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Some topics on Zariski decompositions and restricted base loci of divisors on singular varieties

Tesi di dottorato in matematica

di

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Contents

Chapter 1

Introduction

In algebraic geometry line bundles are particularly interesting as they can be used to define rational maps to projective spaces. In particular, given a line bundle (or equivalently a Cartier divisor) D on a complex projective variety X, the first question arising is if D defines a map (i.e. if it has global sections) and, if this is the case, what are the indeterminacies of the map (i.e. the points where all the global sections vanish). Note that in general we want to investigate the *asymptotic* behavior of D, so that we denote by ϕ_D the rational map induced by a sufficiently high and divisible multiple mD. The classical object that encloses the basic information about ϕ_D is the stable base locus

$$\mathbb{B}(D) = \bigcap_{m \in \mathbb{N}} Bs(|mD|).$$

In fact $\mathbb{B}(D)$ is the (Zariski) closed subset of X where ϕ_D is not defined, so that in particular $\mathbb{B}(D) = X$ if and only if no multiples of the given line bundle have sections, whilst it is empty if mD defines an actual morphism for some $m \in \mathbb{N}$, i.e. D is semiample.

Recently in [ELMNP06] some modifications of the stable case locus have been introduced to address more involved problems. For example the augmented base locus

$$\mathbb{B}_+(D) = \bigcap_{\substack{E \ \mathbb{R}\text{-divisor}, E \geq 0 \\ D-E \ \text{ample}}} \operatorname{Supp}(E)$$

is the closed subset where ϕ_D is not locally a closed immersion, or, in other words, where D is not ample.

In particular $\mathbb{B}_+(D) = \emptyset$ if and only if D is ample and $\mathbb{B}_+(D) = X$ if and only if ϕ_D is not an isomorphism on the generic point of X, i.e. D is not big.

A natural generalization of the property of being ample leads to the notion of nefness. A divisor (or line bundle) D is nef if and only if it non negatively intersects every irreducible curve on X, that is proved to be equivalent to say that its numerical class lies in the closure of the ample cone. Nef divisors are particularly interesting for many reasons: their top self-intersection, for example, asymptotically determine the number of their global sections (see [Laz04, corollary 1.4.41]), while the nefness of the canonical divisor K_X is one of the main issues in the context of the Minimal Model Program, as it corresponds to the minimality of the variety X.

Hence many different tools have been introduced to study "where a divisor fails to be nef". In analogy with the augmented base locus we can consider the restricted base locus

$$\mathbb{B}_-(D) = \bigcup_A \mathbb{B}(D + A),$$

where the union is taken over all ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X (see def. 2.3.5).

Note that $\mathbb{B}_-(D) = \emptyset$ if and only if D is nef. Also $\mathbb{B}_-(D)$ contains every curve C such that $(D \cdot C) < 0$, but it can be strictly bigger than the union of such curves.

Moreover, by definition, $\mathbb{B}_-(D)$ is a countable union of closed subsets; we do not know whether it is closed in general or not.

A more classical approach is to look for a Zariski decomposition of D, i.e. to try to decompose it as the sum of a nef divisor P that encodes the positivity of D in some sense and an effective divisor N that represents the non-nef part of D.

More precisely we say that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a projective variety X admits a $\mathbb{Q}$-Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a $\mathbb{Q}$-CKM Zariski decomposition) $D = P + N$ if

- P and N are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors;

- P is nef and N is effective;

- There exists an integer $k > 0$ such that kD and kP are integral divisors and for every $m \in \mathbb{N}$ the natural map

$$H^0(X, \mathcal{O}_X(kmP)) \to H^0(X, \mathcal{O}_X(kmD))$$

  is bijective.

Every pseudoeffective divisor on a smooth projective surface admits a Zariski decomposition (see [Fuj79]). On the other hand in higher dimension there exist big divisors such that no birational pullbacks admit a Zariski decomposition, as shown by Cutkosky in [Cut86], even if we allow P and N to be $\mathbb{R}$-divisors (see[Nak04]).

Note that the restricted base locus of D is strictly related to the existence of such decompositions. In fact if D is big and $D = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition, then $\mathrm{Supp}(N) = \mathbb{B}_-(D)$, which implies that $\mathbb{B}_-(D)$ is a divisor in this case.

If D admits a $\mathbb{Q}$-CKM Zariski decomposition, then, up to pass to a multiple, the graded ring $R(X, D) := \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mD))$ is isomorphic to $R(X, P)$, so that in order to check its finite generation it suffices to study the finite generation of the ring associated to the nef divisor P. This implies that the semiampleness of P, the so-called positive part of the Zariski decomposition of D, is a sufficient condition for the finite generation of $R(X, D)$. Moreover the semiampleness of P gives that $\mathbb{B}(D) = \mathrm{Supp}(N) = \mathbb{B}_-(D)$. See [Mor87, (9.11)] for the importance of the Zariski decomposition in the context of the *Abundance Conjecture*.

In this work in chapter 3, whose main results appear in the paper [Cac10], we try to extend to the LC case the main theorem of Kawamata's paper [Kaw87], stating that if the canonical divisor K_X of a smooth projective variety admits

a CKM Zariski decomposition then the positive part P of the decomposition is semiample.

More precisely a simple generalization of this theorem says that:

Theorem 1.0.1 (Kawamata)**.** *Let X be a normal projective variety and let Δ be a Weil effective $\mathbb{Q}$-divisor such that (X, Δ) is a KLT pair. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that*

A: *D is big;*

B: *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

C: *D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$;*

then the positive part P is semiample, so that $R(X, D)$ is finitely generated.

The same theorem is no longer true in general if (X, Δ) is an LC pair, as shown in section 3.11, so that we have to assume more. For this reason we use the notion of logbig divisors, introduced by Miles Reid.

A big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is logbig for an LC pair if its restriction to every LC center of the pair is still big (see definition 2.2.7).

We state the following conjecture:

Conjecture 1. *Let (X, Δ) be an LC pair, with Δ effective.*
*If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X which satisfies **A**,**B** and **C**, and P is logbig for the pair (X, Δ), then P is semiample.*

In the case $N = 0$, a very similar result was stated by Miles Reid in [Rei93] and was proved by Florin Ambro in the more general setting of quasi-log varieties in [Amb01, theorem 7.2] (see also [Fuj09c, theorem 4.4]). Moreover, when X is smooth and Δ and N are SNCS divisors Conjecture 1 is true and follows from [Fuj07b, theorem 5.1].

Note that a sufficient condition for the positive part P to be logbig for the pair (X, Δ) is that the augmented base locus $\mathbb{B}_+(D)$ does not contain any LC center of the pair. Hence conjecture 1 would imply the following:

Conjecture 2. *Let (X, Δ) be an LC pair, with Δ effective.*
*If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X which satisfies **A**,**B** and **C**, and $\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ), then P is semiample.*

On the other hand, note that, for a divisor D, the existence of a Zariski decomposition is a very strong property in general, while it is more likely that a birational pullback of D admits one. In other words we want to replace hypothesis **C** with the following:

$\mathbf{C}_f$: There exists a projective birational morphism $f : Z \to X$ such that $f^*(D)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $f^*(D) = P + N$.

We can thus generalize Conjecture 1 and Conjecture 2 as follows (the "b" stands for "birational"):

Conjecture 1b. *Let (X, Δ) be an LC pair, with Δ effective.*
*Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X satisfying **A**,**B** and $\mathbf{C}_f$, for some $f : Z \to X$. If P is logbig for the pair (Z, Δ_Z), where Δ_Z is a $\mathbb{Q}$-divisor on Z such that $K_Z + \Delta_Z = f^*(K_X + \Delta)$, then P is semiample.*

Conjecture 2b. *Let (X, Δ) be an LC pair, with Δ effective.*
Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X satisfying $\mathbf{A}$,$\mathbf{B}$ and $\mathbf{C}_f$, for some $f : Z \to$
X. If $\mathbb{B}_+(f^*D)$ does not contain any LC center of the pair (Z, Δ_Z), where Δ_Z
is a $\mathbb{Q}$-divisor on Z such that $K_Z + \Delta_Z = f^*(K_X + \Delta)$, then P is semiample.

In this thesis we easily deduce from [Fuj07b, theorem 5.1] that Conjecture 1b
holds if a suitable pair on Z is DLT (see theorem 3.3.11), which in particular
implies that Conjecture 1 holds if (X, Δ) is DLT. Moreover, as a corollary of
more general results (see 3.7.5 and 3.7.7), we prove the following:

Theorem 1.0.2 (see corollary 3.7.8). *Conjecture 1b holds if $dim X \leq 3$.*

Also we prove that Conjecture 2 holds if we assume some standard very strong
conjectures concerning the Minimal Model Program, namely the abundance con-
jecture (for semi divisorial log terminal pairs) and the existence of log-minimal
models (in lower dimension) for DLT pairs of log-general type (see theorem
3.8.1). As a corollary we get that

Theorem 1.0.3 (see corollary 3.8.2). *Conjecture 2 holds if $dim X \leq 4$.*

In section 3.9 we consider a relative version of DLT pair:

Definition 1.0.4 (see definition 3.9.1). *Let (X, Δ) be a pair, with $\Delta = \sum a_i D_i$,*
where the D_i's are distinct prime divisors and $a_i \in \mathbb{Q}$ for every i. Let $D \in$
$\mathrm{Div}_{\mathbb{Q}}(X)$.
We define the non-simple normal crossing locus of (X, Δ) as the closed set

$$NSNC(\Delta) = X \setminus U,$$

where U is the biggest open subset of X such that $\Delta_{|_U}$ has simple normal
crossing support.
We say that (X, Δ) is a D-DLT pair if

1. *$V \not\subseteq Sing(X) \cup NSNC(\Delta)$ for every LC center V of the pair (X, Δ) such*
   that $V \cap \mathbb{B}(D) \neq \emptyset$;

2. *$a_i \leq 1$ for every i such that $D_i \cap \mathbb{B}(D) \neq \emptyset$;*

In particular a D-DLT pair is not necessarily LC. Inspired by [Amb05, Theorem
2.1] we prove the following:

Theorem 1.0.5 (see corollary 3.9.10). *Let (X, Δ) be a pair such that Δ is*
effective. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X satisfying $\mathbf{A}$,$\mathbf{B}$ and $\mathbf{C}_f$, for some
$f : Z \to X$, and let Δ_Z be a $\mathbb{Q}$-divisor on Z such that $K_Z + \Delta_Z = f^*(K_X + \Delta)$.
If (Z, Δ_Z) is $f^*(D)$-DLT and $\mathbb{B}_+(f^*(D))$ does not contain any LC center of the
pair (Z, Δ_Z), then P is semiample.

Note that in many statements of chapter 3 we do not need $a \geq 0$ in hypothesis
$\mathbf{B}$. Moreover, as shown in section 3.10, some of the theorems hold under the
more usual hypotheses

$\mathbf{B'}$: $aD - (K_X + \Delta)$ big and nef for some $a \in \mathbb{Q}^+$;

$\mathbf{C}$: D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$.

In chapter 4 we consider a different approach to the asymptotic study of linear series, by means of geometric valuations. The results of this chapter come from a joint work with Lorenzo Di Biagio (see [CD11]).

In [ELMNP06] the authors, inspired by the work of Nakayama in [Nak04], define an asymptotic measure of the singularities of a linear series: if v is a geometric discrete valuation on X then the *asymptotic order of vanishing* of a big $\mathbb{Q}$-divisor D along v is

$$v(\|D\|) := \lim_{p \to \infty} \frac{v(|pD|)}{p}.$$

(see def. 2.6.3).

They notice that $v(\|D\|)$ is a numerical invariant and, by passing to limits, it is possible to define a *numerical order of vanishing* on every pseudoeffective $\mathbb{R}$-Cartier divisor on X that, following the notation of [BBP09], we will denote by $v_{num}(D)$ (see section 2.6). We define the non-nef locus $\text{NNef}(D)$ as the subset of X given by the union of all the centers of the valuations v such that $v_{num}(D) > 0$ (see definition 2.6.1 and [BBP09, definition 1.7]). Note that this corresponds to the numerical base locus $\text{NBs}(D)$ defined by Nakayama in [Nak04, definition III.2.6] in the smooth case.

We know that $\text{NNef}(D) = \emptyset$ if and only if D is nef, as the name itself suggests, and $\text{NNef}(D) = X$ if and only if D is not psuedoeffective. Therefore, as the same holds for the restricted base locus $\mathbb{B}_-(D)$ and considering that $\text{NNef}(D) \subseteq \mathbb{B}_-(D)$ for every $\mathbb{R}$-Cartier divisor D (cf.[BBP09, lemma 1.8]), it is natural to wonder if these two loci coincide in general.

Note that $\text{NNef}(D)$ behaves well with respect to birational morphisms, meaning that $\text{NNef}(\mu^*(D)) = \mu^{-1}(\text{NNef}(D))$ for every μ birational, so that the same would hold for the restricted base locus if we knew the equality of the two sets. By [ELMNP06, theorem 2.8] we have the following:

Theorem 1.0.6 (ELMNP)**.** *Let X be a smooth projective variety and let D be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $\mathbb{B}_-(D) = \text{NNef}(D)$.*

It is then trivial to show that, passing to limits, the same equality holds for pseudoeffective $\mathbb{R}$-divisors.

In this paper we generalize this result to normal varieties that are not too singular:

Theorem 1.0.7 (see theorem 4.2.7)**.** *Let X be a normal projective variety and suppose that there exists an effective $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is a KLT pair.*
Then for every $\mathbb{R}$-Cartier pseudoeffective divisor D on X we have that $\mathbb{B}_-(D) = \text{NNef}(D)$.

This is a partial answer to a conjecture of Boucksom, Broustet and Pacienza (see [BBP09, Conjecture 1.9]).

The hypothesis of the existence of a KLT boundary is necessary in our proof because this is the only context where asymptotic multiplier ideals are not strongly influenced by the singularities of X, so that they reflect the asymptotic behaviour of the base loci. We could avoid it only in the case of surfaces (corollary 4.1.5). See also cor. 4.2.10 and cor. 4.2.12 for slight generalizations.

On the other hand we define *asymptotic* orders of vanishing $v(\|D\|)$ for every effective divisor. These are in general different from *numerical* orders of vanishing (remark 2.6.5) and we have that $v(\|D\|) = 0$ for every geometric discrete

valuation v if and only if D is nef and abundant (lemma 2.6.6). In analogy to the definition of the non-nef locus we use these asymptotic orders of vanishing to define a non-nef abundant locus NNA(D) (see def. 2.6.7). In particular we prove that for every effective Cartier divisor on a variety X that admits a KLT boundary Δ it holds that

$$\mathrm{NNA}(D) = \bigcup_{m \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|mD\|)).$$

(see theorems 4.2.5 and 4.3.1). Note that, when D is big, this means that

$$\mathbb{B}_{-}(D) = \mathcal{Z}(\mathcal{J}((X, \Delta); \|mD\|)),$$

so that, in particular, we get a generalization of [ELMNP06, Cor. 2.10]. Moreover, as a corollary of this result, we give a characterization of nef and abundant divisors in terms of triviality of asymptotic multiplier ideals (see corollary 4.3.2), generalizing to the KLT case the main theorem of F. Russo's paper [Rus09].

Chapter 2

Preliminaries

2.1 Basics

Notation and conventions

We will work over the field of complex numbers $\mathbb{C}$. As in [Laz04] a *scheme* is a separated algebraic scheme of finite type over $\mathbb{C}$ and a *variety* is a reduced, irreducible scheme. A *curve* is a variety of dimension 1. A *surface* is a variety of dimension 2.

Given a variety X and a coherent sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$ we denote by $\mathcal{Z}(\mathcal{J})$ the closed subset of X defined by $\mathcal{J}$, without any scheme structure.

We denote by $\mathrm{WDiv}(X)$ the group of Weil divisors and by $\mathrm{Div}(X)$ the group of Cartier divisors, while, for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, we denote by $\mathrm{Div}(X)_\mathbb{K}$ the group of $\mathbb{K}$-Cartier $\mathbb{K}$-divisors and by $N^1(X)_\mathbb{K}$ the vector space given by the quotient of $\mathrm{Div}(X)_\mathbb{K}$ by the subgroup of numerically trivial divisors. Note that $N^1(X)_\mathbb{R}$ is a finite dimensional $\mathbb{R}$-vector space. Moreover we consider a norm $\| \cdot \|$ on this space.

Given a variety X and a Cartier divisor (or a line bundle) D we denote by $\kappa(X, D)$ its Kodaira dimension. Given a smooth variety X and $D \in \mathrm{Div}_\mathbb{Q}(X)$, we denote by $\mathrm{mult}_x D$ the *multiplicity* at $x \in X$ of D, in the sense of [Laz04, Def. 9.3.1].

Definition 2.1.1. Let X be a normal projective variety over $\mathbb{C}$ and let Δ be a Weil $\mathbb{Q}$-divisor on X. We say that (X, Δ) is a *pair* if $K_X + \Delta \in \mathrm{Div}_\mathbb{Q}(X)$. A pair is *effective* if $\Delta \geq 0$.

Remark 2.1.2. Note that from now on we will always consider normal projective varieties. Then, in our setting, the cycle map

$$\mathrm{Div}(X) \longrightarrow \mathrm{WDiv(X)}$$

is injective.

Hence, with a slight abuse, we will use the same notation to indicate a Cartier divisor and the Weil divisor obtained as the image through the cycle map.

Definition 2.1.3. Let X be a normal projective variety and let D be a Weil divisor on X. We define the sheaf $\mathcal{O}_X(D)$ by putting, for every open subset $U \subseteq X$,

$$\mathcal{O}_X(U) = \{f \in k(X) \ : \ \mathrm{div}_U f + D_{|U} \geq 0\}.$$

Definition 2.1.4. Let X and Y be normal varieties and let $\mu : Y \to X$ be a proper birational morphism.
If $U \subseteq Y$ is the biggest open subset such that $\mu_{|U}$ is an isomorphism, we define

$$\mathrm{exc}(\mu) = Y \setminus U.$$

We say that a prime divisor E on Y is μ-*exceptional* (or simply *exceptional*) if $\mathrm{Supp}(E) \subseteq \mathrm{exc}(\mu)$.
If $E_1, ..., E_t$ are all distinct the prime μ-exceptional divisors on Y, we define

$$\mathrm{divexc}(\mu) = \mathrm{Supp}(E_1 + ... + E_t).$$

Note that, with a slight abuse of notation, we will use the same the notation to indicate the reduced divisor $E_1 + \cdots + E_t$ itself.
Note also that, by convention, we define $\mathrm{Supp}(0) = \emptyset$.

Remark 2.1.5. Note that if E is an exceptional prime divisor for a proper birational morphism $\mu : Y \to X$ between normal varieties, then $codim\mu(\mathrm{Supp}(E)) \geq 2$, that is E is contracted by μ, so that $\mu_*(E) = 0$. Moreover note that $\mu^{-1}\big(\mu(\mathrm{exc}(\mu))\big) = \mathrm{exc}(\mu)$.
For the proof of this facts see [Deb01, 1.40].

Some useful lemmas

In this section we recall some basic well-known lemmas that will be useful in the following chapters:

Lemma 2.1.6. *Let X, Y be normal varieties and let $f : Y \to X$ be a proper birational map. Let $\mathcal{J}$ be an ideal sheaf on Y. Then $\mathcal{Z}(f_*\mathcal{J}) \subseteq f(\mathcal{Z}(\mathcal{J}))$.*

Proof. Let $W := Y \setminus \mathcal{Z}(\mathcal{J})$ and $V := X \setminus f(\mathcal{Z}(\mathcal{J}))$. For every $V' \subseteq V$ open subset of X we have that $f^{-1}(V') \subseteq W$, hence $(f_*\mathcal{J})(V') = \mathcal{J}(f^{-1}(V')) = \mathcal{O}_Y(f^{-1}(V')) = f_*\mathcal{O}_Y(V') \cong \mathcal{O}_X(V')$ by Zariski's main theorem (cf. [Har77, Cor. III.11.4]).
Let $x \in \mathcal{Z}(f_*\mathcal{J})$. By contradiction suppose that $x \in V$. Since $(f_*\mathcal{J})_x \subsetneq \mathcal{O}_{X,x}$ then there exists an open subset $P \subseteq V$ such that $x \in P$ and $(f_*\mathcal{J})(P) \subsetneq \mathcal{O}_X(P)$. But by the previous argument we have also that $(f_*\mathcal{J})(P) = \mathcal{O}_X(P)$. Hence x must be in $f(\mathcal{Z}(\mathcal{J}))$. $\square$

Lemma 2.1.7. *Let X and Y be projective varieties such that Y is smooth and X is normal. Let $\mu : Y \to X$ be a proper birational morphism.*
If $D \in Div(Y)$, $E \in Div(X)$ are such that

$$\mu_* D \leq E,$$

then there exists an effective μ-exceptional Cartier divisor Γ on Y such that

$$D \leq \mu^* E + \Gamma.$$

Proof. Let $E_1, ..., E_t$ be all the prime μ-exceptional divisors on Y. Then, there exists a finite set of prime divisors on X, say $\{D_1, ..., D_s\}$, such that we can write

$$D = \sum_{i=1}^{s} a_i D_i' + \sum_{j=1}^{t} b_j E_j, \qquad \mu^*(E) = \sum_{i=1}^{s} c_i D_i' + \sum_{j=1}^{t} e_j E_j,$$

where the D_i' are the birational transforms of the D_i, and $a_i, c_i, b_j, e_j \in \mathbb{Z}$. By hypothesis we have that

$$\sum_{i=1}^{s} a_i D_i = \mu_* D \le E = \mu_*(\mu^*(E)) = \sum_{i=1}^{s} c_i D_i.$$

which implies that $a_i \le c_i$, for all $i = 1, ..., s$.

Now, for each $j = 1, ..., t$, we define $\gamma_j = \max\{0, b_j - e_j\}$, and let

$$\Gamma := \sum_{j=1}^{t} \gamma_j E_j.$$

Then, by definition, Γ is an effective, μ-exceptional Cartier divisor on Y. Moreover

$$\mu^*(E) + \Gamma \ge \sum_{i=1}^{s} c_i D_i' + \sum_{j=1}^{t} b_j E_j \ge \sum_{i=1}^{s} a_i D_i' + \sum_{j=1}^{t} b_j E_j = D.$$

$\square$

Lemma 2.1.8. *Let X and Y be normal projective varieties, such that Y is smooth and let $\mu : Y \to X$ be a proper birational morphism.*
Suppose $D_Y \in Div(Y)$, D is a Weil divisor on X and B is a Cartier divisor on X. If $\mu_* D_Y \le D$, then

$$h^0(Y, \mathcal{O}_Y(\mu^*(B) + D_Y)) \le h^0(X, \mathcal{O}_X(B + D)).$$

Proof. We use the standard fact that $\mu_* \mathcal{O}_Y(D_Y) \subseteq \mathcal{O}_Y(\mu_* D_Y)$.
Together with projection formula this implies that

$$h^0(Y, \mathcal{O}_Y(\mu^*(B) + D_Y)) \le h^0(X, \mathcal{O}_X(B + \mu_*(D_Y))).$$

But $h^0(X, \mathcal{O}_X(B + \mu_*(D_Y))) \le h^0(X, \mathcal{O}_X(B + D))$ by hypothesis, so that the lemma is proved.

$\square$

Lemma 2.1.9. *Let Y and X be normal projective varieties. Let $\mu : Y \to X$ be a composition of blowings-up of smooth centers of codimension greater than 1. If $A \in Div_{\mathbb{Q}}(X)$ is ample and $E_1, \ldots, E_k$ are all the μ-exceptional prime divisors on Y, then for every $i = 1, \ldots, k$ there exist arbitrarily small positive rational numbers δ_i, such that*

$$\mu^*(A) - \sum \delta_i E_i$$

is an ample divisor on Y.

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Proof. We work by induction on the number k of the blowings-up composing μ. To prove the base of the induction we consider $Z \subseteq X$ a smooth subvariety of codimension greater than one and $\mu : Y = Bl_Z X \to X$ as in the hypothesis.

Let $\mathcal{J} = \mathcal{J}_Z$ be the ideal sheaf defining the subvariety Z. Then there exists an effective Cartier divisor E on Y such that $\mu^{-1}\mathcal{J} = \mathcal{O}_Y(-E)$, so that, in particular, $\mathrm{Supp}(E) = \mu^{-1}(Z)$.

Moreover, by [Har77, II ex. 7.14(b)], we have that $\mu^*(mA) - E$ is ample for every $m \gg 0$, so that

$$\mu^* A - \frac{1}{m} E$$

is ample for every $m \gg 0$.

Note that, by the properties of the blowings-up, μ is an isomorphism outside E, so that $\mathrm{exc}(\mu) \subseteq \mathrm{Supp}(E)$.

Moreover every prime divisor whose support is contained in $\mathrm{Supp}(E)$ is μ-exceptional, because its image is contained in Z, so that it is contracted by μ on a subvariety of codimension greater than 1. Hence $\mathrm{Supp}(E) = \mathrm{divexc}(\mu) = \mathrm{exc}(\mu)$.

Then there exist some integers $a_i > 0$ such that

$$E = \sum a_i^1 E_i^1,$$

where the E_i^1 are all the μ-exceptional prime divisors, so that

$$\mu^*(A) - \sum \frac{a_i^1}{m} E_i^1$$

is ample for every integer $m \gg 0$. Thus we get the base of the induction.

Let us now prove the inductive step:

Let $\mu_1 : X' \to X$ be a composition of $k - 1$ blowings-up as in the hypothesis, let $\mu_2 : Y \to X'$ be the blowing-up along a smooth subvariety $Z' \subseteq X'$ of codimension greater than 1 and let $\mu = \mu_1 \circ \mu_2$.

If $E_1, \ldots, E_s$ are all the prime exceptional divisors of μ_1, by induction we can suppose that there exists arbitrarily small rational numbers $\delta_1, \ldots, \delta_s > 0$ such that

$$\mu_1^*(A) - \delta_1 E_1 - \cdots - \delta_s E_s$$

is ample.

If we denote F_i all the exceptional prime divisors of μ_2, we have to prove that

$$\mu^*(A) - \delta_1 \mu_{2*}^{-1} E_1 - \cdots + \delta_s \mu_{2*}^{-1} E_s - \sum \epsilon_i F_i$$

is ample for positive arbitrarily small rational numbers ϵ_i.

The base of the induction shows that

$$\mu_2^*(\mu_1^*(A) - \delta_1 E_1 - \cdots - \delta_s E_s) - \sum e_i F_i$$

is ample for $0 < \delta_1, \ldots, \delta_s \ll 1$, $\delta_i \in \mathbb{Q}$ and $0 < e_i \ll 1$, $e_i \in \mathbb{Q}$.

But

$$\mu_2^*(\mu_1^*(A) - \delta_1 E_1 - \cdots - \delta_s E_s) - \sum e_i F_i =$$

$$\mu^*(A) - \sum_{j=1}^s \delta_j \mu_{2*}^{-1} E_j - \sum_i (\sum_{j=1}^s a_{ji} \delta_j + e_i) F_i,$$

where the a_{ji} are non-negative integer numbers not depending on the δ_j. Thus if for each i we put

$$\epsilon_i = \sum_{j=1}^{s} a_{ji}\delta_j + e_i$$

we are done because each ϵ_i is a positive rational number if the δ_j's and e_i are such and we can make the ϵ_i's arbitrarily small by suitably choosing the δ_j's and the e_i's. $\qquad\square$

Lemma 2.1.10. *(cf. [Cac08, lemma 5.3]). Let X be a normal projective variety and let $D \in Div_\mathbb{R}(X)$. Then there exists a sequence $\{A_m\}_{m\geq 1}$ of ample $\mathbb{R}$-Cartier divisors such that $\|A_m\| \to 0$ in $N^1(X)_\mathbb{R}$ and $D + A_m$ is a $\mathbb{Q}$-Cartier divisor for every $m \geq 1$.*

Proof. Let A be an ample $\mathbb{R}$-divisor and let $r \in \mathbb{R}^+$ be such that, if we denote by $B_r(A)$ the ball in $N^1(X)$ centered at the class of the divisor A with ray r, then $B_r(A) \subseteq \mathrm{Amp}(X) \subseteq N^1(X)$.
Since $B_r(D + A) \cap N^1(X)_\mathbb{Q} \neq \emptyset$ then there exists a $\mathbb{Q}$-Cartier divisor D' such that $\|D' - (D + A)\| < r$. Therefore $D' - D$ is ample and can be written as $\sum_{i=1}^{s} c_i P_i$ with $c_i \in \mathbb{R}^+$ and P_i is an ample Cartier divisors for every i.
Let $q_{i,m} \in \mathbb{Q}^+$ be such that for every i, $q_{i,m} < c_i$ and $\lim_{m\to\infty} q_{i,m} = c_i$. Let $A_m := \sum_{i=1}^{s} (c_i - q_{i,m})P_i$. It is then easily seen that $\|A_m\| \to 0$ and that $D + A_m$ is $\mathbb{Q}$-Cartier. $\qquad\square$

2.2 Some remarks about singularities of pairs

Definition 2.2.1. Let X be a normal projective variety of dimension n. A reduced divisor $D = \sum D_i$ has *simple normal crossings* (and it is an *SNC divisor*) if each D_i is smooth and if D is defined in a neighborhood of any point by an equation in local analytic coordinates of type

$$z_1 \cdots \cdots z_k = 0$$

for some $k \leq n$. A Weil $\mathbb{Q}$-divisor $\sum a_i D_i$ has *simple normal crossing support* (and it is an *SNCS divisor*) if $\sum D_i$ is an SNC divisor.

Definition 2.2.2. Let (X, D) be a pair, let $|V|$ be a non-empty linear series on X and let $\mathcal{J} \subseteq \mathcal{O}_X$ be a nonzero ideal sheaf on X. Then a *log-resolution* of the pair (X, D) is a projective birational morphism

$$\mu : X' \to X$$

such that the exceptional locus $\mathrm{exc}(\mu)$ is a divisor, X' is smooth and the $\mathbb{Q}$-divisor $\mu_*^{-1}(D) + \mathrm{exc}(\mu)$ is SNCS.
A log-resolution of $|V|$ is a birational morphism $\nu : X'' \to X$ such that $\mathrm{exc}(\nu)$ is a divisor, X'' is smooth and $\nu^*|V| = |W| + F$, where $|W|$ is a linear series without base points and $F + \mathrm{exc}(\nu)$ is SNCS.
A log-resolution of $\mathcal{J} \subseteq \mathcal{O}_X$ is a projective birational morphism $\phi : X''' \to X$ such that $\mathrm{exc}(\phi)$ is a divisor, X''' is smooth and

$$\phi^{-1}(\mathcal{J}) := \mathcal{J} \cdot \mathcal{O}_{X'''} = \mathcal{O}_{X'''}(-F),$$

where F is an effective Cartier divisor on X such that $F + \mathrm{exc}(\mu)$ is SNCS.

LC centers

Definition 2.2.3. Let X and X' be normal projective varieties and let $E \subseteq X'$ be a prime divisor. We say that E is a prime divisor *over* X if there exists a proper birational morphism $\mu : X' \to X$.

Definition 2.2.4. Let (X, Δ) be a pair. Then, for every prime divisor E over X we consider the discrepancy $a(E, X, \Delta) \in \mathbb{Q}$ as in [KM00, 2.3] (note that we will use the same notation of [KM00, Notation 2.26]).

We say that the pair (X, Δ) is *Kawamata log terminal* or *KLT* (resp. *log canonical* or *LC*) if $a(E, X, \Delta) > -1$ (resp. $a(E, X, \Delta) \geq -1$) for every prime divisor E over X. We say that the pair (X, Δ) is *purely log terminal* or *PLT* if (X, Δ) is LC and $a(E, X, \Delta) > -1$ for every exceptional prime divisor E over X.

We say that a subvariety $V \subseteq X$ is a *log canonical center* or a *LC center* of the pair (X, Δ) if it is the image, through a proper birational morphism, of a divisor E over X such that $a(E, X, \Delta) \leq -1$.

We define $CLC(X, \Delta) = \{$LC centers of the pair $(X, \Delta)\}$ and we consider the *non-klt locus* of the pair (X, Δ) defined as the closed subset of X

$$\mathrm{Nklt}(X, \Delta) = \bigcup_{V \in CLC(X, \Delta)} V.$$

Definition 2.2.5. Let X be a normal variety and let D be a Weil $\mathbb{Q}$-divisor on X. If we write $D = \sum d_i D_i$, where the D_i are distinct prime divisors, we define

$$D^{\geq 1} = \sum_{d_i \geq 1} d_i D_i, \qquad D^{=1} = \sum_{d_i = 1} D_i.$$

Remark 2.2.6. Let (X, Δ) be a pair such that X is smooth and Δ is SNCS. Then, set-theoretically,

$$\mathrm{Nklt}(X, \Delta) := \bigcup_{V \in CLC(X, \Delta)} V = \mathrm{Supp}(\Delta^{\geq 1}).$$

In fact on the one hand if E is a prime divisor in $\mathrm{Supp}(\Delta^{\geq 1})$ then $a(E, X, \Delta) = -\mathrm{ord}_E(\Delta) \leq -1$.

On the other hand if $V \subseteq X$ is a subvariety that is not contained in $\mathrm{Supp}(\Delta^{\geq 1})$, $\mu : X' \to X$ is a proper birational morphism and $E \subseteq X'$ is a prime divisor such that $\mu(E) = V$, then

$$a(E, X, \Delta) = a(E, X, \Delta - \Delta^{\geq 1}) - \mathrm{ord}_E(\mu^*(\Delta^{\geq 1})) = a(E, Z, \Delta - \Delta^{\geq 1}),$$

because $V \not\subseteq \mathrm{Supp}(\Delta^{\geq 1})$ implies that $\mathrm{ord}_E(\mu^*(\Delta^{\geq 1})) = 0$. But $(X, \Delta - \Delta^{\geq 1})$ is a KLT pair because X is smooth, $\Delta - \Delta^{\geq 1}$ is SNCS and all its coefficients are smaller than 1.

Thus $a(E, X, \Delta - \Delta^{\geq 1}) > -1$, so that $V \notin CLC(X, \Delta)$.

Now let (X, Δ) be any LC pair, let $\mu : Y \to X$ be a log-resolution of the pair (X, Δ) and let $\Delta_Y \in \mathrm{Div}_{\mathbb{Q}}(Y)$ be such that

$$K_Y + \Delta_Y \equiv \mu^*(K_X + \Delta) \quad \text{and} \quad \mu_* \Delta_Y = \Delta.$$

Then $\Delta_Y{}^{\geq 1} = \Delta_Y{}^{=1}$ and the LC centers of the pair (X, Δ) are exactly the images on X of the irreducible components of the finite intersections of prime divisors in the support of $\Delta_Y^{=1}$.

We give here the definition of logbig divisor with respect to a pair.

Definition 2.2.7. Let (X, Δ) be a pair and let $L \in \mathrm{Div}_{\mathbb{Q}}(X)$.
We say that L is *logbig* for the pair (X, Δ) if L is big and $L_{|V}$ is big for every $V \in CLC(X, \Delta)$.
Moreover given an integer $k \in \{1, \dots, n\}$ we say that L is *logbig in codimension* k if L is big and $L_{|V}$ is big for every $V \in CLC(X, \Delta)$ such that $codim_X V = k$.

In the following three lemmas we remark how the LC centers of a log canonical pair (X, Δ) change when we modify Δ by subtracting an effective divisor.

Lemma 2.2.8. *Let (X, Δ) be a pair.*
If $N \in Div_{\mathbb{Q}}(X)$ is effective, then

1. *$(X, \Delta - N)$ is LC if (X, Δ) is such;*

2. *$CLC(X, \Delta - N) \subseteq CLC(X, \Delta)$.*

Proof. Let F be a prime divisor over X, say F is contained in a normal projective variety Y and $\mu : Y \to X$ is a proper birational morphism.
We have that
$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E;$$

$$K_Y \equiv \mu^*(K_X + \Delta - N) + \sum a(E, X, \Delta - N)E.$$

Hence
$$a(F, X, \Delta - N) = a(F, X, \Delta) + \mathrm{ord}_F \mu^*(N),$$

which implies that
$$a(F, X, \Delta - N) \geq a(F, X, \Delta)$$

thanks to the effectivity of N.
If $a(F, X, \Delta - N) \leq -1$, then $a(F, X, \Delta) \leq -1$ so that every LC center of $(X, \Delta - N)$ is also an LC center of (X, Δ).
Moreover if (X, Δ) is LC then $a(G, X, \Delta) \geq -1$, for every prime divisor G over X, so that, by the previous inequality, we get the same inequalities for the pair $(X, \Delta - N)$, that is $(X, \Delta - N)$ is an LC pair. $\square$

Lemma 2.2.9. *Let (X, Δ) be an LC pair, let $B \in Div_{\mathbb{Q}}(X)$ be effective, $B \neq 0$, and take $V \in CLC(X, \Delta)$.*
Then the following statements are equivalent:

1. *$Supp(B) \supseteq V$;*

2. *$V \notin CLC(X, \Delta - cB)$ for every rational number $c > 0$;*

3. *$V \notin CLC(X, \Delta - cB)$ for some rational number $c > 0$.*

Proof. **(1 $\Rightarrow$ 2)** Let E be a prime divisor over X such that V is the center of E on X.

In other words there exists a projective normal variety Y and a proper birational morphism $\mu : Y \to X$ such that $E \subseteq Y$ and $\mu(E) = V$.

We have to show that $a(E, X, \Delta - cB) > -1$ for every $c \in \mathbb{Q}^+$:

We can write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

$$K_Y \equiv \mu^*(K_X + \Delta - cB) + \sum a(E, X, \Delta - cB)E.$$

Hence

$$a(E, X, \Delta - cB) = a(E, X, \Delta) + c \operatorname{ord}_E(\mu^*(B)).$$

But $V = \mu(E) \subseteq \operatorname{Supp}(B)$, so that $\operatorname{ord}_E(\mu^*(B)) > 0$. Thus

$$a(E, X, \Delta - cB) > a(E, X, \Delta) \geq -1.$$

(2 $\Rightarrow$ 3) Trivial.

(3 $\Rightarrow$ 1) As $V \in CLC(X, \Delta)$ there exists a prime divisor E over X such that $a(E, X, \Delta) = -1$. Using the same notation as before we find again that

$$a(E, X, \Delta - cB) = a(E, X, \Delta) + c \operatorname{ord}_E(\mu^*(B)) = -1 + c \operatorname{ord}_E(\mu^*(B)).$$

Hence $a(E, X, \Delta - cB) > -1$ for some $c > 0$ implies that $\operatorname{ord}_E(\mu^*(B)) > 0$, which in turn implies that $V = \mu(E) \subseteq \operatorname{Supp}(B)$. $\qquad\square$

Lemma 2.2.10. *Let (X, Δ) be an LC pair and let $B \in Div_{\mathbb{Q}}(X)$ be effective, $B \neq 0$.*
Then the following statements are equivalent:

1. *$\operatorname{Supp}(B)$ contains all the LC centers of the pair (X, Δ);*

2. *$(X, \Delta - cB)$ is a KLT pair for every rational number $c > 0$;*

3. *$(X, \Delta - cB)$ is a KLT pair for some rational number $c > 0$.*

Proof. The lemma follows from lemma 2.2.9 and the fact that $CLC(X, \Delta - cB) \subseteq CLC(X, \Delta)$ for every $c > 0$ because B is effective. $\qquad\square$

Standard log-resolutions and applications

It can be sometimes useful to consider log-resolutions of pairs (or ideals) that are isomorphisms on the locus where the pair (or the ideal) is already "log-resolved". For this reason in this subsection we introduce the notion of standard log-resolutions. With this purpose we begin by giving the definition of non-simple normal crossing locus of a divisor:

Definition 2.2.11. Let X be a normal variety and let D be a Weil $\mathbb{R}$-divisor on X. If U is the biggest open subset of X such that $D_{|U}$ is SNCS we define the *non-simple normal crossing locus* of D as the closed set

$$NSNC(D) = X \setminus U.$$

Note that $NSNC(D)$ only depends on the support of D and we always have that $NSNC(D) \subseteq \operatorname{Supp}(D)$.

Lemma 2.2.12. *Let (X, Δ) be an LC pair, let $N \in Div_{\mathbb{Q}}(X)$ be effective, let $W \subseteq X$ be an (irreducible) subvariety. If $W \in CLC(X, \Delta - N)$ and $W \subseteq NSNC(\Delta - N)$, then $W \subseteq NSNC(\Delta)$.*

Proof. We begin by noticing that $(X, \Delta - N)$ is LC by lemma 2.2.8. Moreover note that $NSNC(X, \Delta - N) \subseteq NSNC(\Delta) \cup \mathrm{Supp}(N)$.
Then, as W is irreducible by hypothesis, it suffices to show that $W \not\subseteq \mathrm{Supp}(N)$. Let $\mu : Y \to X$ be a proper birational morphism and let $F \subseteq Y$ be a prime divisor such that $\mu(F) = W$ and $a(F, X, \Delta - N) = -1$. We can write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

$$K_Y \equiv \mu^*(K_X + \Delta - N) + \sum a(E, X, \Delta - N)E,$$

As $N \in \mathrm{Div}_{\mathbb{Q}}(X)$ we can deduce that

$$\sum a(E, X, \Delta)E = \sum a(E, X, \Delta - N)E - \mu^*(N).$$

Hence

$$-1 \leq a(F, X, \Delta) = a(F, X, D - N) - \mathrm{ord}_F\mu^*(N) = -1 - \mathrm{ord}_F\mu^*(N).$$

Thanks to the effectivity of N this implies that $\mathrm{ord}_F\mu^*(N) = 0$, so that $W \not\subseteq \mathrm{Supp}(N)$. $\qquad\square$

Definition 2.2.13. Let X be a normal projective variety, let L be a Cartier divisor on X and let $V \subseteq H^0(X, \mathcal{O}_X(L))$ be a subspace. Consider the evaluation morphism

$$e_V : V \otimes \mathcal{O}_X(-L) \to \mathcal{O}_X.$$

Then the *base ideal* of the linear series $|V|$, denoted by $b(|V|)$, is the ideal sheaf defined as the image of the map e_V.
The *base scheme* of $|V|$, $Bs(|V|)$, is the scheme defined by the base ideal $b(|V|)$, so that, in particular, it is supported on the set $\mathcal{Z}(b(|V|))$

Before giving the definition of standard log-resolution we recall the following easy lemma:

Lemma 2.2.14. *Let X be a normal projective variety, let $|L|$ be a non empty linear series on X and let $b(|L|)$ be its base ideal. Suppose $\mu : X' \to X$ is a log-resolution of the ideal $b(|L|)$. Then*

1. *$b(|\mu^*(L)|) = \mu^{-1}(b(|L|))$;*

2. *μ is a log-resolution of the linear series $|L|$.*

Proof. **1.** By [Gro60, 4.4.5] we have that the ideal $\mu^{-1}(b(|L|))$ defines the scheme $\mu^{-1}(Bs(|L|))$. Now let $|L| = <s_1, \ldots, s_k>$. Then

$$\mu^*(L) = <\mu^*(s_1), \ldots, \mu^*(s_k)> = <s_1 \circ \mu, \ldots, s_k \circ \mu> .$$

Then $Bs(|\mu^*(L)|)$ is the scheme defined by $s_1 \circ \mu, \ldots, s_k \circ \mu$, so that it is the inverse image of the scheme defined by $<s_1, \ldots, s_k>$.
In other words $Bs(|\mu^*(L)|) = \mu^{-1}(Bs(|L|))$.

Therefore we obtain that $\mu^{-1}(b(|L|))$ defines the base scheme $Bs(|\mu^*(L)|)$, that is $\mu^{-1}(b(|L|)) = b(|\mu^*(L)|)$.

2. By hypothesis X' is smooth and there exists a SNCS divisor F on X' such that $\mu^{-1}(b(|L|)) = \mathcal{O}_{X'}(-F)$. Thus, by the first part we have that $b(|\mu^*(L)|) = \mathcal{O}_{X'}(-F)$.
In other words the base scheme $Bs(|\mu^*(L)|) = F$.
Thus we can write $|\mu^*(L)| = |\mu^*(L) - F| + F$, where the linear series $|\mu^*(L) - F|$ is free.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.2.15. Let X be a normal projective variety and let D be a reduced Weil divisor on X.
A *standard log-resolution* of the pair (X, D) is a log-resolution f of the pair (X, D) such that

- f is a composition of blowings-up of smooth subvarieties of codimension greater than 1 up to isomorphisms;

- $f_{|_{f^{-1}(U)}}$ is an isomorphism, where $U = X \setminus (NSNC(D) \cup Sing(X))$.

Definition 2.2.16. Let X be a smooth projective variety and let $\mathcal{J} \subseteq \mathcal{O}_X$ be a non zero ideal sheaf. A *standard log-resolution* of the ideal sheaf $\mathcal{J}$ is a log-resolution g of $\mathcal{J}$ such that g is a composition of blowings-up of smooth subvarieties of codimension greater than 1 contained in $\mathcal{Z}(\mathcal{J})$ up to isomorphisms. In particular $g_{|_{g^{-1}(X \setminus \mathcal{Z}(\mathcal{J}))}}$ is an isomorphism.

Remark 2.2.17. Note that, by definition, a standard log-resolution of a pair or an ideal sheaf has a divisorial exceptional set.
In fact this is true for every composition of blowings-up of smooth subvarieties of codimension greater than 1:
It suffices to prove that for a single blowing-up π of a smooth subvariety Z of codimension greater than 1.
But, in this case, by the properties of the blowing-up we have that $\pi^{-1}(Z)$ is a divisor and $\mathrm{exc}(\pi) \subseteq \pi^{-1}(Z)$. On the other hand every prime divisor in $\pi^{-1}(Z)$ is contracted by π because its image, being contained in Z, has codimension greater than 1.
Thus $\pi^{-1}(Z) \subseteq \mathrm{divexc}(\pi) \subseteq \mathrm{exc}(\pi)$, so that $\pi^{-1}(Z) = \mathrm{divexc}(\pi) = \mathrm{exc}(\pi)$.

Theorem 2.2.18. *Let X be a normal projective variety and let D be a reduced Weil divisor on X. Then there exists a standard log-resolution of the pair (X, D).*

Proof. Let $f : X' \to X$ be a resolution of singularities of X obtained by blowing up smooth subvarieties of X contained in $Sing(X)$ (see for example [Laz04, theorem 4.1.3]). As X is normal $codim Sing(X) \geq 2$, so that f is a composition of blowings-up of subvarieties of codimension greater than 1.
Now, as X' is smooth, thanks to [Fuj07a, theorem 3.5.1] there exists a log-resolution of the pair $(X', f_*^{-1}(D) + \mathrm{divexc}(f))$, say $g : Y \to X'$, such that g is a composition of blowings-up of smooth subvarieties and g is an isomorphism outside $NSNC(f_*^{-1}(D) + \mathrm{divexc}(f))$.

Let $\mu = f \circ g$. Then μ is a log-resolution of (X, D) because

$$\mu_*^{-1}(D) + \mathrm{divexc}(\mu) = g_*^{-1}\big(f_*^{-1}D + \mathrm{divexc}(f)\big) + \mathrm{divexc}(g)$$

is SNC by definition of g.

We have that μ is a composition of blowings-up of smooth subvarieties because the same holds both for f and g and we can suppose their codimension is greater than 1:

In fact we have already proved this for f. As for g, note that every smooth subvariety Z of X' of codimension 1 is a Cartier divisor because X' is smooth. Hence the blowing-up along Z is an isomorphism.

Let us prove now that μ is an isomorphism outside $\mu^{-1}(NSNC(D) \cup Sing(X))$: First of all we note that

$$NSNC(f_*^{-1}(D) + \mathrm{divexc}(f)) \subseteq \mathrm{divexc}(f) \cup f^{-1}(NSNC(D)):$$

In fact if $f_*^{-1}(D) + \mathrm{divexc}(f)$ is not simple normal crossing in a point x outside $\mathrm{divexc}(f)$, then $f_*^{-1}(D)$ is not simple normal crossing in x. But f is an isomorphism in a neighbourhood of x because $x \notin \mathrm{divexc}(f) = \mathrm{exc}(f)$ by remark 2.2.17, hence D is not simple normal crossing in $f(x)$.

Now, by definition, μ is an isomorphism outside

$$\mathrm{exc}(\mu) = \mathrm{divexc}(\mu) = \mathrm{divexc}(g) \cup g^{-1}(\mathrm{divexc}(f)).$$

On the other hand we have that $\mathrm{divexc}(f) \subseteq f^{-1}(Sing(X))$. Hence we get that μ is an isomorphism outside

$$\mathrm{divexc}(g) \cup g^{-1}(\mathrm{divexc}(f)) \subseteq \mathrm{divexc}(g) \cup g^{-1}\big(f^{-1}(Sing(X))\big) \subseteq$$

$$\subseteq g^{-1}\big(NSNC(f_*^{-1}(D) + \mathrm{divexc}(f))\big) \cup \mu^{-1}\big(Sing(X)\big),$$

because of the definition of g. Thus, using the previous observation we have that

$$\mathrm{exc}(\mu) \subseteq g^{-1}\big(\mathrm{divexc}(f)\big) \cup g^{-1}\big(f^{-1}(NSNC(D))\big) \cup \mu^{-1}\big(Sing(X)\big) =$$

$$= \mu^{-1}\big(NSNC(D)\big) \cup \mu^{-1}\big(Sing(X)\big).$$

$\square$

Theorem 2.2.19. *Let X be a smooth projective variety and let $\mathcal{J} \subseteq \mathcal{O}_X$ be a non zero ideal sheaf. Then there exists a standard log-resolution of $\mathcal{J}$.*

Proof. Apart from [Hir64], this is proved, for example, in [Kol05, theorem 35] or [Kol05, theorem 21] (together with theorem 2.2.18). Note that we can suppose the log-resolution being composed, up to isomorphisms, by blowings-up of varieties of codimension greater than 1 because every prime divisor on X is a Cartier divisor, by the smoothness of X itself. Thus blowing up a smooth codimension one subvariety on X gives rise to an isomorphism. $\square$

The following lemma is based on the existence of standard log-resolutions:

Lemma 2.2.20. *Let (X, D) be a pair and let $|L|$ be a linear series on X. Then there exists a common log-resolution of the pair (X, D) and of the base ideal $b(|L|)$, say $\mu : Z \to X$, such that*

1. *μ is an isomorphism on $Z \setminus \mu^{-1}(Bs(|L|) \cup Sing(X) \cup NSNC(D))$;*

2. *μ is a composition of blowings-up of smooth subvarieties of codimension greater than 1, up to isomorphisms.*

In particular, by lemma 2.2.14, μ is a log-resolution of the linear series $|L|$.

Proof. Let $b = b(|L|)$. Let $f : X' \to X$ be a resolution of singularities of X obtained by blowing up smooth subvarieties of X contained in $Sing(X)$ (see for example [Laz04, theorem 4.1.3]). Note that all the varieties we blow up have codimension greater than 1, because $\operatorname{codim} Sing(X) \geq 2$ as X is normal.

As X' is smooth, by theorem 2.2.19 there exists $g : X'' \to X'$ a standard log-resolution of the ideal sheaf $f^{-1}(b)$. Then g is a composition of blowings-up of smooth subvarieties of codimension greater than 1 contained in $\mathcal{Z}(f^{-1}(b))$. Moreover there exists an effective divisor F on X'' such that $g^{-1}(f^{-1}(b)) = \mathcal{O}_{X''}(-F)$.

Now, by theorem 2.2.18 we can consider $h : Z \to X''$, a standard log-resolution of the pair $(X'', \operatorname{Supp}(F) + \operatorname{Supp}(g_*^{-1}(f_*^{-1}D)) + \operatorname{divexc}(f \circ g))$.

Again h is a composition of blowings-up of smooth subvarieties of codimension greater than 1.

Let $\mu = f \circ g \circ h : Z \to X$. Then μ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 because the same holds for f, g and h.

Let us show that μ is a common log-resolution of (X, D) and $|L|$: We begin by noting that Z is smooth by definition.

Now we have that $\mu^{-1}(b) = h^{-1}(\mathcal{O}_{X''}(-F)) = h^*(\mathcal{O}_{X''}(-F))$ by [Har77, II 7.12.2] because $\mathcal{O}_{X''}(-F)$ is an invertible sheaf. Then we have to show that

$$\operatorname{Supp}(\mu_*^{-1}(D)) \cup \operatorname{Supp}(h^*(F)) \cup \operatorname{divexc}(\mu)$$

is SNC. But

$$\operatorname{Supp}(\mu_*^{-1}(D)) \cup \operatorname{Supp}(h^*(F)) \cup \operatorname{divexc}(\mu) =$$

$$= h_*^{-1}\big(\operatorname{Supp}(g_*^{-1}(f_*^{-1}(D)))\big) \cup h_*^{-1}(\operatorname{Supp}(F)) \cup \operatorname{divexc}(h) \cup h_*^{-1}\operatorname{divexc}(f \circ g),$$

so that it is SNC thanks to the choice of h.

It remains to show that μ is an isomorphism on $Z \setminus \mu^{-1}(Bs(|L|) \cup Sing(X) \cup NSNC(D))$.

We will equivalently show that

$$\operatorname{divexc}(\mu) = \operatorname{exc}(\mu) \subseteq \mu^{-1}(Bs(|L|) \cup Sing(X) \cup NSNC(D)).$$

We use that, by definition,

$$\operatorname{divexc}(\mu) = \operatorname{divexc}(h) \cup h^{-1}(\operatorname{divexc}(g)) \cup h^{-1}\big(g^{-1}(\operatorname{divexc}(f))\big).$$

But $\operatorname{divexc}(f) \subseteq f^{-1}(Sing(X))$ by definition of f, so that $h^{-1}\big(g^{-1}(\operatorname{divexc}(f))\big) \subseteq \mu^{-1}(Sing(X))$.

On the other hand, by the definition of g and using [Gro60, 4.4.5], we have that

$$h^{-1}(\text{divexc}(g)) \subseteq h^{-1}\big(g^{-1}(\mathcal{Z}(f^{-1}(b)))\big) \subseteq h^{-1}\big(g^{-1}(f^{-1}(\mathcal{Z}(b)))\big) = \mu^{-1}(Bs(|L|)).$$

It remains to study the exceptional locus of h:

Let $U \subseteq X''$ be the biggest open subset such that

$$\big(\text{Supp}(F) + \text{Supp}(g_*^{-1}(f_*^{-1}D)) + \text{divexc}(f \circ g)\big)_{|U}$$

is SNC.
Then, as h is a standard log-resolution, we have that $\text{divexc}(h) \subseteq Z \setminus h^{-1}(U)$.
Now let V be the biggest open subset of X such that $D_{|V}$ is SNCS, so that $NSNC(D) = X \setminus V$.
We have to show that

$$Z \setminus h^{-1}(U) \subseteq \big(Z \setminus \mu^{-1}(V)\big) \cup \mu^{-1}(Sing(X)) \cup \mu^{-1}(Bs(|L|)).$$

This is equivalent to say that

$$h^{-1}(U) \supseteq \mu^{-1}(V) \cap \big(Z \setminus \mu^{-1}(Sing(X))\big) \cap \big(Z \setminus \mu^{-1}Bs(|L|)\big) =$$
$$= h^{-1}\big(g^{-1}(f^{-1}(V))\big) \cap h^{-1}\big(X'' \setminus g^{-1}(f^{-1}(Sing(X)))\big) \cap$$
$$\cap h^{-1}\big(X'' \setminus g^{-1}(f^{-1}Bs(|L|))\big).$$

Hence we have to show that

$$U \supseteq g^{-1}(f^{-1}(V)) \cap \big(X'' \setminus g^{-1}(f^{-1}(Sing(X)))\big) \cap \big(X'' \setminus g^{-1}(f^{-1}Bs(|L|))\big) =$$
$$=: S.$$

In other words this is equivalent to say that

$$\big(\text{Supp}(F) + \text{Supp}(g_*^{-1}(f_*^{-1}D)) + \text{divexc}(f \circ g)\big)_{|S}$$

is SNC. But

$$\text{divexc}(f \circ g) = \text{divexc}(g) \cup g^{-1}(\text{divexc}(f)) \subseteq g^{-1}\big(f^{-1}(Bs(|L|))\big) \cup$$
$$\cup g^{-1}\big(f^{-1}(Sing(X))\big)$$

and $\text{Supp}(F) \subseteq g^{-1}\big(f^{-1}(Bs(|L|))\big)$ by definition. This implies that

$$\big(\text{Supp}(F) + \text{Supp}(g_*^{-1}(f_*^{-1}D)) + \text{divexc}(f \circ g)\big)_{|S} = \big(\text{Supp}(g_*^{-1}(f_*^{-1}D))\big)_{|S}.$$

Now note that $\text{Supp}(D)_{|(f \circ g)(S)}$ is SNC, as $(f \circ g)(S) \subseteq V$.
On the other hand we have just noticed that $\text{divexc}(f \circ g) \subseteq X'' \setminus S$, so that $f \circ g$ is an isomorphism on S. Thus $(\text{Supp}(g_*^{-1}(f_*^{-1}D)))_{|S}$ remains SNC. $\quad\square$

Remarks on DLT pairs

We recall, first of all, the definition of a DLT pair:

Definition 2.2.21. (cf. [KM00, definition 2.37]). Let (X, Δ) be a pair such that if we write $\Delta = \sum a_i D_i$, with the D_i's distinct prime divisors, then $0 \le a_i \le 1$ for every i. We say that (X, Δ) is *divisorial log terminal* (or DLT) if there exists a closed subset $Z \subseteq X$ such that:

1. $X \setminus Z$ is smooth and $\Delta_{|X \setminus Z}$ is SNCS

2. If $f : Y \to X$ is birational and $E \subseteq Y$ is an irreducible divisor such that $f(E) \subseteq Z$ then the discrepancy $a(E, X, \Delta) > -1$.

Definition 2.2.22. Let (X, Δ) be a pair, let Y be a normal variety and let $f : Y \to X$ be a projective birational morphism. We say that f is a *DLT morphism for the pair* (X, Δ) if

- f is a composition of blowings-up of smooth centers of codimension greater than 1;

- We have that

$$a(E, X, \Delta) > -1 \qquad \text{for every prime divisor } E \subseteq Y \text{ exceptional on } X.$$

Following [Sza95] we give in the next theorem a useful multiple characterization of the property of DLTness:

Theorem 2.2.23 ([Sza95])**.** *Let (X, Δ) be an effective LC pair. Then the following statements are equivalent:*

1. *(X, Δ) is DLT;*

2. *There exists a log-resolution of (X, Δ) that is a DLT morphism for the same pair;*

3. *There exists a* standard *log-resolution of (X, Δ) that is a DLT morphism for the same pair;*

4. *For every $V \in CLC(X, \Delta)$ we have that $V \not\subseteq Sing(X) \cup NSNC(\Delta)$.*

Proof. **(1 $\Rightarrow$ 4)** By definition there exists a closed subset of X, say Z, such that $X \setminus Z$ is smooth, $\Delta_{|X \setminus Z}$ is SNCS and Z does not contain any LC center of (X, Δ). Then $NSNC(\Delta) \subseteq Z$ because $\Delta_{|X \setminus Z}$ is SNCS and $Sing(X) \subseteq Z$ because $X \setminus Z$ is smooth. Thus if $V \in CLC(X, \Delta)$, since $V \not\subseteq Z$ we have that $V \not\subseteq Sing(X) \cup NSNC(\Delta)$.
(4 $\Rightarrow$ 3) Let μ be a standard log-resolution of the pair $(X, \mathrm{Supp}(\Delta))$, whose existence is assured by theorem 2.2.18.
Then, by definition, μ is an isomorphism on $\mu^{-1}(X \setminus (NSNC(\Delta) \cup Sing(X)))$. Hence $\mathrm{divexc}(\mu) \subseteq \mu^{-1}(NSNC(\Delta) \cup Sing(X))$, so that if E is a μ-exceptional prime divisor we have that $\mu(E) \subseteq NSNC(\Delta) \cup Sing(X)$. Thanks to the hypothesis, this implies that $\mu(E) \notin CLC(X, \Delta)$, so that $a(E, X, \Delta) > -1$. In other words μ is a DLT morphism for the pair (X, Δ).
(3 $\Rightarrow$ 2) Trivial.
(2 $\Rightarrow$ 1) Let $f : Y \to X$ be the a log-resolution as in the hypothesis and let U be the biggest open subset of X such that $f_{|_{f^{-1}(U)}}$ is an isomorphism. Then

$$Y \setminus f^{-1}(U) = \mathrm{exc}(f) = \mathrm{divexc}(f)$$

thanks to the properties of DLT morphisms.
Define $Z = X \setminus U$. Then $\Delta_{|X \setminus Z} = \Delta_{|U}$ is SNCS because f is an isomorphism over U and $f_*^{-1}(\Delta)$ is SNCS. Moreover $X \setminus Z = U \simeq f^{-1}(U)$ is smooth because Y is such.

In order to prove that (X, Δ) is DLT it remains to show that there are no LC centers of (X, Δ) contained in Z.

Suppose, by contradiction, there exists $V \in CLC(X, \Delta)$ such that $V \subseteq Z$. Define

$$\Delta_Y = -\sum a(E, X, \Delta)E,$$

where the sum is taken on all prime divisors E on Y. Then, by remark 2.2.6, $V = f(W)$, where W is an irreducible component of a finite intersection of prime divisors in the support of $\Delta_Y^{=1}$. But, by definition of DLT morphism, $a(E, X, \Delta) > -1$ if E is f-exceptional.

Then W is an irreducible component of a finite intersection of strict transforms of divisors in the support of Δ, so that $W \not\subseteq \text{divexc}(f)$ because $\mu_*^{-1}(\Delta) + \text{divexc}(f)$ is a SNCS divisor.

Thus $W \not\subseteq \text{exc}(f)$, so that $V = f(W) \not\subseteq f(\text{exc}(f)) = f(Y) \setminus U = X \setminus U = Z$, and we get the contradiction. $\qquad \square$

In the rest of this subsection we list some good properties of DLT pairs.

Lemma 2.2.24. *Let X be a normal projective $\mathbb{Q}$-Gorenstein variety, and let $\Delta \in Div_{\mathbb{Q}}(X)$ be such that the pair (X, Δ) is DLT. Then $(X, (1 - \epsilon)\Delta)$ is a KLT pair for every rational number $\epsilon > 0$.*

Proof. By theorem 2.2.23 there exists a birational morphism $\mu : Y \to X$ such that μ is a log-resolution of the pair (X, Δ) and a DLT morphism for the same pair.

As Δ is $\mathbb{Q}$-Cartier we have that, for every rational number $\epsilon > 0$, the $\mathbb{Q}$-divisor $K_X + (1 - \epsilon)\Delta$ is $\mathbb{Q}$-Cartier.

Then we can write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

$$K_Y \equiv \mu^*(K_X + (1 - \epsilon)\Delta) + \sum a(E, X, (1 - \epsilon)\Delta) =$$

$$= \mu^*(K_X + \Delta) - \epsilon\mu^*(\Delta) + \sum a(E, X, (1 - \epsilon)\Delta).$$

Hence, if $\Delta_\epsilon := -\sum a(E, X, (1 - \epsilon)\Delta)E$, then, thanks to [KM00, lemma 2.30], we have that

$$a(D, X, (1 - \epsilon)\Delta) = a(D, Y, \Delta_\epsilon) \qquad \forall \, D \text{ prime divisor over } X.$$

Thus, for every $\epsilon \in \mathbb{Q}^+$, $(X, (1 - \epsilon)\Delta)$ is KLT if and only if the same holds for (Y, Δ_ϵ).

But Y is smooth and Δ_ϵ is SNCS, whence (Y, Δ_ϵ) is KLT if and only if all the coefficients of Δ_ϵ are less than 1, that is $a(E, X, (1 - \epsilon)\Delta) > -1$ for each prime divisor E on Y.

But, for every prime divisor E on Y we have that

$$a(E, X, (1 - \epsilon)\Delta) = a(E, X, \Delta) + \epsilon \, \text{ord}_E \mu^*(\Delta).$$

Now, as μ is a DLT morphism, if $a(E, X, \Delta) = -1$, then E is not μ-exceptional. Then, by definition of discrepancies, if $a(E, X, \Delta) = -1$ we have that

$$E \subseteq \text{Supp}(\mu_*^{-1}(\Delta)) \subseteq \text{Supp}(\mu^*(\Delta)),$$

so that $\text{ord}_E \mu^*(\Delta) > 0$.

Thus we obtain that

- if $a(E, X, D) > -1$, then $a(E, X, (1 - \epsilon)\Delta) \geq a(E, X, D) > -1$;

- if $a(E, X, D) = -1$, then $a(E, X, (1 - \epsilon)\Delta) > a(E, X, D) \geq -1$;

so that we are done. $\qquad\square$

Remark 2.2.25. It is very easy to see that the converse of the last lemma does not hold, even if we add the effectiveness of the boundary.
In other words there exist pairs (X, Δ) such that Δ is effective and $(X, (1 - \epsilon)\Delta)$ is KLT for every $\epsilon > 0$, but (X, Δ) is not DLT.
It suffices to consider a smooth variety X and $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ effective, such that (X, Δ) is LC but not DLT. For example take X to be the plane and $\Delta = \frac{5}{6}A$, where A is the cuspidal cubic curve (see [Laz04, Example 9.2.15]).

Lemma 2.2.26. *Let (X, Δ) be an LC pair.*
If $\Delta' \in Div_{\mathbb{Q}}(X)$ is effective and $Supp(\Delta')$ does not contain LC centers of the pair (X, Δ), then there exists $\lambda_0 > 0$ such that for all $\lambda \in \mathbb{Q} \cap [0, \lambda_0)$, we have

1. *$(X, \Delta + \lambda\Delta')$ is an LC pair and $CLC(X, \Delta + \lambda\Delta') = CLC(X, \Delta)$;*

2. *If (X, Δ) is DLT then $(X, \Delta + \lambda\Delta')$ is also DLT.*

Proof. **1.** Let $\mu : Y \to X$ be a common log-resolution of (X, Δ) and (X, Δ'), so that Y is smooth and for every $\lambda \geq 0$ we have that

$$\mathrm{Supp}(\mu^*(\Delta)) \cup \mathrm{Supp}(\mu^*(\lambda\Delta')) \cup \mathrm{exc}(\mu)$$

is SNC.
We can write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E,$$

$$K_Y \equiv \mu^*(K_X + \Delta + \lambda\Delta') + \sum a(E, X, \Delta + \lambda\Delta')E.$$

Hence, if $\Delta_\lambda := -\sum a(E, X, \Delta + \lambda\Delta')E$, thanks to [KM00, lemma 2.30] we have that

$$a(D, X, \Delta + \lambda\Delta') = a(D, Y, \Delta_\lambda) \qquad \forall\, D \text{ prime divisor over } X.$$

Thus, if we show that (Y, Δ_λ) is LC for $0 \leq \lambda \ll 1$, then the same holds for $(X, \Delta + \lambda\Delta')$.
But Y is smooth and Δ_λ is SNCS, because $\mathrm{Supp}(\Delta_\lambda) \subseteq \mathrm{Supp}(\mu^*(\Delta)) \cup \mathrm{Supp}(\mu^*(\lambda\Delta')) \cup \mathrm{exc}(\mu)$, whence it suffices to show that all the coefficients of Δ_λ are less than or equal to 1, that is $a(E, X, \Delta + \lambda\Delta') \geq -1$ for each prime divisor E on Y.
But we have that

$$a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) - \lambda \, \mathrm{ord}_E(\mu^*(\Delta')),$$

and, moreover, if $a(E, X, \Delta) = -1$ then $\mu(E) \in CLC(X, \Delta)$, so that $\mu(E) \not\subseteq \mathrm{Supp}(\Delta')$, that is $\mathrm{ord}_E(\mu^*(\Delta')) = 0$.
Hence, if we define

$$\lambda_0 = \min_{ord_E(\mu^*\Delta') > 0} \left\{ \frac{1 + a(E, X, \Delta)}{\mathrm{ord}_E(\mu^*(\Delta'))} \right\},$$

then $\lambda_0 > 0$, and for each $\lambda \in [0, \lambda_0)$ we have that

- If $a(E, X, \Delta) = -1$ then $a(E, X, \Delta + \lambda\Delta') = -1$;

- If $a(E, X, \Delta) > -1$ then $a(E, X, \Delta + \lambda\Delta') > -1$.

Thus $(X, \Delta + \lambda\Delta')$ is LC and $CLC(X, \Delta + \lambda\Delta') = CLC(X, \Delta)$ thanks to remark 2.2.6, because $\Delta_\lambda^{=1} = \Delta_0^{=1}$.

2. If (X, Δ) is DLT we have that $\Delta + \lambda\Delta'$ is effective and we know, by the first part, that the pair $(X, \Delta + \lambda\Delta')$ is LC. Hence, by theorem 2.2.23, it suffices to show that there exists a log-resolution of the pair $(X, \Delta + \lambda\Delta')$ that is a DLT morphism for the same pair.
As the pair (X, Δ) is DLT, by theorem 2.2.23 there exists a log-resolution $f : X' \to X$ of the pair (X, Δ) that is a DLT morphism for (X, Δ).
Let $g : Y \to X'$ be a standard log-resolution of the pair $(X', \mathrm{Supp}(f^*(\Delta) + f^*(\Delta')) \cup \mathrm{divexc}(f))$.
Then $\mathrm{Supp}(g^*(f^*(\Delta+\Delta'))) \cup \mathrm{divexc}(f \circ g)$ is SNC, so that $f \circ g$ is a log-resolution of the pair $(X, \Delta + \lambda\Delta')$.
Note also that $f \circ g$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1, because the same holds for f and g.
Moreover, if E is a prime $(f \circ g)$-exceptional divisor, then, E is g-exceptional or f-exceptional (identifying E with $g_*(E)$ in the latter case).
If E is f-exceptional, then $a(E, X, \Delta) > -1$ because f is a DLT morphism for (X, Δ), that is $f(E)$ is not a LC center of the pair (X, Δ). Hence (by part 1) $f(E)$ is not a LC center of $(X, \Delta + \lambda\Delta')$, so that $a(E, X, \Delta + \lambda\Delta') > -1$.
Let now E be g-exceptional. Then, by definition of standard log-resolution, as

$$(\mathrm{Supp}(f^*(\Delta) + f^*(\Delta')) \cup \mathrm{divexc}(f))_{|_{X' \setminus Supp(f^*(\Delta'))}} =$$
$$= (\mathrm{Supp}(f^*(\Delta)) \cup \mathrm{divexc}(f))_{|_{X' \setminus Supp(f^*(\Delta'))}}$$

is SNC, we have that g is an isomorphism on $g^{-1}(X' \setminus \mathrm{Supp}(f^*(\Delta')))$.
Hence $g(E) \subseteq \mathrm{Supp}(f^*(\Delta'))$, that is $f(g(E)) \subseteq \mathrm{Supp}(\Delta')$. Then $f(g(E))$ is not a LC center of (X, Δ), which implies, by step 1, that $f(g(E))$ is not a LC center of $(X, \Delta + \lambda\Delta')$, that is $a(E, X, \Delta + \lambda\Delta') > -1$.
We have thus shown that $f \circ g$ is a DLT morphism for the pair $(X, \Delta + \lambda\Delta')$. $\square$

Lemma 2.2.27. *Let (X, Δ) be a DLT pair and let $N \in Div_{\mathbb{Q}}(X)$ be effective. If $|L|$ is a linear series on X such that $Bs(|L|)$ does not contain LC centers of the pair (X, Δ), then there exists $\mu : Z \to X$, a common log-resolution of the pair $(X, \Delta - N)$ and of the linear series $|L|$, such that μ is a DLT morphism for the pair $(X, \Delta - N)$.*
In particular $(X, \Delta - N)$ is DLT if $\Delta \geq N$.

Proof. Let $\mu : Y \to X$ be a common resolution of $(X, \Delta - N)$ and $|L|$ as in lemma 2.2.20. In order to prove that μ is a DLT morphism for $(X, \Delta - N)$ we have only to show that every μ-exceptional prime divisor E on Y has discrepancy $a(E, X, \Delta - N) > -1$, the other property following by lemma 2.2.20.
Let E be a μ-exceptional prime divisor on Y. Lemma 2.2.20 implies that

$$\mu(E) \subseteq Bs(|L|) \cup Sing(X) \cup NSNC(X, \Delta - N).$$

Hence, as E is irreducible, we have that $\mu(E)$ is contained in $Bs(|L|)$ or in $Sing(X)$ or in $NSNC(X, \Delta - N)$.

If $\mu(E) \subseteq Bs(|L|)$, then $\mu(E) \notin CLC(X, \Delta)$ by hypothesis, so that $\mu(E) \notin CLC(X, \Delta - N)$ by lemma 2.2.8. This implies that $a(E, X, \Delta - N) > -1$.

If $\mu(E) \subseteq Sing(X)$, then $\mu(E) \notin CLC(X, \Delta)$ thanks to theorem 2.2.23, because (X, Δ) is DLT. Then, as before, $\mu(E) \notin CLC(X, \Delta - N)$ by lemma 2.2.8, so that $a(E, X, \Delta - N) > -1$.

Suppose now $\mu(E) \subseteq NSNC(\Delta - N)$. If, by contradiction, $\mu(E) \in CLC(X, \Delta - N)$, then, by lemma 2.2.12, $\mu(E) \subseteq NSNC(\Delta)$. This implies, by theorem 2.2.23 that $\mu(E) \notin CLC(X, \Delta)$ so that $\mu(E) \notin CLC(X, \Delta - N)$ using again lemma 2.2.8, and we reach a contradiction.

Therefore $a(E, X, \Delta - N) > -1$ because $\mu(E) \notin CLC(X, \Delta - N)$.

$\square$

2.3 Asymptotic base loci

We recall the following well-known definitions:

Definition 2.3.1. Let X be a normal projective variety, let $D \in \mathrm{Div}_{\mathbb{R}}(X)$.

1. $|D|_{\equiv} := \{E \in \mathrm{Div}_{\mathbb{R}}(X), E \text{ effective}, E \equiv D\}$;

2. $|D|_{\mathbb{R}} := \{E \in \mathrm{Div}_{\mathbb{R}}(X), E \text{ effective}, E \sim_{\mathbb{R}} D\}$, where $E \sim_{\mathbb{R}} D$ means that $E - D$ is an $\mathbb{R}$-linear combination of principal divisors (f), $f \in \mathbb{C}(X)$;

3. $|D|_{\mathbb{Q}} := \{E \in \mathrm{Div}_{\mathbb{R}}(X), E \text{ effective}, E \sim_{\mathbb{Q}} D\}$.

Definition 2.3.2. (cf. [BCHM10, Def. 3.5.1]). Let X be a normal projective variety, let $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The (real) *stable base locus* of D is

$$\mathbb{B}(D) := \bigcap_{E \in |D|_{\mathbb{R}}} \mathrm{Supp}(E),$$

where, by convention, we put $\mathbb{B}(D) = X$ if $|D|_{\mathbb{R}} = \emptyset$.

Remark 2.3.3. Notice that if D is $\mathbb{Q}$-Cartier then by [BBP09, Prop. 1.1] the real stable base locus coincide with the usual stable base locus defined as in [Laz04, Rmk. 2.1.24].

Following [ELMNP06] we introduce two perturbations of the stable base locus: the augmented and the restricted base locus.

Definition 2.3.4. (cf. [ELMNP06, Def. 1.2]). Let X be a normal projective variety, let $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The *augmented base locus* of D is

$$\mathbb{B}_+(D) := \bigcap_{\substack{E \in \mathrm{Div}_{\mathbb{R}}(X), E \geq 0 \\ D - E \text{ ample}}} \mathrm{Supp}(E),$$

if D is big; otherwise $\mathbb{B}_+(D) := X$ by convention.

Definition 2.3.5. (cf. [ELMNP06, Def. 1.12]). Let X be a normal projective variety, let $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The *restricted base locus* of D is

$$\mathbb{B}_-(D) := \bigcup_{\substack{A \in \mathrm{Div}_{\mathbb{R}}(X) \\ A \text{ ample}}} \mathbb{B}(D + A).$$

In the rest of the section we gather some lemmas involving augmented and restricted base locus that we will use in the next chapters.

Lemma 2.3.6. *Let X be a normal projective variety. Let $D \in Div_{\mathbb{R}}(X)$. Let $\{A_m\}_{m \geq 1}$ be any sequence of ample $\mathbb{R}$-Cartier divisors such that $\|A_m\| \to 0$ in $N^1(X)_{\mathbb{R}}$. Then*

$$\mathbb{B}_-(D) = \cup_{m \geq 1} \mathbb{B}_-(D + A_m).$$

Proof. By definition $\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A)$ and hence it can be easily seen that we also have that $\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}_-(D + A)$. For any A ample divisor let m_A be sufficiently large so that $A - A_{m_A}$ is still ample. Hence $\mathbb{B}_-(D + A) = \mathbb{B}_-(D + A_{m_A} + A - A_{m_A}) \subseteq \mathbb{B}_-(D + A_{m_A})$. □

The following lemma allows us to turn big and nef divisors on LC pairs, whose augmented base locus has good properties with respect to the pair itself, into ample ones, by perturbing the pair without worsening its singularities:

Lemma 2.3.7. *Let (X, Δ) be a LC pair. Let $L \in Div_{\mathbb{Q}}(X)$ be big and nef and such that $\mathbb{B}_+(L)$ does not contain any LC center of the pair (X, Δ).*
Then there exists an effective Cartier divisor Γ on X, not containing any LC center of (X, Δ) in its support, and a rational number $\lambda_0 > 0$ such that $Bs(|\Gamma|) = \mathbb{B}_+(L)$ and for each $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$, we have that

1. *$L - \lambda\Gamma \in Div_{\mathbb{Q}}(X)$ and it is ample;*

2. *$(X, \Delta + \lambda\Gamma)$ is an LC pair;*

3. *$CLC(X, \Delta + \lambda\Gamma) = CLC(X, \Delta)$.*

4. *$(X, \Delta + \lambda\Gamma)$ is DLT if (X, Δ) is such.*

Moreover Γ can be chosen generically in its linear series.

Proof. By [ELMNP06, Prop. 1.5] there exists H, an ample $\mathbb{Q}$-divisor on X, such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H).$$

Moreover there exists $m_0 \in \mathbb{N}$ such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Then $Bs(|m_0(L - H)|)$ does not contain LC centers of the pair (X, Δ). Hence, as $CLC(X, \Delta)$ is a finite set (see for example [Kol07, par. 8.1]), we can choose a general divisor Γ in $|m_0(L - H)|$ such that $\text{Supp}(\Gamma)$ does not contain LC centers of (X, Δ).
Thus, by lemma 2.2.26, there exists $\lambda_1 > 0$ such that if $\lambda \in \mathbb{Q} \cap (0, \lambda_1)$, then $(X, \Delta + \lambda\Gamma)$ is an LC pair and $CLC(X, \Delta + \lambda\Gamma) = CLC(X, \Delta)$.
Moreover, by part 2 of the same lemma, it follows that $(X, \Delta + \lambda\Gamma)$ is DLT if (X, Δ) is such.
Now, for all $\lambda \in \mathbb{Q} \cap (0, 1]$, we have that

$$L - \lambda\Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if $\lambda \leq \frac{1}{m_0}$ because L is nef and H is ample.
Thus, if we define $\lambda_0 = \min\{\lambda_1, \frac{1}{m_0}\}$, we are done.

□

Lemma 2.3.8. *Let $\mu : Y \to X$ be a proper birational morphism of normal projective varieties and let $D \in Div_{\mathbb{R}}(X)$ be a big divisor. Then*

$$\mathbb{B}_+(\mu^*(D)) = exc(\mu) \cup \mu^{-1}(\mathbb{B}_+(D)).$$

Proof. See [BBP09, proposition 1.5].

$\square$

Lemma 2.3.9. *Let X be a normal variety and let $P \in Div_{\mathbb{Q}}(X)$ be big and nef.*

1. *If $V \subseteq X$ is a subvariety such that $\mathbb{B}_+(P) \not\supseteq V$, then $P_{|_V}$ is big.*

2. *If E is a prime divisor on X such that $P_{|_E}$ is big, then $\mathbb{B}_+(P) \not\supseteq E$.*

Proof. **1.)** As $V \not\subseteq \mathbb{B}_+(P)$ we can find an ample divisor $A \in \text{Div}_{\mathbb{R}}(X)$ and an effective divisor $F \in \text{Div}_{\mathbb{R}}(X)$ such that $P = A + F$ and $V \not\subseteq \text{Supp}(F)$. Then $P_{|_V} = A_{|_V} + F_{|_V}$, where $A_{|_V}$ is ample and $F_{|_V}$ is effective. Thus $P_{|_V}$ is big.

2.) Let $\mu : X' \to X$ be a resolution of singularities of X and let $\widetilde{E}$ be the strict transform of E through μ. Then

$$(\mu^*(P)^{n-1} \cdot \widetilde{E}) = (P^{n-1} \cdot E) = (P_{|_E})^{n-1} > 0,$$

because $P_{|_E}$ is big and nef.

Now, by [Laz04, 10.3.6], as $\mu^*(P)$ is big and nef, $Null(\mu^*(P))$ is a closed proper subset of X'. Again by [Laz04, 10.3.6], $\widetilde{E}$ is not an irreducible component of $Null(\mu^*(P))$.

But $dim\widetilde{E} = dimX' - 1 \geq dimNull(\mu^*(P))$, so that $\widetilde{E}$ cannot be strictly contained in an irreducible component of $Null(\mu^*(P))$. Thus $\widetilde{E} \not\subseteq Null(\mu^*(P))$. Thanks to Nakamaye's theorem (see [Laz04, theorem 10.3.5]) this implies that $\widetilde{E} \not\subseteq \mathbb{B}_+(\mu^*(P))$.

Then, by lemma 2.3.8, $\widetilde{E} \not\subseteq \mu^{-1}(\mathbb{B}_+(P))$, so that $E = \mu(\widetilde{E}) \not\subseteq \mathbb{B}_+(P)$. $\square$

Remark 2.3.10. The first part of the last lemma implies that if (X, Δ) is a pair and P is a big and nef divisor on X such that $\mathbb{B}_+(P)$ does not contain any LC center of the pair (X, Δ) then P is logbig with respect to the pair (X, Δ).

2.4 Multiplier ideals on singular varieties

We recall here the standard definition of multiplier ideal associated to a pair and a divisor or a line bundle (see [Laz04, Section 9.3 G])

Definition 2.4.1. *(see [Laz04, Definition 9.3.56])* Let (X, Δ) be a pair, let $D \in \text{Div}_{\mathbb{Q}}(X)$ and let us denote by $\mu : Y \to X$ a log-resolution of the pair (X, Δ) such that $\mu^*(D) + \text{divexc}(\mu)$ is SNCS. Then for every prime divisor E on Y we can consider the discrepancy $a(E) = a(E, X, \Delta) \in \mathbb{Q}$ and uniquely defined rational numbers $b(E)$ such that $\mu^*(-D) = \sum b(E)E$, so that

$$K_Y - \mu^*(K_X + \Delta + D) \equiv \sum (a(E) + b(E))E.$$

We define the *multiplier ideal* of D on the pair (X, Δ) as the sheaf

$$\mathcal{J}((X,\Delta);D) = \mu_*\mathcal{O}_Y(\sum \ulcorner a(E) + b(E)\urcorner E).$$

With a slight abuse of notation we can also write

$$\mathcal{J}((X,\Delta);D) = \mu_*\mathcal{O}_Y(K_Y - [\mu^*(K_X + \Delta + D)]).$$

Note that we will use the standard notation $\mathcal{J}(X,\Delta) := \mathcal{J}((X,\Delta);0)$.

Definition 2.4.2. Let (X,Δ) be a pair, let c be a non negative rational number and let $|L|$ be a non empty linear series on X. Consider $\mu : Y \to X$ a log-resolution of the pair (X,Δ) and of the linear series $|L|$, so that we can write

$$\mu^*|L| = |W| + F,$$

where $|W|$ is a base-point free linear series and $F + \mathrm{divexc}(\mu)$ is SNCS.
For every prime divisor E on Y we can consider the discrepancy $a(E) = a(E,X,\Delta) \in \mathbb{Q}$ and uniquely defined rational numbers $b(E)$ such that $-F = \sum b(E)E$.
We define the *multiplier ideal* associated to c and $|L|$ on the pair (X,Δ) as

$$\mathcal{J}((X,\Delta);c|L|) = \mu_*\mathcal{O}_Y(\sum \ulcorner a(E) + cb(E)\urcorner).$$

With a slight abuse of notation we can also write

$$\mathcal{J}((X,\Delta);c|L|) = \mu_*\mathcal{O}_Y(K_Y - [\mu^*(K_X + \Delta) + cF]).$$

Note that we will use the standard notation $\mathcal{J}(X,c|L|) := \mathcal{J}((X,0);c|L|)$ and we use the convention that $\mathcal{J}((X,\Delta);c|L|) = 0$ if $|L| = \emptyset$.

Asymptotic multiplier ideals

In this subsection we define asymptotic multiplier ideals on (possibly singular) pairs. See [Laz04, chapter 11] for details about the same construction in the case of smooth varieties.

Lemma 2.4.3. *(cf. [Laz04, Lemma 11.1.1]) Let (X,Δ) be an effective pair, $c > 0$ a rational number, $p \in \mathbb{N}$ and L a Cartier divisor on X such that $\kappa(X,L) \geq 0$. For every integer $k \geq 1$ one has the inclusion*

$$\mathcal{J}\left((X,\Delta);\frac{c}{p}|pL|\right) \subseteq \mathcal{J}\left((X,\Delta);\frac{c}{pk}|pkL|\right).$$

Proof. We may assume that $|pL| \neq \emptyset$ (and hence also $|pkL| \neq \emptyset$). Choose $\mu : Y \to X$ a log-resolution of the pair (X,Δ) and of the linear series $|pL|$ and $|pkL|$. Hence we can write

$$\mu^*(|pL|) = |W_p| + F_p,$$

$$\mu^*(|pkL|) = |W_{pk}| + F_{pk},$$

where W_p and W_{pk} are base-point free linear series. Let us write $F_p = -\sum b_p(E)E$ and $F_{pk} = -\sum b_{pk}(E)E$, where the sums are taken on all prime

divisors on Y. Note that $k \cdot F_p \geq F_{pk}$, so that $b_p(E) \leq \frac{b_{pk}(E)}{k}$ for every prime divisor E.

Thus, if we denote by $a(E) = a(E, X, \Delta)$ the discrepancies of the pair (X, Δ), we find that

$$\mathcal{J}\left((X, \Delta); \frac{c}{p}|pL|\right) = \mu_* \mathcal{O}_Y\left(\sum \lceil a(E) + \frac{c}{p} b_p(E) \rceil E\right) \subseteq$$

$$\subseteq \mu_* \mathcal{O}_Y\left(\sum \lceil a(E) + \frac{c}{pk} b_{pk}(E) \rceil E\right) = \mathcal{J}\left((X, \Delta); \frac{c}{pk}|pkL|\right).$$

$\square$

Lemma 2.4.4. *Let (X, Δ) be a pair, $c > 0$ a rational number and L a Cartier divisor on X such that $\kappa(X, L) \geq 0$. Then the family of ideals*

$$\left\{\mathcal{J}((X, \Delta); \frac{c}{p}|pL|)\right\}_{p \in \mathbb{N}}$$

has a unique maximal element.

Proof. We first consider the case when (X, Δ) is an effective pair. In this case each of the sheaves $\mathcal{J}((X, \Delta); \frac{c}{p}|pL|)$ is an actual ideal sheaf, i.e. it is contained in $\mathcal{O}_X$. Hence the existence of at least one maximal member follows from the ascending chain condition on ideals. On the other hand if $\mathcal{J}((X, \Delta); \frac{c}{p}|pL|) = \mathcal{J}((X, \Delta); \frac{c}{q}|qL|)$ for some $p, q \in \mathbb{N}$, then, by lemma 2.4.3 they must both coincide with $\mathcal{J}((X, \Delta); \frac{c}{pq}|pqL|)$.

Let us consider now the general case. Let A be an ample Cartier divisor on X and take $m \in \mathbb{N}$ sufficiently high such that $\mathcal{O}_X(mA) \otimes \mathcal{O}_X([\Delta]) \simeq \mathcal{O}_X(mA + [\Delta])$ is a globally generated coherent sheaf. This implies that there exists a Cartier divisor $E \sim mA$ such that $E + [\Delta] \geq 0$, so that $E + \Delta \geq 0$ as well.

Hence if we consider consider the family of ideals

$$\left\{\mathcal{J}((X, \Delta + E); \frac{c}{p}|pL|)\right\}_{p \in \mathbb{N}},$$

by the previous part of the proof we know that it has a unique maximal element, whence there exists $p_0 \in \mathbb{N}$ such that for every integer $p > 0$

$$\mathcal{J}((X, \Delta + E); \frac{c}{p}|pL|) \subseteq \mathcal{J}((X, \Delta + E); \frac{c}{p_0}|p_0L|).$$

Now note that for every integer $p > 0$ we have that $\mathcal{J}((X, \Delta + E); \frac{c}{p}|pL|) = \mathcal{J}((X, \Delta); \frac{c}{p}|pL|) \otimes \mathcal{O}_X(-E)$ by [Laz04, Prop. 9.2.31] (see also pag.185, volume 2), so that we have

$$\mathcal{J}((X, \Delta); \frac{c}{p}|pL|) \otimes \mathcal{O}_X(-E) \subseteq \mathcal{J}((X, \Delta); \frac{c}{p_0}|p_0L|) \otimes \mathcal{O}_X(-E).$$

Thus, tensoring by $\mathcal{O}_X(E)$, we get that

$$\mathcal{J}((X, \Delta); \frac{c}{p}|pL|) \subseteq \mathcal{J}((X, \Delta); \frac{c}{p_0}|p_0L|),$$

whence $\mathcal{J}((X, \Delta); \frac{c}{p_0}|p_0L|)$ is the unique maximal element of our family. $\square$

Definition 2.4.5. Let (X, Δ) be a pair, consider a complete linear series $|L|$ on X such that $\kappa(X, L) \geq 0$ and a rational number $c > 0$. The *asymptotic multiplier ideal sheaf* associated to c and $|L|$ on the pair (X, Δ), denoted by

$$\mathcal{J}((X, \Delta); c\|L\|),$$

is defined as the unique maximal member in the family of ideals $\left\{ \mathcal{J}((X, \Delta); \frac{c}{p} |pL|) \right\}$.

Note that $\mathcal{J}((X, \Delta); c\|L\|) \subseteq \mathcal{O}_X$ if the pair (X, Δ) is effective.

In the following definition we associate to every effective pair (X, Δ) two b-divisors $\mathbf{A}(\Delta)$ and $\mathbf{L}(\Delta)$. This will be useful every time we consider birational transformations of pairs.

Definition 2.4.6. Let (X, Δ) be an effective pair. We define the b-divisors $\mathbf{A}(\Delta)$ and $\mathbf{L}(\Delta)$:
For every proper birational morphism $f : Z \to X$, if E and F are effective Weil $\mathbb{Q}$-divisors on Z without common components such that

$$K_Z + E \equiv f^*(K_X + \Delta) + F \quad \text{and} \quad f_*(E - F) = \Delta,$$

we put the trace $\mathbf{A}(\Delta)_Z = E - F$ and the trace $\mathbf{L}(\Delta)_Z = E$.

Lemma 2.4.7 (Birational transformation rule for asymptotic multiplier ideals)**.** *(cf. [Laz04, Prop. 9.3.62]). Let (X, Δ) be a pair and let $f : Y \to X$ be a proper birational morphism. If D is any Cartier divisor on X such that $\kappa(X, D) \geq 0$, then*

$$\mathcal{J}((X, \Delta); \|D\|) = f_* \left(\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \|f^*D\|) \right)$$

Proof. By definition of asymptotic multiplier ideal we have that $\mathcal{J}((X, \Delta); \|D\|)$ $= \sup_k \{ \mathcal{J}((X, \Delta); \frac{1}{k}|kD|) \}$. When $k \geq 2$ and $|kD| \neq \emptyset$, by [Laz04, Prop. 9.2.26] (see also [Laz04, volume 2, p. 185]) we have that $\mathcal{J}((X, \Delta); \frac{1}{k}|kD|) = \mathcal{J}((X, \Delta); \frac{1}{k}D_k)$, where D_k is a general element in $|kD|$. By [Laz04, Prop. 9.3.62] we have that, for every $k \in \mathbb{N}$ such that $|kD| \neq \emptyset$, $\mathcal{J}((X, \Delta); \frac{1}{k}D_k) = f_* \left(\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \frac{1}{k}f^*D_k) \right)$. Since D_k is general in $|kD|$ then f^*D_k is general in $|kf^*D|$, so that $\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \frac{1}{k}f^*D_k) = \mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \frac{1}{k}|kf^*D|)$.
Since f_* preserves inclusions we also have that

$$\sup_k \left\{ f_* \left(\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \frac{1}{k}|kf^*D|) \right) \right\} = f_* \left(\sup_k \left\{ \mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \frac{1}{k}|kf^*D|) \right\} \right).$$

Hence the lemma follows. □

2.5 Q-CKM Zariski decompositions

For our purposes we need to extend the classical definition of Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki to some non $\mathbb{Q}$-Cartier cases. Hence we will adopt the following definition:

Definition 2.5.1. Let X be a normal projective variety and let D be a Weil $\mathbb{Q}$-divisor on X. We say that D admits a $\mathbb{Q}$-Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a $\mathbb{Q}$-CKM Zariski decomposition) $D = P + N$ if

- $P \in \mathrm{Div}_{\mathbb{Q}}(X)$ and N is a Weil $\mathbb{Q}$-divisor;

- P is nef and N is effective;

- There exists an integer $k > 0$ such that kP is Cartier, kD is an integral Weil divisor and for every $m \in \mathbb{N}$ we have an isomorphism

$$H^0(X, \mathcal{O}_X(kmP)) \simeq H^0(X, \mathcal{O}_X(kmD)).$$

Note that kmD might not be a Cartier divisor but it still makes sense to consider the reflexive sheaf $\mathcal{O}_X(kmD)$ and its H^0 (see definition 2.1.3). In particular if D is $\mathbb{Q}$-Cartier and it admits a CKM Zariski decomposition in the usual sense (see for example [Cut86]) with P and N in $\mathrm{Div}_{\mathbb{Q}}(X)$ then it admits a $\mathbb{Q}$-CKM Zariski decomposition in our sense.

Lemma 2.5.2. *Let X be a normal projective variety and let $D \in Div_{\mathbb{Q}}(X)$. If $D = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and $D' \sim_{\mathbb{Q}} D$ then $D' = P' + N$ is a $\mathbb{Q}$-CKM Zariski decomposition, where $P' = (D' - D) + P$.*

Proof. As $D' \sim_{\mathbb{Q}} D$ there exists an integer $k_0 \in \mathbb{N}$ such that $k_0 D' \sim k_0 D$, so that $k_0 P' \sim k_0 P$, in particular note that P' is nef. Also by our definition of $\mathbb{Q}$-CKM Zariski decomposition there exists $m_0 \in \mathbb{N}$ such that $m_0 D$ and $m_0 P$ are Cartier divisors and $H^0(X, mm_0 D) \simeq H^0(X, mm_0 P)$ for every $m \in \mathbb{N}$. But for every $m \in \mathbb{N}$ we have that $mk_0 m_0 D' \sim mk_0 m_0 D$ and $mk_0 m_0 P' \sim mk_0 m_0 P$, so that $H^0(X, mk_0 m_0 D') \simeq H^0(X, mk_0 m_0 D) \simeq H^0(X, mk_0 m_0 P) \simeq H^0(X, mk_0 m_0 P')$. $\qquad\square$

The following technical lemma will be very useful to treat the case when a birational pullback of a given divisor admits a Zariski decomposition.

Lemma 2.5.3. *Let (X, Δ) be an effective pair, take $D \in Div_{\mathbb{Q}}(X)$ and $a \in \mathbb{Q}$. If there exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition, then there exist Weil $\mathbb{Q}$-divisors D', P', N', Δ_Z such that*

- $\Delta_Z = \mathbf{L}(\Delta)_Z$ *if $a \geq 0$ and $\Delta_Z = \mathbf{L}(\Delta)_Z - aN$ if $a < 0$, so that, in any case, Δ_Z is effective;*

- $D' = P' + N'$ *is a $\mathbb{Q}$-CKM Zariski decomposition;*

- $\Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN$, *so that in particular $(Z, \Delta_Z - N')$ is a pair;*

- $P' = bP$ *for some $b > 0$;*

- $t_0 P' - (K_Z + \Delta_Z - N') = P + f^*(aD - (K_X + \Delta))$ *for some $t_0 \in \mathbb{Q}$.*

*In particular if D is big and $aD - (K_X + \Delta)$ is nef **or** if $aD - (K_X + \Delta)$ is big and nef, then*

$$t_0 P' - (K_Z + \Delta_Z - N')$$

is big and nef.

Proof. Define $a' = -\min\{0, a\}$ and $a'' = \max\{0, a\}$, so that $a' \geq 0$, $a'' \geq 0$ and $a = a'' - a'$. Moreover we can write $\mathbf{A}(\Delta)_Z = A^+ - A^-$, where A^+ and A^- are effective and without common components, so that A^- is f-exceptional. We define $\Delta_Z := A^+ + a'N$, $N' := a''N + A^-$, $P' := (a'' + 1)P$ and $D' := P' + N'$. Then it is immediate to compute Δ_Z and to see that $\Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN$ and P' is a positive rational multiple of P.

Moreover P' is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, N' is effective and by using the hypothesis and Fujita's lemma (see for example [KMM85, lemma 1.3.2]) we can see that there exists $k' \in \mathbb{N}$ such that

$$H^0(Z, \mathcal{O}_Z(k'mP')) \simeq H^0(Z, \mathcal{O}_Z(k'mD'))$$

for every $m \in \mathbb{N}$.

In fact we know that there exists $k \in \mathbb{N}$ such that kD and kP are Cartier and $H^0(Z, \mathcal{O}_Z(f^*(kmD))) \simeq H^0(Z, \mathcal{O}_Z(kmP))$ for every $m \in \mathbb{N}$. Let $c \in \mathbb{N}$ be such that $l := c(a'' + 1) \in \mathbb{N}$ and cN' is integral and define $k' = kc$, so that kP' is Cartier and kN' is an integral Weil divisor.

Then for every $m \in \mathbb{N}$

$$H^0(Z, \mathcal{O}_Z(k'mP')) \simeq H^0(Z, \mathcal{O}_Z(kmlP)) \simeq H^0(Z, \mathcal{O}_Z(f^*(kmlD))) \simeq$$
$$\simeq H^0(Z, \mathcal{O}_Z(f^*(kmlD) + kmlA^-))),$$

by Fujita's lemma, because A^- is effective and f-exceptional. On the other hand

$$H^0(Z, \mathcal{O}_Z(k'mP')) \subseteq H^0(Z, \mathcal{O}_Z(k'mD')) = H^0(Z, \mathcal{O}_Z(kmlP + kcma''N + kcmA^-)) \subseteq H^0(Z, \mathcal{O}_Z(f^*(kmlD) + kmlA^-))).$$

Hence $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition.
Now note that

$$P + f^*(aD - (K_X + \Delta)) = (a+1)P + aN - (K_Z + \mathbf{A}(\Delta)_Z) =$$

$$= (a+1)P + a''N - (K_Z + \mathbf{A}(\Delta)_Z + a'N) = (a+1)P - (K_Z + \Delta_Z - N') =$$

$$= t_0 P' - (K_Z + \Delta_Z - N'),$$

where $t_0 = \frac{a+1}{a''+1} \in \mathbb{Q}$.
$\square$

The following lemma shows that the augmented base locus has good properties with respect to $\mathbb{Q}$-CKM Zariski decompositions.

Lemma 2.5.4. *Let X be a normal projective variety, let $D \in Div_{\mathbb{Q}}(X)$ and suppose there exists a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N.$$

Then $\mathbb{B}_+(D) = \mathbb{B}_+(P)$ and $Supp(N) \subseteq \mathbb{B}_+(D)$.

Proof. If D is not big then P is also not big, so that $\mathbb{B}_+(P) = \mathbb{B}_+(D) = X$. We can thus assume that D is big.
Then the decomposition $D = P + N$ is also a Fujita-Zariski decomposition (see for example [Pro02, proposition 7.4]). Hence, given an ample $\mathbb{R}$-divisor A such

that $A \leq D$, by definition of Fujita-Zariski decomposition we have that $A \leq P$, so that $D - A \geq N$. Now

$$\mathbb{B}_+(D) = \bigcap \mathrm{Supp}(D - A),$$

where the intersection is taken over all the ample $\mathbb{R}$-divisors A such that $A \leq D$. Then

$$\mathbb{B}_+(D) = \bigcap_{A \leq D} \mathrm{Supp}(D - A) = \bigcap_{A \leq D} (\mathrm{Supp}(D - A - N) \cup \mathrm{Supp}(N)) =$$

$$= \mathrm{Supp}(N) \cup \bigcap_{A \leq P} \mathrm{Supp}(P - A) = \mathrm{Supp}(N) \cup \mathbb{B}_+(P).$$

In order to complete the proof it suffices to prove that $\mathrm{Supp}(N) \subseteq \mathbb{B}_+(P)$:

We first prove this in the case when X is smooth:

In particular, denoted by n the dimension of X, we will show that for every divisorial component F of the support of N we have that $(P^{n-1} \cdot F) = 0$. This will imply, as is well known, that $F \subseteq \mathbb{B}_+(P)$, so that $\mathrm{Supp}(N) \subseteq \mathbb{B}_+(P)$.

Let $a \in \mathbb{N}$ be such that aD, aP and aN are Cartier divisors and $H^0(X, \mathcal{O}_X(maD)) \simeq H^0(X, \mathcal{O}_X(maP))$ for all $m \in \mathbb{N}$.

Let us consider now, for all $m \in \mathbb{N}$, the long exact sequence

$$0 \to H^0(X, \mathcal{O}_X(maP)) \to H^0(X, \mathcal{O}_X(maP+F)) \to H^0(F, \mathcal{O}_F((maP+F)_{|_F})) \to$$

$$\to H^1(X, \mathcal{O}_X(maP)) \to .$$

By the choice of a, as $0 \leq F \leq maN$, we have that $H^0(X, \mathcal{O}_X(maP)) \simeq H^0(X, \mathcal{O}_X(maP + F))$.

This implies that the map $H^0(F, \mathcal{O}_F((maP+F)_{|_F})) \to H^1(X, \mathcal{O}_X(maP))$ is an injection, so that

$$h^0(F, \mathcal{O}_F((maP + F)_{|_F})) \leq h^1(X, \mathcal{O}_X(maP))$$

for all $m \in \mathbb{N}$.

But the bigness of D implies that P is also big (because $\kappa(X, P) = \kappa(X, D) = n$). Then, by a theorem of Fujita, there exists a constant $C > 0$ such that $h^1(X, \mathcal{O}_X(maP)) \leq C \cdot m^{n-2}$ for all $m \in \mathbb{N}$ (see [Fta83, corollary 6.7]).

Hence, for every $m \in \mathbb{N}$,

$$h^0(F, \mathcal{O}_F((maP + F)_{|_F})) \leq C \cdot m^{n-2}.$$

On the other hand by asymptotic Riemann-Roch (see [Laz04, 1.4.41]) we get that

$$h^0(F, \mathcal{O}_F((maP + F)_{|_F})) = \frac{(aP_{|_F})^{n-1} \cdot m^{n-1}}{(n-1)!} + O(m^{n-2}).$$

Therefore $0 = (aP_{|_F})^{n-1} = a(P^{n-1} \cdot F)$ and we are done.

In the general case take $\mu : X' \to X$ to be a resolution of singularities of X. Note that $\mu^*(D) = \mu^*(P) + \mu^*(N)$ is a $\mathbb{Q}$-CKM Zariski decomposition of $\mu^*(D)$, so that, by the previous case, we get that $\mathrm{Supp}(\mu^*(N)) \subseteq \mathbb{B}_+(\mu^*(P))$.

Now take F to be a prime divisor contained in the support of N. Then

$$\mu_*^{-1}(F) \subseteq \operatorname{Supp}(\mu^*(N)) \subseteq \mathbb{B}_+(\mu^*(P)) = \operatorname{exc}(\mu) \cup \mu^{-1}(\mathbb{B}_+(P))$$

thanks to lemma 2.3.8.
But $\mu_*^{-1}(F) \not\subseteq \operatorname{exc}(\mu)$ because it is a strict transform, so that, thanks to its irreducibility, $\mu_*^{-1}(F) \subseteq \mu^{-1}(\mathbb{B}_+(P))$, that is $F = \mu(\mu_*^{-1}(F)) \subseteq \mathbb{B}_+(P)$ and we are done. $\qquad\square$

2.6 Asymptotic and numerical orders of vanishing

Let X be a normal projective variety. We can consider a geometric discrete valuation v on X as in [dFH09, section 2].
In other words a *geometric discrete valuation* on X is a discrete valuation of the function field of X of the form $v = q \operatorname{val}_F$, where $q > 0$ is an integer and val_F is the valuation associated to a prime divisor F over X.
Suppose $\mu : X' \to X$ is a projective birational morphism such that $F \subseteq X'$. We denote by $c_X(v)$ the *center* of the valuation v on X, which corresponds to the image on X of the divisor F.
Given $D \in \operatorname{Div}_\mathbb{R}(X)$ we define the valuation $v(D)$ of D as q times the coefficient of F in the divisor $\mu^*(D)$.
If $|L|$ is a non-empty linear series on X we define the valuation $v(|L|)$ as the valuation of a general divisor in $|L|$.
Moreover if $\mathcal{J} \subseteq \mathcal{O}_X$ is a non-trivial ideal sheaf, we define the valuation $v(\mathcal{J})$ of $\mathcal{J}$ as

$$v(\mathcal{J}) = \min\{v(\phi) \mid \phi \in \mathcal{J}(U),\ U \cap c_X(v) \neq \emptyset\}.$$

Note that for every linear series $|L| \neq \emptyset$ we have that $v(|L|) = v(b(|L|))$, where $b(|L|)$ is the base ideal of the linear series $|L|$.

Numerical orders of vanishing: the non-nef locus

If $D \in \operatorname{Div}_\mathbb{R}(X)$ is big we define, as in [BBP09], the *numerical order of vanishing* of D along v as

$$v_{\operatorname{num}}(D) := \inf\{v(E), E \in |D|_\equiv\}.$$

When $D \in \operatorname{Div}_\mathbb{R}(X)$ is pseudoeffective we set

$$v_{\operatorname{num}}(D) := \lim_{\varepsilon \to 0} v_{\operatorname{num}}(D + \varepsilon A),$$

with A ample. It is easy to see that the limit exists and the definition does not depend on the choice of the ample divisor A. See [BBP09, §1.3].

Definition 2.6.1. (cf. [BBP09, Def. 1.7]). Let X be a normal projective variety, let $D \in \operatorname{Div}_\mathbb{R}(X)$ and let us denote by $c_X(v)$ the center on X of a given geometric discrete valuation v on X. The *non-nef locus* of D is

$$\operatorname{NNef}(D) := \bigcup\{c_X(v), v_{\operatorname{num}}(D) > 0\}$$

if D is pseudoeffective. When D is not pseudoeffective we put $\operatorname{NNef}(D) = X$.

Note that, as the name itself suggests, $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is nef if and only if $\mathrm{NNef}(D) = \emptyset$, i.e., if and only if $v_{\mathrm{num}}(D) = 0$ for every geometric discrete valuation v on X (see [BBP09, §1.3]).

Note also that the notation is not univocal and the restricted base locus itself, $\mathbb{B}_-(D)$, is called "non-nef locus" on some papers.

Lemma 2.6.2. *Let X be a normal projective variety and let $D \in Div_{\mathbb{R}}(X)$. Let $\{A_m\}_{m \geq 1}$ be any sequence of ample $\mathbb{R}$-Cartier divisors such that $\|A_m\| \to 0$ in $N^1(X)_{\mathbb{R}}$. Then*

$$\mathrm{NNef}(D) = \cup_{m \geq 1} \mathrm{NNef}(D + A_m).$$

Proof. Since clearly $\mathrm{NNef}(D) = \cup_{A \text{ ample}} \mathrm{NNef}(D + A)$, then we can just apply the proof of lemma 2.3.6, substituting $\mathbb{B}_-$ with NNef. $\square$

Asymptotic orders of vanishing: the non nef-abundant locus

Definition 2.6.3. (cf. [ELMNP06, Def. 2.2] and [Cac08, Def. 5.20]). Let X be a normal projective variety and let D be a Cartier divisor such that $\kappa(X, D) \geq 0$. Let v be a geometric discrete valuation on X. If $e = e(D)$ is the exponent of D (cf. [Laz04, Def. 2.1.1]), we define the *asymptotic order of vanishing* of D along v as

$$v(\|D\|) := \lim_{p \to \infty} \frac{v(|peD|)}{pe}.$$

Remark 2.6.4. The above definition can be generalized to $\mathbb{Q}$-Cartier divisors and it can be easily seen that the limit is also the inf, so that for every divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ we have that

$$v(\|D\|) = \inf\{v(E), E \in |D|_{\mathbb{Q}}\}.$$

See [ELMNP06] for details.

Remark 2.6.5. If D is a big $\mathbb{Q}$-Cartier divisor then by [ELMNP06, Lemma 3.3] we have that $v(\|D\|) = v_{\mathrm{num}}(D) = \inf\{v(E), E \in |D|_{\mathbb{R}}\}$. More generally when D is *abundant* we have that $v(\|D\|) = v_{\mathrm{num}}(D)$ by [Leh11a, Prop. 6.4].
See [Nak04, Def. V 2.23] for the definition of abundant divisor, see [Leh11a] and [Leh11b] for some equivalent definitions and remarks.
In general, if D is an effective $\mathbb{Q}$-Cartier divisor then by definition $v(\|D\|) \geq v_{\mathrm{num}}(D)$ but equality does not hold in general. Take for example a nef irreducible curve D on a smooth surface as in [Rus09, Ex. 1] and set $v := ord_D$. We have that $v_{\mathrm{num}}(D) = 0$ by the nefness of D, while $v(\|D\|) = 1$.

Recall that given a nef Cartier divisor D on a normal projective variety X we can define its *numerical dimension* as $\nu(D) := \max\{k \in \mathbb{N} : D^k \not\equiv 0\}$. If D is nef we have that it is *abundant* if and only if the Kodaira dimension of D equals its numerical dimension (see [Nak04, Prop 2.22 (5)]).
In this paper we will only deal with abundance of nef divisors. The following lemma is a translation of [Rus09, Thm. 1] in terms of discrete valuations:

Lemma 2.6.6. *Let D be a Cartier divisor on a projective normal variety X. Then D is nef and abundant if and only if $v(\|D\|) = 0$ for every geometric discrete valuation v on $\mathbb{C}(X)$.*

Proof. If D is nef and abundant, then by [MR97, Lemma 1] (see also [Kaw85, Prop. 2.1]) there exist a birational morphism $f : Z \to X$, where Z is a smooth projective variety, an integer $k_0 > 0$ and a Cartier divisor N on Z such that $B_m := mk_0 f^*(D) - N$ is semiample for every $m \in \mathbb{N}$. Moreover we can suppose that k_0 is a multiple of the exponent $e(D)$.

Now, given any geometric discrete valuation v on $\mathbb{C}(X)$, we have that $v(\|D\|) = v(\|f^*D\|)$ by [ELMNP06]. But

$$v(\|f^*D\|) = \frac{1}{k_0} v(\|f^*(k_0 D)\|) = \frac{1}{k_0} \cdot \lim_{m \to +\infty} \frac{v(|f^*(mk_0 D)|)}{m} \leq$$

$$\leq \frac{1}{k_0} \cdot \lim_{m \to +\infty} \left(\frac{v(|B_m|)}{m} + \frac{v(|N|)}{m} \right) = 0,$$

because $v(|B_m|) = 0$ by the semiampleness of B_m and $v(|N|)$ does not depend on m.

Now suppose $v(\|D\|) = 0$ for every geometric discrete valuation v on $\mathbb{C}(X)$. If $\mu : X' \to X$ is a desingularization of X, then $v(\|\mu^*D\|) = v(\|D\|) = 0$ for every v. On the other hand, as X' is smooth, we have that for every (not necessarily closed) point $x \in X'$ we can consider the geometric discrete valuation ord_x given by the order of vanishing at x, so that $ord_x(\|\mu^*D\|) = 0$ for every $x \in X'$, which is equivalent to saying that D is almost base-point free (cf. [Rus09, Def. 1]). By [Rus09, Thm. 1] this implies that μ^*D is nef and abundant, which in turn implies that D is nef and abundant. $\qquad\square$

The previous lemma suggests the following definition:

Definition 2.6.7. Let X be a normal projective variety and let D be a Cartier divisor such that $\kappa(X, D) \geq 0$. The *non nef-abundant locus* of D is

$$\mathrm{NNA}(D) := \bigcup_{v \in V} \{c_X(v) : v(\|D\|) > 0\},$$

where V is the set of all geometric discrete valuations on $\mathbb{C}(X)$ and, for any $v \in V$, $c_X(v)$ is the center of v on X.

Remark 2.6.8. Trivially, by lemma 2.6.6, $\mathrm{NNA}(D) = \emptyset$ if and only if D is nef and abundant. When D is a big divisor $\mathrm{NNef}(D) = \mathrm{NNA}(D)$ by remark 2.6.5. By the same remark we see that if D is effective then in general $\mathrm{NNA}(D) \supseteq \mathrm{NNef}(D)$ and that equality does not hold in general if D is not big.

Lemma 2.6.9. *Let X be a normal projective variety and let D be a Cartier divisor such that $\kappa(X, D) \geq 0$. Let $f : X' \to X$ be any birational map from a normal projective variety X'. Then $f(\mathrm{NNA}(f^*D)) = \mathrm{NNA}(D)$.*

Proof. The lemma follows from the easy fact that for any discrete geometric valuation v on $\mathbb{C}(X)$ we have that $v(\|f^*D\|) = v(\|D\|)$ (cf. [ELMNP06]).

$\qquad\square$

Chapter 3

On the semiampleness of the positive part of CKM Zariski decompositions

In this chapter we work to some extensions of theorem 1.0.1 to the case when the given pair (X, Δ) is not KLT. In particular the main results appear in section 3.3, where we prove Conjecture 1b for DLT pairs, in section 3.7 and 3.8, where we consider lower dimensional cases and in section 3.9, where we give some results for relatively DLT pairs. All the main theorems appear in [Cac10].

3.1 The case $N = 0$

In this section we show that in the case when D is nef, that is D admits the trivial Zariski decomposition given by $P = D$ and $N = 0$, Conjecture 2 follows easily by a theorem of Fujino.

We begin the section by stating Fujino's theorem, which generalizes the base-point free theorem to the LC case.

Theorem 3.1.1. *[Fuj09a, theorem 1.2]*
Let (X, B) be an LC pair, with B effective. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the linear system $|mL|$ is base-point free for every $m \gg 0$.

As a simple consequence of this theorem we find the following, which in particular implies Conjecture 2 in the case $N = 0$.

Theorem 3.1.2. *Let (X, Δ) be an effective LC pair. If $D \in Div_{\mathbb{Q}}(X)$, $b \in \mathbb{N}$ is such that bD is a Cartier divisor and*

1. *D is big and nef;*

2. *$aD - (K_X + \Delta)$ is nef for some $a \in \mathbb{Q}^+$;*

3. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

then $|mbD|$ is base-point free for every $m \gg 0$.

Proof. We can assume without loss of generality that D is a Cartier divisor. By lemma 2.3.7 there exist an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $D - \lambda\Gamma$ is ample and $(X, \Delta + \lambda\Gamma)$ is an LC pair. Then

$$(1 + a)D - (K_X + \Delta + \lambda\Gamma) = (D - \lambda\Gamma) + (aD - (K_X + \Delta))$$

is ample because it is the sum of an ample divisor and a nef one.
Thus we can apply theorem 3.1.1 to the pair $(X, \Delta + \lambda\Gamma)$ and we have the assert. $\qquad\square$

Theorem 3.1.3. *Let (X, Δ) be an effective LC pair. If $D \in Div_{\mathbb{Q}}(X)$ is such that*

1. *D is nef;*

2. *$aD - (K_X + \Delta)$ is big and nef for some $a \in \mathbb{Q}^+$;*

3. *$\mathbb{B}_+(aD - (K_X + \Delta))$ does not contain any LC center of the pair (X, Δ);*

then D is semiample.

The proof is similar to the previous theorem.
Note that we cannot remove hypothesis 3. In fact [KMM85, remark 3-1-2] gives a counterexample.
On the other hand note that theorem 3.1.2 and theorem 3.1.3 also follow by [Amb01, theorem 7.2](cfr. [Fuj09c, theorem 4.4]). In particular we can lighten hypothesis 3 in both theorems by assuming in theorem 3.1.2 that D is logbig with respect to (X, Δ) and in theorem 3.1.3 that $aD - (K_X + \Delta)$ is logbig with respect to (X, Δ).

Weak log Fano pairs

Definition 3.1.4. Let (X, Δ) be an effective pair. We say that (X, Δ) is a *weak log Fano pair* if $-(K_X + \Delta)$ is big and nef.

By theorem 3.1.3 we obtain a sufficient condition for the semiampleness of the anticanonical divisors on a LC weak log Fano pair:

Corollary 3.1.5. *Let (X, Δ) be a weak log Fano LC pair. If $\mathbb{B}_+(-(K_X + \Delta))$ does not contain any LC center of the pair (X, Δ), then $-(K_X + \Delta)$ is semiample.*

Remark 3.1.6. Again in the above corollary the hypothesis on the $\mathbb{B}_+$ is necessary: it is not sufficient that some LC centers of (X, Δ) are not contained in $\mathbb{B}_+(-(K_X + \Delta))$. See example 3.11.1 and example 3.11.2.

3.2 Direct consequences of a theorem of Fujino

In this section we easily get from theorem 6.1 of Fujino's paper [Fuj07b] a generalization of Kawamata's theorem 1.0.1 in the LC case that goes in a different direction with respect to Conjecture 1 and Conjecture 2. We require, in fact, as additional hypothesis, that the positive part of the Zariski decomposition is

semiample when restricted to the locus where the given pair is not KLT (see corollary 3.2.3).
The following is a simplified version of theorem 6.1 of [Fuj07b]:

Theorem 3.2.1. *Let (X, B) be an LC pair such that X is smooth and B is SNCS.*
Write $B = B_+ - B_-$, where B_+ and B_- are effective $\mathbb{Q}$-divisors, and they have not common components. Let $P \in Div_{\mathbb{Q}}(X)$ be such that

1. *P is nef;*

2. *$aP - (K_X + B)$ is big and nef for some rational number $a \geq 0$;*

3. *There exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner B_- \urcorner));$$

4. *If (X, B) is not KLT then $P_{|_{B=1}}$ is semiample.*

Then P is semiample.

Corollary 3.2.2. *Let X be a normal projective variety and let Δ be an effective Weil $\mathbb{Q}$-divisor. Suppose D is a Weil $\mathbb{Q}$-divisor that admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$ such that*

1. *$(X, \Delta - aN)$ is an LC pair some rational numbers $a \geq 0$;*

2. *If $(X, \Delta - aN)$ is not KLT then $P_{|_{\mathrm{Nklt}(X, \Delta - aN)}}$ is semiample;*

3. *$m_0 P - (K_X + \Delta - aN)$ is big and nef for some $m_0 \in \mathbb{Q}$.*

Then P is semiample.

Note that in particular we can replace hypotheses 1 and 2 by asking that (X, Δ) is an LC pair such that $\mathrm{Nklt}(X, \Delta) = \emptyset$ or or $P_{|_{\mathrm{Nklt}(X, \Delta)}}$ is semiample.

Proof. Let $\mu : Y \to X$ be a log-resolution of the pair $(X, \Delta - aN)$ and let $B := \mathbf{A}(\Delta - aN)_Y \in \mathrm{Div}_{\mathbb{Q}}(Y)$.
We will show that we can apply theorem 3.2.1 to the pair (Y, B) and the divisor $\mu^*(P)$.
Note that Y is smooth, B is SNCS and (Y, B) is KLT if $(X, \Delta - aN)$ is such. Moreover $\mu^*(P)$ is nef and

$$m_0 \mu^*(P) - (K_Y + B) \equiv \mu^*(m_0 P - (K_X + \Delta - aN)),$$

so that it is big and nef.
Now, by definition of $\mathbb{Q}$-CKM Zariski decomposition, we can choose $k_0 \in \mathbb{N}$ such that $k_0 \geq a$, $k_0 P$ is Cartier, $k_0 D$ is integral and

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 D))$$

for all $m \in \mathbb{N}$.

Furthermore note that if we write $B = B_+ - B_-$, where B_+ and B_- are effective divisors and they have not common components, then $\mu_*(\ulcorner B_- \urcorner) \leq \ulcorner aN \urcorner$, because Δ is effective.

Thus we can apply lemma 2.1.8 and we get that for all $m \in \mathbb{N}$

$$h^0(Y, \mathcal{O}_Y(\mu^*(mk_0 P) + \ulcorner B_- \urcorner) \leq h^0(X, \mathcal{O}_X(mk_0 P) + \ulcorner aN \urcorner) \leq$$

$$\leq h^0(X, \mathcal{O}_X(mk_0 D)) = h^0(X, \mathcal{O}_X(mk_0 P)) = h^0(Y, \mathcal{O}_Y(\mu^*(mk_0 P))).$$

Hence property 3 is verified.

Now suppose (Y, B) is not KLT, define $T = B^{=1}$ and consider the following commutative diagram:

$$
\begin{array}{ccc}
T & \hookrightarrow & Y \\
\downarrow{\scriptstyle \mu_{|T}} & & \downarrow{\scriptstyle \mu} \\
\mathrm{Nklt}(X, \Delta - aN) & \hookrightarrow & X
\end{array}
$$

Then $\mu^*(P)_{|T} = \mu^*_{|T}(P_{|\mathrm{Nklt}(X,\Delta-aN)})$ is semiample, being the pullback of a semiample divisor.

Thus we can apply theorem 3.2.1 and we get that $\mu^*(P)$ is semiample, which in turn implies that P is semiample. $\qquad \square$

Corollary 3.2.3. *Let (X, Δ) be an effective LC pair and let $D \in Div_{\mathbb{Q}}(X)$. consider the following assumptions:*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

2'. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

3. *There exists a projective birational morphism $f : Z \to X$ such that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^*(D) = P + N$ and $P_{|\mathrm{Nklt}(Z,\mathbf{A}(\Delta)_Z)}$ is semiample if (X, Δ) is not KLT.*

If D satisfies 1, 2 and 3, or D satisfies 2' and 3, then P is semiample.

Proof. Let us apply lemma 2.5.3 and consider t_0, D', P', N', Δ_Z as in the lemma, so that $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition, Δ_Z is effective and $t_0 P' - (K_Z + \Delta_Z - N')$ is big and nef.

Note that the pair $(Z, \mathbf{A}(\Delta)_Z)$ is LC because (X, Δ) is such, so that the pair $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$ is also LC by lemma 2.2.8. By the same lemma we get that $P_{|\mathrm{Nklt}(Z,\Delta_Z - N')}$ is semiample whenever $(Z, \Delta_Z - N')$ is not KLT. Therefore the assert follows by applying corollary 3.2.2. $\qquad \square$

3.3 DLT case

In this section we consider some generalizations of Kawamata's thereom 1.0.1 in the case of DLT pairs. In particular by adapting the proof in [Kaw87] we get Conjecture 2 for DLT pairs (see corollary 3.3.4). On the other hand in the second part of the section, as a corollary of [Fuj07b, theorem 5.1], we prove a stronger result (see theorem 3.3.11) which, in particular, implies Conjecture 1 for DLT pairs.

Generalizing Kawamata's proof

Lemma 3.3.1. *Let X be a normal projective variety and let A be a Weil $\mathbb{Q}$-divisor on X such that $(X, -A)$ is a KLT pair.*
Let P and N be Weil $\mathbb{Q}$-divisors such that $P \in Div_{\mathbb{Q}}(X)$ and it is nef, N is effective and $A \leq aN$ for some $a \in \mathbb{Q}$.
If D is a Weil $\mathbb{Q}$-divisor such that

1. *$D = P + N$;*

2. *There exists $t_0 \in \mathbb{Q}$ such that*

$$t_0 P + A - K_X$$

   is big and nef;

then $h^0(X, mD) \geq 0$ for some $m \in \mathbb{N}$.

Proof. Thanks to the nefness of P we can assume $t_0 > 0$.
Now let k_0 be a positive integer such that $k_0 P$ is a Cartier divisor. We have that $k_0 P$ is nef and

$$\frac{t_0}{k_0}(k_0 P) + A - K_X = t_0 P + A - K_X$$

is big and nef.
Then, by Shokurov nonvanishing theorem ([KM00, theorem 3.4]) we get that

$$H^0(X, \mathcal{O}_X(mk_0 P + \lceil A \rceil)) \neq 0$$

for every integer $m \gg 0$.
Consider $k_1 \in \mathbb{N}$ such that $k_1 N$ is an integral divisor and $k_1 \geq a$. Hence $\lceil A \rceil \leq \lceil aN \rceil \leq \lceil k_1 N \rceil = k_1 N$, so that

$$H^0(X, \mathcal{O}_X(mk_0 P + k_1 N)) \neq 0$$

for every $m \gg 0$.
Now take a sufficiently large integer m_0 such that $m_0 k_0 \geq k_1$. Then

$$m_0 k_0 P + k_1 N \leq m_0 k_0 P + m_0 k_0 N = m_0 k_0 D.$$

Thus $H^0(X, \mathcal{O}_X(m_0 k_0 D)) \neq 0$, so that the lemma is proved.

$\square$

Theorem 3.3.2. *Let (X, Δ) be a DLT pair. If $D \in Div_{\mathbb{Q}}(X)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$ such that*

1. *there exist two rational numbers t_0 and a with $a \geq 0$ and*

$$t_0 P + aN - (K_X + \Delta)$$

   is ample;

2. *$\mathbb{B}(P)$ does not contain LC centers of the pair (X, Δ);*

then P is semiample.

Proof.
We define $A := aN - \Delta$ so that $A \leq aN$.
By lemma 2.2.8, the pair $(X, -A)$ is LC and $CLC(X, -A) \subseteq CLC(X, \Delta)$. Then $\mathbb{B}(P)$ does not contain LC centers of the pair $(X, -A)$. Moreover

$$t_0 P + A - K_X = t_0 P + aN - (K_X + \Delta)$$

is ample.
Now if $(X, -A)$ is KLT then $\kappa(X, D) \geq 0$ by lemma 3.3.1. Hence, as $\kappa(X, P) = \kappa(X, D)$ by definition of CKM Zariski decomposition, we have that $\kappa(X, P) \geq 0$, too.
If $(X, -A)$ is not KLT then $CLC(X, -A) \neq \emptyset$, so that $\mathbb{B}(P) \neq X$, that is $\kappa(X, P) \geq 0$.
Then, if we denote

$$\mathbb{N}(P) = \{m \in \mathbb{N} \text{ such that } mP \in Div(X) \text{ e } H^0(X, \mathcal{O}_X(mP)) \neq 0\},$$

we have that $\mathbb{N}(P) \neq \emptyset$.
We suppose, by contradiction, that $\mathbb{B}(P) \neq \emptyset$ and we consider $m_1 \in \mathbb{N}(P)$ such that $Bs(|m_1 P|) = \mathbb{B}(P) \neq \emptyset$.
We will find $m \in \mathbb{N}(P)$ and a subvariety V such that $V \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$ but $V \not\subseteq Bs(|mP|)$, leading, in such a way, to a contradiction.
Let $\{D_j\}_{j=1}^k$ be the finite set of the prime divisors appearing in the support of A or as base components of $|m_1 P|$. We write $A = \sum_{j=1}^k a_j D_j$, where the a_j are possibly zero rational numbers.
As $Bs(|m_1 P|)$ does not contain any element in $CLC(X, \Delta)$, by lemma 2.2.27, there exists $\mu : Y \to X$ such that μ is a log-resolution of the pair $(X, -A)$ and of the linear series $|m_1 P|$ and μ is a DLT morphism for the pair $(X, -A)$.
Let $\{F_j = \tilde{D}_j\}_{j=1}^k$ be the finite set of the strict transforms of the divisors D_j and let $\{F_j = E_j\}_{j=k+1}^l$ be the finite set of the μ-exceptional prime divisors on Y, so that $\sum_{j=1}^l F_j$ is a SNC divisor.
Moreover we can write

$$K_Y \equiv \mu^*(K_X - A) + \sum b_j F_j$$

where

$$b_j \geq -1 \quad \text{if} \quad 1 \leq j \leq k$$

because the pair $(X, -A)$ is LC and

$$b_j > -1 \quad \text{if} \quad k + 1 \leq j \leq l$$

because μ is a DLT morphism for $(X, -A)$.
On the other hand there exists an integral base-point free divisor L and coefficients $r_j \in \mathbb{N} \cup \{0\}$ such that $\mu^*(m_1 P) = L + \sum r_j F_j$ and $\mu^*|m_1 P| = |L| + \sum r_j F_j$.
Hence we have that $Bs(|m_1 P|) = \mu(\bigcup_{r_j \neq 0} F_j)$, so that we can suppose $r_j > 0$ for some j because $Bs(|m_1 P|) \neq \emptyset$.
Moreover if $r_j \neq 0$, then $\mu(F_j) \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$, which implies that $b_j > -1$, as $\mathbb{B}(P)$ does not contain any LC center of the pair $(X, -A)$.

Now, as $t_0 P + A - K_X$ is ample and μ is a DLT morphism, by lemma 2.1.9 there exist, for each $j = k+1, ..., l$, rational, arbitrarily small numbers $\delta_j > 0$, such that

$$\mu^*(t_0 P + A - K_X) - \sum_{j=k+1}^{l} \delta_j F_j$$

is still ample.

Thanks to the openness of the ample cone there exist also, for each $j = 1, ..., k$, positive rational numbers $\delta_j \in \mathbb{Q}$ such that if $0 \leq \delta'_j \leq \delta_j$ then

$$\mu^*(t_0 P + A - K_X) - \sum_{j=1}^{k} \delta'_j F_j - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample.

Now we define

$$c = \min_{\{j : r_j \neq 0\}} \frac{b_j + 1 - \delta_j}{r_j}.$$

By choosing the δ_j's small enough we can suppose that

$$b_j + 1 - \delta_j > 0 \text{ for all } j \text{ such that } b_j > -1.$$

Hence $c > 0$ because $b_j > -1$ if $r_j \neq 0$.

Moreover, perturbing slightly the δ_j's if necessary, we can suppose that the minimum is attained on a unique j, say $j = j_0$. Define $B := F_{j_0}$.

Now take $s := t_0 + c m_1$, and

$$B_m := \mu^*((m - c m_1)P + A - K_X) - \sum_{j=k+1}^{l} \delta_j F_j.$$

Then, if $0 \leq \delta'_j \leq \delta_j$ for each $j = 1, ..., k$, we have that for every integer $m \geq s$

$$B_m - \sum_{j=1}^{k} \delta'_j F_j = \mu^*((m - c m_1)P + A - K_X) - \sum_{j=1}^{k} \delta'_j F_j - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample, because $m - c m_1 \geq t_0$ and $\mu^*(P)$ is nef.

Let us consider now, for each j such that $b_j = -1$, rational, arbitrarily small numbers $\epsilon_j > 0$, and define

$$A' := \sum_{j \neq j_0} (-c r_j + b_j - \delta_j) F_j,$$

$$A'' := \sum_{\substack{b_j > -1 \\ j \neq j_0}} (-c r_j + b_j - \delta_j) F_j + \sum_{b_j = -1} (-1 + \epsilon_j) F_j.$$

As $r_j = 0$ if $b_j = -1$ we have that

$$A'' - A' = \sum_{b_j = -1} (\epsilon_j + \delta_j) F_j.$$

Now we define, for every $m \in \mathbb{N}(P)$, the divisor

$$Q_m := \mu^*(mP) + A'' - B - K_Y.$$

Then

$$Q_m := \mu^*(mP) + A' - B - K_Y + (A'' - A') \equiv$$

$$\equiv \mu^*(mP) + \sum_{j \neq j_0}(-cr_j + b_j - \delta_j)F_j - F_{j_0} - \sum b_j F_j - \mu^*(K_X - A) + (A'' - A') =$$

$$= \mu^*(mP + A - K_X) - \sum_{j \neq j_0} cr_j F_j - \sum_{j \neq j_0} \delta_j F_j + F_{j_0}(-1 - b_{j_0}) + (A'' - A') =$$

$$= \mu^*((m - cm_1)P + A - K_X) + cL + F_{j_0}(cr_{j_0} - 1 - b_{j_0}) - \sum_{j \neq j_0} \delta_j F_j + (A'' - A') =$$

$$= \mu^*((m - cm_1)P + A - K_X) + cL - \sum \delta_j F_j + \sum_{b_j = -1}(\epsilon_j + \delta_j)F_j =$$

$$= \mu^*((m - cm_1)P + A - K_X) - \sum_{b_j > -1} \delta_j F_j + cL + \sum_{b_j = -1} \epsilon_j F_j =$$

$$= B_m - \sum_{j \in J} \delta_j F_j + cL + \sum_{b_j = -1} \epsilon_j F_j,$$

where $J = \{j \in \mathbb{N} \text{ such that } 1 \leq j \leq k \text{ and } b_j > -1\}$.
Let $m_2 = \min\{m \in \mathbb{N}(P) \text{ such that } m \geq s\}$. Then $B_{m_2} - \sum_{j \in J} \delta_j F_j$ is ample,
so that Q_{m_2} is also ample if the ϵ_j are small enough because L is nef.
Hence Q_m is ample for every $m \in \mathbb{N}(P)$ such that $m \geq s$.
Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we
find that $H^1(Y, \mathcal{O}_Y(\mu^*(mP) + \lceil A'' \rceil - B)) = 0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.
This implies that the restriction homomorphism

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \lceil A'' \rceil)) \to H^0(B, \mathcal{O}_B(\mu^*(mP) + \lceil A'' \rceil))$$

is surjective in this case.
Now we notice that $\mu^*(m_2 P)_{|_B} + A''_{|_B} - K_B = Q_{m_2|_B}$ is an ample $\mathbb{Q}$-divisor.
Moreover $A''_{|_B}$ is SNCS because A'' is such and B intersects transversally all the
F_j's with $j \neq j_0$.
Hence it suffices to verify that all the coefficients of A'' are greater than -1 to
show that the pair $(B, -A''_{|_B})$ is KLT.
But we have that

- if $b_j = -1$ then $-1 + \epsilon_j > -1$;

- if $b_j > -1$ and $r_j = 0$ then $-cr_j + b_j - \delta_j = b_j - \delta_j > -1$ (by the choice
  of the δ_j's);

- if $b_j > -1$, $r_j \neq 0$ and $j \neq j_0$ then $-cr_j + b_j - \delta_j > -\frac{b_j + 1 - \delta_j}{r_j}r_j + b_j - \delta_j = -1$.

Hence the pair $(B, -A''_{|_B})$ is KLT. Thus, by Shokurov's nonvanishing theorem ([KM00, theorem 3.4]) for every integer $k > 0$ there exists $\mu_k \in \mathbb{N}$, such that $\mu_k \geq a$, $\mu_k \geq s$, μ_k is a multiple of k and

$$H^0(B, \mathcal{O}_B(\mu^*(\mu_k P) + \ulcorner A'' \urcorner)) \neq 0.$$

In fact this cohomology group is non zero for every sufficiently large multiple of m_2.

Now, by the definition of $\mathbb{Q}$-CKM Zariski decomposition, there exists $k_0 \in \mathbb{N}$ such that $k_0 P$ and $k_0 D$ are integral and $H^0(X, t k_0 P) \simeq H^0(X, t k_0 D)$ for every $t \in \mathbb{N}$. Let us define $m := \mu_{k_0}$. Then

$$B \not\subseteq Bs(|\mu^*(mP) + \ulcorner A'' \urcorner|) \text{ and } H^0(X, \mathcal{O}_X(mP)) \simeq H^0(X, \mathcal{O}_X(mD)).$$

Now we verify the inequalities $\ulcorner A'' \urcorner \geq 0$ and $\mu_* \ulcorner A'' \urcorner \leq \ulcorner aN \urcorner \leq mN$.

The first one holds because all the coefficients of A'' are greater than -1.

The third one is verified because $m \geq a$ and mN is integral (because mP and mD are such).

As for the second one we have that

$$\mu_*(\ulcorner A'' \urcorner) = \sum_{\substack{b_j > -1 \\ j \neq j_0}} \ulcorner -cr_j + b_j - \delta_j \urcorner \mu_*(F_j) + \sum_{b_j = -1} \ulcorner -1 + \epsilon_j \urcorner \mu_*(F_j) =$$

$$= \sum_{\substack{b_j > -1 \\ j \neq j_0}} \ulcorner -cr_j + b_j - \delta_j \urcorner \mu_*(F_j) = \sum_{j \in J'} \ulcorner -cr_j + b_j - \delta_j \urcorner D_j,$$

where $J' = \{j \in \mathbb{N} \text{ such that } 1 \leq j \leq k, \ b_j > -1 \text{ e } j \neq j_0\}$.

But, if $j \in J'$ then $\ulcorner -cr_j + b_j - \delta_j \urcorner = \ulcorner -cr_j + a_j - \delta_j \urcorner \leq \ulcorner a_j \urcorner$.

Hence

$$\mu_*(\ulcorner A'' \urcorner) \leq \sum_{j \in J'} \ulcorner a_j \urcorner D_j \leq \sum_{j=1}^k \max\{0, \ulcorner a_j \urcorner\} D_j \leq \ulcorner aN \urcorner$$

as $\ulcorner aN \urcorner \geq \ulcorner A \urcorner = \sum \ulcorner a_j \urcorner D_j$ and $\ulcorner aN \urcorner \geq 0$.

From these inequalities, thanks to lemma 2.1.7, it follows that

$$\ulcorner A'' \urcorner \leq \mu^*(mN) + T,$$

for a suitable μ-exceptional effective Cartier divisor T.

Moreover, as $\ulcorner A'' \urcorner \geq 0$, we have that

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A'' \urcorner)) \hookrightarrow$$

$$\hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP + mN) + T)) \simeq$$

$$\simeq H^0(X, \mathcal{O}_X(mP + mN) \otimes \mu_* \mathcal{O}_Y(T)) \simeq H^0(X, \mathcal{O}_X(mD))$$

by Fujita's lemma. But

$$H^0(X, \mathcal{O}_X(mD)) \simeq H^0(X, \mathcal{O}_X(mP)) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(mP))).$$

Then the injection

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A'' \urcorner))$$

is an isomorphism. This implies that $B \nsubseteq Bs(|\mu^*(mP)|)$.

Hence $\mu(B) \nsubseteq Bs(|mP|)$.

But $\mu(B) = \mu(F_{j_0}) \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$ and we have the contradiction.

$\square$

Theorem 3.3.3. *Let (X, Δ) be a DLT pair.*
If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *D is big;*

2. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

3. *D admits a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N;$$

then there exists $\beta > 0$ such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

then P is semiample.

Proof. We begin by observing that $0 \leq \dim X = \kappa(X, D)$. Moreover, by definition of CKM Zariski decomposition, $\kappa(X, D) = \kappa(X, P)$, that is P is big.

Note also that, as $\mathbb{B}_+(P) = \mathbb{B}_+(D)$ by lemma 2.5.4, we have that $\mathbb{B}_+(P)$ does not contain any LC center of the pair (X, Δ).

Hence, by lemma 2.3.7, there exists an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $P - \lambda \Gamma$ is ample, $(X, \Delta + \lambda \Gamma)$ is DLT and $CLC(X, \Delta + \lambda \Gamma) = CLC(X, \Delta)$.

Now, as $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in \mathbb{Q}^+$ such that if $0 < \beta' < \beta$, then $(X, \Delta + \lambda \Gamma + \beta' N)$ is still DLT and $CLC(X, \Delta + \lambda \Gamma + \beta' N) = CLC(X, \Delta)$.

Let a be a rational number such that $a > -\beta$ and $aD - (K_X + \Delta)$ is nef.

We have that

$$(1+a)P + aN - \Delta - \lambda \Gamma - K_X = (1+a)P + aN - \lambda \Gamma - aD + (aD - (K_X + \Delta)) =$$

$$= (aP + aN) + (P - \lambda \Gamma) - aD + (aD - (K_X + \Delta)) =$$

$$= (P - \lambda \Gamma) + (aD - (K_X + \Delta))$$

is ample because it is the sum of an ample and a nef divisor.

If $a \geq 0$ we conclude by applying theorem 3.3.2 to the pair $(X, \Delta + \lambda \Gamma)$. In fact, as $\mathbb{B}(P) \subseteq \mathbb{B}_+(P)$, it does not contain any element of $CLC(X, \Delta) = CLC(X, \Delta + \lambda \Gamma)$.

If $-\beta < a < 0$ we can apply theorem 3.3.2 to the pair $(X, \Delta + \lambda \Gamma - aN)$. In fact, again, as $\mathbb{B}(P) \subseteq \mathbb{B}_+(P)$, it does not contain any element of $CLC(X, \Delta) = CLC(X, \Delta + \lambda \Gamma - aN)$.

$\square$

The following corollary corresponds to Conjecture 2 in the case when the pair (X, Δ) is DLT. Note that in the next subsection we prove a stronger result in a different way (see theorem 3.3.11).

Corollary 3.3.4. *Let (X, Δ) be a DLT pair.*
If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

3. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

4. *D admits a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N;$$

then P is semiample.

Remark 3.3.5. Note that in the above theorem the hypothesis that $\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ) is necessary. It is not enough to assume that some LC centers of (X, Δ) are not contained in $\mathbb{B}_+(D)$. See example 3.11.1 and example 3.11.2.

The following is a variant of corollary 3.3.4

Theorem 3.3.6. *Let (X, Δ) be an effective LC pair.*
If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

2. *There exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition;*

3. *$(Z, \mathbf{L}(\Delta)_Z)$ is a DLT pair;*

4. *$\mathbb{B}_+(f^*(aD - (K_X + \Delta)))$ does not contain any LC center of the pair $(Z, \mathbf{L}(\Delta)_Z)$;*

5. *$\mathbb{B}(P)$ does not contain any LC center of the pair $(Z, \mathbf{L}(\Delta)_Z)$;*

then P is semiample.

Proof. Let us apply lemma 2.5.3 and consider t_0, D', P', N', Δ_Z as in the lemma. Note that (Z, Δ_Z) is a pair because $(Z, \mathbf{L}(\Delta)_Z)$ is such. Hence N' and D' are in $Div_{\mathbb{Q}}(X)$. Take $L := f^*(aD - (K_X + \Delta))$, so that by lemma 2.3.7 we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $L - \lambda \Gamma$ is ample, $(Z, \mathbf{L}(\Delta)_Z + \lambda \Gamma)$ is DLT and $CLC(Z, \mathbf{L}(\Delta)_Z) = CLC(Z, \mathbf{L}(\Delta)_Z + \lambda \Gamma)$. Now we have that

$$t_0 P' + N' - (K_Z + \Delta_Z + \lambda \Gamma) = P + L - \lambda \Gamma$$

is ample. Thus we conclude by applying theorem 3.3.2 to the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D' = P' + N'$ and the DLT pair $(Z, \mathbf{L}(\Delta)_Z)$. $\qquad\square$

Nonvanishing theorems

By using the same ingredients of the proof of the main theorem of [Kaw87] (see also [Mor86, step 2 of the proof of thm. 0]), we prove the following non-vanishing, again in the context of DLT pairs.

Theorem 3.3.7. *Let (X, Δ) be a DLT pair.*
Let $P, N \in Div_{\mathbb{Q}}(X)$ be such that P is nef and N is effective. If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

2. *$\mathbb{B}_+(aD - (K_X + \Delta))$ does not contain any LC center of the pair (X, Δ);*

3. *$D = P + N$;*

then $\kappa(X, D) \geq 0$.

Proof. Let $L := aD - (K_X + \Delta)$.
Thanks to lemma 2.3.7 we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $L - \lambda\Gamma$ is ample, the pair $(X, \Delta + \lambda\Gamma)$ is DLT and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$.
Now we define $A := aN - \Delta - \lambda\Gamma$. Then, thanks to lemma 2.2.8, we have that $(X, -A)$ is an LC pair and $CLC(X, -A) \subseteq CLC(X, \Delta)$.
Moreover if $A \neq 0$, we write $A = \sum_{j=1}^{k} a_j D_j$, where the D_j's are prime distinct divisors and the a_j's are non zero rational numbers.
Now for every integer $t \geq a$ we have that

$$tP + A - K_X = tP + aN - \Delta - \lambda\Gamma - K_X =$$

$$(t-a)P + (aP + aN) - aD + aD - \Delta - \lambda\Gamma - K_X = (t-a)P + L - \lambda\Gamma$$

is ample because it is the sum of a nef and an ample divisor.
By lemma 2.2.27 (applied for example to the trivial linear series $|\mathcal{O}_X|$) there exists $\mu : Y \to X$, a log-resolution of the pair $(X, -A)$ such that μ is a DLT morphism for $(X, -A)$.
Let $\{F_j = \tilde{D}_j\}_{j=1}^{k}$ be the finite set of the strict transforms of the divisors D_j and let $\{F_j = E_j\}_{j=k+1}^{l}$ be the finite set of the μ-exceptional prime divisors on Y, so that $\sum_{j=1}^{l} F_j$ is a SNC divisor.
Moreover we can write

$$K_Y \equiv \mu^*(K_X - A) + \sum b_j F_j$$

where

$$b_j \geq -1 \quad \text{if} \quad 1 \leq j \leq k$$

because the pair $(X, -A)$ is LC and

$$b_j > -1 \quad \text{if} \quad k+1 \leq j \leq l$$

because μ is a DLT morphism for $(X, -A)$.
Now, as $aP + A - K_X$ is ample and μ is a DLT morphism, by lemma 2.1.9 there exist, for each $j = k+1, ..., l$, rational, arbitrarily small numbers $\delta_j > 0$, such that

$$\mu^*(aP + A - K_X) - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample.

Thanks to the openness of the ample cone there exist also, for each $j = 1, ..., k$, positive rational numbers $\delta_j \in \mathbb{Q}$ such that if $0 \leq \delta_j' \leq \delta_j$ then

$$\mu^*(aP + A - K_X) - \sum_{j=1}^{k} \delta_j' F_j - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample.

Moreover, by choosing the δ_j's small enough we can suppose that

$$b_j + 1 - \delta_j > 0 \qquad \forall j \, : \, b_j > -1.$$

Now for every $m \in \mathbb{N}$ we define

$$B_m := \mu^*(mP + A - K_X) - \sum_{j=k+1}^{l} \delta_j F_j.$$

Hence, if $0 \leq \delta_j' \leq \delta_j$ for every $j = 1, ..., k$, then for every integer $m \geq a$

$$B_m - \sum_{j=1}^{k} \delta_j' F_j = \mu^*(mP + A - K_X) - \sum_{j=1}^{k} \delta_j' F_j - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample, as $\mu^*(P)$ is nef.

Now, for each j such that $b_j = -1$, we consider arbitrarily small rational numbers $\epsilon_j > 0$, and we define

$$A' := \sum_{b_j > -1} (b_j - \delta_j) F_j + \sum_{b_j = -1} (-1 + \epsilon_j) F_j.$$

We also define, for every $m \in \mathbb{N}$, the $\mathbb{Q}$-divisor

$$Q_m := \mu^*(mP) + A' - K_Y.$$

Then

$$Q_m \equiv \mu^*(mP) + \sum_{b_j > -1} (b_j - \delta_j) F_j + \sum_{b_j = -1} (-1 + \epsilon_j) F_j - \mu^*(K_X - A) - \sum_{j=1}^{l} b_j F_j =$$

$$= \mu^*(mP + A - K_X) - \sum_{b_j > -1} \delta_j F_j + \sum_{b_j = -1} \epsilon_j F_j = B_m - \sum_{j \in J} \delta_j F_j + \sum_{b_j = -1} \epsilon_j F_j,$$

where $J = \{j \in \mathbb{N} \text{ such that } 1 \leq j \leq k \text{ e } b_j > -1\}$.

Let $m_1 = \min\{m \in \mathbb{N} \setminus \{0\} \, : \, m \geq a \text{ and } mP \in Div(X)\}$. Then $B_{m_1} - \sum_{j \in J} \delta_j F_j$ is ample, so that $Q_{m_1} = \mu^*(m_1 P) + A' - K_Y$ is also ample if we take the ϵ_j small enough.

Now we verify that the pair $(Y, -A')$ is KLT: as $-A'$ is a SNCS divisor it suffices to verify that all the coefficients of $-A'$ are less than 1, or, equivalently, that all the coefficients of A' are greater than -1.

But we have that

- if $b_j = -1$ then $-1 + \epsilon_j > -1$;

- if $b_j > -1$ then $b_j - \delta_j > -1$ thanks to our choice of the δ_j's.

Hence, by Shokurov nonvanishing theorem ([KM00, theorem 3.4]), we find that $H^0(Y, \mu^*(sm_1P) + \ulcorner A'\urcorner) \neq 0$ for every integer $s \gg 0$.

In particular, there exists a positive integer $d \geq a$ such that dP and dN are Cartier divisors (hence also dD) and

$$H^0(Y, \mu^*(dP) + \ulcorner A'\urcorner) \neq 0.$$

Now we verify the inequalities $0 \leq \mu_* \ulcorner A'\urcorner \leq \ulcorner aN\urcorner \leq dN$.

The first one holds because all the coefficients of A' are greater than -1. The third one holds because $d \geq a$ and dN is integral.

As for the second we have that, if $A = 0$, then $\mu_*(\ulcorner A'\urcorner) = 0$ because A' is μ-exceptional, due to the fact that all the F_j's are μ-exceptional prime divisors. Hence the inequality holds because $\ulcorner aN\urcorner \geq 0$.

If $A \neq 0$ then

$$\mu_*(\ulcorner A'\urcorner) = \sum_{b_j > -1} \ulcorner b_j - \delta_j \urcorner \mu_*(F_j) + \sum_{b_j = -1} \ulcorner -1 + \epsilon_j \urcorner \mu_*(F_j) =$$

$$= \sum_{b_j > -1} \ulcorner b_j - \delta_j \urcorner \mu_*(F_j) = \sum_{\substack{j \leq k \\ b_j > -1}} \ulcorner b_j - \delta_j \urcorner D_j \leq \sum_{\substack{j \leq k \\ a_j > -1}} \ulcorner a_j \urcorner D_j \leq$$

$$\leq \sum_{j=1}^{k} \max\{0, \ulcorner a_j \urcorner\} D_j \leq \ulcorner aN\urcorner$$

because $\ulcorner aN\urcorner \geq \ulcorner A\urcorner = \sum \ulcorner a_j \urcorner D_j$ and $\ulcorner aN\urcorner \geq 0$.

Now, as $\mu_* \ulcorner A'\urcorner \leq dN$, by lemma 2.1.7 we get that there exists a μ-exceptional divisor $T \geq 0$ on Y such that

$$\ulcorner A'\urcorner \leq \mu^*(dN) + T.$$

Hence

$$0 \neq h^0(Y, \mathcal{O}_Y(\mu^*(dP) + \ulcorner A'\urcorner)) \leq h^0(Y, \mathcal{O}_Y(\mu^*(dP + dN) + T) =$$

$$= h^0(X, \mathcal{O}_X(dD) \otimes \mu_* \mathcal{O}_Y(T)) = h^0(X, \mathcal{O}_X(dD))$$

thanks to Fujita's lemma. Thus $\kappa(X, D) \geq 0$. $\qquad \square$

Definition 3.3.8. *[Bir09, Def 1.3]*

Let X be a normal projective variety and let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$.

We say that D admits a weak $\mathbb{Q}$-Zariski decomposition if there exists a projective birational morphism $g : Y \to X$, where Y is a normal projective variety, such that

1. $g^*(D) = P + N$;

2. $P, N \in \mathrm{Div}_{\mathbb{Q}}(X)$;

3. P is nef and N is effective.

Corollary 3.3.9. *Let (X, Δ) be an effective PLT pair.*
If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a > 0$;*

2. *$\mathbb{B}_+(aD - (K_X + \Delta))$ does not contain any LC center of the pair (X, Δ);*

3. *D a admits a weak $\mathbb{Q}$-Zariski decomposition;*

then $\kappa(X, D) \geq 0$.

Proof. Let $g : Y \to X$ be a projective birational morphism of normal projective varieties such that $g^*(D) = P + N$ is a weak $\mathbb{Q}$-Zariski decomposition.
Let $h : Z \to Y$ be a log-resolution of the pair $(Y, g_*^{-1}(\Delta) + \text{exc}(g))$, so that Z is smooth and $g \circ h : Z \to X$ is a log-resolution of the pair (X, Δ).
Let us denote $f := g \circ h : Z \to X$.
Then
$$f^*(D) = h^*(g^*(D)) = h^*(P) + h^*(N),$$
where $h^*(P)$ is nef and $h^*(N)$ is effective.
Now let $F := \mathbf{L}(\Delta)_Z - \mathbf{A}(\Delta)_Z$. Then we have that

- $\mathbf{A}(\Delta)_Z = \mathbf{L}(\Delta)_Z - F$;

- $\mathbf{L}(\Delta)_Z$ and F are both effective;

- $\text{Supp}(F) \subseteq \text{divexc}(f)$;

- $(Z, \mathbf{L}(\Delta)_Z)$ is a PLT pair and $CLC(Z, \mathbf{L}(\Delta)_Z) = CLC(Z, \mathbf{A}(\Delta)_Z)$.

The first three assertions are obvious. As for the last one we have that $CLC(Z, \mathbf{L}(\Delta)_Z) = CLC(Z, \mathbf{A}(\Delta)_Z)$ by remark 2.2.6 because $\mathbf{L}(\Delta)_Z$ and $\mathbf{A}(\Delta)_Z$ are SNCS and $(\mathbf{L}(\Delta)_Z)^{=1} = (\mathbf{A}(\Delta)_Z)^{=1}$. Hence $(Z, \mathbf{L}(\Delta)_Z)$ is PLT because $(Z, \mathbf{A}(\Delta)_Z)$ is such by [KM00, 2.30].
In particular $(Z, \mathbf{L}(\Delta)_Z)$ is a DLT pair, because $\mathbf{L}(\Delta)_Z$ is effective.
Now $(f^*(aD) + F) - (K_Z + \mathbf{L}(\Delta)_Z) = f^*(aD) - (K_Z + \mathbf{A}(\Delta)_Z) \equiv f^*(aD - (K_X + \Delta))$ is big and nef.
Moreover we can write
$$f^*(aD) + F = h^*(aP) + (h^*(aN) + F) = P' + N',$$
where $P' := h^*(aP)$ is nef and $N' := h^*(aN) + F$ is effective.
We show now that $\mathbb{B}_+((f^*(aD) + F) - (K_Z + \mathbf{L}(\Delta)_Z))$ does not contain any LC center of the pair $(Z, \mathbf{L}(\Delta)_Z)$.
We use that, as the $\mathbb{B}_+$ only depends by the numerical equivalence class,
$$\mathbb{B}_+((f^*(aD) + F) - (K_Z + \mathbf{L}(\Delta)_Z)) = \mathbb{B}_+(f^*(aD - (K_X + \Delta))) \subseteq$$
$$\subseteq \text{exc}(f) \cup f^{-1}(\mathbb{B}_+(aD - (K_X + \Delta))),$$
thanks to lemma 2.3.8.
Let $V \in CLC(Z, \mathbf{L}(\Delta)_Z) = CLC(Z, \mathbf{A}(\Delta)_Z)$. Then V is a prime divisor because $(Z, \mathbf{L}(\Delta)_Z)$ is PLT and it is not contracted by f because $f(V) \in CLC(X, \Delta)$ and (X, Δ) is also PLT. Then, by remark 2.1.5, $V \not\subseteq \text{exc}(f)$.

Moreover, as $f(V) \not\subseteq \mathbb{B}_+(aD - (K_X + \Delta))$ by hypothesis, we have that $V \not\subseteq f^{-1}(\mathbb{B}_+(aD - (K_X + \Delta)))$.

Thus, as V is irreducible, it follows that $V \not\subseteq \mathbb{B}_+((f^*(aD) + F) - (K_Z + \mathbf{L}(\Delta)_Z))$. We can then apply theorem 3.3.7 and we find that $\kappa(Z, f^*(aD) + F) \geq 0$, so that there exists $m \in \mathbb{N}$ such that $m(f^*(aD) + F)$ is integral and

$$H^0(Z, \mathcal{O}_Z(m(f^*(aD) + F))) \neq 0.$$

Hence

$$0 \neq h^0(Z, \mathcal{O}_Z(f^*(maD) + mF)) = h^0(X, \mathcal{O}_X(maD) \otimes f_*\mathcal{O}_Z(mF)) =$$

$$= h^0(X, \mathcal{O}_X(maD))$$

thanks to Fujita's lemma because F is f-exceptional. Thus $\kappa(X, D) \geq 0$. $\qquad \square$

Logbig DLT case

The following theorem is a simplified version of theorem 5.1 of [Fuj07b]:

Theorem 3.3.10. *Let (X, B) be an LC pair such that X is smooth and B is SNCS.*
Write $B = B_+ - B_-$, where B_+ and B_- are effective $\mathbb{Q}$-divisors, and they have not common components. Let $P \in Div_{\mathbb{Q}}(X)$ be such that

1. *P is nef;*

2. *$aP - (K_X + B)$ is logbig for the pair (X, B) and nef for some rational number $a \geq 0$;*

3. *There exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner B_- \urcorner));$$

Then P is semiample.

From the above theorem we deduce in the DLT case the following result, that generalizes corollary 3.3.4:

Theorem 3.3.11. *Let (X, Δ) be an effective LC pair and let $D \in Div_{\mathbb{Q}}(X)$. Consider the following assumptions:*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

(2') *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

3. *There exists a projective birational morphism $f : Z \to X$ such that*

$$f^*(D) = P + N$$

is a $\mathbb{Q}$-CKM Zariski decomposition and the pair $(Z, \mathbf{L}(\Delta)_Z)$ is a DLT pair;

4. *P is logbig for the pair $(Z, \mathbf{L}(\Delta)_Z)$;*

(4') $f^*(aD - (K_X + \Delta))$ is logbig for the pair $(Z, \mathbf{L}(\Delta)_Z)$.

If D satisfies 1,2,3,4 or 2',3,4', then P is semiample.

Proof. Let us apply lemma 2.5.3 and take $t_0 \in \mathbb{Q}$ and D', P', N', Δ_Z Weil $\mathbb{Q}$-divisors on Z as in the lemma, so that in particular $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition.

Define $B := \Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN \leq \mathbf{L}(\Delta)_Z$, so that $t_0 P' - (K_Z + B) = P + f^*(aD - (K_X + \Delta))$ is big and nef.

Moreover $t_0 P' - (K_Z + B)$ is logbig for the pair $(Z, \mathbf{L}(\Delta)_Z)$ because, by hypothesis, its restriction to every LC center of the pair $(Z, \mathbf{L}(\Delta)_Z)$ is the sum of a big and a nef divisor. Then we have that $t_0 P' - (K_Z + B)$ is logbig for the pair (Z, B), because $CLC(Z, B) \subseteq CLC(Z, \mathbf{L}(\Delta)_Z)$ by lemma 2.2.8.

Now thanks to the theorem 2.2.23, the DLTness of $(Z, \mathbf{L}(\Delta)_Z)$ implies that every LC center of $(Z, \mathbf{L}(\Delta)_Z)$ is not contained in $Sing(Z) \cup NSNC(\mathbf{L}(\Delta)_Z)$. Thus by lemma 2.2.12 we have that every LC center of the pair (Z, B) is not contained in $Sing(Z) \cup NSNC(B)$.

Let $\mu : Z' \to Z$ be a standard log-resolution of the pair (Z, B), so that $(Z', \mathbf{A}(B)_{Z'})$ is an LC pair, Z' is smooth and $\mathbf{A}(B)_{Z'}$ is SNCS.

Now we choose $k_0 \in \mathbb{N}$ such that $k_0 P'$ is a Cartier divisor, $k_0 D'$ is integral and

$$H^0(Z, \mathcal{O}_Z(mk_0 P')) \simeq H^0(Z, \mathcal{O}_Z(mk_0 D'))$$

for all $m \in \mathbb{N}$.

Moreover if we write $\mathbf{A}(B)_{Z'} = (B')_+ - (B')_-$, where $(B')_+$ and $(B')_-$ are effective divisors and they have not common components, then $\mu_*(\ulcorner (B')_- \urcorner) \leq \ulcorner N' \urcorner$, because Δ_Z is effective. Thus by lemma 2.1.8 we get that for all $m \in \mathbb{N}$

$$h^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0 P') + \ulcorner (B')_- \urcorner)) \leq h^0(Z, \mathcal{O}_Z(mk_0 P' + \ulcorner N' \urcorner)) \leq$$

$$\leq h^0(Z, \mathcal{O}_Z(mk_0 D')) = h^0(Z, \mathcal{O}_Z(mk_0 P')) = h^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0 P'))),$$

so that $H^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0 P'))) \simeq H^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0 P') + \ulcorner (B')_- \urcorner))$.
Note also that

$$t_0 \mu^*(P') - (K_{Z'} + \mathbf{A}(B)_{Z'}) \equiv \mu^*(t_0 P' - (K_Z + B))$$

is big and nef, being the birational pullback of a big and nef divisor.

We will prove that $\mu^*(t_0 P' - (K_Z + B))$ is logbig for the pair $(Z', \mathbf{A}(B)_{Z'})$:
Let $V \in CLC(Z', \mathbf{A}(B)_{Z'})$. Then $\mu(V) \not\subseteq Sing(Z) \cup NSNC(B)$. Thanks to the choice of μ this implies that $V \not\subseteq exc(\mu)$. Then $\mu_{|V}$ is birational.
Consider the following commutative diagram:

$$
\begin{array}{ccc}
V & \hookrightarrow & Z' \\
\downarrow {\scriptstyle \mu_{|V}} & & \downarrow {\scriptstyle \mu} \\
\mu(V) & \hookrightarrow & Z
\end{array}
$$

We know that $t_0 P' - (K_Z + B)$ is logbig for the pair (Z, B), which implies that $\left(t_0 P' - (K_Z + B) \right)_{|\mu(V)}$ is big.
Then, by birationality of $\mu_{|V}$, we have that $\mu_{|V}^* \left((t_0 P' - (K_Z + B))_{|\mu(V)} \right)$ is a big $\mathbb{Q}$-divisor on V.

But, by commutativity of the diagram, we have that

$$\mu_{|_V}^* \big((t_0 P' - (K_Z + B))_{|_{\mu(V)}}\big) = \big(\mu^*(t_0 P' - (K_Z + B))\big)_{|_V}.$$

Thus we have proved that $\mu^*(t_0 P' - (K_Z + B))$ is big when restricted to each LC center of the pair $(Z', \mathbf{A}(B)_{Z'})$, whence it is logbig for the pair $(Z', \mathbf{A}(B)_{Z'})$.
Hence $t_0 \mu^*(P') - (K_{Z'} + \mathbf{A}(B)_{Z'})$ is logbig for the pair $(Z', \mathbf{A}(B)_{Z'})$.
Therefore we can apply theorem 3.3.10 to the divisor $\mu^*(P')$ and the pair $(Z', \mathbf{A}(B)_{Z'})$, so that $\mu^*(P')$ is semiample, which implies that P is semiample.
$\square$

3.4 LC pairs "approximated" by KLT pairs

The following theorem implies Conjecture 2 when X is a $\mathbb{Q}$-Gorenstein variety and we can "approximate" the pair (X, Δ) by KLT pairs.

Theorem 3.4.1. *Let X be a normal projective $\mathbb{Q}$-Gorenstein variety and let $\Delta \in Div_{\mathbb{Q}}(X)$ be effective and such that (X, Δ) is an LC pair and $(X, (1-b)\Delta)$ is a KLT pair for some rational number $b > 0$.*
If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *D is big;*

2. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

3. *D admits a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N;$$

then there exists $\beta > 0$ such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

then P is semiample.

Proof. First of all we can assume that $\Delta \neq 0$:
In fact, if $\Delta = 0$ the hypotheses imply that the pair $(X, 0)$ is KLT. In particular $(X, 0)$ is a DLT pair, so that we are under the hypotheses of theorem 3.3.3 and we are done.

Being $\Delta \neq 0$, we can apply lemma 2.2.10 and we have that $\text{Supp}(\Delta)$ contains all the LC centers of the pair (X, Δ).
Moreover note that P is big because D is such and $\mathbb{B}_+(P) = \mathbb{B}_+(D)$ by lemma 2.5.4. Then, thanks to lemma 2.3.7, we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $P - \lambda\Gamma$ is ample, the pair $(X, \Delta + \lambda\Gamma)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$.
Now, as $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in \mathbb{Q}^+$ such that if $0 \leq \beta' < \beta$, then the pair $(X, \Delta + \lambda\Gamma + \beta'N)$ is LC and $CLC(X, \Delta + \lambda\Gamma + \beta'N) = CLC(X, \Delta)$.
This implies that $\text{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta + \lambda\Gamma + \beta'N)$.

Hence, by applying again lemma 2.2.10, we have that for every rational number $\epsilon \in (0, 1)$, for every $\beta' \in [0, \beta)$, the pair $(X, (1 - \epsilon)\Delta + \lambda\Gamma + \beta'N)$ is KLT and $(1 - \epsilon)\Delta + \lambda\Gamma + \beta'N$ is effective.

Now we have that

$$(1 + a)P + aN - (K_X + (1 - \epsilon)\Delta + \lambda\Gamma) = (P - \lambda\Gamma) + (aD - (K_X + \Delta)) + \epsilon\Delta$$

is ample, thanks to the openness of the ample cone, for $\epsilon > 0$ small enough. In other words there exists a rational number $\epsilon_0 \in (0, 1)$ such that

$$(1 + a)P + aN - (K_X + (1 - \epsilon_0)\Delta + \lambda\Gamma)$$

is ample.

If $a \geq 0$ we conclude by applying theorem 3.3.2 to the pair $(X, (1 - \epsilon_0)\Delta + \lambda\Gamma)$. If $-\beta < a < 0$ we can apply theorem 3.3.2 to the pair $(X, (1 - \epsilon_0)\Delta + \lambda\Gamma - aN)$. $\qquad\square$

Remark 3.4.2. Note that lemma 2.2.24 implies that, in the $\mathbb{Q}$-factorial case, theorem 3.4.1 is a generalization of corollary 3.3.4.

3.5 Reducing LC to DLT via DLT blow-up

In this section we try to reduce Conjecture 2 to the case when the pair (X, Δ) is DLT, already proved in section 3.3. In order to do this we use a DLT blow-up (see theorem 3.5.1) to turn our pair into a DLT one. The existence of DLT blow-ups has been recently proved as a consequence of the main results of the paper [BCHM10]. Unfortunately in the end we need to add supplementary hypotheses on the LC centers of the pair (see theorem 3.5.3).

Theorem 3.5.1 (DLT blow-up)**.** *Let (X, Δ) be an effective LC pair. Then there exists a projective birational morphism $\mu : Y \to X$, from a normal projective variety, with the following properties:*

1. *Y is $\mathbb{Q}$-factorial;*

2. *$a(E, X, \Delta) = -1$ for every μ-exceptional divisor E on Y;*

3. *If we put*
$$\Gamma = \mu_*^{-1}\Delta + \sum_{E \subseteq exc(\mu)} E,$$

    then (Y, Γ) is a DLT pair and $K_Y + \Gamma \sim_{\mathbb{Q}} \mu^*(K_X + \Delta)$.

μ is called a DLT blow-up of the pair (X, Δ).

For the proof see [Fuj09b, theorem 10.5].

Lemma 3.5.2. *Let $\mu : Y \to X$ be a projective birational morphism of normal projective varieties and suppose X is $\mathbb{Q}$-factorial. Let L be an effective Cartier divisor on Y such that L is μ-exceptional and $-L$ is μ-ample. Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have that*

1. *$\mu(Supp(L)) = Supp(\mu_*(\mathcal{O}_L))$;*

2. $R^j\mu_*\mathcal{O}_Y(-mL) = 0$ *for every* $j > 0$;

3. $\mu(\text{Supp}(L)) = \mathcal{Z}(\mu_*(\mathcal{O}_Y(-mL)))$.

Proof. **1)** We prove separately the two inclusions:

$\supseteq$) Take $x \in \text{Supp}(\mu_*(\mathcal{O}_L))$. Then $(\mu_*\mathcal{O}_L)_x \neq 0$.

This implies that for every open set V containing x there exists an open subset $U \subseteq V$ such that $x \in U$ and $\mathcal{O}_L(\mu^{-1}(U)) = \mu_*\mathcal{O}_L(U) \neq 0$.

If, by contradiction, $x \notin \mu(\text{Supp}(L))$, then take $V = X \setminus \mu(\text{Supp}(L))$, and take $U \subseteq V$.

Hence $\mu^{-1}(U) \subseteq \mu^{-1}(V) = \mu^{-1}(X \setminus \mu(\text{Supp}(L))) \subseteq Y \setminus \text{Supp}(L)$.

Thus $\mathcal{O}_L(\mu^{-1}(U)) = 0$ and we find the contradiction.

$\subseteq$) Consider the morphism of schemes

$$\mu_{|L} : L \to \mu(L).$$

and consider the induced morphism

$$\mu_{|L}^\sharp : \mathcal{O}_{\mu(L)} \to (\mu_{|L})_*\mathcal{O}_L$$

Then, by [Gro60, 9.5.2] we have that $ker\mu_{|L}^\sharp = 0$, that is $\mu_{|L}^\sharp$ is injective.

Now note that $\text{Supp}(L) \supseteq \text{divexc}(\mu)$, because L is effective and $-L$ is μ-ample, and $\text{Supp}(L) \subseteq \text{divexc}(\mu)$ by hypothesis.

Then $\text{Supp}(L) = \text{divexc}(\mu) = \text{exc}(\mu)$ because X is $\mathbb{Q}$-factorial (see [KM00, corollary 2.63]).

Hence $\text{Supp}(L) = \mu^{-1}(\mu(\text{Supp}(L)))$ by remark 2.1.5, which implies that $\mu_*\mathcal{O}_L = (\mu_{|L})_*\mathcal{O}_L$.

Thus

$$\text{Supp}(\mu_*\mathcal{O}_L) = \text{Supp}((\mu_{|L})_*\mathcal{O}_L) \supseteq \text{Supp}(\mathcal{O}_{\mu(L)}) = \mu(\text{Supp}(L)).$$

2) Thanks to [KM00, Prop. 1.45], given an ample divisor A on X, we have that $\mu^*(k_0A) - L$ is ample for some $k_0 \in \mathbb{N}$. Moreover $\mu^*(k_0A)$ is nef, as k_0A is such. Then, thanks to Fujita's vanishing theorem (see[Laz04, theorem 1.4.35]), there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$H^j(Y, \mathcal{O}_Y(m(k_0\mu^*(A) - L) + s\mu^*(k_0A))) = 0 \qquad \forall j > 0, \ \forall s \in \mathbb{N}.$$

Note that the integer m_0 does not depend on s.

Equivalently we have that, for all $m \geq m_0$,

$$H^j(Y, \mathcal{O}_Y(\mu^*((m+s)k_0A) - mL)) = 0 \qquad \forall j > 0, \ \forall s \in \mathbb{N},$$

that is nothing but saying that for all $m \geq m_0$ we have

$$H^j(Y, \mathcal{O}_Y(\mu^*(tk_0A)) \otimes \mathcal{O}_Y(-mL)) = 0 \qquad \forall j > 0, \ \forall t \geq m$$

Hence, as k_0A is ample, by [Laz04, lemma 4.3.10] we get that $R^j\mu_*\mathcal{O}_Y(-mL) = 0$ for all $j > 0$ and for all $m \geq m_0$.

3) Fix an integer $m \geq m_0$ and consider the short exact sequence

$$0 \to \mathcal{O}_Y(-mL) \to \mathcal{O}_Y \to \mathcal{O}_{mL} \to 0.$$

By applying μ_* we get a long exact sequence

$$0 \to \mu_*\mathcal{O}_Y(-mL) \to \mu_*\mathcal{O}_Y \to \mu_*\mathcal{O}_{mL} \to R^1\mu_*\mathcal{O}_Y(-mL) \to \ldots$$

But $\mu_*\mathcal{O}_Y = \mathcal{O}_X$ because μ is birational and $R^1\mu_*\mathcal{O}_Y(-mL) = 0$ by part 2 of the lemma. Hence we get a short exact sequence

$$0 \to \mu_*\mathcal{O}_Y(-mL) \to \mathcal{O}_X \to \mu_*\mathcal{O}_{mL} \to 0.$$

Thus, by the first part of the lemma it follows that $\mathcal{Z}(\mu_*(\mathcal{O}_Y(-mL))) = \mathrm{Supp}(\mu_*\mathcal{O}_{mL}) = \mu(\mathrm{Supp}(mL)) = \mu(\mathrm{Supp}(L))$. $\qquad\square$

Theorem 3.5.3. *Let (X, Δ) be an effective LC pair such that X is $\mathbb{Q}$-factorial. Let $D \in Div_{\mathbb{Q}}(X)$.*
If

1. *D is big;*

2. *D has a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$ such that*

   - *$\mathbb{B}_+(D) \not\supseteq V$, for every $V \in CLC(X, \Delta)$;*
   - *$\mathbb{B}_+(D) \cap V = \emptyset$, for every $V \in CLC(X, \Delta)$ such that $V \not\subseteq Supp(\Delta)$;*

then there exists $\beta > 0$ such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

then P is semiample.

Proof. Note that, by definition of Zariski decomposition, $\kappa(X, P) = \kappa(X, D) = dimX$, so that P is big.
Moreover by lemma 2.5.4 we have that $\mathbb{B}_+(P) = \mathbb{B}_+(D)$, so that $\mathbb{B}_+(P)$ does not contain any LC center of the pair (X, Δ).
Then, thanks to lemma 2.3.7 there exist a Cartier divisor $M > 0$ and $\lambda \in \mathbb{Q}^+$ such that $P - \lambda M$ is ample, $(X, \Delta + \lambda M)$ is an LC pair and $CLC(X, \Delta) = CLC(X, \Delta + \lambda M)$.
Now, as $\mathrm{Supp}(N) \subseteq \mathbb{B}_+(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in (0, 1)$ such that if $0 \le \beta' < \beta$, then the pair $(X, \Delta + \lambda M + \beta' N)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda M + \beta' N)$.
Suppose $a > -\beta$ is a rational number such that $aD - (K_X + \Delta)$ is nef and define $a' = -\min\{0, a\}$, so that $0 \le a' < \beta$. Note that $a + 1 > 0$, because $a > -\beta > -1$.
If $\Delta \ne 0$, then, thanks to lemma 2.2.9, for every $\delta \in \mathbb{Q}^+$, if $V \in CLC(X, \Delta + \lambda M + a'N)$ is such that $V \subseteq \mathrm{Supp}(\Delta)$, then $V \notin CLC(X, \Delta + \lambda M - \delta\Delta + a'N) = CLC(X, (1 - \delta)\Delta + \lambda M + a'N)$.
The same thing trivially holds if $\Delta = 0$, because $\mathrm{Supp}(0) = \emptyset$.
Let us define

$$\Delta'_\delta = (1 - \delta)\Delta + \lambda M + a'N,$$

and let

$$\{V_1, \ldots, V_s\} = \{V \in CLC(X, \Delta) : V \not\subseteq \mathrm{Supp}(\Delta)\}.$$

Then, by lemma 2.2.8, for every $\delta \in \mathbb{Q}^+$ we get that (X, Δ'_δ) is an LC pair such that

$$CLC(X, \Delta'_\delta) \subseteq \{V_1, \ldots, V_s\}.$$

Moreover if we define $a'' = \max\{0, a\}$ we have that $a'' \geq 0$ and $a'' - a' = a$, so that

$$(1+a)P + a''N - (K_X + \Delta'_\delta) = (1+a)P + a''N - (K_X + (1-\delta)\Delta + \lambda M + a'N) =$$

$$= (P - \lambda M) + (aD - (K_X + \Delta)) + \delta\Delta$$

is ample for δ sufficiently small thanks to the openness of the ample cone. Let us denote $\Delta' = \Delta'_\delta$, for a sufficiently small fixed δ such that the above ampleness holds and Δ' is effective.
Let $\mu : Y \to X$ be a DLT blow-up of the pair (X, Δ'), whose existence is assured by theorem 3.5.1.
Take

$$\Delta'_Y = \mu_*^{-1}\Delta' + \sum_{E\ \mu-exc} E,$$

so that $K_Y + \Delta'_Y \sim_\mathbb{Q} \mu^*(K_X + \Delta')$ and (Y, Δ'_Y) is a DLT pair.

If μ is an isomorphism, then the pair (X, Δ') is DLT. Hence we can apply theorem 3.3.2 to the pair (X, Δ') because we have verified that all the hypotheses are satisfied and we are done.
Thus we can assume from now on that μ is not an isomorphism.

By [KM00, lemma 2.62] there exists an effective μ-exceptional Cartier divisor G on Y such that $-G$ is μ-ample.
Hence $\mathrm{Supp}(G) \subseteq \mathrm{divexc}(\mu)$ and $G \neq 0$ because $-G$ is μ-ample and μ contracts some curve.
Thus there exists a set $J \subseteq \{1, \ldots, s\}$ such that $J \neq \emptyset$ and

$$\mu(\mathrm{Supp}(G)) = \bigcup_{j \in J} V_j,$$

because, thanks to the properties of the DLT blow-up, every μ-exceptional divisor on Y maps onto a LC center of the pair (X, Δ').
Take

$$L = \mu^*(D) - \epsilon G,$$

for a sufficiently small rational number $\epsilon > 0$.
We will show that the pair (Y, Δ'_Y) and the $\mathbb{Q}$-divisor L on Y satisfy the hypotheses of theorem 3.3.2.

We begin by proving that the decomposition

$$L = (\mu^*(P) - \epsilon G) + (\mu^*(N))$$

is a $\mathbb{Q}$-CKM Zariski decomposition.
The effectivity of $\mu^*(N)$ is trivial. Let us prove that $\mu^*(P) - \epsilon G$ is nef:

Let A' be an ample divisor on X such that

$$\mathbb{B}_+(P) = \mathbb{B}(P - A') = Bs(|m_0(P - A')|)$$

for some $m_0 \in \mathbb{N}$, so that, by lemma 2.5.4, $\mathbb{B}_+(D) = Bs(|m_0(P - A')|)$.

Let Γ be a general divisor in the linear series $|m_0(P - A')|$, so that $Bs(|\Gamma|) = \mathbb{B}_+(P)$.

Thanks to the hypotheses on $\mathbb{B}_+(D)$, we can suppose that $\mathrm{Supp}(\Gamma)$ does not contain any LC center of the pair (X, Δ).

Take $A = P - \frac{1}{m_0}\Gamma$, so that

$$\mu^*(P) - \epsilon G = \mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma) - \epsilon G.$$

Then $A \equiv A'$, so it is ample.

Hence, thanks to [KM00, 1.45], we get that $\mu^*(sA) - G$ is ample for every $s \gg 0$, which implies that $\mu^*(A) - \epsilon G$ is ample for sufficiently small $\epsilon \in \mathbb{Q} \cap (0, 1)$.

Moreover, $\mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma) = \mu^*(P)$ is nef.

Let C be an irreducible curve on Y. We want to verify the nefness of $\mu^*(P) - \epsilon G$ by checking that its intersection with C is non-negative.

If $C \nsubseteq \mathrm{Supp}(\mu^*(\Gamma))$, then, by effectivity of Γ, we get that

$$((\mu^*(P) - \epsilon G) \cdot C) = ((\mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma) - \epsilon G) \cdot C) \geq ((\mu^*(A) - \epsilon G) \cdot C) > 0$$

because $\mu^*(A) - \epsilon G$ is ample.

If $C \subseteq \mathrm{Supp}(\mu^*(\Gamma))$ and C is contracted by μ, then

$$((\mu^*(P) - \epsilon G) \cdot C) = ((\mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma) - \epsilon G) \cdot C) =$$

$$= ((\mu^*(A) - \epsilon G) \cdot C) + \frac{1}{m_0}(\Gamma \cdot \mu_*(C)) = ((\mu^*(A) - \epsilon G) \cdot C) > 0.$$

Suppose $C \subseteq \mathrm{Supp}(\mu^*(\Gamma))$ and C is not contracted by μ.

If $C \cap \mathrm{Supp}(G) = \emptyset$, then

$$((\mu^*(P) - \epsilon G) \cdot C) = ((\mu^*(P) \cdot C) \geq 0.$$

In order to conclude the check we just lack the case when $C \subseteq \mathrm{Supp}(\mu^*(\Gamma))$, C is not contracted by μ and $C \cap \mathrm{Supp}(G) \neq \emptyset$:

In this case

$$\mu(C \cap \mathrm{Supp}(G)) \subseteq \mu(\mathrm{Supp}(G)) = \bigcup_{j \in J} V_j \subseteq \bigcup_{j=1}^{s} V_j.$$

Hence

$$Bs(|\Gamma|) \cap \mu(C \cap \mathrm{Supp}(G)) = \mathbb{B}_+(P) \cap \mu(C \cap \mathrm{Supp}(G)) \subseteq \mathbb{B}_+(P) \cap (\bigcup_{j=1}^{s} V_j) = \emptyset$$

by hypothesis.

Then we can choose an effective divisor $\Gamma' \sim \Gamma$ such that $\mathrm{Supp}(\Gamma') \nsupseteq \mu(C \cap \mathrm{Supp}(G))$, so that $\mathrm{Supp}(\Gamma') \nsupseteq \mu(C)$, which in turn implies that $(\Gamma' \cdot \mu_*(C)) \geq 0$.

Hence

$$((\mu^*(P) - \epsilon G) \cdot C) = ((\mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma') - \epsilon G) \cdot C) =$$

$$= ((\mu^*(A) - \epsilon G) \cdot C) + \frac{1}{m_0}(\Gamma' \cdot \mu_*(C)) \geq ((\mu^*(A) - \epsilon G) \cdot C) > 0.$$

Thus we have that $\mu^*(P) - \epsilon G$ is nef.

In order to prove that we have a $\mathbb{Q}$-CKM Zariski decomposition it remains to show that there exists an integer $k > 0$ such that kL and $k(\mu^*(P) - \epsilon G)$ are Cartier divisors on Y and for every $p \in \mathbb{N}$ the natural injective maps

$$H^0(Y, \mathcal{O}_Y(pk(\mu^*(P) - \epsilon G))) \longrightarrow H^0(Y, \mathcal{O}_Y(pkL))$$

are bijective.

But we know, by hypothesis, that there exists $k_0 \in \mathbb{N}$ such that $k_0 P$ and $k_0 D$ are Cartier divisors on X and, for every integer $p > 0$ we have that

$$H^0(X, \mathcal{O}_X(pk_0 P)) \simeq H^0(X, \mathcal{O}_X(pk_0 D)).$$

Moreover, by lemma 3.5.2 there exists an integer k_1 such that for all $t \geq k_1$

$$\mathcal{Z}(\mu_*(\mathcal{O}_Y(-tG))) = \mu(\text{Supp}(G)).$$

We choose an integer b such that $\frac{b}{\epsilon}$ is an integer and we define $k := \frac{b}{\epsilon} \cdot k_0 \cdot k_1$. Then $k\mu^*(P)$, $k\epsilon G$ and $k\mu^*(D)$ are Cartier divisors. Hence, the same holds for kL and $k(\mu^*(P) - \epsilon G)$.

Moreover, for every $m \in k\mathbb{N}$, we get that

$$H^0(Y, \mathcal{O}_Y(m(\mu^*(P) - \epsilon G))) \simeq H^0(X, \mu_*(\mathcal{O}_Y(m(\mu^*(P) - \epsilon G)))) \simeq$$
$$\simeq H^0(X, \mathcal{O}_X(mP) \otimes \mu_* \mathcal{O}_Y(-m\epsilon G)).$$

In the same way

$$H^0(Y, \mathcal{O}_Y(mL)) \simeq H^0(X, \mathcal{O}_X(mD) \otimes \mu_* \mathcal{O}_Y(-m\epsilon G)).$$

Now, define $\mathcal{J} = \mu_* \mathcal{O}_Y(-m\epsilon G)$, so that, thanks to our choice of k, for all $m \in k\mathbb{N}$ we have that

$$\mathcal{Z}(\mathcal{J}) = \mu(\text{Supp}(G)) = \bigcup_{j \in J} V_j.$$

In other words, if we denote by $\mathcal{Z}$ the scheme defined by the sheaf of ideals $\mathcal{J}$, then, set-theoretically,

$$\mathcal{Z} = \bigcup_{j \in J} V_j.$$

Now note that $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ by lemma 2.5.4. Then, thanks to the hypotheses, we find that $\text{Supp}(N) \not\supseteq V_j$ for all $j \in J$. Hence $mN_{|\mathcal{Z}}$ is effective for $m \in k\mathbb{N}$.

Thus, for every $m \in k\mathbb{N}$, we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^0(X, \mathcal{O}_X(mD) \otimes \mathcal{J}) & \longrightarrow & H^0(X, \mathcal{O}_X(mD)) & \xrightarrow{\phi} & H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(mD)) \\
& & & & \uparrow g & & \uparrow h \\
0 & \longrightarrow & H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{J}) & \longrightarrow & H^0(X, \mathcal{O}_X(mP)) & \xrightarrow{\psi} & H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(mP))
\end{array}
$$

where the rows are exact, g is an isomorphism thanks to the choice of k and h is injective.

Commutativity of the above diagram leads, on the one hand, to the existence of a map

$$f : H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{J}) \longrightarrow H^0(X, \mathcal{O}_X(mD) \otimes \mathcal{J}),$$

obtained by restricting g.

On the other hand if we denote by $\Phi \subseteq H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(mD))$ the image of ϕ and by $\Psi \subseteq H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(mP))$ the image of ψ, it provides the existence of an injective map $h' : \Psi \to \Phi$, obtained by restricting h.

We obtain, in a such a way, the existence of the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X, \mathcal{O}_X(mD) \otimes \mathcal{J}) & \longrightarrow & H^0(X, \mathcal{O}_X(mD)) & \overset{\phi}{\longrightarrow} & \Phi & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle g} & & \Big\uparrow{\scriptstyle h'} & & \\
0 & \longrightarrow & H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{J}) & \longrightarrow & H^0(X, \mathcal{O}_X(mP)) & \overset{\psi}{\longrightarrow} & \Psi & \longrightarrow & 0
\end{array}
$$

where again the rows are exact, g is an isomorphism and h' is injective because h is such.

Therefore, thanks to the snake lemma, we get that the map f is surjective. This implies that for every $m \in k\mathbb{N}$

$$h^0(Y, \mathcal{O}_Y(m(\mu^*(P) - \epsilon G))) = h^0(X, \mathcal{O}_X(mP) \otimes \mathcal{J}) \geq h^0(X, \mathcal{O}_X(mD) \otimes \mathcal{J}) =$$

$$= h^0(Y, \mathcal{O}_Y(mL)).$$

Thus, for all $m \in k\mathbb{N}$, the injective maps

$$H^0(Y, \mathcal{O}_Y(m(\mu^*(P) - \epsilon G))) \longrightarrow H^0(Y, \mathcal{O}_Y(mL))$$

are bijective.

Now we prove the additional hypothesis of theorem 3.3.2:
We have that

$$(1 + a)(\mu^*P - \epsilon G) + a''\mu^*(N) - (K_Y + \Delta'_Y) \equiv$$

$$\equiv (1 + a)(\mu^*P - \epsilon G) + a''\mu^*(N) - \mu^*(K_X + \Delta') =$$

$$= \mu^*[(1 + a)P + a''N - (K_X + \Delta')] - (1 + a)\epsilon G.$$

Then it is ample if ϵ is sufficiently small thanks to [KM00, 1.45] because we checked that $(1+a)P + a''N - (K_X + \Delta')$ is ample, $-G$ is μ-ample and $1 + a > 0$.

In order to apply theorem 3.3.2 it remains to show that $\mathbb{B}(\mu^*(P) - \epsilon G)$ does not contain any LC center of the pair (Y, Δ'_Y):
Let $V \in CLC(Y, \Delta'_Y)$. Then $\mu(V) \in CLC(X, \Delta') \subseteq CLC(X, \Delta)$.
Now

$$\mathbb{B}(\mu^*(P) - \epsilon G) \subseteq \mathbb{B}_+(\mu^*(P) - \epsilon G) = \bigcap \text{Supp}(\mu^*(P) - \epsilon G - H),$$

where the intersection is taken over all the ample $\mathbb{R}$-divisors H such that $\mu^*(P) - \epsilon G - H \geq 0$.

We proved that $\mu^*(P) - \epsilon G = \mu^*(A) + \frac{1}{m_0}\mu^*(\Gamma) - \epsilon G$, where $\mu^*(A) - \epsilon G$ is ample and Γ is effective.

Hence

$$\mathbb{B}(\mu^*(P) - \epsilon G) \subseteq \mathrm{Supp}(\mu^*(\Gamma)).$$

But we have that $\mathrm{Supp}(\Gamma)$ does not contain any LC center of the pair (X, Δ). Hence $\mu(V) \nsubseteq \mathrm{Supp}(\Gamma)$, so that $V \nsubseteq \mathrm{Supp}(\mu^*(\Gamma))$. Thus $\mathbb{B}(\mu^*(P) - \epsilon G) \nsupseteq V$.

All the hypotheses being satisfied, we can now apply theorem 3.3.2, so that we get the semiampleness of $\mu^*(P) - \epsilon G$.

In other words, there exists $l \in \mathbb{N}$ such that $l(\mu^*(P) - \epsilon G)$ is a Cartier divisor and $Bs(|l(\mu^*(P) - \epsilon G)|) = \emptyset$. This implies that

$$Bs(|l\mu^*(P)|) \subseteq \mathrm{Supp}(G) \Rightarrow \mu^{-1}(Bs(|lP|)) \subseteq \mathrm{Supp}(G).$$

But μ is surjective because it is a projective birational morphism. Thus

$$\mathbb{B}(P) \subseteq Bs(|lP|) \subseteq \mu(\mathrm{Supp}(G)) = \bigcup_{j \in J} V_j.$$

But $\mathbb{B}(P) \subseteq \mathbb{B}_+(P)$. Hence, by hypothesis, $\mathbb{B}(P) \cap V_j = \emptyset$ for all $j \in J$. Therefore $\mathbb{B}(P) = \emptyset$, that is P is semiample. $\qquad\square$

3.6 Relatively KLT case

In this section we present a simplified version of the main theorem of [Amb05] (see theorem 3.6.4) that corresponds to a generalization of Kawamata's theorem 1.0.1 in the particular setting of relatively KLT pairs (see definition 3.6.2).

In the second part of the section we use Ambro's theorem to generalize our results of the previous sections in the case of $\mathbb{Q}$-Gorenstein varieties.

Note that we will apply Ambro's theorem 3.6.4 also in the proof of theorem 3.7.4.

Definition 3.6.1. Let (X, Δ) be a pair and let $S \subseteq X$ be a closed subset. We define

$$CLC(X, \Delta, S) = \{V \in CLC(X, \Delta) \ : \ V \cap S \neq \emptyset\}.$$

Definition 3.6.2. Let (X, Δ) be a pair, let $S \subseteq X$ be a closed subset, let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$.

We say that the pair (X, Δ) is S-KLT if $CLC(X, \Delta, S) = \emptyset$.

We say that (X, Δ) is D-KLT if $CLC(X, \Delta, \mathbb{B}(D)) = \emptyset$

We remark that, given a closed subset $S \subsetneq X$, an S-KLT pair might not be LC in general.

Ambro's theorem

Lemma 3.6.3. *Let X be a normal variety and let $V \subseteq X$ be a subvariety. Suppose P, L and M are Cartier divisors on X such that $P \leq M$ and $P + L \leq M$.*

Let $v \in H^0(M - P)$ be a section defining the effective divisor $M - P$ and let $\tau \in H^0(M - P - L)$ be a section defining the effective divisor $M - P - L$. Denote by

$$\Phi_v : H^0(P) \hookrightarrow H^0(M)$$

and

$$\Phi_\tau : H^0(P + L) \hookrightarrow H^0(M)$$

the injective maps defined by multiplication by v and by τ respectively. If

1. $\Phi_v(H^0(P)) \supseteq \Phi_\tau(H^0(P + L))$;

2. $V \nsubseteq Bs(|P + L|)$;

3. $V \nsubseteq Supp(L)$;

then $V \nsubseteq Bs(|P|)$.

Proof. By hypothesis there exists an effective divisor $\Gamma \in |P + L|$ such that $\mathrm{ord}_V(\Gamma) = 0$. Let $\sigma \in H^0(P + L)$ be a section defining Γ.
Then $\sigma\tau \in \Phi_\tau(H^0(P + L)) \subseteq \Phi_v(H^0(P))$, that is there exists $u \in H^0(P)$ such that $\sigma\tau = uv$. This implies that

$$\Gamma + M - P - L = \{\sigma\tau = 0\} = \{uv = 0\} = \{u = 0\} + M - P.$$

Hence $P' := \{u = 0\}$ is a divisor in $|P|$ such that $P' = \Gamma - L$, so that $\mathrm{ord}_V P' = \mathrm{ord}_V \Gamma - \mathrm{ord}_V L = 0$. Thus $V \nsubseteq Bs(|P|)$. $\qquad\square$

The following theorem is a simplified version of [Amb05, theorem 2.1]:

Theorem 3.6.4 (Ambro)**.** *Let* (X, Δ) *be a pair and let* $P \in Div_{\mathbb{Q}}(X)$ *be such that*

1. *P is nef*

2. *There exists a rational number t_0 such that*

$$t_0 P - (K_X + \Delta)$$

   is big and nef;

3. *There exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have that*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner \Delta_- \urcorner)),$$

   for some $\mathbb{Q}$-divisor Δ_- such that $\Delta_- \geq 0$ and $\Delta + \Delta_- \geq 0$;

4. (X, Δ) *is P-KLT;*

then P is semiample.

Proof. Let $\mathbb{N}(P) = \{m \in \mathbb{N}$ such that $mP \in Div(X)$ and $H^0(X, \mathcal{O}_X(mP)) \neq 0\}$. Then $\mathbb{N}(P) \neq \emptyset$:

In fact if $CLC(X, \Delta) \neq \emptyset$, then $\mathbb{B}(P) \neq X$, while if $CLC(X, \Delta) = \emptyset$, that is (X, Δ) is KLT, then $H^0(X, \mathcal{O}_X(mk_0P + \ulcorner \Delta_- \urcorner)) \neq 0$ for all $m \gg 0$ by Shokurov's nonvanishing (see [KM00, theorem 3.4]), so that, by hypothesis, $H^0(X, \mathcal{O}_X(mk_0P)) \neq 0$.

Now suppose, by contradiction, $\mathbb{B}(P) \neq \emptyset$ and let $m_1 \in \mathbb{N}(P)$ be such that $Bs(|m_1P|) = \mathbb{B}(P) \neq \emptyset$.

We claim that there exists $\mu : Y \to X$, a log-resolution of the pair (X, Δ) and of the linear series $|m_1P|$, and a SNC divisor $F = \sum_{j=1}^{N} F_j$ on Y such that

1. $\text{Supp}(\mu_*^{-1}(\Delta)) \cup \text{divexc}(\mu) \subseteq \text{Supp}(F)$;

2. For every $j = 1, \ldots, N$ there exist $r_j \geq 0$ such that $\mu^*(m_1P) = L + \sum r_j F_j$ and $\mu^*|m_1P| = |L| + \sum r_j F_j$, where L is an integral base-point free divisor on Y.

3. For every $j = 1, \ldots, N$ there exist arbitrarily small rational numbers $\delta_j > 0$ such that $\mu^*(t_0 P - (K_X + \Delta)) - \sum \delta_j F_j$ is ample.

In fact, thanks to [KMM85, corollary 0.3.6], there exists a proper birational morphism $f : X' \to X$, with X' smooth, and a SNC divisor $\sum A_k$ on X' such that $f^*(t_0 P - (K_X + \Delta)) - \sum \epsilon_k A_k$ is ample for $\epsilon_k \in \mathbb{Q}^+$ arbitrarily small.

Let $g : X'' \to X'$ be a log-resolution of the linear series $|f^*(m_1P)|$, so that $|g^*(f^*(m_1P))| = |M| + \sum s_l B_l$, where M is an integral free divisor on X'' and $\sum B_l$ is a SNC divisor on X''.

Now take

$$F' = \text{Supp}(g_*^{-1}(f_*^{-1}\Delta)) + \text{Supp}(\sum s_l B_l) + \text{Supp}(g_*^{-1} \sum \epsilon_k A_k) + \text{divexc}(f \circ g)$$

and let $h : Y \to X''$ be a log-resolution of the pair (X'', F').

We define $F = h_*^{-1}F' + \text{divexc}(h)$ and let $\mu = f \circ g \circ h : Y \to X$. Then F is SNC thanks to its definition, μ is a log-resolution of (X, Δ) because $\text{Supp}(\mu_*^{-1}(\Delta)) \cup \text{divexc } \mu \subseteq \text{Supp}(F)$. In particular note that the assertion 1 holds. Moreover

$$|\mu^*(m_1P)| = |h^*(M)| + h^*(\sum s_l B_l),$$

where $|h^*(M)|$ is free and $\text{Supp}(h^*(\sum s_l B_l)) \subseteq \text{Supp}(F)$. Hence μ is a log-resolution of $|m_1P|$ and 2 holds.

As for the assertion 3, let $\text{divexc}(g \circ h) = \sum C_s$. Note that

$$h^* g^* (f^*(t_0 P - (K_X + \Delta)) - \sum \epsilon_k A_k) - \sum \gamma_s C_s =$$

$$= \mu^*(t_0 P - (K_X + \Delta)) - (h^*(g^*(\sum \epsilon_k A_k)) + \sum \gamma_s C_s)$$

is ample for rational numbers $0 \leq \epsilon_k \ll 1$, $0 \leq \gamma_s \ll 1$, because $g \circ h$ is a morphism between smooth projective varieties (so that we can apply [KM00, lemma 2.62+ Prop. 1.45]).

But $\text{Supp}(h^*(g^*(\sum \epsilon_k A_k)) + \sum \gamma_s C_s) \subseteq \text{Supp}(F)$, and the coefficients can be made arbitrarily small by suitably choosing the ϵ_k's and the γ_s's.

In particular we can suppose all the δ_j's are strictly positive thanks to the openness of the ample cone. Thus the claim is proved.

Now, thanks to the claim, we can write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum_{j=1}^{N} b_j F_j,$$

where $\mu(F_j) \in CLC(X, \Delta)$ if $b_j \leq -1$.
On the other hand we have that $Bs(|m_1 P|) = \mu(\bigcup_{r_j \neq 0} F_j)$, so that we can suppose $r_j > 0$ for some j because $Bs(|m_1 P|) \neq \emptyset$.
Moreover if $r_j \neq 0$, then $\mu(F_j) \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$, which implies that $b_j > -1$, as $\mathbb{B}(P)$ does not contain any LC center of the pair (X, Δ), as (X, Δ) is P-KLT.
Now we define
$$c = \min_{\{j : r_j \neq 0\}} \frac{b_j + 1 - \delta_j}{r_j}.$$

By choosing the δ_j's small enough we can suppose that

$$b_j + 1 - \delta_j > 0 \text{ for all } j \text{ such that } b_j > -1.$$

Hence $c > 0$ because $b_j > -1$ if $r_j \neq 0$.
Moreover, perturbing slightly the δ_j's if necessary, we can suppose that the minimum is attained on a unique j, say $j = j_0$. Define $B := F_{j_0}$.
Now let $s := t_0 + cm_1$, and let

$$B_m := \mu^*((m - cm_1)P - (K_X + \Delta)) - \sum \delta_j F_j.$$

Then B_m is ample for every integer $m \geq s$, because $m - cm_1 \geq t_0$ and $\mu^*(P)$ is nef.
Define now
$$A' := \sum_{j \neq j_0} (-cr_j + b_j - \delta_j) F_j,$$

and, for every $m \in \mathbb{N}(P)$, let

$$Q_m := \mu^*(mP) + A' - B - K_Y$$

Then
$$Q_m := \mu^*(mP) + A' - B - K_Y \equiv$$

$$\equiv \mu^*(mP) + \sum_{j \neq j_0} (-cr_j + b_j - \delta_j) F_j - F_{j_0} - \sum b_j F_j - \mu^*(K_X + \Delta) =$$

$$= \mu^*(mP - (K_X + \Delta)) - \sum_{j \neq j_0} cr_j F_j - \sum_{j \neq j_0} \delta_j F_j + F_{j_0}(-1 - b_{j_0}) =$$

$$= \mu^*((m - cm_1)P - (K_X + \Delta)) + cL + F_{j_0}(cr_{j_0} - 1 - b_{j_0}) - \sum_{j \neq j_0} \delta_j F_j =$$

$$= \mu^*((m - cm_1)P - (K_X + \Delta)) + cL - \sum \delta_j F_j = B_m + cL.$$

Hence Q_m is ample for every integer $m \geq s$ because B_m is such and L is free.

Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that $H^1(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A' \urcorner - B)) = 0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.

This implies that the restriction homomorphism

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A' \urcorner)) \to H^0(B, \mathcal{O}_B(\mu^*(mP) + \ulcorner A' \urcorner))$$

is surjective in this case.

Let $m_2 = \min\{m \in \mathbb{N}(P) \text{ such that } m \geq s\}$. Then $\mu^*(m_2 P)_{|_B} + A'_{|_B} - K_B = Q_{m_2|_B}$ is an ample $\mathbb{Q}$-divisor.

Moreover $A'_{|_B}$ is SNCS because A' is such and B intersects transversally all the F_j's with $j \neq j_0$. Hence, in order to show that the pair $(B, -A'_{|_B})$ is KLT, it suffices to verify that all the coefficients of $A'_{|_B}$ are greater than -1.

But we have that

- if $b_j > -1$ and $r_j = 0$ then $-cr_j + b_j - \delta_j = b_j - \delta_j > -1$ (by the choice of the δ_j's);

- if $b_j > -1$, $r_j \neq 0$ and $j \neq j_0$ then $-cr_j + b_j - \delta_j > -\frac{b_j + 1 - \delta_j}{r_j} r_j + b_j - \delta_j = -1$.

- if $b_j \leq -1$ then $\mu(F_j) \in CLC(X, \Delta)$. On the other hand $\mu(B) \subseteq \mathbb{B}(P)$, so that $\mu(B) \cap \mu(F_j) = \emptyset$ because $CLC(X, \Delta, \mathbb{B}(P)) = \emptyset$. Then $B \cap F_j = \emptyset$, that is $F_{j|_B} = 0$.

Hence the pair $(B, -A'_{|_B})$ is KLT.

Thus, by Shokurov's nonvanishing theorem ([KM00, theorem 3.4]), for every integer $k > 0$ there exists $\mu_k \in \mathbb{N}$, such that $\mu_k \geq a$, $\mu_k \geq s$, μ_k is a multiple of k and

$$H^0(B, \mathcal{O}_B(\mu^*(\mu_k P) + \ulcorner A' \urcorner)) \neq 0.$$

In fact this cohomology group is non zero for every sufficiently large multiple of m_2.

Now, by hypothesis, there exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is integral and

$$H^0(X, \mathcal{O}_X(tk_0 P)) \simeq H^0(X, \mathcal{O}_X(tk_0 P + \ulcorner \Delta_- \urcorner))$$

for every $t \in \mathbb{N}$.

Let us define $m := \mu_{k_0}$. Then

$$B \not\subseteq Bs(|\mu^*(mP) + \ulcorner A' \urcorner|) \text{ and } H^0(X, \mathcal{O}_X(mP)) \simeq H^0(X, \mathcal{O}_X(mP + \ulcorner \Delta_- \urcorner)).$$

Now define

$$T = \sum_{j \neq j_0} \max\{0, \ulcorner -cr_j + b_j - \delta_j \urcorner\} F_j.$$

Then T is an effective Cartier divisor on X', $\ulcorner A' \urcorner \leq T$ by definition and we have that $\mu_* T \leq \ulcorner \Delta_- \urcorner$:

In fact note that, by its choice in the hypothesis, $\ulcorner \Delta_- \urcorner \geq \ulcorner -\Delta \urcorner$ and $\Delta_- \geq 0$.

Moreover by definition of the b_j's it follows that

$$-\Delta = \sum_{j=1}^{N} b_j \mu_*(F_j).$$

Hence

$$\ulcorner \Delta_- \urcorner \geq \sum_{j=1}^{N} \max\{0, \ulcorner b_j \urcorner\} \mu_*(F_j)$$

Thus

$$\mu_*(T) = \sum_{j \neq j_0} \max\{0, \ulcorner -cr_j + b_j - \delta_j \urcorner\} \mu_*(F_j) \leq \sum_{j \neq j_0} \max\{0, \ulcorner b_j \urcorner\} \mu_*(F_j) \leq \ulcorner \Delta_- \urcorner$$

From these inequalities, thanks to lemma 2.1.8 it follows that

$$h^0(Y, \mathcal{O}_Y(\mu^*(mP) + T)) \leq h^0(X, \mathcal{O}_X(mP + \ulcorner \Delta_- \urcorner)).$$

But, by hypothesis,

$$h^0(X, \mathcal{O}_X(mP + \ulcorner \Delta_- \urcorner)) = h^0(X, \mathcal{O}_X(mP)) = h^0(Y, \mathcal{O}_Y(\mu^*(mP))).$$

Then the injection

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP) + T))$$

is an isomorphism. Note also that $B \not\subseteq \mathrm{Supp}(\ulcorner A' \urcorner)$ by definition.
Hence we can apply lemma 3.6.3 and we get that $B \not\subseteq Bs(|\mu^*(mP)|)$, which implies that $\mu(B) \not\subseteq Bs(|mP|)$. Therefore, as $\mu(B) \subseteq Bs(|m_1 P|)$, we get that $Bs(|mP|)$ does not contain $Bs(|m_1 P|) = \mathbb{B}(P)$, and we find a contradiction. $\quad\square$

We can easily apply the previous theorem to the positive part of a $\mathbb{Q}$-CKM Zariski decomposition:

Theorem 3.6.5 (Ambro)**.** *Let X be a normal projective variety and let Δ be an effective Weil $\mathbb{Q}$-divisor. Let D be a Weil $\mathbb{Q}$-divisor such that*

1. *There exists a $\mathbb{Q}$-CKM Zariski decomposition*

   $$D = P + N;$$

2. *There exist a and t_0, rational numbers, with $a \geq 0$, such that $(X, \Delta - aN)$ is a P-KLT pair and*

   $$t_0 P - (K_X + \Delta - aN)$$

   is big and nef,

then P is semiample.

Proof. Let $B = \Delta - aN$ and let $B_- = aN$. Then $B + B_- = \Delta$ is effective and $t_0 P - (K_X + B)$ is big and nef.
Moreover, by definition of $\mathbb{Q}$-CKM Zariski decomposition there exists $k_0 \in \mathbb{N}$ such that $k_0 > a$, $k_0 P$ is a Cartier divisor, $k_0 D$ is integral and

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 D))$$

for all $m \in \mathbb{N}$.
But $\ulcorner B_- \urcorner = \ulcorner aN \urcorner \leq k_0 N$. Hence, for all $m \in \mathbb{N}$, we get that

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner B_- \urcorner)).$$

Thus we can apply theorem 3.6.4 and we get the semiampleness of P.

$\square$

Corollary 3.6.6. *Let (X, Δ) be an effective pair and let $D \in Div_{\mathbb{Q}}(X)$. Consider the following assumptions:*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \in \mathbb{Q}$;*

(2') *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \in \mathbb{Q}$;*

3. *There exists a projective birational morphism $f : Z \to X$ such that $f^*(D)$ admits a $\mathbb{Q}$-CKM Zariski decomposition*

$$f^*(D) = P + N$$

and $(Z, \mathbf{A}(\Delta)_Z - aN)$ is P-KLT.

If D satisfies 1,2, and 3, or D satisfies 2' and 3, then P is semiample.

Proof. Let us apply lemma 2.5.3, and consider $t_0 \in \mathbb{Q}$ and D', P', N', Δ_Z Weil $\mathbb{Q}$-divisors on Z as in the lemma. Then $t_0 P' - (K_Z + \Delta_Z - N')$ is big and nef and $(Z, \Delta_Z - N')$ is P'-KLT.
Thus we can apply theorem 3.6.5 and we are done. □

Using Ambro's theorem in the $\mathbb{Q}$-Gorenstein case

Let us consider now the $\mathbb{Q}$-Gorenstein case. Making use of Ambro's theorem we can give the following more general version of theorem 3.5.3:

Theorem 3.6.7. *Let (X, Δ) be an effective LC pair such that X is $\mathbb{Q}$-Gorenstein. Let $D \in Div_{\mathbb{Q}}(X)$ be such that*

1. *D is big;*

2. *$\mathbb{B}_+(D) \not\supseteq V$, for every $V \in CLC(X, \Delta)$;*

3. *D has a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N$$

such that $\mathbb{B}(P) \cap V = \emptyset$ for every $V \in CLC(X, \Delta)$ such that $V \not\subseteq Supp(\Delta)$;

then there exists $\beta > 0$ such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

then P is semiample.

Proof. Note that P is big because D is such and $\mathbb{B}_+(P) = \mathbb{B}_+(D)$ by lemma 2.5.4. Then, thanks to lemma 2.3.7, we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $P - \lambda\Gamma$ is ample, the pair $(X, \Delta + \lambda\Gamma)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$.
Now, as $Supp(N) \subseteq \mathbb{B}_+(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in \mathbb{Q}^+$ such that if $0 \le \beta' < \beta$, then the pair $(X, \Delta + \lambda\Gamma + \beta'N)$ is LC and $CLC(X, \Delta + \lambda\Gamma + \beta'N) = CLC(X, \Delta)$.
Suppose $a > -\beta$ is a rational number such that $aD - (K_X + \Delta)$ is nef.

Define $a' = -\min\{0, a\}$, $a'' = \max\{0, a\}$, so that $a = a'' - a'$, $a'' \geq 0$, $0 \leq a' < \beta$. Moreover we define $\Delta' = \Delta + \lambda\Gamma + a'N$, so that Δ' is effective, (X, Δ') is LC and $CLC(X, \Delta') = CLC(X, \Delta)$.

Hence, if $\Delta \neq 0$, thanks to lemma 2.2.9 and lemma 2.2.8, we get that for every $\epsilon \in \mathbb{Q}^+$

$$CLC(X, \Delta' - \epsilon\Delta - a''N) \subseteq \{V \in CLC(X, \Delta) \text{ such that } V \not\subseteq \mathrm{Supp}(\Delta)\},$$

so that, by hypothesis, $\mathbb{B}(P)$ does not intersect any LC center of the pair $(X, \Delta' - \epsilon D - a''N)$.

If $\Delta = 0$, then $\mathrm{Supp}(0) = \emptyset$, so that the same result holds by hypothesis because, by lemma 2.2.8, $CLC(X, \Delta' - \epsilon\Delta - a''N) \subseteq CLC(X, \Delta') = CLC(X, 0)$.

Moreover

$$(1+a)P + a''N - (K_X + \Delta' - \epsilon\Delta) = (1+a)P + a''N - (K_X + \Delta + \lambda\Gamma + a'N - \epsilon\Delta) =$$

$$= (P - \lambda\Gamma) + (aD - (K_X + \Delta)) + \epsilon\Delta$$

is ample if ϵ is sufficiently small thanks to the openness of the ample cone. Thus we obtain the semiampleness of P by applying theorem 3.6.5 to the pair $(X, \Delta' - \epsilon\Delta)$.

$\square$

3.7 $\quad \widetilde{\mathrm{Nklt}}$ and Nklt_2: Dimension 3

In this section we generalize the results obtained so far towards Conjecture 1 and Conjecture 2 by making a more careful study of the non-klt locus of the given pair (X, Δ). As a corollary we obtain Conjecture 1b for varieties of dimension less than or equal to 3 (cf. corollary 3.7.8).

Definition 3.7.1. Let (X, Δ) be a pair, then we define

$$\widetilde{\mathrm{Nklt}}(X, \Delta) = \bigcup_{\substack{V \in CLC(X, \Delta) \\ V \subseteq NSNC(\Delta) \cup Sing(X)}} V.$$

Definition 3.7.2. Let (X, Δ) be a pair such that $dim X = n$. For every integer $k \in \{1, \ldots, n\}$ we define

$$\mathrm{Nklt}_k(X, \Delta) = \bigcup_{\substack{V \in CLC(X, \Delta) \\ dim V \leq n-k}} V.$$

Remark 3.7.3. Given a pair (X, Δ) note that, by definition, $\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}_1(X, \Delta)$.

Note also that $\widetilde{\mathrm{Nklt}}(X, \Delta) \subseteq \mathrm{Nklt}_2(X, \Delta)$ because of the normality of X. Moreover if (X, Δ) is DLT then $\widetilde{\mathrm{Nklt}}(X, \Delta) = \emptyset$ by theorem 2.2.23.

Theorem 3.7.4. *Let (X, Δ) be a pair and suppose that $\Delta = \sum_{i \in I} d_i D_i$, where all the D_i's are distinct prime divisors and $d_i \leq 1$ for every $i \in I$.*
Moreover suppose that $P \in Div_{\mathbb{Q}}(X)$ and we can write $\Delta = \Delta_+ - \Delta_-$, where Δ_+ and Δ_- are effective $\mathbb{Q}$-divisors and the following properties are satisfied:

1. *P is nef;*

2. *$t_0 P - (K_X + \Delta)$ is ample for some $t_0 \in \mathbb{Q}^+$;*

3. *There exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is a Cartier divisor and for all $m \in \mathbb{N}$ it holds that*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner \Delta_- \urcorner));$$

4. *$\widetilde{\mathrm{Nklt}}(X, \Delta) = \emptyset$, or $P|_{\widetilde{\mathrm{Nklt}}(X,\Delta)}$ is semiample;*

5. *There exists $\mu : X' \to X$, a standard log-resolution of the pair (X, Δ) such that $a(E, X, \Delta) > -2$ for every prime divisor $E \subseteq X'$.*

Then P is semiample.

Proof. Let μ be as in the hypothesis. Note that $\mathrm{Nklt}(X', \mathbf{A}(\Delta)_{X'}) = \mathrm{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1})$, because X' is smooth and $\mathbf{A}(\Delta)_{X'}$ is SNCS.

Now by the ampleness of $t_0 P - (K_X + \Delta)$, thanks to lemma 2.1.9, for all μ-exceptional divisors $E_1, \ldots, E_s$ on X' there exist arbitrarily small coefficients $\delta_1, \ldots, \delta_s \in \mathbb{Q}^+$, such that

$$\mu^*(t_0 P - (K_X + \Delta)) - \sum_{j=1}^{s} \delta_j E_j$$

is ample.

Then, if $0 \leq \epsilon \ll 1$ we have that

$$\mu^*(t_0 P) - (K_{X'} + (1 - \epsilon)\mathbf{A}(\Delta)_{X'} + \sum_{j=1}^{s} \delta_j E_j)$$

is still ample.

Now for every ϵ sufficiently small such that the above condition holds we define

$$\widehat{\Delta}_\epsilon = (1 - \epsilon)\mathbf{A}(\Delta)_{X'} + \sum \delta_j E_j,$$

so that $\mu^*(mP) - (K_{X'} + \widehat{\Delta}_\epsilon)$ is ample for every integer $m \geq t_0$ thanks to the nefness of P.

Now we can write

$$\mathbf{A}(\Delta)_{X'} = \sum_{k \in K} c_k X_k + \sum_{l \in L} a_l Y_l - \sum_{m \in M} b_m Z_m,$$

where, for every $k \in K$, $l \in L$ and $m \in M$, we have that X_k, Y_l, Z_m are pairwise distinct prime divisors, and

- $b_m > 0 \quad \forall m \in M$;

- $0 \leq a_l < 1 \quad \forall l \in L$;

- $1 \leq c_k < 2 \quad \forall k \in K$:

In fact all the coefficients of $\mathbf{A}(\Delta)_{X'}$ are smaller than 2 because of the choice of μ.

Moreover we can suppose that $\mathrm{divexc}(\mu) \subseteq \mathrm{Supp}(\sum X_k + \sum Y_l + \sum Z_m)$, by considering among the Y_l's also the prime μ-exceptional divisors not appearing in $\mathrm{Supp}(\mathbf{A}(\Delta)_{X'})$, with coefficient 0.

Now let us define

$$\Delta'_+ := \sum_{k \in K} c_k X_k + \sum_{l \in L} a_l Y_l; \qquad \Delta'_- := \sum_{m \in M} b_m Z_m,$$

so that Δ'_+ and Δ'_- are effective, they have no common components and $\mathbf{A}(\Delta)_{X'} = \Delta'_+ - \Delta'_+$. Moreover for every $k \in K$, $l \in L$ and $m \in M$ we define

$$\gamma_k = \begin{cases} \delta_j & \text{if } X_k = E_j \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \gamma_l = \begin{cases} \delta_j & \text{if } Y_l = E_j \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \gamma_m = \begin{cases} \delta_j & \text{if } Z_m = E_j \\ 0 & \text{otherwise} \end{cases}$$

so that we can write

$$\widehat{\Delta}_\epsilon = \sum_{k \in K} ((1-\epsilon)c_k + \gamma_k) X_k + \sum_{l \in L} ((1-\epsilon)a_l + \gamma_l) Y_l - \sum_{m \in M} ((1-\epsilon)b_m - \gamma_m) Z_m.$$

Now we choose ϵ and the δ_j's small enough such that the following inequalities hold:

- $c'_k := (1-\epsilon)c_k + \gamma_k < 2 \quad \forall k \in K$;

- $a'_l := (1-\epsilon)a_l + \gamma_l < 1 \quad \forall l \in L$;

- $b'_m := (1-\epsilon)b_m - \gamma_m > 0 \quad \forall m \in M$,

and we define $\widehat{\Delta} := \widehat{\Delta}_\epsilon$. Hence $\widehat{\Delta} = \sum c'_k X_k + \sum a'_l Y_l - \sum b'_m Z_m$, and

- $0 < c'_k < 2 \quad \forall k \in K$;

- $0 \leq a'_l < 1 \quad \forall l \in L$;

- $0 < b'_m \leq b_m \quad \forall m \in M$.

Note that

$$\mu^*(mP) - (K_{X'} + \widehat{\Delta})$$

is ample for every integer $m \geq t_0$ and $\widehat{\Delta}$ is a SNCS divisor because $\mathrm{Supp}(\widehat{\Delta}) \subseteq \mathrm{Supp}(\mathbf{A}(\Delta)_{X'}) \cup \mathrm{divexc}(\mu)$, so that $\mathrm{Nklt}(X', \widehat{\Delta}) = \mathrm{Supp}((\widehat{\Delta})^{\geq 1})$.

Now let us define

$$\widehat{\Delta}_+ := \sum c'_k X_k + \sum a'_l Y_l; \quad \widehat{\Delta}_- := \sum b'_m Z_m,$$

so that $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$ are effective and $\widehat{\Delta} = \widehat{\Delta}_+ - \widehat{\Delta}_-$.

We claim that $\mu_* \ulcorner \widehat{\Delta}_- \urcorner \leq \ulcorner \Delta_- \urcorner$:

In fact, note that $\widehat{\Delta}_- \leq \Delta'_-$, so that it suffices to show that $\mu_* \ulcorner \Delta'_- \urcorner \leq \ulcorner \Delta_- \urcorner$. In particular we will show that $\mu_* \Delta'_- \leq \Delta_-$, which implies that $\mu_* \ulcorner \Delta'_- \urcorner = \ulcorner \mu_* \Delta'_- \urcorner \leq \ulcorner \Delta_- \urcorner$.

Now the required inequality holds because, by definition,

$$\Delta'_- = \sum_{a(E,X,\Delta)>0} a(E,X,\Delta)E.$$

Hence

$$\mu_*(\Delta'_-) = \sum_{a(\mu_*^{-1}D_i,X,\Delta)>0} a(\mu_*^{-1}D_i,X,\Delta)D_i = \sum_{d_i<0} -d_i D_i \leq \Delta_-,$$

because Δ_- is effective and $\Delta_- = \Delta_+ - \Delta$, so that, for every i, we have that

$$\mathrm{ord}_{D_i} \Delta_- = \mathrm{ord}_{D_i} \Delta_+ - d_i \geq -d_i.$$

Thus the claim is proved.

Thanks to the claim we can use lemma 2.1.8 and we obtain that if k_0 is as in the hypothesis, then

$$h^0(Y, \mathcal{O}_Y(\mu^*(k_0 mP) + \ulcorner \widehat{\Delta}_- \urcorner)) \leq h^0(X, \mathcal{O}_X(k_0 mP + \ulcorner \Delta_- \urcorner))$$

for all $m \in \mathbb{N}$. But, by hypothesis,

$$h^0(Y, \mathcal{O}_Y(\mu^*(k_0 mP))) = h^0(X, \mathcal{O}_X(k_0 mP)) = h^0(X, \mathcal{O}_X(k_0 mP + \ulcorner \Delta_- \urcorner))$$

for all $m \in \mathbb{N}$. Therefore

$$H^0(Y, \mathcal{O}_Y(\mu^*(k_0 mP))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(k_0 mP) + \ulcorner \widehat{\Delta}_- \urcorner))$$

for all $m \in \mathbb{N}$.

We will show the semiampleness of P by applying theorem 3.6.4 to the pair $(X', \widehat{\Delta})$ and the divisor $\mu^*(P)$. This will imply the semiampleness of $\mu^*(P)$, leading, in such a way, to the semiampleness of P itself.
In particular, in order to apply the theorem it remains to show that $(X', \widehat{\Delta})$ is a $\mu^*(P)$-KLT pair. In other words we have to prove that $\mathbb{B}(\mu^*(P)) \cap \mathrm{Nklt}(X', \widehat{\Delta}) = \emptyset$:
Note that

$$\mathrm{Nklt}(X', \widehat{\Delta}) = \mathrm{Supp}((\widehat{\Delta})^{\geq 1}) \subseteq \bigcup_{k \in K} X_k = \mathrm{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1}) =$$
$$= \mathrm{Nklt}(X', \mathbf{A}(\Delta)_{X'}).$$

Moreover $\mathrm{Nklt}(X', \widehat{\Delta}) \subseteq \mathrm{exc}(\mu)$:
In fact if $k \in K$ is such that X_k is not exceptional, then $X_k = \mu_*^{-1}G$, for some prime divisor G on X. Then $c_k = a(\mu_*^{-1}G, X, \Delta) = -\mathrm{ord}_G \Delta \geq -1$, thanks to the hypotheses on Δ.
On the other hand $\gamma_k = 0$ because X_k is not exceptional, so that

$$c'_k = (1-\epsilon)c_k < c_k \leq 1.$$

Thus we get that $\mathrm{Nklt}(X', \widehat{\Delta}) \subseteq \mathrm{Nklt}(X', \mathbf{A}(\Delta)_{X'}) \cap \mathrm{exc}(\mu)$.

Now we define

$$T = \sum_{c'_k \geq 1} X_k,$$

so that T is reduced and $T = \mathrm{Supp}((\widehat{\Delta})^{\geq 1}) = \mathrm{Nklt}(X', \widehat{\Delta})$.
In particular $T \subseteq \mathrm{Nklt}(X', \mathbf{A}(\Delta)_{X'}) \cap \mathrm{exc}(\mu)$.
Let T_0 be a prime divisor in the support of T. Then, on the one hand, $T_0 \subseteq \mathrm{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1})$, that is $a(T_0, X, \Delta) \leq -1$, which implies that $\mu(T_0) \in CLC(X, \Delta)$.
On the other hand $T_0 \subseteq \mathrm{exc}(\mu)$ implies that $\mu(T_0) \subseteq Sing(X) \cup NSNC(\Delta)$, because μ is a standard log-resolution of the pair (X, Δ).
Hence we get that $\mu(T_0) \subseteq \widetilde{\mathrm{Nklt}}(X, \Delta)$. But the same holds for every component of T, so that we have

$$\mu(T) \subseteq \widetilde{\mathrm{Nklt}}(X, \Delta).$$

If $\widetilde{\mathrm{Nklt}}(X, \Delta) = \emptyset$, then $\mu(T) = \emptyset$, so that $T = \mathrm{Nklt}(X', \widehat{\Delta}) = \emptyset$ and there is nothing to prove. We can thus assume that $\widetilde{\mathrm{Nklt}}(X, \Delta) \neq \emptyset$.
Then, as by hypothesis $P_{|_{\widetilde{\mathrm{Nklt}}(X, \Delta)}}$ is semiample, we get that $P_{|_{\mu(T)}}$ is semiample.
Now we consider the commutative diagram:

$$
\begin{array}{ccc}
T & \hookrightarrow & X' \\
\downarrow{\scriptstyle \mu_{|_T}} & & \downarrow{\scriptstyle \mu} \\
\mu(T) & \hookrightarrow & X
\end{array}
$$

As $P_{|_{\mu(T)}}$ is semiample, we have that $\mu^*_{|_T}(P_{|_{\mu(T)}})$ is semiample, which, by commutativity of the diagram, implies that $\mu^*(P)_{|_T}$ is semiample.

Now we claim that $\ulcorner -\widehat{\Delta} \urcorner = \ulcorner \widehat{\Delta}_- \urcorner - T$:
In fact

$$\ulcorner -\widehat{\Delta} \urcorner = \sum_{m \in M} \ulcorner b'_m \urcorner Z_m + \sum_{k \in K} \ulcorner -c'_k \urcorner X_k + \sum_{l \in L} \ulcorner -a'_l \urcorner Y_l.$$

But, for all $l \in L$, we have that $0 \geq -a'_l > -1$, so that $\ulcorner -a'_l \urcorner = 0$.
Moreover for all $k \in K$, $0 > -c'_k > -2$, so that

$$\ulcorner -c'_k \urcorner = \begin{cases} -1 & \text{if } c'_k \geq 1 \\ 0 & \text{if } c'_k < 1 \end{cases}$$

Thus

$$\ulcorner -\widehat{\Delta} \urcorner = \sum_{m \in M} \ulcorner b'_m \urcorner Z_m - \sum_{c'_k \geq 1} X_k = \ulcorner \widehat{\Delta}_- \urcorner - T,$$

and the claim is proved.

Take $k_1 \in \mathbb{N}$ such that $k_1 > t_0$ and k_1 is a multiple of k_0, so that $k_1 P$ is a Cartier divisor and

$$H^0(Y, \mathcal{O}_Y(\mu^*(k_1 m P))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(k_1 m P) + \ulcorner \widehat{\Delta}_- \urcorner))$$

for every $m \in \mathbb{N}$. Let us consider, for every $k \in k_1 \mathbb{N}$, the following commutative diagram:

$$
\begin{array}{ccc}
H^0(X', \mathcal{O}_{X'}(\mu^*(kP) + \ulcorner\widehat{\Delta}_-\urcorner)) & \xrightarrow{\ \beta_k\ } & H^0(T, \mathcal{O}_T(\mu^*(kP)_{|_T} + \ulcorner\widehat{\Delta}_-\urcorner_{|_T})) \\
\simeq\big\uparrow & & i_k\big\uparrow \\
H^0(X', \mathcal{O}_{X'}(\mu^*(kP))) & \xrightarrow{\ \ \alpha_k\ \ } & H^0(T, \mathcal{O}_T(\mu^*(kP)_{|_T}))
\end{array}
$$

where the vertical arrow on the left is an isomorphism thanks to the choice of k_1.

Note that i_k is injective for every $k \in k_1 \mathbb{N}$ because $\ulcorner\widehat{\Delta}_-\urcorner_{|_T}$ is effective:

In fact $\ulcorner\widehat{\Delta}_-\urcorner$ is effective and $\mathrm{Supp}(\ulcorner\widehat{\Delta}_-\urcorner) = \mathrm{Supp}(\widehat{\Delta}_-) = \cup Z_m$ does not contain any component of T.

Let us prove that β_k is surjective for every $k \in k_1\mathbb{N}$. In particular we prove that $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \ulcorner\widehat{\Delta}_-\urcorner - T)) = 0$:

Note that $\mu^*(kP) - (K_{X'} + \widehat{\Delta})$ is ample, thanks to the choice of k_1, and $\{\mu^*(kP) - (K_{X'} + \widehat{\Delta})\} = \{-\widehat{\Delta}\}$ is SNCS.

Then, by Kawamata-Viehweg vanishing theorem (see [Laz04, 9.1.20]), we get that $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \ulcorner-\widehat{\Delta}\urcorner)) = 0$.

But $\ulcorner-\widehat{\Delta}\urcorner = \ulcorner\widehat{\Delta}_-\urcorner - T$. Then $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \ulcorner\widehat{\Delta}_-\urcorner - T)) = 0$, as required.

By the commutativity of the diagram, the surjectivity of β_k implies that i_k is surjective, that is i_k is an isomorphism. Thus α_k is also surjective for every $k \in k_1\mathbb{N}$.

But $\mu^*(P)_{|_T}$ is semiample, whence there exists $k_2 \in k_1\mathbb{N}$ such that $\mu^*(k_2 P)_{|_T}$ is base-point free.

Then the surjectivity of α_{k_2} implies that $Bs(\mu^*(k_2 P)) \cap T = \emptyset$.

Therefore $\mathbb{B}(\mu^*(P)) \cap \mathrm{Nklt}(X', \widehat{\Delta}) = \emptyset$, so that we can apply theorem 3.6.4 to the pair $(X, \widehat{\Delta})$ and the divisor $\mu^*(P)$ and we are done. $\qquad\square$

Corollary 3.7.5. *Let (X, Δ) be an effective pair.*
Let $D \in Div_{\mathbb{Q}}(X)$ be such that

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some $a \in \mathbb{Q}$;*

3. *There exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and*

   - *$(Z, \mathbf{A}(\Delta)_Z - aN)$ is an LC pair;*
   - *$\mathbb{B}_+(f^*(D))$ does not contain any LC center of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$;*
   - *$\widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$, or $P_{|_{\widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN)}}$ is semiample.*

Then P is semiample.

We remark that if $a \geq 0$ the LCness of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$ holds if we suppose that (X, Δ) is an LC pair.

Proof. Let us apply lemma 2.5.3 and consider t_0, D', P', N', Δ_Z as in the lemma, so that $t_0 P' - (K_Z + \Delta_Z - N')$ is big and nef.

Note that $\mathbb{B}_+(P') = \mathbb{B}_+(P) = \mathbb{B}_+(f^*(D))$ by lemma 2.5.4. Hence we can apply lemma 2.3.7 to the big and nef $\mathbb{Q}$-divisor P' and to the pair $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$ and we find a Cartier divisor Γ and a rational number $\lambda > 0$ such that $P' - \lambda\Gamma$ is ample, $(Z, \Delta_Z - N' + \lambda\Gamma)$ is LC and $CLC(Z, \Delta_Z - N' + \lambda\Gamma) = CLC(Z, \Delta_Z - N')$.

Furthermore, we can choose Γ generically in its linear series and we have that $Bs(|\Gamma|) = \mathbb{B}_+(P')$. Then, by Bertini's theorem, we can suppose that, outside $\mathbb{B}_+(P')$, Γ is smooth and it intersects $\Delta_Z - N'$ in a simple normal crossing way. Let us put $B = \Delta_Z - N' + \lambda\Gamma$. We will show that the pair (Z, B) and the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor P' satisfy the hypotheses of theorem 3.7.4.

First of all we have that $(t_0 + 1)P' - (K_Z + B) = (P' - \lambda\Gamma) + (t_0 P' - (K_Z + \Delta_Z - N'))$ is ample, so that property 2 holds.

By the LCness of the pair (Z, B) we get that all the coefficients of B are less than or equal to 1 and property 5 holds. Moreover property 1 is trivially verified and property 3 follows by the definition of $\mathbb{Q}$-CKM Zariski decomposition because Δ_Z is effective.

In order to prove that property 4 holds we will show that $\widetilde{\mathrm{Nklt}}(Z, B) \subseteq \widetilde{\mathrm{Nklt}}(Z, \Delta_Z - N') = \widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN)$, so that we can use the hypothesis of the corollary:

By the choice of Γ we have that $CLC(Z, \Delta_Z - N') = CLC(Z, B)$ and $NSNC(B) \subseteq NSNC(\Delta_Z - N') \cup \mathbb{B}_+(P')$.

Then, if $V \in CLC(Z, B)$ and $V \subseteq Sing(Z) \cup NSNC(B)$, we get that $V \in CLC(Z, \Delta_Z - N')$ and $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N') \cup \mathbb{B}_+(P')$. This implies that $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N')$. Hence $V \subseteq \widetilde{\mathrm{Nklt}}(Z, \Delta_Z - N')$, and we get the required inclusion. Therefore we can apply theorem 3.7.4. $\qquad \square$

Theorem 3.7.6. *Let (X, Δ) be an LC pair, with $\dim X \geq 2$. Suppose that $P \in Div_{\mathbb{Q}}(X)$ and we can write $\Delta = \Delta_+ - \Delta_-$, where Δ_+ and Δ_- are effective $\mathbb{Q}$-divisors, and the following are satisfied:*

1. *P is nef;*

2. *$t_0 P - (K_X + \Delta)$ is nef for some $t_0 \in \mathbb{Q}^+$;*

3. *There exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \ulcorner \Delta_- \urcorner));$$

4. *$\mathrm{Nklt}_2(X, \Delta) = \emptyset$, or $P|_{\mathrm{Nklt}_2(X, \Delta)}$ is semiample.*

5. *P is logbig in codimension 1 for the pair (X, Δ), or $t_0 P - (K_X + \Delta)$ is logbig in codimension 1 for the pair (X, Δ)*

Then P is semiample.

Proof. Let

$$L = \begin{cases} P & \text{if } P \text{ is logbig in codimension 1 for the pair } (X, \Delta) \\ t_0 P - (K_X + \Delta) & \text{otherwise} \end{cases}$$

Then L is nef and logbig in codimension 1 for the pair (X, Δ), so that, thanks to lemma 2.3.9, we have that $\mathbb{B}_+(L)$ does not contain any divisorial LC center of the pair (X, Δ).

By [ELMNP06, Prop. 1.5] there exists $H \in \text{Div}_{\mathbb{Q}}(X)$, ample, such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H).$$

Moreover there exists $m_0 \in \mathbb{N}$ such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Then $Bs(|m_0(L - H)|)$ does not contain any divisorial LC center of the pair (X, Δ). Hence, we can choose a general divisor Γ in $|m_0(L - H)|$ such that $\text{Supp}(\Gamma)$ does not contain any divisorial LC center of (X, Δ).

Note that we have

$$L - \lambda \Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if $\lambda \in (0, \frac{1}{m_0}]$ because L is nef and H is ample.

Now, for every $\lambda \in (0, \frac{1}{m_0}]$, let us define $\Delta_\lambda = \Delta + \lambda \Gamma$. We will prove that there exists $\lambda_0 \in \mathbb{Q}^+$ such that if $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$, then P and the pair (X, Δ_λ) satisfy the hypotheses of the theorem 3.7.4.

First of all note that

$$(t_0+1)P - (K_X + \Delta_\lambda) = (t_0+1)P - (K_X + \Delta + \lambda \Gamma) = P + (t_0 P - (K_X + \Delta)) - \lambda \Gamma =$$

$$= \begin{cases} L - \lambda \Gamma + (t_0 P - (K_X + \Delta)) & \text{if } P \text{ is logbig in codimension 1} \\ P + (L - \lambda \Gamma) & \text{otherwise} \end{cases}$$

is ample in both cases for every $\lambda \in (0, \frac{1}{m_0}]$, being the sum of an ample and a nef divisor.

Now let us define

$$(\Delta_\lambda)_+ := \Delta_+ + \lambda \Gamma; \quad (\Delta_\lambda)_- := \Delta_-.$$

Then $\Delta_\lambda = (\Delta_\lambda)_+ - (\Delta_\lambda)_-$, and $(\Delta_\lambda)_+$ and $(\Delta_\lambda)_-$ are effective $\mathbb{Q}$-divisors for every $\lambda > 0$, because Γ is effective.

Moreover note that, with these definitions, hypotheses 1 and 3 of theorem 3.7.4 are trivially verified.

Now take a rational number $\lambda' > 0$ such that $\text{Supp}(\Delta) + \text{Supp}(\Gamma) = \text{Supp}(\Delta + \lambda \Gamma)$ for every $\lambda \in (0, \lambda')$. and let $\mu : X' \to X$ be a standard log-resolution of the pair $(X, \Delta + \lambda \Gamma)$.

For every prime divisor $E \subseteq X'$ we have that

$$a(E, X, \Delta_\lambda) = a(E, X, \Delta + \lambda \Gamma) = a(E, X, \Delta) - \lambda \text{ord}_E(\mu^*(\Gamma)),$$

where $a(E, X, \Delta) \geq -1$ because (X, Δ) is an LC pair.

Suppose E is a divisor on X' such that E is not μ-exceptional and $a(E, X, \Delta) = -1$.

Then $\mu(E)$ is a divisorial LC center of (X, Δ), so that $\mathrm{ord}_{\mu(E)}\Gamma = 0$, that is $\mathrm{ord}_E(\mu^*(\Gamma)) = 0$, which implies $a(E, X, \Delta_\lambda) = -1$.

Now define
$$\lambda_1 := \min_{\substack{ord_E(\mu^*(\Gamma))>0 \\ a(E,X,\Delta)>-1}} \left\{ \frac{1 + a(E, X, \Delta)}{\mathrm{ord}_E(\mu^*(\Gamma))}, 1 \right\}.$$

Then $\lambda_1 > 0$ and, if $\lambda \in \mathbb{Q} \cap (0, \lambda_1)$, we have that $a(E, X, \Delta_\lambda) > -1$ for every prime divisor $E \subseteq X'$ such that $a(E, X, \Delta) > -1$.

Define
$$\lambda_2 := \min_{\substack{ord_E(\mu^*(\Gamma))>0 \\ a(E,X,\Delta)=-1}} \left\{ \frac{2 + a(E, X, \Delta)}{\mathrm{ord}_E(\mu^*(\Gamma))}, 1 \right\}.$$

Then $\lambda_2 > 0$ and, if $\lambda \in \mathbb{Q} \cap (0, \lambda_2)$, we have that $a(E, X, \Delta_\lambda) > -2$ for every prime divisor $E \subseteq X'$ such that $a(E, X, \Delta) = -1$.

We put $\lambda_0 = \min\{\lambda', \lambda_1, \lambda_2, \frac{1}{m_0}\}$, so that if $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$ then (X, Δ_λ) satisfies hypothesis 5 of theorem 3.7.4.

Furthermore we can write

$$\Delta_\lambda = \sum -a(\mu_*^{-1} B_i, X, \Delta_\lambda) B_i,$$

where the B_i's are distinct prime divisors on X. By definition, for every i, $\mu_*^{-1} B_i$ is not an exceptional divisors, so that, it follows by the previous calculation that $-a(\mu_*^{-1} B_i, X, \Delta_\lambda) \leq 1$.

Thus, in order to apply theorem 3.7.4, it just remains to prove that hypothesis 4 is satisfied for every $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$:

Let us consider

$$\mathbf{A}(\Delta)_{X'} = \sum_{E \subseteq X'} -a(E, X, \Delta)E; \quad \mathbf{A}(\Delta_\lambda)_{X'} = \sum_{E \subseteq X'} -a(E, X, \Delta_\lambda)E.$$

Thanks to the choice of μ and λ we have that Δ' and Δ'_λ are SNCS.

Let us put
$$F := \sum_{a(E,X,\Delta_\lambda)<-1} (-a(E, X, \Delta_\lambda) - 1)E;$$

$$\widetilde{\Delta} := \mathbf{A}(\Delta_\lambda)_{X'} - F = \sum_{a(E,X,\Delta_\lambda)\geq-1} -a(E, X, \Delta_\lambda)E + \sum_{a(E,X,\Delta_\lambda)<-1} E.$$

Then we have that F is effective, $\mathrm{Supp}(\widetilde{\Delta}) \subseteq \mathrm{Supp}(\mathbf{A}(\Delta_\lambda)_{X'})$ and all the coefficients of $\widetilde{\Delta}$ are less than or equal to 1. In particular the pair $(X, \widetilde{\Delta})$ is LC.

Moreover, by the previous calculations, we have that F is exceptional, $\mathrm{Supp}(F) \subseteq \mathrm{Supp}((\mathbf{A}(\Delta)_{X'})^{=1})$ and $\widetilde{\Delta}^{=1} = (\mathbf{A}(\Delta)_{X'})^{=1}$.

Let us show that $\widetilde{\mathrm{Nklt}}(X, \Delta_\lambda) \subseteq \mathrm{Nklt}_2(X, \Delta)$:

We will actually prove that $\mathrm{Nklt}_2(X, \Delta_\lambda) \subseteq \mathrm{Nklt}_2(X, \Delta)$.

Let V be an LC center of the pair (X, Δ_λ) of codimension greater than one.

Then $V = \mu(W)$ for some $W \in CLC(X, \mathbf{A}(\Delta_\lambda)_{X'})$.

If $W \nsubseteq \text{Supp}(F)$, then $W \in CLC(X', \widetilde{\Delta})$. But X' is smooth, $\widetilde{\Delta}$ is SNCS and the pair $(X, \widetilde{\Delta})$ is LC, whence W is an irreducible component of a finite intersection of prime divisors in the support of $\widetilde{\Delta}^{=1} = (\mathbf{A}(\Delta)_{X'})^{=1}$.

Hence $W \in CLC(X', \mathbf{A}(\Delta)_{X'})$, which implies that $V = \mu(W) \in CLC(X, \Delta)$, so that $V \subseteq \text{Nklt}_2(X, \Delta)$, because the codimension of V is greater than 1.

If $W \subseteq \text{Supp}(F)$ then there exists a prime divisor $F_0 \subseteq \text{Supp}(F)$ such that $W \subseteq F_0$.

Then $F_0 \subseteq \text{Supp}(F) \subseteq \text{Supp}((\mathbf{A}(\Delta)_{X'})^{=1})$. Hence $F_0 \in CLC(X', \mathbf{A}(\Delta)_{X'})$, so that $\mu(F_0) \in CLC(X, \Delta)$. Moreover $codim\,\mu(F_0) \geq 2$, because F_0 is exceptional, as F is exceptional.

Thus

$$V = \mu(W) \subseteq \mu(F_0) \subseteq \text{Nklt}_2(X, \Delta).$$

This shows that $\widetilde{\text{Nklt}}(X, \Delta_\lambda) \subseteq \text{Nklt}_2(X, \Delta_\lambda) \subseteq \text{Nklt}_2(X, \Delta)$, which implies, by the hypotheses, that $\widetilde{\text{Nklt}}(X, \Delta_\lambda) = \emptyset$ or $P_{|\widetilde{\text{Nklt}}(X,\Delta_\lambda)}$ is semiample.

Therefore all the hypotheses of theorem 3.7.4 are satisfied and we get the semiampleness of P. $\qquad\square$

Corollary 3.7.7. *Let (X, Δ) be a pair and $dimX \geq 2$ and let $a \in \mathbb{Q}$. Let $D \in Div_\mathbb{Q}(X)$ be such that:*

1. *D is big*

2. *$aD - (K_X + \Delta)$ is nef;*

3. *There exists a projective birational morphism $f : Z \to X$ such that*

$$f^*(D) = P + N$$

   is a $\mathbb{Q}$-CKM Zariski decomposition and

   - *$(Z, \mathbf{A}(\Delta)_Z - aN)$ is an LC pair;*
   - *$\mathbb{B}_+(f^*(D))$ does not contain divisorial LC centers of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$;*
   - *$\text{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$, or $P_{|\text{Nklt}_2(Z,\mathbf{A}(\Delta)_Z-aN)}$ is semiample;*

Then P is semiample.

Note that in the case $a \geq 0$ we can just assume that the pair (X, Δ) is LC in order to have the LCness of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$.

Proof. Let us apply lemma 2.5.3 and consider t_0, D', P', N', Δ_Z as in the lemma, so that $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition and $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$. Then the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor P' and the pair $(Z, \Delta_Z - N')$ satisfy all the hypotheses of theorem 3.7.6. In fact we can take Δ_Z as the positive part $(\Delta_Z - N')_+$ and N' as the negative part. Conditions 1 and 2 are direct consequences of lemma 2.5.3, condition 3 holds because $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition and condition 4 holds by hypothesis. As for condition 5 note that $\mathbb{B}_+(P') = \mathbb{B}_+(P) = \mathbb{B}_+(f^*(D))$ by lemma 2.5.4, so that P' is logbig for the pair $(Z, \Delta_Z - N')$ thanks to lemma 2.3.9. Therefore theorem 3.7.6 applies and we are done.

$\qquad\square$

Corollary 3.7.8. *Let (X, Δ) be an effective pair such that $dim X \leq 3$, let $a \in \mathbb{Q}$. Let $D \in Div_{\mathbb{Q}}(X)$ be such that*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef;*

3. *There exists a projective birational morphism $f : Z \rightarrow X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and*

   - *(X, Δ) is an LC pair and $a \geq 0$ (resp. $(Z, \mathbf{A}(\Delta)_Z - aN)$ is an LC pair);*
   - *P is logbig for the pair $(Z, \mathbf{A}(\Delta)_Z)$ (resp. P is logbig for the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$).*

Then P is semiample.

Proof. Begin by noting that if $dim X \leq 1$ then the theorem is trivial because every big divisor on a curve is ample. We can thus assume that $2 \leq dim X \leq 3$. Note also that if $a \geq 0$ and (X, Δ) is LC then, by lemma 2.2.8, $(Z, \mathbf{A}(\Delta)_Z - aN)$ is LC and $CLC(Z, \mathbf{A}(\Delta)_Z - aN) \subseteq CLC(Z, \mathbf{A}(\Delta)_Z)$. Thus we can assume that P is logbig for the LC pair $(Z, \mathbf{A}(\Delta)_Z - aN)$.

Hence by lemma 2.3.9 we get that $\mathbb{B}_+(P)$ does not contain divisorial LC centers of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$, so that the same holds for $\mathbb{B}_+(f^*(D))$ by lemma 2.5.4.

Then, in order to apply corollary 3.7.7, it just remains to show that $P_{|\mathrm{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN)}$ is semiample if $\mathrm{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN) \neq \emptyset$.

Equivalently we will show that P is semiample when restricted to each connected component of this closed subset:

Let C be a connected component of $\mathrm{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN)$. Then, by hypothesis, we have that $0 \leq dim C \leq 1$.

If $dim C = 0$ then $P_{|C}$ is trivially semiample.

If $dim C = 1$ then we can write $C = \cup_{j=1}^k C_j$, where the C_j's are irreducible curves.

Then we have that $C_j \in CLC(Z, \mathbf{A}(\Delta)_Z - aN)$ for every $j \in \{1, \ldots, k\}$, so that $P_{|C_j}$ is big, because P is logbig for the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$.

But, as C_j is an irreducible curve, this implies that $P_{|C_j}$ is ample for every $j = \{1, \ldots, k\}$. Hence $P_{|C}$ is ample, because ampleness can be checked on the irreducible components (see [Laz04, 1.2.16]), so that in particular it is semiample.

Therefore we can apply corollary 3.7.7 and we are done. $\qquad \square$

3.8 Dimension 4

In this section we show in theorem 3.8.1 that Conjecture 2 holds if we assume some strong standard conjectures in the field of the Minimal Model Program. By using that these conjectures hold true in low dimension we obtain Conjecture 2 in dimension less than or equal to 4 (cf. corollary 3.8.2).

Before stating the theorems let us fix some notation and definitions:

- We say that a pair (X, Δ) is of *log-general type* if $K_X + \Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ is big;

- We refer to [KM00, definition 3.50] for the definition of *log minimal model* of a DLT pair. More precisely given a DLT pair (X, Δ) we say that the pair (X', Δ') is a log minimal model of the pair (X, Δ) if there exists a birational map $\phi : X \dashrightarrow X'$ such that

  1. ϕ^{-1} has no exceptional divisors;
  2. $\Delta' = \phi_*(\Delta)$;
  3. $K_{X'} + \Delta'$ is nef;
  4. $a(E, X, \Delta) < a(E, X'\Delta')$ for every ϕ-exceptional divisor E.

- We refer to [Fuj00, Definition 1.1] for the definition of *semi divisorial log terminal* (or *sDLT*) *n*-fold: More precisely let X be a reduced S_2 scheme and assume it is pure n-dimensional and normal crossing in codimension 1. Let Δ be a Weil $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $X = \bigcup X_i$ be a decomposition into irreducible components and let $\mu : \coprod X_i' \to X = \bigcup X_i$ be the normalization. A $\mathbb{Q}$-divisor Θ on X' is defined by $K'_X + \Theta := \mu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X_i' by $\Theta_i = \Theta_{|_{X_i'}}$.

  We say that (X, Δ) is semi divisorial log terminal n-fold if X_i is normal for every i (so that X_i' is isomorphic to X_i) and (X', Θ) is DLT.

- We say that *sDLT-abundance* holds in dimension n if for every sDLT n-fold (X, Δ) such that $K_X + \Delta$ is nef we have that $K_X + \Delta$ is semiample.

Theorem 3.8.1. *Let (X, Δ) be an effective LC pair of dimension n. Let $D \in Div_\mathbb{Q}(X)$ be such that*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

3. *D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$;*

4. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

Also suppose that log minimal models exist for every $\mathbb{Q}$-factorial DLT pair of dimension n of log-general type and that sDLT-abundance holds in dimension $n - 1$.
Then P is semiample.

Proof. Note that $\mathbb{B}_+(P) = \mathbb{B}_+(D)$ by lemma 2.5.4. Then we can apply lemma 2.3.7 to P and we find a Cartier divisor Γ and a rational number $\lambda > 0$ such that $P - \lambda\Gamma$ is ample, $(X, \Delta + \lambda\Gamma)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$. Then $aD + P - (K_X + \Delta + \lambda\Gamma) = (aD - (K_X + \Delta)) + (P - \lambda\Gamma)$ is ample.
Thus if $D' = aD + P$, then D' is big and D' admits a $\mathbb{Q}$-CKM Zariski decomposition $D' = P' + N'$, where $P' := (a + 1)P$ and $N' := aN$. Furthermore $D' - (K_X + \Delta + \lambda\Gamma)$ is ample, P is semiample if and only if P' is such and $\mathbb{B}_+(D) = \mathbb{B}_+(D')$.
This implies that, if we replace D by D' and Δ by $\Delta + \lambda\Gamma$, we can suppose that $D - (K_X + \Delta)$ is ample.
Hence by [KM00, lemma 5.17] there exists an effective ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H such that

$$D - (K_X + \Delta) \sim_\mathbb{Q} H,$$

and $(X, \Delta + H)$ is an LC pair.

In other words, if we put $\Delta_0 := \Delta + H$, then $D \sim_{\mathbb{Q}} K_X + \Delta_0$ and (X, Δ_0) is an LC pair.

Therefore, thanks to lemma 2.5.2, we are reduced to show that if $K_X + \Delta = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition, $K_X + \Delta$ is big and (X, Δ) is LC, then P is semiample. Moreover up to performing a DLT blow-up (see theorem 3.5.1) we can suppose that X is $\mathbb{Q}$-factorial and the pair (X, Δ) is DLT.

Then by hypothesis we have a log minimal model (X', Δ') of the pair (X, Δ), so that there exists $\phi : X \dashrightarrow X'$ a birational map, $\Delta' = \phi_*(\Delta)$, $K_{X'} + \Delta'$ is nef and (X', Δ') is LC. If we resolve the indeterminacies of ϕ we find two birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that

$$f^*(K_X + \Delta) = g^*(K_{X'} + \Delta') + E,$$

where E is g-exceptional and effective.

Hence, by Fujita's lemma (see for example [KMM85, lemma 1.3.2]), this is a $\mathbb{Q}$-CKM Zariski decomposition of $f^*(K_X + \Delta)$. By the uniqueness of the $\mathbb{Q}$-CKM Zariski decomposition for big divisors (see [Pro02, Proposition 7.4]) this implies that $f^*(P) = g^*(K_{X'} + \Delta')$. Thus we are reduced to prove that $K_{X'} + \Delta'$ is semiample. Note that we can assume that the pair (X', Δ') is DLT by performing again a DLT blow-up if necessary.

Let $V = \mathrm{Nklt}(X', \Delta')$. Then there exists a $\mathbb{Q}$-divisor Δ'_V on V such that

$$(K_{X'} + \Delta')_{|V} = K_V + \Delta'_V$$

and (V, Δ'_V) is an sDLT $(n-1)$-fold (see for example [Fuj00, Remark 1.2(3)]). Hence by sDLT-abundance we have that $(K_{X'} + \Delta')_{|V} = K_V + \Delta'_V$ is semiample. Moreover for every sufficiently divisible $m \geq 2$ we have that

$$H^1(X', \mathcal{I}_V(m(K_{X'} + \Delta'))) = 0$$

by Nadel vanishing (see [Laz04, theorem 9.4.17]), because $K_{X'} + \Delta'$ is big and nef.

Therefore we can lift sections and we find that $\mathbb{B}(K_{X'} + \Delta') \cap \mathrm{Nklt}(X', \Delta') = \emptyset$. Thus $K_{X'} + \Delta'$ is semiample (see for example theorem 3.6.5) and we are done. $\square$

Corollary 3.8.2. *Let (X, Δ) be an LC pair of dimension less than or equal to 4. If $D \in Div_{\mathbb{Q}}(X)$ is such that*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some rational number $a \geq 0$;*

3. *D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$;*

4. *$\mathbb{B}_+(D)$ does not contain any LC center of the pair (X, Δ);*

Then P is semiample.

Proof. sDLT-abundance in dimension 3 holds by [Fuj00, theorem 0.1], while every DLT $\mathbb{Q}$-factorial pair of dimension 4 of log-general type has a log minimal model by [AHK07, corollary 3.6]. Hence we can apply theorem 3.8.1 and we are done. $\square$

3.9   Relatively DLT case

In this section we consider a relative version of DLT pairs (see definition 3.9.1). After investigating some properties of this notion we prove some generalizations of Kawamata's theorem 1.0.1 in the context of relatively DLT pairs (cf. theorem 3.9.9 and 3.9.10).

Relatively DLT pairs

Definition 3.9.1. Let (X, Δ) be a pair, with $\Delta = \sum a_i D_i$, where the D_i's are distinct prime divisors and $a_i \in \mathbb{Q}$ for every i. Suppose $S \subseteq X$ is a closed subset.
We say that (X, Δ) is a *S-DLT pair* if

1. $V \nsubseteq Sing(X) \cup NSNC(\Delta)$ for every $V \in CLC(X, \Delta, S)$;

2. $a_i \leq 1$ for every i such that $D_i \cap S \neq \emptyset$;

Now let $D \in \text{Div}_{\mathbb{Q}}(X)$. We say that (X, Δ) is a *D-DLT pair* if (X, Δ) is a $\mathbb{B}(D)$-DLT pair.

Remark 3.9.2. Let $S \subseteq X$ be a closed subset. Then it is immediate to see that a S-KLT pair is S-DLT. Moreover by theorem 2.2.23 a DLT pair is S-DLT.

Lemma 3.9.3. *Let (X, Δ) be a pair and let $D \in \text{Div}_{\mathbb{Q}}(X)$. Suppose $m \in \mathbb{N}$ is such that mD is a Cartier divisor and, set-theoretically, $Bs(|mD|) = \mathbb{B}(D)$.*
If (X, Δ) is a D-DLT pair and $\mathbb{B}(D)$ does not contain any LC center of the pair (X, Δ), then there exists a common log-resolution of (X, Δ) and of the linear series $|mD|$, say $\mu : Y \to X$, such that

1. *$a(E, X, \Delta) > -1$ for every μ-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$;*

2. *$a(E, X, \Delta) \geq -1$ for every non-μ-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$;*

3. *μ is a composition of blowings-up of smooth subvarieties of codimension greater than 1*

Proof. Fix $m \in \mathbb{N}$ as in the hypothesis and consider $\mu : Y \to X$ a log-resolution of (X, Δ) and $|mD|$ as in lemma 2.2.20.
Let us prove that μ satisfies all the given conditions:
Write $\Delta = \sum a_i D_i$, where the D_i's are distinct prime divisors and $a_i \in \mathbb{Q}$.
We have that
$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E.$$

1) Suppose E is a μ-exceptional prime divisor on Y such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$. As E is μ-exceptional, by the choice of μ, we get that

$$\mu(E) \subseteq \mathbb{B}(D) \cup Sing(X) \cup NSNC(\Delta).$$

If, by contradiction, $a(E, X, \Delta) \leq -1$, then $\mu(E) \in CLC(X, \Delta, \mathbb{B}(D))$. Then, by hypothesis, $\mu(E) \nsubseteq Sing(X) \cup NSNC(\Delta)$. Thus, by the irreducibility of

$\mu(E)$, we get that $\mu(E) \subseteq \mathbb{B}(D)$, so that we get a contradiction, because $\mathbb{B}(D)$ does not contain LC centers of the pair (X, Δ).

2) Let E be a non μ-exceptional prime divisor on Y such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$, then $E = \mu_*^{-1}(D_i)$ for some i, whence $\mu(E) = D_i$, so that $D_i \cap \mathbb{B}(D) \neq \emptyset$.
Then $-a(E, X, \Delta) = a_i \leq 1$, that is $a(E, X, \Delta) \geq -1$.

3) Follows by the choice of μ.

$\square$

In the following lemmas we prove some good properties of S-DLT pairs:

Lemma 3.9.4. *Let (X, Δ) be a pair and let $S \subseteq X$ be a closed subset.*
If (X, Δ) is a S-DLT pair, then $CLC(X, \Delta, S)$ is a finite set.

Proof. Write $\Delta = \sum a_i D_i$, where the a_i's are rational numbers and the D_i's are distinct prime divisors on X.
Let $\mu : Y \to X$ be a standard log-resolution of the pair (X, Δ).
If $E \subseteq Y$ is a μ-exceptional prime divisor, then $\mu(E) \subseteq Sing(X) \cup NSNC(\Delta)$, so that, by hypothesis $\mu(E) \notin CLC(X, \Delta, S)$.
Thus, if $\mu(E) \cap S \neq \emptyset$, then $a(E, X, \Delta) > -1$.
Now define

$$\Delta_Y := -\sum a(E, X, \Delta)E = -\sum a(\mu_*^{-1}(D_i), X, \Delta)\mu_*^{-1}(D_i) - \sum_{E \text{ exc.}} a(E, X, \Delta)E,$$

so that for every $V \in CLC(X, \Delta, S)$ we have that $V = \mu(Z)$, where $Z \in CLC(Y, \Delta_Y)$ is such that $\mu(Z) \cap S \neq \emptyset$.
But, if Z is such, then $\mu(Z) \not\subseteq \mu(E)$ for all prime divisor E on Y such that $\mu(E) \cap S = \emptyset$, so that $Z \not\subseteq E$ if $\mu(E) \cap S = \emptyset$.
Now, if F is a prime divisor on a normal variety Y' such that $h : Y' \to Y$ is birational and $h(F) = Z$, then

$$a(F, Y, \Delta_Y) = a\big(F, Y, -\sum a(\mu_*^{-1}(D_i), X, \Delta)\mu_*^{-1}(D_i) - \sum_{E \text{ exc.}} a(E, X, \Delta)E\big) =$$

$$= a\big(F, Y, -\sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta)\mu_*^{-1}(D_i) - \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta)E\big) +$$

$$+ \sum_{\mu(E) \cap S = \emptyset} a(E, X, \Delta)\mathrm{ord}_F(h^*(E)).$$

But, if $\mu(E) \cap S = \emptyset$, then we have seen that $Z = h(F) \not\subseteq E$, that is $\mathrm{ord}_Z(E) = 0$, so that $\mathrm{ord}_F(h^*(E)) = 0$. Thus if we define

$$\Delta_Y' := -\sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta)\mu_*^{-1}D_i - \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta)E,$$

then

$$a(F, Y, \Delta_Y) = a\big(F, Y, -\sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta)\mu_*^{-1}D_i -$$

$$- \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta)E\big) = a(F, Y, \Delta_Y').$$

Hence, $Z \in CLC(Y, \Delta_Y)$ implies that $Z \in CLC(Y, \Delta'_Y)$, so that V is the image of an element in $CLC(Y, \Delta'_Y)$.

But (Y, Δ'_Y) is an LC pair because Y is smooth, Δ'_Y is SNCS and its coefficients are all less than or equal to 1:

In fact, by hypothesis of S-DLTness of (X, Δ), we have that $-a(\mu_*^{-1}(D_i), X, \Delta) = a_i \leq 1$ if $D_i \cap S \neq \emptyset$.

On the other hand we have proved that if E is μ-exceptional and $\mu(E) \cap S \neq \emptyset$, then $a(E, X, \Delta) > -1$.

Therefore $CLC(Y, \Delta'_Y)$ is finite, so that $CLC(X, \Delta, S)$ is finite as well, because every element in $CLC(X, \Delta, S)$ is the image of an element in $CLC(Y, \Delta'_Y)$. $\square$

Lemma 3.9.5. *Let (X, Δ) be a pair, let $S \subseteq X$ be a closed subset and let $\Delta' \in Div_{\mathbb{Q}}(X)$ be an effective divisor.*
If (X, Δ) is S-DLT and, $Supp(\Delta') \not\supseteq V$, for every $V \in CLC(X, \Delta, S)$, then there exists a rational number $\lambda_0 > 0$ such that for every $\lambda \in \mathbb{Q} \cap [0, \lambda_0]$, we have that $(X, \Delta + \lambda\Delta')$ is S-DLT and $CLC(X, \Delta + \lambda\Delta', S) = CLC(X, \Delta, S)$.

Proof. Let us write

$$\Delta = \sum a_i D_i, \qquad \Delta' = \sum b_i D_i,$$

where the D_i's are distinct prime divisors on X, the a_i's and the b_i's are (possibly zero) rational numbers. In particular, as Δ' is effective, $b_i \geq 0$ for all i. Hence, for every $\lambda > 0$,

$$\Delta + \lambda\Delta' = \sum (a_i + \lambda b_i) D_i.$$

Now let $\mu : Y \to X$ be a standard log-resolution of the pair $(X, Supp(\Delta) + Supp(\Delta'))$.
Define, for every $\lambda \geq 0$,

$$\widetilde{\Delta_\lambda} := -\sum_{E \subseteq Y} a(E, X, \Delta + \lambda\Delta')E,$$

so that, the LC centers of $(X, \Delta + \lambda\Delta')$ are the images of the LC centers of the pair $(Y, \widetilde{\Delta_\lambda})$.
We have that

$$\widetilde{\Delta_\lambda} = -\sum_{E \subseteq Y} a(E, X, \Delta)E + \mu^*(\lambda\Delta'),$$

so that, for every prime divisor $E \subseteq Y$, we get

$$a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) - \lambda\,\mathrm{ord}_E(\mu^*(\Delta')).$$

Claim 1 *There exists $\lambda' \in \mathbb{Q} \cap (0, 1]$ such that for every rational number $\lambda \in [0, \lambda')$ we have the following:*
If $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S \neq \emptyset$, then

1. *$a(E, X, \Delta) \geq -1$;*

2.
   - *$a(E, X, \Delta) > -1 \Rightarrow a(E, X, \Delta + \lambda\Delta') > -1$;*
   - *$a(E, X, \Delta) = -1 \Rightarrow a(E, X, \Delta + \lambda\Delta') = -1$.*

Proof of claim 1.
1) If $E \subseteq Y$ is a μ-exceptional prime divisor and $\mu(E) \cap S \neq \emptyset$, then, by choice of μ, we have that $\mu(E) \subseteq Sing(X) \cup NSNC(\mathrm{Supp}(\Delta) + \mathrm{Supp}(\Delta')) \subseteq Sing(X) \cup NSNC(\Delta) \cup \mathrm{Supp}(\Delta')$.
If, by contradiction $a(E, X, \Delta) < -1$, then $\mu(E) \in CLC(X, \Delta, S)$.
By S-DLTness of (X, Δ) this implies that $\mu(E) \not\subseteq Sing(X) \cup NSNC(\Delta)$. But $\mu(E) \not\subseteq \mathrm{Supp}(\Delta')$ by hypothesis, so that we find a contradiction.
On the other hand if $E \subseteq Y$ is a prime divisor, it is not μ-exceptional and $\mu(E) \cap S \neq \emptyset$, then $E = \mu_*^{-1}(D_i)$ for some prime divisor D_i on X such that $D_i \cap S \neq \emptyset$. Then, as (X, Δ) is S-DLT, we have that $a(E, X, \Delta) = -a_i \geq -1$.

2) Let $A = \{E \subseteq Y : E$ is a prime divisor, $a(E, X, \Delta) > -1,\ \mu(E) \cap S \neq \emptyset,\ \mathrm{ord}_E \mu^*(\Delta') \neq 0\}$.
Then we put

$$\lambda' = \begin{cases} \min_{E \in A} \left\{ \frac{1 + a(E, X, \Delta)}{\mathrm{ord}_E \mu^*(\Delta')} \right\} & \text{if } A \neq \emptyset \\ 1 & \text{if } A = \emptyset \end{cases}$$

Let E be a prime divisor on Y such that $\mu(E) \cap S \neq \emptyset$ and $a(E, X, \Delta) > -1$. If $0 \leq \lambda < \lambda'$, then $a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) - \lambda \mathrm{ord}_E \mu^*(\Delta') > -1$.

Now suppose $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S \neq \emptyset$ and $a(E, X, \Delta) = -1$. Then $\mu(E) \in CLC(X, \Delta, S)$. Hence $\mathrm{Supp}(\Delta') \not\supseteq \mu(E)$, that is $\mathrm{ord}_E(\mu^*(\Delta')) = 0$. Thus, for every $\lambda \geq 0$,

$$a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) = -1.$$

This proves claim 1.
Claim 2 *For every $\lambda \in \mathbb{Q} \cap [0, \lambda')$, we have that*

$$CLC(X, \Delta + \lambda\Delta', S) \subseteq CLC(X, \Delta, S).$$

If the claim holds, then, for every $\lambda \in \mathbb{Q} \cap [0, \lambda')$, we have that

$$CLC(X, \Delta + \lambda\Delta', S) = CLC(X, \Delta, S)$$

because $CLC(X, \Delta + \lambda\Delta') \supseteq CLC(X, \Delta)$ by lemma 2.2.8, so that $CLC(X, \Delta + \lambda\Delta', S) \supseteq CLC(X, \Delta, S)$.

Moreover we can deduce the S-DLTness of $(X, \Delta + \lambda\Delta')$, for $\lambda \in \mathbb{Q} \cap [0, \lambda')$:
Property 1 holds because if $V \in CLC(X, \Delta + \lambda\Delta', S)$, then $V \in CLC(X, \Delta, S)$. Then, thanks to the hypotheses,

$$V \not\subseteq Sing(X) \cup NSNC(\Delta) \cup \mathrm{Supp}(\Delta').$$

But $NSNC(\Delta + \lambda\Delta') \subseteq NSNC(\mathrm{Supp}(\Delta) + \mathrm{Supp}(\Delta')) \subseteq NSNC(\Delta) \cup \mathrm{Supp}(\Delta')$. Therefore $V \not\subseteq Sing(X) \cup NSNC(\Delta + \lambda\Delta')$.

Now suppose $D_i \cap S \neq \emptyset$. Then

$$a_i + \lambda b_i = -a(\mu_*^{-1}(D_i), X, \Delta + \lambda\Delta') \leq 1,$$

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thanks to claim 1. Thus property 2 is satisfied for $\lambda \in \mathbb{Q} \cap [0, \lambda')$.

The lemma follows by choosing $\lambda_0 \in \mathbb{Q}$ such that $0 \leq \lambda_0 < \min\{\frac{1}{m_0}, \lambda'\}$.

Proof of claim 2. Suppose $V \in CLC(X, \Delta + \lambda\Delta', S)$.
Then $V = \mu(W)$, for some $W \in CLC(Y, \widetilde{\Delta_\lambda})$, and $\mu(W) \cap S \neq \emptyset$.
Hence, if $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S = \emptyset$, then $W \nsubseteq E$, that is $\mathrm{ord}_W(E) = 0$.
This implies that, for every prime divisor F over W, for every $x \in \mathbb{Q}$,

$$a(F, Y, \widetilde{\Delta_\lambda} + xE) = a(F, Y, \widetilde{\Delta_\lambda}).$$

Therefore, if we define

$$\widetilde{\Delta'_\lambda} := - \sum_{\mu(E) \cap S \neq \emptyset} a(E, X, \Delta + \lambda\Delta')E,$$

we get that

$$W \in CLC(Y, \widetilde{\Delta_\lambda} + \sum_{\mu(E) \cap S = \emptyset} a(E, X, \Delta + \lambda\Delta')E) =$$

$$= CLC(Y, - \sum_{\mu(E) \cap S \neq \emptyset} a(E, X, \Delta + \lambda\Delta')E) = CLC(Y, \widetilde{\Delta'_\lambda}).$$

Note that the pair $(Y, \widetilde{\Delta'_\lambda})$ is LC, because Y is smooth, $\widetilde{\Delta'_\lambda}$ is SNCS and all its coefficients are less than or equal to 1 by claim 1.
Thus, by remark 2.2.6, all the LC centers of this pair are irreducible components of intersections of prime divisors in the support of $(\widetilde{\Delta'_\lambda})^{=1}$.
But, again by claim 1, we get that

$$\mathrm{Supp}((\widetilde{\Delta'_\lambda})^{=1}) \subseteq \mathrm{Supp}\big((-\sum a(E, X, \Delta)E)^{=1}\big) = \mathrm{Supp}((\widetilde{\Delta_0})^{=1}).$$

Thus W is an irreducible component of a finite intersection of prime divisors in $\mathrm{Supp}((\widetilde{\Delta_0})^{=1})$, that is $W \in CLC(Y, (\widetilde{\Delta_0})^{=1})$.
Moreover, as $\widetilde{\Delta_0}$ is SNCS, W is not contained in any other prime divisor in the support of $\widetilde{\Delta_0}$, so that $W \in CLC(Y, \widetilde{\Delta_0})$.
This implies that $V = \mu(W) \in CLC(X, \Delta, S)$, because we had chosen V such that $V \cap S \neq \emptyset$.
$\square$

The following lemma is an improvement of lemma 2.3.7.

Lemma 3.9.6. *Let (X, Δ) be a pair and let $S \subseteq X$ be a closed subset such that (X, Δ) is an S-DLT pair. Suppose $L \in Div_{\mathbb{Q}}(X)$ is big and nef and $\mathbb{B}_+(L)$ does not contain any element in $CLC(X, \Delta, S)$.*
Then there exists an effective Cartier divisor Γ on X, and a rational number $\lambda_0 > 0$ such that $Bs(|\Gamma|) = \mathbb{B}_+(L)$ and for each $\lambda \in \mathbb{Q} \cap (0, \lambda_0]$, we have that

1. *$L - \lambda\Gamma \in Div_{\mathbb{Q}}(X)$ and is ample;*

2. *$CLC(X, \Delta + \lambda\Gamma, S) = CLC(X, \Delta, S)$;*

3. $(X, \Delta + \lambda\Gamma)$ *is an S-DLT pair.*

Proof. By [ELMNP06, Prop. 1.5] there exists H, an ample $\mathbb{Q}$-divisor on X, such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H).$$

Moreover there exists $m_0 \in \mathbb{N}$ such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Let $\Gamma \in |m_0(L - H)|$ be a general divisor, so that $Bs(|\Gamma|) = \mathbb{B}_+(L)$.
Now, for all $\lambda \in \mathbb{Q} \cap (0, 1]$, we have that

$$L - \lambda\Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if $\lambda \leq \frac{1}{m_0}$ because L is nef and H is ample.

As $CLC(X, \Delta, S)$ is a finite set by lemma 3.9.4, and as, by hypothesis, $Bs(|\Gamma|) = \mathbb{B}_+(L)$ does not contain any element in $CLC(X, \Delta, S)$, we can choose Γ such that $\mathrm{Supp}(\Gamma)$ does not contain any element of $CLC(X, \Delta, S)$, as well.
Then we can apply lemma 3.9.5 and we find a rational number $\lambda_0 > 0$ such that for every rational number $\lambda \in (0, \lambda_0]$ the pair $(X, \Delta + \lambda\Gamma)$ is S-DLT and $CLC(X, \Delta + \lambda\Gamma, S) = CLC(X, \Delta, S)$. $\square$

Lemma 3.9.7. *Let (X, Δ) be a pair, let $S \subseteq X$ be a closed subset and let $N \in Div_{\mathbb{Q}}(X)$ be effective.*
If (X, Δ) is S-DLT, then $(X, \Delta - N)$ is also S-DLT.

Proof. Trivially, $(X, \Delta - N)$ satisfies the property 2, because, by effectivity of N, the coefficients of $\Delta - N$ are not bigger than the coefficients of Δ.

Let us prove that $(X, \Delta - N)$ satisfies the property 1:
Suppose there exists $V \in CLC(X, \Delta - N, S)$, (otherwise there is nothing to prove). Then, as $V \in CLC(X, \Delta, S)$, $V \nsubseteq Sing(X) \cup NSNC(\Delta)$.
In order to prove the property 1 we have to show that $V \nsubseteq NSNC(\Delta - N)$:

Claim *There exists a proper birational morphism $f : Y \to X$, and an irreducible divisor F on Y such that $f(F) = V$ and*

$$a(F, X, \Delta) = a(F, X, \Delta - N) = -1$$

If the claim holds, then, as usual, we have that

$$a(F, X, \Delta) = a(F, X, \Delta - N) - \mathrm{ord}_F(f^*(N)).$$

Then, by the claim, $\mathrm{ord}_F(f^*(N)) = 0$, so that $V = f(F) \nsubseteq \mathrm{Supp}(N)$.
Hence $V \nsubseteq \mathrm{Supp}(N) \cup NSNC(\Delta)$. As $NSNC(\Delta - N) \subseteq \mathrm{Supp}(N) \cup NSNC(\Delta)$, we get that $V \nsubseteq NSNC(\Delta - N)$, so that property 1 holds. Thus the lemma will be proved once we prove the claim.

Proof of the claim. Let $\mu : X' \to X$ be a standard log-resolution of the pair $(X, \mathrm{Supp}(\Delta))$.

Let $E_1, \ldots, E_k$ be prime divisors on X' such that, for all $j \in \{1, \ldots, k\}$, we have

$$a(E_j, X, \Delta) \neq 0 \quad \text{and} \quad \mu(E_j) \supseteq V.$$

Note that the set of the prime divisors on X' with this properties is nonempty because $V \in CLC(X, \Delta)$.

Suppose, furthermore, that $E_1, \ldots, E_k$ are the only prime divisor on X' with both these properties.

Now suppose there exists $j \in \{1, \ldots, k\}$ such that E_j is μ-exceptional.

Then, by definition of standard log-resolution,

$$\mu(E_j) \subseteq Sing(X) \cup NSNC(\Delta) \Longrightarrow V \subseteq Sing(X) \cup NSNC(\Delta).$$

But we have proved before the claim that this is not possible.

Thus, all the E_j are non μ-exceptional. Moreover, as $V \cap S \neq \emptyset$, we have that $\mu(E_j) \cap S \neq \emptyset$ for all $j = 1, \ldots, k$.

Then

$$a(E_j, X, \Delta) \geq -1 \qquad \forall j = 1, \ldots k,$$

thanks to the S-DLTness of (X, Δ) (property 2).

Let $\nu : X'' \to X'$ be a proper birational morphism such that there exists a prime divisor $F \subseteq X''$ such that $\mu(\nu(F)) = V$ and $a(F, X, \Delta - N) \leq -1$.

Note that we can find such a morphism because $V \in CLC(X, \Delta - N)$.

Moreover, composing, if necessary, ν with a suitable log-resolution, we can suppose that X'' is smooth and that the divisor

$$\nu_*^{-1} \mu_*^{-1} \Delta + \nu_*^{-1} \mathrm{exc}(\mu) + \mathrm{exc}(\nu)$$

is SNCS.

Now, as usual, we find that

$$a(F, X, \Delta) = a(F, X', -\sum a(E, X, \Delta)E) =$$

$$a(F, X', -\sum_{j=1}^{k} a(E_j, X, \Delta)E_j) + \sum_{E \neq E_j} a(E, X, \Delta)\mathrm{ord}_F \nu^*(E).$$

Let us show that if $E \subseteq X'$ is a prime divisor such that $E \neq E_j$ for all $j = \{1, \ldots, k\}$, then, either $a(E, X, \Delta) = 0$, or $\mathrm{ord}_F \nu^*(E) = 0$:

In fact, if $\mathrm{ord}_F \nu^*(E) \neq 0$ then

$$F \subseteq \mathrm{Supp}(\nu^*(E)) \subseteq \nu^{-1}(E) \Longrightarrow \nu(F) \subseteq E \Longrightarrow V = \mu(\nu(F)) \subseteq \mu(E).$$

Then $a(E, X, \Delta) = 0$ because of the choice of the E_j's. This shows that

$$a(F, X, \Delta) = a(F, X', -\sum_{j=1}^{k} a(E_j, X, \Delta)E_j).$$

But $(X', -\sum_{j=1}^{k} a(E_j, X, \Delta)E_j)$ is an LC pair, because the divisor $-\sum_{j=1}^{k} a(E_j, X, \Delta)E_j$ is SNCS (as μ is a log-resolution of (X, Δ)), and we have seen that $a(E_j, X, \Delta) \geq -1$ for all $j \in \{1, \ldots, k\}$.

Therefore

$$-1 \geq a(F, X, \Delta - N) \geq a(F, X, \Delta) = a(F, X', -\sum_{j=1}^{k} a(E_j, X, \Delta)E_j) \geq -1,$$

so that $a(F, X, \Delta - N) = a(F, X, \Delta) = -1$, and the claim is proved by putting $f := \mu \circ \nu$ and $Y := X''$. $\square$

Main theorems

Theorem 3.9.8. *Let X be a normal projective variety and let Δ be an effective Weil $\mathbb{Q}$-divisor. If D is a Weil $\mathbb{Q}$-divisor such that*

1. *D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$;*

2. *$(X, \Delta - aN)$ is a P-DLT pair;*

3. *There exist two rational numbers t_0 and a, with $a \geq 0$ such that*

$$t_0 P - (K_X + \Delta - aN)$$

   is ample;

4. *For every $V \in CLC(X, \Delta - aN)$ we have that $V \nsubseteq \mathbb{B}(P)$;*

then P is semiample.

Proof. We define $A := aN - \Delta$. Then $(X, -A)$ is a pair and

$$t_0 P + A - K_X = t_0 P - (K_X + \Delta - aN)$$

is ample.

Now if $(X, -A)$ is KLT then $\kappa(X, D) \geq 0$ by lemma 3.3.1. Hence, as $\kappa(X, P) = \kappa(X, D)$ by definition of CKM Zariski decomposition, we have that $\kappa(X, P) \geq 0$, as well.

If $(X, -A)$ is not KLT then $CLC(X, -A) \neq \emptyset$, so that, $\mathbb{B}(P) \neq X$, because of 4. Hence, again, $\kappa(X, P) \geq 0$.

Then, if we denote

$$\mathbb{N}(P) = \{m \in \mathbb{N} \text{ such that } mP \in Div(X) \text{ e } H^0(X, \mathcal{O}_X(mP)) \neq 0\},$$

we have that $\mathbb{N}(P) \neq \emptyset$. Take $m_1 \in \mathbb{N}$ such that $m_1 P$ is a Cartier divisor and $Bs(|m_1 P|) = \mathbb{B}(P)$, as closed sets.

We suppose, by contradiction, that $\mathbb{B}(P) \neq \emptyset$.

We will find $m \in \mathbb{N}(P)$ and a subvariety $V \subseteq X$ such that, set-theoretically, $V \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$ but $V \nsubseteq Bs(|mP|)$, leading, in such a way, to a contradiction.

Let $\{D_j\}_{j=1}^{k}$ be the finite set of the prime divisors appearing in the support of A or as base components of $|m_1 P|$. We write $A = \sum_{j=1}^{k} a_j D_j$, where the a_j are possibly zero rational numbers.

Now, as $(X, -A)$ is P-DLT and $\mathbb{B}(P)$ does not contain LC centers of the pair $(X, \Delta - aN)$, we can apply lemma 3.9.3, so that we find $\mu : Y \to X$, a log-resolution of the pair $(X, -A)$ and of the linear series $|m_1 P|$ such that:

- μ is a composition of blowings-up of smooth subvarieties of codimension greater than 1.

- $a(E, X, -A) > -1$ for every μ-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(P) \neq \emptyset$;

- $a(E, X, -A) \geq -1$ for every non-μ-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(P) \neq \emptyset$;

Let $\{F_j = \tilde{D}_j\}_{j=1}^k$ be the finite set of the strict transforms of the divisors D_j and let $\{F_j = E_j\}_{j=k+1}^l$ be the finite set of the μ-exceptional prime divisors on Y, so that $\sum_{j=1}^l F_j$ is a SNC divisor.
We can write

$$K_Y \equiv \mu^*(K_X - A) + \sum_{j=1}^l b_j F_j,$$

where $b_j = a(F_j, X, -A)$ for every $j = 1, \ldots, l$.
Moreover we can consider an integral base-point free divisor L and coefficients $r_j \in \mathbb{N} \cup \{0\}$ such that $\mu^*(m_1 P) = L + \sum r_j F_j$ and $\mu^*|m_1 P| = |L| + \sum r_j F_j$.
Hence we have that $Bs(|m_1 P|) = \mu(\bigcup_{r_j \neq 0} F_j)$, so that we can suppose $r_j > 0$ for some j because $Bs(|m_1 P|) \neq \emptyset$.
Moreover if $r_j \neq 0$, then $\mu(F_j) \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$, which implies that $b_j > -1$, as, by hypothesis, $\mathbb{B}(P)$ does not contain any LC center of the pair $(X, -A)$.
Now, as $t_0 P + A - K_X$ is ample and μ satisfies the hypotheses of lemma 2.1.9, there exist, for each $j = k + 1, \ldots, l$, arbitrarily small, rational numbers $\delta_j > 0$, such that

$$\mu^*(t_0 P + A - K_X) - \sum_{j=k+1}^l \delta_j F_j$$

is still ample.
Thanks to the openness of the ample cone there exist also, for each $j = 1, \ldots, k$, positive rational numbers δ_j such that if $0 \leq \delta_j' \leq \delta_j$ then

$$\mu^*(t_0 P + A - K_X) - \sum_{j=1}^k \delta_j' F_j - \sum_{j=k+1}^l \delta_j F_j$$

is ample.
Now we define

$$c = \min_{\{j : r_j \neq 0\}} \frac{b_j + 1 - \delta_j}{r_j}.$$

By choosing the δ_j's small enough we can suppose that

$$b_j + 1 - \delta_j > 0 \text{ for all } j \text{ such that } b_j > -1.$$

Hence $c > 0$ because $b_j > -1$ for every j such that $r_j \neq 0$.
Moreover, perturbing slightly the δ_j's if necessary, we can suppose that the minimum is attained on a unique j, say $j = j_0$. Let $B := F_{j_0}$.
Now we define

- $J_1 = \{j \in \{1, \ldots, k\}$ such that $b_j > -1\}$,

- $J_2 = \{j \in \{1, \ldots, k\}$ such that $b_j = -1$ and $D_j \cap \mathbb{B}(P) \neq \emptyset\}$,

- $J_3 = \{k+1, \ldots, l\}$,

- $J_4 = \{j \in \{1, \ldots, k\}$ such that $b_j \leq -1$ and $D_j \cap \mathbb{B}(P) = \emptyset\}$.

Note that $J_1 \amalg J_2 \amalg J_3 \amalg J_4 = \{1, \ldots, l\}$, because, by choice of μ, $b_j \geq -1$ if $1 \leq j \leq k$ and $D_j \cap \mathbb{B}(P) \neq \emptyset$.

Moreover $j_0 \in J_1 \amalg J_3$ as $b_0 > -1$, being $r_0 \neq 0$.

Now let $s := t_0 + cm_1$, and let

$$B_m := \mu^*((m - cm_1)P + A - K_X) - \sum_{j \in J_3} \delta_j F_j.$$

Then, if $0 \leq \delta_j' \leq \delta_j$ for each $j \in \{1, \ldots, k\} = J_1 \amalg J_2 \amalg J_4$, we have that for every integer $m \geq s$

$$B_m - \sum_{J_1 \amalg J_2 \amalg J_4} \delta_j' F_j = \mu^*((m - cm_1)P + A - K_X) - \sum_{j=1}^{k} \delta_j' F_j - \sum_{j=k+1}^{l} \delta_j F_j$$

is ample, because $m - cm_1 \geq t_0$ and $\mu^*(P)$ is nef.

Let us consider now, for each $j \in J_2$, rational, arbitrarily small numbers $\epsilon_j > 0$, and define

$$A' := \sum_{j \neq j_0} (-cr_j + b_j - \delta_j)F_j,$$

$$A'' := A' + \sum_{j \in J_2} (\epsilon_j + \delta_j)F_j.$$

As $r_j = 0$ if $b_j \leq -1$ we get that

$$A'' = \sum_{(J_1 \amalg J_3 \amalg J_4) \setminus \{j_0\}} (-cr_j + b_j - \delta_j)F_j + \sum_{J_2} (-1 + \epsilon_j)F_j,$$

Now we define, for every $m \in \mathbb{N}(P)$, the divisor

$$Q_m := \mu^*(mP) + A'' - B - K_Y.$$

Then

$$Q_m := \mu^*(mP) + A' - B - K_Y + (A'' - A') \equiv$$

$$\equiv \mu^*(mP) + \sum_{j \neq j_0} (-cr_j + b_j - \delta_j)F_j - F_{j_0} - \sum b_j F_j - \mu^*(K_X - A) + (A'' - A') =$$

$$= \mu^*(mP + A - K_X) - \sum_{j \neq j_0} cr_j F_j - \sum_{j \neq j_0} \delta_j F_j + F_{j_0}(-1 - b_{j_0}) + (A'' - A') =$$

$$= \mu^*((m - cm_1)P + A - K_X) + cL + F_{j_0}(cr_{j_0} - 1 - b_{j_0}) - \sum_{j \neq j_0} \delta_j F_j + (A'' - A') =$$

$$= \mu^*((m - cm_1)P + A - K_X) + cL - \sum \delta_j F_j + \sum_{J_2} (\epsilon_j + \delta_j)F_j =$$

$$= \mu^*((m - cm_1)P + A - K_X) - \sum_{J_1 \amalg J_3 \amalg J_4} \delta_j F_j + cL + \sum_{J_2} \epsilon_j F_j =$$

$$= B_m - \sum_{J_1 \amalg J_4} \delta_j F_j + cL + \sum_{J_2} \epsilon_j F_j.$$

Let $m_2 = \min\{m \in \mathbb{N}(P) \text{ such that } m \geq s\}$. Then $B_{m_2} - \sum_{J_1 \amalg J_4} \delta_j F_j$ is ample, so that Q_{m_2} is also ample if the ϵ_j are small enough because L is nef.

Hence Q_m is ample for every $m \in \mathbb{N}(P)$ such that $m \geq s$ thanks to the nefness of P.

Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that $H^1(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A'' \urcorner - B)) = 0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.

This implies that the restriction homomorphism

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \ulcorner A'' \urcorner)) \to H^0(B, \mathcal{O}_B(\mu^*(mP) + \ulcorner A'' \urcorner))$$

is surjective in this case.

Now we notice that $\mu^*(m_2 P)_{|B} + A''_{|B} - K_B = Q_{m_2|B}$ is an ample $\mathbb{Q}$-divisor.

Moreover $A''_{|B}$ is SNCS because A'' is such and B intersects transversally all the F_j's with $j \neq j_0$.

Hence it suffices to verify that all the coefficients of $A''_{|B}$ are greater than -1 to show that the pair $(B, -A''_{|B})$ is KLT:

Note that if $j \in J_4$, then $\mu(F_j) \cap \mathbb{B}(P) = D_j \cap \mathbb{B}(P) = \emptyset$, so that $\mu(F_j) \cap \mu(B) = \emptyset$, as $\mu(B) \subseteq \mathbb{B}(P)$.

This implies that $B \cap F_j = \emptyset$, that is $F_{j|B} = 0$.

Moreover if $j \in J_3$ and $b_j \leq -1$, then $\mu(F_j) \cap \mathbb{B}(P) = \emptyset$, because of the properties of μ, so that, as before, we obtain again that $F_{j|B} = 0$.

Thus

$$A''_{|B} = \sum_{(J_1 \amalg J_3) \backslash \{j_0\}} (-cr_j + b_j - \delta_j) F_{j|B} + \sum_{J_2} (-1 + \epsilon_j) F_{j|B} =$$

$$= \sum_{J_1 \backslash \{j_0\}} (-cr_j + b_j - \delta_j) F_{j|B} + \sum_{\substack{J_3 \backslash \{j_0\} \\ b_j > -1}} (-cr_j + b_j - \delta_j) F_{j|B} + \sum_{J_2} (-1 + \epsilon_j) F_{j|B}.$$

In particular we can suppose that $j \in J_2$ or $b_j > -1$. But we have that

- if $b_j > -1$ and $r_j = 0$ then $-cr_j + b_j - \delta_j = b_j - \delta_j > -1$ (by the choice of the δ_j's);

- if $b_j > -1$, $r_j \neq 0$ and $j \neq j_0$ then $-cr_j + b_j - \delta_j > -\frac{b_j + 1 - \delta_j}{r_j} r_j + b_j - \delta_j = -1$.

- if $j \in J_2$ then $-1 + \epsilon_j > -1$;

Therefore the pair $(B, -A''_{|B})$ is KLT.

This enables us to use Shokurov's nonvanishing theorem ([KM00, theorem 3.4]), so that for every integer $k > 0$ we can find $\mu_k \in \mathbb{N}(P)$, such that $\mu_k \geq m_2$, $\mu_k \geq a$, μ_k is a multiple of k and

$$H^0(B, \mathcal{O}_B(\mu^*(\mu_k P) + \ulcorner A'' \urcorner)) \neq 0.$$

In fact this cohomology group is non zero for every sufficiently large multiple of m_2.

Now, by the definition of Zariski decomposition, there exists $k_0 \in \mathbb{N}$ such that $k_0 P$ is Cartier, $k_0 D$ is integral and $H^0(X, tk_0 P) \simeq H^0(X, tk_0 D)$ for every $t \in \mathbb{N}$. Let us define $m := \mu_{k_0}$. Then

$$B \not\subseteq Bs(|\mu^*(mP) + \ulcorner A'' \urcorner|) \text{ and } H^0(X, \mathcal{O}_X(mP)) \simeq H^0(X, \mathcal{O}_X(mD)).$$

Now let us write $\ulcorner A'' \urcorner = A^+ - A^-$, where A^+ and A^- are effective divisors without common components. Note that $\ulcorner A'' \urcorner = \sum_{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\}} \ulcorner -cr_j + b_j - \delta_j \urcorner F_j$, so that if we put $x_j := \ulcorner -cr_j + b_j - \delta_j \urcorner$ for every $j = 1, \ldots, l$, we have that

$$A^+ = \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j F_j, \qquad A^- = - \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j < 0}} x_j F_j.$$

Note that $B \not\subseteq \operatorname{Supp}(A^+)$ and $B \not\subseteq \operatorname{Supp}(A^-)$, so that in particular $B \not\subseteq Bs(|\mu^*(mP) + A^+|)$.

Moreover we have that $\mu_*(A^+) \leq \ulcorner aN \urcorner \leq mN$:

In fact

$$\mu_*(A^+) = \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j \mu_*(F_j) = \sum_{\substack{J_1 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j D_j.$$

But, if $j \in J_1 \amalg J_4$ then $x_j = \ulcorner -cr_j + a_j - \delta_j \urcorner \leq \ulcorner a_j \urcorner$. Hence

$$\mu_*(A^+) \leq \sum_{\substack{J_1 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} \ulcorner a_j \urcorner D_j \leq \sum_{j=1}^{k} \max\{0, \ulcorner a_j \urcorner\} D_j \leq \ulcorner aN \urcorner$$

as $\ulcorner aN \urcorner \geq \ulcorner A \urcorner = \sum \ulcorner a_j \urcorner D_j$ and $\ulcorner aN \urcorner \geq 0$.

From these inequalities it follows that $h^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+)) \leq h^0(X, \mathcal{O}_X(mD))$, by using lemma 2.1.8.

But

$$H^0(X, \mathcal{O}_X(mD)) \simeq H^0(X, \mathcal{O}_X(mP)) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \hookrightarrow$$

$$\hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+)),$$

so that $H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+))$.

Therefore we see that $B \not\subseteq Bs(|\mu^*(mP)|)$, which implies that $\mu(B) \not\subseteq Bs(|mP|)$, giving a contradiction. $\qquad \square$

Theorem 3.9.9. *Let (X, Δ) be an effective pair. If $D \in Div_{\mathbb{Q}}(X)$ is such that*

1. *D is big;*

2. *$aD - (K_X + \Delta)$ is nef for some $a \in \mathbb{Q}$;*

3. *There exists a projective birational morphism $f : Z \to X$ such that $f^*(D)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $f^*(D) = P + N$ and*

   - *$(Z, \mathbf{A}(\Delta)_Z - aN)$ is a P-DLT pair;*
   - *For every $V \in CLC(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(P))$ we have that $V \not\subseteq \mathbb{B}_+(f^*(D))$;*

then P is semiample.

Proof. We apply lemma 2.5.3 and we consider t_0, D', P', N', Δ_Z as in the lemma, so that in particular $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition and $t_0 P' - (K_Z + \Delta_Z - N')$ is big and nef. Moreover P' is big and nef, $\mathbb{B}_+(P') = \mathbb{B}_+(P) = \mathbb{B}_+(f^*(D))$ by lemma 2.5.4 and the pair $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$ is P'-DLT.

Then, by lemma 3.9.6, there exists an effective Cartier divisor Γ on Z and a rational number $\lambda > 0$ such that $P' - \lambda\Gamma$ is ample, $(Z, \Delta_Z + \lambda\Gamma - N')$ is P'-DLT and $CLC(Z, \Delta_Z + \lambda\Gamma - N') = CLC(Z, \Delta_Z - N') = CLC(Z, \mathbf{A}(\Delta)_Z - aN)$. Thus

$$(1 + t_0)P' - (K_Z + \Delta_Z + \lambda\Gamma - N') = P' - \lambda\Gamma + t_0 P' - (K_Z + \Delta_Z - N')$$

is ample, being the sum of an ample and a nef divisor.

Moreover, as $\mathbb{B}(P') \subseteq \mathbb{B}_+(P') = \mathbb{B}_+(f^*(D))$, we have that $\mathbb{B}(P')$ does not contain any element in $CLC(Z, \Delta_Z + \lambda\Gamma - N', \mathbb{B}(P'))$, so that $\mathbb{B}(P')$ does not contain any LC center of the pair $(Z, \Delta_Z + \lambda\Gamma - N')$.

Therefore we can apply theorem 3.9.8 and we get that P is semiample. $\square$

The following corollary is a generalization of corollary 3.3.4.

Corollary 3.9.10. *Let (X, Δ) be an effective pair. If $D \in Div_{\mathbb{Q}}(X)$ is such that*

1. *D is big;*

2. *There exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and*

   - *$(Z, \mathbf{A}(\Delta)_Z)$ is a $f^*(D)$-DLT pair;*
   - *For every $V \in CLC(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$ we have that $V \not\subseteq \mathbb{B}_+(f^*(D))$;*

then there exists $\beta > 0$ such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta$$

then P is semiample.

Proof. Note that $\mathrm{Supp}(N) \subseteq \mathbb{B}_+(f^*(D))$ by lemma 2.5.4. Then by lemma 3.9.5 we can find $\beta > 0$ such that if $0 \geq a > -\beta$, then the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$ is $f^*(D)$-DLT and $CLC(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(f^*(D))) = CLC(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$. On the other hand if $a > 0$ then $(Z, \mathbf{A}(\Delta)_Z - aN)$ is a $f^*(D)$-DLT pair by lemma 3.9.7 and $CLC(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(f^*(D))) \subseteq CLC(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$ by lemma 2.2.8 because of the effectivity of N.

Thus, as $\mathbb{B}(P) \subseteq \mathbb{B}(f^*(D))$, for every $a > -\beta$ we have that $(Z, \mathbf{A}(\Delta)_Z - aN)$ is a P-DLT pair and $\mathbb{B}_+(f^*(D))$ does not contain any LC center in $CLC(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(P))$. Therefore we can apply theorem 3.9.9. $\square$

Remark 3.9.11. Note that in theorem 3.9.9 we may change our hypothesis 3 by assuming that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^*(D) = P + N$ such that

- $(Z, \mathbf{A}(\Delta)_Z - aN)$ is a $f^*(D)$-DLT pair;

- $\mathbb{B}_+(f^*(D))$ does not contain any element in $CLC\big(Z, \mathbf{A}(\Delta)_Z - aN,$ $\mathbb{B}(f^*(D))\big)$.

In fact we have that $\mathbb{B}(P) \subseteq \mathbb{B}(f^*(D))$, so that these assumptions imply hypothesis 3 of the theorem.

On the other hand note that in corollary 3.9.10 we may replace the hypothesis 2 with the assumptions that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^*(D) = P + N$ such that $(Z, \mathbf{A}(\Delta)_Z)$ is a P-DLT pair and $\mathbb{B}_+(f^*(D))$ does not contain any LC center in $CLC(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(P))$. This follows by the proof of the corollary itself.

Corollary 3.9.12. *Let (X, Δ) be a weak log Fano pair. Suppose that*

- *(X, Δ) is a $-(K_X + \Delta)$-DLT pair;*

- *for every $V \in CLC(X, \Delta, \mathbb{B}(-(K_X + \Delta)))$ we have $V \not\subseteq \mathbb{B}_+(-(K_X + \Delta))$;*

then $-(K_X + \Delta)$ is semiample.

3.10 Alternative hypotheses

In this section we state some of the theorems in the previous chapters with more common "base-point free type" hypotheses. Proofs are very similar.

The following is a different version of theorem 3.4.1:

Theorem 3.10.1. *Let X be a normal projective $\mathbb{Q}$-Gorenstein variety and let $\Delta \in Div_{\mathbb{Q}}(X)$ be effective and such that (X, Δ) is an LC pair and $(X, (1-b)\Delta)$ is a KLT pair for some rational number $b > 0$.*

If $D \in Div_{\mathbb{Q}}(X)$ is such that

1. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

2. *$\mathbb{B}_+(aD - (K_X + \Delta))$ does not contain any LC center of the pair (X, Δ)*

3. *D admits a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N$$

   such that $\mathbb{B}(P)$ does not contain any LC center of the pair (X, Δ).

Then P is semiample.

Proof. First of all we can assume that $\Delta \neq 0$:

In fact, if $\Delta = 0$ the hypotheses implies that the pair $(X, 0)$ is KLT. In particular $(X, 0)$ is a DLT pair, so that we are under the hypotheses of theorem 3.3.6.

Being $\Delta \neq 0$, we can apply lemma 2.2.10 and we have that $\text{Supp}(\Delta)$ contains all the LC centers of the pair (X, Δ).

Now, thanks to lemma 2.3.7, we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $aD - (K_X + \Delta) - \lambda\Gamma$ is ample, the pair $(X, \Delta + \lambda\Gamma)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$.

This implies that $\mathrm{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta + \lambda\Gamma)$. Hence, by applying again lemma 2.2.10, we have that for every $\epsilon \in \mathbb{Q}^+$ the pair $(X, (1 - \epsilon)\Delta + \lambda\Gamma)$ is KLT.

Now we have that

$$aP + aN - (K_X + (1 - \epsilon)\Delta + \lambda\Gamma) = (aD - (K_X + \Delta) - \lambda\Gamma) + \epsilon\Delta$$

is ample, thanks to the openness of the ample cone, for $\epsilon > 0$ small enough. In other words there exists $\epsilon_0 \in \mathbb{Q}^+$ such that

$$aP + aN - (K_X + (1 - \epsilon_0)\Delta + \lambda\Gamma)$$

is ample.

Thus we can apply theorem 3.3.2 to the pair $(X, (1 - \epsilon_0)\Delta + \lambda\Gamma)$.

$\square$

The following is a different version of theorem 3.6.7:

Theorem 3.10.2. *Let (X, Δ) be an effective LC pair such that X is $\mathbb{Q}$-Gorenstein. Let $D \in Div_{\mathbb{Q}}(X)$ be such that*

1. *$aD - (K_X + \Delta)$ is big and nef for some $a \in \mathbb{Q}^+ \cup \{0\}$;*

2. *$\mathbb{B}_+(aD - (K_X + \Delta)) \not\supseteq V$, for every $V \in CLC(X, \Delta)$;*

3. *D has a $\mathbb{Q}$-CKM Zariski decomposition*

$$D = P + N$$

such that $\mathbb{B}(P) \cap V = \emptyset$ for every $V \in CLC(X, \Delta)$ such that $V \not\subseteq Supp(\Delta)$;

then P is semiample.

Proof. Thanks to lemma 2.3.7, we can find an effective Cartier divisor Γ and a rational number $\lambda > 0$ such that $aD - (K_X + \Delta) - \lambda\Gamma$ is ample, the pair $(X, \Delta + \lambda\Gamma)$ is LC and $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$.

Hence, thanks to lemma 2.2.9 and lemma 2.2.8, we get that for every $\epsilon \in \mathbb{Q}^+$

$$CLC(X, (1 - \epsilon)\Delta + \lambda\Gamma - aN) \subseteq \{V \in CLC(X, \Delta) \text{ such that } V \not\subseteq \mathrm{Supp}(\Delta)\},$$

so that, by hypothesis, $\mathbb{B}(P)$ does not intersect any LC center of the pair $(X, (1 - \epsilon)\Delta + \lambda\Gamma)$. Moreover

$$aP + aN - (K_X + (1 - \epsilon)\Delta + \lambda\Gamma) = (aD - (K_X + \Delta) - \lambda\Gamma) + \epsilon\Delta$$

is ample if ϵ is sufficiently small thanks to the openness of the ample cone. Thus we obtain the semiampleness of P by applying theorem 3.6.5 to the pair $(X, (1 - \epsilon)\Delta + \lambda\Gamma)$.

$\square$

The following is a different version of corollary 3.7.5:

Corollary 3.10.3. *Let (X, Δ) be an effective pair.*
Let $D \in Div_{\mathbb{Q}}(X)$ be such that

1. *$aD - (K_X + \Delta)$ is big and nef for some $a \in \mathbb{Q}$;*

2. *there exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and*

   - *$(Z, \mathbf{A}(\Delta)_Z - aN)$ is an LC pair;*
   - *$\mathbb{B}_+(f^*(aD - (K_X + \Delta)))$ does not contain any LC center of the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$;*
   - *$\widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$, or $P_{|\widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN)}$ is semiample.*

Then P is semiample.

Proof. Define $L := f^*(aD - (K_X + \Delta))$. Then we can apply lemma 2.3.7 to the big and nef $\mathbb{Q}$-divisor L and to the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$ and we find a Cartier divisor Γ and a rational number $\lambda > 0$ such that $L - \lambda\Gamma$ is ample, $(Z, \mathbf{A}(\Delta)_Z - aN + \lambda\Gamma)$ is LC and $CLC(Z, \mathbf{A}(\Delta)_Z - aN + \lambda\Gamma) = CLC(Z, \mathbf{A}(\Delta)_Z - aN)$.
Furthermore, we can choose Γ generically in its linear series and we have that $Bs(|\Gamma|) = \mathbb{B}_+(L)$. Then, by Bertini's lemma, we can suppose that, outside $\mathbb{B}_+(L)$, Γ is smooth and it intersects $\mathbf{A}(\Delta)_Z - aN$ in a simple normal crossing way.
Now we apply lemma 2.5.3, we consider t_0, D', P', N', Δ_Z as in the lemma and define $B := \Delta_Z - N' + \lambda\Gamma$. We will show that the pair (Z, B) and the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor P' satisfy the hypotheses of theorem 3.7.4:
First of all we have that $t_0 P' - (K_Z + B) = t_0 P' - (K_Z + \Delta_Z + \lambda\Gamma - N') = P + L - \lambda\Gamma$ is ample, being the sum of a nef and an ample divisor.
By the LCness of the pair (Z, B) we get that all the coefficients of B are less than or equal to 1 and property 5 holds. Moreover property 1 is trivially verified and property 3 follows by the definition of $\mathbb{Q}$-CKM Zariski decomposition because Δ_Z is effective.

In order to prove that property 4 holds we will show that $\widetilde{\mathrm{Nklt}}(Z, B) \subseteq \widetilde{\mathrm{Nklt}}(Z, \Delta_Z - N') = \widetilde{\mathrm{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN)$, so that we can use the hypothesis of the corollary:
By the choice of Γ we have that $CLC(Z, \Delta_Z - N') = CLC(Z, B)$ and $NSNC(B) \subseteq NSNC(\Delta_Z - N') \cup \mathbb{B}_+(L)$.
Then, if $V \in CLC(Z, B)$ and $V \subseteq Sing(Z) \cup NSNC(B)$, we get that $V \in CLC(Z, \Delta_Z - N')$ and $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N') \cup \mathbb{B}_+(L)$. This implies that $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N')$, because by hypothesis $\mathbb{B}_+(L)$ does not contain LC centers of the pair $(Z, \Delta_Z - N')$. Hence $V \subseteq \widetilde{\mathrm{Nklt}}(Z, \Delta_Z - N')$, and we get the required inclusion. Therefore we can apply theorem 3.7.4 and we are done. $\square$

The following is an alternative version of corollary 3.7.7:

Corollary 3.10.4. *Let (X, Δ) be an effective pair, with $dim X \geq 2$.*
Let $D \in Div_{\mathbb{Q}}(X)$ be such that:

1. *$aD - (K_X + \Delta)$ is big and nef for some rational number $a \geq 0$;*

2. *there exists a projective birational morphism $f : Z \to X$ such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and*

   - *$(Z, \mathbf{A}(\Delta)_Z - aN)$ is an LC pair;*

- $\mathbb{B}_+(f^*(aD - (K_X + \Delta)))$ *does not contain divisorial LC centers of the pair* $(Z, \mathbf{A}(\Delta)_Z - aN)$;

- $\mathrm{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$, *or* $P_{|\mathrm{Nklt}_2(Z, \mathbf{A}(\Delta)_Z - aN)}$ *is semiample.*

Then P is semiample.

Note that if $a \geq 0$ it suffices that (X, Δ) is an LC pair for the pair $(Z, \mathbf{A}(\Delta)_Z - aN)$ to be LC.

Proof. Let us apply lemma 2.5.3 and consider t_0, D', P', N', Δ_Z as in the lemma, so that $D' = P' + N'$ is a $\mathbb{Q}$-CKM Zariski decomposition, Δ_Z is effective and the pair $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$ is LC.
Note that by lemma 2.3.9, we have that $f^*(aD - (K_X + \Delta))$ is logbig in codimension 1 for the pair $(Z, \Delta_Z - N')$, so that $t_0 P' - (K_Z + \Delta_Z - N') = P + f^*(aD - (K_X + \Delta))$ is nef and logbig in codimension 1 for the pair $(Z, \Delta_Z - N')$. Thus we can apply theorem 3.7.6 to the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor P' and to the pair $(Z, \Delta_Z - N')$ because all the hypotheses are satisfied and we get the semiampleness of P. $\qquad\square$

3.11   Examples

Basic construction

The following general construction is due to Hacon and McKernan (see [Laz09, theorem A.6]). The choice of the surface S is due to Gongyo (see [Gon09, Example 5.2]).

Let S be the surface obtained by blowing up $\mathbb{P}^2$ in 9 very general points, so that $-K_S$ is nef but not semiample. Let $S \subseteq \mathbb{P}^N$ be a projectively normal embedding.
Let X_0 be the cone over S and let $\phi : X \to X_0$ be the blowing-up at the vertex. We have that $X \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(-H))$, where H is a sufficiently ample divisor on S. Now we denote by $\pi : X \to S$ the natural projection, and by E the ϕ-exceptional divisor, so that $E \simeq S$.
Note that $-(K_X + E)$ is big and nef. Hence (X, E) is a weak log Fano DLT pair of dimension 3 and E is the only LC center of (X, E); in particular it is a PLT pair.
Now, by adjunction, we have that

$$-(K_X + E)_{|E} = -K_E,$$

whence $-(K_X + E)$ is not semiample because $-K_S$ is not semiample.

Applications

In example 3.11.1 we will show that, with the notation of the previous subsection, $E \subseteq \mathbb{B}_+(-(K_X + E))$, but $E \nsubseteq \mathbb{B}(-(K_X + E))$.
Then we have that (X, E) is a PLT (hence DLT) pair such that

1. $-(K_X + E)$ is big and nef;

2. $\mathbb{B}(-(K_X + E))$ does not contain the only LC center of the pair (X, E);

3. $-(K_X + E)$ is not semiample.

Thus it shows that in many of our theorems, e.g. theorem 3.3.3, theorem 3.3.6, theorem 3.1.2, theorem 3.5.3 and corollary 3.7.5, the hypothesis about the $\mathbb{B}_+$ cannot be removed or replaced with the same hypothesis on the stable base locus.

In example 3.11.2 we will construct, for arbitrary large $k \in \mathbb{N}$, a $\mathbb{Q}$-divisor P and a $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is DLT and the following conditions are satisfied:

1. P is big and nef;

2. $P - (K_X + \Delta)$ is big and nef;

3. The pair (X, Δ) has $m \geq k$ LC centers and $m - 1$ LC centers among these are not contained in $\mathbb{B}_+(P)$;

4. P is not semiample.

Note that, by lemma 2.3.9, property 3 implies that there is one LC center of (X, Δ), say V, such that P remains big when restricted to every LC center in $CLC(X, \Delta) \setminus \{V\}$.

This shows that in many of our theorems, e.g. theorem 3.3.3, theorem 3.5.3 and corollary 3.7.5, we cannot lighten the hypothesis on $\mathbb{B}_+$, in the sense that we must assume that it does not contain any LC center.
Similarly we cannot sharpen the hypothesis of logbigness of P in corollary 3.3.11, in theorem 3.7.6 and in corollary 3.7.8.

Example 3.11.1. Let us show that $E \subseteq \mathbb{B}_+(-(K_X + E))$:
We identify $E \simeq S = Bl_{p_1,\ldots,p_9}\mathbb{P}^2$, we denote by $\epsilon : E \to \mathbb{P}^2$ the blowing-up in 9 very general points, we denote by H an hyperplane section on $\mathbb{P}^2$ and by $F_1, \ldots, F_9$ the 9 distinct prime ϵ-exceptional divisors.
As $-(K_X + E)$ is big and nef, thanks to Nakamaye's theorem we have that

$$\mathbb{B}_+(-(K_X + E)) = Null(-(K_X + E)).$$

Hence, it suffices to show that $(-(K_X + E)^2 \cdot E) = 0$.
But $(-(K_X+E)^2 \cdot E) = (-(K_X+E)_{|_E})^2 = (-K_E)^2 = (3\epsilon^*(H)-F_1-\ldots-F_9)^2 = 0$.

On the other hand we have that $E \not\subseteq \mathbb{B}(-(K_X + E))$:
Note that

$$h^0(E, -(K_X + E)_{|_E}) = h^0(E, -K_E) = h^0(E, 3\epsilon^*(H) - F_1 - \ldots - F_9) =$$

$$= h^0(\mathbb{P}^2, \mathcal{J}_{\{p_1,\ldots p_9\}}(3)) \neq 0.$$

Now consider the exact sequence

$$0 \to H^0(X, -(K_X + E) - E) \to H^0(X, -(K_X + E)) \to H^0(E, -(K_X + E)_{|_E}) \to$$

$$\to H^1(X, -(K_X + E) - E) \to \ldots$$

We have that $-(K_X + E) - E = K_X - 2(K_X + E)$, where $-2(K_X + E)$ is big and nef. Hence, by Kawamata-Viehweg vanishing (see [Laz04, theorem 4.3.1]), $H^1(X, -(K_X + E) - E) = 0$.

Thus we obtain the surjectivity of the restriction map

$$H^0(X, -(K_X + E)) \to H^0(E, -(K_X + E)_{|_E}) \neq 0,$$

which implies that $E \not\subseteq Bs(| - (K_X + E)|)$, so that in particular $E \not\subseteq \mathbb{B}(-(K_X + E))$.

Example 3.11.2. Let $A_1, \ldots, A_k$ be smooth hyperplane sections on X_0 such that $v \notin A_i$ for every $i = 1, \ldots, k$ and the ample divisor $A := \sum A_i$ is SNC. Bertini's theorem assures the existence of such divisors. Let

$$P := -(K_X + E) + \phi^*(A).$$

Moreover define $\Delta := E + \phi^*(A) = E + \phi_*^{-1}(A)$.

Note that the pair (X, Δ) is DLT, because X is smooth and $E + \phi_*^{-1}(A)$ is a SNC divisor, and the LC centers of (X, Δ) are exactly the irreducible components of finite intersections of prime divisors in the support of Δ, namely E and $\phi^*(A_i)$ for every $i \in \{1, \ldots k\}$.

Note also that P is big and nef because it is the sum of two big and nef divisors and

$$P - (K_X + \Delta) = -(K_X + E) + \phi^*(A) - (K_X + E + \phi^*(A)) = 2(-(K_X + E))$$

is also big and nef.

Now, by lemma 2.1.9 there exists $\epsilon > 0$ such that $\phi^*(A) - \epsilon E$ is ample.

Then we can write

$$P = \left(-(K_X + E) + \phi^*(A) - \epsilon E \right) + \epsilon E,$$

where $-(K_X + E) + \phi^*(A) - \epsilon E$ is ample.

This implies that $\mathbb{B}_+(P) \subseteq E$. On the other hand $\phi^*(A) \cap E = \emptyset$, so that the only LC center of the pair (X, Δ) contained in E is E itself.

Thus the only LC center of the pair (X, Δ) contained in $\mathbb{B}_+(P)$ is E.

Moreover $\phi^*(A)_{|_E} = 0$ because $\phi^*(A) \cap E = \emptyset$.

Then $P_{|_E} = -(K_X + E)_{|_E} = -K_E$ is not semiample, because $E \simeq S$. Therefore P is not semiample.

Chapter 4

Asymptotic base loci on singular varieties

The aim of this chapter is to generalize theorem 1.0.6 to the case when the variety X is not smooth. In particular in section 4.1 we study some particular cases, we prove the main theorems in section 4.2, while in section 4.3 we give, as an application, a characterizarion of nef and abundant divisors. All the main results of this chapter appear in [CD11].

Throughout the chapter, unless otherwise stated, by *divisor* we mean a Cartier divisor and, for $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, by $\mathbb{K}$-*divisor* we mean an element of $\mathrm{Div}_{\mathbb{K}}(X)$.

4.1 Some special cases

In this section we investigate the relationship between $\mathbb{B}_-(D)$ and $\mathrm{NNef}(D)$ just exploiting the fact that, by [ELMNP06, Prop. 2.8], we already know that they are equal on smooth varieties. After few lemmas about the behaviour of the restricted base locus under birational maps, we prove that $\mathbb{B}_-(D)$ and $\mathrm{NNef}(D)$ agree on the smooth locus of X. Some considerations will then allow us to conclude that $\mathbb{B}_-(D) = \mathrm{NNef}(D)$ on any normal surface.

Lemma 4.1.1. *Let X and Y be normal projective varieties and let $f : Y \to X$ be a birational morphism. If A is an ample $\mathbb{R}$-divisor on X and $x \in X$ is not contained in the image of the exceptional locus $exc(f)$, then $f^{-1}(x) \notin \mathbb{B}_+(f^*A)$.*

Proof. The lemma immediately follows from lemma 2.3.8. $\qquad\qquad\square$

Lemma 4.1.2. *Let X and Y be normal projective varieties and let $f : Y \to X$ be a birational morphism. If D is an $\mathbb{R}$-divisor on X and $x \in X$ is not contained in $f(exc(f))$, then $x \in \mathbb{B}_-(D) \Rightarrow f^{-1}(x) \in \mathbb{B}_-(f^*D)$.*

Proof. Take $x \in \mathbb{B}_-(D)$. By definition, there exists A_X, an ample $\mathbb{R}$-divisor on X, such that $x \in \mathbb{B}(D + A_X)$. This implies that $f^{-1}(x) \in \mathbb{B}(f^*D + f^*A_X)$: In fact, if by contradiction there exists E', an effective $\mathbb{R}$-divisor on Y, such that $f^{-1}(x) \notin \mathrm{Supp}(E')$ and $E' \sim_{\mathbb{R}} f^*(D + A_X)$ then $f_*E' \sim_{\mathbb{R}} D + A_X$ as $\mathbb{R}$-Weil divisor; but $D + A_X$ is an $\mathbb{R}$-Cartier divisor, hence f_*E' is an effective $\mathbb{R}$-Cartier divisor such that $x \notin \mathrm{Supp}(f_*E')$, i.e., $x \notin \mathbb{B}(D + A_X)$.

By lemma 4.1.1 there exists A_Y, an ample $\mathbb{R}$-divisor on Y, and E_Y, an effective $\mathbb{R}$-divisor on Y, such that $f^*A_X = A_Y + E_Y$ and $f^{-1}(x) \notin \mathrm{Supp}(E_Y)$. Hence $f^{-1}(x) \in \mathbb{B}(f^*D + A_Y) \subseteq \mathbb{B}_-(f^*D)$. $\qquad \square$

We can now compare $\mathbb{B}_-(D)$ and $\mathrm{NNef}(D)$ on the smooth locus of X. To this purpose define X_{sm} to be the *smooth locus* of X, i.e., $X_{\mathrm{sm}} := X \setminus \mathrm{Sing}(X)$. The following holds:

Proposition 4.1.3. *Let X be a normal projective variety and let D be an $\mathbb{R}$-divisor on X. We have that $\mathbb{B}_-(D) \cap X_{\mathrm{sm}} = \mathrm{NNef}(D) \cap X_{\mathrm{sm}}$.*

Proof. In general $\mathrm{NNef}(D) \subseteq \mathbb{B}_-(D)$ (see [BBP09, lemma 1.8]), thus it is enough to show that $\mathbb{B}_-(D) \cap X_{\mathrm{sm}} \subseteq \mathrm{NNef}(D)$.
Let $f : Y \to X$ be a resolution of the singularities of X constructed as a series of blowings-up along smooth centers contained in $\mathrm{Sing}(X)$ (this is possible by Hironaka's theorem - cf. [Laz04, Th. 4.1.3]). By lemma 4.1.2, $f(\mathbb{B}_-(f^*(D)) \supseteq \mathbb{B}_-(D) \cap X_{\mathrm{sm}}$. Since Y is smooth then $\mathbb{B}_-(f^*D) = \mathrm{NNef}(f^*D)$ by [ELMNP06, Prop. 2.8], therefore $\mathrm{NNef}(D) = f(\mathrm{NNef}(f^*(D))) = f(\mathbb{B}_-(f^*D)) \supseteq \mathbb{B}_-(D) \cap X_{\mathrm{sm}}$, where the first equality is a straightforward consequence of [BBP09, Lemma 1.6]. $\qquad \square$

Note that in the following section we will give a generalization of this result (see corollary 4.2.10 and remark 4.2.11).
Recall that for every normal variety X and $\mathbb{R}$-divisor D on X we have that both $\mathbb{B}_-(D)$ and $\mathrm{NNef}(D)$ are at most a countable union of Zariski closed subsets of X (see, for example, [BBP09]).
In particular if the W_i's are irreducible subvarieties of X such that $\mathbb{B}_-(D) = \bigcup_{i \in \mathbb{N}} W_i$, we say that the components of $\mathbb{B}_-(D)$ are the subvarieties W_i that are maximal with respect to inclusion. It is easy to see that this definition does not depend on initial the choice of the W_i's. In the same way we define the components of $\mathrm{NNef}(D)$.
We have that the following holds:

Corollary 4.1.4. *Let X be a normal projective variety and let D be an $\mathbb{R}$-divisor on X. If $\mathbb{B}_-(D)$ has only divisorial components, then $\mathbb{B}_-(D) = \mathrm{NNef}(D)$.*

Proof. As before, we need only to show that $\mathbb{B}_-(D) \subseteq \mathrm{NNef}(D)$. Take $x \in \mathbb{B}_-(D)$. By hypothesis, there exists a prime divisor E such that $x \in E$ and $E \subseteq \mathbb{B}_-(D)$, upon identifying E with its support. Write $\mathrm{NNef}(D) = \cup_{i \in \mathbb{N}} V_i$, where the V_i's are the components of $\mathrm{NNef}(D)$. By prop. 4.1.3, $E \cap X_{\mathrm{sm}} \subseteq \cup_{i \in \mathbb{N}}(V_i \cap E)$, but, since $\mathrm{codim}(\mathrm{Sing}(X)) \geq 2$, we also have that $E \cap X_{\mathrm{sm}}$ is countably dense in E (cf. [DiB11, Def. 2.1, Lemma 2.2(2)]). Therefore there must exist $j \in \mathbb{N}$ such that $V_j = E$, i.e., $E \subseteq \mathrm{NNef}(D)$. Thus $x \in \mathrm{NNef}(D)$. $\qquad \square$

Corollary 4.1.5. *Let X be a normal projective surface and let D be an $\mathbb{R}$-divisor on X. Then $\mathbb{B}_-(D) = \mathrm{NNef}(D)$.*

Proof. We can assume that D is pseudoeffective, so that $\mathbb{B}_-(D)$ is strictly contained in X. We can write $\mathbb{B}_-(D) = \bigcup \mathbb{B}(D + A)$, where the union is taken over all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor. By [ELMNP09, Prop. 1.1] it follows that for every ample A as above $\mathbb{B}(D + A)$ has no isolated points, that is $\mathbb{B}(D + A)$ has no irreducible components of dimension 0, which implies that the same holds for $\mathbb{B}_-(D)$. Since X is a surface this is equivalent to saying

that $\mathbb{B}_-(D)$ has only divisorial components, so that we can conclude by cor. 4.1.4. $\qquad\square$

We can also prove the following:

Proposition 4.1.6. *Let X be a normal projective variety and let $D \in Div_{\mathbb{Q}}(X)$ be such that D is big and D admits a $\mathbb{Q}$-CKM Zariski decomposition $D = P + N$. Then $\mathbb{B}_-(D) = \mathrm{NNef}(D) = Supp(N)$.*

Proof. We have that $\mathrm{NNef}(D) \subseteq \mathbb{B}_-(D)$ by [BBP09, lemma 1.8], whence it is enough to show that $\mathbb{B}_-(D) \subseteq \mathrm{Supp}(N)$ and $\mathrm{Supp}(N) \subseteq \mathrm{NNef}(D)$.

We begin by showing that $\mathbb{B}_-(D) \subseteq \mathrm{Supp}(N)$:

In fact if $A \in Div_{\mathbb{Q}}(X)$ is ample, then $P + A$ is also ample, so that the stable base locus $\mathbb{B}(P + A) = \emptyset$. Hence $\mathbb{B}(D + A) \subseteq \mathbb{B}(P + A) \cup \mathrm{Supp}(N) \subseteq \mathrm{Supp}(N)$. But $\mathbb{B}_-(D) = \bigcup_A \mathbb{B}(D + A)$, where the union is taken over all ample $A \in Div_{\mathbb{Q}}(X)$, so that $\mathbb{B}_-(D) \subseteq \mathrm{Supp}(N)$.

Let us show now that $\mathrm{Supp}(N) \subseteq \mathrm{NNef}(D)$:

We can suppose $N \neq 0$, otherwise there is nothing to prove. Write $N = \sum a_i N_i$, where the a_i's are rational positive numbers and the N_i's are prime divisors. We will show that, for every i, $(\mathrm{ord}_{N_i})_{\mathrm{num}}(D) > 0$, which implies that $\mathrm{Supp}(N) = \bigcup N_i \subseteq \mathrm{NNef}(D)$.

As D is big by [Pro02, proposition 7.4] we know that $D = P + N$ is a Fujita-Zariski decomposition. Moreover if $E \in |D|_{\equiv}$, Then $D - E \equiv 0$ is nef and $D - E \leq D$. Hence $D - E \leq P$, so that $E \geq N$. Thus for every i we have that $\mathrm{ord}_{N_i}(E) \geq \mathrm{ord}_{N_i}(N) = a_i > 0$, which implies that $(\mathrm{ord}_{N_i})_{\mathrm{num}}(D) > 0$. $\qquad\square$

Corollary 4.1.7. *Let X be a normal projective variety and suppose $D \in Div_{\mathbb{Q}}(X)$ is big. If $f : Z \to X$ is a projective birational morphism from a normal variety Z such that $f^*(D) = P + N$ is a $\mathbb{Q}$-CKM Zariski decomposition and P is semiample, then $\mathrm{NNef}(D) = \mathbb{B}_-(D) = \mathbb{B}(D) = f(Supp(N))$.*

Proof. $\mathrm{NNef}(D) = f(\mathrm{Supp}(N))$ because $\mathrm{NNef}(D) = f(\mathrm{NNef}(f^*(D)))$ by [BBP09, lemma 1.6] and $\mathrm{NNef}(f^*(D)) = \mathrm{Supp}(N)$ by proposition 4.1.6. Moreover $\mathbb{B}(D) \subseteq f(\mathrm{Supp}(N))$ because $\mathbb{B}(D) = f(\mathbb{B}(f^*(D)))$ and $\mathbb{B}(f^*(D)) \subseteq \mathrm{Supp}(N)$ by the semiampleness of P. Therefore the statement follows by noting that by [BBP09, lemma 1.8] we always have that $\mathrm{NNef}(D) \subseteq \mathbb{B}_-(D) \subseteq \mathbb{B}(D)$. $\qquad\square$

Remark 4.1.8. In most of the main theorems of chapter 3 we prove the semi-ampleness of the positive part of a $\mathbb{Q}$-CKM Zariski decomposition of a birational pullback of a big $\mathbb{Q}$-divisor D enjoying some good given properties.

Corollary 4.1.7 shows that in all these cases the stable base locus of D coincides with the restricted base locus and the non-nef locus.

4.2 Main results

In this section we will prove theorem 1.0.7, this is done in theorem 4.2.7.

The idea is to prove that given an effective KLT pair (X, Δ) and an effective integral divisor D then we have that

$$\mathbb{B}_-(D) \subseteq \bigcup_p \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)) \subseteq \mathrm{NNA}(D).$$

The former inclusion is a consequence of Nadel's vanishing theorem and the proof is just an easy generalization to singular varieties of some arguments in [ELMNP06, Prop. 2.8]. This is the content of lemma 4.2.1.

To prove the latter inclusion we notice that by considering a suitable log-resolution we can reduce to the smooth case and get rid of the boundary Δ at the same time (see lemma 4.2.2 and proposition 4.2.3), so that the result follows by some considerations in [ELMNP06] (see prop. 4.4 and theorem 4.2.5). The rest of the section is devoted to slight generalizations and resumptive corollaries.

Lemma 4.2.1. *Let (X, Δ) be an effective pair. Let D be an integral divisor. Then $\mathbb{B}_-(D) \subseteq \bigcup_p \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|))$.*

Proof. We will follow [ELMNP06, proof of Prop. 2.8], taking into account the fact that X may be singular. Take $x \in X$ and suppose that, for every $p \geq 1$, $\mathcal{J}((X, \Delta); \|pD\|)_x = \mathcal{O}_{X,x}$. Let A be a fixed very ample divisor such that $A - (K_X + \Delta)$ is ample.

Recall that for every $p \geq 1$, $\mathcal{J}((X, \Delta); \|pD\|) = \mathcal{J}((X, \Delta); \frac{1}{k}D_{pk})$ where k is sufficiently large and divisible and D_{pk} is a general element of $|pkD|$. In particular $\frac{1}{k}D_{pk} \sim_{\mathbb{Q}} pD$. Hence, if $\dim X = n$, by Nadel's vanishing theorem in the singular setting (see [Laz04, Theorem 9.4.17]), we have that for every $i \geq 1$

$$H^i\left(X, \mathcal{O}_X\left((n+1)A + pD\right) \otimes \mathcal{O}_X\left(-iA\right) \otimes \mathcal{J}((X, \Delta); \|pD\|)\right) = 0.$$

Thus, by Mumford's theorem (see [Laz04, Thm. 1.8.5] or [Laz04, volume 2, p. 194]), we have that $\mathcal{O}_X\left((n+1)A + pD\right) \otimes \mathcal{J}((X, \Delta); \|pD\|)$ is globally generated. In particular, since $\mathcal{J}((X, \Delta); \|pD\|)_x = \mathcal{O}_{X,x}$ and $\mathcal{J}((X, \Delta); \|pD\|)$ is an ideal sheaf, this implies that $x \notin \mathrm{Bs}|(n+1)A + pD|$ for every $p \geq 1$, i.e., $x \notin \mathbb{B}_-(D)$. $\square$

Lemma 4.2.2. *Let X be a normal variety and let Δ be a $\mathbb{Q}$-Weil divisor such that (X, Δ) is a KLT pair. Then there exists a log-resolution of (X, Δ), say $f : Y \to X$, such that for every $y \in Y$*

$$\mathrm{mult}_y(\mathbf{L}(\Delta)_Y) < 1.$$

Proof. By [KM00, Proposition 2.36] there exists a log-resolution $f : Y \to X$ such that $\mathrm{Supp}(\mathbf{L}(\Delta)_Y)$ is smooth. By definition

$$\mathbf{L}(\Delta)_Y = \sum_{\substack{E \subseteq Y \\ a(E, X, \Delta) < 0}} -a(E, X, \Delta)E,$$

and, by hypothesis, $-a(E, X, \Delta) < 1$ for any prime divisor $E \subseteq Y$. Since the support of $\mathbf{L}(\Delta)_Y$ is smooth, for any $y \in Y$ either y does not belong to $\mathrm{Supp}(\mathbf{L}(\Delta)_Y)$ or y belongs to only one irreducible component of $\mathrm{Supp}(\mathbf{L}(\Delta)_Y)$, say E_y. In the latter case this means that $\mathrm{mult}_y(\mathbf{L}(\Delta)_Y) = -a(E_y, X, \Delta) < 1$. $\square$

Proposition 4.2.3. *Let (X, Δ) be an effective KLT pair such that X is smooth of dimension n and for every $x \in X$ $\mathrm{mult}_x(\Delta) < 1$. Let D be a Cartier divisor on X such that $\kappa(X, D) \geq 0$. For every $x \in \mathcal{Z}(\mathcal{J}((X, \Delta); \|D\|))$ there exists $p_x \in \mathbb{N}$ such that $x \in \mathcal{Z}(\mathcal{J}(X, \|p_x D\|))$. In particular $\mathcal{Z}(\mathcal{J}((X, \Delta); \|D\|)) \subseteq \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}(X, \|pD\|))$.*

Proof. Take $x \in \mathcal{Z}(\mathcal{J}((X, \Delta); \|D\|))$. By definition, $\mathcal{J}((X, \Delta); \|D\|)$ is the unique maximal element of the family of ideals $\{\mathcal{J}((X, \Delta); \frac{1}{k}|kD|)\}_{k \in \mathbb{N}^+}$. Moreover for every $k \geq 2$, by [Laz04, Prop. 9.2.26] (see also p. 185, volume 2) and [Laz04, Ex. 9.3.57(i)], we have that $\mathcal{J}((X, \Delta); \frac{1}{k}|kD|) = \mathcal{J}(X, \Delta + \frac{1}{k}D_k)$ where D_k is general in $|kD|$. Therefore $x \in \mathcal{Z}(\mathcal{J}(X, \Delta + \frac{1}{k}D_k))$ for every $k \geq 2$.

Hence, by [Laz04, Prop. 9.5.13], we must have that $\mathrm{mult}_x(\Delta + \frac{1}{k}D_k) \geq 1$. Since mult_x is additive and, by the hypothesis on Δ, $\mathrm{mult}_x(\Delta) = 1 - c_x$ for a certain $c_x > 0$, then for any $k \geq 2$, $\mathrm{mult}_x(\frac{1}{k}D_k) \geq c_x$.

Set p_x be any positive natural number such that $p_x \geq \frac{n}{c_x}$. By [Laz04, Th. 11.1.8(i)] and arguing as before $\mathcal{J}(X, \|p_xD\|) = \mathcal{J}(X, p_x\|D\|) = \mathcal{J}(X, \frac{p_x}{h}|hD|) = \mathcal{J}(X, \frac{p_x}{h}D_h)$ for h sufficiently large. Since $\mathrm{mult}_x(\frac{p_x}{h}D_h) = p_x\mathrm{mult}_x(\frac{1}{h}D_h) \geq p_xc_x = n$ then by [Laz04, Prop. 9.3.2], $\mathcal{J}(X, \frac{p_x}{h}D_h)$ is non-trivial at x, i.e., $x \in \mathcal{Z}(\mathcal{J}(X, \|p_xD\|))$. $\square$

Proposition 4.2.4. *Let X be a smooth variety and let D be a Cartier divisor on X such that $\kappa(X, D) \geq 0$. We have that, for every $p \in \mathbb{N}$, $\mathcal{Z}(\mathcal{J}(X, \|pD\|)) \subseteq \mathrm{NNA}(D)$.*

Proof. Note that by [ELMNP06, section 2] we have that for every geometric discrete valuation v on X

$$\sup_p \frac{v(\mathcal{J}(X, \|pD\|))}{p} = \lim_{p \to \infty} \frac{v(\mathcal{J}(X, \|pD\|))}{p} \leq v(\|D\|),$$

where the first equality strongly relies on the subadditivity theorem of Demailly, Ein, Lazarsfeld (cf. [DEL00]). Hence if $x \in \mathcal{Z}(\mathcal{J}(X, \|pD\|))$ for some $p \in \mathbb{N}$, then $\mathrm{ord}_x(\mathcal{J}(X, \|pD\|)) > 0$, so that $\mathrm{ord}_x(\|D\|) > 0$, i.e., $x \in \mathrm{NNA}(D)$. $\square$

Theorem 4.2.5. *Let (X, Δ) be an effective KLT pair. If D is a Cartier divisor on X such that $\kappa(X, D) \geq 0$, then*

$$\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)) \subseteq \mathrm{NNA}(D).$$

Proof. We will show that for every integer p we have that

$$\mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)) \subseteq \mathrm{NNA}(D). \tag{4.1}$$

Since by definition $\mathrm{NNA}(pD) = \mathrm{NNA}(D)$, without loss of generality we can furthermore assume that $p = 1$.

Let $f : Y \to X$ be a log-resolution of (X, Δ) chosen as in lemma 4.2.2. By lemma 2.4.7, $\mathcal{J}((X, \Delta); \|D\|) = f_*(\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \|f^*D\|))$. Moreover since $\mathbf{L}(\Delta)_Y \geq \mathbf{A}(\Delta_Y)$ then $\mathcal{J}((Y, \mathbf{A}(\Delta)_Y); \|f^*D\|) \supseteq \mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)$. Recall that f_* preserves inclusions, therefore $\mathcal{J}((X, \Delta); \|D\|) \supseteq f_*(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|))$ and hence, as by effectivity of $\mathbf{L}(\Delta)_Y$ the latter is an actual ideal sheaf, we have that $\mathcal{Z}(\mathcal{J}((X, \Delta); \|D\|)) \subseteq \mathcal{Z}(f_*(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)))$. Since, by lemma 2.1.6, $\mathcal{Z}(f_*(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|))) \subseteq f(\mathcal{Z}(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)))$ and, by lemma 2.6.9, $f(\mathrm{NNA}(f^*D)) = \mathrm{NNA}(D)$, then in order to prove (4.1) it is sufficient to prove that

$$\mathcal{Z}(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)) \subseteq \mathrm{NNA}(f^*D).$$

This last inclusion comes from the combination of proposition 4.2.3 and proposition 4.2.4.

$\square$

If the effective pair (X, Δ) is not KLT, the same statement does not hold in general, because the zeros of the asymptotic multiplier ideals depend also on the singularities of the pair. In fact we have that $\mathrm{Nklt}(X, \Delta) = \mathcal{Z}(\mathcal{J}(X, \Delta))$. Anyway, we can still recover the previous result outside the non-klt locus. The following holds:

Corollary 4.2.6. *Let (X, Δ) be an effective pair. If D is a Cartier divisor on X such that $\kappa(X, D) \geq 0$, then*

$$\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)) \setminus \mathrm{Nklt}(X, \Delta) \subseteq \mathrm{NNA}(D).$$

Proof. As in the proof of theorem 4.2.5 we can suppose $p = 1$, i.e., it is enough to show that $\mathcal{Z}(\mathcal{J}((X, \Delta); \|D\|) \setminus \mathrm{Nklt}(X, \Delta) \subseteq \mathrm{NNA}(D)$. Let $f : Y \to X$ be a log-resolution of (X, Δ). Set $\mathbf{A}(\Delta)_Y = \sum a(E)E$, where the sum is taken on all prime divisors on Y, and define

$$\Delta_Y^{\geq 1} := \sum_{a(E) \geq 1} a(E)E, \ \Delta_Y^+ := \sum_{0 \leq a(E) < 1} a(E)E, \ \Delta_Y^- = \sum_{a(E) < 0} -a(E)E,$$

so that $\mathbf{L}(\Delta)_Y := \Delta_Y^{\geq 1} + \Delta_Y^+$, and $\mathbf{A}(\Delta)_Y = \mathbf{L}(\Delta)_Y - \Delta_Y^-$. Note also that $f(\mathrm{Nklt}(Y, \mathbf{L}(\Delta)_Y)) = f(\mathrm{Nklt}(Y, \mathbf{A}(\Delta)_Y)) = \mathrm{Nklt}(X, \Delta)$.
As in the proof of theorem 4.2.5, by the birational transformation rule, lemma 2.1.6 and lemma 2.6.9, it is then enough to prove that

$$\mathcal{Z}(\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)) \setminus \mathrm{Nklt}(Y, \mathbf{L}(\Delta)_Y) \subseteq \mathrm{NNA}(f^*D).$$

At this point, notice that for any $y \notin \mathrm{Nklt}(Y, \mathbf{L}(\Delta)_Y) = \mathrm{Supp}(\Delta_Y^{\geq 1})$ we have that $\mathcal{J}((Y, \mathbf{L}(\Delta)_Y); \|f^*D\|)_y \cong \mathcal{J}((Y, \Delta_Y^+); \|f^*D\|)_y$ (see [DiB10, proposition 1.35]). Therefore we are just left to prove that

$$\mathcal{Z}(\mathcal{J}((Y, \Delta_Y^+); \|f^*D\|)) \subseteq \mathrm{NNA}(f^*D),$$

but, since (Y, Δ_Y^+) is an effective KLT pair, this is just an instance of thm. 4.2.5, and we are done. $\qquad\square$

Theorem 4.2.7. *Let (X, Δ) be an effective pair and let D be an $\mathbb{R}$-divisor on X. Then $\mathrm{NNef}(D) \setminus \mathrm{Nklt}(X, \Delta) = \mathbb{B}_-(D) \setminus \mathrm{Nklt}(X, \Delta)$. In particular if (X, Δ) is an effective KLT pair then $\mathrm{NNef}(D) = \mathbb{B}_-(D)$.*

Proof. If D is not pseudoeffective there is nothing to prove. Thus let us assume that D is pseudoeffective. Let $\{A_m\}$ be a sequence of ample $\mathbb{R}$-divisors as in lemma 2.1.10. By lemmas 2.3.6, 2.6.2 it is then clear that we can furthermore assume that D is a big $\mathbb{Q}$-divisor. Since, clearly, for every $c > 0$, $\mathbb{B}_-(cD) = \mathbb{B}_-(D)$, and analogously for NNef, we can also assume that D is integral. By [BBP09, Lemma 1.8] and lemma 4.2.1 we have that $\mathrm{NNef}(D) \subseteq \mathbb{B}_-(D) \subseteq \bigcup_p \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)$. Since D is big, by remark 2.6.8, $\mathrm{NNA}(D) = \mathrm{NNef}(D)$, whence by corollary 4.2.6 we have that $\bigcup_p \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)) \setminus \mathrm{Nklt}(X, \Delta) \subseteq \mathrm{NNef}(D)$ and we are done. $\qquad\square$

Remark 4.2.8. When X is smooth theorem 4.2.7 follows by [ELMNP06, theorem 2.8].
If (X, Δ) is a KLT pair the theorem 4.2.7 has been proved to hold for the divisor $K_X + \Delta$ by Boucksom, Broustet, Pacienza in [BBP09, Proposition 1.10] using [BCHM10].
In general it has been conjectured that theorem 4.2.7 holds for every normal projective variety: see, for example, [BBP09, Conj. 1.9].

Remark 4.2.9. Note also that theorem 4.2.7 implies that $\mathbb{B}_-(D) = \mathrm{NNef}(D)$ for every $\mathbb{R}$-divisor D on a variety X such that $(X, 0)$ is a log-terminal pair in the sense of [dFH09, definition 7.1]. In fact by [dFH09, proposition 7.2] this property is equivalent to the existence of an effective $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is a KLT pair.

Since $\mathrm{NNef}(D)$ and $\mathbb{B}_-(D)$ do not depend on the chosen boundary divisor Δ, if we set $X_{\mathrm{Nklt}} := \bigcap_{\Delta \in \mathcal{F}} \mathrm{Nklt}(X, \Delta)$, where

$$\mathcal{F} := \{E \ \mathbb{Q}\text{-Weil divisor s.t. } (X, E) \text{ is an effective pair}\},$$

then the following holds:

Corollary 4.2.10. *Let X be a normal projective variety and let D be an $\mathbb{R}$-divisor on X. Then $\mathrm{NNef}(D) \setminus X_{\mathrm{Nklt}} = \mathbb{B}_-(D) \setminus X_{\mathrm{Nklt}}$.*

Remark 4.2.11. As Y. Gongyo kindly pointed out to us, for every normal variety X and any smooth closed point $x \in X$ we can find an effective $\mathbb{Q}$-Weil divisor Δ_x such that (X, Δ_x) is a pair and $x \notin \mathrm{Supp}(\Delta_x)$.
In fact if H is an ample Cartier divisor then there exists $m \in \mathbb{N}$ such that the coherent sheaf $\mathcal{O}_X(-K_X + mH) = \mathcal{O}_X(-K_X) \otimes \mathcal{O}_X(mH)$ is globally generated. On the other hand, as $x \in X$ is a smooth point, we have that $\mathcal{O}_X(-K_X + mH)_x \simeq \mathcal{O}_{X,x}$, so that there exists a section $s_x \in H^0(X, \mathcal{O}_X(-K_X + mH))$ such that s_x does not vanish on x. Take $\Delta_x = \{s_x = 0\}$. Then $K_X + \Delta_x \sim mH$ is a Cartier divisor, so that (X, Δ_x) is a pair and $x \notin \mathrm{Supp}(\Delta_x)$.
Therefore prop. 4.1.3 follows also directly by cor. 4.2.10.

Note that as $\mathbb{B}_-(D)$ does not contain isolated points (see [ELMNP09, Prop.1.1]) by corollary 4.2.10 we deduce the following:

Corollary 4.2.12. *Let X be a normal projective variety such that X_{Nklt} has dimension 0. Then for every $\mathbb{R}$-divisor D on X we have that $\mathrm{NNef}(D) = \mathbb{B}_-(D)$.*

4.3 Nef and abundant divisors

Characterization of nef-abundant divisors

In [Rus09, Theorem 2] F. Russo states a characterization of nef and abundant divisors on a smooth projective variety X by means of asymptotic multiplier ideals. Given theorem 4.2.5 and its counterpart below (theorem 4.3.1) we can extend the same characterization to KLT pairs.

Theorem 4.3.1. *Let (X, Δ) be an effective pair. If D is a Cartier divisor on X such that $\kappa(X, D) \geq 0$, then*

$$\mathrm{NNA}(D) \subseteq \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)).$$

Proof. Note that if $e = e(D)$ is the exponent of D, then, by definition, $v(\|eD\|) = e \cdot v(\|D\|)$, so that $v(\|D\|) > 0$ if and only if $v(\|eD\|) > 0$. Moreover

$$\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|p(eD)\|)) \subseteq \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|)).$$

Hence, potentially multiplying D by its exponent $e(D)$, we can suppose that D has exponent 1.

Let $x \in X$ be such that there exists a discrete geometric valuation v with $v(\|D\|) = S > 0$ and $x \in c_X(v)$.

Then

$$0 < S = v(\|D\|) = \inf_{k \in \mathbb{N}} \left\{ \frac{v(|kD|)}{k} \right\}.$$

We can suppose that v corresponds to a prime divisor E_v over X, so that there exists a birational morphism $\mu : X' \to X$ such that $E_v \subseteq X'$ and for every $k \in \mathbb{N}$, if $D_k \in |kD|$ is a general divisor, we have that $v(|kD|) = v(D_k) = \operatorname{ord}_{E_v} \mu^*(D_k)$. Thus $\inf_{k \in \mathbb{N}} \{ \frac{ord_{E_v} \mu^*(D_k)}{k} \} = S > 0$, so that for every $k \in \mathbb{N}$, $\operatorname{ord}_{E_v} \mu^*(D_k) \geq kS$. Let $a(E_v, X, \Delta)$ be the discrepancy of the pair (X, Δ) along the divisor E_v and let

$$p_0 = \begin{cases} \left\lceil \frac{1+a}{S} \right\rceil & \text{if } a := a(E_v, X, \Delta) > -1 \\ 1 & \text{if } a(E_v, X, \Delta) \leq -1. \end{cases}$$

Then, in any case, $a(E_v, X, \Delta) - p_0 S \leq -1$. Let $k \in \mathbb{N}$ be such that $\mathcal{J}((X, \Delta); \|p_0 D\|) = \mathcal{J}\left((X, \Delta); \frac{1}{k} D_{kp_0}\right)$, where D_{kp_0} is a general divisor in $|kp_0 D|$. We have that

$$a\left(E_v, X, \Delta + \frac{1}{k} D_{kp_0}\right) = a(E_v, X, \Delta) - \frac{1}{k} \operatorname{ord}_{E_v} \mu^*(D_{kp_0}) \leq$$
$$\leq a(E_v, X, \Delta) - S p_0 \leq -1.$$

This implies that the subvariety $\mu(E_v) = c_X(v)$ is an LC center of the pair $(X, \Delta + \frac{1}{k} D_{kp_0})$, so that $\mathcal{Z}(\mathcal{J}((X, \Delta + \frac{1}{k} D_{kp_0}); 0)) \supseteq c_X(v)$.
Therefore $x \in \mathcal{Z}(\mathcal{J}((X, \Delta + \frac{1}{k} D_{kp_0}); 0)) = \mathcal{Z}(\mathcal{J}((X, \Delta); \frac{1}{k} D_{kp_0}))$. $\qquad \square$

Corollary 4.3.2. *Let (X, Δ) be an effective KLT pair. Suppose D is a Cartier divisor on X such that $\kappa(X, D) \geq 0$. Then D is nef and abundant if and only if $\mathcal{J}((X, \Delta); \|pD\|) = \mathcal{O}_X$ for every $p \in \mathbb{N}$.*

Proof. We know that D is nef and abundant if and only if $\operatorname{NNA}(D) = \emptyset$. On the other hand by theorem 4.3.1 and corollary 4.2.6 it follows that $\operatorname{NNA}(D) = \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta); \|pD\|))$. $\qquad \square$

It is also interesting to notice that, by the same token, also the following holds:

Corollary 4.3.3. *Let D be a Cartier divisor on a normal projective variety X such that $\kappa(X, D) \geq 0$. Let Δ, Δ' be any two effective divisors such that (X, Δ), (X, Δ') are KLT pairs. Then*

$$\bigcup_{p \in \mathbb{N}} \mathcal{Z}\left(\mathcal{J}((X, \Delta); \|pD\|)\right) = \bigcup_{p \in \mathbb{N}} \mathcal{Z}\left(\mathcal{J}((X, \Delta'); \|pD\|)\right).$$

Applications

It is a general simple trick, when dealing with big and nef divisors up to linear equivalence on a klt pair (X, Δ), to reduce oneself to studying a slightly different klt pair, in which the boundary Δ "has absorbed" the divisor. In view of corollary 4.3.2 this is pretty much the same if we just have nef and abundant divisors instead of big and nef ones. In fact when L is nef and abundant we get, in particular, that $\mathcal{J}((X, \Delta), \|L\|) = \mathcal{O}_X$, so that the hypotheses of the following lemma are verified:

Lemma 4.3.4. *Let (X, Δ) be an effective KLT pair and let L be a line bundle on X. Then $\mathcal{J}((X, \Delta), \|L\|) = \mathcal{O}_X$ if and only if there exists an effective $\mathbb{Q}$-Cartier divisor D such that $D \sim_{\mathbb{Q}} L$ and $(X, \Delta + D)$ is a KLT pair.*

Proof. By definition $\mathcal{J}((X, \Delta), \|L\|) = \mathcal{J}((X, \Delta), \frac{1}{p}|pL|)$ for some $p \in \mathbb{N}$. Moreover by [Laz04, Prop. 9.2.26] (see also p. 185, volume 2) and [Laz04, Ex. 9.3.57(ii)], we have that $\mathcal{J}((X, \Delta); \frac{1}{p}|pL|) = \mathcal{J}(X, (\Delta + \frac{1}{p}L_p); 0)$, where L_p is a general divisor in $|pL|$. Hence $\frac{L_p}{p} \sim_{\mathbb{Q}} L$ and, if $\mathcal{J}((X, \Delta), \|L\|) = \mathcal{O}_X$, then $\mathcal{J}(X, (\Delta + \frac{1}{p}L_p); 0) = \mathcal{O}_X$, so that $(X, \Delta + \frac{1}{p}L_p)$ is a KLT pair.

On the other hand if there exists D such that $D \sim_{\mathbb{Q}} L$ and $(X, \Delta + L)$ is a KLT pair, then $kD \sim kL$ for some $k \in \mathbb{N}$. Thus $\mathcal{O}_X = \mathcal{J}(X, (\Delta + D); 0) = \mathcal{J}((X, \Delta), \frac{1}{k}kD) \subseteq \mathcal{J}((X, \Delta), \frac{1}{k}|kL|)$. In fact if $\mu : X' \to X$ is a suitable log-resolution, so that in particular $|\mu^*(kL)| = |W| + F$, where $|W|$ is base point free and F is effective, we have that $\mu^*(kD) \geq F$, so that

$$\mathcal{J}((X, \Delta), \frac{1}{k}kD) = \mu_* \mathcal{O}_{X'}(\ulcorner \sum_E a(E, X, \Delta)E - \frac{1}{k}\mu^*(kD) \urcorner) \subseteq$$

$$\subseteq \mu_* \mathcal{O}_{X'}(\ulcorner \sum_E a(E, X, \Delta)E - \frac{1}{k}F \urcorner) = \mathcal{J}((X, \Delta), \frac{1}{k}|kL|).$$

Hence $\mathcal{J}((X, \Delta), \frac{1}{k}|kL|) = \mathcal{O}_X$, which implies that $\mathcal{J}((X, \Delta), \|L\|) = \mathcal{O}_X$ as well.

$\square$

To illustrate the general principle touched on before we give a slight generalization of a theorem by F. Campana, V. Koziarz, M. Păun, but only in the case of KLT pairs (see [CKP10, corollary 1]).

Theorem 4.3.5 (Campana, Koziarz, Păun)**.** *Let (X, Δ) be an effective KLT pair, let ρ be a $\mathbb{Q}$-Cartier divisor on X such that $\rho \equiv 0$ and let L be a nef and abundant line bundle on X. Then $\kappa(K_X + \Delta + L) \geq \kappa(K_X + \Delta + L + \rho)$.*

Proof. Since L is nef and abundant then by lemma 4.3.4 there exists Δ' such that (X, Δ') is an effective KLT pair and $K_X + \Delta' \sim_{\mathbb{Q}} K_X + \Delta + L$. Hence $\kappa(K_X + \Delta + L) = \kappa(K_X + \Delta') \geq \kappa(K_X + \Delta' + \rho)$, by [CKP10, corollary 1]. $\square$

Therefore we just have the following corollary:

Corollary 4.3.6. *Let (X, Δ) be an effective KLT pair of dimension n. Let L be a nef line bundle such that $\kappa(L) \geq n - 1$ and let ρ be a numerically trivial $\mathbb{Q}$-Cartier divisor. Then $\kappa(K_X + \Delta + L) \geq \kappa(K_X + \Delta + L + \rho)$.*

Proof. The hypothesis on the Kodaira dimension actually implies that L is nef and abundant. Hence the corollary follows by theorem 4.3.5. $\qquad\square$

Notice that the hypothesis $\kappa(L) \geq n-1$ is necessary. In fact for every n we can find examples of smooth varieties of dimension n and line bundles of Kodaira dimension $n-2$ for which corollary 4.3.6, with $\Delta = 0$, does not hold:

Example 4.3.7. *We will first of all construct an example for $n = 2$.*
Let C be a smooth elliptic curve and let $\eta \in \mathrm{Pic}^0(C)$ be a non-torsion divisor on C. Let $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{O}_C(-\eta)$. Take $X := \mathbb{P}(\mathcal{E})$ and let $\pi : X \to C$ be the related projection. As in [Har77, V.2.8.1] let C_0 be a section $\sigma_0 : C \to X$. Set $\rho := -\pi^*(\eta)$ and $L := -K_X + \pi^*(\eta)$.
By [Har77, Lemma V.2.10], $K_X \sim -2C_0 + \rho$ and $L \sim 2C_0 + 2\pi^*(\eta)$, so that L is nef because $C_0^2 = 0$ and $\eta \equiv 0$.
Hence for any $m \geq 1$, $H^0(X, mL) = H^0(X, 2mC_0 + 2m\pi^*(\eta))$. By projection formula $H^0(X, mL) = H^0(C, S^{2m}(\mathcal{E}) \otimes 2m\eta) = H^0(C, (\mathcal{O}_C \oplus \mathcal{O}_C(-\eta) \oplus \cdots \oplus \mathcal{O}_C(-2m\eta)) \otimes 2m\eta) = \mathbb{C}$. Therefore $\kappa(L) = 0$.
Moreover $K_X + L + \rho = 0$, hence $\kappa(K_X + L + \rho) = 0$. On the contrary $\kappa(K_X + L) = -\infty$, because for any $m \geq 1$, $H^0(X, m\pi^*(\eta)) = H^0(C, m\eta) = 0$.
To produce examples in every dimension just build them up inductively taking products $X \times C$. Call π_1 the first projection and π_2 the second one. Fix any point q on C and define $\rho_{X \times C} := \pi_1^*(\rho)$, $L_{X \times C} := \pi_1^*(L) \otimes \pi_2^*(q)$.

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