given by some electronvolts, that is the typical frequencies are of the order of $\epsilon_F/\hbar \sim 10^{16}\,\text{s}^{-1}$. As we will see, this mean that we can expand D in frequencies:

$$D(\omega) \sim D_0 + \omega\, D_1;$$

we will consider the first order in ω in the appendix A on page 93 (even if it could be a negligible term, we will consider it, as done also in [53], since it gives rise to a Gilbert damping term – see equation 2.3b on page 11).

The terms $\Im D_0^A = \Im D_0^R$ give rise to an action of the form:

$$\int dt\, \tilde{B}^{\text{cl-q}}\, \bar{b}^{\text{cl}}\, b^{\text{q}} + \text{h.c.}$$

and we can repeat the same considerations for the action 9.1 on the preceding page.

The terms $\Re D_0^A = -\Re D_0^R$ give rise to an action of the form:

$$i \int dt\, \Re D_0^A(\theta)\, \bar{b}^{\text{cl}}\, b^{\text{q}} + \text{h.c.}$$

produce a correction of the form $\Re D_0^R\, J_x$ and $\Re D_0^R\, J_y$ to the right side of the first and, respectively, the second equation in 8.3 on page 70.[2]

1 the case with two classical bosonic legs is forbidden because of the equation 4.15 on page 47.

2 Observe that, in our expression 9.2, $D_0^{A/R}$ is an addend of $-i\,[G^{-1}]^{A/R}$. In particular, the fact that $\Re D_0^A = -\Re D_0^R$ and $\Im D_0^A = \Im D_0^R$ guarantees that the action component $S_{\text{cl-q}}$ is real.

In particular, if $\vec{J}\cdot\partial_t\vec{J}=0$, we must have again that the term is zero. Indeed in the moving reference frame we chose, it must be

$$J_z = J, \qquad J_x = J_y = \partial_t J_z = 0.$$

Finally we consider the $S_{\text{q-q}}$ component of the action; for simplicity we consider the low temperature and differential potential limits, but the generalization is easy. For compactness, we write

$$c^{\mathrm{q}} = \sqrt{\frac{\pi\, m\, \lambda^2\, J\, N^4}{8}\left(|\Delta\mu^L_{\text{spin}}| + |\Delta\mu^R_{\text{spin}}|\right)}\, e^{-i\,\phi}\, \frac{t_\uparrow\, t_\downarrow^*}{\sqrt{\epsilon_F}}\, b^{\mathrm{q}},$$

then:

$$S_{\text{q-q}} = i\int dt\left[4\,\bar{c}^{\mathrm{q}}\,c^{\mathrm{q}} - \sin^2\theta\,(\bar{c}^{\mathrm{q}}+c^{\mathrm{q}})^2\right].$$

This term is not more linear in b^{q} and then we cannot apply the considerations in the section 8.3 on page 71 directly. To linearize this term we will use the Hubbard–Stratonovich transformation (you can compare the following calculations with the simpler case in [53]).

We have:

$$e^{i\,S_{\text{q-q}}} = e^{-\int dt\left[4\,\bar{c}^{\mathrm{q}}\,c^{\mathrm{q}} - \sin^2\theta\,(\bar{c}^{\mathrm{q}}+c^{\mathrm{q}})^2\right]} = e^{-\frac{1}{2}\begin{pmatrix}c^{\mathrm{q}} & \bar{c}^{\mathrm{q}}\end{pmatrix} A \begin{pmatrix}\bar{c}^{\mathrm{q}} \\ c^{\mathrm{q}}\end{pmatrix}}$$

where

$$A = 2\begin{pmatrix}2-\sin^2\theta & -\sin^2\theta \\ -\sin^2\theta & 2-\sin^2\theta\end{pmatrix}\otimes I_t,$$

$$A = U^\dagger\left[4\begin{pmatrix}\cos^2\theta & 0 \\ 0 & 1\end{pmatrix}\otimes I_t\right]U,$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\otimes I_t$$

and I_t is the identity over times. Then, if we put

$$c_1 := 2\,\cos\theta\,(c^{\mathrm{q}}+\bar{c}^{\mathrm{q}})\,/\sqrt{2},$$
$$c_2 := 2\,(c^{\mathrm{q}}-\bar{c}^{\mathrm{q}})\,/(i\sqrt{2}),$$

we obtain

$$\begin{aligned}
e^{i\,S_{\text{q-q}}} &= e^{-\frac{1}{2}\int dt\left(|c_1|^2+|c_2|^2\right)} = \\
&= \int \mathcal{D}[y^*,y]\, e^{-\int dt\,\frac{|y_1|^2+|y_2|^2}{2}}\, e^{-\frac{i}{2}\int dt(\bar{c}_1\,y_1+\bar{c}_2\,y_2+\text{h.c.})} = \\
&= \int \mathcal{D}[y^*,y]\, e^{-\int dt\,\frac{|y_1|^2+|y_2|^2}{2}}\, e^{-\frac{i}{2}\int dt\left[\frac{4}{\sqrt{2}}(\cos\theta\,\Re y_1 + i\,\Re y_2)\bar{c}^{\mathrm{q}}+\text{h.c.}\right]} = \\
&\qquad = \int \mathcal{D}[I_1,I_2]\, e^{-\int dt\,\frac{I_1^2+I_2^2}{2}}\, e^{-\left[(C_2^*\,I_1+i\,C_2^*\,\cos\theta\,I_2)\sqrt{J}\,e^{-i\,\phi}\,b^{\mathrm{q}}+\text{h.c.}\right]},
\end{aligned}$$

where in the second equality we used the Hubbard–Stratonovich transformation, in the last equality we integrated over $\Im y_i$ putting $I_i := \Re y_i$ and

$$C_2 := \sqrt{\frac{\pi\, m\, \lambda^2\, N^4}{4\,\epsilon_F}\left(|\Delta\mu^L_{\text{spin}}| + |\Delta\mu^R_{\text{spin}}|\right)\, t^*_\uparrow\, t_\downarrow}.$$

By comparing with the 8.1 on page 69 and the Martin-Siggia-Rose actions, we see that $S_{\text{q-q}}$ introduces a correction

$$(-\cos\theta\, \Im C_2\, I_1 + \Re C_2\, I_2)\,\frac{1}{\sin\theta}\,\hat{z}' \times \vec{J} + \\ + (\cos\theta\, \Re C_2\, I_1 + \Im C_2\, I_2)\,\frac{1}{\sin\theta\, J}\,\vec{J} \times (\hat{z}' \times \vec{J}),$$

in the right side of equation 8.4 on page 70, where I_i are Gaussian stochastic processes with zero mean values and

$$\langle I_i(t_1)\, I_j(t_2)\rangle = \delta(t_1 - t_2)\,\delta_{ij}.$$

In the next sections we will show the calculations to obtain the quadratic corrections to the action.

9.2 ONE FERMIONIC PROPAGATOR

The non-vanishing one propagator term is given by:

$$\begin{aligned}
S_{2-1} &= -i\,\text{tr}\left\{\check{G}_0\left[-\frac{1}{2}\left(\bar{b}^{\text{cl}}\, b^{\text{q}} + \bar{b}^{\text{q}}\, b^{\text{cl}}\right)\hat{\mathcal{M}}(\|) \otimes \hat{\gamma}^{\text{q}}\right]\right\} = \\
&= \frac{i}{2}\sum_{s'd}\text{tr}\left[\left(\bar{b}^{\text{cl}}\, b^{\text{q}} + \bar{b}^{\text{q}}\, b^{\text{cl}}\right) G^K_{0s'd}\, \mathcal{M}^{s's'}_{dd}(\|)\right] = \\
&= \frac{i}{2}\sum_{s'd}\int dt\, \bar{b}^{\text{cl}}\, b^{\text{q}}\,\text{tr}\left[G^K_{0s'd}\,\mathcal{M}^{s's'}_{dd}(\|)\right] + \text{h.c.} = \\
&= \frac{1}{2}\int dt\, \bar{b}^{\text{cl}}\, b^{\text{q}} \sum_{|k_1||k_2|s'd}\delta_{21}\left[1 - 2\, n^{s'd}_F(\epsilon_{k_1})\right]\mathcal{M}^{s's'}_{d|k_2|,d|k_1|}(\|) + \\
&\quad + \text{h.c.} = \\
&= \frac{\lambda\, N^2}{4}\int dt\, \bar{b}^{\text{cl}}\, b^{\text{q}}\sum_{|k|d}\left\{\left[n^{\uparrow'd}_F(\epsilon_k) - n^{\downarrow'd}_F(\epsilon_k)\right]\cdot\right. \\
&\quad \cdot\left[|t_\uparrow(k)|^2(\cos\theta + 1) + |t_\downarrow(k)|^2(\cos\theta - 1)\right] + \\
&\qquad \left. - \left[1 - 2\, n^{\downarrow'd}_F(\epsilon_k)\right]\left[|t_\uparrow(k)|^2 - |t_\downarrow(k)|^2\right]\right\} + \text{h.c.},
\end{aligned}$$

where we have used the property

$$\mathcal{M}^{\uparrow'\uparrow'}_{|k||k|}(\|) = -\mathcal{M}^{\downarrow'\downarrow'}_{|k||k|}(\|) - \frac{\lambda}{2}\, N^2\left[|t_\uparrow(k)|^2 - |t_\downarrow(k)|^2\right].$$

Then, comparing with the term 7.1 on page 63, we see that it can be considered a correction to the z-component of the external magnetic field. The low temperature and low differential potentials limit can be evaluated easily.

9.3 CL-Q TWO FERMIONIC PROPAGATOR

The second order, two fermionic-propagator terms are:

$$\begin{aligned}
&S_{2-2} = \\
&= \frac{i}{2}\,\mathrm{tr}\Bigg\{ \check{G}_0 \left[-\frac{1}{\sqrt{2}} \sum_{\alpha_2=\mathrm{q,cl}} \left[b^{\alpha_2}\,\hat{\mathcal{M}}(\perp) + \bar{b}^{\alpha_2}\,\hat{\mathcal{M}}^{\dagger}(\perp) \right] \otimes \hat{\gamma}^{\alpha_2} \right] \cdot \\
&\qquad \cdot \check{G}_0 \left[-\frac{1}{\sqrt{2}} \sum_{\alpha_1=\mathrm{q,cl}} \left[b^{\alpha_1}\,\hat{\mathcal{M}}(\perp) + \bar{b}^{\alpha_1}\,\hat{\mathcal{M}}^{\dagger}(\perp) \right] \otimes \hat{\gamma}^{\alpha_1} \right] \Bigg\}.
\end{aligned}$$

Terms that do not contain at least a b^{q} or a $\bar{b}^{\mathrm{q}}$ vanish (see e. g. 7.6 on page 67) and it remains a cl-q term and a q-q one: $S_{2-2} = S_{\text{cl-q}} + S_{\text{q-q}}$.

For the cl-q term, if we write

$$\check{A}^{\alpha} := \left[b^{\alpha}\,\hat{\mathcal{M}}(\perp) + \bar{b}^{\alpha}\,\hat{\mathcal{M}}^{\dagger}(\perp) \right] \otimes \hat{\gamma}^{\alpha} = \hat{A}^{\alpha} \otimes \hat{\gamma}^{\alpha}, \qquad \alpha = \mathrm{q,cl},$$

we have

$$\begin{aligned}
&S_{\text{cl-q}} = \frac{i}{4}\,\mathrm{tr}\left[\check{G}_0\,\check{A}^{\mathrm{cl}}\,\check{G}_0\,\check{A}^{\mathrm{q}} + \check{G}_0\,\check{A}^{\mathrm{q}}\,\check{G}_0\,\check{A}^{\mathrm{cl}} \right] = \\
&= \frac{i}{2}\,\mathrm{tr}\left[\check{G}_0\,\check{A}^{\mathrm{cl}}\,\check{G}_0\,\check{A}^{\mathrm{q}} \right] = \\
&= \frac{i}{2} \sum_{1,2,3,4} [\hat{G}_0^R(1,2)\,\hat{A}^{\mathrm{cl}}(2,3)\,\hat{G}_0^K(3,4)\,\hat{A}^{\mathrm{q}}(4,1) + \\
&+ \hat{G}_0^K(1,2)\,\hat{A}^{\mathrm{cl}}(2,3)\,\hat{G}_0^A(3,4)\,\hat{A}^{\mathrm{q}}(4,1)] = \\
&\qquad\qquad = i \int dt_1 \int dt_2 \sum_{ab} b_a^{\mathrm{cl}}(t_1)\,D_{ab}(t_1,t_2)\,b_b^{\mathrm{q}}(t_2),
\end{aligned}$$

where e. g. $1 = (s_1', d_1, |k_1|, t_1)$ and $b_a, b_b = b, \bar{b}$; in particular:

$$\begin{aligned}
&D_{ab}(t_1,t_2) = \\
&= \frac{1}{2} \sum_{\substack{s_1',d_1,|k_1|,\\ s_2',d_2,|k_2|,\\ 3,4}} [\hat{G}_0^K(2,1)\,\hat{\mathcal{M}}_a(\perp;1,3)\,G_0^A(3,4)\,\hat{\mathcal{M}}_b(\perp;4,2) + \\
&+ \hat{G}_0^R(2,1)\,\hat{\mathcal{M}}_a(\perp;1,3)\,G_0^K(3,4)\,\hat{\mathcal{M}}_b(\perp;4,2)] = \\
&= \sum_{\substack{d_1,s_1',|k_1|,\\ s_2',|k_2|}} S(t_1-t_2)\,\mathcal{M}^{s_1's_2'}_{a|k_1||k_2|}(\perp,t_1)\,\mathcal{M}^{s_2's_1'}_{b|k_2||k_1|}(\perp,t_2) \sim \\
&\qquad\qquad \sim \sum_{\substack{d_1,s_1',|k_1|,\\ s_2',|k_2|}} S(t_1-t_2)\,\mathcal{M}^{s_1's_2'}_{a|k_1||k_2|}(\perp)\,\mathcal{M}^{s_2's_1'}_{b|k_2||k_1|}(\perp);
\end{aligned}$$

in the last approximation we used the fact that the fermionic dynamics is much faster than the bosonic one[3] and we have defined:

$$
\begin{aligned}
S(t_1-t_2) := & \frac{1}{2}\Big[K^{s_1'}_{|k_1|d_1}(t_2-t_1)\, A_{|k_2|}(t_1-t_2)+\\
& + R_{|k_1|}(t_2-t_1)\, K^{s_2'}_{|k_2|d_1}(t_1-t_2)\Big] \to\\
& \xrightarrow{\text{F.T.}} \frac{1}{2}\int dt\, e^{i\,\omega\, t}\left[K^{s_1'}_{|k_1|d_1}(-t)\, A_{|k_2|}(t)\, + R_{|k_1|}(-t)\, K^{s_2'}_{|k_2|d_1}(t)\right] =\\
& = \frac{1}{2}\int \frac{d\epsilon}{2\,\pi}\left[K^{s_1'}_{|k_1|d_1}(\epsilon)\, A_{|k_2|}(\epsilon+\omega)\, + R_{|k_1|}(\epsilon-\omega)\, K^{s_2'}_{|k_2|d_1}(\epsilon)\right] =\\
& = i\,\frac{n_F^{s_1'd_1}(\epsilon_{|k_1|}) - n_F^{s_2'd_1}(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega - i\,0^+} =\\
& \qquad = i\left[n_F^{s_1'd_1}(\epsilon_{|k_1|}) - n_F^{s_2'd_1}(\epsilon_{|k_2|})\right]\hat{f}(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)
\end{aligned}
$$

where we used relations 7.3, 7.4 and 7.5 on page 67. Since $[\mathcal{L}(\perp)]^2 = 0$, the terms that multiplies $b\,b$ and $\bar{b}\,\bar{b}$ disappear and we may adjust the surviving terms to obtain the expression 9.2 on page 74, where:

- the term that multiplies $\bar{b}^{\text{cl}}\, b^{\text{q}}$ is

$$
\begin{aligned}
D^A(\omega) = & \sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} i\,\frac{n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega - i\,0^+}\cdot\\
& \cdot (\mathcal{M}^\dagger)^{s_1's_2'}_{|k_1||k_2|}(\perp)\, \mathcal{M}^{s_2's_1'}_{|k_2||k_1|}(\perp) =\\
& = i\,\frac{\lambda^2\, J^2\, N^4}{2}\sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, \frac{n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega - i\,0^+}\cdot\\
& \qquad \cdot (\mathcal{L}^\dagger)^{s_1's_2'}(\perp)\, \mathcal{L}^{s_2's_1'}(\perp);
\end{aligned}
\tag{9.3}
$$

- the term that multiplies $b^{\text{cl}}\,\bar{b}^{\text{q}}$:

$$
\begin{aligned}
D^R(\omega) = & \sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} i\,\frac{n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}-\omega - i\,0^+}\cdot\\
& \cdot \mathcal{M}^{s_1's_2'}_{|k_1||k_2|}(\perp)\, (\mathcal{M}^\dagger)^{s_2's_1'}_{|k_2||k_1|}(\perp) =\\
& = i\,\frac{\lambda^2\, J^2\, N^4}{2}\sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, \frac{n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega + i\,0^+}\cdot\\
& \qquad \cdot (\mathcal{L}^\dagger)^{s_1's_2'}(\perp)\, \mathcal{L}^{s_2's_1'}(\perp)
\end{aligned}
\tag{9.4}
$$

3 a similar approximation is made in [63], since only one electron scattering per time is considered.

We may now assume that $\epsilon \gg \omega$, where ϵ are the typical electrons energies:[4]

$$\begin{aligned}
S(\omega) &= i\left[n_F^{s_1' d_1}(\epsilon_{|k_1|}) - n_F^{s_2' d_1}(\epsilon_{|k_2|})\right] \hat{f}(\epsilon_{|k_1|} - \epsilon_{|k_2|} + \omega) \sim \\
&\sim i\left[n_F^{s_1' d_1}(\epsilon_{|k_1|}) - n_F^{s_2' d_1}(\epsilon_{|k_2|})\right] \cdot \\
&\cdot \left[\hat{f}(\epsilon_{|k_1|} - \epsilon_{|k_2|}) + \omega\, \hat{f}'(\epsilon_{|k_1|} - \epsilon_{|k_2|})\right] \xleftarrow{\text{F.T.}} \\
&\leftarrow i\left[n_F^{s_1' d_1}(\epsilon_{|k_1|}) - n_F^{s_2' d_1}(\epsilon_{|k_2|})\right] \cdot \\
&\cdot \left[\delta(t_1 - t_2)\, \hat{f}(\epsilon_{|k_1|} - \epsilon_{|k_2|}) + i\, \delta'(t_1 - t_2)\, \hat{f}'(\epsilon_{|k_1|} - \epsilon_{|k_2|})\right] = \\
&= S^0 + S^1, \qquad (9.5)
\end{aligned}$$

where S^1 is the term that contain the first order Dirac delta derivative.

We concentrate here on the term S^0. The suppressed term S^1 is considered in the appendix A on page 93. By using the formula of Sokhotski–Plemelj, it gives rise to:

$$\begin{aligned}
D^0_{ab}(t_1 - t_2) &= \\
&= i \sum_{d_1, s_1', s_2'} \int_0^\infty d|k_1| \int_0^\infty d|k_2| \left[n_F^{s_1' d_1}(\epsilon_{|k_1|}) - n_F^{s_2' d_1}(\epsilon_{|k_2|})\right] \cdot \\
&\cdot \delta(t_1 - t_2)\left[i\,\pi\,\delta(\epsilon_{|k_1|} - \epsilon_{|k_2|}) + P\frac{1}{\epsilon_{|k_1|} - \epsilon_{|k_2|}}\right] \cdot \\
&\cdot \mathcal{M}^{s_1' s_2'}_{a|k_1||k_2|}(\perp)\, \mathcal{M}^{s_2' s_1'}_{b|k_2||k_1|}(\perp).
\end{aligned}$$

The non vanishing terms are:

- the term D_0^A that multiplies $\bar{b}^{\text{cl}}\, b^{\text{q}}$, with:

$$\begin{aligned}
\Re D_0^A(t_1 - t_2) &= -\pi\,\delta(t_1 - t_2) \cdot \\
&\cdot \sum_{d, s_1', s_2'} \int_0^\infty d\epsilon \left[n_F^{s_1' d}(\epsilon) - n_F^{s_2' d}(\epsilon)\right] \cdot \\
&\cdot \frac{m}{2\epsilon} \left(\mathcal{M}^\dagger\right)^{s_1' s_2'}_{|k_\epsilon||k_\epsilon|}(\perp)\, \mathcal{M}^{s_2' s_1'}_{|k_\epsilon||k_\epsilon|}(\perp) = \\
&= -\delta(t_1 - t_2)\, \frac{\pi\,\lambda^2\, J\, N^4\, m}{4} \cos\theta \cdot \\
&\cdot \sum_d \int_0^\infty d\epsilon \left[n_F^{\uparrow' d}(\epsilon) - n_F^{\downarrow' d}(\epsilon)\right] \frac{|t_\uparrow(k)\, t_\downarrow(k)|^2}{\epsilon}
\end{aligned}$$

4 In particular, for sufficiently small values of ω, we have that the the contribution to $S(\omega)$ is non negligeable only for $\epsilon_{|k_1|} \sim \epsilon_{|k_2|}$; but in that case, for temperatures and differential potentials sufficiently small, $n_F^{s_1' d_1}(\epsilon_{|k_1|}) - n_F^{s_2' d_1}(\epsilon_{|k_2|})$ is non zero only for $\epsilon_{|k_1|} \sim \epsilon_{|k_2|} \sim \epsilon_F$. So we have to assume $\epsilon_F \gg \omega$.

and in the low temperature and low differential potentials limit:

$$\Re D_0^A(t_1-t_2) = -\delta(t_1-t_2)\,\frac{\pi\,\lambda^2\,J\,N^4\,m}{4}\,\cos\theta\cdot \cdot\frac{|t_\uparrow\,t_\downarrow|^2}{\epsilon_F}\left[\Delta\mu^L_{\text{spin}}+\Delta\mu^R_{\text{spin}}\right],$$

where $t_{\uparrow\downarrow}$ are evaluated at the Fermi wavenumber. In the same way, the principal value of the integral gives rise to:

$$\Im D_0^A(t_1-t_2) = \delta(t_1-t_2)\,\frac{\lambda^2\,J\,N^4\,m}{8}\sum_d\int_0^\infty\frac{d\epsilon_1}{\sqrt{\epsilon_1}}\int_0^\infty\frac{d\epsilon_2}{\sqrt{\epsilon_2}}\cdot \cdot\Big\{n_F^{\uparrow\prime d}(\epsilon_1)+n_F^{\downarrow\prime d}(\epsilon_1)+\left[n_F^{\uparrow\prime d}(\epsilon_1)-n_F^{\downarrow\prime d}(\epsilon_1)\right]\cos\theta+ -n_F^{\uparrow\prime d}(\epsilon_2)-n_F^{\downarrow\prime d}(\epsilon_2)+\left[n_F^{\uparrow\prime d}(\epsilon_2)-n_F^{\downarrow\prime d}(\epsilon_2)\right]\cos\theta\Big\}\cdot \cdot|t_\uparrow(k_1)\,t_\downarrow(k_2)|^2\,P\frac{1}{\epsilon_1-\epsilon_2};$$

in the low temperature limit, we may integrate analytically. For example:

$$\int_0^{\bar\mu}\frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\int_0^\infty\frac{d\epsilon_2}{\sqrt{\epsilon_2}}\,|t_\downarrow(k_2)|^2\,P\frac{1}{\epsilon_1-\epsilon_2} = = \int_0^{\bar\mu}\frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\cdot \cdot P\left[\frac{\sqrt{\epsilon_1}\,\log\left|\frac{\sqrt{\epsilon_2}+\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}-\sqrt{\epsilon_1}}\right|-2\,g_\downarrow\,\arctan\frac{\sqrt{\epsilon_2}}{g_\downarrow}}{\epsilon_1+g_\downarrow^2}\right]_{\epsilon_2=0}^{\infty} = = -\int_0^{\bar\mu}\frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\,\frac{g_\downarrow\,\pi}{\epsilon_1+g_\downarrow^2} = = 2\,g_\downarrow\,\pi\,\frac{g_\uparrow\,\arctan\frac{\sqrt{\bar\mu}}{g_\uparrow}-g_\downarrow\,\arctan\frac{\sqrt{\bar\mu}}{g_\downarrow}}{g_\downarrow^2-g_\uparrow^2},$$

where $g_{\uparrow\downarrow}=\sqrt{m}\,(\lambda_0\pm\lambda\,J/2)/\sqrt{2}$. Collecting all terms together, in the low differential potentials limit:

$$\Im D_0^A(t_1-t_2) = \delta(t_1-t_2)\,\frac{\lambda^2\,J\,N^4\,m\,\pi}{8}\cdot \cdot\Bigg\{4\,\frac{g_\uparrow\,\arctan\left(\frac{\sqrt{\epsilon_F}}{g_\uparrow}\right)-g_\downarrow\,\arctan\left(\frac{\sqrt{\epsilon_F}}{g_\downarrow}\right)}{g_\downarrow-g_\uparrow} +\sum_d\Bigg[\frac{\sqrt{\epsilon_F}\,(g_\uparrow-g_\downarrow)\,(\mu^{\uparrow\prime d}-\mu^{\downarrow\prime d})\,\cos\theta}{\left(g_\downarrow^2+\epsilon_F\right)\left(g_\uparrow^2+\epsilon_F\right)}+ +\frac{\sqrt{\epsilon_F}\,(g_\uparrow+g_\downarrow)(2\,\epsilon_F-\mu^{\uparrow\prime d}-\mu^{\downarrow\prime d})}{\left(g_\downarrow^2+\epsilon_F\right)\left(g_\uparrow^2+\epsilon_F\right)}\Bigg]\Bigg\};$$

- similarly for the term D_0^R that multiplies $b^{\text{cl}}\,\bar{b}^{\text{q}}$, we have:

$$\Re D_0^R = -\Re D_0^A, \qquad \Im D_0^R = \Im D_0^A.$$

9.4 Q-Q TWO FERMIONIC PROPAGATOR

Reproducing the passages analogous to the previous case, we get

$$S_{\text{q-q}} = \frac{i}{4}\,\text{tr}\left[\check{G}_0\,\check{A}^{\text{q}}\,\check{G}_0\,\check{A}^{\text{q}}\right] = \\ = i\int dt_1 \int dt_2 \sum_{ab} b_a^{\text{q}}(t_1)\,D_{ab}(t_1-t_2)\,b_b^{\text{q}}(t_2)$$

where

$$D_{ab}(t_1-t_2) = \sum_{\substack{d,s'_1,s'_2\\|k_1|,|k_2|}} S(t_1-t_2)\,\mathcal{M}^{s'_1 s'_2}_{a|k_1||k_2|}(\perp)\,\mathcal{M}^{s'_2 s'_1}_{b|k_2||k_1|}(\perp)$$

and

$$\begin{aligned}
&S(t) := \\
&\frac{1}{4}\left[K^{s'_1 d}_{|k_1|}(-t)\,K^{s'_2 d}_{|k_2|}(t) + R_{|k_1|}(-t)\,A_{|k_2|}(t) + A_{|k_1|}(-t)\,R_{|k_2|}(t)\right] = \\
&= \frac{1}{4}\Big\{K^{s'_1 d}_{|k_1|}(-t)\,K^{s'_2 d}_{|k_2|}(t) + \\
&- \left[R_{|k_1|}(-t) - A_{|k_1|}(-t)\right]\left[R_{|k_2|}(t) - A_{|k_2|}(t)\right]\Big\} = \\
&= \frac{1}{4}\left[K^{s'_1 d}_{|k_1|}(-t)\,K^{s'_2 d}_{|k_2|}(t) - \frac{K^{s'_1 d}_{|k_1|}(-t)}{1-2\,n_F^{s'_1 d}(\epsilon_{|k_1|})}\,\frac{K^{s'_2 d}_{|k_2|}(t)}{1-2\,n_F^{s'_2 d}(\epsilon_{|k_2|})}\right] \\
&\xrightarrow{\text{F.T.}} \frac{1}{4}\int\frac{d\epsilon}{2\,\pi}\Big[K^{s'_1 d}_{|k_1|}(\epsilon)\,K^{s'_2 d}_{|k_2|}(\epsilon+\omega) + \\
&- \frac{K^{s'_1 d}_{|k_1|}(\epsilon)}{1-2\,n_F^{s'_1 d}(\epsilon_{|k_1|})}\,\frac{K^{s'_2 d}_{|k_2|}(\epsilon+\omega)}{1-2\,n_F^{s'_2 d}(\epsilon_{|k_2|})}\Big] = \\
&= \frac{\pi}{2}\,\delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)\cdot \\
&\qquad\cdot\left\{1-\left[1-2\,n_F^{s'_1 d}(\epsilon_{|k_1|})\right]\left[1-2\,n_F^{s'_2 d}(\epsilon_{|k_2|})\right]\right\}.
\end{aligned}$$

We may simplify the expression by observing that:

$$1-2\,n_F^{\mu}(\epsilon) = \tanh\frac{\beta\,(\epsilon-\mu)}{2},$$

$$1-\tanh x\,\tanh y = \coth(x-y)\left[\tanh x - \tanh y\right],$$

from which

$$S(\omega) = \pi\,\delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)\cdot \\ \cdot\coth\frac{\beta\,(\mu^{s'_1 d}-\mu^{s'_2 d}+\omega)}{2}\left[n_F^{s'_1 d}(\epsilon_{|k_1|}) - n_F^{s'_2 d}(\epsilon_{|k_2|})\right];$$

in particular, $S_{\text{q-q}}$ is given by the sum of three terms:

- a term $i\int dt_1 \int dt_2\, b^{\mathrm{q}}(t_1)\, \tilde{D}_+(t_1-t_2)\, b^{\mathrm{q}}(t_2)$, where:

$$\tilde{D}_+(\omega) = \pi\, \frac{\lambda^2\, J\, N^4}{2} \sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} t_\downarrow^*(k_1)\, t_\downarrow^*(k_2)\, t_\uparrow(k_1)\, t_\uparrow(k_2)\, \delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)\cdot \cdot \coth\frac{\beta\,(\mu^{s_1'd}-\mu^{s_2'd}+\omega)}{2}\cdot \cdot \left[n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})\right] \mathcal{L}^{s_1's_2'}\, \mathcal{L}^{s_2's_1'};$$

- a term $i\int dt_1 \int dt_2\, \bar{b}^{\mathrm{q}}(t_1)\, \tilde{D}_-(t_1-t_2)\, \bar{b}^{\mathrm{q}}(t_2)$, where $\tilde{D}_-(\omega)$ is the complex conjugate of $\tilde{D}_-(\omega)$:

$$\tilde{D}_-(\omega) = \tilde{D}_+^*(\omega)$$

- a term $i\int dt_1 \int dt_2\, \bar{b}^{\mathrm{q}}(t_1)\, D^K(t_1-t_2)\, b^{\mathrm{q}}(t_2)$, where:

$$D^K(\omega) = \pi\, \frac{\lambda^2\, J\, N^4}{2} \sum_{\substack{d,s_1',|k_1|,\\ s_2',|k_2|}} \left[n_F^{s_1'd}(\epsilon_{|k_1|}) - n_F^{s_2'd}(\epsilon_{|k_2|})\right]\cdot \cdot \Bigg[\coth\frac{\beta\,(\mu^{s_1'd}-\mu^{s_2'd}+\omega)}{2}\, \delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)\cdot \cdot |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, (\mathcal{L}^\dagger)^{s_1's_2'}\, \mathcal{L}^{s_2's_1'} + + \coth\frac{\beta\,(\mu^{s_1'd}-\mu^{s_2'd}-\omega)}{2}\, \delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}-\omega)\cdot \cdot |t_\downarrow(k_1)\, t_\uparrow(k_2)|^2\, \mathcal{L}^{s_1's_2'}\, (\mathcal{L}^\dagger)^{s_2's_1'}\Bigg].$$

If we consider the equilibrium limit, that is $\mu^{s'd} = \epsilon_F$, since $\mathcal{L}^2(\perp) = 0$ and $\mathrm{tr}[L^\dagger L] = 1$, we have (see 9.3 and 9.4 on page 78):

$$\tilde{D}_\pm^{\mathrm{eq}}(\omega) = 0,$$

$$D_{\mathrm{eq}}^A(\omega) = i\, \frac{\lambda^2\, J^2\, N^4}{2} \sum_{d,|k_1|,|k_2|} |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, \frac{n_F(\epsilon_{|k_1|}) - n_F(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega - i\,0^+}$$

$$D_{\mathrm{eq}}^R(\omega) = i\, \frac{\lambda^2\, J^2\, N^4}{2} \sum_{d,|k_1|,|k_2|} |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, \frac{n_F(\epsilon_{|k_1|}) - n_F(\epsilon_{|k_2|})}{\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega + i\,0^+}$$

and

$$D_{\mathrm{eq}}^K(\omega) = \pi\, \lambda^2\, J\, N^4\, \coth\frac{\beta\,\omega}{2}\cdot \cdot \sum_{d,|k_1|,|k_2|} |t_\uparrow(k_1)\, t_\downarrow(k_2)|^2\, \left[n_F(\epsilon_{|k_1|}) - n_F(\epsilon_{|k_2|})\right] \delta(\epsilon_{|k_1|}-\epsilon_{|k_2|}+\omega)$$

from which we obtain the fluctuation-dissipation theorem:

$$D_{\mathrm{eq}}^K(\omega) = \coth\frac{\beta\,\omega}{2}\left[D_{\mathrm{eq}}^R(\omega) - D_{\mathrm{eq}}^A(\omega)\right].$$

Far from equilibrium instead, that is in the limit $\omega \to 0$, we have:

$$S(t_1 - t_2) = \frac{\pi}{2}\,\delta(t_1 - t_2)\,\delta(\epsilon_{|k_1|} - \epsilon_{|k_2|})\cdot \\ \cdot\left\{1 - \left[1 - 2\,n_F^{s_1' d}(\epsilon_{|k_1|})\right]\left[1 - 2\,n_F^{s_2' d}(\epsilon_{|k_2|})\right]\right\}$$

and then for D_{ab} we have

- the term proportional to $b^{\mathrm{q}}\,b^{\mathrm{q}}$:

$$\begin{aligned}
\tilde{D}_+(t_1 - t_2) &= \delta(t_1 - t_2) \sum_{\substack{d, s_1', s_2' \\ |k_1|, |k_2|}} \frac{\pi}{2}\,\delta(\epsilon_{|k_1|} - \epsilon_{|k_2|})\cdot \\
&\cdot\left\{1 - \left[1 - 2\,n_F^{s_1' d}(\epsilon_{|k_1|})\right]\left[1 - 2\,n_F^{s_2' d}(\epsilon_{|k_2|})\right]\right\}\cdot \\
&\cdot \mathcal{M}_{|k_1||k_2|}^{s_1' s_2'}(\perp)\,\mathcal{M}_{|k_2||k_1|}^{s_2' s_1'}(\perp) = \\
&= -\delta(t_1 - t_2)\,\frac{\pi\, m\, \lambda^2\, J\, N^4}{8}\, e^{-2i\phi}\, \sin^2\theta\cdot \\
&\cdot \sum_d \int_0^\infty \frac{d\epsilon}{\epsilon}\left[t_\downarrow^*(k)\, t_\uparrow(k)\right]^2 \left[n_F^{\uparrow' d}(\epsilon) - n_F^{\downarrow' d}(\epsilon)\right]^2
\end{aligned}$$

and in the low temperature and differential potentials limit

$$\begin{aligned}
&\left[n_F^{\uparrow' d}(\epsilon) - n_F^{\downarrow' d}(\epsilon)\right]^2 = \\
&= \left[n_F^{\uparrow' d}(\epsilon) - n_F^{\downarrow' d}(\epsilon)\right]\mathrm{sign}(\mu^{\uparrow' d} - \mu^{\downarrow' d}) \sim \\
&\qquad \sim \delta(\epsilon - \epsilon_F)|\mu^{\uparrow' d} - \mu^{\downarrow' d}|;
\end{aligned}$$

- the term proportional to $\bar{b}^{\mathrm{q}}\,\bar{b}^{\mathrm{q}}$, that is $\tilde{D}_-(t_1 - t_2) = \tilde{D}_+^*(t_1 - t_2)$;
- the term proportional to $\bar{b}^{\mathrm{q}}\,b^{\mathrm{q}}$:

$$\begin{aligned}
D^K(t_1 - t_2) &= 2\,\delta(t_1 - t_2) \sum_{\substack{d, s_1', s_2' \\ |k_1|, |k_2|}} \frac{\pi}{2}\,\delta(\epsilon_{|k_1|} - \epsilon_{|k_2|})\cdot \\
&\cdot\left\{1 - \left[1 - 2\,n_F^{s_1' d}(\epsilon_{|k_1|})\right]\left[1 - 2\,n_F^{s_2' d}(\epsilon_{|k_2|})\right]\right\}\cdot \\
&\cdot \left(\mathcal{M}^\dagger\right)_{|k_1||k_2|}^{s_1' s_2'}(\perp)\,\mathcal{M}_{|k_2||k_1|}^{s_2' s_1'}(\perp) = \\
&= \delta(t_1 - t_2)\,\frac{\pi\, m\, \lambda^2\, J\, N^4}{8}\cdot \\
&\cdot \sum_d \int_0^\infty \frac{d\epsilon}{\epsilon}|t_\downarrow(k)\, t_\uparrow(k)|^2 \left\{\left[n_F^{\uparrow' d}(\epsilon) - n_F^{\downarrow' d}(\epsilon)\right]^2\cdot \right. \\
&\left. \cdot \left(-2\sin^2\theta\right) + 4\left[n_F^{\uparrow' d}(\epsilon) + n_F^{\downarrow' d}(\epsilon) - 2\,n_F^{\uparrow' d}(\epsilon)\, n_F^{\downarrow' d}(\epsilon)\right]\right\}
\end{aligned}$$

and in the low temperature and differential potentials limit, where $\left(n_F^{s'd}\right)^2 = n_F^{s'd}$:

$$\begin{aligned}&\left[n_F^{\uparrow'd}(\epsilon) - n_F^{\downarrow'd}(\epsilon)\right]^2 (-2\sin^2\theta) + \\ &+ 4\left[n_F^{\uparrow'd}(\epsilon) + n_F^{\downarrow'd}(\epsilon) - 2\, n_F^{\uparrow'd}(\epsilon)\, n_F^{\downarrow'd}(\epsilon)\right] = \\ &= 2\left[n_F^{\uparrow'd}(\epsilon) - n_F^{\downarrow'd}(\epsilon)\right]^2 (2 - \sin^2\theta) = \\ &\qquad\qquad = 2\,|\mu^{\uparrow'd} - \mu^{\downarrow'd}|\,(2-\sin^2\theta)\,\delta(\epsilon - \epsilon_F).\end{aligned}$$

9.5 COMPARISON WITH BROWN THERMAL NOISE

Consider for simplicity the case $\vec{B} = 0$ and $\Delta\mu^R = 0$. In particular, if we assume that the magnetization of the left layer in figure 5.3 on page 56 interacts significantly with the nano-magnet only by means of the interaction mediated by the polarized current and we assume that we have thermal fluctuations that change the magnitude and the direction of the left layer, we simply have to operate the substitution

$$\hat{z}' \to \hat{z}' + \vec{h}(t)$$

in the dynamics equation 1.7 on page 6, where $\vec{h}(t)$ is a Brown stochastic term. As discussed in [62], the quantum noise must be dominant with respect to the Brown fluctuations at the low temperature; in a realistic case the two kind of fluctuations are comparable at a temperature of some Kelvin degrees.

9.6 NUMERICAL EXAMPLE

We consider the numerical example proposed in [63]. It is convenient to use the SI units. In this example the external magnetic field $\vec{B}$ and the temperature are relevant only for the initial state but, following [63], they can be neglected in the dynamical equations. Furthermore it is considered only a polarized current coming from left to right and we can put: $\Delta\mu^R_{\text{spin}} = 0$ and $\Delta\mu^L_{\text{spin}} =: \Delta\mu_{\text{spin}}$. It is also convenient to write the equation 1.7 on page 6 in the form

$$\begin{aligned}\partial_t \vec{M} = &\left(\Re C_1 + \frac{-\cos\theta\,\Im C_2\, I_1 + \Re C_2\, I_2}{|C_2|\,\sin\theta}\right)\hat{z}' \times \vec{M} + \\ &+ \left(\frac{\Im C_1}{M} + \frac{\cos\theta\,\Re C_2\, I_1 + \Im C_2\, I_2}{|C_2|\,\sin\theta\, M}\right)\vec{M} \times (\hat{z}' \times \vec{M})\end{aligned} \quad (9.6)$$

where $\vec{M} = |\gamma|\,\hbar\,\vec{J}$ is the magnetization ($\gamma \simeq e/m$ is the electron gyromagnetic ratio),

$$\langle I_i(t_1)\, I_j(t_2)\rangle = |C_2|^2\,\delta_{ij}\,\delta(t_1 - t_2)$$

Element	Value	Unit
$t_\uparrow(k_F)$	$0.067 - 0.251\,i$	
$t_\downarrow(k_F)$	$0.924 - 0.265\,i$	
$\Delta\mu_{\text{spin}}$	$9.990 \cdot 10^{-29}/t_n$	J
C_1	$(3.312 + 5.509\,i)/t_n$	s^{-1}
C_2	$(0.009\ + 0.014\,i)/\sqrt{t_n}$	$s^{-1/2}$
B	0.05	T
$\vec{B}$ direction	$(\theta, \phi) = (2.8, 1.0)$	rad
T	1	K

Table 9.1: To reproduce the simulation in [63], we consider $\lambda_0 = 3.36 \cdot 10^{-28}\text{J m}$, $\lambda = 5.76 \cdot 10^{-32}\text{J m}$, $J = 10^4$ and $k_F = 13.6\,\text{nm}^{-1}$; furthermore $\Delta\mu_{\text{spin}} = 2\,\pi\ 1.5 \cdot 10^5\,\hbar/t_n$.

and

$$C_1 = \frac{\Delta\mu_{\text{spin}}\,\lambda\,m\,t_\downarrow(k_F)\,t^*_\uparrow(k_F)}{4\,\pi\,\hbar^3\,k_F},$$

$$C_2 = t_\downarrow(k_F)\,t^*_\uparrow(k_F)\,\sqrt{\frac{\Delta\mu_{\text{spin}}\,\lambda^2\,m}{\pi\,16\,\epsilon_F\,\hbar^3}}.$$

In [63] it is assumed that $n_e = 1.5 \cdot 10^5$ electrons are coming from left to right with spin up (and then are partially transmitted and partially reflected by the magnet) in a time t_n. In particular, comparing with equation 6.3 on page 60, we must have

$$\Delta\mu_{\text{spin}} = \frac{2\,\pi\,n_e\,\hbar}{t_n}.$$

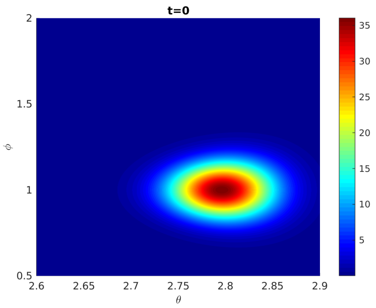
The other values are described in the table 9.1. The initial state is given by

$$P(\hat{z}, t = 0) = C\,e^{-\beta\,E} = C\,e^{\beta\,\vec{M}\cdot\vec{B}} = C\,e^{\beta\,|\gamma|\,\hbar\,J\,\hat{z}\cdot\vec{B}}$$

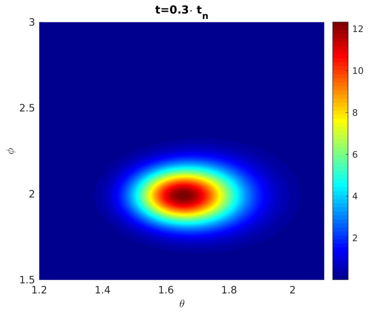
where $\hat{z}$ is the $\vec{J}$ direction, E is the magnetic energy and C is the normalization constant:

$$C^{-1} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\ \sin\theta\,e^{\beta\,|\gamma|\,\hbar\,J\,B\,\cos\theta} = \\ = \frac{4\,\pi\,\sinh(\beta\,B\,|\gamma|\,\hbar\,J)}{\beta\,B\,|\gamma|\,\hbar\,J};$$

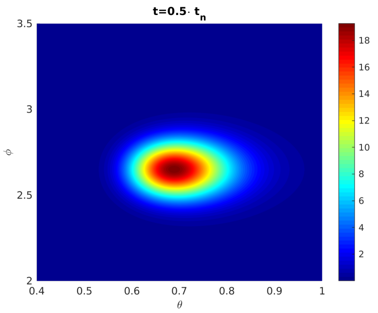
the temperature and the magnetic field chosen in [63] are reported



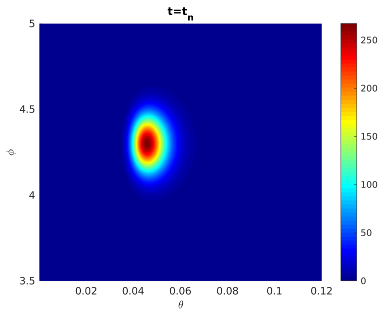
(a) The initial probability distribution.



(b) The distribution for $t = 0.3 \cdot t_n$.



(c) The distribution for $t = 0.5 \cdot t_n$.



(d) The distribution for $t = t_n$.

Figure 9.1: The probability distribution for (θ, ϕ) in radians.

in the table 9.1 on the previous page. In particular, the figure 9.1a represents the initial distribution probability for (θ, ϕ):

$$p_0(\theta, \phi) = p(\theta, \phi; t = 0) = P(\hat{z}, t = 0) \sin\theta.$$

Comparing C_1 and C_2^2, we see that we may neglect the quantum corrections at a first time. For $C_2 = 0$, choosing the $\theta = 0$ axis parallel to the current polarization, the equation 9.6 on page 84 gives us

$$\begin{aligned} &\sin[\theta(t)]\{\Im C_1 \cos[\theta(t)] \sin[\phi(t)] - \Re C_1 \cos[\phi(t)] + \\ &\qquad + \phi'(t) \cos[\phi(t)]\} + \theta'(t) \cos(\theta(t)) \sin(\phi(t)) = 0, \\ &\sin[\theta(t)] \left\{\Im C_1 \sin[\theta(t)] + \theta'(t)\right\} = 0 \end{aligned}$$

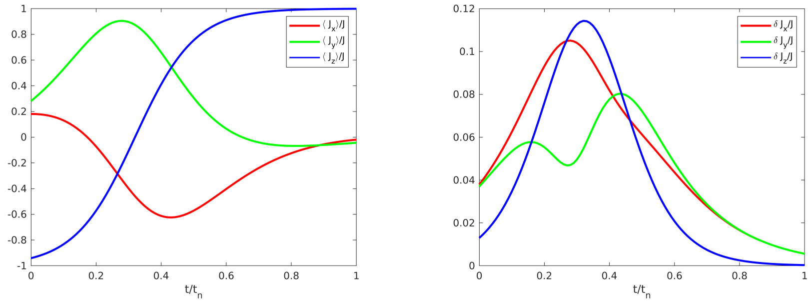
from which we have the solution:

$$\begin{cases} \theta_S(t, \theta_0) = \theta(t) = 2 \cot^{-1}\left[\cot\left(\frac{\theta_0}{2}\right) e^{\Im C_1 t}\right], \\ \phi_S(t, \phi_0) = \phi(t) = \Re C_1 t + \phi_0, \end{cases}$$

where θ_0 and ϕ_0 are the angles for $t = 0$. Observe that $\theta(t)$ is a decreasing function that, for $t \to +\infty$, goes to 0 (this represents the damping effect).

In particular, we can evaluate the time evolution for $p(\theta, \phi)$ simply by considering the evolution of each trajectory:

$$p(\theta, \phi; t) = p_0[\theta_S(-t, \theta), \phi_S(-t, \phi)] \left|\frac{\partial[\theta_S(-t, \theta), \phi_S(-t, \phi)]}{\partial(\theta, \phi)}\right|,$$



(a) Mean value. (b) Fluctuation.

Figure 9.2: Mean value and fluctuation of $\vec{J}/J$ during the time interval $[0, t_n]$.

where $|\partial(\theta_S, \phi_S)/\partial(\theta, \phi)|$ is the absolute value of the Jacobian determinant. $p(\theta, \phi; t)$ is depicted in the figures 9.1 on the facing page for different values of time; the mean value and the fluctuation of $\vec{J}$ are depicted in the figures 9.2. As expected (comparing the order magnitude of C_2^2 and C_1), these figures are similar to the figures in [63]. Only the probability density depicted in 9.1d on the facing page shows a smaller fluctuation with respect to the Wang and Sham corresponding figure (in any case, observe that this figure is much more zoomed with respect to the other three along θ). This is not surprising: for large t the damping produces a fluctuation decrease (all trajectories converge to $\theta = 0$) and then the quantum noise becomes relevant.

9.6.1 *Quantum corrections*

In the same way we can consider also the quantum corrections. By using the equation 2.10 on page 14, it is also possible to obtain the corresponding Fokker-Planck equation.

Because of the damping effect, it is interesting to consider the small θ limit. From 9.6 on page 84 we get:

$$\theta'(t) = -\Im C_1 \sin\theta - \frac{I_1(t)\,\Re C_2 \cos\theta + I_2(t)\,\Im C_2}{|C_2|} \simeq$$
$$\simeq -\Im C_1\,\theta - \frac{I_1(t)\,\Re C_2 + I_2(t)\,\Im C_2}{|C_2|}; \quad (9.7)$$

the last term is a linear combination of Gaussian stochastic processes and then

$$I(t) := -\frac{I_1(t)\,\Re C_2 + I_2(t)\,\Im C_2}{|C_2|} \quad (9.8)$$

is a Gaussian stochastic process with

$$\langle I(t)\rangle = 0, \qquad \langle I(t_1)\,I(t_2)\rangle = |C_2|^2\,\delta(t_1 - t_2). \quad (9.9)$$

In particular, equation 9.7 on the previous page reduces to the Ornstein–Uhlenbeck process. The corresponding Fokker-Plank equation is:

$$\partial_t p(\theta, t) = \Im C_1 \, \partial_\theta [\theta \, p(\theta, t)] + \frac{|C_2|^2}{2} \, \partial_\theta^2 p(\theta, t), \tag{9.10}$$

whose stationary solution is a Gaussian distribution with zero mean value and $|C|^2/(2\,\Im C_1)$ variance ($\simeq 5 \cdot 10^{-5}$, in our numerical example; this means that the standard deviation is of the order of $\simeq 1/100$ radians, that is consistent with the difference between the figure 9.1d on page 86 and the numerical solution in [63]):

$$p_\infty(\theta) = \sqrt{\frac{\Im C_1}{\pi \, |C|^2}} \, e^{-\Im C_1 \, \theta^2 / |C_2|^2}.$$

10

POSSIBLE MODEL EXTENSIONS

In conclusion we considered a relatively simple model that allowed us to describe the quantum noise in the spin transfer torque effect. We used the Keldysh technique, that lends itself particularly well to various kinds of generalizations.

Furthermore, as noted in [62], "the development of quantum optics after the laser operation was understood by semiclassical theory provides perhaps an optimistic historical guide for the development of coherent magnetization dynamics after the successes of the semiclassical STT theory".

Then, as possible further applications, we can consider some quantum optics setups and try to find a correlation in the spintronics field. An example could be the setup proposed in the reference [36] and depicted in figure 10.1. Here a strong $\hat{y}$-polarization laser beam goes trough two atomic ensembles placed in a magnetic field $\vec{B}$ along $\hat{x}$. By tracing over the light degrees of freedom and using the master equation, in [36] it is shown that the steady-state is entangled. That could inspire a similar setup for the spintronics model that we considered, in which two nano-magnets are coupled by a polarized equation.

More in general, the model that we have considered is interesting because it could be a step in a cross-fertilization framework between the open quantum physics and the spintronics. Indeed, in recent years, the techniques typically used in the open quantum physics – such as the master equation – and those typically used in the solid state

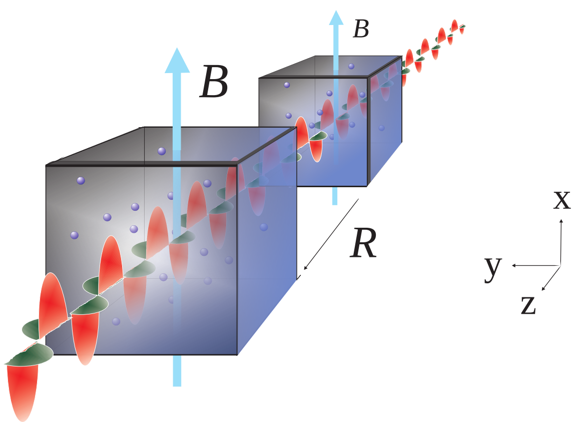


Figure 10.1: In this figure, taken from the reference [36], a laser beam couples two atomic ensembles.

physics – such as the Keldysh approach – came into contact, producing some interesting results and showing a strong correlation between them [35, 49].

APPENDIX

A GILBERT DAMPING TERM

In this appendix we consider the first order term in ω in the expansion 9.5 on page 79 and the corrections produced by this term in the magnet dynamics equation.

A.1 ACTION S^1

In the same way followed for the calculation of D^0 in section 9.3 on page 77, we may proceed for D^1:

$$D^1_{ab}(t) = \delta'(t)\,\frac{m}{2}\sum_{d_1,s'_1,s'_2}\int_0^\infty \frac{d\epsilon_1}{\sqrt{\epsilon_1}}\int_0^\infty \frac{d\epsilon_2}{\sqrt{\epsilon_2}}\left[n_F^{s'_1 d_1}(\epsilon_1) - n_F^{s'_2 d_1}(\epsilon_2)\right]\cdot \\ \cdot\,\mathcal{M}^{s'_1 s'_2}_{a|k_1||k_2|}(\perp)\,\mathcal{M}^{s'_2 s'_1}_{b|k_2||k_1|}(\perp)\,\frac{1}{(\epsilon_1-\epsilon_2-i\,0^+)^2}$$

and the non vanishing terms are:

- the term that multiplies $\bar{b}^{\mathrm{cl}}\,b^{\mathrm{q}}$ is:

$$D^A_1(t) = \delta'(t)\,\frac{\lambda^2\,J\,N^4\,m}{8}\sum_d\int_0^\infty \frac{d\epsilon_1}{\sqrt{\epsilon_1}}\int_0^\infty \frac{d\epsilon_2}{\sqrt{\epsilon_2}}\cdot \\ \cdot\left\{n_F^{\uparrow' d}(\epsilon_1) + n_F^{\downarrow' d}(\epsilon_1) + \left[n_F^{\uparrow' d}(\epsilon_1) - n_F^{\downarrow' d}(\epsilon_1)\right]\cos\theta + \right. \\ \left. - n_F^{\uparrow' d}(\epsilon_2) - n_F^{\downarrow' d}(\epsilon_2) + \left[n_F^{\uparrow' d}(\epsilon_2) - n_F^{\downarrow' d}(\epsilon_2)\right]\cos\theta\right\}\cdot \\ \cdot\,|t_\uparrow(k_1)\,t_\downarrow(k_2)|^2\,\frac{1}{(\epsilon_1-\epsilon_2-i\,0^+)^2};$$

  then in the low temperature limit, we have terms, for example, of the form

$$\int_0^{\bar{\mu}} \frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\int_0^\infty \frac{d\epsilon_2}{\sqrt{\epsilon_2}}|t_\downarrow(k_2)|^2\,\frac{1}{(\epsilon_1-\epsilon_2-i\,0^+)^2} = \\ = \int_0^{\bar{\mu}} \frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\int_0^\infty \frac{d\epsilon_2}{\sqrt{\epsilon_2}}|t_\downarrow(k_2)|^2\,\partial_{\epsilon_2}\frac{1}{\epsilon_1-\epsilon_2-i\,0^+} = \\ = -\int_0^{\bar{\mu}} \frac{d\epsilon_1}{\sqrt{\epsilon_1}}\,|t_\uparrow(k_1)|^2\cdot \\ \cdot\int_0^\infty d\epsilon_2\left[i\,\pi\,\delta(\epsilon_1-\epsilon_2) + P\frac{1}{\epsilon_1-\epsilon_2}\right]\partial_{\epsilon_2}\frac{|t_\downarrow(k_2)|^2}{\sqrt{\epsilon_2}},$$

where the Sokhotski–Plemelj formula has been used; the imaginary part of the integral gives

$$
\begin{aligned}
& -\pi \int_0^{\bar{\mu}} d\epsilon \, \frac{|t_\uparrow(k)|^2}{\sqrt{\epsilon}} \, \partial_\epsilon \frac{|t_\downarrow(k)|^2}{\sqrt{\epsilon}} = \\
& = -\pi \left[\frac{\bar{\mu}}{\left(\bar{\mu} + g_\downarrow^2\right)\left(g_\uparrow^2 - g_\downarrow^2\right)} + \right. \\
& \qquad \left. + \frac{\left(g_\uparrow^2 + g_\downarrow^2\right)\left(\log \frac{\bar{\mu}+g_\uparrow^2}{g_\uparrow^2} - \log \frac{\bar{\mu}+g_\downarrow^2}{g_\downarrow^2}\right)}{2\left(g_\uparrow^2 - g_\downarrow^2\right)^2} \right]
\end{aligned}
$$

while the real one:

$$
\begin{aligned}
& - \int_0^{\bar{\mu}} \frac{d\epsilon_1}{\sqrt{\epsilon_1}} \, |t_\uparrow(k_1)|^2 \, \frac{1}{\left(g_\downarrow^2 + \epsilon_1\right)^2} \cdot \\
& \cdot P \left[\frac{(g_\downarrow^2 - \epsilon_1) \, \log \left| \frac{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}} \right|}{2\sqrt{\epsilon_1}} + \right. \\
& \left. + \frac{\sqrt{\epsilon_2}\left(g_\downarrow^2 + \epsilon_1\right)}{g_\downarrow^2 + \epsilon_2} + 2\, g_\downarrow \, \arctan \frac{\sqrt{\epsilon_2}}{g_\downarrow} \right]_{\epsilon_2=0}^{\infty} = \\
& = - \int_0^{\bar{\mu}} \frac{d\epsilon_1}{\sqrt{\epsilon_1}} \, |t_\uparrow(k_1)|^2 \, \frac{\pi \, g_\downarrow}{\left(g_\downarrow^2 + \epsilon_1\right)^2} = \\
& = -\pi \, g_\downarrow \left[\frac{\sqrt{\bar{\mu}}}{\left(g_\downarrow^2 - g_\uparrow^2\right)\left(g_\downarrow^2 + \bar{\mu}\right)} + \right. \\
& \qquad \left. + \frac{\left(g_\downarrow^2 + g_\uparrow^2\right) \arctan\left(\frac{\sqrt{\bar{\mu}}}{g_\downarrow}\right)}{g_\downarrow \left(g_\downarrow^2 - g_\uparrow^2\right)^2} - \frac{2\, g_\uparrow \arctan\left(\frac{\sqrt{\bar{\mu}}}{g_\uparrow}\right)}{\left(g_\uparrow^2 - g_\downarrow^2\right)^2} \right].
\end{aligned}
$$

At the end, in the low differential potentials limit, we obtain:

$$
\begin{aligned}
& \Im D_1^A(t) = -\delta'(t) \, \frac{\lambda^2 \, J \, N^4 \, m \, \pi}{8} \, \frac{1}{\left(\epsilon_F + g_\uparrow^2\right)^2 \left(\epsilon_F + g_\downarrow^2\right)^2} \cdot \\
& \cdot \sum_d \left[4\, \epsilon_F^3 + \epsilon_F^2 \left(2\, g_\uparrow^2 + 2\, g_\downarrow^2 - \mu^{\uparrow' d} - \mu^{\downarrow' d}\right) + \right. \\
& \left. - \epsilon_F \, \cos\theta \left(g_\uparrow^2 - g_\downarrow^2\right) (\mu^{\uparrow' d} - \mu^{\downarrow' d}) + g_\uparrow^2 \, g_\downarrow^2 \, (\mu^{\uparrow' d} + \mu^{\downarrow' d}) \right]
\end{aligned}
$$

and

$$\Re D_1^A(t) = -\delta'(t)\,\frac{\lambda^2 J N^4 m \pi}{8}\,\frac{1}{(g_\downarrow - g_\uparrow)^2}\cdot$$
$$\cdot\sum_d\Bigg\{\frac{\sqrt{\epsilon_F}\,(g_\downarrow - g_\uparrow)}{\left(\epsilon_F + g_\downarrow^2\right)^2\left(\epsilon_F + g_\uparrow^2\right)^2}\Big[2\,\epsilon_F^3 + 6\,\epsilon_F^2\,g_\downarrow\,g_\uparrow +$$
$$-\left(g_\downarrow^2 - g_\uparrow^2\right)\cos\theta\left(\mu^{\downarrow\prime d} - \mu^{\uparrow\prime d}\right)(\epsilon_F + g_\downarrow\,g_\uparrow) +$$
$$+\,\epsilon_F\Big(4\,g_\downarrow^3\,g_\uparrow + g_\downarrow^2\left(-2\,g_\uparrow^2 + \mu^{\downarrow\prime d} + \mu^{\uparrow\prime d}\right) + 4\,g_\downarrow\,g_\uparrow^3 +$$
$$-\,2\,g_\downarrow\,g_\uparrow\left(\mu^{\downarrow\prime d} + \mu^{\uparrow\prime d}\right) + g_\uparrow^2\left(\mu^{\downarrow\prime d} + \mu^{\uparrow\prime d}\right)\Big) +$$
$$+\,g_\downarrow\,g_\uparrow\Big(g_\downarrow^2\left(2\,g_\uparrow^2 - \mu^{\downarrow\prime d} - \mu^{\uparrow\prime d}\right) + 2\,g_\downarrow\,g_\uparrow\left(\mu^{\downarrow\prime d} + \mu^{\uparrow\prime d}\right) +$$
$$-\,g_\uparrow^2\left(\mu^{\downarrow\prime d} + \mu^{\uparrow\prime d}\right)\Big)\Big] + 2\arctan\frac{\sqrt{\epsilon_F}}{g_\downarrow} - 2\arctan\frac{\sqrt{\epsilon_F}}{g_\uparrow}\Bigg\};$$

- similarly for the term D_1^R that multiplies $b^{\mathrm{cl}}\,\bar{b}^{\mathrm{q}}$:

$$\Re D_1^R = \Re D_1^A, \qquad \Im D_1^R = -\Im D_1^A.$$

A.2 S^1 CORRECTIONS IN THE DYNAMICS EQUATION

Finally we may write the action as:

$$S^1 = i\int dt_1 \int dt_2 \begin{pmatrix}\bar{b}^{\mathrm{cl}}(t_1) & \bar{b}^{\mathrm{q}}(t_1)\end{pmatrix}\cdot$$
$$\cdot\begin{pmatrix} 0 & D_1^A(t_1 - t_2) \\ D_1^R(t_1 - t_2) & 0\end{pmatrix}\begin{pmatrix} b^{\mathrm{cl}}(t_2) \\ b^{\mathrm{q}}(t_2)\end{pmatrix} =$$
$$= \int dt \begin{pmatrix}\bar{b}^{\mathrm{cl}}(t) & \bar{b}^{\mathrm{q}}(t)\end{pmatrix}\begin{pmatrix} 0 & -\alpha^*\,\partial_t \\ \alpha\,\partial_t & 0\end{pmatrix}\begin{pmatrix} b^{\mathrm{cl}}(t) \\ b^{\mathrm{q}}(t)\end{pmatrix} =$$
$$= \alpha\int dt\,\bar{b}^{\mathrm{q}}\,\partial_t b^{\mathrm{cl}} + \mathrm{h.c.}$$

and it is easy to check that it gives rise to a correction in the right side of the dynamics equation 8.4 on page 70 of the form:

$$-\frac{\Re\alpha}{J}\,\vec{J}\times\partial_t\vec{J} + \Im\alpha\,\partial_t\vec{J},$$

that is, a damping term plus a time re-scaling for the nano-magnet dynamics.

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Final Version as of February 15, 2019 (classicthesis v4.6).