Dipartimento di Matematica e Fisica
Sezione di Matematica

Dottorato di Ricerca in Matematica

XXVI Ciclo

Sigmoidal Functions Approximation and Applications

Candidate
Dr. Danilo Costarelli

Danilo Costarelli

Advisor
Prof. Renato Spigler

Coordinator of the Ph.D. Program:

Prof. Luigi Chierchia

ACADEMIC YEAR 2013/2014

Nessuna umana scienza si può
dimandare vera scienza se essa non
passa per le matematiche dimostrazioni.

Leonardo da Vinci (1452-1519)

A special thanks to my advisor
Prof. Renato Spigler
and to all my Masters of Research.
An heartfelt thanks to Marta
and to all my loved ones.

In our country,
when the funds have to be cut,
the funds for culture and research are cut,
as usual,
since they are evidently considered
an unnecessary luxury.

Margherita Hack (1922-2013)

Contents

Introduction

The main purpose of this Thesis is to explore the world of the possible applications of sigmoidal functions approximation, both from the theoretical point of view and for the applications.

Sigmoidal functions are merely measurable, real-valued functions, say $\sigma(x)$, such that $\lim_{x \to -\infty} \sigma(x) = 0$ and $\lim_{x \to +\infty} \sigma(x) = 1$. The most known examples are, the *logistic* function $\sigma_\ell(x) := (1+e^{-x})^{-1}$, $x \in \mathbb{R}$, the hyperbolic tangent sigmoidal function $\sigma_h(x) := (1/2)(\tanh x + 1)$, $x \in \mathbb{R}$, and the Gompertz functions $\sigma_{\alpha\beta}(x) := e^{\alpha e^{-\beta x}}$, $x \in \mathbb{R}$, α, $\beta > 0$. For instance, σ_ℓ and σ_h are well-known for their applications to demography, economics and statistics (see e.g. [25, 78, 129]); these functions also occur in many differential models of population growth, Gompertz functions are also used to describe tumor growth models, see e.g. [110, 3, 55].

Applications of sigmoidal functions in Approximation Theory arise, e.g., from the theory of neural networks (NNs), where they are introduced as a generalization of the *perceptron* function (the Heaviside or unit step function) which plays the role of the activation function of the network.

NNs are fundamental in many fields. In particular, they have been introduced to model the human brain, since they are "able to learn from a training process" based on mathematical algorithms, see e.g. [74, 23, 124, 66, 24]. By "training", it is meant that the weights of an artificial neural network can be chosen in such a way that the network is capable to represent the values of a certain set of samples. The first proposed training algorithm was the *Delta-Rule* [76], while the most popular was the *back-propagation* algorithm [75].

Moreover, NNs with sigmoidal functions are well suited to applications in circuit theory and filter design, where simple nonlinear devices are used to synthesize or approximate desired transfer functions [63].

The NNs are classified in base of their architecture and can be mainly divided in cyclic and acyclic networks. For instance, feed-forward neural networks (of the acyclic type), the kind of NNs considered in this Thesis, are usually represented by

$$\sum_{j=0}^{N} c_j \, \sigma(\underline{w}_j \cdot \underline{x} - \theta_j), \qquad \underline{x} \in \mathbb{R}^n, \; n \in \mathbb{N}^+, \tag{I}$$

where, for every $0 \leq j \leq N$, the $\theta_j \in \mathbb{R}$ are the threshold values, the $\underline{w}_j \in \mathbb{R}^n$ are the weights, and the c_j are the coefficients; $\underline{w}_j \cdot \underline{x}$ denotes the inner product in $\mathbb{R}^n$, and σ is the activation function of the network, see [80, 53, 94, 97, 100, 116, 121, 99, 58].

As it can be observed in (I), artificial neural networks possess a rather complex structure. In particular, they involve many variables and parameters, which provide a very flexible tool. This fact allows us to construct networks very well suited to approximate functions. This explain the reason for using neural networks in Approximation Theory.

On the other hand, the idea of superposing univariate functions to approximate multivariate functions, is strictly connected to the 13th Hilbert problem studied by Kolmogorov and Lorentz, see e.g. [86, 102]. The relation between the Kolmogorov representation theorem and neural networks approximation was also studied by Kurková in [87].

The first results concerning approximation by sigmoidal functions are due to G. Cybenko. In [63], he showed by non-constructive arguments that the space of the neural networks of the form (I), is dense with respect to the uniform topology in the set of all continuous functions defined on the rectangle $[0, 1] \times ... \times [0, 1]$ of $\mathbb{R}^n$.

The problem of the density of NNs in a certain space of functions was studied by many authors in various setting. The approach used for solving this problem is mainly non-constructive, especially in the multivariate case. This fact makes the application of these approximation methods very difficult. However, some constructive results were obtained but by very complex techniques. The constructive proofs were given in many cases for functions of one variable and only rarely for functions of several variables.

Along this Thesis, we present the main approximation results concerning this subject, and the main relevant difficulties are pointed out. In particular, convergence results are established in both, the univariate and the multivariate cases. Also, results and estimates concerning the order of approximation achieved by superposing sigmoidal functions are analyzed. Usually, the error estimates are given in terms of number, N, of the "neuron" making the network, or both, the number of neurons and the values of the weights, which are often related to each other. In this sense, the results obtained from different authors showed that, in order to approximate Lipschitz continuous functions, or functions of bounded variation, by finite linear combinations of sigmoidal functions, the approximation errors decreases as $1/N$, as N tends to $+\infty$, N being the number of superposed sigmoidal functions, see e.g. [1, 116, 72]. In some cases, also the constants in the error estimates can be determined (in the case of functions of bounded variation, e.g.), see [72, 95, 96]. Moreover, when we approximate functions in a certain subspace of $L^2(\mathbb{R}^n)$, the approximation error decreases as $N^{-1/2}$, as $N \to +\infty$ [20]. All results summarized above are presented in Chapter 1, together with a brief introduction on neural networks and their history.

One of the main purposes of this Thesis is to develop constructive approximation algorithms based on sigmoidal functions, useful for the theory of NNs, and also to develop applications, e.g., to Numerical Analysis, for solving numerically a number of problems, such as Volterra integral equations. The general approximation problem was treated in [56]; our approach was inspired by the paper of H. Chen, T. Chen and R. Liu [48].

In [56], a constructive theory is developed for approximating functions of one as well as several variables, by superposition of sigmoidal functions. This is done in the uniform norm as well as in the L^p norm. Results for the simultaneous approximation, with the same order of accuracy, of a given function and its derivatives (whenever these exists), are obtained. The relation with NNs and radial basis functions approximation is discussed. These results are presented in Chapter 2.

Several applications of the theory developed in [56] were proposed in [57, 59]. In [57], a numerical collocation method was developed for solving linear and nonlinear Volterra integral equations of the second kind, of the form

$$y(t) = f(t) + \int_a^t K(t,s)\, y(t)\, ds, \quad t \in [a,b], \tag{II}$$

and

$$y(t) = f(t) + \int_a^t K(t,s;y(s))\, ds, \quad t \in [a,b], \tag{III}$$

where $f : [a,b] \to \mathbb{R}$ and the kernels K are sufficiently smooth. The method is based on the approximation of the exact solution to such equations by sigmoidal functions, by the techniques developed in [56]. By our method, we can solve a large class of integral equations having either continuous or even L^p solutions. Special computational advantages are obtained using unit step functions, and analytical representations of the solutions are also at hand. The numerical errors are discussed, and *a priori* as well as *a posteriori* estimates are derived for them. Many numerical examples are given to test the method, that has been compared with the classical piecewise polynomial collocation techniques, see, e.g., [117, 26, 14]. We recall that, a *collocation solution* to a Volterra integral equation on an interval $[a,b]$, is an element of some finite-dimensional function space (the *collocation space*), which satisfies the equation on an appropriate finite subset of points in $[a,b]$. The latter is the set of *collocation points*, whose cardinality is the dimension of the collocation spaces. Our collocation method based on sigmoidal functions is described in Chapter 3.

To complete the set of applications to integral equations, the collocation method given in [57] has been extended in [59] to solve Volterra integro-differential equations (VIDEs) of the neutral type, of the form

$$y'(t) = f(t, y(t)) + \int_a^t K(t, s, y(s), y'(s))\, ds, \qquad y(a) = y_0, \tag{IV}$$

for $t \in I := [a, b]$, where f and K are sufficiently smooth given functions, see [32, 118, 119, 28]. The method is also suited to solve classical VIDEs, i.e.,

$$y'(t) = f(t, y(t)) + \int_a^t K(t, s, y(s))\, ds, \qquad y(a) = y_0, \tag{V}$$

(where the integral term in (V) does *not* depend on $y'(s)$) as well as non-standard VIDEs like

$$y'(t) = f(t, y(t)) + \int_a^t K(t, s, y(s), y(t))\, ds, \qquad y(a) = y_0, \tag{VI}$$

(where the integral term depends, in addition, on $y(t)$) see [117, 27]. The most known example of a non-standard VIDE is perhaps given by the *logistic equation with a memory term*, see [62, 15, 16].

Computational advantages are gained using unit step functions, but in many important applications other sigmoidal functions, such as logistic and Gompertz functions, are used. The method allows us to obtain a simultaneous approximation of the solution to a given VIDE and its first derivative, by means of an explicit formula. A priori as well as a posteriori estimates are derived for the errors, and numerical examples are given for the purpose of illustration. A comparison is made with other classical collocation methods as for accuracy and CPU time. The collocation method for integro-differential equations based on sigmoidal functions is discussed in Chapter 4.

The numerical methods that we proposed, are very easy to implement and present several advantages. These are discussed in the following chapters in more details. We stress the main peculiarity of our numerical methods, that is that *nonlinear* integral and integro-differential equations can be solved by means of finite sequences of linear recursive formulae, without using iterative methods for solving nonlinear algebraic systems such as, e.g., Newton's method, as usually happens in numerical methods for solving nonlinear problems.

However, accuracy should be improved in our methods. This is because by sigmoidal functions we are able to approximate functions both uniformly (hence, a fortiori, pointwise) and also in L^p, but the approximation method is merely one-order accurate.

Therefore, we have proposed new kinds of approximation techniques by sigmoidal functions, aimed at enhancing the performance of our methods. In Chapter 5 (see [58]), we develop a new constructive theory for approximating absolutely continuous (univariate) functions by series of certain sigmoidal functions. Estimates for the approximation error are derived and the relation with neural networks approximation is discussed. The connection between sigmoidal functions and the scaling functions of r-regular multiresolution approximations are investigated. In this setting, we show that the approximation error for C^1-functions decreases as 2^{-j}, as $j \to +\infty$. As a future work, we plan to use the latter approach to obtain numerical methods for solving integral and differential equations with high accuracy, thus improving the performance of the methods developed in Chapters 3 and 4.

Finally, in Chapter 6, we consider the modern theory of NN operators activated by sigmoidal functions. Anastassiou [5, 6] was the first to establish neural networks approximation to continuous functions with a specific rate, by special NN operators of the Cardalignet-Euvrard and Squashing types [45]. He employed the modulus of continuity of the engaged function or its high order derivatives. In his papers, Anastassiou produced Jackson-type inequalities. In these theorems, the logistic and the hyperbolic tangent functions are used as activation functions. In Sections 6.1 and 6.2, these results, concerning NN operators are recalled. Then, in the last part of the chapter, the results proved by Anastassiou are generalized for certain sigmoidal functions, belonging to suitable classes. We studied the convergence and the order of approximation for the family of operators defined (in the univariate case) by

$$F_n^h(f,x) \ := \ \frac{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \phi_h(nx-k)}{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_h(nx-k)}, \qquad x \in [a,b], \tag{VII}$$

where the density function is $\phi_\sigma(x) := (1/2)(\sigma(x+1)-\sigma(x-1))$, $x \in \mathbb{R}$, for a suitable sigmoidal function σ and for any continuous functions $f : [a,b] \to \mathbb{R}$, with $n \in \mathbb{N}^+$ sufficiently large. Here, $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the "ceiling" and the "floor" (the integral part) of a number, respectively. In some cases, by our operators in (VII) and with some specific choice of σ, we were able to improve the order of approximation with respect to that achieved by Anastassiou. The NN operators activated by sigmoidal functions are studied both in the univariate and the multivariate cases. These results can be found in [60, 61]. By the NN operators we can approximate functions defined on bounded rectangles. To approximate functions defined in the whole space, *quasi-interpolation* operators were introduced and studied (see Subsections 6.5.1 and 6.7.1). Some final remarks are given in Chapter 7, and a discussion about future work and possible extensions of the results established in this Thesis is made.

Chapter 1

Sigmoidal Functions and Neural Networks in Approximation Theory

1.1 A brief introduction to Neural Networks

There is no universally accepted definition of neural network (NN) in the literature. There is a general agreement that NNs are a collection of models of computation, very loosely based on biological motivations. According to Haykin [74],

"a neural network is a massively parallel distributed processor that has a natural propensity for storing experiential knowledge and making it available for use. It resembles the brain in two respects:

1. *knowledge is acquired by the network through a learning process;*

2. *inter-neuron connection strengths known as synaptic weights are used to store the knowledge."*

This is a highly non-mathematical definition, but it describes very well the ideas for which NNs have been introduced.

NNs are models useful to represent the behavior of the *human brain*. The brain is made by billions of *neurons*. Every neuron is a cellular body with branched extensions called *dendrites*, through which the neuron receives electrical impulses from other neurons (see Fig. 1.1) . Moreover, each neuron has a filamentous extension called *axon* with a variable length (from 1 cm to 1 m). At the end of the axon there are ramifications, which transmit the electrical signals to other cells, for instance, to the dendrites of other neurons. Between the terminal of an axon and the receiving cell there is some space. The electrical impulses pass through these spaces by means

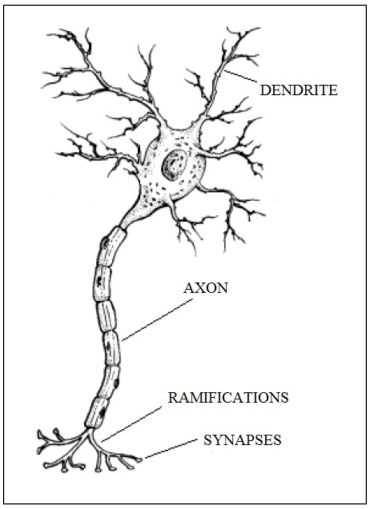

Figure 1.1: The biological neuron.

of chemicals called *neurotransmitters*. The connection point between the terminal and a dendrite is called *synapses*.

A neuron is *activated*, i.e., it transmits an electrical impulse along its axon, when a difference of electric potential between the inside and the outside of the cell occurs. The electrical impulse causes the release of a neurotransmitter from the axon terminals, which can also affect other neurons.

The human brain is a complex, parallel computer and although it consists of very simple processing elements (the neurons), it is able to perform complex tasks, such as recognition, perception, and control of movements. Moreover, it is able to modify the connections among the neurons based on their experience, that is, it is able to learn.

In the brain, there is no centralized control, in the sense that the various areas of the brain work simultaneously, effecting each others and contributing to the realization of specific tasks. Finally, the brain is fault tolerant, that is, when a neuron or one of its connections is damaged, it keeps working, though with a slightly degraded performance. In particular, the performance of the brain processes degrades gradually while the neurons are being destroyed (graceful degradation).

Therefore, to reproduce artificially the human brain behavior, a network of very simple elements should be built up, which is a distributed structure, massively parallel, able to learn, and then, to *generalize*, i.e., to produce outputs, correspondingly to inputs not encountered during the previous training.

1.1.1 The artificial neuron model

The artificial neuron model has a multivariate structure, having many inputs and one output, where each input has an associated weight, which determines

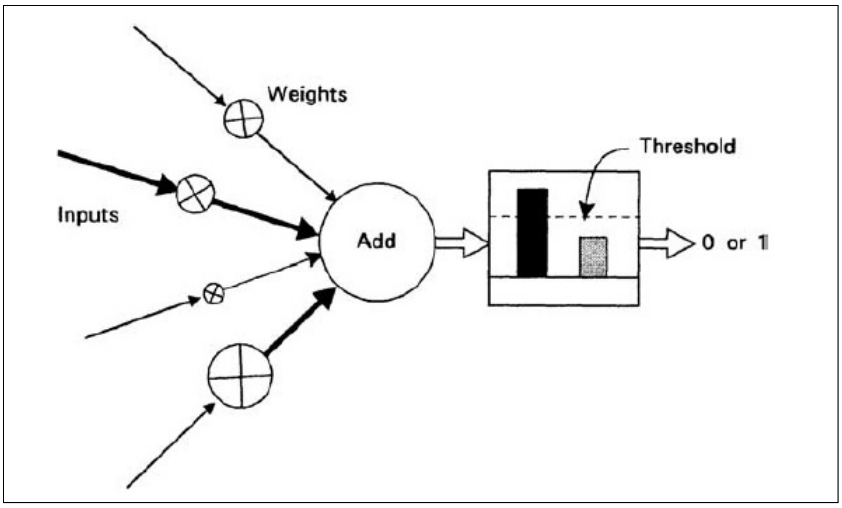

Figure 1.2: The artificial neuron model proposed by McCulloch-Pitts.

the conductivity of the input channel and reproduces the synapses.

The inputs provide a weighted contribution to the neuron and represents the electrical impulse that every neuron receives from the others to which it is linked. From the mathematical point of view, we will denote the inputs by the real parameters, $x_1, ...x_n$, and the associate weights by $w_1, ..., w_n$, $w_k \in \mathbb{R}$. The weighted sum of the input values is given by $\sum_{k=1}^n x_k w_k$. To every neuron a specific *threshold* value $\theta \in \mathbb{R}$ (also called *bias*) is associated as well as an *activation function*, having usually the form of a *Heaviside* function (or unit step function), $H(x)$, $H(x) = 1$, for $x \geq 0$ and $H(x) = 0$ otherwise. Then, the full model of an artificial neuron takes on the form

$$F(x_1, ..., x_n) \; := \; H\left(\sum_{k=1}^n x_k w_k - \theta\right), \qquad (x_1, ..., x_n) \in \mathbb{R}^n. \tag{1.1}$$

If the value $\sum_{k=1}^n x_k w_k$ exceeds the threshold θ, i.e., $\sum_{k=1}^n x_k w_k - \theta \geq 0$, then the function in (1.1) yields 1 as output. This situation represents the activated neuron. In case that $\sum_{k=1}^n x_k w_k - \theta < 0$, we have $F(x_1, ..., x_n) = 0$, which models a neuron that does not transmit any electrical impulse. This neuron model is very simple and it is also known as the *perceptron* or also as the *McCulloch-Pitts* model [111]. The name perceptron was coined by F. Rosenblatt in [125]. Such a model is represented in Fig. 1.2. Afterwards, Rosenblat extended the McCulloch-Pitts model so as to build neural networks consisting of artificial neurons and capable to learn from experience [126, 127].

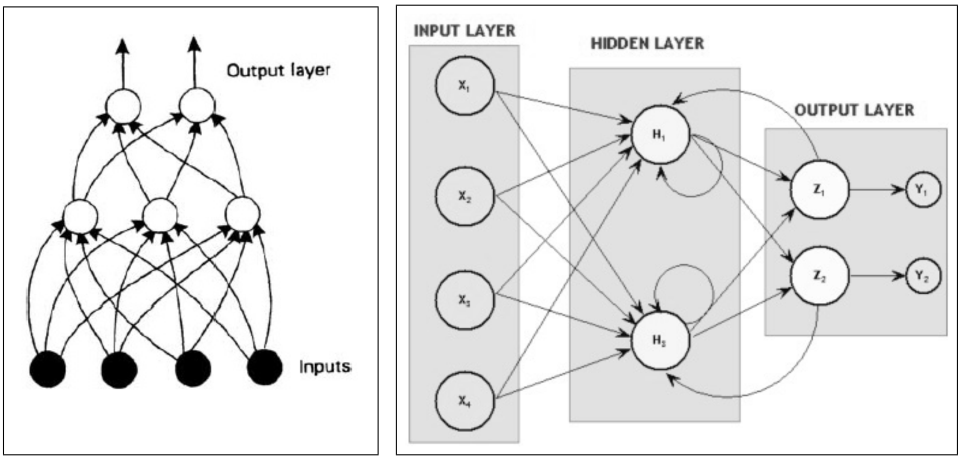

Figure 1.3: The multi-layers feed-forward neural network architecture (left) and the recursive neural network architecture (right).

1.1.2 The artificial neural network model

The design of a NN requires determining several components, such as the number of neurons, the activation functions of the neurons and their architecture, i.e., their connections.

The distinction between the various kinds of NNs is made according to the architecture that characterizes them. A classification can be made in: (i) *feed-forward*, and (ii) *recursive* NNs. The feed-forward NNs (see Fig.1.3, left) have an acyclic structure organized in *layers*, where each neuron is connected in input only with the previous layer and in output with the next layer. This kind of networks computes an input function depending only on the weights. Sometimes, it could have some hidden layers, i.e., layers in which the neurons do not communicate with the outside.

The recursive NNs (see Fig.1.3, right), on the other hand, have a cyclic structure, and the activation of the neurons determine the internal layer. The most common recursive NNs are the *Hopfield* neural networks and the *Boltzmann machines*. In the Hopfield NNs, every neuron appears in both, input and output, and has bidirectional connections with symmetric weights. The Boltzmann machines have hidden neurons, which are not involved in the input and output phases.

On the basis of the NNs models, one neuron can be activated at a time, or all neurons can be activated simultaneously. In the first case, we will talk of *asynchronous* activation, in the second case of *synchronous* or *parallel* activation.

From the mathematical point of view, for instance, a perceptron feed-forward neural network with one hidden layer (the kind of NNs that we will consider in this Thesis), can be represented by

$$\sum_{i=1}^{N} \alpha_i \, H \left(\sum_{k=1}^{n} x_k \, w_k^i - \theta_i \right), \tag{1.2}$$

i.e., as a finite linear combination of perceptrons, H being the unit step function. Defining $\underline{x} := (x_1, ..., x_n)$ and $\underline{w}_i := (w_1^i, ..., w_n^i)$, we denote by $\underline{x} \cdot \underline{w}_i := \sum_{k=1}^{n} x_k \, w_k^i$ the usual inner product in $\mathbb{R}^n$, and then we can rewrite (1.2) in the more compact form

$$\sum_{i=1}^{N} \alpha_i \, H \left(\underline{x} \cdot \underline{w}_i - \theta_i \right). \tag{1.3}$$

As observed in the previous paragraphs, since NNs are expected to behave as the human brain, they should be able to acquire some knowledge by means of a learning process. Mathematically speaking, this amounts to make a suitable choice of the weights w_k^i's of the NN in (1.2). For further details concerning neural networks see, e.g., the monographs [74, 23, 124, 66, 24, 85, 13].

1.1.3 Training the networks

The main task of training a network is to adjust its weights so to produce the desired responses. One of the most used training method is the *supervised learning*, which plans to present to the network the corresponding desired output for each training example. In general, at the beginning of the training, the weights are initialized using some random values. Then, all values contained in the *training set* are presented to the network. For each element of the training set, the *error* made by the NN, i.e., the difference between the desired and its real output, is computed. The size of the error is used to adjust the weights. This process is repeated several times, changing the order of presentation of the elements of the training set to the network. This process ends when the errors made by the NN in all training values are smaller than a given threshold.

After the training, the behavior of the network is tested against the values of a suitable *test set*, made by different elements (with respect to the training set). This phase allows us to consider the ability of the NN to generalize, i.e., to produce outputs, correspondingly to inputs not encountered during the previous training.

The performance of the NN strongly depends on the choice of the training set. These samples should be representative of the pattern that the network is intended to represent. Training is an ad hoc process, depending on the specific problem being treated.

One of the most used training algorithm is the *Widrow-Hoff rule* (or *Delta rule*) [76]. Let $\underline{x} := (x_1, ..., x_n)$ the input value for a given artificial

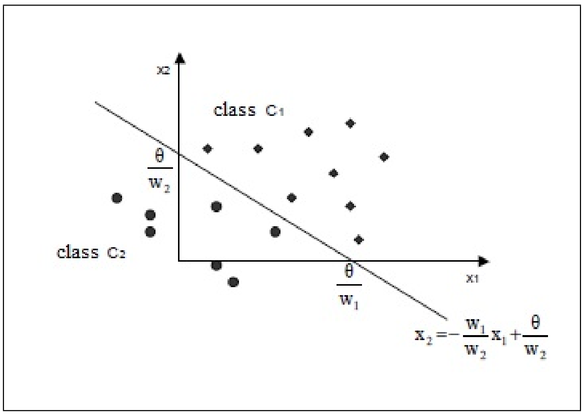

Figure 1.4: Example of classification problem.

neuron, and t and y the desired and the neural network output, respectively. The error is given by $\delta := t - y$. For the Widrow-Hoff rule, the variation of each weight should be computed by

$$\Delta w_k = \eta\, \delta\, x_k,$$

where $0 < \eta \leq 1$ is the so-called *learning rate* (or learning speed).

For instance, one of the possible applications of the training of a perceptron NN is the so-called *classification problems* of a class of objects, consisting in associating each object to the correct class. Suppose that we wish to classify some objects represented by points of the plane, in two distinct classes. If such classes are linearly separable, we can use a NN which approximates a straight line of separation between the two classes. Then, an object will be ranked by representing it as a point in the plane, and moreover, we can assign it to one of the two classes identified by the half-plane where the point is located (see Fig.1.4). This kind of classification can be obtained using, e.g., the Widrow-Hoff rule, a bivariate NN with one perceptron. We recall, for the sake of completeness, that a classification problem is termed *linear* if a line (on the plane), or an hyperplane (on the n-dimensional space, $n > 2$) can be used to separate all points. Otherwise, the problem is called *nonlinear*. In general, the perceptron model can be used to solve linear classification problem, in which the input values must be separated putting them in different classes.

Nonlinear problems can be solved by multi-layers NNs, see, e.g., [21]. In this case, it can be useful to set up an algorithm for the supervised training. The problem appearing in training of multi-layers NNs is that a technique of adjustment of the weights, similar to the Widrow-Hoff rule, allows to update only the weights related to the output neurons. This fact is reasonable, since only for the neurons of the output layers we know the desired output.

This problem was solved, relatively recently, in 1986, when was intro-

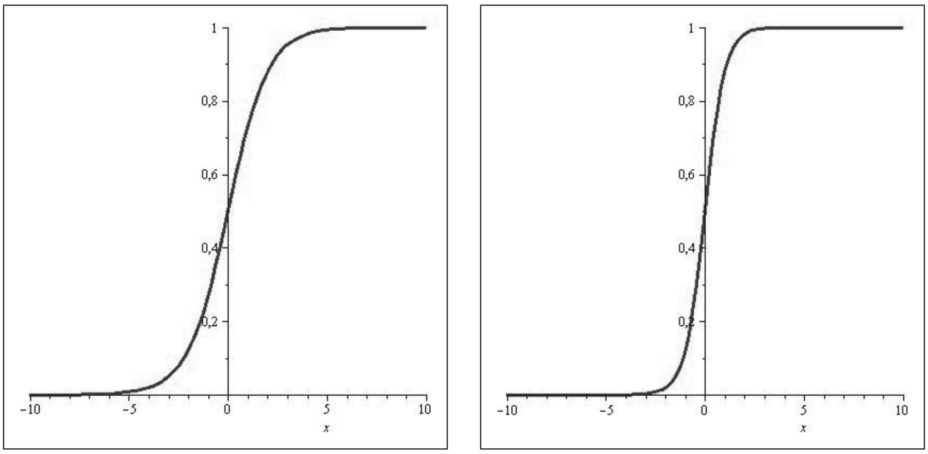

Figure 1.5: The logistic function σ_ℓ (left) and the hyperbolic tangent sigmoidal function σ_h (right).

duced the *back-propagation algorithm* [75]. This algorithm consists in computing the error of the last hidden layer neuron, and propagating back the error computed on the output neurons linked to the last but one hidden layer, and so on.

The back-propagation algorithm consists of two phases. First the elements of the training set go from the input to the output of the network and then the errors are propagated back. In this second phase a weights adjustment is made. At the beginning, the weights are initialized by random values. The back-propagation algorithm is a generalization of the Widrow-Hoff rule, and is a *gradient descent method*. Moreover, it requires that the activation function of the network be differentiable, then the Heaviside function (discontinuous at $x = 0$) is usually replaced by the the smooth, nonlinear function σ_ℓ, defined by

$$\sigma_\ell(x) \ := \ (1 + e^{-x})^{-1}, \qquad x \in \mathbb{R},$$

called *logistic* (or *sigmoid*) function (see Fig.1.5, left). It is clear that σ_ℓ has the same behavior of the unit step function, and may play the role of activation function as well. Another instance often used, for smooth, activation function is given by the *hyperbolic tangent sigmoidal* function, defined by

$$\sigma_h(x) \ := \ (1/2)(\tanh(x) + 1), \qquad x \in \mathbb{R},$$

where $\tanh(x) := (e^{2x} - 1)/(e^{2x} + 1)$, see Fig.1.5 right.

1.2 Sigmoidal functions

Training by the back-propagation algorithm of multilayer neural networks requires considering neural networks with smooth activation functions. Clearly, these functions must have a graph with the same behavior of the unit step function, and this leads to the introduction of a new class of functions, the *sigmoidal* functions. We have the following.

Definition 1.1. *A measurable function $\sigma : \mathbb{R} \to \mathbb{R}$ is called "a sigmoidal function" whenever*

$$\lim_{x \to -\infty} \sigma(x) = 0 \quad \textit{and} \quad \lim_{x \to +\infty} \sigma(x) = 1.$$

Sometimes, boundedness, continuity and/or monotonicity may be prescribed in addition.

Obviously, the Heaviside (unit step) function is a sigmoidal function. Moreover, the logistic and hyperbolic tangent sigmoidal functions introduced in the previous section are clearly examples of smooth and bounded sigmoidal functions.

In particular, logistic functions are largely used in many fields, such as Biology, Physics, Biomathematics, Statistics, Economics, and Demography (see [25, 78], e.g.) and indeed they were first introduced in the 19th Century as a model to describe population growth.

Other important examples of smooth sigmoidal functions are given by the following:

$$\sigma_G(x) \ := \ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-t^2/2}\, dt, \qquad x \in \mathbb{R}, \tag{1.4}$$

the well-known *Gaussian* sigmoidal function (see e.g. [121]), useful in probabilistic and statistical applications,

$$\sigma_{arc}(x) \ := \ \frac{1}{\pi} \arctan(x) + \frac{1}{2}, \qquad x \in \mathbb{R}, \tag{1.5}$$

the *arctan* sigmoidal function (see e.g. [121, 56]) and finally, the *Gompertz* functions

$$\sigma_{\alpha,\beta}(x) \ := \ e^{-\alpha e^{-\beta x}}, \qquad x \in \mathbb{R}, \tag{1.6}$$

where α, $\beta > 0$ represent an effective translation and a scaling term, respectively (see Fig.1.6).

Gompertz functions, have been introduced by Benjamin Gompertz for the study of his demographic model, which represents a refinement of the Malthus model. These functions are characterized by unsymmetrical growth unlike σ_ℓ, σ_h, σ_G and σ_{arc}, moreover, they find applications in modeling tumor growth and in population aging description, see e.g. [110, 3, 55] and [129], respectively.

A further important example of non-smooth, continuous, sigmoidal function is given by

$$\sigma_R(x) = \begin{cases} 0, & \text{for } x < -1/2, \\ x + 1/2, & \text{for } -1/2 \leq x \leq 1/2, \\ 1, & \text{for } x > 1/2, \end{cases} \tag{1.7}$$

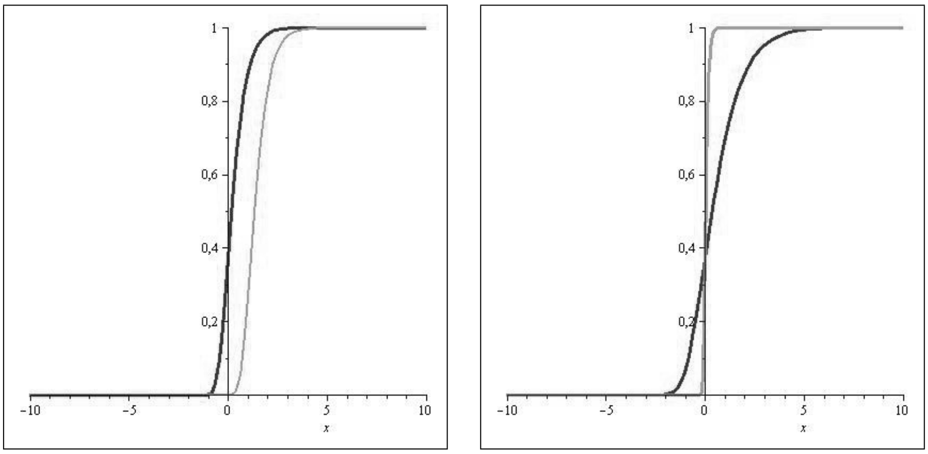

Figure 1.6: Left: The Gompertz functions $\sigma_{\alpha,2}$ with $\alpha = 1$ (black) and $\alpha = 10$ (gray). Right: The Gompertz functions $\sigma_{1,\beta}$ with $\beta = 1$ (black) and $\beta = 10$ (gray).

the so-called *ramp* function (see e.g. [46, 39]).

A further generalization of the concept of sigmoidal function can be given introducing a mathematical formulation for the activation functions. We have the following.

Definition 1.2. *A measurable function $s : \mathbb{R} \to \mathbb{R}$ is called "activation function" whenever*

$$\lim_{x \to -\infty} s(x) = a \quad \textit{and} \quad \lim_{x \to +\infty} s(x) = b,$$

with $a \neq b$.

1.3 A survey on approximation by sigmoidal functions

In Section 1.1.3 the problem of the training neural networks is introduced. Mathematically, this fact can be reviewed as an *approximation problem*.

The adaptable structure of a feed-forward neural network with one hidden layer, composed many neurons, weights and various parameters, results very suitable to approach this kind of problems.

Clearly, in order to obtain applications to the training of NNs, the approximation problems should be solved by *constructive methods*, but as we shall see in the next sections, whenever it is possible, it results to be very difficult.

In fact, the approximation results proved in the last 25 years are mainly obtained by non constructive arguments.

One of the most studied problems of approximation by neural networks with sigmoidal functions, is the so-called *density problem*.

Let σ be a given sigmoidal function. We denote by

$$\mathcal{M}_N(\sigma) \ := \ \left\{ \sum_{i=1}^{N} \alpha_i \sigma(\underline{x} \cdot \underline{w}_i - \theta_i) : \ \alpha_i, \theta_i \in \mathbb{R}, \ \underline{w}_i \in \mathbb{R}^n \right\}, \tag{1.8}$$

for some fixed $N \in \mathbb{N}^+$. We ask the following question:

for which σ is it true that, for any $f \in C^0(\mathbb{R}^n)$, any compact subset K of $\mathbb{R}^n$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$ and a $g \in \mathcal{M}_N(\sigma)$ such that

$$\max_{\underline{x} \in K} |f(\underline{x}) - g(\underline{x})| \ < \ \varepsilon \quad ? \tag{1.9}$$

In other words, when do we have density of the linear space containing all the $\mathcal{M}_N(\sigma)$, for every $N \in \mathbb{N}^+$, in the space $C^0(\mathbb{R}^n)$ with the topology of the uniform convergence on compact sets ?

Clearly, the problem above can also be studied directly on some fixed compact set, i.e., we can study the density in the space $C^0(K)$, with $K \subset \mathbb{R}^n$ compact, in the uniform topology.

This result was first established by G. Cybenko [63] for continuous σ, although his arguments were arranged for showing density in $C^0(I_n)$, where $I_n := [0,1] \times ... \times [0,1]$.

1.3.1 The Cybenko's approximation theorem

The result obtained by Cybenko in [63] is based on the particular concept of *discriminatory* function. In this section we denote by $M(I_n)$, the space of signed regular Borel measures on I_n (see e.g. [128]).

Definition 1.3. *We say that $\sigma : \mathbb{R} \to \mathbb{R}$ is "discriminatory" if for a measure $\mu \in M(I_n)$*

$$\int_{I_n} \sigma(\underline{y} \cdot \underline{x} + \theta)\, d\mu(x) \ = \ 0,$$

for all $\underline{y} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$ implies that $\mu = 0$.

Theorem 1.4 ([63])**.** *Let σ be any continuous discriminatory function. Then finite sums of the form*

$$G(\underline{x}) \ = \ \sum_{j=1}^{N} \alpha_j\, \sigma(\underline{y}_j \cdot \underline{x} + \theta_j), \qquad \underline{x} \in \mathbb{R}^n,$$

are dense in $C^0(I_n)$, according to the definition given in (1.9).

The proof of Theorem 1.4 is non-constructive and is based on the Hahn-Banach theorem and Riesz representation theorem.

The Cybenko's approximation theorem by superposition of sigmoidal functions can be immediately deduced by the following lemma.

Lemma 1.5 ([63])**.** *Any bounded, measurable sigmoidal function, σ, is discriminatory. In particular, any continuous sigmoidal function is discriminatory.*

The problem studied by Cybenko was suitable to applications in circuit theory and filter design, where simple nonlinear devices are used to synthesize or approximate desired transfer functions. Thus, for instance, a fundamental result in digital signal processing is the fact that digital filters made from unit delay and constant multipliers can approximate any continuous transfer functions arbitrary well.

Moreover, the theory above is related to the Hilbert's 13th problem, solved by Kolmogorov, which showed that all continuous functions of n variables have an exact representation in terms of finite superposition and composition of a small number of functions of one variable [86, 102]. However, the Kolmogorov representation involved different nonlinear functions. The relation between the Kolmogorov representation theorem and neural networks approximation was also studied by Kurková in [87].

Funahashi [67] (independently of Cybenko) proves the density in the uniform norm on the compact sets, for any continuous monotone sigmoidal functions. An analogous results for L^p metrics can be found in [130].

The paper of Cybenko has been superseded by a paper of Cheney and Sun [51] and by two papers by Chui and Li [53, 54]. The papers [51] and [53] deal with the density of the space containing all the sets

$$\left\{ \sum_{i=1}^{N} \alpha_i \sigma(\underline{x} \cdot \underline{k}_i - \theta_i) : \; \alpha_i, \in \mathbb{R}, \; \theta_i \in \mathbb{Z}, \; \underline{k}_i \in \mathbb{Z}^n \right\}, \tag{1.10}$$

for every $N \in \mathbb{N}^+$, in the space $C^0(K)$, where $K \subset \mathbb{R}^n$ is compact. The real benefits over Cybenko's results is that the set in (1.10) is considerably smaller to $\mathcal{M}_N(\sigma)$, for every $N \in \mathbb{N}^+$.

1.3.2 The constructive convolution approach

We now move on to consider constructive proofs of density results showed in the previous section. We discuss about the analysis of Cheney, Light and Xu, as found in [49, 50]. As a general principle, the idea of convolution kernel from a sigmoidal function is used. However, this kind of approach is very difficult due the nature of the problem.

First of all, we need to introduce the concept of *ridge function*.

Definition 1.6. *Let $\underline{a} \in \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ be fixed. The function $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(\underline{x}) := g(\underline{a} \cdot \underline{x})$, $\underline{x} \in \mathbb{R}^n$, is called a "ridge function" (or "plane wave" function).*

Now, let $\phi \in C^0(\mathbb{R})$ and consider the ridge function

$$f(\underline{t}) \;:=\; \phi(\underline{x} \cdot \underline{t}), \qquad \underline{t} \in \mathbb{R}^n. \tag{1.11}$$

>From (1.11) we want to construct a second function which is in $L^1(\mathbb{R}^n)$ and whose integral is non-zero. Whit any function whose behavior for large values of $\underline{x} \in \mathbb{R}^n$ is uneven, in the sense that as $\underline{x}$ moves around on the surface of a sphere of large radius, the function varies considerably, it is a sound principle to average the function over the sphere. To this aim, let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and w_{n-1} the surface area of S^{n-1}. We set

$$g(\underline{x}) \;=\; \frac{1}{w_{n-1}} \int_{S^{n-1}} \phi(\underline{x} \cdot \underline{u})\, dS^{n-1}(\underline{u}), \qquad \underline{x} \in \mathbb{R}^n. \tag{1.12}$$

The following result can be found in Madych [103].

Lemma 1.7 ([103]). *Let $\phi \in C^0(\mathbb{R})$, $n \in \mathbb{N}^+$, $n \geq 2$. Define $g : \mathbb{R}^n \to \mathbb{R}$ by (1.12). Then g is a radial function and if $g(\underline{x}) = g_0(\|\underline{x}\|_2)$ (here $\|\cdot\|_2$ denotes the usual Euclidean norm in $\mathbb{R}^n$) we have*

$$g_0(r) \;=\; w_{n-2}\, w_{n-1}^{-1} \int_{-1}^{1} \phi(rs)(1 - s^2)^{(n-3)/2}\, ds.$$

We want to be able to tell not only when a ridge function is $L^1(\mathbb{R}^n)$ but also when a function can be integrated against $\|\underline{x}\|_2^m$. The next lemma helps in this process.

Lemma 1.8 ([50]). *Let $\phi \in C^0(\mathbb{R})$, $n \in \mathbb{N}^+$, $n \geq 2$. Define $g : \mathbb{R}^n \to \mathbb{R}$ as in Lemma 1.7. Then if $\|\cdot\|_2^m g(\cdot) \in L^1(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \|\underline{x}\|_2^m |g(\underline{x})|\, d\underline{x} \;=\; w_{n-2} \int_0^{+\infty} r^{m+1} \left| \int_{-r}^{r} \phi(t)(r^2 - t^2)^{(n-3)/2}\, dt \right| dr.$$

For proof of Lemma 1.8, Lemma 1.7 was used. We see from Lemma 1.8 that $\int_{\mathbb{R}^n} \|\underline{x}\|_2^m |g(\underline{x})|\, d\underline{x} = 0$ if ϕ is an odd function and the effect of the integral on the right-hand side of the equation in Lemma 1.8 is to "remove" the odd part of the function ϕ. Henceforward, we can and do assume ϕ even. We now tackle the integrability of g.

Theorem 1.9 ([50]). *Let $\phi \in C^0(\mathbb{R})$ and let g defined in (1.12). Moreover, let $n \in \mathbb{N}^+$ be odd and let ϕ satisfy*
(i) $\int_{\mathbb{R}} \phi(t) t^{2j}\, dt = 0$, $0 \leq j \leq (n-3)/2$;
(ii) $\int_{\mathbb{R}} |\phi(t) t^{n+m-1}\, dt < +\infty$, for some $m \geq 0$.
Then the function $\underline{x} \mapsto \|\underline{x}\|_2^m g(\underline{x})$ is integrable to $\mathbb{R}^n$.

Of course, it may turns out that $g \in L^1(\mathbb{R}^n)$ but that $\int_{\mathbb{R}^n} g(\underline{x})\, d\underline{x} = 0$, which will preclude our using g to construct a convolution kernel. The next two results address this matter.

Lemma 1.10 ([100]). *Let ϕ and g as in Theorem 1.9 and let $n \in \mathbb{N}^+$ be odd. In order that $\int_{\mathbb{R}^n} g(\underline{x})\, d\underline{x} \neq 0$, it is necessary and sufficient that*

$$\int_{\mathbb{R}^n} \phi(t) t^{n-1}\, dt \neq 0.$$

The situation when n is even is very different. We investigate first conditions for the function g to belong to $L^1(\mathbb{R}^n)$ and then we analyze the value of $\int_{\mathbb{R}^n} g(\underline{x})\, d\underline{x}$.

Lemma 1.11 ([50])**.** *Let $\phi \in C^0(\mathbb{R})$ and g as in (1.12). Moreover, let $n \in \mathbb{N}^+$ be even. Suppose in addition that ϕ satisfy the following assumptions:*
(i) $\int_{\mathbb{R}} \phi(t)t^{2j}\, dt = 0,\ 0 \le j \le (n-2)/2$;
(ii) $\int_{\mathbb{R}} |\phi(t)t^{n-1}\, dt < +\infty$.
Then $g \in L^1(\mathbb{R}^n)$.

Lemma 1.12 ([50])**.** *Let ϕ and g as in Lemma 1.11 and let $n \in \mathbb{N}^+$ be even. Then we have $\int_{\mathbb{R}^n} g(\underline{x})\, d\underline{x} = 0$*

It is worth noting before we proceed that the moment conditions on ϕ are not only sufficient for $g \in L^1(\mathbb{R}^n)$, but also in a sense necessary (see [50] for the details). The problems caused by Lemma 1.12 can be avoided by averaging the kernel in one dimension higher.

Lemma 1.13 ([100])**.** *Let n be an even integer, $n \ge 2$. Let $\phi \in C^0(\mathbb{R})$ satisfy*

1. *$\int_{\mathbb{R}} \phi(t)t^{2j}\, dt = 0,\ 0 \le j \le (n-3)/2$;*
2. *$\int_{\mathbb{R}} |\phi(t)t^{n+m-1}\, dt < +\infty$, for some $m \ge 0$.*

For $\underline{x} \in (x_1, ..., x_n) \in \mathbb{R}^n$ define $\overline{x} := (x_1, ..., x_n, 0) \in \mathbb{R}^{n+1}$. Put

$$h(\underline{x}) = \frac{1}{w_n} \int_{S^n} \phi(\overline{u} \cdot \overline{x})\, dS^n(\overline{u}), \qquad \overline{u} \in \mathbb{R}^{n+1}.$$

Then $\int_{\mathbb{R}^n} \|\underline{x}\|_2^m |h(\underline{x})|\, d\underline{x} < +\infty$. The condition $\int_{\mathbb{R}} \phi(t)\, t^{n-1}\, dt \neq 0$ is necessary and sufficient for $\int_{\mathbb{R}^n} h(\underline{x})\, d\underline{x} \neq 0$.

We now recall the major results towards which we were aiming.

Theorem 1.14 ([50])**.** *Let n be odd, $n \ge 3$. Let ϕ be a uniformly continuous function on $\mathbb{R}$ satisfying:*

1. $\displaystyle\int_0^{+\infty} |\phi(t)\, t^{n-1}|\, dt < +\infty$;
2. $\displaystyle\int_{\mathbb{R}} \phi(t)\, t^{2j}\, dt = 0,\ 0 \le j \le (n-3)/2$;
3. $\displaystyle\int_{\mathbb{R}} \phi(t)\, t^{n-1}\, dt \neq 0$.

Then the set of functions

$$\left\{ \sum_{i=1}^{N} \alpha_i \, \phi(\underline{x} \cdot \underline{u}_i + \theta_i) : \ \alpha_i, \in \mathbb{R}, \ \theta_i \in \mathbb{R}, \ \underline{u}_i \in \mathbb{R}^n, \ N \in \mathbb{N}^+ \right\},$$

is dense in $C^0(\mathbb{R}^n)$ with respect to the uniform topology on the compact subset of $\mathbb{R}^n$.

We now show the sketch of the proof of Theorem 1.14. By the results showed in this section we can deduce that, under the assumptions of Theorem 1.14, the function g defined on (1.12) belongs to $L^1(\mathbb{R})$ and $\int_{\mathbb{R}^n} g(\underline{x})\, d\underline{x} = 1$. Now, by g we can define the sequence

$$g_m(\underline{x}) \ := \ m^n g(m\underline{x}), \qquad m \in \mathbb{N}^+.$$

Then, for every function f with compact support $K \subset \mathbb{R}^n$, we have

$$\|g_m * f - f\|_\infty \to 0, \qquad as \qquad m \to +\infty,$$

on K, where $*$ denotes the usual convolution product (see e.g. [26]) and

$$(g_m * f)(\underline{x}) \ := \ \int_K g_m(\underline{x} - \underline{y}) f(\underline{y}) \, d\underline{y}.$$

The above formulae can be uniformly approximated by a sum of the form

$$\sum_{A \in P} b_A g(m\underline{x} - z_a),$$

where b_A and z_A are suitable coefficients and P is a suitable finite set. Then, applying a quadrature rule to each term $g(m\underline{x} - z_a)$, we can obtain the following approximation formula:

$$\sum_{A \in P} \sum_{B \in Q} c_A e_B \phi(m\underline{x} u_B - z_A u_B),$$

where

$$e_B \ := \ \omega_{n-1}^{-1} \int_B dS^{n-1}(\underline{u}),$$

$u_B \in B$, $B \in Q$, where Q is a partition of S^{n-1} into a finite, disjoint, collection of Borel sets. A similar theorem can be established for n even. The reader is referred to [50] for details.

Now, it is possible to show how the theory above can be applied to obtain results concerning approximation by sigmoidal functions.

Let σ be a continuous sigmoidal function and we set $\phi(t) = \sigma(t+1) - \sigma(t)$, $t \in \mathbb{R}$. Then $\phi(t) \to 0$ as $t \to \pm\infty$. The rate at which $\phi(t) \to 0$ as $t \to \pm\infty$ depends on the rate at which $\sigma(t) \to 0$ as $t \to -\infty$ and $\sigma(t) \to 1$ as $t \to +\infty$.

However, with quite mild assumptions on σ, such rate may be improved by differencing.

Let Δ be the forward difference operator $(\Delta\phi)(t) = \phi(t+1) - \phi(t)$. Suppose ϕ is even and has the power series expansion at $+\infty$,

$$\phi(t) \;=\; \sum_{k=1}^{\infty} a_k\, t^{-k-\alpha}, \qquad for \quad 0 < \alpha \leq 1, \qquad t > T.$$

Then we will denote

$$(\Delta^{n-1}\phi)(t) \;:=\; \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} \phi(t+j),$$

and so for $t > T$,

$$(\Delta^{n-1}\phi)(t) \;:=\; (n-1)! \sum_{k=1}^{+\infty} a_k\, \xi_t^{-k-n+1-\alpha},$$

for some $t < \xi_t < t+n-1$. It follows that $(\Delta^{n-1}\phi)(t) = \mathcal{O}(t^{-n-\alpha})$ as $t \to +\infty$. Consequently, the function $t \mapsto t^{n-1}(\Delta^{n-1}\phi)(t)$ is in $L^1(\mathbb{R})$. Now put $\psi := \Delta^{n-1}\phi$. Then for $0 \leq k \leq n-1$,

$$\int_{\mathbb{R}} \psi(t) t^k\, dt \;=\; \int_{\mathbb{R}} (\Delta^{n-1}\phi)(t) t^k\, dt \;=\; (-1)^{n-1}\, k! \int_{\mathbb{R}} \phi(t+n-1)(\Delta^{n-1}V_k)(t)\, dt$$

$$= \begin{cases} 0, & if\ 0 \leq k \leq n-2, \\ (-1)^{n-1}\, k! \int_{\mathbb{R}} \phi(t)\, dt, & k = n-1. \end{cases}$$

Here we have used the notation V_k for the normalized monomial $V_k(t) = t^k/k!$. This lead to the following result.

Theorem 1.15 ([100]). *Let $\phi \in C^0(\mathbb{R})$ be an even function and let n be odd. Suppose:*
(a) $\int_{\mathbb{R}} \phi(t)\, dt \;\neq\; 0$;
(b) ϕ has a descending power series expansion at $+\infty$ of the following form:

$$\phi(t) \;=\; \sum_{k=1}^{+\infty} a_k\, t^{-k-\alpha}, \qquad 0 < \alpha \leq 1, \qquad t > T.$$

Then the set of functions

$$\left\{ \sum_{k=1}^{N} \alpha_k\, \phi(\underline{x} \cdot \underline{u} + \theta_k) : \; \underline{u} \in \mathbb{R}^n,\; \theta_k \in \mathbb{R},\; N \in \mathbb{N}^+ \right\},$$

is dense in $C^0(\mathbb{R}^n)$ with respect to the uniform topology on the compact subsets of $\mathbb{R}^n$.

Theorem 1.15 may be applied to obtain a constructive proof of the desired density result for suitable sigmoidal functions σ, using the convolution techniques of Theorem 1.14. An example of sigmoidal function for which the above approach can be applied is the logistic function σ_ℓ [100].

Given $f \in C^0(\mathbb{R}^n)$, K compact in $\mathbb{R}^n$ and $\varepsilon > 0$ we can construct $(\underline{u}_k)_{k=1,\dots,N}$ in $\mathbb{R}^n$ and $(c_{k\,j})_{k,j=1,\dots,N}$, $(\theta_j)_{j=1,\dots,N}$ in $\mathbb{R}$ such that

$$\sup_{\underline{x}\in K}\left| f(\underline{x}) - \sum_{k,\,j=1}^{N} c_{k\,j}\psi(\underline{x}\cdot\underline{u}_k+\theta_j)\right| \;<\; \varepsilon.$$

Unscrambling all this gives the approximation

$$\begin{aligned}
&\sum_{k,\,j=1}^{N} c_{k\,j}\psi(\underline{x}\cdot\underline{u}_k+\theta_j)\\
&\quad= \sum_{k,\,j=1}^{N} c_{k\,j}\sum_{\ell=1}^{n} e_\ell\left[\sigma(\underline{x}\,\underline{u}_k+\theta_j+\ell+1)-\sigma(\underline{x}\,\underline{u}_k+\theta_j+\ell)\right]\\
&\quad= \sum_{k,\,j=1}^{N} c_{k\,j}\left\{\sum_{\ell=0}^{n+1} e'_\ell\,\sigma(\underline{x}\,\underline{u}_k+\theta_j+\ell)+e''_\ell\,\sigma(\underline{x}\,\underline{u}_k+\theta_j+\ell)\right\}\\
&\quad= \sum_{k,\,j=1}^{N} c_{k\,j}\left\{\sum_{\ell=0}^{n+1}(e'_\ell+e''_\ell)\,\sigma(\underline{x}\,\underline{u}_k+\theta_j+\ell)\right\}.
\end{aligned}$$

This is the required sigmoidal approximation. Note that even if the decay properties of σ at $\pm\infty$ are not very strong, it can be expected that the differencing applied to generate the function ψ will greatly improve this decay.

There are alternative approaches to these ideas of generating in a practical way appropriate sigmoidal function approximation. An algorithm is proposed by Chui and Li in [54], where they first approach the approximation problem by using the Bernstein operators and then applying again the convolution technique.

Other constructive approach has been proposed in [48], but we will analyze it in detail in the next chapters.

1.3.3 The Mhaskar-Micchelli results and the k-th sigmoidal functions

The idea proposed by Mhaskar and Micchelli in [115], is to construct a suitable operator which is polynomial preserving. The present approach focuses on only to approximation of univariate functions. First of all, we recall the following definition.

Definition 1.16. *Given a continuous sigmoidal function σ and $n \in \mathbb{N}^+$, we define A_n to be the smallest positive integer such that:*

(a) $|\sigma(x)| \leq n^{-1}$ *for* $x \leq -A_n$*;*

(b) $1 - n^{-1} \leq \sigma(x) \leq 1 + n^{-1}$ *for* $x \geq A_n$*.*

The operator introduced by Mhaskar and Micchelli are defined by

$$(G_n f)(x) \ = \ f(0) + \sum_{\nu=1}^{n} \left\{ f\left(\frac{\nu}{n}\right) - f\left(\frac{\nu-1}{n}\right) \right\} \sigma\left(A_n(nx-\nu)\right),$$

$x \in \mathbb{R}$, where σ is any continuous sigmoidal function and $f \in C^0(\mathbb{R})$.

In order to recall the main results obtained in [115] concerning sigmoidal function approximation, we must introduce the well-known notion of *modulus of continuity*, very important in Approximation Theory.

Definition 1.17. *For any continuous function f defined on the interval $[a,b]$, we denote by*

$$\omega(f,\delta) \ := \sup_{x,y\in[a,b],\ |x-y|\leq\delta} |f(x) - f(y)|,$$

the "modulus of continuity of f".

Note that, the above definition can be easily extended to the case of functions defined on the whole real line. Now, we have the following.

Theorem 1.18 ([115])**.** *There exists a constant $c > 0$ such that for $f \in C^0(\mathbb{R})$*

$$\sup_{x\in[0,1]} |f(x) - (G_n f)(x)| \ \leq \ c\,\omega(f, 1/n),$$

for every $n \in \mathbb{N}^+$.

Theorem 1.18 is a constructive approximation result with rate.

Always in [115], a generalization of the concept of sigmoidal function has been introduced.

Definition 1.19. *A "k-th degree sigmoidal function" is a function $\sigma : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{x\to-\infty} \frac{\sigma(x)}{x^k} = 0 \text{ and } \lim_{x\to+\infty} \frac{\sigma(x)}{x^k} = 1,$$

and σ is bounded by a polynomial of degree at most k on $\mathbb{R}$.

Mhaskar and Micchelli provide constructive proof, drawing upon ideas from the area of spline functions in the univariate case. However, this kind of approach does not generate a pure sigmoidal approximation, since the sigmoidal part is augmented by a polynomial. For further details see [115].

To approximate multivariate functions, the method proposed in [115] consists, first, to approximate a given function by means of a suitable average of Fourier series. The average is taken so as to generate an approximation with a sufficiently high degree of accuracy. Then the univariate result is used to obtain a sigmoidal approximation.

Others important papers concerning neural networks approximation are [116, 114, 99]. These results are obtained for 2π-periodic functions and also for non necessarily periodic functions (see e.g. [116]). In [99] neural networks are constructed by wavelet recovery formula and wavelet frame. Finally, the optimal order of approximation for NNs with a single hidden layer was proved in [114] for functions assumed to posses a given number of derivatives.

1.3.4 Multivariate approximation based on Lebesgue-Stieltjes-type convolution operators

In Lenze [94] an original constructive approach for multivariate approximation with sigmoidal functions has been developed. The basic idea was of use an hyperbolic-type argument in σ in place of ridge like typical argument $\underline{x} \cdot \underline{u}_k - \theta_k$. This results were obtained by using the theory of the Lebesgue-Stieltjes convolution operators. We start with the following notations.

Let $\mathcal{R} \subset \mathbb{R}^n$, $\mathcal{R} := [a_1, b_1] \times ... \times [a_n, b_n]$. Denote by

$$Cor(\mathcal{R}) \ := \ \{\underline{x} \in \mathbb{R}^n : \ x_i = a_i \ or \ x_i = b_i, \ for\ every\ i = 1, ..., n\},$$

and by

$$\gamma(\underline{x}, \underline{a}) \ := \ |\{i \in \{1, ..., n\}, \ x_i = a_i\}|, \qquad \underline{x} \in Cor(\mathcal{R}),$$

where $\underline{a} := (a_1, ..., a_n)$ and here, the $|\cdot|$ denotes the number of distinct elements of the set under consideration. Now, for every $f : \mathbb{R}^n \to \mathbb{R}$ define

$$\Delta_f(\mathcal{R}) \ := \sum_{\underline{x} \in Cor(\mathcal{R})} (-1)^{\gamma(\underline{x}, \underline{a})} f(\underline{x}).$$

Definition 1.20. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be of "(uniform) bounded variation" on $\mathbb{R}^n$, if there exists a finite constant $K \geq 0$ such that $\overline{\Delta}_f(\mathcal{R})$, $\mathcal{R} \subset \mathbb{R}^n$, defined by*

$$\overline{\Delta}_f(\mathcal{R}) := \sup \left\{ \sum_{i=1}^{r} |\Delta_f(\mathcal{R}_i)| : \ \mathcal{R}_i \subset \mathcal{R}, \ \mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \ i \neq j, \ 1 \leq i, \ j \leq r, \ r \in \mathbb{N}^+ \right\},$$

satisfies

$$\sup \left\{ \overline{\Delta}_f(\mathcal{R}) : \ \mathcal{R} \subset \mathbb{R}^n \right\} \ = \ K < +\infty. \tag{1.13}$$

In this case we will write $f \in BV(\mathbb{R}^n)$.

As this is well-known, a function $f \in BV(\mathbb{R}^n)$ induces a signed Borel measure m_f, the so-called Lebesgue-Stieltjes measure associated to f. This measure determines the Lebesgue-Stieltjes integral with respect to f, and then, all the integrals in this subsection have to be interpreted in Lebesgue-Stieltjes sense. For major details see [94].

We now introduce the Lebesgue-Stieltjes convolution operators induced by a *continuous* sigmoidal function.

Theorem 1.21 ([94]). *For $f \in BV(\mathbb{R}^n)$ with $\lim_{\|\underline{t}\| \to +\infty} f(\underline{t}) = 0$ (here $\|\underline{t}\| := \max\{|t_i| : \ i = 1, ..., n\}$), the operators Ω_ρ, $\rho > 0$, defined by*

$$\Omega_\rho(f)(\underline{x}) \ := \ (-1)^n \, 2^{1-n} \int_{\mathbb{R}^n} \sigma \left(\rho \prod_{k=1}^{n} (t_k - x_k) \right) df(\underline{t}),$$

for every $\underline{x} \in \mathbb{R}^n$, are well-defined and map into the space $C^0(\mathbb{R}^n)$. Moreover, they are linear and with $K \geq 0$ given by (1.13), they are bounded by

$$\sup_{\underline{x} \in \mathbb{R}^n} |\Omega_\rho(f)(\underline{x})| \ \leq 2^{1-n} K \|\sigma\|_\infty,$$

where $\|\sigma\|_\infty := \sup_{t \in \mathbb{R}} |\sigma(t)|$.

In case of f continuous at $\underline{x} \in \mathbb{R}^n$ we have the local approximation results

$$\lim_{\rho \to +\infty} \Omega_\rho(f)(\underline{x}) \ = \ f(\underline{x}).$$

Since we are interested in practical approximation method, in order to approximate f we have to evaluate the convolution integral appearing in Theorem 1.21 approximatively, using e.g. a simple Riemann-type midpoint quadrature rule and get the discrete operators $\Omega^{(h)}$ (here $\rho = h^{-n}$ in order to obtain operators depending only from the parameter $h > 0$), defined by

$$\Omega^{(h)}(f)(\underline{x}) := (-1)^n \, 2^{1-n} \sum_{\underline{k} \in \mathbb{Z}^n} \sigma \left(\prod_{j=1}^{n} (k_j + \frac{1}{2} - \frac{x_j}{h}) \right) \Delta_f(\widetilde{\mathcal{R}}^h_{j,\underline{k}})$$

where $\widetilde{\mathcal{R}}^h_{j,\underline{k}} := [hk_1, h(k_1+1)] \times ... \times [hk_n, h(k_n+1)]$. In the following, we examine the properties of the discrete operators $\Omega^{(h)}$.

Theorem 1.22 ([94]). *Let σ be a continuous sigmoidal function. For $f \in BV(\mathbb{R}^n)$ with $\lim_{\|\underline{t}\| \to +\infty} f(\underline{t}) = 0$, the operators $\Omega^{(h)}$, $h > 0$, are well-defined and map into the space $C^0(\mathbb{R}^n)$. Moreover, they are linear and with $K \geq 0$ given by (1.13), they are bounded by*

$$\sup_{\underline{x} \in \mathbb{R}^n} \left| \Omega^{(h)}(f)(\underline{x}) \right| \ \leq 2^{1-n} K \|\sigma\|_\infty.$$

We now pass over the approximation properties of the operators $\Omega^{(h)}$, $h > 0$. We introduce the following definition:

$$BVC_0(\mathbb{R}^n) \ := \ \left\{ f \in BV(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) : \lim_{\|\underline{t}\| \to +\infty} f(\underline{t}) = 0 \right\}.$$

We obtain the following.

Theorem 1.23 ([94]). *Let σ be a continuous sigmoidal function and consider $\Omega^{(h)}$, $h > 0$. For all $f \in BVC_0(\mathbb{R}^n)$ and for every $\underline{x} \in \mathbb{R}^n$ we have*

$$\lim_{h \to 0^+} \|\Omega^{(h)}(f) \ - \ f\|_\infty \ = \ 0.$$

For a discussion on the practical applications to neural networks of the present theory see [94].

1.3.5 About the order of approximation

A problem of interest in Approximation Theory is to study the order of approximation.

Bounds for the error of approximation by neural networks with sigmoidal functions are usually given in terms of the number of neurons, and sometimes are related both to the number of the neuron and the weights. In some cases, these estimate are given in term of the modulus of continuity of the function f that we wish to approximate.

This subject has been addressed by many authors. Debao Chen [47] proved an estimate for univariate functions $f \in C^0[0,1]$. We have the following.

Theorem 1.24 ([47]). *Let σ a given sigmoidal function. For every $f \in C^0[0,1]$ we have*

$$dist(f, \mathcal{M}_N(\sigma)) \ \leq \ \|\sigma\|_\infty \, \omega(f, \, 1/N),$$

for every $N \in \mathbb{N}^+$, where $\mathcal{M}_N(\sigma)$ is the set of functions defined in (1.8) and $dist(f, \mathcal{M}_N(\sigma)) = \inf_{g \in \mathcal{M}_N(\sigma)} \|f - g\|_\infty$.

Clearly, from the error of approximation estimates given by the modulus of continuity of f also the convergence of the approximation method can be deduced. In fact, if a general function $f : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous, we have

$$\lim_{N \to +\infty} \omega(f, \, 1/N) \ = \ 0. \tag{1.14}$$

An extension of Theorem 1.24 for suitable functions defined on the whole real line has been proved in [72]. The order of approximation was studied for functions belonging to the following set:

$$\overline{C}^{\,0}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} : f \, continuous \, and \, \lim_{x \to +\infty} f(x), \, \lim_{x \to -\infty} f(x) \, are finite \right\}.$$

The following result has been proved.

Theorem 1.25 ([72]). *Let σ be a bounded sigmoidal function and $f \in \overline{C}^{\,0}(\mathbb{R})$. For every $N \in \mathbb{N}^+$ we have*

$$dist(f, \mathcal{M}_N(\sigma)) \;\leq\; \|\sigma\|_\infty\, \omega\left(f,\, \frac{2A_1}{N+1}\right),$$

where the positive constant A_1 depends on f and σ only.

As a corollary of Theorem 1.25 we can deduce an order of approximation for continuous functions defined on bounded intervals $[a, b]$. In this case the constant $2A_1$ can be replaced by the constant $b-a$. If $[a, b] = [0, 1]$ we obtain again Theorem 1.24.

Moreover, in [72] is also proved that the approximation error derived in Theorem 1.25 for certain continuous functions, when $\sigma(x) = H(x)$ (the Heaviside function), is *almost the best possible*, i.e., a *tight bound* for the approximation error is derived.

We recall that, a tight bound is a simultaneous upper and lower estimate of the approximation error given either with the same modulus of continuity (or $\mathcal{O}$ symbol) and different constants.

A similar result to that proved by Debao Chen in Theorem 1.24 has been proved by Gao and Xu [68] for univariate functions of bounded variation, defined on the interval $[0, 1]$. In particular, they showed that in this case the error of approximation depends on the total variation of f. Recall that, for univariate functions defined on bounded intervals of $\mathbb{R}$ the definition of function with bounded variation can be simplified with respect to that given in Definition 1.20.

Definition 1.26. *Let $f : [a, b] \to \mathbb{R}$. The "total variation of f" in $[a, b] \subset \mathbb{R}$ is defined by:*

$$V(f)[a,b] := \sup\left\{ \sum_{j=0}^{r-1} |f(x_{j+1}) - f(x_j)| : a = x_0 < x_1 < \ldots < x_r = b,\ r \in \mathbb{N}^+ \right\}.$$

If $V(f)[a, b] < +\infty$, f is called a "bounded variation function on $[a, b]$". The set of all functions of bounded variation on $[a, b]$ will be denoted by $BV[a, b]$.

We have the following.

Theorem 1.27 ([68]). *Let σ be a bounded sigmoidal function. Then, there is a positive constant c, depending on σ only, such that for every $N \in \mathbb{N}^+$ and for each $f \in BV[0, 1]$,*

$$dist(f, \mathcal{M}_N(\sigma)) \;\leq\; c\, V(f)[0, 1]\, N^{-1}.$$

The idea of the proof is very similar to that proposed by Chen, Chen and Liu [48], in their convergence theorem for neural networks activated by sigmoidal functions in one-dimension.

Theorem 1.27 has been extended in [95] by Lewicki and Marino for functions of bounded ϕ-variation. We recall the following.

Definition 1.28. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, strictly increasing function such that $\phi(0) = 0$. Moreover, let $f : [a, b] \to \mathbb{R}$. The "$\phi$-variation of f" in $[a, b] \subset \mathbb{R}$ is defined by:*

$$V_\phi(f)[a,b] := \sup \left\{ \sum_{j=0}^{r-1} \phi\left(|f(x_{j+1}) - f(x_j)|\right) : a = x_0 < x_1 < \ldots < x_r = b,\ r \in \mathbb{N}^+ \right\}.$$

If $V_\phi(f)[a, b] < +\infty$, f is called a "bounded ϕ-variation function on $[a, b]$". The set of all functions of bounded ϕ-variation on $[a, b]$ will be denoted by $BV_\phi[a, b]$.

The first remarkable result proved in [95] was the following.

Theorem 1.29 ([95]). *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, strictly increasing function such that $\phi(0) = 0$. Moreover, let $f \in C^0[a, b]$ satisfy the following property*

(P) *there exists a constant $C > 0$ such that for every $N \in \mathbb{N}^+$ we can select a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that for every $i = 1, \ldots, N$, if $x, y \in I_i = [x_{i-1},\ x_i]$, then*

$$|f(x) - f(y)| \ \leq \ \phi^{-1}\left(\frac{C}{N}\right),$$

and let σ be a bounded sigmoidal function. Then

$$dist(f, \mathcal{M}_N(\sigma)) \ \leq \ (1 + 8\|\sigma\|_\infty)\, \phi^{-1}\left(\frac{C}{N}\right).$$

Note that the class of functions satisfying property **(P)** is larger then the class of functions of bounded ϕ-variation. Moreover, if a function f is α-Holder continuous, with $0 < \alpha \leq 1$, condition **(P)** is fulfilled with $\phi(t) = t^\alpha$, then the estimate provided by Theorem 1.29 becomes

$$dist(f, \mathcal{M}_N(\sigma)) \ \leq \ (1 + 8\|\sigma\|_\infty) \left(\frac{C}{N}\right)^{1/\alpha}.$$

Finally, again as a consequence of Theorem 1.29 and by the above observations it is immediate to prove the following.

Theorem 1.30 ([95]). *Let σ be a bounded sigmoidal function and let $f \in C^0[a, b] \cap BV_\phi[a, b]$. Then*

$$dist(f, \mathcal{M}_N(\sigma)) \ \leq \ (1 + 8\|\sigma\|_\infty)\, \phi^{-1}\left(\frac{V_\phi(f)[a, b]}{N}\right),$$

for every $N \in \mathbb{N}^+$.

The results concerning the order of approximation studied in this subsection deal only for univariate functions. To find results in a multivariate setting is certainly more difficult.

For instance, some approximation bounds for the superposition of sigmoidal function in several variables has been proved by Barron [20].

In order to obtain results in this sense, Barron introduced a smoothness property of the function to be approximated, expressed in terms of its Fourier representation.

In its paper [20], Barron considered the class of functions $f : \mathbb{R}^n \to \mathbb{R}$ for which there is the following representation:

$$f(\underline{x}) \;=\; \int_{\mathbb{R}^n} e^{i\,\underline{w}\cdot\underline{x}}\, \widehat{f}(\underline{w})\, d\underline{w},$$

where $\widehat{f}$ denotes the Fourier transform of f for which $\underline{w}\,\widehat{f}(\underline{w})$ is integrable. Moreover, the following term was introduced:

$$C_f \;:=\; \int_{\mathbb{R}^n} (\underline{w}\cdot\underline{w})^{1/2} |\widehat{f}(\underline{w})|\, d\underline{w},$$

where in the above formula $|\widehat{f}(\underline{w})|$ is the modulus of a complex-valued function. We will denote by Γ_C the set of all functions f such that $C_f \leq C$. Functions with C_f finite are continuously differentiable on $\mathbb{R}^n$ and the gradient of f has the Fourier representation

$$\Delta f(\underline{x}) \;=\; \int_{\mathbb{R}^n} e^{i\,\underline{w}\cdot\underline{x}}\, \widehat{\Delta f}(\underline{w})\, d\underline{w},$$

where $\widehat{\Delta f}(\underline{w}) = i\,\underline{w}\,\widehat{f}(\underline{w})$.

The finite linear combination of sigmoidal functions considered by Barron are of the form

$$f_N(\underline{x}) \;:=\; \sum_{i=1}^{N} \alpha_i\, \sigma(\underline{x}\cdot\underline{w}_i - \theta_i) \;+\; \alpha_0, \tag{1.15}$$

where the shifted term α_0 is introduced. Similar to $\mathcal{M}_N(\sigma)$, we define by $\mathcal{G}_N(\sigma)$ the set of all functions of the form in (1.15), for every α_i, $\theta_i \in \mathbb{R}$, $\underline{w}_i \in \mathbb{R}^n$ and $N \in \mathbb{N}^+$. The order of approximation achieved in [20] is measured by the integrated squared error with respect to an arbitrary probability measure μ on the closed ball $B_r \subset \mathbb{R}^n$ centered in the origin and with radius $r > 0$.

Theorem 1.31 ([20]). *Let σ a bounded sigmoidal function. For every function f with C_f finite and every $N \in \mathbb{N}^+$, there exists a linear combination of sigmoidal functions $f_N(\underline{x})$ of the form in (1.15), such that*

$$\int_{B_r} (f(\underline{x}) - f_N(\underline{x}))^2\, d\mu(\underline{x}) \;\leq\; \frac{c'_f}{N},$$

where $c'_f = (2rC_f)^2$. *Moreover, for functions in* Γ_C, *the coefficients of the linear combination in (1.15) may be restricted to satisfy* $\sum_{i=1}^{N} |\alpha_i| \leq 2rC$ *and* $\alpha_0 = f(0)$.

Refinements of the above results are also showed in [20] using elements of functional analysis. Substantially, the main goal of the paper is the proof that the L^2-error of approximation for certain functions, goes to zero as $1/\sqrt{N}$, for $N \to +\infty$. Moreover, the following theorem is also proved in [20].

Theorem 1.32 ([20]). *Let* $S^{n-1} \subset \mathbb{R}^n$ *denotes the unit Euclidean ball of* $\mathbb{R}^n$ *and let* $\widetilde{\nu}$ *a probability Borel measure on* S^{n-1}. *Moreover, let* μ *a real valued Radon measure on* $\mathbb{R}^n$ *and assume additionally that*

$$\int_{\mathbb{R}^n} \|\underline{t}\|_2 d\mu(\underline{t}) \; < \; +\infty,$$

where here $|\mu|$ *denotes the variation of* μ. *Define for* $\underline{x} \in S^{n-1}$,

$$h(\underline{x}) \; := \; \int_{\mathbb{R}^n} e^{i\underline{t}\cdot\underline{x}}\, d\mu(\underline{t}).$$

Then there is a constant $C > 0$, *such that*

$$dist_{L^2}(h, \mathcal{G}_N(\sigma)) \; \leq \; C/\sqrt{N},$$

for every $N \in \mathbb{N}^+$, σ *bounded sigmoidal function, where* $dist_{L^2}(h, \mathcal{G}_N(\sigma)) = \inf_{g \in \mathcal{G}_N(\sigma)} \|h - g\|_{L^2}$ *and* $L^2 = L^2(S^{n-1})$ *with respect to the measure* $\widetilde{\nu}$.

A generalization of Theorem 1.32 is produced by Lewicki and Marino [96]. In particular, they consider functions of the ridge-type

$$h(\underline{x}) \; := \; \int_{\mathbb{R}^n} f(\underline{t} \cdot \underline{x})\, d\mu(\underline{t}), \tag{1.16}$$

where μ is a finite Radon measure on $\mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function of bounded variation on $\mathbb{R}$ (the theory showed in [96] is valid also for complex-valued function). Lewicki and Marino proved that functions of the form in (1.16) can be approximated in L^2-norm by elements of the form (1.15), with σ bounded.

Theorem 1.33 ([96]). *Let* B, $D \subset \mathbb{R}^n$ *and* D *be a compact set. Suppose that* μ *be a real-valued Radon measure on* B *with variation* $|\mu| < +\infty$. *Let* ν *be a finite Borel measure on* D. *Assume* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous and bounded function such that* $f \in BV[a, b]$, *for every* $[a, b] \subset \mathbb{R}$ *and*

$$V(f) \; := \; \sup\{V(f)[a, b] : \; [a, b] \subset \mathbb{R}\} \; < \; +\infty.$$

Set for $\underline{x} \in D$, $h(\underline{x}) = \int_B f(\underline{t} \cdot \underline{x})\, d\mu(\underline{t})$. *Then there is* $C > 0$, *depending only on* f, D *and* ν, *such that*

$$dist_{L^2}(h, \mathcal{G}_N(\sigma)) \;\le\; C/\sqrt{N},$$

where $L^2 = L^2(D)$ *with respect to the measure* ν.

The same result holds true for $f : \mathbb{R} \to \mathbb{R}$ satisfying the Lipschitz condition under the additional assumption that $\int_{\mathbb{R}^n} \|\underline{t}\|_2 d|\mu(\underline{t})| < +\infty$, see [96] again.

For more results concerning the order of approximation see e.g. [80, 1, 88, 89, 82, 83, 91, 92, 73, 41].

A further generalization in the theory of NNs with sigmoidal functions in Approximation Theory is the study of the order of approximation in the general framework of Hilbert spaces. Results in this context were firstly given in Barron [20] and Makovoz [104, 105]. Upper bounds are derived by many authors, e.g., Gnecco, Kainen, Kurkova and Sanguineti in [69, 71, 70, 93, 81, 90].

In particular, tight bounds for the error of approximation were derived in [104, 93].

Chapter 2

Constructive Approximation by Superposition of Sigmoidal Functions

In this chapter, a constructive theory is developed for approximating functions of one or more variables by superposition of sigmoidal functions. This is done in the uniform norm as well as in the L^p norm. Results for the simultaneous approximation, with the same order of accuracy, of a function and its derivatives (whenever these exist), are obtained. The relation with neural networks and radial basis functions approximations is discussed. Numerical examples are given for the purpose of illustration. Comparisons with the results of Chapter 1 are made.

The readers, can be found the present theory in [56].

2.1 Notation and preliminary results

In this chapter, by $\widehat{C}^n[a,b]$, $n \in \mathbb{N}^+$, we will denote the set of all functions f such that $f \in C^n(a',b')$ for some open real interval (a',b') such that $[a,b] \subset (a',b')$. Furthermore, $Q \subset \mathbb{R}^2$ will denote the square $Q := [a,b] \times [c,d]$, with $b-a = d-c$, and $\|(x,y)\|_2 := (x^2+y^2)^{1/2}$, $(x,y) \in \mathbb{R}^2$, the Euclidean norm in $\mathbb{R}^2$.

Now we give a constructive proof of Cybenko's approximation theorem, inspired to the proof given in [48] by Chen, Chen and Liu, for functions belonging to $C^0[a,b]$. In particular, in our approach some little changes have been introduced with respect to [48]. We will prove these density theorem since the proof it is useful to let better understand our proofs below, in next sections, concerning the constructive multivariate theory.

The following statement, that we display as a lemma, can be obtained as an immediate consequence of Definition 1.1.

Lemma 2.1. *Let $x_0, x_1, ..., x_N \in \mathbb{R}$, $N \in \mathbb{N}^+$, be fixed. For every ε, $h > 0$, there exists $\overline{w} := \overline{w}(\varepsilon, h) > 0$ such that, for every $w \geq \overline{w}$ and $k = 0, 1, ..., N$, we have*

1. $|\sigma(w(x - x_k)) - 1| < \varepsilon$, *for every $x \in \mathbb{R}$ such that $x - x_k \geq h$;*
2. $|\sigma(w(x - x_k))| < \varepsilon$, *for every $x \in \mathbb{R}$ such that $x - x_k \leq -h$.*

Now we are able to prove the following

Theorem 2.2. *Let σ be a bounded sigmoidal function and let $f \in C^0[a, b]$ be fixed. For every $\varepsilon > 0$, there exist $N \in \mathbb{N}^+$ and $w > 0$ (depending on N and σ), such that, if*

$$(G_N f)(x) := \sum_{k=1}^{N} [f(x_k) - f(x_{k-1})]\, \sigma(w(x - x_k)) + f(x_0)\, \sigma(w(x - x_{-1})) \quad (2.1)$$

for $x \in [a, b]$, $h := (b - a)/N$, and $x_k := a + kh$, $k = -1, 0, 1, ..., N$, then

$$\|G_N f - f\|_\infty < \varepsilon.$$

We stress that *continuity* of σ is *not* required and its boundedness in the sup-norm, $\|\sigma\|_\infty := \sup_{x \in \mathbb{R}} |\sigma(x)|$, suffices.

Proof. Let $\varepsilon > 0$ be fixed. Since f is uniformly continuous, correspondingly to $\eta := \varepsilon/(\|f\|_\infty + 2\|\sigma\|_\infty + 2)$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ for every x, $y \in [a, b]$, with $|x - y| < \delta$. We fix $N \in \mathbb{N}^+$, $N > 3$, such that $h := (b - a)/N < \delta/2$ and $1/N < \eta$. Moreover, we fix $w \geq \overline{w}(1/N, h) \equiv \overline{w}(1/N) > 0$, where $\overline{w}(1/N)$ is obtained using Lemma 2.1 with $\frac{1}{N}$, $h > 0$ and with $x_k = a + hk$, $k = -1, 0, 1, ..., N$. Now, consider $G_N f$ defined in (2.1) with w. Let $x \in [a, b]$ be fixed and i, $i = 1, ..., N$, such that $x \in [x_{i-1}, x_i]$. Set

$$L_i(x) := f(a) + [f(x_2) - f(x_1)]\, \sigma(w(x - x_2)) + [f(x_1) - f(x_0)]\, \sigma(w(x - x_1))$$

for $i = 1, 2$, and

$$\begin{aligned} L_i(x) \quad &:= \quad \sum_{k=1}^{i-2} [f(x_k) - f(x_{k-1})] + f(a) + [f(x_{i-1}) - f(x_{i-2})]\, \sigma(w(x - x_{i-1})) \\ &+ \quad [f(x_i) - f(x_{i-1})]\, \sigma(w(x - x_i)) \end{aligned}$$

for $i \geq 3$. In any case,

$$|(G_N f)(x) - f(x)| \leq |(G_N f)(x) - L_i(x)| + |L_i(x) - f(x)| =: I_1 + I_2.$$

We now estimate I_1 and I_2 only for $i \geq 3$, since similar estimates can be obtained also for $i = 1, 2$. Being $x - x_k \geq h$ for $k = -1, 0, 1, ..., i - 2$ and

$x - x_k \leq -h$ for $k = i+1, ..., N$, by conditions 1 and 2 of Lemma 2.1, it follows that

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^{i-2} |f(x_k) - f(x_{k-1})| \, |\sigma(w(x - x_k)) - 1| \\
&+ |f(a)| \, |\sigma(w(x - x_{-1})) - 1| \\
&+ \sum_{k=i+1}^{N} |f(x_k) - f(x_{k-1})| \, |\sigma(w(x - x_k))| < \sum_{k=1}^{N} \eta \, \frac{1}{N} + \frac{1}{N} \, |f(a)| \\
&\leq (1 + \|f\|_\infty) \, \eta.
\end{aligned}$$

It may be observed that here above just $|f(a)|$ in place of $\|f\|_\infty$ would suffice. Using the identity

$$\sum_{k=1}^{i-2} [f(x_k) - f(x_{k-1})] + f(a) = f(x_{i-2}),$$

and being $|x_{i-2} - x| \leq |x_{i-2} - x_{i-1}| + |x_{i-1} - x| \leq 2h < \delta$, we can estimate I_2 as

$$\begin{aligned}
I_2 &= |f(x_{i-2}) + [f(x_{i-1}) - f(x_{i-2})] \, \sigma(w(x - x_{i-1})) \\
&+ [f(x_i) - f(x_{i-1})] \, \sigma(w(x - x_i)) - f(x)| \\
&\leq |f(x_{i-1}) - f(x_{i-2})| \; |\sigma(w(x - x_{i-1}))| + |f(x_i) - f(x_{i-1})| \; |\sigma(w(x - x_i))| \\
&+ |f(x_{i-2}) - f(x)| < (2 \, \|\sigma\|_\infty + 1) \, \eta.
\end{aligned}$$

From such estimates for I_1 and I_2, it follows that

$$|(G_N f)(x) - f(x)| \leq I_1 + I_2 < (\|f\|_\infty + 2 \, \|\sigma\|_\infty + 2) \, \eta = \varepsilon.$$

Therefore, being $x \in [a, b]$ arbitrary, we conclude that $\|G_N f - f\|_\infty < \varepsilon$. $\square$

Note that, when σ is continuous, Theorem 2.2 can be viewed as a *density result* in $C^0[a, b]$ for the set containing all functions of $\mathcal{M}_N(\sigma)$, for every $N \in \mathbb{N}^+$, with respect to the uniform norm. In addition, as a consequence of Lemma 2.1, we also observe that $\|G_N f - f\|_\infty < \varepsilon$ for every $w \geq \overline{w}$, where $\overline{w} = \overline{w}(1/N) > 0$ is chosen as in Theorem 2.2.

2.2 Constructive approximation in $L^p[a, b]$

Theorem 2.2 was extended also to L^p-functions in [63] and then in [48], but by non-constructive methods. Results in L^p were also showed in [67, 1, 43, 44].

In this section, we give a constructive proof of an approximation theorem for functions $f \in L^p[a, b]$, $1 \leq p < \infty$, by elements of $\mathcal{M}_N(\sigma)$, with σ bounded. First of all, we have the following theorem.

Theorem 2.3. *Let σ be a bounded sigmoidal function and let $1 \leq p < \infty$ be fixed. For every $f \in C^0[a,b]$ and $\varepsilon > 0$, there exist $N \in \mathbb{N}^+$ and $w > 0$ (depending on N and σ), such that the function $G_N f$ defined in* (2.1) *with w as scaling parameter, is such that*

$$\|G_N f - f\|_{L^p[a,b]} < \varepsilon.$$

Proof. Let $f \in C^0[a,b]$ and $\varepsilon > 0$ be fixed. By Theorem 2.2, correspondingly to $\eta := \varepsilon/(b-a)^{1/p}$, there exist $N \in \mathbb{N}^+$, $N > 3$, and $w > 0$, depending on N, such that $\|G_N f - f\|_\infty < \eta$. Therefore,

$$\|G_N f - f\|_{L^p[a,b]} = \left(\int_a^b |(G_N f)(x) - f(x)|^p \, dx \right)^{1/p} < \left(\int_a^b \eta^p \, dx \right)^{1/p} = \varepsilon.$$

$\square$

Now we can prove the following constructive approximation result in $L^p[a,b]$.

Theorem 2.4. *Let σ be a bounded sigmoidal function and let $f \in L^p[a,b]$, $1 \leq p < \infty$, be fixed. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}^+$ and a function $G_N \in \mathcal{M}_N(\sigma)$, such that*

$$\|G_N - f\|_{L^p[a,b]} < \varepsilon.$$

Proof. The proof is constructive. Define the function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{f}(x) := \begin{cases} f(x), & x \in [a,b], \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

Note that $\widetilde{f} \in L^p(\mathbb{R})$ and $\widetilde{f} = f$ on $[a,b]$. Let $\{\rho_n\}_{n \in \mathbb{N}^+}$, $\rho_n : \mathbb{R} \to \mathbb{R}$, be a sequence of mollifiers, i.e., $\rho_n \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \rho_n \subseteq [-1/n, 1/n]$, $\int_{\mathbb{R}} \rho_n(x)\, dx = 1$, and $\rho_n(x) \geq 0$ for every $x \in \mathbb{R}$, $n \in \mathbb{N}^+$. Define the family $\{f_n\}_{n \in \mathbb{N}^+}$ by

$$f_n(x) := (\rho_n * \widetilde{f})(x) = \int_{\mathbb{R}} \rho_n(x-y)\, \widetilde{f}(y)\, dy, \quad x \in \mathbb{R}, \tag{2.3}$$

where $*$ denotes convolution. By the general properties of sequences of mollifiers and of convolution products [26], it turns out that $f_n = \rho_n * \widetilde{f} \in C^0(\mathbb{R})$ for every $n \in \mathbb{N}^+$, and $f_n \to \widetilde{f}$ in $L^p(\mathbb{R})$ as $n \to +\infty$. Let $\varepsilon > 0$ be fixed. Then, there exists $\overline{n} \in \mathbb{N}^+$ such that

$$\|f_n - f\|_{L^p[a,b]} = \left\| f_n - \widetilde{f} \right\|_{L^p[a,b]} \leq \left\| f_n - \widetilde{f} \right\|_{L^p(\mathbb{R})} < \frac{\varepsilon}{2},$$

for every $n \geq \overline{n}$. Let now $n \geq \overline{n}$ be fixed. Being $f_n \in C^0(\mathbb{R}) \subset C^0[a,b]$, as a consequence of Theorem 2.3, correspondingly to $\varepsilon/2$ there exist $N \in \mathbb{N}^+$,

$N > 3$, and $w > 0$ (depending on N and σ), such that the function $G_N f_n$ defined in (2.1) with w as scaling parameter, is such that

$$\|G_N f_n - f_n\|_{L^p[a,b]} = \left\|G_N(\rho_n * \widetilde{f}) - (\rho_n * \widetilde{f})\right\|_{L^p[a,b]} < \frac{\varepsilon}{2}.$$

Hence, we can conclude that

$$\|G_N f_n - f\|_{L^p[a,b]} \leq \|G_N f_n - f_n\|_{L^p[a,b]} + \|f_n - f\|_{L^p[a,b]} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Setting $G_N(x) := (G_N f_n)(x)$ completes the proof. □

We stress that the proof of Theorem 2.4 is *constructive* and by choosing a specific sequence of mollifiers a precise analytic form of the sums $G_N f_n$ (f_n defined in (2.3)) would be obtained, which approximates $f \in L^p[a,b]$.

One could construct a sequence of mollifiers starting from a single function $\rho \in C_c^\infty(\mathbb{R})$, such that $\operatorname{supp}\rho \subset [-1,1]$, and $\rho \geq 0$ on $\mathbb{R}$. For instance, choosing

$$\rho(x) := \begin{cases} e^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2.4}$$

we obtain a sequence of mollifiers setting

$$\rho_n(x) := C\, n\, \rho(nx), \quad x \in \mathbb{R}, \tag{2.5}$$

where $C := (\int_{\mathbb{R}} \rho(x)\, dx)^{-1}$, [26, 97]. In this case, the approximating sums take on the form

$$G_N(\rho_n * \widetilde{f})(x) = \sum_{k=1}^{N} \alpha_k\, \sigma(w(x - x_k)) + \alpha_0\, \sigma(w(x - x_{-1})), \quad x \in [a,b],$$

where $N \in N^+$, $w > 0$, $x_k = a + kh$, $k = -1, 0, 1, ..., N$, for $h = (b-a)/N$, $\widetilde{f}$ defined in (2.2) and then the coefficients are

$$\alpha_k = \int_a^b [\rho_n(x_k - y)\ - \rho_n(x_{k-1} - y)]\, f(y)\, dy,$$

for $k = 1, ..., N$, while $\alpha_0 = \int_a^b \rho_n(a-y)\, f(y)\, dy$, for n sufficiently large.

2.3 Simultaneous approximation of functions and their derivatives

The theory presented in Section 2.1, allows to approximate any given function $f \in C^0[a,b]$ by means of functions like $G_N f$, defined in (2.1) and hence, if f is sufficiently smooth, say, e.g., $f \in \widehat{C}^1[a,b]$, then its derivative, f', could also be approximated by means of $G_N f'$. In particular, we can choose the same number $N \in \mathbb{N}^+$ to construct simultaneous approximations of f and f', using $G_N f$ and $G_N f'$. In this way, we can also choose the same value for $w > 0$. However, $2(N+1)$ coefficients, namely the values $f(x_k)$ and $f'(x_k)$ for $k = 0, 1, ..., N$, will be needed, in this case.

Similarly, we could construct simultaneous approximations to f, f', ..., $f^{(n)}$, by means of approximants like $G_N f, G_N f', ..., G_N f^{(n)}$, respectively, for every given $f \in \widehat{C}^n[a,b]$, requiring $(n+1)(N+1)$ coefficients.

In view of a number of applications, it could be useful to be able to approximate f *along with* some of its derivatives, by superposing sigmoidal functions but using *only* the $N+1$ coefficients entering the approximation formula for f.

Remark 2.5. Let $f \in \widehat{C}^1[a,b]$ be fixed and $\sigma \in C^1(\mathbb{R})$ be a bounded sigmoidal function. Moreover, let $\varepsilon > 0$ such that $\|G_N f - f\|_\infty < \varepsilon$, for suitable values N and $w > 0$. Consider $(G_N f)'$, the first derivative of $G_N f$, which is of the form

$$(G_N f)'(x) = \sum_{k=1}^{N} w\ [f(x_k) - f(x_{k-1})]\ \sigma'(w(x-x_k)) + w\, f(x_0)\, \sigma'(w(x-x_{-1})), \tag{2.6}$$

for $x \in [a,b]$. First of all, note that, in general, $(G_N f)'$ is *not* a sum of sigmoidal functions, but its coefficients, which are the same appearing in the sum $G_N f$, multiplied by the scaling term $w > 0$, are known. Besides, $\sigma'(w(x-x_k))$, $k = -1, 1, ..., N$, are also known. Hence, f' could be uniformly approximated by $(G_N f)'$, but in general the condition $\|(G_N f)' - f'\|_\infty < \varepsilon$ will not be fulfilled. A similar problem would arise if we try to approximate f' by a sum of sigmoidal functions like $G_M[(G_N f)']$.

Consider for instance $f(x) = x$ on $[0,1]$ and $\sigma_\ell(x) = (1+e^{-x})^{-1}$. Then, $f'(x) = 1$ and $\sigma_\ell'(x) = e^{-x}/(1+e^{-x})^2$, while the sum $G_N f$ takes on the form

$$(G_N f)(x) = h \sum_{k=1}^{N} \sigma_\ell(w_0(x - x_k)), \quad x \in [0,1],$$

where $h = 1/N$ and $x_k = k/N$, $k = 1, ..., N$. Assume that $\|G_N f - f\|_\infty < \varepsilon$ for $\varepsilon > 0$, $N \in \mathbb{N}^+$, and a fixed value of $w_0 > 0$, depending on N. As noted in Section 2.1, we also have $\|G_N f - f\|_\infty < \varepsilon$ if we replace w_0 in $G_N f$ with $w \geq w_0$. Moreover, consider

$$(G_N f)'(x) = w\, h \sum_{k=1}^{N} \sigma_\ell'(w(x - x_k)), \quad x \in [0,1].$$

Let now $\overline{x} \in [0,1]$ be fixed. Then,

$$(G_N f)'(\overline{x}) = w\, h \sum_{k=1}^{N} \sigma_\ell'(w(\overline{x} - x_k)) =: w\, C,$$

where $C = C(w)$ is such that $0 < C \leq 1$ for every $w > 0$. Since $\lim_{w\to+\infty}(w\,C - 1) = +\infty$, a condition like

$$\left|(G_N f)'(\overline{x}) - f'(\overline{x})\right| = |w\,C - 1| < \varepsilon$$

cannot be satisfied, in general, if $w > 0$ is sufficiently large.

We now introduce some notation. For any fixed $f \in C[a,b]$, and for any given uniform partition $\{x_0, x_1, ..., x_N\}$ of the interval $[a,b]$, with $x_0 = a$, $x_N = b$ and $h = x_k - x_{k-1}$, $k = 1, ..., N$, we define

$$\Delta_k^j f := \frac{1}{h^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu f(x_{k+j-\nu}), \tag{2.7}$$

for $j \in \mathbb{N}$, $j \leq N$, and $k = 0, 1, ..., N-j$. We can establish the following

Theorem 2.6. *(*Simultaneous approximation of f and its derivatives*). Let σ be a bounded sigmoidal function and let $f \in \widehat{C}^{n+1}[a,b]$, $n \in \mathbb{N}^+$, be fixed. For every $\varepsilon > 0$, there exist $N \in \mathbb{N}^+$ and $w > 0$ (depending on N and σ), such that, for every $j = 1, ..., n$, defining*

$$(G_N^j f)(x) := \sum_{k=1}^{N-j} \left(\Delta_k^j f - \Delta_{k-1}^j f\right) \sigma(w(x - x_k)) + \Delta_0^j f\, \sigma(w(x - x_{-1})), \tag{2.8}$$

for $x \in [a,b]$, $h := (b-a)/N$ and $x_k := a + kh$, $k = -1, 0, 1, ..., N$, it turns out that

$$\left\| G_N^j f - f^{(j)} \right\|_\infty < \varepsilon.$$

Proof. Let $j = 1, ..., n$ and $\varepsilon > 0$ be fixed. Since $f^{(j)} \in \widehat{C}^1[a,b]$ is uniformly continuous, correspondingly to

$$\eta := \left(4C_j \|\sigma\|_\infty + 2\|\sigma\|_\infty + 3C_j + \left\|f^{(j)}\right\|_\infty + 2\right)^{-1} \varepsilon,$$

where $C_j = C_j(f^{(j+1)}, a, b)$ is a fixed constant that will be determined later, there exists $\delta > 0$ such that, for every x, $y \in [a,b]$ with $|x - y| < \delta$, we have $\left|f^{(j)}(x) - f^{(j)}(y)\right| < \eta$. We choose $N \in \mathbb{N}^+$, $N > j + 3$ sufficiently large so that $h := (b-a)/N < \delta/\max\{2, j\}$ and $1/N < \eta$. Moreover, we fix $w \geq \overline{w}(1/N, h) \equiv \overline{w}(1/N) > 0$, where $\overline{w}(1/N)$ is obtained using Lemma 2.1 with $1/N$, $h > 0$ and with the points $x_k = a + hk$, $k = -1, 0, 1, ..., N$. Now consider the sum $G_N^j f$ defined in (2.8) with w, and let $x \in [a,b]$ be fixed. Then, there exists $i = 1, ..., N$ such that $x \in [x_{i-1}, x_i]$. Set

$$L_i(x) := \Delta_0^j f + \left(\Delta_2^j f - \Delta_1^j f\right) \sigma(w(x - x_2)) + \left(\Delta_1^j f - \Delta_0^j f\right) \sigma(w(x - x_1))$$

for $i = 1, 2$,

$$\begin{aligned} L_i(x) \; &:= \; \sum_{k=1}^{i-2} \left(\Delta_k^j f - \Delta_{k-1}^j f \right) + \Delta_0^j f + \left(\Delta_{i-1}^j f - \Delta_{i-2}^j f \right) \sigma(w(x - x_{i-1})) \\ &+ \; \left(\Delta_i^j f - \Delta_{i-1}^j f \right) \sigma(w(x - x_i)), \end{aligned}$$

for $i = 3, ..., N - j$,

$$L_i(x) := \sum_{k=1}^{N-j-1} \left(\Delta_k^j f - \Delta_{k-1}^j f \right) + \Delta_0^j f + \left(\Delta_{N-j}^j f - \Delta_{N-j-1}^j f \right) \sigma(w(x - x_{N-j}))$$

for $i = N - j + 1$, and

$$L_i(x) := \sum_{k=1}^{N-j} \left(\Delta_k^j f - \Delta_{k-1}^j f \right) + \Delta_0^j f,$$

for $i = N - j + 2, ..., N$. We now write

$$\left| (G_N^j f)(x) - f^{(j)}(x) \right| \le \left| (G_N^j f)(x) - L_i(x) \right| + \left| L_i(x) - f^{(j)}(x) \right| =: J_1 + J_2$$

and start estimating J_1. We confine only to $i = 3, ..., N - j$, since similar estimates can be obtained in the same way in all the other cases.

Being $x - x_k \ge h$ for $k = -1, 0, 1, ..., i - 2$ and $x - x_k \le -h$ for $k = i + 1, ..., N - j$, it follows by conditions 1 and 2 of Lemma 2.1 that

$$\begin{aligned} J_1 \; &\le \; \sum_{k=1}^{i-2} \left| \Delta_k^j f - \Delta_{k-1}^j f \right| \, |\sigma(w(x - x_k)) - 1| + \left| \Delta_0^j f \right| \, |\sigma(w(x - x_{-1})) - 1| \\ &+ \; \sum_{k=i+1}^{N-j} \left| \Delta_k^j f - \Delta_{k-1}^j f \right| \, |\sigma(w(x - x_k))| \\ &< \; \frac{1}{N} \sum_{k=1}^{N-j} \left| \Delta_k^j f - \Delta_{k-1}^j f \right| + \frac{1}{N} \left| \Delta_0^j f \right| \\ &\le \; \frac{1}{N} \sum_{k=1}^{N-j} \left| \Delta_k^j f - f^{(j)}(x_k) \right| + \frac{1}{N} \sum_{k=1}^{N-j} \left| f^{(j)}(x_k) - f^{(j)}(x_{k-1}) \right| \\ &+ \; \frac{1}{N} \sum_{k=1}^{N-j} \left| f^{(j)}(x_{k-1}) - \Delta_{k-1}^j f \right| + \frac{1}{N} \left| \Delta_0^j f - f^{(j)}(x_0) \right| + \frac{1}{N} \left| f^{(j)}(x_0) \right|. \end{aligned}$$

We now observe that, for every $k = 0, 1, ..., N - j$, the terms $\Delta_k^j f$ provide an approximation to $f^{(j)}(x_k)$, obtained by forward finite differences. It is well known that there exists a positive constant, $\widetilde{C_j} > 0$, depending only on $f^{(j+1)}$, such that

$$\left|\Delta_k^j f - f^{(j)}(x_k)\right| \le \widetilde{C_j}\, h = \widetilde{C_j}\, \frac{(b-a)}{N} =: \frac{C_j}{N}, \tag{2.9}$$

for every $k = 0, 1, ..., N-j$, where $C_j = C_j(f^{(j+1)}, a, b)$. Then, using (2.9), the uniform continuity of $f^{(j)}$, and the previous inequality for J_1, we obtain

$$J_1 \le 2 \sum_{k=1}^{N-j+1} \frac{C_j}{N^2} + \frac{1}{N} \sum_{k=1}^{N-j} \eta + \frac{1}{N} \left\|f^{(j)}\right\|_\infty < 2\, C_j\, \eta + \eta + \left\|f^{(j)}\right\|_\infty \eta.$$

Finally, we estimate J_2, separately in four cases, resorting to the same arguments followed to establish the previous inequalities.

Case 1: $i = 1, 2$. Being $|x_0 - x| \le 2\, h \le \max\{2, j\}\ h < \delta$, we have

$$\begin{aligned} J_2 &\le \left(\left|\Delta_2^j f - f^{(j)}(x_2)\right| + \left|f^{(j)}(x_2) - f^{(j)}(x_1)\right| + \left|f^{(j)}(x_1) - \Delta_1^j f\right|\right) \|\sigma\|_\infty \\ &+ \left(\left|\Delta_1^j f - f^{(j)}(x_1)\right| + \left|f^{(j)}(x_1) - f^{(j)}(x_0)\right| + \left|f^{(j)}(x_0) - \Delta_0^j f\right|\right) \|\sigma\|_\infty \\ &+ \left|\Delta_0^j f - f^{(j)}(x_0)\right| + \left|f^{(j)}(x_0) - f^{(j)}(x)\right| \\ &\le (4C_j\eta + 2\eta)\ \|\sigma\|_\infty + C_j\, \eta + \eta. \end{aligned}$$

Case 2: $i = 3, ..., N-j$. We obtain, as above,

$$J_2 < (4C_j\eta + 2\eta)\ \|\sigma\|_\infty + C_j\eta + \eta.$$

Case 3: $i = N - j + 1$.

$$\begin{aligned} J_2 &\le \left(\left|\Delta_{N-j}^j f - f^{(j)}(x_{N-j})\right| + \left|f^{(j)}(x_{N-j}) - f^{(j)}(x_{N-j-1})\right|\right. \\ &+ \left.\left|f^{(j)}(x_{N-j-1}) - \Delta_{N-j-1}^j f\right|\right) \|\sigma\|_\infty + \left|\Delta_{N-j-1}^j f - f^{(j)}(x_{N-j-1})\right| \\ &+ \left|f^{(j)}(x_{N-j-1}) - f^{(j)}(x)\right| < (2C_j\eta + \eta)\ \|\sigma\|_\infty + C_j\eta + \eta. \end{aligned}$$

Case 4: $i = N - j + 2, ..., N - j$.

$$J_2 \le \left|\Delta_{N-j}^j f - f^{(j)}(x_{N-j})\right| + \left|f^{(j)}(x_{N-j}) - f^{(j)}(x)\right| \le C_j\eta + \eta,$$

since $|x_{N-j} - x| \le j\, h \le \max\{2, j\}\ h < \delta$. Thus,

$$\begin{aligned} &\left|(G_N^j f)(x) - f^{(j)}(x)\right| \le J_1 + J_2 < 2C_j\eta + \eta + \left\|f^{(j)}\right\|_\infty \eta\ + C_j\eta + \eta \\ &+ \ (4C_j\eta + 2\eta)\ \|\sigma\|_\infty = \left(4C_j\, \|\sigma\|_\infty + 2\, \|\sigma\|_\infty + 3C_j + \left\|f^{(j)}\right\|_\infty + 2\right) \eta = \varepsilon. \end{aligned}$$

In particular, we have $\left|(G_N^j f)(x) - f^{(j)}(x)\right| < \varepsilon$ for every $x \in [a, b]$, hence $\left\|G_N^j f - f^{(j)}\right\|_\infty < \varepsilon$. This completes the proof. $\square$

For example, if we want to approximate f', being $f \in \widehat{C}^2[a, b]$, by a superposition of sigmoidal functions, with the same coefficients used for approximating f, we can consider the sum $G_N^1 f$, which takes on the form

$$
\begin{aligned}
(G_N^1 f)(x) &= \sum_{k=1}^{N-1} \frac{f(x_{k+1}) - 2f(x_k) + f(x_{k-1})}{h} \sigma(w(x - x_k)) \\
&+ \frac{f(x_1) - f(x_0)}{h} \sigma(w(x - x_{-1})), \quad x \in \mathbb{R}.
\end{aligned}
$$

At this point, we want to estimate the *error* made approximating a given function as well as its derivatives, by a superposition of sigmoidal functions.

For results concerning the order of approximation by sigmoidal functions approximation see Subsection 1.3.5 (see also [112, 42, 38, 40]). There, it was proved that the error made approximating f by linear combinations of N sigmoidal functions is of order $\mathcal{O}(1/N)$ (for N sufficiently large). Here we can prove that *the same* order of approximation can be established approximating the jth derivative of f with sums of the form $G_N^j f$, for every j. In fact,

Theorem 2.7. *Let σ be a bounded sigmoidal function, and let $f \in \widehat{C}^{n+1}[a, b]$, $n \in \mathbb{N}^+$, and $j = 1, ..., n$ be fixed. For every $N \in \mathbb{N}^+$, $N > j + 3$, there exists $\overline{w} > 0$ (depending on N and σ), such that, for every $w \geq \overline{w}$ and $G_N^j f$ defined in (2.8) with w, we obtain*

$$
\begin{aligned}
\left\|G_N^j f - f^{(j)}\right\|_\infty &< \frac{1}{N} \left[L_j(b-a)\left(2\|\sigma\|_\infty + 1 + \max\{2, j\}\right)\right. \\
&+ \left.\widetilde{C}_j(b-a)\left(4\|\sigma\|_\infty + 3\right) + \left\|f^{(j)}\right\|_\infty\right],
\end{aligned}
$$

where $\widetilde{C}_j > 0$ is a constant, depending only on $f^{(j+1)}$, and $L_j > 0$ is the Lipschitz constant of $f^{(j)}$.

Proof. Let $j = 1, ..., n$ and $N \in \mathbb{N}^+$, $N > j + 3$, be fixed. Set $h := (b - a)/N$ and $x_k := a + hk$, for $k = -1, 0, 1, ..., N$. Moreover, let $\overline{w} = \overline{w}(1/N, h) = \overline{w}(1/N) > 0$ obtained using Lemma 2.1 with $1/N$, $h > 0$ and with the points $x_k = a + hk$, $k = -1, 0, 1, ..., N$. Consider now $G_N^j f$ defined in (2.8) for $w \geq \overline{w}$, and let $x \in [a, b]$ be fixed. Then, there exists $i = 1, ..., N$ such that $x \in [x_{i-1}, x_i]$. We set, as in Theorem 2.6,

$$
L_i(x) := \Delta_0^j f + \left(\Delta_2^j f - \Delta_1^j f\right) \sigma(w(x - x_2)) + \left(\Delta_1^j f - \Delta_0^j f\right) \sigma(w(x - x_1))
$$

for $i = 1, 2$,

$$\begin{aligned} L_i(x) \;&:=\; \sum_{k=1}^{i-2} \left(\Delta_k^j f - \Delta_{k-1}^j f\right) + \Delta_0^j f + \left(\Delta_{i-1}^j f - \Delta_{i-2}^j f\right) \sigma(w(x - x_{i-1})) \\ &+\; \left(\Delta_i^j f - \Delta_{i-1}^j f\right) \sigma(w(x - x_i)) \end{aligned}$$

for $i = 3, ..., N - j$,

$$L_i(x) := \sum_{k=1}^{N-j-1} \left(\Delta_k^j f - \Delta_{k-1}^j f\right) + \Delta_0^j f + \left(\Delta_{N-j}^j f - \Delta_{N-j-1}^j f\right) \sigma(w(x - x_{N-j}))$$

for $i = N - j + 1$, and

$$L_i(x) := \sum_{k=1}^{N-j} \left(\Delta_k^j f - \Delta_{k-1}^j f\right) + \Delta_0^j f$$

for $i = N - j + 2, ..., N$. We then write

$$\left|(G_N^j f)(x) - f^{(j)}(x)\right| \le \left|(G_N^j f)(x) - L_i(x)\right| + \left|L_i(x) - f^{(j)}(x)\right| =: J_1 + J_2$$

and estimate J_1 (only for $i = 3, ..., N - j$, the other cases being similar), using the same arguments as in deriving the estimates in Theorem 2.6. We obtain

$$\begin{aligned} J_1 \;&<\; \frac{1}{N} \sum_{k=1}^{N-j} \left|\Delta_k^j f - f^{(j)}(x_k)\right| + \frac{1}{N} \sum_{k=1}^{N-j} \left|f^{(j)}(x_k) - f^{(j)}(x_{k-1})\right| \\ &+\; \frac{1}{N} \sum_{k=1}^{N-j} \left|f^{(j)}(x_{k-1}) - \Delta_{k-1}^j f\right| + \frac{1}{N} \left|\Delta_0^j f - f^{(j)}(x_0)\right| + \frac{1}{N} \left|f^{(j)}(x_0)\right|. \end{aligned}$$

By inequality (2.9) and being $f^{(j)} \in \widehat{C}^1[a, b]$ Lipschitz continuous, if $L_j > 0$ is the Lipschitz constant of $f^{(j)}$, we have

$$J_1 < \frac{1}{N} \left(2\widetilde{C}_j(b - a) + L_j(b - a) + \left\|f^{(j)}\right\|_\infty\right),$$

where $\widetilde{C}_j = \widetilde{C}_j(f^{(j+1)}) > 0$. We now estimate J_2 in the four different cases.

Case 1: $i = 1, 2$.

$$\begin{aligned} J_2 \;&\le\; \left(\left|\Delta_2^j f - f^{(j)}(x_2)\right| + \left|f^{(j)}(x_2) - f^{(j)}(x_1)\right| + \left|f^{(j)}(x_1) - \Delta_1^j f\right|\right) \|\sigma\|_\infty \\ &+\; \left(\left|\Delta_1^j f - f^{(j)}(x_1)\right| + \left|f^{(j)}(x_1) - f^{(j)}(x_0)\right| + \left|f^{(j)}(x_0) - \Delta_0^j f\right|\right) \|\sigma\|_\infty \\ &+\; \left|\Delta_0^j f - f^{(j)}(x_0)\right| + \left|f^{(j)}(x_0) - f^{(j)}(x)\right| \\ &\le\; \left(4\frac{\widetilde{C}_j(b - a)}{N} + 2L_j\frac{(b - a)}{N}\right) \|\sigma\|_\infty + \frac{\widetilde{C}_j(b - a)}{N} + 2L_j\frac{(b - a)}{N}. \end{aligned}$$

Case 2: $i = 3, ..., N - j$. Proceeding as above,

$$J_2 \leq \left(4\frac{\widetilde{C}_j(b-a)}{N} + 2L_j\frac{(b-a)}{N} \right) \|\sigma\|_\infty + \frac{\widetilde{C}_j(b-a)}{N} + 2L_j\frac{(b-a)}{N}.$$

Case 3: $i = N - j + 1$. Similarly,

$$J_2 \leq \left(2\frac{\widetilde{C}_j(b-a)}{N} + L_j\frac{(b-a)}{N} \right) \|\sigma\|_\infty + \frac{\widetilde{C}_j(b-a)}{N} + 2L_j\frac{(b-a)}{N}.$$

Case 4: $i = N - j + 2, ..., N - j$.

$$\begin{aligned} J_2 \quad &\leq \quad \left| \Delta^j_{N-j} f - f^{(j)}(x_{N-j}) \right| + \left| f^{(j)}(x_{N-j}) - f^{(j)}(x) \right| \leq \frac{\widetilde{C}_j(b-a)}{N} \\ &+ \quad L_j \left| x_{N-j} - x \right| \leq \frac{\widetilde{C}_j(b-a)}{N} + j\, L_j\, \frac{(b-a)}{N}. \end{aligned}$$

Then,

$$\begin{aligned} \left| (G^j_N f)(x) - f^{(j)}(x) \right| \quad &\leq \quad J_1 + J_2 < \frac{1}{N} \left[L_j(b-a)\left(2\|\sigma\|_\infty + 1 + \max\{2, j\}\right) \right. \\ &+ \quad \left. \widetilde{C}_j(b-a)\left(4\|\sigma\|_\infty + 3\right) + \left\| f^{(j)} \right\|_\infty \right]. \end{aligned}$$

In particular, the estimates above holds for every $x \in [a, b]$. Therefore,

$$\begin{aligned} \left\| G^j_N f - f^{(j)} \right\|_\infty \quad &< \quad \frac{1}{N} \left[L_j(b-a)\left(2\|\sigma\|_\infty + 1 + \max\{2, j\}\right) \right. \\ &+ \quad \left. \widetilde{C}_j(b-a)\left(4\|\sigma\|_\infty + 3\right) + \left\| f^{(j)} \right\|_\infty \right]. \end{aligned}$$

□

2.4 Constructive multivariate approximation

In this section, we propose a multivariate extension of the constructive theory developed in Sections 2.1 and 2.2. For simplicity, the proofs will be given only for functions of two variables, but their extension to higher dimensions is straightforward.

We first establish the following lemma, obtained as an easy consequence of Definition 1.1. This represents a generalization of Lemma 2.1.

Lemma 2.8. *Let* $(x_0, y_0), (x_1, y_1), ..., (x_N, y_N) \in \mathbb{R}^2$*, for some fixed* $N \in \mathbb{N}^+$*. For every* ε *and* $h > 0$*, there exists* $\overline{w} := \overline{w}(\varepsilon, h) > 0$ *such that, for every* $w \geq \overline{w}$ *and* $k = 0, 1, ..., N$*, we have*

1. $|\sigma(w\,\|(x,y)-(x_k,y_k)\|_2)-1|<\varepsilon$;

2. $|\sigma(-w\,\|(x,y)-(x_k,y_k)\|_2)|<\varepsilon$,

for every $(x,y)\in\mathbb{R}^2$ *such that* $\|(x,y)-(x_k,y_k)\|_2\geq h$.

Note that here the function $\sigma(\|(x,y)\|_2)$ is actually a *radial basis* function (RBF). We can now prove the following theorem, where we stress that no continuity assumption on σ is made.

Theorem 2.9. *Let σ be a bounded sigmoidal function, and let $f\in C^0(Q)$, where $Q:=[a,b]\times[c,d]\subset\mathbb{R}^2$ with $b-a=d-c=l$ fixed. For every $\varepsilon>0$, there exist $N\in\mathbb{N}^+$ and $w>0$ (depending on N and σ), such that, if*

$$(\widetilde{G}_N f)(x,y)$$

$$:=\sum_{i=1}^{N}\sum_{j=1}^{N}[f(x_i,y_j)-f(x_i,y_{j-1})]\;\sigma\left(w\,\chi_{ij}(x,y)\;\left\|(x,y)-(t_{x_i},t_{y_j})\right\|_2\right)$$

$$+\sum_{i=1}^{N}f(x_i,y_0)\,\sigma\left(w\,\chi_{i0}(x,y)\;\|(x,y)-(t_{x_i},t_{y_0})\|_2\right), \tag{2.10}$$

where $(x,y)\in Q$, $h:=l/N$, $x_i:=a+h\,i$, $y_j:=c+h\,j$, for $i,j=-1,0,1,...,N$ and $t_{x_i}:=(x_{i-1}+x_i)/2$, $t_{y_j}:=(y_{j-1}+y_j)/2$ for $i,j=0,1,...,N$, and moreover

$$\chi_{ij}(x,y):=\begin{cases}+1, & \textit{if } x\in(x_{i-1},x_i] \textit{ and } y\geq y_j,\\ -1, & \textit{otherwise.}\end{cases}$$

for $i=2,...,N$ and $j=0,...,N$, while

$$\chi_{1j}(x,y):=\begin{cases}+1, & \textit{if } x\in[x_0,x_1] \textit{ and } y\geq y_j,\\ -1, & \textit{otherwise.}\end{cases}$$

for $j=0,...,N$, then

$$\left\|\widetilde{G}_N f-f\right\|_\infty<\varepsilon.$$

Proof. Let $\varepsilon>0$ be fixed. Since f is uniformly continuous, correspondingly to $\eta:=\varepsilon/(\|f\|_\infty+\|\sigma\|_\infty+2)$, there exists $\delta>0$ such that $|f(x,y)-f(z,t)|<\eta$ for every (x,y), $(z,t)\in Q$ with $\|(x,y)-(z,t)\|_2<\delta$. Now, choose $N\in\mathbb{N}^+$ such that $h:=l/N<\delta/\sqrt{2}$ and $1/N<\eta$. Moreover, fix $w\geq\overline{w}(1/N^2,h/2)=\overline{w}(1/N^2)>0$, where $\overline{w}(1/N^2)$ is obtained from Lemma 2.8 with $1/N^2$, $h/2>0$ and with the points (t_{x_i},t_{y_j}), $i,j=0,1,...,N$. Consider $\widetilde{G}_N f$ defined in (2.10) for w, and let $(x,y)\in Q$ be fixed. Thus, there exist $k,\mu=1,...,N$, such that $(x,y)\in(x_{k-1},x_k]\times[y_{\mu-1},y_\mu]$ provided $k\geq 2$ or $(x,y)\in[x_0,x_1]\times[y_{\mu-1},y_\mu]$ otherwise. Set

$$\mathcal{L}_{k\mu}(x,y):=f(x_k,y_0)$$

$$+ [f(x_k, y_1) - f(x_k, y_0)]\ \sigma(w\, \chi_{k\mu}(x,y)\ \|(x,y) - (t_{x_k}, t_{y_1})\|_2),$$

if $\mu = 1$, and

$$\mathcal{L}_{k\mu}(x,y) := \sum_{j=1}^{\mu-1} [f(x_k, y_j) - f(x_k, y_{j-1})] + f(x_k, y_0)$$

$$+ [f(x_k, y_\mu) - f(x_k, y_{\mu-1})]\ \sigma(w\, \chi_{k\mu}(x,y)\ \left\|(x,y) - (t_{x_k}, t_{y_\mu})\right\|_2),$$

if $\mu \geq 2$. In both cases, let write

$$\begin{aligned} \left|(\widetilde{G}_N f)(x,y) - f(x,y)\right| &\leq \left|(\widetilde{G}_N f)(x,y) - \mathcal{L}_{k\mu}(x,y)\right| \\ &+ \ |\mathcal{L}_{k\mu}(x,y) - f(x,y)| =: H_1 + H_2. \end{aligned}$$

We first estimate H_1 when $\mu \geq 2$, obtaining

$$\begin{aligned} H_1 &\leq \sum_{\substack{i=1 \\ i \neq k}}^{N} \sum_{j=1}^{N} |f(x_i, y_j) - f(x_i, y_{j-1})|\ \left|\sigma(w\, \chi_{ij}(x,y)\ \left\|(x,y) - (t_{x_i}, t_{y_j})\right\|_2)\right| \\ &+ \sum_{\substack{i=1 \\ i \neq k}}^{N} |f(x_i, y_0)|\ \left|\sigma(w\, \chi_{i0}(x,y)\ \|(x,y) - (t_{x_i}, t_{y_0})\|_2)\right| \\ &+ \sum_{j=1}^{\mu-1} |f(x_k, y_j) - f(x_k, y_{j-1})|\ \left|\sigma(w\, \chi_{kj}(x,y)\ \left\|(x,y) - (t_{x_k}, t_{y_j})\right\|_2) - 1\right| \\ &+ \sum_{j=\mu+1}^{N} |f(x_k, y_j) - f(x_k, y_{j-1})|\ \left|\sigma(w\, \chi_{kj}(x,y)\ \left\|(x,y) - (t_{x_k}, t_{y_j})\right\|_2)\right| \\ &+ \ |f(x_k, y_0)|\ \left|\sigma(w\, \chi_{k0}(x,y)\ \|(x,y) - (t_{x_k}, t_{y_0})\|_2) - 1\right|. \end{aligned}$$

Being $\|(x_i, y_j) - (x_i, y_{j-1})\|_2 < \delta$ for every $i, j = 1, ..., N$, $\left\|(x,y) - (t_{x_i}, t_{y_j})\right\|_2 \geq h/2$ for every $(t_{x_i}, t_{y_j}) \neq (t_{x_k}, t_{y_\mu})$, by conditions 1 and 2 of Lemma 2.8 and the definition of χ_{ij}, we obtain

$$\begin{aligned} H_1 &< \frac{1}{N^2} \sum_{\substack{i=1 \\ i \neq k}}^{N} \sum_{j=1}^{N} \eta + \frac{1}{N^2} \sum_{\substack{i=1 \\ i \neq k}}^{N} \|f\|_\infty + \frac{1}{N^2} \sum_{j=1}^{\mu-1} \eta + \frac{1}{N^2} \sum_{j=\mu+1}^{N} \eta \\ &+ \frac{1}{N^2} \|f\|_\infty \leq \eta + \frac{1}{N} \|f\|_\infty < (1 + \|f\|_\infty)\, \eta. \end{aligned}$$

Note that the same estimate for H_1 also holds if $\mu = 1$. Finally, if we note that

$$\sum_{j=1}^{\mu-1} [f(x_k, y_j) - f(x_k, y_{j-1})] + f(x_k, y_0) = f(x_k, y_{\mu-1})$$

and that $\|(x,y)-(x_k,y_{\mu-1})\|_2 \leq \sqrt{2}\,h < \delta$, we obtain

$$H_2 < |f(x_k,y_{\mu-1}) - f(x,y)| + \eta\, \|\sigma\|_\infty = (\|\sigma\|_\infty + 1)\,\eta, \tag{2.11}$$

((2.11) holds also in case $\mu = 1$). We conclude that

$$\left|(\widetilde{G}_N f)(x,y) - f(x,y)\right| \leq H_1 + H_2 < (\|\sigma\|_\infty + \|f\|_\infty + 2)\,\eta = \varepsilon,$$

and since $(x,y) \in Q$ is arbitrary, it follows that $\left\|\widetilde{G}_N f - f\right\|_\infty < \varepsilon$. □

Remark 2.10. We remark that the results obtained in Theorem 2.9 differ substantially from those established by G. Cybenko in [63] and by B. Lenze in [94]. In particular, others than Cybenko's theory, ours is *constructive*, while Lenze considered a different argument of the sigmoidal functions, aiming at describing some special kind of neural networks. Our results lead to *RBF neural networks* [120, 115, 122, 116, 98, 99] (see $(\widetilde{G}_N f)(x,y)$ in (2.10)), and reduce, essentially, to the one-dimensional case of Section 2.1 (where the nodes x_k should be replaced by the midpoints of the kth subinterval, as chosen in [48]).

Remark 2.11. Note that, the functions χ_{ij} in the arguments of the functions σ allow us to update the values of the weights of networks, on the basis of the input values received. This fact is quite natural since, as discussed in Chapter 1, neural networks should be able to learn from the experience, i.e., the weights must be chosen in base of the values that the network wants to represent.

Remark 2.12. Theorem 2.9 holds true also when instead of $\widetilde{G}_N f$, $N \in \mathbb{N}^+$, we have sums of the form

$$\begin{aligned}
&(\overline{G}_N f)(x,y) \\
&:= \sum_{j=1}^{N}\sum_{i=1}^{N} [f(x_i,y_j) - f(x_{i-1},y_j)]\ \sigma(w\,\widetilde{\chi}_{ij}(x,y)\ \left\|(x,y)-(t_{x_i},t_{y_j})\right\|_2) \\
&\qquad + \sum_{j=1}^{N} f(x_0,y_j)\,\sigma(w\,\widetilde{\chi}_{0j}(x,y)\ \left\|(x,y)-(t_{x_0},t_{y_j})\right\|_2),
\end{aligned} \tag{2.12}$$

where $(x,y) \in Q := [a,b]\times[c,d]$, $h := l/N$ with $l = b-a = d-c$, $x_i := a + h\,i$, $y_j := c + h\,j$ for $i,j = -1,0,1,...,N$ and $t_{x_i} := (x_{i-1}+x_i)/2$, $t_{y_j} := (y_{j-1}+y_j)/2$ for $i,j = 0,1,...,N$, and moreover

$$\widetilde{\chi}_{ij}(x,y) := \begin{cases} +1, & \text{for } x \geq x_i \text{ and } y \in (y_{j-1},y_j], \\ -1, & \text{otherwise.} \end{cases}$$

for $i = 0,...,N$ and $j = 2,...,N$, while

$$\widetilde{\chi}_{i1}(x,y) := \begin{cases} +1, & \text{for } x \geq x_j \text{ and } y \in [y_0,y_1], \\ -1, & \text{otherwise.} \end{cases}$$

for $i = 0, ..., N$, when $f \in C(Q)$. The proof is similar to that of Theorem 2.9, and again equation (2.12) contains RBF neural networks with sigmoidal activation function.

By simple modifications in the proof of Theorem 2.9, the following estimate for the approximation error, i.e., for $\left\| \widetilde{G}_N f - f \right\|_\infty$, can be obtained.

Theorem 2.13. *Let σ be a bounded sigmoidal function, and $f \in C^0(Q)$ be an Hölder-continuous function of order α, $0 < \alpha \leq 1$ and Hölder constant $L > 0$. Then, for every $N \in \mathbb{N}^+$, $N > 2$, there exists $\overline{w} > 0$ (depending on N and σ), such that for every $w \geq \overline{w}$, $\widetilde{G}_N f$ defined in (2.10) with w, is such that*

$$\left\| \widetilde{G}_N f - f \right\|_\infty < \frac{1}{N^\alpha} \left[L\, 2^{\alpha/2+1} (b-a)^\alpha + 2^{\alpha/2} (b-a)^\alpha \, \|\sigma\|_\infty + \|f\|_\infty \right].$$

Proof. Let $N \in \mathbb{N}^+$, $N > 2$, be fixed. Set $h := (b-a)/N = (d-c)/N$, $x_i := a + h\, i$, $y_j := c + h\, j$ for $i, j = -1, 0, 1, ..., N$ and $t_{x_i} := (x_{i-1} + x_i)/2$, $t_{y_j} := (y_{j-1} + y_j)/2$ for $i, j = 0, 1, ..., N$. Moreover, let $\overline{w} = \overline{w}(1/N^2, h/2) = \overline{w}(1/N^2) > 0$ obtained using Lemma 2.8 with $1/N^2$, $h/2 > 0$ and with the points (t_{x_i}, t_{y_j}), $i, j = 0, 1, ..., N$. Consider $\widetilde{G}_N f$ defined in (2.10) for $w \geq \overline{w}$ and let $(x, y) \in Q$ be fixed. Adopting the same notation and following the same steps as in the proof of Theorem 2.9, we obtain

$$\begin{aligned} \left| (\widetilde{G}_N f)(x,y) - f(x,y) \right| &\leq \left| (\widetilde{G}_N f)(x,y) - \mathcal{L}_{k\mu}(x,y) \right| \\ &+ \left| \mathcal{L}_{k\mu}(x,y) - f(x,y) \right| =: H_1 + H_2. \end{aligned}$$

Now, we can estimate H_1 and H_2 as in Theorem 2.9, obtaining

$$\begin{aligned} H_1 &< \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |f(x_i, y_j) - f(x_i, y_{j-1})| + \frac{1}{N^2} \sum_{i=1}^{N} \|f\|_\infty \\ &\leq L\, 2^{\alpha/2} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{(b-a)^\alpha}{N^\alpha} + \frac{1}{N} \|f\|_\infty \\ &\leq L\, 2^{\alpha/2} \frac{(b-a)^\alpha}{N^\alpha} + \frac{1}{N} \|f\|_\infty . \end{aligned}$$

Moreover,

$$H_2 < L\, 2^{\alpha/2} \frac{(b-a)^\alpha}{N^\alpha} + 2^{\alpha/2} \frac{(b-a)^\alpha}{N^\alpha} \, \|\sigma\|_\infty .$$

Hence,

$$\begin{aligned} & \left| (\widetilde{G}_N f)(x,y) - f(x,y) \right| \leq H_1 + H_2 < L\, 2^{\alpha/2+1} \frac{(b-a)^\alpha}{N^\alpha} \\ + \;& 2^{\alpha/2} \frac{(b-a)^\alpha}{N^\alpha} \, \|\sigma\|_\infty + \frac{1}{N} \|f\|_\infty \\ \leq \;& \frac{1}{N^\alpha} \left[L\, 2^{\alpha/2+1} \, |b-a|^\alpha + 2^{\alpha/2} (b-a)^\alpha \, \|\sigma\|_\infty + \|f\|_\infty \right], \end{aligned}$$

and since $(x,y) \in Q$ is arbitrary, it follows that

$$\left\| \widetilde{G}_N f - f \right\|_\infty < \frac{1}{N^\alpha} \left[L\, 2^{\alpha/2+1} (b-a)^\alpha + 2^{\alpha/2} (b-a)^\alpha \, \|\sigma\|_\infty + \|f\|_\infty \right].$$

□

Aiming at building a constructive theory also in $L^p(Q)$, $1 \leq p < \infty$, we prove the following theorem, which parallels that established in Section 2.2 for the univariate case.

Theorem 2.14. *Let σ be a bounded sigmoidal function, and $1 \leq p < \infty$ be fixed. For any $f \in C^0(Q)$, $Q \subset \mathbb{R}^2$, and $\varepsilon > 0$, there exist $N \in \mathbb{N}^+$ and $w > 0$ (depending on N and σ), such that $\widetilde{G}_N f$ defined in* (2.10) *with w, is such that*

$$\left\| \widetilde{G}_N f - f \right\|_{L^p(Q)} < \varepsilon.$$

Proof. Let $f \in C^0(Q)$ and $\varepsilon > 0$ be fixed. By Theorem 2.9, correspondingly to $\eta := \varepsilon / |Q|^{1/p}$ ($|Q|$ denoting the Lebesgue measure of Q), there exist $N \in \mathbb{N}^+$ and $w > 0$, depending on N, such that $\left\| \widetilde{G}_N f - f \right\|_\infty < \eta$. Therefore,

$$\begin{aligned} \left\| \widetilde{G}_N f - f \right\|_{L^p(Q)} &= \left(\int_Q \left| (\widetilde{G}_N f)(x,y) - f(x,y) \right|^p \, dxdy \right)^{1/p} \\ &< \left(\int_Q \eta^p \, dxdy \right)^{1/p} = \varepsilon. \end{aligned}$$

□

We are now able to prove an approximation theorem in $L^p(Q)$.

Theorem 2.15. *Let σ be a bounded sigmoidal function and let $f \in L^p(Q)$, $1 \leq p < \infty$, $Q \subset \mathbb{R}^2$ be fixed. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$ and $\widetilde{G}_N$, that is a linear combination of sigmoidal functions based on σ, having the bivariate form introduced in Theorem* 2.9, *such that*

$$\left\| \widetilde{G}_N - f \right\|_{L^p(Q)} < \varepsilon.$$

Proof. Define the function $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widetilde{f}(x,y) := \begin{cases} f(x,y), & \text{for } (x,y) \in Q, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Now, $\widetilde{f} \in L^p(\mathbb{R})$ and $\widetilde{f} = f$ on Q. Let $\{\rho_n\}_{n \in \mathbb{N}^+}$, $\rho_n : \mathbb{R}^2 \to \mathbb{R}$, be a sequence of (bivariate) mollifiers (ρ_n enjoys the same properties listed in Theorem 2.4 for the univariate case). Define the family of functions $\{f_n\}_{n \in \mathbb{N}}$ by

$$f_n(x,y) = (\rho_n * \widetilde{f})(x,y) := \int_{\mathbb{R}^2} \rho_n(x-z, y-t) \, \widetilde{f}(z,t) \, dz \, dt,$$

where $(x, y) \in \mathbb{R}^2$ and $*$ denotes, as usually, the convolution product. By general properties of the sequences of mollifiers and of convolution [26], it turns out that $f_n = \rho_n * \widetilde{f} \in C(\mathbb{R}^2)$ for every $n \in \mathbb{N}^+$, and $f_n \to \widetilde{f}$ in $L^p(\mathbb{R}^2)$ as $n \to \infty$. Let $\varepsilon > 0$ be fixed. Then, there exist $\overline{n} \in \mathbb{N}$ such that

$$\|f_n - f\|_{L^p(Q)} = \left\|f_n - \widetilde{f}\right\|_{L^p(Q)} \leq \left\|f_n - \widetilde{f}\right\|_{L^p(\mathbb{R}^2)} < \frac{\varepsilon}{2}$$

for every $n \geq \overline{n}$. Let now $n \geq \overline{n}$ be fixed. Since $f_n \in C(\mathbb{R}^2) \subset C(Q)$, it follows that, as a consequence of Theorem 2.14, correspondingly to $\varepsilon/2$ there exist $N \in \mathbb{N}^+$ and $w > 0$, depending on N, such that $\widetilde{G}_N f_n$ defined in (2.10) with w, is such that

$$\left\|\widetilde{G}_N f_n - f_n\right\|_{L^p(Q)} = \left\|\widetilde{G}_N(\rho_n * \widetilde{f}) - (\rho_n * \widetilde{f})\right\|_{L^p(Q)} < \frac{\varepsilon}{2}.$$

Therefore, we obtain by the previous estimates

$$\left\|\widetilde{G}_N f_n - f\right\|_{L^p(Q)} \leq \left\|\widetilde{G}_N f_n - f_n\right\|_{L^p(Q)} + \|f_n - f\|_{L^p(Q)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Setting $\widetilde{G}_N(x, y) := (\widetilde{G}_N f_n)(x, y)$ completes the proof. □

Examples of sequences of bivariate mollifiers (or, more generally, of multivariate mollifiers) are given in [26, 97], e.g. In particular, if we consider $\widetilde{\rho}(x_1, ..., x_n) = \rho(\|(x_1, ..., x_n)\|_2)$, $n \in \mathbb{N}^+$, the natural extension to the multivariate case of the function ρ defined in (2.4), we can build the sequence of mollifiers $\widetilde{\rho}_k(x_1, ..., x_n) := C\, k^n\, \widetilde{\rho}(kx_1, ..., kx_n)$, where

$$C := \left(\int_{\mathbb{R}^n} \widetilde{\rho}(x_1, ..., x_n)\, dx_1 ... \, dx_n\right)^{-1}.$$

2.5 Applications based on specific sigmoidal functions

As a first example of sigmoidal function we can consider the logistic-function σ_ℓ, which was already used in Remark 2.5 and in the previous chapter, defined by $\sigma_\ell(x) := (1 + e^{-x})^{-1}$, $x \in \mathbb{R}$. Using the logistic function above, we can see that, if $x_0, x_1, ..., x_M \in \mathbb{R}$, $M \in \mathbb{N}^+$, for every $N \in \mathbb{N}^+$, $N > 2$, and $h > 0$, there exists $\overline{w} := \frac{1}{h} \log(N-1) > 0$ such that, for every $w > \overline{w}$ and $k = 0, 1, ..., M$, we have

1. $|\sigma_\ell(w(x - x_k)) - 1| < \frac{1}{N}$, for every $x \in \mathbb{R}$ such that $x - x_k \geq h$;
2. $|\sigma_\ell(w(x - x_k))| < \frac{1}{N}$, for every $x \in \mathbb{R}$ such that $x - x_k \leq -h$.

In fact, let $N \in \mathbb{N}^+$, $N > 2$, be fixed. Then, being $0 < \sigma_\ell(x) < 1$, for every $x \in \mathbb{R}$,

$$|\sigma_\ell(x) - 1| = 1 - \frac{1}{1+e^{-x}} < \frac{1}{N} \quad \text{for} \quad x > \log(N-1),$$

and

$$|\sigma_\ell(x)| = \frac{1}{1+e^{-x}} < \frac{1}{N} \quad \text{for} \quad x < -\log(N-1).$$

Therefore, for every $w > \overline{w} := \frac{1}{h}\log(N-1)$ and for every $x \in \mathbb{R}$ with $x - x_k \geq h$, $k = 0, ..., M$, we have $w(x - x_k) > \overline{w}\, h = \log(N-1)$, hence $|\sigma_\ell(w(x-x_k)) - 1| < \frac{1}{N}$. Similarly, for every $w > \overline{w}$ and $x \in \mathbb{R}$ with $x - x_k \leq -h$, we have $|\sigma(w(x-x_k))| < \frac{1}{N}$.

The previous inequalities provide an estimate for $w > 0$ in case of approximations made by finite linear combination of sigmoidal functions based on the logistic function. Consequently, by the above estimates, using Theorem 2.2 and Theorem 2.6 we obtain the following.

Corollary 2.16. *Let $\sigma_\ell(x) = (1+e^{-x})^{-1}$ and $f \in \widehat{C}^{n+1}[a,b]$, $n \in \mathbb{N}^+$, be fixed. Denote by*

$$(G_N f)(x) := \sum_{k=1}^{N} [f(x_k) - f(x_{k-1})]\, \sigma_\ell\left(w\left(x - x_k\right)\right) + f(x_0)\, \sigma_\ell\left(w\left(x - x_{-1}\right)\right),$$

and

$$(G_N^j f)(x) := \sum_{k=1}^{N-j} \left(\Delta_k^j f - \Delta_{k-1}^j f\right) \sigma_\ell\left(w\left(x - x_k\right)\right) + \Delta_0^j f\, \sigma_\ell\left(w\left(x - x_{-1}\right)\right),$$

$x \in [a,b]$, $N \in \mathbb{N}^+$, $j = 1, ..., n$, with $w > \frac{N}{(b-a)} \log(N-1)$, and $x_k = a + k\, \frac{(b-a)}{N}$, $k = -1, 0, 1, ..., N$. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}^+$, $N > n+3$, such that

(i) $\|G_N f - f\|_\infty < \varepsilon$;

(ii) $\left\|G_N^j f - f^{(j)}\right\|_\infty < \varepsilon$, for every $j = 1, ..., n$,

for every $w > \frac{N}{(b-a)} \log(N-1)$.

Note that, the above estimates for the scaling parameter w hold also for the case of multivariate approximation with sigmoidal functions discussed in Section 2.4.

Other interesting examples of sigmoidal functions are provided by the Gompertz functions $\sigma_{\alpha\beta}$, defined by

$$\sigma_{\alpha\beta}(x) := e^{-\alpha\, e^{-\beta x}}, \quad x \in \mathbb{R},$$

with $\alpha,\ \beta > 0$. Note that, if $x_0, x_1, ..., x_M \in \mathbb{R}$, and $M \in \mathbb{N}^+$, for every $N \in \mathbb{N}^+$, $N > 2$ and $h > 0$, there exists

$$\overline{w} := \frac{1}{h\beta} \max\left\{ \left| \log\left(-\frac{1}{\alpha} \log\left(\frac{N-1}{N} \right) \right) \right|, \left| \log\left(\frac{1}{\alpha} \log(N) \right) \right| \right\}$$

such that, for every $w > \overline{w}$ and $k = 0, 1, ..., M$, we have

1. $|\sigma_{\alpha\beta}(w(x - x_k)) - 1| < \dfrac{1}{N}$, for every $x \in \mathbb{R}$ such that $x - x_k \geq h$,
2. $|\sigma_{\alpha\beta}(w(x - x_k))| < \dfrac{1}{N}$, for every $x \in \mathbb{R}$ such that $x - x_k \leq -h$.

In fact, let $N \in \mathbb{N}^+$, $N > 2$ be fixed. Then, being $0 < \sigma_{\alpha\beta}(x) < 1$ for every $x \in \mathbb{R}$, we have

$$|\sigma_{\alpha\beta}(x) - 1| = 1 - e^{-\alpha\, e^{-\beta x}} < \frac{1}{N} \quad \text{for } x > -\frac{1}{\beta} \log\left(-\frac{1}{\alpha} \log\left(\frac{N-1}{N} \right) \right),$$

and

$$|\sigma_{\alpha\beta}(x)| = e^{-\alpha\, e^{-\beta x}} < \frac{1}{N} \quad \text{for } x < -\frac{1}{\beta} \log\left(\frac{1}{\alpha} \log(N) \right).$$

Set

$$\overline{w} := \frac{1}{h\beta} \max\left\{ \left| \log\left(-\frac{1}{\alpha} \log\left(\frac{N-1}{N} \right) \right) \right|, \left| \log\left(\frac{1}{\alpha} \log(N) \right) \right| \right\}.$$

For every $w > \overline{w}$ and for every $x \in \mathbb{R}$ such that $x - x_k \geq h$, $k = 0, ..., M$, we have $w(x - x_k) > \overline{w}\, h \geq \left| \frac{1}{\beta} \log\left(-\frac{1}{\alpha} \log\left(\frac{N-1}{N} \right) \right) \right|$, then $|\sigma_{\alpha\beta}(w(x - x_k)) - 1| < \frac{1}{N}$. Similarly, for every $w > \overline{w}$ and $x \in \mathbb{R}$ such that $x - x_k \leq -h$, we have $|\sigma_{\alpha\beta}(w(x - x_k))| < \frac{1}{N}$. Consequently, by the above estimates, using Theorem 2.2 and Theorem 2.6 we obtain the following

Corollary 2.17. *Let* $\sigma_{\alpha\beta}(x) = e^{-\alpha\, e^{-\beta x}}$, $\alpha,\ \beta > 0$, *and* $f \in \widehat{C}^{n+1}[a, b]$, $n \in \mathbb{N}^+$, *be fixed. Define*

$$(G_N f)(x) := \sum_{k=1}^{N} [f(x_k) - f(x_{k-1})]\, \sigma_{\alpha\beta}(w(x - x_k)) + f(x_0)\, \sigma_{\alpha\beta}(w(x - x_{-1})),$$

and

$$(G_N^j f)(x) := \sum_{k=1}^{N-j} \left(\Delta_k^j f - \Delta_{k-1}^j f \right) \sigma_{\alpha\beta}(w(x - x_k)) + \Delta_0^j f\, \sigma_{\alpha\beta}(w(x - x_{-1})),$$

where

$$w > \overline{w} := \frac{N}{(b-a)\beta} \max\left\{ \left| \log\left(-\frac{1}{\alpha} \log\left(\frac{N-1}{N} \right) \right) \right|, \left| \log\left(\frac{1}{\alpha} \log(N) \right) \right| \right\},$$

$x \in [a,b]$, $N \in \mathbb{N}^+$, $j = 1,...,n$, $\delta > 0$, and $x_k = a + k\frac{(b-a)}{N}$, $k = -1,0,1,...,N$. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}^+$, $N > n+3$, such that

(i) $\|G_N f - f\|_\infty < \varepsilon$;

(ii) $\left\|G_N^j f - f^{(j)}\right\|_\infty < \varepsilon$, *for every* $j = 1,...,n$,

for every $w > \overline{w}$.

The estimates given in Corollary 2.17 for w, can be easily extended to the multivariate case.

Another example of sigmoidal function, very useful in the theory of neural networks, is the unit step function (or Heaviside function). In this case, all the linear combination of unit step function of the form considered in this chapter are independent of the choice of the parameter $w > 0$.

2.6 Some numerical examples

In this section, we give some examples to illustrate applications of approximations of functions by means of linear combination of sigmoidal functions.

Example 2.18. Consider the function

$$f(x) := (\cos^2 x + 2)\sin x + 2x + \frac{1}{8}x^2 + 4, \quad x \in \mathbb{R}. \tag{2.14}$$

We first construct approximations of f on the interval $[-5,5]$, obtained by the superposition of logistic sigmoidal functions σ_ℓ (Fig.s 2.1 and 2.2), for $N = 25$ and $N = 50$ and by the Gompertz function $\sigma_{\alpha\beta}$ with $\alpha = 8$ and $\beta = 0.5$ (Fig. 2.3), for $N = 50$. The choice of the parameter $w > 0$ was done according to Corollaries 2.16 and 2.17, i.e., $w = N^2/(b-a)$ in the case of $\sigma(x) = (1+e^{-x})^{-1}$, and $w = N^2/((b-a)\alpha\beta)$, in case of $\sigma_{\alpha\beta}(x)$ ($\alpha = 8$, $\beta = 0.5$).

Note that the choice of a *specific* sigmoidal function, σ, affects the quality of the approximation, which turns out to be better in case of $\sigma_\ell(x) = (1+e^{-x})^{-1}$. In addition, we have

$$f'(x) := \left(\cos^2 x + 2\right)\cos x - 2\sin^2 x\cos x + \frac{1}{4}x + 2 \quad x \in \mathbb{R},$$

and, as a consequence of Theorem 2.6, we can also approximate f' on the interval $[-5,5]$ by means of the functions $G_N^1 f$, $N \in \mathbb{N}^+$. The graphs of f' and its approximation obtained for $N = 25$ with logistic functions, $w = N^2/(b-a)$, are plotted in Fig. 2.4.

Note that the approximation obtained for f' is better compared to those for f, according to the results concerning the size of the approximation errors

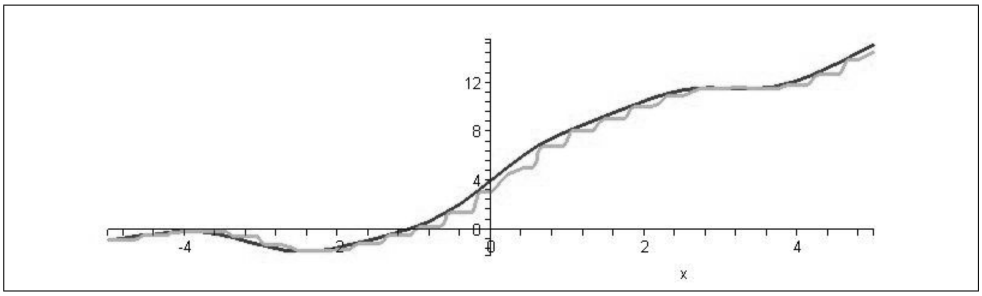

Figure 2.1: Approximation of f (black) of Example 2.18 by $G_N f$ (grey), $N = 25$, defined by σ_ℓ and with $w = N^2/(b-a)$. Here $\|G_N f - f\|_\infty / \|f\|_\infty \approx 1.08 \times 10^{-1}$.

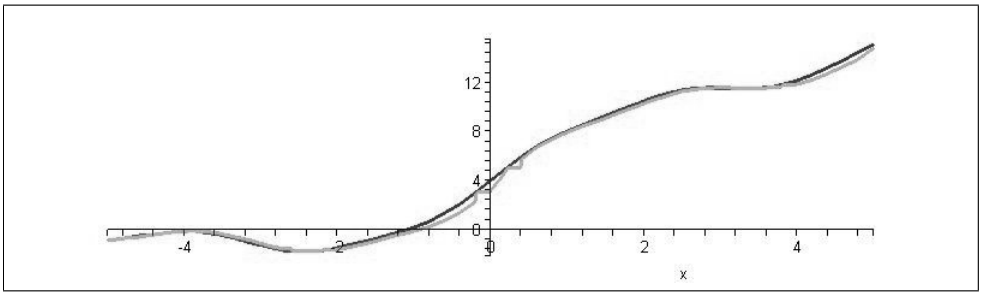

Figure 2.2: Approximation of f (black) of Example 2.18 by $G_N f$ (grey), $N = 50$, defined by σ_ℓ and with $w = N^2/(b-a)$. Here $\|G_N f - f\|_\infty / \|f\|_\infty \approx 5.88 \times 10^{-2}$.

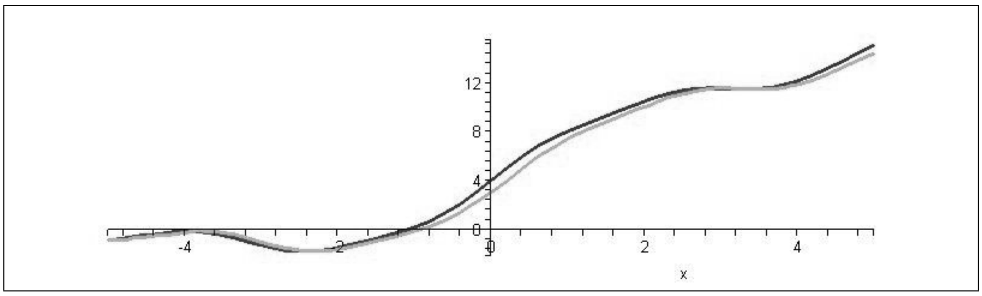

Figure 2.3: Approximation of f (black) of Example 2.18 by $G_N f$ (grey), $N = 50$, defined by $\sigma_{\alpha\beta}$, $\alpha = 8$, $\beta = 0.5$ and with $w = N^2/((b-a)\alpha\beta)$. Here $\|G_N f - f\|_\infty / \|f\|_\infty \approx 7.27 \times 10^{-2}$.

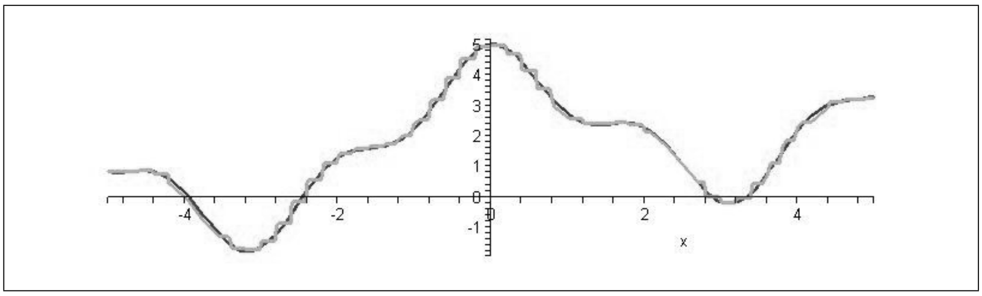

Figure 2.4: Approximation of f' (black) by $G_N^1 f$ (grey) of Example 2.18, $N = 25$ with σ_ℓ and with $w = N^2/(b-a)$. Here $\|G_N^1 f - f'\|_\infty / \|f'\|_\infty \approx 9.40 \times 10^{-2}$.

in Theorem 2.7. In fact, this states that such an error depends on the sup-norm (on the fixed interval $[a, b]$) of the function being approximated, and here $\|f\|_\infty \approx 15.13$, while $\|f'\|_\infty \approx 5$.

Now, we show an application of the constructive theory developed in Section 2.2 for L^p-functions.

Example 2.19. Let $g \in L^1(\mathbb{R})$ be defined by

$$g(x) := \begin{cases} \dfrac{4}{x^2 - 2}, & x < -2 \\[2ex] -3, & -2 \leq x < 0 \\[2ex] \dfrac{5}{2}, & 0 \leq x < 2 \\[2ex] \dfrac{3x + 2}{x^3 - 1}, & x \geq 2. \end{cases} \tag{2.15}$$

We consider approximations of g on the interval $[-5, 5]$ by finite linear combinations of sigmoidal functions of the form $G_N(\rho_n * \widetilde{g})$, where ρ_n are the mollifiers defined in (2.5), and $\widetilde{g}$ is the extension of g defined by

$$\widetilde{g}(x) := \begin{cases} g(x), & -5 \leq x < 5, \\ 0, & \text{otherwise}. \end{cases}$$

In Fig.s 2.5 and 2.6 such approximations, obtained with $G_N(\rho_n * \widetilde{g})$ for $n = 10$ and, respectively, $N = 50$ and $N = 100$, using logistic functions with $w = N^2/(b-a)$, are shown. In Fig. 2.7, the approximation obtained by the Gompertz function $\sigma_{\alpha\beta}(x)$, with $\alpha = 8$, $\beta = 0.5$, $n = 10$, $N = 100$ and with $w = N^2/((b-a)\alpha\beta)$, is shown.

We can observe that the approximation improves as N increases, as one would expect. In addition, the error made approximating g with the same

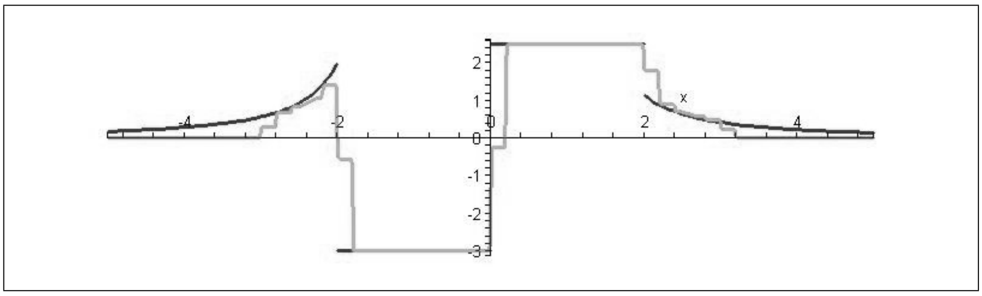

Figure 2.5: Approximation of the function g (black) of Example 2.19, by $G_N(\rho_n * \widetilde{g})$ (grey), $n = 10$, $N = 50$, $w = N^2/(b-a)$, and σ_ℓ.

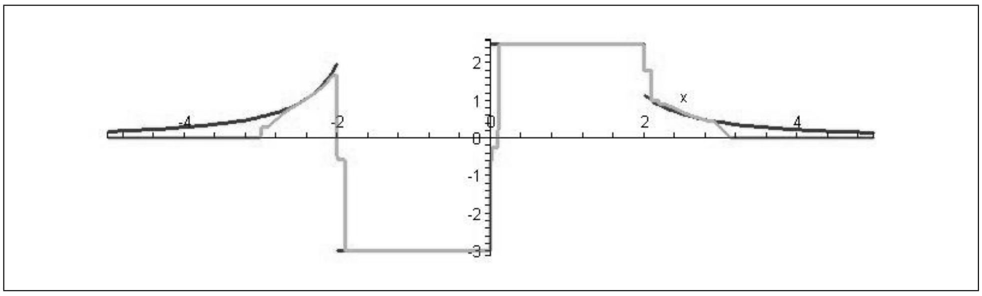

Figure 2.6: Approximation of the function g (black) of Example 2.19 by $G_N(\rho_n * \widetilde{g})$ (grey), $n = 10$, $N = 100$, $w = N^2/(b-a)$, and σ_ℓ.

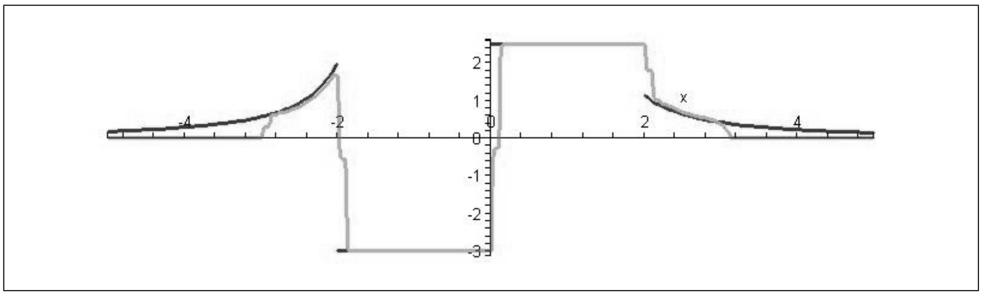

Figure 2.7: Approximation of the function g (black) of Example 2.19 by $G_N(\rho_n * \widetilde{g})$ (grey), $n = 10$, $N = 100$, $\sigma_{\alpha\beta}$, $\alpha = 8$, $\beta = 0.5$ and $w = N^2/((b-a)\alpha\beta)$.

values of N is larger than that made in the cases of regular functions, as were f and f' in the previous example.

Finally, here is an example of multivariate approximation.

Example 2.20. Consider the function of two variables, $h : [-4, 4] \times [-4, 4] \to \mathbb{R}$ (see Fig. 2.8), defined as

$$h(x, y) := \big(y^4 - 2y\big) \sin x - xy^3 + \frac{x}{3} + \frac{1}{x^{16}/30 + e^{-3|y|}/50 + 1/100}.$$

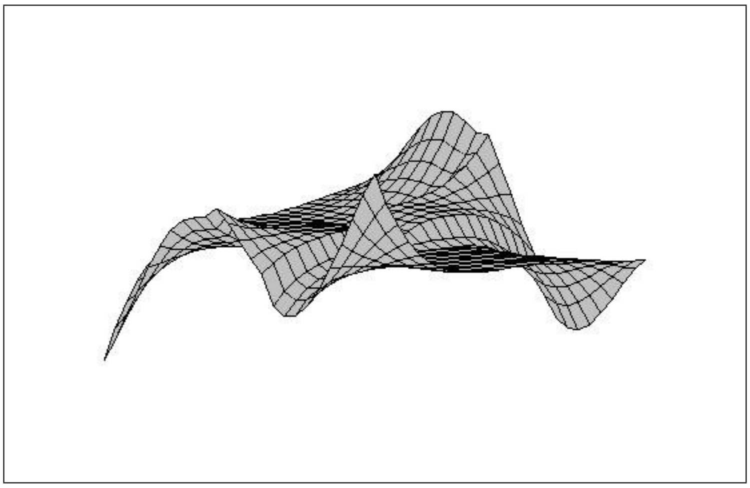

Figure 2.8: Graph of the bivariate function h of Example 2.20.

Fig.s 2.9 and 2.10 was obtained using the functions $\widetilde{G}_N h$ defined in (2.10) for $N = 25$ and $N = 50$, respectively, where σ_ℓ is the logistic function. Similar approximations can be obtained using sums $\overline{G}_N h$ like those in (2.12).

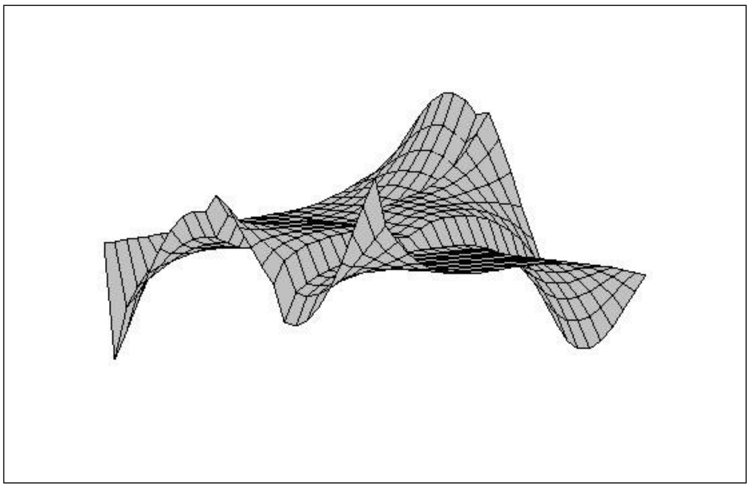

Figure 2.9: Approximation for the function h in Example 2.20 by $\widetilde{G}_N h$, for $N = 25$, $w = N^2/8$ and σ_ℓ.

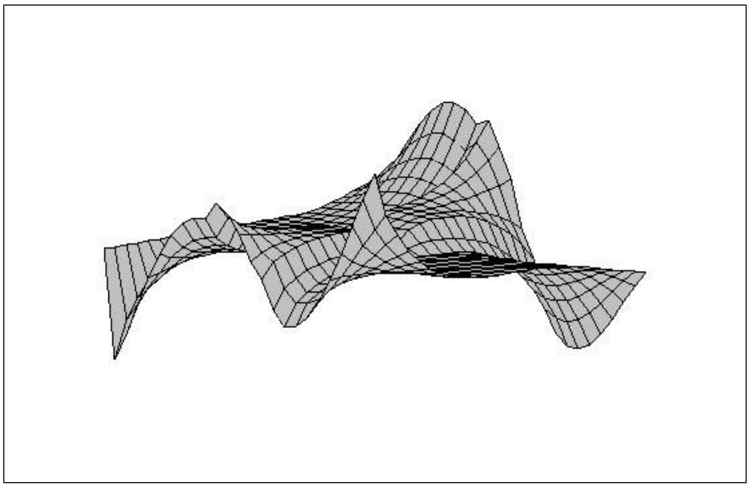

Figure 2.10: Approximation for the function h in Example 2.20 by $\widetilde{G}_N h$, for $N = 50$, $w = N^2/8$, σ_ℓ.

Chapter 3

Applications to the Numerical Solution of Volterra Integral Equations of the Second Kind

In this chapter, a numerical collocation method is developed for solving *linear* and *nonlinear* Volterra integral equations of the second kind. The method is based on the approximation of the (exact) solution by superposition of *sigmoidal* functions and allows us to solve a large class of integral equations having either continuous or L^p solutions. Special computational advantages are obtained using unit step functions, and analytical approximations of the solutions are also at hand. The numerical errors are discussed, and *a priori* as well as *a posteriori* estimates are derived for them. Numerical examples are given for the purpose of illustration.

The readers, can be found the present theory in [57].

We recall that, a *collocation solution* to a Volterra integral equation on an interval $[a, b]$, is an element from some finite-dimensional function space (the *collocation space*), which satisfies the equation on an appropriate finite subset of points in $[a, b]$. The latter is the set of *collocation points*, whose cardinality matches the dimension of the collocation spaces.

We will study linear as well as nonlinear Volterra integral equations of the second kind, of the form

$$y(t) = f(t) + \int_a^t K(t, s)\, y(t)\, ds, \quad t \in [a, b],$$

and

$$y(t) = f(t) + \int_a^t K(t, s; y(s))\, ds, \quad t \in [a, b],$$

where $f : [a, b] \to \mathbb{R}$ and the kernels K are sufficiently smooth. Collocation methods have been widely used to solve integral equations like those above.

The most popular of these methods are based on piecewise polynomial collocation spaces (see [14, 117, 27], e.g.). Other methods are based on wavelets or on Bernstein polynomials approximation (see [135, 107], e.g.).

A neural network approach for solving numerically integral equations of the Fredholm type and differential equations was used in [106, 37].

Here, the choice of the collocation spaces generated by the unit step functions, allows us to solve a large class of integral equations, having either continuous or L^p solutions, with some computational advantages. In fact, in case of linear equation, the method reduces merely to solve a linear *lower triangular* algebraic system. In the nonlinear case, the method instead leads to a nonlinear system, that can be always solved explicitly by means of a direct formula, without using any iteration (such as, e.g., Newton's methods). In both cases, an analytical form for the approximate solutions can be obtained and the numerical algorithm is very fast.

Moreover, for linear equations with a *convolution* kernel, $K(t, s) \equiv K(t-s)$, we can show that the square matrices of the related linear algebraic system, turn out to be lower triangular *Toeplitz* matrices.

For the sake of completeness, we recall that a matrix A is said to be a *Toeplitz matrix* if its entries are constants along all the principal diagonals.

Furthermore, under suitable conditions on K, some estimates for the condition number of such a matrices in the infinity norm, can also be derived.

3.1 Preliminary remarks

Before introducing our collocation method for solving second kind Volterra integral equations, we point out some preliminary observations concerning the approximation results showed in the previous chapter, that will be useful in this chapter.

Remark 3.1. The form of the coefficients in (2.1) of Theorem 2.2 is independent of the choice of the sigmoidal function σ. Therefore, one can provide various approximations of f using different sigmoidal functions, keeping the same coefficients. The scaling parameter $w > 0$ in (2.1) depends on N and σ.

Now we consider the special case of unit step (or Heaviside) functions, $H(t) := 1$ for $t \geq 0$, and $H(t) := 0$ for $t < 0$. In this case, the results established in Theorem 2.2 and in Theorem 2.4 hold true. In case of Theorem 2.2, sums of the form (2.1) reduce to

$$(G_N f)(t) := \sum_{k=1}^{N} [f(t_k) - f(t_{k-1})] H(t - t_k) + f(t_0) H(t - t_{-1}), \qquad (3.1)$$

$t \in \mathbb{R}$, where $f \in C^0[a, b]$, $h := (b-a)/N$, and $t_k := a+hk$, $k = -1, 0, 1, ..., N$. Note that in (3.1) $G_N f$ is independent of the scaling parameter, $w > 0$ and the same happens in Theorem 2.4 when applied to the case of Heaviside functions.

Remark 3.2. Set $H_k(t) := H(t - t_k)$, with $H_k : [a, b] \to \mathbb{R}$, $t_k := a + hk$, $h := (b - a)/N$, for $k = -1, 1, ..., N$, and

$$\Sigma_N := \text{ span}\{H_k : k = -1, 1, 2, ..., N\}.$$

Then, the vector function space Σ_N is an $N + 1$ dimensional space and the set $\{H_k : \; k = -1, 1, 2, ..., N\}$ is a basis for Σ_N. Indeed, it can be proved that the functions H_k's are linearly independent: if $\sum_{k=1}^{N} \alpha_k H_k + \alpha_0 H_{-1} \equiv 0$, i.e.,

$$\sum_{k=1}^{N} \alpha_k H(t - t_k) + \alpha_0 H(t - t_{-1}) = 0,$$

for every $t \in [a, b]$, we have, in particular,

$$\sum_{k=1}^{N} \alpha_k H(t_i - t_k) + \alpha_0 H(t_i - t_{-1}) = 0,$$

for every $i = 0, 1, ..., N$. Then, for $i = 0$ we have $\alpha_0 = 0$, and for $i > 0$ we obtain $\sum_{k=i}^{N} \alpha_k + \alpha_0 = 0$, hence, necessarily $\alpha_0 = \alpha_1 = ... = \alpha_N = 0$.

3.2 A collocation method for linear Volterra integral equations based on unit step functions

In this section, we describe a collocation method for solving linear Volterra integral equations of the second kind, of the form

$$y(t) = f(t) + \int_a^t K(t, s)\, y(s)\, ds, \quad t \in [a, b], \tag{3.2}$$

$a, b \in \mathbb{R}$, where the function $f : [a, b] \to \mathbb{R}$ and the kernel $K : D \to \mathbb{R}$, $D := \{(t, s) : \; a \le t, s \le b\}$, are sufficiently smooth.

Our method, based on unit step functions, consists of determining approximate solutions to equation (3.2), of the form $G_N y$ as defined in (3.1), i.e., $G_N y$ belonging to the collocation space Σ_N, $N \in N^+$. Set

$$(G_N y)(t) = \sum_{k=1}^{N} y_k H(t - t_k) + y_0 H(t - t_{-1}), \quad t \in [a, b], \tag{3.3}$$

where the coefficients $y_0, ..., y_N$, $N \in \mathbb{N}^+$, in (3.3) are unknowns, and $t_k := a + hk$, $k = -1, 0, ..., N$, $h := (b - a)/N$. Inserting $G_N y$ in place of the exact solution y in (3.2), we obtain

$$(G_N y)(t) = f(t) + \int_a^t K(t, s)\, (G_N y)(s)\, ds, \quad t \in [a, b],$$

and rearranging all terms, we have

$$\sum_{k=1}^{N} y_k \left[H(t-t_k) - \int_a^t K(t,s)\, H(s-t_k)\, ds \right]$$

$$+y_0 \left[H(t-t_{-1}) - \int_a^t K(t,s)\, H(s-t_{-1})\, ds \right] = f(t),$$

for every $t \in [a,b]$. If $\mathcal{C}_N := \{t_0, t_1, ..., t_N\}$ is the set of the *collocation points*, we can evaluate the equation above at such points. Set

$$m_{i0} := H(t_i - t_{-1}) - \int_a^{t_i} K(t_i, s)\, H(s-t_{-1})\, ds,$$

$$m_{ik} := H(t_i - t_k) - \int_a^{t_i} K(t_i, s)\, H(s-t_k)\, ds,$$

for $t_i \in \mathcal{C}_N$, $i = 0, ..., N$, and $k = 1, 2, ..., N$. We obtain the following linear algebraic system of $N+1$ equations,

$$\sum_{k=0}^{N} m_{ik}\, y_k = f(t_i), \tag{3.4}$$

for $i = 0, 1, ..., N$. Now, setting $M_N := (m_{ik})_{i,k=0,1,...,N}$, $Y_N := (y_0, y_1, ..., y_N)^t$, and $F_N := (f(x_0), f(x_1), ..., f(x_N))^t$, the linear system (3.4) can be written as $M_N Y_N = F_N$, $N \in \mathbb{N}^+$. Solving (3.4), we can determine $y_0, y_1, ..., y_N$, the coefficients providing an analytical representation of the solution, $y(t)$, of (3.2), as a superposition of unit step functions as in (3.3).

Remark 3.3. By Theorems 2.2 and 2.4, the collocation method based on unit step functions can be applied to *linear* Volterra integral equations with either regular solutions on $[a,b]$, or solutions in $L^p[a,b]$, $1 \le p < \infty$, such as equations with singular kernels.

We can now prove the following.

Theorem 3.4. *The collocation method for solving* (3.2)*, based on Heaviside functions, admits a unique solution. Moreover, the square matrix M_N of the linear system associated with the method is* lower triangular*, for every $N \in \mathbb{N}^+$.*

Proof. Let $i = 0, 1, ..., N$ be fixed. We have $H(t_i - t_k) = 0$ for every $k > i$, and $H(t_i - t_k) = 1$ for $k \le i$. Besides,

$$\int_a^{t_i} K(t_i, s)\, H(s-t_k)\, ds = 0, \tag{3.5}$$

for $k > i$, since $H(\cdot - t_k) = 0$ on $[a, t_i]$ (equation (3.5) also holds for $i = 0$, i.e., for $t_0 = a$). Furthermore,

$$\int_a^{t_i} K(t_i, s)\, H(s - t_{-1})\, ds = \int_a^{t_i} K(t_i, s)\, ds,$$

and,

$$\int_a^{t_i} K(t_i, s)\, H(s - t_k)\, ds = \int_{t_k}^{t_i} K(t_i, s)\, ds,$$

for $k \leq i$, with $k \neq 0$, since $H(\cdot - t_k) = 0$ on $[a, t_k)$ and $H(\cdot - t_k) = 1$ on $[t_k, t_i]$. Hence, we obtain

$$m_{ik} := \begin{cases} 0, & \text{for } k > i, \\ 1, & \text{for } k = i, \\ 1 - \int_{t_k}^{t_i} K(t_i, s)\, ds, & \text{for } k < i, \end{cases} \tag{3.6}$$

for $i, k = 0, 1, ..., N$. Then, the $(N+1) \times (N+1)$ matrix M_N of our method is lower triangular, for every $N \in \mathbb{N}^+$, i.e.,

$$M_N := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{10} & 1 & 0 & \cdots & 0 \\ m_{20} & m_{21} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ m_{N0} & m_{N1} & \cdots & m_{NN-1} & 1 \end{bmatrix} \tag{3.7}$$

Hence, $\det(M_N) = 1$, and the linear system $M_N Y_N = F_N$ admits a unique solution, for every $N \in \mathbb{N}^+$. □

Note that, in Theorem 3.4, an integrability assumption on the kernel K of the integral equation in (3.2) is needed. The entries of M_N of the form $1 - \int_{t_k}^{t_i} K(t_i, s)\, ds$ can be evaluated by exact (analytical) integration, in many instances, or, more generally, upon numerical quadratures.

The method based on unit step functions can be implemented easily, and, in addition, it is definitely characterized by an extremely low computational cost.

In the special (but noteworthy) case of integral equations of the *convolution* type, the linear Volterra integral equations of the second kind,

$$y(t) = f(t) + \int_a^t K(t - s)\, y(s)\, ds, \quad t \in [a, b], \tag{3.8}$$

we have the following

Corollary 3.5. *The collocation method for solving* (3.8)*, based on unit step functions, admits a unique solution. Moreover, the real-valued matrix* M_N *is a* lower triangular Toeplitz *matrix, for every* $N \in \mathbb{N}^+$.

Proof. By Theorem 3.4, M_N is lower triangular with $\det(M_N) = 1$, for every $N \in \mathbb{N}^+$, then the method admits a unique solution. Moreover, if $h := (b-a)/N$ is the step-size separating the collocation points t_i, $i = 0, 1, ..., N$, we obtain changing variable, $s = z - h$,

$$\int_{t_k}^{t_i} K(t_i - s)\, ds = \int_{t_k+h}^{t_i+h} K(t_i + h - z)\, dz = \int_{t_{k+1}}^{t_{i+1}} K(t_{i+1} - z)\, dz.$$

Then, $m_{i,k} = m_{i+1,k+1}$ for every $k < i$ (see (3.6)), and thus M_N is seen to be constant along all its diagonals. This means that M_N is a Toeplitz matrix, and it can be represented as

$$M_N := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_1 & 1 & 0 & \ddots & 0 \\ m_2 & m_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ m_N & \cdots & m_2 & m_1 & 1 \end{bmatrix}, \tag{3.9}$$

where

$$m_i := 1 - \int_a^{t_i} K(t_i - s)\, dt,$$

for every $i = 1, 2, ..., N$. □

In the case of equations with convolution kernel, the required computational cost is much lower than in case of general kernels. In fact, by Corollary 3.5, for every $N \in \mathbb{N}^+$, the linear systems to be solved is characterized by lower triangular Toeplitz matrices, and thus to compute all entries of M_N it suffices to evaluate only the N terms $m_1, ..., m_N$.

Remark 3.6. The approach proposed above can also be useful to determine approximate solutions to initial value problems (IVP) for ordinary differential equations. In fact, generally speaking, every linear IVP, e.g., of the second order, say

$$y'' + A(t)y' + B(t)y = g(t), \quad y(a) = c_1,\ y'(a) = c_2,$$

where A, B and g are sufficiently smooth functions, is equivalent to a linear Volterra integral equation of the second kind (see [79], e.g.), like that in (3.2), where

$$f(t) := \int_a^t (t-s)\, g(s)\, ds + (t-a)[c_1 A(a) + c_2] + c_1,$$

and

$$K(t,s) := (s-t)[B(s) - A'(s)] - A(s).$$

At this point, we introduce some notation. Given the kernel K, we define

$$\mathcal{K}(t) := \int_a^t K(t,s)\,ds, \quad t \in [a,b]. \tag{3.10}$$

Under suitable conditions on $\mathcal{K}$, we can obtain some estimates for $\kappa(M_N)$, the *condition number* of M_N in the infinity norm, in case of integral equations of the convolution type, (3.8). Recall that for every nonsingular real-valued matrix $A := (a_{i,j})_{i,j=0,1,\ldots,N}$,

$$\kappa(A) := \|A\|_\infty \|A^{-1}\|_\infty,$$

where $\|A\|_\infty := \max_{i=0,1,\ldots,N} \sum_{j=0}^N |a_{i,j}|$.

We can establish the following.

Theorem 3.7. *Let* (3.8) *be a Volterra integral equation with convolution kernel,* K*. Let* $\mathcal{K}$ *be defined in* (3.10)*, and such that*
(i) $0 \le \mathcal{K}(t) \le 1$*, for every* $t \in [a,b]$*;*
(ii) $\mathcal{K}$ *is non decreasing.*

Then, the square matrix M_N*,* $N \in \mathbb{N}^+$*, obtained applying our collocation method to* (3.8)*, enjoys the property*

$$\kappa(M_N) \le \begin{cases} \dfrac{2}{1-\mathcal{K}(b)} \left(1 - \mathcal{K}(b)^{\left[\frac{N}{2}\right]+1}\right) \left(N(1-\mathcal{K}(t_1)) + 1\right), & \mathcal{K}(b) < 1, \\ \\ 2\left(\left[\frac{N}{2}\right]+1\right)\left(N(1-\mathcal{K}(t_1)) + 1\right), & \mathcal{K}(b) = 1, \end{cases}$$

where $[\cdot]$ *means taking the integer part, for every* $N \in \mathbb{N}^+$*. In particular, if* $\mathcal{K}(b) < 1$*, we have*

$$\kappa(M_N) < \frac{2}{1-\mathcal{K}(b)} \left(N(1-\mathcal{K}(t_1)) + 1\right).$$

Proof. By Corollary 3.5, M_N is a lower triangular Toeplitz matrix of the form in (3.9). By (i) and (ii), we have $0 \le \mathcal{K}(t_0) \le \mathcal{K}(t_1) \le \ldots \le \mathcal{K}(t_N) \le 1$, where t_i, $i = 0,1,\ldots,N$,are the collocation points, and then, the elements of M_N satisfy the inequalities

$$1 \ge m_1 \ge m_2 \ge \ldots \ge m_N \ge 0.$$

Hence, by well-known results concerning lower triangular Toeplitz matrices with non-increasing monotonic entries (see [22], Theorem 1.1, and [132]), we obtain the bounds

$$\|M_N^{-1}\|_\infty \le \begin{cases} \dfrac{2}{1-\mathcal{K}(b)} \left(1 - \mathcal{K}(b)^{\left[\frac{N}{2}\right]+1}\right), & \mathcal{K}(b) < 1, \\ \\ 2\left(\left[\frac{N}{2}\right]+1\right), & \mathcal{K}(b) = 1. \end{cases}$$

Now, since $\|M_N\|_\infty \leq N(1-\mathcal{K}(t_1))+1$, and $\kappa(M_N) := \|M_N\|_\infty \|M_N^{-1}\|_\infty$, the proof of the first part of the theorem is complete. Moreover, if we note that in case of $\mathcal{K}(b) < 1$, there is $\left(1-\mathcal{K}(b)^{\left[\frac{N}{2}\right]+1}\right) < 1$, we have

$$\kappa(M_N) < \frac{2}{1-\mathcal{K}(b)} \left(N(1-\mathcal{K}(t_1))+1\right).$$

□

3.3 A collocation method for solving nonlinear Volterra integral equations

In this section, we use the collocation method based on unit step functions to solve general nonlinear Volterra integral equations of the second kind, like

$$y(t) = f(t) + \int_a^t K(t,s;y(s))\,ds, \quad t \in [a,b], \tag{3.11}$$

$a, b \in \mathbb{R}$, where the function $f : [a,b] \to \mathbb{R}$ and the kernel $K : \Omega \to \mathbb{R}$, $\Omega := D \times \mathbb{R}$, are sufficiently smooth. Proceeding as in Section 3.2, we can approximate the solution to (3.11) on the interval $[a,b]$ by means of (3.3), i.e.,

$$(G_N y)(t) := \sum_{k=1}^{N} y_k H(t-t_k) + y_0 H(t-t_{-1}),$$

where H is the Heaviside function, $t_k := a + hk$ with $h := (b-a)/N$, $k = -1, 0, 1, ..., N$, and the coefficients $y_0, y_1, ..., y_N$ are unknown. Inserting $G_N y$ for y in (3.11), we obtain

$$(G_N y)(t) = f(t) + \int_a^t K(t,s;(G_N y)(s))\,ds, \quad t \in [a,b].$$

Again, to obtain an approximate solution to (3.11) in the form of a superposition of unit step functions, we should determine the unknown coefficients $y_0, y_1, ..., y_N$. Given the set $\mathcal{C}_N := \{t_0, t_1, ..., t_N\}$ of collocation points, we evaluate the equation at such points as in the linear case. Now we obtain the system of $N+1$ nonlinear equations

$$\sum_{k=1}^{N} y_k H(t_i - t_k) + y_0 H(t_i - t_{-1})$$

$$= f(t_i) + \int_a^{t_i} K\left(t_i, s; \sum_{k=1}^{N} y_k H(s-t_k) + y_0 H(s-t_{-1})\right) ds, \quad i = 0, 1, ..., N. \tag{3.12}$$

It is easy to check that, for $i = 0$, (3.12) reduces to $y_0 = f(t_0)$. Let now $i > 0$ be fixed. In this case, $H(t_i - t_k) = 1$ for every $k = 0, ..., i$ and $H(t_i - t_k) = 0$ for $k > i$. Moreover,

$$
\begin{aligned}
&\int_a^{t_i} K\left(t_i, s; \sum_{k=1}^{N} y_k H(s - t_k) + y_0 H(s - t_{-1})\right) dt \\
= & \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s; \sum_{k=1}^{N} y_k H(s - t_k) + y_0\right) ds \\
= & \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s; \sum_{k=0}^{\nu-1} y_k\right) ds,
\end{aligned}
$$

since, for every $\nu = 1, ..., i$, there is $H(\cdot - t_k) = 1$ on $[t_{\nu-1}, t_\nu]$ for $k = 0, ..., \nu - 1$ and $H(\cdot - t_k) = 0$ on $[t_{\nu-1}, t_\nu]$ for $k \geq \nu$. Therefore, (3.12) reduces to the nonlinear system

$$
y_0 = f(t_0),
$$

$$
\sum_{k=0}^{i} y_k = f(t_i) + \left[\sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s; \sum_{k=0}^{\nu-1} y_k\right) ds\right], \quad i = 1, ..., N,
$$

which admits a unique solution that can be given by the formula

$$
\begin{cases}
y_0 = f(t_0), \\
y_i = f(t_i) + \displaystyle\sum_{\nu=1}^{i} \left[\int_{t_{\nu-1}}^{t_\nu} K(t_i, s; \sum_{k=0}^{\nu-1} y_k)\, ds\right] - \sum_{k=0}^{i-1} y_k, \quad i = 1, ..., N,
\end{cases}
\tag{3.13}
$$

for every $N \in \mathbb{N}^+$.

We stress that the nonlinear system in (3.12) can be solved explicitly, and its solution does not required any iterative method (such as the Newton's method, e.g.). Note also that (3.13) provides an algorithm for solving a large class of nonlinear Volterra integral equations. Only an integrability assumption on the kernel $K(t, \cdot; y)$ on $[a, t]$ for every $y \in \mathbb{R}$ and $t \in [a, b]$ is required.

In the special case of Volterra-Hammerstein integral equations of the second kind, i.e., when the kernel is of the form $K(t, s; y(s)) := \widetilde{K}(t, s) G(y(s))$, where G and $\widetilde{K}$ are sufficiently smooth functions, (3.13) reduces to

$$
\begin{cases}
y_0 = f(t_0), \\
y_i = f(t_i) + \displaystyle\sum_{\nu=1}^{i} \left[G\left(\sum_{k=0}^{\nu-1} y_k\right) \int_{t_{\nu-1}}^{t_\nu} \widetilde{K}(t_i, s)\, ds\right] - \sum_{k=0}^{i-1} y_k, \quad i = 1, ..., N.
\end{cases}
$$

Also in this case, the integrals $\int_{t_{\nu-1}}^{t_\nu} \widetilde{K}(t_i, s)\, ds$ can be evaluated by an exact (analytical) integration, in many specific instances, or, more generally by numerical quadrature. As in case of linear integral equations, our collocation method can be applied to nonlinear Volterra integral equations having either regular or $L^p[a,b]$ solutions (with $1 \le p < \infty$), in view of the results on approximation through superposition of sigmoidal functions discussed in Chapter 2.

3.4 Error analysis

In this section, we discuss the various sources of errors which affect our method. Our numerical method is based on using for the sought solution y its approximate representation in terms of bounded sigmoidal functions, (see $G_N f$ in (2.1)). In view of Theorem 2.2 (and also Theorem 2.4), we can write

$$y(t) = (G_N y)(t) + e_N(t), \tag{3.14}$$

provided that $y(t)$ is continuous (or in $L^p[a,b]$), where the error term, $e_N(t)$, can be estimated uniformly (or in L^p-norm), for every $\varepsilon > 0$, as

$$\|e_N(t)\|_\infty < \varepsilon, \tag{3.15}$$

for a suitable $N \in \mathbb{N}^+$ (and $w > 0$, depending on N).

Clearly, (3.15) holds when $G_N y$ is written with the coefficients computed in Chapter 2. In the following theorem, we establish an estimate for e_N, defined in (3.14), when $G_N y$ is represented in terms of unit step functions (recall that in this case $G_N y$ is independent of $w > 0$), and with the coefficients y_k determined applying our collocation method to the nonlinear integral equation in (3.11).

Theorem 3.8. *Let* (3.11) *be a given nonlinear Volterra integral equation of the second kind. Suppose that the function f is Lipschitz continuous, with Lipschitz constant $L_f > 0$, and that the kernel $K \in C^0(\Omega)$ satisfies the following conditions:*

(i) there exist $L_1 > 0$ and $L_2 > 0$ such that

$$|K(t_1, s, y) - K(t_2, s, y)| \le L_1 |t_1 - t_2|, \quad \textit{for all } (t_1, s, y),\ (t_2, s, y) \in \Omega,$$

$$|K(t, s, y_1) - K(t, s, y_2)| \le L_2 |y_1 - y_2|, \quad \textit{for all } (t, s, y_1),\ (t, s, y_2) \in \Omega;$$

(ii) for every bounded function $y : [a,b] \to \mathbb{R}$, there exists $C = C(y) > 0$ such that

$$|K(t, s, y(s))| \le C, \quad \textit{for all } \ (t, s) \in D.$$

Then, for every $N \in \mathbb{N}^+$, we have

$$|e_N(t)| \le \frac{(b-a)}{N} [L_f + L_1(b-a) + C]\, e^{L_2(t-a)}, \quad t \in [a,b],$$

where e_N *is defined in* (3.14)*,* $G_N y$ *being represented in terms of* unit step *functions and with coefficients,* y_k*, determined applying our collocation method to* (3.11)*.*

Proof. Let $N \in \mathbb{N}^+$ and $t \in [a,b]$ be fixed. Define $j := \max\{i : t_i \leq t, i = 0,1,...,N\}$, where $t_i := a + ih$, $h := (b-a)/N$, $i = 0,1,...,N$ are the collocation points. We can write

$$|e_N(t)| = |y(t) - (G_N y)(t)| \leq |y(t) - y(t_j)| + |y(t_j) - (G_N y)(t)|.$$

Now, observing that $(G_N y)(t) = (G_N y)(t_j)$ (since $G_N y$ is written in terms of unit step functions), we obtain

$$\begin{aligned}
|e_N(t)| &\leq |y(t) - y(t_j)| + |y(t_j) - (G_N y)(t_j)| \\
&= \left| f(t) + \int_a^t K(t,s,y(s))\,ds - f(t_j) - \int_a^{t_j} K(t_j,s,y(s))\,ds \right| \\
&+ \left| \int_a^{t_j} K(t_j,s,y(s))\,ds - \int_a^{t_j} K(t_j,s,(G_N y)(s))\,ds \right| \\
&\leq |f(t) - f(t_j)| + \int_a^{t_j} |K(t,s,y(s)) - K(t_j,s,y(s))|\,ds \\
&+ \int_{t_j}^t |K(t,s,y(s))|\,ds + \int_a^{t_j} |K(t_j,s,y(s)) - K(t_j,s,(G_N y)(s))|\,ds.
\end{aligned}$$

Using condition (i), (ii) and the Lipschitz continuity of f, we have

$$\begin{aligned}
|e_N(t)| &\leq L_f(t-t_j) + L_1(t-t_j)(t_j - a) + C(t-t_j) \\
&+ L_2 \int_a^{t_j} |y(s) - (G_N y)(s)|\,ds \\
&\leq (t-t_j)[L_f + L_1(b-a) + C] + L_2 \int_a^{t_j} |e_N(s)|\,ds \\
&\leq \frac{(b-a)}{N}[L_f + L_1(b-a) + C] + L_2 \int_a^t |e_N(s)|\,ds.
\end{aligned}$$

The inequality above holds for every $t \in [a,b]$, and then, by Gronwall's lemma we obtain

$$|e_N(t)| \leq \frac{(b-a)}{N}(L_f + L_1(b-a) + C)[1 + L_2 \int_a^t e^{L_2(t-s)}\,ds],$$

for every $t \in [a,b]$. Being $L_2 \int_a^t e^{L_2(t-s)}\,ds = e^{L_2(t-a)} - 1$, it follows that

$$|e_N(t)| \leq \frac{(b-a)}{N}(L_f + L_1(b-a) + C)\,e^{L_2(t-a)}, \quad t \in [a,b].$$

□

Theorem 3.8 provides an *a priori* estimate for the approximation errors of our collocation method *applied to* the nonlinear equations in (3.11). In addition, we can infer from Theorem 3.8 that $\|e_N\|_\infty \to 0$ as $N \to +\infty$, and then that the sequence approximations for the solution to (3.11), as determined by our method, converges uniformly to the (exact) solution y.

Remark 3.9. In Theorem 3.8, the Lipschitz condition on the kernel K, with respect to y, is global. Hence, Theorem 3.8 *cannot* cover, e.g., the case of nonlinear equations like that in (3.11) with kernels of the form $K(t, s; y) = \widetilde{K}(t, s)\, y^p$, with $p > 1$.

Clearly, in the special case $K(t, s, y(s)) = \widetilde{K}(t, s)\, y(s)$, equation (3.11) reduces to the linear equation (3.2) with kernel $\widetilde{K}$. Therefore, if f is Lipschitz continuous with Lipschitz constant L_f and $\widetilde{K} \in C^0(D)$ is such that

$$|\widetilde{K}(t_1, s) - \widetilde{K}(t_2, s)| \leq L_1 |t_1 - t_2|,$$

for all (t_1, s), $(t_2, s) \in D$ and some positive constants L_1, we infer from Theorem 3.8 that

$$|e_N(t)| \leq \frac{(b-a)}{N} (L_f + L_1(b-a) + M\|y\|_\infty)\, e^{M(t-a)}, \quad t \in [a, b], \tag{3.16}$$

where y is the (exact) continuous solution to (3.2) with kernel $\widetilde{K}$ and $M := \max_{(t,s)\in D} |\widetilde{K}(t, s)|$, for every $N \in \mathbb{N}^+$. Also in this case we obtain that $\|e_N\|_\infty \to 0$ as $N \to +\infty$. Now, being $\|y\|_\infty = \|G_N y + e_N\|_\infty$, (3.16) becomes

$$\|e_N\|_\infty \leq \frac{(b-a)}{N} (L_f + L_1(b-a) + M\|G_N y\|_\infty + M\|e_N\|_\infty)\, e^{M(b-a)},$$

and we have

$$\|e_N\|_\infty \left(1 - M\frac{(b-a)}{N} e^{M(b-a)}\right) \leq \frac{(b-a)}{N} (L_f + L_1(b-a) + M\|G_N y\|_\infty)\, e^{M(b-a)}.$$

Now, for N sufficiently large, we have $M\frac{(b-a)}{N} e^{M(b-a)} < 1$, and hence

$$\|e_N\|_\infty \leq \frac{(b-a)}{N - M(b-a)\, e^{M(b-a)}} (L_f + L_1(b-a) + M\|G_N y\|_\infty)\, e^{M(b-a)}, \tag{3.17}$$

which represents an *a posteriori* estimate for the approximation error made in case of the linear equations (3.2), when $G_N y$ is given in terms of unit step functions.

Now we can use the estimate provided by Theorem 3.8 *to derive* another interesting estimate for $e_N(t)$. This can be obtained when the approximate solution is expressed by a superposition of general bounded sigmoidal functions. We denote with $G_N^H y$ the collocation solution to (3.11) represented in terms of unit step functions, and with $G_N^\sigma y$ the solution obtained by superposing general bounded sigmoidal functions. Note that, the collocation solution can also be represented by $G_N^\sigma y$, in view of Remark 3.1. Setting

$$(G_N^\sigma y)(t) = (G_N^H y)(t) + s_N(t), \quad t \in [a,b],$$

we obtain

$$\begin{aligned} & |s_N(t)| \leq |(G_N^\sigma y)(t) - (G_N^H y)(t)| \\ = \ & |\sum_{k=1}^{N} y_k[\sigma(w(t-t_k)) - H(t-t_k)] + y_0[\sigma(w(t-t_{-1})) - 1]|. \end{aligned}$$

Observing that

$$\sigma(w(t-t_k)) - H(t-t_k) := \begin{cases} \sigma(w(t-t_k)) - 1, & t \geq t_k \\ \sigma(w(t-t_k)), & t < t_k, \end{cases}$$

for every $k = 1, ..., N$, we have

$$|s_N(t)| \leq \sum_{k:t_k \leq t} |y_k||\sigma(w(t-t_k)) - 1| + \sum_{k:t_k > t} |y_k||\sigma(w(t-t_k))|.$$

Therefore, under the conditions of Theorem 3.8, we obtain

$$\begin{aligned} |e_N(t)| &= |y(t) - (G_N^\sigma y)(t)| = |y(t) - (G_N^H y)(t) - s_N(t)| \\ &\leq |y(t) - (G_N^H y)(t)| + |s_N(t)| \\ &\leq \frac{(b-a)}{N}\left[L_f + L_1(b-a) + C\right] e^{L_2(t-a)} \\ &+ \sum_{k:t_k \leq t} |y_k||\sigma(w(t-t_k)) - 1| \\ &+ \sum_{k:t_k > t} |y_k||\sigma(w(t-t_k))|, \end{aligned} \tag{3.18}$$

for every $t \in [a,b]$. Now, we know by Definition 1.1 that for $w > 0$ sufficiently large, the terms $|\sigma(w(t-t_k)) - 1|$ and $|\sigma(w(t-t_k))|$ in (3.18) are small.

In (3.18), a bound is given for the approximation error, in terms of the y_k's. These have been computed in our collocation method, and thus this can be view as an *a posteriori* estimate. Note that, in general, from (3.18) we cannot infer that $G_N^\sigma y$ converges to y.

Using (3.16), considerations similar to those above can be made, to obtain an a posteriori bound for e_N in the case of linear equations, where the approximate solution is given by $G_N^\sigma y$.

3.5 Numerical examples for Volterra integral equations of the second kind

In this section, we apply the method developed above, in this chapter, to solve numerically some linear as well as nonlinear Volterra integral equations.

3.5.1 Linear Volterra integral equations of the second kind

Here are some examples of linear Volterra equations of the second kind.

Example 3.10. Consider the Volterra equation (3.2) with

$$K(t,s) = e^{ts}, \quad f(t) = e^{2t} - \frac{1}{t+2}(e^{t(t+2)} - e^{-(t+2)}),$$

on the interval $[a,b] = [-1,1]$. Its solution is $y(t) = e^{2t}$.

We test our collocation method based on step functions for solving this equation. By Remark 3.1, we know that various approximation of $y(t)$ can be obtained using different sigmoidal functions, using the same coefficients. We denoted by

$$\varepsilon_N^i := \frac{\|G_N^i y - y\|_\infty}{\|y\|_\infty}, \quad i = 1,2,3, \tag{3.19}$$

where y is the exact solution of the integral equation and $G_N^i y$, $i = 1,2,3$, is its approximation obtained as a superposition of sigmoidal functions, of the Heaviside, logistic and Gompertz (with $\alpha = 0.85$ and $\beta = 0.1$) type, respectively. The $G_N^i y$'s are all obtained evaluating the coefficients $y_0, y_1, ..., y_N$, solution of the linear system $M_N Y_N = F_N$. In the cases of $G_N^2 y$ and $G_N^3 y$, the scaling parameter w was chosen accordingly to Corollaries 2.16 and 2.17, respectively, yielding $w = N^2/(b-a)$ for $G_N^2 y$, and $w = N^2/[(b-a)\alpha\beta]$ for $G_N^3 y$. In Table 3.1, the relative errors ε_N^i are shown. The condition numbers in the infinity norm of the matrices M_N, say $\kappa(M_N)$, are for instance, $\kappa(M_{10}) \approx 39.84$, $\kappa(M_{50}) \approx 125.73$, $\kappa(M_{500}) \approx 816,39$. In Fig.s 3.1 and 3.2, the approximate solutions $G_N^1 y$, $G_N^2 y$ for $N = 20$ and $N = 60$ are plotted, respectively.

N	ε_N^1	ε_N^2	ε_N^3
10	4.45×10^{-1}	3.80×10^{-1}	4.15×10^{-1}
20	2.73×10^{-1}	2.59×10^{-1}	2.73×10^{-1}
30	1.95×10^{-1}	1.94×10^{-1}	1.95×10^{-1}
40	1.50×10^{-1}	1.50×10^{-1}	1.50×10^{-1}
50	1.20×10^{-1}	1.20×10^{-1}	1.20×10^{-1}
60	9.98×10^{-2}	9.98×10^{-2}	9.98×10^{-2}
500	1.16×10^{-2}	1.21×10^{-2}	1.26×10^{-2}
1000	4.10×10^{-3}	6.10×10^{-3}	6.40×10^{-2}

Table 3.1: Numerical results for Example 3.10.

Example 3.11. Consider the following initial value problem of the second order,

$$y'' - \frac{t}{2}y' + y = \frac{t}{2}\sin t, \quad y(0) = -1, \; y'(0) = 0.$$

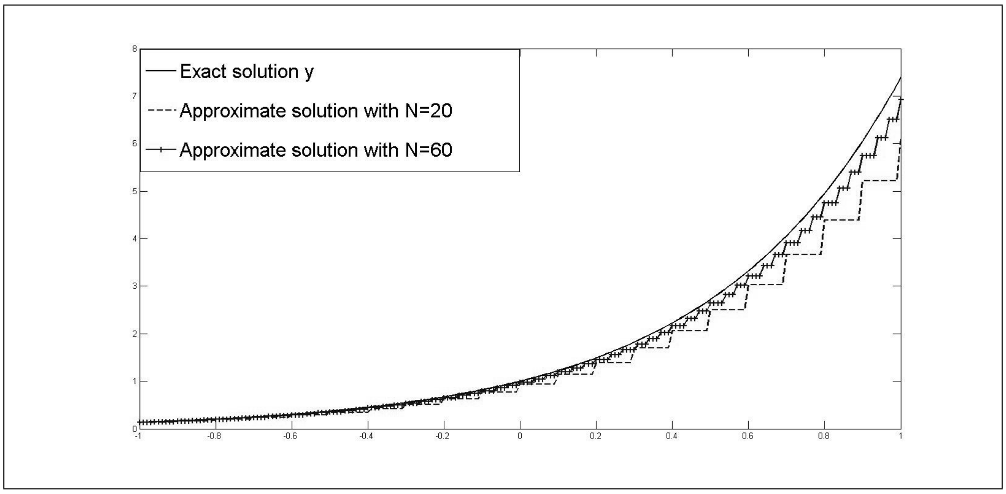

Figure 3.1: Approximate solution $G_N^1 y$ of Example 3.10, for $N = 20$ and $N = 60$.

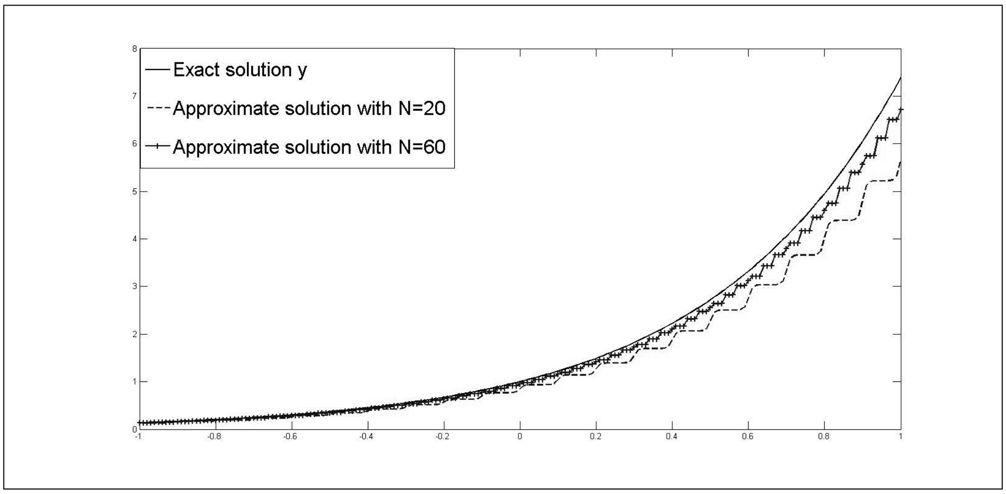

Figure 3.2: Approximate solution $G_N^2 y$ of Example 3.10, for $N = 20$ and $N = 60$.

Its solution is $y(t) = t^2 + \cos t - 2$.

To such an IVP, a linear Volterra integral equation like that in (3.2) can be associated (indeed, it is equivalent to it), with

$$K(t,s) := 2s - \frac{3}{2}t, \quad f(t) := -\frac{t}{2}\sin t - \cos t,$$

see Remark 3.6. Consider such an integral equation on the interval $[a,b] = [0,1]$. In Table 3.2, the relative errors ε_N^i, $i = 1,2,3$ are shown, as in Example 3.10. The same observation can be made on the condition number of the M_N's. In Fig.s 3.3 and 3.4, the approximate solutions $G_N^1 y$, $G_N^2 y$ for $N = 10$ and $N = 50$ are shown.

Our method can be compared with other classical collocation methods, e.g., those based on piecewise polynomials [27]. We have compared the numerical errors made applying both methods to Example 3.11, choosing for the latter method quadratic polynomials on the subintervals of $[0,1]$, when the same number of collocation points are used. Taking M subintervals, we need $N = 3M$ collocation points. The relative approximation errors, ε_M^p, made with $M = 7$, $M = 10$, and $M = 15$, turn out to be $\varepsilon_7^p = 3.03 \times 10^{-2}$, $\varepsilon_{10}^p = 2.82 \times 10^{-2}$, and $\varepsilon_{15}^p = 2.55 \times 10^{-2}$, respectively. These should be compared with the results shown in Table 3.2.

N	ε_N^1	ε_N^2	ε_N^3
10	9.52×10^{-2}	7.08×10^{-2}	8.82×10^{-2}
20	4.31×10^{-2}	4.21×10^{-2}	4.31×10^{-2}
30	3.02×10^{-2}	2.86×10^{-2}	3.02×10^{-2}
40	2.12×10^{-2}	2.12×10^{-2}	2.12×10^{-2}
50	1.06×10^{-2}	1.07×10^{-2}	1.24×10^{-2}
500	8.09×10^{-5}	1.10×10^{-3}	1.20×10^{-3}
1000	4.06×10^{-5}	5.38×10^{-4}	6.22×10^{-4}

Table 3.2: Numerical results for Example 3.11.

Example 3.12. Consider the following *singular* Volterra integral equation (of the Abel's type), with convolution kernel as in (3.8), with

$$K(t,s) = -\frac{1}{\sqrt{t-s}}, \quad f(t) = t^2 + \frac{16}{15}t^{5/2},$$

whose solution is $y(t) = t^2$, see [135] e.g.

We consider this equation on the interval $[a,b] = [0,1]$. The numerical results obtained by our collocation method with unit step functions are described in Table 3.3. The computed relative errors ε_N^i, $i = 1,2,3$ are those defined in (3.19).

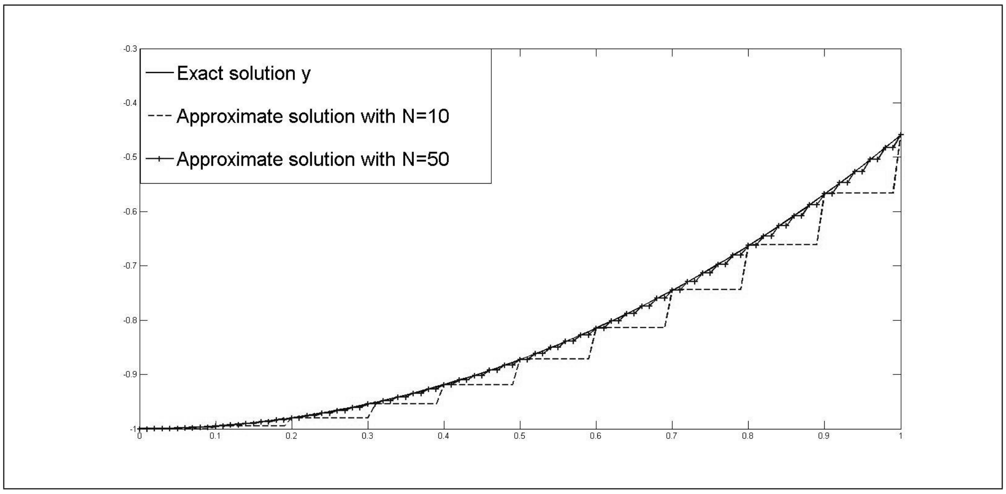

Figure 3.3: Approximate solution $G_N^1 y$ of Example 3.11, for $N = 10$ and $N = 50$.

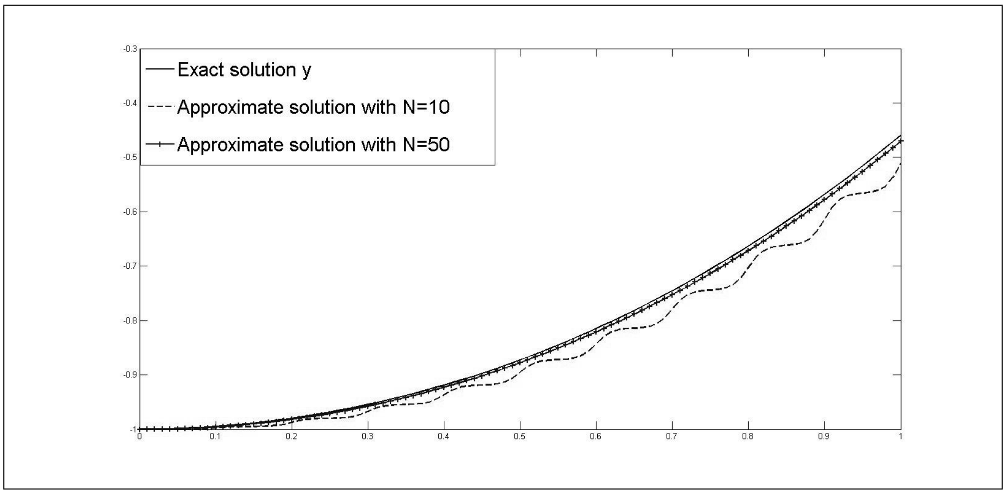

Figure 3.4: Approximate solution $G_N^2 y$ of Example 3.11, for $N = 10$ and $N = 50$.

N	ε_N^1	ε_N^2	ε_N^3
10	1.07×10^{-1}	6.47×10^{-2}	9.53×10^{-2}
20	4.61×10^{-2}	4.43×10^{-2}	4.61×10^{-2}
30	3.56×10^{-2}	3.27×10^{-2}	3.56×10^{-2}
40	2.32×10^{-2}	2.32×10^{-2}	2.32×10^{-2}
50	1.28×10^{-2}	7.20×10^{-3}	1.00×10^{-2}
500	1.20×10^{-3}	7.85×10^{-4}	1.10×10^{-3}
1000	6.02×10^{-4}	3.97×10^{-4}	5.42×10^{-4}

Table 3.3: Numerical results for Example 3.12.

Example 3.13. Finally, consider the singular Volterra integral equation (3.2) with

$$K(t,s) = -\frac{1}{\sqrt{t-s}}, \quad f(t) = 1 + 3\sqrt{t} + \frac{\pi}{2}t,$$

on the interval $[a,b] = [0,1]$. Its solution is $y(t) = 1 + \sqrt{t}$, which has an unbounded derivative at $t = 0$.

The numerical results for such example, obtained by our collocation method with unit step functions, are given in Table 3.4. From the tables,

N	ε_N^1	ε_N^2	ε_N^3
10	1.5×10^{-1}	1.21×10^{-1}	1.41×10^{-1}
20	10^{-1}	9×10^{-2}	10^{-1}
30	8.66×10^{-2}	8.1×10^{-2}	8.66×10^{-2}
40	7.07×10^{-2}	7.07×10^{-2}	7.07×10^{-2}
50	5×10^{-2}	5×10^{-2}	5×10^{-2}
500	8×10^{-4}	1.8×10^{-3}	2.2×10^{-3}
1000	3.78×10^{-4}	9.01×10^{-4}	1.1×10^{-3}

Table 3.4: Numerical results for Example 3.13.

one can observe that the convergence of the method is rather slow, and its accuracy poor. This is due to the basic approximation result based on sigmoidal functions, see Chapter 1. One should note, however, that the method can be applied under very weak assumptions on the kernel and the data. A large class of integral equations can then be solved in this way, and analytical representations of the solutions can also be obtained (as a superposition of sigmoidal functions), at a low computational cost. In fact, all coefficients (of $G_N y$) needed to approximate a given solution, are evaluated just solving a lower triangular algebraic system. In the special case of integral equations of the convolution type, such matrices are Toeplitz matrices, hence only N integrals must be computed. It follows that the methods we propose may actually be very fast, but the so-obtained solution could also be used as a starting point of other methods.

3.5.2 Nonlinear Volterra integral equations of the second kind

Here are some examples of nonlinear Volterra equations.

Example 3.14. Consider the nonlinear Volterra-Hammerstein equation (3.11) with

$$K(t,s;y(s)) := \widetilde{K}(t,s)\,G(y(s)) := e^{y(s)}\cos s, \quad f(t) := \sin t - e^{\sin t} + 1,$$

on the interval $[a,b] = [0,1]$. The solution is $y(t) = \sin t$, see [108], e.g..

The numerical results obtained applying our method with unit step functions (introduced in Section 3.3), are shown in Table 3.5. As above, we computed the relative errors ε_N^i, $i = 1,2,3$ defined in (3.19). In Fig.s 3.5 and 3.6, the approximate solutions $G_N^1 y$, $G_N^2 y$ are depicted, for $N = 30$ and $N = 80$.

N	ε_N^1	ε_N^2	ε_N^3
10	1.66×10^{-1}	1.53×10^{-1}	1.63×10^{-1}
20	8.76×10^{-2}	8.70×10^{-2}	8.76×10^{-2}
30	6.00×10^{-2}	5.90×10^{-2}	6.00×10^{-2}
40	4.53×10^{-2}	4.53×10^{-2}	4.53×10^{-2}
50	3.34×10^{-2}	3.37×10^{-2}	3.46×10^{-2}
60	3.09×10^{-2}	3.09×10^{-2}	3.09×10^{-2}
70	2.74×10^{-2}	2.74×10^{-2}	2.74×10^{-2}
80	2.31×10^{-2}	2.31×10^{-2}	2.31×10^{-2}
500	2.90×10^{-3}	3.50×10^{-3}	3.60×10^{-3}
1000	1.40×10^{-3}	1.80×10^{-3}	1.80×10^{-3}

Table 3.5: Numerical results for Example 3.14.

Example 3.15. Consider the nonlinear Volterra-Hammerstein equation (3.11) with

$$K(t,s;y(s)) := \widetilde{K}(t,s)\,G(y(s)) = e^{s-t}\left(e^{-y(s)} + y(s)\right), \quad f(t) := e^{-t},$$

on the interval $[a,b] = [0,1]$. The solution is $y(t) = \ln(t+e)$, see [108], e.g.

As above, we show in Table 3.6 the relative numerical errors ε_N^i, $i = 1,2,3$. Again, we plotted in Fig.s 3.7 and 3.8 the approximate solutions $G_N^1 y$, $G_N^2 y$, for $N = 10$ and $N = 50$.

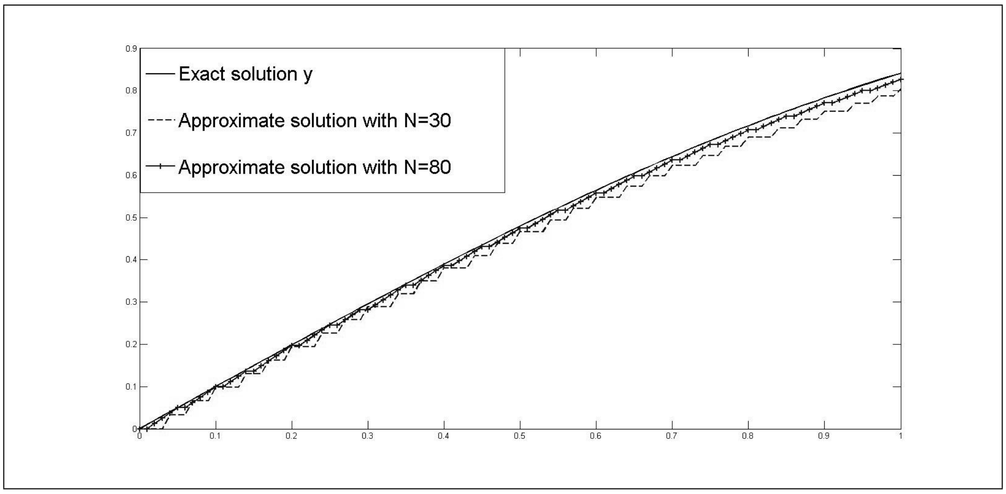

Figure 3.5: Approximate solution $G_N^1 y$ of Example 3.14, for $N = 30$ and $N = 80$.

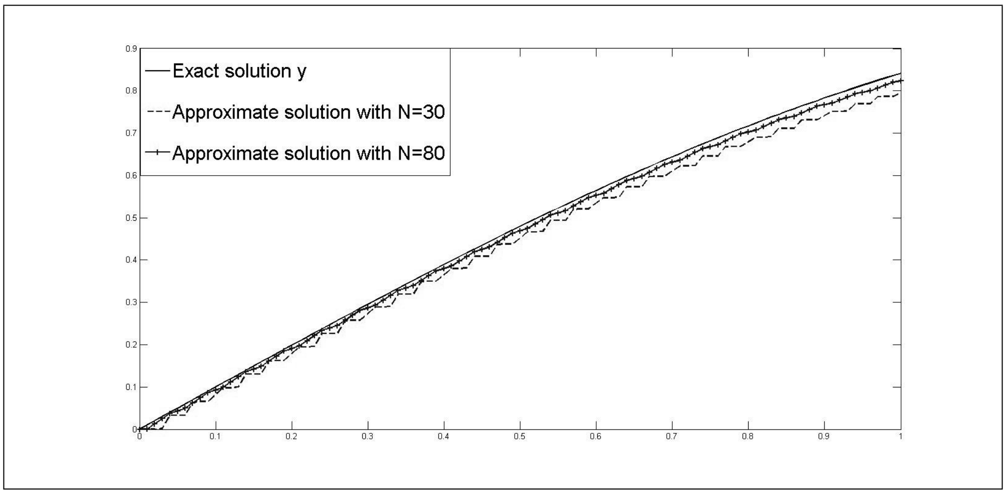

Figure 3.6: Approximate solution $G_N^2 y$ of Example 3.14, for $N = 30$ and $N = 80$.

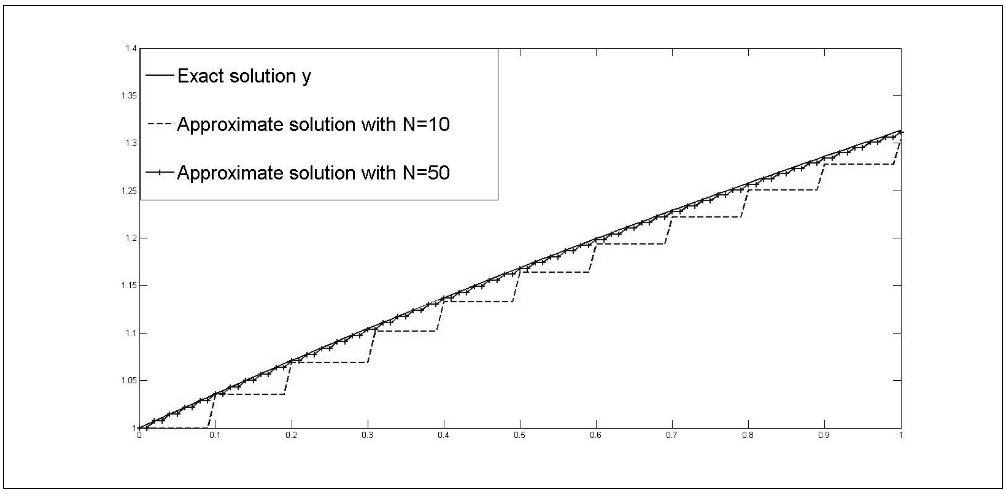

Figure 3.7: Approximate solution $G_N^1 y$ of Example 3.15, for $N = 10$ and $N = 50$.

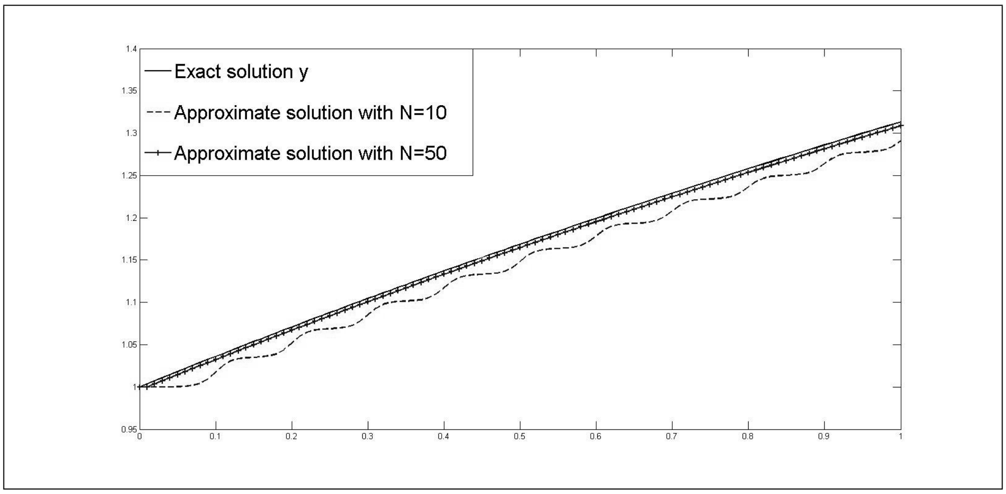

Figure 3.8: Approximate solution $G_N^2 y$ of Example 3.15, for $N = 10$ and $N = 50$.

N	ε_N^1	ε_N^2	ε_N^3
10	2.73×10^{-2}	2.06×10^{-2}	2.38×10^{-2}
20	1.38×10^{-2}	1.14×10^{-2}	1.16×10^{-2}
30	8.40×10^{-3}	8.10×10^{-3}	8.40×10^{-3}
40	6.90×10^{-3}	5.80×10^{-3}	5.80×10^{-3}
50	3.40×10^{-3}	3.40×10^{-3}	3.70×10^{-3}
500	1.40×10^{-4}	3.45×10^{-4}	3.75×10^{-4}
1000	7.03×10^{-5}	1.72×10^{-4}	1.87×10^{-4}

Table 3.6: Numerical results for Example 3.15.

As in case of linear equations, our collocation method exhibits slow convergence and poor accuracy. However, the procedure in (3.13) yields fast all coefficients that can be used in an analytical approximate representation for the solutions in terms of sigmoidal functions, to a large class of nonlinear equations. The low computational cost of the algorithm allows to increase considerably the number N of collocation points, and hence the number of the superposed sigmoidal functions, so to obtain an higher accuracy.

Chapter 4

Applications to the Numerical Solution of Volterra Integro-Differential Equations of the Neutral Type

Continuing with the study of the applications in Numerical Analysis of sigmoidal functions approximation, in this chapter, a numerical collocation method is developed for solving nonlinear Volterra integro-differential equations (VIDEs) of the neutral type, as well as other non-standard and classical VIDEs. Sigmoidal functions approximation is used to suitably represent the solutions. Special computational advantages are obtained using unit step functions, and important applications can be obtained also using other sigmoidal functions, such as logistic and Gompertz functions. The method allows to obtain a *simultaneous* approximation of the solution to a given VIDE and its first derivative, by means of an explicit formula. *A priori* as well as *a posteriori* estimates are derived for the numerical errors, and numerical examples are given for the purpose of illustration. A comparison is made with the classical piecewise polynomial collocation method as for accuracy and CPU time.

The present theory can be found by the readers in [59].

We solve numerically by collocation, nonlinear Volterra integro-differential equations (VIDEs) of the neutral type, of the form

$$y'(t) = f(t, y(t)) + \int_a^t K(t, s, y(s), y'(s))\, ds, \qquad y(a) = y_0,$$

for $t \in I := [a, b]$, where f and K are sufficiently smooth given functions, see [32, 118, 119, 28]. The method is also suited to solve classical VIDEs, i.e., equations where the integral term in (I) does *not* depend on $y'(s)$, as well as non-standard VIDEs, i.e., classical equations where the integral term depends in addition on $y(t)$, see [117, 27]. The most known example of a non-standard VIDE is perhaps given by the *logistic equation with a memory term*, see [62, 15, 16]. Collocation methods are widely used to solve integral equations, see, e.g., [2, 14, 30, 31, 27].

Our collocation method consists in first approximating y' by neural networks with unit step (Heaviside) sigmoidal functions, H. Upon integrating the NN that approximate y', we then obtain an approximation for y as well. At this point, replacing y and y' in the integro-differential equations with their approximations and evaluating the equations at suitable collocation points on the interval I, we can determine the coefficients of the collocation solutions.

The choice of using unit step functions allows to solve a large class of integral equations with some computational advantages. In particular, we can determine an *explicit formula* for calculating the coefficients of the collocation solutions. In this way, an analytical representation for the collocation solutions can be obtained. Moreover, we can show that approximate solutions can also be given in terms of other sigmoidal functions, such as for instance logistic and Gompertz functions.

The numerical errors made approximating y and y' are analyzed, and some *a priori* as well as *a posteriori* estimates are derived for such errors. A number of numerical examples are presented, and the results compared with those obtained by the classical piecewise polynomial collocation method. The collocation method based on sigmoidal functions developed here, seems to be competitive regarding the CPU time it requires. As for its accuracy, it performs better than piecewise polynomial collocation when integro-differential equations with weakly singular kernels are involved.

4.1 A collocation method for Volterra integro-differential equations

In this section, we introduce a collocation method aimed at solving nonlinear Volterra integro-differential equations (VIDEs) of the neutral type.

In what follows, we consider initial value problems of the form

$$y'(t) = f(t, y(t)) + \int_a^t K(t, s, y(s), y'(s))\, ds, \qquad y(a) = y_0, \tag{4.1}$$

for $t \in I := [a, b]$, where $f : I \times \mathbb{R} \to \mathbb{R}$, and $K : \Omega \to \mathbb{R}$, being $\Omega := I \times I \times \mathbb{R} \times \mathbb{R}$, are sufficiently smooth functions.

Suppose that (4.1) admits of a classical solution y, with $y \in C^1(I)$. Hence, $y' \in C^0(I)$ and then, as a consequence of Theorem 2.2, $y(t)$ could be uniformly approximated on I by a superposition of bounded sigmoidal functions. We choose to use unit step (sigmoidal) functions, and set

$$(G_N y')(t) := \sum_{k=1}^{N} \alpha_k H(t - t_k) + \alpha_0 H(t - t_{-1}), \quad t \in I, \tag{4.2}$$

with $G_N y' \in \Sigma_N$, $N \in \mathbb{N}^+$. This will be a neural network which approximates y'. Here, the coefficients α_k, $k = 0, 1, \ldots, N$, are unknowns, and $t_k = a + kh$, $h = (b - a)/N$, for $k = -1, 0, \ldots, N$. Integrating $G_N y'$, we define

$$\begin{aligned} (S_N y)(t) &:= \int_a^t (G_N y')(s) ds + y_0 \\ &= \sum_{k=1}^{N} \alpha_k \int_a^t H(s - t_k) ds + \alpha_0 \int_a^t H(s - t_{-1})\, ds + y_0, \end{aligned} \tag{4.3}$$

for $t \in I$, where y_0 is the initial data in (4.1). Clearly, $S_N y$ approximates y. Note that $G_N y'$ and $S_N y$ are both characterized by the same unknowns coefficients α_k. By the definition of Heaviside functions, we have for every $t \in I$

$$\int_a^t H(s - t_{-1})\, ds = t - a, \quad \text{and} \quad \int_a^t H(s - t_k)\, ds = t - t_k,$$

for every k such that $t_k \leq t$, and

$$\int_a^t H(s - t_k)\, ds = 0,$$

for every k with $t_k > t$, $k = 1, \ldots, N$. Then (4.3) becomes

$$(S_N y)(t) := \sum_{k: t_k \leq t} \alpha_k (t - t_k) + y_0, \quad t \in I. \tag{4.4}$$

Inserting $S_N y$ and $G_N y'$ in (4.1), in place of y and y', respectively, we obtain the *collocation equation*

$$(G_N y')(t) = f(t, (S_N y)(t)) + \int_a^t K(t, s, (S_N y)(s), (G_N y')(s))\, ds. \tag{4.5}$$

If $\mathcal{C}_N := \{t_0, t_1, \ldots, t_N\}$ denotes the set of the *collocation points*, we can evaluate (4.5) at such points, obtaining

$$(G_N y')(t_i) = f(t_i, (S_N y)(t_i)) + \int_a^{t_i} K(t_i, s, (S_N y)(s), (G_N y')(s))\, ds, \tag{4.6}$$

for every fixed i, $i = 0, 1, \ldots, N$. This is a nonlinear algebraic system of $N + 1$ equations where the unknowns are the α_k's, $k = 0, 1, \ldots, N$. Solving such nonlinear system we obtain the α_k's and then $S_N y$ and $G_N y'$.

We will show that $S_N y$ and $G_N y'$ do provide a *simultaneous* approximation of the solution, y, to (4.1) and of its first derivative y'. We can prove that system (4.6) always has a unique solution. Indeed, the following theorem holds.

Theorem 4.1. *The nonlinear algebraic system (4.6) has a unique solution $\alpha_0, \ldots, \alpha_N$, i.e., the collocation method based on unit step functions, used for solving the nonlinear integro-differential equations of the neutral type in (4.1), admits of a unique solution, $S_N y$, for every $N \in \mathbb{N}^+$. In particular, the coefficients α_i of $S_N y$ (and $G_N y'$), can be determined by the following explicit formula:*

$$\alpha_0 := f(a, y_0), \tag{4.7}$$

$$\alpha_1 := f\left(t_1, \alpha_0(t_1 - a) + y_0\right) - \alpha_0 + \int_a^{t_1} K\left(t_1, s, \alpha_0(s-a) + y_0, \alpha_0\right) ds, \tag{4.8}$$

and

$$\begin{aligned}\alpha_i := f\left(t_i, \sum_{k=0}^{i-1} \alpha_k (t_i - t_k) + y_0\right) - \sum_{k=0}^{i-1} \alpha_k \\ + \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s, \sum_{k=0}^{\nu-1} \alpha_k (s - t_k) + y_0, \sum_{k=0}^{\nu-1} \alpha_k\right) ds,\end{aligned} \tag{4.9}$$

for every i, $i = 2, \ldots N$.

Proof. By equation (4.6), for $i = 0, 1, \ldots, N$, $N \in \mathbb{N}^+$, and the definition of $S_N y$ and $G_N y'$, we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha_k H(t_i - t_k) + \alpha_0 H(t_i - t_{-1}) = f\left(t_i, \sum_{k=0}^{i} \alpha_k (t_i - t_k) + y_0\right) \\ + \int_a^{t_i} K\Bigg(t_i, s, \sum_{k=1}^{N} \alpha_k \int_a^s H(z - t_k)\, dz + \alpha_0 \int_a^s H(z - t_{-1})\, dz \\ + y_0, \sum_{k=1}^{N} \alpha_k H(s - t_k) + \alpha_0 H(s - t_{-1})\Bigg) ds,\end{aligned}$$

that can also be rewritten as

$$\begin{aligned}\sum_{k=1}^{N} \alpha_k H(t_i - t_k) + \alpha_0 H(t_i - t_{-1}) = f\left(t_i, \sum_{k=0}^{i} \alpha_k (t_i - t_k) + y_0\right) \\ + \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\Bigg(t_i, s, \sum_{k=1}^{N} \alpha_k \int_a^s H(z - t_k)\, dz + \alpha_0 \int_a^s H(z - t_{-1})\, dz\end{aligned}$$

$$+y_0, \sum_{k=1}^{N} \alpha_k H(s-t_k) + \alpha_0 H(s-t_{-1})\Bigg) \, ds. \tag{4.10}$$

For $i=0$, equation (4.10) reduces to $\alpha_0 = f(a, y_0)$. For $i=1$, (4.10) reduces to

$$\alpha_0 + \alpha_1 = f\left(t_1, \alpha_0(t_1-a)+y_0\right) + \int_a^{t_1} K\left(t_1, s, \alpha_0(s-a)+y_0, \alpha_0\right) \, ds,$$

and then we have

$$\alpha_1 = f\left(t_1, \alpha_0(t_1-a)+y_0\right) - \alpha_0 + \int_a^{t_1} K\left(t_1, s, \alpha_0(s-a)+y_0, \alpha_0\right) \, ds,$$

that can be immediately evaluated since α_0 is known from the previous step. Note that, in general, for every fixed i, $i=2,\dots,N$, $H(t_i - t_k) = 0$ for every $k > i$, and $H(t_i - t_k) = 1$ for every $k \le i$. Moreover, it is easy to see that, for every $\nu = 1,\dots,i$, there is $H(\cdot - t_k) = 1$ on $[t_{\nu-1}, t_\nu]$ for $k = 0,\dots,\nu-1$ and $H(\cdot - t_k) = 0$ on $[t_{\nu-1}, t_\nu]$ for $k \ge \nu$. Finally note that, again for $\nu = 1,\dots,i$, if $s \in [t_{\nu-1}, t_\nu]$, we have

$$\sum_{k=1}^{N} \alpha_k \int_a^s H(z-t_k)\, dz + \alpha_0 \int_a^s H(z-t_{-1})\, dz = \sum_{k=0}^{\nu-1} \alpha_k (s-t_k).$$

Thus, for the general case $i=2,\dots,N$, (4.10) becomes

$$\sum_{k=0}^{i} \alpha_k = f\left(t_i, \sum_{k=0}^{i-1} \alpha_k (t_i - t_k) + y_0\right)$$
$$+ \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s, \sum_{k=0}^{\nu-1} \alpha_k (s-t_k) + y_0, \sum_{k=0}^{\nu-1} \alpha_k\right) ds,$$

so we can conclude that

$$\alpha_i = f\left(t_i, \sum_{k=0}^{i-1} \alpha_k (t_i - t_k) + y_0\right) - \sum_{k=0}^{i-1} \alpha_k$$
$$+ \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s, \sum_{k=0}^{\nu-1} \alpha_k (s-t_k) + y_0, \sum_{k=0}^{\nu-1} \alpha_k\right) ds.$$

This shows that (4.6) admits of a unique solution, $\alpha_0, \dots, \alpha_N$. □

Note that, in Theorem 4.1 integrability of the kernel, K, in the integro-differential equation (4.1) is required. We emphasize the peculiarities of the present collocation method: (i) one can determine all coefficients α_k in a very simple way, and (ii) a simultaneous approximation of y and y' is provided by $S_N y$ and $G_N y'$, respectively, as well as the analytical form of $S_N y$ and $G_N y'$.

Remark 4.2. Note that, if the kernel in equation (4.1) is of the convolution type, i.e., $K(t,s,y(t),y'(s)) = k(t-s)\,G(y(t),y'(s))$, and it does not depend on $y(s)$ (for *all* $s \in [0,t]$), then, formulae (4.8) and (4.9) in Theorem 4.1 reduce to

$$\alpha_1 := f\left(t_1, \alpha_0(t_1-a)+y_0\right) - \alpha_0 + G\left(\alpha_0(t_1-a)+y_0, \alpha_0\right) \int_a^{t_1} k\left(t_1-s\right) ds,$$

and

$$\alpha_i := f\left(t_i, \sum_{k=0}^{i-1} \alpha_k(t_i-t_k)+y_0\right) - \sum_{k=0}^{i-1}\alpha_k$$
$$+\sum_{\nu=1}^{i} G\left(\sum_{k=0}^{\nu-1}\alpha_k(t_i-t_k)+y_0 \sum_{k=0}^{\nu-1}\alpha_k\right)\int_{t_{\nu-1}}^{t_\nu} k\left(t_i-s\right)\,ds,$$

for every i, $i=2,\dots N$. Now, changing variable in the integrals, setting $z=s+h$ (where h is step-size of the collocation points), we obtain

$$\int_{t_{\nu-1}}^{t_\nu} k\left(t_i-s\right)\,ds = \int_{t_{\nu-1}+h}^{t_\nu+h} k\left(t_i+h-z\right)\,dz = \int_{t_\nu}^{t_{\nu+1}} k\left(t_{i+1}-z\right)\,dz. \tag{4.11}$$

Relation (4.11) can be used to simplify the implementation of the collocation method in this case. In fact, in order to determine the coefficients α_i, now we need to compute *only one* additional integral. In general we evaluate i integrals at each step, while in the present approach we are able to reduce significantly the CPU time needed by our method.

The method can also be applied to other problems, such as classical *nonlinear* integro-differential equations of the form

$$y'(t) = f(t,y(t)) + \int_a^t K(t,s,y(s))\,ds, \qquad y(0)=y_0, \tag{4.12}$$

for $t \in I := [a,b]$, where $f : I\times\mathbb{R}\to\mathbb{R}$ and $K : I\times I\times\mathbb{R}\to\mathbb{R}$ are sufficiently smooth functions.

Moreover, a further class of VIDEs to which our collocation method can be applied is that of *non-standard* Volterra integro-differential equations having the typical general form

$$y'(t) = f(t,y(t)) + \int_a^t K(t,s,y(t),y(s))\,ds, \qquad y(0)=y_0, \tag{4.13}$$

where $K : I\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is sufficiently smooth and the integrand depends on both, $y(s)$ and $y(t)$. The best known example of such a non-standard VIDE is given by the so-called *logistic equation with memory term*, see [117, 62, 15, 16, 27]. Assume as above, that (4.12) or (4.13) admits of a classical solution. Proceeding as with neutral VIDEs, that is replacing y

and y' with $S_N y$ and $G_N y'$ in (4.12) or (4.13), and evaluating the ensuing collocation equations at the collocation points, we obtain *nonlinear algebraic* systems similar to those in (4.6). The following theorems can be established by a straightforward adaptation of the proof of Theorem 4.1. For clarity, we state these results explicitly here.

Theorem 4.3. *The collocation method based on unit step functions for solving the nonlinear VIDE (4.12), has a unique solution, $S_N y$, for every $N \in \mathbb{N}^+$. In particular, the coefficients α_i of $S_N y$ (and $G_N y'$), can be determined by the following explicit formula:*

$$\alpha_0 := f(a, y_0), \tag{4.14}$$

$$\alpha_1 := f\left(t_1, \alpha_0(t_1 - a) + y_0\right) - \alpha_0 + \int_a^{t_1} K\left(t_1, s, \alpha_0(s - a) + y_0\right) ds; \tag{4.15}$$

and

$$\begin{aligned} \alpha_i := {} & f\left(t_i, \sum_{k=0}^{i-1} \alpha_k(t_i - t_k) + y_0\right) - \sum_{k=0}^{i-1} \alpha_k \\ & + \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s, \sum_{k=0}^{\nu-1} \alpha_k(s - t_k) + y_0\right) ds, \end{aligned} \tag{4.16}$$

for every i, $i = 2, \ldots N$.

Theorem 4.4. *The collocation method based on unit step functions, for solving the non-standard VIDE (4.13), has a unique solution, $S_N y$, for every $N \in \mathbb{N}^+$. In particular, the coefficients α_i of $S_N y$ (and $G_N y'$), can be determined by the following explicit formula:*

$$\alpha_0 := f(a, y_0), \tag{4.17}$$

$$\begin{aligned} \alpha_1 := {} & f\left(t_1, \alpha_0(t_1 - a) + y_0\right) - \alpha_0 \\ & + \int_a^{t_1} K\left(t_1, s, \alpha_0(t_1 - a) + y_0, \alpha_0(s - a) + y_0\right) ds, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \alpha_i := {} & f\left(t_i, \sum_{k=0}^{i-1} \alpha_k(t_i - t_k) + y_0\right) - \sum_{k=0}^{i-1} \alpha_k \\ & + \sum_{\nu=1}^{i} \int_{t_{\nu-1}}^{t_\nu} K\left(t_i, s, \sum_{k=0}^{i-1} \alpha_k(t_i - t_k) + y_0 \sum_{k=0}^{\nu-1} \alpha_k(s - t_k) + y_0\right) ds, \end{aligned} \tag{4.19}$$

for every i, $i = 2, \ldots N$.

Remark 4.5. Suppose that $y \in C^1(I)$ is the classical solution of a VIDE of the form in (4.1), and that $S_N y$ is its collocation solution, written with coefficients α_k provided by Theorem 4.1. Moreover, $G_N y'$ is given in term of unit step functions, and this represents an approximation of y'. Let now σ be a fixed bounded sigmoidal function. By Remark 3.1, we know that various approximation of any given continuous function can be obtained using various different sigmoidal functions, still retaining the same coefficients. Therefore, using the same α_k, we can define

$$(G_N^\sigma y')(t) := \sum_{k=1}^{N} \alpha_k \sigma(w(t-t_k)) + \alpha_0 \sigma(w(t-t_{-1})), \qquad t \in I, \tag{4.20}$$

where w is a suitable positive parameter depending on σ and N, and the t_k's are the uniformly spaced nodes in the interval $[a, b]$ defined above. It follows that $G_N^\sigma y'$ provides a further approximation to y', and hence we obtain, by integration,

$$(S_N^\sigma y)(t) := \sum_{k=1}^{N} \alpha_k \int_a^t \sigma(w(s-t_k))ds + \alpha_0 \int_a^t \sigma(w(s-t_{-1}))ds + y_0, \quad t \in I, \tag{4.21}$$

which provides a further approximation to y on I, for every $N \in \mathbb{N}^+$.

The same observation can be made when our collocation method is applied to VIDEs of the form (4.12) or (4.13), using the same coefficients α_k determined in Theorem 4.3 and Theorem 4.4, respectively.

4.2 Analysis of the numerical errors

In this section, we analyze the various sources of numerical errors which affect our collocation method.

4.2.1 *A priori* estimates

We start considering our collocation method with unit step functions, applying it to the VIDEs of neutral type in (4.1).

First of all, we define the *error function* for y',

$$e_N(t) := y'(t) - (G_N y')(t), \qquad t \in I := [a, b], \tag{4.22}$$

where y is the classical solution of (4.1) and $G_N y'$ is the neural network given in (4.2) and written with coefficients α_k determined by our collocation method for (4.1). Integrating e_N, we obtain

$$E_N(t) := \int_a^t e_N(z)dz = y(t) - (S_N y)(t), \quad t \in I := [a, b], \tag{4.23}$$

that is the *error function* for y. The following theorem provides some *a priori* estimates for e_N and E_N.

Theorem 4.6. *Let (4.1) a given VIDE of the neutral type, which admits of a classical solution $y \in C^1(I)$. Assume that, there exist the positive constants L_f, C, L_1 and L_2, such that*
(i) For every (t_1, s_1), $(t_2, s_2) \in I \times \mathbb{R}$,

$$|f(t_1, s_1) - f(t_2, s_2)| \le L_f \, \|(t_1, s_1) - (t_2, s_2)\|_2,$$

where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$;
(ii) For every $(t, s, x, z) \in I \times I \times [-\|y\|_\infty, +\|y\|_\infty] \times [-\|y'\|_\infty, +\|y'\|_\infty]$, we have $|K(t, s, x, z)| \le C$;
(iii) for every (t_1, s, x, z), $(t_2, s, x, z) \in I \times I \times [-\|y\|_\infty, \|y\|_\infty] \times [-\|y'\|_\infty, \|y'\|\infty]$,

$$|K(t_1, s, x, z) - K(t_2, s, x, z)| \le L_1 \, |t_1 - t_2|;$$

(iv) For every (t, s, x_1, z_1), $(t, s, x_2, z_2) \in I \times I \times \mathbb{R} \times \mathbb{R}$,

$$|K(t, s, x_1, z_1) - K(t, s, x_2, z_2)| \le L_2 \, \|(x_1, z_1) - (x_2, z_2)\|_2.$$

Then,

$$|e_N(t)| \le \frac{(b-a)\, M_1}{N} \, e^{M_2\,(t-a)}, \qquad t \in I := [a, b],$$

and

$$|E_N(t)| \le \frac{(b-a)\, M_1}{N M_2} \, \left(e^{M_2\,(t-a)} - 1\right), \qquad t \in I,$$

for every $N \in \mathbb{N}^+$, where M_1 and M_2 are suitable positive constants not depending on N.

Proof. Let $N \in \mathbb{N}^+$ and $t \in I := [a, b]$ be fixed. Define

$$j := \max\{i : t_i \le t, t_i \in \mathcal{C}_N, i = 0, 1, \ldots, N\},$$

where $t_i = a + kh$, $h := (b-a)/N$, $k = 0, 1, ..., N$, are the collocation points. We can write

$$|e_N(t)| \le |e_N(t) - e_N(t_j)| + |e_N(t_j)| \le |y'(t) - y'(t_j)|$$

$$+|(G_N y')(t) - (G_N y')(t_j)| + |e_N(t_j)| = |y'(t) - y'(t_j)| + |e_N(t_j)|,$$

because $(G_N y')(t) = (G_N y')(t_j)$, since the function $G_N y'$ is written as a superposition of unit step functions and hence it is piecewise constant. Now,

$$|e_N(t)| \le |y'(t) - y'(t_j)| + |e_N(t_j)| = |f(t, y(t))$$

$$+ \int_a^t K(t, s, y(s), y'(s))\, ds - f(t_j, y(t_j)) - \int_a^{t_j} K(t_j, s, y(s), y'(s))\, ds\Big|$$

$$+\Big| f(t_j, y(t_j)) + \int_a^{t_j} \left[K(t_j, s, y(s), y'(s)) - K(t_j, s, (S_N y)(s), (G_N y')(s))\right] ds$$

$$-f(t_j,(S_N y)(t_j))| \leq |f(t,y(t)) - f(t_j,y(t_j))|$$

$$+\int_{t_j}^{t} \left|K(t,s,y(s),y'(s))\right| ds + \int_a^{t_j} \left|K(t,s,y(s),y'(s)) - K(t_j,s,y(s),y'(s))\right| ds$$

$$+|f(t_j,y(t_j)) - f(t_j,(S_N y)(t_j))| + \int_a^{t_j} |K(t_j,s,y(s),y'(s))$$

$$-K(t_j,s,(S_N y)(s),y'(s))|\, ds + \int_a^{t_j} |K(t_j,s,(S_N y)(s),y'(s))$$

$$-K(t_j,s,(S_N y)(s),(G_N y')(s))|\, ds =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

We first estimate J_1 and J_4. Being $y \in C^1(I)$, y is Lipschitz continuous on I, with some Lipschitz constant, say, $L_y > 0$. Thus, we obtain from condition (i)

$$J_1 \leq L_f \sqrt{(t-t_j)^2 + (y(t)-y(t_j))^2} \leq L_f \sqrt{h^2(1+L_y^2)} =: hM,$$

and moreover

$$J_4 \leq L_f \,|y(t_j) - (S_N y)(t_j)| = L_f \,|E_N(t_j)| \leq L_f \int_a^{t_j} |e_N(z)|\, dz.$$

As for J_2, we can easily infer from condition (ii) that $J_2 \leq C\,(t-t_j) \leq Ch$. Turning our attention to J_3, we have by condition (iii)

$$J_3 \leq L_1 \,|t-t_j|\,(t_j - a) \leq L_1\,(t_j-a)\,h \leq L_1\,(b-a)\,h.$$

We finally estimate J_5 and J_6. From condition (iv), we have

$$J_5 \leq L_2 \int_a^{t_j} |y(s) - (S_N y)(s)|\, ds = L_2 \int_a^{t_j} |E_N(s)|\, ds,$$

and similarly

$$J_6 \leq L_2 \int_a^{t_j} \left|y'(s) - (G_N y')(s)\right| ds = L_2 \int_a^{t_j} |e_N(s)|\, ds.$$

Therefore, combining all such estimates we conclude that

$$|e_N(t)| \leq h\,[M+C+L_1\,(b-a)] + L_2 \int_a^{t_j} |E_N(s)|\, ds + (L_f+L_2) \int_a^{t_j} |e_N(z)|\, dz$$

$$= h\,[M+C+L_1\,(b-a)] + L_2 \int_a^{t_j} \left|\int_a^{s} e_N(z)\, dz\right| ds + (L_f+L_2) \int_a^{t_j} |e_N(z)|\, dz$$

$$\leq h\,[M+C+L_1\,(b-a)] + L_2 \int_a^{t_j} \left[\int_a^{t_j} |e_N(z)|\, dz\right] ds + (L_f+L_2) \int_a^{t_j} |e_N(z)|\, dz$$

$$\le h\,[M + C + L_1\,(b-a)] + [L_2\,(b-a+1) + L_f] \int_a^{t_j} |e_N(z)|\; dz$$

$$\le h\,[M + C + L_1\,(b-a)] + [L_2\,(b-a+1) + L_f] \int_a^{t} |e_N(z)|\; dz$$

$$=: h\,M_1 + M_2 \int_a^t |e_N(z)|\; dz.$$

The previous inequality holds for every $t \in I$, and thus we obtain, by Gronwall's lemma,

$$|e_N(t)| \le h\,M_1\,e^{M_2\,(t-a)} = \frac{(b-a)\,M_1}{N}\,e^{M_2\,(t-a)}, \qquad t \in I,$$

for every $N \in \mathbb{N}^+$. Moreover,

$$|E_N(t)| \le \int_a^t |e_N(s)|\,ds \le \frac{(b-a)\,M_1}{N} \int_a^t e^{M_2\,(s-a)}\,ds,$$

$$= \frac{(b-a)\,M_1}{NM_2}\left(e^{M_2\,(t-a)} - 1\right),$$

for every $N \in \mathbb{N}^+$. □

Remark 4.7. We can infer from Theorem 4.6 that

$$\|e_N\|_\infty \le \frac{(b-a)\,M_1}{N}\,e^{M_2\,(b-a)},$$

and

$$\|E_N\|_\infty \le \frac{(b-a)\,M_1}{NM_2}\left(e^{M_2\,(b-a)} - 1\right),$$

so that we can conclude that $\|e_N\|_\infty \to 0$ and $\|E_N\|_\infty \to 0$ as $N \to +\infty$: $S_N y$ and $G_N y'$ converge both *uniformly* on I, to y and y', respectively.

Remark 4.8. In Theorem 4.6, the Lipschitz conditions (i) and (iv) are *global*. This is a technical assumption, which circumvents the need of having an *a priori* boundedness of $S_N y$ and $G_N y'$. For such a reason, Theorem 4.6 *cannot* cover, e.g., the case of VIDEs like (4.1) with kernels of the form $K(t,s,x,z) = \widetilde{K}(t,s)\,x^p z^p$, or $f(t,s) = \widetilde{f}(t)\,s^p$, with $p > 1$, see also [57].

Theorem 4.6 can also be extended in such a way to apply our collocation method to (4.12). Clearly, the coefficients α_k of $S_N y$ (and $G_N y'$) in (4.12) are the same obtained in Theorem 4.3 (or in Theorem 4.4, in case of the equations of the kind in (4.13)). The following can be proved.

Theorem 4.9. *Let be given a VIDE like in (4.12), which admits of a classical solution $y \in C^1(I)$. Assume that, there exist the positive constants L_f, C, L_1 and L_2, such that*
(i) For every (t_1, s_1), $(t_2, s_2) \in I \times \mathbb{R}$,

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_f \, \|(t_1, s_1) - (t_2, s_2)\|_2;$$

(ii) For every $(t, s, x) \in I \times I \times [-\|y\|_\infty, +\|y\|_\infty]$, we have $|K(t, s, x)| \leq C$;
(iii) for every (t_1, s, x), $(t_2, s, x) \in I \times I \times [-\|y\|_\infty, \|y\|_\infty]$,

$$|K(t_1, s, x) - K(t_2, s, x)| \leq L_1 \, |t_1 - t_2|;$$

(iv) For every (t, s, x_1), $(t, s, x_2) \in I \times I \times \mathbb{R}$,

$$|K(t, s, x_1) - K(t, s, x_2)| \leq L_2 \, |x_1 - x_2|.$$

Then,

$$|e_N(t)| \leq \frac{(b-a)\, M_1}{N} \, e^{M_2 \, (t-a)}, \qquad t \in I := [a, b],$$

and

$$|E_N(t)| \leq \frac{(b-a)\, M_1}{N M_2} \left(e^{M_2 \, (t-a)} - 1 \right), \qquad t \in I,$$

for every $N \in \mathbb{N}^+$, where M_1 and M_2 are suitable positive constants not depending on N.

The proof of this theorem is similar to that of Theorem 4.6, and thus is omitted. Moreover, in case of non-standard VIDEs like those in (4.13), we have the following.

Theorem 4.10. *Let be a fixed non-standard VIDE like that in (4.13), which admits of a classical solution $y \in C^1(I)$. Assume that, there exist the positive constants L_f, C, L_1 and L_2, such that*
(i) For every (t_1, s_1), $(t_2, s_2) \in I \times \mathbb{R}$,

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_f \, \|(t_1, s_1) - (t_2, s_2)\|_2;$$

(ii) For every $(t, s, x, z) \in I \times I \times [-\|y\|_\infty, +\|y\|_\infty] \times [-\|y\|_\infty, +\|y\|_\infty]$, we have $|K(t, s, x, z)| \leq C$;
(iii) for every (t_1, s, x, z), $(t_2, s, x, z) \in I \times I \times [-\|y\|_\infty, \|y\|_\infty] \times [-\|y\|_\infty, \|y\|\infty]$,

$$|K(t_1, s, x, z) - K(t_2, s, x, z)| \leq L_1 \, |t_1 - t_2|;$$

(iv) For every (t, s, x_1, z_1), $(t, s, x_2, z_2) \in I \times I \times \mathbb{R} \times \mathbb{R}$,

$$|K(t, s, x_1, z_1) - K(t, s, x_2, z_2)| \leq L_2 \, \|(x_1, z_1) - (x_2, z_2)\|_2.$$

Then,

$$|e_N(t)| \leq \frac{(b-a)\, M_1}{N} \, e^{M_2 \, (t-a)}, \qquad t \in I := [a, b],$$

and

$$|E_N(t)| \leq \frac{(b-a)\, M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1 \right), \qquad t \in I,$$

for every $N \in \mathbb{N}^+$, where M_1 and M_2 are suitable positive constants not depending on N.

Proof. As in the proof of Theorem 4.6 (and with the same notation), we can write, for every fixed $t \in I$,

$$|e_N(t)| \leq |y'(t) - y'(t_j)| + |e_N(t_j)| = |f(t, y(t))$$

$$+ \int_a^t K(t, s, y(t), y(s))\, ds - f(t_j, y(t_j)) - \int_a^{t_j} K(t_j, s, y(t_j), y(s))\, ds\Big|$$

$$+ \Big| f(t_j, y(t_j)) + \int_a^{t_j} [K(t_j, s, y(t_j), y(s)) - K(t_j, s, (S_N y)(t_j), (S_N y)(s))]\; ds$$

$$- f(t_j, (S_N y)(t_j))| \leq |f(t, y(t)) - f(t_j, y(t_j))|$$

$$+ \int_{t_j}^t |K(t, s, y(t), y(s))|\; ds + \int_a^{t_j} |K(t, s, y(t), y(s)) - K(t_j, s, y(t_j), y(s))|\; ds$$

$$+ |f(t_j, y(t_j)) - f(t_j, (S_N y)(t_j))| + \int_a^{t_j} |K(t_j, s, y(t_j), y(s))$$

$$- K(t_j, s, (S_N y)(t_j), y(s))|\, ds + \int_a^{t_j} |K(t_j, s, (S_N y)(t_j), y(s))$$

$$- K(t_j, s, (S_N y)(t_j), (S_N y)(s))|\, ds =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

The terms J_1, J_2, and J_4 can be estimated as in Theorem 4.6. As for J_3, J_5, and J_6, we obtain, exploiting (iii) and (iv) and the fact that y is Lipschitz continuos with Lipschitz constant $L_y > 0$,

$$J_3 \leq L_1\, (t_j - a)\, \sqrt{(t - t_j)^2 + (y(t) - y(t_j))^2} \leq L_1\, (t_j - a)\, \sqrt{h^2 (1 + L_y^2)}$$

$$\leq L_1\, (b - a)\, \sqrt{(1 + L_y^2)}\; h,$$

and

$$J_5 \leq L_2\, |E_N(t_j)| \int_a^{t_j} ds \leq L_2\, (b-a)\, |E_N(t_j)| \leq L_2\, (b-a) \int_a^t |e_N(z)|\, dz,$$

and finally,

$$J_6 \leq L_2 \int_a^{t_j} |E_N(s)|\, ds \leq L_2\, (b-a) \int_a^{t_j} \int_a^s |e_N(z)|\, dz\, ds$$

$$\leq L_2\, (b-a)^2 \int_a^{t_j} |e_N(z)|\, dz \leq L_2\, (b-a)^2 \int_a^t |e_N(z)|\, dz.$$

The proof then follows as in Theorem 4.6. □

4.2.2 *A posteriori* estimates

As noted in Remark 4.8, the a priori estimates made in Section 4.2.1 hold only for equations with kernels K and data f which are *globally* Lipschitz. However, if we replace conditions (i) and (iv) in Theorem 4.6 with a *local* Lipschitz condition, some *a posteriori* error estimates can be established. Indeed, the following can be proved.

Theorem 4.11. *Let a given VIDE of the neutral type like that in (4.1) have a classical solution $y \in C^1(I)$, and assume that conditions (ii) and (iii) of Theorem 4.6 hold for some positive constants C and L_1, respectively. Suppose in addition that:*

(a) *There exist a constant $L_f > 0$ and a function $\mathcal{L}_f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that, for every $\gamma > 0$,*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq \mathcal{L}_f(\gamma)\, |f(t_1, s_1/\gamma) - f(t_2, s_2/\gamma)|,$$

for every (t_1, s_1), $(t_2, s_2) \in I \times \mathbb{R}$, and such that

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_f\, \|(t_1, s_1) - (t_2, s_2)\|_2,$$

for every (t_1, s_1), $(t_2, s_2) \in I \times [-1, 1]$;

(b) *There exist a constant $L_2 > 0$ and a function $\mathcal{L}_K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that, for every constant $\gamma > 0$,*

$$\begin{aligned} &|K(t_1, s_1, x_1, z_1) - K(t_2, s_2, x_2, z_2)| \\ &\leq \mathcal{L}_K(\gamma)\, |K(t_1, s_1, x_1/\gamma, z_1/\gamma) - K(t_2, s_2, x_2/\gamma, z_2/\gamma)|, \end{aligned}$$

for every (t_1, s_1, x_1, z_1), $(t_2, s_2, x_2, z_2) \in I \times I \times \mathbb{R} \times \mathbb{R}$, and such that

$$|K(t, s, x_1, z_1) - K(t, s, x_2, z_2)| \leq L_2\, \|(x_1, z_1) - (x_2, z_2)\|_2,$$

for every (t, s, x_1, z_1), $(t, s, x_2, z_2) \in I \times I \times [-1, 1] \times [-1, 1]$.
Then,

$$|e_N(t)| \leq \frac{(b-a)\, M_1}{N}\, e^{\mathcal{M}_N\,(t-a)}, \qquad t \in I = [a, b],$$

and,

$$|E_N(t)| \leq \frac{(b-a)\, M_1}{N \mathcal{M}_N} \left(e^{\mathcal{M}_N\,(t-a)} - 1\right), \ \ t \in I,$$

where

$$\mathcal{M}_N := \frac{M_2}{\gamma_N}\, \{\mathcal{L}_f(\gamma_N) + \mathcal{L}_K(\gamma_N)\},$$

M_1 and M_2 being suitable positive constants not depending on N, and

$$\gamma_N := \max\left\{\|y\|_\infty,\ \|y'\|_\infty,\ \|S_N y\|_\infty,\ \|G_N y'\|_\infty\right\}, \tag{4.24}$$

for every $N \in \mathbb{N}^+$.

Proof. We define J_i, $i = 1, \dots, 6$ as in the proof of Theorem 4.6, for $t \in I$ fixed. We first estimate J_1, J_2, and J_3, as in the proof of Theorem 4.6. Now, we estimate J_4, J_5 and J_6. Let $\gamma_N > 0$ be the constant defined in (4.24). Using condition (a), we obtain

$$J_4 \le \mathcal{L}_f(\gamma_N)\, |f\,(t_j, y(t_j)/\gamma_N) - f\,(t_j, (S_N y)(t_j)/\gamma_N)|$$

$$\le L_f\, \frac{\mathcal{L}_f(\gamma_N)}{\gamma_N} |E_N(t_j)| \le L_f\, \frac{\mathcal{L}_f(\gamma_N)}{\gamma_N} \int_a^t |e_N(s)|\, ds,$$

with $|y(t_j)|/\gamma_N$, $|(S_N y)(t_j)|/\gamma_N \le 1$. Furthermore, we have from condition (b)

$$J_5 \le \mathcal{L}_K(\gamma_N) \int_a^{t_j} \left| K(t_j, s, \frac{y(s)}{\gamma_N}, \frac{y'(s)}{\gamma_N}) - K(t_j, s, \frac{(S_N y)(s)}{\gamma_N}, \frac{y'(s)}{\gamma_N}) \right| ds$$

$$\le L_2\, \mathcal{L}_K(\gamma_N) \int_a^{t_j} |y(s)/\gamma_N - (S_N y)(s)/\gamma_N|\; ds = L_2\, \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \int_a^{t_j} |E_N(s)|\, ds$$

$$\le L_2\, \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \int_a^{t_j} \int_a^{s} |e_N(z)|\; dz\, ds \le L_2\, (b-a)\, \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \int_a^{t} |e_N(z)|\; dz.$$

Similarly,

$$J_6 \le L_2\, \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \int_a^{t_j} |e_N(s)|\, ds \le L_2\, \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \int_a^{t} |e_N(s)|\, ds.$$

Proceeding as in Theorem 4.6,

$$|e_N(t)| \le h\, M_1 + M_2 \left\{ \frac{\mathcal{L}_f(\gamma_N)}{\gamma_N} + \frac{\mathcal{L}_K(\gamma_N)}{\gamma_N} \right\} \int_a^t |e_N(z)|\, dz$$

for every $t \in I = [a, b]$, with $h = (b-a)/N$, the constant M_1 being defined as in Theorem 4.6 and $M_2 := 2\, \max\{L_f, L_2\,(b-a), L_2\}$. Then, we obtain from Gronwall's lemma

$$|e_N(t)| \le \frac{(b-a)\, M_1}{N}\, e^{\mathcal{M}_N\,(t-a)}, \qquad t \in I = [a, b],$$

where

$$\mathcal{M}_N := \frac{M_2}{\gamma_N}\, \{\mathcal{L}_f(\gamma_N) + \mathcal{L}_K(\gamma_N)\},$$

and moreover,

$$|E_N(t)| \le \frac{(b-a)\, M_1}{N \mathcal{M}_N} \left(e^{\mathcal{M}_N\,(t-a)} - 1 \right), \;\; t \in I,$$

for every $N \in \mathbb{N}^+$. $\square$

Note that all examples of functions f and kernels K mentioned in Remark 4.8 do satisfy conditions (a) and (b) of Theorem 4.11.

Remark 4.12. Theorem 4.11 can be easily extended to provide estimates for the error affecting our method for VIDEs like (4.12) and (4.13), as in Theorem 4.9 and Theorem 4.10.

We can now use the estimate provided by Theorem 4.6 (or Theorem 4.11) to derive other estimates for the errors affecting the variant of our method introduced in Remark 4.5 for equations like (4.1).

Let σ be a fixed bounded sigmoidal function. In what follows, we denote by

$$e_N^\sigma(t) := y'(t) - (G_N^\sigma y')(t), \qquad t \in I := [a, b], \tag{4.25}$$

where $G_N^\sigma y'$ is defined in (4.20) and it is written with the coefficients α_k determined by our collocation method for equation (4.1). Furthermore, we denote by

$$E_N^\sigma(t) := \int_a^t e_N^\sigma(z)\, dz = y(t) - (S_N^\sigma y)(t), \quad t \in I := [a, b], \tag{4.26}$$

that is the *error function* for y, when the approximate solution is expressed by $S_N^\sigma y$ defined in (4.21). The following theorem provides some *a posteriori* estimates for the error functions e_N^σ and E_N^σ.

Theorem 4.13. *Let σ be a bounded sigmoidal function, and let (4.1) be a given VIDE of the neutral type, which admits of a classical solution, $y \in C^1(I)$. Under the assumptions (i), (ii), (iii), and (iv) of Theorem 4.6, we have*

$$|e_N^\sigma(t)| \le \frac{(b-a)\, M_1}{N}\, e^{M_2 (t-a)}$$
$$+ \sum_{k: t_k \le t} |\alpha_k|\, |\sigma(w\,(t - t_k)) - 1| + \sum_{k: t_k > t} |\alpha_k|\, |\sigma(w\,(t - t_k))|,$$

for every $t \in I$. Furthermore,

$$|E_N^\sigma(t)| \le \frac{(b-a)\, M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1\right)$$
$$+ |\alpha_0| \int_a^t |\sigma(w\,(s - t_{-1})) - 1|\, ds + \sum_{k=1}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s - t_k))|\, ds,$$

for every $t \in [a, t_1]$, and

$$|E_N^\sigma(t)| \le \frac{(b-a)\, M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1\right) + |\alpha_0| \int_a^t |\sigma(w\,(s - t_{-1})) - 1|\, ds$$
$$+ \sum_{k=1}^{i-1} |\alpha_k| \left[\int_a^{t_k} |\sigma(w\,(s - t_k))|\, ds + \int_{t_k}^t |\sigma(w\,(s - t_k)) - 1|\, ds\right]$$

$$+\sum_{k=i}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s-t_k))|\, ds.$$

for every $t \in [t_{i-1}, t_i]$*,* $i = 2, \ldots, N$*, and* $N \in \mathbb{N}^+$*, where* M_1 *and* M_2 *are suitable positive constants not depending on* N*.*

Proof. We first write $(G_N^{\sigma} y')(t) =: (G_N y')(t) + R_N(t)$, where $G_N y'$ is defined in (4.2) with the coefficients α_k determined in Theorem 4.1, and R_N is a suitable function. Let $t \in I := [a, b]$ be fixed. We have

$$|R_N(t)| = |(G_N^{\sigma} y')(t) - (G_N y')(t)|$$

$$= \left| \sum_{k=1}^{N} \alpha_k \left[\sigma(w\,(t-t_k)) - H(t-t_k)\right] + \alpha_0 \left[\sigma(w\,(t-t_{-1})) - 1\right] \right|,$$

where $w > 0$ is a suitable parameter depending on σ and $N \in \mathbb{N}^+$. We can observe that $\sigma(w\,(t-t_k)) - H(t-t_k) = \sigma(w\,(t-t_k)) - 1$ if $t \geq t_k$, and $\sigma(w\,(t-t_k)) - H(t-t_k) = \sigma(w\,(t-t_k))$ if $t < t_k$, for every $k = 1, \ldots, N$. Thus,

$$|R_N(t)| \leq \sum_{k:t_k \leq t} |\alpha_k|\, |\sigma(w\,(t-t_k)) - 1| + \sum_{k:t_k > t} |\alpha_k|\, |\sigma(w\,(t-t_k))|.$$

By the inequality above and Theorem 4.6, we can write

$$|e_N^{\sigma}(t)| = |y'(t) - (G_N^{\sigma} y')(t)| \leq |y'(t) - (G_N y')(t)| + |(G_N y')(t) - (G_N^{\sigma} y')(t)|$$

$$= |e_N(t)| + |R_N(t)| \leq \frac{(b-a)\,M_1}{N}\, e^{M_2\,(t-a)}$$

$$+ \sum_{k:t_k \leq t} |\alpha_k|\, |\sigma(w\,(t-t_k)) - 1| + \sum_{k:t_k > t} |\alpha_k|\, |\sigma(w\,(t-t_k))|,$$

for every $t \in I$. Finally, we can also obtain

$$|E_N^{\sigma}(t)| \leq \int_a^t |e_N^{\sigma}(s)|\, ds \leq \frac{(b-a)\,M_1}{N} \int_a^t e^{M_2\,(s-a)}\, ds$$

$$+|\alpha_0| \int_a^t |\sigma(w\,(s-t_{-1})) - 1|\, ds + \sum_{k=1}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s-t_k))|\, ds,$$

for every $t \in [a, t_1]$, i.e.,

$$|E_N^{\sigma}(t)| \leq \frac{(b-a)\,M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1\right)$$

$$+|\alpha_0| \int_a^t |\sigma(w\,(s-t_{-1})) - 1|\, ds + \sum_{k=1}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s-t_k))|\, ds,$$

for every $t \in [a, t_1]$, where the positive numbers M_1 and M_2 are those determined in the proof of Theorem 4.6. Moreover, we have, for $t \in (t_{i-1}, t_i]$, $i = 2, \ldots, N$,

$$|E_N^\sigma(t)| \le \frac{(b-a)\,M_1}{N} \int_a^t e^{M_2\,(s-a)}\,ds + |\alpha_0| \int_a^t |\sigma(w\,(s-t_{-1})) - 1|\,ds$$
$$+ \sum_{k=1}^{i-1} |\alpha_k| \left[\int_a^{t_k} |\sigma(w\,(s-t_k))|\,ds + \int_{t_k}^t |\sigma(w\,(s-t_k)) - 1|\,ds \right]$$
$$+ \sum_{k=i}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s-t_k))|\,ds,$$

or, equivalently,

$$|E_N^\sigma(t)| \le \frac{(b-a)\,M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1\right) + |\alpha_0| \int_a^t |\sigma(w\,(s-t_{-1})) - 1|\,ds$$
$$+ \sum_{k=1}^{i-1} |\alpha_k| \left[\int_a^{t_k} |\sigma(w\,(s-t_k))|\,ds + \int_{t_k}^t |\sigma(w\,(s-t_k)) - 1|\,ds \right]$$
$$+ \sum_{k=i}^{N} |\alpha_k| \int_a^t |\sigma(w\,(s-t_k))|\,ds.$$

$\square$

Note that, from the definition itself of sigmoidal functions, the terms $|\sigma(w\,(t-t_k)) - 1|$ and $|\sigma(w\,(t-t_k))|$ in the estimates of Theorem 4.13 are small, when the positive parameter w is sufficiently large.

Now, we apply Theorem 4.13 to the special case of *logistic* sigmoidal functions. This yields the following.

Corollary 4.14. *If $\sigma_\ell(t) = (1+e^{-t})^{-1}$, $t \in \mathbb{R}$, and (4.1) is a given VIDE of the neutral type, having a classical solution, $y \in C^1(I)$, we have, under the assumptions (i), (ii), (iii), and (iv) of Theorem 4.6, for every $w > \frac{N}{b-a} \ln(N-1)$,*

$$|e_N^\sigma(t)| \le \frac{(b-a)\,M_1}{N}\, e^{M_2(t-a)}$$
$$+ \sum_{k:t_k \le t} |\alpha_k| \frac{e^{-w\,(t-t_k)}}{1+e^{-w\,(t-t_k)}} + \sum_{k:t_k > t} |\alpha_k|\,(1+e^{-w\,(t-t_k)})^{-1},$$

for every $t \in I$. Furthermore,

$$|E_N^\sigma(t)| \le \frac{(b-a)\,M_1}{N M_2} \left(e^{M_2\,(t-a)} - 1\right)$$

$$+\left\{|\alpha_0| \ln\left(\frac{1+e^{-w\,(b-a)/N}}{1+e^{-w\,(t-t_{-1})}}\right) + \sum_{k=1}^{N} |\alpha_k| \ln\left(1+\frac{e^{w\,(t-a)}}{1+e^{w\,(t_k-a)}}\right)\right\} \frac{b-a}{N\ln(N-1)},$$

for every $t \in [a, t_1]$, *and*

$$|E_N^\sigma(t)| \leq \frac{(b-a)\,M_1}{NM_2}\left(e^{M_2\,(t-a)}-1\right) + \left\{|\alpha_0| \ln\left(\frac{1+e^{-w\,(b-a)/N}}{1+e^{-w\,(t-t_{-1})}}\right)\right.$$

$$\left. + 2\ln 2\left[\sum_{k=1}^{i-1}|\alpha_k|\right] + \sum_{k=i}^{N} |\alpha_k| \ln\left(1+\frac{e^{w\,(t-a)}}{1+e^{w\,(t_k-a)}}\right)\right\} \frac{b-a}{N\ln(N-1)}.$$

for every $t \in (t_{i-1}, t_i]$, $i = 2, \dots, N$, *and* $N \in \mathbb{N}^+$, $N > 2$, *where* M_1 *and* M_2 *are suitable positive constants not depending on* N.

Proof. The proof is a direct consequence of Theorem 4.13 and Corollary 2.16, just observing that

$$\int_a^{t_k} |\sigma_\ell(w\,(s-t_k))|\,ds + \int_{t_k}^{t} |\sigma_\ell(w\,(s-t_k))-1|\,ds \leq 2\ln 2\,w^{-1} \leq \frac{2\ln 2\,(b-a)}{N\ln(N-1)},$$

for $k = 1, \dots, i-1$, where $i = 2, \dots, N$, is such that $t \in (t_{i-1}, t_i]$, for $w > \frac{N}{b-a}\ln(N-1)$, and $N \in \mathbb{N}^+$, $N > 2$. □

Clearly, the analogue of Corollary 4.14 can be established for the case of the Gompertz sigmoidal functions.

Remark 4.15. Theorem 4.13 and Corollary 4.14 can also be extended to the case of VIDEs like (4.12) and (4.13), thus obtaining a posteriori estimates for the corresponding errors, e_N^σ and E_N^σ. Further estimates can be obtained replacing the assumptions of Theorem 4.6 with those of Theorem 4.11, in Theorem 4.13 (and in Corollary 4.14)

4.3 Numerical examples

In this section, we apply the collocation method developed earlier in this paper, to solve numerically some VIDEs of the form (4.1), (4.12), and (4.13).

Example 4.16. Consider the nonlinear VIDE, of the form (4.1),

$$y'(t) = 2e^{-t} - e^{-y(t)} + 2\int_0^t e^{s-t-y(s)}\left[2 - y'(s) - \ln\left(e\,(1+s)y'(s)\right)\right]\,ds,$$

with initial condition $y(0) = 0$, for $t \in [0, 1]$. Its solution is $y(t) = \ln(1+t)$.

N	$\|E_N\|_\infty$	$\|E_N^{\sigma_\ell}\|_\infty$	$\|E_N^{\sigma_{\alpha\beta}}\|_\infty$	$1/(N \ln N)$
5	2.11×10^{-2}	1.86×10^{-2}	2.6×10^{-2}	1.24×10^{-1}
10	9.9×10^{-3}	9.6×10^{-3}	1.13×10^{-2}	4.34×10^{-2}
15	6.4×10^{-3}	6.4×10^{-3}	7.1×10^{-3}	2.46×10^{-2}
30	3.2×10^{-3}	3.2×10^{-3}	3.3×10^{-3}	9.8×10^{-3}
40	2.4×10^{-3}	2.4×10^{-3}	2.5×10^{-3}	6.8×10^{-3}
100	9.44×10^{-4}	9.44×10^{-4}	9.60×10^{-4}	2.2×10^{-3}
200	4.71×10^{-4}	4.71×10^{-4}	4.75×10^{-4}	9.43×10^{-4}

Table 4.1: Numerical results for Example 4.16. E_N, $E_N^{\sigma_\ell}$, and $E_N^{\sigma_{\alpha\beta}}$ are the errors on y, see Section 4.2. The scaling parameters, w of the collocation solutions $S_N^{\sigma_\ell} y$ and $S_N^{\sigma_{\alpha\beta}} y$ are $w = N^2$ and $w = N^2/(\alpha\beta)$, respectively, see Corollary 2.16 and Corollary 2.17.

In Table 4.1, the corresponding numerical errors, obtained by our collocation method with unit step functions, are shown. In the same table, we also show the numerical errors made when logistic functions, σ_ℓ, and Gompertz functions, $\sigma_{\alpha\beta}$ with $\alpha = 0.85$ and $\beta = 0.1$, are used. >From the results of Table 4.1 it seems that the numerical errors pertaining to Example 4.16 decay to zero roughly as $1/(N \ln N)$, when N gets large, hence *faster* than shown by the theoretical results given in Section 4.2.

In order to assess accuracy and performance of our collocation method, based on sigmoidal functions, we compared it with the classical piecewise polynomial collocation method. In what follows, such a comparison is made considering the collocation solutions in the space of piecewise polynomials whose degree does not exceed $m = 2$, with collocation parameters $c_1 = 0$ and $c_2 = 1$, i.e., the well-known *Lobatto points*, see [27]. Here, $h_N := (b - a)/N$ is the uniform mesh size, and N is the number of subintervals of $[a, b]$ where the collocation is accomplished. In Table 4.2, the absolute errors for such a piecewise collocation method are given, for several values of N, for the VIDE of the neutral type in Example 4.16.

N :	5	10	15	30	100
Errors:	1.5×10^{-3}	1.13×10^{-3}	1.11×10^{-3}	6.47×10^{-4}	2.09×10^{-4}

Table 4.2: Numerical errors for the piecewise polynomial collocation method for the equation in Example 4.16.

Comparing the numerical errors in Table 4.2 with those in Table 4.1, we see, in general, the piecewise polynomial collocation methods, which are local in nature, are more accurate than the method based on sigmoidal functions. Moreover, for piecewise polynomial collocation method, $mN = 2N$ collocation points are needed, where N is the number of subintervals of $[a, b]$.

We should then compare the numerical results in Table 4.2 with those of Table 4.1, when the same number of collocation points are used.

However, our method is simpler and offers some computational advantages. Indeed, to solve *nonlinear* equations, we do *not need* to compute the solutions of a sequence of nonlinear algebraic systems, as it happens using the piecewise polynomial collocation method. To determine the coefficients of our collocation solutions, we can merely apply the explicit linear recursive formulae given in Theorem 4.1. All integrals in such formulae can be computed by numerical quadrature. In Table 4.3, we show the *CPU* times needed to compute the solutions with the sigmoidal and piecewise polynomial collocation methods for Example 4.16, for various N.

N	sigmoidal functions	piecewise polynomial
5	0.012895	0.069862
10	0.037928	0.097894
15	0.076565	0.149001
30	0.292096	0.314933

Table 4.3: Comparison between the CPU time (in seconds) for sigmoidal and piecewise collocation methods for the problem in Example 4.16.

Comparing the CPU times given in Table 4.3 we can observe that the numerical solutions by the collocation method with sigmoidal functions are computed in a shorter time than the piecewise polynomial collocation method.

In addition, our method provides a *simultaneous* approximation of the first derivative of the solution by a superposition of sigmoidal functions. In Example 4.16, $y'(t) = (1+t)^{-1}$, $t \in [0,1]$. In Table 4.4, the numerical errors made in the approximation of y' with $G_N y'$, $G_N^{\sigma_\ell} y'$ and $G_N^{\sigma_{\alpha\beta}} y'$, $\alpha = 0.85$ and $\beta = 0.1$ are shown.

N	$\|e_N\|_\infty$	$\|e_N^{\sigma_\ell}\|_\infty$	$\|e_N^{\sigma_{\alpha\beta}}\|_\infty$
5	1.50×10^{-1}	8.37×10^{-2}	1.25×10^{-1}
10	8.26×10^{-2}	6.17×10^{-2}	7.6×10^{-2}
15	5.66×10^{-2}	4.60×10^{-2}	5.61×10^{-2}
30	2.91×10^{-2}	2.75×10^{-2}	2.91×10^{-2}
40	1.96×10^{-2}	1.96×10^{-2}	1.96×10^{-2}
100	1.9×10^{-3}	4.9×10^{-3}	5.6×10^{-3}
200	9.70×10^{-4}	2.4×10^{-3}	2.8×10^{-3}

Table 4.4: Numerical results for Example 4.16. e_N, $e_N^{\sigma_\ell}$ and $e_N^{\sigma_{\alpha\beta}}$ are the errors made computing y', see Section 4.2. The scaling parameters, w, of the $G_N^{\sigma_\ell} y'$ and $G_N^{\sigma_{\alpha\beta}} y'$ are $w = N^2$ and $w = N^2/(\alpha\beta)$, respectively.

Here the numerical errors in the approximation of y' are larger than those made approximating y.

Example 4.17. Let be a classical nonlinear VIDE like (4.12), with a weakly singular kernel, K, i.e.,

$$y'(t) = \frac{1}{8\,y(t)} + \sqrt{t}\left(\frac{t}{3} + \frac{1}{2}\right) - \int_0^t \frac{y^2(s)}{\sqrt{t-s}}\,ds,$$

with initial condition $y(0) = 1/2$, for $t \in [0,1]$. Its solution is $y(t) = (\sqrt{t+1})/2$.

Weakly singular kernels are not Lipschitz continuous near the right-points of the interval $[0,t]$, with $t \in [0,1]$. This fact represents a problem for the convergence of the numerical method. The numerical errors for such example are shown in Table 4.5.

N	$\|E_N\|_\infty$	$\|E_N^{\sigma_\ell}\|_\infty$	$\|E_N^{\sigma_{\alpha\beta}}\|_\infty$	$1/(N\ln N))$
5	4.1×10^{-3}	3.6×10^{-3}	4.6×10^{-3}	1.24×10^{-1}
10	1.8×10^{-3}	1.8×10^{-3}	2×10^{-3}	4.34×10^{-2}
15	1.2×10^{-3}	1.2×10^{-3}	1.3×10^{-3}	2.46×10^{-2}
30	5.73×10^{-4}	5.71×10^{-4}	5.92×10^{-4}	9.8×10^{-3}
40	4.26×10^{-4}	4.25×10^{-4}	4.37×10^{-4}	6.8×10^{-3}
100	1.67×10^{-4}	1.67×10^{-4}	1.69×10^{-4}	2.2×10^{-3}

Table 4.5: Numerical results for Example 4.17. E_N, $E_N^{\sigma_\ell}$, and $E_N^{\sigma_{\alpha\beta}}$ with $\alpha = 0.85$ and $\beta = 0.1$, are the errors made evaluating y. The scaling parameters, w, of $S_N^{\sigma_\ell}y$ and $S_N^{\sigma_{\alpha\beta}}y$ are $w = N^2$ and $w = N^2/(\alpha\beta)$, respectively.

In Example 4.17, the numerical errors seem to decay roughly as $1/(N\ln N)$, as well as in Example 4.16.

In Example 4.17, again, it is more interesting to compare the numerical errors of Table 4.5 with those obtained by piecewise collocation. In Table 4.6, we show the absolute errors of the piecewise polynomial method applied to the equation in Example 4.17 on $[0,1]$.

N :	5	10	15	30	100
Errors:	5.44×10^{-2}	1.7×10^{-2}	9×10^{-3}	3.1×10^{-3}	5.03×10^{-4}

Table 4.6: Numerical errors for the piecewise polynomial collocation method for the equation in Example 4.17.

Comparing the results in Tables 4.5 and 4.6, we can observe that the collocation method with sigmoidal functions is more accurate than the piecewise polynomial collocation method. Therefore, our numerical method seems to be competitive in case of equations with weakly singular kernels.

Example 4.18. We consider the non-standard VIDE of the form (4.13),

$$y'(t) = y(t)\left(\widetilde{f}(t) + \int_0^t e^{-(t-s)} y(s)\, ds\right),$$

where

$$\widetilde{f}(t) := e^{-2t} - e^{-t} - 2,$$

and initial condition $y(0) = 1$, $t \in [0,1]$, whose solution is $y(t) = e^{-2t}$, see [29].

The corresponding numerical errors are given in Table 4.7.

N	$\|E_N\|_\infty$	$\|E_N^{\sigma_\ell}\|_\infty$	$\|E_N^{\sigma_{\alpha\beta}}\|_\infty$
5	8.99×10^{-2}	7.85×10^{-2}	1.01×10^{-1}
10	4.09×10^{-2}	3.98×10^{-2}	4.47×10^{-2}
15	2.63×10^{-2}	2.61×10^{-2}	2.82×10^{-2}
30	1.29×10^{-2}	1.28×10^{-2}	1.34×10^{-2}
40	9.6×10^{-3}	9.6×10^{-3}	9.9×10^{-3}
100	3.8×10^{-3}	3.8×10^{-3}	3.8×10^{-3}
200	1.9×10^{-3}	1.9×10^{-3}	1.9×10^{-3}

Table 4.7: Numerical results for Example 4.18. E_N, $E_N^{\sigma_\ell}$ and $E_N^{\sigma_{\alpha\beta}}$ with $\alpha = 0.85$ and $\beta = 0.1$ are the errors made evaluating y. The scaling parameters, w, of $S_N^{\sigma_\ell} y$ and $S_N^{\sigma_{\alpha\beta}} y$ are $w = N^2$ and $w = N^2/(\alpha\beta)$, respectively.

Again, in Example 4.18, the same observation made for Example 4.16 and Example 4.17 applies, concerning the decay rate of the numerical errors. Here, $\|E_N\|_\infty$, $\|E_N^{\sigma_\ell}\|_\infty$ and $\|E_N^{\sigma_{\alpha\beta}}\|_\infty$ seem to decrease as $C/(N \ln N)$, being $C > 1$ a suitable constant.

Similar considerations can be made for Example 4.18, as for Example 4.16. In Table 4.8, the numerical errors obtained using piecewise polynomial collocation on $[0,1]$ are given. Also in this case, the errors of Table 4.8 turn out to be smaller then those in Table 4.7.

N :	5	10	15	30	100
Errors:	1.33×10^{-2}	6.40×10^{-3}	4.2×10^{-3}	2.1×10^{-3}	6.34×10^{-4}

Table 4.8: Numerical errors for the piecewise polynomial collocation method for the equation in Example 4.18.

Example 4.19. Consider the nonlinear VIDE of the neutral type:

$$y'(t) = 2t^3 - 2y^2(t) - 6t^2 + 13t + 12e^{-t} - 12 + \int_0^t e^{t-s} y(s)(y'(s))^2\, ds,$$

for $t \in [0,1]$, subject to the initial condition $y(0) = 0$. Its solution is $y(t) = (1/2)t^2$.

The numerical errors for such example are shown in Table 4.9.

N	$\|E_N\|_\infty$	$\|E_N^{\sigma_\ell}\|_\infty$	$\|E_N^{\sigma_{\alpha\beta}}\|_\infty$
5	6.28×10^{-2}	2.59×10^{-2}	5.57×10^{-2}
10	5×10^{-3}	3.9×10^{-3}	6.8×10^{-3}
15	2.8×10^{-3}	2.5×10^{-3}	3.9×10^{-3}
30	1.4×10^{-3}	1.3×10^{-3}	1.6×10^{-3}
40	1×10^{-3}	1×10^{-3}	1.1×10^{-3}
100	4.07×10^{-4}	4.06×10^{-4}	4.21×10^{-4}
200	2.03×10^{-4}	2.03×10^{-4}	2.07×10^{-4}

Table 4.9: Numerical results for Example 4.19. E_N, $E_N^{\sigma_\ell}$ and $E_N^{\sigma_{\alpha\beta}}$ with $\alpha = 0.85$ and $\beta = 0.1$ are the errors made evaluating y. The scaling parameters, w, of $S_N^{\sigma_\ell} y$ and $S_N^{\sigma_{\alpha\beta}} y$ are $w = N^2$ and $w = N^2/(\alpha\beta)$, respectively.

N :	5	10	15	30	100
Errors:	3.8×10^{-3}	1.7×10^{-3}	1.1×10^{-3}	5.33×10^{-4}	1.54×10^{-4}

Table 4.10: Numerical errors for the piecewise polynomial collocation method for the equation in Example 4.19.

In Example 4.19, we can determine all parameters required, to obtain the a posteriori error estimates of Section 4.2.2. The constants M_1 and M_2 in Theorems 4.6 and 4.11 (which *do depend neither* on the number N of sigmoidal functions *nor* on the mesh size, h) are

$$M_1 = M + C + L_1, \quad M_2 = 2L_2 + L_f,$$

where $M = \sqrt{L_y^2 + 1} = \sqrt{2}$, $C = L_1 = e/2$, $L_2 = 2e$, and $L_f = 16$, hence,

$$M_1 = \sqrt{2} + e, \quad M_2 = 4e + 16.$$

Moreover, the functions introduced in conditions (a) and (b) of Theorem 4.11 are $\mathcal{L}_f(\gamma) = \gamma^2$ and $\mathcal{L}_K(\gamma) = \gamma^3$, $\gamma \geq 0$, and finally

$$\gamma_N := \max\left\{1, \|S_N y\|_\infty, \|G_N y'\|_\infty\right\},$$

and,

$$\mathcal{M}_N = \frac{(4e+16)}{N}\left(\gamma_N + \gamma_N^2\right).$$

Therefore, the a posteriori estimate provided by Theorem 4.11 for the collocation solutions of such neutral equation is given by

$$|E_N(t)| \;\leq\; \frac{(\sqrt{2}+e)}{N\mathcal{M}_N}\left(e^{\mathcal{M}_N t}-1\right) \;=:\; R_N(t),$$

for every $t \in [0,1]$. This estimate is very sharp, especially when t is near to zero. In Table 4.11, a comparison between the numerical errors $|E_N(t)|$ and the *a posteriori* error estimates $R_N(t)$ of Theorem 4.11, for Example 4.19, is given. As a rule, a priori estimates provide overestimates for the numerical

N	t	$\lvert E_N(t)\rvert$	$R_N(t)$
5	0.05	0.0013	0.2106
5	0.07	0.0025	0.6466
30	0.05	5.8332×10^{-4}	0.0351
30	0.07	8.5885×10^{-4}	0.1078
30	0.1	0.0011	0.5507
40	0.05	5.3125×10^{-4}	0.0263
40	0.1	7.9136×10^{-4}	0.4130
100	0.05	1.9677×10^{-4}	0.0105
100	0.1	3.0052×10^{-4}	0.1652
100	0.13	3.3732×10^{-4}	0.8315

Table 4.11: Comparison between the numerical errors $|E_N(t)|$ and the *a posteriori* error estimates $R_N(t)$ of Theorem 4.11, for Example 4.19.

errors. Clearly, since in the error inequality appear an exponential function with exponent $\mathcal{M}_N t$ depending on the (large) constant $\mathcal{M}_N$, for larger values of t the previous estimate cannot be very sharp. Similar considerations can be made in case of a posteriori estimate obtained approximating solutions to neutral integro-differential equations by logistic or Gompertz functions.

Chapter 5

Approximation by Series of Sigmoidal Functions with Applications to Neural Networks

As shown in the last two chapters, the approximation results of Chapter 2 can be used to obtain numerical solutions of Volterra integral and integro-differential equations. The numerical methods that have been proposed, are very easy to implement and they present various advantages, which we have been previously described in details.

However, our numerical methods have not high accuracy. Then, we study new kinds of approximation techniques by sigmoidal functions, in order to use them to improve the accuracy of our numerical methods.

In this chapter, we develop a constructive theory for approximating absolutely continuous functions by series of certain sigmoidal functions. Estimates for the approximation error are also derived. The relation with neural networks approximation is discussed. The connection between sigmoidal functions and the scaling functions of r-regular multiresolution approximations are investigated. In this setting, we show that the approximation error for C^1-functions decreases as 2^{-j}, as $j \to +\infty$. Examples with sigmoidal functions of several kinds, such as logistic, hyperbolic tangent, and Gompertz functions, are given.

For the present theory the readers refer to [58].

The main idea introduced in this chapter, is to start from appropriate real valued functions, ϕ, normalized so that $\int_{\mathbb{R}} \phi(t)\,dt = 1$, and to construct sigmoidal functions having the integral form $\sigma_\phi(x) := \int_{-\infty}^{x} \phi(t)\,dt$, $x \in \mathbb{R}$.

In this way, we can define the operators

$$(S_w^{\sigma_\phi} f)(x) := \sum_{k\in\mathbb{Z}} \left[\int_a^b \phi(wy-k) f'(y)\, dy\right] \sigma_\phi(wx-k) + f(a),$$

$x \in [a,b]$, where f is an absolutely continuous function on $[a,b] \subset \mathbb{R}$, and $w > 0$.

We can show that, the family $(S_w^{\sigma_\phi} f)_{w>0}$ converges to f uniformly on $[a,b]$. Moreover, we derive estimates for the *approximation error* and the *truncation error* of the series.

A remarkable result is obtained when ϕ is the real-valued wavelet scaling function associated to an r-regular multiresolution approximation of $L^2(\mathbb{R})$, constructed by a suitable procedure, see [109, 52, 64, 113]. In this setting, we replace the weights w with 2^j, $j \in \mathbb{N}^+$, as it seems more natural in view of the relation that ϕ has with the multiresolution approximation. Also in this case, we can show that the family of the operators $(S_w^{\sigma_\phi} f)_{w>0}$ converges to f as $j \to +\infty$, uniformly on $[a,b]$ Approximating C^1-functions, we obtain an approximation error decreasing to zero as 2^{-j} when $j \to +\infty$.

Such a theory, in the present form, however, does not cover the important cases of NNs activated by either logistic, hyperbolic tangent or Gompertz sigmoidal functions. Therefore, we propose an extension of the theory previously developed, which includes such cases, also providing estimates for the approximation errors for functions belonging to suitable Lipschitz class.

5.1 Approximation by series of sigmoidal functions

In what follows, we denote by $AC[a,b]$ the sets of all absolutely continuous functions, $f : [a,b] \to \mathbb{R}$, on the bounded closed nonempty interval $[a,b]$. Moreover, we recall that, by $\widehat{C}^n[a,b]$, $n \in \mathbb{N}^+$, we will denote the set of all functions $f \in C^n(a',b')$, for some open real interval (a',b'), such that $[a,b] \subset (a',b')$.

Let introduce the class of functions we will work with.

Definition 5.1. *The function $\phi : \mathbb{R} \to \mathbb{R}_0^+$ is said to belong to the class Φ, if it satisfies the following conditions:*

$(\varphi 1)$ *ϕ is continuous on $\mathbb{R}$ and there exists $C > 0$ such that*

$$\phi(x) \leq C(1+|x|)^{-\alpha},$$

for every $x \in \mathbb{R}$, and for some $\alpha \geq 2$;

$(\varphi 2)$ $\displaystyle\sum_{k\in\mathbb{Z}} \phi(x-k) = 1$, *for every $x \in \mathbb{R}$.*

Remark 5.2. The condition ($\varphi 2$) is equivalent to

$$\widehat{\phi}(k) := \begin{cases} 0, & k \in \mathbb{Z} \setminus \{0\}, \\ 1, & k = 0, \end{cases}$$

where $\widehat{\phi}(v) := \int_{\mathbb{R}} \phi(t)\, e^{-ivt}\, dt$, $v \in \mathbb{R}$, is the Fourier transform of ϕ; see [36]. In particular, it turns out that $\widehat{\phi}(0) = \int_{\mathbb{R}} \phi(t)\, dt = 1$.

For any fixed $\phi \in \Phi$, the function $K_\phi : \mathbb{R}^2 \to \mathbb{R}_0^+$, defined by

$$K_\phi(x, y) := \sum_{k \in \mathbb{Z}} \phi(x - k)\, \phi(y - k), \quad (x, y) \in \mathbb{R}^2, \tag{5.1}$$

will be called the *kernel* associated to ϕ. Clearly, it follows from condition ($\varphi 2$) and by Remark 5.2 that

$$\int_{\mathbb{R}} K_\phi(x, y)\, dy = 1, \quad \text{for every } x \in \mathbb{R}. \tag{5.2}$$

Moreover, using ($\varphi 1$), it is easy to see that

$$K_\phi(x, y) \leq L\, (1 + |x - y|)^{-\alpha}, \quad \text{for every } x,\ y \in \mathbb{R}, \tag{5.3}$$

for some positive constant L. Under the previous assumptions on K_ϕ, the following lemma, which will turn out to be useful later, could be established. Its proof is classical and can be found in [113].

Lemma 5.3. *Let $(T_w)_{w>0}$ be the family of operators defined explicitly by*

$$(T_w f)(x) := w \int_{\mathbb{R}} K(wx, wy)\, f(y)\, dy, \quad x \in \mathbb{R},$$

for $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$), and where the kernel $K : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{C}$) meets the conditions (5.2) *and* (5.3)*. Then, for any uniformly continuous and bounded function f, we have*

$$\lim_{w \to +\infty} \|T_w f - f\|_\infty = 0.$$

Moreover, for every $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$, it results

$$\lim_{w \to +\infty} \|T_w f - f\|_p = 0.$$

Let now $\phi \in \Phi$ be fixed and define the function $\sigma_\phi : \mathbb{R} \to \mathbb{R}_0^+$ as

$$\sigma_\phi(x) := \int_{-\infty}^{x} \phi(t)\, dt, \quad x \in \mathbb{R}. \tag{5.4}$$

Clearly, from condition ($\varphi 2$) and Remark 5.2, such a function σ_ϕ is a sigmoidal function. We can now give the following

Definition 5.4. *For every fixed function $\phi \in \Phi$, we define the family of operators $(S_w^{\sigma_\phi})_{w>0}$ by*

$$(S_w^{\sigma_\phi} f)(x) := \sum_{k\in\mathbb{Z}} \left[\int_a^b \phi(wy-k)\, f'(y)\, dy\right] \sigma_\phi(wx-k) + f(a), \quad x \in [a,b], \tag{5.5}$$

for every $f \in AC[a,b]$ and $w > 0$. We call $S_w^{\sigma_\phi} f$ the "series of sigmoidal functions for f, based on ϕ", for the given value of $w > 0$.

Clearly, when f is a constant function, the Definition 5.4 becomes trivial. Now we can prove the following

Theorem 5.5. *Let $\phi \in \Phi$ be fixed. For any given $f \in AC[a,b]$, the family $(S_w^{\sigma_\phi} f)_{w>0}$ converges uniformly to f on $[a,b]$, i.e.,*

$$\lim_{w\to\infty} \|S_w^{\sigma_\phi} f - f\|_\infty = 0.$$

Moreover, if $f \in \widehat{C}^1[a,b]$, we have

$$\|S_w^{\sigma_\phi} f - f\|_\infty \leq \widetilde{C} w^{-1},$$

for some positive constant $\widetilde{C}$ and for every $w > 0$.

Proof. Since $f \in AC[a,b]$, $f(x) = \int_a^x f'(z)\, dz\ + f(a)$ for every $x \in [a,b]$. Then, setting $\widetilde{f'}(z) = f'(z)$ for $z \in [a,b]$ and $\widetilde{f'}(z) = 0$ for $z \notin [a,b]$, we obtain

$$|(S_w^{\sigma_\phi} f)(x) - f(x)| = \left|\sum_{k\in\mathbb{Z}} \left[\int_a^b \phi(wy-k)\, f'(y)\, dy\right] \sigma_\phi(wx-k) - \int_a^x f'(z)\, dz\right|$$

$$= \left|\sum_{k\in\mathbb{Z}} \left[\int_{\mathbb{R}} \phi(wy-k)\, \widetilde{f'}(y)\, dy\right] \int_{-\infty}^{wx-k} \phi(t)\, dt - \int_{-\infty}^x \widetilde{f'}(z)\, dz\right|.$$

Changing variable, by setting $t = wz - k$, we get

$$\begin{aligned} & |(S_w^{\sigma_\phi} f)(x) - f(x)| \\ \leq\ & \int_{-\infty}^x \left|\sum_{k\in\mathbb{Z}} \left[w \int_{\mathbb{R}} \phi(wy-k)\, \widetilde{f'}(y)\, dy\right] \phi(wz-k) - \widetilde{f'}(z)\right| dz \\ =\ & \int_{-\infty}^x \left|w \int_{\mathbb{R}} K_\phi(wz, wy)\, \widetilde{f'}(y)\, dy - \widetilde{f'}(z)\right| dz \\ \leq\ & \int_{-\infty}^{+\infty} \left|w \int_{\mathbb{R}} K_\phi(wz, wy)\, \widetilde{f'}(y)\, dy - \widetilde{f'}(z)\right| dz. \end{aligned} \tag{5.6}$$

Being $\widetilde{f'} \in L^1(\mathbb{R})$, we obtain by Lemma 5.3 and inequality (5.6)

$$\lim_{w\to+\infty} \|S_w^{\sigma_\phi} f - f\|_\infty \leq \lim_{w\to+\infty} \|T_w \widetilde{f'} - \widetilde{f'}\|_1 = 0,$$

which completes the proof of the first part of the theorem.

Consider now $f \in \widehat{C}^1[a,b]$. Note that, by conditions $(\varphi 2)$ and (5.2), we have

$$w \int_{\mathbb{R}} K_\phi(wz, wy)\, dy = 1, \quad \text{for every } z \in \mathbb{R} \text{ and } w > 0.$$

Then, again from inequality (5.6), we obtain

$$\begin{aligned}
& |(S_w^{\sigma_\phi} f)(x) - f(x)| \\
\le\ & \int_{\mathbb{R}} \left| w \int_{\mathbb{R}} K_\phi(wz, wy)\, \widetilde{f}'(y)\, dy - \widetilde{f}'(z) w \int_{\mathbb{R}} K_\phi(wz, wy)\, dy \right| dz \\
\le\ & w \int_{\mathbb{R}} \int_{\mathbb{R}} K_\phi(wz, wy)\, |\widetilde{f}'(y) - \widetilde{f}'(z)|\, dy\, dz \\
\le\ & 2w \|f'\|_\infty \int_{\mathbb{R}} \int_{\mathbb{R}} K_\phi(wz, wy)\, dy\, dz. \qquad (5.7)
\end{aligned}$$

Changing the variables z and y in the last integral in (5.7) with z_1/w and y_1/w, respectively, we obtain, in view of condition (5.3),

$$\begin{aligned}
\|S_w^{\sigma_\phi} f - f\|_\infty \ \le\ & 2w^{-1} \|f'\|_\infty \int_{\mathbb{R}} \int_{\mathbb{R}} K_\phi(z_1, y_1)\, dy_1\, dz_1 \\
\le\ & 2w^{-1} \|f'\|_\infty\, L \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |z_1 - y_1|)^{-\alpha}\, dy_1\, dz_1 =: \widetilde{C} w^{-1},
\end{aligned}$$

for every $w > 0$, for some $\widetilde{C} > 0$, and where $\alpha \ge 2$ is the constant of condition $(\varphi 1)$. This completes the proof of the second part of the theorem. $\square$

Examples of functions $\phi \in \Phi$ will be given in the next sections.

5.2 Application to neural networks

Here we give some applications of the theory developed in the previous sections to NNs activated by the sigmoidal functions generated by (5.4).

We will denote by Φ_C the subset of Φ of functions having a compact support.

Let $\phi \in \Phi_C$ be fixed, and let M_1, $M_2 > 0$ such that $\operatorname{supp} \phi \subseteq [-M_1, M_2]$. In this case, we have for any $f \in AC[a,b]$ and $w > 0$,

$$\int_a^b \phi(wy - k) f'(y)\, dy = 0,$$

for every $k < wa - M_2$ and $k > wb + M_1$, $k \in \mathbb{Z}$, since for these values of k, $[wa-k, wb-k] \cap [-M_1, M_2] = \emptyset$. Then, the series appearing in the definition of the operator $S_w^{\sigma_\phi} f$ reduces to a finite sum, i.e.,

$$(S_w^{\sigma_\phi} f)(x) = \sum_{k=\lfloor wa - M_2 \rfloor}^{\lceil wb + M_1 \rceil} \left[\int_a^b \phi(wy - k)\, f'(y)\, dy \right] \sigma_\phi(wx - k) + f(a), \quad (5.8)$$

for every $x \in [a, b]$, where the functions $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the upper and the lower integer part of $x \in \mathbb{R}$, respectively. Now, we introduce the following modification in Definition 5.4 for the case $\phi \in \Phi_C$. For any $f \in AC[a, b]$, set

$$(G_w^{\sigma_\phi} f)(x) :=$$
$$\sum_{k=\lfloor wa - M_2 \rfloor}^{\lceil wb + M_1 \rceil} \left[\int_a^b \phi(wy - k)\, f'(y)\, dy \right] \sigma_\phi(wx - k) + f(a)\, \sigma_\phi(w(x - a + 1)),$$

for every $x \in [a, b]$ and $w > 0$. The $G_w^{\sigma_\phi} f$'s are a kind of NNs. They approximate f, uniformly on $[a, b]$, as $w \to +\infty$. The proof of this claim follows from the same arguments made in Theorem 5.5, taking into account that

$$\sup_{x \in [a,b]} |f(a)||1 - \sigma_\phi(w(x - a + 1))| \leq |f(a)||1 - \sigma_\phi(w)| \ = \ 0, \qquad (5.9)$$

for $w > 0$ sufficiently large. Indeed, by the definition of σ_ϕ, for every $w > M_2$ we have

$$\sigma_\phi(w) \ = \ \int_{-\infty}^{w} \phi(x) \ dx \ = \ \int_{\mathbb{R}} \phi(x) \ dx = 1. \qquad (5.10)$$

Moreover, again by Theorem 5.5, if $f \in \widehat{C}^1[a, b]$ we obtain the convergence rate given by $\|G_w^{\sigma_\phi} f - f\|_\infty \leq \widetilde{C} w^{-1}$, for some positive constants $\widetilde{C}$ and for every sufficiently large $w > 0$.

In addition, our proofs are *constructive* in nature, and allow us to determine explicitly the form of the NN. In particular, we show that the set of NNs $G_w^{\sigma_\phi} f$ is dense in the set $AC[a, b]$, with respect to the uniform norm.

Now, we show that we can obtain NNs also starting from functions $\phi \in \Phi$ which are not necessarily compactly supported. Let first prove the following

Lemma 5.6. *The series $\sum_{k \in \mathbb{Z}} \phi(wx - k)$ converges uniformly on the compact subsets of $\mathbb{R}$, for every fixed $w > 0$.*

In particular, we have for every $[a, b] \subset \mathbb{R}$

$$\sup_{x \in [a,b]} \sum_{|k| > N} \phi(wx - k) \leq \overline{C} \left\{ (N - wb + 1)^{-(\alpha - 1)} + (N + wa + 1)^{-(\alpha - 1)} \right\},$$

for some $\overline{C} > 0$, for every $N > w \max\{|a|, |b|\}$, $N \in \mathbb{N}^+$, where $\alpha \geq 2$ is the constant of condition $(\varphi 1)$.

Proof. Let $[a, b] \subset \mathbb{R}$ be fixed. By condition $(\varphi 1)$ and for $N > w \max\{|a|, |b|\}$ we have

$$\sup_{x \in [a,b]} \sum_{|k| > N} \phi(wx - k) \leq C \sup_{x \in [a,b]} \sum_{|k| > N} (1 + |wx - k|)^{-\alpha}$$

$$= C\left\{\sup_{x\in[a,b]}\sum_{k>N}(1+|wx-k|)^{-\alpha}+\sup_{x\in[a,b]}\sum_{k>N}(1+|wx+k|)^{-\alpha}\right\}$$

$$\leq C\left\{\sum_{k>N}(1+k-wb)^{-\alpha}+\sum_{k>N}(1+wa+k)^{-\alpha}\right\}\leq C\left\{\int_N^{+\infty}(1+x-wb)^{-\alpha}dx\right.$$

$$\left.+\int_N^{+\infty}(1+wa+x)^{-\alpha}dx\right\} =: \overline{C}\left\{(N-wb+1)^{-(\alpha-1)}+(N+wa+1)^{-(\alpha-1)}\right\}.$$

The proof then follows. □

We can now establish the following

Theorem 5.7. *(i) For any $f \in AC[a,b]$, we denote by*

$$(G^{\sigma_\phi}_{N,w}f)(x) := \sum_{k=-N}^{N}\left[\int_a^b \phi(wy-k)\,f'(y)\,dy\right]\sigma_\phi(wx-k)$$

$$+f(a)\,\sigma_\phi(w(x-a+1)), \tag{5.11}$$

for $x \in [a,b]$, $w > 0$, and $N \in \mathbb{N}^+$. Then, for every $\varepsilon > 0$ there exist $w > 0$ and $N \in \mathbb{N}^+$ such that

$$\|G^{\sigma_\phi}_{N,w}f - f\|_\infty < \varepsilon.$$

(ii) Moreover, for any $f \in \widehat{C}^1[a,b]$ we have

$$\|G^{\sigma_\phi}_{N,w}f - f\|_\infty \leq C_1\left\{(N-wb+1)^{-(\alpha-1)}+(N+wa+1)^{-(\alpha-1)}\right\}$$

$$+C_2\,w^{-1}+C_3\,w^{-(\alpha-1)},$$

for some constants C_1, C_2, $C_3 > 0$, and for every $w > 0$ with $N > w\max\{|a|,\ |b|\}$, $N \in \mathbb{N}^+$, where $\alpha \geq 2$ is the constant appearing in condition $(\varphi 1)$.

Proof. (i) Let $\varepsilon > 0$ be fixed. For every $x \in [a,b]$ we have

$$|(G^{\sigma_\phi}_{N,w}f)(x) - f(x)| \leq |(G^{\sigma_\phi}_{N,w}f)(x) - (S^{\sigma_\phi}_w f)(x)| + |(S^{\sigma_\phi}_w f)(x) - f(x)|$$

$$\leq \sum_{|k|>N}\left[\int_a^b \phi(wy-k)\,|f'(y)|\,dy\right]\sigma_\phi(wx-k) + |f(a)||1-\sigma_\phi(w(x-a+1))|$$

$$+ \quad \|S^{\sigma_\phi}_w f - f\|_\infty =: S_1 + S_2 + S_3. \tag{5.12}$$

Proceeding as in (5.9) and using $(\varphi 1)$, we can write

$$S_2 \leq |f(a)||1-\sigma_\phi(w)| = |f(a)|\int_w^{+\infty}\phi(x)\,dx$$

$$\leq |f(a)| C \int_w^{+\infty} (1+x)^{-\alpha} dx =: \underline{C}\,(1+w)^{-(\alpha-1)}, \tag{5.13}$$

where $\alpha \geq 2$ is the constant appearing in condition ($\varphi 1$), and $\underline{C} > 0$, then $S_2 < \varepsilon$ for $w > 0$ sufficiently large. Moreover, we obtain from Theorem 5.5 that $S_3 < \varepsilon$ for $w > 0$ sufficiently large. Finally, we can estimate S_1. Being $\|\sigma_\phi\|_\infty \leq 1$, we obtain for S_1

$$\begin{aligned} S_1 \;&\leq\; \|\sigma_\phi\|_\infty \sum_{|k|>N} \left[\int_a^b \phi(wy-k)\, |f'(y)|\, dy \right] \\ &\leq \left[\sup_{y\in[a,b]} \sum_{|k|>N} \phi(wy-k) \right] \int_a^b |f'(y)|\, dy. \end{aligned} \tag{5.14}$$

We have by Lemma 5.6, for every fixed and sufficiently large $w > 0$,

$$\sup_{y\in[a,b]} \sum_{|k|>N} \phi(wy-k) \leq \overline{C} \left\{ (N-wb+1)^{-(\alpha-1)} + (N+wa+1)^{-(\alpha-1)} \right\}, \tag{5.15}$$

for some constant $\overline{C} > 0$ and for every $N > w \max\{|a|, |b|\}$ with $N \in \mathbb{N}^+$. Then, for N sufficiently large we obtain $S_1 < \varepsilon$. This completes the proof of (i).

(ii) For any $f \in \widehat{C}^1[a,b]$, Theorem 5.5 shows that $S_3 \leq \widetilde{C} w^{-1}$ uniformly with respect to $x \in [a,b]$, for every $w > 0$. Moreover, we obtain by (5.13) and (5.15)

$$\begin{aligned} S_1 + S_2 + S_3 &\leq \overline{C} \left[\int_a^b |f'(y)|\, dy \right] \Big\{ (N-wb+1)^{-(\alpha-1)} \\ &\quad + (N+wa+1)^{-(\alpha-1)} \Big\} + \widetilde{C}\, w^{-1} + \underline{C}\,(1+w)^{-(\alpha-1)} \\ &\leq C_1 \left\{ (N-wb+1)^{-(\alpha-1)} + (N+wa+1)^{-(\alpha-1)} \right\} + C_2\, w^{-1} + C_3\, w^{-(\alpha-1)}, \end{aligned}$$

uniformly with respect to $x \in [a,b]$, for some constants C_1, C_2, $C_3 > 0$, and for $w > 0$ sufficiently large, with $N > w \max\{|a|,\ |b|\}$. □

Remark 5.8. Setting $C_3 = 0$ in Theorem 5.7 (ii), we also obtain an estimate for the *truncation error* for the series of sigmoidal functions introduced in Section 5.1. Note that, when the weight, w, increases, we need a higher number of neurons, N, which depends on w.

We now construct few examples of sigmoidal functions, σ_ϕ, providing first some examples of functions $\phi \in \Phi_C$ satisfying all hypotheses of our theory. Recall that the "central B-splines" of order $n \in \mathbb{N}^+$, are defined as

$$M_n(x) := \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{n}{2} + x - i \right)_+^{n-1},$$

where $(x)_+ := \max\{x, 0\}$ is the positive part of $x \in \mathbb{R}$ [34]. The Fourier transform of M_n is given by

$$\widehat{M_n}(v) := \mathrm{sinc}^n\left(\frac{v}{2\pi}\right), \quad v \in \mathbb{R},$$

where the *sinc* function is defined by

$$\mathrm{sinc}(x) := \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0. \end{cases}$$

The M_n's are bounded and continuous on $\mathbb{R}$ for all $n \in \mathbb{N}^+$, and are compactly supported on $[-n/2, n/2]$. This implies that $M_n \in L^1(\mathbb{R})$ and satisfies condition $(\varphi 1)$ for every $\alpha \geq 2$. Finally, condition $(\varphi 2)$ holds, in view of Remark 5.2, hence, $M_n \in \Phi_C$ for every $n \in \mathbb{N}^+$. Therefore, we can construct explicitly the NNs $G_w^{\sigma_{M_n}} f$, $n \in \mathbb{N}^+$.

As an example of function $\phi \in \Phi$ which is not compactly supported, consider the continuous function

$$F(x) := \frac{1}{2\pi}\mathrm{sinc}^2\left(\frac{x}{2\pi}\right), \qquad x \in \mathbb{R}.$$

Clearly, $F(x) = \mathcal{O}(x^{-2-\varepsilon})$ as $x \to \pm\infty$, $\varepsilon > 0$, hence F satisfies condition $(\varphi 1)$ with $\alpha = 2$, see [34]. Moreover, its Fourier transform is

$$\widehat{F}(v) := \begin{cases} 1 - |v|, & |v| \leq 1, \\ 0, & |v| > 1, \end{cases}$$

(see [34] again). By Remark 5.2, F satisfies also condition $(\varphi 2)$, and then $F \in \Phi$.

Remark 5.9. Note that the theory developed in this section cannot be applied to the case of NNs activated by the logistic functions, $\sigma_\ell(x) := (1 + e^{-x})^{-1}$, or to the hyperbolic tangent sigmoidal functions, $\sigma_h(x) := \frac{1}{2} + \frac{1}{2}\tanh(x) = \frac{1}{2} + \frac{e^{2x}-1}{2(e^{2x}+1)}$. In fact, σ_ℓ and σ_h can be generated by (5.4) from $\phi_\ell(x) := e^{-x}(1 + e^{-x})^{-2}$ and $\phi_h(x) := 2e^{2x}(e^{2x} + 1)^{-2}$, respectively. However, $\widehat{\phi_\ell}(v) = \pi v / \sinh(\pi v)$ and $\widehat{\phi_h}(v) = \pi v/(2\sinh(\pi v/2))$, respectively, which do not meet the condition in Remark 5.2, i.e., do not satisfy condition $(\varphi 2)$. In Section 5.4 below, an extension of the theory developed above is proposed, which allows to use NNs activated by σ_ℓ or σ_h.

5.3 Sigmoidal functions and multiresolution approximation

In this section, we will show a connection between the theory of multiresolution approximation and our theory for approximating functions by series of sigmoidal functions. We first recall some basic facts concerning the multiresolution approximation. For the detailed theory, see [109, 52, 64, 113, 134]. We start recalling the following.

Definition 5.10. *A "multiresolution approximation" of $L^2(\mathbb{R})$ is an increasing sequence, V_j, $j \in \mathbb{Z}$, of linear closed subspaces of $L^2(\mathbb{R})$, enjoying the following properties:*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}\,, \qquad \bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}); \tag{5.16}$$

for all $f \in L^2(\mathbb{R})$ and all $j \in \mathbb{Z}$,

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}; \tag{5.17}$$

for all $f \in L^2(\mathbb{R})$ and all $k \in \mathbb{Z}$,

$$f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0; \tag{5.18}$$

there exists a function, $h(x) \in V_0$, such that the sequence

$$(h(x-k))_{k\in\mathbb{Z}} \text{ is a Riesz basis of } V_0. \tag{5.19}$$

Recall that a sequence of functions $(h_k)_{k\in\mathbb{Z}}$ is a Riesz basis of an Hilbert space, $H \subseteq L^2(\mathbb{R})$, if there exist two constants, C_1 and C_2, with $C_1 > C_2 > 0$, such that, for every sequence of real or complex numbers $(a_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, it turns out that

$$C_2 \left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{1/2} \leq \left\|\sum_{k\in\mathbb{Z}} a_k h_k\right\|_{L^2(\mathbb{R})} \leq C_1 \left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{1/2},$$

and the vector space of finite linear combinations of h_k, is dense in H.

Definition 5.11. *A multiresolution approximation, V_j, $j \in \mathbb{Z}$, is called "r-regular" ($r \in \mathbb{N}^+$), if the function h in (5.19) is such that $h \in C^r(\mathbb{R})$ and*

$$|h^{(i)}(x)| \leq C_m (1+|x|)^{-m}, \quad x \in \mathbb{R}, \tag{5.20}$$

for each integer $m \in \mathbb{N}^+$ and for every positive index $i \leq r$.

For every r-regular multiresolution approximation V_j, $j \in \mathbb{Z}$, we can define the function $\phi \in L^2(\mathbb{R})$, called *scaling function*, as

$$\widehat{\phi}(v) := \widehat{h}(v) \left(\sum_{k\in\mathbb{Z}} |\widehat{h}(v+2\pi k)|^2\right)^{-1/2}, \; v \in \mathbb{R}. \tag{5.21}$$

In [113, Ch. 2] it is proved that $\sum_{k\in\mathbb{Z}} |\widehat{h}(v+2\pi k)|^2 \geq c > 0$, hence ϕ is well-defined. Moreover, by the regularity of h, we have, as a consequence of the

Sobolev's embedding theorem, that $\sum_{k\in\mathbb{Z}} |\widehat{h}(v+2\pi k)|^2$ is a $C^\infty(\mathbb{R})$ function. Furthermore, the family $(\phi(x-k))_{k\in\mathbb{Z}}$ turns out to be an orthonormal basis of V_0, [64, 113], and from (5.17) and (5.18) we obtain by a simple change of scale, that $(2^{j/2}\phi(2^j x-k))_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_j.

Now, by smoothness and periodicity of $\left(\sum_{k\in\mathbb{Z}} |\widehat{h}(v+2\pi k)|^2\right)^{-1/2}$, the latter can be written by means of its Fourier series $\sum_{k\in\mathbb{Z}} \alpha_k e^{ikv}$, where the coefficients α_k decrease rapidly. We thus obtain $\widehat{\phi}(v) = \left(\sum_{k\in\mathbb{Z}} \alpha_k e^{ikv}\right)\widehat{h}(v)$ which gives $\phi(x) = \sum_{k\in\mathbb{Z}} \alpha_k\, h(x+k)$, and then it follows that the scaling function ϕ satisfies the estimates in (5.20). In particular, we have

$$|\phi(x)| \leq \widetilde{C}_\alpha\,(1+|x|)^{-\alpha}, \qquad x\in\mathbb{R}, \tag{5.22}$$

for some $\widetilde{C}_\alpha > 0$, for every integer $\alpha\in\mathbb{N}^+$, i.e., ϕ satisfies condition $(\varphi 1)$ for for every $\alpha\in\mathbb{N}^+$.

Let now E_j be the orthogonal projection of $L^2(\mathbb{R})$ onto V_j, given by

$$(E_j f)(x) := \sum_{k\in\mathbb{Z}} \left[2^j \int_{\mathbb{R}} f(y)\,\overline{\phi}(2^j y-k)\,dy\right] \phi(2^j x-k), \quad f\in L^2(\mathbb{R}), \tag{5.23}$$

where $\overline{\phi}$ is the complex conjugate of ϕ. Let define $E(x,y) := \sum_{k\in\mathbb{Z}} \overline{\phi}(y-k)\,\phi(x-k)$, the kernel of the projection operator E_0, hence $2^j\,E(2^j x, 2^j y)$, $j\in\mathbb{Z}$ will be the kernel of the projection operator E_j.

Again in [113], it is proved the following remakable property for the kernel, E,

$$\int_{\mathbb{R}} E(x,y)\,y^\alpha\,dy = x^\alpha, \quad \text{for every } x\in\mathbb{R}, \tag{5.24}$$

for every integer $\alpha\in\mathbb{N}$ and $\alpha\leq r$. From (5.24) with $\alpha=0$, the integral property

$$\int_{\mathbb{R}} E(x,y)\,dy = 1, \quad \text{for every } x\in\mathbb{R},$$

follows. Moreover, since ϕ satisfies (5.22), it is easy to see that

$$|E(x,y)| \leq \overline{C}_\alpha(1+|x-y|)^{-\alpha}, \quad \forall\ (x,y)\in\mathbb{R}^2, \text{ and } \ \forall\ \alpha\in\mathbb{N}^+,$$

where $\overline{C}_\alpha > 0$. Hence, E is a bivariate kernel satisfying conditions (5.2) and (5.3). Then, by Lemma 5.3, we infer that $\|E_j f - f\|_p \to 0$ as $j\to+\infty$, for every $f\in L^p(\mathbb{R})$ and $1\leq p<\infty$. Moreover, exploiting the properties of the projection operators E_j, the quantity

$$\Sigma(x,v) := \sum_{k\in\mathbb{Z}} e^{i2\pi kx}\,\widehat{\phi}(v+2k\pi)\,\overline{\widehat{\phi}}(v), \quad x,\ v\in\mathbb{R},$$

can be defined, which satisfies the condition $\Sigma(x,0)=1$, for every $x\in\mathbb{R}$, [113]. This yields

$$\sum_{k\in\mathbb{Z}} e^{i2\pi kx}\widehat{\phi}(2k\pi)\,\overline{\widehat{\phi}}(0) = 1. \tag{5.25}$$

Now, we can adjust the scaling function ϕ merely multiplying $\widehat{\phi}$ by a suitable constant of modulus 1, so that $\widehat{\phi}(0) = \int_{\mathbb{R}} \phi(t)\, dt = 1$, while preserving all the other properties, [113]. By the regularity of ϕ, the Poisson summation formula holds, and from (5.25) we obtain

$$1 = \sum_{k\in\mathbb{Z}} e^{i2\pi kx}\, \widehat{\phi}(2k\pi) = \sum_{k\in\mathbb{Z}} \phi(x+k) = \sum_{k\in\mathbb{Z}} \phi(x-k), \quad x \in \mathbb{R},$$

i.e., the scaling function ϕ satisfies condition ($\varphi 2$). Using (5.4), we can now consider the function σ_ϕ constructed by the scaling function ϕ. Clearly, if ϕ is *real valued*, σ_ϕ turns out to be a sigmoidal function. Then, we have the following

Theorem 5.12. *Let ϕ be a real valued* scaling *function like that constructed above, associated to an r*-regular multiresolution *approximation of $L^2(\mathbb{R})$.*
(i) Then, for any $f \in AC[a,b]$, the sequence of operators $(S_j^{\sigma_\phi} f)_{j\in\mathbb{N}^+}$, defined by

$$(S_j^{\sigma_\phi} f)(x) := \sum_{k\in\mathbb{Z}} \left[\int_a^b \phi(2^j y - k)\, f'(y)\, dy\right] \sigma_\phi(2^j x - k) + f(a),$$

for every $x \in [a,b]$, converges uniformly to f on $[a,b]$.
In particular, if $f \in \widehat{C}^1[a,b]$, we have

$$\|S_j^{\sigma_\phi} f - f\|_\infty \le C\, 2^{-j},$$

for some positive constant C and for every positive integer j.
(ii) Denote by $S_{N,j}^{\sigma_\phi} f$, $N \in \mathbb{N}^+$, the truncated series $S_j^{\sigma_\phi} f$, i.e.,

$$(S_{N,j}^{\sigma_\phi} f)(x) := \sum_{k=-N}^{N} \left[\int_a^b \phi(2^j y - k)\, f'(y)\, dy\right] \sigma_\phi(2^j x - k) + f(a).$$

Then, for every $f \in \widehat{C}^1[a,b]$, we have

$$\|S_{N,j}^{\sigma_\phi} f - f\|_\infty \le C_1\, 2^{-j} + C_{2,\alpha} \left\{ (N - 2^j b + 1)^{-(\alpha-1)} + (N + 2^j a + 1)^{-(\alpha-1)} \right\},$$

for some positive constants C_1 and $C_{2,\alpha}$, for every $j \in \mathbb{N}^+$, and $N > 2^j \max\{|a|, |b|\}$, where $\alpha \in \mathbb{N}^+$ is an arbitrary integer.

The proof of Theorem 5.12 (i) follows as the proof of Theorem 5.5, taking into account that, the sequence $(E_j f)_{j\in\mathbb{Z}}$, $f \in L^1(\mathbb{R})$, converges to f in $L^1(\mathbb{R})$. Moreover, the proof of Theorem 5.12 (ii) follows, as the proof of Theorem 5.7 (ii), using condition (5.22) and Lemma 5.6, where we have 2^j in place of w.

Remark 5.13. Note that, in the special setting of r-regular multiresolution approximations, we are able to prove that the real-valued scaling functions ϕ, constructed above, are such that $\phi \in \Phi$. Moreover, condition (5.17) in Definition 5.10 allows us to consider the *weights* in the basis $(2^{j/2}\phi(2^j x - k))_{k\in\mathbb{Z}}$, and then in the series $S_j^{\sigma_\phi} f$, as 2^j, i.e., the weights increase exponentially with respect to j. Then, the error of approximation of C^1-functions decreases as 2^{-j}. Moreover, conditions (5.20) and (5.22) are crucial to prove that the *truncation error* also decrease rapidly.

Examples of r-regular multiresolution analysis satisfying the conditions above can be given, assuming h to be generated by spline wavelets of order $r+1$ (see e.g. Fig. 5.1). These are defined by

$$h_r(x) := \frac{1}{r!}\sum_{i=0}^{r+1}(-1)^i \binom{r+1}{i} (x-i)_+^r\,, \quad x \in \mathbb{R}, \tag{5.26}$$

which can be viewed just as shifted central B-spline M_n. Generally speaking, the definition of h_n is given in terms of convolution, i.e., h_n can be defined as the convolution of $r+1$ characteristic functions of the interval $[0,1)$, see [131]. Note that, also the central B-spline can be defined similarly, in terms of convolutions of the characteristic functions of the interval $[-1/2, 1/2)$, see [34]. The Fourier transform of h_r can be easily obtained by

$$\widehat{h}_r(v) := e^{-iv(r+1)/2}\,\mathrm{sinc}^{r+1}\left(\frac{v}{2\pi}\right), \quad v \in \mathbb{R}.$$

The scaling function ϕ associated to the spline wavelet multiresolution approximation can be obtained using (5.21) and the normalization procedure described above, see [113, 131, 65]. In Fig. 5.2 the sigmoidal function $\sigma_{\phi_{h_2}}$ obtained from ϕ_{h_2} is plotted.

5.4 An extension of the theory for neural network approximation

The theory developed in the previous sections of this chapter, concerning the approximation by means of series of sigmoidal functions based on σ_ϕ, is beset by the technical difficulty of checking that ϕ satisfies condition $(\varphi 2)$. To this purpose, we could use the condition given in Remark 5.2. However, this does not simplify the problem. In fact, evaluating the Fourier transform of a given function is often a difficult task. Moreover, as noticed in Remark 5.9, the sigmoidal functions most used for NN approximation do not satisfy $(\varphi 2)$. Below, we propose an extension of the theory developed in the previous sections, aiming at obtaining approximations with NNs activated by sigmoidal functions σ_ϕ, without assuming that condition $(\varphi 2)$ be satisfied by ϕ.

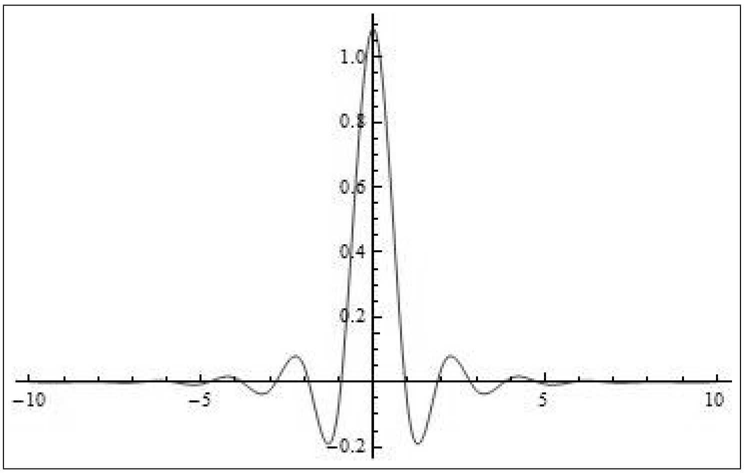

Figure 5.1: The scaling function ϕ_{h_2}, plotted using *Mathematica*.

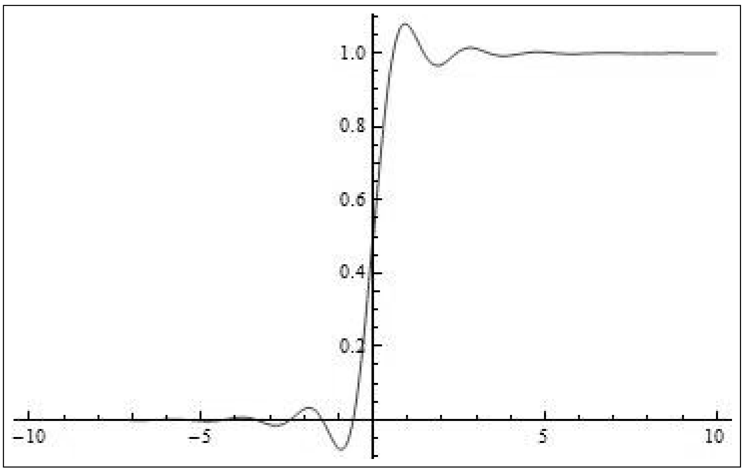

Figure 5.2: The sigmoidal function $\sigma_{\phi_{h_2}}$ obtained from ϕ_{h_2}, plotted using *Mathematica*.

Through this section, we consider functions $\phi : \mathbb{R} \to \mathbb{R}_0^+$, with $\int_{\mathbb{R}} \phi(t)\, dt = 1$ and satisfying condition $(\varphi 1)$ with $\alpha > 2$. Moreover, we set

$$\psi_\phi(t) := \sigma_\phi(t+1) - \sigma_\phi(t) > 0, \quad t \in \mathbb{R},$$

and assume in addition that ψ_ϕ satisfies:

$$(\Psi 1) \qquad \psi_\phi(t) \leq A(1+|t|)^{-\alpha},$$

for every $t \in \mathbb{R}$ and some $A > 0$. We denote by $\mathcal{T}$ the set of all functions ϕ satisfying such conditions. We can now prove the following

Lemma 5.14. *For any given $\phi \in \mathcal{T}$, the relation*

$$\sum_{k \in \mathbb{Z}} \psi_\phi(x-k) = 1, \quad x \in \mathbb{R}$$

holds.

Proof. Let $x \in \mathbb{R}$ be fixed. Then,

$$\sum_{k=-N}^{N} \psi_\phi(x-k) = \sum_{k=-N}^{N} [\sigma_\phi(x-k+1) - \sigma_\phi(x-k)] = \sigma_\phi(x+N+1) - \sigma_\phi(x-N),$$

since the sum is telescopic. Passing to the limit for $N \to +\infty$, we obtain immediately

$$\sum_{k=-\infty}^{+\infty} \psi_\phi(x-k) = \lim_{N \to +\infty} [\sigma_\phi(x+N+1) - \sigma_\phi(x-N)] = 1.$$

$\square$

Let now introduce the bivariate kernel

$$K_{\phi,\psi}(x,y) := \sum_{k \in \mathbb{Z}} \psi_\phi(x-k)\, \phi(y-k), \quad (x,y) \in \mathbb{R}^2.$$

As made in Section 5.1 for the kernel K_ϕ, we can show, using Lemma 5.14 and conditions $(\varphi 1)$ and $(\Psi 1)$, that $K_{\phi,\psi}$ satisfy both, (5.2) and (5.3). Now, for any given $\phi \in \mathcal{T}$, we consider the family of operators $(F_w^\phi)_{w>0}$, defined by

$$\begin{aligned} (F_w^\phi f)(x) &:= \sum_{k \in \mathbb{Z}} w \left[\int_{\mathbb{R}} \phi(wy-k) f(y)\, dy \right] \psi_\phi(wx-k) \\ &:= w \int_{\mathbb{R}} K_{\phi,\psi}(wx, wy) f(y)\, dy, \quad x \in \mathbb{R}, \end{aligned}$$

for every bounded $f : \mathbb{R} \to \mathbb{R}$, $w > 0$.

Remark 5.15. Note that, by Lemma 5.3, for every uniformly continuous and bounded function f, the family of operators $(F_w^{\phi} f)_{w>0}$ converges uniformly to f on $\mathbb{R}$, as $w \to +\infty$.

To study the order of approximation for the operators above, we define the Lipschitz class of the Zygmund type we will work with. Let us define

$$\mathrm{Lip}(\nu) := \left\{ f : \mathbb{R} \to \mathbb{R} : \; f \in C^0(\mathbb{R}), \; \|f(\cdot) - f(\cdot + t)\|_\infty = \mathcal{O}(|t|^\nu) \text{ as } t \; \to 0 \right\},$$

for every $0 < \nu \leq 1$. We can now prove the following lemma concerning the order of approximation of $(F_w^{\sigma_\phi} f)_{w>0}$ to $f(x)$:

Lemma 5.16. *Let $f \in$ Lip(ν), $0 < \nu \leq 1$, be a fixed bounded function. Then, there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$\sup_{x \in \mathbb{R}} |(F_w^{\phi} f)(x) - f(x)| \leq C_1 \, w^{-\nu} + C_2 \, w^{-(\alpha - 1)},$$

for every sufficiently large $w > 0$, where $\alpha > 2$ is the constant of condition $(\varphi 1)$.

Proof. Let $x \in \mathbb{R}$ be fixed. Since $f \in$ Lip(ν), there exist $M > 0$ and $\gamma > 0$ such that

$$\|f(\cdot) - f(\cdot + t)\|_\infty \leq M \, |t|^\nu,$$

for every $|t| \leq \gamma$. Moreover, we infer from condition (5.2)

$$w \int_{\mathbb{R}} K_{\phi,\psi}(wx, wy) \, dy = 1, \quad x \in \mathbb{R}, \tag{5.27}$$

and then we can write

$$|(F_w^{\phi} f)(x) - f(x)| \leq w \int_{\mathbb{R}} K_{\phi,\psi}(wx, wy) \, |f(y) - f(x)| \, dy$$

$$= \left[\int_{|y-x| \leq \gamma} + \int_{|y-x| > \gamma} \right] w \, K_{\phi,\psi}(wx, wy) \, |f(y) - f(x)| \, dy =: J_1 + J_2.$$

Let first estimate J_1. From (5.3) and (5.27), by the change of variable $y = (t/w) + x$, and being $f \in$ Lip(ν), we obtain for $w > 0$ sufficiently large

$$J_1 = \int_{w^{-1}|t| \leq \gamma} K_{\phi,\psi}(wx, t + wx) \, |f\left(x + t/w\right) - f(x)| \, dt$$

$$\leq M \left[\int_{|t| \leq w\,\gamma} K_{\phi,\psi}(wx, t + wx) \left| \frac{t}{w} \right|^\nu \, dt \right] \leq \widetilde{L} w^{-\nu} \int_{\mathbb{R}} (1 + |t|)^{-\alpha} \, |t|^\nu \, dt,$$

where $\widetilde{L} > 0$ is a suitable constant. Now, since $\alpha > 2$, we have $\widetilde{L} \int_{\mathbb{R}}(1 + |t|)^{-\alpha} |t|^{\nu} \, dt =: C_1 < +\infty$, then $J_1 \leq C_1 \, w^{-\nu}$, for $w > 0$ sufficiently large. Moreover, setting $t = wy$ and using again condition (5.3), we have

$$J_2 = \int_{|t-wx|>w\gamma} K_{\phi,\psi}(wx,t)|f(t/w) - f(x)| \, dt$$

$$\leq 2\|f\|_{\infty} \int_{|t-wx|>w\gamma} K_{\phi,\psi}(wx,t) \, dt \leq \overline{L} \int_{|t-wx|>w\gamma} (1 + |t - wx|)^{-\alpha} \, dt,$$

where $\overline{L}$ is a suitable positive constant. Changing now the variable t into z, setting $z = t - wx$ in the last integral, we obtain

$$J_2 \leq \overline{L} \int_{|z|>w\gamma} (1 + |z|)^{-\alpha} \, dz \leq C_2 \, w^{-(\alpha-1)},$$

for every $w > 0$. This completes the proof. $\square$

We can now prove the following

Theorem 5.17. *Let $\phi \in \mathcal{T}$ be fixed. Define the NNs*

$$(N^{\phi}_{N,w} f)(x) := \sum_{k=-N}^{N} w \left[\int_{\mathbb{R}} \phi(wy - k) \, f(y) \, dy \right] \psi_{\phi}(wx - k), \quad x \in \mathbb{R},$$

where $w > 0$, $N \in \mathbb{N}^{+}$, and $f : \mathbb{R} \to \mathbb{R}$ is a bounded function on $\mathbb{R}$.

(i) Let $f \in C^0[a,b]$ be fixed. Then, for every $\varepsilon > 0$, there exist $w > 0$ and $N > w \max\{|a|, |b|\}$, such that

$$\|N^{\phi}_{N,w} \widetilde{f} - f\|_{\infty} = \sup_{x \in [a,b]} |(N^{\phi}_{N,w} \widetilde{f})(x) - f(x)| < \varepsilon,$$

where $\widetilde{f}$ is a continuous extensions of f such that $\widetilde{f}$ has compact support and $\widetilde{f} = f$ on $[a,b]$.

(ii) Let $f \in Lip(\nu)$, $0 < \nu \leq 1$, and $[a,b] \subset \mathbb{R}$ be fixed. Then, we have

$$\|N^{\phi}_{N,w} f - f\|_{\infty} = \sup_{x \in [a,b]} |(N^{\phi}_{N,w} f)(x) - f(x)|$$

$$\leq C_1 \, w^{-\nu} + C_2 \, w^{-(\alpha-1)} + C_3 \left\{ (N - wb + 1)^{-(\alpha-1)} + (N + wa + 1)^{-(\alpha-1)} \right\},$$

for every sufficiently large $w > 0$ and $N > w \max\{|a|, |b|\}$, for some positive constants C_1, C_2, and C_3.

Proof. (i) Suppose for the sake of simplicity that $\|f\|_\infty = \|\widetilde{f}\|_\infty$, and note that $\widetilde{f}$ is uniformly continuous. Let now $\varepsilon > 0$ and $x \in [a,b]$ be fixed. We can write

$$|(N^{\phi}_{N,w}\widetilde{f})(x) - f(x)| \leq |f(x) - (F^{\phi}_{w}\widetilde{f})(x)| + |(F^{\phi}_{w}\widetilde{f})(x) - (N^{\phi}_{N,w}\widetilde{f})(x)| \\ =: I_1 + I_2.$$

By Remark 5.15 we have $I_1 < \varepsilon$ for $w > 0$ sufficiently large. Moreover,

$$I_2 \leq \sum_{|k|>N} w \left[\int_{\mathbb{R}} \phi(wy-k)\, |\widetilde{f}(y)|\, dy \right] \psi_\phi(wx-k).$$

Hence, $w \int_{\mathbb{R}} \phi(wy-k)\ dy = 1$, and since $(\Psi 1)$ holds, we obtain for ψ_ϕ the same estimate given in Lemma 5.6 for ϕ, then for every fixed sufficiently large $w > 0$ we have

$$\begin{aligned} I_2 &\leq \|f\|_\infty \sup_{x\in[a,b]} \sum_{|k|>N} \left[w \int_{\mathbb{R}} \phi(wy-k)\, dy \right] \psi_\phi(wx-k) \\ &= \|f\|_\infty \left[\sup_{x\in[a,b]} \sum_{|k|>N} \psi_\phi(wx-k) \right] \\ &< \|f\|_\infty \widetilde{C} \left\{ (N - wb + 1)^{-(\alpha-1)} + (N + wa + 1)^{-(\alpha-1)} \right\} < \varepsilon, \end{aligned} \tag{5.28}$$

for some positive constant $\widetilde{C}$, $N \in \mathbb{N}^+$, $N > w \max\{|a|, |b|\}$, and then, (i) is proved being $\varepsilon > 0$ arbitrary.

(ii) Let now $f \in \mathrm{Lip}(\nu)$ be a fixed. We have by Lemma 5.16

$$I_1 \leq C_1\, w^{-\nu} + C_2\, w^{-(\alpha-1)},$$

for every sufficiently large $w > 0$ and for some positive constants C_1 and C_2. Moreover, we obtain from (5.28)

$$I_2 \leq C_3 \left\{ (N - wb + 1)^{-(\alpha-1)} + (N + wa + 1)^{-(\alpha-1)} \right\},$$

for a suitable constant $C_3 > 0$. Then, the second part of the theorem is proved. □

As a first example, we can consider the case of the *logistic* function, σ_ℓ (see e.g. [38]), generated by $\phi_\ell(x) := e^{-x}(1+e^{-x})^{-2}$. Clearly, conditions $(\varphi 1)$ and $(\Psi 1)$, are fulfilled, since ϕ_ℓ and

$$\psi_\ell(x) := \sigma_\ell(x+1) - \sigma_\ell(x) = \frac{e\,(e-1)\,e^{-x}}{(1+e^{-x-1})(1+e^{-x})},$$

decay exponentially as $x \to \pm\infty$. A second example, is given by the *hyperbolic tangent* sigmoidal function (see, e.g., [7, 8]),

$$\sigma_h(x) := \frac{1}{2} + \frac{1}{2}\tanh(x) = \frac{1}{2} + \frac{e^{2x}-1}{2(e^{2x}+1)}.$$

This can be generated by $\phi_h(x) = 2\, e^{2x}\,(e^{2x}+1)^{-2}$, whose associated function ψ_h is

$$\psi_h(x) = \frac{(e^2-1)\, e^{2x}}{(e^{2x+2}+1)(e^{2x}+1)}.$$

It can be easily checked that such a function ϕ_h belongs to $\mathcal{T}$.

Finally, we recall that another remarkable example of sigmoidal function, for which the theory can be applied, is provided by the class of Gompetz functions $\sigma_{\alpha\beta}$.

Remark 5.18. Note that, in closing, in order to approximate functions by the NNs $G^{\sigma_\phi}_{N,w}$, the half of the number of sigmoidal functions needed to approximate functions by the NNs $N^{\phi}_{N,w}$, would now suffice. The theory developed in this section, however, can be applied to important sigmoidal functions for which the theory earlier discussed in Sections 5.1 and 5.2 cannot be applied.

Chapter 6

Neural Network Operators

In this chapter, we introduce a modern way to study neural network approximation by means of the Operator Theory.

Our purpose is again to study other kinds of sigmoidal functions approximation, in order to develop numerical methods able to achieve an higher accuracy with respect to the methods developed in Chapters 3 and 4.

6.1 Cardalignet-Euvrard and Squashing neural network operators

In [5, 6], Anastassiou was the first to establish neural networks approximation to continuous functions with rates by very specifically neural network operators of the Cardalignet-Euvrard and squashing types [45], by employing the modulus of continuity of the engaged function or its high order derivative. In his papers, Anastassiou produced a Jackson-type inequalities. We recall the following.

Definition 6.1. *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be "bell-shaped" if $b \in L^1(\mathbb{R})$ and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and non-increasing in $[a, +\infty)$, with $a \in \mathbb{R}$. In particular, $b(x)$ is non-negative an at the point a the function b take a global maximum (a is said to be the "center" of the bell-shaped function). A bell-shaped function is said "centered" if its center is zero.*

The function b may have jump discontinuities. Now, for every continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$, the *Cardalignet-Euvrard neural network* operators ([45]) are defined by:

$$(F_n(f))(x) \ := f_n(x) \ := \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{B_T\, n^\alpha} b\left(n^{1-\alpha}(x - k/n)\right), \tag{6.1}$$

where $0 < \alpha < 1$, b is a bell-shaped function with compact support, $B_T := \int_{-T}^{T} b(x)\, dx$, $supp\, b \subset [-T, T]$, $x \in \mathbb{R}$, $n \in \mathbb{N}^+$.

In what follows, we will denote by $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ the ceiling and the integral part of a number, respectively.

In [4] the following results are proved.

Theorem 6.2 ([4])**.** *Let $x \in \mathbb{R}$, $n \in \mathbb{N}^+$ such that $n \geq \max\left\{T + |x|,\, T^{-1/\alpha}\right\}$. Then*

$$|f_n(x) - f(x)| \;\leq\; |f(x)| \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T\, n^\alpha}\, b\left(n^{1-\alpha}(x - k/n)\right) - 1 \right| + \frac{b^*}{B_T}(2T + 1/n^\alpha)\, \omega(f,\, T/n^{1-\alpha}),$$

where ω is the modulus of continuity of f, and $b^* = b(0)$ is the maximum of b. The inequality above becomes equality over constant functions.

To completely understand the theorem above we need of the following lemma.

Lemma 6.3 ([4])**.** *It holds that*

$$S_n(x) \;:=\; \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T\, n^\alpha}\, b\left(n^{1-\alpha}(x - k/n)\right) \to 1$$

pointwise, as $n \to +\infty$, where $x \in \mathbb{R}$.

Another important result for these operators is obtained for functions with high order derivatives.

Theorem 6.4 ([4])**.** *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}^+$ such that $n \geq \max\left\{T + |x|,\, T^{-1/\alpha}\right\}$. Let $f \in C^N(\mathbb{R})$ such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|f_n(x) - f(x)| \;\leq\; |f(x)| \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T\, n^\alpha}\, b\left(n^{1-\alpha}(x - k/n)\right) - 1 \right|$$

$$+ \frac{b^*}{I}(2T{+}1/n^\alpha) \left(\sum_{j=1}^{N} \frac{|f^{(j)}(x)| T^j}{n^{j(1-\alpha)} j!} \right) + \omega(f^{(N)},\, T/n^{1-\alpha}) \frac{T^N\, b*}{B_T\, N! n^{N(1-\alpha)}}(2T{+}1/n^\alpha).$$

The inequality above is attained by constant functions.

We now recall the following definition (see [45, 4]).

Definition 6.5. *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support contained in $[-T, T]$, $T > 0$, and is non decreasing there and it can be continuously only on either $(-\infty, T]$ or $[-T, T]$. S can have jump discontinuities. We call S the “squashing function”.*

Let $f : \mathbb{R} \to \mathbb{R}$ be either uniformly continuous or continuous and bounded. Assume that

$$B_T^* \; := \; \int_{-T}^{T} S(t)\, dt \; > 0.$$

For $x \in \mathbb{R}$ define the *squashing operator*

$$(G_n(f))(x) \; := \; \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{B_T^*\, n^\alpha} b\left(n^{1-\alpha}(x - k/n)\right), \tag{6.2}$$

where $0 < \alpha < 1$, S is a squashing function, $x \in \mathbb{R}$, $n \in \mathbb{N}^+$. Similarly to the case of the Cardalignet-Euvrard neural network operators, convergence theorems with rate can be proved for the squashing operators.

Theorem 6.6 ([4]). *Let $x \in \mathbb{R}$, $n \in \mathbb{N}^+$ such that $n \geq \max\left\{T + |x|,\, T^{-1/\alpha}\right\}$. Then, under the above assumptions we obtain*

$$|(G_n(f))(x) - f(x)| \; \leq \; |f(x)| \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T^*\, n^\alpha}\, S\left(n^{1-\alpha}(x - k/n)\right) - 1 \right|$$

$$+ \; \frac{S(T)}{B_T^*}(2T + 1/n^\alpha)\, \omega(f,\, T/n^{1-\alpha}),$$

where ω is the modulus of continuity of f, and $S(T)$ is the maximum of S. The inequality above becomes equality over constant functions.

To completely understand the theorem above we need of the following lemma.

Lemma 6.7 ([4]). *It holds that*

$$D_n(x) \; := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T^*\, n^\alpha}\, S\left(n^{1-\alpha}(x - k/n)\right) \to 1$$

pointwise, as $n \to +\infty$, where $x \in \mathbb{R}$.

Another important result for these operators is obtained for functions with high order derivatives.

Theorem 6.8 ([4]). *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}^+$ such that $n \geq \max\left\{T + |x|,\, T^{-1/\alpha}\right\}$. Let $f \in C^N(\mathbb{R})$ such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(G_n(f))(x) - f(x)| \; \leq \; |f(x)| \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lfloor nx + Tn^\alpha \rfloor} \frac{1}{B_T^*\, n^\alpha}\, S\left(n^{1-\alpha}(x - k/n)\right) - 1 \right|$$

$$+ \frac{S(T)}{B_T^*}(2T{+}1/n^\alpha) \left(\sum_{j=1}^{N} \frac{|f^{(j)}(x)| T^j}{n^{j(1-\alpha)} j!} \right) + \omega(f^{(N)},\, T/n^{1-\alpha}) \frac{T^N\, S(T)}{B_T^*\, N! n^{N(1-\alpha)}}(2T{+}1/n^\alpha).$$

The inequality above is attained by constant functions.

6.2 The neural network operators with logistic and hyperbolic tangent functions

In order to study operators of the Cardalignet-Euvrard type, and motivated by the applications to the theory on neural networks, in Anastassiou [7] the hyperbolic tangent neural network operators has been introduced. Moreover, the results of [7] are also inspired by the paper of Cao and Chen [38] concerning the neural network operators activated by the logistic function.

Here, we now consider only the case of [7]. Let $\phi_h(x) := \frac{1}{2}(\sigma_h(x+1) - \sigma_h(x-1))$, where σ_h is the well-known hyperbolic tangent sigmoidal function, and $x \in \mathbb{R}$. We have the following (see [7]).

Definition 6.9. *Let $f \in C^0[a,b]$ and $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the "positive linear neural network operator"*

$$F_n^h(f,x) \;:=\; \frac{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \phi_h(nx-k)}{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_h(nx-k)}, \qquad x \in [a,b].$$

For sufficiently large $n \in \mathbb{N}^+$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$, and since f is bounded, $F_n^h(f,x)$ is well-defined, for all $x \in [a,b]$.

Note that the definition of neural network operators with hyperbolic tangent was extended by Anastassiou to the multivariate case in [8].

For the operators in Definition 6.9 the following approximation theorem with rates has been obtained.

Theorem 6.10 ([7])**.** *Let $f \in C^0[a,b]$, $0 < \alpha < 1$, $x \in \mathbb{R}$ and $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$|F_n^h(f,x) - f(x)| \;\leq\; (4.1488766)\left[\omega(f,\, 1/n^\alpha) + 2e^4 \|f\|_\infty e^{-2n^{1-\alpha}}\right].$$

Clearly, the estimates in Theorem 6.10 holds uniformly for every $x \in [a,b]$.

The high order of approximation was studied by using the smoothness of f.

Theorem 6.11 ([7])**.** *Let $f \in C^N[a,b]$, $0 < \alpha < 1$, $x \in \mathbb{R}$ and N, $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$|F_n^h(f,x) - f(x)| \;\leq\; (4.1488766)\left\{\sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!}\left[\frac{1}{n^{\alpha j}} + e^4(b-a)^j e^{-2n^{1-\alpha}}\right]\right.$$

$$+ \left[\omega(f^{(N)},\, 1/n^{\alpha})\, \frac{1}{n^{\alpha N} N!} + \frac{2e^4 \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{-2n^{1-\alpha}}\right]\Bigg\}.$$

In particular

$$|F_n^h(f,x) - f(x)| \leq (4.1488766) \left\{ \sum_{j=1}^{N} \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + e^4 (b-a)^j e^{-2n^{1-\alpha}}\right] \right.$$

$$\left. + \left[\omega(f^{(N)},\, 1/n^{\alpha})\, \frac{1}{n^{\alpha N} N!} + \frac{2e^4 \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{-2n^{1-\alpha}}\right]\right\}.$$

Furthermore, in [7] also the complex-value neural network operators are studied and convergence estimates with rates are derived also in this case.

In [8], all the results above are generalized in a multivariate setting, as required for the applications to NNs.

Finally, the special case of the so-called *quasi-interpolation* operators with hyperbolic tangent functions is also considered ([7]).

Definition 6.12. *Let $f \in C^0(\mathbb{R})$ and $n \in \mathbb{N}^+$. We introduce and define the "quasi-interpolation operator with hyperbolic tangent function":*

$$\overline{F}_n^h(f,x) \;:=\; \sum_{k \in \mathbb{Z}} f\left(\frac{k}{n}\right) \phi_h(nx-k), \qquad x \in [a,b].$$

Also for the quasi-interpolation operators in Definition 6.12, results similar to that proved in Theorem 6.10 and Theorem 6.11 are proved in [7].

The operators $\overline{F}_n^h$ are a particular case of the *generalized sampling operators* introduced by the German mathematician P.L. Butzer (see e.g. [35, 36, 33]) and studied by many others authors (see e.g. [123, 18, 19, 133]). These operators are very important for their applications to Sampling Theory and Signal Processing.

Moreover, Anastassiou studied neural network operators of the kind defined in this section, but activated by the logistic function, both in univariate and multivariate setting in [9, 11]. These operators, that we will denote by F_n^ℓ are defined like that with the hyperbolic tangent function, but replacing σ_h (i.e., ϕ_h) with σ_ℓ (i.e., ϕ_ℓ). Similarly we will have the quasi-interpolation operators with logistic functions $\overline{F}_n^\ell$.

The approach of Anastassiou of studying neural network operators with logistic functions is little different by the approach used in [38] by Cao and Chen.

In [11] the following results were proved.

Theorem 6.13 ([11])**.** *Let $f \in C^0[a,b]$, $0 < \alpha < 1$, $x \in \mathbb{R}$ and $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$|F_n^\ell(f,x) - f(x)| \;\leq\; (5.250312578) \left[\omega(f,\, 1/n^{\alpha}) + 6.3984\, e^4 \|f\|_\infty e^{-n^{1-\alpha}}\right].$$

Also in this case, the estimate in Theorem 6.13 holds uniformly for every $x \in [a, b]$.

Theorem 6.14 ([11]). *Let $f \in C^N[a,b]$, $0 < \alpha < 1$, $x \in \mathbb{R}$ and N, $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$|F_n^\ell(f,x) - f(x)| \leq (5.250312578) \left\{ \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j \, e^{-n^{1-\alpha}} \right] \right.$$

$$\left. + \left[\omega(f^{(N)}, 1/n^\alpha) \frac{1}{n^{\alpha N} N!} + \frac{(6.3984) \, \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{-n^{1-\alpha}} \right] \right\}.$$

In particular

$$\|F_n^\ell(f,\cdot) - f(\cdot)\|_\infty \leq (5.250312578) \left\{ \sum_{j=1}^{N} \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + (3.1992)(b-a)^j e^{-n^{1-\alpha}} \right] \right.$$

$$\left. + \left[\omega(f^{(N)}, 1/n^\alpha) \frac{1}{n^{\alpha N} N!} + \frac{(6.3984) \, \|f^{(N)}\|_\infty (b-a)^N}{N!} e^{-n^{1-\alpha}} \right] \right\}.$$

Furthermore, in [11] also the complex-value neural network operators are studied and convergence estimates with rates are derived also in this case.

In [9], all the results above for the neural network operators with logistic functions are generalized in a multivariate setting, as it is typical in NNs applications.

NN operators activated by the *ramp* sigmoidal functions has been studied in [39].

Concerning the results discussed above the readers can consult the book [10]. Others important kinds of neural network operators, such as *fuzzy-random* and *fractional* type have been studied in [6] and [12], respectively.

6.3 Neural network operators activated by sigmoidal functions

In the next sections, we study pointwise and uniform convergence, as well as the order of approximation, for a family of linear positive *neural network* operators activated by certain sigmoidal functions. Both the cases of functions of one and several variables are considered. Our approach allows to extend the results discussed in the previous sections about NNs operators. The order of approximation is studied for functions belonging to suitable Lipschitz classes and using a moment-type approach, i.e., we introduce the *discrete absolute moments* for suitable density functions, ϕ_σ, defined by the sigmoidal functions σ. The approximation error is analyzed in connection to both the weights and the number of neurons of the network, in the sup norm. The special cases of neural network operators activated by *logistic*, *hyperbolic tangent*, and *ramp* sigmoidal functions are considered and compared with the results obtained by Anastassiou and others discussed in the previous sections. In particular, we show that for C^1-functions, the order of approximation for our operators with logistic and hyperbolic tangent functions achieved by our approach, is higher with respect to that established in some previous papers (see Section 6.4).

The case of univariate and multivariate *quasi-interpolation* operators constructed with sigmoidal functions is also considered.

For the univariate theory of neural network operators activated by sigmoidal functions, the readers can see [60], while, for the multivariate theory see [61].

6.4 Univariate theory: preliminary results

In this section, we establish some preliminary results that will be useful in the next sections. In what follows, we consider non-decreasing sigmoidal functions, σ, such that $\sigma(2) > \sigma(0)$, and such that all the following assumptions are satisfied:

($\Sigma 1$) $g_\sigma(x) := \sigma(x) - 1/2$, is an odd function;

($\Sigma 2$) $\sigma \in C^2(\mathbb{R})$, and is concave for $x \geq 0$;

($\Sigma 3$) $\sigma(x) = \mathcal{O}(|x|^{-1-\alpha})$, as $x \to -\infty$, for some $\alpha > 0$.

The condition $\sigma(2) > \sigma(0)$ is merely technical.

For any given non-decreasing function σ, satisfying all such assumptions, define

$$\phi_\sigma(x) := \frac{1}{2}[\sigma(x+1) - \sigma(x-1)], \qquad (x \in \mathbb{R}). \tag{6.3}$$

In the following lemmas, we prove a number of important properties of ϕ_σ.

Lemma 6.15. *(i) $\phi_\sigma(x) \geq 0$ for every $x \in \mathbb{R}$, and in particular, $\phi_\sigma(1) > 0$; (ii) $\lim_{x\to\pm\infty} \phi_\sigma(x) = 0$; (iii) $\phi_\sigma(x)$ is an even function.*

Proof. (i) Trivially, $\phi_\sigma(x) \geq 0$ since σ is non-decreasing, and the assumption $\sigma(2) > \sigma(0)$ implies that $\phi_\sigma(1) > 0$.

(ii) By the definition of sigmoidal function, it follows that $\lim_{x\to\pm\infty} \phi_\sigma(x) = 0$.

(iii) By condition ($\Sigma 1$), we have for every $x \geq 0$

$$2[\phi_\sigma(x) - \phi_\sigma(-x)] = \sigma(x+1) - \sigma(x-1) - \sigma(-x+1) + \sigma(-x-1)$$

$$= \left[\sigma(x+1) + \sigma(-x-1) - \frac{1}{2} - \frac{1}{2}\right] - \left[\sigma(x-1) + \sigma(-x+1) - \frac{1}{2} - \frac{1}{2}\right]$$

$$
\begin{aligned}
&= [g_\sigma(x+1) + g_\sigma(-x-1)] - [g_\sigma(x-1) + g_\sigma(-x+1)] \\
&= [g_\sigma(x+1) - g_\sigma(x+1)] - [g_\sigma(x-1) - g_\sigma(x-1)] = 0.
\end{aligned}
$$

□

Lemma 6.16. *For every $x \in \mathbb{R}$, we have $\sum_{k\in\mathbb{Z}} \phi_\sigma(x-k) = 1$.*

Proof. Let $x \in \mathbb{R}$ be fixed. For every fixed and sufficiently large $n \in \mathbb{N}^+$, we have

$$
\sum_{k=-n}^{n} [\sigma(x+1-k) - \sigma(x-k)] = \sigma(x+n+1) - \sigma(x-n),
$$

$$
\sum_{k=-n}^{n} [\sigma(x-k) - \sigma(x-1-k)] = \sigma(x+n) - \sigma(x-n-1),
$$

then,

$$
\begin{aligned}
&\sum_{k=-n}^{n} \phi_\sigma(x-k) \\
&= \frac{1}{2}\left\{ \sum_{k=-n}^{n} [\sigma(x+1-k) - \sigma(x-k)] + \sum_{k=-n}^{n} [\sigma(x-k) - \sigma(x-1-k)] \right\} \\
&= \frac{1}{2}\left\{ \sigma(x+n+1) - \sigma(x-n) + \sigma(x+n) - \sigma(x-n-1) \right\}.
\end{aligned}
$$

Passing to the limit for $n \to +\infty$, we obtain

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} \phi_\sigma(x-k) = \lim_{n\to+\infty} \sum_{k=-n}^{n} \phi_\sigma(x-k) \\
&= \lim_{n\to+\infty} \frac{1}{2}\left\{ \sigma(x+n+1) - \sigma(x-n) + \sigma(x+n) - \sigma(x-n-1) \right\} = 1.
\end{aligned}
$$

□

Lemma 6.17. *$\phi_\sigma(x)$ is non-decreasing for $x < 0$ and non-increasing for $x \geq 0$.*

Proof. By condition $(\Sigma 2)$ and being σ non-decreasing, we have for every $x \geq 0$ that $\sigma'(x) \geq 0$ is a non-increasing function. Then, for every $x \geq 1$, $\phi'_\sigma(x) = (1/2)[\sigma'(x+1) - \sigma'(x-1)] \leq 0$. Moreover, using condition $(\Sigma 1)$ we obtain that $\sigma(x) = 1 - \sigma(-x)$, and we can write

$$
\phi_\sigma(x) = \frac{1}{2}[\sigma(x+1) + \sigma(1-x) - 1].
$$

Then, we have for every $0 \leq x < 1$, $\phi'_\sigma(x) = (1/2)[\sigma'(x+1) - \sigma'(1-x)] \leq 0$ being $0 \leq 1-x \leq x+1$. Finally, $\phi_\sigma(x)$ is even by Lemma 6.15 (iii), and this completes the proof. □

We recall that, $\lfloor \cdot \rfloor$ denotes the integral part of any given number and $\lceil \cdot \rceil$ is the ceiling.

Lemma 6.18. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}^+$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(nx-k)} \leq \frac{1}{\phi_\sigma(1)}.$$

Proof. >From Lemma 6.15 (iii) and Lemma 6.16 we see that, for every $x \in [a,b]$,

$$1 = \sum_{k \in \mathbb{Z}} \phi_\sigma(nx-k) \geq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(|nx-k|) \geq \phi_\sigma(|nx-k_0|),$$

for every $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$. Now, we can choose k_0 such that $|nx-k_0| \leq 1$, and using Lemma 6.15 (i) and Lemma 6.17 we obtain $\phi_\sigma(|nx-k_0|) \geq \phi_\sigma(1) > 0$, then the proof follows. $\square$

Remark 6.19. Note that smoothness and concavity of σ are only used to prove Lemma 6.17. However, Lemma 6.17 is also crucial to prove Lemma 6.18. Then, the assumption $(\Sigma 2)$ could be omitted, and we could directly assume that the conditions in Lemma 6.17 are satisfied. In this case, Lemma 6.18 still holds true, and we can apply our theory also to non-smooth sigmoidal functions. In fact, proving Lemma 6.18, only the fact that ϕ_σ is *non-increasing* for $x \geq 0$ was exploited, to infer that $\phi_\sigma(|nx-k_0|) \geq \phi_\sigma(1)$.

Lemma 6.20. *$\phi_\sigma(x) = \mathcal{O}(|x|^{-1-\alpha})$, as $x \to \pm\infty$.*

Proof. By assumption $(\Sigma 3)$, there exist $C > 0$, $M > 0$ such that $\sigma(x) \leq C|x|^{-1-\alpha}$, for every $x < -M$, for some given $\alpha > 0$. Then,

$$\phi_\sigma(x) \leq \sigma(x+1) \leq C\,|x|^{-1-\alpha},$$

for every $x < -M-1$. Moreover, using assumptions $(\Sigma 1)$ and $(\Sigma 3)$, we have for every $x > M+1$

$$\phi_\sigma(x) \leq 1 - \sigma(x-1) = \sigma(1-x) \leq C\,|x|^{-1-\alpha},$$

and then the proof follows. $\square$

Finally, we can prove the following

Lemma 6.21. *For every $\gamma > 0$, we have*

$$\lim_{n \to +\infty} \sum_{|x-k| > \gamma n} \phi_\sigma(x-k) = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

Proof. For every $x \in \mathbb{R}$ we denote by $x_0 = x - \lfloor x \rfloor$, with $0 \leq x_0 < 1$. Moreover, $\widetilde{k} := k - \lfloor x \rfloor$ be integer for every $k \in \mathbb{Z}$. Let now $\gamma > 0$ be fixed. Then, by Lemma 6.20, we can write for every $n \in \mathbb{N}^+$ sufficiently large,

$$\begin{aligned}\sum_{|x-k|>\gamma n} \phi_\sigma(x-k) &= \sum_{|\widetilde{k}-x_0|>\gamma n} \phi_\sigma(x_0 - \widetilde{k}) \leq \sum_{|\widetilde{k}|>\gamma n-1} \phi_\sigma(x_0 - \widetilde{k}) \\ &\leq \sup_{u\in[0,1]} \sum_{|\widetilde{k}|>\gamma n-1} \phi_\sigma(u-\widetilde{k}) \leq C \sup_{u\in[0,1]} \sum_{|\widetilde{k}|>\gamma n-1} |u-\widetilde{k}|^{-1-\alpha} \\ &\leq C \left\{ \sum_{\widetilde{k}>\gamma n-1} |1-\widetilde{k}|^{-1-\alpha} + \sum_{\widetilde{k}<1-\gamma n} |\widetilde{k}|^{-1-\alpha} \right\},\end{aligned}$$

and passing to the limit for $n \to +\infty$, the assertion follows. □

Remark 6.22. Following the line of the proof of Lemma 6.21, it is easy to show that the series $\sum_{k\in\mathbb{Z}} \phi_\sigma(x-k)$ converges uniformly on the compact subsets of $\mathbb{R}$. In fact, by Lemma 6.20 and for every subset $[a,b] \subset \mathbb{R}$, we have for every $N \in \mathbb{N}^+$ sufficiently large,

$$\begin{aligned}\sup_{u\in[a,b]} \sum_{|k|>N} \phi_\sigma(u-k) &\leq C \sup_{u\in[a,b]} \sum_{|k|>N} |u-k|^{-1-\alpha} \\ &\leq C \left\{ \sum_{k>N} |b-k|^{-1-\alpha} + \sum_{k<-N} |a-k|^{-1-\alpha} \right\},\end{aligned}$$

and passing to the limit for $N \to +\infty$, the claim holds.

Remark 6.23. The results of this section are new, and some of them provide a generalization of the properties established in [7] for the hyperbolic tangent sigmoidal functions, to other functions σ satisfying conditions $(\Sigma 1)$, $(\Sigma 2)$, and $(\Sigma 3)$.

6.5 Univariate theory: the main results

Recall the operators that we will study in this section.

Definition 6.24. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and $n \in \mathbb{N}^+$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$. The *positive linear neural network operators* activated by the sigmoidal function σ, are defined as

$$F_n(f,x) := \frac{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \phi_\sigma(nx-k)}{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(nx-k)}, \qquad x \in [a,b].$$

For sufficiently large $n \in \mathbb{N}^+$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$, and since f is bounded, $F_n(f, x)$ is well-defined, for all $x \in [a, b]$.

We recall that, a linear operator $T : E \to F$ between two ordered vector spaces, E and F, is termed *positive* if and only if $T(f) \geq 0$ for every $f \geq 0$, $f \in E$.

First of all, we study the pointwise and uniform convergence of such operators. We can prove the following.

Theorem 6.25. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then,*

$$\lim_{n \to +\infty} F_n(f, x) = f(x),$$

at any point $x \in [a, b]$ of continuity of f. Moreover, if $f \in C^0[a, b]$, then

$$\lim_{n \to +\infty} \sup_{x \in [a,b]} |F_n(f, x) - f(x)| = 0.$$

Proof. Let $x \in [a, b]$ be a fixed point of continuity of f. By Lemma 6.18 we have

$$|F_n(f, x) - f(x)| = \frac{\left| \displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \phi_\sigma(nx - k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(nx - k) \right|}{\displaystyle\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_\sigma(nx - k)}$$

$$\leq \frac{1}{\phi_\sigma(1)} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k),$$

for every $n \in \mathbb{N}^+$ sufficiently large. Let now $\varepsilon > 0$ be fixed. From the continuity of f at x, there exists $\gamma > 0$ such that, for every $y \in [x - \gamma, x + \gamma] \cap [a, b]$, $|f(y) - f(x)| < \varepsilon$. Hence,

$$|F_n(f, x) - f(x)| \leq \frac{1}{\phi_\sigma(1)} \left\{ \sum_{\substack{k=\lceil na \rceil \\ |\frac{k}{n} - x| \leq \gamma}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k) \right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ |\frac{k}{n} - x| > \gamma}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k) \right\} =: \frac{1}{\phi_\sigma(1)} \left\{ I_1 + I_2 \right\}.$$

We first estimate I_1. Using Lemma 6.16 and the continuity of f at x, we obtain

$$I_1 < \varepsilon \sum_{k\in\mathbb{Z}} \phi_\sigma(nx-k) = \varepsilon.$$

Furthermore, by Lemma 6.21, we have for n sufficiently large

$$I_2 \leq 2\|f\|_\infty \sum_{|nx-k|>n\gamma} \phi_\sigma(nx-k) < 2\|f\|_\infty \varepsilon,$$

uniformly with respect to $x \in \mathbb{R}$. The proof of the first part of the theorem then follows since ε is arbitrary. When $f \in C^0[a,b]$, the second part of the theorem follows similarly, replacing $\gamma > 0$ with the parameter of the uniform continuity of f on $[a,b]$. □

Now, we want to study the order of approximation for the family of the linear positive neural network operators in $C^0[a,b]$. It is natural to introduce the Lipschitz classes in which we will work. We define, for $0 < \nu \leq 1$,

$$Lip(\nu) := \big\{ f \in C^0[a,b] : \text{such that there exist } \gamma,\ C > 0 \text{ such that, for each}$$

$$x \in [a,b], |f(x) - f(x+t)| \leq C\,|t|^\nu, \text{ for every } |t| \leq \gamma \text{ with } x+t \in [a,b] \big\}.$$

Moreover, for $\nu > 0$, we introduce for the functions ϕ_σ, the *discrete absolute moment of order* ν, defined by

$$m_\nu(\phi_\sigma) := \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |x-k|^\nu \phi_\sigma(x-k).$$

Now, we are able to prove the following

Theorem 6.26. *Suppose that $m_\nu(\phi_\sigma) < +\infty$ for some $\nu > 0$. Then, if $0 < \nu \leq 1$, we have for every $f \in Lip(\nu)$*

$$\sup_{x\in[a,b]} |F_n(f,x) - f(x)| = \mathcal{O}(n^{-\nu}), \qquad \text{as } n \to +\infty.$$

While, if $\nu > 1$, we have for every $f \in Lip(1)$

$$\sup_{x\in[a,b]} |F_n(f,x) - f(x)| = \mathcal{O}(n^{-1}), \qquad \text{as } n \to +\infty.$$

Proof. Observe first that if $\nu > 1$, we have $m_1(\phi_\sigma) \leq m_\nu(\phi_\sigma) < +\infty$. Indeed,

$$\sum_{k\in\mathbb{Z}} |x-k|\phi_\sigma(x-k) = \sum_{|x-k|\leq 1} |x-k|\phi_\sigma(x-k) + \sum_{|x-k|>1} |x-k|\phi_\sigma(x-k)$$

$$\leq \sum_{|x-k|\leq 1} \phi_\sigma(x-k) + \sum_{|x-k|>1} |x-k|^\nu \phi_\sigma(x-k) \leq 1 + m_\nu(\phi_\sigma) < +\infty.$$

Then we can lead us back to study the case $\nu = 1$. Therefore, we consider $0 < \nu \leq 1$ and let $f \in Lip(\nu)$ be fixed. As made in the proof of Theorem 6.25, we can write, for every $x \in [a, b]$,

$$F_n(f, x) - f(x)| \leq \frac{1}{\phi_\sigma(1)} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k).$$

Let now γ, $C > 0$ the constants whereby f belongs to $Lip(\nu)$. We have

$$|F_n(f, x) - f(x)| \leq \frac{1}{\phi_\sigma(1)} \left\{ \sum_{\substack{k=\lceil na \rceil \\ |\frac{k}{n} - x| \leq \gamma}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k) \right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ |\frac{k}{n} - x| > \gamma}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| \phi_\sigma(nx - k) \right\} =: \frac{1}{\phi_\sigma(1)} \{J_1 + J_2\}.$$

Since f belongs to $Lip(\nu)$, if $|(k/n) - x| \leq \gamma$, we have $|f(k/n) - f(x)| \leq C\,|(k/n) - x|^\nu$, and we obtain for J_1

$$J_1 \leq C \sum_{|\frac{k}{n} - x| \leq \gamma} \left| \frac{k}{n} - x \right|^\nu \phi_\sigma(nx - k) \leq C m_\nu(\phi_\sigma) n^{-\nu} < +\infty,$$

for every fixed sufficiently large $n \in \mathbb{N}^+$. Moreover,

$$J_2 \leq 2\|f\|_\infty \sum_{|nx - k| > \gamma n} \phi_\sigma(nx - k) = 2\|f\|_\infty \sum_{|nx - k| > \gamma n} \frac{|nx - k|^\nu}{|nx - k|^\nu} \phi_\sigma(nx - k)$$

$$< \frac{2\|f\|_\infty}{\gamma^\nu n^\nu} \sum_{|nx - k| > \gamma n} |nx - k|^\nu \phi_\sigma(nx - k) \leq \frac{2\|f\|_\infty}{\gamma^\nu} m_\nu(\phi_\sigma) n^{-\nu} < +\infty,$$

for every fixed $n \in \mathbb{N}^+$. This completes the proof. □

Remark 6.27. Note that, if $\sigma(x) = \mathcal{O}(|x|^{-1-\beta-\alpha})$, as $x \to -\infty$, by Lemma 6.20 we have $\phi_\sigma(x) = \mathcal{O}(|x|^{-1-\beta-\alpha})$, as $x \to \pm\infty$, for some α, $\beta > 0$. In this case, it is easy to show that $m_\nu(\phi_\sigma) < +\infty$ for every $0 < \nu \leq \max\{\alpha, \beta\}$, see e.g. [34].

Examples of smooth sigmoidal functions satisfying all the assumptions of our theory are provided by the well-known *logistic function*, $\sigma_\ell(x) = (1 + e^{-x})^{-1}$, $x \in \mathbb{R}$ [38, 9, 10, 11, 56], and by the *hyperbolic tangent* sigmoidal function, $\sigma_h(x) := (1/2)(\tanh x + 1)$, $x \in \mathbb{R}$ [7, 8, 10, 56]. In particular, σ_ℓ

and σ_h satisfy condition (Σ3) for all $\alpha > 0$, and then, in view of Remark 6.27, $m_\nu(\phi_{\sigma_\ell})$, $m_\nu(\phi_{\sigma_h}) < +\infty$, for every $\nu > 0$.

An example of non-smooth sigmoidal function is also provided by the *ramp function*, $\sigma_R(x)$, defined by $\sigma_R(x) := 0$ for $x < -1/2$, $\sigma_R(x) := 1$ for $x > 1/2$, and $\sigma_R(x) := x + (1/2)$ for $-1/2 \leq x \leq 1/2$ [46, 39]. Condition (Σ3) is satisfied for every $\alpha > 0$, and the corresponding function $\phi_{\sigma_R}(x)$ is compactly supported. The discrete absolute moments are now $m_\nu(\phi_{\sigma_R}) < +\infty$, for every $\nu > 0$, by Remark 6.27.

For the sigmoidal functions $\sigma_\ell(x)$, $\sigma_h(x)$, and $\sigma_R(x)$, the following corollary holds.

Corollary 6.28. *For every $f \in Lip(\nu)$, $0 < \nu \leq 1$, we have*

$$\sup_{x\in[a,b]} |F_n(f,x) - f(x)| = \mathcal{O}(n^{-\nu}), \qquad \text{as } n \to +\infty.$$

Remark 6.29. Note that, when $f \in C^1[a,b]$ we have $f \in Lip(1)$, then, for all the sigmoidal functions considered in Corollary 6.28, $\sup_{x\in[a,b]} |F_n(f,x) - f(x)| = \mathcal{O}(n^{-1})$ as $n \to +\infty$. This fact improves the result concerning the order of approximation for neural network operators activated by hyperbolic tangent and logistic functions, when $f \in C^1[a,b]$, established by Anastassiou in [7, 11]. In fact, Anastassiou proved that $\sup_{x\in[a,b]} |F_n(f,x) - f(x)| = \mathcal{O}(n^{-\theta})$ as $n \to +\infty$, where $0 < \theta < 1$ and $f \in C^1[a,b]$.

Finally, others examples of sigmoidal functions satisfying the assumptions of our theory, can be constructed starting from

$$M_n(x) := \frac{1}{(n-1)!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\frac{n}{2}+x-i\right)_+^{n-1}, \qquad x \in \mathbb{R},$$

the well-known B-spline of order $n \in \mathbb{N}^+$ ([34]), by a simple procedure described in Chapter 5 ([58]). Here, the function $(x)_+ := \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$. We define the sigmoidal function $\sigma_{M_n}(x)$ by

$$\sigma_{M_n}(x) := \int_{-\infty}^{x} M_n(t)\,dt, \qquad x \in \mathbb{R}.$$

Note that, $\sigma_{M_2}(x)$ coincides exactly with the ramp function, $\sigma_R(x)$.

Examples of sigmoidal functions to which the present theory cannot be applied, are provided by the class of Gompertz functions $\sigma_{\alpha,\beta}$. The Gompertz functions are characterized by unsymmetrical growth, hence they fail to satisfy condition (Σ1).

6.5.1 Univariate quasi-interpolation operators with sigmoidal functions

In the present context, we can use sigmoidal functions, σ, also to study convergence and order of approximation of a class of *quasi-interpolation* operators, defined through the function ϕ_σ (see Section 6.4). Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. We define the so-called quasi-interpolation operators

$$G_n(f,x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{n}\right)\phi_\sigma(nx-k), \qquad x\in\mathbb{R},$$

for every $n>0$, see, e.g., [38, 7, 11]. Using Lemma 6.16 and the boundedness of f, it easy to show that $G_n(f,x)$ are well-defined, for all $x\in\mathbb{R}$ and $n>0$. Note that, for every $x\in\mathbb{R}$, using Lemma 6.16 again, we can write

$$\begin{aligned}|G_n(f,x)-f(x)| &= \left|G_n(f,x)-f(x)\sum_{k\ \in\mathbb{Z}}\phi_\sigma(nx-k)\right| \\ &\le \sum_{k\ \in\mathbb{Z}}\left|f\left(\frac{k}{n}\right)-f\left(x\right)\right|\phi_\sigma(nx-k).\end{aligned}$$

Then, using this inequality, we can prove pointwise convergence for the family $(G_n(f,\cdot))_{n>0}$, for bounded functions $f\in C^0(\mathbb{R})$, as well as uniform convergence, whenever the function f is uniformly continuous and bounded on $\mathbb{R}$. The proof follows as that of Theorem 6.25. Note that condition $(\Sigma 2)$ can be dropped since Lemma 6.18 is not used for studying quasi-interpolation operators. Finally, recall that the definition of Lipschitz classes for functions defined on the whole $\mathbb{R}$ becomes

$$Lip(\nu) := \left\{f\in C^0(\mathbb{R}) : \text{such that} \|f(\cdot)-f(\cdot+t)\|_\infty = \mathcal{O}(|t|^\nu),\ \text{as } t\to 0\right\},$$

for $0<\nu\le 1$, then Theorem 6.26 could also be easily extended to the case of quasi-interpolation operators.

6.6 Multivariate theory: preliminary lemmas

In what follows, we consider non-decreasing functions σ and ϕ_σ, such that considered in Section 6.4. In the next sections we will denote the multivariate spaces by $\mathbb{R}^s$ and $\mathbb{Z}^s$, $s\in\mathbb{N}^+$.

We now define the multivariate function

$$\Psi_\sigma(\underline{x}) := \phi_\sigma(x_1)\cdot\phi_\sigma(x_2)\cdot\ldots\cdot\phi_\sigma(x_s), \qquad \underline{x} := (x_1,\ldots,x_s)\in\mathbb{R}^s. \tag{6.4}$$

In the following lemmas, we prove some important properties of such a function. We denote with $\underline{k}=(k_1,\ldots,k_s)\in\mathbb{Z}^s$ the vectors whose components are $k_i\in\mathbb{Z}$.

Lemma 6.30. *For every $\underline{x}\in\mathbb{R}^s$, $\sum_{\underline{k}\in\mathbb{Z}^s}\Psi_\sigma(\underline{x}-\underline{k})=1$.*

Proof. Note that, for every $\underline{x}\in\mathbb{R}^s$, the series $\sum_{\underline{k}\in\mathbb{Z}^s}\Psi_\sigma(\underline{x}-\underline{k})$ can be rewritten as

$$\sum_{\underline{k}\in\mathbb{Z}^s}\Psi_\sigma(\underline{x}-\underline{k}) = \sum_{k_1\in\mathbb{Z}}\phi_\sigma(x_1-k_1)\cdot\ldots\cdot\sum_{k_s\in\mathbb{Z}}\phi_\sigma(x_s-k_s), \tag{6.5}$$

then the proof follows immediately from Lemma 6.16. □

In what follows, we denote by $\|\cdot\|$ the usual maximum norm of $\mathbb{R}^s$, i.e., $\|\underline{x}\| := \max\{|x_i|,\, i = 1, ..., s\}$, for every $\underline{x} \in \mathbb{R}^s$. We recall that in $\mathbb{R}^s$ all norms are equivalent, hence all results below are independent of the choice of the specific norm we may work with.

Lemma 6.31. *The series $\sum_{\underline{k}\in\mathbb{Z}^s} \Psi_\sigma(\underline{x}-\underline{k})$ converges uniformly on the compact sets of $\mathbb{R}^s$.*

Proof. Let $K \subset \mathbb{R}^s$ be a given compact subset of $\mathbb{R}^s$. For every $\underline{x} \in \mathbb{R}^s$, we can write

$$\sum_{\underline{k}\in\mathbb{Z}^s} \Psi_\sigma(\underline{x}-\underline{k}) = \sum_{k_j\in\mathbb{Z}} \phi_\sigma(x_j - k_j) \left[\sum_{\underline{k}_{[j]}\in\mathbb{Z}^{s-1}} \Psi_\sigma^{[j]}(\underline{x}_{[j]} - \underline{k}_{[j]}) \right],$$

where

$$\Psi_\sigma^{[j]}(\underline{x}_{[j]}-\underline{k}_{[j]}) := \phi_\sigma(x_1-k_1)\cdot ... \cdot\phi_\sigma(x_{j-1}-k_{j-1})\cdot\phi_\sigma(x_{j+1}-k_{j+1})\cdot ... \cdot\phi_\sigma(x_s-k_s),$$

being $\underline{x}_{[j]} := (x_1, ..., x_{j-1}, x_{j+1}, ..., x_s) \in\mathbb{R}^{s-1}$, $\underline{k}_{[j]} := (k_1, ..., k_{j-1}, k_{j+1}, ..., k_s) \in \mathbb{Z}^{s-1}$, for every $j = 1, ..., s$. Let now be $K_{[j]} \subset \mathbb{R}$ a compact set containing the j-th projection of all elements of K. For every fixed and sufficiently large $N \in \mathbb{N}^+$ and by Lemma 6.30, we have

$$\begin{aligned} &\sup_{\underline{x}\in K} \sum_{\|\underline{k}\|>N} \Psi_\sigma(\underline{x}-\underline{k}) \\ &\leq \sup_{\underline{x}\in K} \sum_{j=1}^{s} \left\{ \sum_{|k_j|>N} \phi_\sigma(x_j - k_j) \left[\sum_{\underline{k}_{[j]}\in\mathbb{Z}^{s-1}} \Psi_\sigma^{[j]}(\underline{x}_{[j]} - \underline{k}_{[j]}) \right] \right\} \\ &\leq \sum_{j=1}^{s} \left\{ \sup_{x_j\in K_{[j]}} \sum_{|k_j|>N} \phi_\sigma(x_j - k_j) \right\}, \end{aligned}$$

then the proof follows from Remark 6.22. $\square$

Lemma 6.32. *For every $\gamma > 0$, we have*

$$\lim_{n\to+\infty} \sum_{\|\underline{x}-\underline{k}\|>\gamma n} \Psi_\sigma(\underline{x}-\underline{k}) = 0,$$

uniformly with respect to $\underline{x} \in \mathbb{R}^s$. In particular, for every $\gamma > 0$ and for every $0 < \nu < \alpha$,

$$\sum_{\|\underline{x}-\underline{k}\|>\gamma n} \Psi_\sigma(\underline{x}-\underline{k}) = \mathcal{O}(n^{-\nu}), \quad \text{as} \quad n \to +\infty,$$

where $\alpha > 0$ is the constant appearing in condition $(\Sigma 3)$.

Proof. For every fixed $\underline{x} \in \mathbb{R}^s$, we define $\underline{x}_0 := \underline{x} - \lfloor \underline{x} \rfloor$, where $\lfloor \underline{x} \rfloor := (\lfloor x_1 \rfloor, ..., \lfloor x_s \rfloor)$, and note that $\underline{x}_0 \in I := [0,1] \times ... \times [0,1] \subset \mathbb{R}^s$. Moreover, $\widetilde{\underline{k}} := \underline{k} - \lfloor \underline{x} \rfloor$ belongs to $\mathbb{Z}^s$ for every $\underline{k} \in \mathbb{Z}^s$. Let now $\gamma > 0$ be fixed. For every $n \in \mathbb{N}^+$ sufficiently large, we can write

$$\sum_{\|\underline{x}-\underline{k}\|>\gamma n} \Psi_\sigma(\underline{x} - \underline{k}) = \sum_{\|\widetilde{\underline{k}}-\underline{x}_0\|>\gamma n} \Psi_\sigma(\,\underline{x}_0 - \widetilde{\underline{k}}\,)$$

$$\leq \sum_{\|\widetilde{\underline{k}}\|>\gamma n-1} \Psi_\sigma(\,\underline{x}_0 - \widetilde{\underline{k}}\,) \leq \sup_{\underline{x}_0 \in I} \sum_{\|\widetilde{\underline{k}}\|>\gamma n-1} \Psi_\sigma(\,\underline{x}_0 - \widetilde{\underline{k}}\,),$$

and passing to the limit for $n \to +\infty$, the first part of the lemma holds in view of Lemma 6.31.

Let now $0 < \nu < \alpha$ be fixed. Using Lemma 6.30 and adopting the same notation of Lemma 6.31, we obtain, for $n \in \mathbb{N}^+$ sufficiently large,

$$\sum_{\|\underline{x}-\underline{k}\|>\gamma n} \Psi_\sigma(\underline{x} - \underline{k})$$

$$\leq \sum_{j=1}^{s} \left\{ \sum_{|x_j-k_j|>\gamma n} \phi_\sigma(x_j - k_j) \left[\sum_{\underline{k}_{[j]} \in \mathbb{Z}^{s-1}} \Psi_\sigma^{[j]}(\underline{x}_{[j]} - \underline{k}_{[j]}) \right] \right\}$$

$$\leq \sum_{j=1}^{s} \left\{ \sum_{|x_j-k_j|>\gamma n} \phi_\sigma(x_j - k_j) \right\} = \sum_{j=1}^{s} \left\{ \sum_{|x_j-k_j|>\gamma n} \phi_\sigma(x_j - k_j) \, \frac{|x_j - k_j|^\nu}{|x_j - k_j|^\nu} \right\}$$

$$< \frac{1}{\gamma^\nu \, n^\nu} \sum_{j=1}^{s} \left\{ \sum_{|x_j-k_j|>\gamma n} \phi_\sigma(x_j - k_j) \, |x_j - k_j|^\nu \right\}. \tag{6.6}$$

We now recall the definition of the *discrete absolute moment* of order ν of the function ϕ_σ, defined by

$$m_\nu(\phi_\sigma) := \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \phi_\sigma(x - k) \, |x - k|^\nu. \tag{6.7}$$

By Lemma 6.20, $\phi_\sigma(x) = \mathcal{O}(|x|^{-1-\alpha})$ as $x \to \pm\infty$, and it is well known that, under this assumption, $m_\nu(\phi_\sigma) < +\infty$ for $0 < \nu < \alpha$, see [34, 60]. Therefore, we infer from (6.6)

$$\sum_{\|\underline{x}-\underline{k}\|>\gamma n} \Psi_\sigma(\underline{x} - \underline{k}) < n^{-\nu} \, \frac{s}{\gamma^\nu} \, m_\nu(\phi_\sigma) < +\infty,$$

and thus the second part of the lemma is also proved. $\square$

Remark 6.33. Note that, since in $\mathbb{R}^s$ all norms are equivalent, both, convergence and order of approximation results established in Lemma 6.32 are independent on the choice of the specific norm being used. In fact, if $\| \cdot \|_*$ denotes a specific norm in $\mathbb{R}^s$, we have $\|\underline{x}\|_* \leq C\|\underline{x}\|$ for every $\underline{x} \in \mathbb{R}^s$, for a suitable positive constant C, and hence we can write, for every fixed $n \in \mathbb{N}$,

$$\sum_{\|\underline{x}-\underline{k}\|_* > \gamma n} \Psi_\sigma(\underline{x}-\underline{k}) \leq \sum_{\|\underline{x}-\underline{k}\| > \gamma n / C} \Psi_\sigma(\underline{x}-\underline{k}),$$

from which the claim follows.

Finally, we show that, as a direct consequence of Lemma 6.18, the following lemma holds true:

Lemma 6.34. *Let $\underline{x} \in [a_1, b_1] \times ... \times [a_s, b_s] \subset \mathbb{R}^s$ and $n \in \mathbb{N}^+$, so that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$ for every $i = 1, ..., s$. Then,*

$$\frac{1}{\displaystyle\prod_{i=1}^{s} \sum_{k=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \phi_\sigma(nx_i - k_i)} \leq \frac{1}{[\phi_\sigma(1)]^s}.$$

Proof. Since σ is non-decreasing and $0 \leq \sigma(x) \leq 1$, it is easy to see that $\sigma(2) - \sigma(0) < 2$, hence we infer that $0 < \phi_\sigma(1) < 1$, and thus $1/\phi_\sigma(1) > 1$. Then, by Lemma 6.18,

$$\frac{1}{\displaystyle\sum_{k=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \phi_\sigma(nx_1 - k_1)} \cdot ... \cdot \frac{1}{\displaystyle\sum_{k=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} \phi_\sigma(nx_s - k_s)} \leq \frac{1}{\phi_\sigma(1)} \cdot ... \cdot \frac{1}{\phi_\sigma(1)},$$

hence what had to be proved. □

Remark 6.35. We stress that the results of this section are new, and some of them represent a generalization of the properties established in [60] for the univariate case.

6.7 Multivariate theory: the main results

In what follows, we denote by $\mathcal{R}$ the s-dimensional interval $\mathcal{R} := [a_1, b_1] \times ... \times [a_s, b_s] \subset \mathbb{R}^s$. Let us define now the operators that will be studied in this section.

Definition 6.36. Let $f : \mathcal{R} \to \mathbb{R}$ be a bounded function, and $n \in \mathbb{N}^+$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$ for every $i = 1, ..., s$. The *linear positive multivariate* NN *operators* $F_n(f, \underline{x})$, activated by the sigmoidal function σ, and acting on f, are defined by

$$F_n^s(f, \underline{x}) := \frac{\displaystyle\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} f\left(\underline{k}/n\right) \Psi_\sigma(n\underline{x} - \underline{k})}{\displaystyle\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} \Psi_\sigma(n\underline{x} - \underline{k})},$$

for every $\underline{x} \in \mathcal{R}$ and $\underline{k}/n := (k_1/n, ..., k_s/n)$.

For $n \in \mathbb{N}^+$ sufficiently large, we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., s$. Moreover, $a_i \leq \frac{k_i}{n} \leq b_i$ if and only if $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, and since f is bounded, $F_n^s(f, \underline{x})$ turns out to be well defined for all $\underline{x} \in \mathcal{R}$. Note that $F_n^s(1, \underline{x}) = 1$, for every n sufficiently large.

First of all, we study pointwise and uniform convergence of sequences of such operators. We can prove the following

Theorem 6.37. *Let $f : \mathcal{R} \to \mathbb{R}$ be bounded. Then,*

$$\lim_{n \to +\infty} F_n^s(f, \underline{x}) = f(\underline{x})$$

at each point $\underline{x} \in \mathcal{R}$ of continuity of f. Moreover, if $f \in C^0(\mathcal{R})$, then

$$\lim_{n \to +\infty} \sup_{\underline{x} \in \mathcal{R}} |F_n^s(f, \underline{x}) - f(\underline{x})| = \lim_{n \to +\infty} \|F_n^s(f, \cdot) - f(\cdot)\|_\infty = 0.$$

Proof. Let $\underline{x} \in \mathcal{R}$ be a point where f is continuous. We obtain from Lemma 6.34

$$|F_n^s(f, \underline{x}) - f(\underline{x})| = \frac{\left|\displaystyle\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} \left[f(\underline{k}/n) - f(\underline{x})\right] \Psi_\sigma(n\underline{x} - \underline{k})\right|}{\displaystyle\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} \Psi_\sigma(n\underline{x} - \underline{k})}$$

$$\leq \frac{1}{[\phi_\sigma(1)]^s} \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor} |f(\underline{k}/n) - f(\underline{x})| \Psi_\sigma(n\underline{x} - \underline{k}),$$

for every $n \in \mathbb{N}^+$ sufficiently large. Let now $\varepsilon > 0$ be fixed. From the continuity of f at $\underline{x}$, there exists $\gamma > 0$ such that $|f(\underline{y}) - f(\underline{x})| < \varepsilon$ for every $\underline{y} \in \mathcal{R}$ with $\|\underline{y} - \underline{x}\|_2 < \gamma$, $\|\cdot\|_2$ being the euclidean norm. Hence, defining

$$S_1 := \left\{\underline{k} : \lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor, \ i = 1, ..., s, \text{ and } \|\underline{k}/n - \underline{x}\| \leq \gamma/\sqrt{s}\right\},$$

and

$$S_2 := \left\{\underline{k} : \lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor,\ i = 1, ..., s,\ \text{and } \|\underline{k}/n - \underline{x}\| > \gamma/\sqrt{s}\right\},$$

we can write,

$$|F_n^s(f, \underline{x}) - f(\underline{x})| \leq \frac{1}{[\phi_\sigma(1)]^s} \left\{ \sum_{\underline{k} \in S_1} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}) \right.$$

$$\left. + \sum_{\underline{k} \in S_2} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}) \right\} =: \frac{1}{[\phi_\sigma(1)]^s} \left(I_1 + I_2\right).$$

Let first estimate I_1. Using Lemma 6.30, the continuity of f at $\underline{x}$, and that $\|\underline{k}/n - \underline{x}\|_2 \leq \sqrt{s} \, \|\underline{k}/n - \underline{x}\| \leq \gamma$ whenever $\underline{k} \in S_1$, we obtain

$$I_1 < \varepsilon \sum_{\underline{k} \in S_1} \Psi_\sigma(n\underline{x} - \underline{k}) \leq \varepsilon.$$

Furthermore, by the boundedness of f and Lemma 6.32, we have, for n sufficiently large,

$$I_2 \leq 2 \, \|f\|_\infty \sum_{\|n\underline{x} - \underline{k}\| > n\gamma/\sqrt{s}} \Psi_\sigma(n\underline{x} - \underline{k}) < 2 \, \|f\|_\infty \, \varepsilon,$$

uniformly with respect to $\underline{x} \in \mathbb{R}^s$. The proof of the first part of the theorem then follows by the arbitrarity of ε. When $f \in C^0(\mathcal{R})$, the second part of the theorem follows similarly, replacing $\gamma > 0$ with the parameter of the uniform continuity of f in $\mathcal{R}$. $\square$

Now, we want to study the *order* of approximation for the family of the linear positive NN operators in $C^0(\mathcal{R})$. It is natural to introduce the Lipschitz classes for multivariate functions, within which we will work. We define, for $0 < \nu \leq 1$,

$$Lip(\nu) := \left\{ f \in C^0(\mathcal{R}) : \text{such that there exist } \gamma > 0,\ C > 0 \text{ so that, for each} \right.$$

$$\left. \underline{x} \in \mathcal{R},\ |f(\underline{x} + \underline{t}) - f(\underline{x})| \leq C \, \|\underline{t}\|_2^\nu \text{ for every } \|\underline{t}\|_2 \leq \gamma \text{ with } \underline{x} + \underline{t} \in \mathcal{R} \right\}.$$

Now, we are able to prove the following

Theorem 6.38. *Suppose that σ meets the decay condition ($\Sigma 3$) for some $\alpha > 1$, and let be $f \in Lip(\nu)$ for some ν, with $0 < \nu \leq 1$. Then,*

$$\|F_n^s(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\nu}), \qquad as \qquad n \to +\infty.$$

Proof. Let be $f \in Lip(\nu)$ for some ν, with $0 < \nu \leq 1$. As in the proof of Theorem 6.37, we can write, for every $\underline{x} \in \mathcal{R}$,

$$|F_n^s(f, \underline{x}) - f(\underline{x})| \leq \frac{1}{[\phi_\sigma(1)]^s} \sum_{k_1 = \lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_s = \lceil na_s \rceil}^{\lfloor nb_s \rfloor} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}).$$

If $\gamma > 0$ and $C > 0$ are the constants related to f, in the definition of $Lip(\nu)$, we have

$$|F_n^s(f, \underline{x}) - f(\underline{x})| \leq \frac{1}{[\phi_\sigma(1)]^s} \left\{ \sum_{\underline{k} \in S_1} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}) \right.$$

$$\left. + \sum_{\underline{k} \in S_2} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}) \right\} =: \frac{1}{[\phi_\sigma(1)]^s} \left(J_1 + J_2 \right),$$

where S_1 and S_2 are those defined in the proof of Theorem 6.37 with the value of $\gamma > 0$ fixed above, (in the definition of $Lip(\nu)$). Since $f \in Lip(\nu)$, we have for $\underline{k} \in S_1$, $\|(\underline{k}/n) - \underline{x}\|_2 \leq \sqrt{s} \, \|(\underline{k}/n) - \underline{x}\| \leq \gamma$, and hence

$$|f(\underline{k}/n) - f(\underline{x})| \leq C \|(\underline{k}/n) - \underline{x}\|_2^\nu \leq C s^{\nu/2} \, \|(\underline{k}/n) - \underline{x}\|^\nu. \qquad (6.8)$$

Proceeding as in the proof of Lemma 6.32 and by (6.8), we obtain for J_1 the estimate

$$J_1 \leq \; n^{-\nu} \, C \, s^{\nu/2} \sum_{\underline{k} \in S_1} \Psi_\sigma(n\underline{x} - \underline{k}) \, \|n\underline{x} - \underline{k}\|^\nu$$

$$\leq \; n^{-\nu} \, C \, s^{\nu/2} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \phi_\sigma(x_j - k_j) \, |nx_j - k_j|^\nu \left[\sum_{\underline{k}_{[j]} \in \mathbb{Z}^{s-1}} \Psi_\sigma^{[j]}(\underline{x}_{[j]} - \underline{k}_{[j]}) \right] \right\}$$

$$\leq \; n^{-\nu} \, C \, s^{\nu/2} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \phi_\sigma(x_j - k_j) \, |nx_j - k_j|^\nu \right\} \leq n^{-\nu} \, C \, s^{1+\nu/2} \, m_\nu(\phi_\sigma),$$

where $m_\nu(\phi_\sigma)$ is defined as in (6.7). As it was observed in the proof of Lemma 6.32, $m_\nu(\phi_\sigma) < +\infty$, since $\nu < \alpha$. It then follows that $J_1 = \mathcal{O}(n^{-\nu})$ as $n \to +\infty$. Moreover, by the second part of Lemma 6.32, it turns out that

$$J_2 \leq 2 \, \|f\|_\infty \sum_{\underline{k} \in S_2} \Psi_\sigma(n\underline{x} - \underline{k}) = \mathcal{O}(n^{-\nu}), \qquad as \qquad n \to +\infty,$$

and this completes the proof. □

The most important instances of sigmoidal functions do satisfy condition ($\Sigma 3$) with $\alpha > 1$, and this explains why it was reasonable to make such assumption in Theorem 6.38. However, an order of approximation can be determined even when $0 < \alpha \leq 1$. In fact, we can establish the following

Theorem 6.39. *Assume that σ meets the decay condition* ($\Sigma 3$) *for some* $0 < \alpha \leq 1$, *and let be $f \in Lip(\nu)$ for some ν, with $0 < \nu \leq 1$. Then,*

(i) if $\nu < \alpha$,

$$\|F_n^s(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\nu}), \qquad as \qquad n \to +\infty; \tag{6.9}$$

(ii) if $\alpha \leq \nu \leq 1$, we have

$$\|F_n^s(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\alpha+\varepsilon}), \qquad as \qquad n \to +\infty; \tag{6.10}$$

for every $0 < \varepsilon < \alpha$.

Proof. (i) For functions $f \in Lip(\nu)$ with $0 < \nu < \alpha$, we have

$$\|F_n^s(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\nu}), \qquad as \qquad n \to +\infty, \tag{6.11}$$

and this can be proved exactly as it was done in Theorem 6.38.

(ii) Every function $f \in Lip(\nu)$ with $\alpha \leq \nu \leq 1$ is, in particular, such that $f \in Lip(\beta)$ with $\beta := \alpha - \varepsilon$, for every fixed but arbitrary ε, $0 < \varepsilon < \alpha$. Thus, we obtain $0 < \beta < \alpha$ and, in view of (i),

$$\|F_n^s(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\beta}) = \mathcal{O}(n^{-\alpha+\varepsilon}), \qquad as \qquad n \to +\infty.$$

This completes the proof. $\square$

Remark 6.40. The approach adopted in this paper (and in [60]) to study the order of approximation differs from those of [104, 93]. Here we obtain estimates in the uniform norm for neural network operators applied to functions belonging to Lipschitz classes, and exploiting a moment-type approach. In Makovoz [104], tight bounds for multivariate neural networks with sigmoidal functions were derived using techniques of functional analysis, in the more general context of L^p-spaces, and particular estimates were derived for functions belonging to $L^2(D)$, where $D \subset \mathbb{R}^2$ is an open convex set. Kurková and Sanguineti [93] have established tight bounds for NNs in Hilbert spaces, extending some results earlier obtained in [104]. Our operators F_n are affected by the pointwise behavior of the functions f, and hence they are not suited to L^p-approximations.

Examples of smooth sigmoidal functions, satisfying all assumptions required by our theory, are provided by the well known *logistic function*, σ_ℓ, and by the *hyperbolic tangent* function, $\sigma_h(x)$. In particular, σ_ℓ and σ_h satisfy condition ($\Sigma 3$) for all $\alpha > 0$, in view of their exponential decay to zero as $x \to -\infty$.

An example of a non-smooth sigmoidal function is provided by the *ramp* function, $\sigma_R(x)$, defined by $\sigma_R(x) := 0$ for $x < -1/2$, $\sigma_R(x) := 1$ for $x > 1/2$, and $\sigma_R(x) := x + 1/2$ for $-1/2 \leq x \leq 1/2$ [46, 39]. Condition ($\Sigma 3$) is also satisfied for every $\alpha > 0$ and the corresponding function $\phi_{\sigma_R}(x)$ is compactly supported.

For the sigmoidal functions $\sigma_\ell(x)$, $\sigma_h(x)$, and $\sigma_R(x)$, the following corollary holds.

Corollary 6.41. *For every $f \in Lip(\nu)$, $0 < \nu \leq 1$, we have*

$$\|F_n(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\nu}), \qquad \textit{as} \qquad n \to +\infty.$$

Remark 6.42. Note that, when $f \in C^1(\mathcal{R})$, also $f \in Lip(1)$, and then, for all sigmoidal functions considered in Corollary 6.41, $\|F_n(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-1})$ as $n \to +\infty$. This fact improves the result concerning the order of approximation for NN operators activated by hyperbolic tangent or by logistic functions when $f \in C^1(\mathcal{R})$, earlier established by Anastassiou [8, 9]. In fact, Anastassiou proved that, for every $f \in C^1(\mathcal{R})$, $\|F_n(f, \cdot) - f(\cdot)\|_\infty = \mathcal{O}(n^{-\theta})$ as $n \to +\infty$, where $0 < \theta < 1$.

Other examples of sigmoidal functions satisfying the assumptions made in our theory can be constructed starting from the well-known B-splines of order $n \in \mathbb{N}^+$ [34],

$$M_n(x) := \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{n}{2} + x - i\right)_+^{n-1}, \qquad x \in \mathbb{R}.$$

This can be done by means of a simple procedure described in Chapter 5 ([58]). In the previous formula, the function $(x)_+ := \max\{x, 0\}$ denotes "the positive part" of $x \in \mathbb{R}$.

Finally, as an example of sigmoidal functions to which the present theory *cannot* be applied, we mention the (class of) Gompertz functions. These functions are characterized by unsymmetrical growth, hence they fail to satisfy condition ($\Sigma 1$).

6.7.1 Multivariate quasi-interpolation operators with sigmoidal functions

In the present context, we can use sigmoidal functions also to study convergence and order of approximation of a class of *quasi-interpolation* operators, defined using the function Ψ_σ introduced in (6.4) in Section 6.6. If $f : \mathbb{R}^s \to \mathbb{R}$ is a bounded function, we define the so-called quasi-interpolation operators

$$G_n^s(f, \underline{x}) := \sum_{k \in \mathbb{Z}^s} f(\underline{k}/n)\, \Psi_\sigma(n\underline{x} - \underline{k}), \qquad \underline{x} \in \mathbb{R}^s,$$

for every $n > 0$, see e.g., [38, 7, 11, 60]. Using Lemma 6.30 and the boundedness of f, it easy to show that all $G_n^s(f, x)$'s are well-defined for every $\underline{x} \in \mathbb{R}^s$

and for every $n > 0$. Note that, using Lemma 6.30 again, we can write, for every $\underline{x} \in \mathbb{R}^s$,

$$|G_n^s(f, \underline{x}) - f(\underline{x})| = \left| G_n^s(f, \underline{x}) - f(\underline{x}) \sum_{\underline{k} \in \mathbb{Z}^s} \Psi_\sigma(n\underline{x} - \underline{k}) \right|$$
$$\leq \sum_{\underline{k} \in \mathbb{Z}^s} |f(\underline{k}/n) - f(\underline{x})| \, \Psi_\sigma(n\underline{x} - \underline{k}).$$

Using this inequality, we can then establish *pointwise* convergence for the family $(G_n^s(f, \cdot))_{n>0}$, for every bounded function $f \in C^0(\mathbb{R}^s)$, as well as *uniform* convergence whenever f is uniformly continuous and bounded on $\mathbb{R}^s$. The proof can be obtained as in Theorem 6.37. Moreover, note that condition $(\Sigma 2)$ can be dropped. This can be done since Lemma 6.34 is *not* used for studying quasi-interpolation operators and the proof of such lemma is entirely based on the claim of Lemma 6.18 (see also Remark 6.19). Finally, observe that the definition of Lipschitz class for functions defined on the whole of $\mathbb{R}^s$ becomes

$$Lip(\nu) := \left\{ f \in C^0(\mathbb{R}^s) : \text{so that} \, \|f(\cdot) - f(\cdot + \underline{t})\|_\infty = \mathcal{O}(\|t\|_2^\nu), \text{ as } \|t\|_2 \to 0 \right\},$$

for $0 < \nu \leq 1$. Therefore, Theorem 6.38 and Theorem 6.39 could also be extended immediately to the case of quasi-interpolation operators.

Chapter 7

Conclusions and Future Developments

7.1 Final remarks and conclusions

In this Thesis, we studied the problem of sigmoidal functions approximation. The relation with the theory of neural networks (NNs) has been analyzed. NNs are widely used as models for the human brain and their flexibility fits very well to a bunch of real world applications.

Sigmoidal functions arise naturally from the theory of NNs, where they play the role of activation functions of the networks. From a theoretical point of view, we stress that it is very difficult to obtain constructive approximation algorithms, especially in the multivariate case, which is the most important and the most natural when dealing with NNs.

Along the last twenty years, some constructive results have been obtained, mainly based on a convolution approach. In fact, as shown in Chapter 1, certain convolution operators have been constructed by using sigmoidal functions, and following a ridge-type approach. Then, NNs were constructed applying suitable quadrature rules [94, 100, 121].

In this Thesis, we propose various kinds of approximation algorithms, to construct both univariate and multivariate approximations. One of our main purposes is to devise an easier way to approximate functions by means of sigmoidal functions, compared to the usual and well-known methods discussed in Chapter 1. Moreover, we would like to obtain high-order approximations, and finally to use these results for applications to Numerical Analysis, to develop new competitive numerical methods.

Concerning the univariate approximation, we have followed two approaches. The first one (see Chapter 2), is inspired to a paper of H. Chen, T. Chen, and R. Liu [48]. The second one, is completely new, see Chapter 5. In particular, in the latter case we consider special kinds of sigmoidal functions, defined by suitable integral forms. Also the coefficients of the NNs that we introduced are of the integral type.

The approach proposed in Chapter 5 is related to the theory of multiresolution approximations of $L^2(\mathbb{R})$ [113, 64]. In fact, if the sigmoidal function is generated by the scaling function of an r-regular multiresolution approximation, then the approximation error for approximating C^1-functions decreases as 2^{-j}, when $j \to +\infty$, i.e., the approximation error decreases exponentially.

Moreover, also in case of multivariate approximation we propose two possible approaches. The first one consists of introducing a kind of radial basis functions (RBF) networks, composing sigmoidal functions σ with the Euclidean norm in $\mathbb{R}^n$ (see Section 2.4). In the second approach, we use products of univariate sigmoidal functions (see Chapter 6). Clearly, these results are easier to apply with respect to those based on convolution showed in Chapter 1.

Finally, concerning the theoretical analysis performed in this Thesis, we should mention the theory of NNs operators, a modern approach to study NNs approximation. This theory was first introduced by G.A. Anastassiou in [4, 7, 8, 9, 10]. The basic idea there is to study operators of Cardaliagnet-Euvrard and squashing type [45]. In Chapter 6, we study and generalize some theorems concerning NN operators, both in a univariate and in a multivariate setting. In the latter context, constructive approximation algorithms are more easily obtained, even in the multivariate setting. We stress that some results established in this Thesis represent an improvement of those proved by Anastassiou in its papers. In particular, the order of approximation for C^1-functions can be improved by means of our operators, compared to certain existing results given in [7, 8, 9].

From the NN operators activated by sigmoidal functions, useful to approximate functions defined on bounded intervals, the theory of quasi-interpolation operators can be easily derived. With the latter operators, we are able to approximate functions defined on the whole real line. In particular, this approach is similar to and reminiscent of that approximation made by the generalized sampling operators introduced by P.L. Butzer [35, 36, 33].

Furthermore, several applications to Numerical Analysis are given to show the flexibilities and the peculiarities of sigmoidal functions approximation (see Chapters 3 and 4). For instance, applications to the solution of linear and nonlinear Volterra integral equations of the second kind by collocation, are very easy to obtain, the method performs very well, and is competitive from the computational point of view in comparison to piecewise polynomials collocation methods. In some cases, e.g., in case of integral equations with weakly singular kernels, our method allows to achieve an higher accuracy with respect to the classical piecewise polynomial collocation methods (see Chapter 3).

Analogous advantages can be obtained by our collocation method for solving Volterra integro-differential equations. In particular, the CPU time required by our collocation method is smaller with respect to the case of piecewise polynomial methods. Basing on collocation methods for Volterra integral and integro-differential equations, ordinary differential equations can also be solved. Numerical examples are given in Chapters 3 and 4.

7.2 Future developments

As future works, starting from the problems analyzed in this Thesis, we intend to continue studying sigmoidal functions approximation by constructive multivariate algorithms and applications.

Several open problems still wait for a satisfactory solution. Constructive approximations results are available only for functions defined on special domains, such as intervals of $\mathbb{R}^n$. Extensions of the multivariate theory to approximate functions defined on domains which are not necessarily intervals of $\mathbb{R}^n$ would be desirable. Moreover, a relevant problem is to develop numerical methods which allows us to approximate solutions of integral or differential equations with an high-order of accuracy. To this purpose, the theory developed in Chapter 5 could help. The numerical methods proposed in this Thesis are typically *global* in nature, then one possibility to enhance their accuracy is to apply the methods *locally*, i.e., to every subset of a partition of the original domain of the problem. This is what happens in case of classical "piecewise" methods, such as, for instance, for the piecewise polynomial collocation methods to solve Volterra integral and integro-differential equations [117, 26].

One of the possible field of applications of the approximation results by sigmoidal functions is the numerical solution of partial differential equations, such as conservation and balance laws. Concerning the latter, since hyperbolic problems often possess solutions with sharp fronts or even jump discontinuities (shocks), we may expect that smooth sigmoidal functions, might be well suited to represent this kind of solutions. Clearly, since large errors can be made, due to the rapid variation of such functions, it would be desirable to track the sharp front or the discontinuities of the solutions. This idea reminds that developed in [77], where the discontinuities of the solutions of certain hyperbolic conservation laws are tracked to obtain high accurate numerical solutions for this kind of problems.

Moreover, various aspects of the modern theory of NN operators could be also analyzed. For instance, the high order of approximation can be investigated for functions with high order derivatives, i.e., the results obtained by G.A. Anastassiou in this direction could be extended for the operators activated by sigmoidal functions belonging to the class of these functions introduced in Chapter 6. The idea is to apply Taylor formula with a reminder in the integral form to estimate the error of approximation. In addition, the theory of neural network operators could be extended to approximate functions belonging to L^p-spaces. In this case, the form of the coefficients of such operators should be adapted. In fact, in we consider for instance the univariate case, for functions f belonging to L^p, the coefficients $f(k/n)$, which depend on the pointwise behavior of f, should be replaced by the mean values of the function f into the rectangles $[k/n,\,(k+1)/n]$, i.e., with $n\int_{k/n}^{(k+1)/n} f(u)\,du$. Then, the neural network operators in the L^p (univariate) setting should assume the following form:

$$K_n^\sigma(f,x) \;:=\; \frac{\displaystyle\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor-1}\left[n\int_{k/n}^{(k+1)/n} f(u)\,du\right]\phi_\sigma(nx-k)}{\displaystyle\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor-1}\phi_\sigma(nx-k)}, \qquad x\in[a,b] \tag{7.1}$$

for $n\in\mathbb{N}^+$ sufficiently large, $f:[a,b]\to\mathbb{R}$ belonging to $L^p[a,b]$, and where ϕ_σ is the density function defined in Chapter 6 with the sigmoidal function σ.

Replacing the value $f(k/n)$ with the mean values $n\int_{k/n}^{(k+1)/n} f(u)\,du$ to switch from the study of pointwise operators to operators in L^p, is quite natural. Indeed, such replacements have been made by Kantorovich in [84] when he generalized the Bernstain polynomials to the L^p setting to obtain a proof of the Weierstrass approximation theorem in $L^p(0,1)$. Moreover, the same approach has been adopted in [17] where the authors introduced sampling operators of the Kantorovich type to obtain applications for the reconstruction of non necessarily continuous signals.

Convergence and order of approximation of neural network operators of the form in (7.1) could be studied in this new framework. For the latter problem we should introduce suitable Lipschitz classes related to the norms of the space $L^p[a,b]$.

Another problem that can be investigated concerns interpolation by neural networks and sigmoidal functions. Interpolation of functions on certain finite sets of points by NNs (see e.g. [101]) is a problem strictly connected to the training of networks and to the exact representation of the values of certain test set (see Subsection 1.1.3).

Further applications of the theory of approximating functions by neural network operators could be developed. A typical field of application for multivariate approximation algorithms falls in the context of image reconstruction and image enhancement. In particular, the multivariate form of the operators in (7.1) could be well suited to applications concerning images, since images are usually represented by discontinuous functions.

Bibliography

[1] Adrianov, A.V., *An analog of the Jackson-Nikol'skij theorem on the approximation by superposition of sigmoidal functions*, Mathematical Notes [transl. of Matematicheskie Zametki, Russian Academy of Sciences], 59 (6) (1996), 661-664.

[2] Aguilar, M., Brunner, H., *Collocation method for second-order Volterra integro-differential equations*, Appl. Numer. Math., 4 (6) (1988), 455-470.

[3] Albano, G., Giorno, V., *On the First Exit Time Problem for a Gompertz-Type Tumor Growth*, Lecture Notes in Computer Science, 5717 (2009), 113-120.

[4] Anastassiou, G. A., *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appl., 212 (1997), 237-262.

[5] Anastassiou, G. A., *Quantitative approximation*, Chapmann & Hall/CRC, Boca Raton, New York, 2001.

[6] Anastassiou, G. A., *Univariate fuzzy-random neural network approximation operators*, Computer & Mathematics with Applications, 48 (2004), 1263-1283.

[7] Anastassiou, G. A., *Univariate hyperbolic tangent neural network approximation*, Math. Comput. Modelling, 53 (5-6) (2011), 1111-1132.

[8] Anastassiou, G. A., *Multivariate hyperbolic tangent neural network approximation*, Comput. Math. Appl., 61 (4) (2011), 809-821.

[9] Anastassiou, G. A., *Multivariate sigmoidal neural network approximation*, Neural Networks, 24 (2011), 378-386.

[10] Anastassiou, G. A., *Intelligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library 19, Berlin, Springer-Verlag, 2011.

[11] Anastassiou, G. A., *Univariate sigmoidal neural network approximation*, J. Comput. Anal. Appl., 14 (4) (2012), 659-690.

[12] Anastassiou, G. A., *Fractional neural network approximation*, Computer & Mathematics with Applications, 64 (2012), 1655-1676.

[13] Anthony, M., Bartlett, P.L., *Neural Network Learning: Theoretical foundations*, Cambridge University Press, New York, 2009.

[14] Atkinson, K.E., *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Monogr. Appl. Comput. Math. 4, Cambridge Univ. Press, Cambridge, 1997.

[15] Aves, M.A., Davies, P.J., Higham, D.J., *Fixed points and spurious modes of a nonlinear infinite-step map*, Numerical Analysis: A.R. Mitchel 75th Birthday Volume (D.F. Griffiths and G.A. Watson, eds.), Singapore, Word Scientific Publ. (1996), 21-38.

[16] Aves, M.A., Davies, P.J., Higham, D.J., *The effect of quadrature on the dynamics of a discretized nonlinear integro-differential equation*, Appl. Numer. Math., 32 (2000), 1-20.

[17] Bardaro, C., Butzer, P.L., Stens, R.L., Vinti, G., *Kantorovich-type generalized sampling series in the setting of Orlicz spaces*, Sampling Theory in Signal and Image Processing, 6 (1) (2007), 29-52.

[18] Bardaro, C., Vinti, G., *A general approach to the convergence theorems of generalized sampling series*, Applicable Analysis, 64 (1997), 203-217.

[19] Bardaro, C., Vinti, G., *Uniform convergence and rate of approximation for a nonlinear version of the generalized sampling operator*, Results in Mathematics, special volume dedicated to Professor P.L. Butzer 34 (3/4) (1998), 224-240.

[20] Barron, A.R., *Universal Approximation Bounds for Superpositions of a Sigmoidal Function*, IEEE Transactions on Information Theory, 39 (3) (1993), 930-945.

[21] Baum, E.B, *On the capabilities of multilayer perceptrons*, J. Complexity 4 (1988), 193-215.

[22] Berenhaut, K.S., Morton, D.C., Fletcher, P.T., *Bounds for inverses of triangular Toeplitz matrices*, SIAM. J. Matrix Anal. Appl., 27 (1) (2005), 212-217.

[23] Bishop, C.M., *Neural Networks for Pattern Recognition*, Oxford University Press, Oxford, 1995.

[24] Bos, D., Ellacott, S.W., *Neural Networks: Deterministic Methods of Analysis*, International Thomson Computer Press, London, 1996.

[25] Brauer, F., Castillo Chavez, C., *Mathematical Models in Population Biology and Epidemiology*, New York, Springer, 2001.

[26] Brezis, H., *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, New York, Springer, 2011.

[27] Brunner, H., *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge Monogr. Appl. Comput. Math., 15, Cambridge Univ. Press, Cambridge, 2004.

[28] Brunner, H., *High-order collocation methods for singular Volterra functional equations of neutral type*, Appl. Numer. Math., 57 (5-7) (2007), 533-548.

[29] Brunner, H., Ma, J., *A posteriori error estimates of discontinuous Galerkin methods for non-standerd Volterra integro-differential equations*, IMA J. Numer. Anal., 26 (2006), 78-95.

[30] Brunner, H., Pedas, A., Vainikko, G., *A spline collocation method for linear Volterra integro-differential equations with weakly singular kernels*, BIT 40th Anniversary Meeting. BIT, 41 (5) (2001), 891-900.

[31] Brunner, H., Pedas, A., Vainikko, G., *Piecewise polynomial collocation method for linear Volterra integro-differential equations with weakly singular kernels*, SIAM J. Numer. Anal., 39 (3) (2001), 957-982.

[32] Buhmann, M., Iserles, A., *On the Dynamics of a Discretized neutral equation*, IAM J. Numer. Anal., 12 (1992), 339-363.

[33] Butzer, P.L., Fisher, A., Stens, R.L., *Generalized sampling approximation of multivariate signals: theory and applications*, Note di Matematica, 10 (1) (1990), 173-191.

[34] Butzer, P.L., Nessel, R.J., *Fourier Analysis and Approximation*, Birkhauser Verlag, Bassel, and Academic Press, New York, 1971.

[35] Butzer, P.L., Ries, S., Stens, R.L., *Approximation of continuous and discontinuous functions by generalized sampling series*, J. Approx. Theory, 50 (1987), 25-39.

[36] Butzer, P.L., Splettstößer, W., Stens, R.L., *The sampling theorem and linear prediction in signal analysis*, Jahresber. Deutsch. Math.-Verein 90 (1988), 1-70.

[37] Buzhabadi, R., Effati, S., *A neural network approach for solving Fredholm integral equations of the second kind*, Neural Computing & Applications, 21 (2012), 843-852.

[38] Cao, F., Chen, Z., *The approximation operators with sigmoidal functions*, Comput. Math. Appl., 58 (4) (2009), 758-765.

[39] Cao, F., Chen, Z., *The construction and approximation of a class of neural networks operators with ramp functions*, J. Comput. Anal. Appl., 14 (1) (2012), 101-112.

[40] Cao, F.L., He, Z.R., Zhang, Y., *Interpolation and rates of convergence for a class of neural networks*, Applied Math. Modelling, 33 (3) (2009), 1441-1456.

[41] Cao, F.L., Xie, T., Xu, Z.B., *The estimate for approximation error of neural networks: A constructive approach*, Neurocomputing, 71 (4-6) (2008), 626-630.

[42] Cao, F.L., Xu, Z.B., *The essential order of approximation for neural networks*, Sci. China, Ser. F: Information Sciences, 47 (1) (2004), 97-112.

[43] Cao, F.L., Xu, Z.B., *Simultaneous L^p-approximation order for neural networks*, Neural Networks, 18 (7) (2005), 914-923.

[44] Cao, F.L., Zhang, R., *The errors of approximation for feedforward neural networks in the L^p metric*, Mathematical and Computer Modelling, 49 (7-8) (2009), 1563-1572.

[45] Cardaliaguet, P., Euvrard, G., *Approximation of a function and its derivative with a neural network*, Neural Networks, 5 (2) (1992), 207-220.

[46] Cheang Gerald, H. L. *Approximation with neural networks activated by ramp sigmoids*, J. Approx. Theory, 162 (2010), 1450-1465.

[47] Chen, D., *Degree of approximation by superpositions of a sigmoidal function*, Approx. Theory Appl., 9 (3) (1993), 17-28.

[48] Chen, H., Chen, T., Liu, R., *A constructive proof and an extension of Cybenko's approximation theorem*, Computing Science and Statistics, New York, Springer-Verlag, (1992), 163-168.

[49] Cheney, E.W., Light, W.A., Xu, Y., *On kernels and approximation orders*, Approximation Theory, G. Anastassiou (Ed.), Marcel Dekker, New York (1992), 227-242.

[50] Cheney, E.W., Light, W.A., Xu, Y., *Constructive methods of approximation by rigde functions and radial functions*, Numerical Algorithms, 4 (2) (1993), 205-223.

[51] Cheney, E.W., Sun, X., *The fundamentality of sets of ridge functions*, Aequationes Math. 44 (2-3) (1992), 226-235.

[52] Chui, C.K., *An Introduction to Wavelets*, Wavelet Analysis and its Applications 1, Academic Press Inc., Boston, 1992.

[53] Chui, C. K., Li, X., *Approximation by ridge functions and neural networks with one hidden layer*, J. Approx. Theory, 70 (1992), 131-141.

[54] Chui, C. K., Li, X., *Realization of neural networks with one hidden layer*, Multivariate Approximation: From CAGD to Wavelets, World Scientific, Singapore (1993), 77-89.

[55] Chumerina, E.S., *Choice of Optimal Strategy of Tumor Chemotherapy in Gompertz Model*, J. Computer and Systems Sciences International, 48 (2) (2009), 325-331.

[56] Costarelli, D., Spigler, R., *Constructive approximation by superposition of sigmoidal functions*, Analysis in Theory and Applications, 29 (2) (2013), 169-196.

[57] Costarelli, D., Spigler, R., *Solving Volterra integral equations of the second kind by sigmoidal functions approximations*, Journal of Integral Equations and Applications, 25 (2) (2013), 193-222.

[58] Costarelli, D., Spigler, R., *Approximation by series of sigmoidal functions with applications to neural networks*, to appear in: Annali di Matematica Pura ed Applicata, (2013), DOI: 10.1007/s10231-013-0378-y.

[59] Costarelli, D., Spigler, R., *A collocation method for solving nonlinear Volterra integro-differential equations of the neutral type by sigmoidal functions*, to appear in: Journal of Integral Equations and Applications, (2013).

[60] Costarelli, D., Spigler, R., *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks, 44 (2013), 101-106.

[61] Costarelli, D., Spigler, R., *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks, 48 (2013), 72-77.

[62] Cushing, J. M., *Integrodifferential Equations and Delay Models in Population Dynamics*, Lecture notes in Biomathematics 20, Berlin-New York, Springer-Verlag, 1977.

[63] Cybenko, G., *Approximation by Superpositions of a Sigmoidal Function*, Mathematics of Control, Signals, and Systems, 2 (1989), 303-314.

[64] Daubechies, I., *Ten lectures on Wavelets*, Regional Conference Series in Applied Mathematics 61. Society for Industrial and Applied Mathematics, SIAM, Philadelphia, 1992.

[65] De Boor, C., *A Practical Guide to Spline*, Applied Mathematical Sciences 27, Springer-Verlag, New York, 2001.

[66] Devroye, L., Gyorfi, L., Lugosi, G., *A Probabilistic Theory of Pattern Recognition*, Sprinder, New York, 1996.

[67] Funahashi, K., *On the approximate realization of continuous mapping by neural networks*, Neural Networks 2 (1989), 183-192.

[68] Gao, B., Xu, Y., *Univariant Approximation by Superpositions of a Sigmoidal Function*, J. Math. Anal. Appl., 178 (1993), 221-226.

[69] Girosi, F., Anzellotti, G., *Rates of convergence for radial basis functions and neural networks*, Artificial neural networks for speech and vision, London, Chapman & Hall, R. J. Mammone (Ed.) (1993), 97-113.

[70] Gnecco, G., *A comparison between fixed-basis and variable-basis schemes for function approximation and functional optimization*, J. Appl. Math., ID 806945, (2012) 17 pp.

[71] Gnecco, G., Sanguineti, M., *On a Variational Norm Tailored to Variable-Basis Approximation Schemes*, IEEE Trans. Inform. Theory, 57 (2011), 549-558.

[72] Hahm, N., Hong, B., *Approximation Order to a Function in* $\overline{C}(\mathbb{R})$ *by Superposition of a Sigmoidal Function*, Applied Math. Lett., 15 (2002), 591-597.

[73] Hahm, N., Hong, B., *An approximation by neural networks with a fixed weight*, Computers & Mathematics with Applications, 47 (2004), 1897-1903.

[74] Haykin, S., *Neural Networks*, New York, MacMillan, 1994.

[75] Hinton, R. J., Rumelhart, D. E., Williams, G. E., *Learning internal representations by error propagation*, Parallel Distributed Processing 1, D. E. Rumelhart et al., Eds., MIT Press, Cambridge, Mass., 1986.

[76] Hoff, M. E., Widrow, B., *Adaptive switching circuits*, 1960 IRE WESCON Convention Record, IRE, New York, (1960), 96-104.

[77] Holden, H., Risebro, N. H., *Front Tracking for Hyperbolic Conservation Laws*, Applied Mathematical Sciences 152, Springer, 2011.

[78] Hritonenko, N., Yatsenko, Y., *Mathematical Modelling in Economics*, Ecology and the Environment, Beijing, Science Press, 2006.

[79] Jerry, A.J., *Introduction to Integral Equations with Applications*, A serie of Monographs and Textbooks, Pure and applied mathematics, Marcel Dekker, INC., New York - Bessel, 1985.

[80] Jones, L.K., *Constructive approximations for neural networks by sigmoidal functions*, Proceedings of the IEEE, 78 (10) (1990), 1586-1589.

[81] Kainen, P. C., Kurková, V., *An Integral Upper Bound for Neural Network Approximation*, Neural Comput., 21 (2009), 2970-2989.

[82] Kainen, P. C., Kurková, V., Vogt, A., *Approximation by neural networks is not continuous*, Neurocomputing, 29 (1) (1999), 47-56.

[83] Kainen, P. C., Kurková, V., Vogt, A., *Best approximation by Heaviside perceptron networks*, Neural Networks, 17 (7) (2000), 695-697.

[84] Kantorovich, L.V., *Sur certains développements suivant les polynomes de la forme de S. Bernstain*, I.C.R. Acad. Sc. URSS (in Russian) (1930), 563-568.

[85] Keller, E.P., Priddy, K.L., *Artificial neural networks: An introduction*, SPIE-The international Society for Optical Engineering, Bellingham-Washington, 2005.

[86] Kolmogorov, A.N., *On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition*, Dolk. Akad. Nauk. SSSR 114 (1957), 953-956.

[87] Kurková, V., *Kolmogorov's theorem and multilayer neural networks*, Neural Networks, 5 (3) (1992), 501-506.

[88] Kurková, V., *Dimension-independent rates of approximation by neural networks*, Computer Intensive Methods in Control and Signal Processing, Birkhauser Boston, (1997), 261-270.

[89] Kurková, V., *Incremental approximation by neural networks*, Complexity: Neural Network approach, (1998) 177-188.

[90] Kurková, V., *Complexity estimates based on integral transforms induced by computational units*, Neural Networks, 33 (2012), 160-167.

[91] Kurková, V., Sanguineti, M., *Bounds on rate of variable-basis and neural-network approximation*, IEEE Trans. on Information Theory, 47 (6) (2001), 2659-2665.

[92] Kurková, V., Sanguineti, M., *Comparison of worst case errors in linear and neural network approximation*, IEEE Trans. on Information Theory, 48 (1) (2002), 264-275.

[93] Kurková, V., Sanguineti, M., *Estimates of covering numbers of convex sets with slowly decaying orthogonal subsets*, Discrete Applied Mathematics, 155 (2007), 1930-1942.

[94] Lenze, B., *Constructive Multivariate Approximation with Sigmoidal Functions and Applications to Neural Networks*, Numerical Methods of Approximation Theory, Birkhauser Verlag, Basel-Boston-Berlin, (1992), 155-175.

[95] Lewicki, G., Marino, G., *Approximation by Superpositions of a Sigmoidal Function*, Zeitschrift für Analysis und ihre Anwendungen Journal for Analysis and its Applications, 22 (2) (2003), 463-470.

[96] Lewicki, G., Marino, G., *Approximation of Functions of Finite Variation by Superpositions of a Sigmoidal Function*, Appl. Math. Lett., 17 (2004), 1147-1152.

[97] Li, X., *Simultaneous approximations of multivariate functions and their derivatives by neural networks with one hidden layer*, Neurocomputing, 12 (1996), 327-343.

[98] Li, X., *On Simultaneous approximations by radial basis function neural networks*, Appl. Math. Comput., 95 (1998), 75-89.

[99] Li, X., Micchelli, C.A., *Approximation by Radial Bases and Neural Networks*, Numerical Algorithms, 25 (2000), 241-262.

[100] Light, W., *Ridge functions, sigmoidal functions and neural networks*, Approximation Theory VII (Austin, TX, 1992), Academic Press Boston, MA, (1993), 163-206.

[101] Llanas, B., Ward, J.D., *Constructive approximate interpolation by neural networks*, J. Comput. Appl. Math., 188 (2006), 283-308.

[102] Lorentz, G.G., *The 13th problem of Hilbert*, Mathematical Developments Arising from Hilbert problems (F. Browder, ed.) 2, American Mathematical Society, Providence, RI (1976) 419-430.

[103] Madych, W.R., *Summability and approximate reconstruction from Radon transform data*, Contemp. Math. 113 (1990), 189-219.

[104] Makovoz, Y., *Random approximants and neural networks*, J. Approx. Theory, 85 (1996), 98-109.

[105] Makovoz, Y., *Uniform approximation by neural networks*, J. Approx. Theory, 95 (2) (1998), 215-228.

[106] Malek, A., Shekari Beidokhti, R., *Numerical solution for high order differential equations using a hybrid neural network - Optimization method*, Appl. Math. Comput., 183 (2006) 260-271.

[107] Maleknejad, K., Hashemizadeh, E., Ezzati, R., *A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation*, Commun. Nonlinear Sci. Numer. Simul., 16 (2) (2011), 647-655.

[108] Maleknejad, K, Najafi, E., *Numerical solution of nonlinear Volterra integral equations using the idea of quasilinearization*, Commun. Nonlinear Sci. Numer. Simul., 16 (1) (2011), 93-100.

[109] Mallat, S.G., *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., 315 (1989) 69-87.

[110] Mantysaari, E.A., Sevon Aimonen, M.L., Stranden, I., Vuori, K., *Estimation of non-linear growth models by linearization: a simulation study using a Gompertz function*, Genet. Sel. Evol., 38 (2006), 343-358.

[111] McCulloch, W. S., Pitts, W., *A logical calculus of the ideas immanent in nervous activity*, Bull. Math. Biophysics, 5 (1943), 115-133.

[112] Medvedeva, M.V., *On sigmoidal functions*, Mosc. Univ. Math. Bull., 53 (1) (1998), 16-19.

[113] Meyer, Y., *Wavelets and Operators*, Cambridge studies in advanced mathematics 37, Cambridge, 1992.

[114] Mhaskar, H.N., *Neural networks for optimal approximation of smooth and analytic functions*, Neural Computation, 8 (1996), 164-177.

[115] Mhaskar, H.N., Micchelli, C.A., *Approximation by Superposition of Sigmoidal and Radial Basis Functions*, Adv. in Appl. Math., 13 (1992), 350-373.

[116] Mhaskar, H.N., Micchelli, C.A., *Degree of Approximation by Neural and Translation Networks with a Single Hidden Layer*, Adv. in Appl. Math., 16 (1995), 151-183.

[117] Miller, R. K., *Nonlinear Volterra Integral Equations*, W.A. Benjamin, Menlo Park CA, 1971.

[118] Pachpatte, B. G., *Implicit type Volterra integrodifferential equation*, Tamkang J. Math., 41 (1) (2010), 97-107.

[119] Pachpatte, B. G., *Approximate solutions for integrodifferential equations of the neutral type*, Comment. Math. Univ. Carolin., 51 (3) (2010), 489-501.

[120] Park, J., Sandberg, I.W., *Universal Approximation using Radial-Basis-Function Networks*, Neural Computation, 3 (2) (1991), 246-257.

[121] Pinkus, A., *Approximation theory of the MLP model in neural networks*, Acta Numer., 8 (1999), 143-195.

[122] Powell, M.J.D., *The theory of radial basis function approximation*, Adv. in Num. Anal. III, Wavelets, Subdivision Algorithms and Radial Basis Functions (W.A. Light Ed.), Oxford, Clarendon Press, (1992), 105-210.

[123] Ries, S., Stens, R.L., *Approximation by generalized sampling series,* Constructive Theory of Functions'84, Sofia, 1984, 746-756.

[124] Ripley, B.D., *Pattern Recognition and Neural Networks*, Cambridge University Press, Cambridge, 1996.

[125] Rosenblatt, F., *The perceptron: a probabilistic model for information storage and organization in the brain*, Psychol. Review, 65 (1958), 386-408.

[126] Rosenblatt, F., *Two theorems of statistical separability in the perceptron*, in Mechanization of Thought Processes. Proceedings of a symposium held at the National Physical Laboratory, HM Stationery Office, London, I (1958), 421-456.

[127] Rosenblatt, F., *Principles of Neurodynamics: Perceptrons and the Theory of Brain Mechanisms*, Spartan Books, Washington, DC, 1961.

[128] Rudin, W., *Functional Analysis*, McGraw-Hill, New York, 1973.

[129] Sas A.A., Snieder H., Korf J. *Gompertz' survivorship law as an intrinsic principle of aging*, Medical Hypotheses, 78 (5) (2012), 659-663.

[130] Stinchcombe, M., White, H., *Universal approximation using feedforward networks with non sigmoid hidden layer activation functions*, Proceedings of International Conference on Neural Networks (1989), 613-617.

[131] Unser, M., *Ten good reasons for using spline wavelets*, Wavelets Applications in Signal and Image Processing, 3169 (5) (1997), 422-431.

[132] Vecchio, A., *A bound for the inverse of a lower triangular Toeplitz matrix*, SIAM J. Matrix Anal. Appl., 24 no. 4 (2003), 1167-1174.

[133] Vinti, G., *Approximation in Orlicz spaces for linear integral operators and applications*, Rendiconti del Circolo Matematico di Palermo, Serie II 76 (2005), 103-127.

[134] Xiehua, S., *On the degree of approximation by wavelet expansions*, Approx. Theory Appl., 14 (1) (1998), 81-90.

[135] Yousefi, S.A., *Numerical solution of Abel's integral equation by using Legendre wavelets*, Appl. Math. Comput., 175 (1) (2006), 574-580.