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Universality in interacting dimers: a Renormalization Group approach

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Abstract

In this dissertation we study the effect of perturbations that break integrability in two-dimensional dimer models. We examine universal behaviors through fermionic constructive Renormalization Group methods. Two different phenomenologies are considered: perturbations of a model in the bulk of the so-called rough phase and perturbations of a model at the edge between rough and frozen phases. In the former case, integrability as provided by Kasteleyn's solution is destroyed via the addition of an extensive number of extra edges that break planarity (but not bipartiteness) of the underlying square lattice. We prove that, if the weight λ of the non-planar edges is small enough, a suitably defined height function scales on large distances to the Gaussian Free Field with a λ-dependent amplitude, that coincides with the anomalous exponent of dimer-dimer correlations. This represents a weak-universality result as conceived by Kadanoff. In the latter case the probability measure is perturbed by adding a weak, local interaction between dimers that breaks integrability. We prove that approaching the boundary of the rough phase from the inside, at distance ϵ, the Ronkin function, a.k.a. free energy, displays a so called Pokrovsky-Talapov (PT) type of law: if the strength of the interaction, λ, and ϵ, are small enough, then $R(\epsilon) - R(0) \sim c(\lambda)\epsilon^{3/2}$ for some analytic function $c(\lambda) = \frac{4\sqrt{2}}{3\pi} + O(\lambda)$. This constitutes a strong-universality result for PT behavior, as the critical exponent $3/2$ is not renormalized by the interaction. As such, this suggests a connection with a KPZ Universality Class for the fluctuation of the level lines of the associated height function.

CONTENTS

OUTLINE OF THE THESIS

Chapter 1 summarizes the models studied in this thesis, the results obtained for them, and the main strategies, contextualized in a broader introductory framework. It is structured as follows.

- In **Section 1.1**, we provide an introduction of the topic of universality, particularly for critical phenomena in two-dimensional equilibrium statistical mechanics and motivate how the Renormalization Group enters the discussion. Then we provide a more specific overview of the non-interacting dimer model and related results.
- In **Section 1.2** we give the motivations for the study of interacting dimer models in connection with the previously given overview, justifying the interest in specific issues: in particular we motivate two distinct phenomenologies of interest that we intend to study. The first case concerns a dimer model in the bulk of the rough phase with interaction that breaks the planarity of the underlying graph, and hence integrability, where we study the fluctuations of the associated random surface. The second case concerns a dimer model in the rough phase at the rough-to-frozen transition, where we study the asymptotics of the free energy of the model. Each of the next sections in Chapter 1 is bipartite to distinguish these two cases, which we name "Bulk" and "Edge".
- In **Section 1.3** we introduce the necessary definitions of the models used to study these phenomenologies.
- In **Section 1.4** we state the results we obtain for the two cases together with perspectives and generalizations. We also give, in **Section 1.4.1**, an idea of the strategy of the proof in both cases. These are based on the combination of rigorous fermionic Renormalization Group techniques with the integrability structure of the dimer model.

In **Chapter 2** we discuss the construction required to prove the results of the model in Section 1.3 (bulk). This chapter is a transposition of the contents of the article [**1**] published in 2023 in collaboration with A. Giuliani and F. L. Toninelli.

- In **Section 2.1** we review some useful aspects of Kasteleyn's theory on toroidal graphs and derive the Grassmann representation of the weakly non-planar dimer

model, which is a novelty and turns out to be more involved with respect to the previous works [**2**, **3**].

- In **Section 2.2** we prove one of the main results of our work, concerning the logarithmic behavior of the height covariance at large distances and the Haldane-Kadanoff scaling relation, assuming temporarily a sharp asymptotic result on the correlation functions of the dimer model.
- In **Section 2.3** we describe the proof of the Haldane-Kadanoff relation, based on a generalization of the analysis carried out in [**2**].
- **Sections 2.3.1-2.3.3** contains the main novel aspect of this part of the work, which consists in the identification and integration of the massive degrees of freedom.
- **Section 2.3.4** contains the integration of the massless degrees of freedom, for which the discussion is reduced to the one described in [**2**].
- In **Section 2.4**, we complete the proof of the convergence of the height function to the GFF.

Chapter 3 contains the technical analysis of the model defined in Section 1.3 (edge) and the proof of the results in Theorems 1.4.5-1.4.4. The original aspect of this chapter is the implementation of multiscale Renormalization Group techniques, inspired by those used in [**4**], for the study of the asymptotic free energy behavior of the interacting dimer model. More precisely, this turns out to be, to the best of our knowledge, the first proof of the universality of the Pokrovsky-Talapov critical behavior for an interacting dimer model near the rough-frozen transition curve. This chapter is structured as follows.

- In **Section 3.1** we recall known results for the non-interacting dimer model. In particular we prove Theorem 1.4.5 in the non-interacting case.
- **Section 3.2.1** contains the Grassmannian representation of the model. Our case is a direct Corollary of the one obtained in [**2**, Proposition 1].
- In **Section 3.3** we set up the multiscale analysis. This section is standard and is used to construct the inductive structure of the Renormalization Group analysis. We choose to describe the rough-to-frozen transition via a specific parametrization in the phase diagram which has the convenience of reducing our setting to the existent literature. The technical analysis becomes essentially the same as the one of [**4**]; generalization are then commented (see Section 1.4 (edge)). In **Section 3.3.1** we set up the multiscale structure to deal with the shift of the rough-frozen transition point caused by the interaction: this fact is one the main differences with [**4**], where the transition point instead is not modified by the interaction. In **Section 3.3.2** we set up the double-regime multiscale structure, which is necessary to control the thermodynamic limit of the theory uniformly as we approach the edge of the rough phase.
- **Section 3.4** is the core of the technical analysis. It is standard and rooted in an extensive literature. The main references for us in this context are [**2**, **4**, **5**]. The goal of this section is to combine the ideas of [**4**] with the formalism developed in [**2**] for an interacting dimer model. In **Section 3.4.2** there are quantitative estimates,

output of the multiscale structure, necessary to check the convergence of the theory in multiscale analysis; **Section 3.4.3** contains the Gallavotti-Nicolò tree formalism and a comparison with [**4**] in order to export the estimates into our framework.

- **Section 3.4.1** collects the information arising from the inductive renormalization procedure of the previous section, in terms of finite difference equations for the "running coupling constants", the so called Beta function equations. These equations are solved in **Section 3.4.4** via a fixed point theorem in a suitable space of sequences; the existence and smallness of the solution then implies the convergence of renormalized perturbation theory.
- **Section 3.5** contains the proof of the main Theorems 1.4.4-1.4.5

CHAPTER 1

INTRODUCTION

1.1 The framework

UNIVERSALITY

Universal behaviors are widespread in mathematics and physics. We vaguely say that such a behavior occurs if, starting with an object $\mathcal{O}(P, N)$, characterized by a set of properties P and a parameter N: (1) $\mathcal{O}(P, N)$ admits a limit $\mathcal{O}(P; \infty)$ as $N \to \infty$; (2) $\mathcal{O}(P; \infty)$ is independent of some of the properties of $\mathcal{O}(P, N)$ and it is possibly characterized by new "emerging" ones.

Just to make things concrete, in Probability theory, the fundamental Central Limit Theorem proves that the Gaussian distribution is the limiting universal object of the diffusively-rescaled N-step of a wide class of random walks. Expanding the point of view and endowing the object with more structure, Donsker's Theorem promotes Brownian motion as the continuous limiting random function of such a class of walks. In particular, special properties like scale invariance of the process, emerge *only* in the limiting procedure. Modern developments of these ideas have led to studying and characterizing more complex "limiting" objects, as for instance: scaling limit of random geometries (Random Planar maps and Brownian geometry [**6**]), spectrum of large random matrices (Tracy-Widom and Semicircle laws [**7**], [**8**]), random interfaces (Schramm Loewner evolutions, Gaussian Free Fields.. [**9**–**11**]), etc.

The understanding of such universal behaviors is rewarding from both a theoretical and practical perspective. As inhabitants of the (finite) world, i.e. $N < \infty$, it provides a chance to capture only the crucial properties of the object that we observe, at large but finite N. Furthermore, it allows answering concrete questions arising from the study of real world problems: such models are naturally found in the attempt to understand physical problems (e.g., models for 2D quantum gravity are associated to random geometry of random planar maps, energy levels of nuclear physics heavy atoms behavior to the spectrum of large random matrices, etc.) and, in more recent times, also to understand complex phenomena arising in computer science challenges.

Benefiting from the lively exchange between physics and probability, the field of statistical mechanics aims to describe the macroscopic behavior of a system starting from the

fundamental laws among its many microscopic constituents. This includes enhancing the understanding of phenomena such as phase transitions.

In equilibrium statistical mechanics, the concept of universality, also known as the *universality hypothesis*, has the following meaning: near a second-order phase transition, it is expected that critical exponents do not depend on microscopic details of the Hamiltonian but only on some of its structural properties, such as symmetries, dimensionality, etc. Therefore, models with the same "structural properties", after suitably identifying the parameters describing the systems and possibly changing the value of these critical parameters, share exactly the same critical exponents [**12**]. This leads to the interest in studying different universality classes, i.e., classes of models with the same critical behavior, in the sense just defined.

From the mid-20th century onwards, several experimental confirmations began to highlight the validity of this hypothesis, demonstrating that models describing completely different physical situations manifest the same critical behavior, i.e., the same critical exponents with a suitable identification of the parameters [**13**]. On the theoretical front, the formal known analogy between statistical mechanics and field theory allowed to export ideas from the so-called Renormalization Group (RG), originally used to handle ultraviolet divergences in particle physics, to problems of critical points in phase transitions.

The development of the RG, starting from the late 60s, clarified the concept of the universality class and provided a fundamental explanation for the phenomenological expectation that different models, even describing completely different physical situations, could show the same critical behavior [**14**–**18**]. However, due to the mathematical complexity emerging from any reasonable description of realistic models, a rigorous explanation of such phenomena, and an implementation of RG ideas, were still out of reach.

Surprisingly, extremely simplified models turned out to be good candidates for capturing the crucial properties even of more realistic ones. One of the most famous, the Ising model, proposed by Lenz in 1925 as a simple model for magnetism [**19**,**20**], provided the first rigorous demonstration of a phase transition for short-range (nearest neighbor) interactions in two dimensions [**21**]. It also proved that in low dimensions the predictions on critical exponents of mean-field theories were not correct.

The flourishing theory of integrable models provided new solvable examples, such as 1D quantum spin chains and 2D vertex models (see [**22**] for a review), among which some showing the existence of new universality classes different from the Ising's one. In particular, six-vertex and eight-vertex models (as shown by Lieb's and Baxter's solutions, resp. [**22**–**24**]) emphasized the need for a reformulation of the original universality hypothesis in a "weaker" sense [**22**, 10.12], [**25**]: indeed the same model could have critical exponents that change continuously with an appropriate combination of the energy values of vertex configurations. This was the first example of a continuous family of universality classes "parametrized" by a single constant. In this context, universality means that all the critical exponents are simple, rational, functions of just on of them: once one of the exponents is known, all the others can be computed via simple "universal" relations, mainly due to Kadanoff [**25**].

The recent developments of techniques for integrable models led to outstanding results in the field, to a deeper and broader understanding of critical phenomena, by establishing fascinating connections between them and promoting the development of mathematics itself [**26**–**32**] etc. Unfortunately the (prevalent) impossibility of extending these results to slightly more realistic scenarios remains unsatisfactory due to the fragility of integrability.

To quantitatively understand the broader framework of universality, an ideal approach, unifying the structure of integrability of simplified models together with structures surviving even when an exact solution is no longer present for more realistic ones, was needed. Although the goal is still distant, the RG is a valid candidate for this purpose. Indeed, at least for non-integrable models obtained as perturbations of exactly solvable ones, constructive fermionic RG techniques have shed light on the weak universality phenomenon as conceived by Kadanoff [**12**, **25**, **33**], where critical exponents may vary continuously with the intensity of the perturbation, *but*, are still mutually related by simple algebraic relations that allow expressing them all in terms of a single one.

These techniques, rooted first in the ideas of [**34**, **35**] and formalized in the 1990s [**36**–**39**], have been successfully applied to various models since then. Examples include quantum 1D models [**40**, **41**], quantum systems for charge transport [**42**–**44**], classic spin models [**45**–**47**], and dimer models [**2**, **3**].[1] They allowed, among other things, a rigorous quantification of the original concepts of relevance, marginality, and irrelevance of the RG, which contribute to producing a change in the critical exponents of a model.

While these methods extend the verification of universality to a broader class of non-integrable models, they are still limited to small perturbations of integrable models and are not fully developed to address problems with general geometries, despite some progress has been made in recent times [**45**, **47**]. These are, anyway, the obstacles and the challenges in this field of research and we hope that the above mentioned interplay between integrability and renormalization group could lead to many more spectacular results.

From now on, we will examine the 2D dimer model and perturbations thereof. The dimer model represents an important framework not only for its rich integrable structure, but also because it can be seen as a simple model for random surfaces. Moreover, since the standard dimer model has a determinantal structure, it provides the perfect ground to apply ideas and techniques of the renormalization group in order to test universality in the presence of integrability-breaking perturbations, as explained above.

Dimer models

Perhaps you happened to wonder how many ways you could place domino pieces on a chessboard, or perhaps more likely you happened to be at a party and were bound to be paired up with someone for the final dance. In either case, you were the "victim" of a dimer model: the big difference between the two is the underlying graph structure.

Monomer-dimer models appeared in the physics literature in 1935 in a paper by Fowler and Rushbrooke [**49**] and were introduced as simplified models for liquids of anisotropic molecules. Later on, in 1971, Heilmann and Lieb [**50**] proved that, in the presence of a positive density x of monomers (i.e. vacancies on the underlying lattice), the pressure of the model $p(x)$ is analytic, i.e. no phase transition can occur. Moreover, they proved that monomer-dimer correlations decay exponentially in the distance. When $x = 0$, the so called "fully packed" regime (also known in graph theory as *perfect matching* problem) things are more interesting. Kasteleyn [**51**, **52**] and independently, Temperley and Fisher [**53**], solved the model by computing the partition function in terms of Pfaffians, thus

[1] See also [**48**] about the state of the art.

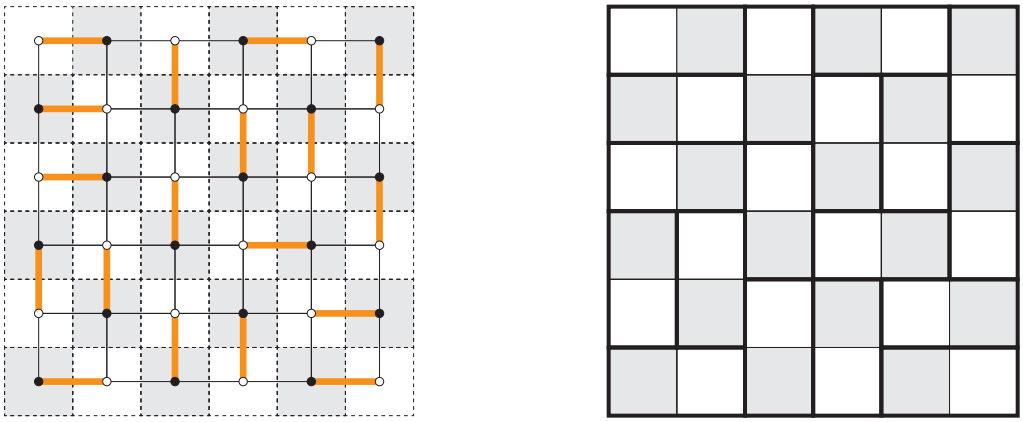

Figure 1.1: *Left: dimer configuration on a portion of $\mathbb{Z}^2$, dual graph dashed; Right: domino tiling of a square.*

showing its determinantal structure, when the underlying graph is planar or embedded in the 2 dimensional torus.[2] When the graph is also bipartite, a natural notion of height [**58**] can be associated to each dimer configuration, which defines a discrete surface that can be seen from the associated tiling in the dual lattice (see Figure 1.2). Such a description is important in connection with the understanding of crystal shapes and growth [**59**], [**60**] [**61**], and more in general, for the broader context of random interfaces arising in systems with coexisting phases [**62**]. It also turns out to be a powerful tool to analize the model itself: as a very first application it gives sufficient conditions for tileability of a domain [**63**].

Despite local dimer statistics were known at least since 1963 [**64**], in the early 2000's, such a discrete-surface representation, provided characterizations of local dimers statistics for domino tilings in terms of variational principles of certain entropy functionals [**65**]: these results were related to the first rigorous proof about the so called *limit shape*[3] of the height field for uniform tilings, robust under geometric deformations of the model. Few years later in the translationally invariant case, more specifically in the case (of our interest in this thesis) of weighted doubly-periodic bipartite and planar graphs, a first complete characterization of the dimer model's phase diagram, in connection with the fluctuations property of the associated surface, was pointed out [**32**]. The set of translationally invariant Ergodic Gibbs Measures was completely characterized in terms of the slope of the height field. They unveiled a very deep algebraic structure of the model connected to its critical behavior. Depending on the choice of the edge weights, we can move between the so called "rough, frozen, smooth" phases of the model[4].

A central object in this scenario is the so-called characteristic polynomial $P(z, w)$,

[2] For higher genus surfaces, Kasteleyn postulated [**54**] that the partition function could be expressed as a sum of 4^g determinants but a full combinatorial proof of this fact came only quite later [**55**, **56**]; see also [**57**].

[3] The first proof of a limit shape existence was obtained in [**66**] and extended then in [**65**] via variational methods and for general geometries.

[4] In the dimer model literature, these are sometimes also known as "liquid, frozen and gas" phases.

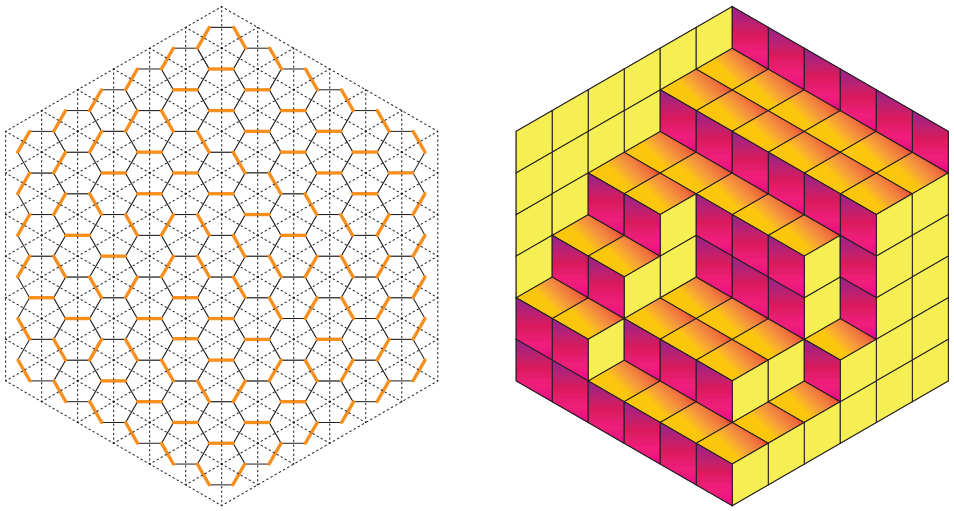

Figure 1.2: *Left: dimer configuration of an hexagonal domain (dual graph dashed); Right: associated tiling configuration in the dual graph. Emerging stepped surface, obtained by coloring by yellow, orange and red the lozenges respectively oriented as north-east, north-west and horizontal. The lozenge tiling is obtained by canceling a dual dashed edge iff it is crossed by a dimer.*

where z, w are complex variables. For instance, the infinite-volume free energy is given by an integral of $\log|P(z,w)|$ over the torus $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\}$. Also, the large-distance decay of correlations is dictated by the so-called spectral curve, i.e. the algebraic curve $\mathcal{C}(P) = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$. When the edge weights are such that the spectral curve intersects $\mathbb{T}^2$ transversally one is in the "rough" or "massless" phase, where the two-point dimer-dimer correlation of the model decays like the inverse distance squared. Correspondingly the height field scales to a Gaussian Free Field (GFF) and the variance grows like the logarithm of the distance times $1/\pi^2$. Remarkably, this pre-factor is independent of the graph weights and of the specific choice of the bipartite periodic planar lattice. In this sense, this model displays a very strong form of universality [**32**].

These universality results concern the dimer model on an infinite periodic lattice. Let us now focus on a version of the model in finite domains in the plane: here things are more subtle. It was already pointed out in Kasteleyn's original paper [**51**], even if not discussed, and then quantitatively proved in [**65**] that boundaries can deeply affect dimer statistics with a long range effect. The first simple geometry for which a heterogeneity, or phase separation, was shown, is the so called Aztec diamond, first introduced in [**67**, **68**]. Here, for uniformly random domino tilings, the presence of boundary was shown to produce a macroscopic phase separation in the domain, between the so called *frozen* and *rough* regions, where the tiling behavior changes from deterministic to random, respectively [**69**]: also, in the scaling limit, the separating 1D interface tends to a deterministic curve, the so called *Arctic circle*. The first results to demonstrate heterogeneity of dimer statisics well inside the arctic circle, also called *local* statistics, are in [**66**], later extended in [**65**]. For dimer statistics near the arctic circle, also referred to as *edge* statistics, results of [**70**] show locally a connection to the Tracy Widom distribution and that the entire boundary

process (the so called north polar boundary), after an appropriate rescaling[5], converges to the so-called Airy process. Such a process, a universal scaling limit introduced in [**71**], is believed to govern various phenomena related to the Kardar–Parisi–Zhang (KPZ) universality class [**72**]. See [**73**–**75**]. Upon these, similar results have been obtained in subsequent years for the edge process for lozenge tilings models in several specific families of geometries[6]. These results, however, rely heavily on the explicit covariance structure (Kasteleyn matrix asymptotics) of the model and are not robust even under small variations of the domain's geometry. Only in very recent times, [**76**, **77**] a form of universality w.r.t. the domain's geometry, has been obtained. Far from specific singular points of the arctic curve[7], convergence of the lines boundary process of lozenge tilings to the Airy's line ensemble has been achieved. The techniques do not rely on the explicit structure of the Kasteleyn matrix but rather are based on a representation of the problem in terms of non-intersecting walks and use technical results from [**78**] for the arctic curve's regularity.

Let us also comment on the general expected effect of boundary for tiling statistics well inside the rough region. A precise conjecture about this question was proposed in [**65**], which suggests that these local statistics around some point in the bulk of the rough region should be given by the infinite-volume, translation-invariant, extremal Gibbs measure with slope equal to the gradient of the limit shape on that point.

This explains the relevance, for domains with boundary, of the work of [**32**]. This conjecture was proved for a large class of geometries for lozenge and domino tilings[8]. Most of them rely on the structure of the Kasteleyn solution, while in the direction of universality, with techniques that are more flexible under geometry variations for lozenge tilings, we mention: [**80**] for stability of the local statistics under perturbations of the boundary and [**79**, **81**, **82**]. Particularly, [**79**] attains the same level of universality for convergence of lozenge tiling local statistics as was shown for the global height profile in [**65**].

At the end of this sketchy picture of the efforts made in the field, we mention some related results, which will not be discussed further here. For more general distributions of tilings (i.e. choice of the edge weights on the graph), one expects more than two phases to appear. In fact, for the so called doubly periodic Aztec diamond [**73**], also a smooth phase occurs where tiles correlations decay exponentially in space. For this model, [**83**] discusses the dimer statistics in the transition region between rough and smooth. In addition, they found [**84**] that a certain kind of line process at the boundary between rough and smooth region converges, if properly rescaled, to an Airy process, in an appropriate limit of the tiling measure (when certain edge weights vanish in the limit). See also [**85**] for a rough-smooth boundary discussion.

Lastly, even if with very different methods from the ones which will be used here, we cannot avoid to mention the recent breakthrough result towards a geometry-universality and limit shape regularity for random tilings obtained in [**78**]. These results are a-sort-of

[5]For a domain of diameter of order L, the correct rescaling is $L^{2/3}$ and $L^{1/3}$ in the tangential and normal direction to the arctic cirle respectively.

[6]See [**76**] for more details.

[7]The behavior of the process at a singular point of the arctic curve is expected to be, in general, different from Airy's.

[8]See [**79**] for more details.

universality analogous of [**32**] in the non translation invariant case. They extend known results relating dimers and PDEs [**32**, **86**] to a greater level of generality. Among other things, they obtain, for more general domains and doubly periodic weighted bipartite underlying graphs: (1) regularity results about the limit shape at the frozen boundary *(Pokrovsky-Talapov law)*; (2) that the arctic curves are algebraic; (3) as a consequence of (1), a conjectured characterization of the limiting processes describing the lines ensembles near the Arctic curve (including Airy's behavior in specific cases). The techniques are based on complex analytical PDE methods and calculus of variations.

After this brief overview about the properties of dimer models, we come back to the translation invariant setting as in [**32**] to study perturbations of the model that destroy integrability as given by Kasteleyn's solution.

Since we want to study two qualitatively different aspects of the dimer model, namely, the universality of fluctuations of the height field in the bulk of the rough phase for a dimer model on the square grid, and the scaling property of the free energy near the rough-frozen transition for a dimer model on the hexagonal lattice, from now on we split in two parts the whole discussion. In each of the following sections, there will be two distinguished discussions, the first referred to as *Bulk case*, and the second as *Edge case*. The names refer to the position in the phase diagram of the model, and must not be confused with the location in some planar domain in $\mathbb{R}^2$ as described in the lines above. We mention here, and clarify below that the choice of grid geometries (resp. square and hexagonal) for the two problems we study is made for simplicity and it is not strictly necessary.

1.2 Motivations

BULK

The understanding of the rough phase of two-dimensional random interfaces is important in connection with the macroscopic fluctuation properties of equilibrium crystal shapes and of the separation surface between coexisting thermodynamic phases. A classical instance arises when studying the low temperature properties of the three-dimensional (3D) Ising model in the presence of Dobrushin Boundary Conditions (DBC). If DBC are fixed so to induce a horizontal interface between the + and − phases, it is well known [**87**] that at low enough temperatures the interface is rigid[9]. It is conjectured that between the so-called roughening temperature and the Curie temperature, the interface displays fluctuations with unbounded variance (the variance diverges logarithmically with the system size), and the height profile supposedly has a massless Gaussian Free Field (GFF) behavior at large scales. This conjecture is completely open, in fact not even the existence of the roughening temperature has been proved. A connected result [**88**] is the logarithmic divergence of fluctuations of the 2D SOS interface at large enough temperature; however, the result comes with no control of the scaling limit. If DBC are chosen so to induce a 'tilted' interface, say orthogonal to the $(1, 1, 1)$ direction, then things are different: fluctuations of the interface are logarithmic already at zero temperature; an exact mapping of the height profile and of its distribution into the dimer model on the hexagonal lattice,

[9] I.e. with bounded fluctuations.

endowed with the uniform measure, allows one to get a full control on the large scale properties of the interface fluctuations, which are now proved to behave like a GFF (see [**32**] for the covariance structure, and [**89**], as well as [**90**, Section 3] for the full Gaussian limit). It is very likely that the GFF behavior survives the presence of a small but positive temperature; however, the techniques underlying the proof at zero temperature, based on the exact solvability of the planar dimer problem, break down. At positive temperatures, the very notion of height of the interface is not well defined, because of overhangs; these will have a low but non-zero density at low temperature. It is likely, though, that the height, even if not defined everywhere at a microscopic level, may be well-defined in a coarse grained sense; therefore, one can still ask about the large-scale behavior of its fluctuations. The coarse-grained height should admit an effective description in terms of a dimer model, whose distribution, however, is not expected to be uniform: temperature induces an effective 'interaction' among dimers.

In a series of works [**2**, **3**, **90**, **91**], the authors started developing methods for the treatment of non-solvable dimer models via constructive, fermionic, Renormalization Group (RG) techniques. They exhibited an explicit class of models, which include the 6-vertex model close to its free-fermion point as well as several non-integrable versions thereof, for which they proved scaling to the GFF, as well as the validity of a 'Kadanoff' or 'Haldane' scaling relation connecting the critical exponent of the so-called electric correlator with the one of the dimer-dimer correlation. Such a scaling relation is the counterpart, away from the free-fermion point, of the universality of the stiffness coefficient of the GFF first observed in [**32**], in connection with the fact that the spectral curve of a planar bipartite dimer model is a Harnack curve.

In this thesis, motivated by our wish to understand the height fluctuations in situations where the height function is not locally well-defined at a microscopic level but only in a coarse-grained sense, as in the case of the 3D Ising interface discussed above, and in situations where the planarity assumption on the underlying graph fails to be satisfied[10], we generalize the analysis of [**2**] to a new setting, inspired by a problem proposed by S. Sheffield a few years ago[11]. Namely, we study the large scale properties of a suitably defined height function, for a dimer model that is *two-dimensional but non-planar*. In short, we introduce a 'weakly non-planar' dimer model, by adding non-planar edges with small weights to a reference planar square lattice. We do so in a periodic fashion, and in such a way that non-planar edges are restricted to belong to cells, separated among each other by corridors of width one, which are crossed by none of the non-planar edges. The fact that non-planar edges avoid these corridors allows us to define a notion of height function on the faces belonging to the corridors themselves. We prove that this height function scales at large distances to a GFF with stiffness coefficient that is *equal* to the anomalous critical exponent of the two-point dimer-dimer correlation.

[10] The interacting dimer model with plaquette interaction studied in [**92**] and in [**93**], which motivated the series of works [**2**, **3**, **90**, **91**], is a toy model for short range Resonance Valence Bond ground states and for liquid crystals in two dimensions. As clear from its definition, such a model is based on drastic simplifications of the physical phenomena one intends to study. In particular, the planarity assumption is physically unjustified: in realistic situations, nothing prevents the presence of defects allowing the dimers to arrange on a bond connecting pairs of sites beyond the nearest neighbors. In fact, non planar dimer models were known already before [**93**] because of their connection to the 6V models with staggered weights, see [**94**] and Appendix C.2. for completeness.

[11] Open problem session at the workshop: "Dimers, Ising Model, and their Interactions", BIRS, 2019

The broader context: height delocalization for discrete interface models Recently, remarkable progress has been made on (logarithmic) delocalization of discrete, two-dimensional random interfaces. We start with the result which is maybe the closest in spirit to our work, that is [**95**, **96**]. These works prove, by means of bosonic, constructive RG methods, that the height function of the discrete Gaussian interface model (that is the lattice GFF conditioned to be integer-valued) has, at sufficiently high temperature, the continuum GFF as scaling limit. In a way, this result is quite complementary to ours, since the model considered there is a perturbation of a *free bosonic* model (the lattice GFF), while in our case we perturb around a *free fermionic* one (the non-interacting dimer model). For closely related results on the 2D lattice Coulomb gas, see also [**97**, **98**].

In a broader perspective, there has been a number of recent results (e.g. [**99**, **100**, **102**]) that prove delocalization of discrete, two-dimensional interface models at high temperature, even though they fall short of proving convergence to the GFF. Let us mention in particular the recent [**100**], which proves with a rather soft argument a (non-quantitative) delocalization statement for rather general height models, under the restriction, however, that the underlying graph has maximal degree three.[12] For the particular case of the 6-vertex model, delocalization of the height function is known to hold in several regions of parameters [**103**–**106**] but full scaling to the GFF has been proven only in a neighborhood of the free-fermion point [**2**].

Edge

As we tried to address in the Universality picture, the height associated with lozenge tilings on finite regions admits a limit shape, as we scale out the lattice, which is the solution of a variational problem.

Now we want to establish a connection between the height of the limit shape of a volume constrained measure and the so called Ronkin function associated to the dimer model without constraints. For definiteness, we consider the model on the hexagonal lattice, where the phase diagram is particularly simple and explicit. As we will recall in Chapter 3 (rough-frozen transition) the Ronkin function $R(x, y)$ is nothing but the free energy of the model, seen in "magnetic field" coordinates: see [**32**, Theorem 3.5, Section 3.2.3], namely

$$R(x,y) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{dz}{z} \frac{dw}{w} \log|1 + ze^x + we^y| \tag{1.2.1}$$

where $\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$; we used that for the uniform dimer model on the hexagonal graph the characteristic polynomial is given by $P(z, w) = 1 + z + w$.

Now consider the dimer model on the *infinite* hexagonal graph, sample a configuration D uniformly among configurations with volume cL^3 for $c > 0$, $L > 0$[13], where the volume (under the surface) is meant by viewing a dimer configuration as a discrete monotone surface in $\mathbb{R}^3$ (see Fig. 1.2 and Fig. 1.3), see also [**107**]. Rescaling the graph so that the lattice mesh is $1/L$ and letting $L \to \infty$, the dimer configuration has a *deterministic* limit shape (see Fig. 1 in [**108**]). As explained in [**32**, Sec. 6], the height function of the limit

[12] See also [**101**].

[13] Alternatively, one can sample D with probability proportional to the exponential of $-\alpha \mathrm{Vol}(D)/L$, with $\alpha > 0$ and $\mathrm{Vol}(D)$ the volume below the surface; the two procedures are asymptotically equivalent for $L \to \infty$, provided that α is chosen as a suitable function of c.

shape is exactly the Ronkin function R defined in (3.1.14). That is, the height of the limit shape at point (x, y) is nothing but $R(x, y)$.

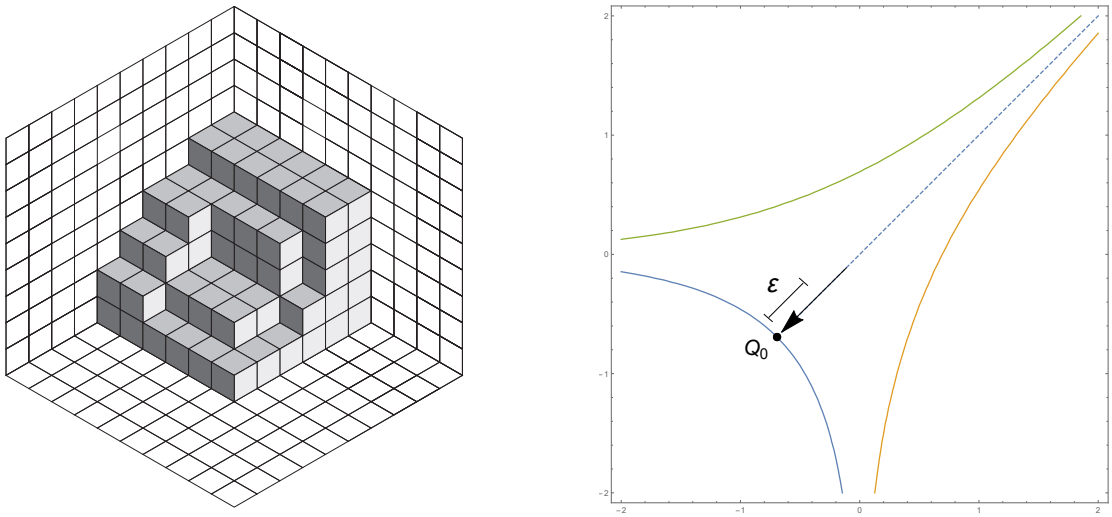

Figure 1.3: *Left: a volume constrained dimer configuration, seen by the surface associated to the tiling; Right: in the xy plane we approach the boundary of the amoeba from the concave region.*

The reason is that the limit shape H of the measure with volume constraint satisfies [**32**, Sec. 6]

$$\operatorname{div}(\nabla\sigma \circ \nabla H) = C, \tag{1.2.2}$$

with C a constant that can be fixed by properly choosing c in the volume tilt. Here, σ is the surface tension of the non-interacting dimer model, see [**32**, Sec 2.2.3]. On the other hand, the Ronkin function (or free energy) is the Legendre transform of the surface tension [**32**, Sec. 3.2.4], so that it satisfies $\nabla\sigma \circ \nabla R = Id$, with Id the 2×2 identity matrix. This implies that indeed R solves (1.2.2) (just choose c so that $C = 2$), that is $H = R$.

It is known that the Ronkin function R is strictly concave in the amoeba

$$\mathcal{A}(P) = \{(\log|z|, \log|w|) : P(z, w) = 0\}$$

and it is affine in the three components of the complement of the amoeba (frozen regions), see [**32**, Sec 3.2.4].

We now want to establish a heuristic connection between the scaling behavior of the Ronkin function near a frozen boundary and the size of fluctuations of the level lines of the height field near such boundary.

Suppose to approach a point Q_0 on the frozen boundary (as shown in Figure 1.3) where $R(Q_0) = 0$ [14] and that $R(Q) \sim \varepsilon^\gamma$ with Q being a point in the interior of the amoeba at a distance $\varepsilon > 0$ from Q_0. Before the rescaling of the graph, this means that the typical height at distance $L\varepsilon$ from the frozen boundary is of order $L\varepsilon^\gamma$. That is, the number of

[14] Since $R = H$ and the height is defined up to a constant, we can set it to zero on a given component of the frozen boundary

level lines between this point and the frozen region (viewing the dimer configuration as a discrete interface made of unit square faces, the height changes by 1 when a single level line is crossed) is of the order $L\varepsilon^\gamma$. Extrapolating to $\varepsilon \approx L^{-1/\gamma}$, this argument suggests that the level lines close to the frozen region have fluctuations of the order $L \times \varepsilon \approx L^\nu, \nu = 1 - 1/\gamma$.

This heuristic argument is actually not specific of the non interacting dimer model. If we infinitesimally perturb the volume constrained measure, it is reasonable to expect that the exact form of the limit shape of the dimer measure with volume tilt will be modified by the interaction and the region of convexity will in general be different from $\mathcal{A}(P)$ but the considerations above about the relation between surface tension, free energy and limit height should still hold, since they are of purely thermodynamic nature. Of course, the perturbation may a priori also affect the value of the critical exponent γ. For the standard, non interacting dimer model on the hexagonal lattice, near frozen boundaries of the arctic curve in bounded domains of $\mathbb{R}^2$ it is known that $\gamma = 3/2$ [**78**, Sec. 1.3]. This behavior goes under the name of *Pokrovsky-Talapov*[15] law or critical behavior (PT). In this scenario, the above reasoning suggests that the expected exponent of fluctuation of the level lines is $\nu = 1/3$. As already commented in Section 1.1, this fact together with the convergence of the $L^{1/3}$ rescaled process of the level lines towards the Airy process have been proved in specific situations, for domino and lozenge random tilings in certain bounded domain of $\mathbb{R}^2$, by means of techniques relying on the integrable structure of the model. However the convergence of the rescaled level lines towards the Airy line ensemble is believed to hold in much more general settings (see [**78**, Sec. 9] and also [**111**]). When we consider perturbations that break integrability of the model, if on the one hand the understanding of the limiting rescaled process of level lines becomes out of reach, on the other hand, in virtue of the previous discussion, it is interesting to understand whether the exponent γ is affected or not by the interaction: this will suggest the fluctuation properties of the interacting level lines.

Thus, the key question motivating our study of the "edge case" is whether γ is, in fact, modified or not by the integrability-breaking perturbation.

1.3 The models

Bulk

To construct the graph G_L on which our dimer model is defined, we let L, m be two positive integers with $m \geq 4$ even, and we start with $G_L^0 = (\mathbb{Z}/(Lm\,\mathbb{Z}))^2$, which is just the toroidal graph obtained by a periodization of $\mathbb{Z}^2$ with period $L\,m$ in both horizontal and vertical directions. We can partition G_L^0 into L^2 square cells B_x, $x = (x_1, x_2) \in \Lambda := (-L/2, L/2]^2 \cap \mathbb{Z}^2$, of sidelength m. The graph G_L^0 is plainly bipartite and we color vertices of the two sub-lattices black and white (each cell contains $m^2/2$ vertices of each color). Black (resp. white) vertices are denoted b (resp. w). We let $\mathbf{e_1}$ (resp. $\mathbf{e_2}$) denote the horizontal (resp. vertical) vectors of length m and we note that translation by $\mathbf{e_1}$ (resp. by $\mathbf{e_2}$) maps the cell B_x into $B_{((x_1+1) \mod L, x_2)}$ (resp. $B_{(x_1,(x_2+1) \mod L)}$). A natural choice of coordinates for vertices is the following one: a vertex is identified by its color (black or white) and by a pair of coordinates (x, ℓ) where x identifies the label of the cell the vertex

[15] Originally appeared in [**109**] in the context of commensurability-incommensurability transition; see also [**110**] for relations between crystal shapes and (PT) behavior.

belongs to, and the "type" $\ell \in \mathcal{I} := \{1, \dots, m^2/2\}$ identifies the vertex within the cell. It does not matter how we label vertices within a cell, but we make the natural choice that if two vertices are related by a translation by a multiple of $\mathbf{e_1}, \mathbf{e_2}$, then they have the same type index ℓ.

The graph G_L is obtained from G_L^0 by adding in each cell B_x a finite number of edges among vertices of opposite color (so that G_L is still bipartite), with the constraint that G_L is invariant under translations by multiples of $\mathbf{e_1}, \mathbf{e_2}$ (i.e., vertex w of coordinates (x, ℓ) is joined to vertex b of coordinates (x, ℓ') if and only if the same holds for any other $x' \in \Lambda$). See Fig. 1.4 for an example.

Remark 1.3.1. *It is easy to see that we need that $m \geq 4$ for this construction to work: if $m = 2$, the two black edges in the cell are already connected to the two black vertices and there are no non-planar edges that can be added.*

Note that G_L is in general non-planar, even in the full-plane limit $L \to \infty$.

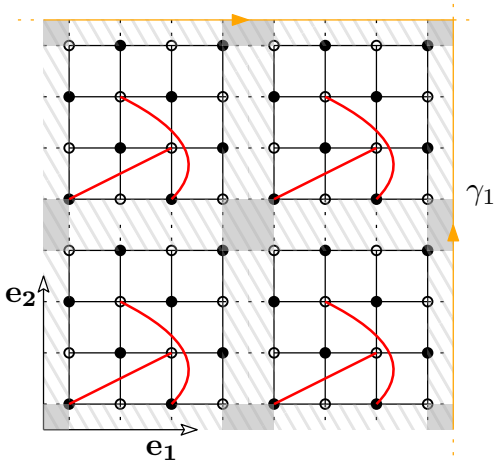

Figure 1.4: *An example of graph G_L with $L = 2$, $m = 4$ and two non-planar edges (in red) per cell. The height function is defined on the set F of dashed faces outside of cells. Faces colored gray are those in $\bar{F}$ (they share a vertex with four different cells).*

Let E_L denote the set of edges of G_L: we write E_L as the disjoint union $E_L = E_L^0 \cup N_L$ where E_L^0 are the edges of G_L^0 (we call these "planar edges") and N_L (we call these "non-planar edges") are the extra ones. Each edge $e \in E_L$ is assigned a positive weight: since we are interested in the situation where the weights of non-planar edges are small compared to those of planar edges, we first take a collection of weights $\{\tilde{t}_e\}_{e \in E_L}, \tilde{t}_e > 0$ that is invariant under translations by multiples of $\mathbf{e_1}, \mathbf{e_2}$, and then we establish that the weight of a planar edge e is $t_e = \tilde{t}_e$ while that of a non-planar edge is $t_e = \lambda \tilde{t}_e$, where λ is a real parameter, that will be taken small later.

To simplify expressions that follow, we will sometimes write $\mathbf{x} \in \mathbf{\Lambda}$ instead of $(x, \ell) \in \Lambda \times \mathcal{I}$ for the coordinate of a vertex of G_L. Also, we label the collection of edges in E_L whose black vertex has type ℓ with a label $j \in \mathcal{J}_\ell = \{1, \dots, |\mathcal{J}_\ell|\}$. The labeling is done in such a way that two edges that are obtained one from the other via a translation by a multiple of $\mathbf{e_1}, \mathbf{e_2}$ have the same label. Note that $|\mathcal{J}_\ell| \geq 4$, and it is strictly larger than four if there are non-planar edges incident to the black vertex of type ℓ. By convention, we label $j = 1, \dots 4$ the four edges of G_L^0 belonging to $\mathcal{J}_\ell$, starting from the horizontal one whose left endpoint is black, and moving anti-clockwise.

The set of perfect matchings (or dimer configurations) of G_L is denoted Ω_L. Each $M \in \Omega_L$ is a subset of E_L and the set of perfect matchings that contain only planar edges is denoted Ω_L^0. Our main object of study is the probability measure on Ω_L given by

$$\mathbb{P}_{L,\lambda}(M) = \frac{w(M)}{Z_{L,\lambda}} \mathbb{1}_{M \in \Omega_L},\ w(M) = \prod_{e \in M} t_e,\ Z_{L,\lambda} = \sum_{M \in \Omega_L} w(M). \tag{1.3.1}$$

We are interested in the limit where L tends to infinity while m (the cell size) is fixed. In this limit, the graph G_L^0 becomes the (planar) graph $G_\infty^0 = \mathbb{Z}^2$ while G_L becomes a periodic, bipartite, non-planar graph G_∞. Cells B_x of the infinite graphs are labelled by $x \in \mathbb{Z}^2$. We let Ω (resp. Ω^0) denote the set of perfect matchings of G_∞ (resp. of $\mathbb{Z}^2$).

In the case $\lambda = 0$, the measure $\mathbb{P}_{L,0}$ is supported on Ω_L^0: in fact, $\mathbb{P}_{L,0}$ is just the Boltzmann-Gibbs measure of the dimer model on the (periodized) square grid, with edge weights of periodicity m (we will refer to this as the "non-interacting dimer model"). The non-interacting model is well understood via Kasteleyn's [**51**, **52**, **112**] (and Temperley-Fisher's [**53**]) theory, that allows to write its partition and correlation functions in determinantal form. According to the choice of the edge weights $\{t_e\}$, the non-interacting model can be either in a rough (massless), smooth (massive) or frozen phase, see [**32**]. In particular, in the rough phase correlations decay like the squared inverse distance (see for instance (2.1.21)-(2.1.22) for a more precise statement). *In this work, we assume that the edge weights are such that for $\lambda = 0$, the model is in a massless phase.*

The essential facts from Kasteleyn's theory that are needed for the present work are recalled in Section 2.1.1. In particular, we emphasize that all the statistical properties of the non-interacting model are encoded in the so-called *characteristic polynomial* μ (see (3.1.6)), that is nothing else but the determinant of the Fourier transform of the so-called Kasteleyn matrix. Then, the assumption that the $\lambda = 0$ model is in the massless phase, can be more precisely stated as follows:

Assumption 1. *The edge weights $\{t_e\}$ are such that the "characteristic polynomial" $\mu : [-\pi, \pi]^2 \mapsto \mathbb{C}$ (see formula (2.1.9) below and the discussion following it) of the non-interacting dimer model has exactly two zeros p_0^+, p_0^- (distinct and simple).*

We recall from [**32**] that this is a non-empty condition on the edge weights (in fact, this set of edge weights is a non-trivial open set). We also remark that if Assumption 1 is not satisfied, then we are in one of the following situations:

1. The edge weights are such that μ has no zeros on $[-\pi, \pi]^2$, corresponding to the frozen or to the gaseous phases of the non-interacting dimer model. In this case, the dimer model can be easily shown to be stable under the addition of dimer interaction such as those treated in [**2**] or of non-planar edges of small weight such as those treated in this paper. By 'easily', we mean 'via a single-step fermionic cluster expansion', which shows that the fluctuations of the perturbed model have the same qualitative behavior of the unperturbed ones. In this case, the height function displays no interesting behavior in the scaling limit.

2. The edge weights are such that μ has one double zero p_0 and: either the system is at the boundary separating the rough from the gaseous or frozen phases (in which case $\nabla\mu(p_0) \neq 0$), or it is at a degenerate point within the rough phase (in which case, called 'real node' in [**32**], $\nabla\mu(p_0) = 0$). These cases display rich and interesting

behaviors, particularly in connection with bounded domains or volume constrained measures, as described in the *edge* case sections above.

EDGE

Here the model has a more straightforward definition. We consider a graph $G_L = (V_L, E_L), L \in 2\mathbb{N}$, embedded in the two-dimensional torus, defined as

$$G_L := \mathcal{H}/(L\mathbb{Z} \times L\mathbb{Z}) \tag{1.3.2}$$

where $\mathcal{H}$ stands for the infinite bipartite hexagonal graph and $L\mathbb{Z} \times L\mathbb{Z}$ denotes the action of the translations with respect to directions $L\vec{e_1}, L\vec{e_2}$ (see Figure 1.5).

We remark that the choice of considering $\mathcal{H}$ rather than $\mathbb{Z}^2$ is unimportant and it is pursued only to simplify the description and structure of the phase diagram. The choice of $\mathcal{H}$ make the description the simplest possible.

Since G_L is bipartite, we split the vertex set as the set of black and of white vertices: $V_L = V_L^W \cup V_L^B$. The sets V_L^B and V_L^W are both isomorphic to $\Lambda := \{(x_1, x_2) : x_i = 0, \dots, L-1\}$. We assume that a black and a white vertex share the same coordinates if and only if they are in the same copy of the fundamental domain G_1. Given a vertex v, we write $x(v) = (x_1(v), x_2(v)) \in \Lambda$ for its coordinates. We define Ω_L as the space of dimer configurations or perfect matchings of G_L.

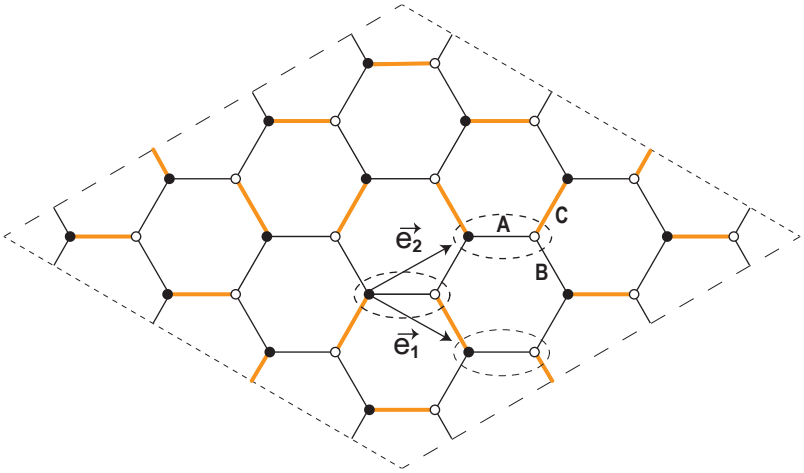

Figure 1.5: *The graph G_L with $L = 4$; the fundamental domain, containing one black and one white vertex which share the same coordinate x, is encircled. Weights of horizontal, north-east and north-west oriented edges are A, B, C, respectively. In orange an example of a dimer configuration.*

To each $M \in \Omega_L$ we can associate a statistical weight $w_\lambda(M)$, where $\lambda \in \mathbb{R}$. First we introduce a weight function $t : E_L \to \mathbb{R}^+$ which assigns a positive number to each edge of the graph. We work in a translationally invariant setting, so we let $t_e = A$ if e is horizontal, $t_e = B$ is e is north-west oriented and $t_e = C$ otherwise. Note that one can define $w_0(M) = \prod_{e \in M} t_e$, which is nothing but the statistical weight of a non interacting dimer model. In order to study the effect of the breaking of integrability of the model, we introduce an interaction term with strength λ. We anticipate here that the specific choice

of the following interaction is not necessary: it is essentially made only to mildly simplify the so called "Grassmann representation" of the model.

Definition 1.3.1. The interacting model is then defined by the following probability measure

$$\begin{aligned} &\mathbb{P}_{\lambda,L}(M) := \frac{w_\lambda(M)}{Z_{\lambda,L}}, \quad w_\lambda(M) := A^{N_A} B^{N_B} C^{N_C} e^{\lambda V} \\ &Z_{\lambda,L} = \sum_{M\in\Omega_L} w_\lambda(M), \quad V = \sum_{f\in G_L^*} \mathbb{1}_f^{(2)} \end{aligned} \tag{1.3.3}$$

where $N_A(M), N_B(M), N_C(M)$ are respectively the number of dimers of type A, B, C in the configuration $M \in \Omega_L$. Given a face $f \in G_L^*$, we define $\mathbb{1}_f^{(2)}(M) = 1$ iff exactly two "parallel" dimers appear on the edges surrounding f in the configuration M while it evaluates 0 otherwise (see Figure below). One can write

$$\mathbb{1}_f^{(2)} = \sum_{\sigma=1}^{3} \mathbb{1}_{f,\sigma}^{(2)} \tag{1.3.4}$$

where σ specifies which type of pair of dimers occupies f, see Fig. 1.6.

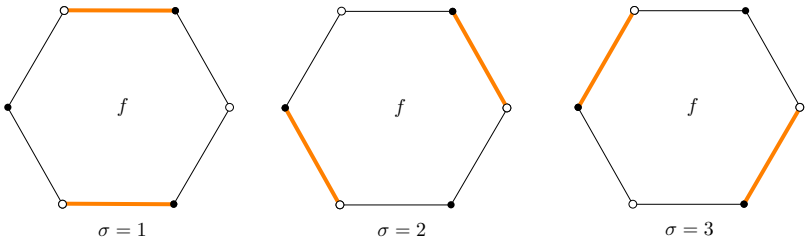

Figure 1.6: *Three local dimer configurations contributing with an extra factor e^λ.*

Let us also define the *free energy* per unit volume and the generating functional as,

$$\begin{aligned} &f_{\lambda,L} = \frac{1}{L^2} \log Z_{\lambda,L} \\ &\mathcal{W}_{\lambda,L}(\mathcal{A}) = \log \sum_{M\in\Omega_L} w_\lambda(M) e^{\sum_{e\in E_L} \mathbb{1}_e A_e} \end{aligned} \tag{1.3.5}$$

where $\mathcal{A} = \{A_e\}_{e\in E_L}$, $A_e \in \mathbb{R}$, are sometimes called external fields, while $\mathbb{1}_e(M)$ is the indicator function of the edge e, i.e. $\mathbb{1}_e(M) = 1$ if $e \in M$ and 0 otherwise. Let us also denote by

$$\rho_{\#,\lambda,L} = \mathbb{E}_{\lambda,L}[L^{-2} N_\#] \tag{1.3.6}$$

the densities of dimers of type $\#$, for $\# = A, B, C$.

In order to state our results in the next section we need few more preliminaries. First, since the model is invariant under rescaling of the edge weights[16], we can set $A = 1$.

As described in Section 1.2(edge), we are interested in studying a perturbation of a dimer model, when its Ronkin function $R(x, y)$ is evaluated at a point in the interior of

[16] This follows from the fact that $N_A + N_B + N_C = L^2$ (perfect matching regime).

the amoeba $\mathcal{A}(P)$, very close to its boundary; recall that $P(z,w) = 1 + z + w$. In order to do this, we need to tune properly the edge weights (B, C). It is known [**113**] that the free energy of the non interacting dimer model is given by

$$F(B,C) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{dz}{z} \frac{dw}{w} \log|1 + Bz + Cw| \tag{1.3.7}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$; a comparison with (1.2.1) gives $F(e^x, e^y) = R(x,y)$. In order to obtain the picture in the right of Figure 1.3, it is enough to choose

$$B = C = e^{-\log 2 + \varepsilon} \Leftrightarrow x = y = -\log 2 + \varepsilon \tag{1.3.8}$$

for $\varepsilon \to 0^+$. To see why this is the case, recall that the amoeba $\mathcal{A}(P)$ given below Figure 1.3, it is known to be the closure of the strictly concavity region of $R(x,y)$. This means that the amoeba must coincide with the locus C

$$C = \Big\{(x,y) : \mathbb{R}^2 : \{(z,w) \in \mathbb{T}^2 : 1 + e^x z + e^y w = 0\} \neq \emptyset\Big\} \tag{1.3.9}$$

since outside C, the Ronkin function is affine[17]. Note that C is symmetric in $x \leftrightarrow y$ and a simple geometric consideration shows that C is the locus where $1, e^x, e^y$ satisfy the triangular inequality. In particular the blue boundary in Figure 1.3 is given by $e^x + e^y = 1$. To conclude, it is enough to observe that for $\varepsilon > 0$ the strict triangular condition is satisfied.

Let us finally give the following

Definition 1.3.2. The interacting dimer model *close to the liquid-frozen transition*, is given by the probability measure in (1.3.3) with the choice of edge weights $A = 1, B = C = e^{-\log 2 + \varepsilon}$ for ε small enough, namely

$$\begin{aligned} &\mathbb{P}_{\varepsilon,\lambda,L}(M) := \frac{w_{\varepsilon,\lambda}(M)}{Z_{\varepsilon,\lambda,L}}, \quad w_{\varepsilon,\lambda}(M) := \Big(\frac{e^\varepsilon}{2}\Big)^{N_B+N_C} e^{\lambda V} \\ &Z_{\varepsilon,\lambda,L} = \sum_{M \in \Omega_L} w_{\varepsilon,\lambda}(M), \quad V = \sum_{f \in G_L^*} \mathbb{1}_f^{(2)} \end{aligned} \tag{1.3.10}$$

where N_A, N_B, N_C and $\mathbb{1}^{(2)}$ are as above. Correspondingly we will denote by $\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})$ the generating function, with $f_{\varepsilon,\lambda,L} = L^{-2}\mathcal{W}_{\varepsilon,\lambda,L}(0)$ the free energy and we will add a label ε to all the quantities of interest to specify such choice of weights.

The infinite volume counterpart of our definitions will be denoted by the same symbols, without the label L, e.g., $f_{\varepsilon,\lambda} = \lim_{L\to\infty} f_{\lambda,\varepsilon,L}$ for the infinite volume free energy, $\mathbb{P}_{\varepsilon,\lambda} = \lim_{L\to\infty} \mathbb{P}_{\varepsilon,\lambda,L}$ for the infinite volume measure, $\rho_{A,\varepsilon,L}$ for the densities of dimers of type A, and so on.

Note that, of course, one could approach any other point on the boundary, not necessarily imposing $B = C$. We make this choice for simplicity: the symmetry $B \leftrightarrow C$ simplifies a little bit the technical analysis in that it reduces the number of "running couplings constant" to be considered in the multiscale analysis of the model. The peculiarity of this choice is that such technical analysis is very close to the one carried in [**4**] in the

[17] This is seen by differentiating twice under integral sign.

context of quantum spin chains. We expect that the methods introduced in this thesis, combined with those of [**2**], will allow us to extend the analysis to a generic point of the boundary, but this goes beyond the purpose of this work.

Finally let us recall also the definition of the height function. Let $h : G_L^* \to \mathbb{R}$, where G_L^* is the dual graph of G_L, defined as $h(\eta_0) = 0$ at some arbitrary reference $\eta_0 \in G_L^*$, and given through its gradients by

$$h(\eta) - h(\eta') = \sum_{e \in C_{\eta' \to \eta}} \sigma_e \left(\mathbb{1}_{\{e \in M\}} - \frac{1}{3} \right) \tag{1.3.11}$$

where: $C_{\eta' \to \eta}$ is a path in the dual graph from η' to η, $\sigma_e = 1$ if the edge $e \in C_{\eta' \to \eta}$ is crossed with the black vertex on the right and $\sigma_e = -1$ otherwise; $\mathbb{1}_{e \in M}$ is the indicator function of the edge $e \in G_L$. [18]

1.4 Results

Bulk

Our main goal is to understand the large-scale properties of the height function under the limit measure $\mathbb{P}_\lambda$, which is the weak limit as $L \to \infty$ of $\mathbb{P}_{L,\lambda}$. The fact that this limit exists, provided that $|\lambda| \leq \lambda_0$ for a sufficiently small λ_0, is a byproduct of the proof.

Our first main result concerns the large distance asymptotics of the truncated dimer-dimer correlations. We use the notation $\mathbb{1}_e$ for the indicator function that the edge e is occupied by a dimer, and $\mathbb{E}_\lambda(f;g)$ for $\mathbb{E}_\lambda(fg) - \mathbb{E}_\lambda(f)\mathbb{E}_\lambda(g)$.

Theorem 1.4.1. *Choose the dimer weights on the planar edges as in Assumption 1. There exists $\lambda_0 > 0$ and analytic functions $\nu : [-\lambda_0, \lambda_0] \mapsto \mathbb{R}^+$, $\alpha_\omega, \beta_\omega : [-\lambda_0, \lambda_0] \mapsto \mathbb{C} \setminus \{0\}$ (labelled by $\omega \in \{+,-\}$ and satisfying $\overline{\alpha_+} = -\alpha_-, \overline{\beta_+} = -\beta_-$ and $\alpha_\omega(\lambda)/\beta_\omega(\lambda) \notin \mathbb{R}$), $K_{\omega,j,\ell}, H_{\omega,j,\ell} : [-\lambda_0, \lambda_0] \mapsto \mathbb{C}$ (labelled by $\omega \in \{+,-\}$, $\ell \in \mathcal{I}$, $j \in \mathcal{J}_\ell$ and satisfying $K_{\omega,j,\ell} = \overline{K_{-\omega,j,\ell}}$, $H_{\omega,j,\ell} = \overline{H_{-\omega,j,\ell}}$) and $p^\omega : [-\lambda_0, \lambda_0] \mapsto [-\pi, \pi]^2$ (labelled by $\omega \in \{-1,+1\}$ and satisfying $p^+ = -p^-$) such that, for any two edges e, e' with black vertices $(x,\ell), (x',\ell') \in \mathbb{Z}^2 \times \mathcal{I}$ such that $x \neq x'$ and labels $j \in \mathcal{J}_\ell$, $j' \in \mathcal{J}_{\ell'}$,*

$$\mathbb{E}_\lambda(\mathbb{1}_e; \mathbb{1}_{e'}) = \sum_\omega \left[\frac{K_{\omega,j,\ell} K_{\omega,j',\ell'}}{(\phi_\omega(x - x'))^2} + \frac{H_{\omega,j,\ell} H_{-\omega,j',\ell'}}{|\phi_\omega(x-x')|^{2\nu}} e^{2ip^\omega \cdot (x - x')} \right] + Err(e,e'), \tag{1.4.1}$$

where, letting $x = (x_1, x_2)$,

$$\phi_\omega(x) := \omega(\beta_\omega(\lambda) x_1 - \alpha_\omega(\lambda) x_2) \tag{1.4.2}$$

and $|Err(e,e')| \leq C|x - x'|^{-3+O(\lambda)}$ for some $C > 0$ and $O(\lambda)$ independent of x, x'. Moreover, $\nu(\lambda) = 1 + O(\lambda)$.

Even if not indicated explicitly, the functions $\nu, \alpha_\omega, \beta_\omega, K_{\omega,j,\ell}, H_{\omega,j,\ell}, p^\omega$ all depend non-trivially on the edge weights $\{t_e\}$. In particular, generically, $\nu = 1 + c_1\lambda + O(\lambda^2)$, with c_1

[18] To be precise, the height is a well defined function on the dual of $\mathcal{H}$ (the full plane triangular lattice); on the torus is only a function of the homology class $\mathbb{H}_1(G_L, \mathbb{Z})$ see [**32**, Sec. 2.1.2].

a non zero coefficient, which depends upon the edge weights (this was already observed in [**3**] for interacting dimers on planar graphs); therefore, generically, ν is larger or smaller than 1, depending on the sign of λ.

At $\lambda = 0$, (1.4.1) reduces to the known asymptotic formula for the truncated dimer-dimer correlation of the standard planar dimer model, which is reviewed in Chapter 2. The most striking difference between the case $\lambda \neq 0$ and $\lambda = 0$ is the presence of the critical exponent ν in the second term in square brackets in the right hand side of (1.4.1). It shows that the presence of non-planar edges in the model qualitatively changes the large distance decay properties of the dimer-dimer correlations. Therefore, naive universality, in the strong sense that all critical exponents of the perturbed model are the same as those of the reference unperturbed one, fails. In the present context the correct notion to be used is that of 'weak universality', due to Kadanoff, on the basis of which we expect that the perturbed model is characterized by a number of exact scaling relations; these should allow us to reduce all the non-trivial critical exponents of the model (i.e., those depending continuously on the strength of the perturbation) to just one of them, for instance ν itself. A rigorous instance of such a scaling is discussed in Remark 1.4.1 below.

The weak universality picture is formally predicted by bosonization methods (see the introduction of [**91**] for a brief overview), which allow one to express the large distance asymptotics of all correlation functions in terms of a single, underlying, massless GFF. Such a GFF is nothing but the scaling limit of the height function of the model, as discussed in the following. Given a perfect matching $M \in \Omega^0$ of the infinite graph $G^0_\infty = \mathbb{Z}^2$, there is a standard definition of height function on the dual graph: given two faces ζ, η of $\mathbb{Z}^2$, one defines

$$h(\eta) - h(\zeta) = \sum_{e \in C_{\zeta \to \eta}} \sigma_e \Big(\mathbb{1}_{e \in M} - \frac{1}{4} \Big) \tag{1.4.3}$$

together with $h(\zeta_0) = 0$ at some reference face ζ_0. Here, $C_{\zeta \to \eta}$ is a nearest-neighbor path from ζ to η and σ_e is a sign which equals $+1$ if the edge e is crossed with the white vertex on right and -1 otherwise). The definition is well-posed since it is independent of the choice of the path. We recall that under $\mathbb{P}_0$, the height function is known to admit a GFF scaling limit [**89**, **90**].

A priori, on a non-planar graph such as G, there is no canonical bijection between perfect matchings and height functions. However, since the non-planarity is "local" (non-planar edges do not connect different cells), there is an easy way out. Namely, let F denote the set of faces of $\mathbb{Z}^2$ that do not belong to any of the cells B_x (see Figure 1.4). Given a perfect matching $M \in \Omega$, define an integer-valued height function h on faces $\zeta \in F$ by setting it to zero at some reference face $\zeta_0 \in F$ and by imposing (1.4.3) for any path $C_{\zeta \to \eta}$ that uses only faces in F. It is easy to check that h is then independent of the choice of path.

Our second main result implies in particular that the variance of the height difference between faraway faces in F grows logarithmically with the distance. For simplicity, let us restrict our attention to the subset $\bar{F} \subset F$ of faces that share a vertex with four cells (see Fig. 1.4): if a face in $\bar{F}$ shares a vertex with $B_x, B_{x-(0,1)}, B_{x-(1,0)}, B_{x-(1,1)}$, then we denote it by η_x.

Theorem 1.4.2. *Under the same assumptions as Theorem 1.4.1, for $x^{(1)},\dots,x^{(4)} \in \mathbb{Z}^2$,*

$$\begin{aligned}\mathbb{E}_\lambda\left[(h(\eta_{x^{(1)}}) - h(\eta_{x^{(2)}}));(h(\eta_{x^{(3)}}) - h(\eta_{x^{(4)}}))\right] \\ = \frac{\nu(\lambda)}{2\pi^2}\Re\left[\log\frac{(\phi_+(x^{(4)})-\phi_+(x^{(1)}))(\phi_+(x^{(3)})-\phi_+(x^{(2)}))}{(\phi_+(x^{(4)})-\phi_+(x^{(2)}))(\phi_+(x^{(3)})-\phi_+(x^{(1)}))}\right] \\ + O\left(\frac{1}{\min_{i\neq j\leq 4}|x^{(i)}-x^{(j)}|^{1/2}+1}\right) \quad (1.4.4)\end{aligned}$$

where ν and ϕ_+ are the same as in Theorem 1.4.1

Note that in particular, taking $x^{(1)} = x^{(3)} = x, x^{(2)} = x^{(4)} = y$ we have

$$\mathrm{Var}_{\mathbb{P}_\lambda}(h(\eta_x) - h(\eta_y)) = \frac{\nu(\lambda)}{\pi^2}\Re\log(\phi_+(x) - \phi_x(y)) + O(1) = \frac{\nu(\lambda)}{\pi^2}\log|x-y| + O(1) \quad (1.4.5)$$

as $|x-y| \to \infty$.

Remark 1.4.1. *The remarkable fact of this result is that the 'stiffness' coefficient $\nu(\lambda)/\pi^2$ of the GFF is the same, up to the $1/\pi^2$ factor, as the critical exponent of the oscillating part of the dimer-dimer correlation. There is no a priori reason that the two coefficients should be the same, and it is actually a deep implication of our proof that this is the case. Such an identity is precisely one of the scaling relations predicted by Kadanoff and Haldane in the context of the 8-vertex, Ashkin-Teller, XXZ, and Luttinger liquid models, which are different models in the same universality class as our non-planar dimers (see [**91**] for additional discussion and references).*

Building upon the proof of Theorem 1.4.2, we also obtain bounds on the higher point cumulants of the height; these, in turn, imply convergence of the height profile to a massless GFF:

Theorem 1.4.3. *Assume that $|\lambda| \leq \lambda_0$, with $\lambda_0 > 0$ as in Theorem 1.4.2. For every C^∞, compactly supported test function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ of zero average and $\epsilon > 0$, define*

$$h^\epsilon(f) := \epsilon^2 \sum_{x\in\mathbb{Z}^2} \big(h(\eta_x) - \mathbb{E}_\lambda(h(\eta_x))\big) f(\epsilon x). \quad (1.4.6)$$

Then, one has the convergence in distribution

$$h^\epsilon(f) \overset{\epsilon\to 0}{\Longrightarrow} \mathcal{N}(0,\sigma^2(f)) \quad (1.4.7)$$

where $\mathcal{N}(0,\sigma^2(f))$ denotes a centered Gaussian distribution of variance

$$\sigma^2(f) := \frac{\nu(\lambda)}{2\pi^2}\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy f(x)\Re[\log\phi_+(x-y)]f(y).$$

Remark 1.4.2 (Universality in the graph geometry)**.** *As a last Remark, we emphasize that the massive modes would arise also in the "planar interacting dimer models" of [**2**, **90**], if one worked on a graph whose fundamental cell contains $\ell \geq 2$ black/white vertices (an example is the square-octagon graph, see Fig. 5 in [**32**], where $\ell = 4$). In contrast, one has*

$\ell = 1$ in the context of [2, 90]. Our procedure consisting of (i) integrating out the massive degrees of freedom and (ii) reducing to an effective massless model, implies in particular that the results proved in [2, 90] for planar interacting dimer models on the square lattice extend to the case of general $\mathbb{Z}^2$-periodic two-dimensional bipartite lattices.

EDGE

The interacting dimer model makes sense for every $\lambda \in \mathbb{R}$, but for our results we will actually need $|\lambda|$ to be small enough (but independent of L). From now on, $\lambda_0 > 0$ will be a sufficiently small (but independent of L and of the edge weights) and $B(0, \lambda_0)$ will denote the ball of radius λ_0 centered at 0 in the complex plane.

Our results states the existence of the thermodynamic limit for the interacting model and a control on the asymptotic behavior of the free energy close to the interacting frozen boundary of the theory.

The first result says that, like for the non-interacting model, the limit of the free energy per unit volume and the weak limit of the measure as $L \to \infty$ exist. Moreover the model still displays a frozen behavior in such a limit, characterized by the concentration of the dimers densities.

Theorem 1.4.4. *There exists $\bar{\epsilon} > 0$ and an analytic function $B(0, \lambda_0) \ni \lambda \mapsto \varepsilon_0(\lambda)$ satisfying $\varepsilon_0(0) = 0$ and $\varepsilon_0(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$, such that for every edge $e \in E_L$, the function*

$$B(0, \lambda_0) \ni \lambda \mapsto \partial_{A_e} \mathcal{W}_{L,\epsilon+\varepsilon_0(\lambda),\lambda}(A)|_{A\equiv 0} \tag{1.4.8}$$

is analytic and bounded uniformly in L and $\epsilon \le \bar{\epsilon}$; admits a limit $L \to \infty$ that is analytic in λ and is a continuous function of ϵ for $\epsilon \le \bar{\epsilon}$. The limit $f_{\lambda,\epsilon+\epsilon_0(\lambda)} = \lim_{L\to\infty} f_{\epsilon+\varepsilon_0(\lambda),\lambda,L}$ exists for $\epsilon \le \bar{\epsilon}$ and is analytic in $\lambda \in B(0, \lambda_0)$.

Moreover, for $\epsilon \le 0$, the density of horizontal dimers (i.e. associated to statistical weight $A = 1$) is maximal

$$\rho_{A,\epsilon+\varepsilon_0(\lambda),\lambda} = 1, \quad \epsilon \le 0. \tag{1.4.9}$$

Beyond the existence of the thermodynamic limit of the interacting theory, this results has to be interpreted as the persistence of a liquid-frozen transition, at a critical point $\varepsilon_0(\lambda)$, which is *different* from the one of the non-interacting model $\varepsilon_0(0) = 0$. Indeed, as the next Corollary 1.4.5.1 shows, for $\epsilon > 0$ the measure is no longer in a frozen phase, i.e. the dimer densities are no longer concentrated: $\rho_{A,\epsilon+\varepsilon_0(\lambda),\lambda} < 1$. The fact that the transition point is affected by the interaction is a general feature of interacting models and here it is one of the main differences with the close setting of [**4**], where instead the transition point is "protected against renormalization", i.e. it is not modified by the presence of interaction. A first order calculation of $\varepsilon_0(\lambda)$ is done in Section B.3.3 which shows that $\varepsilon_0'(0) = -2$.

Remark 1.4.3. *Note that to show the existence of the weak limit of the interacting measure $\mathbb{P}_{\varepsilon,\lambda,L}$, one should study the more general case of the function*

$$\lambda \to \prod_{e\in\mathcal{E}} \partial_{A_e} \mathcal{W}_{L,\epsilon+\varepsilon_0(\lambda),\lambda}(\mathcal{A})_{|\mathcal{A}=0},$$

for any finite $\mathcal{E} \subset E_L$. This is avoided here just because not needed for our next, main, result. If desired, this can be done as in [2] where the analysis is performed for a dimer

model in the bulk of the rough phase. Note that when $|\mathcal{E}| \geq 2$, translation invariance of such function is lost and the discussion becomes a bit more involved. The interested reader can look at [2, Sec. 6.5].

The next result concerns the asymptotic behavior of the free energy close to the interacting critical point.

Theorem 1.4.5. *Let $\lambda_0, \bar{\varepsilon} > 0$ and $\varepsilon_0(\cdot)$ be as in Theorem 1.4.4. The function $\epsilon \mapsto f_{\epsilon+\varepsilon_0(\lambda),\lambda}$ is C^∞ in $(-\infty, 0) \cup (0, \bar{\epsilon})$. Moreover, as $\epsilon \to 0$,*

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lambda + \mathbb{1}_{\{\epsilon \geq 0\}} c(\lambda) \epsilon^{\frac{3}{2}} (1 + O(\epsilon^{\theta'})) \tag{1.4.10}$$

for some analytic function $\lambda \mapsto c(\lambda)$ satysfying $c(\lambda) = \frac{4\sqrt{2}}{3\pi} + O(\lambda)$. Here θ' can be chosen in $(0, 1/2)$[19]*.*

This result must be seen as a universality result for the scaling behavior of the free energy as explained in the next Remark: before we comment let us collect an immediate consequence on of the previous theorems on the dimers densities for $\epsilon > 0$.

Corollary 1.4.5.1. Let $\lambda_0, \bar{\varepsilon}$ as in the hypotheses of Theorem 1.4.5. If $\epsilon > 0$, we have that

$$\rho_{B,\epsilon+\varepsilon_0(\lambda),\lambda} = \rho_{C,\epsilon+\varepsilon_0(\lambda),\lambda} = \frac{3}{2} c(\lambda) \epsilon^{1/2} (1 + O(\epsilon^{\theta})), \qquad \epsilon \to 0^+ \tag{1.4.11}$$

where θ and $c(\lambda)$ are the same of (1.4.10).

Proof. Note that from the definitions in (1.3.10), we have that $\rho_{B,\epsilon+\varepsilon_0(\lambda),L} = \rho_{C,\epsilon+\varepsilon_0(\lambda),L} = \partial_\epsilon f_{\epsilon+\varepsilon_0(\lambda),\lambda,L}$. From Theorem 1.4.4 we have the existence of $f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lim_{L\to\infty} f_{\epsilon+\varepsilon_0(\lambda),\lambda,L}$ which is convex in ϵ. Since by Theorem 1.4.5, the limit is differentiable, we can exchange limit and derivative and obtain

$$\rho_{B,\epsilon+\varepsilon_0(\lambda)} = \lim_{L\to\infty} \rho_{B,\epsilon+\varepsilon_0(\lambda)} = \partial_\epsilon f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \frac{3}{2} c(\lambda) \epsilon^{1/2} (1 + O(\epsilon^{\theta})), \quad \epsilon \to 0^+ \tag{1.4.12}$$

and by symmetry the same holds for $\rho_{C,\epsilon+\varepsilon_0(\lambda)}$. ∎

Remark 1.4.4. *(Pokrovsky-Talapov's strong universality)*

- *At $\lambda = 0$ we saw that (see (1.3.7)and below) $f_{\epsilon,0} = R(\log 1/2 + \epsilon, \log 1/2 + \epsilon) = H(\log 1/2 + \epsilon, \log 1/2 + \epsilon)$, where R and H are respectively the Ronkin function and, the limit shape of a volume constrained dimer model. Thus (1.4.10) at $\lambda = 0$, is exactly the Pokrosvky-Talapov law (PT) (as in [78, Th. 1.4], see also discussion below Fig. 1.3) for the limit shape: as $\epsilon \to 0^+$*

$$H(\log 1/2 + \epsilon, \log 1/2 + \epsilon) = \frac{4\sqrt{2}}{3\pi} \epsilon^{3/2} (1 + O(\epsilon^{\theta})). \tag{1.4.13}$$

- *When $\lambda \neq 0$ small, (1.4.10) can be seen as a universality result for (PT) critical behavior: the effect of the interaction is, indeed, to change the coefficient $c(0)$ into*

[19] As a by product of the technical analysis necessary to obtain the result, bounds are not uniform as $\theta \to 1/2^-$.

some non trivial analytic function $c(\lambda)$, and to shift the liquid-frozen transition point by $\varepsilon_0(\lambda)$, but not the scaling exponent $3/2$. This is a highly non trivial result in virtue of the fact that, in general, perturbations changes the critical exponents of the model, as shown for the dimer-dimer correlation in the bulk of the rough phase (see for instance Theorem 1.4.1). In virtue of the weak universality picture given in the introduction, this result stands instead as a strong universality result.[20] *Moreover, in view of the discussion in Section 1.2 (edge), our result supports the conjecture that the fluctuation exponents of the level lines close to the liquid-frozen boundary is $1/3$, independently of the interaction parameter λ, and thus a universal scaling towards the Airy's process of such lines is expected.*

As already anticipated in Section 1.3 (edge), the specific choice of the parametrization to approach the boundary of the liquid frozen transition, or, the specific choice of the interaction we made, are both not strictly necessary.

Remark 1.4.5. *(Generalizations)*

- *The results are given for the explicit parametrization $B = C = e^{\log 1/2+\epsilon}$ of the non interacting model. Approaching the frozen boundary along the symmetry axis of the rough phase as $\epsilon \to 0^+$ (see Figure 1.3-3.3) has the advantage to keep alive an extra symmetry of the model which is generally not present. From the Renormalization Group point of view, this turns out to be the simplest case to analyze and the analysis becomes almost the same as the one carried out in a quantum spin chain model belonging to the same universality class, see [4]. In the general, asymmetric case, one can approach the blue boundary of Fig. 1.3 at a generic point $Q_0(t)$ via the parametrization $Q_\epsilon(t) = (\log t + \epsilon, \log(1-t) + \epsilon)$ for $t \in (0,1)$ and obtain*

$$f_{\lambda,\epsilon+\varepsilon_{0,t}(\lambda)} = \lambda + \mathbb{1}_{\{\epsilon \geq 0\}} c_t(\lambda) \epsilon^{3/2} (1 + O(\epsilon^\theta)) \tag{1.4.14}$$

where $\varepsilon_{0,t}(\lambda)$ is the analogous of $\varepsilon_0(\lambda)$ for t generic. The breaking of the symmetry $B \leftrightarrow C$, for $t \neq 1/2$, implies that the structure of the interacting dimer model no longer resembles that of [4], and the analysis must be carried out in a different way. Technically speaking, the symmetry loss reflects in a more complicated structure of the so-called "Beta function equations" due to the emergence of extra "running coupling constants", and which, in the multiscale analysis, has to be controlled via the choice of suitable counter-terms.[21] *In order to obtain (1.4.14), the setting of [4] it is therefore not sufficient and an implementation of the double regime multiscale analysis of [4] with the RG asymmetric tractation of [2] becomes necessary. We postpone the detailed discussion of this technical point to a future work.*

- *The specific choice of the interaction (see (1.3.3)), belongs to a wide class on interactions studied in [2]. From a technical point of view the analysis we made can be performed as well for a generic element of this class. The specific form of the result will depend weakly on such a choice: namely, the analytic functions $a(\lambda), c(\lambda), \varepsilon_0(\lambda)$*

[20] The presence of two kind of universalities was stressed in related models in [**114**].

[21] An analogous situation appears when considering a perturbation of dimer model with flat (uniform measure) [**3**] or tilted [**2**] associated surface (non uniform measure): in the first case the presence of extra symmetries reduces significantly the complexity of the RG analysis; see the discussion after (1.2) of [**2**].

such that

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = a(\lambda) + \mathbb{1}_{\epsilon\geq 0} c(\lambda)\epsilon^{3/2}(1 + O(\epsilon^{\theta})) \tag{1.4.15}$$

will in general depend on the interaction chosen.

As last, we connect our results with some recent close literature. It is known that dimer models are closely related to vertex models [**22**]. An instance of this connection was already discussed in [**2**] for an interacting dimer model on the square lattice $\mathbb{Z}^2$ in the bulk of the rough phase: when the interaction is of the "plaquette type" on staggered faces of the dual square lattice [22], it is known that the model is equivalent[23] to a six-vertex (6V) model close to the free-fermion point (FP). This connection allowed to extablish the GFF nature of the height field of the 6V model in a small neighborhood of the FP, [**2**].
In our case, the interacting dimer model on the hexagonal lattice can be mapped to an interacting five-vertex model (5V).[24] Indeed, the non interacting dimer model on the hexagonal lattice is known to be equivalent[25] to a 5V model at the FP, when the vertex weights are suitable functions of A, B, C in (1.3.3) (in appendix C we recall the mapping between the two models). In [**115**] (then extended to [**116**]) the authors study a version of the 5V away from the FP by tuning with a parameter r (such that for $r = 1$ the model is at the FP) certain local vertex configurations[26] and give an exhaustive characterization of the thermodynamic properties of the model, by computing in particular surface tension, free energy and limit shape. What emerges is that regions of phase coexistence ($r < 1$, given by regions of non-strict surface tension convexity) and new frozen regions ($r > 1$) are observed in a FP neighbourhood (see [**115**, Fig. 21], [**116**, Fig. 5-6]). The situation becomes even more interesting when higher periodicity is allowed on local energies configurations [**116**, Fig. 10]. This shows that the FP is a sensitive point and that perturbations around it can change the nature of the model (free energy particularly); this is where our result fits in. Despite our techniques do not give access to the same amount of informations, they stand on a ground where integrability-based techniques, as the Bethe-Ansatz, cannot be applied.[27].

1.4.1 Outline of the proofs

BULK

As in [**2, 90**], the proof is based on an exact representation of the dimer model as a system of interacting lattice fermions and in a rigorous multiscale analysis of the effective

[22] This is the same kind of interaction chosen here, see Figure 1.6, with the difference that: (i) on the square lattice there are only two possible parallel configurations at a face; (ii) the interaction V runs only over faces with, e.g., black top-right vertex.

[23] There is a map between dimer configurations and vertex configurations such that their partition functions equal after a suitable identification of the parameters of the model. See [**2**, Section 2.3] and references therein.

[24] The 5V model can be obtain as a limiting case of the 6V, when the energy value of a local arrow configuration diverges.

[25] Dimer configurations on the hexagonal lattice can be bijectively mapped to five vertex configurations, also known as monotone non-intersecting lattice paths (MNLP). Equivalence again as identification of partition functions of the models after suitable identification of the parameters.

[26] In the MNLP picture, corners of lattice paths are weighted with statistical factor $r > 0$.

[27] The mapping between our dimer model and the 5V it is such that we cannot embedd the interaction parameter in the vertex energies (as done in [**2**]).

fermionic model, which has the structure of a lattice regularization of a Luttinger-type model. With respect to the previous works [**2**, **90**], obtaining a fermionic representation turns out to be much less trivial, due to the loss of planarity. The infrared (i.e., large-scale) analysis of the lattice fermionic model is performed thanks to a comparison with a solvable reference continuum fermionic model, which has been studied and constructed in a series of works by G. Benfatto and V. Mastropietro [**40**, **41**, **117**–**121**], partly in collaboration also with P. Falco [**117**, **118**, **122**]. The GFF behavior and the Kadanoff-Haldane scaling relation of the dimer model follow from a careful comparison between the emergent chiral Ward Identities of the reference model with exact lattice Ward Identities of the dimer model.

The first novelty of the present work, as compared to [**2**, **90**, **91**], is related to the fermionic representation of the weakly non-planar model. The presence of non-planar edges requires a quite non-trivial adaptation of Kasteleyn's theory, which is needed for the very formulation of the finite-volume model in terms of a non-Gaussian Grassmann integral. In fact, our non-planar model can in general be embedded on a surface of minimal genus $g \approx L^2$ (of the order of the number of non-planar edges) and Kasteleyn's theory for the dimer model on general surfaces [**55**, **56**] would express its partition function as the sum of 4^g determinants, i.e. of 4^g Gaussian Grassmann integrals, a rewriting that is not very useful for extracting thermodynamic properties. In this respect, the remarkable aspect of Proposition 2.1.1 below is that it expresses the partition function of *just four Grassmann integrals, which are, however, non-Gaussian.* The second novel ingredient of our construction is the identification (via the block-diagonalization procedure of Section 2.3.2) of massive modes associated with the Grassmann field which enters the fermionic representation of the model. The fact that the elementary cell of our model consists of m^2 sites, with m an even integer larger or equal to 4, implies that the basic Grassmann field of our effective model has a minimum of 16 components. It is well known [**123**, **124**] that multi-component Luttinger models, such as the 1D Hubbard model [**125**], to cite the simplest possible example, do not necessarily display the same qualitative large distance features as the single-component one: new phenomena and quantum instabilities, such as spin-charge separation and metal-insulator transitions accompanied with the opening of a Mott gap may be present and may drastically change the resulting picture. Therefore, it is a priori unclear whether the height function of our model should still display a GFF behavior at large scales. Remarkably, however, the fact that the characteristic polynomial of the reference model has at most two simple zeros, as proven in [**32**], directly implies that all but two of the components of the effective Grassmann field are massive, and they can be preliminarily integrated out. This way, one can at last re-express the effective massless model in terms of just two massless fields (quasi-particle fields), in a way suitable for the application of the multi-scale analysis developed in [**2**, **90**]. At this point, a large part of the multi-scale analysis is based on the tools developed in our previous works, which we will refer to for many technical aspects, without repeating the analysis in the present slightly different setting.

Edge

The logic of the proof goes as follows: again, the starting point is a representation of the model (1.3.10) in terms of a non Gaussian Grassmann integral. As already mentioned, the interaction taken under examination belongs to a wider class of interactions for which Grassmann representations for dimer models on the square lattice $\mathbb{Z}^2$ are known [**2**]. This

representation, in our case, is a direct Corollary of [**2**, Proposition 1]: indeed, our doubly periodic hexagonal lattice with minimal fundamental cell can be obtained from the one in [**2**] by setting an edge weight to zero.

With such a representation at hand we set a multiscale Renormalization Group analysis to show that quantities such as $f_{\lambda,\varepsilon}, \partial_{A_e} \mathcal{W}_{\lambda,\varepsilon}(\mathcal{A})|_{\mathcal{A}=0}$ exist and satisfy the desired properties of Theorem 1.4.4 (analiticity in λ, differentiability in ϵ etc.), when $\varepsilon = \epsilon + \varepsilon_0$ and ε_0 is uniquely fixed as a function of λ. The existence part is a relatively standard result, based on the literature on the fermionic constructive Renormalization. The function $\varepsilon_0(\lambda)$ takes into account the effect of the interaction on the location of the critical point, which is shifted from $\varepsilon = 0$ to $\varepsilon = \varepsilon_0(\lambda)$. The fact that $\varepsilon_0(\lambda) \neq \varepsilon_0(0)$ is a main difference with the close setting of [**4**]. The multiscale analysis depends on the sign and on the size of ϵ: in the most interesting case, for $\epsilon > 0$ small, to obtain analiticity in λ of the theory, uniformly in ϵ, one really needs to distinguish between two different regimes of scales. This discussion is inspired by the analogous double regime integration of [**4**]. When $\epsilon < 0$ fixed, the existence of the thermodynamic limit of the theory can be proved via a single step of integration (no multiscale) provided that λ is sufficiently small: however as $\epsilon \to 0^-$ a single step of integration is not enough to obtain a radius of analyticity in λ uniform in ϵ. In this case a multiscale analysis is needed, but a single regime of scales turns out to be sufficient.

Once the existence of $\partial_{A_e} \mathcal{W}_{\lambda,\varepsilon}(\mathcal{A})|_{\mathcal{A}=0}$ is known, one can obtain the last statement of Theorem 1.4.4 concerning dimer densities. The idea is to export the freezing property of the non interacting measure $\mathbb{P}_{\epsilon,0}$, which is well known for $\epsilon \leq 0$, to the interacting one by means of "naive" perturbation theory, i.e., without the multiscale Grassmannian framework.

With Theorem 1.4.4 at hand, one can deduce a portion of the statement of Theorem 1.4.5: namely the result in (1.4.10) for $\epsilon \leq 0$. Indeed the concentration of dimers densities of Theorem 1.4.4 implies, via thermodynamic arguments not requiring the multiscale expansion, that the free energy does not depend on ϵ and indeed equals λ (cf. (1.4.10)).

To obtain the remaining part of Theorem 1.4.5, it is necessary to exploit the informations carried by the multiscale analysis, particular for the part of (1.4.10) concerning $\epsilon > 0$. The idea is that we can write

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \sum_{h \leq 0} F^{(h)}_{\epsilon,\lambda} \tag{1.4.16}$$

where $F^{(h)}_{\epsilon,\lambda}$ is the "contribution on scale h" to the free energy. The multiscale analysis provides a way to bound and control the contributions $F^{(h)}_{\epsilon,\lambda}$. In particular, for $\epsilon > 0$ small, this bound depends on the scale we are looking at: there exists a separating scale $h^* = O(\log_2 \epsilon)$ for which such bound changes behavior. By appropriately combining the multiscale expansion with these estimates, using (1.4.16), one can find that

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = a(\lambda) + \epsilon b(\lambda) + c(\lambda)\epsilon^{3/2}(1 + O(\epsilon^\theta)) \tag{1.4.17}$$

for some analytic function a, b, c of λ for $|\lambda|$ small enough. Then, the idea is to repeat this strategy for $\epsilon < 0$, by recalling that the multiscale structure is a bit simpler: this provides an expression analogous to the one above with coefficients $a'(\lambda), b'(\lambda), c'(\lambda)$. If on the one hand these coefficients are not explicit analytic function of λ, which can be computed order by order in perturbation theory, a comparison between the multiscale structures of

the cases $\epsilon > 0$ and $\epsilon < 0$ shows that they must satisfy $a = a', b = b'$. Using then, as previously discussed, that $a' = \lambda, b' = 0$, one obtains the full statement in (1.4.10). Finally, the regularity property of $f_{\epsilon+\varepsilon_0(\lambda),\lambda}$ away from $\epsilon \neq 0$ follows from a detailed analysis of the multiscale construction.

CHAPTER 2

INTERACTING DIMERS I: BULK OF THE ROUGH PHASE

In order to prove Theorem 1.4.2, we organize this chapter as follows. In Section 2.1 we review some useful aspects of Kasteleyn's theory on toroidal graphs and derive the Grassmann representation of the weakly non-planar dimer model. In Section 2.2 we prove one of the main results of our work, concerning the logarithmic behavior of the height covariance at large distances and the Kadanoff-Haldane scaling relation, assuming temporarily a sharp asymptotic result on the correlation functions of the dimer model. The proof of the latter is based on a generalization of the analysis of [**2**], described in Section 2.3. As mentioned above, the novel aspect of this part consists in the identification and integration of the massive degrees of freedom (Sections 2.3.1-2.3.3), while the integration of the massless ones (Section 2.3.4) is completely analogous to the one described in [**2**]. Finally, in Section 2.4, we complete the proof of the convergence of the height function to the GFF.

2.1 Grassmann representation of the generating function

In this section we rewrite the partition function $Z_{L,\lambda}$ of (1.3.1) in terms of Grassmann integrals (see Sect.2.1.3). As a byproduct of our construction, we obtain a similar Grassmann representation for the generating function of correlations of the dimer model. We also observe that the Grassmann integral for the generating function is invariant under a lattice gauge symmetry, whose origin has to be traced back to the local conservation of the number of incident dimers at lattice sites, and which implies exact lattice Ward Identities for the dimer correlations (see Sect.2.1.4).

Before diving into the proof of the Grassmann representation, it is convenient to recall some preliminaries about the planar dimer model, its Gaussian Grassmann representation and the structure of its correlation functions in the thermodynamic limit. This will be done in the next two subsection, Sect.2.1.1, 2.1.2

2.1.1 A brief reminder of Kasteleyn theory

Here we recall a few basic facts of Kasteleyn theory for the dimer model on a bipartite graph $G = (V, E)$ embedded on the torus, with edge weights $\{t_e > 0\}_{e\in E}$. For later purposes, we need this for more general such graphs than just G_L^0. For details we refer to [**55**], which considers the more general case where the graph is not bipartite and it is embedded on an orientable surface of genus $g \geq 1$. For the considerations of this section, we do not need the edge weights to display any periodicity, so here we will work with generic, not necessarily periodic, edge weights.

As in [**55**], we assume that G can be represented as a planar connected graph $G_0 = (V, E_0)$ (we call this the "basis graph of G"), embedded on a square, with additional edges that connect the two vertical sides of the square (edges E_1) or the two horizontal sides (edges E_2). Note that $E = E_0 \cup E_1 \cup E_2$. See Figure 2.1. We always assume that the basis graph G_0 is connected and actually[1] that it is 2−connected (i.e. removal of any single vertex together with the edges attached to it does not make G_0 disconnected). We also assume that G_0 admits at least one perfect matching and we fix a reference one, which we call M_0.

Following the terminology of [**55**], we introduce the following definition.

Definition 2.1.1 (Basic orientation)**.** We call an orientation D_0 of the edges E_0 a "basic orientation of G_0" if all the internal faces of the basis graph G_0 are clock-wise odd, i.e. if running clockwise along the boundary of the face, the number of co-oriented edges is odd (since G_0 is 2-connected, the boundary of each face is a cycle).

A basic orientation always exists [**55**], but in general it is not unique. Next, one defines 4 orientations of the full graph G as follows (these are called "relevant orientations" in [**55**]). First, one draws the planar graphs $G_j, j = 1, 2$ whose edge sets are $E_0 \cup E_j$, as in Fig. 2.1.

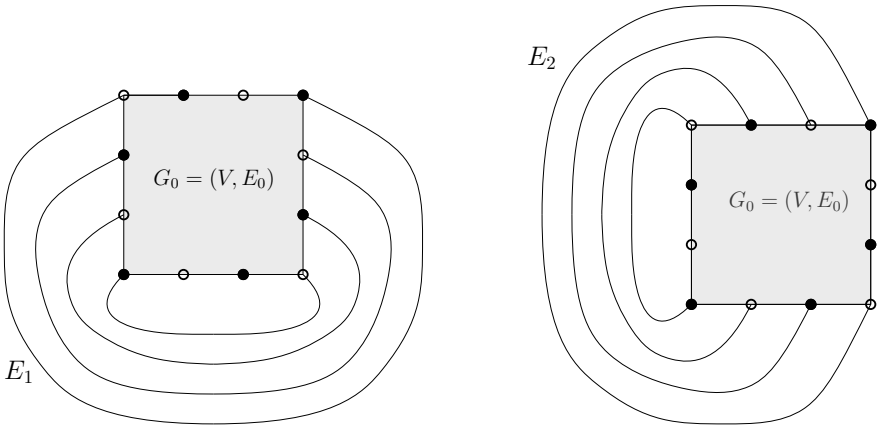

Figure 2.1: *The basis graph G_0 is schematically represented by the gray square (the vertices and edges inside the square are not shown). In the left (resp. right) drawing is pictured the planar graph G_1 (resp. G_2)*

[1] [**55**] develops Kasteleyn's theory without assuming that G_0 is 2−connected. We will avoid below having to deal with non-2-connected graphs, which would entail several useless complications

Note that there is a unique orientation D_j of the edges in $E_0 \cup E_j$ that coincides with D_0 on E_0 and such that all the internal faces of G_j are clockwise odd. Then, we define the relevant orientation D_θ of type $\theta = (\theta_1, \theta_2) \in \{-1, +1\}^2$ of G as the unique orientation of the edges E that coincides with D_0 on G_0 and with $\theta_1 D_j$ on the edges in $E_j, j = 1, 2$. Given one of the four relevant orientations D_θ of G, we define a $|V| \times |V|$ antisymmetric matrix A_{D_θ} by establishing that for $v, v' \in V$, $A_{D_\theta}(v, v') = 0$ if $(v, v') \notin E$, while $A_{D_\theta}(v, v') = t_e$ if v, v' are the endpoint of the edge e oriented from v to v', and $A_{D_\theta}(v, v') = -t_e$ if e is oriented from v' to v. Then, [**55**, Corollary 3.5] says that

$$Z_G = \sum_{M \in \Omega_G} w(M) = \sum_{\theta \in \{-1,+1\}^2} \frac{c_\theta}{2} \frac{\mathrm{Pf}(A_{D_\theta})}{s(M_0)} \tag{2.1.1}$$

where Ω_G is the set of the perfect matchings of G, $w(M) = \prod_{e \in M} t_e$, and

$$c_{(-1,-1)} = -1 \text{ and } c_\theta = 1 \text{ otherwise.} \tag{2.1.2}$$

In (2.1.1), $\mathrm{Pf}(A)$ denotes the Pfaffian of an anti-symmetric matrix A and $s(M_0)$ denotes the sign of the term corresponding to the reference matching M_0 in the expansion of the Pfaffian $\mathrm{Pf}(A_{D_\theta})$. Since by assumption M_0 contains only edges from E_0 whose orientation does not depend on θ, $s(M_0)$ is indeed independent of θ.

In our case, in contrast with the general case considered in [**55**], the graph G is bipartite. By labeling the vertices so that the first $|V|/2$ are black and the last $|V|/2$ are white, the matrices $A_{D_\theta}(v, v')$ have then a block structure of the type

$$A_{D_\theta} = \left(\begin{array}{c|c} 0 & +K_\theta \\ \hline -K_\theta & 0 \end{array}\right) \tag{2.1.3}$$

We view the $|V|/2 \times |V|/2$ "Kasteleyn matrices" K_θ as having rows indexed by black vertices and columns by white vertices.

By using the relation [**126**, Eq. (16)] between Pfaffians and determinants, one can then rewrite the above formula as

$$Z_G = \sum_{\theta \in \{-1,+1\}^2} \frac{\tilde{c}_\theta}{2} \det(K_\theta), \quad \tilde{c}_\theta = c_\theta \frac{(-1)^{(|V|/2-1)|V|/4}}{s(M_0)}. \tag{2.1.4}$$

Remark 2.1.1. *Note that changing the order in the labeling of the vertices changes the sign $s(M_0)$. We suppose henceforth that the choice is done so that the ratio in the definition of $\tilde{c}_\theta$ equals 1, so that $\tilde{c}_\theta = c_\theta$.*

2.1.2 Thermodynamic limit of the planar dimer model

In the previous section, Kasteleyn's theory for rather general toroidal bipartite graphs was recalled, without assuming any type of translation invariance. In this subsection, instead, we specialize to $G = G_L^0$ (the periodized version of $\mathbb{Z}^2$ introduced in Section 1.3 (edge)) and, as was the case there, we assume that the edge weights are invariant under translations by multiples of $\mathbf{e_1}, \mathbf{e_2}$.

With Kasteleyn's theory at hand, one can compute the thermodynamic and large-scale properties of the dimer model on G_L^0 as $L \to \infty$. We refer to [**32**, **89**, **107**] for details. In the case where $G = G_L^0$, the basis graph G_0 is a square grid with Lm vertices per side

and we choose its basic orientation D_0 so that horizontal edges are oriented from left to right, while vertical edges are oriented from bottom to top on every second column and from top to bottom on the remaining columns. With this choice, the orientations D_1, D_2 of G_1, G_2 are like in Fig. 2.2. Note that, if $e = (b, w) \in E_L^0$ is an edge of G_L^0, then for $\theta = (\theta_1, \theta_2) \in \{-1, +1\}^2$, $K_\theta(b, w)$ equals $K_{(+1,+1)}(b, w)$ multiplied by $(-1)^{(\theta_1-1)/2}$ if e belongs to E_1 (see Fig. 2.1) and by $(-1)^{(\theta_2-1)/2}$ if e belongs to E_2. Observe also that the

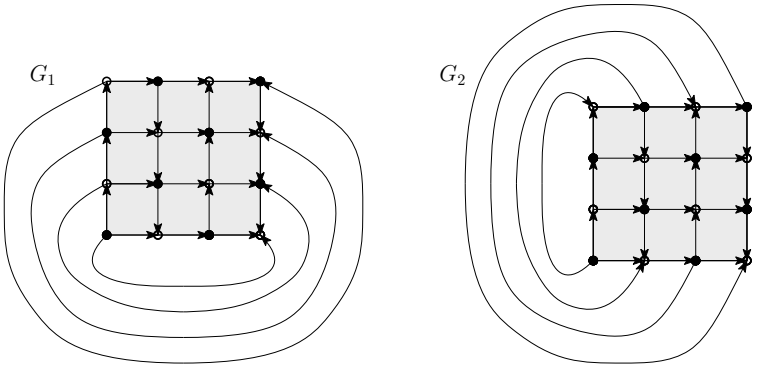

Figure 2.2: *The graphs G_1, G_2 corresponding to the basis graph of G_L^0 (for $L\,m = 4$), together with their orientations D_1, D_2.*

matrix $K_{(-1,-1)}$ is invariant under translations by multiples of $\mathbf{e_1}, \mathbf{e_2}$. Define

$$\mathcal{P}(\theta) := \left\{ k = (k_1, k_2) : k_j = \frac{2\pi}{L}\left(n_j + \frac{\theta_j + 1}{4}\right), -L/2 < n_j \leq L/2 \right\}. \tag{2.1.5}$$

Let P_θ be the orthogonal $(Lm)^2/2 \times (Lm)^2/2$ matrix whose columns are indexed by $(k, \ell), k \in \mathcal{P}(\theta), \ell \in \mathcal{I} = \{1, \ldots, m^2/2\}$, whose rows are indexed by $(x, \ell), x \in \Lambda, \ell \in \mathcal{I}$, and such that the column indexed (k, ℓ) is the vector

$$f_{\ell,k} : ((x, \ell') \in \Lambda \times \mathcal{I}) \mapsto f_{\ell,k}(x, \ell') = \frac{1}{L} e^{-ikx} \mathbf{1}_{\ell'=\ell}. \tag{2.1.6}$$

Then, $P_\theta^{-1} K_\theta P_\theta$ is block-diagonal with blocks of size $|\mathcal{I}|$ labelled by $k \in \mathcal{P}(\theta)$. The block corresponding to the value k is a $|\mathcal{I}| \times |\mathcal{I}|$ matrix $M(k)$ of elements $[M(k)]_{\ell,\ell'}$ with $\ell, \ell' \in \mathcal{I}$ and

$$[M(k)]_{\ell,\ell'} = \sum_{e : \ell \overset{e}{\sim} \ell'} K_{(-1,-1)}(b, w) e^{-ikx_e}. \tag{2.1.7}$$

In this formula, the sum runs over all edges e joining the black vertex b of type ℓ in the cell of coordinates $x = (0, 0)$ to some white vertex w of type ℓ' (w can be either in the same fundamental cell or in another one); $x_e \in \mathbb{Z}^2$ is the coordinate of the cell to which w belongs.

The thermodynamic and large-scale properties of the measure $\mathbb{P}_{L,0}$ are encoded in the matrix M: for instance the infinite volume free energy exists and it is given by [**32**]

$$F =: \lim_{L\to\infty} \frac{1}{L^2} \log Z_{L,0} = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \log |\mu(k)| dk \tag{2.1.8}$$

where μ (the "characteristic polynomial") is

$$\mu(k) := \det M(k), \tag{2.1.9}$$

which is a polynomial in e^{ik_1}, e^{ik_2}. Kasteleyn's theory allows one to write multi-point dimer correlations (in the $L \to \infty$ limit) in terms of the so-called "infinite-volume inverse Kasteleyn matrix" K^{-1}: if w (resp. b) is a white (resp. black) vertex of type ℓ in cell $x = (x_1, x_2) \in \mathbb{Z}^2$ (resp. of type ℓ' and in cell 0), then one has

$$K^{-1}(w, b) := \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} [(M(k))^{-1}]_{\ell,\ell'} e^{-ikx} dk. \tag{2.1.10}$$

As can be guessed from (2.1.10), the long-distance behavior of K^{-1} is related to the zeros of the determinant of $M(k)$, that is, to the zeros of μ on $[-\pi, \pi]^2$. It is a well known fact [**32**] that, for any choice of the edge weights, μ can have at most two zeros. Our Assumption 1 means that we restrict to a choice of edge weights such that μ has exactly two zeros, named p_0^+, p_0^-, with $p_0^+ \neq p_0^- \mod (2\pi, 2\pi)$. We also define the complex numbers

$$\alpha_\omega^0 := \partial_{k_1} \mu(p_0^\omega), \quad \beta_\omega^0 := \partial_{k_2} \mu(p_0^\omega), \quad \omega = \pm. \tag{2.1.11}$$

Note that, since the Kasteleyn matrix elements $K_\theta(b, w)$ are real[2], from (2.1.7) we have the symmetry

$$[M(-k)]_{\ell,\ell'} = \overline{[M(k)]_{\ell\ell'}} \tag{2.1.12}$$

and in particular

$$p_0^+ + p_0^- = 0 \tag{2.1.13}$$

$$\alpha_-^0 = -\overline{\alpha_+^0}, \quad \beta_-^0 = -\overline{\beta_+^0}. \tag{2.1.14}$$

It is also known [**32**] that $\alpha_\omega^0, \beta_\omega^0$ are not collinear as elements of the complex plane:

$$\alpha_\omega^0 / \beta_\omega^0 \notin \mathbb{R}. \tag{2.1.15}$$

Note that from (2.1.14) it follows that $\mathrm{Im}(\beta_+^0/\alpha_+^0) = -\mathrm{Im}(\beta_-^0/\alpha_-^0)$. From now on, with no loss of generality, we assume that

$$\mathrm{Im}(\beta_+^0/\alpha_+^0) > 0, \tag{2.1.16}$$

which amounts to choosing appropriately the labels $+, -$ associated with the two zeros of $\mu(k)$.

If we denote by $\mathrm{adj}(A)$ the adjugate of the matrix A, so that $A^{-1} = \mathrm{adj}(A)/\det A$, the long-distance behavior of the inverse Kasteleyn matrix is given [**32**] as

$$K^{-1}(w, b) \overset{|x|\to\infty}{=} \frac{1}{2\pi} \sum_{\omega=\pm} [\mathrm{adj}(M(p^\omega))]_{\ell,\ell'} \frac{e^{-ip_0^\omega x}}{\phi_\omega^0(x)} + O(|x|^{-2}) \tag{2.1.17}$$

[2] In [**2**,**91**] etc, a different choice of Kasteleyn matrix was done, with complex entries. As a consequence, in that case one had $p_0^+ + p_0^- = (\pi, \pi)$ instead.

where

$$\phi^0_\omega(x) = \omega(\beta^0_\omega x_1 - \alpha^0_\omega x_2). \tag{2.1.18}$$

Note that since the zeros p^ω_0 of $\mu(k)$ are simple, the matrix $\operatorname{adj} M(p^\omega_0)$ has rank 1. This means that we can write

$$\operatorname{adj} M(p^\omega_0) = U^\omega \otimes V^\omega \tag{2.1.19}$$

for vectors $U^\omega, V^\omega \in \mathbb{C}^{|\mathcal{I}|}$, where $\otimes$ is the Kronecker product. Let $e = (b, w), e' = (b', w')$ be two fixed edges of G^0_L: we assume that the black endpoint of e (resp. of e') has coordinates $\mathbf{x} = (x, \ell)$ (resp. $\mathbf{x}' = (x', \ell')$) and that the white endpoint of e (resp. e') has coordinates $(x + v(e), m)$ with $m \in \mathcal{I}$ (resp. coordinates $(x' + v(e'), m')$). Of course, $v(e)$ is either $(0, 0)$ or $(0, \pm 1)$ or $(\pm 1, 0)$, and similarly for $v(e')$. Note that the coordinates of the white endpoint of e are uniquely determined by the coordinates of the black endpoint and the orientation label[3] $j \in \{1, \dots, 4\}$ of e: in this case we will write $v(e) =: v_{j,\ell}$, $K(b, w) =: K_{j,\ell}$ and in (2.1.19), $U_m =: U_{j,\ell}$. The (infinite-volume) truncated dimer-dimer correlation under the measure $\mathbb{P}_{L,0}$ is given as[4]

$$\mathbb{E}_0(\mathbb{1}_e; \mathbb{1}_{e'}) = \lim_{L\to\infty} \mathbb{E}_{L,0}(\mathbb{1}_e; \mathbb{1}_{e'}) = -K(b, w)K(b', w')K^{-1}(w', b)K^{-1}(w, b'). \tag{2.1.20}$$

where recall $\mathbb{E}_\lambda(f; g) = \mathbb{E}_\lambda(fg) - \mathbb{E}_\lambda(f)\mathbb{E}_\lambda(g)$. As a consequence of the asymptotic expression (2.1.17), we have that as $|x' - x| \to \infty$,

$$\mathbb{E}_0[\mathbb{1}_e; \mathbb{1}_{e'}] = A_{j,\ell,j',\ell'}(x, x') + B_{j,\ell,j',\ell'}(x, x') + R^0_{j,\ell,j',\ell'}(x, x') \tag{2.1.21}$$

with

$$\begin{aligned} A_{j,\ell,j',\ell'}(x, x') &= \sum_{\omega=\pm} \frac{K^0_{\omega,j,\ell} K^0_{\omega,j',\ell'}}{(\phi^0_\omega(x - x'))^2} \\ B_{j,\ell,j',\ell'}(x, x') &= \sum_{\omega=\pm} \frac{H^0_{\omega,j,\ell} H^0_{-\omega,j',\ell'}}{|\phi^0_\omega(x - x')|^2} e^{2ip^\omega_0(x-x')} \\ |R^0_{j,\ell,j'\ell'}(x, x')| &\le C|x - x'|^{-3}. \end{aligned} \tag{2.1.22}$$

where

$$\begin{aligned} K^0_{\omega,j,\ell} &:= \frac{1}{2\pi} K_{j,\ell} e^{-ip^\omega_0 v_{j,\ell}} U^\omega_{j,\ell} V^\omega_\ell \\ H^0_{\omega,j,\ell} &:= \frac{1}{2\pi} K_{j,\ell} e^{ip^\omega_0 v_{j,\ell}} U^{-\omega}_{j,\ell} V^\omega_\ell. \end{aligned} \tag{2.1.23}$$

[3] recall the conventions on labeling the type of edges, in Section 1.3 (edge).

[4] the index $\theta \in \{-1, +1\}^2$ in $K_\theta(b, w)$ is dropped, since the dependence on r is present only for edges at the boundary of the basis graph G_0 (see Figure 2.2, so that for fixed (b, w) and L large, $K_\theta(b, w)$ is independent of r

2.1.3 A fermionic representation for $Z_{L,\lambda}$

In this subsection, we work again with generic edge weights, i.e., we do not assume that they have any spatial periodicity.

Determinants and Grassmann integrals

We refer for instance to [**5**] for an introduction to Grassmann variables and Grassmann integration; here we just recall a few basic facts. To each vertex v of G_L we associate a Grassmann variable. Recall that vertices are distinguished by their color and by coordinates $\mathbf{x} = (x, \ell) \in \mathbf{\Lambda} = \Lambda \times \mathcal{I}$. We denote the Grassmann variable of the black (resp. white) vertex of coordinate $\mathbf{x}$ as $\psi^+_{\mathbf{x}}$ (resp. $\psi^-_{\mathbf{x}}$). We denote by $\int D\psi f(\psi)$ the Grassmann integral of a function f and since the variables $\psi^\pm_{\mathbf{x}}$ anti-commute among themselves and there is a finite number of them, we need to define the integral only for polynomials f. The Grassmann integration is a linear operation that is fully defined by the following conventions:

$$\int D\psi \prod_{\mathbf{x}\in\mathbf{\Lambda}} \psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}} = 1, \tag{2.1.24}$$

the sign of the integral changes whenever the positions of two variables are interchanged (in particular, the integral of a monomial where a variable appears twice is zero) and the integral is zero if any of the $2|\mathbf{\Lambda}|$ variables is missing. We also consider Grassmann integrals of functions of the type $f(\psi) = \exp(Q(\psi))$, with Q a sum of monomials of even degree. By this, we simply mean that one replaces the exponential by its finite Taylor series containing only the terms where no Grassmann variable is repeated.

For the partition function $Z_{L,0} = Z_{G^0_L}$ of the dimer model on G^0_L we have formula (2.1.4) of previous subsection where the Kasteleyn matrices K_θ are fixed as in Section 2.1.2, recall also Remark 2.1.1. Using the standard rewriting of determinants as Gaussian Grassmann integrals (i.e. Grassmann integrals where the integrand is the exponential of the corresponding quadratic form), one immediately obtains

$$Z_{L,0} = \frac{1}{2} \sum_{\theta\in\{-1,+1\}^2} c_\theta \int D\psi\, e^{-\psi^+ K_\theta \psi^-}, \tag{2.1.25}$$

where $\psi^+ K_\theta \psi^-$ is a short notation for $\sum_{\mathbf{x},\mathbf{y}\in\mathbf{\Lambda}} \psi^+_{\mathbf{x}} K_\theta(\mathbf{x},\mathbf{y})\psi^-_{\mathbf{y}}$ and $K_\theta(\mathbf{x},\mathbf{y})$ stands for $K_\theta(b,w)$ if the black vertex b (white v. w) has coordinates $\mathbf{x} \in \mathbf{\Lambda}$ (resp. $\mathbf{y} \in \mathbf{\Lambda}$).

The partition function as a non-Gaussian, Grassmann integral

The reason why the r.h.s. of (2.1.4) is the sum of four determinants (and $Z_{L,0}$ is the sum of four Gaussian Grassmann integrals) is that G^0_L is embedded on the torus, which has genus 1: for a dimer model embedded on a surface of genus g, the analogous formula would involve the sum of 4^g such determinants [**55**, **56**]. This is clearly problematic for the graph G_L with non-planar edges, since in general it can be embedded only on surfaces of genus g of order L^2 (i.e. of the order of the number of non-planar edges) and the resulting formula would be practically useless for the analysis of the thermodynamic limit. Our first crucial result is that, even when the weights of the non-planar edges N_L are non-zero, the partition function can again be written as the sum of just four Grassmann integrals, but these are non Gaussian (that is, the integrand is the exponential is a polynomial of order

higher than 2). To emphasize that the following identity holds for generic edge weights, we will write $Z_{L,\underline{t}}$ for the partition function.

Proposition 2.1.1. *One has the identity*

$$Z_{L,\underline{t}} = \sum_{M\in\Omega_L}\prod_{e\in M} t_e = \frac{1}{2}\sum_{\theta\in\{-1,+1\}^2} c_\theta \int D\psi e^{-\psi^+ K_\theta \psi^- + V_{\underline{t}}(\psi)} \tag{2.1.26}$$

where c_θ are given in (2.1.2), $\Lambda = (-L/2, L/2]^2 \cap \mathbb{Z}^2$ *as above,*

$$V_{\underline{t}}(\psi) = \sum_{x\in\Lambda} V^{(x)}(\psi|_{B_x}) \tag{2.1.27}$$

and $V^{(x)}$ is a polynomial with coefficients depending on the weights of the edges incident to the cell B_x, $\psi|_{B_x}$ denotes the collection of the variables $\psi^\pm$ associated with the vertices of cell B_x (as a consequence, the order of the polynomial is at most m^2). When the edge weights $\{t_e\}$ are invariant by translations by $\mathbf{e_1}, \mathbf{e_2}$, then $V^{(x)}$ is independent of x.

The form of the polynomial $V^{(x)}$ is given in formula (2.1.35) below; the expression in the r.h.s. can be computed easily when either the cell size m is small, or each cell contains a small number of non-planar edges. For an explicit example, see Appendix A.1.

Proof. We need some notation. If (b, w) is a pair of black/white vertices joined by the edge e of weight t_e, let us set

$$\psi_\theta(e) := \begin{cases} -t_e \psi_b^+ \psi_w^- & \text{if} \quad e \in N_L \\ -K_\theta(e)\psi_b^+\psi_w^- & \text{if} \quad e \in E_L^0 \end{cases} \tag{2.1.28}$$

with $K_\theta(e) = K_\theta(b, w)$ the Kasteleyn matrix element corresponding to the pair (b, w), which are the endpoints of e. We fix a reference dimer configuration $M_0 \in \Omega_L^0$, say the one where all horizontal edges of every second column are occupied, see Fig. 2.3.

Then, we draw the non-planar edges on the two-dimensional torus on which G_L^0 is embedded, in such a way that they do not intersect (i.e. do not cross) any edge in M_0 (the non-planar edges will in general intersect each other and will intersect some edges in E_L^0 that are not in M_0). Given $J \subset N_L$, we let P_J be the set of edges in E_L^0 that are intersected by edges in J. The drawing of the non-planar edges can be done in such a way that resulting picture is still invariant by translations of $\mathbf{e_1}, \mathbf{e_2}$, the non-planar edges do not exit the corresponding cell and the graph obtained by removing the edges in $N_L \cup P_{N_L}$ (i.e. all the non-planar edges and the planar edges crossed by them) is $2-$connected. See Figure 2.3.

Reorganizing the set of configurations with these definitions we start by rewriting

$$Z_{L,\underline{t}} = \sum_{J\subset N_L}\sum_{S\subset P_J}\sum_{M\in\Omega_{J,S}} w(M) \tag{2.1.29}$$

where $\Omega_{J,S}$ is the set of dimer configurations M such that a non-planar edge belongs to M iff it belongs to J, and an edge in P_J belongs to M iff it belongs to S. Given $M \in \Omega_{J,S}$, we write M as the disjoint union $M = J \cup S \cup M'$ and $w(M) = w(M')w(S)w(J)$, since by its definition (see just below (2.1.1)) $w(M)$ factorizes on such decomposition. In this

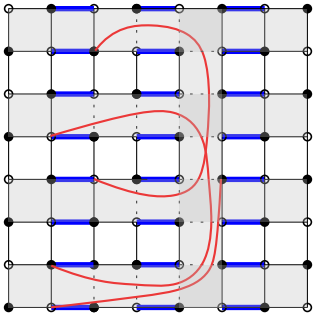

Figure 2.3: *A single cell B_x, with the reference configuration M_0 (thick, blue edges). The non-planar edges (red) are drawn in a way that they do not intersect the edges of M_0 and do not exit the cell. Note that non-planar edges can cross each other. The dotted edges, crossed by the planar edges, belong to P_{N_L}. If the non-planar edges cross only horizontal edges in the same column (shaded) of the cell and vertical edges from every second row (shaded), the graph obtained by removing red edges and dotted edges is 2-connected.*

way (2.1.30) becomes

$$Z_{L,\underline{t}} = \sum_{J \subset N_L} w(J) \sum_{S \subset P_J} w(S) \sum_{M' \sim J,S} w(M') \tag{2.1.30}$$

where $M' \sim S, J$ means that $M' \cup S \cup J$ is a dimer configuration in $\Omega_{J,S}$. To proceed, we use the following

Lemma 2.1.0.1. *There exists $\epsilon_S^J = \pm 1$ such that*

$$\sum_{M' \sim J,S} w(J)w(S)w(M') = \epsilon_S^J \sum_{\theta \in \{-1,+1\}^2} \frac{c_\theta}{2} \int D\psi \, e^{-\psi^+ K_\theta \psi^-} \prod_{e \in J \cup S} \psi(e). \tag{2.1.31}$$

Here, $\psi(e), e \in J \cup S$ is the same as $\psi_\theta(e)$: we have removed the index θ because, since the endpoints b, w of e belong to the same cell, the right hand side of (2.1.28) is independent of θ. If $J = S = \emptyset$, the product of $\psi(e)$ in the right hand side of (2.1.31) should be interpreted as being equal to 1. Moreover, $\epsilon_\emptyset^\emptyset = 1$ and, letting J_x (resp. S_x) denote the collection of edges in J (resp. S) belonging to the cell $B_x, x \in \Lambda$, one has

$$\epsilon_S^J = \prod_{x \in \Lambda} \epsilon_{S_x}^{J_x}. \tag{2.1.32}$$

Let us assume for the moment the validity of Lemma 2.1.0.1 and conclude the proof of Proposition 2.1.1. Going back to (2.1.30), we deduce that

$$Z_{L,\underline{t}} = \sum_\theta \frac{c_\theta}{2} \int D\psi \, e^{-\psi^+ K_\theta \psi^-} \prod_{x \in \Lambda} \left[\sum_{J_x} \sum_{S_x \subset P_{J_x}} \epsilon_{S_x}^{J_x} \prod_{e \in J_x \cup S_x} \psi(e) \right]. \tag{2.1.33}$$

The expression in brackets in (2.1.33) can be written as

$$1 + F_x(\psi) = e^{V^{(x)}(\psi|_{B_x})} \tag{2.1.34}$$

where $F_x(\psi)$ is a polynomial in the Grassmann fields of the box B_x, such that $F_x(0) = 0$ and containing only monomials of even degree, and

$$V^{(x)}(\psi|_{B_x}) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \left(F_x(\psi)\right)^n . \tag{2.1.35}$$

■

Proof of Lemma 2.1.0.1. First of all, let us define a 2−connected graph $G_{J,S}$, embedded on the torus, obtained from G_L as follows:

1. the edges belonging to $N_L \cup P_J$ are removed. At this point, every cell B_x contains a certain number (possibly zero) of faces that are not elementary squares, and the graph is still 2-connected, recall the discussion in the caption of Figure 2.3.

2. the boundary of every such non-elementary face η contains an even number of vertices that are endpoints of edges in $J \cup S$. We connect these vertices pairwise via new edges that do not cross each other, stay within η and have endpoints of opposite color. See Figure 2.4 for a description of a possible procedure. We let $E_{J,S}$ denote the collection of the added edges.

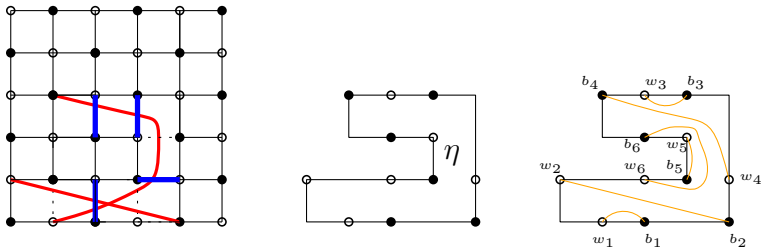

Figure 2.4: *Left drawing: a cell with a collection J of non-diagonal edges (red) and of edges $S \subset P_J$ (thick blue edges). The dotted edges are those in $P_J \setminus S$. Center drawing: the non-elementary face η obtained when the edges in $N_L \cup P_J$ are removed. Only the endpoints of edges in $J \cup S$ are drawn. Right drawing: a planar, bipartite pairing of the endpoints of $J \cup S$. The edges in $E_{J,S}$ are drawn in orange. A possible algorithm for the choice of the pairing is as follows: choose arbitrarily a pair (w_1, b_1) of white/black vertices that are adjacent along the boundary of η and pair them. At step $n > 1$, choose arbitrarily a pair (w_n, b_n) that is adjacent once the vertices $w_i, b_i, i < n$ are removed. Note that some of the edges in $E_{J,S}$ may form double edges with the edges of G_L^0 on the boundary of η (this is the case for (b_1, w_1) and (b_3, w_3) in the example in the figure).*

The first observation is that the l.h.s. of (2.1.31) can be written as

$$\Big(\prod_{e\in J\cup S} t_e\Big)\Big(\sum_{\substack{M\in\Omega_{G_{J,S}}:\\ M\supset E_{J,S}}} w(M)\Big)\Big|_{t_e=1, e\in E_{J,S}} \tag{2.1.36}$$

where $\Omega_{G_{J,S}}$ is the set of perfect matchings of the graph $G_{J,S}$ and as usual $w(M)$ is the product of the edge weights in M. The new edges $E_{J,S}$ are assigned a priori arbitrary weights $\{t_e\}_{e\in E_{J,S}}$, to be eventually replaced by 1, and the partition function on $G_{J,S}$ is called $Z_{G_{J,S}}$.

Let $K_\theta^{J,S}, \theta \in \{-1,+1\}^2$ denote the Kasteleyn matrices corresponding to the four relevant orientations D_θ of $G_{J,S}$, for some choice of the basic orientation on $G_{J,S}$ (recall Definition 2.1.1). Since $G_{J,S}$ is embedded on the torus and is 2-connected, Eq.(2.1.4) guarantees that the sum in the second parentheses in (2.1.36) can be rewritten (before setting $t_e = 1$ for all $e \in E_{J,S}$) as

$$\sum_{\substack{M\in\Omega_{G_{J,S}}\\ M\supset E_{J,S}}} w(M) = \left(\prod_{e\in E_{J,S}} t_e\partial_{t_e}\right) Z_{G_{J,S}} = \frac{1}{2}\sum_\theta c_\theta \left(\prod_{e\in E_{J,S}} t_e\partial_{t_e}\right) \det K_\theta^{J,S}. \quad (2.1.37)$$

In fact, the suitable choice of ordering of vertices mentioned in Remark 2.1.1 (and therefore the value of signs c_θ) is independent of J, S, because the reference configuration M_0 is independent of J, S.

Using the basic properties of Grassmann variables, the r.h.s. of (2.1.37) equals

$$\frac{1}{2}\sum_\theta c_\theta \left(\prod_{e\in E_{J,S}} t_e\partial_{t_e}\right) \int D\psi\, e^{-\psi^+ K_\theta^{J,S}\psi^-} = \frac{1}{2}\sum_\theta c_\theta \int D\psi\, e^{-\psi^+ K_\theta^{J,S}\psi^-} \left(\prod_{e\in E_{J,S}} \psi_\theta^{J,S}(e)\right) \quad (2.1.38)$$

where, in analogy with (2.1.28), $\psi_\theta^{J,S}(e) = -K_\theta^{J,S}(b,w)\psi_b^+\psi_w^-$. We claim:

Lemma 2.1.0.2. *The choice of the basic orientation of $G_{J,S}$ can be made so that the Kasteleyn matrices $K_\theta^{J,S}$ satisfy:*

(i) if $e = (b,w) \in G_{J,S} \setminus E_{J,S}$, then $K_\theta^{J,S}(b,w) = K_\theta(b,w)$, with K_θ the Kasteleyn matrices of the graph G_L^0, fixed by the choices explained in Section 2.1.2.

(ii) if instead $e = (b,w) \in E_{J,S}$ and is contained in cell B_x, then $K_\theta^{J,S}(b,w) = t_e\sigma_e^{J_x,S_x}$ with $\sigma_e^{J_x,S_x} = \pm 1$ a sign that depends only on J_x, S_x.

Assuming Lemma 2.1.0.2, and letting E_{J_x,S_x} denote the subset of edges in $E_{J,S}$ that belong to cell B_x, we rewrite (2.1.38) as

$$\frac{1}{2}\sum_\theta c_\theta \int D\psi\, e^{-\psi^+ K_\theta\psi^-} \prod_x \prod_{e=(b,w)\in E_{J_x,S_x}} (-t_e\sigma_e^{J_x,S_x}\psi_b^+\psi_w^-), \quad (2.1.39)$$

where we could replace $K_\theta^{J,S}$ by K_θ at exponent, because

$$e^{-\psi^+ K_\theta^{J,S}\psi^-}\left(\prod_{e\in E_{J,S}} \psi_\theta^{J,S}(e)\right) = \Big(\prod_{e=(b,w)\in G_{J,S}\setminus E_{J,S}} e^{-\psi_b^+ K_\theta^{J,S}(b,w)\psi_w^-}\Big)\left(\prod_{e\in E_{J,S}} \psi_\theta^{J,S}(e)\right)$$
$$= \Big(\prod_{e=(b,w)\in G_{J,S}\setminus E_{J,S}} e^{-\psi_b^+ K_\theta(b,w)\psi_w^-}\Big)\left(\prod_{e\in E_{J,S}} \psi_\theta^{J,S}(e)\right) = e^{-\psi^+ K_\theta\psi^-}\left(\prod_{e\in E_{J,S}} \psi_\theta^{J,S}(e)\right), \quad (2.1.40)$$

thanks to the Grassmann anti-commutation properties and the fact that $K_\theta^{J,S}(b,w) = K_\theta(b,w)$ for any $(b,w) \in G_{J,S} \setminus E_{J,S}$. Eq.(2.1.39) can be further rewritten as

$$\frac{\prod_{e\in E_{J,S}} t_e}{\prod_{e\in J\cup S} t_e} \sum_\theta \frac{c_\theta}{2} \int D\psi\, e^{-\psi^+ K_\theta \psi^-} \prod_x \Big(\epsilon_{S_x}^{J_x} \prod_{e\in J_x\cup S_x} \psi(e) \Big), \tag{2.1.41}$$

where $\epsilon_{S_x}^{J_x}$ is a sign, equal to

$$\pi(J_x,S_x)\Big(\prod_{e\in E_{J_x,S_x}} \sigma_e^{J_x,S_x} \Big)\Big(\prod_{e\in S_x} \mathrm{sign}(K_\theta(e)) \Big), \tag{2.1.42}$$

and $\pi(J_x,S_x)$ is the sign of the permutation needed to recast $\prod_{(b,w)\in E_{J_x,S_x}} \psi_b^+\psi_w^-$ into the form $\prod_{(b,w)\in J_x\cup S_x} \psi_b^+\psi_w^-$; note also that, for $e \in S_x$, $K_\theta(e)$ is independent of θ. Putting things together, the statement of Lemma 2.1.0.1 follows. ∎

Proof of Lemma 2.1.0.2. Recall that $G_{J,S}$ is a 2-connected graph, with the same vertex set as G_L^0, and edge set obtained, starting from E_L, by removing the edges in $N_L \cup P_J$ and by adding those in $E_{J,S}$. We introduce a sequence of 2-connected graphs $G^{(n)}, n = 0,\dots,z = |E_{J,S}|$ embedded on the torus, all with the same vertex set. Label the edges in $E_{J,S}$ as $e_1,\dots,e_z$ (in an arbitrary order). Then, $G^{(0)}$ is the graph G_L^0 with the edges in $N_L \cup P_J$ removed and $G^{(n)}, 1 \le n \le z$ is obtained from $G^{(0)}$ by adding edges $e_1,\dots,e_n$. Note that $G^{(z)} = G_{J,S}$. We will recursively define the basic orientation $D^{(n)}$ of $G^{(n)}$, in such a way that for $n = z$ the properties stated in the Lemma hold for the Kasteleyn matrices $K_\theta^{(z)} = K_\theta^{J,S}$. The construction of the basic orientation is such that for $n > m$, $D^{(n)}$ restricted to the edges of $G^{(m)}$ is just $D^{(m)}$. That is, at each step $n > 1$ we just need to define the orientation of e_n.

For $n = 0$, $G^{(0)}$ is a sub-graph of G_L^0 and we simply define $D^{(0)}$ to be the restriction of D (the basic orientation of G_L^0) to the edges of the basis graph of $G^{(0)}$. Since the orientation of these edges will not be modified in the iterative procedure, point (i) of the Lemma is automatically satisfied. We need to show that $D^{(0)}$ is indeed a basic orientation for $G^{(0)}$, in the sense of Definition 2.1.1. In fact, an inner face η of the basis graph of $G^{(0)}$ is either an elementary square face (which belongs also to the basis graph of G_L^0), or it is a non-elementary face as in the middle drawing of Fig. 2.4. In the former case, the fact that the boundary of η is clockwise odd is trivial, since its orientation is the same as in the basic orientation of G_L^0. In the latter case, the boundary of η is a cycle Γ of $\mathbb{Z}^2$ that contains no vertices in its interior. The fact that Γ is clockwise odd for D then is well-known [**127**, Sect.V.D].

Assume now that the basic orientation $D^{(n)}$ of $G^{(n)}$ has been defined for $n \ge 0$ and that the choice of orientation of each $e = (b,w) \in E_{J,S}$ that is an edge of $G^{(n)}$ contained in the cell B_x, has been done in a way that depends on J,S only through J_x, S_x. If $n = z$, recalling how Kasteleyn matrices K_θ are defined in terms of the orientations, claim (ii) of the Lemma is proven. Otherwise, we proceed to step $n+1$, that is we define the orientation of e_{n+1} as explained in Figure 2.5. This choice is unique and, again, depends on J,S only through J_x, S_x. The proof of the Lemma is then concluded. ∎

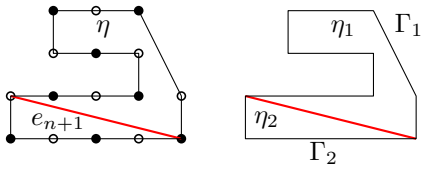

Figure 2.5: *An inner face η of $G^{(n)}$ and the edge e_{n+1}. After adding e_{n+1}, η split into two inner faces η_1, η_2 of $G^{(n+1)}$. By assumption, the boundary Γ of η is clockwise-odd for the orientation $D^{(n)}$. Therefore, exactly one of the two paths Γ_1, Γ_2 contains an odd number of anti-clockwise oriented edges and there is a unique orientation of e_{n+1} such that the boundaries of both η_1, η_2 are clockwise odd. Since, by induction, the orientation of Γ depends on J, S only through J_x, S_x, with x the label of the cell the face belongs to, the same is true also for the orientation of e_{n+1}.*

2.1.4 Generating function and Ward Identities

In this subsection we consider again dimer weights that are periodic under translations by integer multiples of $\mathbf{e_1}, \mathbf{e_2}$.

In view of Proposition 2.1.1, the generating function $W_L(A)$ of dimer correlations, defined, for $A : E_L \to \mathbb{R}$, by

$$e^{W_L(A)} := \sum_{M \in \Omega_L} w(M) \prod_{e \in E_L} e^{A_e \mathbb{1}_e(M)}, \tag{2.1.43}$$

can be equivalently rewritten as $e^{W_L(A)} = \frac{1}{2} \sum_{\theta \in \{1,-1\}^2} c_\theta e^{\mathcal{W}_L^{(\theta)}(A)}$, where

$$e^{\mathcal{W}_L^{(\theta)}(A)} = \int D\psi e^{S_\theta(\psi) + V(\psi, A)}, \tag{2.1.44}$$

where $S_\theta(\psi) = -\psi^+ K_\theta \psi^-$ and $V(\psi, A) := -\psi^+ K_\theta^A \psi_- - S_\theta(\psi) + V_{\underline{t}(A)}(\psi)$[5]. Here, K_θ^A (resp. $V_{\underline{t}(A)}(\psi)$) is the Kasteleyn matrix as in Section 2.1.2 (resp. the potential as in (2.1.27)) with edge weights $\underline{t}(A) = \{t_e e^{A_e}\}_{e \in E_L}$.

As in [**2**, Sect.3.2], it is convenient to introduce a generalization of the generating function, in the presence of an external Grassmann field coupled with ψ. Namely, letting $\phi = \{\phi_{\mathbf{x}}^\pm\}_{\mathbf{x} \in \mathbf{\Lambda}}$ a new set of Grassmann variables, we define

$$\begin{aligned} e^{W_L(A,\phi)} &:= \frac{1}{2} \sum_{\theta \in \{1,-1\}^2} c_\theta e^{\mathcal{W}_L^{(\theta)}(A,\phi)}, \\ \text{with} \qquad e^{\mathcal{W}_L^{(\theta)}(A,\phi)} &:= \int D\psi \, e^{S_\theta(\psi) + V(\psi,A) + (\psi,\phi)} \end{aligned} \tag{2.1.45}$$

and $(\psi, \phi) := \sum_{\mathbf{x} \in \mathbf{\Lambda}} (\psi_{\mathbf{x}}^+ \phi_{\mathbf{x}}^- + \phi_{\mathbf{x}}^+ \psi_{\mathbf{x}}^-)$. The generating function is invariant under a local gauge symmetry, which is associated with the local conservation law of the number of incident dimers at each vertex of Λ. By a gauge symmetry we mean invariance of the generating functional under a local phase change of Grassmann variables $\psi_{\mathbf{x}}^\pm \to \psi_{\mathbf{x}}^\pm e^{i\alpha_{\mathbf{x}}^\pm}$

[5]The arbitrary choice of inserting $S_\theta(\psi)$ in $V(\psi, A)$ is so that the quadratic Grassmannian part does not depend on A.

as described in the following

Proposition 2.1.2 (Chiral gauge symmetry)**.** *Given two functions* $\alpha^+ : \mathbf{\Lambda} \to \mathbb{R}$ *and* $\alpha^- : \mathbf{\Lambda} \to \mathbb{R}$*, we have*

$$W_L(A,\phi) = -i \sum_{\mathbf{x}\in\mathbf{\Lambda}} (\alpha^+_{\mathbf{x}} + \alpha^-_{\mathbf{x}}) + W_L(A + i\alpha, \phi e^{i\alpha}) \tag{2.1.46}$$

where, if $e = (b,w) \in E_L$ *with* $\mathbf{x}$ *and* $\mathbf{y}$ *the coordinates of* b *and* w*, respectively,* $(A+i\alpha)_e := A_e + i(\alpha^+_{\mathbf{x}} + \alpha^-_{\mathbf{y}})$*, while* $(\phi e^{i\alpha})^\pm_{\mathbf{x}} := \phi^\pm_{\mathbf{x}} e^{i\alpha^\mp_{\mathbf{x}}}$*.*

The proof simply consists in performing a change of variables in the Grassmann integral, see [**91**, Proof of Prop.1].

The gauge symmetry (2.1.46), in turn, implies exact identities among correlation functions, known as Ward Identities. Given edges $e_1,\dots,e_k$ and a collection of coordinates $\mathbf{x_1},\dots,\mathbf{x_n},\mathbf{y_1},\dots,\mathbf{y_n}$, define[6] the truncated multi-point correlation associated with the generating function $\mathcal{W}_L(A,\phi)$:

$$\begin{aligned} &g_L(e_1,\dots,e_k;\mathbf{x_1},\dots,\mathbf{x_n};\mathbf{y_1},\dots,\mathbf{y_n}) \\ &\quad := \partial_{A_{e_1}} \cdots \partial_{A_{e_k}} \partial_{\phi^-_{\mathbf{y_1}}} \cdots \partial_{\phi^-_{\mathbf{y_n}}} \partial_{\phi^+_{\mathbf{x_1}}} \cdots \partial_{\phi^+_{\mathbf{x_n}}} W_L(A,\phi)\big|_{A\equiv 0,\phi\equiv 0}. \end{aligned} \tag{2.1.47}$$

Three cases will play a central role in the following: the *interacting propagator* $G^{(2)}$, the *interacting vertex function* $G^{(2,1)}$ and the *interacting dimer-dimer correlation* $G^{(0,2)}$, which deserve a distinguished notation: letting $\mathbf{x} = (x,\ell), \mathbf{y} = (y,\ell'), \mathbf{z} = (z,\ell'')$, and denoting by e (resp. e') the edge with black vertex $\mathbf{x} = (x,\ell)$ (resp. $\mathbf{y} = (y,\ell')$) and label $j \in \mathcal{J}_\ell$ (resp. $j' \in \mathcal{J}_{\ell'}$), we define

$$\begin{aligned} G^{(2)}_{\ell,\ell';L}(x,y) &:= g_L(\emptyset;\mathbf{x};\mathbf{y}) \\ G^{(2,1)}_{j,\ell,\ell',\ell'';L}(x,y,z) &:= g_L(e;\mathbf{y};\mathbf{z}) \\ G^{(0,2)}_{j,j',\ell,\ell';L}(x,y) &:= g_L(e,e';\emptyset;\emptyset). \end{aligned} \tag{2.1.48}$$

As a byproduct of the analysis of Section 2.3, the $L\to\infty$ of all multi-point correlations $g_L(e_1,\dots,e_k,\mathbf{x_1},\dots,\mathbf{x_n},\mathbf{y_1},\dots,\mathbf{y_n})$ exist; we denote the limit simply by dropping the index L. Let us define the Fourier transforms of the interacting propagator and interacting vertex function via the following conventions: for $\ell,\ell',\ell'' \in \mathcal{I}$ and $j \in \mathcal{J}_\ell$, we let

$$\begin{aligned} \hat G^{(2)}_{\ell,\ell'}(p) &:= \sum_{x\in\mathbb{Z}^2} e^{ipx} G^{(2)}_{\ell,\ell'}(x,0) \\ \hat G^{(2,1)}_{j,\ell,\ell',\ell''}(k,p) &:= \sum_{x,y\in\mathbb{Z}^2} e^{-ipx-ik\cdot y} G^{(2,1)}_{j,\ell,\ell',\ell''}(x,0,y). \end{aligned} \tag{2.1.49}$$

Proposition 2.1.3 (Ward identity)**.** *Given* $\ell',\ell'' \in \mathcal{I}$*, we have*

$$\sum_{e\in\mathcal{E}} \hat G^{(2,1)}_{j(e),\ell(e),\ell',\ell''}(k,p)(e^{-ip\cdot v(e)} - 1) = \hat G^{(2)}_{\ell',\ell''}(k+p) - \hat G^{(2)}_{\ell',\ell''}(k) \tag{2.1.50}$$

[6] We refer e.g. to [**91**, Remark 5] for the meaning of the derivative with respect to Grassmann variables

where $\mathcal{E}$ is the set of edges $e = (b(e), w(e))$ having an endpoint in the cell $B_{(0,0)}$ and the other in $B_{(0,-1)} \cup B_{(-1,0)}$. Also, $\ell(e) \in \mathcal{I}$ is the type of $b(e)$, $j(e) \in \mathcal{J}_{\ell(e)}$ is the label associated with the edge e, while $v(e) \in \{(0, \pm 1), (\pm 1, 0)\}$ is the difference of cell labels of $w(e)$ and $b(e)$, see discussion after (2.1.19).

Proof. We start by differentiating both sides of the gauge invariance equation (2.1.46): fix $\mathbf{x} = (x, \ell) \in \mathbf{\Lambda}$, differentiate first with respect to $\alpha^+_{\mathbf{x}}$ and set $\alpha \equiv 0$:

$$1 = \sum_{\substack{e=(b,w)\in E_L \\ \mathbf{x}(b)=\mathbf{x}}} \partial_{A_e} W_L(A, \phi) + \phi^-_{\mathbf{x}} \partial_{\phi^-_{\mathbf{x}}} W_L(A, \phi) \tag{2.1.51}$$

where $\mathbf{x}(b) = (x(b), \ell(b))$ is the coordinate of the black endpoint b of the edge e. The above sum thus contains as many terms as the number of edges incident to the black site of coordinate $\mathbf{x}$, i.e. as the number of elements in $\mathcal{J}_{\ell(b)}$. Then, differentiate with respect to $\phi^-_{\mathbf{z}}$ and $\phi^+_{\mathbf{y}}$ and set $A \equiv \phi \equiv 0$:

$$\sum_{\substack{e=(b,w)\in E_L \\ \mathbf{x}(b)=\mathbf{x}}} g_L(e; \mathbf{y}; \mathbf{z}) + \delta_{\mathbf{x},\mathbf{z}} g_L(\emptyset; \mathbf{y}; \mathbf{z}) = 0. \tag{2.1.52}$$

Repeating the same procedure but differentiating first with respect to $\alpha^-_{\mathbf{x}}$ rather than $\alpha^+_{\mathbf{x}}$, and setting $\alpha \equiv 0$ we obtain the analogous of (2.1.51):

$$1 = \sum_{\substack{e=(b,w)\in E_L \\ \mathbf{x}(w)=\mathbf{x}}} \partial_{A_e} W_L(A, \phi) + \phi^+_{\mathbf{x}} \partial_{\phi^+_{\mathbf{x}}} W_L(A, \phi). \tag{2.1.53}$$

Then we differentiate with respect to $\phi^-_{\mathbf{z}}$ and $\phi^+_{\mathbf{y}}$, set $\phi \equiv 0$ to obtain the analogous of (2.1.52):

$$\sum_{\substack{e=(b,w)\in E_L \\ \mathbf{x}(w)=\mathbf{x}}} g_L(e; \mathbf{y}; \mathbf{z}) + \delta_{\mathbf{x},\mathbf{y}} g_L(\emptyset; \mathbf{y}; \mathbf{z}) = 0 \tag{2.1.54}$$

where $\mathbf{x}(w)$ is the coordinate of the white vertex of e. Now we sum both (2.1.54) and (2.1.52) over $\ell \in \mathcal{I}$ (the type of the vertex $\mathbf{x}$) with the cell index x fixed; then we take the difference of the two expressions thus obtained and we send $L \to \infty$. When taking the difference, the contribution from edges whose endpoints both belong to cell B_x cancel and we are left with

$$\sum_{e=(x',j,\ell)\in E_{\partial B_x}} (-1)^{\delta_{x,x'}} G^{(2,1)}_{j,\ell,\ell',\ell'}(x', y, z) = (\delta_{x,z} - \delta_{x,y}) G^{(2)}_{\ell',\ell''}(y, z), \tag{2.1.55}$$

where we used the notation in (2.1.48), and we denoted by $E_{\partial B_x}$ the set of edges of E^0 having exactly one endpoint in the cell B_x. Note that in the first sum, in writing $e = (x', j, \ell)$, we used the usual labeling of the edge e in terms of the coordinates (x', ℓ) of its black site and of the label $j \in \mathcal{J}_\ell$. Note also that, if $e = (x', j, \ell) \in E_{\partial B_x}$, then x' is either x or $x \pm (0, 1), x \pm (1, 0)$. See Figure 2.6. Using the last remark in the caption of Fig.2.6, we can rewrite the sum in the left hand side of (2.1.55) as a sum over edges in $\mathcal{E}_{1,x} \cup \mathcal{E}_{2,x}$, each term containing the difference of two vertex functions $G^{(2,1)}_{j,\ell,\ell',\ell''}$. Passing to Fourier space via (2.1.49), we obtain (2.1.50), as desired. ∎

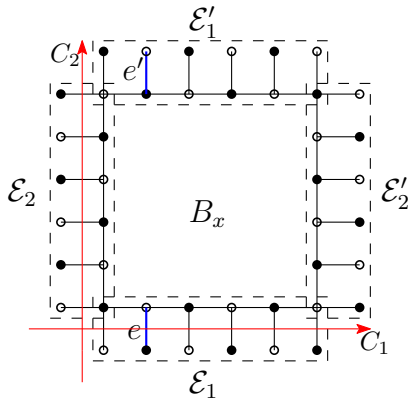

Figure 2.6: *The cell B_x (only vertices on its boundary are drawn) together with the edges in $E_{\partial B_x} = \mathcal{E}_{1,x} \cup \mathcal{E}_{2,x} \cup \mathcal{E}'_{1,x} \cup \mathcal{E}'_{2,x}$. To each edge e in $\mathcal{E}_{1,x}$ (resp. in $\mathcal{E}_{2,x}$) there corresponds a unique edge e' in $\mathcal{E}'_{1,x}$ (resp. $\mathcal{E}'_{2,x}$) whose endpoints are of the same type.*

Remark 2.1.2. *For later reference, note that, if e crosses the path C_1 (resp. C_2) of Figure 2.6, i.e., if $e \in \mathcal{E}_{1,x}$ (resp. $e \in \mathcal{E}_{2,x}$), then, for any $p = (p_1, p_2) \in \mathbb{R}^2$ and $v(e)$ defined as in the statement of Proposition 2.1.3,*

$$p \cdot v(e) = \begin{cases} -p_2\sigma_e & \text{if } e \text{ crosses } C_1 \\ +p_1\sigma_e & \text{if } e \text{ crosses } C_2, \end{cases} \tag{2.1.56}$$

with $\sigma_e = \pm 1$ the same sign appearing in the definition (1.4.3) of height function.

2.2 Proof of Theorem 1.4.2

One important conclusion of the previous section is Proposition 2.1.3, which states the validity of exact identities among the (thermodynamic limit of) correlation functions of the dimer model. In this section we combine these exact identities with a result on the large-distance asymptotics of the correlation functions, which includes the statement of Theorem 1.4.1, and use them to prove Theorem 1.4.2. The required fine asymptotics of the correlation functions is summarized in the following proposition, whose proof is discussed in Section 2.3. In the following we will denote with $\lambda \to f$ a function f of λ.

Proposition 2.2.1. *There exists $\lambda_0 > 0$ such that, for $|\lambda| \le \lambda_0$, the interacting dimer-dimer correlation for $x \neq y$ can be represented in the following form:*

$$\begin{aligned} G^{(0,2)}_{j,j',\ell,\ell'}(x,y) = {} & \frac{1}{4\pi^2 Z^2(1-\tau^2)} \sum_{\omega=\pm} \frac{K^{(1)}_{\omega,j,\ell} K^{(1)}_{\omega,j',\ell'}}{(\phi_\omega(x-y))^2} \\ & + \frac{B}{4\pi^2} \sum_{\omega=\pm} \frac{K^{(2)}_{\omega,j,\ell} K^{(2)}_{-\omega,j',\ell'}}{|\phi_\omega(x-y)|^{2(1-\tau)/(1+\tau)}} e^{2i\, p^\omega \cdot (x-y)} + R_{j,j',\ell,\ell'}(x,y)\ , \end{aligned} \tag{2.2.1}$$

where: $\lambda \mapsto Z$, $\lambda \mapsto \tau$ and $\lambda \mapsto B$ are real-valued analytic functions satisfying $Z = 1 + O(\lambda)$, $\tau = O(\lambda)$ and $B = 1 + O(\lambda)$; $\phi_\omega(x) := \omega(\beta_\omega x_1 - \alpha_\omega x_2)$ where $\lambda \mapsto \alpha_\omega, \lambda \mapsto \beta_\omega$ are complex-valued analytic functions satisfying $\overline{\alpha_+} = -\alpha_-, \overline{\beta_+} = -\beta_-$; $\lambda \mapsto K^{(i)}_{\omega,j,\ell}$ with

$i \in \{1,2\}$ are complex-valued analytic functions of λ satisfying $K^{(i)}_{+,j,\ell} = \overline{K^{(i)}_{-,j,\ell}}$; $\lambda \mapsto p^\omega$ are analytic functions with values in $[-\pi,\pi]^2$ for λ real, satisfying $p^+ = -p^-$ and $2p^+ \neq 0$ mod $(2\pi, 2\pi)$; the correction term $R_{j,j',\ell,\ell'}(x,y)$ is translational invariant and satisfies $|R_{j,j',\ell,\ell'}(x,0)| \leq C|x|^{-1+C|\lambda|}$ for some $C > 0$.

Moreover, there exists an additional set of complex-valued analytic function $\lambda \mapsto I_{\omega,\ell,\ell'}, \omega = \pm 1, \ell, \ell' \in \mathcal{I}$, such that the Fourier transforms of the interacting propagator and of the interacting vertex function satisfy:

$$\hat{G}^{(2)}_{\ell,\ell'}(k+p^\omega) \stackrel{k\to 0}{=} I_{\omega,\ell,\ell'} \hat{G}^{(2)}_{R,\omega}(k)[1+O(|k|^{1/2})], \tag{2.2.2}$$

and, if $0 < \mathfrak{c} \leq |p|, |k|, |k+p| \leq 2\mathfrak{c}$,

$$\hat{G}^{(2,1)}_{j,\ell,\ell',\ell''}(k+p^\omega, p) \stackrel{\mathfrak{c}\to 0}{=} -\sum_{\omega'=\pm} K^{(1)}_{\omega',j,\ell} I_{\omega,\ell',\ell''} \hat{G}^{(2,1)}_{R,\omega',\omega}(k,p)[1+O(\mathfrak{c}^{1/2})]\,, \tag{2.2.3}$$

where $K^{(1)}_{\omega,j,\ell}$ is the same as in (2.2.1) and $\hat{G}^{(2)}_{R,\omega}(k)$,$\hat{G}^{(2,1)}_{R,\omega,\omega'}(k,p)$ are two functions satisfying, for $D_\omega(p) = \alpha_\omega p_1 + \beta_\omega p_2$,

$$\sum_{\omega'=\pm} D_{\omega'}(p) \hat{G}^{(2,1)}_{R,\omega',\omega}(k,p) = \frac{1}{Z(1-\tau)} \Big[\hat{G}^{(2)}_{R,\omega}(k) - \hat{G}^{(2)}_{R,\omega}(k+p)\Big] \Big(1+O(\lambda|p|)\Big)\,, \tag{2.2.4}$$

with Z, τ the same as in (2.2.1), and

$$\hat{G}^{(2,1)}_{R,-\omega,\omega}(k,p) = \tau \frac{D_\omega(p)}{D_{-\omega}(p)} \hat{G}^{(2,1)}_{R,\omega,\omega}(k,p) \Big(1+O(|p|)\Big). \tag{2.2.5}$$

Finally, $\hat{G}^{(2)}_{R,\omega}(k) \sim c_1 |k|^{-1+O(\lambda^2)}$ as $k \to 0$, and, if $0 < \mathfrak{c} \leq |p|, |k|, |k+p| \leq 2\mathfrak{c}$, $\hat{G}^{(2,1)}_{R,\omega,\omega'}(k,p) \sim c_2 \mathfrak{c}^{-2+O(\lambda^2)}$ as $\mathfrak{c} \to 0$, for two suitable non-zero constants c_1, c_2.

A few comments are in order. First of all, the statement of Theorem 1.4.1, (1.4.1), follows from (2.2.1), which is just a way to rewrite it: it is enough to identify $K_{\omega,j,\ell}$ with $(2\pi Z\sqrt{1-\tau^2})^{-1} K^{(1)}_{\omega,j,\ell}$, $H_{\omega,j,\ell}$ with $(\sqrt{B}/2\pi) K^{(2)}_{\omega,j,\ell}$, and ν with $(1-\tau)/(1+\tau)$.

Moreover, we emphasize that Proposition 2.2.1 is the analogue of [**2**, Prop.2] and its proof, discussed in the next section, is a generalization of the corresponding one. The main ideas behind the proof remain the same: in order to evaluate the correlation functions of the non-planar dimer model we start from the Grassmann representation of the generating function, (2.1.45), and we compute it via an iterative integration procedure, in which we first integrate out the degrees of freedom associated with a length scale 1, i.e., the scale of the lattice, then those on length scales $2, 4, \ldots, 2^{-h}, \ldots$, with $h < 0$. The output of the integration of the first $|h|$ steps of this iterative procedure can be written as a Grassmann integral similar to the original one, with the important difference that the 'bare potential' $V(\psi, A) + (\psi, \phi)$ is replaced by an effective one, $V^{(h)}(\psi, A, \phi)$, that, after appropriate rescaling, converges to a non-trivial infrared fixed point as $h \to -\infty$. The large-distance asymptotics of the correlation functions of the dimer model can thus be computed in terms of those of such an infrared fixed-point theory, or of those of any other model with the same fixed point (i.e., of any other model in the same *universality class*, the *Luttinger* universality class). The reference model we choose for this asymptotic comparison is described in [**2**, Section 4], which we refer the reader to for additional details.

It is very similar to the Luttinger model, and differs from it just for the choice of the quartic interaction: it describes a system of Euclidean chiral fermions in $\mathbb{R}^2$ (modeled by Grassmann fields denoted $\psi^{\pm}_{x,\omega}$, with $x \in \mathbb{R}^2$ the space label and $\omega \in \{+,-\}$ the chirality label), with relativistic propagator and a non-local (in both space dimensions, contrary to the case of the Luttinger model) density-density interaction[7]. The bare parameters of the reference model, in particular the strength of its density-density interaction, are chosen in such a way that its infrared fixed point coincides with the one of our dimer model of interest. The remarkable feature of the reference model is that, contrary to our dimer model, it is exactly solvable in a very strong sense: its correlation functions can all be computed in closed form. For our purposes, the relevant correlations are those denoted[8] $G^{(2,1)}_{R,\omega,\omega'}$ (the vertex function of the reference model, corresponding to the correlation of the density of chirality ω with a pair of Grassmann fields of chirality ω'), $G^{(2)}_{R,\omega}$ (the interacting propagator, corresponding to the correlation between two Grassmann fields of chirality ω), $S^{(1,1)}_{R,\omega,\omega}$ (the density-density correlation between two densities with the same chirality ω) and $S^{(2,2)}_{R,\omega,-\omega}$ (the mass-mass correlation between two masses – see footnote 7 – of opposite chiralities): these are the correlations, in terms of which the asymptotics of the vertex function, interacting propagator and dimer-dimer correlation of our dimer model can be expressed.

Remark 2.2.1. *The connection between the interacting propagator of the dimer model and that of the reference model can be read from* (2.2.2)*; similarly, the one between the vertex functions of the two models can be read from* (2.2.3)*. Moreover, in view of the asymptotics of* $S^{(1,1)}_{R,\omega,\omega}$ *and of* $S^{(2,2)}_{R,\omega,-\omega}$*, see [**2**, Eqs.(4.17) and (4.19)],* (2.2.1) *can be rewritten as*

$$\sum_{\omega=\pm} [K^{(1)}_{\omega,j,\ell} K^{(1)}_{\omega,j',\ell'} S^{(1,1)}_{R,\omega,\omega}(x,y) + K^{(2)}_{\omega,j,\ell} K^{(2)}_{-\omega,j',\ell'} S^{(2,2)}_{R,\omega,-\omega}(x,y) e^{2ip^{\omega}\cdot(x-y)}] \tag{2.2.6}$$

plus a faster decaying remainder, which explains the connection between the dimer-dimer correlation and the density-density and mass-mass correlations of the reference model.

The fact that the infrared behavior of the dimer model discussed in this paper can be described via the same reference model used for the dimer model in [**2**] is a priori non-obvious. In fact, the Grassmann representation of our non-planar dimer model involves Grassmann fields labelled by $x \in \Lambda$ and $\ell \in \mathcal{I} = \{1, \dots, m^2/2\}$: therefore, one could expect that the infrared behavior of the system is described in terms of a reference model involving fields labelled by an index $\ell \in \mathcal{I}$. This, a priori, could completely change at a qualitative level the nature of the infrared behavior of the system, which crucially depends on the number of mutually interacting massless fermionic fields. For instance, it is well known that 2D chiral fermions with an additional spin degree of freedom (which is the case of interest for describing the infrared behavior of the 1D Hubbard model), behaves differently, depending on the *sign* of the density-density interaction: for repulsive interactions it behaves qualitatively in the same way as the Luttinger model [**122**], while

[7] By 'density' of fermions with chirality ω we mean the quadratic monomial $\psi^{+}_{x,\omega}\psi^{-}_{x,\omega}$; the reference model we consider has an interaction coupling the density of fermions with chirality + with that of fermions with opposite chirality, see [**2**, Eq.(4.11)]. For later reference, we also introduce the notion of fermionic 'mass' of chirality ω, associated with the off-diagonal (in the chirality index) quadratic monomial $\psi^{+}_{x,\omega}\psi^{-}_{x,-\omega}$.

[8] The label R stands for 'reference' or 'relativistic'.

for attractive interaction the model dynamically generates a mass and enters a 'Mott-insulator' phase [**125**]. In our setting, remarkably, despite the fact that the number of Grassmann fields used to effectively describe the model is large for a large elementary cell (and, in particular, is always larger than 1), the number of massless fields is the same as in the case of [**2**]: in fact, out of the $m^2/2$ fields $\psi^{\pm}_{(x,\ell)}$ with $\ell \in \{1, \dots, m^2/2\}$, all but one of them are *massive*, i.e., their correlations decay exponentially to zero at large distances, with rate proportional to the inverse lattice scaling (this is a direct consequence of the fact that, as proven in [**32**], the characteristic polynomial μ has only two zeros). Therefore, for the purpose of computing the generating function, we can integrate out the massive fields in one single step of the iterative integration procedure, after which we are left with an effective theory of a single massless Grassmann field with chirality index ω associated with the two zeros of μ, see (3.1.6), completely analogous to the one studied in [**2**, Section 6]. See the next section for details.

While the proof of the fine asymptotic result summarized in Proposition 2.2.1 is hard, and based on the sophisticated procedure just described, the proof of Theorem 1.4.2 *given* Proposition 2.2.1 is relatively easy, and close to the analogous proof discussed in [**2**, Section 5]. We provide it here. Let us start with one definition. Given the face $\eta_0 \in \bar{F}$ ($\bar{F}$ and $\eta_x, x = (x_1, x_2)$ were defined in Section 1.4, just before Theorem 1.4.2), let $\mathcal{E}_{1,0}$ (resp. $\mathcal{E}_{2,0}$), be the set of vertical (resp. horizontal) edges crossed by the horizontal (resp. vertical) path $C_{\eta_0 \to \eta'}$ connecting η_0 to the face $\eta' \in \bar{F}$ given by $\eta' = \eta_{(1,0)}$ (resp. $\eta' = \eta_{(0,1)}$). See Fig.2.6, where the same paths and edge sets around the cell B_x rather than B_0 are shown. For $e \in \mathcal{E}_{q,0}$, $q = 1, 2$, we let $(x(e), \ell(e))$ denote the coordinates of its black vertex and $j(e) \in \mathcal{J}_{\ell(e)}$ the type of the edge. We also recall from Section 1.4 that $\sigma_e = \pm 1$ is defined in (1.4.3).

Proposition 2.2.2. *For $q = 1, 2$ and $\omega = \pm$, one has*

$$\sum_{e \in \mathcal{E}_{q,0}} \sigma_e \frac{K^{(1)}_{\omega, j(e), \ell(e)}}{Z\sqrt{1-\tau^2}} = -i\omega\sqrt{\nu}\, \mathrm{d}_q \phi_\omega \tag{2.2.7}$$

where $\nu = (1-\tau)/(1+\tau)$, and

$$\begin{aligned} \mathrm{d}_1 \phi_\omega &:= \phi_\omega(x + (1,0))) - \phi_\omega(x) = \omega \beta_\omega, \\ \mathrm{d}_2 \phi_\omega &:= \phi_\omega(x + (0,1))) - \phi_\omega(x) = -\omega \alpha_\omega. \end{aligned} \tag{2.2.8}$$

Proof. Start with the Ward Identity in Fourier space (2.1.50) evaluated for k replaced by $k + p^\omega$ and substitute (2.2.2) and (2.2.3) in it for $\mathfrak{c} \to 0$. Recalling that $0 < \mathfrak{c} < |k|, |p|, |k+p| < 2\mathfrak{c}$ we obtain for $\mathfrak{c}$ small

$$\sum_{\omega' = \pm} \mathcal{D}_{\omega'}(p) G^{(2,1)}_{R, \omega', \omega}(k, p) = (G^{(2)}_{R,\omega}(k) - G^{(2)}_{R,\omega}(k+p))(1 + O(\mathfrak{c}^{1/2})) \tag{2.2.9}$$

where $\mathcal{D}_\omega(p) := -i \sum_{e \in \mathcal{E}} K^{(1)}_{\omega, j(e), \ell(e)} p \cdot v(e)$, with $\mathcal{E} = \mathcal{E}_{1,0} \cup \mathcal{E}_{2,0}$ the set of edges defined in Proposition 2.1.3. Now comparing the above relation with the identity (2.2.4), by using (2.2.5) and by identifying terms at dominant order for $|p|$ small we obtain (recall the definition of $D_\omega(p)$ right before (2.2.4)):

$$\mathcal{D}_\omega(p) D_{-\omega}(p) + \tau \mathcal{D}_{-\omega}(p) D_\omega(p) = Z(1-\tau^2) D_\omega(p) D_{-\omega}(p). \tag{2.2.10}$$

Letting $p=(p_1,p_2), v(e)=(v_1(e),v_2(e))$, imposing $p_2=0, p_1\neq 0$ first and $p_1=0, p_2\neq 0$ then, we find a linear system for the coefficients $-i\sum_{e\in\mathcal{E}} K_{\omega,\ell(e),j(e)}v_q(e)$, for $q=1,2$ and $\omega=\pm$ whose solution is

$$\begin{aligned}\sum_{e\in\mathcal{E}} K^{(1)}_{\omega,j(e),\ell(e)}v_1(e) &= iZ(1-\tau)\alpha_\omega,\\ \sum_{e\in\mathcal{E}} K^{(1)}_{\omega,j(e),\ell(e)}v_2(e) &= iZ(1-\tau)\beta_\omega.\end{aligned} \tag{2.2.11}$$

Note that, by the very definition of $\mathcal{E}=\mathcal{E}_{1,0}\cup\mathcal{E}_{2,0}$, if $e\in\mathcal{E}$, then $v_1(e)\neq 0$ iff $e\in\mathcal{E}_{2,0}$, while $v_2(e)\neq 0$ iff $e\in\mathcal{E}_{1,0}$. Recall also the relation between $v(e)$ and σ_e outlined in Remark 2.1.2: in view of this, (2.2.11) is equivalent to

$$\sum_{e\in\mathcal{E}_{1,0}} \frac{K^{(1)}_{\omega,j(e),\ell(e)}}{Z\sqrt{1-\tau^2}}\sigma_e = -i\sqrt{\frac{1-\tau}{1+\tau}}\,\beta_\omega = -i\omega\sqrt{\nu}\,\mathrm{d}_1\phi_\omega \tag{2.2.12}$$

$$\sum_{e\in\mathcal{E}_{2,0}} \frac{K_{\omega,j(e),\ell(e)}}{Z\sqrt{1-\tau^2}}\sigma_e = i\sqrt{\frac{1-\tau}{1+\tau}}\,\alpha_\omega = -i\omega\sqrt{\nu}\,\mathrm{d}_2\phi_\omega, \tag{2.2.13}$$

where we used $\nu=(1-\tau)/(1+\tau)$ and the definition (2.2.8). ∎

Proof of Theorem 1.4.2. Given Proposition 2.2.2, the proof of Theorem 1.4.2 is essentially identical to that of [**2**, Eq.(2.47)] and of [**90**, Proof of (7.26)]. Here we give only a sketch and we emphasize only the role played by the relation (2.2.7) that we have just proven.

First of all, we choose a path $C_{\eta_{x^{(1)}}\to\eta_{x^{(2)}}}$ from face $\eta_{x^{(1)}}$ to $\eta_{x^{(2)}}$ that crosses only edges that join different cells. Since $\eta_{x^{(1)}},\eta_{x^{(2)}}\in\bar F$, the path $C_{\eta_{x^{(1)}}\to\eta_{x^{(2)}}}$ visits a sequence of faces $\eta_{y^{(1)}},\dots,\eta_{y^{(k)}}\in\bar F$, with $y^{(1)}=x^{(1)}, y^{(k)}=x^{(2)}$ and $|y^{(a)}-y^{(a+1)}|=1$. The set of edges crossed by the path between $\eta_{y^{(a)}}$ and $\eta_{y^{(a+1)}}$, denoted $\mathcal{E}_{(a)}$, is a translation of either $\mathcal{E}_{1,0}$ (if $y^{(a+1)}-y^{(a)}$ is horizontal) or $\mathcal{E}_{2,0}$ (if $y^{(a+1)}-y^{(a)}$ is vertical). Similarly, one defines a path $C_{\eta_{x^{(3)}}\to\eta_{x^{(4)}}}$ and correspondingly a sequence of faces $\eta_{z^{(1)}},\dots,\eta_{z^{(k')}}\in\bar F$ and $\mathcal{E}'_{(a)}$ the set of edges crossed by the path between $\eta_{z^{(a)}}$ and $\eta_{z^{(a+1)}}$. The two paths can be chosen so that $C_{\eta_{x^{(1)}}\to\eta_{x^{(2)}}}$ is of length $O(|x^{(1)}-x^{(2)}|)$ and $C_{\eta_{x^{(3)}}\to\eta_{x^{(4)}}}$ is of length $O(|x^{(3)}-x^{(4)}|)$, while they are at mutual distance at least of order $\min(|x^{(i)}-x^{(j)}|, i\neq j)$. See [**90**] for more details.

From the definition (1.4.3) of height function, we see that

$$\mathbb{E}_\lambda\left[(h(\eta_{x^{(1)}})-h(\eta_{x^{(2)}}));(h(\eta_{x^{(3)}})-h(\eta_{x^{(4)}}))\right] = \sum_{\substack{1\le a<k,\\ 1\le a'<k'}}\ \sum_{\substack{e\in\mathcal{E}_{(a)},\\ e'\in\mathcal{E}'_{(a')}}} \sigma_e\sigma_{e'}\mathbb{E}_\lambda[\mathbb{1}_e;\mathbb{1}_{e'}]. \tag{2.2.14}$$

As a consequence of Proposition 2.2.1, for edges e,e' with black sites of coordinates $(x,\ell),(x',\ell')$ and with orientations j,j', respectively, we have that

$$\begin{aligned}\mathbb{E}_\lambda[\mathbb{1}_e;\mathbb{1}_{e'}] = &\sum_{\omega=\pm}\frac{K^{(1)}_{\omega,j,\ell}}{Z\sqrt{1-\tau^2}}\frac{K^{(1)}_{\omega,j',\ell'}}{Z\sqrt{1-\tau^2}}\frac{1}{4\pi^2(\phi_\omega(x-x'))^2}\\ &+\frac{B}{4\pi^2}\sum_{\omega=\pm}\frac{K^{(2)}_{\omega,j,\ell}K^{(2)}_{-\omega,j',\ell'}}{|\phi_\omega(x-x')|^{2(1-\tau)/(1+\tau)}}e^{2ip^\omega(x-x')}+R_{j,j',\ell,\ell'}(x,x').\end{aligned} \tag{2.2.15}$$

At this point we plug this expression into (2.2.14). The oscillating term in (2.2.15), proportional to B, and the error term $R_{j,\ell,j',\ell}(x,x')$, once summed over e,e', altogether end up in the error term in (1.4.4) (see the analogous argument in [**90**, Section 3.2 and 7.3]). As for the main term involving $K^{(1)}_{\omega,j,\ell}$, we observe that if we fix a,a', then for $e\in\mathcal{E}_{(a)}, e'\in\mathcal{E}'_{(a')}$ we can replace in (2.2.15) $\phi_\omega(x-x')$ by $\phi_\omega(y^{(a)}-z^{(a')})$, up to an error term of the same order as $R_{j,j',\ell,\ell'}(x,x')$, which again contributes to the error term in (1.4.4). We are thus left with

$$\begin{aligned}
&\sum_{\omega=\pm}\sum_{\substack{1\le a<k,\\ 1\le a'<k'}}\frac{1}{4\pi^2(\phi_\omega(y^{(a)}-z^{(a')}))^2}\sum_{e\in\mathcal{E}_{(a)}}\sigma_e\frac{K_{\omega,j(e),\ell(e)}}{Z\sqrt{1-\tau^2}}\sum_{e'\in\mathcal{E}'_{(a')}}\sigma_{e'}\frac{K_{\omega,j(e'),\ell(e')}}{Z\sqrt{1-\tau^2}}\\
&=-\nu\sum_{\omega=\pm}\sum_{\substack{1\le a<k,\\ 1\le a'<k'}}\frac{(y^{(a+1)}-y^{(a)})\cdot \mathrm{d}\phi_\omega\,(z^{(a'+1)}-z^{(a')})\cdot\mathrm{d}\phi_\omega}{4\pi^2(\phi_\omega(y^{(a)}-z^{(a')}))^2}\qquad(2.2.16)\\
&=-\frac{\nu}{2\pi^2}\Re\Big[\sum_{\substack{1\le a<k,\\ 1\le a'<k'}}\frac{(y^{(a+1)}-y^{(a)})\cdot \mathrm{d}\phi_+\,(z^{(a'+1)}-z^{(a')})\cdot\mathrm{d}\phi_+}{(\phi_+(y^{(a)}-z^{(a')}))^2}\Big]
\end{aligned}$$

where in the first step we used Proposition 2.2.2 and defined $\mathrm{d}\phi_\omega := (\mathrm{d}_1\phi_\omega, \mathrm{d}_2\phi_\omega)$. As explained in [**2**, Section 5.2] and [**90**, Section 7.3] (see also [**32**, Section 4.4.1] in the non-interacting case), this sum equals the integral in the complex plane

$$-\frac{\nu}{2\pi^2}\Re\int_{\phi_+(x^{(1)})}^{\phi_+(x^{(2)})}dz\int_{\phi_+(x^{(3)})}^{\phi_+(x^{(4)})}dz'\frac{1}{(z-z')^2}\qquad(2.2.17)$$

(which equals the main term in the r.h.s. of (1.4.4)), plus an error term (coming from the Riemann approximation) estimated as in the r.h.s. of (1.4.4). ∎

2.3 Proof of Proposition 2.2.1

In this section we give the proof of Proposition 2.2.1 (which immediately implies Theorem 1.4.1, as already commented above), via the strategy sketched after its statement. As explained there, the novelty compared to the proof in [**2**, Section 6] is the reduction to an effective model involving a *single* Grassmann critical field φ, of the same form as the one analyzed in [**2**, Section 6]. Therefore, most of this section will be devoted to the proof of such reduction, which consists of the following steps. Our starting point is the generating function of correlations in its Grassmann form, see (2.1.45). In (2.1.45), we first integrate out the 'ultraviolet' degrees of freedom at the lattice scale, see Section 2.3.1 below; the resulting effective theory can be conveniently formulated in terms of a collection of chiral fields $\{\psi^\pm_{x,\omega}\}_{x\in\Lambda}^{\omega\in\{+,-\}}$, where $\psi^\pm_{x,\omega}$ are Grassmann vectors with $|\mathcal{I}|$ components, which represent fluctuation fields supported in momentum space close to the unperturbed Fermi points p^ω_0. Next, we perform a 'rigid rotation' of these Grassmann vectors via a matrix B that is independent of x but may depend on the chirality index ω; the rotation is chosen so to block-diagonalize the reference quadratic part of the effective action, in such a way that the corresponding covariance is the direct sum of two terms, a one-dimensional one, which is singular at p^ω_0, and a non-singular one, of dimension $|\mathcal{I}|-1$; the components associated with this non-singular $(|\mathcal{I}|-1)\times(|\mathcal{I}|-1)$ block are referred to as the 'massive

components', which can be easily integrated out in one step, see Section 2.3.2 below (this is the main novel contribution of this section, compared with the multiscale analysis in [**2**]). In Section 2.3.3 below we reduce essentially to the setting of [**2**], that is, to an effective theory that involves one single-component "quasi-particle" chiral massless field, which can be analyzed along the same lines as [**2**, Section 6]. Finally, in Section 2.3.4 we conclude the proof of Proposition 2.2.1.

2.3.1 Integration of the ultraviolet degrees of freedom

We intend to compute the generating function (2.1.45) with θ boundary conditions. We introduce Grassmann variables in Fourier space via the following transformation:

$$\hat{\psi}^{\pm}_k := \sum_{x\in\Lambda} e^{\mp ikx}\psi^{\pm}_x, \qquad \psi^{\pm}_x = \frac{1}{L^2}\sum_{k\in\mathcal{P}(\theta)} e^{\pm ikx}\hat{\psi}^{\pm}_k, \tag{2.3.1}$$

where we recall that each $\psi^{\pm}_x$ and each $\hat{\psi}^{\pm}_k$ has $|\mathcal{I}|$ components and indeed we assume that $\psi^{+}_x = (\psi^{+}_{x,1},\dots,\psi^{+}_{x,|\mathcal{I}|})$ is a row vector while similarly ψ^{-}_x is a column vector (whenever unnecessary, we shall drop the 'color' index $\ell\in\mathcal{I}$); in this way the transformation above is performed component-wise.

For each $\theta\in\{-1,+1\}^2$, we let p^{ω}_θ, $\omega=\pm1$ denote the element of $\mathcal{P}(\theta)$ that is closest to p^{ω}_0[9], we rewrite

$$\psi^{\pm}_x = \psi'^{\pm}_x + \Psi^{\pm}_x$$

with

$$\psi'^{\pm}_x = \frac{1}{L^2}\sum_{k\notin\{p^+_\theta,p^-_\theta\}} e^{\pm ikx}\hat{\psi}^{\pm}_k, \quad \Psi^{\pm}_x = \frac{1}{L^2}\sum_{k\in\{p^+_\theta,p^-_\theta\}} e^{\pm ikx}\hat{\psi}^{\pm}_k. \tag{2.3.2}$$

Noting that

$$S_\theta(\psi) = S_\theta(\Psi) + S_\theta(\psi') := -\frac{1}{L^2}\sum_{k\in\{p^+_\theta,p^-_\theta\}} \hat{\psi}^{+}_k M(k)\hat{\psi}^{-}_k - \frac{1}{L^2}\sum_{k\notin\{p^+_\theta,p^-_\theta\}} \hat{\psi}'^{+}_k M(k)\hat{\psi}'^{-}_k,$$

we rewrite

$$e^{\mathcal{W}^{(\theta)}_L(A,\phi)} = \Big(\prod_{k\notin\{p^+_\theta,p^-_\theta\}}\mu(k)\Big)\int D\Psi\, e^{S_\theta(\Psi)}\int P(D\psi')\, e^{V(\psi,A)+(\psi,\phi)}, \tag{2.3.3}$$

where $D\Psi = \prod_{k\in\{p^+_\theta,p^-_\theta\}}\big(L^{2|\mathcal{I}|}D\hat{\Psi}_k\big)$ and the Grassmann "measure" $D\hat{\Psi}_k$ is defined, as usual, so that

$$\int\Big(\prod_{k\in\{p^+_\theta,p^-_\theta\}} D\hat{\Psi}_k\Big)\Big(\prod_{k\in\{p^+_\theta,p^-_\theta\}}\prod_{\ell\in\mathcal{I}}\hat{\Psi}^{-}_{k,\ell}\hat{\Psi}^{+}_{k,\ell}\Big) = 1,$$

while we have $\int\Big(\prod_{k\in I} D\hat{\Psi}_k\Big)Q(\Psi) = 0$ whenever $Q(\Psi)$ is a monomial in $\{\hat{\Psi}^{\pm}_{k,\ell}\}_{k\in\{p^+_\theta,p^-_\theta\},\ell\in\mathcal{I}}$ of degree strictly lower or strictly larger than $4|\mathcal{I}|$. Moreover, $P(D\psi')$ is the Grassmann

[9] In the case of more than one momentum at minimum distance, any choice of $p^{\pm}_\theta$ will work. The dependence on L of $p^{\pm}_\theta$ is understood.

Gaussian integration, normalized so that $\int P(D\psi') = 1$, associated with the propagator

$$g'(x,y) = \int P(D\psi')\psi'^{-}_x \psi'^{+}_y = L^{-2} \sum_{k \notin \{p_\theta^+, p_\theta^-\}} e^{-ik(x-y)} (M(k))^{-1}. \tag{2.3.4}$$

Note that, since $\psi'^{\pm}_x$ is a vector with $|\mathcal{I}|$ components, $g'(x,y)$ is an $|\mathcal{I}| \times |\mathcal{I}|$ matrix, for fixed x, y.

Remark 2.3.1. *We emphasize also that, since the zeros of μ are simple, $\mu(k) \neq 0$ for every $k \notin \{p_\theta^+, p_\theta^-\}$ (this is the reason why we singled out the two momenta p_θ^ω where μ possibly vanishes and M is not invertible).*

Next we introduce the following

Definition 2.3.1. We let $\chi_\omega : \mathbb{R}^2 \longrightarrow [0,1], \omega = \pm 1$ be two C^∞ functions in the Gevrey class of order 2, see [**90**, App.C], with the properties that:

(i) $\chi_\omega(k) = \chi_{-\omega}(-k)$,

(ii) $\chi_\omega(k) = 1$ if $|k - p_0^\omega| \leq c_0/2$, and $\chi_\omega(k) = 1$ if $|k - p_0^\omega| > c_0$, with c_0 a small enough positive constant, such that in particular the support of χ_+ is disjoint from the support of χ_-.

We will specify later a more explicit definition of χ_ω. We rewrite $g' = g^{(0)} + g^{(1)}$, with

$$\begin{aligned} g^{(0)}(x,y) &= L^{-2} \sum_{\omega = \pm} \sum_{k \notin \{p_\theta^+, p_\theta^-\}} e^{-ik(x-y)} \chi_\omega(k) (M(k))^{-1}, \\ g^{(1)}(x,y) &= L^{-2} \sum_{k \in \mathcal{P}(\theta)} e^{-ik(x-y)} (1 - \chi_+(k) - \chi_-(k)) (M(k))^{-1}. \end{aligned} \tag{2.3.5}$$

Since the cutoff functions χ_ω are Gevrey functions of order 2, the propagator $g^{(1)}$ has stretched-exponential decay at large distances

$$\| g^{(1)}(x,y) \| \leq C e^{-\kappa \sqrt{|x-y|}}, \tag{2.3.6}$$

for suitable L-independent constants $C, \kappa > 0$, cf. with [**2**, Eq. (6.21)] $|\mathcal{I}| \times |\mathcal{I}|$ (recall that the propagators are $|\mathcal{I}| \times |\mathcal{I}|$ matrices; the norm in the l.h.s. is any matrix norm). In (2.3.6), $|x-y|$ denotes the graph distance between x and y on G_L.

Using the addition principle for Grassmann Gaussian integrations [**90**, Proposition 1], we rewrite (2.3.3) as

$$\begin{aligned} e^{\mathcal{W}_L^{(\theta)}(A,\phi)} &= \Big(\prod_{k \notin \{p_\theta^+, p_\theta^-\}} \mu(k) \Big) \int D\Psi \, e^{S_\theta(\Psi)} \int P_{(0)}(D\psi^{(0)}) \\ &\qquad \times \int P_{(1)}(D\psi^{(1)}) \, e^{V(\Psi + \psi^{(0)} + \psi^{(1)}, A) + (\Psi + \psi^{(0)} + \psi^{(1)}, \phi)} \\ &= \Big(\prod_{k \notin \{p_\theta^+, p_\theta^-\}} \mu(k) \Big) e^{L^2 E^{(0)} + S^{(0)}(J,\phi)} \int D\Psi \, e^{S_\theta(\Psi)} \int P_{(0)}(D\psi^{(0)}) \, e^{V^{(0)}(\Psi + \psi^{(0)}, J, \phi)} \end{aligned} \tag{2.3.7}$$

where: $P_{(0)}$ and $P_{(1)}$ are the Grassmann Gaussian integrations with propagators $g^{(0)}$ and $g^{(1)}$, respectively, i.e., letting $O_\omega = \{k \in \mathcal{P}(\theta) \setminus \{p_\theta^+, p_\theta^-\} : \chi_\omega(k) \neq 0$,

$$P_{(0)}(D\psi) = \prod_\omega \frac{\Big(L^{2|\mathcal{I}||O_\omega|} \prod_{k\in O_\omega} D\hat\psi_k\Big) \exp\Big(- L^{-2} \sum_{k\in O_\omega} (\chi_\omega(k))^{-1} \hat\psi_k^+ M(k) \hat\psi_k^-\Big)}{\Big(\prod_{k\in O_\omega} \mu(k) (\chi_\omega(k))^{-|\mathcal{I}|}\Big)}, \tag{2.3.8}$$

and a similar explicit expression for $P_{(1)}$ holds; $J = \{J_e\}_{e\in E_L}$ with $J_e = e^{A_e} - 1$; $E^{(0)}$, $S^{(0)}$ and $V^{(0)}$ are defined via

$$L^2 E^{(0)} + S^{(0)}(J,\phi) + V^{(0)}(\psi, J, \phi) = \log \int P_{(1)}(D\psi^{(1)}) e^{V(\psi+\psi^{(1)}, A) + (\psi+\psi^{(1)}, \phi)}, \tag{2.3.9}$$

with $E^{(0)}, S^{(0)}$ fixed uniquely by the condition that $V^{(0)}(0, J, \phi) = S^{(0)}(0,0) = 0$. Proceeding as in the proof of [**2**, Eq.(6.24)], one finds that the effective potential $V^{(0)}$ can be represented as follows:

$$V^{(0)}(\psi, J, \phi) = \sum_{\substack{n>0\\ m,q\geq 0\\ n+q\in 2\mathbb{N}}} {\sum_{\substack{\underline{x},\underline{y},\underline{z}\\ \underline{\ell},\underline{\ell}',\underline{\ell}',\\ \underline{s},\underline{\sigma},\underline{\sigma}'}}}^{*} \psi^{\underline{\sigma}}_{\underline{x},\underline{\ell}} J_{\underline{y},\underline{\ell}',\underline{s}} \phi^{\underline{\sigma}'}_{\underline{z},\underline{\ell}''} W_{n,m,q;\boldsymbol{a}}(\underline{x},\underline{y},\underline{z}) \tag{2.3.10}$$

where the second sum runs over $\underline{x} \in \Lambda^n, \underline{y} \in \Lambda^m, \underline{z} \in \Lambda^q$, $\underline{\ell} \in \mathcal{I}^n, \underline{\ell}' \in \mathcal{I}^m, \underline{\ell}'' \in \mathcal{I}^q$, $\underline{s} \in \mathcal{J}_{\ell_1} \times \cdots \times \mathcal{J}_{\ell_m}$, $\underline{\sigma} \in \{+,-\}^n, \underline{\sigma}' \in \{+,-\}^q$ (the $*$ on the sum indicates the constraint that $\sum_{i=1}^n \sigma_i + \sum_{i=1}^q \sigma_i' = 0$), and we defined $J_{\underline{y},\underline{\ell}',\underline{s}} := \prod_{i=1}^m J_{y_i,\ell_i',s_i}$ (here $J_{y,\ell,s}$ stands for J_e when the edge $e \in E_L$ has black site of coordinates (y,ℓ) and orientation $s \in \mathcal{J}_\ell$), $\psi^{\underline{\sigma}}_{\underline{x},\underline{\ell}} := \prod_{i=1}^n \psi^{\sigma_i}_{x_i,\ell_i}$, and similarly for $\phi^{\underline{\sigma}'}_{\underline{z},\underline{\ell}''}$; finally, $\boldsymbol{a} := (\underline{\ell},\underline{\sigma},\underline{\ell}',\underline{s},\underline{\ell}'',\underline{\sigma}')$. Without loss of generality, we can assume that the kernels $W_{n,m,q;(\underline{\ell},\underline{\sigma},\underline{\ell}',\underline{s},\underline{\ell}'',\underline{\sigma}')}$ are symmetric under permutations of the indices $(\underline{y},\underline{\ell}',\underline{s})$ and antisymmetric both under permutations of $(\underline{x},\underline{\ell},\underline{\sigma})$ and of $(\underline{z},\underline{\ell}'',\underline{\sigma}')$. A representation similar to (2.3.10) holds also for $S^{(0)}(J,\varphi)$ with kernels $W^{0,m,q}_{\boldsymbol{a}}(\underline{y},\underline{z})$, where $\boldsymbol{a} = (\underline{\ell}',\underline{s},\underline{\ell}'',\underline{\sigma}')$. As discussed after [**2**, Eq.(6.27)], using the Battle-Brydges-Federbush-Kennedy determinant formula and the Gram-Hadamard bound [**5**, Sec. 4.2] one finds that $E^{(0)}$ and the values of the kernels $W_{n,m,q;\boldsymbol{a}}(\underline{x},\underline{y},\underline{z})$ at fixed positions $\underline{x},\underline{y},\underline{z}$ are real analytic functions of the parameter λ, for $|\lambda| \leq \lambda_0$ and λ_0 sufficiently small but independent of L. Moreover, in the analyticity domain, $|E^{(0)}| \leq C|\lambda|$, and

$$\|W_{n,m,q}\|_{\kappa,0} \leq C^{n+m+q} |\lambda|^{\mathbb{1}_{n+q>2} \max\{1, c(n+q)\}} \tag{2.3.11}$$

for suitable positive constants C, c independent of L. Here the weighted norm $\|\cdot\|_{\kappa,0}$ is defined as

$$\|W_{n,m,q}\|_{\kappa,0} := L^{-2} \sup_{\boldsymbol{a}} \sum_{\underline{x},\underline{y},\underline{z}} |W_{n,m,q;\boldsymbol{a}}(\underline{x},\underline{y},\underline{z})| e^{\frac{\kappa}{2}\sqrt{\delta(\underline{x},\underline{y},\underline{z})}}, \tag{2.3.12}$$

where $\kappa > 0$ is the same as in (2.3.6), and $\delta(\cdot)$ denotes the tree distance, that is the length of the shortest tree on the torus connecting points with the given coordinates.

Remark 2.3.2. *The kernels of the effective potential $V^{(0)}$, of $S^{(0)}$, as well as the constant $E^{(0)}$, depend on θ, because both the interaction $V(\psi, A)$ in* (2.3.9) *and the propagator $g^{(1)}$ involved in the integration do. Both these effects can be thought of as being associated with boundary conditions assigned to the Grassmann fields, periodic in both coordinate*

*directions for $\theta = (-,-)$, anti-periodic in both coordinate directions for $\theta = (+,+)$, and mixed (periodic in one direction and anti-periodic in the other) in the remaining two cases. Therefore, using Poisson summation formula (see e.g. [**90**, App. A.2], where notations are different), both $g^{(1)}$ and the kernels of $V^{(0)}$ and $S^{(0)}$ can be expressed via an 'image rule', analogous to the summation over images in electrostatics, of the following form:*

$$g^{(1)}(x,y) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} (-1)^{\frac{\theta_1+1}{2}n_1+\frac{\theta_2+1}{2}n_2} g^{(1),\infty}(x-y+nL), \tag{2.3.13}$$

*where $g^{(1),\infty}(x) = \lim_{L\to\infty} g^{(1)}(x,0)$ (an analogous sum rule holds for the kernels of $V^{(0)}$ and $S^{(0)}$). From this representation, together with the decay bounds mentioned above on $g^{(1)}$ and on the kernels of the effective potential, it readily follows that the dependence upon θ of these functions is a finite-size effect that is stretched-exponentially small in L. Similarly, the dependence upon θ of $E^{(0)}$ corresponds to a stretched-exponentially small correction as $L\to\infty$ (see also [**90**, Appendix A.2]). Therefore, all these corrections are irrelevant for the purpose of computing the thermodynamic limit of thermodynamic functions and correlations. For this reason and for ease of notation, here and below we will not indicate the dependence upon θ explicitly in most of the functions and constants involved in the multiscale construction.*

2.3.2 Integration of the massive degrees of freedom

Using (2.3.8) in (2.3.7) and renaming $\Psi + \psi^{(0)} \equiv \psi$, we get

$$e^{\mathcal{W}_L^{(\theta)}(A,\phi)} = e^{L^2(t^{(0)}+E^{(0)})+S^{(0)}(J,\phi)} \times \int D\psi\, e^{-L^{-2}\sum_\omega \sum_{k\in\mathcal{B}_\omega} (\chi_\omega(k))^{-1}\hat\psi^+_k M(k)\hat\psi^-_k}\, e^{V^{(0)}(\psi,J,\phi)} \tag{2.3.14}$$

where, recalling that O_ω was defined right before (2.3.8),

$$\mathcal{B}_\omega := O_\omega \cup \{p^\omega_\theta\} = \{k\in\mathcal{P}(\theta) : \chi_\omega(k)\neq 0\},$$

$D\psi := \prod_{\omega=\pm}\left(L^{2|\mathcal{I}||\mathcal{B}_\omega|}\prod_{k\in\mathcal{B}_\omega} D\hat\psi_k\right)$ and we have set

$$t^{(0)} := \frac{1}{L^2}\sum_{k\in(\cup_\omega\mathcal{B}_\omega)^c}\log\mu(k) + \frac{|\mathcal{I}|}{L^2}\sum_\omega\sum_{k\in O_\omega}\log\chi_\omega(k).$$

Since p_0^+ is a simple zero of $\mu(k)$, there exists an invertible complex matrix B_+ such that

$$B_+ M(p_0^+) B_+^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & A_+ \end{pmatrix} \tag{2.3.15}$$

for an invertible $(|\mathcal{I}|-1)\times(|\mathcal{I}|-1)$ matrix A_+. Clearly, B_+ (and, therefore, A_+) is not defined uniquely; we choose it arbitrarily, in such a way that (2.3.15) holds, and fix it once and for all. Taking the complex conjugate in the above equation and using the symmetry of M, see (2.1.13), one finds that the same relation holds at p_0^- with matrices $B_- := \overline{B_+}$, $A_- := \overline{A_+}$. Let $\mathbf{M}_\omega(k) := B_\omega M(k) B_\omega^{-1}$, and define the matrices $T_\omega(k), W_\omega(k), U_\omega(k)$ and

$V_\omega(k)$ of sizes 1×1, $(|\mathcal{I}|-1)\times(|\mathcal{I}|-1)$, $1\times(|\mathcal{I}|-1)$ and $(|\mathcal{I}|-1)\times 1$, respectively, via

$$\begin{pmatrix} T_\omega(k) & U_\omega(k) \\ V_\omega(k) & W_\omega(k)\end{pmatrix} := \mathbf{M}_\omega(k). \tag{2.3.16}$$

Analyticity of $M(k)$ in k implies, in particular, that $T_\omega(k+p_0^\omega)$, $U_\omega(k+p_0)$ and $V_\omega(k+p_0^\omega)$ are all $O(k)$ as $k\to 0$, while $W_\omega(k+p_0^\omega)=A_\omega+O(k)$. Let $\mathcal{B}_\omega^{(2)}\supset\mathcal{B}_\omega$ be the ball centered at p_0^ω with radius $2c_0$, and assume that c_0 is so small that $\inf_{k\in\mathcal{B}_\omega^{(2)}}|\det W_\omega(k)|$ is positive. Taking the determinant at both sides of (2.3.16), letting $\rho_\omega := \det A_\omega$, we find that

$$\mu(k) \overset{k\to p_0^\omega}{=} \rho_\omega T_\omega(k) + O((k-p_0^\omega)^2)$$

so that, recalling (2.1.11),

$$T_\omega(k+p_0^\omega) \overset{k\to 0}{=} \frac{\alpha_\omega^0 k_1+\beta_\omega^0 k_2}{\rho_\omega} + O(k^2). \tag{2.3.17}$$

Since $W_\omega(k)$ is non singular on $\mathcal{B}_\omega$, for $k\in\mathcal{B}_\omega$ we can block diagonalize $\mathbf{M}_\omega$ as

$$\mathbf{M}_\omega(k) = \begin{pmatrix} 1 & U_\omega(k)W_\omega^{-1}(k) \\ 0 & \mathbb{1}\end{pmatrix}\begin{pmatrix} \mathbf{T}_\omega(k) & 0 \\ 0 & W_\omega(k)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ W_\omega^{-1}(k)V_\omega(k) & \mathbb{1}\end{pmatrix} \tag{2.3.18}$$

where $\mathbf{T}_\omega(k) := T_\omega(k) - U_\omega(k)W_\omega^{-1}(k)V_\omega(k)$ is the Schur complement of the block W_ω. Note that from the properties of $U_\omega, V_\omega, W_\omega$, the function $\mathbf{T}_\omega$ satisfies

$$\mathbf{T}_\omega(k+p_0^\omega) \overset{k\to 0}{=} \frac{\alpha_\omega^0 k_1+\beta_\omega^0 k_2}{\rho_\omega} + O(k^2),$$

like T_ω. In view of this decomposition, we perform the following change of Grassmann variables: for $k\in\mathcal{B}_\omega$ we define

$$\begin{aligned} (\hat\varphi_k^+,\hat\zeta_{k,1}^+,\dots,\hat\zeta_{k,|\mathcal{I}|-1}^+) &:= \hat\psi_k^+ B_\omega^{-1}\begin{pmatrix} 1 & U_\omega(k)W_\omega^{-1}(k) \\ 0 & \mathbb{1}\end{pmatrix} \\ (\hat\varphi_k^-,\hat\zeta_{k,1}^-,\dots,\hat\zeta_{k,|\mathcal{I}|-1}^-)^T &:= \begin{pmatrix} 1 & 0 \\ W_\omega^{-1}(k)V_\omega(k) & \mathbb{1}\end{pmatrix} B_\omega\hat\psi_k^-. \end{aligned} \tag{2.3.19}$$

For later convenience, we give the following

Lemma 2.3.0.1. *Define* $\psi_x^\pm := L^{-2}\sum_\omega\sum_{k\in\mathcal{B}_\omega} e^{\pm ikx}\hat\psi_k^+$ *and*

$$\zeta_x^\pm(\omega) := L^{-2}\sum_{k\in\mathcal{B}_\omega} e^{\pm ikx}\hat\zeta_k^+, \quad \varphi_x^\pm(\omega) := L^{-2}\sum_{k\in\mathcal{B}_\omega} e^{\pm ikx}\hat\varphi_k^+.$$

Then, the inverse of the transformation (2.3.19) *in* x *space is*

$$\begin{aligned} \psi_{x,\ell}^+ &= \sum_\omega\Big(\varphi_x^+(\omega)(B_\omega)_{1\ell} + (\varphi^+(\omega)*\tau_{\omega,\ell}^+)_x + \sum_{j=2}^{|\mathcal{I}|}\zeta_{x,j-1}^+(\omega)(B_\omega)_{j\ell}\Big) \\ \psi_{x,\ell}^- &= \sum_\omega\Big((B_\omega^{-1})_{\ell 1}\varphi_x^-(\omega) + (\tau_{\omega,\ell}^-*\varphi^-(\omega))_x + \sum_{j=2}^{|\mathcal{I}|}(B_\omega^{-1})_{\ell j}\zeta_{x,j-1}^-(\omega)\Big) \end{aligned} \tag{2.3.20}$$

where

$$\begin{aligned}\tau^+_{\omega,\ell}(x) &:= -L^{-2}\sum_{k\in\mathcal{P}(\theta)}\sum_{j=2}^{|\mathcal{I}|} e^{ikx}\chi_\omega\big(\tfrac{k+p_0^\omega}{2}\big)(U_\omega(k)\cdot W_\omega^{-1}(k))_j(B_\omega)_{j\ell}\\ \tau^-_{\omega,\ell}(x) &:= -L^{-2}\sum_{k\in\mathcal{P}(\theta)}\sum_{j=2}^{|\mathcal{I}|} e^{-ikx}\chi_\omega\big(\tfrac{k+p_0^\omega}{2}\big)(B_\omega^{-1})_{\ell j}(W_\omega^{-1}(k)\cdot V_\omega(k))_j.\end{aligned} \tag{2.3.21}$$

Proof. The proof is essentially an elementary computation (one inverts the linear relation (2.3.19) for given k and then takes the Fourier transform to obtain the expression in real space) but there is a slightly delicate point, that is to see where the cut-off function $\chi_\omega(\frac{k+p_0^\omega}{2})$ comes from. After a few elementary linear algebra manipulations, one finds that $\psi^+_{x,\ell}$ equals an expression like in the r.h.s. of (2.3.20), where the term $(\varphi^+(\omega)*\tau^+_{\omega,\ell})_x$ is replaced by

$$\frac{1}{L^2}\sum_\omega\sum_{k\in\mathcal{B}_\omega}\hat\varphi^+_k f^\omega_\ell(k)e^{ikx},\quad f^\omega_\ell(k) := -\sum_{j=2}^{|\mathcal{I}|}\big(U_\omega(k)W_\omega^{-1}(k)\big)_j(B_\omega)_{j\ell}.$$

Since the sum is restricted to $k\in\mathcal{B}_\omega$, we can freely multiply the summand by $\chi_\omega(\frac{k+p_0^\omega}{2})$, which is identically equal 1 there, since the argument is at distance at most $c_0/2$ from p_0^ω. At that point, we use the fact that

$$\hat\varphi^+_k\mathbf{1}_{k\in\mathcal{B}_\omega} = \sum_x\varphi^+_x(\omega)e^{-ikx}$$

and we immediately obtain that (2.3.2) coincides with $(\varphi^+(\omega)*\tau^+_{\omega,\ell})_x$, with $\tau^+_{\omega,\ell}$ as in (2.3.21). ∎

At this point we go back to (2.3.14), that we rewrite as

$$\begin{aligned}e^{\mathcal{W}_L^{(\theta)}(A,\phi)} &= e^{L^2(\mathbf{t}^{(0)}+E^{(0)})+S^{(0)}(J,\phi)}\\ &\times\int D\varphi\, e^{-L^{-2}\sum_\omega\sum_{k\in\mathcal{B}_\omega}(\chi_\omega(k))^{-1}\hat\varphi^+_k\mathbf{T}_\omega(k)\hat\varphi^-_k}\int P_W(D\zeta)\, e^{\widetilde V^{(0)}(\varphi,\zeta,J,\phi)}\end{aligned} \tag{2.3.22}$$

where

$$\mathbf{t}^{(0)} := L^{-2}\sum_{k\in O'}\log\mu(k) + L^{-2}\sum_{\omega=\pm}\sum_{k\in\mathcal{B}_\omega}\big(\log\det W_\omega(k)+\log\chi_\omega(k)\big),$$

$D\varphi := \prod_{\omega=\pm}\Big(L^{2|\mathcal{B}_\omega|}\prod_{k\in\mathcal{B}_\omega}D\hat\varphi_k\Big)$, $P_W(D\zeta)$ is the normalized Gaussian Grassmann integration with propagator (which is a $(|\mathcal{I}|-1)\times(|\mathcal{I}|-1)$ matrix)

$$g^W_{\omega,\omega'}(x,y) := \int P(D\zeta)\,\zeta^-_x(\omega)\zeta^+_y(\omega') = \frac{\delta_{\omega,\omega'}}{L^2}\sum_{k\in P(\theta)}e^{-ik(x-y)}\chi_\omega(k)(W_\omega(k))^{-1}, \tag{2.3.23}$$

and $\widetilde V^{(0)}(\varphi,\zeta,J,\phi)$ is the same as $V^{(0)}(\psi,J,\phi)$, once ψ is re-expressed in terms of the new variables (φ,ζ), as in Lemma 2.3.0.1.

Remark 2.3.3. *Note that, because of $\chi(\cdot)$, the sums* (2.3.21) *defining $\tau^\pm_{\omega,\ell}(x)$ are restricted to momenta $k\in\mathcal{B}^{(2)}_\omega$ where W_ω is indeed invertible. Note also that, from the smoothness*

of $\hat\tau^{\pm}_{\omega,\ell}(k)$ it follows that $\tau^{\pm}_{\omega,\ell}(x)$ decays to zero in a stretched-exponential way, similar to (2.3.6). *That is, ψ is essentially a local function of φ,ζ. As a consequence, the kernels of $\widetilde V^{(0)}$ satisfy qualitatively the same bounds as those of $V^{(0)}$.*

Since $W_\omega(\cdot)$ is smooth and invertible in the support of χ_ω, we see from (2.3.23) that the propagator of the variables $\{\zeta_x(\omega)\}$ decays as

$$\|g^W(x,y)\| \le Ce^{-\kappa\sqrt{|x-y|}}$$

uniformly in L, a behavior analogous to (2.3.6). For this reason, we call the variables $\{\zeta_x(\omega)\}$ *massive*. On the other hand, we call *critical* the remaining $\{\varphi_x(\omega)\}$ variables.

The integration of the massive fields ζ, which is performed in a way completely analogous to the one of $\psi^{(1)}$ in (2.3.9), produces an expression for the generating functional in terms of a Grassmann integral involving only the critical fields φ:

$$e^{\mathcal{W}^{(\theta)}_L(A,\phi)} = e^{L^2E^{(-1)}+S^{(-1)}(J,\phi)}\int D\varphi e^{-L^{-2}\sum_\omega\sum_{k\in\mathcal{B}_\omega}\hat\varphi^+_k(\chi_\omega(k))^{-1}\mathbf{T}_\omega(k)\hat\varphi^-_k}e^{V^{(-1)}(\varphi,J,\phi)} \tag{2.3.24}$$

where

$$L^2(E^{(-1)}-\mathbf{t}^{(0)}-E^{(0)})+S^{(-1)}(J,\phi)-S^{(0)}(J,\phi)+V^{(-1)}(\varphi,J,\phi) = \log\int P_W(D\zeta)e^{\widetilde V^{(0)}(\varphi,\zeta,J,\phi)} \tag{2.3.25}$$

and $V^{(-1)}, S^{(-1)}$ are fixed in such a way that $V^{(-1)}(0,J,\phi)=S^{(-1)}(0,0)=0$. The effective potential $V^{(-1)}$ can be represented in a way similar to (2.3.10), namely

$$V^{(-1)}(\varphi,J,\phi)=\sum_{\substack{n>0\\ m,q\ge0\\ n+q\in2\mathbb{N}}}{\sum_{\substack{\underline{x},\underline{y},\underline{z}\\ \underline{\ell}',\underline{\ell}'',\underline{\omega}\\ \underline{s},\underline{\sigma},\underline{\sigma}'}}}^{*}\varphi^{\underline\sigma}_{\underline x}(\underline\omega)J_{\underline{y},\underline{\ell}',\underline{s}}\phi^{\underline{\sigma}'}_{\underline{z},\underline{\ell}''}W^{(-1)}_{n,m,q;\boldsymbol{a}}(\underline x,\underline y,\underline z;\underline\omega) \tag{2.3.26}$$

where $\underline\omega\in\{-1,+1\}^n$ and $\varphi^{\underline\sigma}_{\underline x}(\underline\omega):=\prod_{i=1}^n\varphi^{\sigma_i}_{x_i}(\omega_i)$, while the other symbols and labels have the same meaning as in (2.3.10). In virtue of the decay properties of the propagator $g^W_{\omega,\omega'}$, the kernels $W^{(-1)}_{n,m,q;\boldsymbol{a}}(\underline x,\underline y,\underline z;\underline\omega)$ of $V^{(-1)}(\varphi,J,\phi)$ satisfy the same bounds as (2.3.11).

2.3.3 Reduction to the setting of [2]

We are left with the integral of the critical variables, which we want to perform in a way analogous to that discussed in [**2**, Section 6]. In order to get to a point where we can literally apply the results of [**2**], a couple of extra steps are needed. First, in order to take into account the fact that, in general, the interaction has the effect of changing the location of the singularity in momentum space of the propagator of φ, as well as the value of the residues at the singularity, we find it convenient to rewrite the 'Grassmann action' in (2.3.24),

$$-L^{-2}\sum_\omega\sum_{k\in\mathcal{B}_\omega}\hat\varphi^+_k(\chi_\omega(k))^{-1}\mathbf{T}_\omega(k)\hat\varphi^-_k+V^{(-1)}(\varphi,J,\phi),$$

in the form of a reference quadratic part, with the 'right' singularity structure, plus a remainder, whose specific value will be fixed a posteriori via a fixed-point argument. More

precisely, we proceed as described in [**2**, Section 6.1]: we introduce

$$N(\varphi) = L^{-2} \sum_{\omega} \sum_{k \in \mathcal{B}_\omega} \hat{\varphi}^+_k (-\mathbf{T}_\omega(p^\omega) + a_\omega(k_1 - p^\omega_1) + b_\omega(k_2 - p^\omega_2)) \hat{\varphi}^-_k \tag{2.3.27}$$

where $p^\omega, a_\omega, b_\omega$ will be fixed a posteriori, and are assumed to satisfy

$$|p^\omega - p^\omega_0| \ll 1 \tag{2.3.28}$$

for λ small, $p^+ = -p^-$ and $\overline{a_+} = -a_-, \overline{b_+} = -b_-$. Define also

$$C_\omega(k) := \mathbf{T}_\omega(k) - \chi_\omega(k)\Big(\mathbf{T}_\omega(p^\omega) - a_\omega(k_1 - p^\omega_1) - b_\omega(k_2 - p^\omega_2)\Big) \tag{2.3.29}$$

and note that it satisfies $C_\omega(p^\omega) = 0$,

$$\partial_{k_1} C_\omega(p^\omega) = \partial_{k_1} \mathbf{T}_\omega(p^\omega) + a_\omega =: \alpha_\omega, \qquad \partial_{k_2} C_\omega(p^\omega) = \partial_{k_2} \mathbf{T}_\omega(p^\omega) + b_\omega =: \beta_\omega, \tag{2.3.30}$$

as well as the symmetry $C_{-\omega}(-k) = \overline{C_\omega(k)}$.

Let us introduce the matrix $\mathcal{M}$ (the same as in [**2**, Eq.(4.1)]) given by

$$\mathcal{M} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} \beta^1 & \beta^2 \\ -\alpha^1 & -\alpha^2 \end{pmatrix} \tag{2.3.31}$$

where α^1 and α^2 (resp. β^1 and β^2) are, respectively, the real and imaginary part of α_+ (resp. β_+), see (2.3.30), and $\Delta := \alpha^1\beta^2 - \alpha^2\beta^2$ is a positive real number, in agreement with (2.1.16): note, in fact, that at $\lambda = 0$ the sign of Δ is the same as the sign of $\mathrm{Im}(\beta_+/\alpha_+)$. At this point, we can finally fix the cut-off functions χ_ω of Definition 2.3.1 as follows:

$$\chi_\omega(k) := \chi(|\mathcal{M}^{-1}(k - p^\omega)|)$$

where $\chi : \mathbb{R} \mapsto [0, 1]$ is a compactly supported function in the Gevrey class of order 2. It is immediate to verify that χ can be chosen so that that properties (i)-(ii) of Definition 2.3.1 are verified.

Given this, we rewrite (2.3.24) as

$$e^{\mathcal{W}^{(\theta)}_L(A,\phi)} = e^{L^2 E^{(-1)} + S^{(-1)}(J,\phi)} \int D\varphi e^{-L^{-2} \sum_\omega \sum_{k \in \mathcal{B}_\omega} \hat{\varphi}^+_k (\chi_\omega(k))^{-1} C_\omega(k) \hat{\varphi}^-_k} e^{N(\varphi) + V^{(-1)}(\varphi, J, \phi)}. \tag{2.3.32}$$

In the above integration, the momenta closest to the zeros of C_ω (i.e., close to p^ω) play a special role and have to be treated at the end of the multiscale procedure, as discussed in [**2**, Section 6.5]. For a given $\theta \in \{-1, +1\}^2$, denote by $k^\pm_\theta \in \mathcal{B}_\omega$ the closest momenta to $p^\pm$ respectively (with the same remark as in footnote 9 in case of several possible choices) and note that they satisfy $k^+_\theta = -k^-_\theta$. Next we define $\hat{\Phi}^\pm_\omega := \varphi^\pm_{k^\omega_\theta}$, $\Phi^\pm_{\omega,x} := L^{-2} e^{\pm i k^\omega_\theta x} \hat{\Phi}^\pm_\omega$ and $\mathcal{P}'(\theta) := \mathcal{P}(\theta) \setminus \{k^\pm_\theta\}$. Since C_ω does not vanish on $\mathcal{P}'(\theta)$, we can rewrite (2.3.32) as

$$\begin{aligned} e^{\mathcal{W}^{(\theta)}_L(A,\phi)} &= e^{L^2 \mathbf{E}^{(-1)} + S^{(-1)}(J,\phi)} \\ &\times \int D\Phi e^{-L^{-2} \sum_\omega \Phi^+_\omega C_\omega(k^\omega_\theta) \Phi^-_\omega} \int \tilde{P}_{(\le -1)}(D\varphi) e^{N(\varphi,\Phi) + V^{(-1)}(\varphi,\Phi,J,\phi)} \end{aligned} \tag{2.3.33}$$

where, letting $\mathcal{B}'_\omega := \mathcal{B}_\omega \cap \mathcal{P}'(\theta)$,

$$\mathbf{E}^{(-1)} = E^{(-1)} + L^{-2} \sum_\omega \sum_{k \in \mathcal{B}'_\omega} (\log C_\omega(k) - \log \chi_\omega(k)).$$

Moreover, $D\Phi := L^4 D\hat{\Phi}_+ \, D\hat{\Phi}_-$ and $\tilde{P}_{(\le -1)}(D\varphi)$ is the normalized Grassmann Gaussian integration with propagator

$$\int \tilde{P}_{(\le -1)}(D\varphi) \varphi^-_{\omega,x} \varphi^+_{\omega',y} = \delta_{\omega,\omega'} \frac{1}{L^2} \sum_{k \in \mathcal{B}'_\omega} e^{-ik(x-y)} \chi_\omega(k) (C_\omega(k))^{-1}. \tag{2.3.34}$$

Finally, we remark that since the momenta k in (2.3.34) are close to p^ω, the propagator (2.3.34) has an oscillating prefactor $e^{-ip^\omega(x-y)}$ that it is convenient to extract. To this end, we define *quasi-particle* fields $\varphi^{\pm,(\le -1)}_{x,\omega}$ via

$$\varphi^\pm_x(\omega) =: e^{\pm ip^\omega x} \varphi^{\pm,(\le -1)}_{x,\omega}.$$

Note that the propagator of the quasi-particle fields equals

$$\begin{aligned} &\int P_{(\le -1)}(d\varphi^{\le -1}) \varphi^{-,(\le -1)}_{x,\omega} \varphi^{+,(\le -1)}_{y,\omega'} = \delta_{\omega,\omega'} g^{(\le -1)}_\omega(x,y) \\ &\qquad g^{(\le -1)}_\omega(x,y) := \frac{1}{L^2} \sum_{k \in \mathcal{P}'_\omega(\theta)} \frac{e^{-ik(x-y)} \chi(k + p^\omega - p^\omega_0) \chi(|\mathcal{M}^{-1} k|)}{C_\omega(k + p^\omega)}, \end{aligned} \tag{2.3.35}$$

where $\mathcal{P}'_\omega(\theta) = \{k : k + p^\omega \in \mathcal{P}'(\theta)\}$. Of course, the r.h.s. of (2.3.35) is just the r.h.s. of (2.3.34) multiplied by $e^{ip^\omega(x-y)}$. We now rewrite (2.3.33) as

$$\begin{aligned} &e^{\mathcal{W}^{(\theta)}_L(A,\phi)} \\ &= e^{L^2 \mathbf{E}^{(-1)} + S^{(-1)}(J,\phi)} \int D\Phi e^{-L^{-2} \sum_\omega \Phi^+_\omega C_\omega(k^\omega_\theta) \Phi^-_\omega} \int P_{(\le -1)}(D\varphi^{(\le -1)}) e^{\mathcal{V}^{(-1)}(\varphi^{(\le -1)}, \Phi, J, \phi)}, \end{aligned} \tag{2.3.36}$$

where

$$\mathcal{V}^{(-1)}(\varphi, \Phi, J, \phi) := N(\Phi, \varphi) + V^{(-1)}(\Phi, \varphi, J, \phi), \tag{2.3.37}$$

and in the r.h.s. it is meant that the φ variables are expressed in terms of the quasi-particle fields as in (2.3.3). That is, we have simply re-expressed $V^{(-1)}$ in terms of the quasi-particle fields and we included the counter-terms in the definition of effective potential. After this rewriting, we find that the following representation holds for $\mathcal{V}^{(-1)}$:

$$\mathcal{V}^{(-1)}(\varphi, J, \phi) = \sum_{\substack{n>0 \\ m,q \ge 0 \\ n+q \in 2\mathbb{N}}} \sum^*_{\substack{\underline{x},\underline{y},\underline{z} \\ \underline{\ell}',\underline{\ell}'',\underline{\omega} \\ \underline{s},\underline{\sigma},\underline{\sigma}'}} \varphi^{\underline{\sigma}}_{\underline{x},\underline{\omega}} J_{\underline{y},\underline{\ell}',\underline{s}} \phi^{\underline{\sigma}'}_{\underline{z},\underline{\ell}''} \mathcal{W}^{(-1)}_{n,m,q;\underline{\omega},\boldsymbol{a}}(\underline{x},\underline{y},\underline{z}), \tag{2.3.38}$$

with kernels $\mathcal{W}^{(-1)}_{n,m,q;\underline{\omega},\boldsymbol{a}}$ satisfying the same estimates as in (2.3.11). The kernels $\mathcal{W}^{(-1)}_{n,m,q;\underline{\omega},\boldsymbol{a}}$ are the analogues of $W^{(-1)}_{n,m;\underline{\omega},\underline{r}}$ in [**2**, Eq.(6.24)] and satisfy the same properties spelled in [**2**, Eq.(6.25)] and following lines. Here the labels $\boldsymbol{a}$ denote the collection of labels $(\underline{\sigma}, \underline{\ell}', \underline{s}, \underline{\ell}'', \underline{\sigma}')$.

At this point, we have reduced precisely to the fermionic model studied in [**2**, Sec. 6].

2.3.4 Infrared integration and conclusion of the proof of Proposition 2.2.1

Once the partition function is re-expressed as in (2.3.36), we are in the position of applying the multiscale analysis of [**2**, Section 6]: note in fact that (2.3.36) has exactly the same form as [**2**, Eq. (6.19)] with its second line written as in [**2**, Eq.(6.22)]. Therefore, at this point, we can integrate out the massless fluctuation field φ via the same iterative procedure described in [**2**, Section 6.2.1] and following sections. Such a procedure allows us to express the thermodynamic and correlation functions of the theory in terms of an appropriate sequence of *effective potentials* $\mathcal{V}^{(h)}$, $h < 0$. The discussion in [**2**, Section 6.4] implies that we can fix $p^\omega, a_\omega, b_\omega$ uniquely as appropriate *analytic* functions of λ, for λ sufficiently small (so that, in particular, (2.3.28) is satisfied), in such a way that the whole sequence of the effective potentials is well defined for λ sufficiently small, their kernels are analytic in λ uniformly in the system size, and they admit a limit as $L \to \infty$. In particular, the running coupling constants characterizing the local part of the effective potentials are analytic functions of λ and the associated critical exponents are analytic functions of λ, see [**2**, Sects.6.4.5 to 6.4.9]. The existence of the thermodynamic limit of correlation functions follows from [**2**, Section 6.5]. The proofs of (2.2.1), (2.2.2) and (2.2.3) in Proposition 2.2.1 follow from the discussion in [**2**, Section 6.6] (they are the analogues of [**2**, Eqs. (5.1),(5.2) and (5.3) in Proposition 2]) and this, together with the fact that (2.2.4) and (2.2.5) are just restatements of [**2**, Eq.(4.24)] and [**2**, Eq.(5.8)], respectively, concludes the proof of Proposition 2.2.1.

A noticeable, even though mostly aesthetic, difference between the statements of Proposition 2.2.1 and [**2**, Proposition 2] is in the labeling of the constants $K^{(1)}_{\omega,j,\ell}$ and $K^{(2)}_{\omega,j,\ell}$ in (2.2.1), as compared to those in [**2**, Eq.(5.1)], which are called there $\hat{K}_{\omega,r}$ and $\hat{H}_{\omega,r}$, and in the presence of the constants $I_{\omega,\ell,\ell'}$ in (2.2.2)-(2.2.3), which are absent in their analogues in [**2**, Eq.(5.2)-(5.3)]. This must be traced back to the different labeling of the sites and edges and, correspondingly, of the external fields ϕ and A, used in this paper, as compared to [**2**].

First of all, in this paper the edges and the external fields of type A are labelled (x, j, ℓ), with (j, ℓ) playing the same role as the index r in [**2**]; correspondingly, the analogues of the running coupling constants $Y_{h,r,(\omega_1,\omega_2)}$ defined in [**2**, Eq.(6.49)] should now be labelled $Y_{h,(j,\ell),(\omega_1,\omega_2)}$; by repeating the discussion in [**2**, Section 6.6] leading to [**2**, Eq.(6.160)], it is apparent that the analogues of the constants $\hat{K}_{\omega,r}, \hat{H}_{\omega,r}$ should now be labelled (ω, j, ℓ), as anticipated.

Concerning the constants $I_{\omega,\ell,\ell'}$, they come from the local part of the effective potentials in the presence of the external fields ϕ. After having integrated out the massive degrees of freedom, the infrared integration procedure involves at each step a splitting of the effective potential into a sum of its local part $\mathcal{L}\mathcal{V}^{(h)}$ and of its 'renormalized', or 'irrelevant', part $\mathcal{R}\mathcal{V}^{(h)}$, as discussed in [**2**, Section 6.2.3]. In [**2**, Section 6], for simplicity, we discussed the infrared integration only in the absence of external ϕ fields. In their presence, the definition of localization must be adapted accordingly. When acting on the ϕ-dependent

part of the effective potential, using a notation similar to [**2**, Eq.(6.37)], we let

$$\mathcal{L}\Big(\mathcal{V}^{(h)}(\varphi,J,\phi)-\mathcal{V}^{(h)}(\varphi,J,0)\Big)=\sum_{x\in\Lambda}\sum_{\omega,\ell}\Big(\varphi^{+}_{x,\omega}\phi^{-}_{x,\ell}e^{ip^{\omega}\cdot x}\hat{\mathcal{W}}^{(h),\infty}_{1,0,1;\omega,(+,\ell,-)}(0) + \varphi^{-}_{x,\omega}\phi^{+}_{x,\ell}e^{-ip^{\omega}\cdot x}\hat{\mathcal{W}}^{(h),\infty}_{1,0,1;\omega,(-,\ell,+)}(0)\Big). \tag{2.3.39}$$

Next, in analogy with [**2**, Eq.(6.49)], we let

$$I^{\pm}_{h,\omega,\ell}:=\frac{1}{\sqrt{Z_{h-1}}}\hat{W}^{(h),\infty}_{1,0,1;\omega,\ell,(\pm,\mp)}(0), \tag{2.3.40}$$

where Z_h is a real, scalar, function of λ, called the 'wave function renormalization', recursively defined as in [**2**, Eq.(6.45)]. Eq.(2.3.40) defines the running coupling constant (r.c.c.) associated with the external field ϕ. Note that such r.c.c. naturally inherit the label ℓ from the corresponding label of the external field ϕ. A straightforward generalization of the discussion in [**2**, Section 6.4] shows that $I^{\pm}_{h,\omega,\ell}$ are analytic in λ and converge as $h\to-\infty$ to finite constants $I^{\pm}_{-\infty,\omega,\ell}$, which are, again, analytic functions of λ. Therefore, by repeating the discussion in [**2**, Section 6.6] for $\hat{G}^{(2)}_{\ell,\ell'}(k+p^{\omega})$ and $\hat{G}^{(2,1)}_{j,\ell_0,\ell,\ell'}(k+p^{\omega},p)$, we find that the dominant asymptotic behavior of these correlations as $k,p\to0$ is proportional to $I^{+}_{-\infty,\omega,\ell}I^{-}_{-\infty,\omega,\ell'}$, times a function that is independent of ℓ,ℓ'. Building upon this, we obtain (2.2.2) and (2.2.3), with $I_{\omega,\ell,\ell'}$ proportional to $I^{+}_{-\infty,\omega,\ell}I^{-}_{-\infty,\omega,\ell'}$. Additional details are left to the reader.

2.4 Proof of Theorem 1.4.3

In order to prove Theorem 1.4.3 we proceed as in [**90**, Section 7.3]: using the fact that convergence of the moments of a random variable ζ_n to those of a Gaussian random variable ζ implies convergence in law of ζ_n to ζ, we reduce the proof of (1.4.7) to that of the following identities:

$$\begin{aligned}&\lim_{\epsilon\to0}\mathbb{E}_{\lambda}(h^{\epsilon}(f);h^{\epsilon}(f))=\frac{\nu(\lambda)}{2\pi^2}\int dx\int dy\, f(x)\,f(y)\,\Re[\log\phi_{+}(x-y)].\\&\lim_{\epsilon\to0}\mathbb{E}_{\lambda}(\underbrace{h^{\epsilon}(f);\cdots;h^{\epsilon}(f)}_{n\text{ times}})=0,\qquad n>2\end{aligned} \tag{2.4.1}$$

where the l.h.s. of the second line denotes the n^{th} cumulant of $h^{\epsilon}(f)$. The first equation is a straightforward corollary of Theorem 1.4.2, for additional details see [**90**, p.161, proof of (7.26)]. For the proof of the second equation we need to show that, for any $2n$-ple of distinct points $x_1,\dots,x_{2n}$,

$$\mathbb{E}_{\lambda}(h(\eta_{x_1})-h(\eta_{x_2});\cdots;h(\eta_{x_{2n-1}})-h(\eta_{x_{2n}}))=O((\min_{1\le i<j\le 2n}|x_i-x_j|)^{-\theta}), \tag{2.4.2}$$

for some constant $\theta>0$. In fact, by proceeding as in [**90**, p.162, Proof of (7.27)], Eq. (2.4.2) readily implies the second line of (2.4.1). In order to prove (2.4.2), we first expand each difference within the expectation in the left side as in (1.4.3), thus getting

$$\text{LHS of (2.4.2)}=\sum_{e_1\in C_{\eta_{x_1}\to\eta_{x_2}}}\cdots\sum_{e_n\in C_{\eta_{x_{2n-1}}\to\eta_{x_{2n}}}}\sigma_{e_1}\cdots\sigma_{e_n}\mathbb{E}_{\lambda}(\mathbb{1}_{e_1};\cdots;\mathbb{1}_{e_n}). \tag{2.4.3}$$

At a dimensional level, the truncated n-point correlation in the right hand side decays like $d^{-n(1+O(\lambda))}$, where d is the minimal pairwise distance among the edges $e_1, \dots, e_n$; therefore, the result of the n-fold summation in (2.4.3) is potentially unbounded as $\max_{i<j} |x_i - x_j| \to \infty$. In order to show that this is not the case, and actually the result of the n-fold summation is bounded as in the right hand side of (2.4.2), we need to exhibit appropriate cancellations. Once more, we use the comparison of the dimer lattice model with the infrared reference model, which allows us to re-express the multi-point truncated dimer correlation $\mathbb{E}_\lambda(\mathbb{1}_{e_1}; \cdots; \mathbb{1}_{e_n})$ as a dominant term, which is the multi-point analogue of (2.2.6), plus a remainder, which decays faster at large distances. More precisely, by using a decomposition analogous to [**90**, Eq. (7.7)] and using the analogue of [**90**, Eq. (6.90)], if e_i has labels (x_i, j_i, ℓ_i), we rewrite

$$\mathbb{E}_\lambda(\mathbb{1}_{e_1}; \cdots; \mathbb{1}_{e_n}) = \sum_{\substack{\omega_1,\dots,\omega_n=\pm \\ s_1,\dots,s_n=1,2}} \Big(\prod_{r=1}^{n} K^{(s_r)}_{\omega_r, j_r, \ell_r} \big(\prod_{r:s_r=2} e^{2ip^{\omega_r}\cdot x_r} \big) \Big) S^{(s_1,\dots,s_n)}_{R;\omega_1,\dots,\omega_n}(x_1, \dots, x_n) + \mathrm{Err}(e_1, \dots, e_n), \tag{2.4.4}$$

where $S^{(s_1,\dots,s_n)}_{R;\omega_1,\dots,\omega_n}$ are the multi-point density-mass correlations of the reference model (defined as in [**90**, Eq. (6.85)] or as the multi-point analogue of [**2**, Eq. (4.15)-(4.16)]). Moreover, if $D_{\underline{x}}$ is the diameter of $\underline{x} = (x_1, \dots, x_n)$ and if the minimal separation among the elements of $\underline{x}$ is larger than $c_0 D_{\underline{x}}$ for some positive constant c_0, then, for θ equal to, say, $1/2$ (in general, θ can be any positive constant smaller than $1 - O(\lambda)$) the remainder term is bounded as $|\mathrm{Err}(e_1, \dots, e_n)| \le C_{n,\theta}(c_0) D_{\underline{x}}^{-n-\theta}$. The latter bound is the analogue of [**90**, Eq. (6.90)]. Moreover[10], the functions $S^{(s_1,\dots,s_n)}_{R;\omega_1,\dots,\omega_n}$ are non-zero only if the quasi-particle indices satisfy the constraint $\sum_{i:s_i=2} \omega_i = 0$ (this is the multi-point generalization of [**90**, Eq. (6.92)]). Finally, and most importantly, if $s_1 = \cdots = s_n = 1$, then

$$S^{(1,\dots,1)}_{R;\omega_1,\dots,\omega_n}(x_1, \dots, x_n) \equiv 0, \qquad n > 2, \tag{2.4.5}$$

which is the analogue of [**90**, Eq.(6.94)] and is an instance of 'bosonization' for the reference model: in fact, (2.4.5) can be interpreted by saying that the n-point (with $n > 2$) truncated density correlations of the reference model (recall Footnote 7 for the definition of 'density' and 'mass' observables) are all identically equal to zero.

In conclusion, in the right hand side of (2.4.3) we can replace $\mathbb{E}_\lambda(\mathbb{1}_{e_1}; \cdots; \mathbb{1}_{e_n})$ by the right hand side of (2.4.4), where the term with $s_1 = \cdots = s_n = 1$ vanishes. Therefore, all the terms we are left with either involve oscillating factors $\prod_{r:s_r=2} e^{2ip^{\omega_r}\cdot x_r}$ or the remainder term $\mathrm{Err}(e_1, \dots, e_n)$. In both cases, exactly like in the case $n = 2$, the contribution of these terms to the n-fold summation over $e_1, \dots, e_n$ in (2.4.3) is bounded better than the naive dimensional estimate, and we are led to the bound in (2.4.2). For a detailed discussion of how the estimate of the summation is performed, we refer the reader to [**90**, Section 7.2].

[10] See the discussion after [**90**, Eq. (6.94)] for references about the properties of $S^{(s_1,\dots,s_n)}_{R;\omega_1,\dots,\omega_n}$ that are discussed in this paragraph.

CHAPTER 3

INTERACTING DIMERS II: LIQUID-FROZEN TRANSITION

In order to prove Theorems 1.4.4-1.4.5, this chapter is structured as follows.

In Section 3.1 we recall known results for the non-interacting dimer model. In particular we prove Theorem 1.4.5 in the non-interacting case. Section 3.2.1 contains the Grassmannian representation of the model. Our case is a direct Corollary of [**2**]. In Section 3.3 we set up the multiscale analysis. This section is standard and is used to set up the inductive structure of the Renormalization Group analysis. It is essentially the same as the analysis in [**4**]. In Section 3.3.1 we set up the multiscale structure to deal with the shift of the liquid-frozen transition point caused by the interaction: this fact is one the main differences with [**4**]. In Section 3.3.2 we set the double-regime multiscale structure, which is necessary to control the thermodynamic limit of the theory uniformly as we approach the edge of the frozen phase. Section 3.4 is the core of the technical analysis. It is standard and rooted in an extensive literature. The main references for us in this context are [**2**, **4**, **5**]. The goal of this section is to combine the ideas of [**4**] with the formalism developed in [**2**] for an interacting dimer model. In Section 3.4.2 there are quantitative estimates, output of the multiscale structure, necessary to check the convergence of the theory in multiscale analysis; Section 3.4.3 contains the Gallavotti-Nicolo' tree formalism and a comparison with [**4**] in order to export the estimates into our framework. Section 3.4.1 collects the information arising from the inductive renormalization procedure in terms of a system of equations in a space of sequences, the so called Beta function; this is used in Section 3.4.4 to reformulate the convergence of the perturbative theory in terms of a fixed point solution for the Beta function. Section 3.5 contains the proof of the main Theorems 1.4.4-1.4.5

3.1 The non-interacting dimer model and the liquid-frozen transition

In this section, we recall some basic definitions and results about the integrable (or non-interacting) dimer model on the infinite honeycomb bipartite graph $\mathcal{H}$. For the reader's

convenience we also report here some of the definitions given in Sec. 1.3.

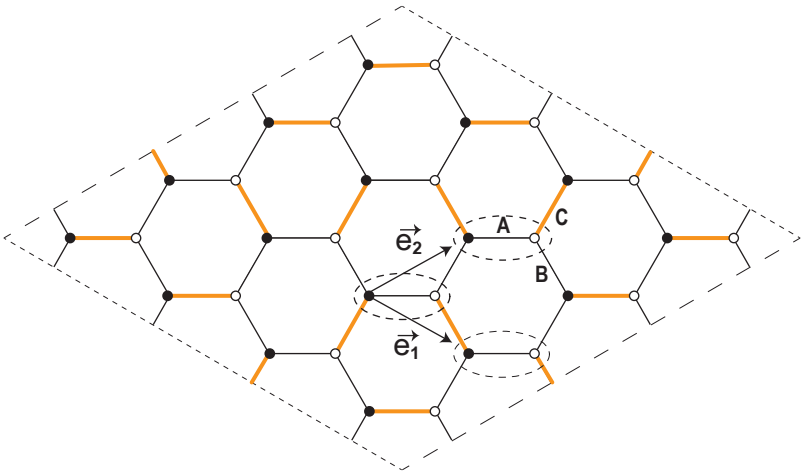

Figure 3.1: *The graph G_L with $L = 4$; the fundamental domain, containing one black and one white vertex which share the same coordinate x, is encircled. Weights of horizontal, north-east and north-west oriented edges are A, B, C, respectively. In orange an example of a dimer configuration.*

- We work on the graph $G_L = (V_L, E_L), L \in 2\mathbb{N}$, embedded in the two-dimensional torus, defined as
$$G_L := \mathcal{H}/(L\mathbb{Z} \times L\mathbb{Z}) \tag{3.1.1}$$
where $\mathcal{H}$ stands for the infinite bipartite hexagonal graph and $L\mathbb{Z} \times L\mathbb{Z}$ denotes the action of the translations with respect to directions $L\vec{e_1}, L\vec{e_2}$ (see Figure 3.1).

- G_L is bipartite, so we color with black and white its vertices and we decompose $V_L = V_L^W \cup V_L^B$ where, the sets V_L^B and V_L^W are both isomorphic to $\Lambda := \{(x_1, x_2) : x_i = 0, \dots, L-1\}$. We assume that a black and a white vertex share the same coordinates if and only if they are in the same copy of the fundamental domain G_1.Given a vertex v, we write $x(v) = (x_1(v), x_2(v)) \in \Lambda$ for its coordinates in the basis $\vec{e_1}, \vec{e_2}$.

- We denoted by Ω_L the space of dimer configurations or perfect matchings of G_L. To each $M \in \Omega_L$ we associate a statistical weight $w(M) = \prod_{e \in M} t_e = A^{N_A} B^{N_B} C^{N_C}$ where we recall that $t : E_L \to \mathbb{R}^+$ assigns a positive number to each edge of the graph: since we work in a translationally invariant setting, we let $t_e = A$ if e is horizontal, $t_e = B$ is e is north-west oriented and $t_e = C$ otherwise.

- The probability measure $\mathbb{P}_L$ (with corresponding expectation $\mathbb{E}_L$) on Ω_L is defined by
$$\mathbb{P}_L(M) := \frac{w(M)}{Z_L}, \quad w(M) = \prod_{e \in M} t_e = A^{N_A} B^{N_B} C^{N_C} \tag{3.1.2}$$
with partition function $Z_L = \sum_{M \in \Omega_L} w(M)$. Here, N_A, N_B, N_C denote the number of dimers of type A, B, C in the configuration M, respectively. The free energy per

unit volume and the generating functional are given by

$$\begin{aligned} f_L &= \frac{1}{L^2}\log Z_L \\ e^{\mathcal{W}_L(\mathcal{A})} &= \sum_{M\in\Omega_L} w(M) e^{\sum_{e\in E_L}\mathbb{1}_{\{e\in M\}}A_e} \end{aligned} \tag{3.1.3}$$

where $\mathcal{A} = \{A_e\}_{e\in E_L}$ is a collection of real parameters.

Definition 3.1.1 (Kasteleyn Matrix)**.** For vertices $b \in V_L^B, w \in V_L^W$ define $b \sim w$ iff $(b,w) \in E_L$. Denote by K the t-weighted adjacency matrix of G_L, i.e. the matrix whose rows and columns are indexed by black and white vertices of V_L respectively and whose entries are given for $b \in V_L^B, w \in V_L^W$ by

$$K(b,w) = \begin{cases} t_e & b\sim w,\ e=(b,w) \\ 0 & b \not\sim w. \end{cases} \tag{3.1.4}$$

3.1.1 Kasteleyn theory and thermodynamics of the non-interacting model

The non-interacting dimer model is integrable and there are explicit determinantal formulas for its correlation functions and partition function. In this section, we recall the minimal amount of known facts, needed for the rest of this work. We refer to [**82**, **113**] for more details.

Definition 3.1.2. Given $\theta = (\theta_1,\theta_2) \in \{0,1\}^2$ we define the θ-altered Kasteleyn matrix K_θ as follows: $K_\theta(b,w) = (-1)^{\theta_1}K(b,w)$ if $x_1(w) = 0$ and $x_1(b) = L-1$, $K_\theta(b,w) = (-1)^{\theta_2}K(b,w)$ if $x_2(w) = 0$ and $x_2(b) = L-1$ and $K_\theta(e) = K(e)$ otherwise.

Notice that $K_{00} \equiv K$; we call K_{00} the Kasteleyn matrix with periodic-periodic boundary conditions, K_{10} the one with antiperiodic-periodic b.c. and similarly for K_{01} and K_{11}. The matrix K_θ can be diagonalized by the Fourier basis $\{\frac{e^{-ikx}}{L}\}$ with $x \in \Lambda$ and $k \in \mathcal{D}(\theta)$ given by

$$\mathcal{D}(\theta) := \left\{(k_1,k_2) : k_i = \frac{2\pi}{L}\left(n_i + \frac{\theta_i}{2}\right), -\frac{L}{2} \le n_i < \frac{L}{2}\right\} \subset [-\pi,\pi]^2. \tag{3.1.5}$$

The corresponding eigenvalues are given by

$$\mu(k) = A + Be^{ik_1} + Ce^{ik_2}. \tag{3.1.6}$$

The following proposition summarizes what we need about the determinantal structure of the non-interacting mode:

Proposition 3.1.1. *The partition function can be written as*

$$Z_L = \sum_{\theta\in\{0,1\}^2} \frac{c_\theta}{2}\det K_\theta, \quad \det K_\theta = \prod_{k\in\mathcal{D}_\theta}\mu(k). \tag{3.1.7}$$

where $(c_{00},c_{10},c_{01},c_{11}) = (-1,1,1,1)$ *if* L *is even*[1]*. Assume that* $\det K_\theta \neq 0$ *for all*

[1] see [**113**] for a proof

θ. Given distinct edges $e_1, \dots, e_n$, the multipoint correlation function is given (with $\mathbb{1}_e$ denoting the indicator function of the event that there is a dimer occupying the edge e) by

$$\mathbb{E}_L[\mathbb{1}_{e_1} \cdots \mathbb{1}_{e_n}] = \frac{\sum_\theta \frac{c_\theta}{2} (\prod_{i=1}^n K_\theta(e_i)) \det K_\theta \det \left(K_\theta^{-1}(w_i, b_j)\right)_{i,j=1}^n}{Z_L} \tag{3.1.8}$$

where w_i, b_i are the white/black endpoints of e_i.

The inverse of K_θ is given as

$$K_\theta^{-1}(w, b) = \frac{1}{L^2} \sum_{k \in \mathcal{D}_\theta} \frac{e^{-ik(x-y)}}{\mu(k)} \tag{3.1.9}$$

where the white and black vertices w, b have coordinates $x, y \in \Lambda$ respectively. The following proposition summarizes the results in the infinite volume limit:

Proposition 3.1.2. *The infinite-volume limit of the free energy per unit volume exists and equals*

$$f(A, B, C) = \lim_{L \to \infty} \frac{1}{L^2} \log Z_L = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \log |\mu(k)| dk \tag{3.1.10}$$

The infinite-volume limit of the correlation functions is given by

$$\mathbb{E}[\mathbb{1}_{e_1} \cdots \mathbb{1}_{e_n}] = \left(\prod_{i=1}^n K(e_i) \right) \det \left(K^{-1}(x_i - y_j) \right)_{i,j=1}^n \tag{3.1.11}$$

where x_i / y_i are the coordinates of w_i / b_i and

$$K^{-1}(x) := \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{e^{-ikx}}{\mu(k)} dk = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{dz}{z} \frac{dw}{w} \frac{z^{-x_1} w^{-x_2}}{A + Bz + Cw} \tag{3.1.12}$$

where $\mathbb{T}^2 := \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$.

Since the probability measure $\mathbb{P}_L$ is unchanged when all weights are multiplied by the same factor, we will from now on set $A = 1$.

To describe the phase diagram of the model it is convenient to define the "magnetic fields" $(B_x, B_y) \in \mathbb{R}^2$ via $B = e^{B_x}, C = e^{B_y}$. Also, it is useful to introduce the following

Definition 3.1.3. Let $Log : (\mathbb{C} \setminus \{0\})^2 \to \mathbb{R}^2$ defined by $Log(z, w) = (\log |z|, \log |w|)$. Then the amoeba of the polynomial $P(z, w) = 1 + z + w$ is defined as

$$\mathcal{A}(P) := \{Log(z, w) : P(z, w) = 0\}. \tag{3.1.13}$$

Note that the denominator in (3.1.12) has zeros on $\mathbb{T}^2$ (so that the integrand is singular) if and only if (B_x, B_y) belong to $\mathcal{A}(P)$. The latter is known to be a simply connected subset of $\mathbb{R}^2$ with non-empty interior, such that its complement is made of three infinite connected components. See Fig. 3.3. In terms of the weights $A = 1, B, C$, the condition $(B_x, B_y) \in \mathcal{A}(P)$ is equivalent to the condition that $A = 1, B, C$ satisfy the triangular inequality. The phase diagram of the model is summarized in the following

Proposition 3.1.3. • *(r Rough region) If $A = 1, B, C$ satisfy strictly the triangle inequality (equivalently, if (B_x, B_y) belongs to the interior of $\mathcal{A}(P)$), then the function μ in (3.1.6) has two simple zeros $p^{\pm} \in [-\pi, \pi]^2, p^+ = -p^-$. Explicitly, $p^+ = (\pi - \theta_C, -\pi + \theta_B)$ where θ_B, θ_C are defined geometrically as in Fig. 3.2. Moreover, the so-called "Ronkin function" [**32**]*

$$R(B_x, B_y) := f(1, e^{B_x}, e^{B_y}) \tag{3.1.14}$$

is a strictly convex function of (B_x, B_y) in the interior of $\mathcal{A}(P)$ and the infinite-volume limit ρ_A, ρ_B, ρ_C of the density of dimers of type A, B, C is $\theta_A/\pi, \theta_B/\pi, \theta_C/\pi$, respectively.

- *(Frozen region) In the three connected components of the complement of $\mathcal{A}(P)$, $R(B_x, B_y)$ is an affine function and the function μ has no zeros on $[-\pi, \pi]^2$. In particular, if $A > B + C$ (resp. $B > A + C$ or $C > A + B$) then $R(B_x, B_y) = 0$ (resp. $R(B_x, B_y) = B_x$ or $R(B_x, B_y) = B_y$) and ρ_A (ρ_B or ρ_C) is 1.*

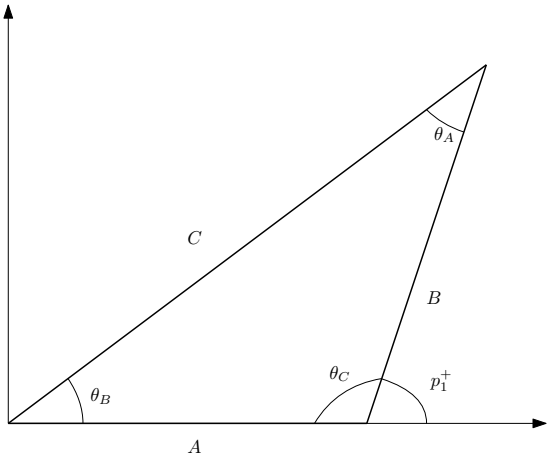

Figure 3.2: *Geometric interpretation of the dimer densities $\theta_A/\pi, \theta_B/\pi, \theta_C/\pi$.*

The liquid-frozen transition

We are interested in the liquid-frozen transition. For the non-interacting model, according to Proposition 3.1.3, this corresponds to the limit where the magnetic fields lie on the boundary of the amoeba. Otherwise stated, one of the three weights equals the sum of the other two: e.g. $A = 1$, $B + C = 1$. Without loss of generality, we consider the limit where (B, C) tends to $(t, 1 - t)$ for some $t \in (0, 1)$. The case where $t = 0$ or $t = 1$ is somewhat degenerate and will not be considered here; it corresponds to (B_x, B_y) tending to infinity along one of the three "tentacles" of the amoeba. We refer the reader to [**128**] for the asymptotic behavior of dimer models in the tentacles of the amoeba in relation to the so called *Bead model*.

To approach the point $(B, C) = (t, 1 - t)$, we can choose for instance $(B, C) = (B(\varepsilon), C(\varepsilon)) = \frac{e^{\varepsilon}}{2}(t, 1 - t)$ and let $\varepsilon \to 0$. Changing ε corresponds to changing linearly the magnetic fields. For $\varepsilon > 0$ (resp. $\varepsilon < 0$) we are in the rough (resp. frozen) region. From Proposition 3.1.3, $R(\log t + \varepsilon, \log(1 - t) + \varepsilon) = f(1, B(\varepsilon), C(\varepsilon))$ is zero for $\varepsilon < 0$ and

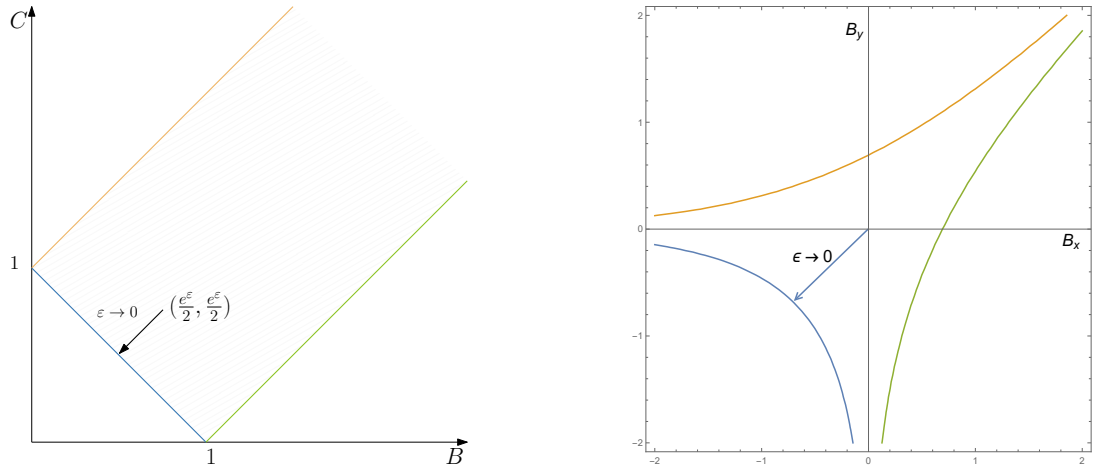

Figure 3.3: *Left: in the shaded region the triangle condition for $A = 1, B, C$ holds. The blue segment corresponds to $(B, C) = (t, 1-t), t \in [0, 1]$. The midpoint can be approached for instance choosing $(B, C) = (\frac{e^\varepsilon}{2}, \frac{e^\varepsilon}{2}), \varepsilon \to 0$. Right: the amoeba $\mathcal{A}(P)$ is the unbounded triangular region delimited by the three colored curves.*

non-zero for $\varepsilon > 0$ (because the free energy is strictly convex in the rough region). The free energy is zero at the transition point (because of convexity). The way it vanishes as $\varepsilon \to 0^+$ is easily determined:

Theorem 3.1.1. *Let $t \in (0, 1)$. As $\varepsilon \to 0$,*

$$R(\log t + \varepsilon, \log(1-t) + \varepsilon) = \mathbb{1}_{\varepsilon \geq 0} \left(\frac{2\sqrt{2}}{3\pi\sqrt{t(1-t)}} \varepsilon^{3/2} + O(\varepsilon^{5/2}) \right). \tag{3.1.15}$$

Note the symmetry $t \leftrightarrow 1 - t$, corresponding to the symmetry $B \leftrightarrow C$ of the dimer model.

Proof of Theorem 3.1.1. Let $\varepsilon > 0$. Note that $\partial_{B_x} R(B_x, B_y) = \rho_B = \theta_B/\pi$ and $\partial_{B_y} R(B_x, B_y) = \rho_C = \theta_C/\pi$, the densities of dimers of type B and C. Therefore,

$$R(\log t + \bar{\varepsilon}, \log(1-t) + \bar{\varepsilon}) = \frac{1}{\pi} \int_0^{\bar{\varepsilon}} (\theta_B + \theta_C) d\varepsilon. \tag{3.1.16}$$

Recalling Fig. 3.2 and applying the law of cosines,

$$\theta_B = \arccos\left(\frac{\cosh \varepsilon}{1-t} - \frac{t e^\varepsilon}{1-t} \right) = \sqrt{2\varepsilon \frac{t}{1-t}} + O(\varepsilon^{3/2}),$$
$$\theta_C = \arccos\left(e^\varepsilon - \frac{\sinh \varepsilon}{t} \right) = \sqrt{2\varepsilon \frac{1-t}{t}} + O(\varepsilon^{3/2}) \tag{3.1.17}$$

and the claim follows from (3.1.16). ∎

Theorem 3.1.1, is the Pokrovsky-Talapov law for the non interacting Ronkin function near a generic point on the boundary between the frozen and rough regions. This proves the statement of Theorem 1.4.5 for $\lambda = 0$ and $t = 1/2$.

3.2 Fermionic setting

In this section we want to derive a Grassmann representation for the partition function of the interacting model, $Z_{\lambda,\varepsilon,L}$ and more generally for the generating functional $\mathcal{W}_{\lambda,\varepsilon,L}(\mathcal{A})$. Let us recall the necessary definitions. Given $\mathcal{A} = \{A_e\}_{e\in E_L}$, let

$$\begin{aligned} e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} &= \sum_{M\in\Omega_L} w_{\varepsilon,\lambda}(M) e^{\sum_{e\in E_L} \mathbb{1}_e A_e}, \quad Z_{\varepsilon,\lambda,L} = e^{\mathcal{W}_{\varepsilon,\lambda,L}(0)} \\ w_{\varepsilon,\lambda}(M) &:= \Big(\frac{e^\varepsilon}{2}\Big)^{N_B+N_C} e^{\lambda V}, \quad V = \sum_{f\in G_L^*} \mathbb{1}_f^{(2)} \end{aligned} \tag{3.2.1}$$

where N_B, N_C are the numbers of dimers of type B, C respectively, in the configuration M; $\mathbb{1}_e$ is the indicator function of the event that a dimer occupies the edge $e \in E_L$ while, given a face f in the dual graph G_L^*, $\mathbb{1}_f^{(2)} = 1$ iff two parallel dimers surround the face f and it is zero otherwise. We can also write

$$\mathbb{1}_f^{(2)} = \sum_{\sigma=1}^{3} \mathbb{1}_{f,\sigma}^{(2)} \tag{3.2.2}$$

where σ specifies which type of dimers occupies f, among types A, B, C (horizontal, North-West, North-East oriented) see Fig. 3.4.

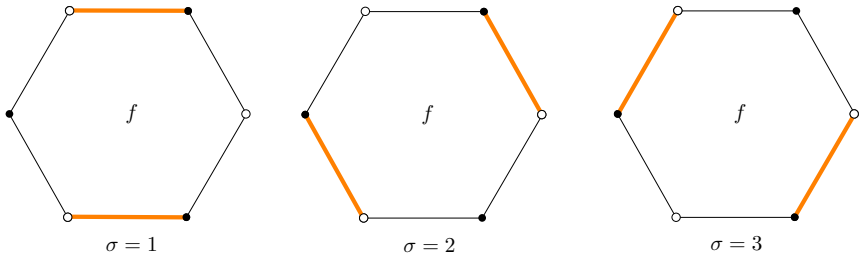

Figure 3.4

Next, we need a Grassmann algebra. We assign to each vertex of the lattice a Grassmann variable $\psi_x^\pm$ where $x \in \Lambda$, with the convention that ψ^+ is associated to black vertices and ψ^- to white ones (recall that the graph is bipartite). They satisfy by definition the anticommutation relation

$$\psi_x^\sigma \psi_{x'}^{\sigma'} = -\psi_{x'}^{\sigma'} \psi_x^\sigma, \qquad x, x' \in \Lambda,\ \sigma, \sigma' = \pm. \tag{3.2.3}$$

Given the Grassmann algebra generated by $\{\psi_x^\sigma\}_{x\in\Lambda}^{\sigma=\pm}$, sometimes here also called Grassmann field, we can consider its Fourier transform, namely the Grassmann algebra generated by $\{\hat\psi_k^\sigma\}_{k\in\mathcal{D}_\theta}^{\sigma=\pm}$ satisfying the following relation

$$\psi_x^\pm = L^{-2} \sum_{k\in\mathcal{D}_\theta} e^{\pm ikx} \hat\psi_k^\sigma. \tag{3.2.4}$$

where $\mathcal{D}_\theta$ is the same as in (3.1.5).

3.2.1 Grassmann representation

We start from the formula (3.1.7) of the partition function of the non-interacting model. It is well known that determinants can be written in terms of Gaussian Grassmann integrals (see e.g. [**5**]) so that we can express $Z_{\varepsilon,0,L}$ as

$$\begin{aligned} Z_{\varepsilon,0,L} &= \frac{1}{2}\sum_{\theta\in\{-1,1\}^2} c_\theta \int D\psi e^{S_{\theta,\varepsilon}(\psi)}, \\ S_{\theta,\varepsilon}(\psi) &:= -\sum_{e\in E_L} E_{\varepsilon,\theta,e}(\psi), \qquad E_{\varepsilon,\theta,e}(\psi) := K_{\varepsilon,\theta}(b,w)\psi_b^+\psi_w^- \end{aligned} \tag{3.2.5}$$

where b, w are the black and white vertices of the edge e; we specified the dependence on ε in the Kasteleyn matrix K_θ (see Section 3.1.1) for the choice of weights $A = 1, B = C = \frac{1}{2}e^\varepsilon$ and c_θ are the same signs as in Proposition 3.1.1. Note that for later convenience, using the conventions in Fourier space, we can also write

$$S_{\varepsilon,\theta}(\psi) = -L^{-2}\sum_{k\in\mathcal{D}_\theta} \hat\psi_k^+ \mu_\varepsilon(k)\hat\psi_k^-, \qquad \mu_\varepsilon(k) = 1 + \frac{1}{2}e^\varepsilon(e^{ik_1} + e^{ik_2}) \tag{3.2.6}$$

In order to obtain a Grassmann representation for the interacting partition function, or more in general for the interacting generating functional, notice that the interaction V (see (3.2.1)) can be written as

$$V = \sum_{x\in\Lambda} f(\tau_x M), \qquad f(M) = \mathbb{1}_{e_1}\mathbb{1}_{e_4} + \mathbb{1}_{e_2}\mathbb{1}_{e_5} + \mathbb{1}_{e_3}\mathbb{1}_{e_6} \tag{3.2.7}$$

where τ_x is a lattice translation by $x_1\vec{e_1} + x_2\vec{e_2}$ and we denote by $e_1, \dots, e_6$ the edges around fixed face, ordered clockwise starting from the horizontal one at the bottom. This is a specific instance of perturbation belonging to a wider class studied in [**2**] (see Sec. 2.3); also, the different lattice structure used here is a sub-case of the one discussed there[2]. There, a Grassmann representation of generating function was obtained (see Prop. 1 [**2**]). We report here the statement and the proof in this special case:

Proposition 3.2.1. *The generating functional $\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})$ (3.2.1) can be written as*

$$e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \sum_{\theta\in\{-1,1\}^2} \frac{c_\theta}{2}\int D\psi e^{S_{\varepsilon,\theta}(\psi)+V(\psi,\mathcal{A})} \tag{3.2.8}$$

where $S_{\varepsilon,\theta}(\psi)$ is given above and

$$V(\psi,\mathcal{A}) = \sum_{e\in E_L}(e^{A_e}-1)E_{\varepsilon,\theta,e}(\psi) + \sum_\gamma c(\gamma)\prod_{e\in\gamma} E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi) \tag{3.2.9}$$

with $E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi) := e^{A_e}E_{\varepsilon,\theta,e}(\psi)$ and the sum runs over collections $\gamma \subset E_L$, $|\gamma| \geq 2$, of parallel adjacent edges, so one can write $\gamma = \{e_1, \dots, e_{|\gamma|}\}$ with e_i and e_{i+1} adjacent, for $i = 1, \cdots, |\gamma| - 1$. We say that two parallel edges are adjacent if they surround the same

[2] This is seen by setting t_4 in [**2**, Sec: 2.1] in which case the square graph becomes isomorphic to the hexagonal one.

face $\eta \in G_L^*$ (as in Figure 3.4); $c(\gamma) = (e^\lambda - 1)^{|\gamma|-1}(-1)^{|\gamma|}$.

Proof. We can rewrite the generating functional (3.2.1) as

$$e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \sum_{M \subset \Omega_L} w_{\mathcal{A}}(M) e^{V(M)}, \qquad w_{\mathcal{A}}(M) = \left(\frac{e^\varepsilon}{2}\right)^{N_B+N_C} e^{\sum_{e\in E_L} A_e \mathbb{1}_e}. \tag{3.2.10}$$

Using the definition of V,

$$\begin{aligned} &e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \\ &= \sum_{M\subset\Omega_L} w_{\mathcal{A}}(M) \prod_{\eta\in G_L^*} (1 + \alpha \sum_{\sigma=1}^{3} \mathbb{1}_{\eta,\sigma}^{(2)}) = \sum_{M\subset\Omega_L} w_{\mathcal{A}}(M) \prod_{\eta\in G_L^*} \prod_{\sigma=1}^{3} (1 + \alpha \mathbb{1}_{\eta,\sigma}^{(2)}) \end{aligned} \tag{3.2.11}$$

where $\alpha = e^\lambda - 1$ and we used the identity, for η fixed, $e^{\lambda \mathbb{1}_\eta^{(2)}} = 1 + (e^\lambda - 1)\mathbb{1}_\eta^{(2)}$; then we used (3.2.2), and the fact that for $\sigma \neq \sigma'$, $\mathbb{1}_{\eta,\sigma}^{(2)} \mathbb{1}_{\eta,\sigma'}^{(2)} = 0$ for all configurations M. Now the following equality holds

$$\prod_{\eta\in G_L^*} \prod_{\sigma=1}^{3} (1 + \alpha \mathbb{1}_{\eta,\sigma}^{(2)}) = \sum_{n\geq 0} \sum_{\gamma_1,\dots,\gamma_n}^{*} \prod_{i=1}^{n} \tilde{c}(\gamma_i) \prod_{e\in\gamma_i} \mathbb{1}_e \tag{3.2.12}$$

where the value of the sum when $n = 0$ is 1 by definition, and:

1. γ_i is a set of edges of E_L such that each of its edges has the same orientation (i.e. it comes from the same choice of σ, see Figure 1.6), meaning that there exist a lattice translation τ such that $e = \tau e'$, for every pair $(e, e') \in \gamma \times \gamma$; moreover for every non trivial partition $\gamma = \gamma_1 \cup \gamma_2$ there exist $e_j \in \gamma_j$, $j = 1, 2$, and $\eta \in G_L^*$ such that e_1, e_2 belongs to η.

2. the $*$ in the sum means that the γ_i's are pairwise disjoint and for every $i \neq j$, $e_j \in \gamma_j$ with $j = 1, 2$, e_1, e_2 do not belong to the same face of G_L^*.

3. $\tilde{c}(\gamma) := \alpha^{|\gamma|-1}$

Then, plugging (3.2.12) in (3.2.11) we can rewrite, changing the order of summation,

$$e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \sum_{n\geq 0} \sum_{\gamma_1,\dots,\gamma_n}^{*} \left(\prod_{i=1}^{n} c(\gamma_i) \prod_{e\in\gamma_i} \partial_{A_e} \right) e^{\mathcal{W}_{\epsilon,0,L}(\mathcal{A})}. \tag{3.2.13}$$

Now, since $e^{\mathcal{W}_{\epsilon,L,0}(\mathcal{A})}$ can be seen as the partition function of a non interacting dimer model with modified edge weights $t_e e^{A_e}$, we can express it in terms of a Gaussian Grassmann integral, in the same spirit as (3.2.5), namely

$$e^{\mathcal{W}_{\epsilon,L,0}(\mathcal{A})} = \sum_{\theta\in\{-1,1\}^2} c_\theta \int D\psi e^{S_\theta^{(\mathcal{A})}(\psi)} \tag{3.2.14}$$

where $S_{\varepsilon,\theta}^{(\mathcal{A})}$ is the same as $S_{\varepsilon,\theta}$ (cf. (3.2.5)) where $E_{\varepsilon,\theta,e}(\psi)$ is replaced by $E_{\varepsilon,\theta,e}^{(\mathcal{A})}(\psi) :=$

$e^{A_e} E_{\varepsilon,\theta,e}(\psi)$; so (3.2.13) becomes

$$e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \sum_{n\geq 0} \sum_{\gamma_1,\dots,\gamma_n}^{*} \sum_{\theta} \frac{c_\theta}{2} \int D\psi e^{S^{(\mathcal{A})}_{\varepsilon,\theta}(\psi)} \prod_{i=1}^{n} c(\gamma_i) \prod_{e\in\gamma_i} E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi), \tag{3.2.15}$$

where $c(\gamma) := (-1)^{|\gamma|}\tilde{c}(\gamma)$. Finally notice that

$$\begin{aligned} & e^{\sum_\gamma c(\gamma)\prod_{e\in\gamma} E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi)} = \\ & = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\gamma} c(\gamma) \prod_{e\in\gamma} E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi) \right)^n = \sum_{n\geq 0} \sum_{\gamma_1,\dots,\gamma_n}^{*} \prod_{i=1}^{n} c(\gamma_i) \prod_{e\in\gamma_i} E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi) \end{aligned} \tag{3.2.16}$$

since the $E^{(\mathcal{A})}_{\varepsilon,\theta,e}(\psi)$ commute among themself and the * comes from $\psi^2 = 0$ for every field ψ. ∎

3.2.2 Symmetries

For $\lambda = 0$, note that that for general weights A, B, C the free model (Proposition 3.1.1 and above) display the symmetry under complex conjugation

$$\mu(-k) = \overline{\mu(k)} \tag{3.2.17}$$

which is a consequence of choosing the Kasteleyn matrix to be real (as it follows from (3.1.9)). Moreover if $B = C$, one also has (cf. (3.1.6))

$$\mu(\tilde{k}) = \mu(k), \quad \tilde{k} := (k_2, k_1) \tag{3.2.18}$$

where $k = (k_1, k_2)$. Let us describe the effect of these symmetries, in terms of transformations $\mathcal{S}_1, \mathcal{S}_2$ on the Grassmann algebra generated by $\{\psi_x\}_{x\in\Lambda}$. Note first that (3.2.17) implies that the action $S_\theta(\psi)$ defined in (3.2.5) is left invariant by $\mathcal{S}_1$ defined by

$$\mathcal{S}_1 : \hat{\psi}^{\pm}_k \longrightarrow \hat{\psi}^{\pm}_{-k}, \qquad c \to \overline{c} \tag{3.2.19}$$

for $c \in \mathbb{C}$, meaning that in every Grassmann polynomial, each Grassmann field $\hat{\psi}$ is replaced as above and every constant c is replaced by its complex conjugate. This is a consequence of $-\mathcal{D}_\theta = \mathcal{D}_\theta$, for every $\theta \in \{0,1\}^2$, and of (3.2.6). Instead, the consequence of (3.2.18) is that, if we define

$$\mathcal{S}_2 : \hat{\psi}^{\pm}_k \longrightarrow \hat{\psi}^{\pm}_{\tilde{k}}, \qquad c \to c \tag{3.2.20}$$

we find $\mathcal{S}_2(S_\theta(\psi)) = S_{\tilde{\theta}}(\psi)$, as a consequence of $\tilde{\mathcal{D}}_\theta = \mathcal{D}_{\tilde{\theta}}$. Notice that since $c_{(0,1)} = c_{(1,0)}$ in (3.2.5), then $\hat{\mathcal{S}}_2$ is a symmetry for the free theory, if $B = C$.

For $\lambda \neq 0$, we need to check also the symmetries of the bare potential $V(\psi)$ defined in (3.2.8). Rewriting the symmetries in terms of the fields $\{\psi^{\pm}_x\}$ instead of their Fourier transform, we find

$$\mathcal{S}_1 : \psi^{\pm}_x \longrightarrow \psi^{\pm}_x, \quad c \to \overline{c}, \qquad \mathcal{S}_2 : \psi^{\pm}_x \longrightarrow \psi^{\pm}_{\tilde{x}}, \quad c \to c \tag{3.2.21}$$

where again $\tilde{x} = (x_2, x_1)$ if $x = (x_1, x_2)$. Since $V(\psi)$ has real coefficients, (cfr. Proposition

3.2.1), then V is symmetric under $\mathcal{S}_1$; moreover making explicit the dependence on θ in $V(\psi)$ (cfr. (3.2.8) and (3.2.5)) one can see that $\mathcal{S}_2(V_\theta(\psi)) = V_{\tilde{\theta}}(\psi)$, if $B = C$. Indeed, it is enough to note that (1) $\mathcal{S}_2$ is an involution that sends NW-oriented edges in NE-oriented ones so that it is a bijection from the set of edges $\{\gamma\}$ in the definition of $V(\psi)$ in itself; (2) $E^\theta_e(\psi)$ does not depend on θ if e is an horizontal edge: (3) $\mathcal{S}_2 E^\theta_e(\psi) = E^{\tilde{\theta}}_e(\psi)$.

3.3 RG approach: multiscale decomposition

In this Section we set the multiscale integration procedure, by exploiting the Grassmann representation obtained in Proposition 3.2.1. We will define an iterative procedure to obtain the value of $\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})$, as well as of $f_{\varepsilon,\lambda,L}$. We first show the construction for the so called critical case in Section 3.3.1, namely we choose $\varepsilon = \epsilon + \varepsilon_0$ with $\epsilon = 0$ and we set temporarily the external fields $\mathcal{A} = 0$ to lighten the notation. Here ε_0 plays the role of a free parameter. It will be fixed, in order to obtain a convergent perturbative expansion, in Section 3.4.4. Next we generalize the iterative procedure to the case $\epsilon \neq 0$, $\mathcal{A} \neq 0$ in Section 3.3.2.

3.3.1 The critical theory

At the rough-frozen transition point $\varepsilon = 0$ (i.e., $B = C = 1/2 = A/2$) of the non-interacting model, the two zeros $p^\pm$ of the function μ coincide modulo $(2\pi, 2\pi)$, and they equal (π, π) (recall Proposition 3.1.3). One of the effects of the interaction V is, in general, to change the location of the critical point. This means that the critical behavior that we observe at $\varepsilon = 0$ for the non interacting $\lambda = 0$ model must be expected, when we switch on the interaction, at some λ-dependent value $\varepsilon_0(\lambda)$, yet to be determined, such that $\varepsilon_0(0) = 0$. For the moment, we take some $\varepsilon_0 \in \mathbb{R}$ which will be suitably fixed later, as a function of λ.
In order to simplify as much as possible the construction, we set the external fields $\mathcal{A} = 0$. We will describe in the next Sections the complete procedure for $\mathcal{A} \neq 0$, and $\varepsilon = \epsilon + \varepsilon_0(\lambda)$ with $\epsilon \neq 0$. Thanks to Proposition 3.2.1 our starting point is $Z_{\lambda,\varepsilon,L,\theta} = \sum_\theta \frac{c_\theta}{2} Z_{\lambda,\varepsilon,L,\theta}$ where for a fixed $\theta \in \{0,1\}^2$

$$\begin{aligned} Z_{\lambda,\varepsilon,L,\theta} &:= e^{L^2\varepsilon_0} \int D\psi e^{\bar{S}_{0,\theta}(\psi) + \nu_0 N(\psi) + \bar{V}(\psi)}, \\ \bar{S}_\theta(\psi) &:= -L^{-2} \sum_{k \in \mathcal{D}_\theta} \psi^+_k \left(1 + \frac{e^{ik_1} + e^{ik_2}}{2}\right) \psi^-_k, \\ N(\psi) &:= -L^{-2} \sum_{k \in \mathcal{D}_\theta} \psi^+_k \psi^-_k, \qquad \nu_0 := e^{-\varepsilon_0} - 1 \end{aligned} \tag{3.3.1}$$

where the factor $e^{L^2\varepsilon_0}$ comes from the rescaling of the Grassmann variables $\psi \mapsto \sqrt{e^{-\varepsilon_0}}\psi$, while $\bar{S}_{0,\theta}$, $\bar{V}(\psi)$ are the rewritings of $S_{0,\theta}, V$ of Proposition 3.2.1 in terms of the rescaled fields. Notice that $\mu_0(k) = 1 + \frac{1}{2}(e^{ik_1} + e^{ik_2})$ (cf. (3.2.6)) has a unique zero at $k_F = (\pi, \pi)$ (mod $(2\pi, 2\pi)$) which belongs to $\mathcal{D}_\theta$ only for $\theta = (0,0)$, for every L. This means that for $\theta = (0,0)$ we must single out such a momenta and study it after the multiscale integration

procedure.[3] Let us postpone the discussion of this technical aspect (see Remark 3.3.6). For $\theta \neq (0,0)$ we can normalize the integration and write

$$Z_{\lambda,\varepsilon,L,\theta} = e^{L^2\varepsilon_0 + F_0^{(0)}} \int P_\theta(d\psi) e^{\nu_0 N(\psi) + \bar{V}(\psi)}, \qquad F_0^{(0)} := L^{-2} \log\det K_{0,\theta} \tag{3.3.2}$$

(cf. (3.2.5) for $K_{0,\theta}$), with the property that for $x, x' \in \Lambda$,

$$\int P_\theta(d\psi) \psi_x^- \psi_{x'}^+ = L^{-2} \sum_{k \in \mathcal{D}_\theta} \frac{e^{-ik(x-x')}}{1 + \frac{e^{ik_1} + e^{ik_2}}{2}} = K_{0,\theta}^{-1}(x,x') \tag{3.3.3}$$

A suitable change of coordinates

For the particular choice of the weights we made, when $B = C$, an extra symmetry is preserved, so that it is convenient to introduce the following change of variables for the discussion to come. Given $k \in \mathcal{D}_\theta$, let

$$\begin{cases} k_1 = q_1 - q_2 \\ k_2 = q_1 + q_2 \end{cases} \tag{3.3.4}$$

which can be written as $k = Rq$. Define then

$$\mathcal{D}_{\theta,R} := R^{-1}\mathcal{D}_\theta = \{q : Rq \in \mathcal{D}_\theta\}. \tag{3.3.5}$$

In terms of the new variables

$$\tilde{\mu}(q) := \mu(Rq) = 1 + e^{iq_1} \cos q_2.$$

This allow us to rewrite K^{-1} in (3.3.3) as

$$K_\theta^{-1}(x,0) = g_{L,\theta}(R^T x, 0), \qquad g_{L,\theta}(y,y') := L^{-2} \sum_{q \in \mathcal{D}_{\theta,R}} \frac{e^{-iq(y-y')}}{\tilde{\mu}(q)}. \tag{3.3.6}$$

Since the covariance naturally depends on x through $R^T x = (x_1 + x_2, x_2 - x_1)$, we introduce the set, $\tilde{\Lambda} := R^T\Lambda = \{(y_1,y_2) : y = R^T x,\, x \in \Lambda\}$, as shown in Figure 3.5. We will use the symbol x (resp. y) for coordintes belonging to Λ (resp. $\tilde{\Lambda}$) and notice that a natural constraint appears on $y = (y_1, y_2)$, i.e. $y_1 = y_2 \pmod 2$. In terms of the Grassmann variables, we denote

$$\hat{\varphi}_q^\pm := \hat{\psi}_{Rq}^\pm, \quad \varphi_y^\pm := \psi_{(R^T)^{-1}y}^\pm \tag{3.3.7}$$

which of course satisfy the analogous of (3.2.4), namely

$$\varphi_y^\pm = L^{-2} \sum_{q \in \mathcal{D}_{R,\theta}} e^{\pm iqy} \hat{\varphi}_q^\pm. \tag{3.3.8}$$

More in general we have a covariant description of (3.3.1) in terms of the new coordinates, obtained by replacing sums over k in $\mathcal{D}_\theta$ with analogous over $q \in \mathcal{D}_{R,\theta}$ (similarly $x \in \Lambda$

[3] Exactly as done in [**2**, Sec. 6.1].

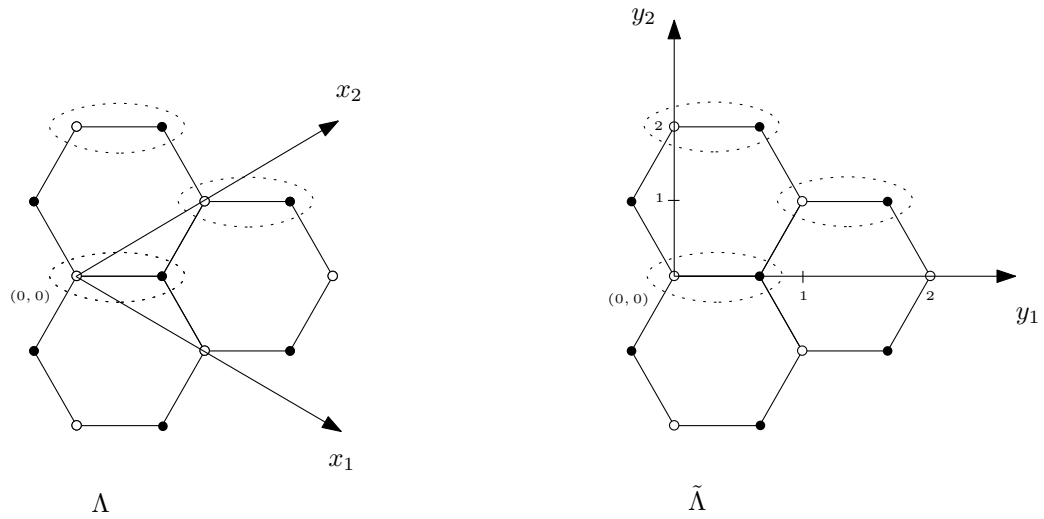

Figure 3.5: *On the left the coordinate system defined in Section 1, on the lattice Λ; on the right the orthogonal coordinates system of $\tilde{\Lambda}$, with the constraint that $y_1 \equiv_2 y_2$, since $y_1 = x_1 + x_2$, $y_2 = x_2 - x_1$.*

with $y \in \tilde{\Lambda}$), every ψ with φ and writing every expression of k as a function of q. In particular note that

$$\begin{aligned} \bar{S}_\theta(\psi(\varphi)) &= L^{-2} \sum_{q \in \mathcal{D}_{\theta,R}} \hat{\varphi}_q^+ \tilde{\mu}(q) \hat{\varphi}_q^- \\ \tilde{N}(\varphi) &= L^{-2} \sum_{q \in \mathcal{D}_{\theta,R}} \hat{\varphi}_q^+ \hat{\varphi}_q^- \end{aligned} \tag{3.3.9}$$

so that we can start the following analysis from (3.3.2), which becomes

$$\frac{Z_{L,\theta,\lambda}}{e^{L^2 \varepsilon_0} Z_{L,\theta,0}} = \int P_\theta(d\varphi) e^{\nu_0 \tilde{N}(\varphi) + \tilde{V}(\varphi)}, \qquad \int P_\theta(d\varphi) \varphi_y^- \varphi_0^+ = g_{L,\theta}(y, 0) \tag{3.3.10}$$

where $\tilde{N}, \tilde{V}$ are the rewritings of $N, \bar{V}$ in terms of the new variables.

Remark 3.3.1. *The choice to introduce $\tilde{\Lambda}$ is not strictly necessary, but we prefer to do so in order not to carry the label $R^T x$ in every expression.*

Integration of the ultraviolet degrees of freedom

Let $\chi : \mathbb{R}^2 \to [0, 1]$ be a smooth, compactly supported function in the *Gevrey*-2 class[4] on the torus, such that

$$\chi(k) = \begin{cases} 1 & |k|_s \leq \delta \\ 0 & |k|_s > 2\delta \end{cases} \tag{3.3.11}$$

[4]We refer to [**3**, Appendix C] and [**129**, Appendix A] for an explicit instance and properties of such class of function.

with $\delta > 0$ small; $|\cdot|_s$ is a *stretched* distance on the torus, i.e.

$$|k|_s^2 = \left|\frac{k_1+k_2}{2}\right|_{\mathbb{T}}^2 + \frac{1}{2}\left|\frac{k_2-k_1}{2}\right|_{\mathbb{T}}^4$$

where we define $|\cdot|_{\mathbb{T}} := \min\{|\cdot - 2\pi n| : n \in \mathbb{N}\}$. Then let

$$\chi_{-1}(q) := \chi(R^{-1}k) \tag{3.3.12}$$

and let also $f_0(q) := 1 - \chi_{-1}(q)$. We introduce a smooth partition of $\mathcal{D}_{R,\theta}$ via the identity $1 = \chi_{-1}(q - q_F) + f_0(q - q_F)$ in such a way that we have the decomposition $g_L(y,0) = (-1)^{y_2}(g_L^{(\le -1)}(y,0) + g_L^{(0)}(y,0))$, where

$$g_L^{(\le -1)}(y,0) = \frac{1}{L^2}\sum_{q\in\mathcal{D}'_{\theta,R}} \frac{e^{-iqy}\chi_{-1}(q)}{\tilde{\mu}(q+q_F)} = \frac{1}{L^2}\sum_{q\in\mathcal{D}'_{\theta,R}} \frac{e^{-iqy}\chi_{-1}(q)}{1 - e^{iq_1}\cos q_2}. \tag{3.3.13}$$

and

$$\mathcal{D}'_{\theta,R} = \mathcal{D}_{\theta,R} - q_F, \qquad q_F := R^{-1}k_F = (\pi, 0) \tag{3.3.14}$$

(cfr. below (3.3.1) and (3.3.6)). Note that, compared to K^{-1}, now $g_{L,\theta}(y,y')$ is still translation invariant, but θ periodic (in the same sense as just after eq. (6.25) of [**2**]) in the directions $\vec{e_1} + \vec{e_2}$ and $\vec{e_2} - \vec{e_1}$. The decomposition above naturally reflects on the Grassmann integration via the Addition Principle of Grassmann measures (see [**5**] Eq. (4.21)): we can write,

$$\varphi_y^\sigma = (-1)^{y_1}(\varphi_y^{\sigma,(\le -1)} + \psi_y^{\sigma,(0)}) \tag{3.3.15}$$

where

$$\int P^{(\le 0)}(d\varphi^{(\le -1)})\varphi_y^{+,(\le -1)}\varphi_{y'}^{-,(\le -1)} = g_L^{(\le -1)}(y,y') \tag{3.3.16}$$

(similarly for $g_L^{(0)}$) and decompose

$$\begin{aligned} &Z_{L,\theta,\lambda} = \\ &= e^{L^2F^{(0)}}\int P^{(\le -1)}(d\varphi^{(\le -1)})\int P^{(0)}(d\varphi^{(0)})e^{\nu_0 N(\varphi^{(\le -1)}+\varphi^{(0)})+V((\varphi^{(\le -1)}+\varphi^{(0)})} \\ &= e^{L^2F^{(-1)}}\int P^{(\le -1)}(d\varphi^{(\le -1)})e^{V^{(-1)}(\varphi^{(\le -1)})} \end{aligned} \tag{3.3.17}$$

with

$$F^{(0)} := \varepsilon_0 + L^{-2}\log Z_{L,\theta,0} \tag{3.3.18}$$

and the effective potential $V^{(-1)}$ and the u.v. contribution to the free energy, $F^{(-1)} - F^{(0)}$, satisfying $V^{(-1)}(0) = 0$ and

$$\begin{aligned} &L^2\left(F^{(-1)} - F_L^{(0)}\right) + V^{(-1)}(\varphi) = \\ &= \sum_{n\ge 1}\frac{1}{n!}\mathcal{E}_{(0)}^T(\underbrace{\nu_0 N(\varphi+\cdot) + V(\varphi+\cdot); \ldots; \nu_0 N(\varphi+\cdot) + V(\varphi+\cdot)}_{\text{n times}}), \end{aligned} \tag{3.3.19}$$

where $\mathcal{E}_{(0)}^T$ is the truncated expectation, (see Sec. 4 of [**5**]) with respect to the integration $P^{(0)}$.

Remark 3.3.2. *Since the cutoff function χ_{-1} in* (3.3.11) *is a Gevrey function of order* 2*, the propagator $g^{(0)}$ has stretched-exponential decay at large distances:*

$$|g^{(0)}(y,y')| \le Ce^{-\kappa\sqrt{|y-y'|}}, \tag{3.3.20}$$

*for suitable L-independent constants $C,\kappa > 0$, if $|y-y'|$ is the distance on the torus $\tilde\Lambda$. This is seen by writing $g^{(0)}$ via the Poisson summation formula as a sum of Fourier integrals, as in [**3**, App. A]; each integral decays in the desired way because it is the Fourier transform of a Gevrey function [**130**], [**129**, App. A]*

From (3.3.19) it follows that $V^{(-1)}$ is a polynomial in the $\varphi^{(\le -1)}$ fields made of even monomials only. In particular it admits the following representation, which is an implicit definition of the kernels $W_n^{(-1)}$: setting $\phi = \varphi^{(\le -1)}$,

$$V^{(-1)}(\phi) = \sum_{\substack{n\ge 2\\ n \text{ even}}} \sum_{\underline{y}\in\tilde\Lambda^n} W_n^{(-1)}(\underline{y})\phi^+_{y_1}\phi^-_{y_2}\cdots\phi^+_{y_{n-1}}\phi^-_{y_n} \tag{3.3.21}$$

where $y_i \in \tilde\Lambda$, and $\underline{y} = (y_1,\dots,y_n)$; note that by definition they include the oscillatory factors, arising from (3.3.15), $W_n(\underline{y}) = (-1)^{\sum_{k=1}^n y_{k1}}\widetilde{W}_n(\underline{y})$ where we denoted $y_i \in \tilde\Lambda$ with $y_i = (y_{i1}, y_{i2})$. The kernels are not uniquely identified by such representation, due to anticommutation of the Grassmann variables. We assume then that they are antisymmetric separately under permutations of the variables y with even and odd index, i.e. $W_n(y_{\pi(1)}, y_2, \dots, y_{\pi(n-1)}, y_n) = W_n(y_1, y_2, \dots, y_{n-1}, y_n)$ for any permutation π, and similarly for coordinates with even position. For later convenience, using the convention of Grassmann variables in Fourier space ((3.3.8) considering the shift around q_F), we can rewrite

$$V^{(-1)}(\phi) = L^{-2(n-1)}\sum_{\underline{q}} \hat W^{(-1)}(q_2,\dots,q_n)\hat\phi^+_{q_1+q_F}\cdots\hat\phi^-_{q_n+q_F}\delta(\underline{q}) \tag{3.3.22}$$

where $\hat W(q_1,\dots,q_n)$ is the Fourier transform of $\widetilde W$ computed at momenta $q_2+q_F,\dots,q_n+q_F$ which depends on $n-1$ variables due to translation invariance; $\underline{q} = (q_1,\dots,q_n)$ and $q_i \in \mathcal{D}'_{R,\theta}$ defined in (3.3.13), and

$$\delta(\underline{q}) = \begin{cases} 1 & \sum_{j=1}^n (-1)^j q_j = 0 \\ 0 & \text{otherwise.}\end{cases} \tag{3.3.23}$$

Then using the so called *Battle-Brydges-Federbush-Kennedy* determinant formula and the Gram Hadamard bound for determinants (see Section 4.2 of [**5**]), applied to the truncated expectation in (3.3.19), we find that $F^{(-1)} - F^{(0)}$ and $W_n^{(-1)}(\underline{y})$ are absolutely convergent series and real analytic function of (ν_0,λ) (recall $\nu_0 = e^{-\varepsilon_0}-1$) if $\max\{|\nu_0|,|\lambda|\} \le \bar\delta$, with $\bar\delta > 0$ small enough. Moreover the following bounds hold,

$$\begin{aligned} &|F^{(-1)} - F_\theta(0)| \le c\bar\delta \\ &\|W_n^{(-1)}\|_{\kappa,-1} \le C^n\bar\delta^{\max\{1,cn\}}, \qquad n>0 \end{aligned} \tag{3.3.24}$$

for L independent constants c, C and where we introduced the norm

$$\|W_n^{(-1)}\|_{\kappa,-1} := L^{-2} \sum_{\underline{y}\in\tilde{\Lambda}^n} |W_n^{(-1)}(\underline{y})| e^{\kappa\sqrt{2^{-1}d(\underline{y})}} \tag{3.3.25}$$

with $d(y_1,\dots,y_m)$ being the length of the shortest tree in $\tilde{\Lambda}$ connecting the m points $y_1,\dots,y_m$. The reason for the use of this norm is related to the exponential decay of the single scale propagator involved in the truncated expectation, see (3.3.20). For a proof of (3.3.24) in a similar context, see [**46**, Sec. III A].

Remark 3.3.3. *(1) Since the kernels $W_n^{(-1)}$ are absolutely convergent series in λ, ν_0, and have estimates uniform in L, their $L\to\infty$ limit is well defined and satisfy the same estimates: this follows from the fact that each term in their expansion admits an $L\to\infty$ limit. Indeed every propagator $g^{(0)}_{L,\theta}$ will be replaced by its $L\to\infty$ limit $g^{(0)}_\infty$*

$$g^{(0)}_\infty(y,0) = \frac{1}{\pi^2}\int_{R^{-1}[0,2\pi]^2} dq \frac{e^{-iqy} f_0(q)}{\tilde{\mu}(q+q_F)} \tag{3.3.26}$$

and every sum of coordinates over the torus $\tilde{\Lambda}$ will be replaced with the sum over $y\in R^T\mathbb{Z}^2$. For later reference we will denote this $L\to\infty$ limit of the kernels by $W_n^{(-1);\infty}$ and similarly for its Fourier transform.

The output of this section is that we have now a description of the partition function $Z_{L,\theta,\lambda}$, see (3.3.17), in terms of an effective interaction $V^{(-1)}$ involving only "infrared degrees of freedom" $\varphi^{(\le -1)}$ and for which we have bounds on its kernels, see (3.3.24). At this point one could iterate the idea by writing $\varphi^{(\le -1)} = \varphi^{(-1)} + \varphi^{(\le -2)}$ and integrating the field $\varphi^{(-1)}$ associated to a propagator $g_L^{(-1)}$, with the same expression as (3.3.26) where f_0 replaced by $f_{-1}(q) := \chi_{-1}(q) - \chi_{-2}(q)$, with $\chi_{-2}(q) := \chi_{-1}(2q)$. This will produce an effective interaction $V^{(-2)}$, for which one could deduce similar bounds as above, involving only fields $\varphi^{(\le -2)}$. Then the iteration stops on a scale $h_L = O(\log_2(L))$ for which the support of $\chi^{(\le h_L)}$ does not intersect anymore the lattice $\mathcal{D}_{\theta,R}$. Integrating the last field $\varphi^{(h_L)}$ then one obtain the partition function Z_λ:

$$Z_L = e^{F^{(h_L)}}, \qquad F^{(h_L)} = F^{(0)} + (F^{(-1)} - F^{(0)}) + \cdots + (F^{(h_L)} - F^{(h_L+1)}) \tag{3.3.27}$$

where for $h = 0,..,h_L$, $F^{(h)} - F^{(h+1)}$ is the contribution coming from the integration on scale h, compare with (3.3.19). The problem of this naive iteration is that it does not allow to obtain uniform bounds in L, as explained in [**3**, Section 5.2] or [**5**, Section 5.4]. In order to cure this apparent divergence, one has to *renormalize* some terms. In order to distinguish which are the dangerous terms, the scaling behavior of the propagator $G^{(h)}$, on scale h, is a key ingredient; $G^{(h)}$ is given by (3.3.26) where $f_0(q)$ is replaced by $f_h(q) = \chi_h - \chi_{h-1}$ and replacing $\tilde{\mu}(q+q_F)$ with its Taylor approximation at $q=0$. By a rescaling argument one has that $G^{(h)}(y_1,y_2) = 2^{h/2}G^{(0)}(2^h y_1, 2^{h/2}y_2)$. With such a scaling at hand, repeating the same discussion of [**3**, Sec. 5.2.1] one finds that a kernel $W_n^{(h)}$ on scale h, can be bounded as $\|W_n^{(h)}\|_{\kappa,h} \le C^n \lambda^{\max\{1,cn\}} 2^{h(\frac{3}{2}-\frac{n}{4})}$ for constants $c, C(L) > 0$, with $C(L)$ a priori not uniformly bounded in L. As explained in [**3**, Sec. 5.2.2], the potentially dangerous terms for a convergent perturbative expansion uniform in L, are those appearing in the effective potentials that are associated to a non-negative *scaling*

dimension $D_{sc}(n) := \frac{3}{2} - \frac{n}{4}$. In the Renormalization Group jargon, terms for which D is positive, negative or zero are respectively called *relevant, marginal* and *irrelevant*. This means that some procedure has to be implemented for the terms with $n = 2, 4, 6$ Grassmann fields (note in fact that $D_{sc}(2) = 1, D_{sc}(4) = 1/2, D_{sc}(6) = 0$ while $D_{sc}(n) < 0$ for $n > 6$). Let us now describe the so called *localization* procedure acting on marginal and relevant terms.

Inductive scheme: localization operator

After the integration of the ultraviolet degrees of freedom we are left with the following representation given in the third line of (3.3.17) where $V^{(-1)}$ is given in (3.3.21). Now we describe a single step of the anticipated *renormalization* procedure which will lead us to represent $Z_{L,\theta,\lambda}$ in terms of an analogous expression as above, with the scale label -1 replaced by -2. In virtue of the previous brief discussion, let us define a *localization* operator $\mathcal{L}_{1,0}$[5], acting on the effective potential. We will write

$$V^{(-1)} = \mathcal{L}_{1,0} V^{(-1)} + \mathcal{R}_{1,0} V^{(-1)} \tag{3.3.28}$$

where $\mathcal{R}_{1,0}$ is called *renormalization* operator and it is a rewriting of $1 - \mathcal{L}_{1,0}$, with $\mathcal{L}_{1,0}$ acting as follows: using the Fourier representation in (3.3.22), we define

$$\begin{aligned} &\mathcal{L}_{1,0} V^{(-1)}(\phi) := \\ &L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} \phi^+_{q+q_F} \phi^-_{q+q_F} \left(\hat{W}_2^{\infty;(-1)}(0) + q \cdot \partial \hat{W}_2^{\infty;(-1)}(0) + \frac{1}{2} q_2^2 \partial_2^2 \hat{W}_2^{\infty;(-1)}(0) \right) \end{aligned} \tag{3.3.29}$$

where $\phi = \varphi^{(\leq -1)}$ and recall $\hat{W}^{\infty;(-1)}$ is the same as in Remark 3.3.3; ∂_i is the partial derivative on the component $i = 1, 2$, and q_F as below (3.3.13).

The reader may notice that $\mathcal{L}_{1,0}$ has no effect on the contributions with 4 or 6 Grassmann variables, even if we claimed in the previous discussion that something has to be done. The point is that, as a consequence of the Pauli exclusion principle, i.e. $\varphi_y^2 = 0$ for every $y \in \tilde{\Lambda}$, one can automatically rewrite, e.g. the quartic term $V_4^{(-1)}(\phi)$ of $V^{(-1)}(\phi)$, as

$$V_4^{(-1)}(\phi) = \sum_{\underline{y} \in \tilde{\Lambda}^4} \phi^+_{y_1} \phi^-_{y_2} (\phi^+_{y_3} - \phi^+_{y_1})(\phi^-_{y_4} - \phi^-_{y_1}) W_4^{(-1)}(\underline{y}) \tag{3.3.30}$$

and similarly for the term with 6 Grassmann variables. Using the convention in Fourier space (3.3.22) this can be rewritten as

$$L^{-6} \sum_{i,j=1,2} \sum_{\underline{q}} \hat{\phi}^+_{q_1+q_F} \hat{\phi}^-_{q_2+q_F} \hat{\phi}^+_{q_3+q_F} \hat{\phi}^-_{q_4+q_F} \delta(\underline{q}) q_{3i} q_{4j} \mathcal{I}_{ij} W_4^{(-1)}(q_2, q_3, q_4) \tag{3.3.31}$$

where we denoted $q_3 = (q_{31}, q_{32})$ and the interpolated kernel $\mathcal{I}_{ij} W_4$ is given by

$$\mathcal{I}_{ij} W_4(q_2, q_3, q_4) := \int_0^1 \int_0^1 ds dt \partial_i \partial_j W_4^{(-1)}(q_2, t q_3, s q_4).$$

[5]The pedices stands respectively for "first regime" and $\epsilon = 0$: they refer to the fact that when $\varepsilon = \varepsilon_0 + \epsilon$, $\epsilon = 0$, only a first regime survives; for a better understanding compare with Sec 3.3.2.

By anti-trasforming the equation above one can obtain (see Appendix B.1) an expression in coordinates space in terms of pseudo differential operators $\hat{\partial}_i$, of the form

$$V_4^{(-1)}(\phi) = \sum_{\underline{y}} \phi^+_{y_1}\phi^+_{y_2}\hat{\partial}_i\phi^+_{y_3}\hat{\partial}_j\phi^-_{y_4}\tilde{\mathcal{R}}_{ij}W_4^{(-1)}(\underline{y}) \tag{3.3.32}$$

where $\tilde{\mathcal{R}}W_4^{(-1)}(\underline{y})$ is the antitransform of the interpolated kernel and

$$\hat{\partial}_i\phi^\pm_y := L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}} e^{\mp iqy}\hat{\phi}^\pm_{q+q_F}\chi_{-1}(q)q_i. \tag{3.3.33}$$

A similar representation holds as well for the kernels associated to 6 Grassmann variables and moreover for the rewriting of $1-\mathcal{L}_{1,0}$ acting on the term with 2 fields, coming from (3.3.29). For more details see Appendix B.1. The point of such rewriting is that it will allow, in the multiscale iterative construction, to obtain "dimensional gains" showing that such terms, originally potentially dangerous, are instead irrelevant. This is discussed in Remark 3.3.5.

Remark 3.3.4. *(1) The above discussion suggests that we have to think at the contributions to effective potentials, when seen in coordinates space $\tilde{\Lambda}$, as labeled not only by the number and the coordinates of the grassmann fields they are associated to, but also by extra labels which carry informations about the derivatives $\hat{\partial}_i$. Indeed, if we intend to iterate the integration procedure, they will naturally appear as a by product of the renormalization operator effect as shown above.*

(2) As explained in Section B.4, since the bare potential $\bar{V}$ (c.f (3.3.1)*) and the quadratic action associated to $P^{(0)}$ (cf.* (3.3.17)*) are both invariant under the symmetries (1) $\varphi_q^{(\leq -1)} \to \varphi_q^{(\leq -1)}$, together with the complex conjugation of the constants $c \to \bar{c}$, and (2) $\varphi_q^{(\leq -1)} \to \varphi_{\hat{q}}^{(\leq -1)}$ with $\hat{q} = (q_1, -q_2)$, we have that $V^{(-1)}$ is invariant under the same symmetries and as a consequence its kernels satisfy*

$$\begin{aligned} &\hat{W}_2^{\infty;(-1)}(0) \in \mathbb{R}, \\ &\partial_1\hat{W}_2^{\infty;(-1)}(0) \in i\mathbb{R}, \\ &\partial_2\hat{W}_2^{\infty;(-1)}(0) = 0, \\ &\partial_2^2 W_2^{\infty;(-1)}(0) \in \mathbb{R}. \end{aligned} \tag{3.3.34}$$

In particular, (3.3.29) *can be rewritten as*

$$L^{-2}\sum_{q\in\mathcal{D}'_{R,\theta}} \hat{\varphi}^+_{q+q_F}\hat{\varphi}^-_{q+q_F}(\hat{W}_2^{\infty;(-1)}(0) - i\tilde{\zeta}_{-1}q_1 + \frac{1}{2}q_2^2\tilde{a}_{-1}) \tag{3.3.35}$$

for some real constants $\tilde{\zeta}_{-1}, \tilde{a}_{-1}$.

(3) A comparison with [4], shows that for kernels with 4 *or* 6 *legs, the renormalization operator is exactly the same, compare* (3.3.30) *with [4, (30)]. Instead note that the localization procedures differs only for two irrelevant contributions, as we now point*

out. If one writes the localization $\mathcal{L}_1$ in [4, (43)] in Fourier space, one finds

$$\mathcal{L}_{1,0}V = \int d\mathbf{k}\psi^+_{\mathbf{k}}\psi^-_{\mathbf{k}}\left(\hat{W}_2(0) - ik_0\partial_0\hat{W}_2(0) + (\cos k - 1)\partial_1^2\hat{W}_2(0)\right) \tag{3.3.36}$$

with the conventions given in [4], e.g. $\mathbf{k} = (k_0, k)$. Then the two differences between $\mathcal{L}_1$ and $\mathcal{L}_{1,0}$ above are given by "finite size effects", $\hat{W}_2^\infty(0) - \hat{W}_2(0)$ and by terms of order $O(q_2^4)$ (the identification is $(k_0, k_1) \leftrightarrow (q_1, q_2)$. Both terms are irrelevant in the sense discussed before the beginning of this section. For a discussion concerning the former, which is not object of further discussion here, the reader can look at [2, Appendix B]. For the latter note that a term of order $O(q_2^4)$ can be rewritten in coordinate space as

$$\sum_y \hat{\partial}_2^2\varphi^+_y\hat{\partial}_2^2\varphi^-_y w_2(y) \tag{3.3.37}$$

where $\hat{\partial}$ is the same as in (3.3.33) *(for a similar computation see Appendix B.1). As shown in the next Remark, the presence of extra derivatives produce dimensional gains associated to irrelevant contributions.*

Now, we start from (3.3.17), decompose $V^{(-1)} = (\mathcal{L}_{1,0} + \mathcal{R}_{1,0})V^{(-1)}$, and use some of the term of the local part $\mathcal{L}_{1,0}V^{(-1)}$ to define a new Grassmann measure $\tilde{P}^{(\leq -1)}(d\varphi^{(\leq -1)})$ such that the following holds

$$(3.3.17) = e^{L^2(F^{(-1)}+t^{(-2)})}\int \tilde{P}^{(\leq -1)}(d\varphi^{(\leq -1)})e^{2^{-1}\nu_{-1}N(\varphi^{(\leq -1)})+\mathcal{R}_{1,0}V^{(-1)}(\varphi^{(\leq -1)})} \tag{3.3.38}$$

where, renaming $\tilde{\mu}_{-1} := \tilde{\mu}$, the Grassmann measure $\tilde{P}^{(\leq -1)}(d\varphi^{(\leq -1)})$ is associated to the following propagator

$$\tilde{g}_L^{(\leq -1)}(y,0) := L^{-2}\sum_{q\in\mathcal{D}'_{R,\theta}} \frac{e^{-iqy}\chi_{-1}(q)}{\tilde{\mu}_{-1}(q+q_F) + \chi_{-1}(q)(-iq_1\tilde{\zeta}_{-1} + \frac{q_2^2}{2}\tilde{a}_{-1})} \tag{3.3.39}$$

where the term $t^{(-2)}$, obtained by the change of measure is given by

$$t^{(-2)} := L^{-2}\sum_{q\in\mathcal{D}'_{R,\theta}} \log\frac{\tilde{\mu}_{-1}(q+q_F) + \chi_{-1}(q)(-iq_1\tilde{\zeta}_{-1} + \frac{q_2^2}{2}\tilde{a}_{-1})}{\tilde{\mu}_{-1}(q+q_F)}. \tag{3.3.40}$$

Note that expanding $\tilde{\mu}_{-1}(q+q_F) = D_{-1}(q)+\rho(q)$, where $D_{-1}(q)$ is the leading contribution for $|q|$ small defined at the beginning of Section 3.3.1, and $\rho(q) = 1 - e^{iq_1}\cos q_2 + iq_1 - \frac{q_2^2}{2}$, one has that the denominator in the r.h.s. of (3.3.39) can be rewritten as $\boldsymbol{D}_{-2}(q) + \rho(q)$ with

$$\begin{aligned}
&\boldsymbol{D}_{-2}(q) := -iq_1(1 + \boldsymbol{\zeta}_{-2}(q)) + \frac{q_2^2}{2}(1 + \boldsymbol{\alpha}_{-2}(q))\\
&\boldsymbol{\zeta}_{-2}(q) := \zeta_{-1} + \chi_{-1}(q)\tilde{\zeta}_{-1}\\
&\boldsymbol{\alpha}_{-2}(q) := \alpha_{-1} + \chi_{-1}(q)\tilde{a}_{-1}
\end{aligned} \tag{3.3.41}$$

where $\zeta_{-1}, \alpha_{-1} = 0$ are introduced just for consistency with the notation of the general case of the next pages and $\tilde{\zeta}_{-1}, \tilde{a}_{-1}$ are given in (3.3.35). Then, we use the addition principle to the decomposition of the propagator $\tilde{g}_L^{(\leq -1)}(y,0) = g_L^{(\leq -2)}(y,0) + \tilde{g}_L^{(-1)}(y,0)$

where

$$\tilde{g}_L^{(-1)}(y,0) := L^{-2} \sum_{q \in \mathcal{D}'_{R,\theta}} e^{-iqy} \frac{f_{-1}(q)}{\boldsymbol{D}_{-2}(q) + \rho(q)} \tag{3.3.42}$$

obtained by replacing $\chi_{-1} = f_{-1} + \chi_{-2}$, with $\chi_{-2}(q) := \chi_{-1}(2q)$, which allows us to rewrite (3.3.38) as

$$\begin{aligned} & e^{L^2(F^{(-2)}+t^{(-1)})} \int P^{(\leq -2)}(d\varphi^{(\leq -2)}) e^{V^{(-2)}(\varphi^{(\leq -2)})} \\ & e^{V^{(-2)}(\varphi)+L^2(F^{(-2)}-F^{(-1)})} := \int \tilde{P}^{(-1)}(d\varphi') e^{\nu_{-1} N(\varphi+\varphi') + \mathcal{R}_{1,0} V^{(-1)}(\varphi+\varphi')} \end{aligned} \tag{3.3.43}$$

where $P^{(\leq -2)}$, $\tilde{P}^{(-1)}$ are the Grassmann measures associated respectively to the propagators $g_L^{(\leq -2)}$, $\tilde{g}_L^{(-1)}$. Note that in the support of χ_{-2}, $\boldsymbol{\zeta}_{-1}(q) = \zeta_{-1}(0) =: z_1$ and $\boldsymbol{\alpha}_{-1}(q) = \alpha_{-1}(0) =: \alpha_1$ so that the propagator $g_L^{(\leq -2)}$ is obtained by (3.3.13) by shifting the scale index $-1 \mapsto -2$, cfr. the definitions of $\tilde{\mu}_0, \tilde{\mu}_{-1}$. Finally ν_{-1} appearing in (3.3.38) it is given by

$$2^{-1}\nu_{-1} := \hat{W}_2^{\infty;(-1)}(0) \tag{3.3.44}$$

The effect of this single scale integration is to give a description of the partition function in terms of a new effective interaction $V^{(-2)}$ integrated against a new covariance $P^{(\leq -2)}$ involving degrees of freedom with smallest momenta in Fourier space. It is natural then to extend this procedure to any finite step, until the covariance is supported on some point of $\mathcal{D}_{\theta,R}$. We now state and discuss what is the expression for $Z_{L,\theta,\lambda}$ after h steps of integration, without explicitly proving such a statement. The proof will be given in a more general setting in Section 3.3.2.

Inductive statement There exists a scale $h_L = O(\log_2 L)$, such that, for any integer $-h_L \leq h \leq -1$, the following identity holds

$$Z_{L,\theta,\lambda} = e^{L^2(F^{(h)}+t^{(h)})} \int P^{(\leq h)}(d\varphi^{(\leq h)}) e^{V^{(h)}(\varphi^{(\leq h)})} \tag{3.3.45}$$

where h_L is given explicitly below and $F^{(h)}$, $V^{(h)}$, $P^{(\leq h)}$ are inductively described as follows.

(1) First, let $\chi_h(q) := \chi_{h+1}(2q)$ and χ_{-1} was defined in (3.3.12). Thus the stopping scale h_L is defined by

$$h_L := \min\{h \leq 0 : \operatorname{supp} \chi_h \cap \mathcal{D}'_{\theta,R} \neq \emptyset\}. \tag{3.3.46}$$

Note that $h_L = O(\log_2 L)$.

The Grassmann integration $P^{(\leq h)}$ is associated to the propagator

$$g_L^{(\leq h)}(y,y') = L^{-2} \sum_{q \in \mathcal{D}'_{R,\theta}} \frac{e^{-iq(y-y')}\chi_h(q)}{\tilde{\mu}_h(q+q_F)} \tag{3.3.47}$$

where $\tilde{\mu}_h$ it is defined by

$$\tilde{\mu}_h(q+q_F) := D_h(q) + \rho(q) \tag{3.3.48}$$

with $\rho(q)$ being the same as below (3.3.41) while $D_h(q) = -iq_1(1+\zeta_h) + \frac{1}{2}q_2^2(1+\alpha_h)$ with

$$\begin{aligned} \zeta_h &= \zeta_{h+1} + i\partial_1 \hat{W}_2^{(h+1);\infty}(0) \\ \alpha_h &= \alpha_{h+1} + \partial_2^2 \hat{W}_2^{(h+1);\infty}(0). \end{aligned} \tag{3.3.49}$$

The kernels $\hat{W}^{(h+1)}$ of the effective potential $V^{(h+1)}$ are known by induction. The effective potential on scale h satisfies $V^{(h)}(0) = 0$ and it is given by

$$V^{(h)}(\varphi) + L^2 F^{(h)} = L^2 F^{(h+1)} + \log \int \tilde{P}^{(h+1)}(d\phi) e^{2^{h+1}\nu_{h+1} N(\varphi+\phi) + \mathcal{R}_{1,0} V^{(h+1)}(\varphi+\phi)} \tag{3.3.50}$$

which is also a definition of $F^{(h)} - F^{(h+1)}$ by setting $\varphi = 0$.

As commented in Remark 3.3.4 and above, if one represents the effective potential in coordinate space, then "derivative" operators $\hat{\partial}$ (see (3.3.33)) can appear on Grassmann variables: indeed if $\phi = \varphi^{(\leq h)}$, then $V^{(h)}(\phi)$ can be written as

$$\begin{aligned} V^{(h)}(\phi) &= \sum_{\substack{n \in 2\mathbb{N} \\ \underline{D}}} V^{(h)}_{n,\underline{D}}(\phi) \\ V^{(h)}_{n,\underline{D}}(\phi) &:= \sum_{\underline{y} \in \tilde{\Lambda}^n} \hat{\partial}^{D_1} \phi^+_{y_1} \cdots \hat{\partial}^{D_n} \phi^-_{y_n} W^{(h)}_{n,\underline{D}}(\underline{y}), \quad \partial^{D_j} \phi^\sigma_y := \hat{\partial}_1^{D_{j1}} \hat{\partial}_2^{D_{j2}} \phi^\sigma_y \end{aligned} \tag{3.3.51}$$

where $\underline{D} = (D_1, \dots, D_n)$ with $D_j = (D_{j1}, D_{j2}) \in \{0,1,2\} \times \{0,1,2,3\}$ keeps track of the amount and direction of the derivatives: The above expression is not unique but the point is that it exists. Anyway, by writing such an expression in Fourier space (see conventions (3.3.22)), we have that

$$V^{(h)}(\phi) = \sum_{n \in 2\mathbb{N}} L^{-2(n-1)} \sum_{\underline{q} \in (\mathcal{D}'_{\theta,R})^n} \hat{\phi}^+_{q_1+q_F} \cdots \hat{\phi}^-_{q_n+q_F} \delta(\underline{q}) \hat{W}^{(h)}_n(q_2, \dots, q_n) \tag{3.3.52}$$

where $\hat{W}^{(h)}_n$ is expressed in terms of a combination of the Fourier transforms of the kernels $W_{n,\underline{D}}$ and includes the factors q_i^D's coming from the effect of $\hat{\partial}$ (cf. (3.3.33)). This representation has the advantage of not distinguishing between the derivatives labels in the Grassmann monomial.

Next, the Grassmann integration $\tilde{P}^{(h+1)}$ in (3.3.50) is associated to the propagator

$$\tilde{g}^{(h+1)}_L(y, y') = L^{-2} \sum_{q \in \mathcal{D}'_{R,\theta}} \frac{e^{-iq(y-y')} f_{h+1}(q)}{\boldsymbol{D}_h(q) + \rho(q)} \tag{3.3.53}$$

with $f_h(q) := \chi_{h+1}(q) - \chi_h(q)$, $\boldsymbol{D}_h(q) := -iq_1(1 + \boldsymbol{\zeta}_h(q)) + \frac{1}{2}q_2^2(1 + \boldsymbol{\alpha}_h(q))$ where

$$\begin{aligned} \boldsymbol{\zeta}_h(q) &= \zeta_{h+1}(q) + i\chi_{h+1}(q)\partial_1 \hat{W}_2^{(h+1);\infty}(0) \\ \boldsymbol{\alpha}_h &= \alpha_{h+1} + \chi_{h+1}(q)\partial_2^2 \hat{W}_2^{(h+1);\infty}(0) \end{aligned} \tag{3.3.54}$$

and observe that $\boldsymbol{D}_h(q) \equiv D_h(q)$ in the support of $\chi_h(q)$. Next, in the argument of the exponential in the r.h.s. of (3.3.50),

$$2^{h+1}\nu_{h+1} := \hat{W}_2^{(h+1);\infty}(0), \tag{3.3.55}$$

and $N(\varphi)$ was defined in (3.3.9); the renormalization operator $\mathcal{R}_{1,0}$ acts on effective potentials of any given scale h, as a rewriting of $1-\mathcal{L}_{1,0}$ in exactly the same way as explained above Remark 3.3.4, where on a generic scale h, the localization operator acts as $\mathcal{L}_{1,0}V_{n,\underline{D}}=0$ if $n>2$ and for $n=2$ as

$$\begin{aligned}&\mathcal{L}_{1,0}\left(\sum_{\underline{D}}V_{2,\underline{D}}(\phi)\right)=\\&L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}\phi^+_{q+q_F}\phi^-_{q+q_F}\left(\hat{W}_2^{\infty;(h)}(0)+q\cdot\partial\hat{W}_2^{\infty;(h)}(0)+\frac{1}{2}q_2^2\partial_2\hat{W}_2^{\infty;(h)}(0)\right)\end{aligned}\tag{3.3.56}$$

where $\phi=\varphi^{(\le h)}$. Finally, the constant $t^{(h)}$ in (3.3.45) satisfies

$$t^{(h)}-t^{(h+1)}=\sum_{\substack{q\in\mathcal{D}'_{R,\theta}\\ \chi_{h+1}(q))\neq 0}}\log\frac{\boldsymbol{D}_h(q)+\rho(q)}{D_{h+1}(q)+\rho(q)}.\tag{3.3.57}$$

Remark 3.3.5. *(1) At any given scale h, the quadratic part of the measure $\tilde{P}^{(h)}$ preserves the symmetries (1)-(2) of Remark 3.3.4 (applied to the fields $\varphi^{(\le h)}$), as it follows from its explicit expression and by induction. As a consequence, the properties listed in* (3.3.29) *are valid replacing the scale label (-1) with (h) and as a particular consequence $\zeta_h,\alpha_h,\nu_h\in\mathbb{R}$. This also implies $\tilde{\mu}_h(q+q_F)$ vanishes only at $q=(0,0)$ for every $h\ge h_L$.*

(2) Note that we can write, for $h\le -1$,

$$\nu_h=2\nu_{h+1}+B^\nu_h\tag{3.3.58}$$

where B^ν_h is the "ν-component" of the so called Beta function, which is implicitly defined as follows. To see the above identity, note that the contributions to the effective potential $V^{(h)}$ given by the second term in the RHS of (3.3.50)*, can be written in terms of truncated expectations, exactly as in* (3.3.19)*, with $\mathcal{E}^T_{(0)}$ replaced by $\mathcal{E}^T_{(h+1)}$ associated to the covariance $\tilde{P}^{(h+1)}$. Then, from $n=1$ in* (3.3.19) *we obtain*

$$\mathcal{E}_{(h+1)}\left(2^{h+1}\nu_{h+1}N(\varphi+\cdot)+\mathcal{R}_{1,0}V^{(h+1)}(\varphi+\cdot)\right).\tag{3.3.59}$$

When applying $\mathcal{L}_{1,0}$ to such terms, it is easy to see that the only surviving is exactly $2^{h+1}\nu_{h+1}N(\varphi)$ (cfr. (3.3.9) *for N), because $\mathcal{L}_{1,0}(1-\mathcal{L}_{1,0})=0$ (see (3) of Remark 3.3.4). In particular the contributions to B^ν_h comes only from the analogous of* (3.3.19) *at scale $(h+1)$ when $n\ge 2$.*

(3) By repeating the discussion in (3.3.30) *and lines below, we have an analogous rewriting of the operator $1-\mathcal{L}_{1,0}$ acting e.g. on the quartic terms $V_4^{(h)}(\phi)$ as obtained by replacing the label -1 with h. We want to motivate, at least with the tools introduced up to now, that such a rewriting has a non trivial effect by extracting dimensional gains. Note first that the two derivatives appearing in the Fourier representation of $\mathcal{R}_{ij}W_4^{(h)}$, act on the Fourier symbol of some propagator* (3.3.53) *on scale $h'\ge h$, with the effect of producing a term of the order $2^{-h'},2^{-h'/2}$, respectively if the derivative is ∂_i with $i=1,2$. This is due to the presence of the cutoff $f_{h'}(q)=f_0(2^{-h'}q)$*

(see (3.3.54)*): either* ∂_i *acts on* $(\boldsymbol{D}_h(q)+\rho(q))^{-1}$ *or on* $f_{h'}(q)$*, and its action can be bounded by using the support properties of* $f_{h'}(q)$ *itself. On the other hand, the derivative fields* $\hat{\partial}_i\varphi$ *(see* (3.3.33)*) will be contracted on some scale* $h''<h$*, producing a* $\hat{\partial}_i\tilde{g}^{(h'')}$*. As shown in* (B.2.6)*, the effect of such a derivative on the bound of the propagator is an extra factor* $2^{h''}, 2^{h''/2}$*, respectively for* $i=1,2$*. This discussion implies that if we have a way to bound* $W_4^{(h)}$*, the rewriting of* $1-\mathcal{L}_{1,0}$ *in the interpolated form provides at least a dimensional improvement* $2^{h''-h'}$ *w.r.t to the original bound*[6]*. This procedure turns out to be enough to cure the apparent divergences arising from the naive multiscale analysis, as described below* (3.3.27)*.*[7]

Kernels estimates and running constants

The inductive construction of the previous section provides a description at any given scale of the observables of the original system. Among other things, this allows in principle to compute the infinite volume interacting free energy via $\lim_{h\to-\infty} F^{(h)}+t^{(h)}$ (see cf. (3.3.45). In order to control such limit, we need to have a control on the size of the single scale contributions $F^{(h)}-F^{(h+1)}$, in the same way as we did for the first scale integration, see (3.3.24). These bounds will be a special case ($\epsilon=0$) of a general discussion for the so called "off-critical" regime, i.e. when $\varepsilon=\epsilon+\varepsilon_0$ (see Section 3.3.2): we postpone the precise construction of the expansion needed to prove such bounds (see Section 3.4.3). We only keep here the implications on the kernels built up to this point and thus on to the running constants α_h,ζ_h,ν_h. For a justification of these facts, see Sections 3.4.2-3.4.4.

As observed in Remark 3.3.2, the scaling properties of the propagators involved in the truncated expectation were crucial to derive bounds on the kernels of a single step of integration (see (3.3.24). The same holds for a generic scale h if one has bounds on the single scale propagators (see (3.3.53)) $\tilde{g}^{(h')}$ on higher scales $h'\geq h$. We have that

Lemma 3.3.0.1. *If there exists* $C>0$*, independent of* L*, such that*

$$\sup_{k\geq h}\max\{|\alpha_k|,|\zeta_k|\}\leq C|\lambda| \tag{3.3.60}$$

then, for λ *small enough, there exist* L,λ *independent constants* $c_0,\kappa>0$ *such that*

$$|\tilde{g}^{(h)}(y)|\leq c_0 2^{\frac{h}{2}} e^{-\kappa\sqrt{d^{(h)}(y)}} \tag{3.3.61}$$

where $d^{(h)}(y):=2^h|y_1|+2^{h/2}|y_2|$.

A more detailed discussion about these Lemma can be found in Appendix B.2, and Lemma B.2.0.1[8]. This Lemma is important in order to obtain the following

Proposition 3.3.1. *Let* $\delta_0>0$ *small enough. Suppose that*

$$\sup_{h'>h}\max\{|\lambda|,|\alpha_{h'}|,|\zeta_{h'}|,2^{-\theta h'}|\nu_{h'}|\}\leq\delta_0, \tag{3.3.62}$$

[6]This is because we have two derivatives in the rewriting of the quartic term in (3.3.30); the same reasoning applies with more derivatives: e.g. for the sextic term we will have at least a factor $2^{\frac{3}{2}(h''-h')}$.

[7]For the counter part of this discussion in coordinate space see also [**3**, Sec. 6.1.4]

[8]In such Lemma, to recover the analogy with the present case it is sufficient to set $r_h=0$. This is because as the next Section shows, the constant r_h arises in the first regime of scale only when $\epsilon\neq 0$.

then there exist $C > 0$ independent of L, h, such that

$$\|W^{(h)}_{n,\underline{D}}\|_{\kappa,h} \leq C^n \delta_0^{\max\{1,cn\}} 2^{h\left(\frac{3}{2}+\theta-\frac{n}{4}-|\underline{D}|^{(1)}\right)} \tag{3.3.63}$$

where $|\underline{D}|^{(1)} := \sum_{i=1}^n \sum_{j=1,2} D_{ij} 2^{-\mathbb{1}_{\{j=2\}}}$ and θ can be chosen in $(0, 1/2)$; the norm is defined by

$$\|W^{(h)}_{n,\underline{D}}\|_{\kappa,h} := L^{-2} \sup_{\underline{s}} \sum_{\underline{y}} |W^{(h)}_{n,m,\underline{D}}(\underline{y})| e^{\kappa\sqrt{d^{(h)}(\underline{y})}} \tag{3.3.64}$$

where $d^{(h)}(\underline{y})$ is a scale weighted tree distance given by $2^h d_1(\underline{y}) + 2^{h/2} d_2(\underline{y})$, where d_1, d_2 are the total displacements of the shortest lattice tree connecting the points $(\underline{y})$ in the directions $1, 2$ respectively.

This result resumes the effect of the localization-renormalization procedure on the bounds of the kernels of the effective potentials. In particular, it translates the original hard problem of potentially divergent contributions to the perturbative expansion into a simpler problem of controlling the flow of a finite number of *running constants* $(\nu_h, \alpha_h, \zeta_h)$. These bounds are a special case ($\epsilon = 0$) of those discussed in Section 3.4.2 and they are the analogous in our context of those of [**4**]: in Sections 3.4.2 and 3.4.3 we discuss the general case and the main differences with [**4**].

As a last point let us state in the following Proposition one of the outputs of Section 3.4.4 about the flow of such constants. This involves the way of fixing the parameter ε_0 as a function of λ as explained at the beginning of Section 3.3.1, which is the same function appearing in Theorem 1.4.5. The point of Section 3.4.4 is that there is a unique way to fix $\varepsilon_0 = \varepsilon_0(\lambda)$, for $|\lambda|$ small enough, such that the hypotheses of the previous proposition hold *uniformly in h*, or otherwise stated:

Proposition 3.3.2. *There exists λ_0 small such that for $|\lambda| \leq \lambda_0$ there exist analytic functions ε_0, $\alpha_{-\infty}$, $\zeta_{-\infty}$ of λ, all of order $O(\lambda)$, such that if one chooses ε_0 in* (3.3.1) *as $\varepsilon_0 = \varepsilon_0(\lambda)$), then*

$$|\nu_h| \leq c_0|\lambda| 2^{\theta h}, \quad |\alpha_h - \alpha_{-\infty}| \leq c_0|\lambda| 2^{\theta h}, \quad |\zeta_h - \zeta_{-\infty}| \leq c_0|\lambda| 2^{\theta h} \tag{3.3.65}$$

for some constant $c_0 > 0$ independent of L, h, λ.

For more details see Proposition 3.4.3 and comments below.

3.3.2 The off critical theory: a double regime

Let us describe now, how to construct the thermodynamic limit of the interacting theory for $\varepsilon = \epsilon + \varepsilon_0$, with $\epsilon \neq 0$ and $\varepsilon_0 = \varepsilon_0(\lambda)$ fixed as in the last Proposition 3.3.2 of previous section. This guarantees that the bounds on the kernels of the critical theory (Proposition 3.3.1) are valid for all the scales $h \leq 0$.

Again, Proposition 3.2.1 implies that we can write

$$e^{\mathcal{W}_{\varepsilon,\lambda,L}(\mathcal{A})} = \sum_{\theta\in\{0,1\}^2} \frac{c_\theta}{2} e^{\mathcal{W}^{(\theta)}_{\varepsilon,\lambda,L}(\mathcal{A})}$$

where for a fixed $\theta \in \{0,1\}^2$

$$\begin{aligned} e^{\mathcal{W}^{(\theta)}_{\varepsilon,\lambda,L}(\mathcal{A})} &:= e^{L^2(\epsilon+\varepsilon_0)} \int D\varphi e^{\tilde{S}_{\epsilon,\theta}(\varphi)+\nu_0 N(\varphi)+\tilde{V}_\epsilon(\varphi,\mathcal{A})}, \\ \tilde{S}_{\epsilon,\theta}(\varphi) &:= -L^{-2} \sum_{q\in\mathcal{D}'_{\theta,R}} \hat{\varphi}^+_{q+q_F}(1+r_0-e^{iq_1}\cos q_2)\hat{\varphi}^-_{q+q_F}, \\ N(\varphi) &:= -L^{-2} \sum_{q\in\mathcal{D}'_{\theta,R}} \hat{\varphi}^+_{q+q_F}\hat{\varphi}^-_{q+q_F}, \qquad r_0 := (e^{-\epsilon}-1)e^{-\varepsilon_0} \end{aligned} \tag{3.3.66}$$

where $\tilde{V}_\epsilon, \tilde{S}_{\epsilon,\theta}$ are the rewritings of V, S given in Proposition 3.2.1, in terms of the composition of the transformations $\psi \to \sqrt{e^{-\varepsilon_0-\epsilon}}\psi$ and $\psi \to \psi(\varphi)$ (cfr. Section 3.3.1 and (3.3.7)); we recall that $\mathcal{D}'_{\theta,R} = \mathcal{D}_{\theta,R} - q_F$, with $q_F = (\pi, 0)$, $\nu_0 = e^{-\varepsilon_0} - 1$.
The first thing to note is that now, in the rhs of the second line in (3.3.66),

$$1 + r_0 - e^{iq_1}\cos q_2 = 0 \tag{3.3.67}$$

may have two distinct solutions, denoted with $q^\pm_\epsilon$, or not, depending on the sign of ϵ.

- **Non massive case:** $\epsilon > 0$:
  If $\epsilon > 0$ then $r_0 < 0$ and the two solutions satisfy $q^+_\epsilon = -q^-_\epsilon$ with $q^+_\epsilon = (0, \arccos(1+r_0))$. It is easy to see from the definition of r_0 that for $\epsilon > 0$ sufficiently small, they belong to a ball of radius $4\sqrt{\epsilon}$ centered at 0. This property is at the base of the *double regime* multiscale analysis that will be performed for $\epsilon > 0$. As it was for the critical case, in order to start the u.v. integration we must take care of the case when possibly $q^\pm_\epsilon \in \mathcal{D}'_{\theta,R}$ (see analogous comment after (3.3.1)). One can see that for $\theta \neq (0,0), (1,1)$ then $q^\pm_\epsilon \notin \mathcal{D}'_{\theta,R}$ for every ϵ, L, so we will fix θ in this way. Anyway, since we are interested only in deriving properties of $L \to \infty$ quantities, we will not be anymore interested in the dependence on L (see Remark 3.3.6).

- **Massive case:** $\epsilon < 0$
  In this case, controlling the thermodynamic limit of the theory turns out to be easier: in particular, for a fixed ϵ, one can integrate all the scales at once, in the same spirit as we did a single step in Section 3.3.1. This is because r_0 now plays the role of a mass and the propagator itself, without any cutoff on momenta, decays exponentially in the distance, allowing a single step to be sufficient to obtain a convergent perturbative expansion. Anyway, when $\epsilon \to 0^-$ (i.e. $r_0 \to 0^+$), since the rate of the exponential decay vanishes, a single step integration is not sufficient to obtain uniform bounds in ϵ. A single regime multiscale analysis turns out to be sufficient for that purpose, and a posteriori useful for a comparison with the analogous regime for $\epsilon > 0$, to obtain Theorem 1.4.5.

In both cases, under the assumption given, we can rewrite the first equation in (3.3.66) as (analogous to (3.3.2) of the critical case)

$$\begin{aligned} e^{\mathcal{W}^{(\theta)}_{\varepsilon,\lambda,L}(A)} &= e^{L^2 F^{(0)}_\epsilon} \int P(\varphi) e^{\nu_0 N(\varphi)+\tilde{V}_\epsilon(\varphi,A)} \\ F^{(0)}_\epsilon &= \epsilon + \varepsilon_0 + L^{-2}\log \prod_{q\in\mathcal{D}'_{\theta,R}} (1+r_0-e^{iq_1}\cos q_2) \end{aligned} \tag{3.3.68}$$

with the integration $P_{\theta,\epsilon}(d\varphi)$, given by

$$\int P_\theta(d\varphi)\varphi^-_y\varphi^+_{y'} = L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}\frac{e^{-iq(y-y')}}{1+r_0-e^{iq_1}\cos q_2} \tag{3.3.69}$$

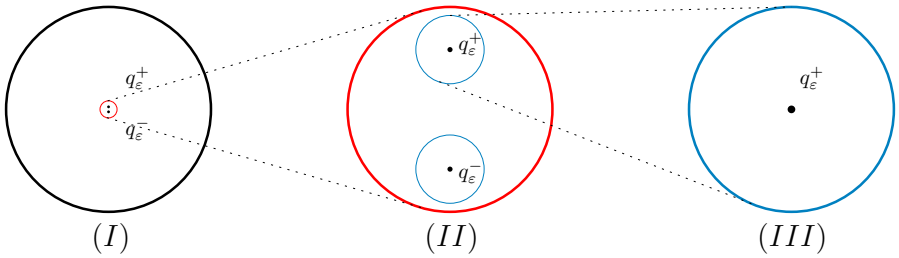

Figure 3.6: *(I): The black circle on the left is on scale of order 1; we start a multiscale decomposition up to a scale h^* (red circle) that is of the same order of the mutual distance between the singularities $q^\pm_\epsilon$. (II)-(III) After that scale we must split around the two singularities (blue circle) and carry on the multiscale down in the infrared $h\to-\infty$.*

Remark 3.3.6. *Since we want to prove results (Theorem 1.4.5) for the infinite volume free energy, and since at every single step, the kernels $W^{(h)}$ admits a natural infinite volume limit $W^{(h);\infty}$ (see Remark 3.3.3), we only care of defining a meaningful rigorous procedure at finite L for at least one boundary condition. From this we can extrapolate the infinite volume contribution which of course will not depend on the specific θ. A discussion of finite size effects is not the objective of this thesis; we defer the interested reader to [2, Appendix C].*

$\epsilon > 0$: The first regime

Ultraviolet integration We intend to proceed in the following way. The u.v. integration is almost the same as in Section 3.3.1, except for the dependence on ϵ and the presence of the external fields $\mathcal{A}$ which were absent there, so we do not report it here the whole discussion but just its consequence: repeating the discussion one obtains the following representation for $\mathcal{W}^{(\theta)}_{\epsilon+\varepsilon_0,L,\lambda}(\mathcal{A})$ (see (3.3.66))

$$e^{\mathcal{W}^{(\theta)}_{\epsilon+\varepsilon_0,L,\lambda}(\mathcal{A})} = e^{F^{(-1)}_\epsilon+S^{(-1)}_\epsilon(J)}\int P^{(\le-1)}_\epsilon(d\varphi^{(\le-1)})e^{V^{(-1)}_\epsilon(\varphi^{(\le-1)},J)} \tag{3.3.70}$$

where

1. $P^{(\le-1)}_\epsilon$ has covariance $g^{(\le-1)}_{L,\epsilon}$ which has the same expression of (3.3.13), with the denominator in the rhs replaced by (3.3.67) (and similarly $g^{(0)}_{L,\epsilon}$).

2. The "effective potentials" $F^{(-1)}_\epsilon, V^{(-1)}_\epsilon(\varphi,J), S^{(-1)}(J)$ are given as in (3.3.17) with the replacements $F^{(-1)}\to F^{(-1)}_\epsilon+S^{(-1)}_\epsilon(J)$, $V^{(-1)}(\varphi)\to V^{(-1)}_\epsilon(\varphi,J)$ and $\nu_0 N + V(\cdot)\to\nu_0 N+\tilde V_\epsilon(\cdot,J)$ where $\tilde V_\epsilon$ is given in (3.3.66) and we expressed the variables $\{A\}_{e\in E_L}$ in terms of $\{J\}_{e\in E_L}$ via

$$J_e := e^{A_e}-1 \tag{3.3.71}$$

for every edge $e \in E_L$. In particular the structure of the equation (3.3.66) implies that $V_\epsilon^{(-1)}(\varphi, J)$ admits the following expression, for $\phi = \varphi^{(\leq -1)}$ (analogous to cfr. (3.3.21), (3.3.22) with the presence of the external fields J)

$$\begin{aligned} V_\epsilon^{(-1)}(\phi, J) &= \sum_{\substack{m\in\mathbb{N}\\ n\in 2\mathbb{N}^+}} \sum_{\underline{y},\underline{x},\underline{s}} W^{(-1)}_{n,m,\epsilon,\underline{s}}(\underline{y},\underline{x})\phi_{\underline{y}} J_{\underline{x},\underline{s}} = \\ &= L^{-2(n+m-1)} \sum_{\underline{q},\underline{p},\underline{s}} \hat{W}^{(-1)}_{n,m,\epsilon,\underline{s}}(q_2,\dots,q_n,\underline{p})\hat{\phi}_{\underline{q}+q_F}\hat{J}_{\underline{p},\underline{s}}\delta(\underline{q},\underline{p}) \end{aligned} \tag{3.3.72}$$

where $\underline{y} \in \tilde{\Lambda}^n$, $\underline{x} \in \Lambda^m$ and $\phi_{\underline{y}} := \phi^+_{y_1}\phi^-_{y_2}\cdots\phi^+_{y_{n-1}}\phi^-_{y_n}$; $\underline{q} + q_F = (q_1 + q_F, \dots, q_n + q_F)$ with $q_i \in \mathcal{D}'_{\theta,R}$; $J_{\underline{x},\underline{s}} = \prod_{i=1}^m J_{x_i,s_i}$ where $J_{x,s}$ denotes J_e when the edge e has black vertex of coordinate $x \in \Lambda$ and is of type $s = A, B, C$ (see Figure 3.1); $\underline{p} \in (\mathcal{D}_0)^m$ (cfr. (3.1.5)) and

$$J_{x,s} = \sum_{p\in\mathcal{D}_0} e^{-ipx}\hat{J}_{p,s}. \tag{3.3.73}$$

finally $\delta(\underline{q},\underline{p})$ enforces global momentum conservation, i.e. equals 1 if $\sum_{i=1}^n (-1)^i R q_i = \sum_{i=1}^m p_i$ and 0 otherwise.

3. the kernels of the effective potential satisfy $\|W^{(-1)}_{n,m}\|_{\kappa,-1} \leq C^{n+m}|\lambda|^{\max\{1,cn\}}$ where

$$\|W^{(-1)}_{n,m,\epsilon}\|_{\kappa,-1} := L^{-2}\sup_{\underline{s}} \sum_{\underline{x},\underline{y}} |W^{(-1)}_{n,m,\epsilon,\underline{s}}(\underline{y},\underline{x})| e^{\kappa\sqrt{2^{-1}d(\underline{y},\underline{x})}} \tag{3.3.74}$$

which is a natural generalization of the norm given below (3.3.24), and $\delta(\underline{y},\underline{x})$ is the tree distance between the two sets of points $\underline{y},\underline{x}$, namely the length of the shortest tree on the torus Λ connecting $\underline{x}$ and $(R^T)^{-1}\underline{y}$[9]. Moreover, in the Fourier representation the satisfy the following symmetries (see Appendix B.4)

$$\hat{W}^{(-1)}_{n,\epsilon,\theta}(\underline{q}) = \overline{W^{(-1)}_{n,\epsilon,\theta}(-\underline{q})}, \qquad \hat{W}^{(-1)}_{n,\epsilon,\theta}(\hat{\underline{q}}) = W^{(-1)}_{n,\epsilon,\tilde{\theta}}(\underline{q}). \tag{3.3.75}$$

Inductive statement There exists an integer h^*, which will be defined along the inductive procedure (see (3.3.101)), satisfying, for $|\lambda|$ small,

$$c_1 \log\epsilon \leq h^* \leq c_2 \log\epsilon \tag{3.3.76}$$

for some constant $c_1, c_2 > 0$ which are L, ϵ, λ independent. For any h s.t. $\max\{h^*, h_L\} \leq h \leq -1$, if we set $\phi = \varphi^{(\leq h)}$, the following identity holds

$$e^{\mathcal{W}^{(\theta)}_{\epsilon+\varepsilon_0,L,\lambda}} = e^{L^2(F^{(h)}_\epsilon + t^{(h)}_\epsilon) + S^{(h)}(J)} \int P^{(\leq h)}_\epsilon(d\phi) e^{V^{(h)}_\epsilon(\phi,J)} \tag{3.3.77}$$

where

[9] The coordinates of the external fields have no reason to be thought in terms of $\tilde{\Lambda}$, while the Grassmann fields are naturally functions of $R^T\Lambda$ via the transformation in Fourier space given in (3.3.4).

1. the Grassmann measure $P_\epsilon^{(\leq h)}$ is associated to the propagator

$$g_{L,\epsilon}^{(\leq h)} = L^{-2} \sum_{q \in \mathcal{D}'_{R,\theta}} \frac{e^{-iqy}\chi_h(q)}{r_h + \tilde{\mu}_h(q+q_F)} \tag{3.3.78}$$

with χ_h, $\tilde{\mu}_h$ being the same functions appearing in (3.3.48), while r_h is a *real* constant that will be defined inductively.

2. The constants $F_\epsilon^{(h)}$ and $t_\epsilon^{(h)}$, as well the effective potentials $V_\epsilon^{(h)}, S^{(h)}$ are inductively described shortly. We will show that $V_\epsilon^{(h)}$ admits an expression analogous to the second line in (3.3.72), with the scale label (-1) replaced by (h), and that its kernels satisfy the same symmetries of item 3 above, namely

$$\hat{W}^{(h)}_{n,\epsilon,\theta}(\underline{q}) = \overline{W^{(h)}_{n,\epsilon,\theta}(-\underline{q})}, \qquad \hat{W}^{(h)}_{n,\epsilon,\theta}(\underline{\hat{q}}) = W^{(h)}_{n,\epsilon,\tilde{\theta}}(\underline{q}). \tag{3.3.79}$$

Remark 3.3.7. *We will look at the kernels of the effective potential $V_\epsilon^{(h)}$, as functions of the running constants of higher scales, $\{r_k, \alpha_k, z_k, \nu_k\}_{k>h}$, thought of as independent variables. In particular, the dependence on ϵ of the kernels will be mediated by single scale propagators $\{\tilde{g}^{(h')}_{L,\epsilon}\}_{h'>h}$*

$$\tilde{g}^{(h')}_{L,\epsilon}(y,y') = L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} \frac{e^{-iq(y-y')}\tilde{f}_{h'}(q)}{\boldsymbol{r}_{h'}(q) + \boldsymbol{D}_{h'}(q) + \rho(q)} \tag{3.3.80}$$

(where $\boldsymbol{D}_{h'}, \rho$ are the same as in the critical case see (3.3.54)*), through a sequence of smooth functions $\boldsymbol{r}_{h'}(q;\underline{r})$ themselves depending on the constants of higher scales $\underline{r} = (r_{h'+1}, \dots, r_0)$ (defined inductively, see* (3.3.90)*), and q being the relative momentum to q_F. In* (3.3.78) *above, $r_{h'} := \boldsymbol{r}_{h'}(0)$. With this in mind we will sometimes use the following notation,*

$$\hat{W}^{(h)}_{2,m,\underline{s}}(\cdot,\cdot;\underline{r}) := \hat{W}^{(h)}_{2,m,\epsilon,\underline{s}}(\cdot,\cdot) \tag{3.3.81}$$

where $\underline{r} = \{r_k\}_{k>h}$.[10]

Base of the induction For $h = -1$, in virtue of the u.v. integration, i.e. (3.3.70) the representation holds if we set $t_\epsilon^{(-1)} = 0$. As for the covariance, recall that $\tilde{\mu}_{-1} := \tilde{\mu}$ (see below (3.3.38)) and that $\tilde{\mu}(q+q_F) = 1 - e^{iq_1}\cos q_2$ so that defining $r_{-1} := r_0$ the representation holds for $h = -1$, Lastly, the dependence on ϵ of the kernels can indeed be seen through r_0 in the single scale propagators $\tilde{g}^{(0)}_{L,\epsilon}$ contributing to it (recall item 2 above and (3.3.19)).

Inductive step The idea behind the localization here is the following: in order to handle the dependence on ϵ we insert, at each scale, not only the *critical* (i.e. evaluated at $\epsilon = 0$) renormalization constants α_h, ζ_h in the measure (which contributes to the function

[10] To avoid heavy notations, the symbol $\underline{r}$, whenever it appears as argument of some $\hat{W}^{(h)}$, will stand for $\{r_k\}_{k>h}$.

D_h appearing in $\tilde{\mu}_h$ in (3.3.48)) as we did in (3.3.38), but also the linear term[11] in ϵ of $\hat{W}^{(h)}_{2,0,\epsilon}(0) - \hat{W}^{(h)}_{2,0,\epsilon}(0)$: this will iteratively define the function $\boldsymbol{r}_h$ vanishing for $\epsilon \to 0$.

More precisely, assume the inductive statement to be true up to scale h. Recall that, as explained in Section 3.3.1, the iterated action of the $\epsilon = 0$ part of the renormalization procedure produce, in the effective potential representation in coordinate space, the presence of pseudo differential operators $\hat{\partial}$ (cf. (3.3.33)) applied to the Grassmann fields. In formulas, if $\phi = \varphi^{(\leq h)}$

$$\begin{aligned} &V^{(h)}_{\epsilon}(\phi, J) = \sum_{\substack{n\in 2\mathbb{N}\\ m\in\mathbb{N}}} V^{(h)}_{\epsilon,n,m}(\phi, J), \quad V^{(h)}_{\epsilon,n,m}(\phi, J) = \sum_{\underline{D},\underline{s}} V^{(h)}_{\epsilon,n,m,\underline{D},\underline{s}}(\phi, J) \\ &V^{(h)}_{\epsilon,n,m,\underline{D},s}(\phi, J) := \sum_{\underline{y}\in\tilde{\Lambda}^n} \hat{\partial}^{D_1}\phi^+_{y_1} \cdots \hat{\partial}^{D_n}\phi^-_{y_n} J_{x_1,s_1} \cdots J_{x_n,s_n} W^{(h)}_{\epsilon,n,m,\underline{D},\underline{s}}(\underline{y}, \underline{x}) \end{aligned} \tag{3.3.82}$$

with the conventions of (3.3.51) and (3.3.72). In Fourier this reads

$$V^{(h)}_{\epsilon}(\phi, J) = L^{-2(n+m-1)} \sum_{\underline{q},\underline{p},\underline{s}} \hat{W}^{(h)}_{n,m,\epsilon,\underline{s}}(q_2, \dots, q_n, \underline{p}) \hat{\phi}_{\underline{q}+q_F} \hat{J}_{\underline{p},\underline{s}} \delta(\underline{q}, \underline{p}) \tag{3.3.83}$$

where $\hat{W}^{(h)}_{\epsilon,n,m,\underline{s}}$ is expressed in terms of a combination of the Fourier transforms of the kernels $W_{\epsilon,n,m,\underline{D},\underline{s}}$ and includes the factors q_i^D's coming from the effect of $\hat{\partial}$, see (3.3.33).

We now split $V^{(h)}_{\epsilon}$ in (3.3.77), as

$$V^{(h)}_{\epsilon} = \mathcal{L}_{1,\epsilon} V^{(h)}_{\epsilon} + \mathcal{R}_{1,\epsilon} V^{(h)}_{\epsilon},$$

where we define $\mathcal{L}_{1,\epsilon} V^{(h)}_{\epsilon,n,m}(\phi, J) = 0$ if $n > 2$ or $m > 1$ and

$$\begin{aligned} &\mathcal{L}_{1,\epsilon} V^{(h)}_{\epsilon,2,0}(\phi, J) = \\ &L^{-2} \sum_{q\in\mathcal{D}'_{\theta,R}} \hat{\phi}^+_{q+q_F} \hat{\phi}^-_{q+q_F} \Big[\hat{W}^{\infty;(h)}_{2,0,0}(0) + q\cdot\partial \hat{W}^{\infty;(h)}_{2,0,0}(0) + \frac{1}{2} q_2^2 \partial_2^2 \hat{W}^{\infty;(h)}_{2,0,0}(0) + \\ &+ \mathcal{T}\left(\hat{W}^{\infty;(h)}_{2,0,\epsilon}(0) - \hat{W}^{\infty;(h)}_{2,0,0}(0) \right) \Big], \end{aligned} \tag{3.3.84}$$

$$\mathcal{L}_{1,\epsilon} V^{(h)}_{\epsilon,2,1}(\phi, J) = \sum_{x\in\Lambda} \sum_{s=A,B,C} \phi^+_{R^T x} \phi^-_{R^T x} J_{x,s} \hat{W}^{\infty;(h)}_{2,1,0,s}(0,0)$$

where, in the first equation, $\mathcal{T}$ is an operator which extract the linear part in ϵ of the infinite volume kernel $\hat{W}^{\infty;(h)}_{2,0,\epsilon}$, through an expansion in the sequence $\underline{r}$ (cfr. (3.3.81)):

$$\mathcal{T}\Big[\hat{W}^{\infty;(h)}_{2,0,\epsilon}(0) - \hat{W}^{\infty;(h)}_{2,0,0}(0) \Big] := \sum_{k>h} r_k \partial_{r_k} \hat{W}^{\infty;(h)}_{2,0}(0; \underline{0}). \tag{3.3.85}$$

[11] Now the Taylor expansion will be performed separately both in momenta q close to q_F and in ϵ.

In the second equation of (3.3.84), since both the Grassmann and the external fields are evaluated at the same point on the lattice, we used that if $y \in \tilde{\Lambda}$, $y = R^T x$ (see (3.3.4)).

Note that the first line of the first equation in (3.3.84) depends only on the critical theory, because ϵ is set to zero: it corresponds to the action of the operator $\mathcal{L}_{1,0}$ as described in Section 3.3.1. The operator $\mathcal{R}_{1,\epsilon}$ is a suitable rewriting of $1 - \mathcal{L}_{1,\epsilon}$, as it was briefly discussed in the lines below (3.3.29) in the case of $\epsilon = 0$. Note that the rewriting of $1 - \mathcal{L}_{1,0}$ produce extra operators $\hat{\partial}$ on the associated Grassmann monomial in coordinate space (as a consequence of the interpolation in Fourier space, e.g. (3.3.31)), instead for $\epsilon \neq 0$ the rewriting of $1 - \mathcal{T}$ it is given by

$$(1-\mathcal{T})\Big[\hat{W}^{\infty;(h)}_{2,0,\epsilon}(0) - \hat{W}^{\infty;(h)}_{2,0,0}(0)\Big] = \int_0^1 dt \sum_{\ell,m\geq h} r_\ell r_m \partial^2_{r_\ell,r_m} W^{\infty;(h)}_{2,0,\epsilon}(0; t\underline{r}). \tag{3.3.86}$$

The renormalizing effect of such term will be discussed in the tree expansion framework of Section 3.4.3 (point (2) below Proposition 3.4.0.1).

Now we insert all but the terms $2^h\nu_h$ (cfr. (3.3.55) for its definition) and the last one in (3.3.84), in the propagator, defining a new measure $\tilde{P}^{(\leq h)}_\epsilon$ in such a way to rewrite

$$(3.3.77) = e^{F^{(h)}_\epsilon + t^{(h-1)}_\epsilon + S^{(h)}(J)} \int \tilde{P}^{(\leq h)}_\epsilon(\phi) e^{2^h\nu_h N(\phi)(\phi) + \sum_{x,s} Y^{(h)}_{s,\varepsilon} F_Y(\phi,J) + \mathcal{R}_{1,\epsilon} V^{(h)}_\epsilon(\phi,J)}. \tag{3.3.87}$$

where we defined

$$F^{(h)}_{Y;\epsilon}(\phi, J) = \sum_{x,s} \phi^+_x \phi^-_x J_{x,s}, \quad Y^{(h)}_{s,0} := W^{\infty;(h)}_{2,1,0,s}(0,0). \tag{3.3.88}$$

The integration $\tilde{P}^{(\leq h)}_\epsilon$ has covariance

$$\tilde{g}^{(\leq h)}_\epsilon(y, y') = L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} \frac{e^{-iq(y-y')}\chi_h(q)}{\boldsymbol{r}_{h-1}(q) + \boldsymbol{D}_{h-1}(q) + \rho(q)} \tag{3.3.89}$$

where $\boldsymbol{D}_{h-1}$ and ρ are the same of the previous Section, see resp. (3.3.53)-(3.3.54), while $\boldsymbol{r}_{h-1}(q)$ satisfies

$$\boldsymbol{r}_{h-1}(q) := r_h + \chi_h(q) \sum_{k>h} r_k \partial_{r_k} \hat{W}^{\infty;(h)}_{2,0,0}(0; \underline{0}) \tag{3.3.90}$$

and finally

$$\tilde{t}^{(h-1)}_\epsilon = t^{(h-1)}_\epsilon - t^{(h)}_\epsilon \tag{3.3.91}$$

is obtained from the change of measure (cfr. rhs of (3.3.57) with the scale index h replaced by $h-1$, where the numerator and the denominator have extra summands, respectively, $\boldsymbol{r}_{h-1}$ and r_h). Now we decompose $\chi_h = f_h + \chi_{h-1}$, with $\chi_{h-1}(q) = \chi_h(2^{-1}q)$, to write

$$\tilde{g}^{(\leq h)}_{L,\epsilon} = \tilde{g}^{(h)}_{L,\epsilon} + g^{(\leq h-1)}_{L,\epsilon} \tag{3.3.92}$$

where the single scale propagator $\tilde{g}^{(h)}_{L,\epsilon}$ has the same expression as (3.3.89) with χ_h replaced

by f_h and admits bounds as in the following Lemma. Then $g^{(\leq h-1)}_{L,\epsilon}$ can be written as

$$g^{(\leq h-1)}_{L,\epsilon}(y,y') := L^{-2} \sum_{q\in\mathcal{D}'_{\theta,R}} \frac{e^{-iq(y-y')}\chi_{h-1}(q)}{r_{h-1}+D_{h-1}(q)+\rho(q)}, \qquad r_{h-1} := \boldsymbol{r}_{h-1}(0) \tag{3.3.93}$$

in fact in the support of χ_{h-1} one has $\chi_h \equiv 1$, cfr. with (3.3.89). Note that the flow of r_h, via (3.3.90), guarantees by inductive hypothesis on the symmetries of the kernels (see (3.3.79))[12] that r_{h-1} is a real constant (see also Appendix B.4). Finally using the addition principle of measures with such decomposition (exactly as in (3.3.43)) we can rewrite, if $\phi = \varphi^{(h)} + \phi'$, $\phi' := \varphi^{(\leq h-1)}$

$$(3.3.87) = e^{F^{(h-1)}_\epsilon + t^{(h-1)}_\epsilon + S^{(h-1)}(J)} \int P^{(\leq h-1)}_\epsilon(\phi') e^{V^{(h-1)}_\epsilon(\phi',J)} \tag{3.3.94}$$

where

$$\begin{aligned} &L^2(F^{(h-1)}_\epsilon - F^{(h)}_\epsilon) + S^{(h-1)}(J) - S^{(h)}(J) + V^{(h-1)}_\epsilon(\phi',J) = \\ &= \sum_{n\geq 1} \frac{1}{n!} \mathcal{E}^T_{(h)} \left(\mathcal{V}^{(h)}_\epsilon(\phi'+\cdot,J); \cdots ; \mathcal{V}^{(h)}_\epsilon(\phi'+\cdot,J) \right) \end{aligned} \tag{3.3.95}$$

where

$$\mathcal{V}^{(h)}_\epsilon(\phi,J) := 2^h \nu_h N(\phi) + \sum_{x,s} Y^{(h)}_{s,\epsilon} F^{(h)}_Y(\phi,J) + \mathcal{R}_{1,\epsilon} V^{(h)}_\epsilon(\phi)$$

(cf. with (3.3.88)) and the truncated expectation $\mathcal{E}^T_{(h)}$ is with respect to the propagator $\tilde{g}^{(h)}$ above. In particular, repeating the discussion after (3.3.19), considering that now $\mathcal{R}_{1,\epsilon}V^{(h)}_\epsilon$ carries fields with derivatives $\hat{\partial}$, one obtains that $V^{(h-1)}_\epsilon$ has an expression (3.3.82)-(3.3.83) with the scale index replaced by $(h-1)$. The kernels of $V^{(h-1)}_\epsilon$ satisfy the same symmetries of those of $V^{(h)}$: this is because by induction the arguments of (3.3.95) satisfy such symmetries and the covariance $\tilde{P}^{(h)}$ is symmetric under the same symmetries, so that one uses the argument of the beginning of Section B.4.

The single scale propagator of the first regime The integration procedure of the first regime, $h > h^*$, produces a single scale propagator given by

$$\tilde{g}^{(h)}_{L,\epsilon}(y,y') = L^{-2} \sum_{q\in\mathcal{D}'_{\theta,R}} e^{-iq(y-y')} \frac{f_h(q)}{\mathbf{r}_h(q)+\mathbf{D}_h(q)+\rho(q)} \tag{3.3.96}$$

[12] From (3.3.79) follows that the kernels are real valued, independently of the value of $\underline{r}$, so that also their derivative with respect to some r_k is still real.

where we recall $f_h(q) = f(2^{-h}|iq_1 + \frac{q_2^2}{2}|)$ with $f(q) = \chi(q) - \chi(2q)$ see (3.3.11); $\boldsymbol{D}_h(q) := -iq_1(1+\boldsymbol{\zeta}_h(q)) + \frac{1}{2}q_2^2(1+\boldsymbol{\alpha}_h(q))$ and

$$\begin{aligned}
\boldsymbol{r}_h(q) &= r_{h+1} + \chi_{h+1}(q)\sum_{k>h} r_k \partial_{\boldsymbol{r}_k} \hat{W}_{2,0,0}^{\infty;(h+1)}(0;\underline{0}) \\
\boldsymbol{\zeta}_h(q) &= \zeta_{h+1} + i\chi_{h+1}(q)\partial_1 \hat{W}_2^{(h+1);\infty}(0) \\
\boldsymbol{\alpha}_h &= \alpha_{h+1} + \chi_{h+1}(q)\partial_2^2 \hat{W}_2^{(h+1);\infty}(0) \\
\rho(q) &= 1 - e^{iq_1}\cos q_2 + iq_1 - \frac{q_2^2}{2}.
\end{aligned} \tag{3.3.97}$$

We have then

Lemma 3.3.0.2. *Let $h \geq h^*$. If there exists $C > 0$, independent of L, such that*

$$\sup_{h'\geq h} \max\{|\alpha_k|, |\zeta_k|\} \leq C|\lambda|, \quad \sup_{h'\geq h} |r_h| \leq C|r_0| \tag{3.3.98}$$

then there exist L, ϵ independent constants $c_0, c_1, \kappa > 0$ such that

$$|\tilde{g}_{L,\epsilon}^{(h)}(y)| \leq c_0 2^{h/2} e^{-\kappa\sqrt{d^{(h)}(y)}} \tag{3.3.99}$$

where $d^{(h)}(y) := 2^h|y_1| + 2^{h/2}|y_2|$.

For more details, see Appendix B.2.

The flow of r_h and the scale h^* Let us begin with a motivation for the stopping scale h^*. Notice that, differently from the critical case, here each "dressing" of the propagator with local terms of $V_\epsilon^{(h)}$ produces a shift in the location of the zeros of the inverse propagator: this is exactly caused by the sequence $\underline{r}$. Since the flow of $\underline{r}$ is linear (cfr. (3.3.90) for $q = 0$), then we can write

$$r_h = r_0 Q_h(\lambda) \tag{3.3.100}$$

for some real function $Q_h(\lambda)$ s.t. $Q_{-1}(\lambda) = 1$[13]. The fact that $Q_h(\lambda)$ depends only on kernels of the critical theory, which admits good bounds (cfr. Prop. 3.3.2) will imply (see next Lemma) that $Q_h(\lambda) = 1 + O(\lambda)$ uniformly in h, so that r_h *stays close to the initial datum uniformly in h.* In other words the shift of the zeros is still of order ϵ, independently of the scale. Anyway the multiscale procedure must stop when the support of χ_h does not contain the zeros anymore. Neglecting the higher order corrections of ρ to D_h in (3.3.78), the shifted zeros at the h^{th} step, $(0, \pm q_{F,h})$, satisfy $r_h + 1/2 q_{F,h}^2(1+\alpha_h) = 0$. Then we will stop the procedure at a scale h^* such that

$$h^* := \min\left\{h \leq -1 : \frac{|r_h|}{(1+\alpha_h)} < 2^h\delta\right\} \tag{3.3.101}$$

where δ enters the definition of χ (cfr. (3.3.11)), and α_h is the same as in (3.3.48)$-$(3.3.49). This choice guarantees that $\chi_{h^*}(q_{F,h^*}) = 1$, which will be used later. Using then the just mentioned property of Q_h, recalling those of α_h, ε_0 in Prop 3.3.2 and the definition of r_0

[13] Recall that we set $r_1 := r_0$.

one can see that indeed h^* satisfies (3.3.76), for ϵ and λ sufficiently small, in particular

$$h^* = \min\{h \leq -1 : \delta 2^h > \epsilon\}, \tag{3.3.102}$$

which implies that $2^{h^*} = \Theta(\epsilon)$.
Let us now state and prove the desired property for the flow of r_h.

Lemma 3.3.0.3. *Under the conditions of Proposition 3.3.2, there exists a constant $C > 0$, independent of h^*, such that, for $|\lambda|$ small*

$$\max\left\{\sup_{h\geq h^*}\left|\frac{r_h}{r_0}-1\right|, \sup_{h\geq h^*}\left|\frac{r_h}{r_{h^*}}-1\right|\right\} \leq C|\lambda|. \tag{3.3.103}$$

As a consequence, writing $r_h = r_0 Q_h(\lambda)$, then $|Q_h(\lambda)-1| \leq C|\lambda|$; finally $|r_h|2^{-h} \leq C2^{h^*-h}$.

Proof. We proceed by induction, assuming that there exists a constant $C' > 0$ such that $\sup_{h'>h}|r_{h'} - r_0| \leq C'|r_0\lambda|$. From the definition of the flow (3.3.90) at $q = 0$, we can bound

$$|r_h - r_{h+1}| \leq |r_0|(1+C|\lambda|)\sum_{k\geq h+1}|\partial_{r_k}\hat{W}_2^{\infty;(h+1)}(0;\underline{0})| \leq \tilde{C}(r_0,\lambda)\sum_{k\geq h+1}2^{(1+\theta)h-k} \tag{3.3.104}$$

for $\tilde{C}(r_0,\lambda) = c_0|r_0\lambda|(1 + C'|\lambda|)$, c_0 as in Proposition 3.3.2, which implies that r_h is a Cauchy sequence. In the second inequality we used that a derivative with respect to r_k of $\hat{W}_2^{\infty;(h+1)}(q_F;\underline{r})$, evaluated at $\underline{r} = 0$, produces an extra factor 2^{-k} with respect to the bound of $W_2^{\infty;(h+1)}(q_F;\underline{0}) = 2^{h+1}\nu_{h+1}$ which is given by Proposition 3.3.2. This is because such derivative acts on some propagator on scale $k \geq h+1$ and its effect can be dimensionally bounded using Lemma B.2.0.1.[14]. Thus using (3.3.104) we can write

$$|r_h - r_0| \leq \tilde{C}(r_0,\lambda)\sum_{j=h}^{-2}2^{\theta j} \leq C'|r_0||\lambda| \tag{3.3.105}$$

where we used that for $|\lambda|$ small enough, since θ is bounded away from 0[15], $(1+C'|\lambda|)c_0 \leq C'2^\theta(2^\theta - 1)$. In particular, writing $r_h = r_0 Q_h(\lambda)$, thanks to the linearity of its flow, for some function $Q_h(\lambda)$, then (3.3.105) immediately implies the property of $Q_h(\lambda)$.
Now we can write

$$\Big|\frac{r_h}{r_{h^*}} - 1\Big| = \Big|\frac{Q_h(\lambda) - Q_{h^*}(\lambda)}{Q_{h^*}(\lambda)}\Big| \leq \frac{2C'|\lambda|}{1 - C'|\lambda|} \leq 4C'|\lambda| \tag{3.3.106}$$

for $|\lambda|$ small. To conclude, notice using the definition of h^*,

$$|r_h 2^{-h}| \leq \Big|\frac{r_h}{r_{h^*}}2^{h^*-h}4(1+\alpha_{h^*})\delta\Big| \leq C''2^{h^*-h} \tag{3.3.107}$$

for some constant C'' which depends on C', δ but not on $|\lambda|, h$. Then one can choose

[14]For a proof this fact we need to know a bit more on the structure one uses to obtain the bounds in Proposition 3.3.2. We will come back to reasoning of this type in a more concrete setting, given by Section 3.4.3.
[15]We recall θ can be chosen within the range $(0, 1/2)$.

$C = \max\{C'', 4C'\}$ to complete the statement. ∎

The transition to the second regime

After the first regime we are left with the following representation for the partition function

$$e^{\mathcal{W}_{\epsilon+\varepsilon_0,L,\theta,\lambda}} = e^{L^2(F_\epsilon^{(h^*)}+t_\epsilon^{(h^*)})+S^{(h^*)}(J)} \int P_\epsilon^{(\le h^*)}(d\varphi^{(\le h^*)}) e^{V_\epsilon^{(h^*)}(\varphi^{(\le h^*)},J)} \tag{3.3.108}$$

where $P_\epsilon(d\varphi^{(\le h^*)})$ has propagator given by the same of (3.3.93) with the scale index replaced by h^*. For $h < h^*$, in order to deal with a shift of the singularities, as well of the analogous of the coefficients of the function D_h, (see (3.3.48)) we now proceed differently: instead of dynamically modifying such constants at each scale, we fix the structure of the propagator, modulo a global factor Z_h, to be inductively defined, by introducing free parameters $\nu_{h^*,\omega}$, $a_{h^*,\omega}$, $b_{h^*,\omega}$, for $\omega = \pm$, also called *counterterms*. Their value will be uniquely determined via a fixed point argument (see Section 3.4.4) in order for the perturbative expansion to be convergent uniformly in L as $L \to \infty$.

More precisely, we start by rewriting the effective interaction in (3.3.108): we split as usual $V_\epsilon^{(h^*)} = \mathcal{L}_{1,\epsilon} V_\epsilon^{(h^*)} + \mathcal{R}_{1,\epsilon} V_\epsilon^{(h^*)}$, and we insert in the Grassmann integration $P_\epsilon^{(\le h^*)}$ not only the usual part of $\mathcal{L}_{1,\epsilon}$ (as described in (3.3.87) and above) but also the quadratic *counterterm* $M(\phi)$, for $\phi = \varphi^{(\le h^*)}$, given by

$$\begin{aligned} M(\phi) &:= L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} \hat\phi^+_{q+q_F} M(q) \hat\phi^-_{q+q_F} \\ M(q) &:= \sum_{\omega=\pm} \tilde\chi_{h^*}(q - \omega \boldsymbol{q}_F)(2^{h^*}\nu_{h^*,\omega} - iq_1 a_{h^*,\omega} + (q_2 - \tilde q_F^\omega) b_{h^*,\omega}). \end{aligned} \tag{3.3.109}$$

This means that we rewrite (3.3.108) as

$$(3.3.108) = e^{L^2(F_\epsilon^{(h^*)}+t_\epsilon^{(h^*-1)})+S^{(h^*)}(J)} \int \tilde{\tilde P}_\epsilon^{(\le h^*)}(d\phi) e^{F_Y^{(h^*)}(\phi,J)+\mathcal{R}_{1,\epsilon}V^{(h^*)}(\phi,J)+M(\phi)} \tag{3.3.110}$$

where $t_\epsilon^{h^*-1} - t_\epsilon^{h^*}$ is obtained by the change of measure $\tilde P \to \tilde{\tilde P}$ (as described in (3.3.57)) where $\tilde{\tilde P}(d\varphi^{(\le h^*)})$ has propagator $\tilde{\tilde g}_{L,\epsilon}^{(\le h^*)}$ which is the same as in (3.3.89), except that in the denominator in the rhs, h^* is replaced by h^*-1 and we added the term $2^{h^*}\nu_{h^*} + \chi_{h^*}(q)M(q)$. In (3.3.109) we have that

- $\boldsymbol{q}_F := (0, \tilde q_F)$ with $\tilde q_F$ to be fixed. It is assumed to satisfy

$$|\tilde q_F - q_{F,h^*}| \le C|\lambda|\sqrt{\epsilon} \tag{3.3.111}$$

  for some $C > 0$; here $\pm q_{F,h^*}$ are the singularities of the propagator of the first regime of scales, after $h^* - 1$ steps of integration (see the lines above (3.3.101)),

- $\nu_{h^*,\omega}, a_{h^*,\omega}, b_{h^*,\omega} \in \mathbb{R}$ are free parameters, to be fixed later, which we assume to satisfy, $\nu_{h^*,\omega}, a_{h^*,\omega}, b_{h^*,\omega} = O(\lambda)$, and

$$\nu_{h^*,\omega} = \nu_{h^*,-\omega}, \quad a_{h^*,\omega} = a_{h^*,-\omega}, \quad b_{h^*,\omega} = -b_{h^*,-\omega}, \quad b_{h^*,+} = O(\lambda\sqrt{\epsilon}). \tag{3.3.112}$$

The first three equations are natural in view of the original symmetries of the model, carried along the first regime of integration (see Appendix B.4). This choice guarantees that this intermediate integration does not break such symmetries. In view of (3.3.112) we have only three independent real parameter to play with $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$. The last assumption, plays a role in the following item.

- $\tilde{\chi}_h(q) := \chi(c_0 2^{-h}|iq_1 + q_2 v_F|)$ where c_0 is a positive, big enough constant, such that $\tilde{\chi}_{h^*}(q - \boldsymbol{q}_F)\tilde{\chi}_{h^*}(q + \boldsymbol{q}_F) = 0$ for every q, and

$$v_F := \frac{\sin \tilde{q}_F + \tilde{q}_F \alpha_{h^*-1} + b_{h^*,+}}{\cos \tilde{q}_F + \zeta_{h^*-1} + a_{h^*,+}}. \tag{3.3.113}$$

Note first that for c_0 big enough and $|\lambda|$ small enough we have that $\cup_\omega \{q \in \mathcal{D}'_{\theta,R} : \tilde{\chi}_{h^*}(q - \omega \mathbf{q}_F) > 0\} \subset \{q \in \mathcal{D}'_{\theta,R} : \chi_{h^*}(q) = 1\}$. Secondly, by the assumption of the previous points, since $q_{F,h^*} = O(\sqrt{\epsilon})$ (see (3.3.101) and lines above) we have that $v_F = O(\sqrt{\epsilon})$.

We can then rewrite

$$\chi_{h^*}(q) = \sum_{\omega=\pm} \tilde{\chi}_{h^*-1}(q - \omega \mathbf{q}_F) + \tilde{f}_{h^*}(q) \tag{3.3.114}$$

which implicitly defines $\tilde{f}_{h^*}(q)$, being such that in its support $q_1 = O(2^{h^*}), q_2 = O(2^{\frac{h^*}{2}})$. Then, using the addition principle coming from (3.3.114) associated to the fields decomposition

$$\varphi_y^{\sigma,(\leq h^*)} = \sum_{\omega=\pm} e^{-i\sigma\omega y \mathbf{q}_F} \varphi_{y,\omega}^{\sigma,(\leq h^*-1)} + \varphi_y^{\sigma,(h^*)} \tag{3.3.115}$$

and to the propagators decomposition

$$\tilde{\tilde{g}}_{L,\epsilon}^{(\leq h^*)} = \sum_{\omega=\pm} e^{-i\sigma\omega y \mathbf{q}_F} g_{L,\epsilon,\omega}^{(\leq h^*-1)} + g_{L,\epsilon}^{(h^*)}, \tag{3.3.116}$$

we can rewrite

$$(3.3.110) = e^{L^2(F_\epsilon^{(h^*-1)} + t_\epsilon^{(h^*-1)}) + S^{(h^*-1)}(J)} \int P_\epsilon^{(\leq h^*-1)}(d\varphi^{(\leq h^*-1)}) e^{\tilde{V}_\epsilon^{(h^*-1)}(\varphi^{(\leq h^*-1)}, J)} \tag{3.3.117}$$

where the effective interaction $S_\epsilon^{(h^*-1)}, \tilde{V}_\epsilon^{(h^*-1)}$ and the energy term $F_\epsilon^{(h^*-1)}$ are related as usual by the analogous formula (3.3.95) with the following replacements: the scale label h is replaced with h^*, in $\mathcal{V}_\epsilon^{(h^*)}$ the term associated to ν is replaced by the counterterm M defined in (3.3.109), and the truncated expectation is taken w.r.t. $g_{L,\epsilon}^{(h^*)}$.

Now the integration $P_\epsilon^{(\leq h^*-1)}$ is diagonal in the ω index, i.e.

$$\int P^{(\leq h^*-1)}(d\phi) \phi_{y,\omega}^- \phi_{z,\omega'}^+ = \delta_{\omega,\omega'} \frac{1}{Z_{h^*-1}} g_{L,\epsilon,\omega}^{(\leq h^*-1)}(y,z) \tag{3.3.118}$$

and its propagator can be rewritten as

$$\begin{aligned} &\frac{1}{Z_{h^*-1}} g_{L,\epsilon,\omega}^{(\leq h^*-1)}(y,z) = L^{-2} \sum_{q \in \mathcal{D}'_{R,\theta,\omega}} \frac{e^{-iq(y-z)} \tilde{\chi}_{h^*-1}(q)}{Z_{h^*-1} D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)}, \\ &Z_{h^*-1} = \cos \tilde{q}_F + \zeta_{h^*-1} + a_{h^*,+}, \qquad D_{h^*-1,\omega}(q) := -iq_1 + \omega q_2 v_F \end{aligned} \tag{3.3.119}$$

where: $\mathcal{D}'_{R,\theta,\omega} = \mathcal{D}'_{R,\theta} - \omega \mathbf{q}_F$; v_F is the same as in (3.3.113); $\rho_{h^*-1}(q)$ is such that $|\rho_{h^*-1}(q)| \leq C 2^{2h^*}$ in the support of $\tilde{\chi}_{h^*-1}$. To obtain the representation (3.3.119), one can proceed in the following way. First note that from the conditions on $\delta_0, |\lambda|$, the denominator in the expression of $g^{(\leq h^*-1)}_{L,\epsilon,\omega}$ (see (3.3.93) for $h = h^*$ where we added the counterterm M) can be rewritten as

$$r_{h^*-1} + D_{h^*-1}(q + \omega \boldsymbol{q}_F) + \rho(q + \omega \boldsymbol{q}_F) + 2^{h^*}(\nu_{h^*} + \nu_{h^*,+}) - i q_1 a_{h^*,+} + \omega q_2 b_{h^*,+}. \quad (3.3.120)$$

Then expanding in Taylor series at $q = 0$, using the definition of D_h and ρ ((3.3.48),(3.3.41)) and imposing the following relation for $\tilde{q}_F$ and $\nu_{h^*,+}$,

$$1 + r_{h^*-1} - \cos \tilde{q}_F + \frac{(\tilde{q}_F)^2}{2} \alpha_{h^*-1} + 2^{h^*}(\nu_{h^*} + \nu_{h^*,+}) = 0, \quad (3.3.121)$$

the expression in (3.3.119) follows.

We now stress how the given equations above, relating the free parameters, must be looked at:

Remark 3.3.8. *First recall that $r_{h^*-1}, \alpha_{h^*-1}, \nu_{h^*}$ are known by the integration of the first regime and they respectively are $O(\epsilon), O(\lambda), O(\lambda)$. Then,* (3.3.121) *must be seen as an equation for $\nu_{h^*,+}$ given $\tilde{q}_F$. This equation fixes the singularities of the propagator $g^{(\leq h^*)}$ exactly at the not yet determined points $\pm \mathbf{q}_F = \pm(0, \tilde{q}_F)$: $\tilde{q}_F$ will be indeed determined by such relation only when $\nu_{h^*,+}$ will be fixed in the multiscale construction in order for the perturbative expansion to be absolutely convergent (uniformly in L, ϵ etc..). More precisely, in order to obtain such a convergence, $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$ will be determined via a fixed point equation in a certain space of sequences, making them functions of the multiscale expansion of the second regime, and thus of $\tilde{q}_F, v_F, Z_{h^*-1}$. Then, with such counterterms fixed, one can solve via the Implicit Function the system of equations*

$$\begin{aligned} 1 + r_{h^*-1} - \cos \tilde{q}_F + \frac{(\tilde{q}_F)^2}{2} \alpha_{h^*-1} + 2^{h^*}(\nu_{h^*} + \nu_{h^*,+}) &= 0 \\ Z_{h^*-1} - \cos \tilde{q}_F - \zeta_{h^*-1} + a_{h^*,+} &= 0 \\ Z_{h^*-1} v_F - \sin \tilde{q}_F - \tilde{q}_F \alpha_{h^*-1} - b_{h^*,+} &= 0 \end{aligned} \quad (3.3.122)$$

finding then the desired functions $\tilde{q}_F(\lambda, \epsilon), v_F(\lambda, \epsilon), Z_{h^*-1}(\lambda, \epsilon)$ with the desired properties. This will be studied in Section 3.4.4, more precisely in Section 3.4.5-3.4.5.

Observe that the assumption of point (1) *above is an automatic consequence of the fact (to be proven later) that $\nu_{h^*,+} = O(\lambda)$; moreover when such a condition is true for $\tilde{q}_F$, proving also that $b_{h^*,+} = O(\sqrt{\epsilon})$ guarantees automatically that $v_F = O(\sqrt{\epsilon})$. This will be crucial in estimating the kernels.*

Localization procedure: second regime

Inductive statement First, since the second regime $h < h^*$ can be present only if $\epsilon > 0$, the dependence on it will be present, *but not explicited*, in every expression. Let us assume that the fields $\varphi^{(h^*-1)}, \dots, \varphi^{(h+1)}$, $h \geq h_L$, have been integrated out, and that after their integration the generating function has the following structure

$$e^{\mathbb{W}^{(\theta)}_{L,\epsilon}(\mathcal{A},0)} = e^{L^2(F^{(h)}+t^{(h)})+S^{(h)}(J)} \int P^{(\le h)}(d\varphi^{(\le h)}) e^{V^{(h)}(\sqrt{Z_h}\varphi^{(\le h)},J)}, \tag{3.3.123}$$

for a suitable real constant $E^{(h)}$ and suitable "effective potentials" $S^{(h)}(J)$, $V^{(h)}(\varphi, J)$, to be defined inductively below, and fixed in such a way that $V^{(h)}(0, J) = S^{(h)}(0) = 0$. $P^{(\le h)}(d\psi)$ is the Grassmann Gaussian integration with propagator (diagonal in the index ω)

$$\frac{1}{Z_h} g^{(\le h)}_{L,\epsilon,\omega}(y, y') = \frac{1}{Z_h} \frac{1}{L^2} \sum_{q \in \mathcal{D}'_{R,\theta,\omega}} e^{-iq(y-y')} \frac{\tilde\chi_h(q)}{\mu_{h,\omega}(q)} \tag{3.3.124}$$

for some real constant Z_h, inductively defined, and

$$\mu_{h,\omega}(q) := D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}/Z_h \tag{3.3.125}$$

where $D_{h^*-1,\omega}, \rho_{h^*-1}$ are the same as in (3.3.119), in particular ρ_{h^*-1} is of order 2^{2h} in the support of $\tilde f_h := \tilde\chi_h - \tilde\chi_{h-1}$. We will also prove inductively that:

1. $V^{(h)}(\varphi, J)$ has the same structure as in the second line of (3.3.72), with the kernels being specified by the extra label $\underline\omega = (\omega_1, \dots, \omega_n)$ carried by the Grassmann variables $\hat\phi_{\underline q,\underline\omega} = \hat\phi^+_{q_1,\omega_1} \cdots \hat\phi^-_{q_n,\omega_n}$, with $\hat\phi^\pm_{q,\omega} := \hat\phi^\pm_{q+q_F+\omega\mathbf{q}_F}$. Note that in terms of lattice coordinates, the dependence on ω in the kernels $W^{(h)}_{n,m;\underline\omega,\underline s}(\underline y, \underline x)$ is due to the oscillatory factors $e^{-i\mathbf{q}_F \sum_{i=1}^n (-1)^i \omega_i y_i}$ coming from (3.3.116): then we denote $W^{(h)}_{n,m;\underline\omega,\underline s}(\underline y, \underline x) = e^{-i\mathbf{q}_F \sum_{i=1}^n (-1)^i \omega_i y_i} \widetilde W^{(h)}_{n,m;\underline s}(\underline y, \underline x)$.

2. the kernels of $V^{(h)}(\varphi, J)$ satisfy the following symmetry, in Fourier space:

$$\begin{aligned} \hat W^{(h)}_{n,m;-\underline\omega,\underline s}(\underline q, \underline p) &= \overline{\hat W^{(h)}_{n,m;\underline\omega,\underline s}(-\underline q, -\underline p)} \\ \hat W^{(h)}_{n,m;-\underline\omega,\underline s}(\underline q, \underline p) &= \hat W^{(h)}_{n,m;\underline\omega,\underline s}(\hat{\underline q}, \underline p) \end{aligned} \tag{3.3.126}$$

   where $\hat W^{(h)}_{n,m;\underline\omega,\underline s}(\underline q, \underline p)$ is the Fourier transform of $\widetilde W^{(h)}_{n,m;\underline s}(\underline y, \underline x)$ computed at momenta $(q_2 + \omega_2\mathbf{q}_F, \dots, q_n + \omega_n\mathbf{q}_F, p_1, \dots, p_m)$; $\hat{\underline q} := (\hat q_2, \dots, \hat q_n)$ where for $q \in \mathbb{R}^2$, $\hat q := (q_1, -q_2)$.[16]

Remark 3.3.9. *As it was for the first regime, we emphasize that we view the kernels $W^{(h)}_{n,m;\underline\omega,\underline s}$, $h \le h^* - 2$, as functions of:*

(i) a sequence of running coupling constants

$$\{\lambda_{h'}, \nu_{h',\omega}, a_{h',\omega}, b_{h',\omega}, Y_{h',r,(\omega,\omega')}\}_{h<h'\le h^*-1}.$$

(ii) a sequence of single-scale propagators $\{g^{(h')}_\omega / Z_{h'-1}\}_{h<h'\le h^*-1}$, of the form

$$\frac{1}{Z_{h-1}} g^{(h)}_\omega(x, y) := \frac{1}{L^2} \sum_{q \in \mathcal{D}'_{R,\theta,\omega}} e^{-iq(y-y')} \frac{\tilde f_h(q)}{\tilde Z_{h-1}(k) D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)}, \tag{3.3.127}$$

[16] See Appendix B.4, for a discussion with $\mathcal{A} = 0$.

where $\tilde{f}_h(q) = \tilde{\chi}_h(q) - \tilde{\chi}_{h-1}(q)$ *and*

$$\tilde{Z}_{h-1}(q) = Z_{h-1}\tilde{\chi}_h(q) + Z_h(1 - \tilde{\chi}_h(q));$$

(iii) the irrelevant part of $V^{(h^*-1)}$*, denoted by* $\mathcal{R}_2 V^{(h^*-1)}$*.*

The actual values of the running coupling constants (RCCs) will be defined via an inductive procedure in Sections 3.3.2-3.3.2, the outcome of which is the beta function equation (3.4.1)*. In Section 3.4.4, we will show that there is only one specific choice of the initial data* $(\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+})$*, which we will determine via a fixed point argument, guaranteeing that the flow of RCCs is uniformly bounded for all* $h \le 0$*. For the fixed point argument itself, it is convenient to allow the beta function, as well as the kernels of the effective potential, to be computed at values of the RCCs different from the final, 'correct', ones. This is what we mean by saying that* $W^{(h)}_{n,m;\underline{\omega},\underline{s}}$ *will be thought of as functions of the RCCs: we will allow ourselves to think of the RCCs as independent variables, which can be varied freely, as long as they remain sufficiently small; similarly for the dependences on* $g^{(h)}_\omega/Z_{h-1}$ *and* $\mathcal{R}_2 V^{(h^*-1)}$ *mentioned in items (ii)-(iii): for certain manipulations discussed below, we will allow ourselves to modify the definition of the kernels by modifying the form of the single-scale propagators or of the kernel of the irrelevant part at scale* $h^* - 1$*, keeping the rest of the iterative definition unchanged.*

Base of the induction: $h = h^* - 1$ The representation (3.3.123) is valid at the initial step, $h = h^* - 1$, with $F^{(h^*-1)} + t^{(h^*)} = F_\epsilon^{(h^*-1)} + t_\epsilon^{h^*-1}, S^{(h^*-1)} = S_\epsilon^{(h^*-1)}$ and $V^{(h^*-1)}(\cdot, J) = \tilde{V}_\epsilon^{(h^*-1)}(Z_{h^*-1}^{-1/2}(\cdot), J)$ as one can check by comparing with (3.3.117) after the integration of the transition modes.

To see that (3.3.126) holds for $h = h^* - 1$, note that it is equivalent to requiring that $V^{(h^*-1)}$ is invariant under the following transformations $(1) : \varphi^\pm_{y,\omega} \to \varphi^\pm_{y,\omega}$ together with complex conjugation of the kernels and $(2) : \varphi^\pm_{y,\omega} \to \varphi^\pm_{\hat{y},-\omega}$, where $\hat{y} = (y_1, -y_2)$. On the other hand the assumption on the counterterms (3.3.112), guarantees that the potential $V^{(h^*)}(\varphi, J)$ is invariant under such symmetries. The statement (3.3.126) for $h = h^* - 1$ easily follows from the relation between $V^{(h^*)}$ and $V^{(h^*-1)}$, compare (3.3.110), (3.3.117) and below, together with the fact that the grasmmann integration $P^{(h^*-1)}$ is invariant under the same transformations $(1) - (2)$, see Appendix B.4.

The inductive step We assume that (3.3.123) holds with $V^{(h)}$ satisfying the properties specified in the inductive statement, and we discuss here how to get the same representation at the next scale $h - 1$. First, we split $V^{(h)}$ into its *local* and *irrelevant* parts: $V^{(h)} = \mathcal{L}_2 V^{(h)} + \mathcal{R}_2 V^{(h)}$ where, denoting by $\hat{W}^{(h),\infty}_{n,m;\underline{\omega},\underline{s}}$ the infinite volume limit of $\hat{W}^{(h)}_{n,m;\underline{\omega},\underline{s}}$,

$$\begin{aligned}
&\mathcal{L}V^{(h)}(\varphi, J) := \qquad (3.3.128)\\
&\quad = L^{-2}\sum_\omega \sum_{q\in\mathcal{D}'_{R,\theta,\omega}} \hat{\phi}^+_{q,\omega}[\hat{W}^{(h),\infty}_{2,0;(\omega,\omega)}(0) + q\cdot\partial_q \hat{W}^{(h),\infty}_{2,0;(\omega,\omega)}(0)]\hat{\phi}^-_{q,\omega}\\
&\quad + \sum_{y\in\tilde{\Lambda}} \sum_{\omega_1,\dots,\omega_4} \phi^+_{y,\omega_1}\phi^-_{y,\omega_2}\phi^+_{y,\omega_3}\phi^-_{y,\omega_4}\hat{W}^{(h),\infty}_{4,0;(\omega_1,\dots,\omega_4)}(0,0,0)\\
&\quad + \sum_{x\in\Lambda}\sum_{\omega_1,\omega_2,r} J_{x,r}\phi^+_{R^T x,\omega_1}\phi^-_{R^T x,\omega_2} e^{i\boldsymbol{k}_F x(\omega_2-\omega_1)}\hat{W}^{(h),\infty}_{2,1;(\omega_1,\omega_2),s}(0, \boldsymbol{k}_F(\omega_2-\omega_1)).
\end{aligned}$$

where $\boldsymbol{k}_F = R\boldsymbol{q}_F$ (cfr. item 1 below (3.3.109).

Remark 3.3.10. *A few remarks about this definition are in order:*

1. *The existence of the limit of $\hat{W}^{(h)}_{n,m;\underline{\omega},\underline{r}}$ as $L \to \infty$ is a corollary of the inductive bounds on the kernels of $V^{(h)}$, which are uniform in L, as it was the case for $h = -1$, cf. with Remark 3.3.3. More details on the inductive bounds on the kernels of $V^{(h)}$ are discussed below.*

2. *The reason why, in the second line of* (3.3.129)*, we only include terms where the Grassmann fields have the same index ω, is that the terms with opposite ω indices give zero contribution to the generating function, due to the support properties of the Grassmann fields. In fact, in* (3.3.123) *we need to compute $V^{(h)}$ at Grassmann fields $\hat{\varphi}^{(\leq h)\pm}_{q,\omega}$ that, in momentum space, have the same support as $\hat{g}^{(\leq h)}_{L,\omega}(q)$, i.e., $|q| = O(2^h)$ (note that the support properties of $\hat{g}^{(\leq h)}_{\omega}$ are the same as those of χ_h (see lines above* (3.3.113) *in Item 3). If $h \leq h^* - 1$ and c_0 is sufficiently big, quadratic terms of the form $\hat{\varphi}^{(\leq h),+}_{q,\omega}\hat{\varphi}^{(\leq h),-}_{q+2\omega\boldsymbol{q}_F,-\omega}$ would involve two fields that cannot both satisfy this support property.*

3. *Due to the Grassmann anti-commmutation rules and the anti-symmetry of the kernels, the quartic term in* (3.3.129) *can be rewritten as*

$$4\sum_{y\in\tilde{\Lambda}} \varphi^+_{y,+}\varphi^-_{y,+}\varphi^+_{y,-}\varphi^-_{y,-}\hat{W}^{(h),\infty}_{4,0;(+,+,-,-)}(0,0,0). \tag{3.3.129}$$

Along the induction step, we will need a function $W^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}(y_1,y_2)$ (the upper index '$\mathtt{R}$' stands for "relativistic", and it is different from the matrix R appearing in the transformation in (3.3.4)) which should be thought of as the kernel for $n = 2, m = 0$ of a relativistic model. More precisely, at step $h = h^* - 1$, one simply let $W^{(h^*-1),\mathtt{R}}_{2,0;(\omega,\omega)}(y_1,y_2) \equiv 0$. For $h < h^* - 1$, $W^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}$ is defined as a suitable modification of $W^{(h),\infty}_{2,0;(\omega,\omega)}$ (that, by the induction hypothesis, has already been defined); more precisely, $W^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}$ is obtained by making the following replacements in $W^{(h),\infty}_{2,0;(\omega,\omega)}$ (which should be thought of as a function of the running coupling constants, of the single scale propagators and of the irrelevant part of $V^{(h^*-1)}$, as explained in Remark 3.3.9):

(i) the running coupling constants $\{\nu_{h',\omega}, a_{h',\omega}, b_{h',\omega}\}_{h'>h}$ are set to zero, (note that the running coupling constants $\lambda_{h'}$ are *not* set equal to zero);

(ii) the single-scale propagators $g^{(h')}_{\omega}/Z_{h'-1}$ are replaced by the 'relativistic' single-scale propagators $g^{(h')}_{\mathtt{R},\omega}/Z_{h'-1}$, for all $h < h' \leq h^* - 1$, where

$$\frac{g^{(h')}_{\mathtt{R},\omega}(y,y')}{Z_{h'-1}} = \int_{\mathbb{R}^2} \frac{dq}{(2\pi)^2} e^{-iq(y-y')} \frac{\tilde{f}_{h'}(q)}{\tilde{Z}_{h'-1}D_{h^*-1,\omega}(q)}; \tag{3.3.130}$$

(iii) $\mathcal{R}V^{(h^*-1)}$ is set to zero.

The function $W^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}$ will be shown to satisfy both the identity (3.3.126) and the extra symmetry

$$W^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}(v_F q_2, v_F^{-1} q_1) = i\omega \overline{\hat{W}^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}(q)} \tag{3.3.131}$$

where v_F is the same of (3.3.119). This symmetry is a special case of the one discussed in [**2**] (see in particular Appendix A and Section 4). Let us assume that $W^{(h'),\mathtt{R}}_{2,0;(\omega,\omega)}, h' \geq h$ has been already shown to satisfy (3.3.131) and below we explain how to prove the same at scale $h-1$.

In order to define the running coupling constants on scale h, we decompose the term containing $\partial_q \hat{W}^{(h),\infty}_{2,0;(\omega,\omega)}(0)$ in (3.3.129), by rewriting

$$\partial_q \hat{W}^{(h),\infty}_{2,0;(\omega,\omega)}(0) = \partial_q \hat{W}^{(h),\mathtt{R}}_{2,0;(\omega,\omega)}(0) + \partial_q \hat{W}^{(h),\mathtt{S}}_{2,0;(\omega,\omega)}(0), \tag{3.3.132}$$

('$\mathtt{S}$' stands for 'subdominant'). From the symmetries (3.3.131), a straightforward computation shows that

$$q \cdot \partial_q \hat{W}^{(h),R}_{2,0;(\omega,\omega)}(0) = -z_h(-iq_1 + \omega v_F q_2) = -z_h D_{h^*-1,\omega}(q), \tag{3.3.133}$$

for some real number z_h[17]. We now combine this term with the Grassmann Gaussian integration $P^{(\leq h)}(d\varphi)$, and define:

$$P^{(\leq h)}(d\varphi) e^{-z_h Z_h L^{-2} \sum_\omega \sum_{q\in\mathcal{D}'_{R,\theta,\omega}} D_{h^*-1,\omega}(q) \hat{\varphi}^+_{q,\omega} \hat{\varphi}^-_{q,\omega}} \equiv e^{L^2(t_{h-1}-t_h)} \tilde{P}^{(\leq h)}(d\varphi), \tag{3.3.134}$$

where $\tilde{P}^{(\leq h)}(d\varphi)$ is the Grassmann Gaussian integration with propagator

$$\frac{\tilde{g}^{(\leq h)}_{L,\omega}(y,y')}{Z_{h-1}} = \frac{1}{L^2} \sum_{q\in\mathcal{D}'_{R,\theta,\omega}} e^{-iq(y-y')} \frac{\tilde{\chi}_h(q)}{\tilde{Z}_{h-1}(q) D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)}, \tag{3.3.135}$$

with

$$\tilde{Z}_{h-1}(q) := Z_h(1 + z_h \tilde{\chi}_h(q)), \qquad Z_{h-1} := \tilde{Z}_{h-1}(0) = Z_h(1+z_h), \tag{3.3.136}$$

and $e^{L^2 t_h}$ is a constant that normalizes $\tilde{P}^{(\leq h)}(d\varphi)$ to 1:

$$t_h = \frac{1}{L^2} \sum_\omega \sum_{q\in\mathcal{D}'_{R,\theta,\omega}} \log\Big(1 + \frac{z_h \tilde{\chi}_h(q) D_{h^*-1,\omega}(q)}{D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)/Z_h}\Big). \tag{3.3.137}$$

By using (3.3.134), we rewrite the Grassmann integral in the right side of (3.3.123) as

$$(3.3.123) = e^{L^{F^{(h)}} + t^{(h-1)}) + S^{(h)}(J)} \int \tilde{P}^{(\leq h)}(d\varphi) e^{\hat{V}^{(h)}(\sqrt{Z_{h-1}}\varphi, J)} \tag{3.3.138}$$

[17] That z_h is real follows from the fact that the relativistic kernels satisfy as well (3.3.126).

where

$$\begin{aligned}\widehat{V}^{(h)}(\varphi,J) &= L^{-2}\sum_{\omega}\sum_{q\in\mathcal{D}'_{R,\theta,\omega}}\hat{\varphi}^{+}_{q,\omega}\left(2^{h}\nu_{h,\omega}+ia_{h,\omega}q_1+b_{h,\omega}q_2\right)\hat{\varphi}^{-}_{q,\omega}\\ &+ \lambda_h\sum_{y\in\tilde{\Lambda}}\varphi^{+}_{y,+}\varphi^{-}_{y,+}\varphi^{+}_{y,-}\varphi^{-}_{y,-}\\ &+ \sum_{\omega_1,\omega_2,s}\frac{Y_{h,s,(\omega_1,\omega_2)}}{Z_{h-1}}\sum_{x\in\Lambda}J_{x,r}e^{i(\omega_2-\omega_1)\boldsymbol{k}_F x}\varphi^{+}_{R^Tx,\omega_1}\varphi^{-}_{R^Tx,\omega_2}\\ &+ \mathcal{R}_2V^{(h)}(\sqrt{Z_h/Z_{h-1}}\,\varphi,J),\end{aligned}\tag{3.3.139}$$

and the running coupling constants at scale h are defined as

$$\begin{aligned}&2^h\nu_{h,\omega}=\frac{Z_h}{Z_{h-1}}\hat{W}^{(h),\infty}_{2,0;(\omega,\omega)}(0),\\ &a_{h,\omega}=-i\frac{Z_h}{Z_{h-1}}\partial_{q_1}\hat{W}^{(h),\mathtt{s}}_{2,0;(\omega,\omega)}(0),\qquad b_{h,\omega}=\frac{Z_h}{Z_{h-1}}\partial_{q_2}\hat{W}^{(h),\mathtt{s}}_{2,0;(\omega,\omega)}(0),\\ &\lambda_h=4\Big(\frac{Z_h}{Z_{h-1}}\Big)^2\hat{W}^{(h),\infty}_{4,0;(+,+,-,-)}(0,0,0),\\ &Y_{h,s,(\omega_1,\omega_2)}=Z_h\hat{W}^{(h),\infty}_{2,1;(\omega_1,\omega_2),s}(0,(\omega_2-\omega_1)\boldsymbol{k}_F).\end{aligned}\tag{3.3.140}$$

Thanks to the symmetry (3.3.126) of the kernels (that by inductive hypothesis holds at step h) the running coupling constants are real valued, $\nu_{h,\omega},a_{h,\omega},b_{h,\omega},\lambda_h,Y_{h,s,(\omega_1,\omega_2)}\in\mathbb{R}$ and satisfy the following (see Appendix B.4):

$$\nu_{h,\omega}=\nu_{h,-\omega},\quad a_{h,\omega}=a_{h,-\omega},\quad b_{h,\omega}=-b_{h,-\omega},\quad Y_{h,r,\underline{\omega}}=Y_{h,r,-\underline{\omega}}.\tag{3.3.141}$$

For later reference, we rewrite the local part of $\widehat{V}^{(h)}(\varphi,J)$ as

$$\begin{aligned}\mathcal{L}\widehat{V}^{(h)}(\varphi,J) &= \sum_{\omega}\Big[2^h\nu_{h,\omega}F_{\nu;\omega}(\varphi)+a_{h,\omega}F_{a;\omega}(\varphi)+b_{h,\omega}F_{b;\omega}(\varphi)\Big]\\ &+ \lambda_hF_\lambda(\varphi)+\sum_{s,\underline{\omega}}\frac{Y_{h,s,\underline{\omega}}}{Z_{h-1}}F_{Y;s,\underline{\omega}}(\varphi,J),\end{aligned}\tag{3.3.142}$$

(for the definitions of $F_{\nu;\omega}(\varphi),F_{a;\omega},F_{b;\omega}$, etc., compare (3.3.142) with the first two lines of (3.3.139)).

We now decompose the propagator (3.3.135) as

$$\tilde{g}^{(\le h)}_{L,\omega}(y,y')=g^{(h)}_{L,\omega}(y,y')+g^{(\le h-1)}_{L,\omega}(y,y'),$$

with $g^{(\le h-1)}_{\omega}$ as in (3.3.124) and $g^{(h)}_{\omega}$ as in (3.3.127). The scaling bounds of $g^{(h)}_{\omega}$, used in the contraction at every scale, are given in the next paragraph. To see that this decomposition holds, note that $\tilde{Z}_{h-1}(k)\equiv Z_{h-1}$ on the support of $\bar{\chi}_{h-1}(\cdot)$.

Then, rewrite (3.3.138) as

$$\int P^{(\le h)}(d\varphi)e^{V^{(h)}(\sqrt{Z_h}(\varphi,J)}=\int P^{(\le h-1)}(d\varphi)\int P^{(h)}(d\varphi')e^{\widehat{V}^{(h)}(\sqrt{Z_{h-1}}(\varphi+\varphi',J)},\tag{3.3.143}$$

which implies the validity of the representation (3.3.123) at scale $h-1$, with $F^{(h-1)}$, $S^{(h-1)}(\cdot)$ and $V^{(h-1)}(\cdot)$ defined by

$$\begin{aligned} e^{L^2F^{(h-1)}+S^{(h-1)}(J)+V^{(h-1)}(\sqrt{Z_{h-1}}\varphi,J)} = \\ = e^{L^2F^{(h)}+S^{(h)}(J)} \int P_{(h)}(d\varphi') e^{\widehat{V}^{(h)}(\sqrt{Z_{h-1}}(\varphi+\varphi'),J)}, \end{aligned} \tag{3.3.144}$$

that is,

$$\begin{aligned} L^2(F^{(h-1)}-F^{(h)}) + (S^{(h-1)}(J)-S^{(h)}(J)) + V^{(h-1)}(\varphi,J) \\ = \sum_{n\geq 1} \frac{1}{n!} \mathcal{E}^T_{(h)}(\underbrace{\widehat{V}^{(h)}(\sqrt{Z_{h-1}}(\varphi+\varphi'),J);\cdots;\widehat{V}^{(h)}(\sqrt{Z_{h-1}}(\varphi+\varphi'),J)}_{n \text{ times}}), \end{aligned} \tag{3.3.145}$$

with $\mathcal{E}^T_{(h)}$ the truncated expectation w.r.t. the Grassmann Gaussian integration $P^{(h)}(d\varphi)$, and $F^{(h-1)}$, $S^{(h-1)}(\cdot)$ fixed as usual by the conditions $S^{(h-1)}(0)=0$ and $V^{(h-1)}(0,J)=0$.

To conclude the proof of the induction step, it remains to prove that the kernels of $V^{(h-1)}$ satisfy (3.3.126) and that (3.3.131) holds, at scale $h-1$. The proof of both statements follow from the inductive hypothesys and the fact that the covariance is still symmetric under the same symmetries, see Appendix B.4.

The single scale propagator of the second regime The integration procedure just described, for $h < h^*-1$, produces a single scale propagator given by

$$\frac{1}{Z_{h-1}} g^{(h)}_{\omega,L}(x,y) := \frac{1}{L^2} \sum_{q\in\mathcal{D}'_{R,\theta,\omega}} e^{-iq(y-y')} \frac{\tilde{f}_h(q)}{\tilde{Z}_{h-1}(q) D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)}, \tag{3.3.146}$$

where $\tilde{f}_h(q) = f(2^{-h}|iq_1 + v_F q_2|)$, and f same as above; $D_{h^*-1,\omega}(q) := -iq_1 + \omega q_2 v_F$ with

$$\begin{aligned} &\tilde{Z}_{h-1}(q) := Z_h(1+z_h\tilde{\chi}_h(q)) \\ &Z_{h-1} := \tilde{Z}_{h-1}(0) = Z_h(1+z_h) \\ &v_F := \frac{\sin\tilde{q}_F + \tilde{q}_F\alpha_{h^*-1} + b_{h^*,+}}{\cos\tilde{q}_F + \zeta_{h^*-1} + a_{h^*,+}} \\ &\rho_{h^*-1}(q) = \rho(q+\omega\mathbf{q}_F) - \rho(\omega\mathbf{q}_F) - q\cdot\partial\rho(\omega\mathbf{q}_F). \end{aligned} \tag{3.3.147}$$

where ρ is the same as in (3.3.97) and $\mathbf{q}_F$ is defined below (3.3.109). Then we have

Lemma 3.3.0.4. *If there exists $C>0$, such that $\sup_{h\leq h'<h^*}|z_h| \leq C\delta_1$ then there exist L,ϵ independent constants $c_0,c_1,\kappa>0$ such that*

$$|\tilde{g}^{(h)}_\omega(y)| \leq c_0 v_F^{-1} 2^{h(1+n_1+n_2)} e^{-\kappa\sqrt{2^h(|y_1|+v_F^{-1}|y_2|)}}. \tag{3.3.148}$$

and

$$\sum_{y\in R^T\mathbb{Z}^2} |y_1^{m_1} y_2^{m_2} \tilde{g}^{(h);\underline{n}}_{\mathtt{R},\omega}(y)| e^{\frac{\kappa}{2}\sqrt{2^h(|y_1|+v_F^{-1}|y_2|)}} \leq c_1 v_F^{-1-n_2-n_0+m_2} 2^{h(1+n_1+n_2-m_2-m_1)}. \tag{3.3.149}$$

Moreover, $\tilde{g}^{(h);\underline{n}}_{\mathtt{S},\omega}$ satisfy the same estimates times an extra factor 2^h.

For more details, see Appendix B.2. See also [**4**] for an analogous bound.

$\epsilon < 0$: The massive case

Proceeding as in Section 3.3.1, it is possible to show that for a fixed $\epsilon < 0$, a single step of integration is sufficient to obtain bounds for $f_{\epsilon,\lambda,L}$ which are uniform in L, and to show the existence of its thermodynamic limit. Indeed one can write

$$\mathcal{W}^{(\theta)}_{\varepsilon_0(\lambda)+\epsilon,L,\lambda}(\mathcal{A}) = L^2(F^{(0)}_\epsilon + \tilde{F}^{(0)}) + S^{(0)}(\mathcal{A}) \tag{3.3.150}$$

where $F^{(0)}_\epsilon$ is the same as in (3.3.68), while

$$L^2\tilde{F}^{(0)} + S^{(0)}(\mathcal{A}) = \sum_{n\geq 1}\frac{1}{n!}\mathcal{E}^T_\epsilon(\underbrace{\nu_0 N(\cdot) + V(\cdot,\mathcal{A}); \dots ; \nu_0 N(\cdot) + V(\cdot,\mathcal{A})}_{\text{n times}}) \tag{3.3.151}$$

and the truncated expectation $\mathcal{E}^T_\epsilon$ is associated to the propagator

$$\begin{aligned} g_\epsilon(y,y') &= L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}\frac{e^{-iq(y-y')}}{1+r_0-e^{iq_1}\cos q_2} \\ r_0 &= e^{-\varepsilon_0(\lambda)}(e^{-\epsilon}-1). \end{aligned} \tag{3.3.152}$$

Since $r_0 > 0$ iff $\epsilon < 0$, then g_ϵ decays exponentially[18] in the distance on the torus, the discussion above (3.3.24) applies and, in particular, we obtain bounds which are uniform in L, but not as $\epsilon \to 0^-$ (the rate of the exponential decay tends to 0). If $\epsilon \to 0^-$, we need to perform a multiscale analysis. In particular we can repeat exactly the same multiscale procedure of Section 3.3.2, by obtaining an inductive structure as in (3.3.101) and below by recalling that now all the quantities depending on ϵ are evaluated at some negative small value. The stopping scale for the first regime of integration has the same definition as in (3.3.76):

$$h^*(\epsilon) = \min\Big\{h \leq -1 : \frac{|r_h|}{(1+\alpha_h)} < 2^h\delta\Big\} \tag{3.3.153}$$

where r_h is given by (3.3.100) and satisfies Lemma 3.3.0.3. Note in particular that since Q_h, α_h do not depend on ϵ, the only difference in h^* arising by changing sign to ϵ comes from r_0. It is easy to see that $h^*(-|\epsilon|) \geq h^*(|\epsilon|)$ and $h^*(-\epsilon) = h^*(\epsilon)$ if $|\epsilon|$ is small enough. Since we are free to choose ϵ as small as we like, we choose it so that the parity holds. When we reach scale h^*, then we can integrate all the lower scale at once. Recall indeed that $r_h > 0$ for every $h < h^*$ since it satisfies $r_h = r_0 Q_h(\lambda)$, with $|Q_h - 1| = O(\lambda)$ by Lemma 3.3.0.3. Then we have that

$$g^{(\leq h^*)}_{L,\epsilon}(y,y') = L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}\frac{e^{-iq(y-y')}\chi_{h^*}(q)}{r_{h^*} + D_{h^*}(q) + \rho(q)} \tag{3.3.154}$$

has the same scaling of the single scale propagator at scale h^* as in Lemma B.2.0.1, as it is seen by rescaling $(q_1, q_2) = (2^h p_1, 2^{h/2}p_2)$ and using the definition of χ and the fact that

[18] It is the Fourier transform of an analytic function.

D is linear in q while $2^{-h^*}\epsilon = O(1)$.

3.4 RG approach: dimensional bounds and flow of the running constants

In this Section we collect the informations coming from the iterative construction of the previous Sections 3.3.1-3.3.2 and we reformulate the problem of obtaining a convergent perturbative expansion in terms of the study of a discrete flow equation (the Beta function equation). The content of this Section is based on a somehow standard approach and we take as references [**2**–**5**]. The definition of the flow equations is given in Section 3.4.1; the bounds on the kernels built in Sections 3.3.1-3.3.2 are discussed in Section 3.4.2 in terms of the flow's properties. Since the iterative construction we made is very similar to the one of [**4**], we discuss only the main differences and we do not report the whole proof of such bounds. In Section 3.4.3, we recall how these kernels bounds are derived and what is the impact of the differences with [**4**] acts by means of the so called Gallavotti-Nicolo tree expansion. Finally in Section 3.4.4 we reformulate the convergence of the perturbative expansion in terms of a fixed point problem for a map on a sequence space. This reformulates the discussion of [**4**] in the direction of the same framework of [**2**, Sec. 6.4].

3.4.1 The beta function

The iterative integration scheme described above in Section 3.3.2 allows us to express the kernels of $V^{(h)}$ and, in particular, the running coupling constants (RCC) at scale h, as functions of the sequence of RCC and of the single-scale propagators on higher scale and of irrelevant part of the interaction at the "first" scale (see Remark 3.3.9). More precisely,

- if $h \geq h^*$, the kernels $W^{(h)}_{n,m,\epsilon,\underline{s}}$, are viewed as functions of the running coupling $\{\nu_k\}_{h<k\leq 0}$, of the renormalization constants $\{r_k, \alpha_k, \zeta_k\}_{h\leq k\leq -1}$, of the single scale propagators $\{\tilde{g}^{(k)}_L\}_{k\geq h}$ (cf. (3.3.92)) and of the irrelevant part of the interaction $\mathcal{R}_1 V^{-1}_\epsilon$.
- If instead $h < h^*$, the kernels $W^{(h)}_{n,m,\underline{\omega},\underline{s}}$ are viewed as functions of the sequence of running coupling $\{(\nu_{j,\omega}, a_{j,\omega}, b_{j,\omega}, \lambda_j)\}_{h\leq j\leq h^*}$, of the renormalization constant $\{z_j\}_{h\leq j<h^*}$, of the single scale propagators $\{Z^{-1}_{h-1} g^{(j)}\}_{h\leq j<h^*}$ (cf. (3.3.127)), and of the irrelevant part of the interaction, at the transition scale, $\mathcal{R}_2 V^{(h^*-1)}$.

In particular we rewrite the relation between contiguous scale constants in the form

$$(h \geq h^*) : \begin{cases} \nu_{h-1} = 2\nu_h + B^\nu_h \\ r_{h-1} = r_h + B^r_h \\ \zeta_{h-1} = \zeta_h + B^\zeta_h \\ \alpha_{h-1} = \alpha_h + B^a_h \\ Y_{h-1,s} = Y_{h,s} + B^Y_{h,s} \end{cases} \qquad (h \leq h^*) : \begin{cases} \nu_{h-1,\omega} = 2\nu_{h,\omega} + B^\nu_{h,\omega} \\ a_{h-1,\omega} = a_{h,\omega} + B^a_{h,\omega} \\ b_{h-1,\omega} = b_{h,\omega} + B^b_{h,\omega} \\ \lambda_{h-1} = \lambda_h + B^\lambda_h \\ Y_{h-1,s,\underline{\omega}} = Y_{h,s,\underline{\omega}} + B^Y_{h,s,\underline{\omega}} \end{cases} \tag{3.4.1}$$

which is just a reformulation of (3.3.49),(3.3.58),(3.3.90)-(3.3.93), (3.3.88) for $h \geq h^*$, and of (3.3.140) for $h < h^*$. Here $B^\#_{h,\cdot}$, $h \leq -1$, is the so-called *Beta function*. One has to think of $B^\#_{h,\cdot}$ as well as a function of the RCC on higher scales. Note the following

(0) the equations in the lhs of (3.4.1) make sense also with $h = 0$ by defining $\alpha_0 = 0, \zeta_0 = 0, Y_{h,s} = 0, B^\zeta_0 = 0, B^\alpha_0 = 0, B^r_0 = 0$: recall indeed that $\zeta_{-1} = \alpha_{-1} = 0$ (cf. (3.3.41)) and that $r_{-1} = r_0$ (cf. below (3.3.81)). The first equation is instead non trivial and expresses the relation between ν_{-1} and the counterterm $\nu_0 = e^{-\varepsilon_0} - 1$. For $h = h^*$ they are meaningful with $\nu_{h^*-1} = Y_{h^*-1,s} = 0$ (which is also a definition of the associated beta function, $B^\nu_{h^*} = -2^{h^*}\nu_{h^*}$ and similarly for B^Y).
The rhs of (3.4.1) is as well meaningful also for $h = h^*$: the first three equations relate the evolution of the counterterms $(a_{h^*,\omega}, b_{h^*,\omega}, \nu_{h^*,\omega})$ after the integration at the transition scale; a simple calculation shows that $Y_{h^*,s,\underline{\omega}} = Y_{h^*,s}$, while the fourth equation is valid with $\lambda_{h^*} = 0$: recall indeed that on scale h^* the local quartic term still vanishes. After that scale, introducing the "quasi-particle decomposition" in (3.3.116), one has istead the first non vanishing quartic local term λ_{h^*-1}.

(1) For $h \geq h^*$, B^ν, B^α, B^ζ do not depend on $\{Y_{h,\cdot}, r_h\}_{h \geq h^*}$,

(2) By construction, the beta function $B^{\#}_{h,\cdot}$ depends on $Z_{h'}$ only via the combinations $Z_{h'}/Z_{h'-1} = (1 + z_{h'})^{-1}$, with $h < h' < h^*$. For later reference, we rewrite the definition of z_h, (3.3.133), in a form analogous to (3.4.1),

$$z_{h-1} = B^z_h, \quad h \leq h^*, \tag{3.4.2}$$

where the right side is thought of as a function of $(\lambda_{h'}, z_{h'})_{h \leq h' < h^*}$, with the convention that $z_{h^*} = z_{h^*-1} = 0$ (the latter is because $W^{(h^*-1),\mathtt{R}}_{2,0;(\omega,\omega)} \equiv 0$).

(3) The components of the beta function for $\nu_{h,\omega}, a_{h,\omega}, b_{h,\omega}, \lambda_h$ are independent of $Y_{h',r,\underline{\omega}}, h' > h$. Therefore, we can first solve the flow equation for $\nu_{h,\omega}, a_{h,\omega}, b_{h,\omega}, \lambda_h$ and then inject the solution into the flow equation for $Y_{h,r,\underline{\omega}}$.

3.4.2 Dimensional bounds for the kernels

Before we proceed in describing the dimensional bounds satisfied by the kernels of the effective potential, let us comment on their structure. We have proven inductively that $V^{(h)}$ has, in momentum space, the same structure as in (3.3.72). If one writes $V^{(h)}$ in real space, due to the iterative action of the $\mathcal{R}$ operator in the inductive procedure, the structure that naturally emerges is that of a polynomial with pseudo differential operators $\hat\partial$ acting on some of the Grassmann fields $\varphi^\pm_{y,\omega}$ if $h < h^*$ or $\varphi^\pm_y$ if $h \geq h^*$. This was discussed for the integration of the first regime of scales ($h \geq h^*$): see Section 3.3.1 (cf. (3.3.51)) and Appendix B.3.4. With an analogous discussion for $h < h^*$, one finds that $V^{(h)}$ can be as well represented as

$$V^{(h)}(\varphi, J) = \sum_{\substack{n,m \geq 0: \\ n \text{ even}, \ n \geq 2}} \sum_{\underline{y}, \underline{x}, \underline{\omega}, \underline{s}, \underline{D}} W^{(h)}_{n,m,\underline{D};\underline{\omega},\underline{s}}(\underline{y}, \underline{x}) \times$$
$$\times \hat\partial^{D_1} \varphi^{(\leq h)+}_{y_1,\omega_1} \cdots \hat\partial^{D_n} \varphi^{(\leq h)-}_{y_n,\omega_n} J_{x_1,s_1} \cdots J_{x_m,s_m}. \tag{3.4.3}$$

The main difference between this formula and the one in the first line of (3.3.72), besides the different scale label, is the presence of the indices $\underline{D} = (D_1, \dots, D_n)$ with $D_j = (D_{j1}, D_{j2}) \in \{0, 1, 2\}^2$ associated to the operators $\hat\partial^{D_j} = \hat\partial_1^{D_{j1}} \hat\partial_2^{D_{j2}}$ acting on the Grassmann fields: we recall that $\hat\partial^D_i$ is a pseudo differential operator, dimensionally equiv-

alent to a derivative of order D in direction i (cf. (3.3.33)). Let us stress that the representation in (3.4.3) is not unique: the claim is that there exists such a representation, with the kernels satisfying natural dimensional estimates, discussed below. Of course a similar representation holds for the kernels of the first regime as well, where in particular the ω label is absent.

In order for the iterative construction to allow us to compute the thermodynamic limit of the generating function, and in particular to prove the existence of the free energy with the desired properties of Theorem 1.4.4, we need to prove that:

(1) The running constants of the first regime, $\nu_h, r_h/r_0 - 1, \alpha_{h,\omega}, \zeta_h$ are small, uniformly in h and in the separating scale h^* (say, smaller than a sufficiently small constant δ_0), provided the function ε_0 (see (3.3.1)), i.e. ν_0, have been properly fixed as a function of λ;

(2) The running constants of the second regime $\nu_{h,}, a_{h,\omega}, b_{h,\omega}, \lambda_h, z_h$ are as well small in the same sense as above, *but also* $\lambda_h, b_{h,\omega} = O(\sqrt{\epsilon})$ uniformly in h, provided now that $(\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+})$ are uniquely fixed as functions of λ (cf. (3.3.109)).

The reason behind these two requirements is the following Proposition, which we comment here and will be the output of the next section. Again, we assume L to be much bigger than ϵ^{-1} so that $h_L < h^*$ (cf. lines above (3.3.45) with (3.3.76)-(3.3.101))

Proposition 3.4.1. *Let $h \geq h^* - 1$ and $\delta_0 > 0$ small enough. Suppose that*

$$\sup_{h'>h} \max\{|\lambda|, |\alpha_{h'}|, |\zeta_{h'}|, 2^{-\theta h'}|\nu_{h'}|, |r_{h'}/r_0 - 1|\} \leq \delta_0, \tag{3.4.4}$$

then there exist $C(\theta) > 0$ independent of L, ϵ, h, such that

$$\|W^{(h)}_{n,m,\epsilon,\underline{D}}\|_{\kappa,h} \leq C^{n+m}\delta_0^{\max\{1,cn\}} 2^{h(\frac{3}{2}+\theta-\frac{n}{4}-m-|\underline{D}|^{(1)})} \left(\sup_{h'>h}|Y_{h',\cdot}|\right)^m \tag{3.4.5}$$

where $|\underline{D}|^{(1)} := \sum_{i=1}^n \sum_{j=1,2} D_{ij} 2^{-\mathbb{1}_{\{j=2\}}}$, $Y_{h',\cdot} := \sup_s Y_{h',s}$ and θ can be chosen in $(0, 1/2)$[19]; the norm is defined by

$$\|W_{n,m,\epsilon,\underline{D}}\|_{\kappa,h} := L^{-2} \sup_{\underline{s}} \sum_{\underline{y},\underline{x}} |W^{(h)}_{n,m,\epsilon,\underline{D},\underline{s}}(\underline{y},\underline{x})| e^{\kappa\sqrt{d^{(h)}(\underline{y},\underline{x})}} \tag{3.4.6}$$

where $d^{(h)}(\underline{y},\underline{x})$ is a scale weighted tree distance, given by $2^h d_1(\underline{x},\underline{y}) + 2^{h/2} d_2(\underline{x},\underline{y})$, where d_1, d_2 are the total displacements of the shortest lattice tree connecting the points $(\underline{x},\underline{y})$ in the directions $1, 2$ respectively[20].

Note that the expression of $|\underline{D}|^{(1)}$ follows from the scaling properties of the single scale propagator of the first regime (Lemma B.2.0.1): indeed recall that derivatives in different directions $i = 1, 2$ produce different scaling factors, respectively $2^h, 2^{h/2}$. With Proposition 3.4.1 at hand, we can obtain also bounds for the second regime of integration. Namely

[19] The bounds are not uniform as $\theta \to \frac{1}{2}^-$, so we can fix, e.g., $\theta = \frac{1}{4}$

[20] In terms of d_1, d_2 the usual tree distance $\delta(\underline{,y})$ is given by $d_1(\underline{x},\underline{y}) + d_2(\underline{x},\underline{y})$.

Proposition 3.4.2. *Let $h_L \leq h < h^* - 1$, $\delta_0 > 0$ small enough and suppose that* (3.4.4) *holds together with*

$$\sup_{h \leq h' < h^*} \{|\nu_{h',+}|, |z_{h'}|\} \leq \delta_0, \qquad \sup_{h \leq h' < h^*} \{|\lambda_{h'}|, |b_{h',+}|\} \leq 2^{h^*/2}\delta_0, \tag{3.4.7}$$

then there exists $C > 0$ independent of L, ϵ, h, such that

$$\|W_{n,m,\underline{D},\underline{s}}\|_{\kappa,h} \leq C^{n+m}\delta_0^{\max\{1,cn\}} 2^{h^*(-\frac{1}{2}+\frac{n}{4})} 2^{h(2-\frac{n}{2}-m-|\underline{D}|)} \left(\sup_{h < h' \leq h^*-1} \frac{|Y_{h',\cdot}|}{Z_{h'}} \right)^m \tag{3.4.8}$$

where $|\underline{D}| = \sum_{i=1}^n \sum_{j=1,2} D_{ij}$ and $Y_{h',\cdot} := \sup_{s,\omega} Y_{h',s,\omega}$ and the norm is given by

$$\|W_{n,m,\underline{D}}\|_{\kappa,h} := L^{-2} \sup_{\underline{s}} \sum_{\underline{y},\underline{x}} |W^{(h)}_{n,m,\underline{D},\underline{s}}(\underline{y},\underline{x})| e^{\kappa\sqrt{2^h \delta(\underline{x},\underline{y})}} \tag{3.4.9}$$

where $\delta(\underline{x},\underline{y})$ is the tree distance of the points $(\underline{x},\underline{y})$.

These results show that having control on the size of the coupling constants, *uniformly in* $h \leq 0$, gives access to bounds of the kernels for *all the scales* $h \leq 0$, which are uniform in L.

Anyway, the boundedness of the complete flow of the running constants will be the final outcome of our technical analysis, and we want to stress the logic behind the proof. Note first the following two facts: both the propositions are true even if one views the running constants of higher scale as independent variables, not necessarily related by the beta equations (3.4.1) (see also Remark 3.3.9): this follows from the scaling properties of the single scale propagators (Lemmas B.2.0.1 and B.2.0.2) and iterated application of the so called Battle-Brydges-Federbush-Kennedy (BBFK) formula in the framework of the so called Gallavotti-Nicolo's *tree expansion* (see next Section 3.4.3); Proposition 3.4.2 is built upon the estimates of Proposition 3.4.1, and, by its very definition, r_h depends only on kernels of the critical theory (see (3.3.90), (3.3.93)), i.e. at $\epsilon = 0$, as well as the other running constant in the hypothesis of Proposition 3.4.1. Thus

(1) We first obtain boundedness for $\{\alpha_h, \nu_h, \zeta_h\}_{h \leq 0}$ of the critical theory (Sec. 3.3.1), i.e. at $\epsilon = 0$. In this case Proposition 3.4.1 applies as well, with $h^* = -\infty$ (cf. (3.3.76)), and as a particular case it provides bounds on the components of the beta function $B^\nu_h, B^\alpha_h, B^\zeta_h$ (cf. (3.4.1)): this opens the way to an inductive proof for the boundedness of such constant, via a fixed point argument, in the spirit of [**2**]. Anyway the discussion is simplified by the special structure of the Beta function: in this regime the local quartic term (the analogous of λ_h for $h < h^*$) is zero. For the argument to work we need λ small enough and to uniquely fix $\nu_0 = \nu_0(\lambda)$, see (3.3.1).

(2) Fix $\nu_0 = \nu_0(\lambda)$. For $\epsilon \neq 0$, the previous point and Lemma 3.3.0.3 implies that the hypothesis of Proposition 3.4.1 are satisfied, uniformly in h^*.

(3) From the previous point, for $h < h^* - 1$, Proposition 3.4.2 implies that, assuming also (3.4.7) up to a given scale, one can obtain bounds on the same scale beta components of such constants, opening a way , through (3.4.1), for an inductive proof of the boundedness of the constants of the second regime. For this

to hold, we need a comparison with a "relativistic model", the kernels of which were introduced below (3.3.129). The special structure of the relativistic model beta functions, in particular its "vanishing property" of the λ-component, allows one to control the beta B_h^λ of our dimer model and obtain finally boundedness of the full flow $\{z_h, \lambda_h, \nu_{h,+}, a_{h,+}, b_{h,+}\}_{h\le h^*-1}$, provided that $|\lambda|$ is small enough and $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$ are uniquely fixed as functions of λ. The discussion is an adaptation of [**2**] to our case; in particular in order to control the small factors arising from the choice of ϵ small, we will follows the ideas first appeared in [**4**].

3.4.3 The tree expansion for the effective potential

As already anticipated, the detailed structure of the kernels of $V^{(h)}$, arising from the iterative construction of the two regimes described in Sections 3.3.1-3.3.2 can be naturally represented in terms of trees. This follows from the fact that each step of integration is characterized by the following relation, for any given scale h,

$$\begin{aligned} &L^2(F^{(h-1)} - F^{(h)}) + S^{(h-1)}(J) - S^{(h)}(J) + V^{(h-1)}(\phi', J) = \\ &= \sum_{n\ge 1} \frac{1}{n!} \mathcal{E}^T_{(h)} \left(\mathbb{V}^{(h)}(\phi' + \cdot, J); \cdots ; \mathbb{V}^{(h)}(\phi' + \cdot, J) \right) \end{aligned} \tag{3.4.10}$$

where $\mathbb{V}^{(h)}$ depends on the regime of integration, namely

$$\mathbb{V}^{(h)}(\phi, J) = \begin{cases} 2^h \nu_h N(\phi) + \sum_{x,s} Y^{(h)}_{s,\epsilon} F^{(h)}_Y(\phi, J) + \mathcal{R}_{1,\epsilon} V^{(h)}_\epsilon(\phi) & (h \ge h^*) \\ \\ \sum_\omega \left(2^h \nu_{h,\omega} F_{\nu;\omega}(\varphi) + a_{h,\omega} F_{a;\omega}(\varphi) + b_{h,\omega} F_{b;\omega}(\varphi) \right) + \\ + \lambda_h F_\lambda(\varphi) + \sum_{s,\underline{\omega}} \frac{Y_{h,s,\underline{\omega}}}{Z_{h-1}} F_{Y;s,\underline{\omega}}(\varphi, J) + \mathcal{R}_2 V^{(h)}(\sqrt{Z_h/Z_{h-1}}\,\varphi, J) & (h < h^*) \end{cases}$$

see (3.3.95) and (3.3.139).

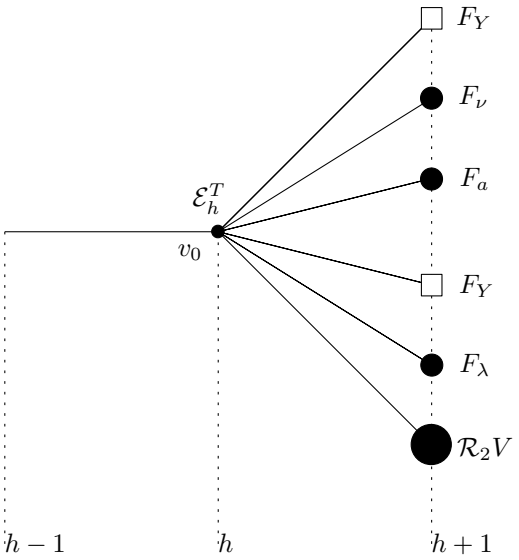

Figure 3.7: *An instance of a single step tree in the second regime.*

The many contributions to the l.h.s. of the above equation can be pictorially schema-

tized as in the Figure below, when we associate the following symbols to the terms appearing in the argument of the truncated expectation:

- $\bullet$ to represent the action of $\mathcal{E}_h^T$,
- ● to the terms corresponding to $N, F_{\nu,\omega}, F_\lambda, F_{a,\omega}, F_{b,\omega}$: the "normal endpoints",
- $\square$ to the terms $F_Y, F_{Y,\omega}$: the "special endpoints",
- ⬤ to the term corresponding to $\mathcal{R}_{1,\epsilon}, \mathcal{R}_2$.

where, the converging lines of Figure 3.7 from the symbols ●, $\square$, ⬤ to $\bullet$, represent the fact that the truncated expectation must connect its arguments. Note that, associated to a graphical representation there are many contrbution coming from the choices of the Grassmann fields $\phi' + \cdot$ in each of the arguments of $\mathcal{E}_h^T$, which in particular already set a constrain on such choices.
Then, one can iterate the above procedure by applying (3.4.10) to every term associated to ⬤, until it is associated to $\mathcal{R}_{1,\epsilon}V^{(0)}$ or $\mathcal{R}_2 V^{(h^*-1)}$, or no more endpoints of such type are present: in both case the procedure stops. The result of the iteration is a tree structure τ as in Figure 3.7 above, with endpoints on all the possible scales.[21] This expansion have been discussed in several contexts, some of which very close to ours. We mention [**5**], for a complete, self consistent discussion (Sec. 5.1 and 8.2 for the renormalized tree expansion); [**3**] (Sec. 6.2) for the expansion in a context closer to ours; [**4**] (Sec. 2.1) for an expansion which considers a double regime integration; [**45**] for an updated discussion treated in greater generality. Since the tree expansion depends in our case on the regime of integration, let us distinguish two cases:

Tree expansion for $h \geq h^*$

For the first regime of scales, we refer to [**4**, Sec. 2.1], for the necessary definitions; we will discuss the main differences with our case[22]. In particular, the trees involved in our construction are characterized by the following, different, features

1. A GN tree τ contributing to $V_\epsilon^{(h)}$, $\tilde{S}_\epsilon^{(h)}(J) := S_\epsilon^{(h)}(J) - S_\epsilon^{(h+1)}(J)$, or to $\tilde{F}_\epsilon^{(h)} = F_\epsilon^{(h)} - F_\epsilon^{(h+1)}$ has root on scale h and can have endpoints (either normal or special, represented as black dots or white squares, as in the Figure above) *on all possible scales* between $h+2$ and 0. The endpoints v on scales $h_v < 0$ are preceded by a node v' of τ, on scale $h_{v'} = h_v - 1$, that is necessarily a branching point. The family of GN trees with root on scale h, N_n normal endpoints and N_s special endpoints is denoted by $\mathcal{T}_{N_n,N_s}^{(h)}$. Note that in [**4**] (see e.g. eq. (53)) the endpoints can be only normal (because no external fields were considered) and on scale 0, because the interaction is only made of renormalized terms[23].

2. A normal endpoint v on scale $h_v \leq 0$ can be of two different types, ν or $\mathcal{R}V^{(-1)}$. If v is of type ν, then it is associated with $2^h \nu_h F_\nu(\varphi^{(\leq h_{v'})})$, ; in this case, the node v'

[21] recall indeed that if $h < h^* - 1$ then the inductive construction stop at scale $h = h^* - 1$.
[22] Recall that in the first regime we specify the dependence on ϵ of the kernels while in the second regime we drop the label, because the second regime it is built upon $\epsilon \neq 0$
[23] The term analogous to our ν is used to dress the integration.

immediately preceding v on τ, of scale $h_{v'} = h_v - 1$, is necessarily a branching point. If v is of type $\mathcal{R}V^{(-1)}$, then $h_v = 0$, and v is associated with (one of the monomials contributing to) $\mathcal{R}_{1,\epsilon}V^{(-1)}(\varphi^{(\leq -1)}, 0)$; in this case, the node immediately preceding v on τ, of scale $h_v - 1$, is not necessarily a branching point.

3. A special endpoint v on scale $h_v \leq 0$ can be either local, or non-local. If v is local, then it is associated with $Y_{h_{v'},s}F_{Y;s}(\varphi^{(\leq h_{v'})}, J)$ for some $s \in \{A, B, C\}$; in the case that v is local, the node v' immediately preceding v on τ, of scale $h_{v'} = h_v - 1$, is necessarily a branching point. If v is non-local, then $h_v = 0$, and v is associated with (one of the monomials contributing to) $V_\epsilon^{(-1)}(\varphi^{(\leq -1)}, J) - V_\epsilon^{(-1)}(\varphi^{(\leq -1)}, 0)$; in this case, the node immediately preceding v on τ, of scale $h_v - 1$, is not necessarily a branching point.

In addition to the items above, let us recall that each vertex of the tree that is not an endpoint and that is not the special vertex v_0 (the leftmost vertex of the tree, immediately following the root on τ) is associated with the action of an $\mathcal{R}_{1,\epsilon}$ operator. In particular, note the following main difference with the renormalization operator, $\mathcal{R}_1$, used in [**4**] (see Eq. (36) and the following lines). From Remark 3.3.4 follows that we can write

$$\mathcal{R}_{1,\epsilon} - \mathcal{R}_1 = \Delta\mathcal{R}_1\mathcal{P}_0 + \mathcal{R}_{\mathcal{T}}(1 - \mathcal{P}_0) \tag{3.4.11}$$

where $\mathcal{P}_0$ is the operator that maps effective potentials in the critical ($\epsilon = 0$) counterpart; $\mathcal{R}_{\mathcal{T}} := (1 - \mathcal{T})$; and $\Delta\mathcal{R}_1 := \mathcal{R}_{1,0} - \mathcal{R}_1$. As already commented, the operator $\Delta\mathcal{R}_1$ has a renormalization effect coming from "finite size effect" due to our choice of localizing infinite volume kernels, see [**2**, Appendix B] for more details. The renormalization effect of the operator $\mathcal{R}_{\mathcal{T}}(1 - \mathcal{P}_0)$ will instead be justified shortly (see point (2) below).

In terms of the tree expansion, we can express the effective potential and the single-scale contributions to the free energy and generating function as

$$L^2\tilde{F}_\epsilon^{(h)} + \tilde{S}_\epsilon^{(h)}(J) + V_\epsilon^{(h)}(\varphi, J) = \sum_{\substack{N_n, N_s \geq 0: \\ N_n + N_s \geq 1}} \sum_{\tau \in \mathcal{T}_{N_n,N_s}^{(h)}} V^{(h)}(\tau, \varphi, J), \tag{3.4.12}$$

where

$$V^{(h)}(\tau, \varphi, J) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{\mathbf{D}} \sum_{\mathbf{x}_{v_0}} W_{\tau,\mathbf{P},T,\mathbf{D}}^{(h)}(\mathbf{x}_{v_0}) \varphi_{\mathbf{D}}(P_{v_0}^\varphi)\, J(P_{v_0}^J)\,. \tag{3.4.13}$$

Where we recall that $P_{v_0}^\psi$ and $P_{v_0}^J$ are two sets of indices that label the Grassmann external fields and the external fields of type J, respectively; moreover, $J(P_{v_0}^J) = \prod_{f \in P_{v_0}^J} J_{x(f),s(f)}$ and

$$\varphi_{\mathbf{D}}(P_{v_0}^\varphi) = \prod_{f \in P_{v_0}^\varphi} \hat{\partial}^{D(f)} \varphi_{y(f),\omega(f)}^{\sigma(f)}\,. \tag{3.4.14}$$

The equation (3.4.13) above is the analogue of [**4**, (80)] or [**3**, (6.64)] (in this formula, the indices $\mathbf{D}$ replace the multi-indices $\beta \in B_T$ [**3**, (6.64)]); we denote $D(f) = (D_1(f), D_2(f))$ so that $\hat{\partial}^{D(f)} := \hat{\partial}_1^{D_1(f)}\hat{\partial}_2^{D_2(f)}$.

When we restrict to trees τ for which $\mathcal{R}_{1,\epsilon}$ acts as the identity at every vertex of the

tree, then $W^{(h)}_{\tau,\mathbf{P},\mathbf{T},\mathbf{D}}(\mathbf{x}_{v_0})$ can be written as

$$W^{(h),*}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = \\ = \left(\prod_{v \text{ e.p.}} K^{(h_v)}_v(\mathbf{x}_v)\right) \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det(M^{h_v,T_v}(\mathbf{t}_v)) \prod_{\ell \in T_v} g^{(h_v)}_\ell \tag{3.4.15}$$

which can be obtained by the iterative application of the so called determinant formula (BBFK) (see [**4**, (75)], [**3**, Lemma 3, (6.63)]), to the truncated expectation $\mathcal{E}^T_{h_v}$ at each v which is not an endpoint ("e.p." stands for endpoints, see). Of course a similar representation, carrying all the extra labels of the non trivial action of $\mathcal{R}_{1,\epsilon}$, exists but in order to not overwhelm the notation we prefer to skip such details (see [**131**, Sec. 3.10] for a complete derivation of such representation).

Then we have the following

Lemma 3.4.0.1. *There exists L-independent constants $\delta_0, C, c, \kappa > 0, \theta \in (0, 1/2)$ such that, if $h \geq h^*$ and*

$$\sup_{h'>h} \{|\lambda|, 2^{-\theta h'}|\nu_{h'}|, |\alpha_{h'}|, |\zeta_{h'}|, |r_{h'}/r_0 - 1|\} \leq \delta_0, \tag{3.4.16}$$

and $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$, then

$$\|W^{(h)}_{\tau,\mathbf{P},T,\mathbf{D}}\|_{\kappa,h} \leq C^{N_s} (C\delta_0)^{\max\{N_n, c|I^\psi_{v_0}|\}} 2^{h\left(\frac{3}{2} - \frac{1}{4}|P^\psi_{v_0}| - |P^J_{v_0}| - |\mathbf{D}|^{(1)}\right)} \tag{3.4.17}$$
$$\times \left(\prod_{\substack{v \text{ n.e.p.} \\ h_v < -1}} 2^{\theta h_v}\right) \left(\prod_{v \text{ s.e.p.}} \sup_s |Y_{h_v-1,s}|\right) \prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{C^{|Q_v|}}{s_v!} 2^{(h_v - h_{v'})\left(\frac{3}{2} - \frac{1}{4}|P^\psi_v| - |P^J_v| - z_1(P_v)\right)},$$

where: $|I^\psi_{v_0}| = \sum_{v \text{ e.p.}} |P^\psi_v|$ is the total number of Grassmann fields associated with the endpoints of the tree; $|Q_v| = \sum_{i=1}^{s_v} |P_{v_i}| - |P_v|$ is the number contracted fields on the vertex v; the first product in the second line runs over the special endpoints, while the second over all the vertices of the tree that are not endpoints. Moreover $|\mathbf{D}|^{(1)} = \sum_{f \in P^\psi_{v_0}} \sum_{j=1,2} D_j(f) 2^{-\mathbb{1}_{j=2}}$ and

$$z_1(P_v) = \begin{cases} 2 & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (6, 0), \\ \frac{3}{2} & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (2, 0), \\ 1 & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (4, 0) \\ \frac{1}{2} & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (2, 1) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.18}$$

The proof of this lemma is very similar to the one in [**4**, Lemma 2], so we do not report it here. Let us comment on the analogies and main differences, setting for simplicity $J = 0$.[24] The main points are the following:

(0) the fact that we have trees which can have endpoints on scales $h < -1$, only of type ν, is related of course to the first factor in the second line, as it follows from the

[24]The presence of the external fields does not imply substancial changes, and we will discuss it when necessary later.

hypothesis (3.4.16). This term is crucial to obtain the factor $2^{\theta h}$ as it appears in Proposition 3.4.1, see the discussion below.

(1) the scaling dimension $d_v := 3/2 - |P_v|/4$ appearing at each vertex v which is not an endpoint, is the same because of same the scaling properties of the propagators, compare [**4**, (60)] with (B.2.6), as follows from an iterative application of the BBFK formula and the Gram-Hadamard bound for determinants (cf. [**4**, (80)]).

(2) the renormalization factor $z_1(P_v)$ is obtained when $\mathcal{R}_{1,\epsilon}$ is applied to a branching vertex $v \in \tau$: if one picks $\mathcal{R}_{1,0}\mathcal{P}_0$ in (3.4.11), then the resulting $z_1(P_v)$ is the same as the one discussed in [**4**] see also the third item of Remark 3.3.4. When instead one chooses $\mathcal{R}_{\mathcal{T}}(1-\mathcal{P}_0)$, only if $\epsilon \neq 0$, then necessarily $|P_v| = 2$ and $|D_v| = 0$. In this case, from the definition of $\mathcal{T}$ (see (3.3.84)) and the definition above of $\tilde{\mathcal{R}}_{\mathcal{T}}$, the resulting term to bound is the Taylor remainder in the sequence $\underline{r}$, namely

$$\int_0^1 dt \sum_{\ell,m \geq h_v} r_\ell r_m \partial^2_{r_\ell, r_m} W^{\infty;(h_v)}_{\tau_v, \mathbf{P}_v, T^{(v)}}(\mathbf{x}_v; t\underline{r}) \tag{3.4.19}$$

where we recall, τ_v is the subtree of τ rooted at v, $\mathbf{P}_v = \cup_{w \geq v} P_w$ ($\geq$ is the partial order of the tree τ), $T^{(v)} := \cup_{w \geq v} T_w$ is the union of the spanning trees coming from the truncated expectations in the subtree τ_v and $\mathbf{x}_v$ is the collection of coordinates of the endpoints following v. Each derivative $\partial_{\mathbf{r}_j}$ apply to one of the propagators involved in the truncated expectation (via the application of BBFK, cf. (3.4.15)) at some vertex $w \geq v$, with $h_w = j$ which dimensionally produces an extra 2^{-j} factor [25] (cf. with Lemma B.2.0.1 with $n_0 = 1$). Thus we can repeat the bound as if no $\mathcal{R}$ operator acted at v, with the extra dimensional factor

$$\sum_{\ell,m \geq h_v} r_\ell r_m 2^{-\ell} 2^{-m} \leq (C' r_0)^2 2^{-2h_v} \leq C'' 2^{2(h^* - h_v)} \tag{3.4.20}$$

with constants $C', C'' > 0$ independent of h_v, ϵ, λ. We used that from the hypothesis, $\sup_{j>h} |r_j| \leq |r_0|(1+\delta_0) \leq |r_0| C'$ and that $|r_0| \leq C''/C' 2^{h^*}$ as it follows from the definition (3.3.101). This extra term is of course enough to renormalize the vertex v, since $2^{2(h^* - h_v)} \leq 2^{2(h_{v'} - h_v)} \leq 2^{(h_{v'} - h_v) z_1(P_v)}$ with $|P_v| = 2$ (cf. (3.4.18)) but also all the vertices $v_0 < w \leq v$, since $h^* - h_v = \sum_{v_0 < w \leq v} (h_{w'} - h_w)$.

(3) the way the intgration over the spatial variables x_v is carried is modified by the presence of the norm $\|\cdot\|^{(1)}_{\kappa,h}$; its definition is related to the analogous decay properties of the propagator of the first regime (cf. Lemma 3.3.0.2). We refer to [**45**, Prop 4.6] (see also (4.46) and lines below) for a discussion on how to include such exponential factors, which can be easily adapted to our context.

Note that, the kernels in (3.4.3) are obtained by summing $W_{\tau,\mathbf{P},T,\mathbf{i},\mathbf{D}}(\mathbf{x}_{v_0})$ over $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$ and over N_n, N_s, under the constraint that the number of external fields of type ψ and J is equal to n and m, respectively, that the elements of $\mathbf{D}$ are the same as $\underline{D}$, etc. Similarly, the kernels of the single scale contribution to the generating function, $\tilde{S}^{(h)}(J)$,

[25] There is also a combinatorial factor $|Q_w|^2/4 \leq C^{|Q_w|/4}$ for $C > 0$ big enough, where recall $|Q_w| = \sum_{i=1}^{s_w} |P_{w_i}| - |P_w|$, which can be reabsorbed in the product of Proposition 3.4.0.1

which we denote by $W^{(h)}_{0,m;\underline{r}}(\underline{y})$, are obtained by summing the tree values $W_{\tau,\mathbf{P},T}$ over $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$ and over N_n, N_s, under the constraint that $P^{\psi}_{v_0} = \emptyset$ and that $\cup_{f\in P^J_{v_0}}\{(y(f), r(f))\}$ matches the tuple $(\underline{y}, \underline{s})$; finally, the single scale contribution to the free energy, $L^2\tilde{E}^{(h)}$ is obtained by an analogous sum over GN trees, under the constraint that $P^{\psi}_{v_0} = P^J_{v_0} = \emptyset$.

Then, to obtain the bound in Proposition 3.4.1, it is enough to perform the sum (3.4.17) over $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$, over $T \in \mathbf{T}$, and over $\mathbf{P} \in \mathcal{P}_\tau$, under the constraint that $|P^{\varphi}_{v_0}| = n$ and $|P^J_{v_0}| = m$. This is possible because all the renormalized scaling dimensions are now negative, thanks to the presence of $z_1(P_v)$ (see [**5**, App. 6.1] for details on how to sum over the labels. Actually, from (3.4.18), also

$$d^{(1)}_{v,\theta',z} := 3/2 - \frac{1}{4}|P^{\varphi}_v| - |P^J_v| - z_1(P_v) + \theta' < 0 \tag{3.4.21}$$

for every $\theta' < 1/2$. Now, a generic τ has at least one endpoint w which carries a factor $2^{\theta h_w}$ (if w is on scale $h = -1$, then up to an overall constant we have such factor) we can denote by $\Gamma_\tau(w) := (v_1, \dots, w)$, where $(v_0, v_1, \dots, w)$ is the ordered ($v_i \geq v_{i-1}$ with respect to the partial order of the tree) collection of vertices from v_0 to w so that we can write

$$2^{\theta h_w}\prod_{\substack{v \text{ not}\\ \text{e.p.}}} 2^{(h_v - h_{v'})d_{v,0,z}} = 2^{\theta h}2^{h_w(\theta-\theta')}\prod_{\substack{v \text{ not}\\ \text{e.p.}}} 2^{(h_v-h_{v'})\left(\frac{3}{2}-\frac{1}{4}|P^{\psi}_v|-|P^J_v|-z_1(P_v)+\theta'\mathbb{1}_{v\in\Gamma_\tau}\right)} \tag{3.4.22}$$

which gives, if $\theta' \geq \theta$ (cf. (3.4.16)), and after the summation over the labels, the desired bound of Proposition 3.4.1.

Tree expansion for $h < h^*$

In the second regime we still refer to [**4**], but also to [**2**], thanks to the similarity of the inductive structure, to set the ground of the next Section. In this case the trees are differently characterized, with respect to the previous case, by the following features

- A tree τ contributing to $V^{(h)}$, $\tilde{S}^{(h)}(J) := S^{(h)}(J) - S^{(h+1)}(J)$, or to $\tilde{F}^{(h)} = F^{(h)} - F^{(h+1)}$ can have endpoints (either normal or special as before) at each scale between $h+2$ and h^*-1. The family of GN trees with root on scale h, N_n normal endpoints and N_s special endpoints is denoted again by $\mathcal{T}^{(h)}_{N_n,N_s}$, $h < h^* - 1$.

- Normal endpoints v on scale $h_v \leq 0$ can be of five different types, ν,a,b,λ or $\mathcal{R}V^{(h^*-1)}$. They are respectively associated to the terms in the second line of (3.4.11): if v is of type ν, then it is associated with $\sum_\omega 2^h \nu_{h,\omega} F_{\nu,\omega}(\varphi^{(\leq h_{v'})})$, and so on for $a, b, \lambda, \mathcal{R}V^{(h^*-1)}$.

- A special endpoint v is now associated to $\sum_{\underline{\omega}} Y_{h_{v'},s,\underline{\omega}} F_{Y;s,\underline{\omega}}(\varphi^{(\leq h_{v'})}, J)$.

Of course we can repeat the above discussion (cf. (3.4.12) and below) and represent the contributions to the lhs of (3.4.10) in terms of such trees: we obtain a representation similar to (3.4.13) where the Grassmann fields φ, and the kernels, have the extra label $\underline{\omega}$. Then the analogous of Proposition 3.4.0.1, on these trees, reads

Lemma 3.4.0.2. *There exists L-independent constants $\delta_0, C, c, \kappa > 0$ such that, if* (3.4.16) *holds with*

$$\sup_{h'>h}\{|\lambda|, |\nu_{h',+}|, |a_{h',+}|, |z_{h'}|\} \le \delta_0, \qquad \{|\lambda_{h'}|, |b_{h',+}|\} \le v_F\delta_0 \tag{3.4.23}$$

and $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$, then

$$\begin{aligned}\|W^{(h)}_{\tau,\mathbf{P},T,\mathbf{D}}\|_{\kappa,h} \le C^{N_s}\,(C\delta_0)^{\max\{N_n, c|I^\psi_{v_0}|\}}\, v_F^{-1+\frac{|P^\varphi_{v_0}|}{2}+|\mathbf{D}_2|} 2^{h\left(2-\frac12|P^\psi_{v_0}|-|P^J_{v_0}|-|\mathbf{D}|\right)} \\ \times\left(\prod_{v\ \text{s.e.p.}}\sup_{s,\underline{\omega}}\left|\frac{Y_{h_v-1,s,\underline{\omega}}}{Z_{h'-1}}\right|\right)\prod_{\substack{v\ \text{not}\\ \text{e.p.}}}\frac{C^{|Q_v|}}{s_v!}\,2^{(h_v-h_{v'})\left(2-\frac12|P^\psi_v|+\delta_0|P_v|-|P^J_v|-z_2(P_v)\right)}\,,\end{aligned} \tag{3.4.24}$$

where $|\mathbf{D}| = \sum_{f\in P^\varphi_{v_0}}\sum_{j=1,2} D_j(f)$ and $|\mathbf{D}_2| := \sum_{f\in P^\varphi_{v_0}} D_2(f)$. Moreover

$$z_2(P_v) = \begin{cases} 2 & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (2,0), \\ 1 & \text{if} \quad (|P^\psi_v|, |P^J_v|) = (4,0), (2,1), \\ 0 & \text{otherwise.}\end{cases} \tag{3.4.25}$$

In this case the proof is a mixture of the results of [**4**, Lemma 3.2] and [**2**, Proposition 3]. To simplify the following discussion, set again $J \equiv 0$: the general case can be treated similarly.

The presence of the small factors $v_F^{-1+|P_{v_0}|/2+|\mathbf{D}_2|}$ is a Corollary of [**4**, Lemma 2] because of:

(1) the propagators have the same scaling properties; particularly their $\|\cdot\|_\infty$ and $\|\cdot\|_1$ norms: compare Lemma B.2.0.2 with [**4**, Lemma 3.1],

(2) the endpoints v of the tree of type a (which are not present there) do not change the bound; each of them produce an extra term $\delta_0 2^{h_{v'}}$ which can be safely included in the last line of [**4**, eq. (128)] (note that our δ_0, b_h play the same role, respectively of, λ, δ_h). The origin of such factors is related to the fact that $a_{h_v,+}$ is associated to a Grassmann term $L^{-2}\sum_{q\in\mathcal{D}_\theta}\hat\varphi^+_{q,+}\varphi^-_{q,+} iq_1$ which can be rewritten[26] as $\sum_{y\in\tilde\Lambda}\varphi^+_y\hat\partial_1\varphi^-_y$.

(3) the difference between the two renormalization operators $\mathcal{R}_{2,\epsilon}$ and $\mathcal{R}_2$ of [**4**] contributes with irrelevant terms[27], in the same spirit as the discussion for the critical case $\Delta\mathcal{R} = \mathcal{R}_{1,0} - \mathcal{R}_1$, made below (3.4.11);

Of course one of the main points is to show that indeed $|\lambda_h|, |b_h| \le \delta_0 v_F$ for all the scales $h \le 0$. Note that, while such a statement for $b_{h,+}$ will follow by suitably fixing the counterterm $b_{h^*,+}$ (which is assumed up to now to satisfy such a condition), as shown in the next Section, we have no freedom to choose λ_{h^*}, and it is not a direct consequence of [**4**] that it satisfies such a condition. Anyway we are lucky enough that $\lambda_{h^*} = O(\lambda v_F)$ also in our case, as we will see in the next Section (see Lemma 3.4.0.4).

[26] See Appendix B.1

[27] In other words, by repeating the estimates using $\Delta\mathcal{R}_2 := \mathcal{R}_{2,\epsilon} - \mathcal{R}_2$, one obtains a renormalization factor $\Delta z_2(P_v) > z_2(P_v)$ for every P_v.

Remark 3.4.1. *Let us comment first the presence of the factor* $v_F^{-1+|P^{\varphi}_{v_0}|+|\mathbf{D}|_{\mathbf{2}}}$*: since* $v_F = O(\sqrt{\epsilon})$*, as explained in Remark 3.3.8 (and after suitably fixing the counterterms, introduced in* (3.3.113)*, as in Sec. 3.4.4) we have that such factor, in particular, plays a crucial role in the* ϵ *asymptotics for the free energy of Theorem 1.4.5. Secondly, note that if we had* $\epsilon = O(1)$*, then* $v_F = O(1)$ *and the result would be exactly the same as in [2] (by a renaming of* δ_0*, reabsorbing the power of* v_F *in the constant factor). If instead* $\epsilon \sim 0$*, the first regime produces a significant contribution to the kernels of the second regime (as explained in [4, Lemma 3.2.]) which otherwise would have poor bounds as* $\epsilon \to 0$*. Note also that, around the transition scale* h^**, since* $2^{h^*} = O(\epsilon)$ *(see* (3.3.101)*), thus* $v_F = O(2^{h^*/2})$ *and we have that the bounds for the two regimes indeed agree: if* $n = |P^{\varphi}_{v_0}|$*,*

$$2^{h^*/2(-1+n/2-|\mathbf{D}_2|)}2^{h^*(2-n/2-|\mathbf{D}}) = 2^{h^*(3/2-n/4-|\mathbf{D}|+|\mathbf{D}_2|/2)} = 2^{h^*(3/2-n/4-|\mathbf{D}|^{(1)})}. \quad (3.4.26)$$

As it was for the first regime of scales, to obtain the bound in Proposition 3.4.0.1, one has to sum (3.4.17) over $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$, over $T \in \mathbf{T}$, and over $\mathbf{P} \in \mathcal{P}_\tau$, under the constraint that $|P^{\varphi}_{v_0}| = n$ and $|P^{J}_{v_0}| = m$. This is possible again because all the renormalized scaling dimensions are now negative, thanks to the presence of $z_2(P_v)$. Now, a generic endpoint v of the tree does not carry a factor 2^{h_v} as in the previous case. Anyway the following remark shows that trees with long branches gains exponentially small factors (the so called *short memory property)* (see also see Remark 16 after [**3**, Proposition 8]):

Remark 3.4.2. *If we sum* (3.4.17) *over* $\tau \in \mathcal{T}^{(h)}_{N_n,N_s}$*,* $T \in \mathbf{T}$*,* $\mathbf{P} \in \mathcal{P}_\tau$*, with* $|P^{\psi}_{v_0}| = n$*,* $|P^{J}_{v_0}| = m$*, under the additional constraint that* τ *has at least one node on scale* $k > h$*, then we get a bound that is the same as* (3.4.24) *times an additional gain factor* $2^{\theta'(h-k)}$*, where* θ' *is a positive constant, smaller than 1 (estimates are not uniform as* $\theta' \to 1^-$*; from here on, we will choose* $\theta' = 3/4$*).*

The fact that here θ' is not bounded by $1/2$ as in the first regime, is related to the fact that in the second regime the renormalization z_2, as one can check from (3.4.0.2), makes the scaling dimension satisfying

$$d^{(2)}_{v,z} := 2 - \frac{1}{2}|P^{\psi}_v| + \delta_0|P_v| - |P^{J}_v| - z_2(P_v) \le -1 + O(\delta_0) \quad (3.4.27)$$

so that in particular for every fixed $\theta' \in (0,1)$, also $d^{(2)}_{v,\theta',z} := d_{v,z} + \theta < 0$ for δ_0 small enough (cf. with the discussion (3.4.21)).

3.4.4 The space of sequences: the flow of the running constants

With the previous Sections at hand, we are ready to discuss how to fix the initial data of the problem, ν_0 first, and $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$ then, in order for the running constants $\nu_h, \alpha_h, \zeta_h, r_h, \nu_{h,+}, a_{h,+}, b_{h,+}, \lambda_h, z_h$ to be small uniformly in $h \le 0$, in the sense of the hypothesis of Proposition 3.4.1 and Proposition 3.4.2.

Definition 3.4.1. Let $\mathbb{S} := \mathbb{R}^{\mathbb{Z}_{\le 0}}$ be the space of sequences. We define operators $\tau, \rho, \pi_* : \mathbb{S} \to \mathbb{S}$ given by

$$\begin{aligned} (\tau \underline{s})_h &:= s_{h-1} \\ \rho \underline{s} &= (v_F)^{-1} \underline{s}. \\ (\pi_* \underline{s})_h &= s_h \mathbb{1}_{\{h<h^*\}} \end{aligned} \quad (3.4.28)$$

where, as usual, $\mathbb{1}_A$ is the characteristic function of the set A. We define also norms $\|\cdot\|_\theta, \|\cdot\|_\theta^* : \mathbb{S} \to \mathbb{R}$,

$$\|\underline{\nu}\|_\theta := \sup_{h\le 0} 2^{-\theta h}|\nu_h|, \qquad \theta \in (0,1/2)$$
$$\|\underline{\nu}\|_\theta^* := \sup_{h\le h^*} 2^{-\theta(h-h^*)}|\nu_h|, \qquad \theta \in (0,1)$$

Next, we consider $\underline{u} = (\underline{u}_0, \underline{u}_1) \in \mathbb{S}^9$, where

$$\underline{u}_0 := (\underline{\nu}, \underline{\alpha}, \underline{\zeta}), \quad \underline{u}_1 := (\underline{r}, \underline{\nu}_+, \underline{a}_+, \underline{b}_+, \underline{\lambda}, \underline{z})$$

and $\underline{\nu} = (\nu_h)_{h\le 0}$, $\underline{\alpha} = (\alpha_h)_{h\le 0}$, $\underline{\zeta} = (\zeta_h)_{h\le 0} \in \mathbb{S}$ etc. are arbitrary sequences: for the moment they do not have to be the sequences driven by the beta equations in (3.4.1). Given $\underline{\mathcal{F}} : \mathbb{S}^9 \longrightarrow \mathbb{S}^9$, we denote $\underline{\mathcal{F}}(\underline{u}) = (\mathcal{F}^\nu(\underline{u}), \dots, \mathcal{F}^z(\underline{u}))$; note that we can rewrite the beta equations in (3.4.1)-(3.4.2), in terms of a fixed point equation

$$\underline{u} = \underline{\mathcal{B}}(\underline{u}) \tag{3.4.29}$$

for a map $\underline{\mathcal{B}} = (\mathcal{B}^\nu, \dots, \mathcal{B}^z)$ such that, e.g., $\mathcal{B}^\#(\underline{u}) = (\mathcal{B}^\#_h(\underline{u}))_{h\le 0}$, for $\# = \nu, \alpha, \zeta$ satisfy

$$\mathcal{B}^\nu_h(\underline{u}) = 2\nu_{h+1} + B^\nu_{h+1}(\underline{u}), \quad \mathcal{B}^\alpha_h(\underline{u}) = \alpha_{h+1} + B^\alpha_{h+1}(\underline{u}), \quad \mathcal{B}^\zeta_h(\underline{u}) = \zeta_{h+1} + B^\zeta_{h+1}(\underline{u}) \tag{3.4.30}$$

where $B^\#_h$ are the same appearing in (3.4.1); or similarly $\mathcal{B}^r(\underline{u}) = (\mathcal{B}^r_h(\underline{u}))_{h\le 0}$

$$\mathcal{B}^r_h(\underline{u}) = \begin{cases} r_{h+1} + B^r_{h+1}(\underline{u}) & h \ge h^* - 1 \\ 0 & h < h^* - 1 \end{cases} \tag{3.4.31}$$

or again $\mathcal{B}^{\nu_+}(\underline{u}) = (\mathcal{B}^{\nu_+}_h(\underline{u}))_{h\le 0}$ with

$$\mathcal{B}^{\nu_+}_h(\underline{u}) = \begin{cases} 2\nu_{h+1,+} + B^\nu_{h+1,+} & h < h^* \\ 0 & h \ge h^* \end{cases} \tag{3.4.32}$$

and a similar representation holds for the other components. The equations above show that depending on the component of $\underline{\mathcal{B}}$ we look at, the image of a sequence $\underline{u}$, is a sequence that can have a finite or infinite number of zero components. This follows from the iterative integration scheme of Sections 3.3.1-3.3.2 and the structure of the Beta equations (3.4.1). In terms of (3.4.28), this means that $\mathcal{B}^\#(\underline{u}) = \pi_* \mathcal{B}^\#(\underline{u})$ for $\# = \nu_+, \alpha_+, b_+, \lambda, Y_+$.

Remark 3.4.3.

- *In order for the fixed point equation $\underline{u} = \underline{\mathcal{B}}(\underline{u})$ to be well defined, we need to restrict our focus to sequences which are small, in the same sense as in Lemma 3.4.0.1-3.4.0.2, which indeed guarantees that the components of the Beta function are well defined. This leads to the definition of a normed space of sequences as defined below.*
- *$\mathcal{B}^\#_h$ depends only on $\underline{u}^{(h)} := (\underline{\nu}^{(h)}, \dots, \underline{z}^{(h)})$ where e.g. $\underline{\nu}^{(h)} := (\nu_{h'})_{h'>h}$ and similarly for the others. This is because $B^\#_h$ does depend only on constants on scales $h' > h$ (see (3.4.1)).*
- *The system of equations can be decoupled, by isolating the critical (i.e. $\epsilon = 0$ part,*

cf. Section 3.3.1): more precisely, we have

$$\begin{cases} \underline{u}_0 = \underline{\mathcal{B}}_0(\underline{u}_0, \underline{0}) \\ \underline{u}_1 = \underline{\mathcal{B}}_1(\underline{u}_0, \underline{u}_1) \end{cases} \tag{3.4.33}$$

where $\underline{\mathcal{B}}_0 := (\mathcal{B}^\nu, \mathcal{B}^\alpha, \mathcal{B}^\zeta), \underline{\mathcal{B}}_1 = (\mathcal{B}^r, \mathcal{B}^{\nu_+}, \mathcal{B}^{a_+}, \mathcal{B}^{b_+}, \mathcal{B}^\lambda, \mathcal{B}^z)$.

- *Even if not explicited,* $\mathcal{B}_\#$ *depends on* λ*; we will write* $\mathcal{B}^\#(\underline{u}, \lambda)$ *when necessary.*

Then the strategy is to first solve the first equation above, finding a solution $\underline{u}_0^*$, then to plug it into $\underline{u}_1 = \underline{\mathcal{B}}_1(\underline{u}_0^*, \underline{u}_1)$, and then solving for $\underline{u}_1$. Recall that we have the freedom to choose the initial data of the flow, but we cannot fix λ, which tunes the interaction.

The general idea (in both the steps) to obtain a bounded solution (in the sense of the hypothesis of Proposition 3.4.1 and 3.4.2), is first to define a suitable Banach space $(\mathbb{S}^{3+6j}, \|\cdot\|^{(j)})$, $j = 0, 1$; on such we can "reverse" the equation, i.e. there exist a map $\underline{\hat{\mathcal{B}}}_j$ such that $\underline{u}_j$ solves (3.4.29) if and only if $\underline{u}_j = \underline{\hat{\mathcal{B}}}_j(\underline{u}_j)$; then one wants to show that if $\mathbb{X}_\delta^{(j)} := \{\underline{u}_0 : \|\underline{u}_0\|^{(j)} \le \delta\}$ and $\delta, |\lambda|$ are small enough then

(1) $\underline{\hat{\mathcal{B}}}_j(\mathbb{X}_\delta^{(j)}) \subset (\mathbb{X}_\delta^{(j)})$

(2) $\underline{\hat{\mathcal{B}}}_j$ is a contraction on the metric space $\mathbb{Y}^{(j)} := (\mathbb{X}_\delta^{(j)}, d)$ where d is a suitable distance related to $\|\cdot\|^{(j)}$.

These two results, otherwise stated allow to: (1) show that the beta function $\underline{\mathcal{B}}$ maps "small" sequences into "small" sequences, with a suitable definition of "size"; (2) via the standard Fixed Point Theorem, there is then a unique solution of the problem $\underline{u}_j^* = \underline{\hat{\mathcal{B}}}_j(\underline{u}_j^*)$, then of (3.4.29), in the set in $\mathbb{X}_\delta^{(j)}$ and thus a unique choice of the counterterms, which are particular elements of the sequence. This can be rephrased in terms of the original problem: there is a unique choice of the counterterms such that the beta function generates a "bounded" flow. Of course the notion of size of such sequences, thus $\|\cdot\|_\theta^{(i)}$, depends on the structure of the flow one needs to consider and in particular on the dimensional scaling properties of theory, thus of the single scale propagator.

We will study now in detail the flow of $\underline{u}_0$. As far as it concerns $\underline{u}_1$, we will refer to [**2**] since the structure of the flow is qualitatively the same: we will discuss the main differences in order to single out the dependence on the small parameter ϵ, following ideas of [**4**].

Boundedness of $\{\nu_h, \alpha_h, \zeta_h\}_{h \le 0}$

In view of the desired properties of such constants, (3.4.4), we define for $\underline{u}_0 \in \mathbb{S}^3$,

$$\|\underline{u}_0\|^{(0)} := \max\left\{ \|\underline{\nu}\|_\theta, K_\theta \|(\tau - 1)\underline{\alpha}\|_\theta, K_\theta \|(\tau - 1)\underline{\zeta}\|_\theta \right\}, \quad K_\theta := 1 - 2^{-\theta} \tag{3.4.34}$$

where $\tau, \|\cdot\|_\theta$ are as in Definition 3.4.1. Note that $(\mathbb{S}^3, \|\cdot\|^{(0)})$ is a Banach space of convergent sequences; moreover, $\nu_h \to 0$ and there exist $\alpha_{-\infty}, \zeta_{-\infty} \in \mathbb{R}$ such that $\alpha_h \to \alpha_{-\infty}$, $\zeta_h \to \zeta_{-\infty}$ exponentially with rate θ; finally $|\alpha_h|, |\zeta_h| \le \delta_0$ for every $h \le 0$, by using $\alpha_h = \sum_{k \ge h} a_k - a_{k+1}$ and similarly for ζ.

We look for a solution of the problem $\underline{u}_0 = \underline{\mathcal{B}}_0(\underline{u}_0)$ in this space[28]. Note that, from (3.4.1), we can rewrite the flow of the constants as

$$\begin{cases} \nu_h = -\sum_{j\le h} 2^{j-h} B^\nu_j(\lambda, \underline{u}_0) \\ \alpha_h = \sum_{j\ge h} B^\alpha_j(\lambda, \underline{u}_0) \\ \zeta_h = \sum_{j\ge h} B^\zeta_j(\lambda, \underline{u}_0), \end{cases} \tag{3.4.35}$$

which can be compactly written as $\underline{u}_0 = \hat{\underline{\mathcal{B}}}_0(\underline{u}_0)$. Observe that now the lhs of the first line depends upon the whole sequence $\underline{u}_0$, while the other two components still depends on a finite number of elements, as in Remark 3.4.3.

For the ν component, we used the following relation, valid for any $k < h$, as it follows from (3.4.1):

$$\nu_h = 2^{k-h}\nu_k - \sum_{k<j\le h} 2^{j-h} B^\nu_j(\lambda_0, \underline{u}), \tag{3.4.36}$$

and then send $k \to -\infty$.

Next we consider a ball of radius δ,

$$\mathbb{X}^{(0)}_\delta := \{\underline{u}_0 : \|\underline{u}_0\|^{(0)} \le \delta\}$$

and we want to show that for $|\lambda|$ sufficiently small, $\hat{\underline{\mathcal{B}}}_0(\mathbb{X}^{(0)}_\delta) \subset \mathbb{X}^{(0)}_\delta$, that is

$$\|\hat{\underline{\mathcal{B}}}_0(\mathbb{X}^{(0)}_\delta)\|^{(0)} \le \delta.$$

Let us focus on the ν component of $\hat{\underline{\mathcal{B}}}_0(\underline{u}_0)$ first.

The bound on $\hat{\mathcal{B}}^\nu_0$: We intend to prove that $|(\hat{\mathcal{B}}^\nu_0)_h| \le 2^{\theta h}\delta$ uniformly in h, if $|\lambda|$ is small enough. In virtue of the tree expansion (cf. Section 3.4.3, (3.4.12)), we can rewrite

$$B^\nu_j(\lambda, \underline{u}_0) = \sum_{N\ge 1} \sum_{\tau\in\mathcal{T}^{(j)}_{N,0}} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} B^\nu_j(\lambda, \underline{u}_0; \tau, \mathbf{P}, T), \tag{3.4.37}$$

which can be further decomposed, by extracting the first order contribution in $(\lambda, \underline{u}_0)$[29],

$$B^\nu_j(\lambda, \underline{u}_0) = C^\nu_j \lambda + \sum_{N\ge 1}^{*} \sum_{\tau\in\mathcal{T}^{(j)}_{N,0}} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} B^\nu_{j,\omega}(\lambda, \underline{u}_0; \tau, \mathbf{P}, T), \tag{3.4.38}$$

where $|C^\nu_j| \le \bar{C} 2^{\theta j}$ from the short memory property since it correspond to an endpoint on scale 0; the $*$ means that the sum is restricted either to trees that have one endpoint of type $\mathcal{R}_1 V^{-1}$ of order λ^2, or to trees with at least two endpoints. In this case one can bound

$$|B^\nu_j| \le 2^{\theta j}\left(\bar{C}|\lambda| + C_1 \max\{\lambda^2, \delta|\lambda|, \delta^2\}\right) \tag{3.4.39}$$

[28] Since $\underline{\mathcal{B}}_0$ does not depend on $\underline{u}_1$, we denote by the same symbol its projection on $\mathbb{S}^3$.

[29] The first order contribution in $(\lambda, \underline{u}_0)$ is of order λ because we have endpoints only of type $\mathcal{R}_1 V^{-1}$ or ν: in the latter case we already extracted the first order term in ν in (3.4.1), which plays no role in the contribution to B^ν.

where the constant C_1 comes from the sum over τ, $\mathbf{P}$ etc, as explained below (3.4.22), while the factor $2^{\theta j}$ comes from Proposition 3.4.1[30]. To conclude,

$$|(\tilde{\mathcal{B}}_0^\nu)_h| \leq 2^{\theta h-2}\delta(1+2C_1\delta)\sum_{k\leq 0}2^k \leq \delta 2^{\theta h} \tag{3.4.40}$$

where we chose $\lambda = \frac{\delta}{2}\min\{1, \bar{C}^{-1}\}$ and δ small enough, e.g. $\delta < (2C_1)^{-1}$.

The bound on $\hat{\mathcal{B}}_0^\alpha$, $\hat{\mathcal{B}}_0^\zeta$: One can reason in a similar way for the components of α, ζ. Since $(\hat{\mathcal{B}}_0^\alpha)_h - (\hat{\mathcal{B}}_0^\alpha)_{h+1} = B_h^\alpha$, we want to prove that $\sup_{h\leq 0} 2^{-\theta h}|B_h^\alpha| \leq \delta$. Also in this case the first order contribution in $(\lambda, \underline{u}_0)$ is independent of $\underline{u}_0$[31], so one obtain an expression analogous to (3.4.38), with constant $|C_j^\alpha| \leq 2^{\theta j}\bar{C}'$. Now to extract the dimensional factor from the second term, we use Proposition 3.4.1 by noting that α_h (cf. (3.3.49)) is associated in the coordinate space, as it follows from Appendix B.1, to a term $\sum_x \varphi_x^+ \hat{\partial}_2^2 \varphi_x^-$, thus $|\mathbf{D}|^{(1)} = 1$ and the $2^{\theta j}$ follows.

At this point we want to show that $\hat{\mathcal{B}}_0$ is a contraction on the metric space $\mathbb{Y}^{(0)} = (\mathbb{X}_\delta^{(0)}, d_0)$ with respect to the metric d_0 induced by the norm

$$\|\underline{u}_0\| := \max\{\|\underline{\nu}\|_\theta, \|\underline{\alpha}\|_0, \|\underline{\zeta}\|_0\} \tag{3.4.41}$$

see Definition 3.4.1: note that $\|\cdot\|_0$ is the usual ℓ^∞ norm. More precisely we prove that, for λ, δ small enough, $\forall \underline{u}_0, \underline{u}_0' \in \mathbb{X}_\delta^{(0)}$,

$$\|\underline{\hat{\mathcal{B}}}_0(\underline{u}_0) - \underline{\hat{\mathcal{B}}}_0(\underline{u}_0')\| \leq \frac{1}{2}\|\underline{u}_0 - \underline{u}_0'\| \tag{3.4.42}$$

We only discuss the bound for the ν component, since it will include all the ideas to discuss similarly the bounds for ζ, α.

The contraction: the ν component By definition (cf. (3.4.35)), we can write

$$\left(\hat{\mathcal{B}}_0^\nu(\lambda, \underline{u}_0) - \hat{\mathcal{B}}_0^\nu(\lambda, \underline{u}_0')\right)_h = \sum_{j\leq h} 2^{j-h}\left(B_j^\nu(\lambda, \underline{u}_0) - B_j^\nu(\lambda, \underline{u}_0')\right) \tag{3.4.43}$$

then using the tree expansion for B_j^ν, we can rewrite

$$B_j^\nu(\lambda, \underline{u}_0) - B_j^\nu(\lambda, \underline{u}_0') = \int_0^1 dt \sum_{N\geq 2}\sum_{\tau\in\mathcal{T}_{N,0}^{(j)}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{T\in\mathbf{T}}^{*}\frac{d}{dt}B_j^\nu(\lambda, \underline{u}_0(t); \tau, \mathbf{P}, T) \tag{3.4.44}$$

where $\underline{u}_0'(t) = t\underline{u}_0 + (1-t)\underline{u}_0$. Note that the sum is restricted to trees with at least two endpoints: contributions coming from trees with exactly one endpoint, $\mathcal{R}V^{(-1)}$, (among which the first term in (3.4.38)) cancel out because they do not depend on $\underline{u}$; moreover, $B_j^\nu(\lambda, \underline{u}_0(t); \tau, \mathbf{P}, T)$ satisfies a bound analogous to the one in (3.4.39), uniformly in $t \in$

[30] Note that the term contributing to $\sum^*$, when summed over the labels, leads to a bound analogous to (3.4.5) with the difference that we already extracted the factor 2^j in the definition of ν, cf. (3.3.55).

[31] A tree with exactly one endpoint v, of type ν, can contribute only to $\nu_{h_v-1}, \tilde{F}^{(h_v-1)}$, where $\tilde{F}^{(h_v-1)} = F^{(h_v-1)} - F^{(h_v)}$ is the single scale contribution to the free energy.

$[0,1]$. The effect of the derivative instead can be of two types:

- When it acts on an endpoint v on scale $k > h$ of type ν, then it replaces $\nu_k(t)$ with $\nu_k - \nu'_k$
- if it acts on some propagator $\tilde{g}^{(k)}$ then its effect is, dimensionally, to multiply by $\alpha_k - \alpha'_k$ or $\zeta_k - \zeta'_k$.[32]

Then in analogy with (3.4.39) we obtain

$$|B^{\nu}_j(\lambda, \underline{u}_0) - B^{\nu}_j(\lambda, \underline{u}'_0)| \leq C_2 2^{\theta j} \|\underline{u}_0 - \underline{u}'_0\| (|\lambda| + 2\delta) \tag{3.4.45}$$

where, C_2 is the analogous of C_1, and we used that since all the trees have at least two endpoints, not both of type λ, we have two kind of contributions: $\lambda\nu$ and ν^2 up to higher order. A factor $\nu - \nu'$ can be bounded by $\|\underline{u}_0 - \underline{u}'_0\|$ while the other by 2δ since $\underline{u}_0, \underline{u}'_0 \in \mathbb{X}^{(0)}_\delta$. Then one chooses λ, δ small and obtains the claim.

Recalling that $\underline{u}_0$ solves (3.4.35) if and only if it solves the first line in (3.4.33), then we have just proven that

Proposition 3.4.3. *There exist $\delta_0, 0 < \lambda_0(\delta_0) < \delta_0$ such that if $|\lambda| \leq \lambda_0$ and $\delta \leq \delta_0$, there exists a unique solution $\underline{u}^*_0 \in \mathbb{X}^{(0)}_\delta$ of the problem*

$$\underline{u}_0 = \underline{\mathcal{B}}_0(\underline{u}_0) \tag{3.4.46}$$

This can be read by saying that fixing ν_0 as ν^*_0, the iterative construction of Section 3.3.1, produces a flow $\{\nu^*_h, \alpha^*_h, \zeta^*_h\} \in \mathbb{X}^{(0)}_\delta$. In particular, since the flow is just a function of the coupling λ, using the analiticity properties in λ of the beta functions, as they follows from the kernel bounds of Proposition 3.4.1, we will be able to implicitly determine the analytic functions $\nu_0(\lambda), \alpha_{-\infty}$, and $\zeta_{-\infty}(\lambda)$ of λ in the ball $|\lambda| \leq \lambda_0$ appearing in Proposition 3.3.2; this fixed also the function $\varepsilon_0(\lambda) = \log(1 + \nu_0(\lambda))$ described at the beginning of Section 3.3.1.

The flow of $\underline{u}_1$: boundedness of $\{\nu_+, a_+, b_+, \lambda, z\}$

We fix $\underline{u}_0 = \underline{u}^*_0$ and $|\lambda| \leq \lambda_0, \delta \leq \delta_0$. As an immediate consequence, since r_h depends only on $\underline{u}^*_0$, we have that Lemma 3.3.0.3 holds for the sequence $r^*_h = r_h(\underline{u}^*_0)$. This implies that Proposition 3.4.1 holds, *uniformly in h^**. In particular to determine the unique solution $\underline{u}^*_1$ it is enough to study $\tilde{\underline{u}}_1 = (\underline{\nu}_+, \underline{a}_+, \underline{b}_+, \underline{\lambda}, \underline{z})$.[33]

For this part we intend to follow closely the construction of [**2**, Sec. 6.4], because of the analogies of the structure of the flow. Recall indeed that we have exactly the same running constants, except for the fact that here $a_{h,\omega}, b_{h,\omega}$ are real instead of complex valued. This is related to our relativistic model, which is a particular case of the one used in [**2**, Sec. 4] for the choice $\bar{\alpha}_\omega = 1$ and $\bar{\beta}_\omega = \omega v_F$ (compare [**2**, Eq. (4.4)] with (3.3.119)). Anyway there is still an important difference with our setting, namely the fact that the trees involved in

[32] The derivatives $\partial_{\alpha_k}, \partial_{\zeta_k}$ applied to $\tilde{g}^{(k)}$ are dimensionless, i.e. they do not change its scaling. This can be seen from (3.3.53) and the support properties of f_k.

[33] Recall $\underline{u}_1 = (\underline{r}, \tilde{\underline{u}}_1)$.

the second regime have endpoints on scale h^*-1 instead of 0 and that v_F is vanishing as $\epsilon \to 0$.

In view of (3.4.7), we now consider the Banach space of sequences $(\mathbb{S}^5, \|\underline{\tilde{u}}_1\|^{(1)})$, where

$$\|\underline{\tilde{u}}_1\|^{(1)} := \Big\{ \|\underline{\nu}_+\|^*_\theta, \|\underline{a}_+\|^*_\theta, \|\rho\underline{b}_+\|^*_\theta, K_\theta\|(\tau-1)\rho\underline{\lambda}\|^*_\theta, \|(\tau-1)\underline{z}\|^*_\theta \Big\}, \tag{3.4.47}$$

where $K_\theta = 1-2^{-\theta}$ is the same as in (3.4.34) and the definitions of $\|\cdot\|^*_\theta, \tau, \rho$ are given in Def. 3.4.1.

Remark 3.4.4. • *Denote $\underline{\tilde{\mathcal{B}}}_1 := (\mathcal{B}^{\nu_+}, \mathcal{B}^{a_+}, \mathcal{B}^{b_+}, \mathcal{B}^{\lambda}, \mathcal{B}^{z})$. From the structure of the flow in* (3.4.1)*, which follows from the inductive scheme of Section 3.3.2, it follows that $\mathcal{B}^{\#}, \# = \nu_+, a_+, \dots$ has the first h^* entries zero, and it actually depends only on the its arguments $\underline{\tilde{u}}_1$ only through the components $h < h^*$. We can still think of $\mathcal{B}^{\#}$ as an element of $\mathbb{S}$ with the following identity $\mathcal{B}^{\#}(\underline{u}_0, \underline{r}, \underline{\nu}_+, \dots, \underline{z}) = \pi_* \mathcal{B}^{\#}(\underline{u}_0, \underline{r}, \pi_*\underline{\nu}_+, \dots \pi_*\underline{z})$. In particular, note e.g.,*

$$\|\mathcal{B}^z\|^*_\theta = \sup_{h\le h^*} 2^{-\theta(h-h^*)} |\mathcal{B}^z_h(\pi_*\underline{\lambda}, \pi_*\underline{z})|.$$

• *Note that from the definition of K_θ, by writing $\lambda_h = \sum_{k=h}^{h^*-1}(\lambda_k - \lambda_{k+1})$ one has that $\sup_{h<h^*}|\lambda_h| \le v_F\delta'$, i.e.*

$$\|\rho\underline{\lambda}\|^*_0 \le \delta'. \tag{3.4.48}$$

If we define as usual, the ball

$$\mathbb{X}^{(1)}_{\delta'} := \{\underline{\tilde{u}}_1 : \|\underline{\tilde{u}}_1\|^{(1)} \le \delta'\},$$

then we have that

Proposition 3.4.4. *Under the hypothesis of the previous Proposition, there exist constants $\delta_1, \delta_0(\delta_1), \lambda_0(\delta_0)$ such that if $\delta' \le \delta_1, \delta \le \delta_0, \lambda \le \lambda_0$, then the problem*

$$\underline{\tilde{u}}_1 = \underline{\tilde{\mathcal{B}}}_1(\underline{u}^*_0, \underline{r}^*, \underline{\tilde{u}}_1) \tag{3.4.49}$$

admits a unique solution $\underline{\tilde{u}}^*_1 \in \mathbb{X}^{(1)}_{\delta'}$, where $\underline{u}^*_0$ is the solution of (3.4.46) *and $\underline{r}^*$ is defined at the beginning of this section.*

In order to skecth a proof of this result, by a comparison with [**2**], let us first explain how to manage the dependence on the small factors v_F for the reference model, which is crucial in order to control the flow of the constants of this regime.

The relativistic beta function Recall that z_h was defined in terms of a suitable modification of the kernels associated to two external Grassmann fields, as explained below (3.3.129). In particular its beta function $\mathcal{B}^z$ depends only on $(\underline{\lambda}, \underline{z})$. The same procedure can be of course made for the kernels with four legs and so on. Let us attach an extra label $\mathtt{R}$ to kernels, as well to the components of the beta function whenever, in their expansion, we apply the substitutions of points $(i), (ii), (iii)$ below (3.3.129); of course $\mathcal{B}^z = \mathcal{B}^{z,\mathtt{R}}$. Let us also denote by $\mathcal{B}^{\#,\mathtt{R}}_v$ the relativistic beta function evaluated at parameter $v_F = v$ (cf. (3.3.130) and (3.3.119)): of course $\mathcal{B}^{\#,\mathtt{R}}_{v_F} = \mathcal{B}^{\#,\mathtt{R}}$. We have then (cf. Definition 3.4.1 for the notations):

Lemma 3.4.0.3. *The beta function of the relativistic model satisfies,*

- *Parity:*
$$\mathcal{B}^{\nu,\mathtt{R}}(\underline{z},\underline{\lambda}) = \underline{0} \tag{3.4.50}$$
- *Shift: If we restrict $\mathcal{B}^{\#;\mathtt{R}}$ to sequences $\underline{\lambda}$ with $\lambda_{h^*-1} = 0$, then we have that*
$$\Big(\mathcal{B}^{\#,\mathtt{R}}(\underline{z},\underline{\lambda})\Big)_{h-1} = \Big(\mathcal{B}^{\#,\mathtt{R}}(\tau\underline{z},\tau\underline{\lambda})\Big)_{h} \tag{3.4.51}$$
or, more compactly, $\tau\mathcal{B}^{\#,\mathtt{R}}(\underline{\lambda},\underline{z}) = \mathcal{B}^{\#,\mathtt{R}}(\tau\underline{\lambda},\tau\underline{z})$; $\# = z,\lambda$.
- *Rescaling:*
$$\mathcal{B}^{z,\mathtt{R}}_{v_F}(\underline{\lambda},\underline{z}) = \mathcal{B}^{z,\mathtt{R}}(\rho\underline{\lambda},\underline{z}), \quad \mathcal{B}^{\lambda,\mathtt{R}}_{v_F}(\underline{\lambda},\underline{z}) = v_F\mathcal{B}^{z,\mathtt{R}}(\rho\underline{\lambda},\underline{z}) \tag{3.4.52}$$
- *"Vanishing":*
$$\Big|\Big(\mathcal{B}^{\lambda,\mathtt{R}}(\underline{\lambda},\underline{z})\Big)_h\Big| \le \|\pi_*\underline{\lambda}\|_0 2^{\theta(h-h^*)}, \tag{3.4.53}$$

where we recall that $\|\pi_*\underline{\lambda}\|_0 = \sup_{h\le 0}|(\pi_*\underline{\lambda})_h| = \sup_{h<h^*}|\lambda_h|$. While the first three items follows directly from the properties of the single scale relativistic propagator in (3.3.130), the last one has a different deep nature: we refer to [**119**, **120**] for its proof.

Sketch of the proof of Proposition 3.4.4.

First we want to prove that $\underline{\tilde{\mathcal{B}}}_1(\mathbb{X}^{(1)}_{\delta'}) \subset \mathbb{X}^{(1)}_{\delta'}$.

The z component: We start with $\mathcal{B}^z$, since it couples only $\underline{z},\underline{\lambda}$: we take any given $\underline{\lambda}$ satysfing $\|(\tau-1)\rho\underline{\lambda}\|^*_\theta \le \delta'$, and note that it implies that $\|\rho\underline{\lambda}\|^*_0 \le K_\theta\delta'$ as described in (3.4.48). In view of the previous Lemma, we can write

$$(\tau-1)\mathcal{B}^z(\underline{\lambda},\underline{z}) = \tau\left(\mathcal{B}^z(\rho\underline{\lambda},\underline{z}) - \mathcal{B}^z(\rho p_*\underline{\lambda},\underline{z})\right) + \int_0^1 dt\frac{d}{dt}\mathcal{B}^z(\rho\underline{\lambda}(t),\underline{z}(t)) \tag{3.4.54}$$

where the integral is meant componentwise; the operator $p_* : \mathbb{S}\to\mathbb{S}$ pins to zero the h^*-1 component; $\underline{\lambda}(t) := t\tau\underline{\lambda} + (1-t)\underline{\lambda}$ and similarly for $\underline{z}(t)$. Now, since $\mathcal{B}^z_h = B^z_h$ contributes to $\partial_1\hat{W}^{\infty;(h),\mathtt{R}}_2$, as it follows from (3.4.2) and (3.4.29), it admits a tree expansion, as in (3.4.37); we write then

$$B^z_h(\underline{\lambda},z) = \sum_{N\ge 2}\sum_{\tau\in\mathcal{T}^{(h)}_{N,0}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{T\in\mathbf{T}} B^z_h(\underline{\lambda},\underline{z},\tau,\mathbf{P},T), \tag{3.4.55}$$

where we used that the trees with only one endpoint, necessarily of type λ, do not contribute to B^z_h.[34] By using such expansion in the r.h.s (3.4.54), we find that the first term is non vanishing only for trees that have at least one endpoint on scale h^*-1: these contributions can be bounded by $C'\delta_1^2 2^{\theta(h-h^*)}$, where we used $\|\rho\underline{\lambda}\|^*_0$ on the endpoints and the

[34] This is related to the fact that either we contract two of the four fields on a scale $k > h$ or on $k = h$. While the latter contributes only to ν_h, the former cannot contribute to a local part because of the action of $\mathcal{R}$ on each vertex in the scales between k and h.

short memory property to obtain the factor $2^{\theta(h-h^*)}$. The latter admits the same bound with an extra factor $c\delta' 2^{\theta(h-h^*}$ coming from the action of the derivative[35]. We have an overall bound, for δ' small enough,

$$|((\tau-1)\mathcal{B}^z(\underline{\lambda},\underline{z}))_h| \le c\delta'^2 2^{\theta(h-h^*)}. \tag{3.4.56}$$

where $c>0$ does not depend on h, L, λ, ϵ.

The ν_+, a_+, b_+, λ components: First we reverse the equations for ν, a, b, as we did in the previous section on for ν (cf. (3.4.35)), by rewriting

$$\begin{cases} \nu_{h,+} = -\sum_{j\le h} 2^{j-h} B^{\nu}_{j,+}(\lambda,\underline{u}), \\ a_{h,+} = -\sum_{j\le h} B^{a}_{j,+}(\lambda,\underline{u}), \\ b_{h,+} = -\sum_{j\le h} B^{b}_{j,+}(\lambda,\underline{u}) \\ \lambda_{h-1} = \lambda_{h^*-1} + \sum_{j=h}^{h^*-1} B^{\lambda}_{j}(\lambda,\underline{u}), \end{cases} \tag{3.4.57}$$

because in the space under consideration $a_{h,+}, b_{h,+} \to 0$ as $h \to -\infty$, and we used the definition of λ_h in (3.4.1). The system is meaningful for $h \le h^*$ by thinking in the last equation the sum in the r.h.s. vanishing for $h = h^*$.

Again, we think of such equations as a fixed point equation $(\underline{\nu}_+, \underline{a}_+, \underline{b}_+, \underline{\lambda}) = \hat{\underline{\mathcal{B}}}_1(\lambda, \underline{u})$. To prove that $\hat{\mathcal{B}}_1(\mathbb{X}^{(1)}_{\delta'}) \subset \mathbb{X}^{(1)}_{\delta'}$, i.e., $\|\hat{\mathcal{B}}^{\nu_+}_1\|^*_\theta \le \delta'$ etc. we comment only the main points in order to apply the strategy of [**2**, Sec. 6.4.3].

The ν_+ component: From the parity property of Lemma 3.4.0.3 one need to show $|\mathcal{B}^{\nu_+}_h - \mathcal{B}^{\nu_+,\mathtt{R}}_h| \le \delta' 2^{-\theta(h-h^*)}$, uniformly in $h \le h^*$. Using, (3.4.57) and the tree expansion together with Lemma 3.4.0.2, one proceeds in the same way of the mentioned paper, and obtains the desired result by noting the small differences: a tree with only endpoints of type λ contributes at least as

$$\sum_{k=h}^{h^*-2} (v_F\delta') \frac{2^{2k}}{v_F^2} 2^{\theta'(h-k)} \le \delta' 2^{3h^*/2} 2^{\theta'(h-h^*)} \le \frac{\delta'}{3} 2^{\theta'(h-h^*)} \tag{3.4.58}$$

where k is the scale of the propagator difference $Z^{-1}_{h-1}(\tilde{g}^{(k)} - \tilde{g}^{(k)}_{\mathtt{R}})$ and we used that it has the same estimates as $\tilde{g}^{(k)}$ times an extra 2^k;[36] the factor $2^{\theta'(h-h^*)}$ comes from the short memory property[37]. For an unique endpoint of type $\mathcal{R}V^{h^*-1}$, we have a bound $C\delta_0 2^{\theta(h-h^*)}$ where δ_0 is the same of Proposition 11 and the other factor comes from the short memory property for an endpoint on scale h^*. So one chooses $C\delta_0 < \delta'/3$[38]. From trees with at least two endpoints, among which one of type $\nu, a, b, \mathcal{R}V^{h^*-1}$, one obtains the bound $C2^{\theta(h-h^*)} \max\{\delta_0^2, \delta'\delta_0\} \le (\delta'/3) 2^{\theta(h-h^*)}$, for δ_0, δ' small.

[35] On the component h, the derivatives produce an extra $\lambda_k - \lambda_{k+1}$, or $z_k - z_{k+1}$ with $k > h$, which can be bounded by the properties of

[36] Indeed the propagator difference has an extra $O(q_1^2)$ or $O(q_2^3 v_F)$ at the numerator and a square at the denominator.

[37] Indeed we must have at least an endpoint on scale k to make there a contraction.

[38] i.e. λ small by Proposition 11

The a, b components: The desired bound on $\mathcal{B}^a_h, \mathcal{B}^b_h$, (3.4.57), follows if show that $|B^a_{h-1} - B^a_h| \leq \delta' 2^{\theta(h-h^*)}, |B^b_{h-1} - B^b_h| \leq v_F \delta' 2^{\theta(h-h^*)}$. Since a, b are defined in terms of $W_2^{\infty;(h)} - W_2^{(h),\mathtt{R}}$, the bounds for B^a, B^b can be discussed in analogy with the previous point, via the tree expansion. Note only that to extract the factor v_F for B^b, one uses that since b_h is associated to the Grassmann monomial $\varphi^+ \hat{\partial}_2 \varphi^-$, then $|\mathbf{D}_2| = 1$ and from Lemma 3.4.0.2 one has an extra v_F.

The λ component: From (3.4.57), we want to show $|B^\lambda_h| \leq v_F \delta' 2^{\theta(h-h^*)}$. It turns out that B^λ_h is at least of second order in $(\lambda, \underline{u})$: indeed, by definition of B^λ_h, the first order in the sequence $\underline{\lambda}$ does not contribute to B^λ_h while the first order contribution in λ coming from the bare potential, contributes in local part to λ_{h^*-1} or to $\mathcal{R}_2 V^{h^*-1}$, which both cannot contribute to a local part on scale $h < h^* - 1$. Thus we already get a factor $\max\{\delta'^2, \delta'\delta_0, \delta_0^2\}$. To obtain the remaining factor $v_F 2^{\theta(h-h^*)}$, we use both the *rescaling* and the *vanishing* of $\mathcal{B}^{\lambda,\mathtt{R}}$, by first rewriting $B^\lambda_h = B^{\lambda,\mathtt{R}}_{v_F,h} + B^\lambda_h - B^{\lambda,\mathtt{R}}_{v_F,h}$. From the just mentioned properties of the relativistic beta we get that, for δ' small enough

$$|B^{\lambda,\mathtt{R}}_{v_F,h}(\underline{\lambda}, \underline{z})| = |v_F B^{\lambda,\mathtt{R}}_{1,h}(\rho\underline{\lambda}, \underline{z})| \leq C 2^{\theta(h-h^*)} v_F \delta'^2 \leq \frac{\delta'}{2} v_F 2^{\theta(h-h^*)}. \tag{3.4.59}$$

Here δ'^2 comes from the norm of the endpoints $\|\rho\underline{\lambda}\|^*_0 \leq \delta'$, see (3.4.48), where $\rho\underline{\lambda}$ follows from the rescaling property in Lemma 3.4.0.3. To obtain the desired factor for the remaining term $B^\lambda_h - B^{\lambda,\mathtt{R}}_{v_F,h}$, one can use Lemma 3.4.0.2: the factor v_F comes from $v_F = v_F^{-1+4/2}$. For the exponential factor, when expanding in trees $B^\lambda_h - B^{\lambda,\mathtt{R}}_h$, among the two endpoints, we always have at least one of type $\nu, a, b, \mathcal{R}_2 V^{(h^*-1)}$ on some scale $h < k < h^*$, as it follows from the definition of the relativistic beta: so, the short memory property (see Remark 3.4.2) and the hypothesis on $\|\nu_+\|^*_\theta$, etc., produces respectively $2^{\theta(h-k)} \times 2^{\theta(k-h^*)}$ so that one recovers also $2^{\theta(h-h^*)}$. Collecting these considerations one obtains the bound

$$|B^\lambda_h - B^{\lambda,\mathtt{R}}_{v_F,h}| \leq c \max\{\delta'^2, \delta_0\delta', \delta_0^2\} v_F 2^{\theta(h-h^*)} \leq \frac{\delta'}{2} v_F 2^{\theta(h-h^*)} \tag{3.4.60}$$

where we chose δ_0, δ' small enough.

Once we showed how to extract the dependence on v_F, to prove that $\hat{\underline{\mathcal{B}}}_1$ is a contraction on the metric space $\mathbb{Y}^{(1)} = (\mathbb{X}^{(1)}_{d'}, d_1)$ with the distance d_1 induced by

$$\|\tilde{\underline{u}}_1\| := \max\{\|\underline{\nu}_+\|_\theta, \|\underline{a}_+\|_\theta, \|\rho\underline{b}_+\|_\theta, \|\rho\underline{\lambda}\|_0, \|\underline{z}\|_0\}, \tag{3.4.61}$$

one can proceed by reasoning as we did for (3.4.43). Then one finds a unique solution of (3.4.57), thus $\underline{z}^*(\underline{\lambda}^*)$ and then the unique solution $\tilde{\underline{u}}_1$ of (3.4.4). ∎

In order to obtain the desired unique solution, there is still a point to be stressed:

Remark 3.4.5. *Proposition 3.4.4 tells us that, fixed λ, δ_0 small enough such that $(\underline{u}^*_0, \underline{r}^*)$ exists, then there exists a unique sequence of running constants of the second regime satisfying the beta equations in* (3.4.1)-(3.4.2)*, which stay in the ball $\mathbb{X}^{(1)}_{\delta'}$, for δ' small, possibly after shrinking λ a bit more. Anyway there is a big difference with the first regime: it is not guaranteed that we can find a datum in the ball $\mathbb{X}^{(1)}_{\delta'}$ w.r.t. the norm $\|\cdot\|^{(1)}_\theta$. This is because while in the h^* component we have the right to fix the counterterms $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$ (for*

the ν, a, b components), in the h^*-1 component we have λ_{h^*-1}, which is the output of the integration of the first regime.

This means that one needs to check that $|\lambda_{h^*}| \leq v_F\delta'$ in order for a solution to exists. This calculation is done in the next Lemma.

Recall that $\underline{u}_0^*$ denotes the unique solution of (3.4.46), and $\underline{r}^* := \underline{r}(\underline{u}_0^*)$ the corresponding sequence $\underline{r}$ of the first regime.

Lemma 3.4.0.4 (Bound on λ_{h^*})**.** *Let λ, δ_0 as in the hypotheses of Proposition 3.4.3, then there exists a costant $C > 0$ independent of h^*, λ, δ_0 such that*

$$|\lambda_{h^*}| \leq C\delta_0 2^{h^*(\frac{1}{2}+\theta)} \tag{3.4.62}$$

where θ can be chosen in $(0, 1/2)$, and estimates are not uniform as $\theta \to 1/2^-$.

Note that in virtue of the assumptions on the counterterms $\tilde{q}_F, b_{h^*,+}$ in the lines below (3.3.110), we have that (see (3.3.113)) $v_F = \Theta(\sqrt{\epsilon}) = \Theta(2^{h^*/2})$ so in particular for δ_0 small enough as a function of δ', we have that $|\lambda_{h^*}| \leq \delta' v_F$.

Proof of Lemma 3.4.0.4. Once we reach the scale h^*, the integration in (3.3.117), produce a generic quartic term in the Grassmann fields of the form

$$V_4^{(h^*-1)}(\phi) = \sum_{h\geq h^*} \sum_{\underline{y},\underline{\omega},\underline{D}} \hat{\partial}^{D_1}\phi^+_{y_1}\hat{\partial}^{D_2}\phi^-_{y_2}\hat{\partial}^{D_3}\phi^+_{y_3}\hat{\partial}^{D_4}\phi^-_{y_4} W^{(h)}_{4,\underline{D}}(\underline{y}) \tag{3.4.63}$$

where $\underline{y} = (y_1, y_2, y_3, y_4)$ and similarly for $\underline{D}$ which are respectively the number and the directions of the derivative operators $\hat{\partial}$ (see (3.3.33)) associated to the fields ϕ (see (3.4.3) for the notations), and we denoted

$$\phi^\sigma_y = \sum_\omega e^{i\omega\sigma\mathbf{q}_F y}\phi^\sigma_{y,\omega}, \quad \phi^\sigma_{y,\omega} = \varphi^{\sigma,\ (\leq h^*-1)}_{y,\omega} \tag{3.4.64}$$

as a consequence of the decomposition (3.3.116). Here h represents the last scale for which the truncated expectation $\mathcal{E}^T_h$ acted non trivially.

Now, let us temporarily assume that once we transform the Grassmann variables with (3.4.64), the operator $\hat{\partial}_j$ acts as follows

$$\hat{\partial}_j(e^{i\omega\mathbf{q}_F y}\phi^+_{y,\omega}) = e^{i\omega\mathbf{q}_F y}\left(\omega(\mathbf{q}_F)_j\phi^+_{y,\omega} + \hat{\partial}_j\phi^+_{y,\omega}\right). \tag{3.4.65}$$

This means that, in order for a term in (3.4.63) to survive the action of the operator $\mathcal{L}_{2,\epsilon}$ and to contribute to λ_{h^*} (local quartic term without derivatives $\hat{\partial}$), it must be associated to a label $\underline{D}$ satisfying $D_{j1} = 0$ for each $j = 1, 2, 3, 4$ and the action of each $\hat{\partial}_2^{D_{j2}}$ must be D_{j2} times on the oscillating factor in (3.4.65). This is because $\mathbf{q}_F = (0, \tilde{q}_F)$ (recall the definition above (3.3.111)). Then we have that

$$\lambda_{h^*} = \sum_{h\geq h^*} \sum_{y_2,y_3,y_4\in R^T\mathbb{Z}^2} \sum_{\underline{\omega}}^{*} (-1)^{\omega_1\omega_2} \prod_{j=1}^{4} (\omega_j\tilde{q}_F)^{D_{j2}} W^{\infty;(h)}_{4,\underline{D}}(\underline{y}) \tag{3.4.66}$$

where we used that λ_{h^*} is defined in terms of infinite volume kernels; the * means that

the sum is restricted to $\underline{\omega} \in \{1,-1\}^4$ such that $\sum_j \omega_j = 0$. Then we can bound

$$|\lambda_{h^*}| \le C'' \sum_{h \ge h^*} |\tilde{q}_F|^{|\underline{D}|} \|W_4^{\infty;(h)}\|_{\kappa,h} \le C' \delta_0 \tilde{q}_F^{|\underline{D}|} 2^{h^*(\frac{1}{2}+\theta-\frac{1}{2}|\underline{D}|)} \le C\delta_0 2^{h^*(\frac{1}{2}+\theta)} \tag{3.4.67}$$

where $|\underline{D}| = \sum_j D_{j2}$, and we used Proposition 3.4.1: note that since $D_{j1} = 0$, then $|\underline{D}|^{(1)} = 2^{-1}|\underline{D}|$ and that $\frac{1}{2} + \theta - \frac{1}{2}|\underline{D}| < 0$ as a consequence of the renormalization procedure. In the last step we used that from the hypotheses on the counterterms, we have that $\tilde{q}_F = \Theta(\sqrt{\epsilon}) = \Theta(2^{h^*/2})$. We are left with showing (3.4.65). From its definition

$$\begin{aligned} &\hat{\partial}_j \phi^+_y = L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} e^{iqy} \chi(q) q_j \hat{\phi}^+_{q+q_F} = \\ &L^{-2} \sum_{\omega} e^{iq\omega \mathbf{q}_F y} \sum_{q \in \mathcal{D}_{\theta,R,\omega}} e^{iqy} \hat{\phi}^+_{q,\omega} \chi(q+\omega\mathbf{q}_F)(q+\omega\mathbf{q}_F)_j + L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R}} e^{iqy} \hat{\phi}^+_{q+q_F,o} \chi(q) q_j \end{aligned} \tag{3.4.68}$$

where we used $\hat{\phi}^+_{q+q_F} = \hat{\phi}^+_{q-\omega\mathbf{q}_F,\omega} + \hat{\phi}^+_{q+q_F,o}$ with $\hat{\phi}^+_{\cdot,\omega}, \hat{\phi}^+_{\cdot,o}$ that are respectively supported in $\tilde{\chi}_{h^*}(\cdot), \tilde{f}_{h^*}(\cdot)$ as a consequence of the decomposition (3.3.114); $\mathcal{D}'_{R,\theta,\omega} = \mathcal{D}'_{R,\theta} - \omega\mathbf{q}_F$ as below (3.3.119). This means that if we define

$$\hat{\partial}_j \phi^+_{y,\omega} = L^{-2} \sum_{q \in \mathcal{D}_{\theta,R,\omega}} e^{iqy} \hat{\phi}^+_{q,\omega} q_j, \quad \phi^+_{y,\omega} = L^{-2} \sum_{q \in \mathcal{D}_{\theta,R,\omega}} e^{iqy} \hat{\phi}^+_{q,\omega}$$

we have (3.4.65)[39]. ∎

Actually, as shown in the Appendix B.3.3, the lowest order in λ in the contribution to λ_{h^*} is $O(\lambda\epsilon)$, showing that $\theta = 1/2$ in (3.4.62) can be reached (at least in this case).

3.4.5 Inversion of the counterterms

In this section we want to show how the free parameters introduced along the multiscale construction, $\varepsilon_0, \mathbf{q}_F, v_F, Z_{h^*1}$ (see Sections 3.3.1-3.3.2) are fixed as functions of the original parameters of the model λ, ϵ. By definition, they are related to the counterterms $\nu_0, a_{h^*,+}, b_{h^*,+}$, namely the initial data of the flow of running constants built in Sections 3.3.1-3.3.2: these relations that we recall shortly are given in (3.3.1) and Section 3.3.2. We now show how, the fixing of such counterterms as described in the previous Sections can be used to determine the desired functions.

If, on the one hand, the tree expansion allowed us to obtain Proposition 3.4.4, as a consequence of Lemma 3.4.0.2-3.4.0.1, by thinking at the sequence of running constants as to independent variables, on the other hand they will be only functions of the parameters of the theory: the bare coupling λ and ϵ.

[39] Note that in the support of $\tilde{\chi}_{h^*}(q)$, $\chi(q+\omega\mathbf{q}_F) \equiv 1$.

Inversion of ν_0^*: fixing $\varepsilon_0(\lambda)$

Let $\underline{u}_0^* = (\underline{\nu}^*, \underline{\alpha}^*, \underline{\zeta}^*)$, with $\underline{\nu}^* = (\nu_h^*)_{h\leq 0}$ etc., as usual denote the unique solution of (3.4.46). From (3.4.35), we have that

$$\nu_0^* = -\sum_{h\leq 0} 2^h B_h^\nu(\lambda, \underline{u}_0^*). \tag{3.4.69}$$

From the iterative construction of Section 3.3.2, and the fact that it solves (3.4.46), one sees that $\underline{u}_0^*$ depends only on (λ, ν_0^*), so B_h^ν does. This means that the above equation can be written as

$$G(\lambda, \nu_0^*) = 0, \qquad G(0,0) = 0 \tag{3.4.70}$$

For $|\lambda| \leq \lambda_0$ with λ_0 small enough, since $\partial_{\nu_0} G(0,0) = 1$, from the Implicit Function Theorem (IFT) we can find a unique $\nu_0^*(\lambda)$ s.t. $\nu_0^*(0) = 0$ and $G(\lambda, \nu_0^*(\lambda)) = 0$. Since the r.h.s. in the equation above is analytic in λ[40], we have that $\nu_0^*(\lambda)$ is analytic in $|\lambda| \leq \lambda_0$. The function $\varepsilon_0(\lambda)$ of Theorem 1.4.5 is given by

$$\varepsilon_0(\lambda) = \log(1 + \nu_0^*(\lambda))$$

as it follows from the definition of ν_0, (3.3.1).

An explicit calculation shows that (see Appendix B.3.3), at lowest order in λ

$$\varepsilon_0(\lambda) = \nu_0(\lambda) = -2\lambda + O(\lambda^2). \tag{3.4.71}$$

Inversion of $\nu_{h^*,+}^*, a_{h^*,+}^*, b_{h^*,+}^*$: fixing $\mathbf{q}_F = (0, \tilde{q}_F), v_F, Z_{h^*-1}$

As mentioned few lines above, in this Section we want to discuss the relation between the counterterms of the second regime of integration and the free parameter introduced at the transition scale h^* integration. Since we want to discuss also the regularity of these functions w.r.t the original parameter of the theory, λ, ϵ, we assume temporarily to work out of the discrete jump set $\mathcal{J}_2$ of the transition scale $h^*(\epsilon)$ (see Appendix B.3.1). This guarantees that $h^*(\epsilon)$ is indeed constant in ϵ. We will discuss how to deal with this problem in Section 3.5.2, about the regularity in ϵ of the free energy.

We denote by $\tilde{\underline{u}}_1 = (\underline{\nu}_+^*, \underline{a}_+^*, \underline{b}_+^*, \underline{\lambda}^*, \underline{z}^*)$ the unique solution of (3.4.4), given that the flow of the critical theory $(\underline{u}_0, \underline{r})$ has been fixed as the solution $(\underline{u}_0^*, \underline{r}^*)$ of (3.4.46). Since $\tilde{\underline{u}}_1$ solves (3.4.57) this means that $\nu_{h^*,+}^*, a_{h^*,+}^*, b_{h^*,+}^*$ solve the implicit equations

$$\begin{cases} \nu_{h^*,+}^* = -\sum_{h\leq h^*} 2^{h-h^*} B_{h,+}^\nu(\lambda, \underline{u}^*), \\ a_{h^*,+}^* = -\sum_{h\leq h^*} B_{h,+}^a(\lambda, \underline{u}^*), \\ b_{h^*,+}^* = -\sum_{h\leq h^*} B_{h,+}^b(\lambda, \underline{u}^*) \end{cases} \tag{3.4.72}$$

where $\underline{u}^* = (\underline{u}_0^*, \underline{r}^*, \tilde{\underline{u}}_1^*)$. Note that, the constants of the first regime, $(\underline{u}_0^*, \underline{r}^*)$ depend only on λ, ϵ: the former because $\underline{u}_0^*$ depends on λ, since we fixed $\nu_0(\lambda)$ as in the previous Section and the latter because recall $r_h = Q_h(\lambda) r_0 = Q_h(\lambda) e^{-\varepsilon_0(\lambda)}(e^{-\epsilon} - 1)$ with $Q_h(\lambda) =$

[40] This follows from the fact that B_h^ν is analytic in the sequence $\underline{u}_0^*$ and the components of $\underline{u}_0^*$ are analytic in λ for $|\lambda| \leq \lambda_0$, as it follows from the dimensional bounds.

$1 + O(\lambda)$ analytic (see (3.3.66), (3.3.100) and Lemma 3.3.0.3). As far as concerns the dependence on the flow of the second regime $\underline{\tilde{u}}_1^*$, since it solves (3.4.4), this means that the constants at any given scale are only function of (λ, ϵ), through the contributions coming from the first regime of integration, and of $(\mathbf{q}_F, v_F, Z_{h*-1}, \nu^*_{h^*,+}, a^*_{h^*,+}, b^*_{h^*,+})$ which characterize the second regime of integration. Using the tree expansion e.g. for the ν component of the beta function

$$B^{\nu}_{h,+}(\lambda, \epsilon, \underline{\tilde{u}}_1^*) = \sum_{N\geq 1} \sum_{\tau\in\mathcal{T}^{(h)}_{N,0}} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} B^{\nu}_{h,+}(\lambda, \epsilon, \underline{\tilde{u}}_1^*; \tau, \mathbf{P}, T) \tag{3.4.73}$$

we can see more explicitly how the dependence on $\lambda, \epsilon, \mathbf{q}_F, v_F, Z_{h^*-1}$ works:

- the ϵ, λ explicit dependence comes, as already anticipated, from the first regime and it is associated with endpoints of τ of type $\mathcal{R}V^{h^*-1}$ on scale $h^* - 1$,
- the dependence on $\mathbf{q}_F$ it is given as well only by endpoints of type $\mathcal{R}V^{h^*-1}$, through the oscillating factors $e^{i\omega\mathbf{q}_F y}$ coming from the field decomposition (3.3.116),[41]
- the dependence on v_F is mediated by the single scale propagators $g^{(h')}$ (see (3.3.127)). Moreover, since $Z_h = Z_{h^*-1}\prod_{k=h}^{h^*-1}(1 + z_k)$ as it follows from (3.3.136), we have that the dependence on Z_{h^*-1} is mediated as well by the single scale propagators $g^{(h')}$.[42]

Now we want to determine the functions $\mathbf{q}_F(\lambda, \epsilon) = (0, \tilde{q}_F(\lambda, \epsilon)), v_F(\lambda, \epsilon)$ and $Z_{h^*-1}(\lambda, \epsilon)$, by applying the (IFT) to the following system

$$\begin{cases} 1 + r^*_{h^*-1} - \cos\tilde{q}_F + \frac{(\tilde{q}_F)^2}{2}\alpha^*_{h^*-1} + 2^{h^*}(\nu^*_{h^*} + \nu^*_{h^*,+}) = 0 \\ Z_{h^*-1}v_F = \sin\tilde{q}_F + \tilde{q}_F\alpha^*_{h^*-1} + b^*_{h^*,+} \\ Z_{h^*-1} = \cos\tilde{q}_F + \zeta^*_{h^*-1} + a^*_{h^*,+} \end{cases} \tag{3.4.74}$$

where $\nu^*_{h^*,+}, a^*_{h^*,+}, b^*_{h^*,+}$ are the solutions of (3.4.72) and, as explained in terms of the tree expansion, are functions of $(\lambda, \epsilon, \tilde{q}_F, v_F, Z_{h^*-1})$. The system above can be written also as

$$\underline{H}(\lambda, \epsilon, \tilde{q}_F, v_F, Z_{h^*-1}) = \underline{0} \tag{3.4.75}$$

We want to find the solution around the point $(0, \bar{\epsilon}, \bar{q}, \bar{v}, \bar{Z})$, with $\epsilon \neq 0$, where it is readily seen that, using $\alpha^*_{h^*-1}, \nu^*_{h^*}, \nu^*_{h^*,+}, b^*_{h^*,+}, \zeta^*_{h^*-1} = O(\lambda)$,

$$\cos\bar{q} = e^{-\bar{\epsilon}}, \quad \bar{Z} = e^{-\bar{\epsilon}}, \quad \bar{v} = \sin\bar{q}. \tag{3.4.76}$$

Now, assume the following

Lemma 3.4.0.5. *Let $\# = a^*_{h^*-1}, \nu^*_{h^*,+}, b^*_{h^*,+}$. Then there exists λ_0 s.t. for $|\lambda| \leq \lambda_0$ there exists a constant $C > 0$ independent of $\lambda, \epsilon, v_F, Z_{h^*-1}, \tilde{q}_F$, such that*

$$|\partial_{\tilde{q}_F}\#| \leq C|\lambda|2^{-h^*}v_F \quad |\partial_{v_F}\#| \leq C|\lambda|v_F^{-1}, \quad |\partial_{Z_{h^*-1}}\#|C|\lambda|v_F^{-2}. \tag{3.4.77}$$

[41] Indeed any other endpoint, since it is local, does not have such oscillating factor.

[42] Note that any endpoint depend on the sequence $\underline{Z}$ only through ratios $Z_h/Z_{h-1} = (1 + z_k)^{-1}$ (see e.g. (3.3.139)) and thus not on Z_{h^*-1}.

Then we have that the Hessian of $\underline{H}$ w.r.t. $(\tilde{q}_F, v_F, Z_{h^*-1})$ evaluated at $(0, \bar{\epsilon}, \bar{q}, \bar{v}, \bar{Z})$ has determinant given by $-\sin\bar{q} \neq 0$ if $\bar{\epsilon} \neq 0$ as can be checked from (3.4.74). This means that in a neighborhood of $(0, \bar{\epsilon})$ we can find uniquely the desired functions

$$(\lambda, \epsilon) \mapsto \tilde{q}_F, v_F, Z_{h^*-1} \tag{3.4.78}$$

satisfying (3.4.74). We are left with proving the Lemma above.

Proof of Lemma 3.4.0.5. There is no qualitative difference in the choice of #, so we restrict to $\nu^*_{h^*,+}$. Let us discuss the effect of the derivatives on such term. Recall that it satisfies (3.4.72), so that a derivative acts on the beta $B^\nu_{h,+}$ at some scale $k \geq h$. Using the tree expansion, if we consider ∂_{v_F} or $\partial_{Z_{h^*-1}}$ we have that they act on some propagator $g^{(k)}$ in the expansion. In the former case, as it follows from Lemma B.2.0.2, ∂_{v_F} modifies the bound of the propagator by adding an extra factor v_F^{-1}. In the latter, a derivative $\partial_{Z_{h^*-1}}$ on $g^{(k)}_\omega$ gives

$$\partial_{Z_{h^*-1}} g^{(k)}_\omega(y) = \frac{1}{Z_k Z_{h^*-1}(2\pi)^2} \int_{\mathbb{R}^2} dq e^{-iqy} \frac{\tilde{f}_k(q)\rho_{h^*-1}(q)}{(D_{h^*-1,\omega}(q) + \rho_{h^*-1}/Z_k)^2}. \tag{3.4.79}$$

where we used that and that $Z_k = Z_{h^*-1}\prod_{j=k}^{h^*-1}(1+z_j)$. Next, recall that $\rho_{h^*-1}(q)$ is quadratic in q, that in the support of $\tilde{f}_k(q)$ one has $q_1 = \Theta(2^h), q_2 = \Theta(2^h v_F^{-1})$. Then using that there exist constants $c, c_1, c_2 > 0$, with c_1, c_2 small as needed if $|\lambda|$ is small enough, such that $ce^{-c_1|k-h^*|} \leq |Z_k| \leq ce^{c_2|k-h^*|}$ we find that

$$|\partial_{Z_{h^*-1}} g^{(k)}_\omega| \leq C\frac{2^k}{v_F}\frac{2^{k+c_1|k-h^*|}}{v_F^2} \leq C\frac{2^k}{v_F}\frac{2^{\theta k}}{v_F^2}. \tag{3.4.80}$$

This means that the derivative produce an extra $2^{\theta k}/v_F^2$, for some $0 < \theta < 1$ if λ is small enough, with respect to the original bound of $g^{(k)}$. When we sum over $k \geq h$, we have that the latter contribution is bounded by a constant, uniformly in h: using the bound for $B^\nu_{h,+}$ of Lemma 3.4.0.2, one finds

$$|\partial_{Z_{h^*-1}} \nu^*_{h^*,+}| \leq \frac{C|\lambda|}{v_F^2}.$$

In the former case, such a sum produce dangerous factor $|h - h^*|$. This can be safely absorbed with the short memory factor $2^{\theta(h-h^*)}$ coming from the bound of $B^\nu_{h,+}$, as discussed in the lines below (3.4.58), in order to perform summations over $h \leq h^*$. Thus we obtain

$$|\partial_{v_F} \nu^*_{h^*,+}| \leq \frac{C|\lambda|}{v_F}.$$

To conclude, it is possible also to show (see Appendix B.3.2 for more details) that $|\partial_{\tilde{q}_F}\nu_{h^*,+}| \leq C|\lambda|2-h^* v_F$. ∎

Regularity and properties of q_F, v_F, Z_{h^*-1}

Properties Let λ small enough and $\tilde{q}_F(\lambda,\epsilon), v_F(\lambda,\epsilon), Z_{h^*-1}(\lambda,\epsilon)$ fixed as above: let us first discuss their properties.

First notice that $\tilde{q}_F(\lambda,\epsilon) \to 0$ as $\epsilon \to 0$. From the first line in (3.4.74), since 2^{h^*} and $r^*_{h^*}$ are $O(\epsilon)$, we have that $1-\cos\tilde{q}_F + \alpha^*_{h^*-1}\tilde{q}_F^2/2 = 0$ iff $\tilde{q}_F = 0$ (remember that $\alpha^*_{h^*-1} = O(\lambda)$). Then, for ϵ, λ small, the reverse triangular inequality applied to the first line of (3.4.74) gives

$$\tilde{q}_F^2 \le C\left(|1-\cos\tilde{q}_F| - \frac{\tilde{q}_F^2}{2}\alpha_{h^*-1}\right) \le C|r^*_{h^*-1} + 2^{h^*}(\nu^*_{h^*-1} + \nu^*_{h^*,+})|$$

for some $C>0$ independent of λ, ϵ, v_F, so that $\tilde{q}_F = O(\sqrt{\epsilon})(1+O(\lambda))$. To prove (3.3.111) note that at $\lambda = 0$ again (3.4.74) reduces to (3.3.67), and the same happens for the equation defining $q_{h^*,F}$[43]: showing that $\tilde{q}_F(0,\epsilon) = q_{h^*,F}(0,\epsilon) = q^+_\epsilon$.

Next, note that from the first line in (3.4.74) $Z_{h^*-1} = 1 + O(\lambda)$. Regarding v_F , to show that $v_F = O(\sqrt{\epsilon})$, recall that $|b^*_{h^*,+}| \le C|\lambda| v_F$[44]; thus for λ small enough

$$|v_F| \le |(1+O(\lambda))\sin\tilde{q}_F + \alpha^*_{h^*-1}\tilde{q}_F| \le C\sqrt{\epsilon}(1+O(\lambda)). \tag{3.4.81}$$

Regularity From the (IFT), the regularity of the functions $(\lambda,\epsilon) \mapsto \tilde{q}_F, v_F, Z_{h^*-1}$ is the same as that of the function $\underline{H}$ in (3.4.75). We now show that

- $(\lambda,\epsilon) \mapsto \tilde{q}_F, v_F, Z_{h^*-1}$ are analytic in λ for $|\lambda| \le \lambda_0$ and C^∞ in ϵ for $\epsilon \notin \mathcal{J}_2$, where $\mathcal{J}_2$ is the discontinuity set of the step function $h^*(\epsilon)$ (see Appendix B.3.1)

From the bounds for the beta function and the flow, it follows that $\underline{H}$ is analytic in λ for $|\lambda| \le \lambda_0$, λ_0 small enough. Let us discuss the regularity in the other variables: we start with ϵ. Then, if $\epsilon \notin \mathcal{J}_2$,

$$\partial^n_\epsilon \underline{H} = (\partial^n_\epsilon r^*_{h^*-1} + 2^{h^*}\partial^n_\epsilon \nu^*_{h^*-1}, \partial^n_\epsilon b^*_{h^*-1}, \partial^n_\epsilon a^*_{h^*-1}). \tag{3.4.82}$$

and similarly for the other derivatives, using (3.4.74). For n large we must control the effect of the derivatives on $a^*_{h^*-1}, \nu^*_{h^*,+}, b^*_{h^*,+}$, since $r^*_{h^*-1}$ depends analitically on ϵ. In this case, using the tree expansion as in (3.4.73), the derivatives can only act on a propagator of the first regime $g^{(k)}$ for some $k \ge h^*$, as explained below (3.4.73). Such a derivative produce dimensionally an extra factor 2^{-k} (see Lemma B.2.0.1): by summing over $k \ge h^*$ these contributions can be bounded with an extra $C\epsilon^{-1}$ wrt to the bound on $B^\nu_{h,+}$. By iteration, this implies that $\nu_{h^*,+}$ is C^∞ in $(0,+\infty)\setminus\mathcal{J}_2$.
A similar reasoning works for $\partial_{v_F}, \partial_{Z_{h^*-1}}$ and $\partial_{\tilde{q}_F}$. Let us discuss only the former: the derivative acts on propagators of the second regime $g^{(k)}$, $k < h^*$ and produce dimensionally an extra v_F^{-1} by Lemma B.2.0.1. Repeating the argument as in the proof of Assumption 3.4.0.5, we find that $\partial^n_{v_F}$ produce an extra v_F^{-n} to the bound of $\nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$.

[43] The equation is the same as the one of $\tilde{q}_F$, without the term proportional to 2^{h^*} and by replacing the scale label h^*-1 with h^*.

[44] $b^*_{h^*,+} = -\sum_{k\le h^*} B^b_k(\lambda,\underline{u})$ (see (3.4.57)) and $|B^b_k| \le v_F$ from (3.3.113)

3.5 Proof of Theorems 1.4.4- 1.4.5

3.5.1 Proof of Theorem 1.4.4

Let us first discuss the existence part, by starting with the infinite volume free energy $f_{\varepsilon,\lambda}$. Recall that $f_{\epsilon+\varepsilon_0,\lambda,L} = L^{-2}\log Z_{\epsilon+\varepsilon_0,\lambda\,L} = e^{\mathcal{W}_{\epsilon+\varepsilon_0,\lambda,L}(0)}$, as it follows from Definition 1.3.1. Then, thanks to the Grassmann representation of Proposition 3.2.1 and the inductive construction of Section 3.3.2 we are able to write (cf. (3.3.66) with (3.3.123))

$$\begin{aligned} e^{\mathcal{W}_{\epsilon+\varepsilon_0,L,\lambda}(\mathcal{A})} &= \sum_{\theta\in\{0,1\}^2} \frac{c_\theta}{2} e^{\mathcal{W}^{(\theta)}_{\epsilon+\varepsilon_0,L,\lambda}(\mathcal{A})} \\ e^{\mathcal{W}^{(\theta)}_{\epsilon+\varepsilon_0,L,\lambda}(\mathcal{A})} &= e^{L^2(F^{(h_L)}_{\epsilon,\lambda}+t^{(h_L)}_{\epsilon,\lambda})+S^{(h_L)}_{\epsilon,\lambda}(J)} \end{aligned} \tag{3.5.1}$$

where of course the r.h.s depends also on θ, L. Recall that h_L is the last scale before the integration procedure stops (cf. (3.3.46)) and $J_e = e^{A_e} - 1$ (cf. (3.3.71)). We can rewrite

$$F^{(h_L)}_{\epsilon,\lambda} = \sum_{k=h_L}^{-1} \tilde F^{(k)}_{\epsilon,\lambda} + F^{(0)}_{\epsilon,\lambda}, \quad t^{(h_L)}_{\epsilon,\lambda} = \sum_{k=h_L}^{-1} \tilde t^{(k)}_{\epsilon,\lambda} \tag{3.5.2}$$

where $F^{(0)}_{\epsilon,\lambda}$ is given in (3.3.68), while $\tilde F^{(k)}_{\epsilon,\lambda} = F^{(k)}_{\epsilon,\lambda} - F^{(k+1)}_{\epsilon,\lambda}\ \tilde t^{(k)}_{\epsilon,\lambda}$, are the single scale contributions to the free energy obtained from the action of the truncated expectation (see e.g. (3.3.95)) and from the change of Grassmann measure (c.f. (3.3.91)) respectively. Since $\tilde F^{(k)}_{\epsilon,\lambda}$ is nothing but the kernel associated with no external fermionic legs at scale k, it admits a tree expansion (cf. Section 3.4.3) and satisfies the bounds, uniform in L, of Lemma 3.4.0.1-3.4.0.2 thus of Proposition 3.4.1-3.4.2, with $n, m = 0$. As mentioned in Remark 3.3.10 we also know that it admits a natural $L \to \infty$ limit, which we denote by $F^{(k);\infty}_{\epsilon,\lambda}$[45]; of course $\tilde t^{(h)}_{\epsilon,\lambda}$ and $F^{(0)}_{\epsilon,\lambda}$ have an $L \to \infty$ limit too, as it follows from their explicit definition. We write $F^{(k)}_{\epsilon,\lambda} = F^{(k);\infty}_{\epsilon,\lambda} + \delta^{(\theta)} F^{(k)}$ and similarly for $\tilde t$, where $\delta^{(\theta)}$ represents the finite size contribution to the free energy. If we choose $\nu_0(\lambda), a_{h^*,+}(\lambda,\epsilon), b_{h^*,+}(\lambda,\epsilon), \nu_{h^*,+}(\lambda,\epsilon)$ uniquely as in Section 3.4.5[46], recalling that ε_0 depends on ν_0 via $\varepsilon_0 = \log(1+\nu_0)$, we can rewrite from (3.5.1)

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lim_{L\to\infty} f_{\epsilon+\varepsilon_0(\lambda),\lambda,L} = \sum_{h<0} (\tilde F^{(h);\infty}_{\epsilon,\lambda} + \tilde t^{(h);\infty}_{\epsilon,\lambda}) + F^{(0);\infty}_{\epsilon,\lambda}. \tag{3.5.3}$$

Let us discuss now the existence of the limit $L \to \infty$ of $\partial_{A_e} \mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(\mathcal{A})|_{A=0}$. Again, the multiscale analysis allows to write

$$\mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(\mathcal{A}) = \sum_{h_L\le h<0} \tilde S^{(h)}_{\epsilon,\lambda}(J) \tag{3.5.4}$$

where recall $e^{A_e} = J_e - 1$ for $e \in E_L$, and $\tilde S^{(h)}_{\epsilon,\lambda}(J) = S^{(h-1)}_{\epsilon,\lambda}(J) - S^{(h)}_{\epsilon,\lambda}(J)$ is the single scale contribution to the generating function, see (3.3.95). Recall that $\partial_{A_e} \mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(\mathcal{A})|_{\mathcal{A}=0}$

[45] Obtained by replacing in its tree expansion, every single scale propagator with its $L \to \infty$ limit, and every sum in $\tilde\Lambda$ with the counterpart in $R^T\mathbb{Z}^2$ (see Section 3.3.1 for the definitions).

[46] That is, we choose $\underline{u} = (\underline{u}^*_0, \underline{u}^*_1)$ as in section 3.4.4 as the unique solution of Proposition 3.4.4.

depends, by translation invariance of the model, only on the orientation of the edge e, namely on the label $s = A, B, C$ of the associated external fields J. If e is of type s, we have $\partial_{A_e} \mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(\mathcal{A})|_{\mathcal{A}=0} = \rho_{s,\epsilon+\varepsilon_0(\lambda),\lambda,L}$ where the densities are the same appearing in Theorem 1.4.4 and Corollary 1.4.5.1. On the other hand, using the multiscale construction we can rewrite, for e of type $s \in \{A, B, C\}$

$$\partial_{A_e} \mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(\mathcal{A})|_{\mathcal{A}=0} = \sum_{h_L \le h < 0} W^{(h)}_{0,1,s}(0) \tag{3.5.5}$$

where the kernels $W^{(h)}_{0,1,s}$ are the same appearing in (3.4.3) for $n = 0, m = 1.$ and without labels $\underline{D}$ which are absent since $n = 0$. In particular, by translation invariance, $W^{(h)}_{0,1,s}(0) \le \|W^{(h)}_{0,1}\|_{\kappa,h}$ where the bounds on this kernel norm are given in Proposition 3.4.1-3.4.2. Finally the existence of the $L \to \infty$ limit in the equation above is a consequence of two facts: (1) the kernels admit bounds uniform in L as a consequence of Propositions 3.4.1-3.4.2; (2) the kernels $W^{(h)}_{0,1,s}$ admits a well defined $L \to \infty$ limit $W^{(h);\infty}_{0,1,s}$ (exactly as used up to now for the kernel with $(n, 0)$ kernels, see Remark 3.3.10). This means that we can write[47]

$$\rho_{s,\epsilon+\varepsilon_0(\lambda)} = \sum_{h<0} W^{(h);\infty}_{0,1,s}(0). \tag{3.5.6}$$

Continuity in ϵ of the limit follows from absolute convergence of the expansion, which is uniform in ϵ, thanks to the double regime multiscale analysis.

We now prove the statement regarding the concentration of the dimers densities (1.4.9). Proving that the limit density of dimers of type A is 1 for $\epsilon < 0$ is equivalent to proving that the density of dimers of type B, C is zero. For any fixed edge,

$$\mathbb{P}_{\lambda,\epsilon+\varepsilon_0(\lambda)}(e \in M) = \lim_{L\to\infty} \partial_{A_e} \mathcal{W}_{\epsilon+\varepsilon_0(\lambda),\lambda,L}(A)|_{A=0}. \tag{3.5.7}$$

Thanks to the previous discussion, the limit exists and can be expanded as

$$\mathbb{P}_{\epsilon+\varepsilon_0(\lambda),\lambda}(e \in M) = \sum_{n\ge 0} \frac{\lambda^n}{n!} c_n(\epsilon) \tag{3.5.8}$$

where

$$\begin{aligned} c_n(\epsilon) &= \lim_{L\to\infty} \partial_{A_e} \partial^n_\lambda \log \mathbb{E}_{\epsilon,0,L}[e^{B_\lambda(M) + \sum_{e\in E_L} A_e \mathbb{1}_e}]_{\underline{A},\lambda=0} \\ B_\lambda(M) &:= \lambda V(M) + \varepsilon_0(\lambda)(N_B(M) + N_C(M)). \end{aligned} \tag{3.5.9}$$

To evaluate c_n in (3.5.8), one can first expand in (λ, ε_0) and then expand $\varepsilon_0(\lambda)$, which is

[47] The reader can look at Sec. 6.5 of [**2**] for more details about finite size effects and higher order correlation functions.

analytic, in powers of λ. Namely, c_n is the coefficient of λ^n in T given by

$$T := \sum_{s,t\geq 0} \frac{\lambda^s \varepsilon_0^t(\lambda)}{s!t!} d_{s,t}(\epsilon), \qquad \varepsilon_0(\lambda) = \sum_{k\geq 1} \frac{\lambda^k}{k!} \varepsilon_0^{(k)}(0) \tag{3.5.10}$$

where

$$d_{s,t}(\epsilon) = \lim_{L\to\infty} \mathbb{E}_{\epsilon,0,L}[\underbrace{V;\cdots;V}_{s\,\text{times}};\underbrace{N_B+N_C;\cdots;N_B+N_C}_{t\,\text{times}};\mathbb{1}_e] \tag{3.5.11}$$

is the joint cumulant, with respect to $\mathbb{P}_{\epsilon,0,L}$, of its $s+t+1$ arguments. Let us now show that for every s,t,

$$d_{s,t}(\epsilon) = 0. \tag{3.5.12}$$

which will imply that $c_n(\epsilon) = 0$. Let us prove the more general result:

Lemma 3.5.0.1. *For $i = 1,..,m$, let $S_i \subset E_L$ be such that $\cup_{i=1}^m S_i$ contains at least one edge $\bar{e}$ of type B or C. Then*

$$\left|\mathbb{E}_{\epsilon,0,L}\Big[\mathbb{1}_{S_1};\cdots;\mathbb{1}_{S_m}\Big]\right| \leq c(m)e^{-\gamma L}, \qquad \mathbb{1}_S := \prod_{e\in S} \mathbb{1}_e \tag{3.5.13}$$

for some $c(m), \gamma > 0$.

Proof of Lemma 3. Expanding the truncated correlation in simple expectations, we find that

$$\mathbb{E}_{\epsilon,0,L}\Big[\mathbb{1}_{S_1};\cdots;\mathbb{1}_{S_m}\Big] = \sum_{\pi} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{Y\in\pi} \mathbb{E}_{\epsilon,0,L}[\mathbb{1}_Y] \tag{3.5.14}$$

where the sum runs over partitions π of $\{1,\dots,m\}$ in sets $Y_1,\dots Y_{|\pi|}$ and $\mathbb{1}_Y = \prod_{i\in Y} \mathbb{1}_{S_i}$[48]. Note that in every π, at least one Y_j in contains the label $k \in \{1,\dots,m\}$ of the set S_k that contains the edge $\bar{e}$ not of type A. Thus we can bound (3.5.14) by $c(m)\mathbb{P}_{\epsilon,0,L}(\bar{e} \in M) \leq c(m)e^{-\gamma L}$ for some $c(m) > 0$. The exponential decay follows from (3.1.8), that can be applied without difficulty since for $\epsilon < 0$, the function μ has no zeros on $[-\pi,\pi]^2$ and the determinants of K_θ are non-zero. Then, if e.g. $\bar{e} = (\bar{w},\bar{b})$ is of type B,

$$\mathbb{P}_{\epsilon,0,L}(\bar{e} \in M) \leq B \sup_\theta |K_\theta^{-1}(\bar{w},\bar{b})| \tag{3.5.15}$$

and the desired exponential decay follows from (B.3.24) and (B.3.25), because the difference of coordinates of $\bar{w}$ and $\bar{b}$ is $(-1,0)$, so that $K^{-1}((-1,0)) = 0$ (similarly $(0,-1)$ for $\bar{e}$ of type C). ∎

To conclude, recall that V is the sum over faces f of $\mathbb{1}_{f,A} + \mathbb{1}_{f,B} + \mathbb{1}_{f,C}$ and note that $\mathbb{1}_{f,A}$ equals 1 minus the indicator function that at least one among the two bottom vertices or the two top vertices of the face f is contained in a dimer of type B or C. Therefore, $\mathbb{E}_{\epsilon,0,L}[\underbrace{V;\cdots;V}_{s\,\text{times}};\underbrace{N_B+N_C;\cdots;N_B+N_C}_{t\,\text{times}};\mathbb{1}_e]$ can be written, by the multilinearity of the cumulant, as a linear combination of terms each having an expression as in the LHS of (3.5.14), with at least one of the S_i containing an edge of type B or C. The number

[48] Of course each set Y_i depends on π but we understand the label to keep the notation lighter.

of terms in this linear combination is $O(L^{2(t+s)})$. In particular, by Lemma 3.5.0.1 the cumulant can be bounded by a constant times $L^{2(t+s)}e^{-\gamma L}$ and (3.5.12) follows.

3.5.2 Proof of Theorem 1.4.5

Recall that we want to prove two facts about $f_{\epsilon+\varepsilon_0(\lambda)}$ (which exists thanks to Theorem 1.4.4): namely that

(1) *(Pokrovsky-Talapov law):* as $\epsilon \to 0$,

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lambda + \mathbb{1}_{\{\epsilon\geq 0\}} c(\lambda)\epsilon^{\frac{3}{2}}(1+O(\epsilon^{\theta'}))$$

for an analytic function $c(\lambda) = \frac{4\sqrt{2}}{3\pi} + O(\lambda)$ and $\theta' \in (0, 1/2)$.

(2) $\epsilon \mapsto f_{\epsilon+\varepsilon_0(\lambda),\lambda}$ is C^∞ in $(-\infty, 0) \cup (0, \bar{\epsilon})$

Note that these two points are proven, when the case $\lambda = 0$, in Theorem 3.1.1. Let us start with item **(1)**. We dinstinguish the case $\epsilon \leq 0$ from $\epsilon > 0$.

Proof of (1): $\epsilon \leq 0$

In this case, point **(1)** follows directly from Theorem 1.4.4 without any use of the multiscale expansion.

Lemma 3.5.0.2. *For $\epsilon \leq 0$, λ as in Theorem 1.4.4, we have that $f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lambda$.*

Proof. Note that the derivative with respect to ϵ of the free energy is the density of dimers of type B plus the density of dimers of type C. We have seen from the previous Section that this is zero in the thermodynamic limit, so the infinite-volume free energy is constant for $\epsilon < 0$. We want to show that this constant is just λ. Restricting the partition sum to the single configuration containing only type-A dimers, one has $f_{\epsilon+\varepsilon_0(\lambda),\lambda} \geq \lambda$. As for the opposite bound, take $\delta > 0$ small. The partition function restricted to configurations where all except at most $L^2\delta$ dimers are of type A is at most $e^{(\lambda+R_\delta)L^2}$ with R_δ tending to zero as $\delta \to 0$, where the term $\exp(R_\delta L^2)$ is a bound on the number of configurations satisfying this constraint. As for the configuration with at least $L^2\delta$ dimers *not* of type A, they contribute at most $2^{L^2}e^{L^2(\lambda-\epsilon\delta)} \leq 1$ to the partition function, where for the last bound we need $|\epsilon|$ to be large enough, as a function of δ. Taking $L \to \infty$ first and $\delta \to$ after wards, and recalling that the free energy is constant as a function of ϵ, the claim follows for $\epsilon < 0$. Since $f_{\epsilon+\varepsilon_0(\lambda),\lambda}$ is continuous, being convex, the statement follows. ∎

We now move to the proof of item **(1)** by means of the multiscale expansion.

Proof of (1): $\epsilon > 0$

Recall that we want to study $f_{\epsilon+\varepsilon_0(\lambda),\lambda} - f_{\lambda,\varepsilon_0(\lambda)}$, where, as it follows from Section 3.5.1 we can write

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = F^{(0);\infty}_{\epsilon,\lambda} + \sum_{h<0}(\tilde{F}^{(h);\infty}_{\epsilon,\lambda} + \tilde{t}^{(h);\infty}_{\epsilon,\lambda}). \tag{3.5.16}$$

We introduce for later convenience

$$\Delta F^{(h);\infty}_{\epsilon,\lambda} := \tilde{F}^{(h);\infty}_{\epsilon,\lambda} - \tilde{F}^{(h);\infty}_{0,\lambda} \tag{3.5.17}$$

and similarly for $\Delta t^{(h);\infty}_{\epsilon,\lambda}$, $\Delta F^{(0);\infty}_{\epsilon,\lambda}$. We can of course rewrite

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} - f_{\varepsilon_0(\lambda),\lambda} = \Delta F^{(0);\infty}_{\epsilon,\lambda} + \sum_{h<0} (\Delta F^{(h);\infty}_{\epsilon,\lambda} + \Delta t^{(h);\infty}_{\epsilon,\lambda}). \tag{3.5.18}$$

We now study separately the contributions coming from different Δ's. We start with $\Delta F^{(h);\infty}_{\epsilon,\lambda}$ and we postpone the discussion of the other two, which is simpler, to the end of this part. Using the fact that for $\epsilon > 0$ the multiscale integration provides estimates which are different, depending in which of the two regimes we are, we split

$$\sum_{h<0} \Delta F^{(0);\infty}_{\epsilon,\lambda} = \underbrace{\sum_{h^*\le h<0} \Delta F^{(0);\infty}_{\epsilon,\lambda}}_{\text{(I)}} + \underbrace{\sum_{h<h^*} \Delta F^{(0);\infty}_{\epsilon,\lambda}}_{\text{(II)}}. \tag{3.5.19}$$

The contribution (II)**:** From Proposition 3.4.1-3.4.2 we know that there exists a constant $c > 0$, (that will change from line to line) independent of λ, h, ϵ, such that

$$\begin{aligned} |\tilde F^{(h);\infty}_{0,\lambda}| &\le c|\lambda| 2^{\left(\frac{3}{2}+\theta\right)h} \qquad h \le 0 \\ |\tilde F^{(h);\infty}_{\epsilon,\lambda}| &\le c|\lambda| v_F^{-1} 2^{2h} \qquad h \le h^* \end{aligned} \tag{3.5.20}$$

so that

$$\sum_{h<h^*} |\Delta^{(h)}_{\epsilon,\lambda}| \le c|\lambda| \sum_{h<h^*} \left(2^{(\frac{3}{2}+\theta)h} + 2^{2h} 2^{-\frac{h^*}{2}} \right) \le c|\lambda| \epsilon^{\frac{3}{2}} (1 + O(\epsilon^\theta)) \tag{3.5.21}$$

where we used that $v_F^2 = \Theta(2^{h^*}) = \Theta(\epsilon)$.

The contribution (I) For the remaining term in (3.5.18) we need to exploit some cancellation between the critical and off critical theories: indeed since now the sum runs over $h^* \le h < 0$ the same reasoning as above would unnecessarily produce an $O(1)$ quantity in ϵ. Notice that for $h \ge h^*$, the dimensional bound for the free energy reads (again Proposition 3.4.1),

$$|\tilde F^{(h);\infty}_{\epsilon,\lambda}| \le c|\lambda| 2^{(\frac{3}{2}+\theta)h} \tag{3.5.22}$$

uniformly in ϵ small, thus in h^*. Using the tree expansion (Section 3.4.3), and as mentioned in Remark 3.3.7 we view $\tilde F^{(h);\infty}_{\epsilon,\lambda}$ as functions of the sequence $\{v_h, r_h, \alpha_h, \zeta_h\}$ and of the infinite volume single scale propagators $g^{(h)}$ (see (B.2.1))[49], which satisfy Lemma 3.1.1. In particular, the dependence on ϵ is only given by such propagators through $\mathbf{r}_h(\underline{r})$ as explained in Remark 3.3.7. Recall also that $\underline{r}$ satisfies the properties of Lemma 3.3.0.3. In this spirit, we write

$$\begin{aligned} \tilde F^{(h);\infty}_{\lambda}(\underline{r}) &= \tilde F^{(h);\infty}_{\epsilon,\lambda} \\ \Delta^{(h);\infty}_{\lambda}(\underline{r}) &= \Delta F^{(h);\infty}_{\epsilon,\lambda}. \end{aligned} \tag{3.5.23}$$

[49] We denoted with $g^{(h)} = \lim_{L\to\infty} g^{(h)}_L$.

So we Taylor expand $\Delta_\lambda^{(h);\infty}(\underline{r})$ in the sequence $\{r_k\}_{k\geq h}$, and obtain the expression

$$\Delta_\lambda^{(h);\infty}(\underline{r}) = \sum_{\ell>h} r_\ell \partial_{r_\ell} \tilde{F}_\lambda^{(h);\infty}(\underline{0}) + \sum_{\ell,m\geq h} r_\ell r_m \int_0^1 dt(1-t)\partial^2_{r_\ell r_m} \tilde{F}_\lambda^{(h);\infty}(t\underline{r}) \tag{3.5.24}$$

Now, in the tree expansion for $\tilde{F}^{(h);\infty}$, a derivative ∂_{r_ℓ} acts on some propagator $g^{(j)}$ with $h<j<\ell$ and it can be bounded dimensionally by $2^{-\ell}$. Indeed a derivative $\partial_{r_\ell} g^{(j)}$ acts as $\partial_{\mathbf{r}_h} g^{(j)}$, which gives a 2^{-j} (cf. Lemma B.2.0.1), times a dimensional factor $2^{(1+\theta)j-k}$ coming from $\partial_{r_\ell}\mathbf{r}_j$. Then sum over j produce the claimed factor: see a similar discussion in (3.4.19).

Thus, the first term in (3.5.24), summed over $h\geq h^*$ can be bounded by

$$\sum_{h^*\leq h<0}\sum_{\ell\geq h} |r_\ell \partial_{r_\ell}\tilde{F}_\lambda^{(h);\infty}(\underline{0})| \leq C|\lambda| \sum_{h^*\leq h\leq 0} 2^{(\frac{3}{2}+\theta)h} \sum_{\ell\geq h} \frac{|r_\ell|}{2^\ell} \leq C'|\lambda|\epsilon \tag{3.5.25}$$

where we used Lemma 3.3.0.3 and $2^{h^*}=\Theta(\epsilon)$. On the other hand, since r_h is exactly linear in r_0, (3.3.100), one finds that

$$\begin{aligned} &\sum_{h^*\leq h\leq 0}\sum_{\ell\geq h} r_\ell \partial_{r_\ell}\tilde{F}_\lambda^{(h);\infty}(\underline{0}) = I(\lambda) r_0 + O(\epsilon^{3/2+\theta}) = I(\lambda)e^{-\varepsilon_0(\lambda)}\epsilon + O(\epsilon^{3/2+\theta}) \\ &I(\lambda) := \sum_{h\leq 0}\sum_{\ell\geq h} Q_\ell(\lambda)\partial_{r_\ell}\tilde{F}_\lambda^{(h);\infty}(\underline{0}) \end{aligned} \tag{3.5.26}$$

where $Q_\ell(\lambda)$ is defined by $r_\ell = Q_\ell(\lambda) r_0$ (3.3.100). To obtain the factor $O(\varepsilon^{3/2+\theta})$ in the middle equation in the first line above, we used that $|Q_\ell| = 1+O(\lambda)$ and that $|\partial_{r_\ell}\tilde{F}_\lambda^{(h);\infty}(\underline{0})| \leq C 2^{h(3/2+\theta)-\ell}$. In the last equation instead, we used the definition of $r_0 = e^{-\varepsilon_0(\lambda)}(e^{-\epsilon}-1)$. Note that $I(\lambda)$ is associated to a linear contribution in ϵ that is not present in (1.4.10). We shortly show that it cancels out with other linear contributions, by using the knowledge on the case $\epsilon\leq 0$ of the previous paragraph. Note also that $I(\lambda)$ depends only on the critical theory, i.e. it does not depend on ϵ.

Let us show now that the contribution coming from the remainder term in (3.5.24) is $O(\epsilon^{\frac{3}{2}+\theta})$. Recalling that $|\partial^2_{r_\ell r_m}\tilde{F}_\lambda^{(h);\infty}(t\underline{r})| \leq c|\lambda| 2^{-\ell-m} 2^{(\frac{3}{2}+\theta)h}$ uniformly in t, we can bound the Taylor remainder in (3.5.24) as

$$\begin{aligned} &\left| \int_0^1 dt(1-t) \sum_{h\geq h^*}\sum_{\ell,m\geq h} r_\ell r_m \partial^2_{r_\ell r_m}\tilde{F}_\lambda^{(h);\infty}(t\underline{r}) \right| \leq \\ &\leq c|\lambda| \sum_{h\geq h^*} 2^{(\frac{3}{2}+\theta)h} \left(\sum_{\ell\geq h} |r_\ell| 2^{-\ell} \right)^2 \leq c|\lambda| 2^{2h^*} \sum_{h\geq h^*} 2^{(-\frac{1}{2}+\theta)h} \leq |\lambda| O(\epsilon^{\frac{3}{2}+\theta}) \end{aligned} \tag{3.5.27}$$

where we used that $|r_h| 2^{-h} \leq C 2^{h^*-h}$ as it follows from Lemma 3.3.0.3.

Note that up to now, a term of exactly order $\epsilon^{3/2}$ may only come from the contribution

$$\sum_{h<h^*} \tilde{F}_{\epsilon,\lambda}^{(h);\infty} \tag{3.5.28}$$

which contributes to the function $c(\lambda)$ appearing in the statement but which we do not

derive here in an explicit form.

We now discuss the contributions to (3.5.18) coming from the scale $h = 0$ of the free energy and from the change of Grassmann integration: let us discuss the former first.

The $h = 0$ contribution of the free energy

From the definition of (3.3.68), it is easy to see that $F^{(0);\infty}_{\epsilon,\lambda}$ is a shifted version of the infinite volume free energy of a non-interacting dimer model with edge weights modified by the interaction. More precisely

$$F^{(0):\infty}_{\epsilon,\lambda} = \epsilon + \varepsilon_0(\lambda) + f\left(1 + r_0, \frac{1}{2}, \frac{1}{2}\right), \qquad r_0 = e^{-\varepsilon_0(\lambda)}(e^{-\epsilon} - 1) \tag{3.5.29}$$

where f is the free energy of the non-interacting model given in (3.1.10). By a rescaling argument we rewrite $f(1+r_0, 1/2, 1/2) = \log(1+r_0) + f(1, 1/2(1+r_0), 1/2(1+r_0))$. Then, defining

$$e^{\beta} := 1 + r_0, \quad \beta \geq 0 \iff \epsilon \geq 0$$

we can apply Theorem 3.1.1 with $t = 1/2$ and find that

$$F^{(0);\infty}_{\epsilon,\lambda} = \varepsilon_0(\lambda) + \epsilon(1 - e^{-\varepsilon_0(\lambda)}) + \mathbb{1}_{\{\epsilon\geq 0\}} \frac{4\sqrt{2}}{3\pi} (e^{-\varepsilon_0(\lambda)}\epsilon)^{3/2}(1 + O(\epsilon))$$

where we used that $\beta = e^{-\varepsilon_0(\lambda)}\epsilon(1 + O(\epsilon))$. This implies that

$$\Delta F^{(0);\infty}_{\epsilon,\lambda} = \epsilon(1 - e^{-\varepsilon_0(\lambda)}) + \mathbb{1}_{\{\epsilon\geq 0\}} \frac{4\sqrt{2}}{3\pi} (e^{-\varepsilon_0(\lambda)}\epsilon)^{3/2}(1 + O(\epsilon)). \tag{3.5.30}$$

The change of measure contribution

Recall (3.5.2) and that $h \geq h^*$

$$\tilde{t}^{(h);\infty}_{\epsilon,\lambda} = \frac{1}{\pi^2} \int dq \log\left(1 + \frac{\chi_{h-1}(q)(iq_1\zeta_{h-1} + q_2^2\alpha_{h-1})}{r_h + D_h(q) + \rho(q)}\right) \tag{3.5.31}$$

while for $h < h^*$

$$\tilde{t}^{(h);\infty}_{\epsilon,\lambda} = \sum_{\omega} \frac{1}{\pi^2} \int dq \log\left(1 + \frac{z_h \tilde{\chi}_h(q) D_{h^*-1,\omega}(q)}{D_{h^*-1,\omega}(q) + \rho_{h^*-1}/Z_h}\right). \tag{3.5.32}$$

Repeating exactly the same discussion as in ((C.8) and lines below, [**2**]) we have

$$|\tilde{t}^{(h);\infty}_{\epsilon,\lambda}| \leq c|\lambda| \begin{cases} 2^{\frac{3}{2}h} & h \geq h^* - 1 \\ v_F^{-1} 2^{2h} & h < h^* - 1 \end{cases} \tag{3.5.33}$$

where the only difference is the support of the cutoff functions χ_h: for $h < h^*$ its size is $O(2^{2h}v_F^{-1})$, while for $h \leq h^*$, we have that $|\operatorname{supp}\chi_h| = O(2^{\frac{3}{2}h})$. Then we can write the contribution of $\Delta t^{(h);\infty}_{\epsilon,\lambda}$ in (3.5.18) by splitting the sum in the two regimes of scales,

$h < h^*$ and $h \geq h^*$ respectively, so that

$$\sum_{h<0} |\Delta t^{(h);\infty}_{\epsilon,\lambda}| \leq c|\lambda| \left(\epsilon^{3/2} + \sum_{h\geq h^*} |\Delta t^{(h);\infty}_{\epsilon,\lambda}| \right) \leq c|\lambda|\epsilon(1+\epsilon^{1/2}) \tag{3.5.34}$$

where in the first step we used the triangular inequality and (3.5.33); for the second inequality above we used (3.5.31) to rewrite $\Delta t^{(h);\infty}_{\epsilon,\lambda}$ and that $(1 + A(B + C)^{-1})(1 + AC^{-1})^{-1} = (1 + B(C + A)^{-1})(1 - B(B + C)^{-1})$ with $B = r_h = O(\epsilon)$. Note that we can then write

$$\Delta t^{(h);\infty}_{\epsilon,\lambda} = T(\lambda)\epsilon + O(\lambda\epsilon^{3/2}) \tag{3.5.35}$$

for some analytic function $T(\lambda) = O(\lambda)$, implicitly defined[50].

Completion of the proof of (1)

Let us collect what we proved for the item **(1)** of the beginning of this Section. For $\epsilon \leq 0$, the statement is proven, while for $\epsilon > 0$ we found

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \lambda + \epsilon \left(1 + T(\lambda) + (I(\lambda) - 1)e^{-\varepsilon_0(\lambda)}\right) + c(\lambda)\epsilon^{3/2}(1 + O(\epsilon^\theta)) \tag{3.5.36}$$

where $T(\lambda), I(\lambda)$ are analytic functions collected only from the *first regime* of scales $h \geq h^*$, respectively from the change of measures contribution $\Delta t^{(h);\infty}_{\epsilon,\lambda}$ (see few line above) and from $\Delta F^{(h);\infty}_{\epsilon,\lambda}$ in (3.5.27); $\epsilon(1 - e^{-\varepsilon_0(\lambda)})$ comes from the first scale contribution $\Delta F^{(0);\infty}_{\epsilon,\lambda}$ in (3.5.30). Instead, $c(\lambda) = \frac{4\sqrt{2}}{3\pi} e^{-\frac{3}{2}\varepsilon_0(\lambda)} + O(\lambda)$ is analytic and the error term $O(\lambda)$ comes from the change of measure contribution and from the *second regime* of scales involving $\Delta F^{(h);\infty}_{\epsilon,\lambda}$, see (3.5.21)-(3.5.28). Note that, correctly, $c(0) = \frac{4\sqrt{2}}{3\pi}$. We now want to show that the coefficient of the linear term in ϵ is indeed 0, to complete the proof of Theorem 1.4.5.

The idea is that such function of λ is the same coefficient of the linear part in ϵ appearing in the multiscale analysis performed for $\epsilon < 0$: this, a posteriori, thanks to Lemma 3.5.0.1, is zero. More precisely:

As explained in Section 3.3.2, when $\epsilon \to 0^-$, as in our case, we have only a first regime integration: once we reach the transition scale h^*, we can integrate the scales $h < h^*$ at once with a propagator that has the same scaling of the single one, namely as given in Lemma B.2.0.1, (B.2.6). In this situation the free energy is given by

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = \sum_{h\geq h^*} (\tilde{F}^{(h);\infty}_{\epsilon,\lambda} + \tilde{t}^{(h);\infty}_{\epsilon,\lambda}) + F^{(0);\infty}_{\epsilon,\lambda} + F^{(<h^*);\infty}_{\epsilon,\lambda}. \tag{3.5.37}$$

where $F^{(0);\infty}_{\epsilon,\lambda}$ is the same as in the previous section. In virtue of the discussion above and of Section 3.3.2, Proposition 3.4.1 implies that $|F^{(h);\infty}_{\epsilon,\lambda}| \leq c|\lambda|2^{h(3/2+\theta)}$ and the same bound is satisfied by $F^{(\leq h^*);\infty}_{\epsilon,\lambda}$ with h replaced by h^*.

Using the same definitions in (3.5.17), we extract from $f_{\epsilon+\varepsilon_0(\lambda)\lambda} - f_{\varepsilon_0(\lambda),\lambda}$, the linear

[50] The dependence on λ is given by ζ_h, α_h, r_h which are analytic of λ for $|\lambda| \leq \lambda_0$

term in ϵ. From the just mentioned bounds for $F_{\epsilon,\lambda}^{(<h^*);\infty}$, we have that

$$\Delta F_{\epsilon,\lambda}^{(\leq h^*);\infty} = O(\epsilon^{3/2+\theta}), \tag{3.5.38}$$

which means that the linear term in ϵ comes from the first regime of scales. By repeating exactly the same arguments above, for $\Delta F_{\epsilon,\lambda}^{(h);\infty}, \Delta t_{\epsilon,\lambda}^{(h);\infty}, \Delta$, for $h > h^*$ and $F_{\epsilon,\lambda}^{(0);\infty}$, we obtain that

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} - f_{\varepsilon_0(\lambda),\lambda} = (1 - e^{-\varepsilon_0(\lambda)} + T(\lambda) + I(\lambda)e^{-\varepsilon_0(\lambda)})\epsilon + O(\epsilon^{3/2}), \quad \epsilon \to 0^- \tag{3.5.39}$$

where $T(\lambda), I(\lambda)$ that are the same functions appearing in the $\epsilon > 0$ case. Then we apply Lemma 3.5.0.1 to find that

$$1 - e^{-\varepsilon_0(\lambda)} + T(\lambda) + I(\lambda)e^{-\varepsilon_0(\lambda)} = 0. \tag{3.5.40}$$

Note that the same reasoning does not show that $c(\lambda) = c(0)$: when $\epsilon > 0$ there are a priori terms coming from the second regime of scales contributing to it (see (3.5.28)).

We are left now with item **(2)**.

Proof of (2)

We want to show that $f_{\epsilon+\varepsilon_0(\lambda),\lambda}$ is indeed C^∞ in $\epsilon \in (-\infty, 0) \cup (0, \bar{\epsilon})$ as in the claim of Theorem 1.4.5. Recall that $\bar{\epsilon}$ is an arbitrary small parameter, chosen in order to deal with the asymptotics of the free energy but it is not strictly necessary for its regularity properties.

The statement is trivially true for $\epsilon < 0$, since $f_{\epsilon+\varepsilon,\lambda} = \lambda$. For $\epsilon \in (0, \bar{\epsilon})$ fixed, we use the multiscale structure. From the previous analysis we know that we can write

$$f_{\epsilon+\varepsilon_0(\lambda),\lambda} = F_{\epsilon,\lambda}^{(0);\infty} + \sum_{h<0} \tilde{F}_{\epsilon,\lambda}^{\infty;(h)} + \tilde{t}_{\epsilon,\lambda}^{\infty;(h)} \tag{3.5.41}$$

where $F_{\epsilon,\lambda}^{\infty;(0)}, F_{\epsilon,\lambda}^{\infty;(h)}$, $\tilde{t}_{\epsilon,\lambda}^{\infty;(h)}$ are the same scale contributions appearing in the proof of item **(1)**. In particular, from its expression $F_{\epsilon,\lambda}^{\infty;(0)}$ it is easily seen to be smooth (3.5.29). For the other contributions, there is a point to be stressed: $h^*(\epsilon)$ does depend on ϵ and it is a step right continuous function with discontinuity set $\mathcal{J}_2$ accumulating at $\epsilon = 0$ (see Appendix B.3.1). So let $\epsilon \in (0, \bar{\epsilon}) \cap \mathcal{J}_2^c$. In this set the dependence it is only given through the single scale contributions in the equation above. In particular from its explicit expression, the contribution associated to $\tilde{t}_{\epsilon,\lambda}^{(h);\infty}$ is seen to be $C^\infty((0, \bar{\epsilon}) \cap \mathcal{J}_2^c)$.

Let us focus on the more interesting term $\Omega_{\epsilon,\lambda} = \sum_{h<0} \tilde{F}_{\epsilon,\lambda}^{(h);\infty}$. Recall that each $\tilde{F}_{\epsilon,\lambda}^{(h);\infty}$ admits a tree expansion, which is absolutely convergent for λ small, uniformly in ϵ in our set. In particular since $\Omega_{\epsilon,\lambda}$ is given by an absolute convergent series, uniformly in λ, ϵ small we study its regularity in ϵ by means of the regularity of each tree in the expansion. We can write

$$\Omega_{\epsilon,\lambda} = \sum_{h<0} \sum_{N\geq 1} \sum_{\tau \in \mathcal{T}_{N,0}^{(h)}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{T \in \mathbf{T}} F_\epsilon^{(h);\infty}(\lambda, \tau, \mathbf{P}, T) \tag{3.5.42}$$

using the notations of Section 3.4.3. Recall that, depending on whether $h \geq h^*$ or not, the dependence on ϵ of $F_\epsilon^{(h);\infty}(\lambda, \tau, \mathbf{P}, T)$ has a different nature (see Sections 3.4.3-3.4.3).

- For $h \geq h^*$, $F_\epsilon^{(h);\infty}(\lambda, \tau, \mathbf{P}, T)$ depends on ϵ only through the sequence $\underline{r} = (r_h)_{h \geq h^*}$ (see (3.3.100)), and the dependence of r_h on ϵ is analytic (3.3.66))
- For $h < h^*$ the dependence on ϵ is instead given through $v_F(\lambda, \epsilon), \tilde{q}_F(\lambda, \epsilon), Z_{h^*-1}$, and $\mathcal{R}V^{h^*-1}$, where the former functions are the same as in (3.4.78), which are $C^\infty((0, \bar{\epsilon}) \cap \mathcal{J}_2^c)$, while the latter arise from trees which have an irrelevant endpoint on scale $h^* - 1$.

We want to show that the effect of a derivative ∂_ϵ on Ω evaluated at $\tilde{\epsilon} \in (0, \bar{\epsilon}) \cap \mathcal{J}_2^c$ can be bounded by $\tilde{\epsilon}^{-1}$.

If the derivative applies to $F_\epsilon^{(h);\infty}(\lambda, \tau, \mathbf{P}, T)$, it acts on some propagator on scale $k > h$ of the first regime. We already discussed below (3.5.24) that its effect is to produce dimensionally an extra factor 2^{-k} to the bound of the tree. The derivative of r_h in ϵ contributes instead as $O(1)$. Summing over $k \geq h$ and $h \geq h^*$ we obtain indeed an extra factor $2^{-h^*} = O(\epsilon^{-1})$ with respect to the bound without derivative.

If the derivative instead acts on $F_\epsilon^{(h);\infty}(\lambda, \tau, \mathbf{P}, T)$ for $h < h^*$ then we have that the desired bound follows from two facts: (1) we already showed in 3.4.5 that the tree expansion of the second regime is C^∞ in $(v_F, \tilde{q}_F, Z)$ away from $(0, 0, Z)$. In particular, e.g., a derivative in v_F produce an extra v_F^{-1}; (2) the functions $v_F, \tilde{q}_F, Z$ are C^∞ in $\epsilon \in (0, \bar{\epsilon}) \cap \mathcal{J}_2^c$. For instance, since $v_F = \Theta(\sqrt{\epsilon})$, when a derivative in ϵ acts on v_F in the tree expansion, we get the desired extra ϵ^{-1}. [51] By iterating the strategy one finds that ∂_ϵ^n produce an extra factor $C(n)\epsilon^{-n}$ with respect to the bound without derivatives. This implies that $\Omega_{\epsilon,\lambda} \in C^\infty(0, \bar{\epsilon}) \cap \mathcal{J}_2^c$.

In order to recover smoothness on the whole $(0, \bar{\epsilon})$, we reason as follows. The way the multiscale analysis was carried along has several degrees of freedom. In particular we could decide to iteratively cutoff momenta by setting, for any $\gamma > 2$,

$$\chi_h(q) = \chi_{h+1}(\gamma q) \tag{3.5.43}$$

instead of $\gamma = 2$ as we did. This means that every dimensional bound changes, as given in Propositions 3.4.1-3.4.2, by replacing $2^h \mapsto \gamma^h$, as a consequence of the different scaling of the propagator (e.g. in the first regime we would have that $|g^{(h)}| \leq C\gamma^{h/2}$ etc.). Anyway the double regime multiscale analysis persists and in particular the scale $h^* = h^*_\gamma(\epsilon)$, for λ sufficiently small, has discontinuity set given by $\mathcal{J}_\gamma = \{\epsilon > 0 : \gamma^h = |\epsilon|\}$ (see Appendix B.3.1). Since, e.g. $\mathcal{J}_3 \cap \mathcal{J}_2 = \emptyset$, this means that in order to study the regularity of $f_{\epsilon+\varepsilon_0(\lambda),\lambda}$, by repeating the multiscale analysis with $\gamma = 3$ and using the same considerations above, we can indeed recover smoothness also on the set $\mathcal{J}_2 \cap (0, \bar{\epsilon}) \subset (0, \bar{\epsilon}) \cap \mathcal{J}_3$. □

[51] The other cases are treated similarly; Anyway the worst case of the effect of ∂_ϵ, as $\epsilon \to 0$, remains a bound of the type ϵ^{-1}.

APPENDIX A

A.1 An explicit example of non-planar dimer model

Here, we work out the Grassmann potential V for the easiest but non-trivial example of non-planar dimer model. Choose $m = 4$ for the cell size and let the edge weights be invariant by translations by multiples of m, so that V_x in Proposition 3.1.1 does not depend on x. In this example we add just one non planar edge per cell, denoted by e_λ, connecting the leftmost black site in the second row to the rightmost white in the same row; it crosses two vertical edges, denoted by e_1, e_2, see Figure A.1. Let $\psi(e_\lambda), \psi(e_1), \psi(e_2)$ be

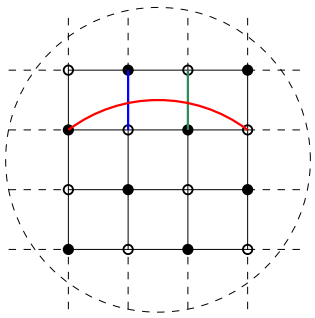

Figure A.1: *A 4×4 cell with the edges e_λ, e_1, e_2 colored in red, blue, green, respectively.*

the Grassmann monomials defined in (2.1.28) (we drop the index θ). From the definition (2.1.35), one can check that the potential satisfies $V(\psi) = F(\psi)$ and that it is given by

$$V(\psi) = \varepsilon_{\emptyset}^{\{e_\lambda\}}\psi(e_\lambda) + \varepsilon_{\{e_1\}}^{\{e_\lambda\}}\psi(e_\lambda)\psi(e_1) + \varepsilon_{\{e_2\}}^{\{e_\lambda\}}\psi(e_\lambda)\psi(e_2) + \varepsilon_{\{e_1,e_2\}}^{\{e_\lambda\}}\psi(e_\lambda)\psi(e_1)\psi(e_2). \quad \text{(A.1.1)}$$

The computation of the signs ε_S^J can be easily done starting from (2.1.42) and with the help of Fig. A.2; details are left to the reader. The final result is that

$$\varepsilon_{\emptyset}^{\{e_\lambda\}} = \varepsilon_{\{e_1,e_2\}}^{\{e_\lambda\}} = 1, \varepsilon_{\{e_1\}}^{\{e_\lambda\}} = \varepsilon_{\{e_2\}}^{\{e_\lambda\}} = -1. \quad \text{(A.1.2)}$$

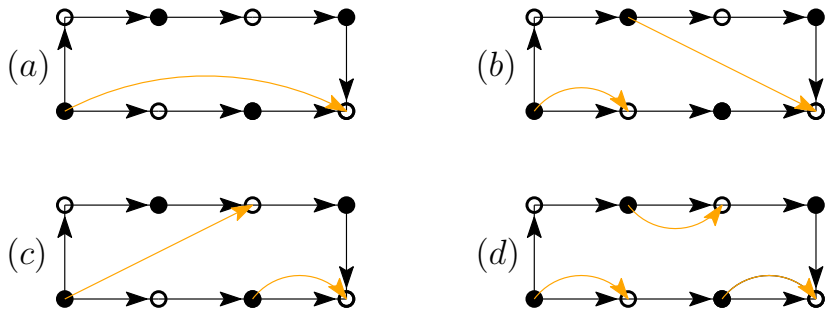

Figure A.2: *The set of edges $E_{J,S}$ with $J = \{e_\lambda\}, S = \emptyset$ (drawing (a)), $J = \{e_\lambda\}, S = \{e_1\}$ (drawing (b)), $J = \{e_\lambda\}, S = \{e_2\}$ (drawing (c)), $J = \{e_\lambda\}, S = \{e_1, e_2\}$ (drawing (d)), colored in orange. Here the orientation of black edges coincides with that on G_L^0 (see Fig. 2.2), while that of orange edges is the one described in Lemma 2.1.0.2 and in the caption of Figure 3.5.*

APPENDIX B

B.1 The renormalization operator

B.1.1 Coordinate space

We want to discuss here how to represent the renormalization operator $\mathcal{R}_{1,0} = 1 - \mathcal{L}_{1,0}$ defined in (3.3.84)-(3.3.29). Even if not necessary[1], let us first discuss the rewriting of the operator $\mathcal{L}_{1,0}$ in coordinate space. Of course the constant term can be rewritten, dropping the scale label of the fields, just by using (2.3.1),(3.3.7),(3.3.8) as $L^{-2}\sum_{q\in\mathcal{D}_{\theta,R}}\hat{\varphi}^+_q\hat{\varphi}^-_q = \sum_y \phi^+_y\phi^-_y$. For the linear term in the Taylor expansion, we can rewrite

$$L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}\hat{\varphi}^+_{q+q_F}\hat{\varphi}^-_{q+q_F}q_i = \sum_{y,y'}\varphi^+_y\varphi^+_{y'}O_i(y-y') = \sum_y \varphi^+_y\hat{\partial}_i\varphi^-_y \tag{B.1.1}$$

which is a definition of the operators[2]

$$\begin{aligned}\hat{\partial}_i\varphi^-_y &:= \sum_{y'}O_i(y-y')\varphi^+_{y'}\\ O(y) &:= L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}}e^{-iqy}\chi_{h+1}(q)q_i.\end{aligned} \tag{B.1.2}$$

One can iterate then to obtain a representation also for the quadratic term. As far as concerns $\mathcal{R}_{1,0}$, first notice that we can rewrite

$$\begin{aligned}\hat{W}^\infty_2(q) - \hat{W}^\infty_2(0) - q\cdot\partial\hat{W}^\infty_2(0) &= (q,\mathcal{H}_q q)\\ \mathcal{H}_q &:= \int_0^1 dt\int_0^t ds\partial^2\hat{W}^\infty_2(qs)\end{aligned} \tag{B.1.3}$$

where we denote by ∂ and ∂^2 the gradient and the Hessian matrix of W^∞_2. If we had not the local term of order q_2^2, (which is the case in the second regime, see (3.3.129)) then we

[1] Recall that the local terms are used to dressed the measure, so we do not need an explicit expression in terms of coordinates in Λ.

[2] We can freely add the function χ_{h+1} in the definition of O because the fields associated to a kernel on scale h (as we are assuming) have the same support of χ_h, over which $\chi_{h+1} = 1$ by definition.

could write

$$\mathcal{R}_{1,0}V = L^{-2}\sum_{q}\varphi^{+}_{q}(q,\mathcal{H}_q q)\varphi^{-}_{q} = \sum_{i,j=1,2}\sum_{y,y'}\hat{\partial}_i\varphi^{+}_{y}\mathcal{R}W(y-y')\hat{\partial}_j\varphi^{-}_{y} \tag{B.1.4}$$

where $\hat{\partial}\varphi$ are the same as above while $\mathcal{R}W(y-y') := L^{-2}\sum_q e^{-iq(y-y')}(\mathcal{H}_q)_{ij}$. To take into account also the quadratic term, then one can iterate (B.1.3), which gives an expression analogous to (B.1.4), where more derivatives has to be considered.

B.2 The propagator

B.2.1 First regime

We recall that the integration procedure of the first regime Section 3.3.2, $h > h^*$, produce a single scale propagator given by

$$\tilde{g}^{(h)}_{L,\epsilon}(y,y') = L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R}} e^{-iq(y-y')}\frac{f_h(q)}{\mathbf{r}_h(q)+\mathbf{D}_h(q)+\rho(q)} \tag{B.2.1}$$

where $f_h(q) = f(2^{-h}|iq_1+\frac{q_2^2}{2}|)$ with $f(q)=\chi(q)-\chi(2q)$ see (3.3.11); $\boldsymbol{D}_h(q) := -iq_1(1+\boldsymbol{\zeta}_h(q))+\frac{1}{2}q_2^2(1+\boldsymbol{\alpha}_h(q))$ and

$$\begin{aligned}
\boldsymbol{r}_h(q) &= r_{h+1}+\chi_{h+1}(q)\sum_{k>h} r_k\partial_{\boldsymbol{r}_k}\hat{W}^{\infty;(h+1)}_{2,0,0}(0;\underline{0})\\
\boldsymbol{\zeta}_h(q) &= \zeta_{h+1}+i\chi_{h+1}(q)\partial_1\hat{W}^{(h+1);\infty}_2(0)\\
\boldsymbol{\alpha}_h &= \alpha_{h+1}+\chi_{h+1}(q)\partial_2^2\hat{W}^{(h+1);\infty}_2(0)\\
\rho(q) &= 1-e^{iq_1}\cos q_2+iq_1-\frac{q_2^2}{2}.
\end{aligned} \tag{B.2.2}$$

Note that by fixing $\nu_0(\lambda)$ as in Proposition 3.3.2,i.e. as done in Section 3.4.5, we have that (recall $\zeta_h := \zeta_h(0)$), $|\zeta_h(q)-\zeta_h|\le c_0|\lambda|$ for every q in the support of f_h, uniformly in h, and similarly for the other constants. Moreover, in the support of f_h, $c_1 2^h \le |q_1|, |q_2|^2 \le c_2 2^h$; also, $r_h = r_0 Q_h(\lambda)$ (see. (3.3.100)) that is $|r_h| = O(\epsilon) = O(2^{h^*}) \le c2^h$. Finally $\rho(q)$ is of second order in q. This considerations implies that the denominator in (B.2.1) is of order 2^h in the support of f_h.

More precisely, given $\underline{n} = (n_0, n_1, n_2) \in \mathbb{N}^3$ define

$$\tilde{g}^{(h);\underline{n}}_{L}(y,y') := \partial^{n_0}_{\boldsymbol{r}_h}\hat{\partial}^{n_1}_1\hat{\partial}^{n_2}_2\tilde{g}^{(h)}_L(y,y') \tag{B.2.3}$$

where $\hat{\partial}_i$ is the same as in the previous section and the action on a propagator is naturally given in terms of Grassmann expectations. Since we are only interested in $L\to\infty$ limit of the theory, we consider the continuum version of the propagator above, which we denote by dropping the L label [3]. For brevity we denote $g(y-y') = g(y,y')$. Next define the

[3] It is obtained by simply replacing the sum with the integral, with the appropriate normalization factor.

usual norms

$$\|g^{(h),\underline{n}}\|_{\tilde{\Lambda},\infty} := \sup_{R^T\mathbb{Z}^2} |g^{(h);\underline{n}}(y)|, \quad \|g^{(h);\underline{n}}\|_1 := \sum_{y\in R^T\mathbb{Z}^2} |g^{(h);\underline{n}}(y)|. \tag{B.2.4}$$

We have the following

Lemma B.2.0.1. *Let $h \geq h^*$. If there exists $C > 0$, independent of L, such that*

$$\sup_{h'\geq h} \max\{|\alpha_k|, |\zeta_k|\} \leq C|\lambda|, \quad \sup_{h'\geq h} |r_h| \leq C|r_0| \tag{B.2.5}$$

then there exist L, ϵ independent constants $c_0, c_1, \kappa > 0$ such that

$$|g^{(h);\underline{n}}(y)| \leq c_0 2^{h\left(\frac{1+n_2}{2}+n_1-n_0\right)} e^{-\kappa\sqrt{d^{(h)}(y)}} \tag{B.2.6}$$

where $d^{(h)}(y) := 2^h|y_1| + 2^{h/2}|y_2|$. Moreover

$$\sum_{y\in R^T\mathbb{Z}^2} |y_1^{m_1} y_2^{m_2} g^{(h);\underline{n}}(y)| e^{\kappa/2\sqrt{2^h|y_1|+2^{h/2}|y_2|}} \leq c_1 2^{h\frac{1+n_2-m_2}{2}+n_1+n_0} \tag{B.2.7}$$

Sketch of the proof. Take first $m_1, m_2 = 0, \underline{n} = \underline{0}$. Set $(q_1, q_2) = (2^h p_1, 2^{h/2} p_2)$ in order to rewrite

$$\tilde{g}^{(h)}(y) = 2^{h/2} G_h^{(0)}(2^h y_1, 2^{h/2} y_2), \qquad G_h^{(0)}(y) = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} dp \frac{e^{-ipy} f_0(p)}{D_h(p)} \tag{B.2.8}$$

where now $D_h(p)$ has a weak dependence on h, namely $|D_h(p)| = O(1)$ uniformly in h[4]. Using then that f_0, is a *Gevray-2* function, one obtains the desired decay

$$|G_h^{(0)}(y)| \leq c e^{-\kappa\sqrt{|y_1|+|y_2|}} \tag{B.2.9}$$

for $c > 0$ independent of h. For more details, one can look at [**129**, Appendix A]. The reader can also look at [**4**, Appendix B] for a derivation, without exponential decay, in a similar context. Recall then that $\bar{\partial}_i$ acts as a multiplication in Fourier space by a factor q_i, so that, using the comments above one obtains an extra factor $2^{n_1 h + n_2 h/2}$. Then, since the dependence on $\mathbf{r}_h$ is only in the denominator, $\partial_{\mathbf{r}_h}$ produces an extra 2^{-h}. To conclude and obtain (B.2.7), it is enough to use (B.2.6) rescale variables $2^h y_1 = z_1, 2^{h/2} y_2 = z_2$. ∎

Remark B.2.1. *Under the validity of Proposition 3.3.2, the hypotesis of the Lemma are satisfied. Thus, fixing the counterterm ν_0 of the critical theory, guarantees that every propagator $\tilde{g}^{(h)}$ satisfy such estimates, uniformly in h, and in particular in h^*.*

[4] We used also that $2^{-h} r_h = O(1)$, uniformly in h.

B.2.2 Second regime

The integration procedure of the second regime in Section 3.3.2, $h < h^* - 1$, produce a single scale propagator given by

$$\frac{1}{Z_{h-1}} g^{(h)}_{\omega,L}(x,y) := \frac{1}{L^2} \sum_{q \in \mathcal{D}'_{R,\theta,\omega}} e^{-iq(y-y')} \frac{\tilde{f}_h(q)}{\tilde{Z}_{h-1}(q) D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q)}, \tag{B.2.10}$$

where $\tilde{f}_h(q) = f(2^{-h}|iq_1 + v_F q_2|)$, and f same as above; $D_{h^*-1,\omega}(q) := -iq_1 + \omega q_2 v_F$ with

$$\begin{aligned} &\tilde{Z}_{h-1}(q) := Z_h(1 + z_h \tilde{\chi}_h(q)) \\ &Z_{h-1} := \tilde{Z}_{h-1}(0) = Z_h(1 + z_h) \\ &v_F := \frac{\sin \tilde{q}_F + \tilde{q}_F \alpha_{h^*-1} + b_{h^*,+}}{\cos \tilde{q}_F + \zeta_{h^*-1} + a_{h^*,+}} \\ &\rho_{h^*-1}(q) = \rho(q + \omega \mathbf{q}_F) - \rho(\omega \mathbf{q}_F) - q \cdot \partial \rho(\omega \mathbf{q}_F). \end{aligned} \tag{B.2.11}$$

where ρ is the same as in the previous section, and $\mathbf{q}_F = (0, \tilde{q}_F)$ is defined below (3.3.109). Note that by fixing the counterterms $\nu_0, \nu_{h^*,+}, a_{h^*,+}, b_{h^*,+}$ for the running constants to converge, as described in Section 3.4.4 and more precisely in Section 3.4.5, from Remark 3.3.8 and Section 3.4.5 one has that $v_F = \tilde{q}_F = O(\sqrt{\epsilon}) = O(2^{h^*/2})$. Now, from the support properties of f_h, $c_1 2^h \le |q_1|, v_F|q_2| \le c_2 2^h$; $\rho(q)$ is of second order in q. Note also that, if $z_{h'}$ in (3.3.136) satisfies $|z_{h'}| \le \delta_0$ uniformly in $L, h' < h^*$, then $e^{-c\delta_0|h-h^*|} \le Z_h \le e^{c\delta_0|h-h^*|}$. As a consequence, on the basis of the previous considerations so that $\rho_{h^*-1}(q)/Z_{h-1}$ is negligible w.r.t. $D_{h^*-1,\omega}(q)$. Denoting with $\tilde{g}^{(h)}_\omega$ the $L \to \infty$ limit of $\tilde{g}(h)_{\omega,L}$, we can write $g^{(h)}_\omega(x,y) = g^{(h)}_{\mathtt{R},\omega}(x,y) + g^{(h)}_{\mathtt{S},\omega}(x,y)$ where we recall from (3.3.130),

$$g^{(h)}_{\mathtt{R},\omega}(x,y) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} e^{-iq(y-y')} \frac{Z_{h-1} \tilde{f}_h(q)}{\tilde{Z}_{h-1}(-iq_1 + v_F \omega q_2)} \tag{B.2.12}$$

and $g^{(h)}_{\mathtt{S},\omega}$ is given by the difference. Recall

$$\tilde{g}^{(h);\underline{n}}_L(y) := \partial^{n_0}_{v_F} \hat{\partial}^{n_1}_1 \hat{\partial}^{n_2}_2 \tilde{g}^{(h)}_L(y) \tag{B.2.13}$$

where $\hat{\partial}_i$ are the same of the previous section. We have the following

Lemma B.2.0.2. *If there exists $C > 0$, such that $\sup_{h \le h' < h^*} |z_h| \le C\delta_1$ then there exist L, ϵ independent constants $c_0, c_1, \kappa > 0$ such that*

$$|\tilde{g}^{(h);\underline{n}}_{\mathtt{R},\omega}(y)| \le c_0 v_F^{-1-n_2-n_0} 2^{h(1+n_1+n_2)} e^{-\kappa \sqrt{2^h(|y_1| + v_F^{-1}|y_2|)}}. \tag{B.2.14}$$

and

$$\sum_{y \in R^T \mathbb{Z}^2} |y_1^{m_1} y_2^{m_2} \tilde{g}^{(h);\underline{n}}_{\mathtt{R},\omega}(y)| e^{\frac{\kappa}{2}\sqrt{2^h(|y_1| + v_F^{-1}|y_2|)}} \le c_1 v_F^{-1-n_2-n_0+m_2} 2^{h(1+n_1+n_2-m_2-m_1)}. \tag{B.2.15}$$

Moreover, $\tilde{g}^{(h);\underline{n}}_{\mathtt{S},\omega}$ satisfy the same estimates times an extra factor 2^h.

Sketch of the proof. Take first $m_1, m_2 = 0, \underline{n} = \underline{0}$. Set, $(q_1, q_2) = 2^h(p_1, p_2 v_F^{-1})$, so one

finds that[5]

$$\tilde{g}^{(h)}_{\mathtt{R},\omega,v_F}(y) = \frac{2^h}{v_F} g^{(0)}_{\mathtt{R},\omega,1}(2^h y_1, 2^h v_F^{-1} y_2) \tag{B.2.16}$$

Using again that f_0, is a *Gevrey-2* function, the desired decay follows: (see analogous discussion in the previous proof)

$$|g^{(0)}_{\mathtt{R},\omega,1}(y)| \leq c e^{-\kappa\sqrt{|y_1|+|y_2|}} \tag{B.2.17}$$

for $c > 0$ independent of h. Again, $\hat{\partial}_i$ acts as a multiplication in Fourier space by a factor q_i, so using the structure of f_h we obtain extra factor $v_F^{-n_2} 2^{h(n_1+n_2)}$ with respect to the previous estimate. Then, a derivative ∂_{v_F} produce dimensionally a factor v_F^{-1} both if it applies to the denominator or to $\tilde{f}_h$. To conclude and obtain (B.2.15), it is enough to use the desired strecthed exponential decay and rescale variables $2^h y_1 = z_1, 2^h v_F^{-1} y_2 = z_2$.

For the last statement we can rewrite

$$\tilde{g}^{(h)}_{\mathtt{S},\omega}(y) = \frac{1}{2\pi^2} \int dq \frac{Z_{h-1} e^{-iqy} \tilde{f}_h(q) \rho_{h^*-1}(q)}{(\tilde{Z}_{h-1} D_{h^*-1,\omega}(q) + \rho_{h^*-1,\omega}(q))(\tilde{Z}_{h-1} D_{h^*-1,\omega}(q))}. \tag{B.2.18}$$

Then, from the definition of $\rho_{h^*-1}(q)$, which is of second order, the leading contribution for q small are given by $O(q_1^2), O(\epsilon q_2^2)$ as it follows from an explicit computation. Then recalling that $v_F q_2, q_1 \sim 2^h$ one obtains the extra 2^h by repeating the above discussion. ■

Note that the factor v_F in 3.3.148 plays an important role. When $v_F(\lambda, \epsilon)$ is fixed in Section 3.4.5, it satisfies $v_F = O(\sqrt{\epsilon})$ which implies that the bounds on the single scale propagator are deeply affected by taking $\epsilon \to 0^+$.

B.3 Some technical facts

B.3.1 The transition scale $h^*(\epsilon)$

Let us define for $\gamma \geq 2$,

$$h^*_\gamma(\epsilon) = \min \left\{ h \leq 0 : \gamma^h > \frac{|e^{-\epsilon} - 1||Q_h|}{(1+\alpha_h)\delta} \right\}. \tag{B.3.1}$$

and note that $h^*_2(\epsilon) = h^*$ (3.3.76). To simplify things, since $Q_h = 1 + O(\lambda)$, $\alpha = 1 + O(\lambda)$ and $\delta = O(1)$, we have that for ϵ, λ small enough, we can approximate h^* with

$$h_\gamma(\epsilon) = \min\{h : \gamma^h > \epsilon\}.$$

Then h_γ is a right continuous step function and has as a discontinuity set $\mathcal{J}_\gamma := \{\epsilon : \epsilon \in \mathcal{D}_\gamma\}$ where $\mathcal{D}_\gamma = \{\gamma^h : h \leq 0\}$. Note that $J_\gamma \cap J_{\gamma'} = \emptyset$ as long as $\gamma' \neq \gamma^z$ for every $z \in \mathbb{Z}$.

B.3.2 Completion of Lemma 3.4.0.5

We want to show that $|\partial_{\tilde{q}_F} \nu_{h^*,+}| \leq C|\lambda| 2^{-h^*} v_F$. As discussed in the second item below (3.4.73), the dependence on $\tilde{q}_F$ is only given by endpoint of the tree on scale h^*-1, through

[5] We specified the dependence on v_F of the propagator $\tilde{g}_{\mathtt{R},\omega}$.

irrelevant terms $\mathcal{R}V^{h^*-1}$. This term is associated to a Grassmann monomial given by

$$e^{-i\mathbf{q}_F \sum_{f\in P^{\varphi}_{v^*}} \omega(f)y(f)} W^{(h^*-1);\infty}_{\tau^*,\mathbf{P}^*,T^*,\mathbf{D}}(\mathbf{y}_{v*})\varphi_{\mathbf{D},\mathbf{i}}(P^{\varphi}_{v^*}) \tag{B.3.2}$$

and we are using the notations of (3.4.14) for the Grassmann part and: τ^* is the sub-tree of τ rooted at v^*, which is the vertex on scale h^*-1 associated to $\mathcal{R}V^{h^*-1}$; $\mathbf{P}^*, T^*, \mathbf{y}^*$ are respectively the set of fields label, the spanning tree and the coordinates of the enpoints of the subtree τ^*; $P^{\varphi}_{v^*}$ is the set of external fields associated to v^*. Now, when the derivative $\partial_{\tilde{q}_F}$ acts, we obtain a "zero" factor $\sum_{i=1}^{n}(-1)^i\omega_i y_{i2}$,[6] which multiplies the above kernel (where we numbered the field labels in $P^{\varphi}_{v^*}$, s.t. $|P^{\varphi}_{v^*}| = n$ even). We distinguish two cases

(1) $\sum_i \omega_i = 0$. In this case the "zero" factor can be written as a sum of gradients coordinates, $y_j - y_{j-1}$, which can be bounded with propagators along the spanning tree T^*. This produces w.r.t the original bound an extra $2^{-h^*}v_F$, as it follows from (B.2.15).

(2) $\sum_i \omega_i \in 2\mathbb{Z}\setminus\{0\}$. In this case, writing (B.3.2) in Fourier we have that the global momentum on the vertex v^* must be preserved, namely

$$\sum_{i=1}(-1)^i(q_i + \omega_i\mathbf{q}_F) = 0. \tag{B.3.3}$$

Using that the fields $\varphi^{(\le h^*-1)}$ are supported on small momenta, i.e. $|q_i| \le O(2^{h^*})$, we can deduce that for some constant $C > 0$

$$n2^{h^*} \ge C\tilde{q}_F \Rightarrow n \ge C\tilde{q}_F/\epsilon \tag{B.3.4}$$

where as usual $2^{h^*} = \Theta(\epsilon)$; this means that as $\epsilon \to 0^+$, this situation occurs only when we have an huge number of external fields associated to v^*. On the other hand, since we are studying contributions to $B^{\nu}_{h,+}$ that preserves the $\underline{\omega}$ on scale h[7], and, since every contraction of Grassmann variables preserves as well the ω's of the contracted fields, this scenario can occur only for trees τ having pairs of vertices with this property. So let's discuss only the case of a pair and denote by $\tilde{v}^*$ the twin vertex of v^*, on scale h^*-1 of irrelevant type.

Now, when we apply $\partial_{\tilde{q}_F}$ to the tree, we produce a zero factor given by

$$\sum_{i=1}^{n}(-1)^i\omega_i y_{i2} + \sum_{i=1}^{\tilde{n}}(-1)^i\tilde{\omega}_i\tilde{y}_{i2} = \sum_{j=1}^{n+\tilde{n}-1} z_{j2} - z_{(j+1)2} \tag{B.3.5}$$

where $\sum_{i=1}^{n}(-1)^i\omega_i y_{i2}$ and $\sum_{i=1}^{\tilde{n}}(-1)^i\tilde{\omega}_i\tilde{y}_{i2}$ come respectively from the oscillatory factors associated to $v^*, \tilde{v}^*$. In the r.h.s we used the conservation of $\omega + \tilde{\omega}$ and we denoted with z the union of the coordinates $y, \tilde{y}$, relabeled in such a way to single out gradients. Now, let $\bar{v}$ the vertex on scale $\bar{h} < h^*-1$ which is the first parent of both $v^*, \tilde{v}^*$ in the tree. If we denote by $\bar{n} < n + \tilde{n}$ the number of Grassmann fields

[6] Recall that $\mathbf{q}_F = (0, \tilde{q}_F)$ and we use the component notation $y_i = (y_{i1}, y_{i2})$.

[7] The same holds for any other running constants which preserves the $\underline{\omega}$ on scale h, and also for the free energy contributions. See item 2 of Remark 3.3.10.

immediately before the contraction on $\bar{v}$, we have that since $\omega, \tilde{\omega}$ are separately not preserved, by reasoning as in (B.3.4) we must have $\bar{n} \geq C2^{-\bar{h}}/\tilde{q}_F$. This means that we have at least a good factor $|\lambda|^{2^{-\bar{h}}-h^*/2}$. This term is more than sufficient to conclude the bound of the effect of $\partial_{\tilde{q}_F}$. Indeed we can estimate the gradients $z_{j2} - z_{(j+1)2}$ together with propagators along the spanning tree $\bar{T}$ associated to $\bar{v}$, as in point (1): this produces an extra dangerous $2^{-\bar{h}}v_F$ which is compensated by $|\lambda|^{2^{-\bar{h}}-h^*/2}$.

Summarizing this shows that point (2) occurs only in extremely high order perturbation theory and that the potentially dangerous factor 2^{-h} coming from the integration of the gradients $z_j - z_{j+1}$ along the spanning trees, are indeed negligible when summed over $h \leq h^*$. This shows that

$$|\partial_{\tilde{q}_F}\nu_{h^*,+}| \leq \frac{C|\lambda|}{v_F}. \tag{B.3.6}$$

B.3.3 Lowest order calculations

First order contribution to λ_h^*

The lowest order contribution in λ comes necessarily from trees of the first regime of scale with only one endpoint of type $\mathcal{R}_{1,\epsilon}V^{(-1)}$ (compare with the proof of Lemma 3.4.0.4. In particular this happens when in the r.h.s. of (3.3.19) we pick the Grassmann monomials of the bare potential $\bar{V}$, which are of lowest order (quartic monomials), and we do not contract any fields in the truncated expectation. This are exactly the Grassmann monomials appearing in the Figure 3.4: in formulas

$$\begin{aligned}(e^{\lambda}-1)\sum_{y\in\tilde{\Lambda}}\Big(E_{H,y}(\phi)E_{H,y+(0,2)}(\phi)+2E_{O,y}(\phi)E_{O,y+(1,1)}(\phi)\Big),\\ E_{H,y}(\phi)=e^{-\epsilon-\varepsilon_0(\lambda)}\phi^+_y\phi^-_y,\qquad E_{O,y}(\phi)=\frac{1}{2}\phi^+_y\phi^-_{y+(-1,1)}.\end{aligned} \tag{B.3.7}$$

H stands for horizontal and O for oblique: recall that from the symmetry of the model, the second and the third contribution in Figure 3.4 are equal. Using (3.4.64) and taking the local part $\mathcal{L}_{1,\epsilon}$ (cf. (3.3.129)) we find

$$\sum_{y\in\tilde{\Lambda}}\sum_{\underline{\omega}}\phi^+_{y,\omega_1}\phi^-_{y,\omega_2}\phi^+_{y,\omega_3}\phi^-_{y,\omega_4}\left(e^{-2\epsilon-2\varepsilon_0}e^{2i\tilde{q}_F(\omega_3-\omega_4)}+\frac{1}{2}e^{i\tilde{q}_F(-\omega_2+\omega_3-2\omega_4)}\right). \tag{B.3.8}$$

From the anticommutation of the Grassmann variables one finds that $\underline{\omega}$ can assume only four values $(+,+,-,-),(-,-,+,+),(+,-,-,+),(-,+,+,-)$, which carry relative signs among them in the associated Grassmann monomial. Putting things together one finds that the contribution to λ_{h^*} it is given by

$$(e^{\lambda}-1)(1-\cos\tilde{q}_F)(2e^{-2\epsilon-2\varepsilon_0}+1) \tag{B.3.9}$$

which is indeed $O(\lambda\epsilon)$, using that by hypothesis, $\tilde{q}_F = \Theta(\sqrt{\epsilon})$ (see below (3.3.110)).

First order contribution to $\varepsilon_0(\lambda)$

Recall that $\varepsilon_0(\lambda) = \log(1+\nu_0(\lambda))$, so that the lowest order of ε_0 coincides with that of ν_0. On the other hand ν_0 satisfies the implicit equation

$$\nu_0 = -\sum_{h\le 0} 2^h B_h^{\nu}(\lambda, \nu_0). \tag{B.3.10}$$

Recall that from (3.4.1),(3.3.55) and (3.3.58), $2^h B_h^{\nu}$ is nothing but the kernel $\hat{W}_2^{(h):\infty}(0)$ where we extracted the linear term in the sequence $\underline{\nu}$ (namely $2^{h+1}\nu_{h+1}$). This means that, as in the previous Section, the lowest order comes from trees having only exactly one endpoint that is of type $\mathcal{R}_{1,0}V^{(-1)}$. In particular in, (3.3.19) we must pick the Grassmann monomials of the bare potential $\bar{V}$, which are of lowest order (quartic monomials), and we do not contract any fields in the truncated expectation on that scale. Indeed to contribute to a local quadratic part on scale h, in virtue of the renormalized tree expansion we must contract exactly two of the four Grassmann variables on scale h. In formulas we have that the first order contribution to ν_0 is given by the coefficient of the grassmann monomial

$$(e^{\lambda}-1)\sum_{h\le 0}\mathcal{E}^T_{h,*}\left(E_{H,y}(\phi+\cdot)E_{H,y+(0,2)}(\phi+\cdot) + 2E_{O,y}(\phi+\cdot)E_{O,y+(1,1)}(\phi+\cdot)\right) \tag{B.3.11}$$

where the * in the expectation means that we contract only two fields and $E_{H,y}, E_{O,y}$ are the same of the previous section. A simple computation shows that we can write the coefficient of (B.3.11)

$$\sum_{h\le 0}\left(2g_\lambda^{(h)}(0) - g_\lambda^{(h)}(-2\hat{e}_2) - g_\lambda^{(h)}(2\hat{e}_2) + \frac{1}{2}\left(2g_\lambda^{(h)}(\hat{e}_1-\hat{e}_2) - g_\lambda^{(h)}(-2\hat{e}_2) - g_\lambda^{(h)}(2\hat{e}_2)\right)\right) \tag{B.3.12}$$

where $\hat{e}_1 = (1,0), \hat{e}_2 = (0,2)$, while the propagator associated to $\mathcal{E}^T_{h,*}$ is given by

$$g_\lambda^{(h)}(y_1,y_2) = \int_{R^{-1}[0,2\pi]^2} dq \frac{e^{-iqy} f_h(q)}{1 - e^{iq_1}\cos q_2 - iq_1\boldsymbol{\zeta}_h + \frac{q_2^2}{2}\boldsymbol{\alpha}_h} \tag{B.3.13}$$

as it follows from (B.2.1), by taking the $L\to\infty$ limit: note indeed in that $\mathbf{D}_h(q)+\rho(q)$ it is nothing but the dressing with the running constant $\boldsymbol{\zeta}, \boldsymbol{\alpha}$ of the original propagator in (3.3.13); R is the same transformation matrix in (3.3.4). Using that $\boldsymbol{\alpha}, \boldsymbol{\zeta} = O(\lambda)$ we finally have that the first order contribution to ν_0 is given by

$$-\lambda\left(2g_0(0) - g_0(-2\hat{e}_2) - g_0(2\hat{e}_2) + \frac{1}{2}\left(2g_0(\hat{e}_1-\hat{e}_2) - g_0(-2\hat{e}_2) - g_0(2\hat{e}_2)\right)\right) \tag{B.3.14}$$

where

$$g_0(y) = \sum_{h\le 0} g_0^{(h)}(y) = \frac{1}{\pi^2}\int_{R^{-1}[0,2\pi]^2} dq \frac{e^{-iqy}}{1-e^{iq_1}\cos q_2} = (-1)^{x_2-x_1}K^{-1}(x,0) \tag{B.3.15}$$

where we used the definition of the cutoff functions $\sum_{h\le 0} f_h = 1$ and the change of variables $k = Rq$, $y = R^T x$ in the last step: K^{-1} is the infinite volume inverse Kasteleyn operator given by the $L\to\infty$ limit of (3.3.3) (see also (3.1.12)). It has an explicit expression

(B.3.25). Thus we have that

$$\nu_0(\lambda) = -2K^{-1}(0)\lambda + O(\lambda^2) = -2\lambda + O(\lambda^2). \tag{B.3.16}$$

B.3.4 Frozen phase for the integrable model

We start with two standard results.

Lemma B.3.0.1. *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be analytic in each argument in the strip $S_\eta := \{z : |\Im z| \leq \eta, \Re z \in [-\pi, \pi]\}$ and be 2π-periodic in each argument. Then,*

$$\Big|\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi} d\phi_1 \dots d\phi_n f(\phi) e^{-ix\cdot\phi}\Big| \leq e^{-\eta|x|_1} \sup_{\phi\in S_\eta\times\cdots\times S_\eta} |f(\phi)| \tag{B.3.17}$$

with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $|x|_1 = |x_1| + \cdots + |x_n|$.

Lemma B.3.0.2 (Poisson summation)**.** *If $\hat{F} : [-\pi, \pi]^2 \to \mathbb{C}$ is C^∞, then*

$$L^{-2} \sum_{k\in\mathcal{P}(\theta)} \hat{F}(k) = \sum_{n\in\mathbb{Z}^2} F(n_1 L, n_2 L)(-1)^{\theta_1 n_1 + \theta_2 n_2} \tag{B.3.18}$$

where

$$F(x) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} dk e^{-ik\cdot x}\hat{F}(k) \tag{B.3.19}$$

and $\mathcal{D}(\theta)$ is the same as in (3.1.5).

If $A = 1$, $B = C = \frac{e^\epsilon}{2}$ with $\epsilon < 0$, we have that

$$f(z_1, z_2) = \frac{1}{1 + \frac{e^\epsilon}{2}(e^{iz_1} + e^{iz_2})} \tag{B.3.20}$$

is separately analytic in the strip $S_{\frac{\epsilon}{2}}$, thus by Lemma B.3.0.1,

$$\begin{aligned} |K^{-1}(x)| &\leq e^{-\frac{\epsilon}{2}|x|_1} \sup_{S_{\frac{\epsilon}{2}}\times S_{\frac{\epsilon}{2}}} |f(z)|, \quad |x|_1 = |x_1| + |x_2| \\ K^{-1}(x) &:= \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \frac{e^{-ikx}}{1 + \frac{e^\epsilon}{2}(e^{ik_1} + e^{ik_2})}. \end{aligned} \tag{B.3.21}$$

Lemma B.3.0.3. *Assume that there exists $\eta > 0$ such that $|K^{-1}(x)| \leq c_\eta e^{-\eta|x|_1}$ for every $x \in \mathbb{Z}^2$. Then,*

$$\sup_{x,y\in\Lambda:|x_1-y_1|,|x_2-y_2|\leq L/2} |K_\theta^{-1}(x, y) - K^{-1}(x - y)| \leq C(\eta) e^{-L/C(\eta)} \tag{B.3.22}$$

for some $C(\eta) < \infty$.

The restriction on x, y does not entail a loss of generality, since for any two vertices on the $L-$torus the horizontal and vertical distances are at most $L/2$. However, the formula written without the restriction woudl be false, since for instance $x = (0, 0)$ and $y = (L - 1, 0)$ are at distance 1 on the torus but at distance L on $\mathbb{Z}^2$.

Proof. Assume without loss of generality that $y = 0$, and set $\hat{F}(k) := \frac{e^{-ikx}}{1+\frac{e^{\varepsilon}}{2}(e^{ik_1}+e^{ik_2})}$, which is C^∞ on $[-\pi,\pi]^2$ if $\varepsilon < 0$. Then we can use Poisson summation to write

$$K_\theta^{-1}(x,0) - K^{-1}(x) = \sum_{n\in\mathbb{Z}^2\setminus\{(0,0)\}} K^{-1}(x_1+n_1L, x_2+n_2L)(-1)^{n_1\theta_1+n_2\theta_2}. \tag{B.3.23}$$

Since $|x_1+n_1L|+|x_2+n_2L| \geq L/2$, the sum is easily seen to be exponentially small in L. ∎

From (B.3.21) we see then that Lemma B.3.0.3 holds with $2\eta = \epsilon$ and $c_\eta = \sup\{|f(z)| : z \in S_\eta \times S_\eta\} = \frac{1}{1-e^{\frac{\epsilon}{2}}}$. Therefore,

$$\sup_{x,y\in\Lambda:|x_1-y_1|,|x_2-y_2|\leq L/2} |K_\theta^{-1}(x,y) - K^{-1}(x-y)| \leq C(\varepsilon)e^{-L/C(\varepsilon)}. \tag{B.3.24}$$

Lemma B.3.0.4. *Let A, B, C such that $A > B + C$, then the infinite volume inverse Kasteleyn operator K^{-1} (cfr.* (3.1.9) *can be explicitly computed:*

$$K^{-1}(x,0) = \mathbb{1}_{\{x_1,x_2\geq 0\}} \frac{(-1)^{x_1+x_2}}{A} \left(\frac{B}{A}\right)^{x_1} \left(\frac{C}{A}\right)^{x_2} \binom{x_1+x_2}{x_1}. \tag{B.3.25}$$

In particular,

$$|K^{-1}(x,0)| \leq \mathbb{1}_{\{x_1,x_2\geq 0\}} \frac{1}{A} \left(\frac{B+C}{A}\right)^{x_1+x_2}. \tag{B.3.26}$$

Proof. We start from

$$K^{-1}(x) = \frac{1}{(2\pi i)^2} \int_{|z|=1} dz \int_{|w|=1} dw \frac{z^{-x_1-1}w^{-x_2-1}}{A+Bz+Cw}. \tag{B.3.27}$$

We perform the integral over z. Since $A > B+C$, the only pole inside the circle $|z| = 1$ is at $z = 0$ (if $x_1 \geq 0$, otherwise there is no pole inside and the integral is zero). The residue theorem then gives

$$\mathbb{1}_{x_1\geq 0}(-B)^{x_1} \frac{1}{2\pi i} \int_{|w|=1} dw \frac{w^{-x_2-1}}{(A+Cw)^{x_1+1}}. \tag{B.3.28}$$

The integral over w is performed similarly: for $x_2 < 0$ it is zero because there is no pole inside $|w| = 1$, and for $x_2 \geq 0$ the residue theorem leads to (B.3.25). As for (B.3.26), it is enough to note that $\left(\frac{B}{B+C}\right)^{x_1} \left(\frac{C}{B+C}\right)^{x_2} \binom{x_1+x_2}{x_1} \leq 1$. ∎

B.4 Symmetries of the effective theory

Let us recall that the original symmetries of the non interacting model, for $\lambda = 0$, are given in terms of mappings $\mathcal{S}_1, \mathcal{S}_2$ on the Grassmann algebra by (see Section 3.2.2)

$$\begin{aligned} &\mathcal{S}_1 : \hat{\psi}_k^\pm \longrightarrow \hat{\psi}_{-k}^\pm \qquad c \to \overline{c}, \qquad &&\mathcal{S}_2 : \hat{\psi}_k^\pm \longrightarrow \hat{\psi}_{\tilde{k}}^\pm \qquad c \to c \\ &\mathcal{S}_1 : \psi_x^\pm \longrightarrow \psi_x^\pm \qquad c \to \overline{c}, \qquad &&\mathcal{S}_2 : \psi_x^\pm \longrightarrow \psi_{\tilde{x}}^\pm \qquad c \to c, \end{aligned} \tag{B.4.1}$$

where $x = (x_1, x_2)$ and $\tilde{x} = (x_2, x_1)$ (similarly for k); they mean that in every Grassmann polynomial, each Grassmann field $\psi, \hat{\psi}$ is replaced as above and every constant $c \in \mathbb{C}$ is

replaced by its complex conjugate only for the action of $\mathcal{S}_1$.

For $\lambda \neq 0$, we showed in Section 3.2.2 that they are also symmetries of the interacting model. Then we want to check the effect of these symmetries on the multiscale construction. Note that in terms of the coordinates in $\tilde{\Lambda}$ and $\mathcal{D}_{\theta,R}$ introduced in (3.3.4), we have that, e.g., $\mathcal{S}_2$ reads

$$\mathcal{S}_2 : \varphi^{\pm}_y \longrightarrow \varphi^{\pm}_{\hat{y}} \qquad c \to c, \qquad \mathcal{S}_2 : \hat{\varphi}^{\pm}_q \longrightarrow \hat{\varphi}^{\pm}_{\hat{q}} \qquad c \to c, \tag{B.4.2}$$

where $\hat{y} = (y_1, -y_2)$ and similarly for $q \in \mathcal{D}_{\theta,R}$. Note that $\hat{q}$ comes $\tilde{k}$ via the relation $\widehat{R^T k} = R^T \tilde{k}$.

Symmetries for the effective potential

We use the following general fact: if a Grassmann integration $P(d\psi)$ and an interaction $V(\psi)$ are symmetric under a symmetry $\mathcal{S}$ acting on the Grassmann algebra $\{\psi\}_{x\in\Lambda}$ as described above, then V' given by the relation

$$e^{V'(\varphi)} = \int P(d\psi) e^{V(\psi+\varphi)} \tag{B.4.3}$$

is still symmetric under $\mathcal{S}$. By a measure to be symmetric we mean that its quadratic part is symmetric, i.e. $P(d\psi) = \mathcal{N}^{-1} D\psi e^{Q(\psi)}$ with $\mathcal{S}Q(\psi) = Q(\psi)$, where $\mathcal{N}$ is the normalization constant. As a consequence of the above general fact we have that the symmetries satisfied by the bare action, $S + V$ of (3.2.8), are preserved at each scale by the multiscale construction of Section (3.3.2), in the following form. *For simplicity let us set the external fields $\mathcal{A}$ to zero*: a similar discussion holds also in that case.[8]

Lemma B.4.0.1. *The kernels of the effective potential $V_\epsilon^{(h)}$ (cfr.* (3.3.77)*) satisfy*

$$(h \geq h^*) : \begin{cases} W^{(h)}_{n,\epsilon,\theta}(\underline{y}) \in \mathbb{R} \\ W^{(h)}_{n,\epsilon,\theta}(\hat{\underline{y}}) = W^{(h)}_{n,\epsilon,\tilde{\theta}}(\underline{y}) \end{cases} \qquad (h < h^*) : \begin{cases} \overline{W^{(h)}_{n,\epsilon,\theta,\underline{\omega}}(\underline{y})} = W^{(h)}_{n,\epsilon,\theta,-\underline{\omega}}(\underline{y}) \\ W^{(h)}_{n,\epsilon,\theta,\underline{\omega}}(\hat{\underline{y}}) = W^{(h)}_{n,\epsilon,\tilde{\theta},-\underline{\omega}}(\underline{y}) \end{cases} \tag{B.4.4}$$

where $\underline{y} = (y_1, \dots, y_n)$ and $\underline{\omega} = (\omega_1, \dots, \omega_n)$ with $y_i \in \tilde{\Lambda}$, and $\omega_i = \pm 1$; $\hat{\underline{y}} := (\hat{y_i}, \dots, \hat{y_n})$ with $\hat{y} := (y_1, -y_2)$; $\theta = (\theta_1, \theta_2)$ and $\tilde{\theta} = (\theta_2, \theta_1)$. In terms of momenta,

$$(h \geq h^*) : \begin{cases} \hat{W}^{(h)}_{n,\epsilon,\theta}(\underline{q}) = \overline{W^{(h)}_{n,\epsilon,\theta}(-\underline{q})} \\ \hat{W}^{(h)}_{n,\epsilon,\theta}(\hat{\underline{q}}) = W^{(h)}_{n,\epsilon,\tilde{\theta}}(\underline{q}) \end{cases} \qquad (h < h^*) : \begin{cases} \overline{\hat{W}^{(h)}_{n,\epsilon,\theta,\underline{\omega}}(\underline{q})} = \hat{W}^{(h)}_{n,\epsilon,\theta,-\underline{\omega}}(-\underline{q}) \\ \hat{W}^{(h)}_{n,\epsilon,\theta,\underline{\omega}}(\hat{\underline{q}}) = \hat{W}^{(h)}_{n,\epsilon,\tilde{\theta},-\underline{\omega}}(\underline{q}) \end{cases} \tag{B.4.5}$$

Proof of Lemma. Let's start with $h \geq h^*$. Since the bare action $S + V$ ((3.2.8)) is symmetric under $\mathcal{S}_1, \mathcal{S}_2$, in virtue of (B.4.3) above, we just need to check that at each scale the integration $P^{(h)}$ associated to the expectation in (3.3.95) is indeed symmetric. The

[8] The external fields are real value, so they do not change the conjugation symmetry and do not interact with the symmetries acting only on the Grassmann algebra.

integration $P^{(h)}(d\varphi^{(h)})$ can be written as, let $\phi = \varphi^{(h)}$

$$\begin{aligned} &P^{(h)}(d\phi) \propto D\phi^{(h)} e^{S_\theta^{(h)}(\phi)}, \\ &S_\theta^{(h)}(\phi) = -L^{-2} \sum_{\substack{q\in\mathcal{D}_{\theta,R} \\ f_h(q-q_F\neq 0)}} \hat\phi_q^+ (f_h(q-q_F))^{-1} \tilde\mu_h(q) \hat\phi_q^- \end{aligned} \tag{B.4.6}$$

where $\tilde\mu_h(q) = \mathbf{r}_h(q) + \mathbf{D}_h(q) + \rho(q)$ see (B.2.1) for the necessary definitions; recall that $q_F = R^{-1}k_F = (0,\pi)$. Since $\widehat{\mathcal{D}_{\theta,R}} = \mathcal{D}_{\tilde\theta,R}$ and $f_h, \tilde\mu_h, q_F$ are symmetric, we have that

$$\mathcal{S}_2(S_\theta^{(h)}(\phi) = S_{\tilde\theta}^{(h)}(\phi^{(h)}). \tag{B.4.7}$$

Note that $\tilde\mu_h$ it symmetric under $\hat q$, since it is given from an even function in q_2 (see below (3.3.4)) by adding even terms in q_2, i.e. $\chi_h(q)$, $\boldsymbol{\alpha}_h(q)$ (cf. (B.2.1). One can repeat the same, in simpler form, for $\mathcal{S}_1$.

This means that the effective interaction $V^{(h)}$, and its kernels are indeed symmetric at each $h \geq h^*$. Since the symmetries preserve the degree of Grassmann monomials, we find the desired statement in the l.h.s. of (B.4.5)(see below for a derivation of this conclusion in the simplest case).

Once we reach scale h^* we introduce (see just above (3.3.114)) the field decomposition

$$\varphi_y^{\sigma,(\leq h^*)} = \varphi_y^{\sigma,(\leq h^*-1)} + \varphi_y^{\sigma,(h^*-1)}, \qquad \varphi_y^{\sigma,(\leq h^*-1)} = \sum_{\omega=\pm} e^{-i\sigma\omega y \mathbf{q}_F} \varphi_{y,\omega}^{\sigma,(\leq h^*-1)} \tag{B.4.8}$$

which shows that in order for the fields $\varphi^{(\leq h^*)}$ to preserve the symmetries $\mathcal{S}_i$, $i = 1,2$, they act on "quasi particles" fields $\phi_{y,\omega}^\sigma = \varphi_{y,\omega}^{\sigma,(\leq h^*-1)}$ as

$$\mathcal{S}_1: \quad \phi_{y,\omega}^\sigma \longrightarrow \phi_{y,-\omega}^\sigma, \qquad c \to \bar c, \qquad \mathcal{S}_2: \quad \phi_{y,\omega}^\sigma \longrightarrow \phi_{\hat y,-\omega}^\sigma, \qquad c \to c. \tag{B.4.9}$$

In Fourier space they reads

$$\mathcal{S}_1: \quad \hat\phi_{q,\omega}^\sigma \longrightarrow \hat\phi_{-q,-\omega}^\sigma, \qquad c \to \bar c, \qquad \mathcal{S}_2: \quad \hat\phi_{q,\omega}^\sigma \longrightarrow \hat\phi_{\hat q,-\omega}^\sigma, \qquad c \to c \tag{B.4.10}$$

where $\hat\phi_{q,\omega}^\sigma := \hat\phi_{q+q_F+\omega\mathbf{q}_F}^\sigma$. This can be seen by writing $\phi_{y,\omega}^\sigma = L^{-2}\sum_{q\in\mathcal{D}'_{\theta,R,\omega}} e^{\sigma iqy}\hat\phi_{q,\omega}^\sigma$ and noting that $\mathcal{D}'_{\theta,R,-\omega} = -\mathcal{D}'_{\theta,R,\omega}$, see (3.3.119) and lines below. Using the representation in Fourier space it is readily seen that the counterterm $M(\phi)$, introduced at the transition scale (3.3.109), is symmetric under these transformations (using (3.3.112)). This means that M preserves separately the symmetry of the Grassmann integration and of the effective interaction on scale h^*-1. Then a repetitive application of (B.4.3) implies that the symmetries (B.4.9)-(B.4.10) are preserved at each scale $h < h^*$. Since the symmetries preserve the degree of every Grassmann monomial, then we have that the statement in the r.h.s. in (B.4.5) also holds. Let us derive for concreteness, using the Fourier representation of the effective potentials (see (3.3.126) and lines above), the case $n = 2$. We have that

the following term is preserved by $\mathcal{S}_1, \mathcal{S}_2$ (let's work out only $\mathcal{S}_2$)

$$\begin{aligned} &L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R,\omega}} \hat{\phi}^+_{q,\omega} \hat{W}^{(h)}_{2,0,\omega,\theta}(q) \hat{\phi}^-_{q,\omega} \overset{\mathcal{S}_2}{=} L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R,\omega}} \hat{\phi}^+_{\hat{q},-\omega} W^{(h)}_{2,0,\omega,\theta}(q) \hat{\phi}^-_{\hat{q},-\omega} = \\ &= L^{-2} \sum_{q \in \mathcal{D}'_{\theta,R,\omega}} \hat{\phi}^+_{q,\omega} \hat{W}^{(h)}_{2,0,-\omega,\tilde{\theta}}(\hat{q}) \hat{\phi}^-_{q,\omega} \end{aligned} \tag{B.4.11}$$

where we used that $\mathcal{D}'_{\theta,R,-\omega} = -\mathcal{D}'_{\theta,R,\omega}$ and $\widehat{\mathcal{D}_{\theta,R,\omega}} = \mathcal{D}_{\tilde{\theta},R,\omega}$ where recall $\hat{q} = (q_1, -q_2)$, $\tilde{\theta} = (\theta_2, \theta_1)$ if $\theta = (\theta_1, \theta_2)$. This implies that

$$\hat{W}^{(h)}_{2,0,-\omega,\tilde{\theta}}(\hat{q}) = \hat{W}^{(h)}_{2,0,\omega,\theta}(q). \tag{B.4.12}$$

Note finally that taking the infinite volume kernels, which do not depend on boundary conditions θ we obtain

$$\hat{W}^{(h);\infty}_{2,0,\omega}(\hat{q}) = \hat{W}^{(h);\infty}_{2,0,-\omega}(q). \tag{B.4.13}$$

■

Corollary B.4.0.1. As a consequence we have that the running coupling constants satisfy:

$$\begin{aligned} \nu_h, r_h, \alpha_h, \zeta_h, z_h, \lambda_h, \nu_{h,+}, ia_{h,+}, b_{h,+} \in \mathbb{R} \\ \nu_{h,\omega} = \nu_{h,-\omega}, \quad a_{h,\omega} = a_{h,-\omega}, \quad b_{h,\omega} = -b_{h,-\omega} \end{aligned} \tag{B.4.14}$$

Proof of Corollary. Note that almost all the running constants are originally defined in Fourier space. See (3.4.1) and lines below for the definition of the constants.

For the first regime constants, we obtain, after taking the $L \to \infty$ of the kernels,

$$\hat{W}^{(h);\infty}_{2,\epsilon}(0) \in \mathbb{R}, \quad \partial_1 \hat{W}^{(h);\infty}_{2,\epsilon}(0) \in i\mathbb{R}, \quad \partial_2 \hat{W}^{(h);\infty}_{2,\epsilon}(0) = 0, \quad \partial_2^2 \hat{W}^{(h);\infty}_{2,\epsilon}(0) \in \mathbb{R} \tag{B.4.15}$$

as it follows by deriving the equations in (B.4.5). This implies the desired statement for the constants $\nu_h, r_h, \alpha_h, \zeta_h$. For the second regime of constants instead we obtain

$$\begin{aligned} &\hat{W}^{(h);\infty}_{2,\omega}(0) \in \mathbb{R}, && \hat{W}^{(h);\infty}_{2,\omega}(0) = \hat{W}^{(h);\infty}_{2,-\omega}(0) \\ &\partial_1 \hat{W}^{(h);\infty}_{2,\omega}(0) \in i\mathbb{R}, && \partial_1 \hat{W}^{(h);\infty}_{2,\omega}(0) = \partial_1 \hat{W}^{(h);\infty}_{2,-\omega}(0) \\ &\partial_2 \hat{W}^{(h);\infty}_{2,\omega}(0) \in \mathbb{R}, && \partial_2 \hat{W}^{(h);\infty}_{2,\omega}(0) = -\partial_2 \hat{W}^{(h);\infty}_{2,-\omega}(0) \end{aligned} \tag{B.4.16}$$

which implies the statement for all the remaining constants but λ_h. In this case notice that the r.h.s. of (B.4.5) for $n = 4, L = \infty$ evaluated at $q_2, q_3, q_4 = 0$ does the job. ■

APPENDIX C

CONNECTION WITH VERTEX MODELS

C.1 Honeycomb dimers and the 5V model

In this appendix we discuss some mapping between dimer models and the 6V model. The 6V model on the torus is defined as follows. Let $L > 0$ even and as usual $\tilde{G}_L = (\mathbb{Z}/L\mathbb{Z})^2$ be the torus of side L[1]. An *arrow configuration* ω (or six-vertex configuration) is a choice of orientation for every edge of $\tilde{G}_L$ such that every vertex of G_L has two incoming and two outgoing edges in ω (*ice rule)*, as shown in Figure C.1. If Ω_{6V} denotes the space of 6V configurations, then the partition function is given by

$$Z_{6V} = \sum_{\omega \in \Omega_{6V}} W(\omega)$$

where the product runs over vertices of $\tilde{G}_L$ and the weight of a configuration $W(\omega)$ is given by

$$W(\omega) = \prod_{v \in \tilde{G}_L} a_{\sigma(v)} = \prod_{i=1}^{6} a_i^{n_i}$$

where $\sigma(\omega) : V \to \{1, \dots, 6\}$ identifies the type of v in the configuration ω and $n_i(\omega)$ is the number of vertices of type i (Fig. C.1) in ω.

Given a six-vertex configuration ω, if we keep track only of North and West going arrows at each vertex, we obtain a lattice path model, as shown Figure C.2.

When we specialize to the case $a_3 = 0$, it is seen that the lattice paths cannot intersect anymore: this model goes under the name of monotone, non-increasing lattice paths model (MNLP) [**115**]. Instead, in terms of the vertex model, it is called *five-vertex* model (5V). It is known that dimer configurations on the hexgonal periodic lattice are bijectively mapped to MNLP configurations. Consider a dimer configuration on the graph G_L (1.3.2) as in the non interacting edge case, as in Figure 1.5. Contracting each horizontal edge in the graph

[1] with respect to translation in the standard orthonormal basis

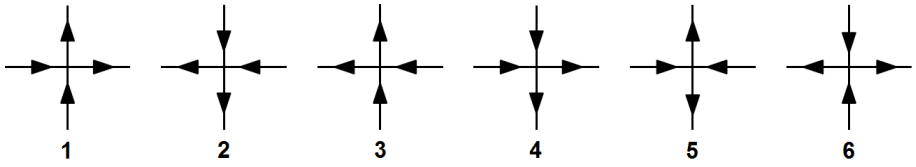

Figure C.1: *The six possible vertex configurations. Each type comes with a weight* $a_1, \dots, a_6$.

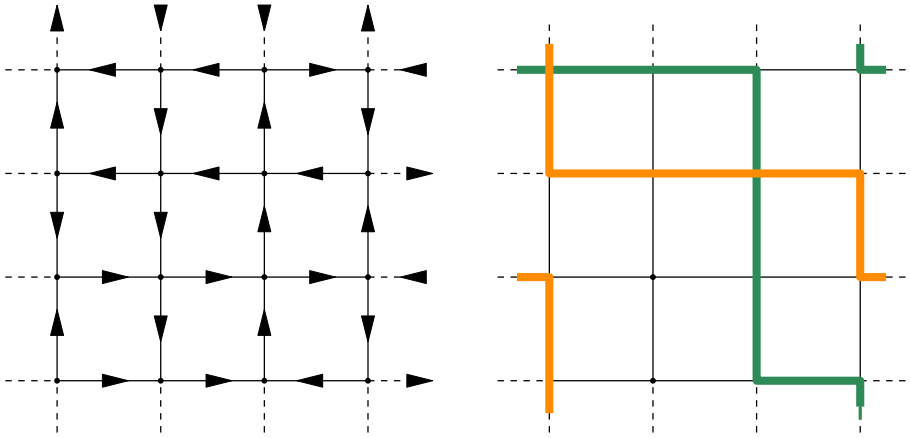

Figure C.2: *Left: six-vertex configuration on* $\tilde{G}_4$. *Right: a lattice paths configuration: paths are colored green and orange.*

G_L, i.e. gluing into a single vertex the black and white vertices of the fundamental cell, we obtain exactly the graph $\tilde{G}_L$ and the dimer configuration becomes a MNLP configuration, see Figure C.3.

Since MNLP are nothing but 5V configurations, we have that if we choose the vertex weights as follows

$$a_1 = B \quad a_2 = C \quad a_4 = A \quad a_5 = a_6 = \sqrt{BC}$$

we obtain that

$$\boxed{Z_L(A, B, C) = Z_{5V}}$$

where $Z_L(A, B, C)$ is the partition function of the dimer model on G_L as below (3.1.2) and Z_{5V} is the same as Z_{6V} given above when $a_3 = 0$. A simple check shows that with the choice above, the weights of the configurations in Figure C.3 are

$$w(M) = A^8 B^4 C^4 \qquad W(\omega) = a_4^8 a_1^2 a_2^2 a_5^2 a_6^2$$

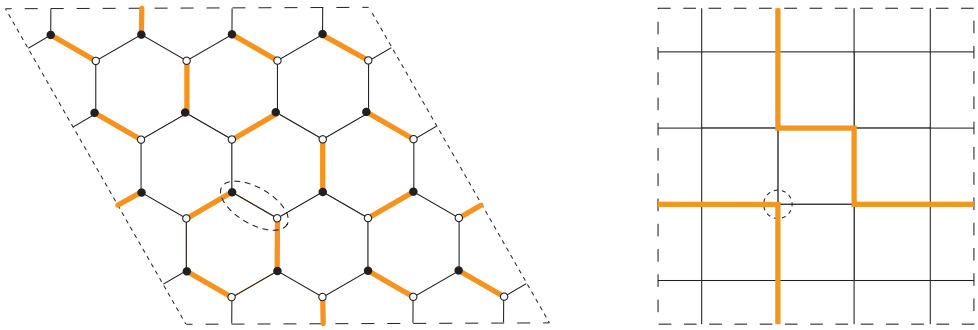

Figure C.3: *The mapping between dimer configurations on G_L and MNLP configurations on $\tilde{G}_L$. The fundamental domains enclosing horizontal edge are mapped, by contraction of the edge, bijectively in the set of vertices of $\tilde{G}_L$.*

and indeed $w(M) = W(\omega)$ with the choice of $(a_1, \ldots, a_6)$ above. Note also that the free-fermion condition (FP) for the 6V model reads

$$a_1 a_2 + a_3 a_4 - a_5 a_6 = 0$$

and for the 5V ($a_3 = 0$)

$$a_1 a_2 - a_5 a_6 = 0$$

which is satisfied with the choice above. Indeed $6V$ it is known to be determinantal exactly at the FP which corresponds to the determinantal structure of the dimer model.
Then, the interacting dimer model of Section 1.3 (edge), see (1.3.3), is equivalent to an interacting 5V (or MNLP) model where interaction acts on faces of the lattice by promoting ($\lambda > 0$) or disfavoring ($\lambda < 0$) empty faces and faces with parallels paths, see Figure C.4.

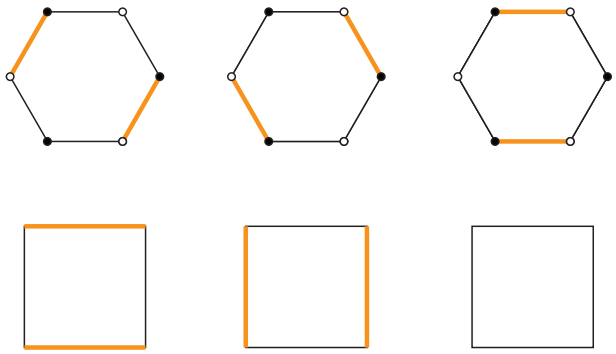

Figure C.4: *The interaction modifies the weight of a configuration by an extra e^{λ} for every of the faces above. In the MNLP model paths prefer or not certain patterns depending on the sign of λ.*

C.2 6V model as non-planar square-octagon dimers

If we allow vertex weights in the 6V to have some periodicity, by varying from site to site, it is known that a connection with non planar dimer models can be extablished (together with connections to other lattice statistical mechanics models), if the family distinct family of weights satisfy the free-fermion condition [**94**]. The simplest way to realize it is to consider *staggered vertex weights*, namely two sets of arrows weights associated to the two sublattices of the graph $\tilde{G}_L$, the color of which are black and white for simplicity as in Figure (*a*) C.5; we denote also by V_B and V_W the set of black and white vertices of $\tilde{G}_L$. Finally, denote $a_1, \dots, a_6$ and $a'_1, \dots, a'_6$ vertex weights associated to black and white vertices, respectively. The free-fermion condition on the weights then reads

$$a_5 a_6 - a_3 a_4 = a_1 a_2, \quad a'_5 a'_6 - a'_3 a'_4 = a'_1 a'_2.$$

The generating function of the model is

$$Z^{\mathrm{S}}_{6V} = \sum_{\omega \in \Omega_{6V}} W(\omega)$$

where now

$$W(\omega) = \left(\prod_{v \in V_B} a_{\sigma_v} \right) \left(\prod_{v \in V_W} a'_{\sigma_v} \right) = \prod_{i=1}^{6} a_i^{n_i^B} (a'_i)^{n_i^W},$$

with $n_i^{\#}(\omega)$ being the number of # vertices of type i, where # = black or white. There is

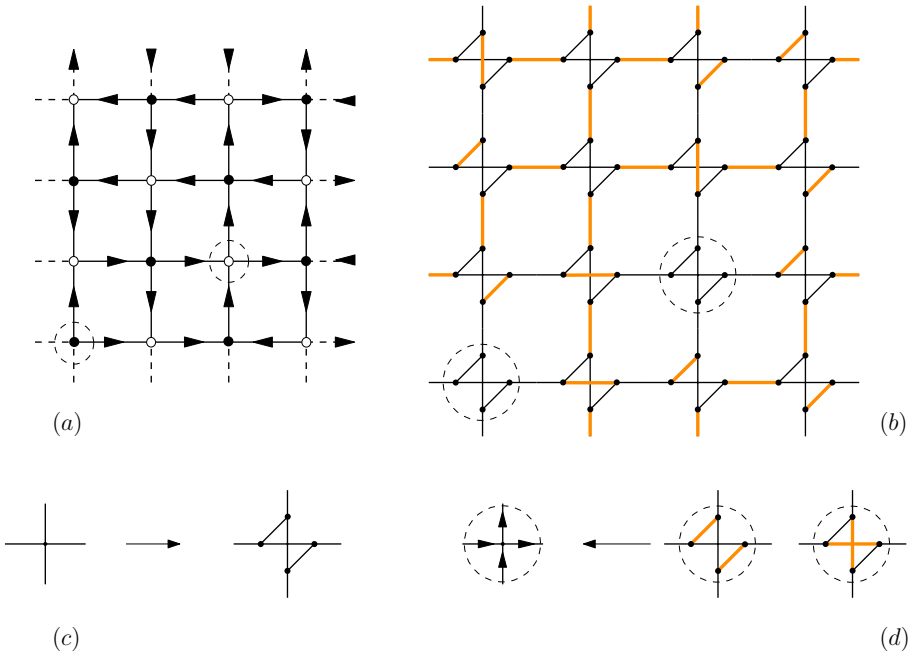

Figure C.5: *(a): a vertex configuration on the bipartite graph (vertex of type* 1 *are encircled). (b): the associated expanded graph and the dimer configuration. (c): the expansion procedure, a vertex becomes a city. (d): type* 1 *vertex configurations correspond to two possible dimer covering of the associated city.*

a way to associate to the 6V model a dimer configuration on an expanded graph. Expand

each vertex of $\tilde{G}_L$ in a "city" of four vertices connected by edges as shown in Figure C.5 (c). This gives the graph structure of Figure C.5 (b). Then, place a dimer on horizontal or vertical edges of an octagon face iff they correspond to edges of $\tilde{G}_L$ with a leftwards of rightwards arrow. This gives a unique dimer configuration on the expanded graph except on cities corresponding to vertices of type $i = 1$ in $\tilde{G}_L$, as shown in Figure C.5 $(b), (d)$,

where two possible dimer configurations occupy the city, one of which is non planar. It is known [**52**] that non-planar graphs G do not have in general a clockwise odd *Kasteleyn orientation* O such that the partition function of the dimer model on G can be written as $\sum_\theta \frac{c_\theta}{2} Pf(D_\theta)$, where Pf is the Pfaffian of the antisymmetric weighted adjacency matrix D_θ associated to the orientation O on the graph G^2, cf. [**52**]. Anyway, there exists an orientation O such that every non self intersecting loop in the expanded graph is indeed counterclockwise odd [**52**, Figure 8]: together with the free-fermion condition on the vertex weights and the local nature of the non planarity, this allows to obtain Z^{S}_{6V} in terms of $\sum_\theta \frac{c_\theta}{2} Pf(D_\theta)$ [**94**], where D_θ is the weighted adjacency matrix on the expanded graph weighted as in [**94**, Figure 5]. This is a generalization of the idea used by Kasteleyn [**52**] to obtain the Ising partition function in terms of dimers on similar non planar graphs[3].

[2] θ here represents the possible 4 boundary conditions (periodic-periodic, periodic-antiperiodic, etc.) as in, e.g., (2.1.1) and χ_θ are signs s.t. $\sum_\theta c_\theta = 1$.

[3] Here the free-fermion condition on the weights plays a crucial role for the argument to work.

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LIST OF FIGURES